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PERSPECTIVE

The stability problem for extremal black holes

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Abstract

I present a series of conjectures aiming to describe the general dynamics of the Einstein equations of classical general relativity in the vicinity of extremal black holes. I will reflect upon how these conjectures transcend older paradigms concerning extremality and near-extremality, in particular, the so-called "third law of black hole thermodynamics", which viewed extremality as an unattainable limit, and the "overspinning/overcharging" scenarios, which viewed extremality as a harbinger of naked singularities. Finally, I will outline some of the difficulties in proving these conjectures and speculate on what it could mean if the conjectures turn out not to be true.

Keywords Extremal black holes · Stability · Einstein equations

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1 Introduction

General relativists have long made their peace with "normal" black holes, even highly spinning ones. *Extremal black holes*, on the other hand, remain a source of extreme uneasiness. Though their name arises from the fact that they are extreme in the literal sense of the word, occupying the extreme end of the allowed parameter space, they turn out to be so also in the figurative sense, exhibiting properties "extreme" even by the standards of black hole physics, in particular *instabilities*. Indeed, both these two facets of the word "extreme" will be essential to the story I want to tell, and in fact it is precisely the coexistence of the two which is responsible for much of the story's rich complication—and the surrounding confusion. For, in this author's view at least, *there is no object more misunderstood in classical general relativity than extremal black holes!*

In this short essay, accompanying a talk given at the "Black holes: inside and out" conference held in August 2024 in Copenhagen, I will describe some recent progress which has conditioned my own—still provisional!—expectations for the general dynamics of the Einstein equations in a neighbourhood of extremal black holes, and I will try to organise these in a set of precisely stated conjectures. I emphasise that the conjectures may or may not be true (hence the word conjecture!), but they represent a mix of reasonable extrapolation from recently proven theorems with a dose of sheer wishful thinking. In particular, I will also discuss the various ways in which the conjectures may turn out to be false, a scenario which would be even more interesting but also more complex than anything previously entertained. Nonetheless, I will stick my head out and commit the conjectures to paper, because having a definite goal to prove or disprove provides a useful starting point for further rational study. If they do in fact turn out to be false, so much the better!

To set the "ground rules" of my discussion, let me say at the outset that I will remain firmly within the confines of *classical* general relativity, indeed, much of what I say will concern the Einstein *vacuum* equations alone, though sometimes it will be useful to also invoke well-established classical matter models, like the Einstein–Maxwell equations governing electrovacuum. I will assume moreover that my reader is familiar with the basic black hole solutions of these equations—those of Schwarzschild, Reissner–Nordström, Kerr and Kerr–Newman, including their extremal cases—and will refer to their standard properties without comment and without ever explicitly writing down the metric. I will also assume the reader has at least nodding familiarity with the fundamental principle that classical general relativity can be understood *dynamically*, i.e. it has a well-posed initial value problem, where an appropriate notion of Cauchy (or alternatively, characteristic) initial data gives rise to a unique solution of the equations of motion. For all this, the reader may refer to standard textbooks, for instance [1].

Of course, extremal black holes are important in considerations connected to *quantum* gravity. Indeed, it is common in interactions with my high energy physics colleagues that even before you finish your sentence they are conjuring up (in real time!) quantum effects that will change whatever classical story you were about to tell them. Let me appeal only for a little bit of the reader's patience! I have no doubt that other essays in this collection, reflecting the wide spectrum of talks at the Copenhagen conference, will dedicate ample space for quantum speculations of various sorts. My purpose here is simply to get the classical story right.

In a similar vein, I will also stay clear from any discussion of extremal or near extremal black holes in real astrophysical environments. Though these of course lie well within the domain of classical general relativity, astrophysics is messy and complicated, whereas the Einstein vacuum equations are clean and simple(r), and much more amenable to mathematical analysis. When it comes to issues of principle, the point of view of the present essay is: the cleaner, the better! I will thus for now leave aside speculation as to what implications the conjectural picture for the vacuum to be described here would have for more realistic astrophysical settings.

Though I hope to eventually convince the reader that the conjectures stated here are reasonable, indeed in some sense inevitable as the "minimalist" statements to hope for, I want to emphasise at the outset that they in fact would completely contradict two paradigms of how to think about extremality which have dominated the literature, what I will call the *third law paradigm* and the *overspinning/overcharging paradigm*. Indeed, I believe that these two—false, in my view!—paradigms have been the biggest hindrance to understanding the problem of extremality, and—*independently of the truth of the conjectures I will propose in their place*—in order to make further progress we must transcend both these paradigms. Thus, I will in fact begin my discussion already in Sect. 2 below from a description of these, followed in Sect. 3 by a critical analysis.

Finally, though the present reflections are in my own words (and thus any errors or unfortunate formulations here are mine alone!), the conjectured picture which I will attempt to sketch is very much influenced by the work of others (especially some surprising recent developments to be described in more detail below), and my own understanding arose in the context of many years of discussion, indeed sometimes vigorous debate, and collaboration. Let me mention, in addition to my own teacher Demetrios Christodoulou, especially the influence of Yannis Angelopoulos, Stefanos Aretakis, Dejan Gajic, Christoph Kehle, Jonathan Luk, Frans Pretorius, Harvey Reall, Rita Teixeira da Costa, Ryan Unger, Bob Wald and Claude Warnick, all of who have worked directly on aspects of this problem. I am of course greatly indebted to my collaborators Gustav Holzegel, Igor Rodnianski and Martin Taylor and have adapted Conjectures 6.1 and 6.2 from the discussion in Section IV.2 of our joint paper [2].

2 The two received paradigms

The study of extremal black holes over the last fifty years has been largely shaped by two distinct paradigms which—implicitly and explicitly—have dominated the way these objects are discussed in the literature. *I believe that both these paradigms are* *wrong*. In order to explain my expectations for the actual dynamics of the Einstein equations in a neighbourhood of extremal black holes, the first order of business is to describe these paradigms.

2.1 The third law paradigm: *extremal black holes as the unattainable*

The first paradigm, which I will dub the "third law paradigm", goes back to the seminal paper [3] of Bardeen, Carter and Hawking which explicitly initiated the thermodynamic analogy in black hole physics. I will not discuss here the history of this analogy (the first hints of which go back to Christodoulou's work [4]) or its incredible successes. I will focus entirely on the final part of the analogy they propose, what they call the third law of black hole mechanics, also known as the third law of black hole *thermodynamics*.

I recall of course that in the analogy with classical thermodynamics, *temperature* corresponds to *surface gravity*, which vanishes in the extremal case. Motivated by the "unattainability" formulation of the third law of thermodynamics, the authors of [3] put forth a conjectured "third law" for black holes, which in their own words read:

> *"It is impossible, by any procedure, no matter how idealized, to reduce [the surface gravity] κ to zero in a finite sequence of operations."*

The law was given a more precise formulation by Israel [5], who interpreted "finite sequence of operations" to mean "finite affine time along the event horizon", added the stipulation that the matter model be reasonable, and, importantly, the condition of regularity, because without the latter, there were already counterexamples, in fact, counterexamples he himself had constructed! Indeed, Israel seems to have explicitly related the existence of his "counterexamples" to failure of regularity—a point to which I will return in Sect. 3.1.

The interesting thermodynamic analogy aside, at first glance, the question of the "third law" per se, namely, whether one can produce an exactly extremal black hole in finite time, may seem like an academic one. After all, even if one could do this, it is clearly something very exceptional (hence the explicit emphasis "no matter how idealized" in the formulation!), not only because one must obtain exact extremality, but one then must keep the black hole at extremality for infinite time thereafter. The point, however, is that the analogous procedure—exceptional and idealised though it may be—is indeed theoretically realisable in the subextremal case, as one can see from elementary examples. The significance of the unattainability of extremality in finite time is thus best understood *relative to the subextremal (fixed spin-to-mass or charge-to-mass ratio) case.* I shall return to this point in Sect. 3.3 where I introduce the "generalised third law paradigm".

2.2 The overspinning/overcharging paradigm: *extremal black holes as the harbinger of super-extremal naked singularities*

The second paradigm I will discuss can also be said to originate from a quote from Bardeen, Carter and Hawking's paper [3]:

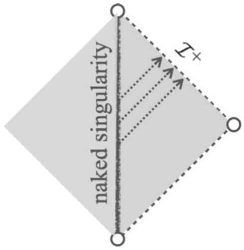

Fig. 1 The Penrose diagram of superextremal Kerr ($|a| > M$)

> *"Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity."*

Wald was the first to entertain this possibility in his paper [6]. What Wald showed, however, was that neither overspinning nor overcharging were possible in the "test particle" approximation.

Let us note that the idea that overspinning or overcharging a black hole has anything in principle to do with naked singularities in the first place arises from the fact that in the global Penrose diagram of *super-extremal* Kerr or Reissner–Nordström, there is indeed a naked singularity as in Fig. 1.

We will revisit this implicit connection later on in Sect. 6.2.

The story could have ended with Wald's work, but the idea of overspinning/overcharging proved too tempting to be given up so easily! The idea was revived in [7, 8], and more recently has been taken up again and again by many authors, the scenarios becoming more and more elaborate and the approximations used harder and harder to make sense of. I will not try to give a survey of these works, but distil from these what I will dub the "overspinning/overcharging paradigm", namely:

> *Near-extremal dynamics can indeed lead to the formation of a naked singularity as in super-extremal Kerr or Reissner–Nordström via an overspinning or overcharging mechanism.*

Of course, the formation of naked singularities—at least if their "nakedness" is moreover stable to perturbation—would contradict another well-known conjecture, Penrose's *weak cosmic censorship* conjecture [9]. Some overspinning/overcharging papers embrace the prospect of falsifying weak cosmic censorship, whereas others present a failed attempt at overspinning/overcharging as more evidence for the validity of the conjecture.

3 Questioning the paradigms?

In this section, I will attempt a first critical analysis of the two paradigms. Let me first dispose of the third law, for which a very definitive theorem can now be stated:

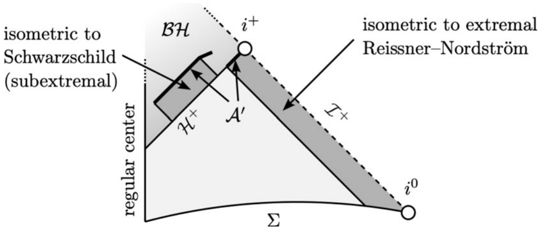

Fig. 2 The third-law violating spacetimes of Kehle–Unger, taken from [10] with permission

3.1 The death of the third law

There is in fact not much to say here, other than quote directly a remarkable recent theorem of Kehle and Unger:

Theorem 3.1 (Kehle–Unger [10]). *There exist regular one-ended Cauchy data for the Einstein-Maxwell-charged scalar field system which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Reissner-Nordström event horizon at a later advanced time.*

Thus, initially subextremal black holes can become extremal in finite time after all, evolving from regular initial data. The matter model is reasonable by all measures (there is now in fact a similar construction for the Einstein–Maxwell–charged Vlasov system [11]), and the process is completely regular, meeting all of the requirements of [5]. The "third law of black hole thermodynamics", as formulated in [3, 5], **is simply false!**

The Penrose diagram of the spacetimes constructed is given by Fig. 2.

I can't do justice to the history of the third law here, but I do want to point out one important point: In the Penrose diagram of Fig. 2, the thicker curve $\mathcal{A}'$ represents an outermost apparent horizon, and as noted, this is disconnected, the event horizon $\mathcal{H}^+$ containing an isolated component. Israel's attempted proof in [5] had implicitly used the global connectivity of an apparent horizon (here understood as a tube of marginally trapped surfaces).

(By unfortunate coincidence, in previous singular "counterexamples" discussed earlier, the existence of which indeed motivated Israel's explicit added assumption of regularity in [5], there were again discontinuities of the outermost apparent horizon which occurred exactly when the singular matter shells crossed it. This may have suggested that for sufficiently regular solutions, apparent horizons would necessarily be connected, something which is not in fact true!)

3.2 Status of the overspinning/overcharging paradigm

At first glance, the fact that extremal black holes can indeed be created in finite time may suggest that the overspinning/overcharging paradigm is all the more relevant. After all, it was precisely the menace of overspinning which provided the authors of [3] one of their motivations for proposing the third law in the first place.

Before discussing the status of the overspinning/overcharging paradigm, it may be useful to emphasise the following point of logic in comparison to the status of the third law: The third law had claimed the *impossibility* of something. Thus, it would have been potentially hard to prove (even had it been true, that is!), but was "easy" to disprove (given that it was false), as it was falsifiable by providing a single example. In contrast, overspinning/overcharging claims the *possibility* of something. Thus, the situation is reversed. It would be proveable by providing a single example, but to falsify one would require a very general kind of mathematical theorem, applying to an infinite dimensional set of solutions of the full non-linear Einstein equations.

In the restricted context of spherical symmetry, one does actually have such a result:

Theorem 3.2 ([12]). *For any reasonable matter model in spherical symmetry (e.g. charged scalar field [13], charged Vlasov [11]), then if spacetime contains at least one trapped or marginally trapped surface, then there are no naked singularities.*

It follows in particular from the above that one cannot create a naked singularity by overcharging either an exactly extremal or near extremal black hole in an entirely spherically symmetric process. (In particular, Theorem 3.2 can be thought to already give a definitive answer, in the negative, to the attempts at overcharging described in [7].)

As we shall see much later, the definitive disproof of overspinning in a very general, fully nonlinear setting *without symmetry* assumptions would follow as a corollary (Corollary 6.1) to a positive resolution of Conjecture 6.2, which I will propose in Sect. 6. But even if that conjecture turns out to be false, I claim (see already Sect. 6.4) that there is no sense that this could be reasonably interpreted as being due to "overspinning"! In any case, to arrive at these conjectures we essentially have to revisit the problem of the near extremal dynamics of the Einstein equations with a fresh lens.

3.3 A generalised third law paradigm?

Before proceeding, however, let us return to the issue of the third law. Though Theorem 3.1 definitively disproves the third law, one could wonder whether its "spirit", as captured in the last paragraph of Sect. 2.1 by the idea of extremality as representing "the (relative) unattainable", might somehow live on. To humour this idea, let me formulate what I will dub the "generalised third law paradigm":

> *Forming extremal black holes (whether in finite or infinite time, and whether starting from a subextremal black hole or directly in collapse) should somehow be more difficult than forming a black hole of any other fixed subextremal spin-to-mass or charge-to-mass ratio.*

Though the remarkable paper [10] put an end to the third law itself, it does not of course address the "generalised third law paradigm" above. Like with the overspinning/overcharging paradigm, such a statement can only be falsified by a general mathematical theorem pertaining to general dynamics of the Einstein equations in a neighbourhood of extremal Kerr, and no such statement is yet available. Indeed, I have conjured up this generalised third law paradigm precisely in order to give the "spirit" of the third law, as exemplified by the idea of "unattainability", a second fighting chance!

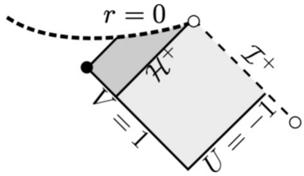

Fig. 3 Schwarzschild as a Cauchy evolution of characteristic initial data, taken from [2]

I must warn the reader already, however, that one who is still betting for this generalised third law paradigm does so at their own risk! In analogy with Theorem 3.2, there is in fact already some evidence in plain view—coming from a pioneering numerical study of Murata–Reall–Tanahashi [14]—that even this generalised third law paradigm is again false, at least when restricted to spherical symmetry. I will discuss this work a little bit later. Indeed, as with the overspinning/overcharging paradigm, a *definitive disproof* of the generalised third law paradigm would in fact follow from Conjecture 6.1 to be discussed later.

4 The Schwarzschild case: a blueprint for extremal Kerr?

To get a first glimpse of the conjectured picture of near extremal dynamics which I will describe in Sect. 6, it is useful to first examine the other "extremal" member of the Kerr family, namely Schwarzschild (extremal now only in the literal "boundary" sense as described in the opening paragraph of this essay, characterised by the rotational parameter taking its *minimum* possible modulus, i.e. vanishing!).

Concerning thus near-Schwarzschild dynamics, it is of course a truism that one can perturb Schwarzschild into the Kerr family by adding a little bit of angular momentum. Thus, Schwarzschild is not asymptotically stable in the strict sense, i.e. if we view Schwarzschild as the Cauchy evolution of "Schwarzschild initial data" prescribed say on two null cones $V = 1$ and $U = -1$ (here U, V are global double null coordinates covering Schwarzschild) as in Fig. 3, then the generic small perturbation of the data will evolve under the Einstein vacuum equations

$$\mathrm{Ric}(g) = 0 \tag{1}$$

to a spacetime which will not settle back down to a Schwarzschild metric. *What is then the best asymptotic stability statement that can be true about the Schwarzschild family?*

It turns out that at the linear level, one can explicitly identify "Kerr perturbations" as a 3-dimensional subfamily [15], spanned by a set of three quantities with fixed spherical harmonic frequency $\ell = 1$. (Note that the dimension is 3 and not 1 because to parametrise smoothly the moduli space near Schwarzschild, one must also encode the unique symmetry axis of $a \neq 0$ Kerr solutions.) Given a linear perturbation for which these three quantities vanish, then it was proven in [15] that the perturbation decays as time goes to infinity to a pure gauge solution, and moreover, gauge normalisations may be chosen ("teleologically", i.e. from the future!) so that this pure gauge solution in fact vanishes identically. This can be recast as the following statement:

In the infinite dimensional space of all linear vacuum perturbations around Schwarzschild, after suitable teleological double null gauge normalisations, there is a unique codimension-3 subspace such that all linear perturbations lying in this space in fact decay to zero as time goes to infinity.

The best fully nonlinear asymptotic stability statement concerning the Schwarzschild family that could be expected under evolution by (1) would then be an analogous nonlinear codimension-3 asymptotic stability statement. This is precisely the main result proven in [2]:

Theorem 4.1 [2] *For all vacuum characteristic initial data prescribed on cones as in Fig. 3, assumed sufficiently close to Schwarzschild data with mass M_{init} and lying on a codimension-3 "submanifold" $\mathfrak{M}_{\text{stable}}$ of the moduli space $\mathfrak{M}$ of initial data, the arising vacuum solution $(\mathcal{M}, g)$ satisfies the following properties:*

(i) $(\mathcal{M}, g)$ *possesses a complete future null infinity $\mathcal{I}^+$, and in fact the future boundary of $J^-(\mathcal{I}^+)$ in $\mathcal{M}$ is a regular, future affine complete event horizon $\mathcal{H}^+$.*
(ii) *The metric g remains close to the Schwarzschild metric with mass M_{init} in $J^-(\mathcal{I}^+)$.*
(iii) *The metric g asymptotes, inverse polynomially, to a Schwarzschild metric with mass $M_{\text{final}} \approx M_{\text{init}}$ as $u \to \infty$ and $v \to \infty$, in particular along $\mathcal{I}^+$ and $\mathcal{H}^+$, where u and v are suitably normalised null coordinates.*

Statement (i) asserts the absence of naked singularities, statement (ii) asserts "orbital stability", while statement (iii) is the statement of asymptotic stability, all for data on $\mathfrak{M}_{\text{stable}}$.

We note that, unlike linear theory, where the codimension-3 subspace is explicit (characterised by vanishing Kerr perturbations), the codimension-3 "submanifold" $\mathfrak{M}_{\text{stable}}$ above *is itself only teleologically determined* (as is the exact value of the final Schwarzschild mass M_{final} and the location of the event horizon $\mathcal{H}^+$). This is reminiscent of the fact mentioned earlier that already in linear theory, the gauge normalisations necessary to obtain decay were themselves determined only teleologically.

For discussion of stability results for Reissner–Nordström as a solution of Einstein–Maxwell, see [16]. For the fate of data as in Theorem 4.1 but *not* lying on $\mathfrak{M}_{\text{stable}}$ above—as expected, they settle down to a very slowly rotating (i.e. $|a| \ll M$) Kerr exterior—see [17]. See also [18] for nonlinear stability for black holes in the $\Lambda > 0$ case.

5 Instabilities

Could the above picture of Theorem 4.1 carry over directly to the extremal Kerr case $a = M$, i.e. can we simply replace *Schwarzschild* above by *extremal Kerr*? Before entertaining this issue, we have to deal with the elephant in the room, related to the other—figurative—meaning of the word "extreme", mentioned in the opening paragraphs of this essay: *Extremal black holes are characterised by extreme behaviour, in particular, by the presence of several well-known instabilities.*

5.1 The Aretakis instability

The Aretakis instability [19] is a remarkable very general property which can be associated to any extremal Killing horizon. It already applies to the linear massless wave equation

$$\Box_g \psi = 0, \tag{2}$$

but more generally applies to the linearised Einstein vacuum equations and Einstein–Maxwell equations, around extremal Kerr and extremal Reissner–Nordström respectively [20, 21]. According to this instability, translation-invariant transversal first derivatives of ψ (say $\partial_r \psi$) along the horizon $\mathcal{H}^+$ *fail to decay*, while second derivatives **blow up polynomially**

$$|\partial_r^2 \psi|(r, v) \to \infty \tag{3}$$

as advanced time v goes to infinity along $\mathcal{H}^+$. (Here (r, v) denote Eddington–Finkelstein type coordinates regular through $\mathcal{H}^+$.) The seed of the instability is an exact conservation law along the horizon $\mathcal{H}^+$, which turns out to have a similar origin to the Newman–Penrose constants at null infinity $\mathcal{I}^+$ (see [22]).

5.2 Weak stability for Reissner–Nordström

An important aspect of the Aretakis instability is that it is *weak*. The blow-up (3) along $\mathcal{H}^+$ is still compatible with good decay properties *away from the horizon* $\mathcal{H}^+$, and moreover, the amplitude of ψ itself (and its *tangential* derivatives, say $\partial_v \psi$) may still decay along $\mathcal{H}^+$ itself. Indeed, in the case of (2) on Reissner–Nordström, Aretakis [23, 24] already showed precisely such stability statements, complementing (3). For the full linearised Einstein–Maxwell system around extremal Reissner–Nordström, such weak stability statements, as well as upper bounds for the growth of the unstable, transversal quantities, were shown recently by Apetroaie [25].

5.3 Nonlinear model problems on fixed extremal Reissner–Nordström backgrounds

Of course, it is not clear at all whether weak stability results at the linear level are sufficient to ensure *nonlinear stability*. For instance, Aretakis has shown [26] that for nonlinear ("semilinear") wave equations of the form (for $n \geq 1$)

$$\Box_g \psi = \psi^{2n} + (\partial_v \psi)^{2n} + (\partial_r \psi)^{2n} \tag{4}$$

on fixed extremal Reissner–Nordström backgrounds, arbitrarily small spherically symmetric data lead to solutions which blow up in <u>finite</u> advanced time on the horizon $\mathcal{H}^+$, whereas it follows from [27] that in the subextremal case, for sufficiently high n, such solutions exist for all time on the black hole exterior, up to and including the horizon. (In (4), coordinate derivatives are again with respect to Eddington–Finkelstein coordinates (v, r) regular across $\mathcal{H}^+$.)

The equation (4) is not so good a model, however, for the nonlinearities occurring in the Einstein vacuum equations (1). A better model would perhaps be a nonlinear (again "semilinear") equation of the form

$$\Box_g \psi = A(\psi, x)\, g^{\alpha\beta} \partial_\alpha \psi\, \partial_\beta \psi, \tag{5}$$

where $A(0, x)$ is potentially nonzero and the nonlinearity is quadratic in first derivatives of ψ. In a remarkable series of works [28, 29], it was shown that solutions of (5) arising from sufficiently small initial data exist globally in the black hole exterior of extremal Reissner–Nordström, up to and including the horizon $\mathcal{H}^+$, despite the Aretakis instability (3).

We note that, even in the case where $A(\psi, x)$ vanishes for large x, the results of [29] depend on the fact that it is precisely the expression $g^{\alpha\beta} \partial_\alpha \psi\, \partial_\beta \psi$ which appears in (5), and not some other quadratic combination of second derivatives of ψ. (This is again in contrast to the subextremal Reissner–Nordström or Kerr case where, for such A, global existence holds replacing $g^{\alpha\beta} \partial_\alpha \psi\, \partial_\beta \psi$ with a general quadratic expression; see for instance [30].) The importance of structure for the quadratic terms is analogous to the well known fact that, if A does not vanish for large x, say if $A(\psi, x) = 1$ identically, then even if g denotes the Minkowski metric, while small data solutions of (5) indeed exist globally (as first observed by Nirenberg), replacing $g^{\alpha\beta} \partial_\alpha \psi\, \partial_\beta \psi$ with say $(\partial_t \psi)^2$ leads to blow up [31]. We will return to this point in Sect. 6.4.

5.4 Extremal Kerr and higher azimuthal instabilities

Turning to extremal Kerr, however, the situation is much less clear. Whereas restricting to axisymmetric solutions of (2), one has results analogous to that of extremal Reissner–Nordström (see [32]), a recent theorem of Gajic [33] proves that for fixed higher azimuthal m-mode solutions, *worse* instabilities arise, confirming a previous heuristic study by Casals, Gralla and Zimmerman [34]. Even for such fixed higher m-modes, however, it is not known whether weak stability statements analogous to those of [32] hold. Indeed, the only general statement presently known is so-called "mode stability", recently shown in the extremal case by Teixeira da Costa [35]. The situation for general data, which would concern the sum over *infinitely* many m-modes, is even more unclear. Thus, even the question of whether all solutions of the linear homogeneous scalar wave equation (2) on extremal Kerr, arising from regular localised initial data, remain bounded for all time (cf. [36–38] where this is shown in the general subextremal case $|a| < M$ for the wave and Teukolsky equations, respectively), even when one restricts considerations to a region well outside the horizon, remains very much open!

6 The stability conjecture for extremal black holes and the phase portrait of near extremal dynamics

The current uncertainty regarding linear theory on extremal Kerr described above might make it feel a bit premature to conjecture anything at the nonlinear level. Indeed,

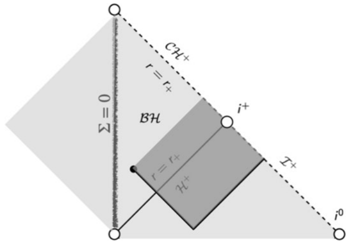

Fig. 4 Extremal Kerr as a Cauchy evolution of characteristic initial data

precisely for this reason, in our [2], we shied away from any such conjecture, preferring to volunteer a conjecture only for the Einstein–Maxwell theory around extremal Reissner–Nordström (see Conjecture IV.2 of [2] and the subsequent paragraph), for which, as described in Sect. 5.2, the linear theory is better understood. Nonetheless, I will here state the analogous conjectures in the extremal Kerr case.

6.1 Codimension-1 stability of extremal Kerr with horizon hair

As with Schwarzschild, we may view extremal Kerr as the Cauchy evolution of characteristic initial data posed on two null cones. Refer to Fig. 4. Unlike in the Schwarzschild case, however, when parametrising linear perturbations around extremal Kerr, non-extremal Kerr perturbations are spanned by a single quantity, not 3. Thus, if one wishes to show asymptotic stability of extremal Kerr, the natural conjecture would now be a codimension-1 nonlinear stability statement. At the same time, the instabilities described in Sects. 5.1 and 5.4 must at the very least also somehow appear. The most optimistic scenario is then:

Conjecture 6.1 *For all vacuum characteristic initial data prescribed on cones as in Fig. 4, assumed sufficiently close to extremal Kerr data with mass $M_{\text{init}} = a_{\text{init}}$ and lying on a codimension-1 "submanifold" $\mathfrak{M}_{\text{stable}}$ of the moduli space $\mathfrak{M}$ of initial data, the arising vacuum solution $(\mathcal{M}, g)$ satisfies the following properties:*

(i) *$(\mathcal{M}, g)$ possesses a complete future null infinity $\mathcal{I}^+$, and in fact the future boundary of $J^-(\mathcal{I}^+)$ in $\mathcal{M}$ is a regular, future affine complete event horizon $\mathcal{H}^+$.*
(ii) *The metric g remains close to the extremal Kerr metric with mass M_{init} in $J^-(\mathcal{I}^+)$.*
(iii) *The metric g asymptotes, inverse polynomially, to an extremal Kerr metric with mass $M_{\text{final}} = a_{\text{final}} \approx M_{\text{init}}$ as $u \to \infty$ and $v \to \infty$, in particular along $\mathcal{I}^+$ and $\mathcal{H}^+$, where u and v are suitably normalised null coordinates.*
(iv) *For generic initial data conditioned to lie on $\mathfrak{M}_{\text{stable}}$, then suitable quantities associated to derivatives of the metric grow without bound ("horizon hair") along $\mathcal{H}^+$ as $v \to \infty$.*

The statement is thus in complete analogy with Theorem 4.1, *except for the additional weak instability statement (iv).* For near extremal Kerr black holes with fixed spin-to-mass ratio, one would again expect a similar codimension-1 stability statement (without of course the instability part (iv)). In particular, the codimension in moduli space $\mathfrak{M}$ of the set $\mathfrak{M}_{\text{stable}}$ of spacetimes evolving to extremal Kerr would be *exactly the same* as those evolving to nearby fixed subextremal spin-to-mass ratio Kerrs. In

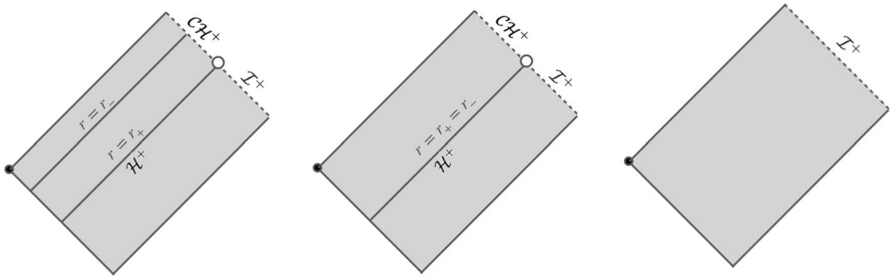

Fig. 5 The Kerr family as Cauchy evolutions of a smooth family characteristic initial data: the subextremal, extremal and superextremal cases, respectively

this sense, it would be no more difficult to form an extremal black hole than it would be a subextremal black hole of some other fixed spin-to-mass ratio. Thus, according to the above, there would be no saving even the "generalised" third law paradigm from the fate of the original third law.

In addition to the semilinear problem (4) discussed in Sect. 5.3, which concerned a fixed extremal Reissner–Nordström background, there is in fact a nonlinear self-gravitating model problem (i.e. where the spacetime is itself dynamical) where a much simplified analogue of the above conjecture can already be studied, namely the Einstein–Maxwell-scalar field system under spherical symmetry. Indeed, a numerical study of this system was conducted by Murata–Reall–Tanahashi [14], and the results reported are entirely consistent with a statement analogous to Conjecture 6.1. Proving these numerical results for the spherically-symmetric system would be an important first step towards a proof of the far more ambituous Conjecture 6.1. See also [39]. *(Note added: A proof of the analogue of Conjecture* 6.1 *for the above spherically symmetric system has in fact just been announced (see the upcoming* [40]).*)*

6.2 The phase portrait of near extremal dynamics

If Conjecture 6.1 is indeed true, then it makes sense to ask what nearby solutions *not lying on this codimension*-1 *hypersurface* $\mathfrak{M}_{\text{stable}}$ evolve to (cf. the comments after Theorem 4.1). It is instructive to first understand the Kerr family itself from the point of view of this initial value problem. Indeed, when we realise the Kerr family (through extremality!) as a smooth family of solutions evolving from initial data as in Conjecture 6.1, then we see that as we approach extremality, the event horizon and the "outgoing" component of the inner horizon coalesce into the extremal event horizon, which then disappears once we pass into the superextremal range. See Fig. 5. Note that the superextremal case has no naked singularity in the domain of development of the characteristic initial data depicted (see already Fig. 8 for the relation of this region to the global Penrose diagram). In this case, the solution lies entirely in $J^-(\mathcal{I}^+)$, but $\mathcal{I}^+$ is now incomplete, since after finite Bondi time observers see the final sphere of the initial ingoing cone.

I claim that the "minimalist" expectation consistent with what we know would be the following:

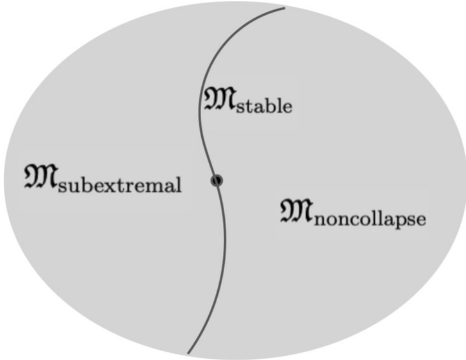

Fig. 6 The moduli space $\mathfrak{M}$ of characteristic initial data near extremal Kerr partitioned as in the statement of Conjecture 6.2

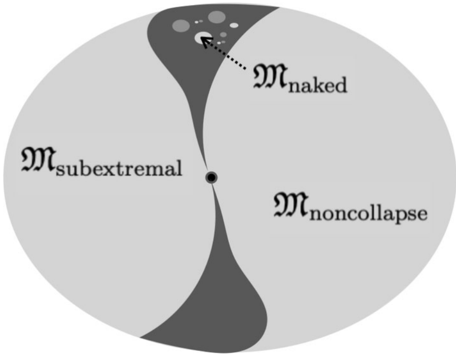

Fig. 7 A scenario where a chaotic region of the moduli space $\mathfrak{M}$ opens up from extremal Kerr, with alternating regions including a set $\mathfrak{M}_{\text{naked}}$ of data leading to naked singularities

Conjecture 6.2 *Under the assumptions of Conjecture 6.1, the codimension-*1 *"sub-manifold"* $\mathfrak{M}_{\text{stable}}$ *is in fact a regular hypersurface which separates the moduli space* $\mathfrak{M}$ *into two open regions, each with boundary* $\mathfrak{M}_{\text{stable}}$*: the set of initial data* $\mathfrak{M}_{\text{subextremal}}$ *evolving to subextremal black holes and the set of initial data* $\mathfrak{M}_{\text{noncollapse}}$ *not collapsing in the domain of dependence of the data, i.e. such that the domain of dependence of the data is entirely contained in* $J^-(\mathcal{I}^+)$ *(but with incomplete* $\mathcal{I}^+$*).*

The structure of the moduli space $\mathfrak{M}$ according to the above conjecture is depicted schematically in Fig. 6.

Let us note that a *necessary* condition for the above to hold is that for all data lying on $\mathfrak{M}_{\text{stable}}$, there is no (strictly) trapped surface in the domain of dependence of the data. For otherwise, by Cauchy stability any nearby solution would necessarily also have such a trapped surface, and thus, by Penrose type monotonicity arguments involving the Raychaudhuri equation (see for instance [1]), could not lie entirely in $J^-(\mathcal{I}^+)$.

Note how the three Kerr spacetimes in Fig. 5 provide explicit examples of data lying on $\mathfrak{M}_{\text{subextremal}}$, $\mathfrak{M}_{\text{stable}}$ and $\mathfrak{M}_{\text{noncollapse}}$, respectively. According to Conjecture 6.2, whereas for data lying in $\mathfrak{M}_{\text{subextremal}}$, the above development includes a complete null infinity $\mathcal{I}^+$, and in this sense describes the complete future of far away observers, in the case of data lying in $\mathfrak{M}_{\text{noncollapse}}$, null infinity $\mathcal{I}^+$ is incomplete. This is not because there is anything singular in the evolution, but simply because, just as for superextremal Kerr itself, the ingoing cone of the data themselves is incomplete, yet no horizon formed to shield the end of this cone from $J^-(\mathcal{I}^+)$. These solutions would in fact be smoothly extendible beyond a complete outgoing null cone emanating from the final sphere of the initial ingoing null cone.

In order to determine what happens "next" to far away observers in spacetimes evolving from data lying in $\mathfrak{M}_{\text{noncollapse}}$, one must "complete" the initial data. An even more ambitious conjecture, due to Kehle and Unger, considers the "larger" moduli space $\mathfrak{M}$ consisting now of the set of one-ended (complete) asymptotically flat initial data. Their conjecture asserts the existence of a codimension-1 submanifold $\mathfrak{M}_{\text{critical}} \subset \mathfrak{M}$ consisting of data evolving to spacetimes settling down to extremal black holes such that $\mathfrak{M}_{\text{critical}}$ *locally* separates $\mathfrak{M}$ into data evolving to spacetimes settling down to subextremal black holes and data evolving to spacetimes $(\mathcal{M}, g)$ with a complete $\mathcal{I}^+$ and no horizon, i.e. with $\mathcal{M} = J^-(\mathcal{I}^+)$, such that g moreover asymptotically settles down to the Minkowski metric. Thus, according to Kehle and Unger's conjecture, extremal black holes can arise at the threshold between black hole formation and dispersion. The authors dub this "extremal critical collapse", in analogy with the more familiar threshold naked singularity solutions which have been studied numerically in the "critical collapse" literature [41, 42], solutions which presumably lie on a separate codimension-1 submanifold of $\mathfrak{M}$ with a similar local separation property. We emphasise, however, that not all extremal black holes are threshold solutions! The data $\mathfrak{M}_{\text{noncollapse}}$ in Conjecture 6.2 can also be suitably completed as one-ended asymptotically flat data in $\mathfrak{M}$ so as to themselves evolve to black hole spacetimes, where the horizon $\mathcal{H}^+$ however does not lie in the domain of dependence of $\mathfrak{M}_{\text{noncollapse}}$. In this case, crossing the submanifold $\mathfrak{M}_{\text{stable}}$ from $\mathfrak{M}_{\text{subextremal}}$ to $\mathfrak{M}_{\text{noncollapse}}$ would correspond not to the horizon $\mathcal{H}^+$ disappearing but to the location of the horizon "jumping". For more details, see [11].

6.3 How would Conjecture 6.2 disprove "overspinning"?

As we have pointed out already, just as in Theorem 4.1, the set $\mathfrak{M}_{\text{stable}}$ of Conjecture 6.2 depicted in Fig. 6 would only be teleologically defined. Thus, in general one would not know which "side" of the $\mathfrak{M}_{\text{stable}}$ hypersurface an initial data set lies on without evolving the data to the future. There would however be an obvious *sufficient* condition on initial data that ensures that the data indeed lie in $\mathfrak{M}_{\text{subextremal}} \cup \mathfrak{M}_{\text{stable}}$, namely the existence of at least one trapped or marginally trapped surface. This is because by Penrose type monotonicity arguments mentioned already in Sect. 6.2, such a surface could not lie in the past of $\mathcal{I}^+$. Thus, we may infer from a positive resolution of Conjecture 6.2 the following Corollary:

Corollary 6.1 (Given Conjecture 6.2). *Consider a vacuum solution that contains a Cauchy hypersurface with a marginally trapped surface such that outside that surface, the spacetime is close to extremal Kerr in a suitable sense. Then the spacetime has a black hole bounded by a regular event horizon $\mathcal{H}^+$ and with complete null infinity $\mathcal{I}^+$. In particular, no naked singularity forms.*

To say it more colloquially, no matter what small incoming gravitational radiation one tries to "throw" into an initially exactly extremal or slightly subextremal black hole, one can in particular never produce a naked singularity. The spacetime will settle back down to a subextremal or extremal black hole. Thus, Corollary 6.1, if indeed true, can be viewed as definitively disproving the "overspinning" paradigm.

An analogous statement to Corollary 6.2 replacing Kerr with Kerr–Newman, and the vacuum equations with those of a suitable self-gravitating charged matter model, would thus similarly definitively disprove "overcharging" scenarios. (One can also of course consider overspinning more generally for Einstein-matter systems, where the analogous statement can again be conjectured, but we emphasise that overspinning, unlike overcharging, is an issue which may be studied for the vacuum equations (1).)

One might ask what about "large perturbations" of extremal Kerr? Remember, however, that in nonlinear theories, *large perturbations are no longer "perturbations"*! If there is some self-gravitating system which collapses to form a naked singularity, then it will presumably still form a naked singularity if the whole system is "thrown" into a large extremal black hole. This would have nothing to do with overspinning/overcharging.

6.4 What if Conjectures 6.1 and 6.2 are not true?

Already Conjecture 6.1 is based on two pieces of wishful thinking: Firstly, that the linear instabilities of extremal Kerr described in Sect. 5.4, though known to be *stronger* than those of extremal Reissner–Nordström, are still in some sense weak, and in particular are still complemented by *stability* statements away from the horizon (and, for tangential quantities, along the horizon). And secondly, that the nonlinearities in the Einstein vacuum equations (1) near the horizon are indeed well modelled by (5). The worse the linear behaviour around extremal Kerr turns out to be, the more compensating the nonlinear structure of (1) near the horizon would have to be for there to be any hope to prove Conjecture 6.1.

In view of the fact that there is a lot of uncertainty as to all of the above, we should already also entertain the possibility that Conjectures 6.1 and 6.2 may turn out to be **false**. *What could the situation look like in that case?*

Perhaps the most exciting possibility to entertain is that the linear instabilities described in Sect. 5.4 indeed lead in the full nonlinear theory to the formation of naked singularities.

In this case, we may further distinguish two scenarios: (i) the case where such naked singularities occur only for an exceptional non-generic set of data, in which case "weak cosmic censorship" would be saved and in fact a modified version of Conjecture 6.2 could still hold, with a codimension-1 submanifold $\mathfrak{M}_{\text{threshold}}$ in place of $\mathfrak{M}_{\text{stable}}$, now containing both all data $\mathfrak{M}_{\text{extremal}}$ evolving to extremal Kerr but also all data $\mathfrak{M}_{\text{naked}}$ leading to naked singularities, and (ii) the case where these naked singularities are in fact stable and thus the set $\mathfrak{M}_{\text{naked}}$ of data leading to naked singularities would have non-empty interior in $\mathfrak{M}$.

Indeed, scenario (ii) would be the even more exciting one, as it would in particular falsify *weak cosmic censorship*. One can for instance imagine a picture of the moduli

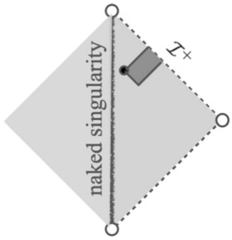

Fig. 8 The evolution of the initial data of Fig. 5 in the superextremal case, superimposed on the spacetime of Fig. 1

space $\mathfrak{M}$ as depicted in Fig. 7. Here, a chaotic region "opens up" from the extremal Kerr solution, containing infinitely many disconnected components of $\mathfrak{M}_{\text{subextremal}}$ and $\mathfrak{M}_{\text{noncollapse}}$ alternating with a (possibly disconnected) set $\mathfrak{M}_{\text{naked}}$ of open interior consisting of data evolving to naked singularities (depicted say as the lightest shaded regions), all surrounded by a complicated residual set $\mathfrak{M}_{\text{extremal}}$ of data evolving to extremal Kerr.

Would any of the above scenarios vindicate the original generalised third law paradigm or the overspinning/overcharging paradigms?

Concerning the generalised third law paradigm, if the scenario was indeed something as depicted in Fig. 7, then, clearly the "residual" set $\mathfrak{M}_{\text{extremal}}$ of data evolving to extremal black holes would have a very different geometry from the analogous set for fixed subextremal spin-to-mass ratio. It isn't clear, however, even in this scenario, whether $\mathfrak{M}_{\text{extremal}}$ would be "smaller" (in the sense of codimension) than the set of analogous subextremal fixed spin-to-mass ratio ones. It could of course indeed be smaller, but, in principle, it could even be that $\mathfrak{M}_{\text{extremal}}$ is in fact larger, filling part of the darker shaded region, say as a fractional codimension subset.

Concerning the overspinning paradigm, whereas according to Fig. 7, naked singularities would indeed occur in an arbitrary small neighbourhood of extremal Kerr, these naked singularities would have nothing to do with the naked singularities of superextremal Kerr, and in fact, they would not arise from "overspinning" *but precisely from trying to preserve extremality as closely as possible.*

Indeed, the connection of "overspinning" with the "naked singularity" of Fig. 1 in the first place was always based on what I believe is simply a misreading of the Penrose diagram, and it is perhaps useful to elaborate some more on this point. If the "overspinning" scenario really envisions a "quasistationary" transition from subextremal to superextremal, then the only relevant regions would be the regions of Fig. 5, as these are the domain of dependence of the relevant initial data. (See also Fig. 8 where this region is superimposed on Fig. 1 in the superextremal case.) For as long as the solution indeed remains near the Kerr family in this domain, the monotonicity of Raychaudhuri's equation would clearly exclude such a transition, just as in the proof of Corollary 6.1. One often reads that, if a naked singularity were to form, one cannot apply this monotonicity, hence evading the argument. For this, however, *long before this purported naked singularity* has formed, one would have to have evolved *far from the Kerr family*, thus thwarting the premise of this having anything to do with "overspinning" in the first place.

But to humour "overspinning" even more, let us even suppose that the Einstein equations didn't happen to enjoy the monotonicity of the Raychaudhuri equation, and something like a quasistationary transition were indeed possible from subextremal to superextremal. For instance, redefine the energy momentum tensor of a usual matter model to be its negative and consider the resulting Einstein-matter system, and assume the validity of Conjecture 6.2. Corollary 6.1 would not now follow, and indeed, initial data with a marginally trapped surface could in principle now be contained in the set $\mathfrak{M}_{\text{noncollapse}}$ of Conjecture 6.2. This in turn could mean that at the final outgoing cone emanating from the data, the solution would be close to an outgoing cone of superextremal Kerr. (Refer again to Fig. 5 or Fig. 8.) This would indeed represent in some sense successful "overspinning"! But what of it? This would be in no sense inconsistent with the solution subsequently dispersing in the future, or recollapsing later to a sub-extremal (or even exactly extremal!) black hole, as angular momentum can happily radiate to infinity (cf. the fate of initially superextremal data in [11]). Thus, even in the absence of the Einstein equations' monotonicity properties, in no scenario is there a "mechanism" by which the naked singularity of Fig. 8 becomes relevant for the evolution of data as in Conjectures 6.1 and 6.2.

In summary, even if it turns out that, in the complexity of Fig. 7, extremal black holes are indeed "more exceptional" than fixed spin-to-mass subextremal ones, or that naked singularities do arise in a neighbourhood of extremality, the considerations leading to this would be entirely different from those underlying the generalised third law and the overspinning/overcharging paradigms.

So as not to end on such a note of criticism, however, and lest I give the impression that I believe the legacy of these two received paradigms to be entirely negative, let me say one important thing in their favour: Both these paradigms, in their slightly contradictory ways, did succeed nonetheless in drawing attention early on to the importance—and the potential complications—of near extremal dynamics, in a period where the stability considerations of black holes in general, even subextremal ones, were not yet well understood. This is unquestionably a positive historical legacy, despite whatever confusion accompanied it. As should be clear, however, going forward, I don't find the paradigms particularly useful any more, and the considerations that led to their formulation are in my view completely transcended by the considerations discussed in Sect. 5. Indeed, irrespectively of whether one is "rooting" for Conjectures 6.1 and 6.2, or one is attracted to the more complicated alternative scenarios described in this section, I hope that I have convinced the reader that the fundamental issue which will determine what is true can be nothing other than the *precise analysis of the interaction between the instabilities of Sects. 5.2 and 5.4 and the nonlinearities of the Einstein equations*.

7 Epilogue

I opened this article remarking how the two natures of the word "extreme", its literal and its figurative sense, are both central to the story of extremal black holes. I hope the conjectural picture of Sect. 6 succeeded in describing a scenario where these two facets of extremality may "peacefully" coexist, Conjecture 6.2 capturing its literal "boundary" aspect, while the instability statement (iv) of Conjecture 6.1 (the "horizon hair") capturing one figurative aspect of these black holes' extreme behaviour. We must not forget, however, that this is the optimistic scenario. The "coexistence" might turn out to be a lot more chaotic, and Fig. 7 gives just one idea of what we might have to come to terms with if these conjectures turn out to be false. While I hope the conjectures described here provide a fruitful framework for further study, given the surprising twists and turns of the story of extremal black holes so far, it is extremely unlikely that anything written here will end up being the last word!

References

1. Wald, R.M.: General Relativity. University of Chicago press, Chicago (2010)
2. Dafermos, M., Holzegel, G., Rodnianski, I., Taylor, M.: (2021). arXiv:2104.08222 [gr-qc]
3. Bardeen, J.M., Carter, B., Hawking, S.W.: Commun. Math. Phys. **31**, 161 (1973)
4. Christodoulou, D.: Phys. Rev. Lett. **25**, 1596 (1970)
5. Israel, W.: Phys. Rev. Lett. **57**, 397 (1986)
6. Wald, R.: Ann. Phys. **82**, 548 (1974)
7. Hubeny, V.E.: Phys. Rev. D **59**, 064013 (1999). https://doi.org/10.1103/PhysRevD.59.064013
8. Jacobson, T., Sotiriou, T.P.: Phys. Rev. Lett. **103**, 141101 (2009)
9. Penrose, R.: Structure of Space-Time. In Penrose, R., DeWitt, C. M. (eds.), Battelle Rencontres, p 222. W. A. Benjamin, New York (1968)
10. Kehle, C., Unger, R.: (2022). arXiv:2211.15742 [gr-qc]
11. Kehle, C., Unger, R.: (2024). arXiv:2402.10190 [gr-qc]
12. Dafermos, M.: Class. Quantum Gravity **22**, 2221 (2005)
13. Kommemi, J.: Commun. Math. Phys. **323**, 35 (2013)
14. Murata, K., Reall, H.S., Tanahashi, N.: Class. Quantum Gravity **30**, 235007 (2013)
15. Dafermos, M., Holzegel, G., Rodnianski, I.: Acta Math. **222**, 1 (2019). https://doi.org/10.4310/ACTA.2019.v222.n1.a1
16. Giorgi, E.: Commun. Math. Phys. **380**, 1313 (2020)
17. Giorgi, E., Klainerman, S., Szeftel, J.: (2022). arXiv:2205.14808 [gr-qc]
18. Hintz, P., Vasy, A.: Acta Math. **220**, 1 (2018). https://doi.org/10.4310/ACTA.2018.v220.n1.a1
19. Aretakis, S.: Adv. Theor. Math. Phys. **19**, 507 (2015). arXiv:1206.6598 [gr-qc]
20. Lucietti, J., Reall, H.S.: Phys. Rev. D **86**, 104030 (2012). https://doi.org/10.1103/PhysRevD.86.104030
21. Lucietti, J., Murata, K., Reall, H.S., Tanahashi, N.: J. High Energy Phys. **2013**, 35 (2013). https://doi.org/10.1007/JHEP03(2013)035
22. Aretakis, S.: Ann. PDE **3**, 3 (2017)
23. Aretakis, S.: Commun. Math. Phys. **307**, 17 (2011)
24. Aretakis, S.: Ann. Henri Poincaré **12**, 1491 (2011)
25. Apetroaie, M.A.: (2022). arXiv:2211.09182 [gr-qc]
26. Aretakis, S.: Phys. Rev. D **87**, 084052 (2013)
27. Dafermos, M., Rodnianski, I.: Journal de mathématiques pures et appliquées **84**, 1147 (2005)
28. Angelopoulos, Y.: Int. Math. Res. Not. **2016**, 3279 (2016)
29. Angelopoulos, Y., Aretakis, S., Gajic, D.: Ann. PDE **6**, 1 (2020)
30. Dafermos, M., Holzegel, G., Rodnianski, I., Taylor, M.: (2022). arXiv:2212.14093 [gr-qc]
31. John, F.: Manuscr. Math. **28**, 235 (1979)
32. Aretakis, S.: J. Funct. Anal. **263**, 2770 (2012)

33. Gajic, D.: (2023). arXiv:2302.06636 [gr-qc]
34. Casals, M., Gralla, S.E., Zimmerman, P.: Phys. Rev. D **94**, 064003 (2016)
35. Teixeira da Costa, R.: Commun. Math. Phys. **378**, 705 (2020)
36. Dafermos, M., Rodnianski, I., Shlapentokh-Rothman, Y.: Ann. Math. **183**, 787 (2016)
37. Shlapentokh-Rothman, Y., da Costa, R.T.: (2020). arXiv:2007.07211 [gr-qc]
38. Shlapentokh-Rothman, Y., da Costa, R.T.: (2023). arXiv:2302.08916 [gr-qc]
39. Gajic, D., Luk, J.: Pure Appl. Anal. **1**, 263 (2019)
40. Angelopoulos, Y., Kehle, C., Unger, R.: (2024). arXiv:2410.16234
41. Choptuik, M.W.: Phys. Rev. Lett. **70**, 9 (1993)
42. Gundlach, C., Martin-Garcia, J.M.: Living Rev. Relativ. **10**, 1 (2007)