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The structure of the $\mathcal{N}=4$ supersymmetric linear $W_\infty[\lambda]$ algebra

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Abstract For the vanishing deformation parameter λ, the full structure of the (anti)commutator relations in the $\mathcal{N}=4$ supersymmetric linear $W_\infty[\lambda=0]$ algebra is obtained for arbitrary weights h_1 and h_2 of the currents appearing on the left hand sides in these (anti)commutators. The $w_{1+\infty}$ algebra can be seen from this by taking the vanishing limit of other deformation parameter q with the proper contractions of the currents. For the nonzero λ, the complete structure of the $\mathcal{N}=4$ supersymmetric linear $W_\infty[\lambda]$ algebra is determined for the arbitrary weight h_1 together with the constraint $h_1-3\leq h_2\leq h_1+1$. The additional structures on the right hand sides in the (anti)commutators, compared to the above $\lambda=0$ case, arise for the arbitrary weights h_1 and h_2 where the weight h_2 is outside of above region.

Contents

On the occasion of my thirtieth Ph.D. anniversary.

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1 Introduction and outlook

In the description of [1,2], the conformal weights of the free bosonic and fermionic operators in the two dimensional conformal field theory do depend on the deformation parameter λ. See also the relevant work [3] for the role of this λ in the similar two dimensional model. The bosonic and fermionic currents made from the above free fields quadratically by including the multiple derivatives have integer or half integer weights. The algebra from these currents has the λ dependent structure constants on the right hand sides of the (anti)commutator relations because the relative coefficients between the free fields in the expression of the currents reveal the λ dependence in nontrivial way. By construction in [1,2], it is a new feature, compared to the previous construction by using free fields (for example, [4–6]), that there exist the bosonic current of weight-1 and the fermionic current of weight-$\frac{1}{2}$. The multiple number of free bosonic and fermionic operators can be introduced. Then it is straightforward to write down the corresponding algebra because the defining operator product expansions(OPEs) between these multiple free fields satisfy independently. See also the relevant work in [7] without a deformation parameter we mentioned above.

The structure of the $\mathcal{N} = 2$ supersymmetric linear $W_{\infty}^{K,K}[\lambda]$ algebra where K is the number of complex bosons or the number of complex fermions is found in [8]. The structure constants for vanishing deformation parameter are generalized to the ones for nonzero λ. However, it turns out that there are some constraints in the weights for the currents on the left hand sides of the algebra according to the result of this paper. Of course, when we go from the algebra (i) at $\lambda = 0$ to the algebra (ii) at $\lambda \neq 0$, once the corresponding structure constants in the latter which lead to the ones in the former when we take $\lambda \to 0$ are determined, then definitely we expect to have the generalized structure constants for nonzero λ in the algebra ii) via above transition. This is the simplest deformed algebra.

On the other hand, suppose that there are some "additional" structure constants having the factor λ in the (new or known) currents appearing in the latter algebra (ii). Then after we take the above $\lambda \rightarrow 0$ limit in this algebra (ii), we still have the same former algebra (i) at $\lambda = 0$ because these "additional" terms vanish in this limit. This is another deformed algebra. Therefore, it is nontrivial to determine the "additional" current terms having the λ factor explicitly for general h_1 and h_2. They can appear as either the new currents with the coefficients having the λ factor or previously known currents with the structure constants having the λ factor explicitly.

Recently [9], by considering the particular number $K = 2$ of free fields above, the $\mathcal{N} = 4$ supersymmetric linear $W_\infty[\lambda]$ algebra is studied. For low weights, the explicit OPEs between the currents of $\mathcal{N} = 4$ multiplet are obtained. All the structure constants appearing on the right hand sides of these OPEs are given in terms of the above deformation parameter λ explicitly but the general behavior of those on the weights is not known.

In this paper, we continue to study the structure of the $\mathcal{N} = 4$ supersymmetric linear $W_\infty[\lambda]$ algebra found in [9]. We try to determine the $\mathcal{N} = 4$ multiplet for any weight h in terms of above free field operators. The corresponding five components are determined explicitly for arbitrary weight h. After the explicit form of the $\mathcal{N} = 4$ multiplet is obtained, we perform their OPEs by using the defining OPEs between the free fields. In doing this, we should use the previous results in [8] for the OPEs between the nonsinglet currents, as an intermediate step. We will observe the appearance of the extra structures (described before) on the right hand sides of the (anti)commutators for the particular weights h_1 and h_2. Eventually we will determine the (anti)commutator relations between the currents of the $\mathcal{N} = 4$ multiplet, as in the abstract. In the various footnotes, we emphasize that the extra structures on the right hand sides of the (anti)commutators arise for the specific h_1 and h_2.

In Sect. 2, we construct the $\mathcal{N} = 4$ multiplet in terms of free fields for general weight h. In Sect. 3, the explicit (anti)commutator relations between the nonsinglet currents for the weights h_1 and h_2 satisfying some constraints are obtained. In Sect. 4, the fundamental commutator relations between the $\mathcal{N} = 4$ multiplets of $SO(4)$ singlets or nonsinglets are determined. In Appendices, the details appearing in Sects. 3 and 4 are described. In particular, the remaining ten (anti)commutators for the $\mathcal{N} = 4$ multiplet are given here. We are using the Thielemans package [10] with a mathematica [11].

We summarize what we have obtained as follows. At the vanishing deformation parameter $\lambda = 0$, the complete structure of the $\mathcal{N} = 4$ supersymmetric linear $W_\infty[\lambda = 0]$ algebra is given by (4.4), (4.6), (4.8), (4.9) and (4.10) in addition to Appendices (G.1), (G.2), (G.3), (G.5), (G.6), (G.9), (G.10), (G.11), (G.13) and (G.14) where all the λ dependence appearing on the right hand sides is gone by putting $\lambda = 0$. Compared to the previous work in [12], due to the presence of the weights 1 and $\frac{1}{2}$ currents mentioned before, in general, they do appear on the right hand sides of above (anti)commutators at $\lambda = 0$. However, in the most of the examples, the structure constants appearing in these currents are vanishing at $\lambda = 0$. See also the footnotes in Sects. 3 and 4. It is an open problem whether the present algebra at $\lambda = 0$ can be reduced to the one of [12] by decoupling the above weights 1 and $\frac{1}{2}$ currents. For nonzero λ, still the above (anti)commutator relations between the currents can be used for the weights h_1 and h_2 satisfying some constraints. From the analysis in the footnotes of Sect. 4, this algebra at nonzero λ is different from the one in [13]. In other words, they have common algebra at $\lambda = 0$ (their structure constants are the same) and for nonzero λ, one deformed algebra is given by [13] and another one is the algebra obtained in this paper.

Then what happens for the generic weights h_1 and h_2? First of all, the $\mathcal{N} = 4$ supersymmetric linear $W_\infty[\lambda]$ algebra is linear in the sense that the right hand sides of the OPEs between the currents contain all the possible current terms which are linear. Furthermore, the $\mathcal{N} = 4$ multiplet is given by (2.7), (2.8), (2.9), (2.10) and (2.11). Then we can perform any OPE inside the Thielemans package [10] using the explicit forms of the $\mathcal{N} = 4$ multiplet. Of course, from the beginning we should fix the weights h_1 and h_2 within the package. Then the possible poles are given by the highest singular term which is $(h_1 + h_2)$th order pole, the next singular term which is $(h_1 + h_2 - 1)$th order pole, and so on until the first order pole. Then the next step is to rewrite all the singular terms in terms of the components of the $\mathcal{N} = 4$ multiplet of $SO(4)$ nonsinglets or singlets. This is straightforward because the possible current terms at the particular singular term are known. The weight h is fixed and the possible currents can be determined. They consist of the current having that weight h, the current having the weight $(h - 1)$ with one derivative, ..., and the current having the weight 1 with $(h - 1)$ derivative. See also Appendix C for the specific examples.

Now we introduce the arbitrary coefficients on these possible current terms and solve the linear equations for these unknown coefficients by requiring that the algebra is closed in the sense that the right hand side of the OPE contains the components of $\mathcal{N} = 4$ multiplet. It will turn out that they can be determined explicitly in terms of the deformation parameter λ. The point is, according to the result of this paper, that there exists a critical singular term where the possible current of weight h_c is allowed. If $h \leq h_c$, then we expect to have the additional structures (either the presence of new currents or the different λ dependent structure constants in the previously known currents) on the right hand side of the OPE. In this case, although the structure constants are known in terms of the λ for fixed h_1 and h_2, their explicit expressions for generic h_1 and h_2 are not known so far. On the other hand, if $h \geq h_c$, then still we can use the previous (anti)commutator relations described in the previous paragraph, for nonzero λ without any modifications. In this case, all the structure constants are known and they are given by those in Appendix A.

Let us list some future directions along the line of the present paper.

- The $\mathcal{N}=4$ superspace OPE.

Although we have found 15 (anti)commutator relations explicitly, it is nice to observe its $\mathcal{N}=4$ superspace description. In order to perform the $\mathcal{N}=4$ superspace approach, we need to rewrite the above fundamental OPEs in (4.4), (4.6), (4.8), (4.9) and (4.10) such that the second element with the coordinate w on the left hand side of the OPE should be the lowest component of the $\mathcal{N}=4$ multiplet. That is, they are given by $\big[(\Phi_0^{(h_1)})_m, (\Phi_0^{(h_2)})_n\big]$, $\big[(\Phi_{\frac{1}{2}}^{(h_1),i})_r, (\Phi_0^{(h_2)})_n\big]$, $\big[(\Phi_1^{(h_1),ij})_m, (\Phi_0^{(h_2)})_n\big]$, $\big[(\Phi_{\frac{3}{2}}^{(h_1),i})_r, (\Phi_0^{(h_2)})_n\big]$, and $\big[(\Phi_2^{(h_1)})_m, (\Phi_0^{(h_2)})_n\big]$ in the commutators. After these are obtained, then it is straightforward to express them in the $\mathcal{N}=4$ superspace. For consistency check, it is obvious to extract the remaining 10 (anti)commutator relations (or its corresponding OPEs) from the above $\mathcal{N}=4$ superspace description. In other words, we do not have to calculate the remaining (anti)commutators separately and this is the power of $\mathcal{N}=4$ supersymmetry.

- The complete structure constants for any h_1 and h_2 for nonzero λ.

One way to determine these is that it is better to consider the modes of the currents in terms of those in the free fields. By simplifying the (anti)commutator relations in terms of the modes of the free fields, we can express them in terms of several (anti)commutator relations according to the decomposition of the (anti)commutator in quantum mechanics. Then we can use the corresponding (anti)commutator relations for (2.1). We can try to obtain the general structure by fixing the weights h_1 and h_2. It is rather nontrivial to determine the structure constants for generic h_1 and h_2 by varying them.

- Realization of the present algebra in the celestial conformal field theory.

At the vanishing deformation parameter $\lambda=0$, the algebra is known completely. In other words, the structure constants are given in Appendix A by inserting the $\lambda=0$. We are left with another deformation parameter q. As described before, we realize that the $w_{1+\infty}$ algebra can be obtained from the $SO(4)$ singlet currents via proper contractions of the currents with vanishing q limit at $\lambda=0$. It would be interesting to observe whether there exists any realization of the present algebra in the celestial conformal field theory, along the line of [14–20], or not.

2 The $\mathcal{N}=4$ multiplet

The $\mathcal{N}=4$ multiplet for any weight h is described by using the free bosonic and fermionic fields.

2.1 Review

The $\beta\,\gamma$ and $b\,c$ systems satisfy the following operator product expansions

$$\gamma^{i,\bar{a}}(z)\,\beta^{\bar{j},b}(w) = \frac{1}{(z-w)}\,\delta^{i\bar{j}}\,\delta^{\bar{a}b}+\cdots\,,$$
$$c^{i,\bar{a}}(z)\,b^{\bar{j},b}(w) = \frac{1}{(z-w)}\,\delta^{i\bar{j}}\,\delta^{\bar{a}b}+\cdots\,. \tag{2.1}$$

The fundamental indices a, b of $SU(2)$ run over $a, b = 1, 2$ while the antifundamental indices $\bar{a}, \bar{b}$ of $SU(2)$ run over $\bar{a}, \bar{b} = 1, 2$. Similarly the fundamental indices i, j of $SU(N)$ run over $i, j = 1, 2, \ldots, N$ and the antifundamental indices $\bar{i}, \bar{j}$ of $SU(N)$ run over $\bar{i}, \bar{j} = 1, 2, \ldots, N$. The (β, γ) fields are bosonic operators and the (b, c) fields are fermionic operators.

Then the $SU(N)$ singlet currents (the generalization of [1,2]) can be obtained by taking the bilinears of above free fields with a summation over the (anti)fundamental indices of $SU(N)$ as follows:

$$\begin{aligned}
V^{(h)+}_{\lambda,\bar{a}b} &= \sum_{i=0}^{h-1} a^i(h,\lambda)\,\partial^{h-1-i}\,((\partial^i\,\beta^{\bar{l}b})\,\delta_{l\bar{l}}\,\gamma^{l\bar{a}}) \\
&\quad + \sum_{i=0}^{h-1} a^i\left(h,\lambda+\frac{1}{2}\right)\,\partial^{h-1-i}\,((\partial^i\,b^{\bar{l}b})\,\delta_{l\bar{l}}\,c^{l\bar{a}})\,, \\
V^{(h)-}_{\lambda,\bar{a}b} &= -\frac{(h-1+2\lambda)}{(2h-1)}\sum_{i=0}^{h-1} a^i(h,\lambda)\,\partial^{h-1-i}\,((\partial^i\,\beta^{\bar{l}b})\,\delta_{l\bar{l}}\,\gamma^{l\bar{a}}) \\
&\quad + \frac{(h-2\lambda)}{(2h-1)}\sum_{i=0}^{h-1} a^i\left(h,\lambda+\frac{1}{2}\right)\,\partial^{h-1-i}\,((\partial^i\,b^{\bar{l}b})\,\delta_{l\bar{l}}\,c^{l\bar{a}}), \\
Q^{(h)+}_{\lambda,\bar{a}b} &= \sum_{i=0}^{h-1} \alpha^i(h,\lambda)\,\partial^{h-1-i}\,((\partial^i\,\beta^{\bar{l}b})\,\delta_{l\bar{l}}\,c^{l\bar{a}}) \\
&\quad - \sum_{i=0}^{h-2} \beta^i(h,\lambda)\,\partial^{h-2-i}\,((\partial^i\,b^{\bar{l}b})\,\delta_{l\bar{l}}\,\gamma^{l\bar{a}}), \\
Q^{(h)-}_{\lambda,\bar{a}b} &= \sum_{i=0}^{h-1} \alpha^i(h,\lambda)\,\partial^{h-1-i}\,((\partial^i\,\beta^{\bar{l}b})\,\delta_{l\bar{l}}\,c^{l\bar{a}}) \\
&\quad + \sum_{i=0}^{h-2} \beta^i(h,\lambda)\,\partial^{h-2-i}\,((\partial^i\,b^{\bar{l}b})\,\delta_{l\bar{l}}\,\gamma^{l\bar{a}})\,.
\end{aligned} \tag{2.2}$$

Note that the weights of these currents are given by h, h, $(h-\frac{1}{2})$ and $(h-\frac{1}{2})$ respectively. The conformal weights of (β, γ) fields are given by $(\lambda, 1-\lambda)$ while conformal weights of (b, c) fields are given by $(\frac{1}{2}+\lambda, \frac{1}{2}-\lambda)$. The above weights for the currents do not depend on the deformation parameter λ due to the particular combinations of the free fields. By counting the number of (anti)fundamental indices, there exist four components labeled by $(\bar{a}b) = (11, 12, 21, 22)$ in each current.

The relative coefficients appearing in (2.2) depend on the conformal weight h and deformation parameter λ explicitly and they are given by the binomial coefficients and the rising Pochhammer symbols where $(a)_n \equiv a(a+1)\cdots(a+n-1)$ as follows [1,2]:

$$
\begin{aligned}
a^i(h,\lambda) &\equiv \binom{h-1}{i} \\
&\quad\times \frac{(-2\lambda-h+2)_{h-1-i}}{(h+i)_{h-1-i}}, \quad 0 \le i \le (h-1), \\
\alpha^i(h,\lambda) &\equiv \binom{h-1}{i} \\
&\quad\times \frac{(-2\lambda-h+2)_{h-1-i}}{(h+i-1)_{h-1-i}}, \quad 0 \le i \le (h-1), \\
\beta^i(h,\lambda) &\equiv \binom{h-2}{i} \\
&\quad\times \frac{(-2\lambda-h+2)_{h-2-i}}{(h+i)_{h-2-i}}, \quad 0 \le i \le (h-2).
\end{aligned}
\tag{2.3}
$$

Let us consider the following currents consisting of (b, c) fields, (β, γ) fields, (γ, b) fields and (β, c) fields respectively by taking the linear combinations of (2.2) with the help of (2.3)

$$
\begin{aligned}
W^{\lambda,\bar{a}b}_{F,h}(b,c) &= \frac{2^{h-3}(h-1)!}{(2h-3)!!}\,\frac{(-1)^h}{\sum_{i=0}^{h-1} a^i(h,\frac{1}{2})} \\
&\quad\times\left[\frac{(h-1+2\lambda)}{(2h-1)}\,V^{(h)+}_{\lambda,\bar{a}b} + V^{(h)-}_{\lambda,\bar{a}b}\right], \\
W^{\lambda,\bar{a}b}_{B,h}(\beta,\gamma) &= \frac{2^{h-3}\,h!}{(2h-3)!!}\,\frac{(-1)^h}{\sum_{i=0}^{h-1} a^i(h,0)} \\
&\quad\times\left[\frac{(h-2\lambda)}{(2h-1)}\,V^{(h)+}_{\lambda,\bar{a}b} - V^{(h)-}_{\lambda,\bar{a}b}\right], \\
Q^{\lambda,\bar{a}b}_{h+\frac{1}{2}}(\gamma,b) &= \frac{1}{2}\,\frac{2^{h-\frac{1}{2}}h!}{(2h-1)!!}\,\frac{(-1)^{h+1}\,h}{\sum_{i=0}^{h-1} \beta^i(h+1,0)} \\
&\quad\times\left[Q^{(h+1)-}_{\lambda,\bar{a}b} - Q^{(h+1)+}_{\lambda,\bar{a}b}\right], \\
\bar{Q}^{\lambda,b\bar{a}}_{h+\frac{1}{2}}(\beta,c) &= \frac{1}{2}\,\frac{2^{h-\frac{1}{2}}h!}{(2h-1)!!}\,\frac{(-1)^{h+1}}{\sum_{i=0}^{h} \alpha^i(h+1,0)} \\
&\quad\times\left[Q^{(h+1)-}_{\lambda,\bar{a}b} + Q^{(h+1)+}_{\lambda,\bar{a}b}\right].
\end{aligned}
\tag{2.4}
$$

The overall coefficients do not depend on the deformation parameter λ. Then we have eight bosonic currents for the weight $h = 1, 2, \ldots$ and eight fermionic currents for the weight $h + \frac{1}{2} = \frac{3}{2}, \frac{5}{2}, \ldots$ as well as four fermionic currents $\bar{Q}^{\lambda,b\bar{a}}_{\frac{1}{2}}$ of the weight $\frac{1}{2}$ in (2.4). Note that four fermionic currents $Q^{\lambda,\bar{a}b}_{\frac{1}{2}}$ of the weight $\frac{1}{2}$ are identically zero.

The stress energy tensor of weight 2 is given by

$$
L = \left(W^{\lambda,\bar{a}a}_{\mathrm{B},2} + W^{\lambda,\bar{a}a}_{\mathrm{F},2}\right),
\tag{2.5}
$$

which can be written as $V^{(2)+}_{\lambda,\bar{a}a}$. The corresponding central charge is

$$
c_{cen} = 6\,N\,(1-4\lambda),
\tag{2.6}
$$

which depends on the deformation parameter λ explicitly. The above bosonic and fermionic currents in (2.4) are quasiprimary operators under the stress energy tensor (2.5) by using the defining OPEs in (2.1). The central charge (2.6) becomes $c_{cen} = 6N$ at $\lambda = 0$.

2.2 The $\mathcal{N} = 4$ multiplet

2.2.1 The lowest component

It is known, in [9], that the lowest components $\Phi^{(h)}_0$ for the weights $h = 1, 2, 3$ and 4 have their explicit λ dependences $(h-2\lambda)$ and $(h-1+2\lambda)$ in their relative coefficients. Then the question is how we determine these relative coefficients for arbitrary weight h. We realize that there exists an additional overall factor -4 from the weight h to the weight $(h + 1)$ in (2.4). Moreover, the denominator of the overall factor can be extracted easily and is given by $\frac{1}{(2h+1)}$ in terms of the weight h. We expect to have the factor $(-4)^h$ from the above analysis. The other numerical (h independent) factor can appear in general. This can be fixed only after we calculate the OPE between this lowest component and itself and obtain the central term. We will compute this central term later. For the time being we simply write down the following form

$$
\Phi^{(h)}_0 = \frac{(-4)^{h-2}}{(2h-1)}\left[-(h-2\lambda)\,W^{\lambda,\bar{a}a}_{\mathrm{F},h} + (h-1+2\lambda)\,W^{\lambda,\bar{a}a}_{\mathrm{B},h}\right].
\tag{2.7}
$$

For the weights $h = 1, 2, 3, 4$, we can observe that the corresponding numerical values appearing in the relative coefficients of (2.7) can be seen from the ones in [9]. The normalization in (2.7) is different from the one in [9] where there appear the additional numerical factors 16, 8, 12, 24 for the weights $h = 1, 2, 3, 4$ respectively. At the moment, it is not easy to figure out the exact h dependence from these values. In other words, we take the h dependence as in (2.7) together with the additional numerical factor $(-4)^{-2}$. As explained before, the overall factor in (2.7) can be determined by the normalization of the highest order singular term in the OPE between the $\Phi^{(h)}_0$ and itself.

2.2.2 The second components

From the observation of [9] for the weights $h = 1, 2, 3, 4$, the second components contain the various fermionic currents and their relative coefficients are common for any weights $h = 1, 2, 3, 4$. This implies that it is natural to take these relative coefficients for any h and the question is how we obtain the overall numerical factor. By taking the normalization of (2.7) we can extract the overall factors for the weights $h = 1, 2, 3, 4$ and they are given by $-\frac{1}{16} \times 1 = -\frac{1}{16}$, $-\frac{1}{8} \times (-2) = \frac{1}{4}$, $-\frac{1}{12} \times 12 = -1$ and $-\frac{1}{24} \times 96 = 4$ respectively. We can easily see that there exists $(-4)^h$ dependence when we increase the weight by 1. Therefore, the general expression for the weight h is given by $4(-4)^{h-4}$ which covers the above numerical values for the weights $h = 1, 2, 3, 4$. Then we can write down the corresponding second components as follows:

$$
\begin{aligned}
\Phi_{\frac{1}{2}}^{(h),1} &= 4\,(-4)^{h-4} \Big[\frac{1}{2} \Big(Q^{\lambda,11}_{h+\frac{1}{2}} + i\sqrt{2}\, Q^{\lambda,12}_{h+\frac{1}{2}} \\
&\quad + 2i\sqrt{2}\, Q^{\lambda,21}_{h+\frac{1}{2}} - 2\, Q^{\lambda,22}_{h+\frac{1}{2}} + 2\, \bar{Q}^{\lambda,11}_{h+\frac{1}{2}} \\
&\quad + 2i\sqrt{2}\, \bar{Q}^{\lambda,12}_{h+\frac{1}{2}} + i\sqrt{2}\, \bar{Q}^{\lambda,21}_{h+\frac{1}{2}} - \bar{Q}^{\lambda,22}_{h+\frac{1}{2}} \Big) \Big], \\
\Phi_{\frac{1}{2}}^{(h),2} &= 4\,(-4)^{h-4} \Big[-\frac{i}{2} \Big(Q^{\lambda,11}_{h+\frac{1}{2}} + 2i\sqrt{2}\, Q^{\lambda,21}_{h+\frac{1}{2}} \\
&\quad - 2\, Q^{\lambda,22}_{\frac{3}{2}} + 2\, \bar{Q}^{\lambda,11}_{h+\frac{1}{2}} + 2i\sqrt{2}\, \bar{Q}^{\lambda,12}_{h+\frac{1}{2}} - \bar{Q}^{\lambda,22}_{h+\frac{1}{2}} \Big) \Big], \\
\Phi_{\frac{1}{2}}^{(h),3} &= 4\,(-4)^{h-4} \Big[-\frac{i}{2} \Big(Q^{\lambda,11}_{h+\frac{1}{2}} + i\sqrt{2}\, Q^{\lambda,12}_{h+\frac{1}{2}} \\
&\quad - 2\, Q^{\lambda,22}_{h+\frac{1}{2}} + 2\, \bar{Q}^{\lambda,11}_{h+\frac{1}{2}} + i\sqrt{2}\, \bar{Q}^{\lambda,21}_{h+\frac{1}{2}} - \bar{Q}^{\lambda,22}_{h+\frac{1}{2}} \Big) \Big], \\
\Phi_{\frac{1}{2}}^{(h),4} &= 4\,(-4)^{h-4} \Big[-\frac{1}{2}\, Q^{\lambda,11}_{h+\frac{1}{2}} - Q^{\lambda,22}_{h+\frac{1}{2}} + \bar{Q}^{\lambda,11}_{h+\frac{1}{2}} \\
&\quad + \frac{1}{2}\, \bar{Q}^{\lambda,22}_{h+\frac{1}{2}} \Big].
\end{aligned}
\tag{2.8}
$$

Or we can calculate the OPEs between the supersymmetry generators and the lowest component and read off the first order pole which will provide the second components in (2.8). We will calculate the central terms coming from the highest order pole between the second components and itself later. As described before, once we fix this central term, then the overall factor in (2.8) where we use the normalization in (2.7) can be determined.

2.2.3 The third components

The third components contain the various bosonic currents and their relative coefficients are equal to each other for any weights $h = 1, 2, 3, 4$ in the analysis of [9]. Then we obtain the following results by replacing them with the corresponding expressions for arbitrary weight h with the above overall factors in previous section

$$
\begin{aligned}
\Phi_1^{(h),12} &= 4\,(-4)^{h-4} \Big[2i\, W^{\lambda,11}_{\mathrm{B},h+1} - \sqrt{2}\, W^{\lambda,12}_{\mathrm{B},h+1} - 2i\, W^{\lambda,22}_{\mathrm{B},h+1} \\
&\quad + 2i\, W^{\lambda,11}_{\mathrm{F},h+1} - 2\sqrt{2}\, W^{\lambda,12}_{\mathrm{F},h+1} - 2i\, W^{\lambda,22}_{\mathrm{F},h+1} \Big], \\
\Phi_1^{(h),13} &= 4\,(-4)^{h-4} \Big[-2i\, W^{\lambda,11}_{\mathrm{B},h+1} + 4\sqrt{2}\, W^{\lambda,21}_{\mathrm{B},h+1} + 2i\, W^{\lambda,22}_{\mathrm{B},h+1} \\
&\quad - 2i\, W^{\lambda,11}_{\mathrm{F},h+1} + 2\sqrt{2}\, W^{\lambda,21}_{\mathrm{F},h+1} + 2i\, W^{\lambda,22}_{\mathrm{F}+h+1} \Big], \\
\Phi_1^{(h),14} &= 4\,(-4)^{h-4} \Big[2\, W^{\lambda,11}_{\mathrm{B},h+1} + i\sqrt{2}\, W^{\lambda,12}_{\mathrm{B},h+1} \\
&\quad + 4i\sqrt{2}\, W^{\lambda,21}_{\mathrm{B},h+1} - 2\, W^{\lambda,22}_{\mathrm{B},h+1} - 2\, W^{\lambda,11}_{\mathrm{F},h+1} \\
&\quad - 2i\sqrt{2}\, W^{\lambda,12}_{\mathrm{F},h+1} - 2i\sqrt{2}\, W^{\lambda,21}_{\mathrm{F},h+1} + 2\, W^{\lambda,22}_{\mathrm{F},h+1} \Big], \\
\Phi_1^{(h),23} &= 4\,(-4)^{h-4} \Big[-2\, W^{\lambda,11}_{\mathrm{B},h+1} - i\sqrt{2}\, W^{\lambda,12}_{\mathrm{B},h+1} \\
&\quad - 4i\sqrt{2}\, W^{\lambda,21}_{\mathrm{B},h+1} + 2\, W^{\lambda,22}_{\mathrm{B},h+1} - 2\, W^{\lambda,11}_{\mathrm{F},h+1} \\
&\quad - 2i\sqrt{2}\, W^{\lambda,12}_{\mathrm{F},h+1} - 2i\sqrt{2}\, W^{\lambda,21}_{\mathrm{F},h+1} + 2\, W^{\lambda,22}_{\mathrm{F},h+1} \Big], \\
\Phi_1^{(h),24} &= 4\,(-4)^{h-4} \Big[-2i\, W^{\lambda,11}_{\mathrm{B},h+1} + 4\sqrt{2}\, W^{\lambda,21}_{\mathrm{B},h+1} + 2i\, W^{\lambda,22}_{\mathrm{B},h+1} \\
&\quad + 2i\, W^{\lambda,11}_{\mathrm{F},h+1} - 2\sqrt{2}\, W^{\lambda,21}_{\mathrm{F},h+1} - 2i\, W^{\lambda,22}_{\mathrm{F},h+1} \Big], \\
\Phi_1^{(h),34} &= 4\,(-4)^{h-4} \Big[-2i\, W^{\lambda,11}_{\mathrm{B},h+1} + \sqrt{2}\, W^{\lambda,12}_{\mathrm{B},h+1} + 2i\, W^{\lambda,22}_{\mathrm{B},h+1} \\
&\quad + 2i\, W^{\lambda,11}_{\mathrm{F},h+1} - 2\sqrt{2}\, W^{\lambda,12}_{\mathrm{F},h+1} - 2i\, W^{\lambda,22}_{\mathrm{F},h+1} \Big].
\end{aligned}
\tag{2.9}
$$

In principle, the OPEs between the supersymmetry generators of $\mathcal{N} = 4$ superconformal algebra and the second components and the first order pole will provide the third components in (2.9). The central terms coming from the highest order pole between the third components and itself will be determined later. Note that we can express the linear combination of $W^{\lambda,\bar{a}b}_{B,h+1}$ and another linear combination of $W^{\lambda,\bar{a}b}_{F,h+1}$ in terms of $\Phi_1^{(h),ij}$ and $\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h),kl}$. For example, the $\Phi_1^{(h),12}$ and the $\Phi_1^{(h),34}$ look similar to each other in the sense that the field contents are the same and half of them have opposite signs. By adding or subtracting these two relations, the two independent field contents can be written in terms of the third components of $\mathcal{N} = 4$ multiplet as above.

2.2.4 The fourth components

The fourth components contain the various fermionic currents and their relative coefficients are equal to each other for any weights $h = 1, 2, 3, 4$ in the analysis of [9]. By replacing them with the corresponding expressions for arbitrary weight h with the above overall factors in previous section we determine the following results as follows:

$$\begin{aligned}
\tilde{\Phi}_{\frac{3}{2}}^{(h),1} &\equiv \Phi_{\frac{3}{2}}^{(h),1} - \frac{1}{(2h+1)}\,(1-4\lambda)\,\partial\,\Phi_{\frac{1}{2}}^{(h),1} \\
&= 4\,(-4)^{h-4}\Bigg[-\frac{1}{2}\Big(Q_{h+\frac{3}{2}}^{\lambda,11} + i\sqrt{2}\,Q_{h+\frac{3}{2}}^{\lambda,12} \\
&\quad +2i\sqrt{2}\,Q_{h+\frac{3}{2}}^{\lambda,21} - 2\,Q_{h+\frac{3}{2}}^{\lambda,22} - 2\,\bar{Q}_{h+\frac{3}{2}}^{\lambda,11} \\
&\quad -2i\sqrt{2}\,\bar{Q}_{h+\frac{3}{2}}^{\lambda,12} - i\sqrt{2}\,\bar{Q}_{h+\frac{3}{2}}^{\lambda,21} + \bar{Q}_{h+\frac{3}{2}}^{\lambda,22}\Big)\Bigg], \\
\tilde{\Phi}_{\frac{3}{2}}^{(h),2} &\equiv \Phi_{\frac{3}{2}}^{(h),2} - \frac{1}{(2h+1)}\,(1-4\lambda)\,\partial\,\Phi_{\frac{1}{2}}^{(h),2} \\
&= 4\,(-4)^{h-4}\Bigg[\frac{i}{2}\Big(Q_{h+\frac{3}{2}}^{\lambda,11} + 2i\sqrt{2}\,Q_{h+\frac{3}{2}}^{\lambda,21} - 2\,Q_{h+\frac{3}{2}}^{\lambda,22} \\
&\quad -2\,\bar{Q}_{h+\frac{3}{2}}^{\lambda,11} - 2i\sqrt{2}\,\bar{Q}_{h+\frac{3}{2}}^{\lambda,12} + \bar{Q}_{h+\frac{3}{2}}^{\lambda,22}\Big)\Bigg], \\
\tilde{\Phi}_{\frac{3}{2}}^{(h),3} &\equiv \Phi_{\frac{3}{2}}^{(h),3} - \frac{1}{(2h+1)}\,(1-4\lambda)\,\partial\,\Phi_{\frac{1}{2}}^{(h),3} \\
&= 4\,(-4)^{h-4}\Bigg[\frac{i}{2}\Big(Q_{h+\frac{3}{2}}^{\lambda,11} + i\sqrt{2}\,Q_{h+\frac{3}{2}}^{\lambda,12} - 2\,Q_{h+\frac{3}{2}}^{\lambda,22} \\
&\quad -2\,\bar{Q}_{h+\frac{3}{2}}^{\lambda,11} - i\sqrt{2}\,\bar{Q}_{h+\frac{3}{2}}^{\lambda,21} + \bar{Q}_{h+\frac{3}{2}}^{\lambda,22}\Big)\Bigg], \\
\tilde{\Phi}_{\frac{3}{2}}^{(h),4} &\equiv \Phi_{\frac{3}{2}}^{(h),4} - \frac{1}{(2h+1)}\,(1-4\lambda)\,\partial\,\Phi_{\frac{1}{2}}^{(h),4} \\
&= 4\,(-4)^{h-4}\Bigg[\frac{1}{2}\Big(Q_{h+\frac{3}{2}}^{\lambda,11} + 2\,Q_{h+\frac{3}{2}}^{\lambda,22} + 2\,\bar{Q}_{h+\frac{3}{2}}^{\lambda,11} + \bar{Q}_{h+\frac{3}{2}}^{\lambda,22}\Big)\Bigg].
\end{aligned} \tag{2.10}$$

Compared to the $\Phi_{\frac{3}{2}}^{(h),i}$ which belongs to the components of the $\mathcal{N}=4$ multiplet, the $\tilde{\Phi}_{\frac{3}{2}}^{(h),i}$ in (2.10) are quasiprimary fields under the stress energy tensor (2.5). The $\Phi_{\frac{3}{2}}^{(h),i}$ and the $\Phi_{\frac{1}{2}}^{(h+1),i}$ by considering that the weight h in (2.8) is replaced with the weight $(h+1)$ look similar to each other in the sense that the field contents are the same and half of them have opposite signs. By adding or subtracting these two relations as before, the two independent field contents can be written in terms of the second components of the $(h+1)$th $\mathcal{N}=4$ multiplet and the fourth components of the h-th $\mathcal{N}=4$ multiplet. We expect that the OPEs between the supersymmetry generators of $\mathcal{N}=4$ superconformal algebra and the third components will provide the fourth components in (2.10).

2.2.5 The last component

Finally we describe the last component for arbitrary weight h as follows:

$$\begin{aligned}
\tilde{\Phi}_{2}^{(h)} &\equiv \Phi_{2}^{(h)} - \frac{1}{(2h+1)}\,(1-4\lambda)\,\partial^2\,\Phi_{0}^{(h)} \\
&= 4\,(-4)^{h-4}\Big[-2\Big(W_{\mathrm{B},h+2}^{\lambda,\bar{a}a} + W_{\mathrm{F},h+2}^{\lambda,\bar{a}a}\Big)\Big].
\end{aligned} \tag{2.11}$$

Under the stress energy tensor (2.5), this is a quasiprimary operator. The OPEs between the supersymmetry generators of $\mathcal{N}=4$ superconformal algebra and the fourth components will provide the last component in (2.11). By replacing h with $(h+2)$ in (2.7), we can express $W_{\mathrm{B},h+2}^{\lambda,\bar{a}a}$ and $W_{\mathrm{F},h+2}^{\lambda,\bar{a}a}$ in terms of $\Phi_0^{(h+2)}$ and $\tilde{\Phi}_2^{(h)}$ by simple linear combinations as before.

Therefore, the $\mathcal{N}=4$ multiplet is summarized by (2.7), (2.8), (2.9), (2.10) and (2.11) together with (2.2), (2.3) and (2.4). Their algebra will be obtained explicitly by using the defining relations in (2.1).

3 The $\mathcal{N}=4$ supersymmetric linear $W_\infty^{2,2}$ algebra between the adjoints and the bifundamentals under the $U(2)\times U(2)$ symmetry

In order to obtain the algebra between (2.7), (2.8), (2.9), (2.10) and (2.11), it is necessary to determine the algebra between the currents in (2.4). In the footnotes, we present some examples where there are extra structures (described in the introduction) on the right hand sides of the (anti)commutator relations for the specific weights h_1 and h_2.

3.1 The (anti)commutator relations between the nonsinglet currents

Let us consider the algebra between the currents consisting of (b, c) fields in (2.4). By multiplying the Pauli matrix of $SU(2)$ with the additional factor $\frac{1}{2}$ properly and summing over the indices $\bar{a}$ and b as in [9], we can construct the three fundamentals of $SU(2)$. By multiplying the Kronecker delta (or 2×2 identity matrix) with the contractions of the indices, we obtain the singlet of $SU(2)$. First of all, in $SU(2)$, there is no symmetric $d^{\hat{A}\hat{B}\hat{C}}$ symbols.

3.1.1 The commutator relation with $h_1 = h_2, h_2 \pm 1$ for nonzero λ

Then we can associate $(W_{\mathrm{F},h}^{\lambda,12} + W_{\mathrm{F},h}^{\lambda,21})$, $i\,(W_{\mathrm{F},h}^{\lambda,12} - W_{\mathrm{F},h}^{\lambda,21})$ and $(W_{\mathrm{F},h}^{\lambda,11} - W_{\mathrm{F},h}^{\lambda,22})$ with the triplets $W_{\mathrm{F},h}^{\lambda,\hat{A}=1}$, $W_{\mathrm{F},h}^{\lambda,\hat{A}=2}$, and $W_{\mathrm{F},h}^{\lambda,\hat{A}=3}$ of $SU(2)$ respectively. Moreover, the $(W_{\mathrm{F},h}^{\lambda,11} +$

$W_{\mathrm{F},h}^{\lambda,22}) = W_{\mathrm{F},h}^{\lambda,\bar{a}a}$ plays the role of the singlet $W_{\mathrm{F},h}^{\lambda,\hat{A}=0} \equiv W_{\mathrm{F},h}^{\lambda}$ of $SU(2)$. One of the commutator relations in [9] can be written as the following commutator relation[1]

$$\begin{aligned}
&\big[(W_{\mathrm{F},h_1}^{\lambda,\hat{A}})_m, (W_{\mathrm{F},h_2}^{\lambda,\hat{B}})_n\big] \\
&= - \sum_{h=-1,\ \text{odd}}^{h_1+h_2-3} q^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda) \\
&\quad \times i\, f^{\hat{A}\hat{B}\hat{C}}\, (W_{\mathrm{F},h_1+h_2-2-h}^{\lambda,\hat{C}})_{m+n} \\
&\quad + \binom{m+h_1-1}{h_1+h_2-1} c_F(h_1,h_2,\lambda)\, \delta^{\hat{A}\hat{B}}\, q^{h_1+h_2-4}\, \delta_{m+n} \\
&\quad + \sum_{h=0,\text{even}}^{h_1+h_2-3} q^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda) \\
&\quad \times \delta^{\hat{A}\hat{B}}\, (W_{\mathrm{F},h_1+h_2-2-h}^{\lambda})_{m+n}. \qquad (3.1)
\end{aligned}$$

The structure constant is given by (A.4).

Let us calculate the central term in (3.1) explicitly. One of the reasons why we are doing this is that we have not seen any literatures which provides all the details in the calculation and it is useful to observe the general structure of the computation of any OPEs including the free fields. In order to calculate the highest singular term in the OPE $V_{\lambda,\bar{a}b}^{(h_1)+}(z)\, V_{\lambda,\bar{c}d}^{(h_2)+}(w)$, we need to calculate the central term in the OPE between $V_{\lambda,\bar{a}b}^{(h_1)+}(z)$ and $\delta_{l\bar{l}}\, \gamma^{l\bar{c}}\, \partial^i\, \beta^{\bar{l}d}(w)$ coming from $V_{\lambda,\bar{c}d}^{(h_2)+}(w)$. It is known that the following OPE satisfies

$$\begin{aligned}
V_{\lambda,\bar{a}b}^{(h_1)+}(z)\, \gamma^{j\bar{c}}(x) &= \delta_{b\bar{c}} \sum_{j=0}^{h_1-1} a^j(h_1,\lambda)\, (-1)^{h_1}\, j! \\
&\quad \times \sum_{t=0}^{j+1} (j+1-t)_{h_1-1-j}\, \frac{1}{t!}\, \frac{1}{(z-x)^{h_1-t}}\, \partial^t\, \gamma^{l\bar{a}}(x) + \cdots .
\end{aligned} \qquad (3.2)$$

The next step is to calculate the OPE between $\partial_x^t\, \gamma^{l\bar{a}}(x)$ appearing in the last factor in (3.2) and $\partial_w^i\, \beta^{\bar{l}d}(w)$. We have the defining OPE relation in (2.1). The multiple derivative with respect to x acting on $\frac{1}{(x-w)}$ can be obtained explicitly and the similar multiple derivative with respect to w acting on $\frac{1}{(x-w)}$ can be rewritten as the corresponding multiple derivative with respect to x acting on $\frac{1}{(x-w)}$ with the number of minus signs. Then we obtain the following result

$$\partial_x^t\, \gamma^{l\bar{a}}(x)\, \partial_w^i\, \beta^{\bar{l}d}(w) = \frac{1}{(x-w)^{t+i+1}}\, (-1)^t\, (t+i)!\, \delta^{l\bar{l}}\, \delta^{d\bar{a}} + \cdots . \qquad (3.3)$$

Now we expand $\frac{1}{(z-x)^{h_1-t}}$ appearing in the second factor from the last in (3.2) around $x = w$ by using the Taylor expansion. Then we obtain that the coefficient of $(x-w)^{t+i+1}$ in the $\frac{1}{(z-x)^{h_1+i+1}}$ evaluated at $x = w$ is given by $\frac{1}{(t+i+1)!}\,(h_1-t)_{t+i+1}$. We are left, by collecting the contributions from (3.2) and (3.3), with

$$\begin{aligned}
&\Bigg[\delta_{b\bar{c}} \sum_{j=0}^{h_1-1} a^j(h_1,\lambda)\,(-1)^{h_1}\, j! \sum_{t=0}^{j+1} (j+1-t)_{h_1-1-j}\, \frac{1}{t!}\Bigg] \\
&\times \Big[(-1)^t\,(t+i)!\,\delta^{l\bar{l}}\,\delta^{d\bar{a}}\Big] \\
&\times \Bigg[\frac{1}{(t+i+1)!}\,(h_1-t)_{t+i+1}\Bigg],
\end{aligned} \qquad (3.4)$$

in the $\frac{1}{(z-w)^{h_1+i+1}}$ term. After acting the derivative $\partial_w^{h_2-1-i}$ from the remaining factor in the first part of $V_{\lambda,\bar{c}d}^{(h_2)+}(w)$ on the $\frac{1}{(z-w)^{h_1+i+1}}$ term, we obtain $(h_1+i+1)_{h_2-1-i}$. By combining with (3.4), the final contribution from the central terms in the OPE between $V_{\lambda,\bar{a}b}^{(h_1)+}(z)$ and the first part of $V_{\lambda,\bar{c}d}^{(h_2)+}(w)$ can be written as

$$\begin{aligned}
&N\, \delta_{b\bar{c}}\, \delta_{d\bar{a}} \sum_{j=0}^{h_1-1} \sum_{i=0}^{h_2-1} \sum_{t=0}^{j+1} a^j(h_1,\lambda)\, a^i(h_2,\lambda) \\
&\times \frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1+t}\,(j+1-t)_{h_1-1-j} \\
&\times (h_1-t)_{t+1+i}\,(h_1+1+i)_{h_2-1-i}.
\end{aligned} \qquad (3.5)$$

We can do the similar calculation for the contribution from the second part of $V_{\lambda,\bar{c}d}^{(h_2)+}(w)$. In this case, we should use the following intermediate result in [9]

$$\begin{aligned}
V_{\lambda,\bar{a}b}^{(h_1)+}(z)\, c^{j\bar{c}}(x) &= \delta_{b\bar{c}} \sum_{j=0}^{h_1-1} a^j\Big(h_1, \lambda+\frac{1}{2}\Big)(-1)^{h_1}\, j! \\
&\quad \times \sum_{t=0}^{j+1} (j+1-t)_{h_1-1-j} \\
&\quad \times \frac{1}{t!}\, \frac{1}{(z-x)^{h_1-t}}\, \partial^t c^{l\bar{c}}(w) + \cdots .
\end{aligned} \qquad (3.6)$$

By starting with (3.6) and following the procedures in (3.3), (3.4) and (3.5) we have described above, we obtain the

[1] We can calculate the OPE between $W_{\mathrm{F},h_1=6}^{\lambda,\hat{A}=1}(z)$ and $W_{\mathrm{F},h_2=4}^{\lambda,\hat{B}=1}(w)$ where $h_1 = h_2+2$ and read off the ninth order pole which has the structure constant $\frac{131072}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)(2\lambda+3)$ appearing in the current $W_{\mathrm{F},h_1+h_2-2-h=1}^{\lambda}(w)$. In this case, the weight h is given by odd number $h = 7$. We do not see this term from the commutator (3.1) because the even h appears in the last term of (3.1). This implies that if the weights h_1 and h_2 do not satisfy the above constraint ($h_1 = h_2$ or $h_1 = h_2 \pm 1$) for nonzero λ, then we cannot use the formula in (3.1) fully and there appear the extra terms on the right hand sides of the corresponding OPE. Because the above extra factor contains λ, there will be no problem for vanishing λ when we use (3.1). It would be interesting to obtain the above commutator for generic h_1 and h_2. It seems that there is a critical singular term in the sense that we still have the structure of (3.1) for the poles less than this critical singular term. Of course, for the poles greater than the critical singular term there exist other extra terms in general.

following contribution

$$-N\,\delta_{b\bar{c}}\,\delta_{d\bar{a}}\sum_{j=0}^{h_1-1}\sum_{i=0}^{h_2-1}\sum_{t=0}^{j+1}a^j\left(h_1,\lambda+\frac{1}{2}\right)a^i\left(h_2,\lambda+\frac{1}{2}\right)\times\frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1+t}\,(j+1-t)_{h_1-1-j}\times(h_1-t)_{t+1+i}\,(h_1+1+i)_{h_2-1-i}. \quad (3.7)$$

Therefore, we obtain the final central term, by adding (3.5) and (3.7), as follows:

$$V^{(h_1)+}_{\lambda,\bar{a}b}(z)\,V^{(h_2)+}_{\lambda,\bar{c}d}(w)\Big|_{\frac{1}{(z-w)^{h_1+h_2}}}=N\,\delta_{b\bar{c}}\,\delta_{d\bar{a}}\sum_{j=0}^{h_1-1}\sum_{i=0}^{h_2-1}\sum_{t=0}^{j+1}\times\left(a^j(h_1,\lambda)\,a^i(h_2,\lambda)-a^j\left(h_1,\lambda+\frac{1}{2}\right)a^i\left(h_2,\lambda+\frac{1}{2}\right)\right)\times\frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1+t}\,(j+1-t)_{h_1-1-j}\times(h_1-t)_{t+1+i}\,(h_1+1+i)_{h_2-1-i}. \quad (3.8)$$

Due to the behavior of two Kronecker deltas, the central term is nonzero only for the case where the second index of the first operator should equal to the first index of the second operator and the first index of the first operator should equal to the second index of the second operator on the left hand side. In Appendix B, we present other central terms.

By realizing that the overall factor appearing in the first current of (2.4) is given by $(-4)^{h-2}$, we can write down the central terms

$$c_F(h_1,h_2,\lambda)\equiv(-4)^{h_1+h_2-4}\,\delta_{b\bar{a}}\,\delta_{d\bar{c}}\Bigg[\frac{(h_1-1+2\lambda)}{(2h_1-1)}\times\frac{(h_2-1+2\lambda)}{(2h_2-1)}\,V^{(h_1)+}_{\lambda,\bar{a}b}(z)\,V^{(h_2)+}_{\lambda,\bar{c}d}(w)+V^{(h_1)-}_{\lambda,\bar{a}b}(z)\,V^{(h_2)-}_{\lambda,\bar{c}d}(w)+\frac{(h_1-1+2\lambda)}{(2h_1-1)}\,V^{(h_1)+}_{\lambda,\bar{a}b}(z)\,V^{(h_2)-}_{\lambda,\bar{c}d}(w)+\frac{(h_2-1+2\lambda)}{(2h_2-1)}\,V^{(h_1)-}_{\lambda,\bar{a}b}(z)\times V^{(h_2)+}_{\lambda,\bar{c}d}(w)\Bigg]_{\frac{1}{(z-w)^{h_1+h_2}}}, \quad (3.9)$$

where the relation (3.8) and the relations in Appendix (B.1) are used. In the last term of (3.9), we can use the central term of $V^{(h_2)+}_{\lambda,\bar{c}d}(z)\,V^{(h_1)-}_{\lambda,\bar{a}b}(w)$ with the extra factor $(-1)^{h_1+h_2}$.

In Appendices (C.1) and (C.2), we present the corresponding OPEs for $h_1=h_2=4$ where the indices $\hat{A}$ and $\hat{B}$ are equal to each other for the former while they are different from each other for the latter. Compared to the commutator relation in (3.1), there exists $(-1)^{h-1}$ factor. Due to the Kronecker delta, the former corresponds to the last two terms in (3.1) while the latter corresponds to the first term in (3.1). The other four cases between the nonsinglet currents are checked explicitly and we do not present them in this paper. The commutator relations between the nonsinglet currents are given in [8]. The corresponding commutator relations between the nonsinglet currents and the singlet currents can be determined similarly.

3.1.2 The second commutator relation with $h_1=h_2,h_2\pm1$ for nonzero λ

Similarly, we obtain the following commutator relation for the currents consisting of (β,γ) fields[2]

$$\left[(W^{\lambda,\hat{A}}_{\mathrm{B},h_1})_m,(W^{\lambda,\hat{B}}_{\mathrm{B},h_2})_n\right]=-\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\times i f^{\hat{A}\hat{B}\hat{C}}\,(W^{\lambda,\hat{C}}_{\mathrm{B},h_1+h_2-2-h})_{m+n}+\binom{m+h_1-1}{h_1+h_2-1}c_B(h_1,h_2,\lambda)\,\delta^{\hat{A}\hat{B}}\,q^{h_1+h_2-4}\times\delta_{m+n}+\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\times\delta^{\hat{A}\hat{B}}\,(W^{\lambda}_{\mathrm{B},h_1+h_2-2-h})_{m+n}. \quad (3.10)$$

The central term appearing in (3.10), by recalling the definition of the second relation of (2.4), can be described by

$$c_B(h_1,h_2,\lambda)\equiv(-4)^{h_1+h_2-4}\,\delta_{b\bar{a}}\,\delta_{d\bar{c}}\Bigg[\frac{(h_1-2\lambda)}{(2h_1-1)}\,\frac{(h_2-2\lambda)}{(2h_2-1)}\times V^{(h_1)+}_{\lambda,\bar{a}b}(z)\,V^{(h_2)+}_{\lambda,\bar{c}d}(w)+V^{(h_1)-}_{\lambda,\bar{a}b}(z)\,V^{(h_2)-}_{\lambda,\bar{c}d}(w)-\frac{(h_1-2\lambda)}{(2h_1-1)}\times V^{(h_1)+}_{\lambda,\bar{a}b}(z)\,V^{(h_2)-}_{\lambda,\bar{c}d}(w)$$

[2] As in the footnote 1, the OPE between $W^{\lambda,\hat{A}=1}_{\mathrm{B},h_1=7}(z)$ and $W^{\lambda,\hat{B}=1}_{\mathrm{B},h_2=3}(w)$ where $h_1=h_2+4$ can be calculated and the ninth order pole contains the structure constant $-\frac{524288}{11}(\lambda-1)\lambda(2\lambda-3)(2\lambda-1)(2\lambda+1)(\lambda^2-\lambda+5)$ appearing in the current $W^{\lambda}_{\mathrm{B},h_1+h_2-2-h=1}(w)$ and the seventh order pole contains the structure constant $-6144(\lambda-1)\lambda(2\lambda-3)(2\lambda-1)(2\lambda+1)$ appearing in the current $W^{\lambda}_{\mathrm{B},h_1+h_2-2-h=3}(w)$. In this case, the weight h is given by odd number $h=7$ or $h=5$ while the corresponding dummy variable h is given by even number in (3.10). If the weights h_1 and h_2 do not satisfy the above constraint ($h_1=h_2$ or $h_1=h_2\pm1$) for nonzero λ, then we cannot use the formula in (3.10) exactly because the extra terms on the right hand sides of the corresponding OPE occur, compared to the $\lambda=0$ case. We observe that for the poles less than the critical singular term mentioned in the footnote 1, the above commutator can be used precisely and for the poles greater than that singular term there exist extra terms on the right hand side of the OPE. Although these extra terms can be obtained for fixed h_1 and h_2, at the moment, those for general h_1 and h_2 are not known. It seems that as the difference between h_1 and h_2 increases, the pole corresponding to the critical singular term decreases.

$$
-\frac{(h_2-2\lambda)}{(2h_2-1)}\, V^{(h_1)-}_{\lambda,\bar{a}b}(z)\, V^{(h_2)+}_{\lambda,\bar{c}d}(w)\Big]_{\frac{1}{(z-w)^{h_1+h_2}}}. \tag{3.11}
$$

The previous relation (3.8) and the previous relations in Appendix (B.1) can be used. As before, in the last term of (3.11), the central term of $V^{(h_2)+}_{\lambda,\bar{c}d}(z)\, V^{(h_1)-}_{\lambda,\bar{a}b}(w)$ with the extra factor $(-1)^{h_1+h_2}$ can be used. The relevant OPEs are given in Appendices (C.3) and (C.4). The additional $(-1)^{h-1}$ factor appears in the OPEs.

3.1.3 Other commutator relations with $h_1 = h_2, h_2+1$ for nonzero λ

The remaining commutator relations between the bosonic currents and the fermionic currents can be described as

$$
\begin{aligned}
\big[(W^{\lambda,\hat{A}}_{\mathrm{F},h_1})_m, (Q^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda) \\
&\quad \times\Big(i\, f^{\hat{A}\hat{B}\hat{C}}\, (Q^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&\quad + \delta^{\hat{A}\hat{B}}\, (Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big), \\
\big[(W^{\lambda,\hat{A}}_{\mathrm{B},h_1})_m, (Q^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda) \\
&\quad \times\Big(-i\, f^{\hat{A}\hat{B}\hat{C}}\, (Q^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&\quad + \delta^{\hat{A}\hat{B}}\, (Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big), \\
\big[(W^{\lambda,\hat{A}}_{\mathrm{F},h_1})_m, (\bar{Q}^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-2} q^h\, (-1)^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda) \\
&\quad \times\Big(-i\, f^{\hat{A}\hat{B}\hat{C}}\, (\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&\quad + \delta^{\hat{A}\hat{B}}\, (\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big), \\
\big[(W^{\lambda,\hat{A}}_{\mathrm{B},h_1})_m, (\bar{Q}^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-2} q^h\, (-1)^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda) \\
&\quad \times\Big(i\, f^{\hat{A}\hat{B}\hat{C}}\, (\bar{Q}^{\lambda,\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} \\
&\quad + \delta^{\hat{A}\hat{B}}\, (\bar{Q}^{\lambda}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big).
\end{aligned} \tag{3.12}
$$

The corresponding OPEs for fixed h_1 and h_2 can be found in Appendices (C.5), (C.6), (C.7), (C.8),(C.9), (C.10), (C.11) and (C.12). In the last two commutator relations of (3.12), the upper limit of h is given by $h = h_1 + h_2 - 2$ due to the presence of the lowest fermionic current $\bar{Q}^{\lambda,b\bar{a}}_{\frac{1}{2}}$. The additional $(-1)^{h-1}$ factor appears when we change the commutator relations into the corresponding OPEs.[3]

3.1.4 The final anticommutator relation with $h_1 = h_2$ for nonzero λ

The final anticommutator relation between the fermionic currents is summarized by

$$
\begin{aligned}
\Big\{(Q^{\lambda,\hat{A}}_{h_1+\frac{1}{2}})_r, (\bar{Q}^{\lambda,\hat{B}}_{h_2+\frac{1}{2}})_s\Big\} &= \sum_{h=0}^{h_1+h_2-1} q^h\, o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda) \\
&\quad \times\Big(i\, f^{\hat{A}\hat{B}\hat{C}}\, (W^{\lambda,\hat{C}}_{F,h_1+h_2-h})_{r+s} \\
&\quad + \delta^{\hat{A}\hat{B}}\, (W^{\lambda}_{F,h_1+h_2-h})_{r+s}\Big) \\
&\quad + \sum_{h=0}^{h_1+h_2-1} q^h\, o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda) \\
&\quad \times\Big(-i\, f^{\hat{A}\hat{B}\hat{C}}\, (W^{\lambda,\hat{C}}_{B,h_1+h_2-h})_{r+s}
\end{aligned}
$$

[3] As in two previous examples, the OPE between $W^{\lambda,\hat{A}=1}_{\mathrm{F},h_1=5}(z)$ and $Q^{\lambda,\hat{B}=1}_{\mathrm{B},h_2+\frac{1}{2}=\frac{7}{2}}(w)$ where $h_1 = h_2 + 2$ can be obtained and the seventh order pole contains the structure constant $-\frac{6144}{35}(\lambda-1)(\lambda+1)(\lambda+2)(2\lambda-3)(2\lambda+1)(2\lambda+3)$ appearing in the current $Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h=\frac{3}{2}}(w)$. On the other hand, the structure constant $q_{\mathrm{F}}^{5,\frac{7}{2},5}(m,r,\lambda)$ contains the λ dependent factor $\frac{4}{1575}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(4\lambda^2-14\lambda-9)$. By subtracting the contribution $\frac{2}{225}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)$ coming from the structure constant $q_{\mathrm{F}}^{4,\frac{7}{2},5}(m,r,\lambda)$, where 4 is realized by (h_1-1), from the above, we obtain $-\frac{2}{525}(\lambda-1)(\lambda+1)(\lambda+2)(2\lambda-3)(2\lambda+1)(2\lambda+3)$. Note that the additional term in the $q_{\mathrm{F}}^{4,\frac{7}{2},5}(m,r,\lambda)$ contains the factor λ. By considering the numerical factor 46080 when we move from the modes in the commutator to the differential operators in the OPE and multiplying this into the above factor, we obtain the previous structure constant in the current $Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h=\frac{3}{2}}(w)$. This implies that if the weights h_1 and h_2 do not satisfy the above constraint ($h_1 = h_2$ or $h_1 = h_2 + 1$) for nonzero λ, then the formula in the first equation of (3.12) cannot be used fully because the extra contribution from the structure constant on the right hand sides of the corresponding OPE occur. As before, for the poles less than the critical singular term, still we can use some of terms in the corresponding expression of the first relation of (3.12). Similarly, let us consider the last equation of (3.12). Then the OPE between $W^{\lambda,\hat{A}=1}_{\mathrm{B},h_1=5}(z)$ and $\bar{Q}^{\lambda,\hat{B}=1}_{\mathrm{B},h_2+\frac{1}{2}=\frac{7}{2}}(w)$ provides the seventh order pole with $-\frac{2048}{35}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)(11\lambda-10)$ appearing in the current $\bar{Q}^{\lambda}_{\frac{3}{2}}(w)$. This can be obtained by adding the extra contribution from $q_{\mathrm{B}}^{4,\frac{7}{2},5}(m,r,\lambda)$ in addition to the one from $q_{\mathrm{B}}^{5,\frac{7}{2},5}(m,r,\lambda)$ as in previous paragraph. Note that the $\bar{Q}^{\lambda}_{\frac{1}{2}}(w)$ term on the right hand side of the OPE can be determined by the contribution from the structure constant $q_{\mathrm{B}}^{5,\frac{7}{2},6}(m,r,\lambda)$ only. Therefore, the last equation of (3.12) should be modified for nonzero λ with more general h_1 and h_2 where the weight h_2 is outside of the above allowed region. The similar behaviors corresponding to the second and the third equations of (3.12) we do not present in this paper can appear.

$$+\delta^{\hat{A}\hat{B}}\,(W^{\lambda}_{B,h_1+h_2-h})_{r+s}\Big) + \binom{r+h_1-\frac{1}{2}}{h_1+h_2} c_Q(h_1,h_2,\lambda) \times \delta^{\hat{A}\hat{B}}\, q^{h_1+h_2-2}\delta_{r+s}. \tag{3.13}$$

The corresponding OPEs for fixed h_1 and h_2 can be found in Appendices (C.13) and (C.14).[4] The additional $(-1)^h$ factor appears when we change the anticommutator relations into the corresponding OPEs. Because the lowest weight of $W^{\lambda,\bar{a}b}_{B,h}$ is given by 1, the upper limit of the second summation is also given by $h = h_1+h_2-1$ which is the same as the one in the first summation of (3.13). Here the central term appearing in (3.13) can be described by

$$\begin{aligned}
c_Q(h_1,h_2,\lambda) &\equiv 8(-4)^{h_1+h_2-4}\delta_{b\bar{a}}\delta_{d\bar{c}} \\
&\times\Big[-Q^{(h_1+1)+}_{\lambda,\bar{a}b}(z)\,Q^{(h_2+1)+}_{\lambda,\bar{c}d}(w) \\
&+Q^{(h_1+1)-}_{\lambda,\bar{a}b}(z)\,Q^{(h_2+1)-}_{\lambda,\bar{c}d}(w) \\
&-Q^{(h_1+1)+}_{\lambda,\bar{a}b}(z)\,Q^{(h_2+1)-}_{\lambda,\bar{c}d}(w) \\
&+Q^{(h_1+1)-}_{\lambda,\bar{a}b}(z)\,Q^{(h_2+1)+}_{\lambda,\bar{c}d}(w) \\
&-(-1)^{h_1+h_2}\,Q^{(h_2+1)+}_{\lambda,\bar{c}d}(z)\,Q^{(h_1+1)+}_{\lambda,\bar{a}b}(w) \\
&+(-1)^{h_1+h_2}\,Q^{(h_2+1)-}_{\lambda,\bar{c}d}(z)\,Q^{(h_1+1)-}_{\lambda,\bar{a}b}(w) \\
&-(-1)^{h_1+h_2}\,Q^{(h_2+1)+}_{\lambda,\bar{c}d}(z)\,Q^{(h_1+1)-}_{\lambda,\bar{a}b}(w) \\
&+(-1)^{h_1+h_2}\,Q^{(h_2+1)-}_{\lambda,\bar{c}d}(z)\,Q^{(h_1+1)+}_{\lambda,\bar{a}b}(w)\Big]_{\frac{1}{(z-w)^{h_1+h_2+1}}},
\end{aligned} \tag{3.14}$$

where we can use Appendix (D.1).

Therefore, the seven (anti)commutator relations between the nonsinglet currents are given by (3.1), (3.10), (3.12) and (3.13).

3.2 The (anti)commutator relations between the nonsinglet currents in explicit forms with $h_1 = h_2, h_2 \pm 1$ for nonzero λ

In order to construct the algebra from the $\mathcal{N}=4$ multiplets, we should rewrite the previous (anti)commutator relations in the basis of four components of $(\bar{a}b) = (11, 12, 21, 22)$ in each current. For example, from (3.1), the commutator relation between $\hat{A}=1$ (sum of (12) and (21)) and $\hat{B}=1$ is known. Moreover, the commutator relation between $\hat{A}=1$ and $\hat{B}=2$ (difference of (12) and (21) up to an overall factor) is known. Then we obtain the commutator relation between $\hat{A}=1$ and the element (12) current by adding the above two commutator relations. By realizing that there is no singular term in the commutator relation between the element (12) current and itself, the above analysis leads to the commutator relation between the element (21) and the element (12) as follows:

$$\begin{aligned}
\big[(W^{\lambda,21}_{\mathrm{F},h_1})_m, (W^{\lambda,12}_{\mathrm{F},h_2})_n\big] &= \frac{1}{2}\sum_{h=0,\mathrm{even}}^{h_1+h_2-3} q^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda) \\
&\times(W^{\lambda,11}_{\mathrm{F},h_1+h_2-2-h} + W^{\lambda,22}_{\mathrm{F},h_1+h_2-2-h})_{m+n} \\
&+\frac{1}{2}\binom{m+h_1-1}{h_1+h_2-1} c_F(h_1,h_2,\lambda)\, q^{h_1+h_2-4}\,\delta_{m+n} \\
&-\frac{1}{2}\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3} q^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,11}_{\mathrm{F},h_1+h_2-2-h} \\
&-W^{\lambda,22}_{\mathrm{F},h_1+h_2-2-h})_{m+n}.
\end{aligned} \tag{3.15}$$

On the other hand, by subtracting the previous two commutator relations and by using the fact that there is no singular term in the commutator relation between the element (21) current and itself, we obtain the following commutator relation between the element (12) and the element (21) as follows:

$$\begin{aligned}
\big[(W^{\lambda,12}_{\mathrm{F},h_1})_m, (W^{\lambda,21}_{\mathrm{F},h_2})_n\big] &= \frac{1}{2}\sum_{h=0,\mathrm{even}}^{h_1+h_2-3} q^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda) \\
&\times(W^{\lambda,11}_{\mathrm{F},h_1+h_2-2-h} + W^{\lambda,22}_{\mathrm{F},h_1+h_2-2-h})_{m+n} \\
&+\frac{1}{2}\binom{m+h_1-1}{h_1+h_2-1} c_F(h_1,h_2,\lambda)\, q^{h_1+h_2-4}\,\delta_{m+n}
\end{aligned}$$

[4] As in the examples appearing in the previous footnotes, the OPE between $Q^{\lambda,\hat{A}=1}_{h_1+\frac{1}{2}=\frac{3}{2}}(z)$ and $\bar{Q}^{\lambda,\hat{B}=1}_{h_2+\frac{1}{2}=\frac{7}{2}}(w)$ where $h_1 = h_2-2$ can be determined and the fourth order pole contains the structure constant $\frac{128}{5}(\lambda-1)(2\lambda-3)(2\lambda-1)$ appearing in the current $W^{\lambda}_{F,h_1+h_2-h=1}(w)$. On the other hand, the structure constant $o_{\mathrm{F}}^{\frac{3}{2},\frac{7}{2},3}(r,s,\lambda)$ contains the λ dependent factor $\frac{8}{15}(2\lambda-1)(4\lambda^2+2\lambda+3)$. We can take the sum (due to the odd h) of these with proper numerical value -48 for the latter and obtain $-\frac{128}{5}\lambda(2\lambda-1)(2\lambda+7)$ which vanishes at $\lambda=0$ and this implies that this extra contribution should appear. Furthermore, the fourth order pole contains the structure constant $\frac{256}{5}\lambda(\lambda+1)(2\lambda+1)$ appearing in the current $W^{\lambda}_{B,h_1+h_2-h=1}(w)$. On the other hand, the structure constant $o_{\mathrm{B}}^{\frac{3}{2},\frac{7}{2},3}(r,s,\lambda)$ contains $\frac{16}{15}\lambda(4\lambda^2-6\lambda+5)$. Obviously they are different from each other and the extra term $-\frac{256}{5}(\lambda-4)\lambda(2\lambda-1)$ should appear from similar analysis. Similarly, the third order pole contains the structure constant $\frac{96}{5}(\lambda-1)(2\lambda-3)$ appearing in the current $W^{\lambda}_{F,h_1+h_2-h=2}(w)$. Moreover, the structure constant $o_{\mathrm{F}}^{\frac{3}{2},\frac{7}{2},2}(r,s,\lambda)$ contains $-\frac{4}{5}(4\lambda^2+10\lambda-9)$. By taking the difference (due to the even h) between the latter with an additional numerical value 8 and the former, we obtain $-32\lambda(2\lambda-1)$ which vanishes at $\lambda=0$ and should appear. The third order pole also contains the structure constant $\frac{96}{5}(\lambda+1)(2\lambda+1)$ appearing in the current $W^{\lambda}_{B,h_1+h_2-h=2}(w)$. On the other hand, the structure constant $o_{\mathrm{B}}^{\frac{3}{2},\frac{7}{2},2}(r,s,\lambda)$ contains $-\frac{4}{5}(4\lambda^2-14\lambda-3)$. We obtain $-32\lambda(2\lambda-1)$ from the difference between the latter and the former as before and this λ dependent contribution should appear. If the weights h_1 and h_2 are not equal to each other for nonzero λ, then the formula in the first equation of (3.13) cannot be used fully because the extra contribution from the structure constant on the right hand sides of the corresponding OPE arises.

$$-\frac{1}{2}\sum_{h=-1,\text{odd}}^{h_1+h_2-3} q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,11}_{\text{F},h_1+h_2-2-h} - W^{\lambda,22}_{\text{F},h_1+h_2-2-h})_{m+n}. \quad (3.16)$$

Similarly, the commutator relation having both $\hat{A}=1$ and $\hat{B}=3$ and the commutator relation having $\hat{A}=2$ and $\hat{B}=3$ provide the commutator relation between the element (12) and the current having $\hat{B}=3$ and the commutator relation between the element (21) and the current having $\hat{B}=3$. Furthermore, the commutator relation having both $\hat{A}=1$ and $\hat{B}=0$ and the commutator relation having $\hat{A}=2$ and $\hat{B}=0$ provide the commutator relation between the element (12) and the current having $\hat{B}=0$ and the commutator relation between the element (21) and the current having $\hat{B}=0$. Then we obtain the following commutator relations by linear combination of previous four commutators

$$\begin{aligned}
\big[(W^{\lambda,12}_{\text{F},h_1})_m, (W^{\lambda,11}_{\text{F},h_2})_n\big] &= \frac{1}{2}\left(\sum_{h=0,\text{even}}^{h_1+h_2-3} - \sum_{h=-1,\text{odd}}^{h_1+h_2-3}\right) \times q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,12}_{\text{F},h_1+h_2-2-h})_{m+n},\\
\big[(W^{\lambda,12}_{\text{F},h_1})_m, (W^{\lambda,22}_{\text{F},h_2})_n\big] &= \frac{1}{2}\left(\sum_{h=0,\text{even}}^{h_1+h_2-3} + \sum_{h=-1,\text{odd}}^{h_1+h_2-3}\right) \times q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,12}_{\text{F},h_1+h_2-2-h})_{m+n},\\
\big[(W^{\lambda,21}_{\text{F},h_1})_m, (W^{\lambda,11}_{\text{F},h_2})_n\big] &= \frac{1}{2}\left(\sum_{h=0,\text{even}}^{h_1+h_2-3} + \sum_{h=-1,\text{odd}}^{h_1+h_2-3}\right) \times q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,21}_{\text{F},h_1+h_2-2-h})_{m+n},\\
\big[(W^{\lambda,21}_{\text{F},h_1})_m, (W^{\lambda,22}_{\text{F},h_2})_n\big] &= \frac{1}{2}\left(\sum_{h=0,\text{even}}^{h_1+h_2-3} - \sum_{h=-1,\text{odd}}^{h_1+h_2-3}\right) \times q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,21}_{\text{F},h_1+h_2-2-h})_{m+n}. \quad (3.17)
\end{aligned}$$

From the results of the commutator relation between the current having the index $\hat{A}=0$ and the current having $\hat{B}=1$ and the commutator relation between the current having the index $\hat{A}=0$ and the current having $\hat{B}=2$, the commutator relation between the current having the index $\hat{A}=0$ and the element (12) and the one between the current having the index $\hat{A}=0$ and the element (21) can be determined. Moreover, from the commutator relation having the index $\hat{A}=3$ and $\hat{B}=1$ and the commutator relation having the index $\hat{A}=3$ and $\hat{B}=2$, the commutator relation between the current having the index $\hat{A}=3$ and the element (12) and the one between the current having the index $\hat{A}=3$ and the element (21) can be fixed. Therefore, by linear combinations of these results, we can determine the following commutator relations

$$\begin{aligned}
\big[(W^{\lambda,11}_{\text{F},h_1})_m, (W^{\lambda,12}_{\text{F},h_2})_n\big] &= \frac{1}{2}\left(\sum_{h=0,\text{even}}^{h_1+h_2-3} + \sum_{h=-1,\text{odd}}^{h_1+h_2-3}\right) \times q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,12}_{\text{F},h_1+h_2-2-h})_{m+n},\\
\big[(W^{\lambda,22}_{\text{F},h_1})_m, (W^{\lambda,12}_{\text{F},h_2})_n\big] &= \frac{1}{2}\left(\sum_{h=0,\text{even}}^{h_1+h_2-3} - \sum_{h=-1,\text{odd}}^{h_1+h_2-3}\right) \times q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,12}_{\text{F},h_1+h_2-2-h})_{m+n},\\
\big[(W^{\lambda,11}_{\text{F},h_1})_m, (W^{\lambda,21}_{\text{F},h_2})_n\big] &= \frac{1}{2}\left(\sum_{h=0,\text{even}}^{h_1+h_2-3} - \sum_{h=-1,\text{odd}}^{h_1+h_2-3}\right) \times q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,21}_{\text{F},h_1+h_2-2-h})_{m+n},\\
\big[(W^{\lambda,22}_{\text{F},h_1})_m, (W^{\lambda,21}_{\text{F},h_2})_n\big] &= \frac{1}{2}\left(\sum_{h=0,\text{even}}^{h_1+h_2-3} + \sum_{h=-1,\text{odd}}^{h_1+h_2-3}\right) \times q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,21}_{\text{F},h_1+h_2-2-h})_{m+n}. \quad (3.18)
\end{aligned}$$

Finally, from the results of the commutator relation between the current having the index $\hat{A}=0$ and the current having the index $\hat{B}=3$ and the commutator relation between the current having the index $\hat{A}=0$ and the current having the index $\hat{B}=0$, we can determine the following commutator relations by realizing that there is no nontrivial commutator relation between the element (11) and the element (22)

$$\begin{aligned}
\big[(W^{\lambda,11}_{\text{F},h_1})_m, (W^{\lambda,11}_{\text{F},h_2})_n\big] &= \sum_{h=0,\text{even}}^{h_1+h_2-3} q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,11}_{\text{F},h_1+h_2-2-h})_{m+n} + \frac{1}{2}\binom{m+h_1-1}{h_1+h_2-1} c_F(h_1,h_2,\lambda) \times q^{h_1+h_2-4}\,\delta_{m+n},\\
\big[(W^{\lambda,22}_{\text{F},h_1})_m, (W^{\lambda,22}_{\text{F},h_2})_n\big] &= \sum_{h=0,\text{even}}^{h_1+h_2-3} q^h\, p_{\text{F}}^{h_1,h_2,h}(m,n,\lambda) \times (W^{\lambda,22}_{\text{F},h_1+h_2-2-h})_{m+n} + \frac{1}{2}\binom{m+h_1-1}{h_1+h_2-1} c_F(h_1,h_2,\lambda) \times q^{h_1+h_2-4}\,\delta_{m+n}. \quad (3.19)
\end{aligned}$$

Therefore, there are (3.15), (3.16), (3.17), (3.18) and (3.19). Other complete relations are summarized by Appendices (E.1), (E.2), (E.3), (E.4), (E.5) and (E.6). By adding the two equations of (3.19), we obtain the commutator between the singlet currents and by taking the contractions of the currents with the vanishing q limit for $\lambda = 0$ we obtain the $w_{1+\infty}$ algebra [7,21,22].

4 The $\mathcal{N}=4$ supersymmetric linear $W_\infty^{2,2}$ algebra between the $\mathcal{N}=4$ multiplets

Based on the results in previous sections, we present the first five commutator relations between the $\mathcal{N}=4$ multiplets. In the footnotes, some particular examples for the specific weights h_1 and h_2 show that there are extra terms on the right hand sides of the commutators for the nonzero λ we explained in the introduction.

4.1 The commutator relation between the lowest components with $h_1 = h_2, h_2 \pm 1$ for nonzero λ

Let us consider the commutator relation between the lowest components $\Phi_0^{(h_1)}(z)$ and $\Phi_0^{(h_2)}(w)$ of the $\mathcal{N}=4$ multiplet. Because there are no singular terms in the OPE of $W_{F,h_1}^{\lambda,\bar{a}a}(z)$ and $W_{B,h_2}^{\lambda,\bar{b}b}(w)$ according to (2.1), the commutator relation consists of two parts. That is, they are given by $\big[(W_{F,h_1}^{\lambda,\bar{a}a})_m, (W_{F,h_2}^{\lambda,\bar{b}b})_n\big]$ and $\big[(W_{B,h_1}^{\lambda,\bar{a}a})_m, (W_{B,h_2}^{\lambda,\bar{b}b})_n\big]$. Then we can use the relation (3.19) and the last two relations of Appendix (E.1). We need to express the right hand sides of these commutator relations in terms of the components of the $\mathcal{N}=4$ multiplet in order to present the complete algebra between them. The two relations in (3.19) and the last two relations of Appendix (E.1) can be added respectively because the structure constants are common at each expression.

As described before, from the relations (2.7) and (2.11), we obtain the following relations

$$
\begin{aligned}
W_{F,h}^{\lambda,\bar{a}a} &= -\frac{1}{(-4)^{h-2}}\,\Phi_0^{(h)} - \frac{(h-1+2\lambda)}{8(2h-1)(-4)^{h-6}}\,\tilde{\Phi}_2^{(h-2)},\\
W_{B,h}^{\lambda,\bar{a}a} &= \frac{1}{(-4)^{h-2}}\,\Phi_0^{(h)} - \frac{(h-2\lambda)}{8(2h-1)(-4)^{h-6}}\,\tilde{\Phi}_2^{(h-2)}.
\end{aligned}
\tag{4.1}
$$

Once we observe the currents on the left hand sides, then we should rewrite them in terms of the components of $\mathcal{N}=4$ multiplet according to (4.1).

On the other hand, there exist the following relations between the lowest bosonic currents and the lowest component $\Phi_0^{(1)}$ of the first $\mathcal{N}=4$ multiplet from (2.7) and the weight-1 current $U = 2(W_{F,1}^{\lambda,\bar{a}a} + W_{B,1}^{\lambda,\bar{a}a})$ of the $\mathcal{N}=4$ stress energy tensor

$$
\begin{aligned}
W_{F,1}^{\lambda,\bar{a}a} &= 4\,\Phi_0^{(1)} + \lambda\,U,\\
W_{B,1}^{\lambda,\bar{a}a} &= -4\,\Phi_0^{(1)} + \frac{1}{2}\,(1-2\lambda)\,U.
\end{aligned}
\tag{4.2}
$$

This implies that by comparing (4.1) with (4.2), we can identify the previous weight-1 current as $\tilde{\Phi}_2^{(-1)} \equiv 256\,U$.

Furthermore, there are relations between the lowest component $\Phi_0^{(2)}$ of the second $\mathcal{N}=4$ multiplet and the weight-2 stress energy tensor L of the $\mathcal{N}=4$ stress energy tensor together with (2.7) and (2.5) as follows:

$$
\begin{aligned}
W_{F,2}^{\lambda,\bar{a}a} &= -\Phi_0^{(2)} + \frac{1}{3}\,(1+2\lambda)\,L,\\
W_{B,2}^{\lambda,\bar{a}a} &= \Phi_0^{(2)} + \frac{2}{3}\,(1-\lambda)\,L.
\end{aligned}
\tag{4.3}
$$

Again, from the relations (4.1) and (4.3), we identify the stress energy tensor in terms of the component of $\mathcal{N}=4$ multiplet as $\tilde{\Phi}_2^{(0)} \equiv -32\,L$.

Therefore, the commutator relation between the lowest component of the h_1-th $\mathcal{N}=4$ multiplet and the lowest component of the h_2-th $\mathcal{N}=4$ multiplet can be described as, by substituting the relation (4.1) into the above right hand sides of the relevant commutator relations described before,

$$
\begin{aligned}
\big[(\Phi_0^{(h_1)})_m, (\Phi_0^{(h_2)})_n\big] &= \frac{(-4)^{h_1-2}}{(2h_1-1)}\,\frac{(-4)^{h_2-2}}{(2h_2-1)}\\
&\times\Bigg[\binom{m+h_1-1}{h_1+h_2-1}\,q^{h_1+h_2-4}\\
&\times\Big((h_1-2\lambda)(h_2-2\lambda)\,c_F(h_1,h_2,\lambda) + (h_1-1+2\lambda)\\
&\times(h_2-1+2\lambda)\,c_B(h_1,h_2,\lambda)\Big)\,\delta_{m+n}\\
&+\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}\Big(-(h_1-2\lambda)(h_2-2\lambda)\,q^h\,p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)\\
&+(h_1-1+2\lambda)(h_2-1+2\lambda)\,q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\Big)\\
&\times\frac{1}{(-4)^{h_1+h_2-h-4}}\,(\Phi_0^{(h_1+h_2-2-h)})_{m+n}\\
&-\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}\Big((h_1-2\lambda)(h_2-2\lambda)(h_1+h_2-h-3+2\lambda)\\
&\times q^h\,p_{\mathrm{F}}^{h_1,h_2,h}(m,n,\lambda)\\
&+(h_1-1+2\lambda)(h_2-1+2\lambda)(h_1+h_2-h-2-2\lambda)\\
&\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\Big)\\
&\times\frac{1}{8(2h_1+2h_2-2h-5)(-4)^{h_1+h_2-h-8}}\\
&\times(\tilde{\Phi}_2^{(h_1+h_2-h-4)})_{m+n}\Bigg].
\end{aligned}
\tag{4.4}
$$

Each central term is given by (3.9) and (3.11). Due to the even weight h in the summation, the currents of even (or odd) weights can occur depending on the weights h_1 and h_2. When we change the above commutator relation (4.4) into the corresponding OPE, due to the factor $(-1)^{h-1}$ for even h, then there exists an extra minus sign on the right hand side of the OPE. We can easily observe that for the maximum value of dummy variable h, $h = h_1 + h_2 - 3$ with odd $(h_1 + h_2)$, on the right hand side of (4.4), there appear the currents $\Phi_0^{(1)}$ and $\tilde{\Phi}_2^{-1}$ which is related to the previous current U of $\mathcal{N} = 4$ stress energy tensor. Note that we have vanishing structure constant $p_B^{h_1,h_2,h_1+h_2-3}$ at $\lambda = 0$ [8]. On the other hand, for even $(h_1 + h_2)$, the maximum value of the weight h is given by $h = h_1 + h_2 - 4$ because the weight h should be even. For this value, there appear the currents $\Phi_0^{(2)}$ and $\tilde{\Phi}_2^{(0)}$, which is related to the previous current L, with proper λ dependent structure constants on the right hand side of (4.4). Let us emphasize that for nonzero λ, the above commutator holds for the arbitrary weight h_1 under the condition $h_1 = h_2$ or $h_1 = h_2 \pm 1$. We take the particular example which shows that if the weights h_1 and h_2 do not satisfy this condition, then there exist other terms on the right hand of the above commutator.[5]

4.2 The commutator relation between the lowest component and the second component with $h_1 = h_2, h_2 + 1$ for nonzero λ

Let us consider the commutator relation between the lowest component $\Phi_0^{(h_1)}(z)$ of the h_1-th $\mathcal{N} = 4$ multiplet in (2.7) and the second component $\Phi_{\frac{1}{2}}^{(h_2),i}(w)$ of h_2-th $\mathcal{N} = 4$ multiplet in (2.8).

We expect to have the fermionic currents $Q_{h+\frac{1}{2}}^{\lambda,\bar{a}b}$ and $\bar{Q}_{h+\frac{1}{2}}^{\lambda,b\bar{a}}$ on the right hand side of the commutator relation because the product of bosonic and fermionic operators produce the fermionic ones. As before, we need to rewrite them in terms of the components of the $\mathcal{N} = 4$ multiplet in order to complete the algebra we are considering. For the index $i = 1$ of these components of the $\mathcal{N} = 4$ multiplets, the following relations are satisfied

$$
\begin{aligned}
&\frac{1}{2}\left(Q_{h+\frac{1}{2}}^{\lambda,11} + i\sqrt{2}\,Q_{h+\frac{1}{2}}^{\lambda,12} + 2i\sqrt{2}\,Q_{h+\frac{1}{2}}^{\lambda,21} - 2\,Q_{h+\frac{1}{2}}^{\lambda,22}\right)\\
&= \frac{1}{2}\left[\frac{1}{4(-4)^{h-4}}\,\Phi_{\frac{1}{2}}^{(h),1} - \frac{1}{4(-4)^{h-5}}\,\tilde{\Phi}_{\frac{3}{2}}^{(h-1),1}\right],\\
&\frac{1}{2}\left(2\,\bar{Q}_{h+\frac{1}{2}}^{\lambda,11} + 2i\sqrt{2}\bar{Q}_{h+\frac{1}{2}}^{\lambda,12} + i\sqrt{2}\bar{Q}_{h+\frac{1}{2}}^{\lambda,21} - \bar{Q}_{h+\frac{1}{2}}^{\lambda,22}\right)\\
&= \frac{1}{2}\left[\frac{1}{4(-4)^{h-4}}\,\Phi_{\frac{1}{2}}^{(h),1} + \frac{1}{4(-4)^{h-5}}\,\tilde{\Phi}_{\frac{3}{2}}^{(h-1),1}\right]. \qquad (4.5)
\end{aligned}
$$

Note that the weight on both sides is given by $(h + \frac{1}{2})$. On the right hand side, the $(h-1)$-th component of $\mathcal{N} = 4$ multiplet appears also. Other relations for $i = 2, 3, 4$ appear in Appendix (F.1). For $h = 0$, the above relation (4.5) with others in Appendix (F.1) implies that there exists the relation between the currents $\tilde{\Phi}_{\frac{3}{2}}^{(-1),i} = -\frac{1}{4}\,\Phi_{\frac{1}{2}}^{(0),i}$ together with (2.8) and (2.10) because the left hand side of the first relation of (4.5) is identically zero and moreover, the weight-$\frac{1}{2}$ current of the $\mathcal{N} = 4$ stress energy tensor has the following relation $-i\,\Gamma^i = 64\,\Phi_{\frac{1}{2}}^{(0),i}$ with (2.8). For $h = 1$, the corresponding weight-$\frac{3}{2}$ currents of $\mathcal{N} = 4$ stress energy tensor can be written as $G^i = 64\,\tilde{\Phi}_{\frac{3}{2}}^{(0),i}$ with (2.10) by subtracting the two relation of (4.5). Therefore, the components, U, Γ^i, G^i and L of $\mathcal{N} = 4$ stress energy tensor, except the weight-1 currents T^{ij} which will appear in next subsection, are given by the components of the $\mathcal{N} = 4$ multiplet, $\tilde{\Phi}_2^{(-1)}$, $\Phi_{\frac{1}{2}}^{(0),i}$, $\tilde{\Phi}_{\frac{3}{2}}^{(0),i}$ and $\tilde{\Phi}_2^{(0)}$ up to normalization respectively.

The result of the commutator relation we are considering, by replacing all the fermionic currents with the $\mathcal{N} = 4$ components appearing in (4.5) and Appendix (F.1), can be summarized by

$$
\begin{aligned}
&[(\Phi_0^{(h_1)})_m, (\Phi_{\frac{1}{2}}^{(h_2),i})_r] = 5\,\frac{(-4)^{h_1-2}}{(2h_1-1)}\,4(-4)^{h_2-4}\\
&\times\Bigg[\Bigg(-(h_1-2\lambda)\sum_{h=-1}^{h_1+h_2-3} q^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\\
&-(h_1-2\lambda)\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\\
&+(h_1-1+2\lambda)\sum_{h=-1}^{h_1+h_2-3} q^h q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\\
&+(h_1-1+2\lambda)\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\Bigg)\\
&\times\frac{1}{8(-4)^{h_1+h_2-h-6}}\,(\Phi_{\frac{1}{2}}^{(h_1+h_2-2-h)})_{m+r}
\end{aligned}
$$

[5] Let us take the OPE between $\Phi_0^{(h_1=1)}(z)$ and $\Phi_0^{(h_2=3)}(w)$ where the weights satisfy $h_1 = h_2 - 2$. The third order pole of this OPE provides the structure constant $2048\lambda(2\lambda-1)$ appearing in the current $\Phi_0^{(1)}(w)$ and the structure constant $\frac{4}{5}\lambda(2\lambda-1)(4\lambda-1)$ appearing in the current $U(w)$. According to (4.4), the exponent of the first current implies $h_1+h_2-2-h = 2-h$. Moreover, h is given by $h = 0$ from the summation and the exponent is 2. Therefore, the current $\Phi_0^{(1)}(w)$ cannot appear from the (4.4). Furthermore, the exponent in the second current implies $h_1+h_2-4-h = -h = -1$ from the presence of U. Moreover, h is given by $h = 0$ from the summation and there is a contradiction. In this case also, the current $U(w)$ cannot appear from the corresponding term in (4.4). Note that the two structure constants contain the λ factor and they become zero at $\lambda = 0$. From the above analysis, there exist extra terms on the right hand side of the commutator (or corresponding OPE) for the weights which do not satisfy the constraint $h_1 = h_2$ or $h_1 = h_2 \pm 1$ we mentioned before. As in previous footnotes in Sect. 3, this feature also arises for the singlet currents.

$$+\Big(\sum_{h=-1}^{h_1+h_2-3}(h_1-2\lambda)\,q^h\,q_{\rm F}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)$$
$$-\sum_{h=-1}^{h_1+h_2-2}(h_1-2\lambda)\,q^h\,(-1)^h\,q_{\rm F}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)$$
$$-\sum_{h=-1}^{h_1+h_2-3}(h_1-1+2\lambda)\,q^h\,q_{\rm B}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)$$
$$+\sum_{h=-1}^{h_1+h_2-2}(h_1-1+2\lambda)\,q^h\,(-1)^h\,q_{\rm B}^{h_1,h_2+\frac{1}{2},h}(m,r,\lambda)\Big)$$
$$\times\frac{1}{8(-4)^{h_1+h_2-h-7}}\,(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-3-h)})_{m+r}\Big]. \tag{4.6}$$

Note that the upper limit of h having the term with the factor $(-1)^h$ is given by $h=h_1+h_2-2$. Except these four terms, the range of h in all the other terms is the same. Then these four terms occur if their structure constants are nonvanishing. Depending on the odd h or even h, some of terms in (4.6) can cancel each other because four kinds of each two terms have the same structure except the factor $(-1)^h$. At $h=h_1+h_2-2$, there appear the current $\Phi_{\frac{1}{2}}^{(0),i}$ and the current $\tilde{\Phi}_{\frac{3}{2}}^{(-1),i}$ which are related to the previous weight-$\frac{1}{2}$ current Γ^i as above, with the relevant structure constants. Moreover, we have vanishing structure constants $q_F^{h_1,h_2+\frac{1}{2},h_1+h_2-2}=0=q_B^{h_1,h_2+\frac{1}{2},h_1+h_2-2}$ at $\lambda=0$ [8]. For the $h\le h_1+h_2-3$ with odd h in (4.6), there appear the current $\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-3-h),i}$ terms because the coefficients of the current $\Phi_{\frac{1}{2}}^{(h_1+h_2-2-h),i}$ terms are identically vanishing. Note that there are relative signs in (4.6). On the other hand, for the $h\le h_1+h_2-3$ with even h, there appear the current $\Phi_{\frac{1}{2}}^{(h_1+h_2-2-h),i}$ terms because the coefficients of the current $\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-3-h),i}$ terms are identically zero. We also present some example where the weights h_1 and h_2 do not satisfy the condition $h_1=h_2$ or $h_1=h_2+1$.[6]

4.3 The commutator relation between the lowest component and the third component with $h_1=h_2, h_2+1, h_2+2$ for nonzero λ

Let us consider the commutator relation between the lowest component $\Phi_0^{(h_1)}(z)$ of the h_1-th $\mathcal{N}=4$ multiplet and the third component $\Phi_1^{(h_2),ij}(w)$ of the h_2-th $\mathcal{N}=4$ multiplet. Because there are no singular terms in the OPE of $W_{F,h_1}^{\lambda,\bar{a}b}(z)$ and $W_{B,h_2}^{\lambda,\bar{c}d}(w)$, the commutator relation consists of two parts. That is, there are $\big[(W_{F,h_1}^{\lambda,\bar{a}a})_m,(W_{F,h_2}^{\lambda,\bar{c}d})_n\big]$ and $\big[(W_{B,h_1}^{\lambda,\bar{a}a})_m,(W_{B,h_2}^{\lambda,\bar{c}d})_n\big]$. Then we can use the previous relations, (3.15), (3.16), (3.17), (3.18) and (3.19) and Appendix (E.1). We need to express the right hand sides of these commutator relations in terms of the components of the $\mathcal{N}=4$ multiplet as before.

We obtain the following relations

$$2i\,W_{{\rm B},h+1}^{\lambda,11}-\sqrt{2}\,W_{{\rm B},h+1}^{\lambda,12}-2i\,W_{{\rm B},h+1}^{\lambda,22}=\frac{1}{8(-4)^{h-4}}\Big[\Phi_1^{(h),12}-\Phi_1^{(h),34}\Big],$$
$$2i\,W_{{\rm F},h+1}^{\lambda,11}-2\sqrt{2}\,W_{{\rm F},h+1}^{\lambda,12}-2i\,W_{{\rm F},h+1}^{\lambda,22}=\frac{1}{8(-4)^{h-4}}\Big[\Phi_1^{(h),12}+\Phi_1^{(h),34}\Big]. \tag{4.7}$$

For other relations, we present them in Appendix (F.2). By using (4.7) and Appendix (F.2), we can express the weight-1 current T^{ij} of the $\mathcal{N}=4$ stress energy tensor as $i\,T^{ij}=64\,\Phi_1^{(0),ij}$ with (2.9). On the right hand sides of (4.7), there are only the components of hth $\mathcal{N}=4$ multiplet. By linear combinations, any component of the hth $\mathcal{N}=4$ multiplet can be written in terms of $W_{F,h+1}^{\lambda,\bar{a}b}$ and $W_{B,h+1}^{\lambda,\bar{c}d}$ explicitly.

It turns out that the corresponding commutator relation, after substituting all the bosonic currents in terms of the components of the $\mathcal{N}=4$ multiplet described above, can be written as

$$\big[(\Phi_0^{(h_1)})_m,(\Phi_1^{(h_2),ij})_n\big]=\frac{(-4)^{h_1-2}}{(2h_1-1)}\,4(-4)^{h_2-4}$$

[6] Along the line of the footnote 5, we consider the OPE between $\Phi_0^{(h_1=4)}(z)$ and $\Phi_{\frac{1}{2}}^{(h_2=2),i}(w)$ where $h_1=h_2+2$. The sixth order pole of this OPE gives us the structure constant $\frac{2048}{7}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)(4\lambda-1)$ appearing in the current $\Phi_{\frac{1}{2}}^{(h_1+h_2-2-h=0),i}(w)$ with the weight $h=4$. By substituting the various expressions in the corresponding terms of (4.6), we can check that we obtain the above structure constant correctly where $q_F^{4,\frac{5}{2},4}$ term corresponds to $\frac{256}{3}(\lambda-1)\lambda(2\lambda-3)(2\lambda-1)(2\lambda+1)$ while $q_B^{4,\frac{5}{2},4}$ term corresponds to $-\frac{512}{3}(\lambda-1)\lambda(\lambda+1)(2\lambda-1)(2\lambda+1)$. There is no current $\tilde{\Phi}_{\frac{3}{2}}^{(-1),i}(w)$ term in the sixth order pole from our calculation and this is consistent with (4.6) because all the coefficients vanish for the even $h=4$.
Now we can move to the fifth order pole of this OPE. The current $G^i(w)$ term of $\mathcal{N}=4$ stress energy tensor has the structure constant $-\frac{512}{7}(\lambda-1)(2\lambda+1)(6\lambda^2-3\lambda-4)$ which can be identified with

Footnote 6 continued
the coefficient appearing in the second current $\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-3-h=0),i}(w)$ of (4.6) by substituting $q_F^{4,\frac{5}{2},3}$ and $q_B^{4,\frac{5}{2},3}$ correctly as before. On the other hand, there exists the current $\Phi_{\frac{1}{2}}^{(h_1+h_2-2-h=1),i}(w)$ term with the structure constant $\frac{8192}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)$ having the λ factor. We can check that this structure constant is equal to the one in the first current terms in (4.6) where the previous $q_F^{4,\frac{5}{2},3}$ and $q_B^{4,\frac{5}{2},3}$ are replaced by $q_F^{3,\frac{5}{2},3}$ and $q_B^{3,\frac{5}{2},3}$ respectively. That is, the h_1 is replaced by (h_1-1). Therefore, we cannot use the Eq. (4.6) for this particular pole fully. We can check that the remaining lower order poles can be described by (4.6) without any extra terms precisely.

$$\times\Bigg[-(h_1-2\lambda)\sum_{h=-1,\text{even}}^{h_1+h_2-2} q^h\, p_{\rm F}^{h_1,h_2+1,h}(m,n,\lambda)$$
$$\times\frac{1}{8(-4)^{h_1+h_2-h-6}}\Bigg(\Phi_1^{(h_1+h_2-2-h),ij}$$
$$+\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-2-h),kl}\Bigg)_{m+n}$$
$$+(h_1-1+2\lambda)\sum_{h=-1,\text{even}}^{h_1+h_2-2} q^h\, p_{\rm B}^{h_1,h_2+1,h}(m,n,\lambda)$$
$$\times\frac{1}{8(-4)^{h_1+h_2-h-6}}\Bigg(\Phi_1^{(h_1+h_2-2-h),ij}$$
$$-\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-2-h),kl}\Bigg)_{m+n}\Bigg]. \tag{4.8}$$

Due to the even weight h, on the right hand side of (4.8), the currents of odd (or even) weights can appear depending on the weights h_1 and h_2. Due to the $SO(4)$ index ij appearing on the left hand side of (4.8), the field contents on the right hand side are different from the ones in (4.4). At the maximum value of h, $h=h_1+h_2-2$, the current $\Phi_1^{(0),ij}$ term and the current $\varepsilon^{ijkl}\,\Phi_1^{(0),kl}$ term, which are related to the previous weight-1 current of the $\mathcal{N}=4$ stress energy tensor, occur on the right hand side of (4.8). Note that as before we have vanishing structure constant $p_B^{h_1,h_2+1,h_1+h_2-2}$ at $\lambda=0$ [8]. In this case also, there some constraints on the weight h_2 for nonzero λ.[7]

4.4 The commutator relation between the lowest component and the fourth component with $h_1=h_2+1, h_2+2$ for nonzero λ

In this case, we can use the previous relation (4.5) and Appendix (F.1) by simply substituting h with $(h+1)$ in order to rewrite the right hand side of this commutator relation in terms of the components of the $\mathcal{N}=4$ multiplet.

Then we can determine the following result for the commutator relation between the lowest component $\Phi_0^{(h_1)}(z)$ of the h_1-th $\mathcal{N}=4$ multiplet and the fourth component (and derivative term) $\tilde{\Phi}_{\frac{3}{2}}^{(h_2),i}(w)$ of the h_2-th $\mathcal{N}=4$ multiplet

$$\big[(\Phi_0^{(h_1)})_m,(\tilde{\Phi}_{\frac{3}{2}}^{(h_2),i})_r\big]=\frac{(-4)^{h_1-2}}{(2h_1-1)}\,4(-4)^{h_2-4}$$

$$\times\Bigg[\Bigg((h_1-2\lambda)\sum_{h=-1}^{h_1+h_2-2} q^h\, q_{\rm F}^{h_1,h_2+\frac{3}{2},h}(m,r,\lambda)$$
$$-(h_1-2\lambda)\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h q_{\rm F}^{h_1,h_2+\frac{3}{2},h}(m,r,\lambda)$$
$$-(h_1-1+2\lambda)\sum_{h=-1}^{h_1+h_2-2} q^h q_{\rm B}^{h_1,h_2+\frac{3}{2},h}(m,r,\lambda)$$
$$+(h_1-1+2\lambda)\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\, q_{\rm B}^{h_1,h_2+\frac{3}{2},h}(m,r,\lambda)\Bigg)$$
$$\times\frac{1}{8(-4)^{h_1+h_2-h-5}}\,(\Phi_{\frac{1}{2}}^{(h_1+h_2-1-h),i})_{m+r}$$
$$+\Bigg(-(h_1-2\lambda)\sum_{h=-1}^{h_1+h_2-2} q^h\, q_{\rm F}^{h_1,h_2+\frac{3}{2},h}(m,r,\lambda)$$
$$-(h_1-2\lambda)\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h q_{\rm F}^{h_1,h_2+\frac{3}{2},h}(m,r,\lambda)$$
$$+(h_1-1+2\lambda)\sum_{h=-1}^{h_1+h_2-2} q^h q_{\rm B}^{h_1,h_2+\frac{3}{2},h}(m,r,\lambda)$$
$$+(h_1-1+2\lambda)\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\, q_{\rm B}^{h_1,h_2+\frac{3}{2},h}(m,r,\lambda)\Bigg)$$
$$\times\frac{1}{8(-4)^{h_1+h_2-h-6}}\,(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2-h),i})_{m+r}\Bigg]. \tag{4.9}$$

The form of this (4.9) looks similar to the previous one in (4.6) because the field contents are the same. The relative signs are different from each other and we observe that after replacing h_2 with (h_2+1) appearing in (4.6), the corresponding expressions occur in (4.9). At $h=h_1+h_2-1$, there appear the $\Phi_{\frac{1}{2}}^{(0),i}$ and the $\tilde{\Phi}_{\frac{3}{2}}^{(-1),i}$ which are related to the weight-$\frac{1}{2}$ current Γ^i. Except four terms having the factor $(-1)^h$ with $h=h_1+h_2-1$, there some cancellation between the currents. Furthermore, we have vanishing structure constants $q_F^{h_1,h_2+\frac{3}{2},h_1+h_2-1}=0=q_B^{h_1,h_2+\frac{3}{2},h_1+h_2-1}$ at $\lambda=0$ [8]. For the $h\le h_1+h_2-2$ with odd h in (4.9), there appear the current $\Phi_{\frac{1}{2}}^{(h_1+h_2-1-h),i}$ terms because the coefficients of the current $\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2-h),i}$ terms are identically vanishing. On the other hand, for the $h\le h_1+h_2-2$ with even h, there appear the current $\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2-h),i}$ terms because the coefficients of the current $\Phi_{\frac{1}{2}}^{(h_1+h_2-2-h),i}$ terms are identically zero. This behavior is different from the one in (4.6) because the appearance of signs behaves differently. We also comment on the possible ranges for the weight h_2 as described before.[8]

[7] For the OPE between $\Phi_0^{(h_1=4)}(z)$ and $\Phi_1^{(h_2=1),ij}(w)$ where the weights satisfy $h_1=h_2+3$, the fifth order pole of this OPE has the structure constant $-\frac{8192}{7}\lambda(2\lambda-1)(4\lambda^2-2\lambda-5)$ in the current $\Phi_1^{(0),ij}(w)$ term and the structure constant $\frac{8192}{7}\lambda(2\lambda-1)(4\lambda-1)$ in the current $\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(0),kl}(w)$ term from our calculation. However, the corresponding structure constants $p_F^{h_1=4,h_2+1=2,h=2}$ and $p_B^{h_1=4,h_2+1=2,h=2}$ do not produce these λ dependent structure constants respectively. This implies that there are extra contributions in the structure constants for the weights which do not satisfy the condition $h_1=h_2$, $h_1=h_2+1$, or $h_1=h_2+2$.

[8] Similar to the previous examples, we consider the OPE between $\Phi_0^{(h_1=4)}(z)$ and $\Phi_{\frac{3}{2}}^{(h_2=1),i}(w)$ where the weights satisfy $h_1=h_2+3$. The sixth order pole of this OPE provides the structure constant

4.5 The commutator relation between the lowest component and the last component with $h_1 = h_2 + 1, h_2 + 2, h_2 + 3$ for nonzero λ

Finally, the commutator relation between the lowest component $\Phi_0^{(h_1)}(z)$ of the h_1-th $\mathcal{N} = 4$ multiplet and the last component (and derivative term) $\tilde{\Phi}_2^{(h_2)}(w)$ of the h_2th $\mathcal{N} = 4$ multiplet can be determined by using (4.1)

$$
\begin{aligned}
&[(\Phi_0^{(h_1)})_m, (\tilde{\Phi}_2^{(h_2)})_n] = \frac{(-4)^{h_1-2}}{(2h_1-1)}\, 4(-4)^{h_2-4} \\
&\times\Bigg[\begin{pmatrix} m+h_1-1 \\ h_1+h_2-1 \end{pmatrix} q^{h_1+h_2-2} \Big(2(h_1-2\lambda)\, c_F \\
&-2(h_1-1+2\lambda)\, c_B\Big)\,\delta_{m+n} \\
&+\sum_{h=0,\text{even}}^{h_1+h_2-1} \frac{2\,q^h}{(-4)^{h_1+h_2-h-2}} \Big(-(h_1-2\lambda)\, p_{\text{F}}^{h_1,h_2+2,h}(m,n,\lambda) \\
&-(h_1-1+2\lambda)\, p_{\text{B}}^{h_1,h_2+2,h}(m,n,\lambda)\Big)\,(\Phi_0^{(h_1+h_2-h)})_{m+n} \\
&+\sum_{h=0,\text{even}}^{h_1+h_2-1} \frac{q^h}{4(2h_1+2h_2-2h-1)(-4)^{h_1+h_2-h-6}} \\
&\times\Big(-(h_1-2\lambda)(h_1+h_2-h-1+2\lambda)\, p_{\text{F}}^{h_1,h_2+2,h}(m,n,\lambda) \\
&+(h_1-1+2\lambda)(h_1+h_2-h-2\lambda)\, p_{\text{B}}^{h_1,h_2+2,h}(m,n,\lambda)\Big) \\
&\times(\tilde{\Phi}_2^{(h_1+h_2-h-2)})_{m+n}\Bigg]. \qquad (4.10)
\end{aligned}
$$

The field contents appearing in (4.10) are the same as the one in (4.4). As before, we observe that for the maximum value of dummy variable h, $h = h_1+h_2-1$ with odd (h_1+h_2), on the right hand side of (4.10), there appear the currents $\Phi_0^{(1)}$ and $\tilde{\Phi}_2^{-1}$ which is related to the previous current U of $\mathcal{N} = 4$ stress energy tensor. On the other hand, for even (h_1+h_2), the maximum value of the weight h is given by $h = h_1 + h_2 - 2$ because the weight h should be even. For this value, there appear the currents $\Phi_0^{(2)}$ and $\tilde{\Phi}_2^{(0)}$, which is related to the previous current L, with proper λ dependent structure constants on the right hand side of (4.10). Note that as before we have vanishing structure constant $p_B^{h_1,h_2+2,h_1+h_2-1}$ at $\lambda = 0$ [8]. We comment on the other possible cases for the weights.[9]

In Appendix G, the remaining (anti) commutator relations are given. Therefore, the (anti)commutator relations between the $\mathcal{N} = 4$ multiplets are summarized by (4.4), (4.6), (4.8), (4.9) and (4.10) in addition to Appendices (G.1), (G.2), (G.3), (G.5), (G.6), (G.9), (G.10), (G.11), (G.13) and (G.14). Among these, the more fundamental (anti)commutator relations are given by (4.4), (4.6), (4.8), Appendices (G.1), and (G.6) in the sense that the field contents on the right hand sides of the remaining ones can be seen from the ones of these fundamental relations up to signs. As mentioned at the end of previous Sect. 3, by using (4.4), (4.10) and Appendix (G.14) together with (4.1), the $w_{1+\infty}$ algebra can be obtained by taking the proper limit on the parameter q at $\lambda = 0$ with the contractions of the currents.

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Footnote 8 continued
$\frac{32768}{7}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)(4\lambda-1)$ appearing in the current $\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2-h=-1),i}(w)$ with the weight $h = 4$. By substituting the various expressions in the corresponding terms of (4.9), we can check that we obtain the above structure constant correctly where $q_F^{4,\frac{5}{2},4}$ term corresponds to the λ dependent factor $\frac{256}{3}(\lambda-1)\lambda(2\lambda-3)(2\lambda-1)(2\lambda+1)$ while $q_B^{4,\frac{5}{2},4}$ term corresponds to the λ dependent factor $-\frac{512}{3}(\lambda-1)\lambda(\lambda+1)(2\lambda-1)(2\lambda+1)$. There is no $\Phi_{\frac{1}{2}}^{(1),i}(w)$ term in the sixth order pole from our calculation and this is consistent with (4.9) because all the coefficients vanish for the even weight $h = 4$.
Now we can move to the fifth order pole of above OPE. The $\Phi_{\frac{1}{2}}^{(h_1+h_2-2-h=1),i}(w)$ term has the structure constant $-\frac{2048}{7}(\lambda-1)(2\lambda+1)(6\lambda^2-3\lambda-4)$ which can be identified with the coefficient appearing in the second current $\Phi_{\frac{1}{2}}^{(h_1+h_2-2-h=1),i}(w)$ of (4.9) by substituting $q_F^{4,\frac{5}{2},3}$ and $q_B^{4,\frac{5}{2},3}$ correctly. On the other hand, there exists the current $G^i(w)$ term of $\mathcal{N} = 4$ stress energy tensor with the structure constant $\frac{128}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)$ having the λ factor. We can check that this structure constant is equal to the one in the first current terms in (4.9) where the previous $q_F^{4,\frac{5}{2},3}$ and $q_B^{4,\frac{5}{2},3}$ are replaced by $q_F^{3,\frac{5}{2},3}$ and $q_B^{3,\frac{5}{2},3}$ respectively. The weight h_1 is replaced by the weight (h_1-1) and this reflects the fact that the weight h_1 is increased by 1 from $h_1 = h_2 + 2$ to $h_1 = h_2 + 3$. Now we have seen the extra structures on the right hand side of this particular pole for the weights we are considering.

[9] From the OPE between $\Phi_0^{(h_1=4)}(z)$ and $\Phi_2^{(h_2=0)}(w)$ where the weights satisfy $h_1 = h_2 + 4$, the fifth order pole of this OPE provides the structure constant $\frac{8192}{7}\lambda(2\lambda-1)(4\lambda-1)$ appearing in the current $\Phi_0^{(1)}(w)$ and the structure constant $\frac{256}{7}\lambda(2\lambda-1)(10\lambda^2-5\lambda-2)$ appearing in the current $U(w)$. In (4.10), the exponent of the first current implies the weight $h_1+h_2-h = 4-h$. Moreover, the weight h is given by $h = 0$ or $h = 2$ from the summation. Therefore, the current $\Phi_0^{(1)}(w)$ cannot appear from the (4.10). Furthermore, the exponent of the second current implies the weight $h_1+h_2-2-h = 2-h = -1$ from the presence of U. Moreover, the weight h is given by $h = 0$ or $h = 2$ from the summation. In this case also, the current $U(w)$ cannot appear from the (4.10). Therefore, there appear the extra terms for the currents having the weights we are considering which do not satisfy the above constraints.

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Appendix A: The structure constants

By introducing the generalized hypergeometric function

$$\phi_r^{h_1,h_2}(\Lambda,a) \equiv {}_4F_3\left[\begin{matrix}\frac{1}{2}+\Lambda & \frac{1}{2}-\Lambda & \frac{1+a-r}{2} & \frac{a-r}{2}\\ \frac{3}{2}-h_1 & \frac{3}{2}-h_2 & \frac{1}{2}+h_1+h_2-r\end{matrix}\middle|1\right], \quad (A.1)$$

and mode dependent function

$$N_h^{h_1,h_2}(m,n) \equiv \sum_{l=0}^{h+1}(-1)^l\binom{h+1}{l} \times[h_1-1+m]_{h+1-l}[h_1-1-m]_l \times[h_2-1+n]_l[h_2-1-n]_{h+1-l}, \quad (A.2)$$

the three kinds of structure constants in [13] can be summarized, together with (A.1) and (A.2), by

$$\begin{aligned}\mathrm{BB}_{r,\pm}^{h_1,h_2}(m,n;\mu) &\equiv -\frac{1}{(r-1)!}N_{r-2}^{h_1,h_2}(m,n)\times\left[\phi_r^{h_1,h_2}(\mu,1)\pm\phi_r^{h_1,h_2}(1-\mu,1)\right],\\ \mathrm{BF}_{r,\pm}^{h_1,h_2+\frac{1}{2}}(m,\rho;\mu) &\equiv -\frac{1}{(r-1)!}N_{r-2}^{h_1,h_2+\frac{1}{2}}(m,\rho)\times\left[\phi_{r+1}^{h_1,h_2+1}\left(\mu,\frac{3\pm1}{2}\right)\pm\phi_{r+1}^{h_1,h_2+1}\left(1-\mu,\frac{3\pm1}{2}\right)\right],\\ \mathrm{FF}_{r,\pm}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\rho,\omega;\mu) &\equiv -\frac{1}{(r-1)!}N_{r-2}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\rho,\omega)\times\left[\phi_{r+1}^{h_1+1,h_2+1}\left(\mu,\frac{3\pm1}{2}\right)\pm\phi_{r+1}^{h_1+1,h_2+1}\left(1-\mu,\frac{3\pm1}{2}\right)\right].\end{aligned} \quad (A.3)$$

Then the structure constants in Sect. 3, from (A.3), are given by

$$\begin{aligned}p_{F,h}^{h_1,h_2}(m,n,\lambda) &= -\frac{1}{4}\left[\mathrm{BB}_{h+2,+}^{h_1,h_2}+\mathrm{BB}_{h+2,-}^{h_1,h_2}\right]_{\mu=2\lambda},\\ p_{B,h}^{h_1,h_2}(m,n,\lambda) &= -\frac{1}{4}\left[\mathrm{BB}_{h+2,+}^{h_1,h_2}-\mathrm{BB}_{h+2,-}^{h_1,h_2}\right]_{\mu=2\lambda},\\ q_{F,2h}^{h_1,h_2+\frac{1}{2}}(m,n,\lambda) &= \left[-\frac{1}{8}\mathrm{BF}_{2h+2,+}^{h_1,h_2+\frac{1}{2}}+\frac{(2h_1-2h-3)}{16(h+1)}\mathrm{BF}_{2h+2,-}^{h_1,h_2+\frac{1}{2}}\right]_{\mu=2\lambda},\\ q_{F,2h+1}^{h_1,h_2+\frac{1}{2}}(m,n,\lambda) &= \left[\frac{1}{8}\mathrm{BF}_{2h+3,+}^{h_1,h_2+\frac{1}{2}}-\frac{(h_1-h-2)}{4(2h+3)}\mathrm{BF}_{2h+3,-}^{h_1,h_2+\frac{1}{2}}\right]_{\mu=2\lambda},\\ q_{B,2h}^{h_1,h_2+\frac{1}{2}}(m,n,\lambda) &= \left[-\frac{1}{8}\mathrm{BF}_{2h+2,+}^{h_1,h_2+\frac{1}{2}}-\frac{(2h_1-2h-3)}{16(h+1)}\mathrm{BF}_{2h+2,-}^{h_1,h_2+\frac{1}{2}}\right]_{\mu=2\lambda},\\ q_{B,2h+1}^{h_1,h_2+\frac{1}{2}}(m,n,\lambda) &= \left[-\frac{1}{8}\mathrm{BF}_{2h+3,+}^{h_1,h_2+\frac{1}{2}}-\frac{(h_1-h-2)}{4(2h+3)}\mathrm{BF}_{2h+3,-}^{h_1,h_2+\frac{1}{2}}\right]_{\mu=2\lambda},\\ o_{F,2h}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(m,n,\lambda) &= \left[-\mathrm{FF}_{2h+1,+}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}-\frac{2(h_1+h_2-h)}{(2h+1)}\mathrm{FF}_{2h+1,-}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}\right]_{\mu=2\lambda},\\ o_{F,2h+1}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(m,n,\lambda) &= \left[\mathrm{FF}_{2h+2,+}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}+\frac{2(h_1+h_2-h)-1}{2(h+1)}\mathrm{FF}_{2h+2,-}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}\right]_{\mu=2\lambda},\\ o_{B,2h}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(m,n,\lambda) &= \left[-\mathrm{FF}_{2h+1,+}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}+\frac{2(h_1+h_2-h)}{(2h+1)}\mathrm{FF}_{2h+1,-}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}\right]_{\mu=2\lambda},\\ o_{B,2h+1}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(m,n,\lambda) &= \left[-\mathrm{FF}_{2h+2,+}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}+\frac{2(h_1+h_2-h)-1}{(2h+2)}\mathrm{FF}_{2h+2,-}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}\right]_{\mu=2\lambda}.\end{aligned} \quad (A.4)$$

At $\lambda=0$, the above structure constants reduce to the ones in [6,12].

Appendix B: The other central terms

As done in (3.8), we obtain the following central terms

$$V_{\lambda,\bar{a}b}^{(h_1)+}(z)\,V_{\lambda,\bar{c}d}^{(h_2)-}(w)\Big|_{\frac{1}{(z-w)^{h_1+h_2}}} = N\,\delta_{b\bar{c}}\,\delta_{d\bar{a}}\sum_{j=0}^{h_1-1}\sum_{i=0}^{h_2-1}\sum_{t=0}^{j+1} \times\left(-\frac{(h_2-1+2\lambda)}{(2h_2-1)}a^j(h_1,\lambda)\,a^i(h_2,\lambda) -\frac{(h_2-2\lambda)}{(2h_2-1)}a^j(h_1,\lambda+\frac{1}{2})\,a^i\left(h_2,\lambda+\frac{1}{2}\right)\right)$$

$$\times \frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1+t}\,(j+1-t)_{h_1-1-j}$$
$$\times (h_1-t)_{t+1+i}\,(h_1+1+i)_{h_2-1-i},$$

$$V^{(h_1)-}_{\lambda,\bar{a}b}(z)\,V^{(h_2)-}_{\lambda,\bar{c}d}(w)\Big|_{\frac{1}{(z-w)^{h_1+h_2}}} = N\,\delta_{b\bar{c}}\,\delta_{d\bar{a}}\sum_{j=0}^{h_1-1}\sum_{i=0}^{h_2-1}\sum_{t=0}^{j+1}$$
$$\times\Bigg(\frac{(h_1-1+2\lambda)}{(2h_1-1)}\,\frac{(h_2-1+2\lambda)}{(2h_2-1)}\,a^j(h_1,\lambda)\,a^i(h_2,\lambda)$$
$$-\frac{(h_1-2\lambda)}{(2h_1-1)}\,\frac{(h_2-2\lambda)}{(2h_2-1)}\,a^j(h_1,\lambda+\frac{1}{2})\,a^i(h_2,\lambda+\frac{1}{2})\Bigg)$$
$$\times \frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1+t}\,(j+1-t)_{h_1-1-j}$$
$$\times (h_1-t)_{t+1+i}\,(h_1+1+i)_{h_2-1-i}, \qquad \text{(B.1)}$$

where the previous relations in (2.3) are used.

Appendix C: The OPEs between the nonsinglet currents

In this Appendix, the three kinds of OPEs corresponding to the ones in Sect. 3 are given explicitly.

C.1 The OPE between the bosonic currents

We present the explicit check for (3.1) and (3.10) with the weights $h_1 = h_2 = 4$.

C.1.1 The OPE between $W^{\lambda,\hat{A}=1}_{F,4}$ and itself

For the same indices $\hat{A} = \hat{B} = 1$ in (3.1), we obtain the following result

$$(W^{\lambda,12}_{F,4}+W^{\lambda,21}_{F,4})(z)\,(W^{\lambda,12}_{F,4}+W^{\lambda,21}_{F,4})(w)$$
$$=\frac{1}{(z-w)^8}\left[-\frac{1536}{5}(112\lambda^6-280\lambda^4+147\lambda^2-9)\right]$$
$$+\frac{1}{(z-w)^6}\left[\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)\right]$$
$$\times(W^{\lambda,11}_{F,2}+W^{\lambda,22}_{F,2})(w)$$
$$+\frac{1}{(z-w)^5}\,\frac{1}{2}\left[\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)\right]$$
$$\times\partial\,(W^{\lambda,11}_{F,2}+W^{\lambda,22}_{F,2})(w)$$
$$+\frac{1}{(z-w)^4}\Bigg[\frac{3}{20}\,\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times\partial^2\left(W^{\lambda,11}_{F,2}+W^{\lambda,22}_{F,2}\right)$$
$$-\frac{96}{5}(4\lambda^2-19)\,(W^{\lambda,11}_{F,4}+W^{\lambda,22}_{F,4})\Bigg](w)$$
$$+\frac{1}{(z-w)^3}\Bigg[\frac{1}{30}\,\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times\partial^3\,(W^{\lambda,11}_{F,2}+W^{\lambda,22}_{F,2})$$
$$-\frac{1}{2}\,\frac{96}{5}(4\lambda^2-19)\,\partial\,(W^{\lambda,11}_{F,4}+W^{\lambda,22}_{F,4})\Bigg](w)$$
$$+\frac{1}{(z-w)^2}\Bigg[\frac{1}{168}\,\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times\partial^4\,(W^{\lambda,11}_{F,2}+W^{\lambda,22}_{F,2})$$
$$-\frac{5}{36}\,\frac{96}{5}(4\lambda^2-19)\,\partial^2\,(W^{\lambda,11}_{F,4}+W^{\lambda,22}_{F,4})$$
$$+6\,(W^{\lambda,11}_{F,6}+W^{\lambda,22}_{F,6})\Bigg](w)$$
$$+\frac{1}{(z-w)}\Bigg[\frac{1}{1120}\,\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+3)\,\partial^5\,(W^{\lambda,11}_{F,2}+W^{\lambda,22}_{F,2})$$
$$-\frac{1}{36}\,\frac{96}{5}(4\lambda^2-19)\,\partial^3\,(W^{\lambda,11}_{F,4}+W^{\lambda,22}_{F,4})$$
$$+\frac{1}{2}\,6\,\partial\,(W^{\lambda,11}_{F,6}+W^{\lambda,22}_{F,6})\Bigg](w)+\cdots$$
$$=\frac{1}{(z-w)^8}\left[-\frac{1536}{5}(112\lambda^6-280\lambda^4+147\lambda^2-9)\right]$$
$$-p^{4,4}_{F,4}(\partial_z,\partial_w,\lambda)\left[\frac{(W^{\lambda,11}_{F,2}+W^{\lambda,22}_{F,2})(w)}{(z-w)}\right]$$
$$-p^{4,4}_{F,2}(\partial_z,\partial_w,\lambda)\left[\frac{(W^{\lambda,11}_{F,4}+W^{\lambda,22}_{F,4})(w)}{(z-w)}\right]$$
$$-p^{4,4}_{F,0}(\partial_z,\partial_w,\lambda)\left[\frac{(W^{\lambda,11}_{F,6}+W^{\lambda,22}_{F,6})(w)}{(z-w)}\right]+\cdots,$$
$$=\frac{1}{(z-w)^8}\,c_F(4,4,\lambda)\,\delta^{11}\,q^4$$
$$+\sum_{h=0,\text{even}}^{4} q^h\,(-1)^{h-1}\,p^{4,4}_{F,h}(\partial_z,\partial_w,\lambda)$$
$$\times\left[\frac{(W^{\lambda,11}_{F,6-h}+W^{\lambda,22}_{F,6-h})(w)}{(z-w)}\right]+\cdots. \qquad \text{(C.1)}$$

The right hand side of (C.1) can be seen from the last two terms in (3.1) by changing the commutator to the corresponding OPE. Note that there is an additional factor $(-1)^{h-1}$ in the above.

C.1.2 The OPE between $W^{\lambda,\hat{A}=1}_{F,4}$ and $W^{\lambda,\hat{B}=2}_{F,4}$

For the different indices $\hat{A} = 1$ and $\hat{B} = 2$ in (3.1), the following result is satisfied

$$(W^{\lambda,12}_{F,4}+W^{\lambda,21}_{F,4})(z)\,i\,(W^{\lambda,12}_{F,4}-W^{\lambda,21}_{F,4})(w)$$
$$=\frac{1}{(z-w)^7}\Bigg[\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)(2\lambda+3)\Bigg]\,i\,(W^{\lambda,11}_{F,1}-W^{\lambda,22}_{F,1})(w)$$
$$+\frac{1}{(z-w)^6}\Bigg[\frac{1}{2}\,\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda-1)$$

$$\times(2\lambda+1)(2\lambda+3)\Big] i\,\partial(W^{\lambda,11}_{F,1} - W^{\lambda,22}_{F,1})(w)$$
$$+\frac{1}{(z-w)^5}\Big[\frac{1}{6}\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)(2\lambda+3)\, i\,\partial^2\,(W^{\lambda,11}_{F,1} - W^{\lambda,22}_{F,1})$$
$$-\frac{256}{25}\,(2\lambda-3)(2\lambda+3)(2\lambda^2-17)\, i\,(W^{\lambda,11}_{F,3}-W^{\lambda,22}_{F,3})\Big](w)$$
$$+\frac{1}{(z-w)^4}\Big[\frac{1}{24}\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)(2\lambda+3)\, i\,\partial^3\,(W^{\lambda,11}_{F,1} - W^{\lambda,22}_{F,1})$$
$$-\frac{1}{2}\frac{256}{25}\,(2\lambda-3)(2\lambda+3)(2\lambda^2-17)\, i$$
$$\partial\,(W^{\lambda,11}_{F,3}-W^{\lambda,22}_{F,3})\Big](w)$$
$$+\frac{1}{(z-w)^3}\Big[\frac{1}{120}\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)(2\lambda+3) i\,\partial^4(W^{\lambda,11}_{F,1}-W^{\lambda,22}_{F,1})$$
$$-\frac{1}{7}\frac{256}{25}(2\lambda-3)(2\lambda+3)(2\lambda^2-17) i\,\partial^2(W^{\lambda,11}_{F,3} - W^{\lambda,22}_{F,3})$$
$$+\frac{4}{5}(4\lambda^2-69) i\,(W^{\lambda,11}_{F,5} - W^{\lambda,22}_{F,5})\Big]$$
$$+\frac{1}{(z-w)^2}\Big[\frac{1}{720}\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)(2\lambda+3) i\,\partial^5(W^{\lambda,11}_{F,1} - W^{\lambda,22}_{F,1})$$
$$-\frac{5}{168}\frac{256}{25}\,(2\lambda-3)(2\lambda+3)(2\lambda^2-17)$$
$$\times i\,\partial^3\,(W^{\lambda,11}_{F,3} - W^{\lambda,22}_{F,3})$$
$$+\frac{1}{2}\frac{4}{5}(4\lambda^2-69)\, i\,\partial\,(W^{\lambda,11}_{F,5} - W^{\lambda,22}_{F,5})\Big](w)$$
$$+\frac{1}{(z-w)}\Big[\frac{1}{5040}\frac{2048}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)(2\lambda+3) i\,\partial^6(W^{\lambda,11}_{F,1} - W^{\lambda,22}_{F,1})$$
$$-\frac{5}{1008}\frac{256}{25}\,(2\lambda-3)(2\lambda+3)(2\lambda^2-17)\, i\,\partial^4$$
$$\times(W^{\lambda,11}_{F,3} - W^{\lambda,22}_{F,3})$$
$$+\frac{3}{22}\frac{4}{5}\,(4\lambda^2-69)\, i\,\partial^2\,(W^{\lambda,11}_{F,5} - W^{\lambda,22}_{F,5})$$
$$-\frac{1}{2}\, i\,(W^{\lambda,11}_{F,7} - W^{\lambda,22}_{F,7})\Big](w) + \cdots$$
$$= -p^{4,4}_{F,5}(\partial_z,\partial_w,\lambda)\Big[\frac{i\,(W^{\lambda,11}_{F,1} - W^{\lambda,22}_{F,1})(w)}{(z-w)}\Big]$$
$$-p^{4,4}_{F,3}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{i\,(W^{\lambda,11}_{F,3} - W^{\lambda,22}_{F,3})(w)}{(z-w)}\Big]$$
$$-p^{4,4}_{F,1}(\partial_z,\partial_w,\lambda)\Big[\frac{i\,(W^{\lambda,11}_{F,5} - W^{\lambda,22}_{F,5})(w)}{(z-w)}\Big]$$
$$-p^{4,4}_{F,-1}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{i\,(W^{\lambda,11}_{F,7} - W^{\lambda,22}_{F,7})(w)}{(z-w)}\Big] + \cdots$$
$$= -\sum_{h=-1,\mathrm{odd}}^{5} q^h\,(-1)^{h-1}\,p^{4,4}_{F,h}(\partial_z,\partial_w,\lambda) i\, f^{123}$$
$$\times\Big[\frac{W^{\lambda,\hat{A}=3}_{F,6-h}(w)}{(z-w)}\Big] + \cdots. \tag{C.2}$$

Therefore, we observe that this can be seen from the first term in (3.1). It turns out that the three elements of $W^{\lambda,\hat{A}}_{F,4}$ are normalized correctly. Here the structure constant is given by $f^{123}=1$. We can also check other cases of (3.1) by choosing the different indices and expect to have similar relations to (C.1) and (C.2). We have checked that the relation (3.1) satisfies for the case of the arbitrary weight h_1 with the restricted weight h_2 ($h_1 = h_2, h_2 \pm 1$).

4.5.1 C.1.3 The OPE between $W^{\lambda,\hat{A}=1}_{B,4}$ and itself

We can consider the second case described by (3.10). It turns out that we obtain the result

$$(W^{\lambda,12}_{B,4} + W^{\lambda,21}_{B,4})(z)\,(W^{\lambda,12}_{B,4} + W^{\lambda,21}_{B,4})(w) =$$
$$\times\frac{1}{(z-w)^8}\Big[\frac{6144}{5}(28\lambda^6 - 84\lambda^5 + 35\lambda^4 + 70\lambda^3$$
$$-42\lambda^2 - 7\lambda + 3)\Big]$$
$$+\frac{1}{(z-w)^6}\Big[\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\Big]$$
$$\times(W^{\lambda,11}_{B,2} + W^{\lambda,22}_{B,2})(w)$$
$$+\frac{1}{(z-w)^5}\frac{1}{2}\Big[\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\Big]$$
$$\times\partial\,(W^{\lambda,11}_{B,2} + W^{\lambda,22}_{B,2})(w)$$
$$+\frac{1}{(z-w)^4}\Big[\frac{3}{20}\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\partial^2$$
$$\times(W^{\lambda,11}_{B,2} + W^{\lambda,22}_{B,2})$$
$$-\frac{192}{5}(2\lambda^2-2\lambda-9)(W^{\lambda,11}_{B,4}+W^{\lambda,22}_{B,4})\Big](w)$$
$$+\frac{1}{(z-w)^3}\Big[\frac{1}{30}\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+1)\partial^3$$
$$\times(W^{\lambda,11}_{B,2} + W^{\lambda,22}_{B,2})$$
$$-\frac{1}{2}\frac{192}{5}(2\lambda^2-2\lambda-9)\partial(W^{\lambda,11}_{B,4} + W^{\lambda,22}_{B,4})\Big](w)$$
$$+\frac{1}{(z-w)^2}\Big[\frac{1}{168}\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)\,\partial^4\,(W^{\lambda,11}_{B,2} + W^{\lambda,22}_{B,2})$$
$$-\frac{5}{36}\frac{192}{5}(2\lambda^2-2\lambda-9)\,\partial^2\,(W^{\lambda,11}_{B,4} + W^{\lambda,22}_{B,4})$$

$$+6\,(W^{\lambda,11}_{B,6}+W^{\lambda,22}_{B,6})\Big](w)$$
$$+\frac{1}{(z-w)}\Big[\frac{1}{1120}\,\frac{2048}{5}(\lambda-2)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)\,\partial^5\,(W^{\lambda,11}_{B,2}+W^{\lambda,22}_{B,2})$$
$$-\frac{1}{36}\,\frac{192}{5}(2\lambda^2-2\lambda-9)\,\partial^3\,(W^{\lambda,11}_{B,4}+W^{\lambda,22}_{B,4})$$
$$+\frac{1}{2}\,6\,\partial\,(W^{\lambda,11}_{B,6}+W^{\lambda,22}_{B,6})\Big](w)+\cdots$$
$$=\frac{1}{(z-w)^8}\Big[\frac{6144}{5}(28\lambda^6-84\lambda^5+35\lambda^4+70\lambda^3$$
$$-42\lambda^2-7\lambda+3)\Big]$$
$$-p^{4,4}_{B,4}(\partial_z,\partial_w,\lambda)\Big[\frac{(W^{\lambda,11}_{B,2}+W^{\lambda,22}_{B,2})(w)}{(z-w)}\Big]$$
$$-p^{4,4}_{B,2}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{(W^{\lambda,11}_{B,4}+W^{\lambda,22}_{B,4})(w)}{(z-w)}\Big]$$
$$-p^{4,4}_{B,0}(\partial_z,\partial_w,\lambda)\Big[\frac{(W^{\lambda,11}_{B,6}+W^{\lambda,22}_{B,6})(w)}{(z-w)}\Big]+\cdots$$
$$=\frac{1}{(z-w)^8}\,c_B(4,4,\lambda)\,\delta^{11}\,q^4$$
$$+\sum_{h=0,\text{even}}^{4}q^h\,(-1)^{h-1}\,p^{4,4}_{B,h}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{(W^{\lambda,11}_{B,6-h}+W^{\lambda,22}_{B,6-h})(w)}{(z-w)}\Big]+\cdots, \tag{C.3}$$

which looks similar to the previous one in (C.1). The corresponding central charge, the structure constants and the currents occur.

C.1.4 The OPE between $W^{\lambda,\hat{A}=1}_{B,4}$ and $W^{\lambda,\hat{B}=2}_{B,4}$

By taking the different indices $\hat{A}=1$ and $\hat{B}=2$ in (3.10), the following result is obtained

$$(W^{\lambda,12}_{B,4}+W^{\lambda,21}_{B,4})(z)\,i\,(W^{\lambda,12}_{B,4}-W^{\lambda,21}_{B,4})(w)$$
$$=\frac{1}{(z-w)^7}\Big[\frac{8192}{5}(\lambda-2)(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)\Big]\,i\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})(w)$$
$$+\frac{1}{(z-w)^6}\,\frac{1}{2}\Big[\frac{8192}{5}(\lambda-2)(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)\Big]\,i\,\partial\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})(w)$$
$$+\frac{1}{(z-w)^5}\Big[\frac{1}{6}\,\frac{8192}{5}(\lambda-2)(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)i\,\partial^2\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{512}{25}(\lambda-2)(\lambda+1)(4\lambda^2-4\lambda-33)\,i$$
$$\times(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})\Big](w)$$
$$+\frac{1}{(z-w)^4}\Big[\frac{1}{24}\,\frac{8192}{5}(\lambda-2)(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)i\,\partial^3\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{1}{2}\,\frac{512}{25}(\lambda-2)(\lambda+1)(4\lambda^2-4\lambda-33)\,i\,\partial$$
$$\times(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})\Big](w)$$
$$+\frac{1}{(z-w)^3}\Big[\frac{1}{120}\,\frac{8192}{5}(\lambda-2)(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)i\,\partial^4\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{1}{7}\,\frac{512}{25}(\lambda-2)(\lambda+1)(4\lambda^2-4\lambda-33)\,i$$
$$\partial^2\,(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})$$
$$+\frac{16}{5}(\lambda^2-\lambda-17)\,i\,(W^{\lambda,11}_{B,5}-W^{\lambda,22}_{B,5})\Big](w)$$
$$+\frac{1}{(z-w)^2}\Big[\frac{1}{720}\,\frac{8192}{5}(\lambda-2)(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)i\,\partial^5\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{5}{168}\,\frac{512}{25}(\lambda-2)(\lambda+1)(4\lambda^2-4\lambda-33)\,i$$
$$\times\partial^3\,(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})$$
$$+\frac{1}{2}\,\frac{16}{5}(\lambda^2-\lambda-17)\,i\,\partial\,(W^{\lambda,11}_{B,5}-W^{\lambda,22}_{B,5})\Big](w)$$
$$+\frac{1}{(z-w)}\Big[\frac{1}{5040}\,\frac{8192}{5}(\lambda-2)(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)i\,\partial^6\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{5}{1008}\,\frac{512}{25}(\lambda-2)(\lambda+1)(4\lambda^2-4\lambda-33)\,i\,\partial^4$$
$$\times(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})$$
$$+\frac{3}{22}\,\frac{16}{5}(\lambda^2-\lambda-17)\,i\,\partial^2\,(W^{\lambda,11}_{B,5}-W^{\lambda,22}_{B,5})$$
$$-\frac{1}{2}\,i\,(W^{\lambda,11}_{B,7}-W^{\lambda,22}_{B,7})\Big](w)+\cdots$$
$$=-p^{4,4}_{B,5}(\partial_z,\partial_w,\lambda)\Big[\frac{i\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})(w)}{(z-w)}\Big]$$
$$-p^{4,4}_{B,3}(\partial_z,\partial_w,\lambda)\Big[\frac{i\,(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})(w)}{(z-w)}\Big]$$
$$-p^{4,4}_{B,1}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{i\,(W^{\lambda,11}_{B,5}-W^{\lambda,22}_{B,5})(w)}{(z-w)}\Big]$$
$$-p^{4,4}_{B,-1}(\partial_z,\partial_w,\lambda)\Big[\frac{i\,(W^{\lambda,11}_{B,7}-W^{\lambda,22}_{B,7})(w)}{(z-w)}\Big]+\cdots$$

$$= -\sum_{h=-1,\text{odd}}^{5} q^h (-1)^{h-1} p_{B,h}^{4,4}(\partial_z, \partial_w, \lambda) i f^{123} \times \left[\frac{W_{B,6-h}^{\lambda,\hat{A}=3}(w)}{(z-w)}\right] + \cdots . \quad \text{(C.4)}$$

Note that there is a λ factor in the structure constants appearing the current $(W_{B,1}^{\lambda,11} - W_{B,1}^{\lambda,22})$ and its descendants. It is obvious that they vanish at $\lambda = 0$.

C.2 The OPE between the bosonic currents and the fermionic currents

We present the explicit check for (3.12) with the weights $h_1 = 4$ and $h_2 = 3$.

C.2.1 The OPE between $W_{F,4}^{\lambda,\hat{A}=1}$ and $Q_{\frac{7}{2}}^{\lambda,\hat{B}=1}$

For the same indices $\hat{A} = \hat{B} = 1$ in the first equation of (3.12), the following result holds

$$\begin{aligned}
&(W_{F,4}^{\lambda,12} + W_{F,4}^{\lambda,21})(z)\,(Q_{\frac{7}{2}}^{\lambda,12} + Q_{\frac{7}{2}}^{\lambda,21})(w) \\
&= \frac{1}{(z-w)^6}\left[\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3) \times(2\lambda+1)(2\lambda+3)\right](Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22})(w) \\
&+ \frac{1}{(z-w)^5}\left[\frac{2}{3}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \times(2\lambda+3)\,\partial\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \right.\\
&\left. - \frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3) \times(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22})\right](w) \\
&+ \frac{1}{(z-w)^4}\left[\frac{1}{4}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \times(2\lambda+3)\,\partial^2\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \right.\\
&- \frac{3}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\partial \times(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22}) \\
&\left. - \frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22})\right](w) \\
&+ \frac{1}{(z-w)^3}\left[\frac{1}{15}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \times(2\lambda+3)\,\partial^3\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \right.\\
&- \frac{1}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3) \times\partial^2\,(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22}) \\
&- \frac{4}{7}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\partial\,(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22}) \\
&\left. + \frac{4}{5}(2\lambda^2-5\lambda-27)\,(Q_{\frac{9}{2}}^{\lambda,11} + Q_{\frac{9}{2}}^{\lambda,22})\right](w) \\
&+ \frac{1}{(z-w)^2}\left[\frac{1}{72}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \times(2\lambda+3)\,\partial^4\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \right.\\
&- \frac{1}{21}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,\partial^3 \times(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22}) \\
&- \frac{5}{28}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\partial^2\,(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22}) \\
&+ \frac{5}{9}\frac{4}{5}(2\lambda^2-5\lambda-27)\,\partial\,(Q_{\frac{9}{2}}^{\lambda,11} + Q_{\frac{9}{2}}^{\lambda,22}) \\
&\left. + \frac{1}{10}(2\lambda+27)\,(Q_{\frac{11}{2}}^{\lambda,11} + Q_{\frac{11}{2}}^{\lambda,22})\right](w) \\
&+ \frac{1}{(z-w)}\left[\frac{1}{420}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \times(2\lambda+3)\,\partial^5\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \right.\\
&- \frac{1}{112}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3) \times\partial^4\,(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22}) \\
&- \frac{5}{126}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\partial^3\,(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22}) \\
&+ \frac{1}{6}\frac{4}{5}(2\lambda^2-5\lambda-27)\,\partial^2\,(Q_{\frac{9}{2}}^{\lambda,11} + Q_{\frac{9}{2}}^{\lambda,22}) \\
&+ \frac{6}{11}\frac{1}{10}(2\lambda+27)\,\partial\,(Q_{\frac{11}{2}}^{\lambda,11} + Q_{\frac{11}{2}}^{\lambda,22}) \\
&\left. - \frac{1}{4}\,(Q_{\frac{13}{2}}^{\lambda,11} + Q_{\frac{13}{2}}^{\lambda,22})\right](w) + \cdots \\
&= -q_{F,4}^{4,\frac{7}{2}}(\partial_z, \partial_w, \lambda)\left[\frac{(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22})(w)}{(z-w)}\right] + q_{F,3}^{4,\frac{7}{2}}(\partial_z, \partial_w, \lambda)\left[\frac{(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22})(w)}{(z-w)}\right] \\
&- q_{F,2}^{4,\frac{7}{2}}(\partial_z, \partial_w, \lambda)\left[\frac{(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22})(w)}{(z-w)}\right] + q_{F,1}^{4,\frac{7}{2}}(\partial_z, \partial_w, \lambda)\left[\frac{(Q_{\frac{9}{2}}^{\lambda,11} + Q_{\frac{9}{2}}^{\lambda,22})(w)}{(z-w)}\right] \\
&- q_{F,0}^{4,\frac{7}{2}}(\partial_z, \partial_w, \lambda)\left[\frac{(Q_{\frac{11}{2}}^{\lambda,11} + Q_{\frac{11}{2}}^{\lambda,22})(w)}{(z-w)}\right] + q_{F,-1}^{4,\frac{7}{2}}(\partial_z, \partial_w, \lambda)\left[\frac{(Q_{\frac{13}{2}}^{\lambda,11} + Q_{\frac{13}{2}}^{\lambda,22})(w)}{(z-w)}\right] + \cdots \\
&= \sum_{h=-1}^{4} q^h (-1)^{h-1} q_{F,h}^{4,\frac{7}{2}}(\partial_z, \partial_w, \lambda)\,\delta^{11}\left[\frac{Q_{\frac{11}{2}-h}^{\lambda,\bar{a}a}(w)}{(z-w)}\right] + \cdots . \quad \text{(C.5)}
\end{aligned}$$

The right hand side of (C.5) can be seen from the last term in the first relation of (3.12) by changing the commutator to the corresponding OPE. Note that there is an additional factor $(-1)^{h-1}$ in the above.

C.2.2 The OPE between $W_{F,4}^{\lambda,\hat{A}=1}$ and $Q_{\frac{7}{2}}^{\lambda,\hat{B}=2}$

The following result is obtained by taking the different indices $\hat{A} = 1$ and $\hat{B} = 2$ in the first equation of (3.12),

$$
\begin{aligned}
&(W_{F,4}^{\lambda,12} + W_{F,4}^{\lambda,21})(z)\, i\, (Q_{\frac{7}{2}}^{\lambda,12} - Q_{\frac{7}{2}}^{\lambda,21})(w) \\
&= \frac{1}{(z-w)^6}\Bigg[\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3) \\
&\quad \times(2\lambda+1)(2\lambda+3)\Bigg] \\
&\quad \times i \left(Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22}\right)(w) \\
&\quad + \frac{1}{(z-w)^5}\Bigg[\frac{2}{3}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \\
&\quad \times(2\lambda+3)\, i\, \partial\, (Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22}) \\
&\quad - \frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\, i \\
&\quad \times \left(Q_{\frac{5}{2}}^{\lambda,11} - Q_{\frac{5}{2}}^{\lambda,22}\right)\Bigg](w) \\
&\quad + \frac{1}{(z-w)^4}\Bigg[\frac{1}{4}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \\
&\quad \times(2\lambda+3)\, i\, \partial^2\, (Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22}) \\
&\quad - \frac{3}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\, i\, \partial \\
&\quad \times \left(Q_{\frac{5}{2}}^{\lambda,11} - Q_{\frac{5}{2}}^{\lambda,22}\right) \\
&\quad - \frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\, i \\
&\quad \times \left(Q_{\frac{7}{2}}^{\lambda,11} - Q_{\frac{7}{2}}^{\lambda,22}\right)\Bigg](w) \\
&\quad + \frac{1}{(z-w)^3}\Bigg[\frac{1}{15}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \\
&\quad \times(2\lambda+3)\, i\, \partial^3 \left(Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22}\right) \\
&\quad - \frac{1}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3) \\
&\quad \times i\, \partial^2 \left(Q_{\frac{5}{2}}^{\lambda,11} - Q_{\frac{5}{2}}^{\lambda,22}\right) \\
&\quad - \frac{4}{7}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\, i\, \partial\, (Q_{\frac{7}{2}}^{\lambda,11} - Q_{\frac{7}{2}}^{\lambda,22}) \\
&\quad + \frac{4}{5}(2\lambda^2-5\lambda-27)\, i\, (Q_{\frac{9}{2}}^{\lambda,11} - Q_{\frac{9}{2}}^{\lambda,22})\Bigg](w) \\
&\quad + \frac{1}{(z-w)^2}\Bigg[\frac{1}{72}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \\
&\quad \times(2\lambda+3)\, i\, \partial^4 \left(Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22}\right) \\
&\quad - \frac{1}{21}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\, i \\
&\quad \times \partial^3\, (Q_{\frac{5}{2}}^{\lambda,11} - Q_{\frac{5}{2}}^{\lambda,22}) \\
&\quad - \frac{5}{28}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\, i\, \partial^2\, (Q_{\frac{7}{2}}^{\lambda,11} - Q_{\frac{7}{2}}^{\lambda,22}) \\
&\quad + \frac{5}{9}\frac{4}{5}(2\lambda^2-5\lambda-27)\, i\, \partial\, (Q_{\frac{9}{2}}^{\lambda,11} - Q_{\frac{9}{2}}^{\lambda,22}) \\
&\quad + \frac{1}{10}(2\lambda+27)\, i \left(Q_{\frac{11}{2}}^{\lambda,11} - Q_{\frac{11}{2}}^{\lambda,22}\right)\Bigg](w) \\
&\quad + \frac{1}{(z-w)}\Bigg[\frac{1}{420}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1) \\
&\quad \times(2\lambda+3)\, i\, \partial^5\, (Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22}) \\
&\quad - \frac{1}{112}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\, i \\
&\quad \times \partial^4\, (Q_{\frac{5}{2}}^{\lambda,11} - Q_{\frac{5}{2}}^{\lambda,22}) \\
&\quad - \frac{5}{126}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\, i\, \partial^3\, (Q_{\frac{7}{2}}^{\lambda,11} - Q_{\frac{7}{2}}^{\lambda,22}) \\
&\quad + \frac{1}{6}\frac{4}{5}(2\lambda^2-5\lambda-27)\, i\, \partial^2\, (Q_{\frac{9}{2}}^{\lambda,11} - Q_{\frac{9}{2}}^{\lambda,22}) \\
&\quad + \frac{6}{11}\frac{1}{10}(2\lambda+27)\, i\, \partial\, (Q_{\frac{11}{2}}^{\lambda,11} - Q_{\frac{11}{2}}^{\lambda,22}) \\
&\quad - \frac{1}{4}\, i\, (Q_{\frac{13}{2}}^{\lambda,11} - Q_{\frac{13}{2}}^{\lambda,22})\Bigg](w) + \cdots \\
&= -q_{F,4}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{i\,(Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22})(w)}{(z-w)}\right] \\
&\quad + q_{F,3}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{i\,(Q_{\frac{5}{2}}^{\lambda,11} - Q_{\frac{5}{2}}^{\lambda,22})(w)}{(z-w)}\right] \\
&\quad - q_{F,2}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{i\,(Q_{\frac{7}{2}}^{\lambda,11} - Q_{\frac{7}{2}}^{\lambda,22})(w)}{(z-w)}\right] \\
&\quad + q_{F,1}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{i\,(Q_{\frac{9}{2}}^{\lambda,11} - Q_{\frac{9}{2}}^{\lambda,22})(w)}{(z-w)}\right] \\
&\quad - q_{F,0}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{i\,(Q_{\frac{11}{2}}^{\lambda,11} - Q_{\frac{11}{2}}^{\lambda,22})(w)}{(z-w)}\right] \\
&\quad + q_{F,-1}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{i\,(Q_{\frac{13}{2}}^{\lambda,11} - Q_{\frac{13}{2}}^{\lambda,22})(w)}{(z-w)}\right] + \cdots \\
&= \sum_{h=-1}^{4} q^h\, (-1)^{h-1}\, q_{F,h}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\, i\, f^{123}\left[\frac{Q_{\frac{11}{2}-h}^{\lambda,\hat{A}=3}(w)}{(z-w)}\right] \\
&\quad + \cdots . \qquad \text{(C.6)}
\end{aligned}
$$

Then we observe the first term of the first equation of (3.12) and the first relation of (3.12) satisfies for the case of the arbitrary weight h_1 with the restricted weight h_2 $(h_1 = h_2, h_2 + 1)$.

C.2.3 The OPE between $W_{B,4}^{\lambda,\hat{A}=1}$ and $Q_{\frac{7}{2}}^{\lambda,\hat{B}=1}$

We obtain the following result for the same indices $\hat{A} = \hat{B} = 1$

$$
\begin{aligned}
&(W_{B,4}^{\lambda,12} + W_{B,4}^{\lambda,21})(z)\,(Q_{\frac{7}{2}}^{\lambda,12} + Q_{\frac{7}{2}}^{\lambda,21})(w) \\
&= \frac{1}{(z-w)^6}\Bigg[-\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1) \\
&\times(2\lambda-3)(2\lambda+1)\Bigg]\left(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}\right)(w) \\
&+\frac{1}{(z-w)^5}\Bigg[-\frac{2}{3}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3) \\
&\times(2\lambda+1)\,\partial\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \\
&+\frac{256}{25}(\lambda-2)(\lambda+1) \\
&\times(2\lambda-3)(2\lambda+7)\,(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22})\Bigg](w) \\
&+\frac{1}{(z-w)^4}\Bigg[-\frac{1}{4}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3) \\
&\times(2\lambda+1)\,\partial^2\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \\
&+\frac{3}{5}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7) \\
&\times\partial\,(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22}) \\
&+\frac{32}{25}(\lambda-2)\left(4\lambda^2-22\lambda-51\right)\,(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22})\Bigg](w) \\
&+\frac{1}{(z-w)^3}\Bigg[-\frac{1}{15}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3) \\
&\times\quad(2\lambda+1)\,\partial^3\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \\
&+\frac{1}{5}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7) \\
&\times\partial^2\,(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22}) \\
&+\frac{4}{7}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\partial\,(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22}) \\
&-\frac{4}{5}(2\lambda^2+3\lambda-29)\,(Q_{\frac{9}{2}}^{\lambda,11} + Q_{\frac{9}{2}}^{\lambda,22})\Bigg](w) \\
&+\frac{1}{(z-w)^2}\Bigg[-\frac{1}{72}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3) \\
&\times(2\lambda+1)\,\partial^4\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \\
&+\frac{1}{21}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\partial^3 \\
&\times(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22}) \\
&+\frac{5}{28}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\partial^2\,(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22}) \\
&-\frac{5}{9}\frac{4}{5}(2\lambda^2+3\lambda-29)\,\partial\,(Q_{\frac{9}{2}}^{\lambda,11} + Q_{\frac{9}{2}}^{\lambda,22}) \\
&+\frac{1}{5}(-\lambda+14)\,(Q_{\frac{11}{2}}^{\lambda,11} + Q_{\frac{11}{2}}^{\lambda,22})\Bigg](w) \\
&+\frac{1}{(z-w)}\Bigg[-\frac{1}{420}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3) \\
&\times(2\lambda+1)\,\partial^5\,(Q_{\frac{3}{2}}^{\lambda,11} + Q_{\frac{3}{2}}^{\lambda,22}) \\
&+\frac{1}{112}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,\partial^4 \\
&\times(Q_{\frac{5}{2}}^{\lambda,11} + Q_{\frac{5}{2}}^{\lambda,22}) \\
&+\frac{5}{126}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\partial^3\,(Q_{\frac{7}{2}}^{\lambda,11} + Q_{\frac{7}{2}}^{\lambda,22}) \\
&-\frac{1}{6}\frac{4}{5}(2\lambda^2+3\lambda-29)\,\partial^2\,(Q_{\frac{9}{2}}^{\lambda,11} + Q_{\frac{9}{2}}^{\lambda,22}) \\
&+\frac{6}{11}\frac{1}{5}(-\lambda+14)\,\partial\,(Q_{\frac{11}{2}}^{\lambda,11} + Q_{\frac{11}{2}}^{\lambda,22}) \\
&+\frac{1}{4}\,(Q_{\frac{13}{2}}^{\lambda,11} + Q_{\frac{13}{2}}^{\lambda,22})\Bigg](w) + \cdots \\
&= -q_{B,4}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{Q_{\frac{3}{2}}^{\lambda,\bar{a}a}(w)}{(z-w)}\right] + q_{B,3}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda) \\
&\times\left[\frac{Q_{\frac{5}{2}}^{\lambda,\bar{a}a}(w)}{(z-w)}\right] - q_{B,2}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{Q_{\frac{7}{2}}^{\lambda,\bar{a}a}(w)}{(z-w)}\right] \\
&+q_{B,1}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{Q_{\frac{9}{2}}^{\lambda,\bar{a}a}(w)}{(z-w)}\right] - q_{B,0}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda) \\
&\times\left[\frac{Q_{\frac{11}{2}}^{\lambda,\bar{a}a}(w)}{(z-w)}\right] + q_{B,-1}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\left[\frac{Q_{\frac{13}{2}}^{\lambda,\bar{a}a}(w)}{(z-w)}\right] \\
&+\cdots \\
&= \sum_{h=-1}^{4} q^h\,(-1)^{h-1}\,q_{B,h}^{4,\frac{7}{2}}(\partial_z,\partial_w,\lambda)\,\delta^{11}\left[\frac{Q_{\frac{11}{2}-h}^{\lambda,\bar{a}a}(w)}{(z-w)}\right] \\
&+\cdots.
\end{aligned}
\tag{C.7}
$$

We observe that the second relation of (3.12) can be seen from (C.7).

C.2.4 The OPE between $W_{B,4}^{\lambda,\hat{A}=1}$ and $Q_{\frac{7}{2}}^{\lambda,\hat{B}=2}$

Similarly, we can calculate the following OPE for different indices $\hat{A} = 1$ and $\hat{B} = 2$

$$
\begin{aligned}
&(W_{B,4}^{\lambda,12} + W_{B,4}^{\lambda,21})(z)\,i\,(Q_{\frac{7}{2}}^{\lambda,12} - Q_{\frac{7}{2}}^{\lambda,21})(w) \\
&= \frac{1}{(z-w)^6}\Bigg[\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3) \\
&\times(2\lambda+1)\Bigg]i\,(Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22})(w) \\
&+\frac{1}{(z-w)^5}\Bigg[\frac{2}{3}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3) \\
&\times(2\lambda+1)\,i\,\partial\,(Q_{\frac{3}{2}}^{\lambda,11} - Q_{\frac{3}{2}}^{\lambda,22}) \\
&-\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,i\,(Q_{\frac{5}{2}}^{\lambda,11} - Q_{\frac{5}{2}}^{\lambda,22})\Bigg](w)
\end{aligned}
$$

$$+\frac{1}{(z-w)^4}\bigg[\frac{1}{4}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)\, i\, \partial^2\, (Q^{\lambda,11}_{\frac{3}{2}}-Q^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{3}{5}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\, i$$
$$\times\partial\, (Q^{\lambda,11}_{\frac{5}{2}}-Q^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{32}{25}(\lambda-2)\left(4\lambda^2-22\lambda-51\right)\, i\, (Q^{\lambda,11}_{\frac{7}{2}}-Q^{\lambda,22}_{\frac{7}{2}})\bigg](w)$$
$$+\frac{1}{(z-w)^3}\bigg[\frac{1}{15}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)\, i\, \partial^3\, (Q^{\lambda,11}_{\frac{3}{2}}-Q^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{1}{5}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\, i$$
$$\times\partial^2\, (Q^{\lambda,11}_{\frac{5}{2}}-Q^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{4}{7}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\, i\, \partial\, (Q^{\lambda,11}_{\frac{7}{2}}-Q^{\lambda,22}_{\frac{7}{2}})$$
$$+\frac{4}{5}(2\lambda^2+3\lambda-29)\, i\, (Q^{\lambda,11}_{\frac{9}{2}}-Q^{\lambda,22}_{\frac{9}{2}})\bigg](w)$$
$$+\frac{1}{(z-w)^2}\bigg[\frac{1}{72}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)\, i\, \partial^4\, (Q^{\lambda,11}_{\frac{3}{2}}-Q^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{1}{21}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)$$
$$\times i\, \partial^3\, (Q^{\lambda,11}_{\frac{5}{2}}-Q^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{5}{28}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\, i\, \partial^2\, (Q^{\lambda,11}_{\frac{7}{2}}-Q^{\lambda,22}_{\frac{7}{2}})$$
$$+\frac{5}{9}\frac{4}{5}(2\lambda^2+3\lambda-29)\, i\, \partial\, (Q^{\lambda,11}_{\frac{9}{2}}-Q^{\lambda,22}_{\frac{9}{2}})$$
$$-\frac{1}{5}(-\lambda+14)\, i\, (Q^{\lambda,11}_{\frac{11}{2}}-Q^{\lambda,22}_{\frac{11}{2}})\bigg](w)$$
$$+\frac{1}{(z-w)}\bigg[\frac{1}{420}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)\, i\, \partial^5\, (Q^{\lambda,11}_{\frac{3}{2}}-Q^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{1}{112}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)i$$
$$\times\partial^4\, (Q^{\lambda,11}_{\frac{5}{2}}-Q^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{5}{126}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\, i\, \partial^3\, (Q^{\lambda,11}_{\frac{7}{2}}-Q^{\lambda,22}_{\frac{7}{2}})$$
$$+\frac{1}{6}\frac{4}{5}(2\lambda^2+3\lambda-29)\, i\, \partial^2\, (Q^{\lambda,11}_{\frac{9}{2}}-Q^{\lambda,22}_{\frac{9}{2}})$$
$$-\frac{6}{11}\frac{1}{5}(-\lambda+14)\, i\, \partial\, (Q^{\lambda,11}_{\frac{11}{2}}-Q^{\lambda,22}_{\frac{11}{2}})$$
$$-\frac{1}{4}\, i\, (Q^{\lambda,11}_{\frac{13}{2}}-Q^{\lambda,22}_{\frac{13}{2}})\bigg](w)+\cdots$$
$$=q^{4,\frac{7}{2}}_{B,4}(\partial_z,\partial_w,\lambda)\left[\frac{i\, (Q^{\lambda,11}_{\frac{3}{2}}-Q^{\lambda,22}_{\frac{3}{2}})(w)}{(z-w)}\right]$$
$$-q^{4,\frac{7}{2}}_{B,3}(\partial_z,\partial_w,\lambda)\left[\frac{i\, (Q^{\lambda,11}_{\frac{5}{2}}-Q^{\lambda,22}_{\frac{5}{2}})(w)}{(z-w)}\right]$$
$$+q^{4,\frac{7}{2}}_{B,2}(\partial_z,\partial_w,\lambda)\left[\frac{i\, (Q^{\lambda,11}_{\frac{7}{2}}-Q^{\lambda,22}_{\frac{7}{2}})(w)}{(z-w)}\right]$$
$$-q^{4,\frac{7}{2}}_{B,1}(\partial_z,\partial_w,\lambda)\left[\frac{i\, (Q^{\lambda,11}_{\frac{9}{2}}-Q^{\lambda,22}_{\frac{9}{2}})(w)}{(z-w)}\right]$$
$$+q^{4,\frac{7}{2}}_{B,0}(\partial_z,\partial_w,\lambda)\left[\frac{i\, (Q^{\lambda,11}_{\frac{11}{2}}-Q^{\lambda,22}_{\frac{11}{2}})(w)}{(z-w)}\right]$$
$$-q^{4,\frac{7}{2}}_{B,-1}(\partial_z,\partial_w,\lambda)\left[\frac{i\, (Q^{\lambda,11}_{\frac{13}{2}}-Q^{\lambda,22}_{\frac{13}{2}})(w)}{(z-w)}\right]+\cdots$$
$$=-\sum_{h=-1}^{4} q^h\, (-1)^{h-1}\, q^{4,\frac{7}{2}}_{B,h}(\partial_z,\partial_w,\lambda)\, i\, f^{123}$$
$$\times\left[\frac{Q^{\lambda,\hat{A}=3}_{\frac{11}{2}-h}(w)}{(z-w)}\right]+\cdots. \qquad \text{(C.8)}$$

We observe that the first term of second relation of (3.12) can be seen from (C.8).

C.2.5 The OPE between $W^{\lambda,\hat{A}=1}_{F,4}$ and $\bar{Q}^{\lambda,\hat{B}=1}_{\frac{7}{2}}$

We can consider the third equation of (3.12) and the following result can be determined

$$(W^{\lambda,12}_{F,4}+W^{\lambda,21}_{F,4})(z)\, (\bar{Q}^{\lambda,12}_{\frac{7}{2}}+\bar{Q}^{\lambda,21}_{\frac{7}{2}})(w)$$
$$=\frac{1}{(z-w)^7}\bigg[-\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)\bigg](\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{(z-w)^6}\bigg[-\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)\, \partial\, (\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})\bigg](w)$$
$$+\frac{1}{(z-w)^5}\bigg[-\frac{1}{2}\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)$$
$$\times(2\lambda-3)(2\lambda-1)$$
$$\times(\quad 2\lambda+1)\, \partial^2\, (\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{2}{3}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times\partial\, (\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+3)\, (\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})\bigg](w)$$

$$+\frac{1}{(z-w)^4}\Bigg[-\frac{1}{6}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$

$$\times(2\lambda-1)(2\lambda+1)\,\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$\times\frac{1}{4}\,\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{3}{5}\,\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times\partial\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})\Bigg](w)$$
$$+\frac{1}{(z-w)^3}\Bigg[-\frac{1}{24}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda-1)(2\lambda+1)\,\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{15}\,\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{5}\,\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{4}{7}\,\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$-\frac{4}{5}(2\lambda^2-5\lambda-27)\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}+\bar{Q}^{\lambda,22}_{\frac{9}{2}})\Bigg](w)$$
$$+\frac{1}{(z-w)^2}\Bigg[-\frac{1}{120}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda-1)(2\lambda+1)\,\partial^5\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{72}\,\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{21}\,\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{5}{28}\,\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$-\frac{5}{9}\,\frac{4}{5}(2\lambda^2-5\lambda-27)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}+\bar{Q}^{\lambda,22}_{\frac{9}{2}})$$
$$+\frac{1}{10}(2\lambda+27)\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}+\bar{Q}^{\lambda,22}_{\frac{11}{2}})\Bigg](w)$$
$$+\frac{1}{(z-w)}\Bigg[-\frac{1}{720}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda-1)(2\lambda+1)\,\partial^6\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{420}\,\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times\partial^5\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{112}\,\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+3)\,\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{5}{126}\,\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)$$

$$\times\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$-\frac{1}{6}\,\frac{4}{5}(2\lambda^2-5\lambda-27)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}+\bar{Q}^{\lambda,22}_{\frac{9}{2}})$$
$$+\frac{6}{11}\,\frac{1}{10}(2\lambda+27)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}+\bar{Q}^{\lambda,22}_{\frac{11}{2}})$$
$$+\frac{1}{4}\,(\bar{Q}^{\lambda,11}_{\frac{13}{2}}+\bar{Q}^{\lambda,22}_{\frac{13}{2}})\Bigg](w)+\cdots$$
$$=-q^{4,\frac{7}{2}}_{F,5}(\partial_z,\partial_w,\lambda)\left[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{1}{2}}(w)}{(z-w)}\right]-q^{4,\frac{7}{2}}_{F,4}(\partial_z,\partial_w,\lambda)$$
$$\times\left[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{3}{2}}(w)}{(z-w)}\right]-q^{4,\frac{7}{2}}_{F,3}(\partial_z,\partial_w,\lambda)$$
$$\times\left[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{5}{2}}(w)}{(z-w)}\right]$$
$$-q^{4,\frac{7}{2}}_{F,2}(\partial_z,\partial_w,\lambda)\left[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{7}{2}}(w)}{(z-w)}\right]-q^{4,\frac{7}{2}}_{F,1}(\partial_z,\partial_w,\lambda)$$
$$\times\left[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{9}{2}}(w)}{(z-w)}\right]-q^{4,\frac{7}{2}}_{F,0}(\partial_z,\partial_w,\lambda)$$
$$\times\left[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{11}{2}}(w)}{(z-w)}\right]$$
$$-q^{4,\frac{7}{2}}_{F,-1}(\partial_z,\partial_w,\lambda)\left[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{13}{2}}(w)}{(z-w)}\right]+\cdots$$
$$=-\sum_{h=-1}^{5}q^h\,q^{4,\frac{7}{2}}_{F,h}(\partial_z,\partial_w,\lambda)\,\delta^{11}\left[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{11}{2}-h}(w)}{(z-w)}\right]$$
$$+\cdots. \qquad \text{(C.9)}$$

There appears the λ factor in the coefficients appearing in the current $\bar{Q}^{\lambda,\bar{a}a}_{\frac{1}{2}}(w)$ and its descendants.

C.2.6 The OPE between $W^{\lambda,\hat{A}=1}_{F,4}$ and $\bar{Q}^{\lambda,\hat{B}=2}_{\frac{7}{2}}$

For different indices, we obtain the following result

$$(W^{\lambda,12}_{F,4}+W^{\lambda,21}_{F,4})(z)\,(-i)\,(\bar{Q}^{\lambda,12}_{\frac{7}{2}}-\bar{Q}^{\lambda,21}_{\frac{7}{2}})(w)$$
$$=\frac{1}{(z-w)^7}\Bigg[-\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)\Bigg](-i)\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{(z-w)^6}\Bigg[-\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)$$

$$\times(2\lambda+1)(-i)\partial(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)(2\lambda+3)(-i)(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})\Big](w)$$
$$+\frac{1}{(z-w)^5}\Big[-\frac{1}{2}\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)(-i)\partial^2(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{2}{3}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times(-i)\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})\Big](w)$$
$$+\frac{1}{(z-w)^4}\Big[-\frac{1}{6}\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)(-i)\partial^3(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$

$$\times\frac{1}{4}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times(-i)\;\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{3}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)$$
$$\times(-i)\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})\Big](w)$$
$$+\frac{1}{(z-w)^3}\Big[-\frac{1}{24}\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda-1)(2\lambda+1)(-i)\partial^4(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{15}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times(-i)\,\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{5}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times(-i)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{4}{7}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,(-i)\,\partial$$
$$(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$-\frac{4}{5}(2\lambda^2-5\lambda-27)\,(-i)\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}-\bar{Q}^{\lambda,22}_{\frac{9}{2}})\Big](w)$$
$$+\frac{1}{(z-w)^2}\Big[-\frac{1}{120}\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)$$
$$\times(2\lambda-3)(2\lambda-1)(2\lambda+1)$$
$$\times\times(-i)\partial^5(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{72}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times(-i)\,\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{21}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)\,(-i)$$
$$\times\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$

$$-\frac{5}{28}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)\,(-i)$$
$$\times\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$-\frac{5}{9}\frac{4}{5}(2\lambda^2-5\lambda-27)\,(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}-\bar{Q}^{\lambda,22}_{\frac{9}{2}})$$
$$+\frac{1}{10}(2\lambda+27)\,(-i)\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}-\bar{Q}^{\lambda,22}_{\frac{11}{2}})\Big](w)$$
$$+\frac{1}{(z-w)}\Big[-\frac{1}{720}\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)$$
$$\times(2\lambda-3)(2\lambda-1)(2\lambda+1)$$
$$\times(-i)\partial^6(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{420}\frac{256}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$$
$$\times(-i)\,\partial^5\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{112}\frac{256}{25}(\lambda-4)(\lambda+1)(2\lambda-3)(2\lambda+3)$$
$$\times(-i)\,\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{5}{126}\frac{16}{25}(2\lambda+3)(4\lambda^2+18\lambda-61)$$
$$\times(-i)\,\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$-\frac{1}{6}\frac{4}{5}(2\lambda^2-5\lambda-27)\,(-i)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}-\bar{Q}^{\lambda,22}_{\frac{9}{2}})$$
$$+\frac{6}{11}\frac{1}{10}(2\lambda+27)\,(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}-\bar{Q}^{\lambda,22}_{\frac{11}{2}})$$
$$+\frac{1}{4}\,(-i)\,(\bar{Q}^{\lambda,11}_{\frac{13}{2}}-\bar{Q}^{\lambda,22}_{\frac{13}{2}})\Big](w)+\cdots$$
$$=-q^{4,\frac{7}{2}}_{F,5}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(-i)(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{F,4}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(-i)(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})(w)}{(z-w)}\Bigg]$$

$$-q^{4,\frac{7}{2}}_{F,3}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{F,2}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{F,1}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}-\bar{Q}^{\lambda,22}_{\frac{9}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{F,0}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}-\bar{Q}^{\lambda,22}_{\frac{11}{2}})(w)}{(z-w)}\Bigg]$$

$$-q^{4,\frac{7}{2}}_{F,-1}(\partial_z,\partial_w,\lambda)\left[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{13}{2}}-\bar{Q}^{\lambda,22}_{\frac{13}{2}})(w)}{(z-w)}\right]+\cdots$$
$$=\sum_{h=-1}^{5} q^h\, q^{4,\frac{7}{2}}_{F,h}(\partial_z,\partial_w,\lambda)\, i\, f^{123}\left[\frac{\bar{Q}^{\lambda,\hat{A}=3}_{\frac{11}{2}-h}(w)}{(z-w)}\right]$$
$$+\cdots. \tag{C.10}$$

The λ factor appears in the structure constant appearing in the current $(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$ and its descendants.

C.2.7 The OPE between $W^{\lambda,\hat{A}=1}_{B,4}$ and $\bar{Q}^{\lambda,\hat{B}=1}_{\frac{7}{2}}$

By considering the last equation of (3.12), the following result is obtained for the same indices

$$(W^{\lambda,12}_{B,4}+W^{\lambda,21}_{B,4})(z)\,(\bar{Q}^{\lambda,12}_{\frac{7}{2}}+\bar{Q}^{\lambda,21}_{\frac{7}{2}})(w)$$
$$=\frac{1}{(z-w)^7}\left[\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)\right.$$
$$\left.\times(2\lambda+1)\right](\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{(z-w)^6}\left[\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)\right.$$
$$\times(2\lambda+1)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$-\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\left.\times(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})\right](w)$$
$$+\frac{1}{(z-w)^5}\left[\frac{1}{2}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)\right.$$
$$\times(2\lambda+1)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$-\frac{2}{3}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times\partial\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)$$
$$\left.\times(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})\right](w)$$
$$+\frac{1}{(z-w)^4}\left[\frac{1}{6}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)\right.$$
$$\times(2\lambda+1)\,\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$-\frac{1}{4}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{3}{5}\,\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)$$
$$\times\partial\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$\left.+\frac{32}{25}(\lambda-2)\left(4\lambda^2-22\lambda-51\right)\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})\right](w)$$
$$+\frac{1}{(z-w)^3}\left[\frac{1}{24}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)\right.$$
$$\times(2\lambda+1)\,\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$-\frac{1}{15}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{1}{5}\,\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)$$
$$\times\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$+\frac{4}{7}\,\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$\left.+\frac{4}{5}(2\lambda^2+3\lambda-29)\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}+\bar{Q}^{\lambda,22}_{\frac{9}{2}})\right](w)$$
$$+\frac{1}{(z-w)^2}\left[\frac{1}{120}\,\frac{2048}{5}\,(\lambda-1)\lambda\right.$$
$$\times(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)\,\partial^5\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$-\frac{1}{72}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{1}{21}\,\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+7)\,\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$+\frac{5}{28}\,\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$+\frac{5}{9}\,\frac{4}{5}(2\lambda^2+3\lambda-29)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}+\bar{Q}^{\lambda,22}_{\frac{9}{2}})$$
$$\left.+\frac{1}{5}(-\lambda+14)\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}+\bar{Q}^{\lambda,22}_{\frac{11}{2}})\right](w)$$
$$+\frac{1}{(z-w)}\left[\frac{1}{720}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)\right.$$
$$\times(2\lambda+1)\,\partial^6\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$-\frac{1}{420}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times\partial^5\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$-\frac{1}{112}\,\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+7)\,\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$+\frac{5}{126}\,\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$+\frac{1}{6}\,\frac{4}{5}(2\lambda^2+3\lambda-29)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}+\bar{Q}^{\lambda,22}_{\frac{9}{2}})$$
$$+\frac{6}{11}\,\frac{1}{5}(-\lambda+14)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}+\bar{Q}^{\lambda,22}_{\frac{11}{2}})$$

$$-\frac{1}{4}\,(\bar{Q}^{\lambda,11}_{\frac{13}{2}}+\bar{Q}^{\lambda,22}_{\frac{13}{2}})\Bigg](w)+\cdots$$

$$=-q^{4,\frac{7}{2}}_{B,5}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(\bar{Q}^{\lambda,11}_{\frac{1}{2}}+\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{B,4}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(\bar{Q}^{\lambda,11}_{\frac{3}{2}}+\bar{Q}^{\lambda,22}_{\frac{3}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{B,3}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(\bar{Q}^{\lambda,11}_{\frac{5}{2}}+\bar{Q}^{\lambda,22}_{\frac{5}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{B,2}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(\bar{Q}^{\lambda,11}_{\frac{7}{2}}+\bar{Q}^{\lambda,22}_{\frac{7}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{B,1}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(\bar{Q}^{\lambda,11}_{\frac{9}{2}}+\bar{Q}^{\lambda,22}_{\frac{9}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{B,0}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(\bar{Q}^{\lambda,11}_{\frac{11}{2}}+\bar{Q}^{\lambda,22}_{\frac{11}{2}})(w)}{(z-w)}\Bigg]$$
$$-q^{4,\frac{7}{2}}_{B,-1}(\partial_z,\partial_w,\lambda)\Bigg[\frac{(\bar{Q}^{\lambda,11}_{\frac{13}{2}}+\bar{Q}^{\lambda,22}_{\frac{13}{2}})(w)}{(z-w)}\Bigg]+\cdots$$
$$=-\sum_{h=-1}^{5}q^h\,q^{4,\frac{7}{2}}_{B,h}(\partial_z,\partial_w,\lambda)\,\delta^{11}\Bigg[\frac{\bar{Q}^{\lambda,\bar{a}a}_{\frac{11}{2}-h}(w)}{(z-w)}\Bigg]$$
$$+\cdots. \tag{C.11}$$

There appears the λ factor in the coefficients of the current $\bar{Q}^{\lambda,\bar{a}a}_{\frac{1}{2}}(w)$ and its descendants.

C.2.8 The OPE between $W^{\lambda,\hat{A}=1}_{B,4}$ and $\bar{Q}^{\lambda,\hat{B}=2}_{\frac{7}{2}}$

For different indices we obtain the following result

$$(W^{\lambda,12}_{B,4}+W^{\lambda,21}_{B,4})(z)\,(-i)\,(\bar{Q}^{\lambda,12}_{\frac{7}{2}}-\bar{Q}^{\lambda,21}_{\frac{7}{2}})(w)$$
$$=\frac{1}{(z-w)^7}\Bigg[-\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)\Bigg](-i)\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)$$
$$+\frac{1}{(z-w)^6}\Bigg[-\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$\times(2\lambda+1)\,(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$+\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(-i)\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})\Bigg](w)$$
$$+\frac{1}{(z-w)^5}\Bigg[-\frac{1}{2}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda-1)(2\lambda+1)\,(-i)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$+\frac{2}{3}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)$$
$$\times(-i)\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})\Bigg](w)$$
$$+\frac{1}{(z-w)^4}\Bigg[-\frac{1}{6}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda-1)(2\lambda+1)\,(-i)\,\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$+\frac{1}{4}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(-i)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{3}{5}\,\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)$$
$$\times(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{32}{25}(\lambda-2)\left(4\lambda^2-22\lambda-51\right)\,(-i)$$
$$\times(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})\Bigg](w)$$
$$+\frac{1}{(z-w)^3}\Bigg[-\frac{1}{24}\,\frac{2048}{5}\,(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$(2\lambda-1)(2\lambda+1)\,(-i)\,\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$+\frac{1}{15}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$(-i)\,\partial^3\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{5}\,\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,(-i)$$
$$\times\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{4}{7}\,\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$-\frac{4}{5}(2\lambda^2+3\lambda-29)$$
$$\times(-i)\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}-\bar{Q}^{\lambda,22}_{\frac{9}{2}})\Bigg](w)$$
$$+\frac{1}{(z-w)^2}\Bigg[-\frac{1}{120}\,\frac{2048}{5}\,(\lambda-1)\lambda$$
$$\times(\lambda+1)(2\lambda-3)(2\lambda-1)$$
$$(2\lambda+1)\,(-i)\,\partial^5\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$+\frac{1}{72}\,\frac{512}{5}\,(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$(-i)\,\partial^4\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{21}\,\frac{256}{25}\,(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,(-i)\,\partial^3$$
$$\times(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{5}{28}\,\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,(-i)$$
$$\times\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$

$$-\frac{5}{9}\frac{4}{5}(2\lambda^2+3\lambda-29)\,(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}-\bar{Q}^{\lambda,22}_{\frac{9}{2}})$$

$$-\frac{1}{5}(-\lambda+14)\,(-i)\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}-\bar{Q}^{\lambda,22}_{\frac{11}{2}})\Big](w)$$
$$+\frac{1}{(z-w)}\Big[-\frac{1}{720}\frac{2048}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda-1)(2\lambda+1)\,(-i)\,\partial^6\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})$$
$$+\frac{1}{420}\frac{512}{5}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(-i)\,\partial^5\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})$$
$$+\frac{1}{112}\frac{256}{25}(\lambda-2)(\lambda+1)(2\lambda-3)(2\lambda+7)\,(-i)\,\partial^4$$
$$\times(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})$$
$$-\frac{5}{126}\frac{32}{25}(\lambda-2)(4\lambda^2-22\lambda-51)\,(-i)\,\partial^3$$
$$\times(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})$$
$$-\frac{1}{6}\frac{4}{5}(2\lambda^2+3\lambda-29)\,(-i)\,\partial^2\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}-\bar{Q}^{\lambda,22}_{\frac{9}{2}})$$
$$-\frac{6}{11}\frac{1}{5}(-\lambda+14)\,(-i)\,\partial\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}-\bar{Q}^{\lambda,22}_{\frac{11}{2}})$$
$$+\frac{1}{4}(-i)\,(\bar{Q}^{\lambda,11}_{\frac{13}{2}}-\bar{Q}^{\lambda,22}_{\frac{13}{2}})\Big](w)+\cdots$$
$$=q^{4,\frac{7}{2}}_{B,5}(\partial_z,\partial_w,\lambda)\Big[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{1}{2}}-\bar{Q}^{\lambda,22}_{\frac{1}{2}})(w)}{(z-w)}\Big]$$
$$+q^{4,\frac{7}{2}}_{B,4}(\partial_z,\partial_w,\lambda)\Big[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{3}{2}}-\bar{Q}^{\lambda,22}_{\frac{3}{2}})(w)}{(z-w)}\Big]$$
$$+q^{4,\frac{7}{2}}_{B,3}(\partial_z,\partial_w,\lambda)\Big[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{5}{2}}-\bar{Q}^{\lambda,22}_{\frac{5}{2}})(w)}{(z-w)}\Big]$$
$$+q^{4,\frac{7}{2}}_{B,2}(\partial_z,\partial_w,\lambda)\Big[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{7}{2}}-\bar{Q}^{\lambda,22}_{\frac{7}{2}})(w)}{(z-w)}\Big]$$
$$+q^{4,\frac{7}{2}}_{B,1}(\partial_z,\partial_w,\lambda)\Big[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{9}{2}}-\bar{Q}^{\lambda,22}_{\frac{9}{2}})(w)}{(z-w)}\Big]$$
$$+q^{4,\frac{7}{2}}_{B,0}(\partial_z,\partial_w,\lambda)\Big[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{11}{2}}-\bar{Q}^{\lambda,22}_{\frac{11}{2}})(w)}{(z-w)}\Big]$$
$$+q^{4,\frac{7}{2}}_{B,-1}(\partial_z,\partial_w,\lambda)\Big[\frac{(-i)\,(\bar{Q}^{\lambda,11}_{\frac{13}{2}}-\bar{Q}^{\lambda,22}_{\frac{13}{2}})(w)}{(z-w)}\Big]+\cdots$$
$$=-\sum_{h=-1}^{5}q^h\,q^{4,\frac{7}{2}}_{B,h}(\partial_z,\partial_w,\lambda)\,i\,f^{123}$$
$$\times\Big[\frac{\bar{Q}^{\lambda,\hat{A}=3}_{\frac{11}{2}-h}(w)}{(z-w)}\Big]+\cdots. \tag{C.12}$$

Similarly, we observe that the last equation of (3.12) can be seen from the above (C.12).

C.3 The OPE between the fermionic currents

We check (3.13) with the weights $h_1=h_2=3$.

C.3.1 The OPE between $Q^{\lambda,\hat{A}=1}_{\frac{7}{2}}$ and $\bar{Q}^{\lambda,\hat{B}=1}_{\frac{7}{2}}$

When the indices $\hat{A}$ and $\hat{B}$ ($\hat{A}=\hat{B}=1$) are the same, then we obtain the following expression

$$(Q^{\lambda,12}_{\frac{7}{2}}+Q^{\lambda,21}_{\frac{7}{2}})(z)\,(\bar{Q}^{\lambda,12}_{\frac{7}{2}}+\bar{Q}^{\lambda,21}_{\frac{7}{2}})(w)$$
$$=\frac{1}{(z-w)^7}\Big[-\frac{6144}{5}(4\lambda-1)(12\lambda^4-12\lambda^3$$
$$-13\lambda^2+8\lambda+3)\Big]$$
$$+\frac{1}{(z-w)^6}\Big[\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$(2\lambda+1)(2\lambda-1)\,W^{\lambda,\bar{a}a}_{F,1}$$
$$+\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,W^{\lambda,\bar{a}a}_{B,1}\Big](w)$$
$$+\frac{1}{(z-w)^5}\Big[\frac{1}{2}\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(2\lambda-1)\,\partial\,W^{\lambda,\bar{a}a}_{F,1}$$
$$+\frac{1}{2}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial\,W^{\lambda,\bar{a}a}_{B,1}$$
$$+\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,W^{\lambda,\bar{a}a}_{F,2}$$
$$+\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,W^{\lambda,\bar{a}a}_{B,2}\Big](w)$$
$$+\frac{1}{(z-w)^4}\Big[\frac{1}{6}\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(2\lambda-1)\,\partial^2\,W^{\lambda,\bar{a}a}_{F,1}$$
$$+\frac{1}{6}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial^2\,W^{\lambda,\bar{a}a}_{B,1}$$
$$+\frac{1}{2}\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,\partial\,W^{\lambda,\bar{a}a}_{F,2}$$
$$+\frac{1}{2}\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial\,W^{\lambda,\bar{a}a}_{B,2}$$
$$-\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)\,W^{\lambda,\bar{a}a}_{F,3}-\frac{512}{25}(\lambda+1)$$
$$\times(4\lambda^2+2\lambda-27)W^{\lambda,\bar{a}a}_{B,3}\Big](w)$$
$$+\frac{1}{(z-w)^3}\Big[\frac{1}{24}\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(2\lambda-1)\,\partial^3\,W^{\lambda,\bar{a}a}_{F,1}$$
$$+\frac{1}{24}\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial^3\,W^{\lambda,\bar{a}a}_{B,1}$$
$$+\frac{3}{20}\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,\partial^2\,W^{\lambda,\bar{a}a}_{F,2}$$
$$+\frac{3}{20}\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial^2\,W^{\lambda,\bar{a}a}_{B,2}$$

$$-\frac{1}{2}\,\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)\,\partial\, W_{F,3}^{\lambda,\bar{a}a}$$
$$-\frac{1}{2}\,\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)\,\partial\, W_{B,3}^{\lambda,\bar{a}a}$$
$$-\frac{64}{25}(4\lambda^2+18\lambda-61)\, W_{F,4}^{\lambda,\bar{a}a}-\frac{64}{25}(4\lambda^2-22\lambda$$
$$-51)\, W_{B,4}^{\lambda,\bar{a}a}\Big](w)$$
$$+\frac{1}{(z-w)^2}\Big[\frac{1}{120}\,\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)(2\lambda-1)\,\partial^4\, W_{F,1}^{\lambda,\bar{a}a}$$
$$+\frac{1}{120}\,\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial^4\, W_{B,1}^{\lambda,\bar{a}a}$$
$$+\frac{1}{30}\,\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,\partial^3\, W_{F,2}^{\lambda,\bar{a}a}$$
$$+\frac{1}{30}\,\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial^3\, W_{B,2}^{\lambda,\bar{a}a}$$
$$-\frac{1}{7}\,\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)\,\partial^2\, W_{F,3}^{\lambda,\bar{a}a}$$
$$-\frac{1}{7}\,\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)\,\partial^2\, W_{B,3}^{\lambda,\bar{a}a}$$
$$-\frac{1}{2}\,\frac{64}{25}(4\lambda^2+18\lambda-61)\,\partial\, W_{F,4}^{\lambda,\bar{a}a}-\frac{1}{2}\,\frac{64}{25}(4\lambda^2$$
$$-22\lambda-51)\,\partial\, W_{B,4}^{\lambda,\bar{a}a}$$
$$+\frac{8}{5}(2\lambda-13)\, W_{F,5}^{\lambda,\bar{a}a}+\frac{16(\lambda+6)}{5}\, W_{B,5}^{\lambda,\bar{a}a}\Big](w)$$
$$+\frac{1}{(z-w)}\Big[\frac{1}{720}\,\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)(2\lambda-1)\,\partial^5\, W_{F,1}^{\lambda,\bar{a}a}$$
$$+\frac{1}{720}\,\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial^5\, W_{B,1}^{\lambda,\bar{a}a}$$
$$+\frac{1}{168}\,\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,\partial^4\, W_{F,2}^{\lambda,\bar{a}a}$$
$$+\frac{1}{168}\,\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,\partial^4\, W_{B,2}^{\lambda,\bar{a}a}$$
$$-\frac{5}{168}\,\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)\,\partial^3\, W_{F,3}^{\lambda,\bar{a}a}$$
$$-\frac{5}{168}\,\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)\,\partial^3\, W_{B,3}^{\lambda,\bar{a}a}$$
$$-\frac{5}{36}\,\frac{64}{25}(4\lambda^2+18\lambda-61)\,\partial^2\, W_{F,4}^{\lambda,\bar{a}a}-\frac{5}{36}\,\frac{64}{25}(4\lambda^2$$
$$-22\lambda-51)\,\partial^2\, W_{B,4}^{\lambda,\bar{a}a}$$
$$+\frac{1}{2}\,\frac{8}{5}(2\lambda-13)\,\partial\, W_{F,5}^{\lambda,\bar{a}a}+\frac{1}{2}\,\frac{16(\lambda+6)}{5}\,\partial\, W_{B,5}^{\lambda,\bar{a}a}$$
$$+2\, W_{F,6}^{\lambda,\bar{a}a}+2\, W_{B,6}^{\lambda,\bar{a}a}\Big](w)+\cdots$$
$$=\frac{1}{(z-w)^7}\Big[-\frac{6144}{5}(4\lambda-1)(12\lambda^4-12\lambda^3-$$
$$13\lambda^2+8\lambda+3)\Big]$$
$$-o_{F,5}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\Big[\frac{W_{F,1}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]-o_{B,5}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{W_{B,1}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]+o_{F,4}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\Big[\frac{W_{F,2}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]$$
$$+o_{B,4}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\Big[\frac{W_{B,2}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]-o_{F,3}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{W_{F,3}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]-o_{B,3}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\Big[\frac{W_{B,3}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]$$
$$+o_{F,2}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\Big[\frac{W_{F,4}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]+o_{B,2}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{W_{B,4}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]-o_{F,1}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\Big[\frac{W_{F,5}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]$$
$$-o_{B,1}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\Big[\frac{W_{B,5}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]+o_{F,0}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)$$
$$\times\Big[\frac{W_{F,6}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]+o_{B,0}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\Big[\frac{W_{B,6}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]$$
$$+\cdots$$
$$=\frac{1}{(z-w)^7}\,c_Q(3,3,\lambda)\,\delta^{11}\,q^4+\sum_{h=0}^{5}q^h\,(-1)^h$$
$$\times o_{F,h}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\,\delta^{11}\Big[\frac{W_{F,6-h}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]$$
$$+\sum_{h=0}^{5}q^h\,(-1)^h\,o_{B,h}^{\frac{7}{2},\frac{7}{2}}(\partial_z,\partial_w,\lambda)\,\delta^{11}\Big[\frac{W_{B,6-h}^{\lambda,\bar{a}a}(w)}{(z-w)}\Big]$$
$$+\cdots. \qquad (C.13)$$

We can check that the corresponding terms in (3.13) can be seen from the above (C.13) and note that there is an additional factor $(-1)^h$ coming from the anticommutator to the OPE.

C.3.2 The OPE between $Q_{\frac{7}{2}}^{\lambda,\hat{A}=1}$ and $\bar{Q}_{\frac{7}{2}}^{\lambda,\hat{B}=2}$

By considering the different indices as before, we determine the following result

$$(Q_{\frac{7}{2}}^{\lambda,12}+Q_{\frac{7}{2}}^{\lambda,21})(z)\,(-i)\,(\bar{Q}_{\frac{7}{2}}^{\lambda,12}-\bar{Q}_{\frac{7}{2}}^{\lambda,21})(w)$$
$$=\frac{1}{(z-w)^6}\Big[-\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(2\lambda-1)\,(-i)\,(W_{F,1}^{\lambda,11}-W_{F,1}^{\lambda,22})$$
$$+\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times(W_{B,1}^{\lambda,11}-W_{B,1}^{\lambda,22})\Big](w)$$
$$+\frac{1}{(z-w)^5}\Big[-\frac{1}{2}\,\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(2\lambda-1)\,(-i)\,\partial\,(W_{F,1}^{\lambda,11}-W_{F,1}^{\lambda,22})$$
$$+\frac{1}{2}\,\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial\,(W_{B,1}^{\lambda,11}-W_{B,1}^{\lambda,22})$$

$$-\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,(-i)$$
$$\times(W^{\lambda,11}_{F,2}-W^{\lambda,22}_{F,2})$$
$$+\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times(W^{\lambda,11}_{B,2}-W^{\lambda,22}_{B,2})\Big](w)$$
$$+\frac{1}{(z-w)^4}\Big[-\frac{1}{6}\,\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$$
$$\times(2\lambda-1)\,(-i)\,\partial^2\,(W^{\lambda,11}_{F,1}-W^{\lambda,22}_{F,1})$$
$$+\frac{1}{6}\,\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial^2\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{1}{2}\,\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,(-i)$$
$$\times\partial\,(W^{\lambda,11}_{F,2}-W^{\lambda,22}_{F,2})$$
$$+\frac{1}{2}\,\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial\,(W^{\lambda,11}_{B,2}-W^{\lambda,22}_{B,2})$$
$$+\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)\,(-i)\,(W^{\lambda,11}_{F,3}-W^{\lambda,22}_{F,3})$$
$$-\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)$$
$$\times(-i)\,(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})\Big](w)$$
$$+\frac{1}{(z-w)^3}\Big[-\frac{1}{24}\,\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)(2\lambda-1)\,(-i)$$
$$\times\partial^3\,(W^{\lambda,11}_{F,1}-W^{\lambda,22}_{F,1})$$
$$+\frac{1}{24}\,\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial^3\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{3}{20}\,\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,(-i)$$
$$\times\partial^2\,(W^{\lambda,11}_{F,2}-W^{\lambda,22}_{F,2})$$
$$+\frac{3}{20}\,\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial^2\,(W^{\lambda,11}_{B,2}-W^{\lambda,22}_{B,2})$$
$$+\frac{1}{2}\,\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)$$
$$\times(-i)\partial\,(W^{\lambda,11}_{F,3}-W^{\lambda,22}_{F,3})$$
$$-\frac{1}{2}\,\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)$$
$$\times(-i)\,\partial\,(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})$$
$$+\frac{64}{25}(4\lambda^2+18\lambda-61)\,(-i)\,(W^{\lambda,11}_{F,4}-W^{\lambda,22}_{F,4})$$
$$-\frac{64}{25}(4\lambda^2-22\lambda-51)\,(-i)\,(W^{\lambda,11}_{B,4}-W^{\lambda,22}_{B,4})\Big](w)$$
$$+\frac{1}{(z-w)^2}\Big[-\frac{1}{120}\,\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)(2\lambda-1)\,(-i)\,\partial^4\,(W^{\lambda,11}_{F,1}-W^{\lambda,22}_{F,1})$$
$$+\frac{1}{120}\,\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial^4\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{1}{30}\,\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,(-i)$$
$$\times\partial^3\,(W^{\lambda,11}_{F,2}-W^{\lambda,22}_{F,2})$$
$$+\frac{1}{30}\,\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial^3\,(W^{\lambda,11}_{B,2}-W^{\lambda,22}_{B,2})$$
$$+\frac{1}{7}\,\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)$$
$$\times(-i)\,\partial^2\,(W^{\lambda,11}_{F,3}-W^{\lambda,22}_{F,3})$$
$$-\frac{1}{7}\,\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)$$
$$\times(-i)\,\partial^2\,(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})$$
$$+\frac{1}{2}\,\frac{64}{25}(4\lambda^2+18\lambda-61)(-i)\partial(W^{\lambda,11}_{F,4}-W^{\lambda,22}_{F,4})$$
$$-\frac{1}{2}\,\frac{64}{25}(4\lambda^2-22\lambda-51)(-i)\partial(W^{\lambda,11}_{B,4}-W^{\lambda,22}_{B,4})$$
$$-\frac{8}{5}(2\lambda-13)\,(-i)\,(W^{\lambda,11}_{F,5}-W^{\lambda,22}_{F,5})$$
$$+\frac{16(\lambda+6)}{5}\,(-i)\,(W^{\lambda,11}_{B,5}-W^{\lambda,22}_{B,5})\Big](w)$$
$$+\frac{1}{(z-w)}\Big[-\frac{1}{720}\,\frac{4096}{5}(\lambda-1)(\lambda+1)(2\lambda-3)$$
$$\times(2\lambda+1)(2\lambda-1)\,(-i)\,\partial^5\,(W^{\lambda,11}_{F,1}-W^{\lambda,22}_{F,1})$$
$$+\frac{1}{720}\,\frac{8192}{5}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial^5\,(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$$
$$-\frac{1}{168}\,\frac{1024}{25}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+17)\,(-i)$$
$$\partial^4\,(W^{\lambda,11}_{F,2}-W^{\lambda,22}_{F,2})$$
$$+\frac{1}{168}\,\frac{1024}{25}(\lambda-9)(\lambda+1)(2\lambda-3)(2\lambda+1)\,(-i)$$
$$\times\partial^4\,(W^{\lambda,11}_{B,2}-W^{\lambda,22}_{B,2})$$
$$+\frac{5}{168}\,\frac{256}{25}(2\lambda-3)(4\lambda^2-6\lambda-25)$$
$$\times(-i)\,\partial^3\,(W^{\lambda,11}_{F,3}-W^{\lambda,22}_{F,3})$$
$$-\frac{5}{168}\,\frac{512}{25}(\lambda+1)(4\lambda^2+2\lambda-27)$$

$$\times (-i)\,\partial^3\,(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})$$
$$+\frac{5}{36}\frac{64}{25}(4\lambda^2+18\lambda-61)\,(-i)\,\partial^2\,(W^{\lambda,11}_{F,4}-W^{\lambda,22}_{F,4})$$
$$-\frac{5}{36}\frac{64}{25}(4\lambda^2-22\lambda-51)\,(-i)\,\partial^2\,(W^{\lambda,11}_{B,4}-W^{\lambda,22}_{B,4})$$
$$-\frac{1}{2}\frac{8}{5}(2\lambda-13)\,(-i)\,\partial\,(W^{\lambda,11}_{F,5}-W^{\lambda,22}_{F,5})$$
$$+\frac{1}{2}\frac{16(\lambda+6)}{5}\,(-i)\,\partial\,(W^{\lambda,11}_{B,5}-W^{\lambda,22}_{B,5})$$
$$-2\,(-i)\,(W^{\lambda,11}_{F,6}-W^{\lambda,22}_{F,6})+2$$
$$\times(-i)\,(W^{\lambda,11}_{B,6}-W^{\lambda,22}_{B,6})\Bigg](w)+\cdots$$
$$= o^{\frac{7}{2},\frac{7}{2}}_{F,5}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{F,1}-W^{\lambda,22}_{F,1})(w)}{(z-w)}\Bigg]$$
$$-o^{\frac{7}{2},\frac{7}{2}}_{B,5}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})(w)}{(z-w)}\Bigg]$$
$$-o^{\frac{7}{2},\frac{7}{2}}_{F,4}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{F,2}-W^{\lambda,22}_{F,2})(w)}{(z-w)}\Bigg]$$
$$+o^{\frac{7}{2},\frac{7}{2}}_{B,4}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{B,2}-W^{\lambda,22}_{B,2})(w)}{(z-w)}\Bigg]$$
$$+o^{\frac{7}{2},\frac{7}{2}}_{F,3}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{F,3}-W^{\lambda,22}_{F,3})(w)}{(z-w)}\Bigg]$$
$$-o^{\frac{7}{2},\frac{7}{2}}_{B,3}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{B,3}-W^{\lambda,22}_{B,3})(w)}{(z-w)}\Bigg]$$
$$-o^{\frac{7}{2},\frac{7}{2}}_{F,2}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{F,4}-W^{\lambda,22}_{F,4})(w)}{(z-w)}\Bigg]$$
$$+o^{\frac{7}{2},\frac{7}{2}}_{B,2}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{B,4}-W^{\lambda,22}_{B,4})(w)}{(z-w)}\Bigg]$$
$$+o^{\frac{7}{2},\frac{7}{2}}_{F,1}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{F,5}-W^{\lambda,22}_{F,5})(w)}{(z-w)}\Bigg]$$
$$-o^{\frac{7}{2},\frac{7}{2}}_{B,1}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{B,5}-W^{\lambda,22}_{B,5})(w)}{(z-w)}\Bigg]$$
$$-o^{\frac{7}{2},\frac{7}{2}}_{F,0}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{F,6}-W^{\lambda,22}_{F,6})(w)}{(z-w)}\Bigg]$$
$$+o^{\frac{7}{2},\frac{7}{2}}_{B,0}(\partial_z,\partial_w,\lambda)\Bigg[(-i)\,\frac{(W^{\lambda,11}_{B,6}-W^{\lambda,22}_{B,6})(w)}{(z-w)}\Bigg]$$
$$+\cdots$$
$$=\sum_{h=0}^{5} q^h\,(-1)^h\,o^{\frac{7}{2},\frac{7}{2}}_{F,h}(\partial_z,\partial_w,\lambda)\,i\,f^{123}\left[\frac{W^{\lambda,\hat{A}=3}_{F,6-h}(w)}{(z-w)}\right]$$
$$-\sum_{h=0}^{5} q^h\,(-1)^h\,o^{\frac{7}{2},\frac{7}{2}}_{B,h}(\partial_z,\partial_w,\lambda)\,i\,f^{123}$$
$$\times\left[\frac{W^{\lambda,\hat{A}=3}_{B,6-h}(w)}{(z-w)}\right]+\cdots\,. \tag{C.14}$$

As before, the λ factor appears in the structure constants appearing in the current $(W^{\lambda,11}_{B,1}-W^{\lambda,22}_{B,1})$ and its descendants.

Appendix D: The other central terms in the OPEs between the fermionic currents

As in (3.8) and (B.1), we can calculate the following OPEs for the highest order poles

$$Q^{(h_1)+}_{\lambda,\bar{a}b}(z)\,Q^{(h_2)+}_{\lambda,\bar{c}d}(w)\Big|_{\frac{1}{(z-w)^{h_1+h_2-1}}}$$
$$= N\,\delta_{b\bar{c}}\,\delta_{d\bar{a}}\Bigg(\sum_{j=0}^{h_1-2}\sum_{i=0}^{h_2-1}\sum_{t=0}^{j+1}\beta^j(h_1,\lambda)\,\alpha^i(h_2,\lambda)$$
$$\times\frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1-1+t}\,(j+1-t)_{h_1-2-j}$$
$$\times(h_1-1-t)_{t+1+i}\,(h_1+i)_{h_2-1-i}$$
$$-\sum_{j=0}^{h_1-1}\sum_{i=0}^{h_2-2}\sum_{t=0}^{j+1}\alpha^j(h_1,\lambda)\,\beta^i(h_2,\lambda)$$
$$\times\frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1+t}\,(j+1-t)_{h_1-1-j}$$
$$\times(h_1-t)_{t+1+i}\,(h_1+1+i)_{h_2-2-i}\Bigg),$$

$$Q^{(h_1)+}_{\lambda,\bar{a}b}(z)\,Q^{(h_2)-}_{\lambda,\bar{c}d}(w)\Big|_{\frac{1}{(z-w)^{h_1+h_2-1}}}$$
$$= N\,\delta_{b\bar{c}}\,\delta_{d\bar{a}}\Bigg(\sum_{j=0}^{h_1-2}\sum_{i=0}^{h_2-1}\sum_{t=0}^{j+1}\beta^j(h_1,\lambda)\,\alpha^i(h_2,\lambda)$$
$$\times\frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1-1+t}\,(j+1-t)_{h_1-2-j}$$
$$\times(h_1-1-t)_{t+1+i}\,(h_1+i)_{h_2-1-i}$$
$$+\sum_{j=0}^{h_1-1}\sum_{i=0}^{h_2-2}\sum_{t=0}^{j+1}\alpha^j(h_1,\lambda)\,\beta^i(h_2,\lambda)$$
$$\times\frac{j!\,(t+i)!}{t!\,(t+i+1)!}\,(-1)^{h_1+t}\,(j+1-t)_{h_1-1-j}$$
$$\times(h_1-t)_{t+1+i}\,(h_1+1+i)_{h_2-2-i}\Bigg),$$

$$Q^{(h_1)-}_{\lambda,\bar{a}b}(z)\,Q^{(h_2)-}_{\lambda,\bar{c}d}(w)\Big|_{\frac{1}{(z-w)^{h_1+h_2-1}}}$$

$$= N\,\delta_{b\bar{c}}\,\delta_{d\bar{a}}\Bigg(-\sum_{j=0}^{h_1-2}\sum_{i=0}^{h_2-1}\sum_{t=0}^{j+1}\beta^j(h_1,\lambda)\,\alpha^i(h_2,\lambda)$$
$$\times\frac{j!\,(t+i)!}{t!\,(t+i+1)!}(-1)^{h_1-1+t}\,(j+1-t)_{h_1-2-j}$$
$$\times(h_1-1-t)_{t+1+i}\,(h_1+i)_{h_2-1-i}$$
$$+\sum_{j=0}^{h_1-1}\sum_{i=0}^{h_2-2}\sum_{t=0}^{j+1}\alpha^j(h_1,\lambda)\,\beta^i(h_2,\lambda)$$
$$\times\frac{j!\,(t+i)!}{t!\,(t+i+1)!}(-1)^{h_1+t}\,(j+1-t)_{h_1-1-j}$$
$$(h_1-t)_{t+1+i}\,(h_1+1+i)_{h_2-2-i}\Bigg). \tag{D.1}$$

These results will be used in (3.13) and (3.14).

Appendix E: The other (anti)commutator relations

We present the remaining (anti)commutator relations discussed in Sect. 3.

E.1 The commutators between the currents consisting of $\beta\,\gamma$ system with $h_1 = h_2, h_2 \pm 1$ for nonzero λ

As done in Sect. 3, we present the final result for the commutators between the currents consisting of $\beta\,\gamma$ system as follows:

$$\big[(W^{\lambda,21}_{\mathrm{B},h_1})_m,(W^{\lambda,12}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)$$
$$\times(W^{\lambda,11}_{\mathrm{B},h_1+h_2-2-h}+W^{\lambda,22}_{\mathrm{B},h_1+h_2-2-h})_{m+n}$$
$$+\frac{1}{2}\binom{m+h_1-1}{h_1+h_2-1}c_B(h_1,h_2,\lambda)\,q^{h_1+h_2-4}\,\delta_{m+n}$$
$$-\frac{1}{2}\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,11}_{\mathrm{B},h_1+h_2-2-h}$$
$$-W^{\lambda,22}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,12}_{\mathrm{B},h_1})_m,(W^{\lambda,21}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)$$
$$\times(W^{\lambda,11}_{\mathrm{B},h_1+h_2-2-h}+W^{\lambda,22}_{\mathrm{B},h_1+h_2-2-h})_{m+n}$$
$$+\frac{1}{2}\binom{m+h_1-1}{h_1+h_2-1}c_B(h_1,h_2,\lambda)\,q^{h_1+h_2-4}\,\delta_{m+n}$$
$$+\frac{1}{2}\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)$$
$$\times(W^{\lambda,11}_{\mathrm{B},h_1+h_2-2-h}-W^{\lambda,22}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,12}_{\mathrm{B},h_1})_m,(W^{\lambda,11}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\Bigg(\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}-\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}\Bigg)$$
$$\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}$$
$$\times(m,n,\lambda)\,(W^{\lambda,12}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,12}_{\mathrm{B},h_1})_m,(W^{\lambda,22}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\Bigg(\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}+\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}\Bigg)$$
$$\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,12}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,21}_{\mathrm{B},h_1})_m,(W^{\lambda,11}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\Bigg(\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}+\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}\Bigg)$$
$$\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,21}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,21}_{\mathrm{B},h_1})_m,(W^{\lambda,22}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\Bigg(\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}-\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}\Bigg)$$
$$\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,21}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,11}_{\mathrm{B},h_1})_m,(W^{\lambda,12}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\Bigg(\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}+\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}\Bigg)$$
$$\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,12}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,22}_{\mathrm{B},h_1})_m,(W^{\lambda,12}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\Bigg(\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}-\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}\Bigg)$$
$$\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,12}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,11}_{\mathrm{B},h_1})_m,(W^{\lambda,21}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\Bigg(\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}-\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}\Bigg)$$
$$\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,21}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,22}_{\mathrm{B},h_1})_m,(W^{\lambda,21}_{\mathrm{B},h_2})_n\big]=\frac{1}{2}\Bigg(\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}+\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3}\Bigg)$$
$$\times q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)\,(W^{\lambda,21}_{\mathrm{B},h_1+h_2-2-h})_{m+n},$$

$$\big[(W^{\lambda,11}_{\mathrm{B},h_1})_m,(W^{\lambda,11}_{\mathrm{B},h_2})_n\big]=\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)$$
$$\times(W^{\lambda,11}_{\mathrm{B},h_1+h_2-2-h})_{m+n}$$
$$+\frac{1}{2}\binom{m+h_1-1}{h_1+h_2-1}c_B(h_1,h_2,\lambda)\,q^{h_1+h_2-4}\,\delta_{m+n},$$

$$\big[(W^{\lambda,22}_{\mathrm{B},h_1})_m,(W^{\lambda,22}_{\mathrm{B},h_2})_n\big]=\sum_{h=0,\mathrm{even}}^{h_1+h_2-3}q^h\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n,\lambda)$$
$$\times(W^{\lambda,22}_{\mathrm{B},h_1+h_2-2-h})_{m+n}$$
$$+\frac{1}{2}\binom{m+h_1-1}{h_1+h_2-1}c_B(h_1,h_2,\lambda)\,q^{h_1+h_2-4}\,\delta_{m+n},$$

(E.1)

where the central term (3.11) can be substituted.

E.2 The commutators between the currents consisting of $b\,c$ system and the currents consisting of $\gamma\,b$ system with $h_1 = h_2, h_2+1$ for nonzero λ

By analyzing the first equation of (3.12), we obtain the following commutators between the currents consisting of $b\,c$ system and the currents consisting of $\gamma\,b$ system

$$\begin{aligned}
\big[(W^{\lambda,21}_{\mathrm{F},h_1})_m, (Q^{\lambda,12}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,11}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,12}_{\mathrm{F},h_1})_m, (Q^{\lambda,21}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,22}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,12}_{\mathrm{F},h_1})_m, (Q^{\lambda,11}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,12}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,21}_{\mathrm{F},h_1})_m, (Q^{\lambda,22}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,21}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,22}_{\mathrm{F},h_1})_m, (Q^{\lambda,12}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,12}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,11}_{\mathrm{F},h_1})_m, (Q^{\lambda,21}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad (m,r,\lambda)\,(Q^{\lambda,21}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,11}_{\mathrm{F},h_1})_m, (Q^{\lambda,11}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad (m,r,\lambda)\,(Q^{\lambda,11}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,22}_{\mathrm{F},h_1})_m, (Q^{\lambda,22}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad (m,r,\lambda)\,(Q^{\lambda,22}_{h_1+h_2-\frac{3}{2}-h})_{m+r}.
\end{aligned} \tag{E.2}$$

As in previous footnote 3, the OPE between $W^{\lambda,\hat{A}=1}_{\mathrm{F},h_1=6}(z)$ and $Q^{\lambda,\hat{B}=1}_{\mathrm{B},h_2+\frac{1}{2}=\frac{7}{2}}(w)$ where the weights satisfy $h_1 = h_2+3$ can be obtained and the seventh order pole of this OPE contains the structure constant $-\frac{2048}{105}(\lambda+1)(\lambda+2)(2\lambda-3)(2\lambda+3)(2\lambda+5)(11\lambda-17)$ appearing in the current $Q^{\lambda}_{h_1+h_2-\frac{3}{2}-h=\frac{5}{2}}(w)$ with weight $h = 5$. On the other hand, the structure constant $q_{\mathrm{F}}^{6,\frac{7}{2},5}(m,r,\lambda)$ contains $\frac{4}{945}(\lambda+1)(2\lambda-3)(4\lambda^4 - 28\lambda^3 - 13\lambda^2 + 49\lambda + 51)$. By subtracting the contribution $\frac{2}{225}(\lambda-1)\lambda(\lambda+1)(2\lambda-3)(2\lambda-1)(2\lambda+1)$ coming from the structure constant $q_{\mathrm{F}}^{4,\frac{7}{2},5}(m,r,\lambda)$ from this, we obtain $-\frac{2}{4725}(\lambda+1)(\lambda+2)(2\lambda-3)(2\lambda+3)(2\lambda+5)(11\lambda-17)$. The weight $h_1 = 6$ is replaced by the weight $(h_1-2) = 4$. Note that the additional term appearing in the $q_{\mathrm{F}}^{4,\frac{7}{2},5}(m,r,\lambda)$ contains the factor λ. By considering the numerical factor 46080 when we move from the modes to the differential operator in the OPE and multiplying this into the above factor, we obtain the above structure constant in the current $Q^{\lambda}_{\frac{5}{2}}(w)$.

E.3 The commutators between the currents consisting of $\beta\,\gamma$ system and the currents consisting of $\gamma\,b$ system with $h_1 = h_2, h_2+1$ for nonzero λ

In this case, by using the second equation of (3.12), we can write down the following commutators between the currents consisting of $\beta\,\gamma$ system and the currents consisting of $\gamma\,b$ system

$$\begin{aligned}
\big[(W^{\lambda,21}_{\mathrm{B},h_1})_m, (Q^{\lambda,12}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,22}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,12}_{\mathrm{B},h_1})_m, (Q^{\lambda,21}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,11}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,12}_{\mathrm{B},h_1})_m, (Q^{\lambda,22}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,12}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\big[(W^{\lambda,21}_{\mathrm{B},h_1})_m, (Q^{\lambda,11}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} q^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h} \\
&\quad \times (m,r,\lambda)\,(Q^{\lambda,21}_{h_1+h_2-\frac{3}{2}-h})_{m+r},
\end{aligned}$$

$$\big[(W^{\lambda,11}_{\mathrm{B},h_1})_m,(Q^{\lambda,12}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3} q^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(Q^{\lambda,12}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,22}_{\mathrm{B},h_1})_m,(Q^{\lambda,21}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3} q^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(Q^{\lambda,21}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,11}_{\mathrm{B},h_1})_m,(Q^{\lambda,11}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3} q^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(Q^{\lambda,11}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,22}_{\mathrm{B},h_1})_m,(Q^{\lambda,22}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3} q^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(Q^{\lambda,22}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\tag{E.3}$$

which look similar to the previous relations in (E.2).

E.4 The commutators between the currents consisting of $b\,c$ system and the currents consisting of $\beta\,c$ system with $h_1=h_2, h_2+1$ for nonzero λ

Similarly, from the third equation of (3.12), we can write down the commutators between the currents consisting of $b\,c$ system and the currents consisting of $\beta\,c$ system

$$\big[(W^{\lambda,21}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,21}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,22}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,12}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,12}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,11}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,12}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,22}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,21}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,21}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,11}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,12}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,11}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,21}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,21}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,22}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,12}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,12}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,11}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,11}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,11}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,22}_{\mathrm{F},h_1})_m,(\bar{Q}^{\lambda,22}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,22}_{h_1+h_2-\frac{3}{2}-h})_{m+r}.\tag{E.4}$$

Note that there exists the factor $(-1)^h$ in (E.4) which appears in the (anti)commutators described in Sect. 4.

E.5 The commutators between the currents consisting of $\beta\,\gamma$ system and the currents consisting of $\beta\,c$ system with $h_1=h_2, h_2+1$ for nonzero λ

Finally, from the analysis of the last equation of (3.12), we obtain the following result for the commutators between the currents consisting of $\beta\,\gamma$ system and the currents consisting of $\beta\,c$ system

$$\big[(W^{\lambda,21}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,21}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,11}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,12}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,12}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,22}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,12}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,11}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,21}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,21}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,22}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,12}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,22}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,21}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\times(m,r,\lambda)\,(\bar{Q}^{\lambda,21}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$
$$\big[(W^{\lambda,11}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,12}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}$$

$$\times(m,r,\lambda)\,(\bar{Q}^{\lambda,12}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$

$$\big[(W^{\lambda,11}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,11}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}$$

$$\times(m,r,\lambda)\,(\bar{Q}^{\lambda,11}_{h_1+h_2-\frac{3}{2}-h})_{m+r},$$

$$\big[(W^{\lambda,22}_{\mathrm{B},h_1})_m,(\bar{Q}^{\lambda,22}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-2} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}$$

$$\times(m,r,\lambda)\,(\bar{Q}^{\lambda,22}_{h_1+h_2-\frac{3}{2}-h})_{m+r}. \tag{E.5}$$

Similar to the previous footnote 3, the OPE between $W^{\lambda,\hat{A}=1}_{\mathrm{B},h_1=7}(z)$ and $\bar{Q}^{\lambda,\hat{B}=1}_{\mathrm{B},h_2+\frac{1}{2}=\frac{7}{2}}(w)$ provides the seventh order with the structure constant $\frac{2048}{1155}(\lambda+2)(2\lambda+3)(2\lambda+5)(662\lambda^3-3165\lambda^2+5434\lambda-3339)$ appearing in the current $\bar{Q}^{\lambda}_{\frac{7}{2}}(w)$. This can be obtained by adding the extra contribution from $q_{\mathrm{B}}^{4,\frac{7}{2},5}(m,r,\lambda)$ where h_1 is replaced by (h_1-3) in addition to the one from $q_{\mathrm{B}}^{7,\frac{7}{2},5}(m,r,\lambda)$. Note that the $\bar{Q}^{\lambda}_{\frac{1}{2}}(w)$ term on the right hand side of the tenth order pole of the OPE can be determined by the contribution from the structure constant $q_{\mathrm{B}}^{5,\frac{7}{2},8}(m,r,\lambda)$ only.

Note that there exists the factor $(-1)^h$ in (E.5) which appears in the (anti)commutators described in Sect. 4 and Appendix G.

E.6 The anticommutators between the currents consisting of $\gamma\, b$ system and the currents consisting of $\beta\, c$ system with $h_1=h_2$ for nonzero λ

By analyzing the equation of (3.13), the following result satisfies

$$\{(Q^{\lambda,21}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,21}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\Big(o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,11}_{F,h_1+h_2-h})_{r+s}$$

$$+o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,22}_{B,h_1+h_2-h})_{r+s}\Big)$$

$$+\frac{1}{2}\binom{r+h_1-\frac{1}{2}}{h_1+h_2}c_Q(h_1,h_2,\lambda)\,q^{h_1+h_2-2}\delta_{r+s},$$

$$\{(Q^{\lambda,12}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,12}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\Big(o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,22}_{F,h_1+h_2-h})_{r+s}$$

$$+o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,11}_{B,h_1+h_2-h})_{r+s}\Big)$$

$$+\frac{1}{2}\binom{r+h_1-\frac{1}{2}}{h_1+h_2}c_Q(h_1,h_2,\lambda)\,q^{h_1+h_2-2}\delta_{r+s},$$

$$\{(Q^{\lambda,12}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,11}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\,o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,12}_{F,h_1+h_2-h})_{r+s},$$

$$\{(Q^{\lambda,12}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,22}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\,o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,12}_{B,h_1+h_2-h})_{r+s},$$

$$\{(Q^{\lambda,21}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,11}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\,o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,21}_{B,h_1+h_2-h})_{r+s},$$

$$\{(Q^{\lambda,21}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,22}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\,o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,21}_{F,h_1+h_2-h})_{r+s},$$

$$\{(Q^{\lambda,11}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,21}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\,o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,12}_{B,h_1+h_2-h})_{r+s},$$

$$\{(Q^{\lambda,22}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,21}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\,o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,12}_{F,h_1+h_2-h})_{r+s},$$

$$\{(Q^{\lambda,11}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,12}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\,o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,21}_{F,h_1+h_2-h})_{r+s},$$

$$\{(Q^{\lambda,22}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,12}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\,o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,21}_{B,h_1+h_2-h})_{r+s},$$

$$\{(Q^{\lambda,11}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,11}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\Big(o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,11}_{F,h_1+h_2-h})_{r+s}$$

$$+o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\,(W^{\lambda,11}_{B,h_1+h_2-h})_{r+s}\Big)$$

$$+\frac{1}{2}\binom{r+h_1-\frac{1}{2}}{h_1+h_2}c_Q(h_1,h_2,\lambda)\,q^{h_1+h_2-2}\delta_{r+s},$$

$$\{(Q^{\lambda,22}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\lambda,22}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1} q^h\Big(o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$$

$$\times(r,s,\lambda)\,(W^{\lambda,22}_{F,h_1+h_2-h})_{r+s}$$

$$+o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\,(W^{\lambda,22}_{B,h_1+h_2-h})_{r+s}\Big)$$

$$+\frac{1}{2}\begin{pmatrix} r+h_1-\frac{1}{2} \\ h_1+h_2 \end{pmatrix} c_Q(h_1,h_2,\lambda)\, q^{h_1+h_2-2}\delta_{r+s}. \tag{E.6}$$

There are the relations when we interchange h_1 and h_2 as $o_{\rm F}^{h_2+\frac{1}{2},h_1+\frac{1}{2},h} = (-1)^h\, o_{\rm F}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$ and $o_{\rm B}^{h_2+\frac{1}{2},h_1+\frac{1}{2},h} = (-1)^h\, o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}$. These will be used in Appendix G.

Appendix F: Some relations which can be used in Sect. 4

Other component result related to (4.5) can be summarized by

$$\begin{aligned}
&-\frac{i}{2}\left(Q^{\lambda,11}_{h+\frac{1}{2}}+2i\sqrt{2}\,Q^{\lambda,21}_{h+\frac{1}{2}}-2\,Q^{\lambda,22}_{h+\frac{1}{2}}\right)=\frac{1}{2}\\
&\quad\times\left[\frac{1}{4(-4)^{h-4}}\,\Phi^{(h),2}_{\frac{1}{2}}-\frac{1}{4(-4)^{h-5}}\,\tilde{\Phi}^{(h-1),2}_{\frac{3}{2}}\right],\\
&-\frac{i}{2}\left(2\,\bar{Q}^{\lambda,11}_{h+\frac{1}{2}}+2i\sqrt{2}\,\bar{Q}^{\lambda,12}_{h+\frac{1}{2}}-\bar{Q}^{\lambda,22}_{h+\frac{1}{2}}\right)=\frac{1}{2}\\
&\quad\times\left[\frac{1}{4(-4)^{h-4}}\,\Phi^{(h),2}_{\frac{1}{2}}+\frac{1}{4(-4)^{h-5}}\,\tilde{\Phi}^{(h-1),2}_{\frac{3}{2}}\right],\\
&-\frac{i}{2}\left(Q^{\lambda,11}_{h+\frac{1}{2}}+i\sqrt{2}\,Q^{\lambda,12}_{h+\frac{1}{2}}-2\,Q^{\lambda,22}_{h+\frac{1}{2}}\right)=\frac{1}{2}\\
&\quad\times\left[\frac{1}{4(-4)^{h-4}}\,\Phi^{(h),3}_{\frac{1}{2}}-\frac{1}{4(-4)^{h-5}}\,\tilde{\Phi}^{(h-1),3}_{\frac{3}{2}}\right],\\
&-\frac{i}{2}\left(2\,\bar{Q}^{\lambda,11}_{h+\frac{1}{2}}+i\sqrt{2}\,\bar{Q}^{\lambda,21}_{h+\frac{1}{2}}-\bar{Q}^{\lambda,22}_{h+\frac{1}{2}}\right)=\frac{1}{2}\\
&\quad\times\left[\frac{1}{4(-4)^{h-4}}\,\Phi^{(h),3}_{\frac{1}{2}}+\frac{1}{4(-4)^{h-5}}\,\tilde{\Phi}^{(h-1),3}_{\frac{3}{2}}\right],\\
&-\frac{1}{2}\,Q^{\lambda,11}_{h+\frac{1}{2}}-Q^{\lambda,22}_{h+\frac{1}{2}}=\frac{1}{2}\\
&\quad\times\left[\frac{1}{4(-4)^{h-4}}\,\Phi^{(h),4}_{\frac{1}{2}}-\frac{1}{4(-4)^{h-5}}\,\tilde{\Phi}^{(h-1),4}_{\frac{3}{2}}\right],\\
&\bar{Q}^{\lambda,11}_{h+\frac{1}{2}}+\frac{1}{2}\,\bar{Q}^{\lambda,22}_{h+\frac{1}{2}}=\frac{1}{2}\\
&\quad\times\left[\frac{1}{4(-4)^{h-4}}\,\Phi^{(h),4}_{\frac{1}{2}}+\frac{1}{4(-4)^{h-5}}\,\tilde{\Phi}^{(h-1),4}_{\frac{3}{2}}\right].
\end{aligned} \tag{F.1}$$

Similarly, other relations associated with (4.7) can be described as

$$\begin{aligned}
&-2i\,W^{\lambda,11}_{{\rm B},h+1}+4\sqrt{2}\,W^{\lambda,21}_{{\rm B},h+1}+2i\,\,W^{\lambda,22}_{{\rm B},h+1}\\
&\quad=\frac{1}{8(-4)^{h-4}}\left[\Phi^{(h),13}_{1}+\Phi^{(h),24}_{1}\right],\\
&-2i\,W^{\lambda,11}_{{\rm F},h+1}+2\sqrt{2}\,W^{\lambda,21}_{{\rm F},h+1}+2i\,W^{\lambda,22}_{{\rm F}+h+1}\\
&\quad=\frac{1}{8(-4)^{h-4}}\left[\Phi^{(h),13}_{1}-\Phi^{(h),24}_{1}\right],\\
&2\,W^{\lambda,11}_{{\rm B},h+1}+i\sqrt{2}\,W^{\lambda,12}_{{\rm B},h+1}+4i\sqrt{2}\,\,W^{\lambda,21}_{{\rm B},h+1}-2\,W^{\lambda,22}_{{\rm B},h+1}\\
&\quad=\frac{1}{8(-4)^{h-4}}\left[\Phi^{(h),14}_{1}-\Phi^{(h),23}_{1}\right],\\
&-2W^{\lambda,11}_{{\rm F},h+1}-2i\sqrt{2}W^{\lambda,12}_{{\rm F},h+1}-2i\sqrt{2}W^{\lambda,21}_{{\rm F},h+1}+2W^{\lambda,22}_{{\rm F},h+1}\\
&\quad=\frac{1}{8(-4)^{h-4}}\left[\Phi^{(h),14}_{1}+\Phi^{(h),23}_{1}\right].
\end{aligned} \tag{F.2}$$

Both relations in (F.1) and (F.2) are used in Sect. 4 and Appendix G.

Appendix G: The remaining (anti)commutator relations between the $\mathcal{N}=4$ multiplets

In this Appendix, the remaining ten (anti)commutators with the particular examples for the specific weights h_1 and h_2 showing that the extra structures on the right hand sides of these (anti)commutators are given explicitly.

G.1 The anticommutator relation between the second components with $h_1=h_2$ for nonzero λ

By using the expressions in (2.8) and (E.6) together with (4.1), (4.7) and (F.2), the following anticommutator can be obtained

$$\begin{aligned}
&\left\{(\Phi^{(h_1),i}_{\frac{1}{2}})_r,(\Phi^{(h_2),j}_{\frac{1}{2}})_s\right\}=16(-4)^{h_1-4}\,(-4)^{h_2-4}\\
&\quad\times\Bigg[\begin{pmatrix} r+h_1-\frac{1}{2} \\ h_1+h_2 \end{pmatrix} q^{h_1+h_2-2}\,c_Q\,\delta^{ij}\,\delta_{r+s}\\
&\quad+\delta^{ij}\sum_{h=0}^{h_1+h_2-1} q^h\,(1+(-1)^h)\Big(o_{\rm F}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\\
&\quad-o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\Big)\\
&\quad\times\frac{1}{2(-4)^{h_1+h_2-h-2}}\,(\Phi^{(h_1+h_2-h)}_{0})_{r+s}+\delta^{ij}\\
&\quad\times\sum_{h=0}^{h_1+h_2-1}\Big((h_1+h_2-h-1+2\lambda)\,q^h\,(1+(-1)^h)\\
&\quad\times o_{\rm F}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)+(h_1+h_2-h-2\lambda)\\
&\quad\times q^h\,(1+(-1)^h)\,o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)\Big)\\
&\quad\times\frac{1}{16(2h_1+2h_2-2h-1)(-4)^{h_1+h_2-h-6}}\\
&\quad\times(\tilde{\Phi}^{(h_1+h_2-h-2)}_{2})_{r+s}\\
&\quad\times-\sum_{h=0}^{h_1+h_2-1} q^h\,(1-(-1)^h)\,o_{\rm F}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}\\
&\quad\times(r,s,\lambda)\frac{1}{32(-4)^{h_1+h_2-h-5}}\Big(\Phi^{(h_1+h_2-h-1),ij}_{1}
\end{aligned}$$

$$
+\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-h-1),kl}\Big)_{r+s}
+\sum_{h=0}^{h_1+h_2-1} q^h\,(1-(-1)^h)\,o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s,\lambda)
\times\frac{1}{32(-4)^{h_1+h_2-h-5}}\Big(\Phi_1^{(h_1+h_2-h-1),ij}
-\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-h-1),kl}\Big)_{r+s}\Big]. \tag{G.1}
$$

The central term is given by (3.14). Due to the factor $(1\pm(-1)^h)$ in the above, the two kinds of the currents, the $SO(4)$ singlets and the $SO(4)$ adjoints appear alternatively. When the weight h is equal to its maximum value $h=h_1+h_2-1$, then the currents $\Phi_0^{(1)}$ and $\tilde{\Phi}_2^{(-1)}$ terms in the $SO(4)$ singlets appear on the right hand side while the currents $\Phi_1^{(0),ij}$ and $\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(0),kl}$ appear similarly.

G.2 The commutator relation between the second component and the third component with $h_1=h_2, h_2+1$ for nonzero λ

By using the relation (2.8), (2.9) together with (E.2), (E.3), (E.4), (E.5), (4.5), and (F.1), It turns out that we obtain

$$
\begin{aligned}
&\big[(\Phi_{\frac{1}{2}}^{(h_1),i})_r,(\Phi_1^{(h_2),jk})_m\big]=4(-4)^{h_1-4}\,4(-4)^{h_2-4}\\
&\times\Big[\delta^{ij}\Big(-\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
+\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\\
&+\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
-\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\Big)
\times\frac{1}{4(-4)^{h_1+h_2-h-5}}\Big(\Phi_{\frac{1}{2}}^{(h_1+h_2-1-h),k}\Big)_{m+r}\\
&+\delta^{ij}\Big(\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
+\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\\
&-\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
-\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\Big)
\times\frac{1}{4(-4)^{h_1+h_2-h-6}}\,(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2-h),k})_{m+r}\\
&-\delta^{ik}\Big(-\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
+\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\\
&+\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
-\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\Big)
\times\frac{1}{4(-4)^{h_1+h_2-h-5}}\,(\Phi_{\frac{1}{2}}^{(h_1+h_2-1-h),j})_{m+r}\\
&-\delta^{ik}\Big(\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
+\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\\
&-\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
-\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\Big)
\times\frac{1}{4(-4)^{h_1+h_2-h-6}}\,(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2-h),j})_{m+r}\\
&+\varepsilon^{ijkl}\Big(-\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
+\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\\
&-\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
+\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\Big)
\times\frac{1}{4(-4)^{h_1+h_2-h-5}}\,(\Phi_{\frac{1}{2}}^{(h_1+h_2-1-h),l})_{m+r}\\
&+\varepsilon^{ijkl}\Big(\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
+\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\\
&+\sum_{h=-1}^{h_1+h_2-2} q^h\,q_{\mathrm{B}}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)
\end{aligned}
$$

$$+\sum_{h=-1}^{h_1+h_2-1} q^h\,(-1)^h\, q_{\rm B}^{h_2+1,h_1+\frac{1}{2},h}(m,r,\lambda)\Big)$$
$$\times\frac{1}{4(-4)^{h_1+h_2-h-6}}(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-2-h),l})_{m+r}\Big]. \tag{G.2}$$

Due to the antisymmetric property in the indices j and k on the left hand side, we observe that there exists antisymmetric property for the interchange of those indices on the right hand side. The field contents in (G.2) look like as the ones in (4.6). Note that the ordering of first two upper elements in the structure constants in the above is opposite to the one in (4.6). As before, in the summation over the dummy variable of the weight h, there are current terms for the case of $h = h_1 + h_2 - 2$ where we see the presence of the current $\Phi_{\frac{1}{2}}^{(1),k}$ and $\tilde{\Phi}_{\frac{3}{2}}^{(0),k}$ terms with Kronecker delta or epsilon tensor of $SO(4)$ and for the weights $h \leq h_1 + h_2 - 3$, there are some cancellations in the current terms due to the factor $(-1)^h$ depending on the even or odd property of the weight h.

Let us consider the OPE between $\Phi_{\frac{1}{2}}^{(h_1=3),i}(z)$ and $\Phi_{1}^{(h_2=4),jk}(w)$ with $i = j$ where $h_1 = h_2 - 1$ as in the footnotes of Sect. 4. The seventh order pole of this OPE gives us the structure constant $\frac{131072}{35}(\lambda-2)(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(2\lambda+3)$ appearing in the current $\Phi_{\frac{1}{2}}^{(h_1+h_2-1-h=1),k}(w)$ with weight $h = 5$. By substituting the various expressions in the corresponding terms of (G.2), we can check that we obtain the above structure constant correctly where $q_F^{5,\frac{7}{2},5}$ term corresponds to $\frac{4096}{35}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(4\lambda^2-14\lambda-9)$ while $q_B^{5,\frac{7}{2},5}$ term corresponds to $-\frac{4096}{35}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)(4\lambda^2+10\lambda-15)$. There appears the current $\tilde{\Phi}_{\frac{3}{2}}^{(0),i}(w)$ term with nonzero structure constant having the λ factor in the seventh order pole of this OPE and this is not consistent with (G.2) because all the coefficients vanish for the odd weight $h = 5$. This implies that there is additional term on the right hand side of the OPE for the weights $h_1 = 3$ and $h_2 = 4$.

G.3 The anticommutator relation between the second component and the fourth component with $h_1 = h_2 + 1$ for nonzero λ

By using the relations (2.8) and (2.10), we can rewrite it in terms of the corresponding anticommutators in (E.6) and using the relations (4.1), (4.7) and (F.2), the following result can be obtained

$$\{(\Phi_{\frac{1}{2}}^{(h_1),i})_r,(\tilde{\Phi}_{\frac{3}{2}}^{(h_2),j})_s\} = 16(-4)^{h_1-4}\,(-4)^{h_2-4}$$
$$\times\Big[-\binom{r+h_1-\frac{1}{2}}{h_1+h_2+1}\,q^{h_1+h_2-1}\,c'_Q\,\delta^{ij}\,\delta_{r+s}$$
$$+\delta^{ij}\sum_{h=0}^{h_1+h_2} q^h\,(1-(-1)^h)\Big(o_{\rm F}^{h_1+\frac{1}{2},h_2+\frac{3}{2},h}(r,s,\lambda)$$
$$-o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{3}{2},h}(r,s,\lambda)\Big)$$
$$\times\frac{1}{2(-4)^{h_1+h_2-h-1}}\,(\Phi_0^{(h_1+h_2+1-h)})_{r+s}$$
$$+\delta^{ij}\sum_{h=0}^{h_1+h_2}\Big((h_1+h_2-h+2\lambda)\,q^h\,(1-(-1)^h)$$
$$\times o_{\rm F}^{h_1+\frac{1}{2},h_2+\frac{3}{2},h}(r,s,\lambda)$$
$$+(h_1+h_2+1-h-2\lambda)\,q^h\,(1-(-1)^h)$$
$$\times o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{3}{2},h}(r,s,\lambda)\Big)$$
$$\times\frac{1}{16(2h_1+2h_2-2h+1)(-4)^{h_1+h_2-h-5}}$$
$$\times(\tilde{\Phi}_2^{(h_1+h_2-h-1)})_{r+s}$$
$$-\sum_{h=0}^{h_1+h_2} q^h\,(1+(-1)^h)\,o_{\rm F}^{h_1+\frac{1}{2},h_2+\frac{3}{2},h}(r,s,\lambda)$$
$$\times\frac{1}{32(-4)^{h_1+h_2-h-4}}\Big(\Phi_1^{(h_1+h_2-h),ij}$$
$$+\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-h),kl}\Big)_{r+s}$$
$$+\sum_{h=0}^{h_1+h_2} q^h\,(1+(-1)^h)\,o_{\rm B}^{h_1+\frac{1}{2},h_2+\frac{3}{2},h}(r,s,\lambda)$$
$$\times\frac{1}{32(-4)^{h_1+h_2-h-4}}\Big(\Phi_1^{(h_1+h_2-h),ij}$$
$$-\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-h),kl}\Big)_{r+s}\Big]. \tag{G.3}$$

The central charge will be given later soon. The right hand side of (G.3) is similar to the one in (G.1) in the sense that the field contents are the same and the relative signs are different from each other. It is obvious to see that some expressions having h_2 in (G.1) are replaced by (h_2+1). As mentioned before, the two kinds of currents occur alternatively depending on the even or odd property of the weight h.

The OPE between $\Phi_{\frac{1}{2}}^{(h_1=4),i}(z)$ and $\Phi_{\frac{3}{2}}^{(h_2=1),j}(w)$ with $i = j$ where the weights satisfy $h_1 = h_2 + 3$ can be calculated. The fifth order pole of this OPE gives us the vanishing structure constant appearing in the current $\Phi_0^{(h_1+h_2+1-h=2)}(w)$ with weight $h = 4$ from our calculation. There appears the current $\tilde{\Phi}_2^{(0)}(w)$ term with nonzero structure constant $\frac{256}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)$ in the fifth order pole of this OPE and this is not consistent with (G.3) because all the coefficients vanish for the even weight $h = 4$. This implies that there is additional term on the right hand side of the OPE for the weights $h_1 = 4$ and $h_2 = 1$ which are outside of the allowed region we consider.

Here the central term is given by

$$c'_Q = 8(-4)^{h_1+h_2-4}\,\delta_{b\bar{a}}\,\delta_{d\bar{c}}\Big[-Q^{(h_1+1)+}_{\lambda,\bar{a}b}(z)\,Q^{(h_2+2)+}_{\lambda,\bar{c}d}(w) +Q^{(h_1+1)-}_{\lambda,\bar{a}b}(z)\,Q^{(h_2+2)-}_{\lambda,\bar{c}d}(w) -Q^{(h_1+1)+}_{\lambda,\bar{a}b}(z)\,Q^{(h_2+2)-}_{\lambda,\bar{c}d}(w) +Q^{(h_1+1)-}_{\lambda,\bar{a}b}(z)\,Q^{(h_2+2)+}_{\lambda,\bar{c}d}(w) +(-1)^{h_1+h_2}\,Q^{(h_2+2)+}_{\lambda,\bar{c}d}(z)\,Q^{(h_1+1)+}_{\lambda,\bar{a}b}(w) -(-1)^{h_1+h_2}\,Q^{(h_2+2)-}_{\lambda,\bar{c}d}(z)\,Q^{(h_1+1)-}_{\lambda,\bar{a}b}(w) +(-1)^{h_1+h_2}\,Q^{(h_2+2)+}_{\lambda,\bar{c}d}(z)\,Q^{(h_1+1)-}_{\lambda,\bar{a}b}(w) -(-1)^{h_1+h_2}\,Q^{(h_2+2)-}_{\lambda,\bar{c}d}(z)\,Q^{(h_1+1)+}_{\lambda,\bar{a}b}(w)\Big]_{\frac{1}{(z-w)^{h_1+h_2+2}}}. \tag{G.4}$$

Note that from the inside of the bracket in (3.14), after the h_2 is replaced with (h_2+1), we obtain the above result (G.4).

G.4 The commutator relation between the second component and the last component with $h_1 = h_2 + 1, h_2 + 2$ for nonzero λ

By using the relations (2.8) and (2.11), we can rewrite it in terms of the corresponding commutators in (E.2), (E.3), (E.4), and (E.5) and using the relations (4.5) and (F.1), the following result can be obtained

$$\Big[(\Phi^{(h_1),i}_{\frac{1}{2}})_r, (\tilde{\Phi}^{(h_2)}_{2})_m\Big] = 4(-4)^{h_1-4}\,4(-4)^{h_2-4} \times\Bigg[\Bigg(\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\mathrm{F}}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\mathrm{F}}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\mathrm{B}}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\mathrm{B}}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda)\Bigg) \times\frac{1}{4(-4)^{h_1+h_2-h-4}}\,(\Phi^{(h_1+h_2-h),i}_{\frac{1}{2}})_{m+r} +\Bigg(-\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\mathrm{F}}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\mathrm{F}}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) - \sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\mathrm{B}}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\mathrm{B}}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda)\Bigg) \frac{1}{4(-4)^{h_1+h_2-h-5}}\,(\tilde{\Phi}^{(h_1+h_2-1-h),i}_{\frac{3}{2}})_{m+r}\Bigg]. \tag{G.5}$$

The field contents on the right hand side in (G.5) look similar to the ones in (4.6). Except the four terms having $(-1)^h$ factor for the weight $h = h_1 + h_2$, there are some cancellation between the currents due to the factor $(-1)^h$. In other words, the two kinds of currents appear alternatively depending on the even or odd property of the weight h. Note the presence of different ordering in the elements of the structure constants.

Let us consider the OPE between $\Phi^{(h_1=5),i}_{\frac{1}{2}}(z)$ and $\Phi^{(h_2=1)}_{2}(w)$ where the weights satisfy $h_1 = h_2 + 4$. The seventh order pole of this OPE gives us the structure constant $\frac{16384}{21}(\lambda-1)(2\lambda+1)(4\lambda-1)(6\lambda^2-3\lambda+5)$ appearing in the current $G^i(w)$ with weight $h = 5$. By substituting the various expressions in the corresponding terms of (G.5), we can check that we obtain the above structure constant correctly where $q_F^{3,\frac{7}{2},5}$ term corresponds to $-\frac{4096}{21}(\lambda-1)(\lambda+1)(2\lambda+1)(8\lambda^3+8\lambda^2+24\lambda-25)$ while $q_B^{3,\frac{7}{2},5}$ term corresponds to $\frac{4096}{21}(\lambda-1)(2\lambda-3)(2\lambda+1)(4\lambda^3-10\lambda^2+19\lambda+5)$. There appears the current $\Phi^{(1),i}_{\frac{1}{2}}(w)$ term with nonzero structure constant having the λ factor in the seventh order pole of the OPE and this is not consistent with (G.5) because all the coefficients vanish for the odd weight $h = 5$. There is additional term on the right hand side of the OPE for the weights $h_1 = 5$ and $h_2 = 1$ which are outside of the above allowed region.

G.5 The commutator between the third components with $h_1 = h_2 - 1, h_2, h_2 + 1$ for nonzero λ

By using the relations (2.9), we can rewrite them in terms of the commutators in (3.15), (3.16), (3.17), (3.18), (3.19), and (E.1). Then we can use (4.1), (4.7) and (F.2), we obtain the following commutator

$$\Big[(\Phi^{(h_1),ij}_{1})_m, (\Phi^{(h_2),kl}_{1})_n\Big] = 16(-4)^{h_1-4}\,(-4)^{h_2-4} \times\Bigg[\binom{m+h_1}{h_1+h_2+1}\, q^{h_1+h_2-2}\Big(-4\,(\delta^{ik}\,\delta^{jl}-\delta^{jk}\,\delta^{il}) \times(c'_F+c'_B) - 4\,\varepsilon^{ijkl}\,(c'_F-c'_B)\Big)\,\delta_{m+n} +(\delta^{ik}\,\delta^{jl}-\delta^{il}\,\delta^{jk})\sum_{h=0,\mathrm{even}}^{h_1+h_2-1} q^h\Big(p_{\mathrm{F}}^{h_1+1,h_2+1,h}(m,n,\lambda) -p_{\mathrm{B}}^{h_1+1,h_2+1,h}(m,n,\lambda)\Big)$$

$$\times \frac{4}{(-4)^{h_1+h_2-h-2}} (\Phi_0^{(h_1+h_2-h)})_{m+n}$$
$$+(\delta^{ik}\,\delta^{jl} - \delta^{il}\,\delta^{jk}) \sum_{h=0,\text{even}}^{h_1+h_2-1} \Big((h_1+h_2-h-1+2\lambda)$$
$$\times q^h\, p_{\mathrm{F}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$+(h_1+h_2-h-2\lambda)\, q^h\, p_{\mathrm{B}}^{h_1+1,h_2+1,h}(m,n,\lambda) \Big)$$
$$\times \frac{1}{2(2h_1+2h_2-2h-1)(-4)^{h_1+h_2-h-6}}$$
$$\times (\tilde{\Phi}_2^{(h_1+h_2-h-2)})_{m+n}$$
$$+\varepsilon^{ijkl} \sum_{h=0,\text{even}}^{h_1+h_2-1}$$
$$\times q^h \left(p_{\mathrm{F}}^{h_1+1,h_2+1,h}(m,n,\lambda) + p_{\mathrm{B}}^{h_1+1,h_2+1,h}(m,n,\lambda) \right)$$
$$\times \frac{4}{(-4)^{h_1+h_2-h-2}} (\Phi_0^{(h_1+h_2-h)})_{m+n}$$
$$+\varepsilon^{ijkl} \sum_{h=0,\text{even}}^{h_1+h_2-1} \Big((h_1+h_2-h-1+2\lambda)$$
$$\times q^h\, p_{\mathrm{F}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$-(h_1+h_2-h-2\lambda)\, q^h\, p_{\mathrm{B}}^{h_1+1,h_2+1,h}(m,n,\lambda) \Big)$$
$$\times \frac{1}{2(2h_1+2h_2-2h-1)(-4)^{h_1+h_2-h-6}}$$
$$\times (\tilde{\Phi}_2^{(h_1+h_2-h-2)})_{m+n}$$
$$+\delta^{ik} \sum_{h=-1,\text{odd}}^{h_1+h_2-1} q^h\, p_{\mathrm{F}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$\times \frac{1}{4(-4)^{h_1+h_2-h-5}} \Big(\Phi_1^{(h_1+h_2-h-1),jl}$$
$$+\frac{1}{2}\,\varepsilon^{jlm_1n_1}\, \Phi_1^{(h_1+h_2-h-1),m_1n_1} \Big)_{m+n}$$
$$+\delta^{ik} \sum_{h=-1,\text{odd}}^{h_1+h_2-1} q^h\, p_{\mathrm{B}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$\times \frac{1}{4(-4)^{h_1+h_2-h-5}} \Big(\Phi_1^{(h_1+h_2-h-1),jl}$$
$$-\frac{1}{2}\,\varepsilon^{jlm_1n_1}\, \Phi_1^{(h_1+h_2-h-1),m_1n_1} \Big)_{m+n}$$
$$-\delta^{il} \sum_{h=-1,\text{odd}}^{h_1+h_2-1} q^h\, p_{\mathrm{F}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$\times \frac{1}{4(-4)^{h_1+h_2-h-5}} \Big(\Phi_1^{(h_1+h_2-h-1),jk}$$
$$+\frac{1}{2}\,\varepsilon^{jkm_1n_1}\, \Phi_1^{(h_1+h_2-h-1),m_1n_1} \Big)_{m+n}$$
$$-\delta^{il} \sum_{h=-1,\text{odd}}^{h_1+h_2-1} q^h\, p_{\mathrm{B}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$\times \frac{1}{4(-4)^{h_1+h_2-h-5}} \Big(\Phi_1^{(h_1+h_2-h-1),jk}$$
$$-\frac{1}{2}\,\varepsilon^{jkm_1n_1}\, \Phi_1^{(h_1+h_2-h-1),m_1n_1} \Big)_{m+n}$$
$$-\delta^{jk} \sum_{h=-1,\text{odd}}^{h_1+h_2-1} q^h\, p_{\mathrm{F}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$\times \frac{1}{4(-4)^{h_1+h_2-h-5}} \Big(\Phi_1^{(h_1+h_2-h-1),il}$$
$$+\frac{1}{2}\,\varepsilon^{ilm_1n_1}\, \Phi_1^{(h_1+h_2-h-1),m_1n_1} \Big)_{m+n}$$
$$-\delta^{jk} \sum_{h=-1,\text{odd}}^{h_1+h_2-1} q^h\, p_{\mathrm{B}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$\times \frac{1}{4(-4)^{h_1+h_2-h-5}} \Big(\Phi_1^{(h_1+h_2-h-1),il}$$
$$-\frac{1}{2}\,\varepsilon^{ilm_1n_1}\, \Phi_1^{(h_1+h_2-h-1),m_1n_1} \Big)_{m+n}$$
$$+\delta^{jl} \sum_{h=-1,\text{odd}}^{h_1+h_2-1} q^h\, p_{\mathrm{F}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$\times \frac{1}{4(-4)^{h_1+h_2-h-5}} \Big(\Phi_1^{(h_1+h_2-h-1),ik}$$
$$+\frac{1}{2}\,\varepsilon^{ikm_1n_1}\, \Phi_1^{(h_1+h_2-h-1),m_1n_1} \Big)_{m+n}$$
$$+\delta^{jl} \sum_{h=-1,\text{odd}}^{h_1+h_2-1} q^h\, p_{\mathrm{B}}^{h_1+1,h_2+1,h}(m,n,\lambda)$$
$$\times \frac{1}{4(-4)^{h_1+h_2-h-5}} \Big(\Phi_1^{(h_1+h_2-h-1),ik}$$
$$-\frac{1}{2}\,\varepsilon^{ikm_1n_1}\, \Phi_1^{(h_1+h_2-h-1),m_1n_1} \Big)_{m+n} \Big]. \tag{G.6}$$

The central terms will be given later. There are antisymmetric properties both in the indices i and j and in the indices k and l on the left hand side above. This will give us rather complicated expressions on the right hand side. We do not use any simplified notations.

Let us consider the OPE between $\Phi_1^{(h_1=3),ij}(z)$ and $\Phi_1^{(h_2=1),kl}(w)$ where the weights satisfy $h_1 = h_2 + 2$. The fifth order pole of this OPE gives us the nonvanishing structure constants in the current $\Phi_0^{(h_1+h_2-h=1)}(w)$ and $\tilde{\Phi}_2^{(h_1+h_2-2-h=-1)}(w)$ with weight $h = 3$. This is not consistent with (G.6) because the dummy variable h can appear as even number. There is additional term on the right hand side of the OPE for the weights $h_1 = 3$ and $h_2 = 1$ which do not satisfy the above constraint between the weights.

The central term can be written in terms of

$$c'_F = (-4)^{h_1+h_2-4}\,\delta_{b\bar{a}}\,\delta_{d\bar{c}} \Big[\frac{(h_1+2\lambda)}{(2h_1+1)} \frac{(h_2+2\lambda)}{(2h_2+1)}$$
$$\times V_{\lambda,\bar{a}b}^{(h_1+1)+}(z)\, V_{\lambda,\bar{c}d}^{(h_2+1)+}(w)$$

$$+V^{(h_1+1)-}_{\lambda,\bar{a}b}(z)\, V^{(h_2+1)-}_{\lambda,\bar{c}d}(w) + \frac{(h_1+2\lambda)}{(2h_1+1)} \times V^{(h_1+1)+}_{\lambda,\bar{a}b}(z)\, V^{(h_2+1)-}_{\lambda,\bar{c}d}(w) + \frac{(h_2+2\lambda)}{(2h_2+1)} V^{(h_1+1)-}_{\lambda,\bar{a}b}(z)\, V^{(h_2+1)+}_{\lambda,\bar{c}d}(w)\Bigg]_{\frac{1}{(z-w)^{h_1+h_2+2}}}, \tag{G.7}$$

and

$$c'_B = (-4)^{h_1+h_2-4}\,\delta_{b\bar{a}}\,\delta_{d\bar{c}}\Bigg[\frac{(h_1+1-2\lambda)}{(2h_1+1)}\frac{(h_2+1-2\lambda)}{(2h_2+1)} \times V^{(h_1+1)+}_{\lambda,\bar{a}b}(z)\, V^{(h_2+1)+}_{\lambda,\bar{c}d}(w) + V^{(h_1+1)-}_{\lambda,\bar{a}b}(z)\, V^{(h_2+1)-}_{\lambda,\bar{c}d}(w) - \frac{(h_1+1-2\lambda)}{(2h_1+1)} \times V^{(h_1+1)+}_{\lambda,\bar{a}b}(z)\, V^{(h_2+1)-}_{\lambda,\bar{c}d}(w) - \frac{(h_2+1-2\lambda)}{(2h_2+1)} V^{(h_1+1)-}_{\lambda,\bar{a}b}(z)\, V^{(h_2+1)+}_{\lambda,\bar{c}d}(w)\Bigg]_{\frac{1}{(z-w)^{h_1+h_2+2}}}. \tag{G.8}$$

We can easily see that after h_1 inside the bracket of (3.9) and h_2 inside the bracket of (3.11) are replaced by (h_1+1) and (h_2+1) respectively, the central terms in (G.7) and (G.8) can be obtained.

G.6 The commutator relation between the third component and the fourth component with $h_1 = h_2, h_2+1$ for nonzero λ

By using the relations (2.9) and (2.10) we can have the commutators in (E.2), (E.3), (E.4) and (E.5). Then by using (4.5) and (F.1), the following result can be determined

$$\begin{aligned}
&[(\Phi_1^{(h_1),ij})_r, (\tilde{\Phi}_{\frac{3}{2}}^{(h_2),k})_m] = 4(-4)^{h_1-4}\,4(-4)^{h_2-4} \\
&\times\Bigg[\delta^{ik}\Bigg(\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm F}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm F}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) - \sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm B}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) - \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm B}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda)\Bigg) \\
&\times\frac{1}{4(-4)^{h_1+h_2-h-4}}(\Phi_{\frac{1}{2}}^{(h_1+h_2-h),j})_{m+r} \\
&+\delta^{ik}\Bigg(-\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm F}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm F}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm B}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) - \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm B}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda)\Bigg) \\
&\frac{1}{4(-4)^{h_1+h_2-h-5}}(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-1-h),j})_{m+r} \\
&-\delta^{jk}\Bigg(\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm F}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm F}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) - \sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm B}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) - \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm B}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda)\Bigg) \\
&\times\frac{1}{4(-4)^{h_1+h_2-h-4}}(\Phi_{\frac{1}{2}}^{(h_1+h_2-h),i})_{m+r} \\
&-\delta^{jk}\Bigg(-\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm F}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm F}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm B}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda) - \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm B}^{h_1+1,h_2+\frac{3}{2},h}(m,r,\lambda)\Bigg) \\
&\times\frac{1}{4(-4)^{h_1+h_2-h-5}}(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-1-h),i})_{m+r} \\
&+\varepsilon^{ijkl}\Bigg(\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm F}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm F}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm B}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda) + \sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm B}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda)\Bigg) \\
&\times\frac{1}{4(-4)^{h_1+h_2-h-4}}(\Phi_{\frac{1}{2}}^{(h_1+h_2-h),l})_{m+r}
\end{aligned}$$

$$+\varepsilon^{ijkl}\Bigg(-\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm F}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda)$$
$$+\sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm F}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda)$$
$$-\sum_{h=-1}^{h_1+h_2-1} q^h\, q_{\rm B}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda)$$
$$+\sum_{h=-1}^{h_1+h_2} q^h\,(-1)^h\, q_{\rm B}^{h_2+2,h_1+\frac{1}{2},h}(m,r,\lambda)\Bigg)$$
$$\times\frac{1}{4(-4)^{h_1+h_2-h-5}}\,(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-1-h),l})_{m+r}\Bigg]. \qquad (G.9)$$

The field contents in (G.9) are similar to the ones in (G.2). The antisymmetric property in the indices i and j on the left hand side is seen on the right hand side. Except the case of $h = h_1 + h_2$, due to the $(-1)^h$ factor, the two kinds of currents appear alternatively.

Let us consider the OPE between $\Phi_1^{(h_1=3),ij}(z)$ and $\tilde{\Phi}_{\frac{3}{2}}^{(h_2=1),k}(w)$ where $h_1 = h_2 + 2$. The fifth order pole of this OPE gives us the structure constant $\frac{32}{5}(\lambda-1)(\lambda+1)(2\lambda-3)(2\lambda+1)$ appearing in the current $G^i(w)$ of $\mathcal{N}=4$ stress energy tensor with weight $h=3$. By substituting the various expressions in the corresponding terms of (G.9), we can check that we obtain the above structure constant correctly where $q_F^{4,\frac{5}{2},3}$ term corresponds to $-\frac{128}{5}(\lambda-1)(2\lambda+1)(2\lambda^2-5\lambda-2)$ while $q_B^{4,\frac{5}{2},3}$ term corresponds to $\frac{128}{5}(\lambda-1)(2\lambda+1)(2\lambda^2+3\lambda-4)$. On the other hand, there exists the current $\Phi_{\frac{1}{2}}^{(h_1+h_2-h=1),i}(w)$ term with the structure constant $-\frac{512}{3}(\lambda-1)\lambda(2\lambda-1)(2\lambda+1)$ having the λ factor. We can check that this structure constant is equal to the one in the first current terms in (G.9) where the previous $q_F^{4,\frac{5}{2},3}$ and $q_B^{4,\frac{5}{2},3}$ are replaced by $q_F^{3,\frac{5}{2},3}$ and $q_B^{3,\frac{5}{2},3}$ respectively. Note that there is a replacement of h_1 by (h_1-1).

G.7 The commutator relation between the third component and the last component with $h_1 = h_2, h_2+1, h_2+2$ for nonzero λ

By using (2.9) and (2.11) together with (3.15), (3.16), (3.17), (3.18), (3.19), (E.1), (4.1), (4.7), and (F.2), we determine the following commutator

$$[(\Phi_1^{(h_1),ij})_m, (\tilde{\Phi}_2^{(h_2)})_n] = 16(-4)^{h_1-4}\,(-4)^{h_2-4}$$
$$-\Bigg[\sum_{h=0,\text{even}}^{h_1+h_2} q^h\, p_{\rm F}^{h_1+1,h_2+2,h}(m,n,\lambda)$$
$$\times\frac{1}{4(-4)^{h_1+h_2-h-4}}\Bigg(\Phi_1^{(h_1+h_2-h),ij}$$
$$+\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-h),kl}\Bigg)_{m+n}$$
$$-\sum_{h=0,\text{even}}^{h_1+h_2} q^h\, p_{\rm B}^{h_1+1,h_2+2,h}(m,n,\lambda)$$
$$\times\frac{1}{4(-4)^{h_1+h_2-h-4}}\Bigg(\Phi_1^{(h_1+h_2-h),ij}$$
$$-\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-h),kl}\Bigg)_{m+n}\Bigg]. \qquad (G.10)$$

The field contents on the right hand side of (G.10) are similar to the ones in (4.8). At the maximum value of the weight $h = h_1 + h_2$, we observe that there exists the current of the weight-1 current having $SO(4)$ indices of the $\mathcal{N}=4$ stress energy tensor.

Let us calculate the OPE between $\Phi_1^{(h_1=5),ij}(z)$ and $\Phi_2^{(h_2=2)}(w)$ where the weights satisfy $h_1 = h_2 + 3$. The ninth order pole of this OPE gives us the nonvanishing structure constants in the current $\Phi_1^{(h_1+h_2-h=1),ij}(w)$ and $\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-2-h=-1),kl}(w)$ with weight $h=7$. They have the λ factor explicitly. This is not consistent with (G.10) because the dummy variable h can appear as even number. This implies that there is additional term on the right hand side of the OPE for the weights $h_1 = 5$ and $h_2 = 2$ which are outside of the above allowed region.

G.8 The anticommutator relation between the fourth components with $h_1 = h_2$ for nonzero λ

By using the relations (2.10), we can rewrite in terms of the in terms of anticommutator in (E.6). The relations (4.1), (4.7), and (F.2) can be used further. We obtain

$$\{(\tilde{\Phi}_{\frac{3}{2}}^{(h_1),i})_r, (\tilde{\Phi}_{\frac{3}{2}}^{(h_2),j})_s\} = 16(-4)^{h_1-4}\,(-4)^{h_2-4}$$
$$\times\Bigg[\binom{r+h_1+\frac{1}{2}}{h_1+h_2+2}\, q^{h_1+h_2-1}\, c_Q''\,\delta^{ij}\,\delta_{r+s}$$
$$-\delta^{ij}\sum_{h=0}^{h_1+h_2+1} q^h\,(1+(-1)^h)\Big(o_{\rm F}^{h_1+\frac{3}{2},h_2+\frac{3}{2},h}(r,s,\lambda)$$
$$-o_{\rm B}^{h_1+\frac{3}{2},h_2+\frac{3}{2},h}(r,s,\lambda)\Big)$$
$$\times\frac{1}{2(-4)^{h_1+h_2-h}}\,(\Phi_0^{(h_1+h_2+2-h)})_{r+s}$$
$$-\delta^{ij}\sum_{h=0}^{h_1+h_2+1}\Big((h_1+h_2+1-h+2\lambda)\,q^h\,(1+(-1)^h)$$
$$\times o_{\rm F}^{h_1+\frac{3}{2},h_2+\frac{3}{2},h}(r,s,\lambda)$$
$$-(h_1+h_2+2-h-2\lambda)\,q^h\,(1+(-1)^h)$$

$$\times o_{\mathrm{B}}^{h_1+\frac{3}{2},h_2+\frac{3}{2},h}(r,s,\lambda)\Bigg)$$
$$\times \frac{1}{8(2h_1+2h_2-2h+3)(-4)^{h_1+h_2-h-4}}$$
$$\times(\tilde{\Phi}_2^{(h_1+h_2-h)})_{r+s}$$
$$+\sum_{h=0}^{h_1+h_2+1} q^h\,(1-(-1)^h)\,o_{\mathrm{F}}^{h_1+\frac{3}{2},h_2+\frac{3}{2},h}(r,s,\lambda)$$
$$\times \frac{1}{32(-4)^{h_1+h_2-h-3}}\Bigg(\Phi_1^{(h_1+h_2+1-h),ij}$$
$$+\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2-h),kl}\Bigg)_{r+s}$$
$$-\sum_{h=0}^{h_1+h_2+1} q^h\,(1-(-1)^h)\,o_{\mathrm{B}}^{h_1+\frac{3}{2},h_2+\frac{3}{2},h}(r,s,\lambda)$$
$$\times \frac{1}{32(-4)^{h_1+h_2-h-3}}\Bigg(\Phi_1^{(h_1+h_2+1-h),ij}$$
$$-\frac{1}{2}\,\varepsilon^{ijkl}\,\Phi_1^{(h_1+h_2+1-h),kl}\Bigg)_{r+s}\Bigg]. \tag{G.11}$$

The central term will be given later. The structure of (G.11) is very similar to the one of (G.1). Once we replace the weights h_1 and h_2 with (h_1+1) and (h_2+1) in the latter respectively, then we can see most of the structure of (G.11). Even the factor $(1\pm(-1)^h)$ appears precisely.

Let us calculate the OPE between $\Phi_{\frac{3}{2}}^{(h_1=3),i}(z)$ and $\Phi_{\frac{3}{2}}^{(h_2=1),j}(w)$ with $i=j$ where the weights satisfy $h_1=h_2+2$. The sixth order pole of this OPE gives us the nonvanishing structure constants in the currents $\Phi_0^{(h_1+h_2+2-h=1)}(w)$ and $\tilde{\Phi}_2^{(h_1+h_2-h=-1),kl}(w)$ with weight $h=5$. They have the λ factor. This is not consistent with (G.11) because all the coefficients are vanishing for odd number h. There is additional term on the right hand side of the OPE for the weights $h_1=3$ and $h_2=1$ which do not satisfy the above constraint between the weights.

The central term contains

$$c''_Q = 8(-4)^{h_1+h_2-4}\,\delta_{b\bar{a}}\,\delta_{d\bar{c}}\Bigg[-Q_{\lambda,\bar{a}b}^{(h_1+2)+}(z)$$
$$\times Q_{\lambda,\bar{c}d}^{(h_2+2)+}(w)+Q_{\lambda,\bar{a}b}^{(h_1+2)-}(z)\,Q_{\lambda,\bar{c}d}^{(h_2+2)-}(w)$$
$$-Q_{\lambda,\bar{a}b}^{(h_1+2)+}(z)\,Q_{\lambda,\bar{c}d}^{(h_2+2)-}(w)$$
$$+Q_{\lambda,\bar{a}b}^{(h_1+2)-}(z)\,Q_{\lambda,\bar{c}d}^{(h_2+2)+}(w)$$
$$-(-1)^{h_1+h_2}\,Q_{\lambda,\bar{c}d}^{(h_2+2)+}(z)\,Q_{\lambda,\bar{a}b}^{(h_1+2)+}(w)$$
$$+(-1)^{h_1+h_2}\,Q_{\lambda,\bar{c}d}^{(h_2+2)-}(z)\,Q_{\lambda,\bar{a}b}^{(h_1+2)-}(w)$$
$$-(-1)^{h_1+h_2}\,Q_{\lambda,\bar{c}d}^{(h_2+2)+}(z)\,Q_{\lambda,\bar{a}b}^{(h_1+2)-}(w)$$
$$+(-1)^{h_1+h_2}\,Q_{\lambda,\bar{c}d}^{(h_2+2)-}(z)\,Q_{\lambda,\bar{a}b}^{(h_1+2)+}(w)\Bigg]_{\frac{1}{(z-w)^{h_1+h_2+3}}}. \tag{G.12}$$

Note that from the inside of the bracket in (3.14), after the h_1 and the h_2 are replaced with (h_1+1) and (h_2+1) respectively, we obtain the above result (G.12).

G.9 The commutator relation between the fourth component and the last component with $h_1=h_2, h_2+1$ for nonzero λ

By using the relations (2.10) and (2.11), we can reexpress them in terms of (E.2), (E.3), (E.4) and (E.5). After using (4.5) and (F.1) the following result can be obtained

$$\big[(\tilde{\Phi}_{\frac{3}{2}}^{(h_1),i})_r,(\tilde{\Phi}_2^{(h_2)})_m\big] = 4(-4)^{h_1-4}\,4(-4)^{h_2-4}$$
$$\times\Bigg[-\Bigg(-\sum_{h=-1}^{h_1+h_2} q^h\,q_{\mathrm{F}}^{h_2+2,h_1+\frac{3}{2},h}(m,r,\lambda)$$
$$+\sum_{h=-1}^{h_1+h_2+1} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_2+2,h_1+\frac{3}{2},h}(m,r,\lambda)$$
$$-\sum_{h=-1}^{h_1+h_2} q^h\,q_{\mathrm{B}}^{h_2+2,h_1+\frac{3}{2},h}(m,r,\lambda)$$
$$+\sum_{h=-1}^{h_1+h_2+1} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_2+2,h_1+\frac{3}{2},h}(m,r,\lambda)\Bigg)$$
$$\times\frac{1}{4(-4)^{h_1+h_2-h-3}}\,(\Phi_{\frac{1}{2}}^{(h_1+h_2+1-h),i})_{m+r}$$
$$-\Bigg(\sum_{h=-1}^{h_1+h_2} q^h\,q_{\mathrm{F}}^{h_2+2,h_1+\frac{3}{2},h}(m,r,\lambda)$$
$$+\sum_{h=-1}^{h_1+h_2+1} q^h\,(-1)^h\,q_{\mathrm{F}}^{h_2+2,h_1+\frac{3}{2},h}(m,r,\lambda)$$
$$+\sum_{h=-1}^{h_1+h_2} q^h\,q_{\mathrm{B}}^{h_2+2,h_1+\frac{3}{2},h}(m,r,\lambda)$$
$$+\sum_{h=-1}^{h_1+h_2+1} q^h\,(-1)^h\,q_{\mathrm{B}}^{h_2+2,h_1+\frac{3}{2},h}(m,r,\lambda)\Bigg)$$
$$\times\frac{1}{4(-4)^{h_1+h_2-h-4}}\,(\tilde{\Phi}_{\frac{3}{2}}^{(h_1+h_2-h),i})_{m+r}\Bigg]. \tag{G.13}$$

In this case we can compare the above with the previous result in (G.5). They look similar to each other. The alternating feature between the currents depending on the property of even or odd in the weight h can be observed.

Let us consider the OPE between $\tilde{\Phi}_{\frac{3}{2}}^{(h_1=1),i}(z)$ and $\tilde{\Phi}_2^{(h_2=2)}(w)$ where the weights satisfy $h_1=h_2-1$. The sixth order pole of this OPE gives us the nonvanishing structure constant having the λ factor in the current $\Gamma^i(w)$ of the $\mathcal{N}=4$ stress energy tensor. This is not consistent with (G.13) because the minimum value of the exponent (h_1+h_2-h) of the first current is given by 3 and those of the second current is 2. There is additional term on the right hand side of the OPE for the weights $h_1 = 1$ and $h_2 = 2$ which are outside of the allowed region between the weights.

G.10 The commutator relation between the last components with $h_1 = h_2 - 1, h_2, h_2 + 1$ for nonzero λ

Finally, by using (2.11) together with (3.15), (3.16), (3.17), (3.18), (3.19), (E.1), (4.1), (4.7), and (F.2) the following result can be determined

$$\begin{aligned}
\big[(\tilde{\Phi}_2^{(h_1)})_m, (\tilde{\Phi}_2^{(h_2)})_n\big] &= 16(-4)^{h_1-4}\,(-4)^{h_2-4} \\
&\times\Bigg[\binom{m+h_1+1}{h_1+h_2+3}\,4\,q^{h_1+h_2}\,(c_F''+c_B'')\,\delta^{ij}\,\delta_{r+s} \\
&+\sum_{h=0,\mathrm{even}}^{h_1+h_2+1} q^h\Big(-p_{\mathrm{F}}^{h_1+2,h_2+2,h}(m,n,\lambda) \\
&+p_{\mathrm{B}}^{h_1+2,h_2+2,h}(m,n,\lambda)\Big) \\
&\times\frac{4}{(-4)^{h_1+h_2-h}}\,(\Phi_0^{(h_1+h_2+2-h)})_{m+n} \\
&+\sum_{h=0,\mathrm{even}}^{h_1+h_2+1}\Big(-(h_1+h_2+1-h+2\lambda)\,q^h \\
&\times p_{\mathrm{F}}^{h_1+2,h_2+2,h}(m,n,\lambda) \\
&-(h_1+h_2+2-h-2\lambda)\,q^h\,p_{\mathrm{B}}^{h_1+2,h_2+2,h}(m,n,\lambda)\Big) \\
&\times\frac{1}{2(2h_1+2h_2-2h+3)(-4)^{h_1+h_2-h-4}} \\
&\times(\tilde{\Phi}_2^{(h_1+h_2-h)})_{m+n}\Bigg].
\end{aligned} \tag{G.14}$$

This has the similar structure to the one in (4.4).

Let us calculate the OPE between $\tilde{\Phi}_2^{(h_1=2)}(z)$ and $\tilde{\Phi}_2^{(h_2=0)}(w)$ where the weights satisfy $h_1 = h_2 + 2$. The fifth order pole of this OPE gives us the nonvanishing structure constants having the λ factor in the currents $\Phi_0^{(h_1+h_2+2-h=1)}(w)$ and $\tilde{\Phi}_2^{(h_1+h_2-h=-1)}(w)$ with weight $h = 3$. This is not consistent with (G.14) because the dummy variable h can appear as even number. This implies that there is additional term on the right hand side of the OPE for the weights $h_1 = 2$ and $h_2 = 0$ we are considering.

The central term has

$$\begin{aligned}
c_F'' &= (-4)^{h_1+h_2-4}\,\delta_{b\bar{a}}\,\delta_{d\bar{c}}\Bigg[\frac{(h_1+1+2\lambda)}{(2h_1+3)} \\
&\times\frac{(h_2+1+2\lambda)}{(2h_2+3)}\,V_{\lambda,\bar{a}b}^{(h_1+2)+}(z)\,V_{\lambda,\bar{c}d}^{(h_2+2)+}(w) \\
&+V_{\lambda,\bar{a}b}^{(h_1+2)-}(z)\,V_{\lambda,\bar{c}d}^{(h_2+2)-}(w)+\frac{(h_1+1+2\lambda)}{(2h_1+3)} \\
&\times V_{\lambda,\bar{a}b}^{(h_1+2)+}(z)\,V_{\lambda,\bar{c}d}^{(h_2+2)-}(w) \\
&+\frac{(h_2+1+2\lambda)}{(2h_2+3)}\,V_{\lambda,\bar{a}b}^{(h_1+2)-}(z)\,V_{\lambda,\bar{c}d}^{(h_2+2)+}(w)\Bigg]_{\frac{1}{(z-w)^{h_1+h_2+4}}},
\end{aligned} \tag{G.15}$$

$$\begin{aligned}
c_B'' &= (-4)^{h_1+h_2-4}\,\delta_{b\bar{a}}\,\delta_{d\bar{c}}\Bigg[\frac{(h_1+2-2\lambda)}{(2h_1+3)} \\
&\times\frac{(h_2+2-2\lambda)}{(2h_2+3)}\,V_{\lambda,\bar{a}b}^{(h_1+2)+}(z)\,V_{\lambda,\bar{c}d}^{(h_2+2)+}(w) \\
&+V_{\lambda,\bar{a}b}^{(h_1+2)-}(z)\,V_{\lambda,\bar{c}d}^{(h_2+2)-}(w)-\frac{(h_1+2-2\lambda)}{(2h_1+3)} \\
&\times V_{\lambda,\bar{a}b}^{(h_1+2)+}(z)\,V_{\lambda,\bar{c}d}^{(h_2+2)-}(w) \\
&-\frac{(h_2+2-2\lambda)}{(2h_2+3)}\,V_{\lambda,\bar{a}b}^{(h_1+2)-}(z)\,V_{\lambda,\bar{c}d}^{(h_2+2)+}(w)\Bigg]_{\frac{1}{(z-w)^{h_1+h_2+4}}}.
\end{aligned} \tag{G.16}$$

We can easily see that after h_1 inside the bracket of (3.9) and h_2 inside the bracket of (3.11) are replaced by $(h_1 + 2)$ and $(h_2 + 2)$ respectively, the central terms in (G.15) and (G.16) can be obtained.

References

1. E. Bergshoeff, M.A. Vasiliev, B. de Wit, The SuperW(infinity) (lambda) algebra. Phys. Lett. B **256**, 199–205 (1991). https://doi.org/10.1016/0370-2693(91)90673-E
2. E. Bergshoeff, B. de Wit, M.A. Vasiliev, The structure of the superW(infinity) (lambda) algebra. Nucl. Phys. B **366**, 315–346 (1991). https://doi.org/10.1016/0550-3213(91)90005-I
3. D. Friedan, E.J. Martinec, S.H. Shenker, Conformal invariance, supersymmetry and string theory. Nucl. Phys. B **271**, 93–165 (1986). https://doi.org/10.1016/0550-3213(86)90356-1
4. I. Bakas, E. Kiritsis, Grassmannian Coset models and unitary representations of W(infinity). Mod. Phys. Lett. A **5**, 2039–2050 (1990). https://doi.org/10.1142/S0217732390002328
5. S. Odake, T. Sano, W(1) + infinity and superW(infinity) algebras with SU(N) symmetry. Phys. Lett. B **258**, 369–374 (1991). https://doi.org/10.1016/0370-2693(91)91101-Z
6. S. Odake, Unitary representations of W(infinity) algebras. Int. J. Mod. Phys. A **7**, 6339–6356 (1992). https://doi.org/10.1142/S0217751X9200288X. arXiv:hep-th/9111058
7. C. Ahn, A deformed supersymmetric $w_{1+\infty}$ symmetry in the celestial conformal field theory. Eur. Phys. J. C **82**(7), 630 (2022). https://doi.org/10.1140/epjc/s10052-022-10582-9. arXiv:2202.02949 [hep-th]
8. C. Ahn, The $\mathcal{N} = 2$ supersymmetric $w_{1+\infty}$ symmetry in the two-dimensional SYK models. JHEP **05**, 115 (2022). https://doi.org/10.1007/JHEP05(2022)115. arXiv:2203.03105 [hep-th]
9. C. Ahn, N=4 supersymmetric linear $W_\infty[\lambda]$ algebra. Phys. Rev. D **106**(2), 026008 (2022). https://doi.org/10.1103/PhysRevD.106.026008. arXiv:2205.04024 [hep-th]
10. K. Thielemans, A Mathematica package for computing operator product expansions. Int. J. Mod. Phys. C **2**, 787–798 (1991). https://doi.org/10.1142/S0129183191001001
11. Wolfram Research, Inc., Mathematica, Version 13.0.0, Champaign (2021)
12. C. Ahn, D. Kim, M.H. Kim, The $mathcal N = 4$ coset model and the higher spin algebra. Int. J. Mod. Phys. A **35**(11n12),

2050046 (2020). https://doi.org/10.1142/S0217751X20500463. arXiv:1910.02183 [hep-th]
13. C. Ahn, M.H. Kim, The $mathcal N = 4$ higher spin algebra for generic μ parameter. JHEP **02**, 123 (2021). https://doi.org/10.1007/JHEP02(2021)123. arXiv:2009.04852 [hep-th]
14. S. Pasterski, M. Pate, A.M. Raclariu, Celestial holography. arXiv:2111.11392 [hep-th]
15. A. Strominger, "w(1+infinity) and the celestial sphere. arXiv:2105.14346 [hep-th]
16. A. Strominger, $w_{1+\infty}$ algebra and the celestial sphere: infinite towers of soft graviton, photon, and gluon symmetries. Phys. Rev. Lett. **127**(22), 221601 (2021). https://doi.org/10.1103/PhysRevLett.127.221601
17. A. Guevara, E. Himwich, M. Pate, A. Strominger, Holographic symmetry algebras for gauge theory and gravity. JHEP **11**, 152 (2021). https://doi.org/10.1007/JHEP11(2021)152. arXiv:2103.03961 [hep-th]
18. J. Mago, L. Ren, A.Y. Srikant, A. Volovich, Deformed $w_{1+\infty}$ algebras in the celestial CFT. arXiv:2111.11356 [hep-th]
19. H. Jiang, Holographic chiral algebra: supersymmetry, infinite Ward identities, and EFTs. JHEP **01**, 113 (2022). https://doi.org/10.1007/JHEP01(2022)113. arXiv:2108.08799 [hep-th]
20. E. Himwich, M. Pate, K. Singh, Celestial operator product expansions and $\mathrm{w}_{1+\infty}$ symmetry for all spins. JHEP **01**, 080 (2022). https://doi.org/10.1007/JHEP01(2022)080. arXiv:2108.07763 [hep-th]
21. I. Bakas, The large n limit of extended conformal symmetries. Phys. Lett. B **228**, 57 (1989). https://doi.org/10.1016/0370-2693(89)90525-X
22. C. Ahn, Towards a supersymmetric $w_{1+\infty}$ symmetry in the celestial conformal field theory. Phys. Rev. D **105**(8), 086028 (2022). https://doi.org/10.1103/PhysRevD.105.086028. arXiv:2111.04268 [hep-th]