On $\mathcal{N} = 6$ Superconformal Field Theories

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Abstract

In this thesis we study $\mathcal{N} = 6$ superconformal field theories (SCFTs) in three dimensions. Such theories are highly constrained by supersymmetry, allowing many quantities to be computed exactly. Yet though constrained, $\mathcal{N} = 6$ SCFTs still exhibit a rich array of behaviors, and in various regimes can be dual holographically to M-theory on $\text{AdS}_4 \times S^7$, IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$, and higher-spin gravity on AdS_4. We will use tools from conformal bootstrap and supersymmetric localization to study $\mathcal{N} = 6$ theories, both in general and in holographic regimes.

We begin in Chapter 2 by deriving the supersymmetric Ward identities and the superconformal block expansion for the four-point correlator $\langle SSSS \rangle$ of stress tensor multiplet scalars S. Chapter 3 then studies the mass-deformed sphere partition function, which can be computed exactly using supersymmetric localization, and relates derivatives of this quantity to specific integrals of $\langle SSSS \rangle$.

In Chapter 4 we study the IIA string and M-theory limits of the ABJ family of $\mathcal{N} = 6$ SCFTs. Using the supersymmetric Ward identities and localization results, we are able to fully determine the R^4 corrections to the $\langle SSSS \rangle$ correlator in both limits. By taking the flat space limit, we can compare to the known R^4 contribution to the IIA and M-theory S-matrix, allowing us to perform a check of AdS/CFT at finite string coupling.

In Chapter 5 we study the higher-spin limit of $\mathcal{N} = 6$ theories. Using the weakly broken higher-spin Ward identity, we completely determine the leading correction to $\langle SSSS \rangle$ in this limit up to two free parameters, which for ABJ theory we then fix using localization. Finally, in Chapter 6 we perform the first numerical bootstrap study of $\mathcal{N} = 6$ superconformal field theories, allowing us to derive non-perturbative bounds on the CFT data contributing to $\langle SSSS \rangle$.

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For Abigail

May Oxford prove closer to Hong Kong

Then Princeton is to Canberra

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Chapter 1

Introduction

In this thesis we study a class of highly symmetric quantum field theories, $\mathcal{N} = 6$ superconformal field theories (SCFTs) in three spacetime dimensions. Much like the quantum field theories we use in particle physics — the Standard Model, QCD, QED, and so on — $\mathcal{N} = 6$ SCFTs have Poincaré symmetry: they are invariant under translations, rotations, and boosts. But $\mathcal{N} = 6$ SCFTs are also invariant under two additional kinds of spacetime symmetries: conformal symmetries and supersymmetries. We will now motivate each of these in turn.

Conformal Symmetry

Physical theories come with characteristic length scales. At distances much greater than this length scale, the precise details of the short-distance physics become unimportant and so we expect the theory to become scale-invariant. Under broad but still not entirely understood circumstances,[1] scale symmetry is enhanced to the larger group of conformal symmetries, which are the spacetime transformations which locally look like rotations and rescalings. We therefore expect that at large distances, quantum field theories (QFTs) should approach conformality. This intuition is formally captured by the renormalization group (RG), which describes how quantum field theories "flow" from short distances (the "UV") to long distances (the "IR"). For UV complete QFTs, the short distance behavior will also become scale-invariant, and so we can often[2] think of QFT as an RG

[1]It has been shown in 2d and 4d that Lorentzian invariance and unitarity, along with certain technical assumptions, are sufficient [1, 2], but in other spacetime dimensions the situation is less clear. In particular, free Maxwell theory in 3d is scale-invariant theory, satisfies all the conditions of the 2d and 4d results, and yet is not conformally-invariant [3]. It is not known, however, if interacting counterexamples exist. A thorough discussion of these issues can be found in [4].

[2]This picture does not quite work for gauge theories in 3d, because as noted in the previous footnote free Maxwell theory is scale-invariant but not conformally-invariant. Nevertheless, we can still think of a UV-complete QFT as an RG flow between scale-invariant theories.

flow between two conformal field theories (CFTs):

$$\mathrm{CFT}_{\mathrm{UV}} \xrightarrow{\mathrm{QFT}} \mathrm{CFT}_{\mathrm{IR}}$$

Studying CFTs thus enables us to map out more generally the space of all QFTs.

Let us give two examples of conformality in physical theories. Consider the Standard Model, which provides our current best description of particles physics. The only massless particle in the Standard Model is the photon. At energies much lower than the electron mass,[3] we can ignore the massive particles and describe just the photons by the most famous conformally invariant theory of them all: Maxwell's theory of electromagnetism. This theory is classically conformally invariant, and remains conformally invariant after quantization.[4]

Another theory believed to be conformally invariant is the critical Ising model, which describes second order phase transitions in water (and other fluids) at its critical point and in uni-axial magnets at their critical points. Although at short distances water and uni-axial magnets have very different behaviors, at long distances they both limit to the same conformal field theory, and so share the same critical exponents.

Due to the greater amount of symmetry available, CFTs are subject to more stringent consistency conditions then regular QFTs. This has led to the "conformal bootstrap", a series of tools for studying CFTs which focus directly on the CFT itself, rather than relying on a specific lattice model or Lagrangian description which only flows to the CFT in the infrared. These methods aim to fully utilize the constraints imposed by conformal symmetry, along with any additional internal symmetries or supersymmetries. Combining these symmetries with consistency conditions coming from unitarity and crossing symmetries, conformal bootstrap can be used to constrain and sometimes even fully solve a CFT. Because conformal bootstrap methods rely on non-perturbative properties of CFTs, they are particular useful for studying strongly-coupled theories. While the basic bootstrap philosophy was first articulated in the 1970s [5, 6], it has only been in the last 15 years that the conformal bootstrap has reached maturity for theories in more than two dimensions.[5] In particular, the numeric conformal bootstrap, first proposed in [7], provides a general method for computing rigorous bounds on conformal field theories and has led to a vast array of new results — see [8] for

[3] Technically, the neutrinos are much lighter than the electron, but interact so weakly with ordinary matter that they can be ignored when studying electromagnetism.

[4] When coupled to charged matter fields, however, conformality is broken due to the scale-anomaly.

[5] In two dimensions the conformal group is infinite dimensional, whereas for $d \geq 3$ the conformal group is finite dimensional. For this reason, 2d CFTs are much better understood than their $d \geq 3$ cousins, and tools such as the Virasoro algebra and modular invariance even allow for theories to be solved exactly. No exactly solvable interacting CFTs in $d \geq 3$ are known.

a recent review. One goal of this thesis will be to perform the first general numeric bootstrap study of $\mathcal{N}=6$ SCFTs.

Supersymmetry

Supersymmetries are generated by charges transforming in spinor representations of the Lorentz group, which anticommute to generate spacetime translations. We can therefore think of them as the "square root" of a translation. One motivation to study supersymmetric theories is provided by the Coleman-Mandula [9] and Haag-Łopuszański-Sohnius theorems [10], which show that the only way to non-trivially extend the Poincaré invariance in $d \geq 3$ is with supersymmetries. For technical reasons these theorems do not directly apply to conformal symmetries, although analogous results still apply [11, 12]. By combining supersymmetry with conformal symmetry, superconformal field theories are amongst the most symmetric theories we can study.

Many tools have been developed to study supersymmetric field theories. Non-renormalization theorems, for instance, can be used to show that the UV divergences which plague most QFTs are either milder, or, in some cases, even absent entirely. Dynamical phenomena such as confinement, electro-magnetic duality, and instantons, which we expect to generically occur in gauge theories, are often amenable to analytic study in supersymmetric settings. Supersymmetric theories therefore provide a window into the behavior of strongly interacting theories more generally.

In this thesis we will focus on $\mathcal{N}=6$ superconformal symmetry, which is the next-to-maximal amount of supersymmetry available in three dimensions.[6] With next-to-maximal supersymmetry comes next-to-maximal analytic control, allowing us to compute many quantities exactly. We will focus on a specific tool called supersymmetric localization, which allows certain partition functions to be computed exactly [14, 15]. $\mathcal{N}=6$ superconformal symmetry also strongly constrains the matter content and interactions possible in a theory. Indeed, only two families of $\mathcal{N}=6$ SCFTs are known [16–19]: Chern-Simons matter theories with gauge groups[7]

$$\begin{aligned} \text{ABJ family:} \quad & U(N)_k \times U(N+M)_{-k}\,, \\ \text{OSp family:} \quad & SO(2)_{2k} \times USp(2+2M)_{-k}\,, \end{aligned}$$

and it is tempting to conjecture that these are the only ones that can exist.

[6] Interacting conformal field theories in 3d must have $\mathcal{N} \leq 8$. In addition to this restriction, all $\mathcal{N}=7$ theories automatically enhance to $\mathcal{N}=8$, and so $\mathcal{N}=6$ is the second greatest amount of supersymmetry available in three-dimensions [13].

[7] We list the gauge groups only up to a specific choice of $U(1)$ factors, because, as we conjecture in Section 1.2, the $U(1)$ factors do not modify the correlators studied in this thesis.

Holography

We have so far motivated the study of $\mathcal{N} = 6$ superconformal fields theories by noting that they are amongst the most symmetric and tractable of all QFTs. There is a separate reason to consider $\mathcal{N} = 6$ superconformal field theories, and this comes from the study not of quantum field theories but of quantum gravity. Supersymmetry appears to be a necessary ingredient of String Theory, which is our leading candidate for a theory of quantum gravity. Solutions of string and M theory on manifolds with boundaries are believed to be described holographically by local quantum field theories living on said boundaries. Just as $\mathcal{N} = 6$ SCFTs are amongst the most symmetric of all quantum field theories, their holographic duals are amongst the most symmetric of all string and M-theory backgrounds. With the analytic control provided by $\mathcal{N} = 6$ supersymmetry, we can perform many checks of conjectured holographic dualities which are not available in less symmetric theories.

When N is large, the ABJ family of theories is holographically dual to both IIA string theory and to M theory on AdS_4, depending on the value of k [16, 17]. This duality is akin to the most famous AdS/CFT duality, between 4d $\mathcal{N} = 4$ super-Yang Mills and IIB string theory on AdS_5 [20]. When M is large, however, the ABJ family of theories are believed to be dual to a completely different kind of quantum gravity, known as higher-spin gravity [21]. In higher-spin theories, the graviton is joined by an infinite number of massless higher-spin particles. While the precise connection between higher-spin gravity and string theory is not yet clear, that fact that both are holographically dual to the same family of $\mathcal{N} = 6$ SCFTs is suggestive. Note that while there are $\mathcal{N} = 8$ SCFTs with M theory duals, SCFTs with IIA or higher-spins duals must have $\mathcal{N} \leq 6$. The ABJ family of theories is the most symmetric family of theories exhibiting all three kinds of holographic duals.

Thesis Overview

Having motivated the study of $\mathcal{N} = 6$ SCTs, let us now give an overview of the rest of the thesis. Our aim is to develop a series of tools to study $\mathcal{N} = 6$ SCFTs. Some of the tools we develop are general and apply to all such theories, while others are specific to either the stringy or higher-spin limits. The observable we focus on is the four-point correlator $\langle SSSS \rangle$, where S is a scalar operator related by supersymmetry to the stress tensor. This scalar is present in any $\mathcal{N} = 6$ SCFT, and in interacting theories is always the operator with lowest scaling dimension.

The rest of this introduction gives a more technical overview of topics discussed thus far. In Section 1.1 we review the properties of conformal and superconformal field theories in three dimensions. We then discuss the construction of super Chern-Simons matter theories in Section 1.2, and describe in detail the ABJ and OSp families of $\mathcal{N} = 6$ SCFTs. Finally in Section 1.3 we review the holographic dualities relating $\mathcal{N} = 6$ SCFTs to theories of quantum gravity.

Chapter 2 discusses the constraints of superconformal invariance on the four-point function $\langle SSSS \rangle$. We derive the superconformal Ward identities, which are linear differential equations satisfied by $\langle SSSS \rangle$, and which further relate $\langle SSSS \rangle$ to other four-point correlators. We then derive the superconformal block expansion for $\langle SSSS \rangle$. The Ward identities and superblocks expansions are fundamental tools which we use throughout the thesis. As they rely only on $\mathcal{N} = 6$ superconformal symmetry, they apply to all $\mathcal{N} = 6$ SCFTs.

In Chapter 3 we study constraints from supersymmetric localization. $\mathcal{N} = 6$ SCFTs can be naturally mapped to the 3-sphere using conformal invariance. On the sphere, they posses certain mass-deformations — so named because in Lagrangian theories they give mass to the otherwise massless scalars and fermions — which preserve certain supersymmetries. We focus on two of these mass-deformations, which we parameterize by m_+ and m_-. In Lagrangian $\mathcal{N} = 6$ SCFTs (which includes all known $\mathcal{N} = 6$ theories), the sphere partition function $Z(m_+, m_-)$ can be computed exactly using supersymmetric localization for all values of $m_\pm$. In Section 3.1 we relate the derivatives of $Z(m_+, m_-)$ to integrals of $\langle SSSS \rangle$. The rest of the chapter then focuses on evaluating derivatives of $Z(m_+, m_-)$ explicitly for the ABJ and OSp families in various regimes.

In Chapter 4 we study the large N limit of ABJ theory, which, depending on the precise large N limit taken, is holographically dual to either M-theory or IIA string theory. To study this limit, we rewrite $\langle SSSS \rangle$ in Mellin space. Holographic Mellin amplitudes have a simple analytic structure, and behave analogously to flat-space scattering amplitudes. Indeed, the Penedones formula [22] provides a direct mapping between Mellin amplitudes in the boundary CFT and flat-space scattering amplitudes in the bulk gravitational theory. Motivated by this connection, we study the $\mathcal{N} = 6$ on-shell spinor-helicity formalism in 4d and use it to classify higher derivative corrections to 4-graviton scattering in flat-space. Combining the results of this classification with the superconformal Ward identities, we fully fix the Mellin amplitudes contributing to $\langle SSSS \rangle$ at large N. To determine the precise contribution of each Mellin amplitude, we use both the supersymmetric localization constraints for ABJ derived in Chapter 3, and the known flat-space string and M-theory scattering amplitudes. Certain coefficients can be computed independently using either localization or from flat space, allowing us to test the AdS/CFT correspondence to several orders in $1/N$.

In Chapter 5 we study $\mathcal{N} = 6$ theories in the higher-spin limit, which are holographically dual to theories of higher-spin gravity. Theories with unbroken higher-spin symmetries cannot be interacting, as the constraints of higher-spin symmetry are so stringent as to force all correlation functions to be equal to those of free field theory [11]. We study $\mathcal{N} = 6$ SCFTs where the higher-spin symmetries are weakly broken. The resulting theories can now have interesting dynamics, but are still strongly constrained by the weakly broken higher-spin symmetries. We use these constraints to fix an ansatz for the leading corrections to $\langle SSSS \rangle$, and then use localization to completely determine the unknown coefficients in the ansatz for both ABJ and OSp theories at large M.

In Chapter 6, we present the first numeric bootstrap study of the $\langle SSSS \rangle$ correlator. This allows us to derive general non-perturbative bounds on the $\langle SSSS \rangle$ correlator in any $\mathcal{N} = 6$ SCFT. To impose further constraints, we combine the numeric bootstrap with the localization results of Chapter 3, allowing us to derive precise bounds on certain physical quantities in specific ABJ theories. We also find that the bootstrap bounds of a certain protected OPE coefficient appears to be saturated by the $U(1)_{2M} \times U(1+M)_{2M}$ family of ABJ theories. Assuming this conjecture is true, we can compute numerically the spectrum of these theories using the extremal functional method. As M becomes large these theories have weakly broken higher-spin symmetry, allowing us to compare numeric bootstrap results to the analytic computations performed in Chapter 5.

We finish with a summary of our results and a discussion of future directions in Chapter 7. Four appendices then follow which discuss various technical details.

This thesis is based on the work with Shai M. Chester and Silviu S. Pufu [23], Shai M. Chester, Max Jerdee and Silviu S. Pufu [24], and with Shai M. Chester and Max Jerdee [25], edited together to form a coherent narrative. Chapters 2 and 6 are primarily based on [24], but also include material from the other two papers. Chapter 4 is based on [23], Chapter 5 is based on [25], and Chapter 3 draws on material from all three papers.

1.1 Conformal Fields Theories with $\mathcal{N} = 6$ Supersymmetry

We will now discuss in more details the implications of conformal and superconformal symmetry in three dimensions. First we consider conformal symmetry and its implications for CFTs, and then extend our discussion to $\mathcal{N} = 6$ superconformal symmetry. We then finish by describing the $\mathcal{N} = 6$ stress tensor multiplet, which is present in any local $\mathcal{N} = 6$ SCFT.

1.1.1 Conformal Symmetry

Conformal field theories are widely studied and their properties are discussed in many places. Here we will focus on the basics needed for the thesis; more detailed discussions aimed towards the conformal bootstrap can be found, for instance, in [26,27]. The 3d conformal group is defined as the group generated by spacetime translations and Lorentz transformations (which together generate the Poincaré group), along with dilatations

$$x^\mu \to \lambda x^\mu \tag{1.1}$$

and special conformal transformations

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x \cdot a + a^2 x^2}\,, \tag{1.2}$$

parameterized by a scalar λ and vector a^μ respectively. Together these generate all transformations $x \to x'$ which preserve the metric $\eta_{\mu\nu} = \text{diag}\{-1, +1, +1\}$ up to a scalar factor:

$$\eta_{\mu\nu} \to \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\eta_{\rho\sigma} = \Omega(\vec{x})^2 \eta_{\mu\nu} \tag{1.3}$$

and which can be continuously connected to the identity map. We will use P_μ, $M_{\mu\nu}$, D and K_μ to denote the (anti-hermitian) infinitesimal generators of translations, Lorentz transformations, dilatations and special conformal transformations respectively, defined by the equations

$$P_\mu = \partial_\mu\,, \qquad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu\,, \qquad D = x^\mu \partial_\mu\,, \qquad K_\mu = x^2 \partial_\mu - 2x_\mu(\vec{x} \cdot \partial)\,. \tag{1.4}$$

Together they satisfy the $\mathfrak{so}(3, 2)$ commutation relations

$$\begin{aligned}
&[M_{\mu\nu}, P_\rho] = \eta_{\rho(\nu}P_{\mu)}\,, \qquad [M_{\mu\nu}, K_\rho] = \eta_{\rho(\nu}K_{\mu)}\,, \qquad [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu(\rho}M_{\sigma)\nu} - \eta_{\nu(\rho}M_{\sigma)\mu} \\
&[D, P_\mu] = -P_\mu\,, \qquad [D, K_\mu] = K_\mu\,, \qquad [K_\mu, P_\nu] = 2M_{\mu\nu} + 2\eta_{\mu\nu}D \\
&[M_{\mu\nu}, D] = 0\,.
\end{aligned} \tag{1.5}$$

As in any Lorentzian theory, we can label local operators $\mathcal{O}(\vec{x})$ in a conformal field theory by their spin $\ell = 0\,, \frac{1}{2}\,, 1\,, \ldots$. Because the dilatation generator D commutes with the Lorentz generators $M_{\mu\nu}$, we can then simultaneously label operators by their conformal dimension Δ, such that

$$[D, \mathcal{O}(\vec{x})] = (-\Delta + \vec{x} \cdot \partial)\mathcal{O}(\vec{x})\,. \tag{1.6}$$

In particular, the operator $\mathcal{O}(0)$ when placed at the origin is an eigenstate of D with eigenvalue Δ. The generators P_μ and K_μ act to raise and lower the scaling dimension of an operator by one. A primary operator $\mathcal{O}(0)$ is an operator which is annihilated by K_μ, while operators constructed from a primary $\mathcal{O}(0)$ by acting with P_μ are called descendants. Note in particular that

$$\mathcal{O}(\vec{x}) = \mathcal{O}(0) + x^\mu \partial_\mu \mathcal{O}(0) + \frac{1}{2} x^\mu x^\nu \partial_\mu \partial_\nu \mathcal{O}(0) + \dots\,, \tag{1.7}$$

so that $\mathcal{O}(\vec{x})$ can be written as a sum of $\mathcal{O}(0)$ and its descendants.

The conformal group admits three kinds of unitary multiplets. *Long multiplets* have conformal dimensions above the unitarity bounds

$$\Delta > \begin{cases} \frac{1}{2} & \ell = 0 \\ 1 & \ell = \frac{1}{2} \\ \ell + 1 & \ell \geq 1 \end{cases} \tag{1.8}$$

Short multiplets occur at the bottom of the continuum in (1.8)

$$\Delta = \begin{cases} \frac{1}{2} & \ell = 0 \\ 1 & \ell = \frac{1}{2} \\ \ell + 1 & \ell \geq 1 \end{cases} \tag{1.9}$$

and satisfy shortening conditions so that certain descendants of the primary vanish. For scalars, the shortening condition is $P^2 \mathcal{O}(0) = 0$, implying that the operator satisfies the Klein-Gordon equation

$$\partial^2 \mathcal{O}(x) = 0\,, \tag{1.10}$$

while for integer spin-ℓ operators, the shortening condition is $P^{\nu_1} \mathcal{O}_{\nu_1 \dots \nu_\ell}(0) = 0$, implying that the symmetric traceless tensor $\mathcal{O}_{\nu_1 \dots \nu_\ell}(x)$ is a conserved current:[8]

$$\partial^{\nu_1} \mathcal{O}_{\nu_1 \dots \nu_\ell}(x) = 0\,. \tag{1.11}$$

Finally, the *trivial multiplet* consists of just the identity operator. This is a scalar with $\Delta = 0$, and is annihilated by all P^μ.

[8]Fermionic operators behave analogously. Short spin-1/2 operators satisfy the massless Dirac equation while higher-spin fermions are conserved.

Conformal symmetry fixes the two and three points functions of scalar operators to take the form

$$\begin{aligned}\langle \mathcal{O}_a(\vec{x}_1)\mathcal{O}_b(\vec{x}_2)\rangle &= \frac{C_{ab}}{x_{12}^{2\Delta_a}}\,,\\ \langle \mathcal{O}_a(\vec{x}_1)\mathcal{O}_b(\vec{x}_2)\mathcal{O}_c(\vec{x}_3)\rangle &= \frac{\lambda_{abc}}{2}\frac{1}{x_{12}^{\Delta_a+\Delta_b-\Delta_c}x_{23}^{\Delta_b+\Delta_c-\Delta_a}x_{31}^{\Delta_c+\Delta_a-\Delta_b}}\,,\end{aligned} \tag{1.12}$$

where C_{ab} is non-zero only if $\Delta_a = \Delta_b$, and where we define $x_{ij} = |\vec{x}_i - \vec{x}_j|$. In a unitary theory we can always redefine our operators such that $C_{ab} = \delta_{ab}$, and so will always make this choice of normalization. Similar formulas hold for spinning operators, whose two-point functions we can always fix to take the form

$$\begin{aligned}\langle \mathcal{O}_a^{\mu_1\dots\mu_\ell}(\vec{x}_1)\mathcal{O}_b^{\nu_1\dots\nu_\ell}(\vec{x}_2)\rangle &= \delta_{ab}\left(\frac{I^{(\mu_1\dots\mu_n)(\nu_1\dots\nu_n)}(x_{12})}{x_{12}^{\Delta_a+\ell_a-2}} - \text{traces}\right)\,,\\ \text{where } I^{\mu_1\dots\mu_n\nu_1\dots\nu_n}(x_{12}) &= \left(\delta^{\mu_1\nu_1} - \frac{x_{12}^{\mu_1}x_{12}^{\nu_1}}{x_{12}^2}\right)\dots\left(\delta^{\mu_n\nu_n} - \frac{x_{12}^{\mu_n}x_{12}^{\nu_n}}{x_{12}^2}\right)\,,\end{aligned} \tag{1.13}$$

and where we use $A_{(\mu_1\dots\mu_n)}$ to denote the symmetrization of the tensor $A_{\mu_1\dots\mu_n}$. Three-point functions between spinning operators may in general include multiple independent conformal structures, but for the case of two scalars and a third spinning operator there is a unique structure:

$$\begin{aligned}\langle \mathcal{O}_a(\vec{x}_1)\mathcal{O}_b(\vec{x}_2)\mathcal{O}_c^{\mu_1\dots\mu_\ell}(\vec{x}_3)\rangle &= \sqrt{\frac{(1/2)_\ell}{2^{\ell+2}\ell!}}\frac{\lambda_{abc}}{x_{12}^{\Delta_a+\Delta_b-\Delta_c+\ell}x_{23}^{\Delta_b+\Delta_c-\Delta_1-\ell}x_{31}^{\Delta_c+\Delta_a-\Delta_b-\ell}}\\ &\times\left[\left(\frac{x_{13}^{\mu_1}}{x_{13}^2} - \frac{x_{23}^{\mu_1}}{x_{23}^2}\right)\dots\left(\frac{x_{13}^{\mu_\ell}}{x_{13}^2} - \frac{x_{23}^{\mu_\ell}}{x_{23}^2}\right) - \text{traces}\right]\end{aligned} \tag{1.14}$$

where the ℓ dependent factors out front are chosen to match with our conventions for conformal blocks, to be introduced shortly.

The operator product expansion (OPE) allows us to expand products of operators in terms of other operators in the theory

$$\mathcal{O}_a(\vec{x})\mathcal{O}_b(0) = \sum_c \lambda_{abc}C_{\Delta_c,\ell_c}(\vec{x},\partial_y)\mathcal{O}_c(\vec{y})\Big|_{\vec{y}=0}\,, \tag{1.15}$$

where the differential operator $C_{\Delta_c,\ell_c}(\vec{x},\partial_y)$ is fully fixed by conformal symmetry, and λ_{abc} are the coefficients which appear in (1.14). By repeatedly taking OPEs, all higher-point functions can be fixed from just the OPE coefficients, conformal dimensions, and spins of local operators. If, for instance, we take the OPE of the first two operators in a four-point function of identical scalars

$\phi(\vec{x})$, we find that

$$\langle \phi(\vec{x}_1)\phi(\vec{x}_2)\phi(\vec{x}_3)\phi(\vec{x}_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}\in\phi\times\phi} \lambda^2_{\phi\phi\mathcal{O}} g_{\Delta_\mathcal{O},\ell_\mathcal{O}}(U,V) \tag{1.16}$$

where U and V are the conformal cross-ratios

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,, \tag{1.17}$$

and where $g_{\Delta,\ell}(U,V)$ are the conformal blocks, normalized such that

$$g_{\Delta,\ell}(U,V) = \frac{\Gamma(\ell+1/2)}{4^\Delta\sqrt{\pi}\ell!} U^{\frac{\Delta-\ell}{2}} \left((1-V)^\ell + O\left((1-V)^{\ell+1}\right)\right) + O\left(U^{\frac{\Delta-\ell}{2}+1}\right)\,. \tag{1.18}$$

At various points throughout the thesis we will need to decompose four-point functions into sums of conformal blocks. The expression (1.18) is the leading term in an expansion about $U \sim 0$ and $V \sim 1$, explicit expressions for which can be found for instance in [28]. As we can see from (1.18), the expansion organizes around the twist $\Delta - \ell$ and spin ℓ of the operators exchanged.

If we work in Euclidean signature, the order of the $\phi(x_i)$s in (1.16) does not matter. We can hence interchange $1 \leftrightarrow 3$, and so derive the crossing equation

$$\sum_{\mathcal{O}\in\phi\times\phi} \lambda^2_{\phi\phi\mathcal{O}} g_{\Delta_\mathcal{O},\ell_\mathcal{O}}(U,V) = \left(\frac{U}{V}\right)^{\Delta_\phi} \sum_{\mathcal{O}\in\phi\times\phi} \lambda^2_{\phi\phi\mathcal{O}} g_{\Delta_\mathcal{O},\ell_\mathcal{O}}(V,U)\,. \tag{1.19}$$

This gives an infinite number of constraints on conformal dimensions, spins, and OPE coefficients which may appear in $\phi\times\phi$. More general crossing equations can be derived when the operators are not identical, and not necessarily scalar. Crossing equations are in general quite difficult to analyze, as the $U \sim 0$, $V \sim 1$ expansion cannot be applied to both sides of (1.19) simultaneously. In spite of this difficulty, they provide one of the most fundamental tools used to study conformal field theories, both analytically and numerically.

1.1.2 $\mathcal{N} = 6$ Superconformal Symmetry

Before introducing the superconformal algebra, let us first recall the basic properties of spinors in three dimensions. They transform in a two dimensional representation of the Lorentz group, which, due to the accidental isomorphism $\mathfrak{so}(3) \approx \mathfrak{sp}(2)$, we can think of as an $\mathfrak{sp}(2)$ fundamental. We will use indices $\alpha\,, \beta\,, \ldots$ to describe such spinors, which we raise and lower with the antisymmetric symbol

$$\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1\,, \tag{1.20}$$

so that for any spinor s^α we define $s_\alpha \equiv \epsilon_{\alpha\beta} s^\beta$, which in turn implies that $s^\alpha = \epsilon^{\alpha\beta} s_\beta$. Spinor and vector representations of $\mathfrak{so}(3)$ are related through the 3d gamma matrices:

$$(\gamma^\mu)_{\alpha\beta} \equiv (\mathbb{1}, \sigma^1, \sigma^3)\,, \tag{1.21}$$

which we can use to rewrite the conformal generators as

$$P_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta} P_\mu\,, \qquad K_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta} K_\mu\,, \qquad M_{\alpha\beta} = \gamma^\mu_{\alpha\gamma} \gamma^\nu_{\beta\delta} \epsilon^{\gamma\delta} M_{\mu\nu}\,. \tag{1.22}$$

The commutator relations (1.5) can then be rewritten as

$$\begin{aligned}
&[M_\alpha{}^\beta, P_{\gamma\delta}] = \delta_\gamma^\beta P_{\alpha\delta} + \delta_\delta^\beta P_{\alpha\gamma} - \delta_\alpha^\beta P_{\gamma\delta}\,, && [D, P_{\alpha\beta}] = P_{\alpha\beta}\,, \\
&[M_\alpha{}^\beta, K_{\gamma\delta}] = \delta_\gamma^\beta K_{\alpha\delta} + \delta_\delta^\beta K_{\alpha\gamma} - \delta_\alpha^\beta K_{\gamma\delta}\,, && [D, K_{\alpha\beta}] = -K_{\alpha\beta}\,, \\
&[M_\alpha{}^\beta, M_\gamma{}^\delta] = -\delta_\alpha^\delta M_\gamma{}^\beta + \delta_\gamma^\beta M_\alpha{}^\delta\,, && [K^{\alpha\beta}, P_{\gamma\delta}] = 4\delta_{(\gamma}^{(\alpha} M_{\delta)}{}^{\beta)} + 4\delta_{(\gamma}^{\alpha}\delta_{\delta)}^{\beta} D\,.
\end{aligned} \tag{1.23}$$

To embed the conformal algebra into a large superconformal algebra, we first note that $\mathfrak{so}(3,2) \approx \mathfrak{sp}(4)$. The conformal algebra is thus a subalgebra of the $\mathfrak{osp}(\mathcal{N}|4)$ superalgebra, which has a maximal bosonic subalgebra $\mathfrak{sp}(4) \oplus \mathfrak{so}(\mathcal{N})_R$ — this turns out to be the only way to extend the 3d conformal algebra with supersymmetric generators.[9] The $\mathfrak{so}(\mathcal{N})_R$ subalgebra generate a global $SO(\mathcal{N})$ symmetry, known as the R-symmetry. We will use indices $I\,, J\,, \ldots$ to denote $SO(\mathcal{N})$ vectors. The $\mathfrak{so}(\mathcal{N})_R$ generators are antisymmetric tensors R_{IJ} which satisfy the commutation relation

$$[R_{IJ}, R_{KL}] = i\left(\delta_{I(L} R_{K)J} - \delta_{J(L} R_{K)I}\right)\,. \tag{1.24}$$

Along with the bosonic $\mathfrak{sp}(4) \oplus \mathfrak{so}(\mathcal{N})$ generators, $\mathfrak{osp}(\mathcal{N}|4)$ also includes $2\mathcal{N}$ fermionic generators $Q_{\alpha I}$ and $S_{\alpha I}$ with conformal dimension $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively, so that

$$[D, Q_{\alpha I}] = \frac{1}{2} Q_{\alpha I}\,, \qquad [D, S_{\alpha I}] = -\frac{1}{2} S_{\alpha I}\,. \tag{1.25}$$

The Q and S operators anticommute to generate all of the bosonic operators via the anticommutation

[9] Unlike supersymmetric extensions of the Poincaré algebra, superconformal algebras are highly constrained. A complete classification for $d \geq 3$ is given in [29, 30], and, in particular, superconformal algebras can exist only when $d \leq 6$.

Irrep	Dynkin Label	Common Name
$\mathbf{1}$	[000]	Trivial
$\mathbf{6}$	[100]	Vector
$\mathbf{20'}$	[200]	Symmetric
$\mathbf{15}$	[011]	Adjoint
$\mathbf{84}$	[022]	

Irrep	Dynkin Label	Common Name
$\mathbf{4}$	[010]	Spinor
$\bar{\mathbf{4}}$	[001]	Spinor
$\mathbf{10}$	[020]	
$\overline{\mathbf{10}}$	[002]	
$\mathbf{45}$	[120]	
$\overline{\mathbf{45}}$	[102]	

Table 1.1: Representations of $\mathfrak{so}(6)$ used in this thesis. The third column gives common names for the representations, where they exist. Note that $\mathfrak{so}(6) \approx \mathfrak{su}(4)$, but different conventions are used for the Dynkin labels. In the $\mathfrak{su}(4)$ notation, the $[a_1 a_2 a_3]$ representation is instead written as $[a_2 a_1 a_3]$.

relations

$$\begin{aligned} &\{Q_{\alpha I}, Q_{\beta J}\} = 2\delta_{IJ} P_{\alpha\beta}\,, \qquad \{S_{\alpha I}, S_{\beta J}\} = -2\delta_{IJ} K_{\alpha\beta}\,, \\ &\{Q_{\alpha I}, S_{\beta J}\} = 2i\delta_{IJ}\left(M_{\alpha\beta} - \epsilon_{\alpha\beta} D\right) - 2\epsilon_{\alpha\beta} R_{IJ}\,. \end{aligned} \tag{1.26}$$

The rest of the commutators in $\mathfrak{osp}(\mathcal{N}|4)$ then follow from $\mathfrak{sp}(4) \oplus \mathfrak{so}(\mathcal{N})_R$ invariance:

$$\begin{aligned} &[K^{\alpha\beta}, Q_{\gamma I}] = -i\left(\delta_\gamma{}^\alpha S^\beta{}_I + \delta_\gamma{}^\beta S^\alpha{}_I\right)\,, && [P_{\alpha\beta}, S^\gamma{}_I] = -i\left(\delta_\alpha{}^\gamma Q_{\beta I} + \delta_\beta{}^\gamma Q_{\alpha I}\right)\,, \\ &[M_\alpha{}^\beta, Q_{\gamma I}] = \delta_\gamma{}^\beta Q_{\alpha I} - \frac{1}{2}\delta_\alpha{}^\beta Q_{\gamma I}\,, && [M_\alpha{}^\beta, S^\gamma{}_I] = -\delta_\alpha{}^\gamma S^\beta{}_I + \frac{1}{2}\delta_\alpha{}^\beta S^\gamma{}_I\,, \\ &[R_{IJ}, Q_{\alpha K}] = i\left(\delta_{IK} Q_{\alpha J} - \delta_{JK} Q_{\alpha I}\right)\,, && [R_{IJ}, S^\alpha{}_K] = i\left(\delta_{IK} S^\alpha{}_J - \delta_{JK} S^\alpha{}_I\right)\,. \end{aligned} \tag{1.27}$$

Much like how P^μ and K^μ raise and lower conformal dimensions by 1, $Q_{\alpha I}$ and $S_{\alpha I}$ raise and lower conformal dimensions by $\frac{1}{2}$. We define a superconformal primary to be a conformal primary annihilated by $S_{\alpha I}$. A superconformal multiplet consists of a superconformal primary, a finite number of other conformal primaries constructed from the superconformal primary using $Q_{\alpha I}$, and all of the conformal descendants of these conformal primaries.

Our focus in this thesis is on $\mathcal{N} = 6$ superconformal symmetry, which as we have already noted is the next-to-maximal amount of supersymmetry possible for an interacting superconformal field theory in three dimensions. The R-symmetry group is $\mathfrak{so}(6)$, We can label $\mathfrak{so}(6)$ irreps by their Dynkin labels $[a_1 a_2 a_3]$, and in Table 1.1 we list the $\mathfrak{so}(6)$ irreps which appear in this thesis. Note that $\mathfrak{so}(6) \approx \mathfrak{su}(4)$, where the spinorial representation $\mathbf{4}$ of $\mathfrak{so}(6)$ is the fundamental representation of $\mathfrak{su}(4)$.

Unitary $\mathfrak{osp}(6|4)$ multiplets are classified in [13,31]. Each multiplet can be labeled by the conformal dimension Δ, spin ℓ, and $\mathfrak{so}(6)$ R-symmetry irrep $\mathbf{r} = [a_1 a_2 a_3]$ of its superconformal primary. Unitary multiplets of $\mathfrak{osp}(6|4)$ fall into three possible classes, depending on their conformal dimension. Long multiplets have conformal dimension above the unitarity bound

$$\Delta > \ell + a_1 + \frac{1}{2}(a_2 + a_3) + 1 \tag{1.28}$$

and do not satisfy any shortening conditions. Semishort, or A-type, multiplets occur at the bottom of the continuum in (1.28)

$$\Delta = \ell + a_1 + \frac{1}{2}(a_2 + a_3) + 1 \tag{1.29}$$

and satisfy shortening conditions. Finally, if $\ell = 0$ we can also have short, or B-type, multiplets with dimension

$$\Delta = a_1 + \frac{1}{2}(a_2 + a_3)\,, \tag{1.30}$$

below the end of the lower continuum in (1.28), also obeying shortening conditions. Note that the division of $\mathcal{N} = 6$ multiplets into long, A-type, and B-type multiplets is analogous to the division of non-supersymmetric conformal multiplets into long, short and trivial multiplets described in the previous section. However, while highly constrained by supersymmetry, B-type multiplets still have non-trivial correlation functions.

Multiplets can furthermore be distinguished by their BPSness, which counts the number of $Q^{\alpha I}$ operators which annihilate the superconformal primary. For generic representations A-type multiplets are 1/12-BPS and B-type multiplets are 1/6-BPS, but for specific R-symmetry representations the multiplets may be higher BPS. We list all possible multiplets in Table 1.2. When describing multiplets we will often find it useful to use the notation $\mathcal{T}_{\Delta,\ell}^{[a_1a_2a_3]}$ to refer to the supermultiplet of type $\mathcal{T}$, whose superconformal primary has spin ℓ, conformal dimension Δ, and transforms in the $[a_1a_2a_3]$ under $\mathfrak{so}(6)_R$.

1.1.3 The $\mathcal{N} = 6$ Stress Tensor Multiplet

All local quantum field theories contain a stress tensor $T^{\mu\nu}(\vec{x})$ which acts as the generator of translations. In a conformal field theory the stress tensor is traceless and has conformal dimension 3. Conformal invariance fixes its two-point function to be

$$\begin{aligned}
\langle T^{\mu_1\mu_2}(\vec{x}_1)T^{\nu_1\nu_2}(\vec{x}_2)\rangle &= \left(\frac{I^{(\mu_1\mu_2)(\nu_1\nu_2)}(x_{12})}{x_{12}^{2\ell-1}} - \text{traces}\right)\,, \\
\text{where } I^{\mu_1\mu_2\nu_1\nu_2}(x_{12}) &= \left(\delta^{\mu_1\nu_1} - \frac{x_{12}^{\mu_1}x_{12}^{\nu_1}}{x_{12}^2}\right)\left(\delta^{\mu_2\nu_2} - \frac{x_{12}^{\mu_2}x_{12}^{\nu_2}}{x_{12}^2}\right)\,.
\end{aligned} \tag{1.31}$$

Type	Δ	Spin	Multiplet	$\mathfrak{so}(6)_R$	BPS
Long	$> \Delta_B + \ell + 1$	ℓ	Long	$[a_1 a_2 a_3]$	0
A	$\Delta_B + \ell + 1$	ℓ	$(A, 1)$	$[a_1 a_2 a_3]$	1/12
			$(A, 2)$	$[0 a_2 a_3]$	1/6
			$(A, +)$	$[0 a_2 0]$	1/4
			$(A, -)$	$[00 a_3]$	1/4
			$(A, \text{cons.})$	$[000]$	1/3
B	Δ_B	0	$(B, 1)$	$[a_1 a_2 a_3]$	1/6
			$(B, 2)$	$[0 a_2 a_3]$	1/3
			$(B, +)$	$[0 a_2 0]$	1/2
			$(B, -)$	$[00 a_3]$	1/2
			Trivial	$[000]$	1

Table 1.2: Multiplets of $\mathfrak{osp}(6|4)$ and the quantum numbers of their superconformal primary, where $\Delta_B = a_1 + \frac{1}{2}(a_2 + a_3)$.

With our choice of normalization, the stress tensor satisfies the Ward identity [32]

$$4\pi\sqrt{\frac{c_T}{3}} \int d^3x \, \langle \nabla^\mu T_{\mu\nu}(\vec{x}) \mathcal{O}_1(\vec{y}_1) \ldots \mathcal{O}_n(\vec{y}_n) \rangle = -\sum_i \langle \mathcal{O}_1(\vec{y}_1) \ldots \partial_\nu \mathcal{O}_i(\vec{y}_i) \ldots \mathcal{O}_n(\vec{y}_n) \rangle \tag{1.32}$$

for any arbitrary string of operators $\mathcal{O}_i(\vec{y}_i)$. Note that we have chosen to normalize the stress tensor using the same conventions used in (1.13) for more general spinning operators, which is not the canonical stress tensor normalization.[10] The value of c_T depends on the specific CFT in question, and is a measure of the number of degrees of freedom in the theory. For a free scalar or free Majorana fermion, $c_T = 1$.

In a superconformal field theory, the stress tensor belongs to a larger supermultiplet, known as the stress tensor multiplet. For an $\mathcal{N} = 6$ superconformal theory, the stress tensor forms part of a $(B, 2)^{[011]}_{1,0}$ multiplet. The superconformal primary for this multiplet is a scalar $S^a{}_b(\vec{x})$ with dimension 1. This operator transforms in the adjoint of $\mathfrak{so}(6)_R$, where we use indices $a, b = 1, \ldots, 4$ to denote $\mathfrak{su}(4) \approx \mathfrak{so}(6)$ fundamental (lower) and anti-fundamental (upper) indices. There are then three fermions with dimensions $\frac{3}{2}$, χ_α, F_α and $\overline{F}_\alpha$, which transform in the **6**, **10**, and $\overline{\mathbf{10}}$ of $\mathfrak{so}(6)_R$ respectively. Next we have an additional scalar $P^a{}_b(\vec{x})$, along with the R-symmetry current $(J^\mu)^a{}_b(\vec{x}, X)$ and a $U(1)$ flavor current $j^\mu(\vec{x})$, each of which have dimension 2. Completing the multiplet is the supercurrent $\psi^{\mu\alpha}(\vec{x})$ with dimension $\frac{5}{2}$, and finally the stress tensor itself, $T^{\mu\nu}(\vec{x})$.

[10] The canonical stress tensor is defined by

$$T^{\mu\nu}_{\text{canonical}}(x) = 4\pi\sqrt{\frac{c_T}{3}} T^{\mu\nu}(x)$$

in order to cancel the factors on the left-hand side of (1.32).

Operator	Δ	Spin	$\mathfrak{so}(6)_R$ irrep
S	1	0	$\mathbf{15} = [011]$
χ	3/2	1/2	$\mathbf{6} = [100]$
F	3/2	1/2	$\mathbf{10} = [020]$
$\bar{F}$	3/2	1/2	$\overline{\mathbf{10}} = [002]$
P	2	0	$\mathbf{15} = [011]$
J	2	1	$\mathbf{15} = [011]$
j	2	1	$\mathbf{1} = [000]$
ψ	5/2	3/2	$\mathbf{6} = [100]$
T	3	2	$\mathbf{1} = [000]$

Table 1.3: The conformal primary operators in the $\mathcal{N} = 6$ stress tensor multiplet. For each such operator, we list the scaling dimension, spin, and $\mathfrak{so}(6)_R$ representation.

We list all of these operators in Table 1.3.

1.2 Known $\mathcal{N} = 6$ Superconformal Field Theories

The simplest $\mathcal{N} = 6$ superconformal field theory, and indeed, the only one for which we can compute correlators exactly, is free field theory. This theory consists of four free complex scalars Φ^a with conformal dimension $\frac{1}{2}$, and four free complex fermions Ψ_a^α with conformal dimension 1, along with their complex conjugates. These scalars and fermions together transform in the $(B,+)^{[010]}_{\frac{1}{2},0}$ and $(B,-)^{[001]}_{\frac{1}{2},0}$ supermultiplets, known as hypermultiplets. We compute all correlation functions via Wick contractions with the two-point functions

$$\langle \Phi^a(\vec{x}_1)\bar{\Phi}_b(\vec{x}_2)\rangle = \frac{\delta^a{}_b}{4\pi x_{12}}\,, \qquad \langle \Psi_a^\alpha(\vec{x}_1)\bar{\Psi}^{\beta b}(\vec{x}_2)\rangle = \frac{i\delta_a{}^b(\gamma_\mu)^{\alpha\beta}x_{12}^\mu}{4\pi x_{12}^2}\,. \tag{1.33}$$

Because the free theory has 8 real scalars and 8 real fermions, it has $c_T = 16$, which is the lowest of all known $\mathcal{N} = 6$ theories. We can also consider a more general free field theory with N hypermultiplets $(\Phi_i^a, \Psi_{ia}^\alpha)$, which has $c_T = 16N$.

Let us now consider interacting theories. All known $\mathcal{N} = 6$ theories are superconformal Chern-Simons theories [33] with additional massless matter multiplets, and so we will begin with a lightning review of these theories more generally.

Let us fix a simple Lie group G, and let A_μ be a gauge field transforming in the adjoint of G. The Chern-Simons action is then

$$S_{\text{CS}} = \frac{k}{4\pi}\int \text{Tr}\left(A\wedge dA + \frac{2}{3}A^3\right)\,, \tag{1.34}$$

where the Chern-Simons level k is quantized to ensure invariance under large gauge transformations. Note that the Chern-Simons action is independent of the metric, and so the theory is topological.

To construct a dynamical $\mathcal{N}=2$ superconformal theory, we couple the gauge field to complex scalars ϕ_i and complex fermions ψ_i transforming in a (generally reducible) representation $\mathbf{r}$ of the gauge group G. Supersymmetry requires that both ϕ and ψ transform in the same representation of G. Suppressing the $\mathbf{r}$ indices, we can write the $\mathcal{N}=2$ superconformal Lagrangian [34–36]

$$\begin{aligned} S_{\mathcal{N}=2} = S_{\rm CS} + \int d^3x \bigg(D_\mu\bar{\phi}D^\mu\phi + i\bar{\psi}\gamma^\mu D_\mu\psi - \frac{16\pi^2}{k^2}(\bar{\phi}\tau^a_{\mathbf{r}}\phi)(\bar{\phi}\tau^b_{\mathbf{r}}\phi)(\bar{\phi}\tau^a_{\mathbf{r}}\tau^b_{\mathbf{r}}\phi) \\ - \frac{4\pi}{k}(\bar{\phi}\tau^a_{\mathbf{r}}\phi)(\bar{\psi}\tau^a_{\mathbf{r}}\psi) - \frac{8\pi}{k}(\bar{\psi}\tau^a_{\mathbf{r}}\phi)(\bar{\phi}\tau^a_{\mathbf{r}}\psi)\bigg)\,, \end{aligned} \tag{1.35}$$

where $\tau^a_{\mathbf{r}}$ are the generators of G acting on the representation $\mathbf{r}$, with adjoint index a, and where $D^\mu \equiv \partial^\mu + \tau^a_{\mathbf{r}}A^\mu_a$ is the usual covariant derivative. Due to the integrality of k, it was argued in [36] that the Lagrangian (1.35) cannot be renormalized except for a possible one-loop shift of k, and so superconformal invariance is maintained in the full quantum theory.

More general $\mathcal{N}=2$ Chern-Simons matter theories can be constructed for any semisimple gauge group $(G_1)_{k_1}\times\cdots\times(G_n)_{k_n}$, where each G_i is a simple Lie group (or $U(1)$) with Chern-Simon level k_i. Once again the matter content consists of complex scalars and fermions transforming in some general representation of the gauge group. $\mathcal{N}=2$ theories also allow for superpotential terms built from the hypermultiplets, although these interactions preserve superconformal invariance if and only if they are both classically marginal and do not break any flavor symmetries [37, 38].

We can construct general $\mathcal{N}=3$ Chern-Simons matter theories as a special case of $\mathcal{N}=2$ theories [39–41]. We must simply pair the matter fields ϕ_i and ψ_i transforming in $\mathbf{r}$ with fields $\tilde{\phi}^i$ and $\tilde{\psi}^i$ transforming in $\bar{\mathbf{r}}$. The Lagrangian is then the same as the $\mathcal{N}=2$ Lagrangian (1.35) (including terms for the $\tilde{\phi}$ and $\tilde{\psi}$ fields), but with an additional interaction which can be compactly written in superspace as a superpotential

$$W_{\mathcal{N}=3\text{-coupling}} = \frac{4\pi}{k}(\tilde{\Phi}\tau^a\Phi)(\tilde{\Phi}\tau^a\Phi)\,. \tag{1.36}$$

Constructing theories with $\mathcal{N}\geq 4$ is more challenging and is possible only for very specific gauge groups, Chern-Simons levels, and matter representations. Theories with manifest $\mathcal{N}=6$ symmetry were first constructed in [16, 17], following previous work on $\mathcal{N}=8$ theories [42–44]. The most general such $\mathcal{N}=6$ theories were classified in [19], up to discrete quotients that do not affect correlators of stress tensor multiplet operators.[11] In $\mathcal{N}=3$ SUSY notation, they are Chern-Simons-matter theories with two matter hypermultiplets. There are two possible families of gauge groups and representations:[12]

$$SU(N)_k \times SU(N+M)_{-k} \times U(1)^L_K \,, \qquad K^{ab} q_a q_b = \frac{1}{k}\left(\frac{1}{M+N} - \frac{1}{N}\right) \,, \tag{1.37}$$

for $N, M \geq 1$ where the hypermultiplets are in the bifundamental[13] of $SU(M) \times SU(N)$, and

$$USp(2+2M)_k \times U(1)^L_K \,, \qquad K^{ab} q_a q_b = -\frac{1}{2k} \,, \tag{1.38}$$

for $M \geq 0$ where the hypermultiplets are in the fundamental of $USp(2+2M)$. In both cases, the hypermultiplets have equal and opposite charges q_i for $i = 1, \ldots, L$ under the $U(1)$'s. The matrix K^{ab} is the inverse of the matrix K_{ab} of Chern-Simons levels for the L $U(1)$ gauge groups, and must satisfy the relations given in (1.37) and (1.38).

As we show in Section 3.4, the S^3 partition function for both families of theories is independent of L, as long as the conditions in (1.37) and (1.38) are obeyed, up to an overall normalization constant. This leads us to conjecture that all these theories have the same stress tensor multiplet correlators, so for this sector we only need consider two families of theories. One is the ABJ(M) family[14]

$$\text{ABJ family:} \qquad U(N)_k \times U(N+M)_{-k} \,, \tag{1.39}$$

with $M \leq |k|$ [16, 17], which is the special case of (1.37) where $L = 2$ with $q_1 = q_2 = 1$ and $K_{11} = kN$, $K_{22} = -k(N+M)$, and $K_{12} = 0$. The other family is what we will dub the "OSp" family,

$$\text{OSp family:} \qquad SO(2)_{2k} \times USp(2+2M)_{-k} \,, \tag{1.40}$$

with $M+1 \leq |k|$ [17,18], is the $L = 1$, $q = 1$ case of (1.38). Sending $k \to -k$ gives a parity-conjugate theory, so without loss of generality we can focus on $k > 0$.

Various Seiberg dualities are believed to relate $\mathcal{N}=2$ superconformal matter theories to each

[11] See [45] for a conjectured classification that takes into account discrete quotients.

[12] The case $SU(N)_k \times SU(N)_{-k}$ describes the BLG theories [43, 44, 46].

[13] Note that when $N = 1$ in (1.37), the hypermultiplets are just in the fundamental of $SU(1+M)$ with appropriate charges under the $U(1)$'s.

[14] The special case of ABJ with $M = 0$ is known as ABJM theory. When $N = 1$, $M = 0$, the ABJM theory describes free field theory, when $M = 0$ and $N > 1$, ABJM flows to the product of a free SCFT and a strongly-coupled SCFT, while for all other parameters ABJM theory has a unique stress tensor.

other [47–50]. These strong-weak dualities exchange regions of strong and weak Chern-Simons coupling, and are analogous to Seiberg duality in $\mathcal{N}=1$ 4d theories [51]. Although conjectural, the proposed dualities have been extensively tested using supersymmetric localization [52–55]. When applied to $\mathcal{N}=6$ theories, Seiberg duality imposes the equivalences

$$\begin{aligned} U(N)_k \times U(N+M)_{-k} \quad &\longleftrightarrow \quad U(N)_{-k} \times U(N+|k|-M)_k \,, \\ SO(2)_{2k} \times USp(2+2M)_{-k} \quad &\longleftrightarrow \quad SO(2)_{-2k} \times USp(2(|k|-M-1)+2)_k \,. \end{aligned} \tag{1.41}$$

The $k=2M$ ABJ theories are self-dual, with the duality transformation acting as a parity symmetry. The OSp theories with $k=2M+1$ are likewise self-dual and parity preserving.

Both families of $\mathcal{N}=6$ theories become tractable in the semiclassical regime, where $k\to\infty$ while M and N are held fixed. In this limit, the gauge couplings become weak and both families approach free field theory, with $N(N+M)$ free hypermultiplets for the ABJ family and $2M+2$ free hypermultiplets for the OSp family. In particular,

$$\begin{aligned} \text{ABJ Family}: \quad & c_T = 16N(N+M) + O(k^{-2}) \,, \\ \text{OSp Family}: \quad & c_T = 16(2M+2) + O(k^{-2}) \,, \end{aligned} \tag{1.42}$$

so that c_T becomes large as we take either M or N to infinity.

We close this section by noting that certain $\mathcal{N}=6$ theories have enhanced $\mathcal{N}=8$ supersymmetry. These include free field theory itself, as the $\mathcal{N}=6$ hypermultiplet is identical in field content with an $\mathcal{N}=8$ hypermultiplet: they both consist of eight real scalars and eight Majorana fermions. Apart from free field theory, we also have (up to Seiberg duality) the interacting families

$$\begin{aligned} \text{BLG:} \quad & SU(2)_k \times SU(2)_{-k} \\ \text{ABJM:} \quad & U(N)_1 \times U(N)_{-1} \text{ and } U(N)_2 \times U(N)_{-2} \\ \text{ABJ:} \quad & U(N)_2 \times U(N+1)_{-2} \end{aligned} \tag{1.43}$$

each of which are special cases of the ABJ family (1.37). Of these theories, the BLG family, introduced in [42, 44], are manifestly invariant under $\mathcal{N}=8$ for any integer Chern-Simons level k, while for the other families, supersymmetry is enhanced from $\mathcal{N}=6$ to $\mathcal{N}=8$ due to monopole operators.

1.3 ABJ Triality

We now review the conjectured ABJ triality relating the large M and large N limits of $U(N)_k \times U(N+M)_{-k}$ ABJ theory to quantum gravity on AdS_4. We will begin with a quick review of AdS/CFT duality, and will then describe the various holographic regimes of ABJ theory.

1.3.1 The AdS/CFT Duality

AdS/CFT duality refers to a number of interconnected dualities between theories of quantum gravity in Anti-de Sitter space (AdS) and conformal field theories living on the boundary of AdS. The first proposed AdS/CFT duality was between $\mathcal{N}=4$ super-Yang Mills in 4d and IIB string theory on AdS_5 [20, 56, 57]. Many more examples are now known, and are reviewed for instance in [58, 59]. Reviews of AdS/CFT focusing on more general aspects of the duality, rather than on specific stringy instantiations, can be found in [60–62].

We can describe AdS_4 with radius L as a hyperboloid in $\mathbb{R}^{3,2}$, that is, as the set of five-dimensional vectors X^A which satisfy the condition

$$\eta_{AB} X^A X^B = -L^2\,, \text{ where } \eta_{AB} = \mathrm{diag}\{-1,1,1,1,-1\}\,. \tag{1.44}$$

The η_{AB} bilinear is preserved by the group $SO(3,2)$, and so the induced metric on (1.44) is automatically invariant under $SO(3,2)$ isometries. AdS_4 is commonly parametrized with the Poincaré patch coordinates

$$X^I(\vec{x}, z) = \frac{L}{z}\left(\vec{x}, \frac{1-\vec{x}^2-z^2}{2}, \frac{1+\vec{x}^2+z^2}{2}\right)\,, \tag{1.45}$$

where $\vec{x}$ is a 3-vector and $z>0$ is the radial coordinate which vanishes at the boundary of AdS_4. As we take $z\to 0$ we limit to a null vector

$$P^I(\vec{x}) = \lim_{z\to 0} z X^I = \left(\vec{x}, \frac{1-\vec{x}^2}{2}, \frac{1+\vec{x}^2}{2}\right)\,. \tag{1.46}$$

The space of vectors $P^I(\vec{x})$ is isometric to $\mathbb{R}^{2,1}$. Note however that the flat metric is an artifact of our choice of coordinates (1.45). If we were to redefine the radial coordinate

$$z \to \Omega(\vec{x}) z \tag{1.47}$$

for some scalar function $\Omega(\vec{x})$, then we would instead limit to a null vector $\Omega(\vec{x})P^I(\vec{x})$. Under this shift, the boundary metric undergoes a Weyl transformation. In particular, the $SO(3,2)$ isometries induce conformal transformations on the boundary, as they relate the boundary metric to itself only up to Weyl transformations.

The AdS/CFT dictionary relates theories of quantum gravity on AdS with conformal field theories on the boundary of AdS, and, more precisely, states that the two theories have equal partition function. To compute the bulk path-integral we must specify boundary conditions for the fields, so that $\phi(\vec{x},z) \to \phi_{\text{bound}}(\vec{x})$ as we take $z \to 0$. We then define

$$Z_{\text{AdS}}[\phi_{\text{bound}}] = \int_{\phi_{\text{bound}}} D\phi \, \exp\left(iS_{\text{AdS}}[\phi]\right) \,, \tag{1.48}$$

so that the partition function is a functional of these boundary field configurations. On the CFT side, we define the partition function as

$$Z_{\text{CFT}}[\phi_{\text{bound}}] = \int_{\text{CFT}} D\chi \, \exp\left(iS_{\text{CFT}}[\chi] - i\int d^3x \, \phi_{\text{bound}}(\vec{x})\mathcal{O}(\vec{x})\right) \,, \tag{1.49}$$

where $\mathcal{O}(\vec{x})$ is some operator on the boundary and $\phi_{\text{bound}}(\vec{x})$ acts a source for the operator. The AdS/CFT duality then states that these two partition functions are equal:

$$Z_{\text{AdS}}[\phi_{\text{bound}}] = Z_{\text{CFT}}[\phi_{\text{bound}}] \,. \tag{1.50}$$

In particular, we can compute local correlation functions on the boundary by taking functional derivatives of the partition function.

When the bulk theory is free, the boundary is described by generalized free field theory. The spin and mass of the bulk fields determining the spin and conformal dimension of their boundary duals. For example, a scalar field $\phi(\vec{x},z)$ of mass M in the bulk is dual to a scalar operator $\mathcal{O}(\vec{x})$ on the boundary, with conformal dimension Δ satisfying

$$M^2L^2 = \Delta(\Delta - 3) \,. \tag{1.51}$$

We can identify the Hilbert space of single particles states in the bulk with "single-trace" states[15] $\mathcal{O}(\vec{x})\,|0\rangle$ on the boundary, while n-particle states are dual to "n-trace" states $\mathcal{O}(\vec{x}_1)\ldots\mathcal{O}(\vec{x}_n)\,|0\rangle$. Higher-spin fields work in much the same way, with spin ℓ bulk particles dual to spin ℓ boundary

[15]The terminology "single-trace" and "multi-trace" originates from the Lagrangian description of boundary CFTs with matrix-like large N limits, where it refers to the number of traces needed to define the operators. But since the bulk dual of n-trace operators are n-particle states, it has become standard to refer to n-particle states as "n-trace" even in theories without matrix-like large N limits.

operators. In particular, a massless spin-ℓ particle, the boundary dual is a conserved current with ℓ and conformal dimension $\Delta = \ell + 1$. The bulk $\mathcal{N} = 6$ supergraviton multiplet is dual to the $\mathcal{N} = 6$ stress tensor multiplet in the boundary, where the gravitons themselves are dual to the stress tensors.

Bulk gravitational theories are tractable in the semiclassical limit, where the bulk Newton constant G_N is much smaller than the squared radius of AdS L^2. In this regime, we can use Witten diagrams (which are essentially Feynman diagrams in AdS, supplemented by a special 'bulk-boundary' propagator) to systematically compute boundary correlators in an G_N/L^2 expansion. In particular, in the semiclassical regime c_T is given by [63]

$$c_T = \frac{32L^2}{8\pi G_N} + O(1) \gg 1\,. \tag{1.52}$$

We thus see that the known $\mathcal{N} = 6$ theories can be dual to semiclassical gravity only if at least one of M or N is large, so that the boundary theories contain a large number of fields.

1.3.2 Holographic Regimes of ABJ Theory

So far our discussion of AdS/CFT has been general. The most well understood examples of AdS/CFT duality, however, fall into one of two specific categories:

1. **Stringy duals:** In these examples, the bulk theory is either a string theory on $\text{AdS}_d \times \mathcal{M}_{10-d}$ or M-theory on $\text{AdS}_d \times \mathcal{M}_{11-d}$, where in either case $\mathcal{M}_D$ is a compact manifold. In the semiclassical regime only the massless string modes survive, and so the bulk is described by supergravity in either 10d or 11d, along with higher derivative corrections. The boundary theory is a supersymmetric gauge theory with a matrix-like large N limit, and all single trace operators have spin ≤ 2. The prototypical example is $\mathcal{N} = 4$ super-Yang-Mills in 4d, which is dual to IIB string theory on $\text{AdS}_5 \times S^5$ [20].

2. **Higher-spin duals:** In these examples, the bulk theory has an infinite number of higher-spin massless particles. The boundary theory is a Chern-Simons matter theory with a vector-like large N limit and an infinite number of higher-spin single trace operators. The prototypical example is the singlet sector of the $O(N)$ model in 3d, which has a massless higher-spin current for each even spin ℓ [64].

The $U(N)_k \times U(N+M)_{-k}$ ABJ theories are particularly interesting because they exhibit both kinds of holographic duals. When N is large, the theories have a matrix-like large N limit and stringy duals, while when M is large they have a vector-like large M limit and are dual to higher-spin gravity. This is known as ABJ triality [21], as the large N limit itself exhibits distinct IIA and M-theory limits.

Let us begin with the large N limit, where we hold M finite. In this limit, the $U(N)_k \times U(N+M)_{-k}$ ABJ theories can be interpreted as effective theories on N coincident M2-branes placed at a $\mathbb{C}^4/\mathbb{Z}_k$ singularity in the transverse directions, together with a discrete flux due to M fractional M2-branes localized at the singularity. We will begin with the $N \gg k^5$ regime, which is dual to weakly coupled M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. At low energies, M-theory can be described by 11d supergravity

$$S = \frac{1}{16\pi^5 \ell_{11}^9} \int d^{11}x \sqrt{-G} \left(R - \frac{1}{2}|F_4|^2 \right) - \frac{1}{192\pi^5 \ell_{11}^9} \int A_3 \wedge F_4 \wedge F_4 + \text{fermions} \tag{1.53}$$

where R is the Einstein-Hilbert term for the 11d metric G, A_3 is a 3-form with field strength $F_4 = dA_3$, and ℓ_{11} is the 11d Planck length. Eleven dimensional supergravity has an $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ solution:

$$\begin{aligned} ds^2 &= H^{-2/3} dx_\mu dx^\mu + H^{1/3} \left(dr^2 + r^2 ds^2_{S^7/\mathbb{Z}_k} \right) \\ F_4 &= dH^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \,, \\ \text{with} \qquad H &= \frac{(2L)^6}{r^6} \,. \end{aligned} \tag{1.54}$$

The boundary quantities c_T and k are related to the AdS radius L and the 11d Planck length ℓ_{11} via the AdS/CFT relation [16, 65]

$$\frac{L^9}{\ell_{11}^9} = \frac{3\pi k}{2^{11}} c_T + \dots \,, \tag{1.55}$$

where the additional terms are subleading at large c_T. When $k = 1, 2$ and $M = 0$, or $k = 2$ and $M = 1$, the theories have enhanced $\mathcal{N} = 8$ supersymmetry.

Next we consider the strong coupling 't Hooft limit of ABJ theory. Let us define the 't Hooft parameter

$$\lambda = \frac{\tilde{N}}{k} - \frac{1}{3k^2} - \frac{1}{24} \,, \qquad \text{where} \qquad \tilde{N} = N + \frac{M}{2} - \frac{M^2}{2k} \,. \tag{1.56}$$

Taking N to infinity while holding λ fixed, and then taking $\lambda \to \infty$, we find that ABJ theory is dual to weakly coupled type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ [16]. Unlike M-theory, IIA string theory has a dimensionless parameter, the string coupling g_s. When $g_s \ll 1$ the theory becomes weakly coupled and can be studied using string perturbation theory. The leading order AdS/CFT relations are [16, 65]

$$\frac{L^8}{\ell_s^8} = 4\pi^4\lambda^2 + \ldots\,, \qquad g_s^2 = \frac{512\lambda^2}{3c_T} + \ldots\,, \tag{1.57}$$

where both the string length scale ℓ_s/L and the string coupling g_s are small in this double expansion. The ellipses in (1.57) stand for terms that are suppressed at large c_T in both expressions.

To interpolate between the M-theory and IIA regimes, we studythe large N limit with

$$\mu \equiv \frac{\tilde{N}}{k^5}\,, \tag{1.58}$$

held fixed. Like the 't Hooft strong coupling limit, ABJ theory in this limit is dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$, except now the string coupling g_s is finite. The AdS/CFT relations are [16, 65]

$$\frac{L^8}{\ell_s^8} = \frac{3c_T\pi^5\sqrt{\mu}}{16\sqrt{2}} + \cdots\,, \qquad g_s^4 = 32\pi^2\mu + \cdots\,, \tag{1.59}$$

with corrections suppressed at large c_T. We can recover both the finite k and strong coupling 't Hooft limit expansions from the finite μ expansion by taking the $\mu \to \infty$ and $\mu \to 0$ limits respectively, as we explain at the end of Section 3.2.3.

The final regime we can study at large N is the weak 't Hooft coupling limit where $\lambda \ll 1$. In this regime, the boundary theory can be computed using a standard perturbative expansion, but the bulk theory is strongly coupled. To summarize, we have the following four distinct regimes at large N and finite M:

$$\begin{aligned} \text{M-theory:} &\qquad k \text{ finite}\,, \\ \text{IIA at Finite String Coupling:} &\qquad \mu = \frac{\tilde{N}}{k^5} \text{ finite}\,, \\ \text{IIA at Weak String Coupling:} &\qquad \lambda \gg 1\,, \\ \text{Semiclassical Boundary:} &\qquad \lambda \ll 1\,. \end{aligned}$$

We will further study the large N limit in Chapter 4.

Let us now turn to the large M limit of $U(N)_k \times U(N+M)_{-k}$ ABJ theory, with N held finite. Because the ABJ theories are only defined for $k > M$, as we take $M \to \infty$ we must simultaneously take $k \to \infty$ as well, and so introduce the parameter

$$\lambda_{\text{HS}} = \frac{M}{k} \in [0,1] \,. \tag{1.60}$$

Seiberg duality relates $\lambda_{\text{HS}} \leftrightarrow 1 - \lambda_{\text{HS}}$. In particular, when $\lambda_{\text{HS}} = \frac{1}{2}$ the theory is parity preserving, while $\lambda = 0$ describes the free field theory. The bulk dual of ABJ theory in this limit is an $\mathcal{N} = 6$ higher-spin theory, where λ_{HS} is dual to the bulk parity-breaking parameter.

We can also study the large M limit of the $SO(2)_{2k} \times USp(2+2M)_{-k}$ OSp theories. This time, the natural definition of λ_{HS} is

$$\lambda_{\text{HS}} = \frac{M+1/2}{k} \in [0,1] \,, \tag{1.61}$$

so that again Seiberg duality maps $\lambda_{\text{HS}} \leftrightarrow 1 - \lambda_{\text{HS}}$. In [18,66] it was argued that the OSp theories are related to the same $\mathcal{N} = 6$ theory of higher-spin gravity as the ABJ theories, but with an additional orientifold present. We will study all of these higher-spin theories further in Chapter 5.

For both the ABJ and OSp theories, we can smoothly take $\lambda_{\text{HS}} \to 0$, which corresponds to taking the semiclassical limit $k \to \infty$. As a result, we can derive the semiclassical expansion from the large M limit. This should be contrasted with the large N limit, in which the SUGRA regime and the semiclassical regime do not overlap.

Chapter 2

The $\langle SSSS \rangle$ Superconformal Block Expansion

In this chapter we derive the superconformal block expansion for the $\langle SSSS \rangle$ four-point correlator. We begin in Section 2.1 by deriving the constraints on $\langle SSSS \rangle$ imposed by superconformal invariance. Supersymmetry also also relates $\langle SSSS \rangle$ to four-point functions of other stress tensor multiplet operators, including both $\langle SSPP \rangle$ and $\langle PPPP \rangle$. In Section 2.2 we restrict the supermultiplets which can appear in the $S \times S$ OPE, and hence exchanged in $\langle SSSS \rangle$, to a small number of possibilities. We then derive the superconformal Casimir equation in Section 2.3, and use it to fix all superconformal blocks which may contribute to $\langle SSSS \rangle$. We close with a discussion of free field theory.

2.1 Superconformal Ward Identities

2.1.1 Conformal and R-symmetry Invariance

Our primary focus in thesis is the four-point functions of scalar operators in the stress tensor multiplet, and in particular, the four-point function $\langle SSSS \rangle$. To impose superconformal invariance on a correlator such as $\langle SSSS \rangle$, it is sufficient to impose conformal invariance, R-symmetry invariance, and invariance under the Poincaré supercharge $Q_{\alpha I}$; invariance under $S_{\alpha I}$ then follows automatically from the commutator relations (1.27). Conformal invariance is straightforward to impose. Embedding space formalisms such as [67, 68] provide a straightforward tool to construct all possible conformally covariant structures which can contribute to a given four-point function.

To describe $\mathfrak{so}(6) \approx \mathfrak{su}(4)$ representations, we use the vector $I\,, J\,, \ldots$ and spinorial $a\,, b\,, \ldots$ indices, as we did in the Introduction. The vector and spinor representations are related by the gamma matrices C^I_{ab} and $\overline{C}^{Iab}$, which convert antisymmetric tensors of the $\mathbf{4}$ and $\overline{\mathbf{4}}$ into the $\mathbf{6}$. A convenient basis for these matrices is:

$$\begin{aligned}
C_1 &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad & C_2 &= \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad & C_3 &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \\
C_4 &= -\,i \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \qquad & C_5 &= -i \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \qquad & C_6 &= -i \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix},
\end{aligned} \tag{2.1}$$

where σ_i are the Pauli matrices. We can now introduce index-free notation for the stress tensor multiplet operators:

$$\begin{aligned}
S(\vec{x}, X) &= X_a{}^b S_b{}^a(\vec{x})\,, \qquad & F(\vec{x}, Y) &= Y^{ab} F_{ab}(\vec{x})\,, \\
\chi(\vec{x}, Z) &= Z^I \chi_I(\vec{x})\,, \qquad & P(\vec{x}, X) &= X_a{}^b P_b{}^a(\vec{x})\,.
\end{aligned} \tag{2.2}$$

with analogous notation for other operators in the multiplet. To implement tracelessness of $S_b{}^a$ we impose the condition $X_a{}^a = 0$, and similarly we impose that the matrix Y^{ab} is symmetric. We can alternatively think of the matrix $X_a{}^b$ as an antisymmetric tensor $\check{X}^{IJ}$ via the mapping

$$\check{X}^{IJ} = X^a_{\ b} C^{[I}_{ac} \overline{C}^{J]bc}\,. \tag{2.3}$$

Similarly, the Z^I can also be written as antisymmetric tensors $\not{Z}_{ab} = C^I_{ab} Z_I$ and $\overline{\not{Z}}^{ab} = \overline{C}^{ab}_I Z^I$. Imposing R-symmetry invariance now consists of finding all linearly independent ways to constraint the polarization tensors X, Y and Z in a four-point function.

Let us now write scalar correlators in terms of manifestly conformal and $\mathfrak{so}(6)_R$ invariant structures. We begin by normalizing the $S(\vec{x}, X)$ and $P(\vec{x}, X)$ two-point functions:

$$\langle S(\vec{x}_1, X_1) S(\vec{x}_2, X_2) \rangle = \frac{\operatorname{tr}(X_1 X_2)}{x_{12}^2}\,, \qquad \langle P(\vec{x}_1, X_1) P(\vec{x}_2, X_2) \rangle = \frac{\operatorname{tr}(X_1 X_2)}{x_{12}^4}\,. \tag{2.4}$$

We then use conformal and R-symmetry invariance to expand:

$$\langle S(\vec{x}_1, X_1) S(\vec{x}_2, X_2) S(\vec{x}_3, X_3) S(\vec{x}_4, X_4) \rangle = \frac{1}{x_{12}^2 x_{34}^2} \sum_{i=1}^{6} \mathcal{S}^i(U, V) \mathcal{B}_i\,, \tag{2.5}$$

where we define the R-symmetry structures

$$
\begin{aligned}
\mathcal{B}_1 &= \operatorname{tr}(X_1 X_2) \operatorname{tr}(X_3 X_4)\,, \\
\mathcal{B}_2 &= \operatorname{tr}(X_1 X_3) \operatorname{tr}(X_2 X_4)\,, \\
\mathcal{B}_3 &= \operatorname{tr}(X_1 X_4) \operatorname{tr}(X_2 X_3)\,, \\
\mathcal{B}_4 &= \operatorname{tr}(X_1 X_4 X_2 X_3) + \operatorname{tr}(X_3 X_2 X_4 X_1)\,, \\
\mathcal{B}_5 &= \operatorname{tr}(X_1 X_2 X_3 X_4) + \operatorname{tr}(X_4 X_3 X_2 X_1)\,, \\
\mathcal{B}_6 &= \operatorname{tr}(X_1 X_3 X_4 X_2) + \operatorname{tr}(X_2 X_4 X_3 X_1)\,,
\end{aligned}
\tag{2.6}
$$

and where $\mathcal{S}^i$ are functions of the conformally-invariant cross-ratios (1.17). We can similarly write

$$
\begin{aligned}
\langle S(\vec{x}_1, X_1) S(\vec{x}_2, X_2) P(\vec{x}_3, X_3) P(\vec{x}_4, X_4)\rangle &= \frac{1}{x_{12}^2 x_{34}^4} \sum_{i=1}^{6} \mathcal{R}^i(U,V)\mathcal{B}_i\,, \\
\langle P(\vec{x}_1, X_1) P(\vec{x}_2, X_2) P(\vec{x}_3, X_3) P(\vec{x}_4, X_4)\rangle &= \frac{1}{x_{12}^4 x_{34}^4} \sum_{i=1}^{6} \mathcal{P}^i(U,V)\mathcal{B}_i\,,
\end{aligned}
\tag{2.7}
$$

for $\langle SSPP\rangle$ and $\langle PPPP\rangle$. The functions $\mathcal{S}^i$, $\mathcal{R}^i$ and $\mathcal{P}^i$ are not independent. By interchanging $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$, we can derive the crossing relations:

$$
\begin{aligned}
&\mathcal{S}^1(U,V) = \mathcal{S}^1\left(U, \frac{1}{V}\right), \quad \mathcal{S}^2(U,V) = U\mathcal{S}^1\left(\frac{1}{U}, \frac{V}{U}\right), \quad \mathcal{S}^3(U,V) = \frac{U}{V}\mathcal{S}^1(V,U)\,, \\
&\mathcal{S}^4(U,V) = \mathcal{S}^4\left(U, \frac{1}{V}\right), \quad \mathcal{S}^5(U,V) = U\mathcal{S}^4\left(\frac{1}{U}, \frac{V}{U}\right), \quad \mathcal{S}^6(U,V) = \frac{U}{V}\mathcal{S}^4(V,U)\,, \\
&\mathcal{R}^3(U,V) = \mathcal{R}^2\left(\frac{U}{V}, \frac{1}{V}\right), \quad \mathcal{R}^6(U,V) = \mathcal{R}^5\left(\frac{U}{V}, \frac{1}{V}\right), \\
&\mathcal{P}^1(U,V) = \mathcal{P}^1\left(U, \frac{1}{V}\right), \quad \mathcal{P}^2(U,V) = U^2\mathcal{P}^1\left(\frac{1}{U}, \frac{V}{U}\right), \quad \mathcal{P}^3(U,V) = \frac{U^2}{V^2}\mathcal{P}^1(V,U)\,, \\
&\mathcal{P}^4(U,V) = \mathcal{P}^4\left(U, \frac{1}{V}\right), \quad \mathcal{P}^5(U,V) = U^2\mathcal{P}^4\left(\frac{1}{U}, \frac{V}{U}\right), \quad \mathcal{P}^6(U,V) = \frac{U^2}{V^2}\mathcal{P}^4(V,U)\,,
\end{aligned}
\tag{2.8}
$$

In particular, these relations imply that $\langle SSSS\rangle$ can be uniquely specified by $\mathcal{S}^1(U,V)$ and $\mathcal{S}^4(U,V)$, $\langle PPPP\rangle$ by $\mathcal{P}^1(U,V)$ and $\mathcal{P}^4(U,V)$, and $\langle SSPP\rangle$ by $\mathcal{R}^1(U,V)$, $\mathcal{R}^2(U,V)$, $\mathcal{R}^4(U,V)$, and $\mathcal{R}^5(U,V)$.

2.1.2 Discrete Symmetries

Before enforcing Q-invariance, let us first discuss discrete symmetries in $\mathcal{N} = 6$ theories. The stress tensor multiplet forms a representation not only of the superconformal group $OSp(6|4)$, but also of a larger supergroup which includes two $\mathbb{Z}_2$ transformations: a parity transformation $\mathcal{P}$ and a discrete R-symmetry transformation $\mathcal{Z}$. Individually $\mathcal{P}$ and $\mathcal{Z}$ may or may not be symmetries of a given

$\mathcal{N} = 6$ theory, though as we shall see they are symmetries of free field theory.

Let us begin with parity $\mathcal{P}$, which maps $x^\mu \to -x^\mu$ and so extends the $SO(3,2)$ conformal group to $O(3,2)$. Under parity the R-symmetry current J^μ and stress tensor $T^{\mu\nu}$ must transforms as a vector and a tensor rather than a pseudovector or a pseudotensor. This then fixes S to transform as a scalar and P to transform as a pseudoscalar. As a result, the correlators $\langle SSSP \rangle$ and $\langle PPPS \rangle$ violate parity and so must vanish in parity preserving theories.

Now we turn to the discrete R-symmetry $\mathcal{Z}$. This symmetry does not commute with the $SO(6)_R$ R-symmetry, but instead extends it from $SO(6)_R$ to $O(6)_R$. Let us define the $\mathcal{Z}$ generator so that it corresponds to the $O(6)$ matrix

$$\mathcal{Z}^{IJ} = \text{diag}\{-1, -1, -1, 1, 1, 1\} \tag{2.9}$$

that is not part of $SO(6)$. The group $O(6)$ has two 6-dimensional representations: the vector representation $\mathbf{6}^+$ under which a vector v^I transforms as $v^I \to \mathcal{Z}^{IJ} v^J$, and the pseudovector representation under which $v^I \to -\mathcal{Z}^{IJ} v^J$. By convention, we take the supercharges to transform as the $\mathbf{6}^+$.[1] The representations of $O(6)$ appearing in the stress tensor multiplet are all antisymmetric products of the $\mathbf{6}^+$, because we can start with the stress-energy tensor, which is a singlet, and obtain all other operators by acting with anti-symmetric products of the superconformal generators. Thus: the rank-0 tensor is the singlet $\mathbf{1}^+$ that is invariant under $\mathcal{Z}$; the rank-1 anti-symmetric tensor is the $\mathbf{6}^+$; the rank-2 anti-symmetric tensor is the adjoint representation $\mathbf{15}^+$; the rank-3 anti-symmetric tensor, the $\mathbf{20}$ is irreducible under $O(6)$ but would have been reducible to $\mathbf{10} + \overline{\mathbf{10}}$ under $SO(6)$; the rank-4 anti-symmetric tensor is the $\mathbf{15}^-$ and can also be represented as a rank-2 anti-symmetric tensor with the same $SO(6)$ transformation properties as the $\mathbf{15}^+$ except for an additional minus sign under $\mathcal{Z}$; the rank-5 anti-symmetric tensor is the $\mathbf{6}^-$ and can also be represented as a pseudovector; and lastly, the rank-6 anti-symmetric tensor $\mathbf{1}^-$ is invariant under $SO(6)$ but it gets multiplied by (-1) under $\mathcal{Z}$. See Table 2.1 for a list of conformal primaries of the stress tensor multiplet and the $O(6)$ representations under which they transform. In particular, note that the superconformal primary S is an $O(6)$ antisymmetric rank-2 pseudotensor. It is not hard to check that $\langle SSSS \rangle$, $\langle SSPP \rangle$, and $\langle PPPP \rangle$ always preserve $\mathcal{Z}$, even if this is not a symmetry of the full theory.

[1] We could have considered the supercharges to transform as a pseudovector, but this choice is related to the first choice by an $SO(6)$ rotation.

Operators	$T^{\mu\nu}$	$\psi^{\alpha\mu}$	J^{μ}	F^{α}	S,P	χ^{α}	j^{μ}
$O(6)$	$\mathbf{1}^+$	$\mathbf{6}^+$	$\mathbf{15}^+$	$\mathbf{20}$	$\mathbf{15}^-$	$\mathbf{6}^-$	$\mathbf{1}^-$
$SO(6)$	$\mathbf{1}$	$\mathbf{6}$	$\mathbf{15}$	$\overline{\mathbf{10}}+\mathbf{10}$	$\mathbf{15}$	$\mathbf{6}$	$\mathbf{1}$

Table 2.1: $O(6)$ and $SO(6)$ assignments for operators in the stress tensor multiplet.

2.1.3 The Q Variations

We now use the Q-supercharge variations to complete our derivation the superconformal Ward identities on $\langle SSSS\rangle$. To impose $Q_{\alpha I}$ invariance, we need to know the action of the Poincaré supercharges $Q_{\alpha I}$ on the operators in the stress tensor multiplet. Using index-free notation, we find

$$\begin{aligned}
\delta^\alpha(Z)S(\vec{x},X) &= \frac{1}{4}\left[F^\alpha(\vec{x},X\cdot\not{Z})+\overline{F}^\alpha(\vec{x},\overline{\not{Z}}\cdot X)\right]+\frac{1}{4}\chi^\alpha(\vec{x},\check{X}\cdot Z)\,,\\
\delta^\alpha(Z)F^\beta(\vec{x},Y) &= \frac{1}{2}\epsilon^{\alpha\beta}P(\vec{x},Y\cdot\overline{\not{Z}})+\gamma^{\alpha\beta}_\mu J^\mu(\vec{x},\not{Z}_1\cdot\overline{\not{Z}}_2-\not{Z}_2\cdot\overline{\not{Z}}_1)\\
&\quad -\frac{i}{2}\gamma^{\alpha\beta}_\mu\partial^\mu S(\vec{x},Y\cdot\overline{\not{Z}})\\
\delta^\alpha(Z_1)\chi^\beta(\vec{x},Z_2) &= \frac{1}{2}\epsilon^{\alpha\beta}P(\vec{x},\not{Z}_1\cdot\overline{\not{Z}}_2-\not{Z}_2\cdot\overline{\not{Z}}_1)+Z_1\cdot Z_2\; i\gamma^{\alpha\beta}_\mu j^\mu(\vec{x})\,,\\
&\quad +\frac{i}{8}\gamma^{\alpha\beta}_\mu\partial^\mu S(\vec{x},\not{Z}_1\cdot\overline{\not{Z}}_2-\not{Z}_2\cdot\overline{\not{Z}}_1)\,,\\
\delta^\alpha(Z)P(\vec{x},X) &= \frac{i}{6}\left(\gamma^{\alpha\beta}_\mu\partial^\mu F_\beta(\vec{x},X\cdot\not{Z})+\gamma^{\alpha\beta}_\mu\partial^\mu F_\beta(\vec{x},\overline{\not{Z}}\cdot X)\right)\\
&\quad -\frac{i}{6}\sigma^{\alpha\beta}_\mu\partial^\mu\chi_\beta(\vec{x},\check{X}\cdot Z)\,,\\
&\text{etc.}
\end{aligned}\tag{2.10}$$

Here, $\delta^\alpha(Z)$ represents the action of $Z_I Q^{\alpha I}$ on the various operators, and γ_μ are the 3d gamma matrices. We have omitted the supersymmetric variations of J, j, ψ, and T as they are not needed in this thesis. Superconformal Ward identities then follow by imposing that the Q variations of four-point correlators vanish. To derive the Ward identities on $\langle SSSS\rangle$, we require that

$$\delta\langle SSS\chi\rangle = 0\,,\qquad \delta\langle SSSF\rangle = 0\,.\tag{2.11}$$

When we expand these expressions, we will find equations relating $\langle SSSS\rangle$ to various two-scalar, two-fermion correlators. We have already seen how to expand $\langle SSSS\rangle$ as a sum of conformal and R-symmetry structures. We can similarly expand two-scalar, two-fermion correlators; explicit expressions are given in Appendix A. Note that we only include conformal and R-symmetry structures which preserve both spacetime parity $\mathcal{P}$ and the discrete R-symmetry transformation $\mathcal{Z}$. This is because $\langle SSSS\rangle$ itself always preserves both discrete symmetries, and so the superconformal Ward

Variation	Correlators Used			Correlators Obtained			
$\delta\langle SSS\chi\rangle$	$\langle SSSS\rangle$			$\langle SS\chi\chi\rangle$	$\langle SS\chi F\rangle$	$\langle SSSj\rangle$	
$\delta\langle SSSF\rangle$	$\langle SSSS\rangle$	$\langle SSFF\rangle$		$\langle SS\chi F\rangle$	$\langle SSF\overline{F}\rangle$	$\langle SSSJ\rangle$	
$\delta\langle SSP\chi\rangle$	$\langle SS\chi\chi\rangle$	$\langle SS\chi F\rangle$	$\langle SSPP\rangle$	$\langle SP\chi\chi\rangle$	$\langle SP\chi F\rangle$	$\langle SSPP\rangle$	$\langle SSPj\rangle$
$\delta\langle PPS\chi\rangle$	$\langle SP\chi\chi\rangle$	$\langle SP\chi F\rangle$		$\langle PP\chi\chi\rangle$	$\langle PP\chi F\rangle$	$\langle PPSj\rangle$	
$\delta\langle PPP\chi\rangle$	$\langle PP\chi\chi\rangle$	$\langle PP\chi F\rangle$		$\langle PPPP\rangle$	$\langle PPPj\rangle$		

Table 2.2: Taking supersymmetric variations to compute correlators. By setting the variation in the first column to zero, we can use the correlators in the second column to compute the correlators in the third column. For each correlator we only compute the $\mathcal{P}$ and $\mathcal{Z}$ invariant structures. In the table we have not included correlators involving $\overline{F}$ which are related to those with F by Hermitian conjugation.

identities can only relate $\langle SSSS\rangle$ to structures in other correlators which also preserve these discrete symmetries. Indeed, all 4-point superconformal invariants (i.e. invariants under $OSp(6|4)$) can be classified as even or odd under $\mathcal{P}$ and $\mathcal{Z}$, as explained for instance in Appendix B of [23].

After expanding these expressions and then writing all correlators in a manifestly conformally invariant and R-symmetry preserving form, we derive the equations

$$\begin{aligned}
\partial_U \mathcal{S}^6(U,V) = \frac{1}{2U^2}\Big[&-(U^3\partial_U + U^2V\partial_V)\mathcal{S}^1 + (1-V+U(V-1)\partial_U + UV\partial_V)\mathcal{S}^2 \\
&+(1-U-V-U(1-2U+U^2-V)\partial_U + U(1-U)V\partial_V)\mathcal{S}^3 \\
&+(2-U-2V+2U(U+V-1)\partial_U + 2UV\partial_V)\mathcal{S}^4 \\
&-U(1+2U(U-1)\partial_U + 2UV\partial_V)\mathcal{S}^5 + U\mathcal{S}^6\Big], \\
\partial_V \mathcal{S}^6(U,V) = \frac{1}{2U}\Big[&U(U\partial_U + (V-1)\partial_V)\mathcal{S}^1 + (1-U\partial_U - U\partial_V)\mathcal{S}^2 \\
&+(1+U(U-1)\partial_U + UV\partial_V)\mathcal{S}^3 + (2-2U\partial_U)\mathcal{S}^4 \\
&+(2U^2\partial_U + 2UV\partial_V)\mathcal{S}^5\Big].
\end{aligned} \tag{2.12}$$

The equations (2.11) also give a number of relations between $\langle SSSS\rangle$ and various two scalar, two fermion correlators. We list these identities in Appendix A.

To derive Ward identities relating $\langle SSPP\rangle$ to $\langle SSSS\rangle$, we need to consider a further variation, $\delta\langle SSP\chi\rangle$. Using the results of (2.11) and the variation $\delta\langle SSP\chi\rangle$, we can fully determine $\langle SSPP\rangle$, along with $\langle SP\chi\chi\rangle$, $\langle SP\chi F\rangle$ and $\langle SP\chi\bar{F}\rangle$, in terms of $\langle SSSS\rangle$. The resulting expressions for $\mathcal{R}^i(U,V)$ in terms of $\mathcal{S}^i(U,V)$ can be found in Appendix A. We can then furthermore compute $\langle PPPP\rangle$ using the additional variations $\langle PPS\chi\rangle$ and $\langle PPP\chi\rangle$. The variations we consider and correlators we can compute from these are listed in Table 2.2.

2.1.4 Parity Odd Superconformal Ward Identity

Now let us turn to the parity odd four-point function $\langle SSSP\rangle$. While this correlator will play no further role in this chapter, it will become important in Chapter 5, where we use it to study the $\mathcal{N}=6$ theories with weakly broken higher-spin symmetry.

Conformal and R-symmetry invariance together imply that

$$\langle S(\vec{x}_1,X_1)S(\vec{x}_2,X_2)S(\vec{x}_3,X_3)P(\vec{x}_4,X_4)\rangle = \frac{x_{13}}{x_{12}^2x_{34}^3x_{14}}\sum_{i=1}^{6}\mathcal{T}^i(U,V)\mathcal{B}_i\,, \tag{2.13}$$

where the $\mathcal{B}_i$ are defined as in (2.6), and where $\mathcal{T}^i(U,V)$ are functions of the cross-ratios (1.17). Crossing under $1\leftrightarrow 3$ and $2\leftrightarrow 3$ relates the different $\mathcal{T}^i(U,V)$:

$$\begin{aligned}
\mathcal{T}^2(U,V) &= U^{3/2}\mathcal{T}^1\left(\frac{1}{U},\frac{V}{U}\right)\,, \quad \mathcal{T}^3(U,V)=\frac{U}{V}\mathcal{T}^1(V,U)\,,\\
\mathcal{T}^5(U,V) &= U^{3/2}\mathcal{T}^4\left(\frac{1}{U},\frac{V}{U}\right)\,, \quad \mathcal{T}^6(U,V)=\frac{U}{V}\mathcal{T}^4(V,U)\,,
\end{aligned} \tag{2.14}$$

so that $\langle SSSP\rangle$ is uniquely specified by $\mathcal{T}^1(U,V)$ and $\mathcal{T}^4(U,V)$. By demanding that the $Q_{\alpha I}$ supersymmetry charge annihilates $\langle SSSF\rangle$ and $\langle SSS\chi\rangle$, but this time expanding the correlators two-scalar, two-fermion correlators using parity violating rather than parity preserving conformal structures, we derive the superconformal Ward identities

$$\begin{aligned}
\mathcal{T}^5(U,V) &= \frac{1}{2U}\left(-U\mathcal{T}^1(U,V)+\mathcal{T}^2(U,V)+(1-U)\mathcal{T}^3(U,V)+2\mathcal{T}^4(U,V)\right)\,,\\
\mathcal{T}^6(U,V) &= \frac{1}{2U}\left(-U\mathcal{T}^1(U,V)+(V-U)\mathcal{T}^2(U,V)+V\mathcal{T}^3(U,V)+2V\mathcal{T}^4(U,V)\right)\,.
\end{aligned} \tag{2.15}$$

2.2 The OPE Expansion

Just all conformally invariant four-point functions, we can expand $\langle SSSS\rangle$ as a sum of conformal blocks. Because the S transforms in the $\mathbf{15}$ of $\mathfrak{so}(6)_R$, any operator which appears in the $S\times S$ OPE must belong to an irreducible representation in the tensor product

$$\mathbf{15}\otimes\mathbf{15} = \mathbf{1}_s\oplus\mathbf{15}_a\oplus\mathbf{15}_s\oplus\mathbf{20'}_s\oplus(\mathbf{45}_a\oplus\overline{\mathbf{45}}_a)\oplus\mathbf{84}_s\,. \tag{2.16}$$

When studying the s-channel OPE, the most convenient basis for the R-symmetry structures is one in which each irreducible representation contributes to a single R-symmetry structure. This leads us to define the functions $\mathcal{S}_{\mathbf{r}}(U,V)$ via the equation

$$\vec{\mathcal{S}}\cdot\mathbf{B} = \begin{pmatrix}\mathcal{S}_{\mathbf{1}_s} & \mathcal{S}_{\mathbf{15}_a} & \mathcal{S}_{\mathbf{15}_s} & \mathcal{S}_{\mathbf{20}'_s} & \mathcal{S}_{\mathbf{45}_a\oplus\overline{\mathbf{45}}_a} & \mathcal{S}_{\mathbf{84}_s}\end{pmatrix}\,,$$
$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{15} & -\frac{1}{8} & \frac{1}{6} & \frac{1}{24} & -\frac{1}{8} & \frac{1}{16} \\ \frac{1}{15} & \frac{1}{8} & \frac{1}{6} & \frac{1}{24} & \frac{1}{8} & \frac{1}{16} \\ -\frac{1}{30} & 0 & -\frac{1}{6} & -\frac{1}{12} & 0 & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}\,. \tag{2.17}$$

These are defined such that in the s-channel each function $\mathcal{S}_{\mathbf{r}}$ receives contributions only from operators transforming in the $\mathbf{r}$, and so can be expanded as a sum of conformal blocks

$$\mathcal{S}_{\mathbf{r}}(U,V) = \sum_{\text{conformal primaries } \mathcal{O}_{\Delta,\ell,\mathbf{r}}} a_{\Delta,\ell,\mathbf{r}}\, g_{\Delta,\ell}(U,V)\,, \tag{2.18}$$

where the sum is taken over all the distinct conformal primary operators $\mathcal{O}_{\Delta,\ell,\mathbf{r}}$ which transform in the representation $\mathbf{r}$ and which appear in the $S\times S$ OPE. As usual, Δ and ℓ are the scaling dimension and spin, respectively, of $\mathcal{O}_{\Delta,\ell,\mathbf{r}}$. The operator P also transforms in the $\mathbf{15}$, and so we can similarly decompose both $\langle SSPP\rangle$ and $\langle PPPP\rangle$ into conformal blocks.

Each supermultiplet contains operators with various spins, conformal dimensions and $\mathfrak{so}(6)_R$ representations. Superconformal symmetry imposes various linear relations on the coefficients $a_{\Delta,\ell,\mathbf{r}}$ for operators in a given supermultiplet. We can thus reorganize the conformal block expansion of $\langle SSSS\rangle$ into a superconformal block expansion, where all operators belonging to a given supermultiplet are grouped together and the superconformal Ward identities are automatically satisfied superblock by superblock. Our ultimate task in this chapter will be to derive these superblocks for $\langle SSSS\rangle$.

Our task in the rest of this section will be to constrain the supermultiplets which may appear in the $S\times S$ OPE. Recall that the full list of unitary supermuliplets of $\mathfrak{osp}(6|4)$ is given in Table 1.2. However, not just any operator can appear in $S\times S$, as there are various selection rules at play. As we have already seen, any operator exchanged must transform one of the representations appearing in (2.16). Due to $1\leftrightarrow 2$ crossing symmetry, even spin operators must be in the $\mathbf{1}$, $\mathbf{15}$, $\mathbf{20}'$, or $\mathbf{84}$ while odd spin operators must be in the $\mathbf{15}$, $\mathbf{45}$, or $\overline{\mathbf{45}}$. A large number of supermultiplets contain operators in at least one of these representations, so by themselves these conditions are not very restrictive.

We can do better by using the fact that $S(\vec{x}, X)$ is a 1/3-BPS operator, and as such is annihilated by certain Poincaré supercharges. If $\mathcal{Q}$ is a Poincaré supercharge annihilating $S(\vec{x}, X)$ (for any $\vec{x}$ but a specific X), then it also annihilates $S(\vec{x}, X)S(\vec{y}, X)$. We will explore the consequences of this fact in the next section, and use it to show that any operator in the $\mathbf{20'}$, $\mathbf{45} \oplus \overline{\mathbf{45}}$ or $\mathbf{84}$ can only belong to one of a limited number of supermultiplets. In Section 2.2.2 we then use the superconformal Ward identities to constrain superblocks in which only operators in the $\mathbf{1}$ and $\mathbf{15}$ are exchanged.

2.2.1 Operators in the $S \times S$ OPE

Let us begin by writing the generators of $\mathfrak{osp}(6|4)$ in terms of the $\mathfrak{so}(6)$ and $\mathfrak{sp}(4)$ Cartan subalgebras. The Lie algebra $\mathfrak{so}(6)$ has a three dimensional Cartan subalgebra, spanned by orthogonal operators[2] H_1, H_2, and H_3. The other twelve R-symmetry generators take the form:

$$R_{\pm1,\pm1,0}\,,\quad R_{\pm1,0,\pm1}\,,\quad R_{0,\pm1,\pm1}\,,\quad R_{\pm1,\mp1,0}\,,\quad R_{\pm1,0,\mp1}\,,\quad R_{0,\pm1,\mp1}\,,$$

where for each R the subscripts are correlated and label the weights of each of these generators under the Cartan subalgebra:

$$[H_i, R_{r_1,r_2,r_3}] = r_i R_{r_1,r_2,r_3}\,, \text{ for } i = 1\,,2\,,3\,. \tag{2.19}$$

We take the simple roots of $\mathfrak{so}(6)$ to be the raising operators

$$\mathcal{R}^+ = \left\{R_{1,-1,0}\,, R_{0,1,1}\,, R_{0,1,-1}\right\},$$

while their corresponding lowering operators are

$$\mathcal{R}^- = \left\{R_{-1,1,0}\,, R_{0,-1,-1}\,, R_{0,-1,1}\right\}.$$

A highest weight state is one that is annihilated by each element of $\mathcal{R}^+$; the highest weight state of the $\mathbf{15}$ is then $R_{1,1,0}$.

[2]For instance, in the $\mathbf{6}$ irrep of $\mathfrak{so}(6)_R$, we can represent the Cartan generators by the matrices

$$H_1 = \begin{pmatrix} \sigma_2 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & & \\ & \sigma_2 & \\ & & 0 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & \sigma_2 \end{pmatrix},$$

where σ_2 is the second Pauli matrix.

We perform a similar procedure with the conformal group $\mathfrak{sp}(4)$. We can take one Cartan element to be the dilatation operator D and the other to be the rotation operator J^0. The other two rotation operators are the raising and lowering operators $J^{\pm}$. The H_i, D, and J^0 span a Cartan subalgebra of $\mathfrak{osp}(6|4)$.

We can now write the $\mathfrak{osp}(6|4)$ supercharges in terms of their charges under this subalgebra. The Qs and Ss can be written as

$$Q^{\pm}_{\pm 1,0,0}\,, \quad Q^{\pm}_{0,\pm 1,0}\,, \quad Q^{\pm}_{0,0,\pm 1}\,, \qquad \text{and } S^{\pm}_{\pm 1,0,0}\,, \quad S^{\pm}_{0,\pm 1,0}\,, \quad S^{\pm}_{0,0,\pm 1}\,,$$

respectively, where the superscript is the J^0 charge and the subscripts are the H_i charges. (The sign in the superscript is uncorrelated with the signs in the subscripts.) Note that the Qs have scaling dimension $+1/2$ and the Ss have scaling dimension $-1/2$, so their charges under dilatation operator are also manifest in this notation.

Given an irreducible representation of $\mathfrak{osp}(6|4)$, the highest weight state $|\Delta, \ell, r\rangle$ is one which is annihilated by the raising operators of $\mathfrak{osp}(6|4)$:

$$K^{\mu}\,|\Delta, \ell, r\rangle = S^{\pm}_{a,b,c}\,|\Delta, \ell, r\rangle = J^{+}\,|\Delta, \ell, r\rangle = R^{+}\,|\Delta, \ell, r\rangle = 0\,, \tag{2.20}$$

where $R^{+} \in \mathcal{R}^{+}$, and is an eigenstate of each of the Cartans:

$$H_i\,|\Delta, \ell, r\rangle = r_i\,|\Delta, \ell, r\rangle\;, \qquad D\,|\Delta, \ell, r\rangle = \Delta\,|\Delta, \ell, r\rangle\;, \qquad J^0\,|\Delta, \ell, r\rangle = 2\ell\,|\Delta, \ell, r\rangle\;. \tag{2.21}$$

Here Δ and ℓ are the conformal dimension and spin of the superconformal primary, and the $r = (r_1, r_2, r_3)$'s are the highest weight states of the R-symmetry representation of the superconformal primary. These weights are related to the Dynkin label $[a_1 a_2 a_3]$ by the equation

$$r_1 = a_1 + \frac{a_2 + a_3}{2}\,, \qquad r_2 = \frac{a_2 + a_3}{2}\,, \qquad r_3 = \frac{a_2 - a_3}{2}\,, \tag{2.22}$$

and always satisfy $r_1 \geq r_2 \geq r_3$.

The highest weight state of the stress tensor multiplet, $\left|\mathcal{S}^H\right\rangle$, has conformal dimension $\Delta = 1$, spin $\ell = 0$, and R-symmetry weights $(1, 1, 0)$. It can be created by acting with the operator[3] $\hat{S}_{1,1,0}(0)$

[3]To avoid confusion between the superconformal generators $S^{\pm}_{r_1,r_2,r_3}$ and components of the stress tensor superconformal primary $\hat{S}_{r_1,r_2,r_3}(\vec{x})$, in this section we adopt the convention that the latter operators are always hatted.

on the vacuum. The stress tensor is a 1/3-BPS multiplet, satisfying the shortening condition

$$Q\left|\mathcal{S}^H\right\rangle = 0 \text{ for all } Q \in \mathcal{Q}^+ = \left\{Q^\pm_{1,0,0}\,, \quad Q^\pm_{0,1,0}\right\}\,, \tag{2.23}$$

which in turn implies that

$$Q\hat{S}_{1,1,0}(\vec{x}) = 0 \text{ for all } \vec{x} \in \mathbb{R}^3 \text{ and all } Q \in \mathcal{Q}^+\,. \tag{2.24}$$

This is equivalent to imposing that $\hat{S}(\vec{x}, X)$ has no fermionic descendant in the **64** of $\mathfrak{so}(6)$. We will find it useful to further define

$$\mathcal{Q}^0 = \left\{Q^\pm_{0,0,\pm1}\right\}\,, \qquad \mathcal{Q}^- = \left\{Q^\pm_{-1,0,0}\,, \quad Q^\pm_{0,-1,0}\right\}\,, \qquad \mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^0 \cup \mathcal{Q}^- \tag{2.25}$$

along with analogous definitions for the S-supercharges.

Let $\Phi_{2,2,0}(\vec{x})$ be any operator which appears in the OPE $\hat{S}_{1,1,0} \times \hat{S}_{1,1,0}$, and $|\Phi\rangle = \Phi_{2,2,0}(0)\,|0\rangle$ the associated state. This is the highest weight state of an **84** multiplet which is annihilated by $\mathcal{R}^+$ and $\mathcal{Q}^+$. Without loss of generality we can take this operator to be a conformal primary which is annihilated by J^+; if it is not, we can act with the raising operators K^μ and J^+ to construct such an operator. Because any operator $S^+ \in \mathcal{S}^+$ is of the form $[K, Q^+]$ for some $Q^+ \in \mathcal{Q}$, we find that $\mathcal{S}^+$ also annihilates $|\Phi\rangle$. So in total, we have the conditions

$$\begin{gathered} Q^+\,|\Phi\rangle = J^+\,|\Phi\rangle = R^+\,|\Phi\rangle = S^+\,|\Phi\rangle = K^\mu\,|\Phi\rangle = 0 \\ \text{for any } R^+ \in \mathcal{R}^+\,,\; Q^+ \in \mathcal{Q}^+\,,\; S^+ \in \mathcal{S}^+\,. \end{gathered} \tag{2.26}$$

Our task it to determine which supermultiplets $|\Phi\rangle$ may belong to.

By acting with operators in $\mathcal{S}$ on $|\Phi\rangle$ we can construct states of lower conformal dimension. Consider first constructing a state $|\mathcal{O}'\rangle$ by acting with all eight supercharges in $\mathcal{S}^0 \cup \mathcal{S}^-$:

$$|\mathcal{O}'\rangle = S^+_{0,+1,0}S^-_{0,+1,0}S^+_{0,-1,0}S^-_{0,-1,0}S^+_{0,0,+1}S^-_{0,0,+1}S^+_{0,0,-1}S^-_{0,0,-1}\,|\Phi\rangle\,. \tag{2.27}$$

By assumption $|\Phi\rangle$ satisfies (2.26), and it is straightforward to see that $|\mathcal{O}'\rangle$ then also satisfies (2.26). Because the $\mathcal{S}$ operators anticommute with themselves, we furthermore find that any operator in $\mathcal{S}^0 \cup \mathcal{S}^-$ annihilates $|\mathcal{O}'\rangle$. The state $|\mathcal{O}'\rangle$ is therefore annihilated by all of the $\mathcal{S}$ and by J^+ and R^+, and so either $|\mathcal{O}'\rangle$ is the highest weight state of the superconformal primary of the supermultiplet, or $|\mathcal{O}'\rangle = 0$. In either case we conclude that there exists some $0 \leq k \leq 8$ for which acting with any $k+1$ operators from $\mathcal{S}^0 \cup \mathcal{S}^-$ annihilates $|\Phi\rangle$, but for which acting with just k operators does not:

$$|\mathcal{O}\rangle = S_1 \cdots S_k \, |\Phi\rangle \neq 0 \tag{2.28}$$

for some string of k operators $S_i \in \mathcal{S}^0 \cup \mathcal{S}^-$. It is again easy to see that $|\Phi\rangle$ satisfies (2.26) and is annihilated by the operators in $\mathcal{S}^0 \cup \mathcal{S}^-$; we hence conclude that $|\mathcal{O}\rangle$ is the highest weight state of the superconformal primary of the multiplet. Note that the different orderings of the operators S_i in (2.28) are equivalent, up to an overall minus sign.

Let us denote the $\mathfrak{so}(6)$ weights of $|\mathcal{O}\rangle$ by

$$w = (2,2,0) + \sum_i v_i \,, \tag{2.29}$$

where $v_i = (v_{i1}, v_{i2}, v_{i3})$ are the $\mathfrak{so}(6)$ Cartans of the S_i we act with in (2.28). Because $|\mathcal{O}\rangle$ is a highest weight state, we must have

$$w_1 \geq w_2 \geq |w_3| \,, \tag{2.30}$$

which provides a useful additional constraint on (2.28).

As discussed in the previous section, $|\mathcal{O}\rangle$ belongs to one of the three types of unitary representations of $\mathfrak{osp}(6|4)$. If $|\mathcal{O}\rangle$ is part of a long multiplet, it is annihilated by all of the raising operators (2.20) but satisfies no other conditions. If instead it belongs to an A-type multiplet it satisfies shortening conditions [31]

$$\left(Q^-_{q_1,q_2,q_3} - \frac{1}{2\ell} Q^+_{q_1,q_2,q_3} J^- \right) |\mathcal{O}\rangle = 0 \tag{2.31}$$

with the specific weights q_i depending on the $\mathfrak{so}(6)$ weights of $|\mathcal{O}\rangle$. Finally, if it is part of a B-type multiplet, it is annihilated by both $Q^+_{q_1,q_2,q_3}$ and $Q^-_{q_1,q_2,q_3}$ for specific weights q_i. Furthermore for B-type multiplets $|\mathcal{O}\rangle$ is always a scalar.

With this information out of the way, we now simply enumerate all possibilities for (2.28), subject to the constraint (2.30). The simplest case is where $|\Phi\rangle$ is itself the highest weight primary. Then we have a $(B, 2)$ multiplet in the **84**.

Next let us extend this reasoning to the case

$$|\mathcal{O}\rangle = S_1 \cdots S_n \, |\Phi\rangle \;\; \text{where } S_i \in \mathcal{S}^0 \,. \tag{2.32}$$

Because $\{\mathcal{Q}^+, \mathcal{S}^0\}$ consists only of positive R-symmetry generators, we see that $|\mathcal{O}\rangle$ is annihilated by $\mathcal{Q}^+$ and hence we still have a $(B, 2)$ multiplet. The possible R-symmetry representations are the [022] (which is the **84**), the [031] and its conjugate [013], and the [040] and its conjugate [004]. We can however eliminate the [031] possibility, as in this case one needs to act with an odd number of supercharges to construct an operator in the **84** from the superconformal primary.

Let us next consider the cases

$$|\mathcal{O}\rangle = S_1 \cdots S_n S^{\pm}_{-1,0,0} |\Phi\rangle \text{ where } S_i \in \mathcal{S}^0 , \tag{2.33a}$$

$$|\mathcal{O}\rangle = S_1 \cdots S_n S^{\pm}_{0,-1,0} |\Phi\rangle \text{ where } S_i \in \mathcal{S}^0 , \tag{2.33b}$$

$$|\mathcal{O}\rangle = S_1 \cdots S_n S^{+}_{-1,0,0} S^{-}_{-1,0,0} |\Phi\rangle \text{ where } S_i \in \mathcal{S}^0 . \tag{2.33c}$$

$$|\mathcal{O}\rangle = S_1 \cdots S_n S^{+}_{0,-1,0} S^{-}_{0,-1,0} |\Phi\rangle \text{ where } S_i \in \mathcal{S}^0 , \tag{2.33d}$$

Cases (2.33a) and (2.33c) violate (2.30) and so are not possible. For the other two possibilities we find that $|\mathcal{O}\rangle$ is annihilated by $S^{\pm}_{1,0,0}$ and so $|\mathcal{O}\rangle$ must be a B-type multiplet. Using (2.30) we find that the possible multiplets for (2.33b) are

$$(B, 1) \text{ in the } [120] , \ [102] , \text{ or a } (B, 2) \text{ in the } [111] ,$$

while for (2.33d) we can only have a $(B, 1)$ in the [200]. We can furthermore eliminate the $(B, 2)$ in the [111] as an option because in this multiplet only fermions transform in the **84**.

The next cases to consider are

$$|\mathcal{O}\rangle = S_1 \cdots S_n S^{\pm}_{-1,0,0} S^{\pm}_{0,-1,0} |\Phi\rangle \text{ where } S_i \in \mathcal{S}^0 , \tag{2.34a}$$

$$|\mathcal{O}\rangle = S_1 \cdots S_n S^{\pm}_{-1,0,0} S^{+}_{0,-1,0} S^{-}_{0,-1,0} |\Phi\rangle \text{ where } S_i \in \mathcal{S}^0 . \tag{2.34b}$$

$$|\mathcal{O}\rangle = S_1 \cdots S_n S^{\pm}_{0,-1,0} S^{+}_{-1,0,0} S^{-}_{-1,0,0} |\Phi\rangle \text{ where } S_i \in \mathcal{S}^0 . \tag{2.34c}$$

Case (2.34c) violates (2.30) and so is forbidden. For other two cases we find some combination of $Q^{-}_{1,0,0}$ and $Q^{+}_{1,0,0}$ annihilate $|\mathcal{O}\rangle$, so $|\mathcal{O}\rangle$ must be either an A-type or B-type multiplet. For (2.34a) we find that (2.30) restricts us to an $(A, +)$ or $(B, +)$ in the [020], an $(A, -)$ or $(B, -)$ in the [002], or an $(A, 2)$ or $(B, 2)$ in the [011]. For (2.34b) we instead find that $|\mathcal{O}\rangle$ is an $(A, 1)$ or $(B, 1)$ multiplet in the [100]. However, we can rule out all B-type multiplets; the $(B, \pm)$ and $(B, 1)$ only contains fermionic operators in the **84**, while due to its shortening conditions the $(B, 2)$ does not contain any operator in the **84**. Thus, only the A-type multiplets are possible.

Finally, we have the case

$$|\mathcal{O}\rangle = S_1 \cdots S_n S^+_{-1,0,0} S^-_{0,-1,0} S^+_{0,-1,0} S^-_{0,-1,0} |\Phi\rangle \text{ where } S_i \in \mathcal{S}^0 \,. \tag{2.35}$$

Now $|\mathcal{O}\rangle$ need not be annihilated by any supercharges, so it can be a long multiplet. The condition (2.30) however forces it to be an $\mathfrak{so}(6)$ singlet. If $|\mathcal{O}\rangle$ satisfies any shortening conditions it must be either a conserved current multiplet or the trivial (vacuum) multiplet, but neither of these contain an operator in the **84** so they are both ruled out.

We summarize our results in the first 11 lines of Table 2.3, where we give the full list of all possible superconformal blocks which contain an operator in the **84**.

Our next task is to extend our arguments to operators Ψ, $\bar{\Psi}$ and Ξ in the **45**, $\overline{\mathbf{45}}$ and **20′** of $\mathfrak{so}(6)$ respectively. The highest weight state under $\mathfrak{so}(6)$ for each of these operators is

$$\Psi_{2,1,1}\,, \qquad \bar{\Psi}_{2,1,-1}\,, \qquad \text{and } \; \Xi_{2,0,0}$$

respectively, and so if these operators appear in the OPE $\hat{S} \times \hat{S}$ they must appear in

$$\Psi_{2,1,1} \in \hat{S}_{1,1,0} \times \hat{S}_{1,0,1}\,, \qquad \bar{\Psi}_{2,1,-1} \in \hat{S}_{1,1,0} \times \hat{S}_{1,0,-1}\,, \qquad \text{and} \qquad \Xi_{2,0,0} \in \hat{S}_{1,1,0} \times \hat{S}_{1,-1,0}\,.$$

The shortening conditions on S imply that $Q^{\pm}_{1,0,0}$ annihilates $\hat{S}_{1,\pm1,0}$ and $\hat{S}_{1,0,\pm1}$, and so must annihilate $\Psi_{2,1,1}$, $\bar{\Psi}_{2,1,-1}$ and $\Xi_{2,0,0}$. We can then repeat the analysis previously performed for $\Phi_{2,2,0}$, and recover the same list of multiplets that we found by analyzing the conditions for the operators in the **84**. We thus conclude that any supermultiplet appearing in $S \times S$ not listed in the first 11 lines of Table 2.3 can contain non-zero contributions only from operators in the adjoint **15** and the singlet **1**. We will analyze this case in the next section using the superconformal Ward identities. The results of this analysis are simple to state. There are only 3 types of supermultiplets in which only singlets and adjoints contribute: the identity supermultiplet (containing just the identity operator), the stress tensor multiplet itself, as well as a conserved multiplet $(A, \text{cons.})$ whose superconformal primary is an $\mathfrak{so}(6)$ singlet scalar with scaling dimension $\ell + 1$.

Table 2.3 shows a summary of our analyses containing all possible supermultiplets which can appear in the $S \times S$ OPE. By using the superconformal Casimir equation we shall find that most of these supermultiplets can in fact be exchanged; we mark those that cannot in red.

Multiplet	$\mathfrak{so}(6)_R$	Δ	ℓ	Case
$(B,2)$	$[022]=\mathbf{84}$	2	0	(2.32)
$(B,1)$	$[200]=\mathbf{20'}$	2	0	(2.33d)
$(A,+)$	$[020]=\mathbf{10}$	$\ell+2$	half-integer	(2.34a)
$(A,-)$	$[002]=\overline{\mathbf{10}}$	$\ell+2$	half-integer	(2.34a)
$(A,2)$	$[011]=\mathbf{15}$	$\ell+2$	integer	(2.34a)
$(A,1)$	$[100]=\mathbf{6}$	$\ell+2$	half-integer	(2.34b)
Long	$[000]=\mathbf{1}$	$>\ell+1$	integer	(2.35)
$(B,+)$	$[040]=\mathbf{35}$	2	0	(2.32)
$(B,-)$	$[004]=\overline{\mathbf{35}}$	2	0	(2.32)
$(B,1)$	$[120]=\mathbf{45}$	2	0	(2.33b)
$(B,1)$	$[102]=\overline{\mathbf{45}}$	2	0	(2.33b)
$(A,\text{cons.})$	$[000]=\mathbf{1}$	$\ell+1$	integer	Section 2.2.2
$(B,2)$	$[011]=\mathbf{15}$	1	0	Section 2.2.2
Trivial	$[000]=\mathbf{1}$	0	0	Section 2.2.2

Table 2.3: Table of superconformal blocks not eliminated by our analysis. The $\mathfrak{so}(6)_R$, Δ and ℓ given the R-symmetry, conformal dimension and spin of the superconformal primary of the exchanged multiplet. The rows in red are for multiplets which we do not eliminate, but for which the superconformal Casimir equation cannot be solved and so no superconformal block exists.

2.2.2 Constraining $\mathfrak{so}(6)$ Singlets and Adjoints

We will now finish our justification of Table 2.3, finding all superblocks in which the only exchanged operators are in the $\mathbf{1}$ or $\mathbf{15}$. We will analyze this possibility using the superconformal Ward identities.

Let us fix some supermultiplet $\mathcal{M}$ and define

$$\mathcal{S}_{\mathbf{r}}^{(\mathcal{M})}(U,V)=\sum_{(\Delta,\ell,\mathbf{r})\in\mathcal{M}} a_{\Delta,\ell,\mathbf{r}}\, g_{\Delta,\ell}(U,V) \tag{2.36}$$

to be the contribution from s-channel $\mathcal{M}$ exchange to $\mathcal{S}_{\mathbf{r}}(U,V)$. The superconformal Ward identities apply to each superblock independently, and so $\mathcal{S}_{\mathbf{r}}^{(\mathcal{M})}(U,V)$ must satisfy the Ward identities (2.12). If we demand that no operators in the $\mathbf{84}$, $\mathbf{45}$, $\overline{\mathbf{45}}$ or $\mathbf{20'}$ are exchanged, we then find that

$$\begin{aligned} U^2\partial_U\left(\mathcal{S}_{\mathbf{1}}^{(\mathcal{M})}+\mathcal{S}_{\mathbf{15}_s}^{(\mathcal{M})}\right)+\left[2U(U+2V-2)\partial_U+4UV\partial_V-2(V-1)\right]\mathcal{S}_{\mathbf{15}_a}^{(\mathcal{M})}&=0\\ U\partial_V(\mathcal{S}_{\mathbf{1}}^{(\mathcal{M})}+\mathcal{S}_{\mathbf{15}_s}^{(\mathcal{M})})+\left[-4U\partial_U-2U\partial_V+2\right]\mathcal{S}_{\mathbf{15}_a}^{(\mathcal{M})}&=0\,. \end{aligned} \tag{2.37}$$

To make further progress we can consider the correlator $\langle SSPP\rangle$. Because both S and P transform in the $\mathbf{15}$ of $\mathfrak{so}(6)$, the correlators $\langle SSSS\rangle$ and $\langle SSPP\rangle$ have the same R-symmetry structures. Since we are interested in s-channel conformal block expansion, we are led to define $\mathcal{P}_{\mathbf{r}}(U,V)$ in an analogous fashion to $\mathcal{S}_{\mathbf{r}}(U,V)$ in (2.17).

If s-channel exchange of $\mathcal{M}$ only contributes to the **1** and **15** channels in the $\langle SSSS\rangle$ correlator, this must also be true of $\langle SSPP\rangle$ and so

$$\mathcal{P}^{(\mathcal{M})}_{\mathbf{20'}}(U,V)=\mathcal{P}^{(\mathcal{M})}_{\mathbf{45}\oplus\overline{\mathbf{45}}}(U,V)=\mathcal{P}^{(\mathcal{M})}_{\mathbf{84}}(U,V)=0\,. \tag{2.38}$$

Combining this with (2.37) and the Ward identities for $\mathcal{R}^i(U,V)$ given in Appendix A, we find that

$$\mathcal{D}\mathcal{S}^{(\mathcal{M})}_{\mathbf{1}}(U,V)=\mathcal{D}\mathcal{S}^{(\mathcal{M})}_{\mathbf{15}_s}(U,V)=\mathcal{D}\mathcal{S}^{(\mathcal{M})}_{\mathbf{15}_a}(U,V)=0\,, \tag{2.39}$$

where $\mathcal{D}$ is the differential operator

$$\mathcal{D}=2U^2\partial_U^2+2(U+V-1)\partial_U\partial_V+2UV\partial_V^2+U\partial_U+(1+2U-V)\partial_V\,. \tag{2.40}$$

Our next step is to rewrite the cross-ratios (U,V) using radial coordinates (r,η)

$$U=\frac{16r^2}{(1+r^2+2r\eta)^2}\,,\qquad V=\frac{(1+r^2-2r\eta)^2}{(1+r^2+2r\eta)^2}\,. \tag{2.41}$$

Conformal blocks have a relatively simple form in radial coordinates:

$$g_{\Delta,\ell}(r,\eta)=r^{\Delta}\sum_{k=0}^{\infty}r^{2k}p_{\Delta,\ell,k}(\eta)\,, \tag{2.42}$$

where each $p_{\Delta,\ell,k}(\eta)$ is polynomial in η [69]. In particular, the leading term is given by

$$p_{\Delta,\ell,0}(\eta)=P_\ell(\eta)\,, \tag{2.43}$$

where $P_n(x)$ is the n^{th} Legendre polynomial. Since $\mathcal{S}^{(\mathcal{M})}_{\mathbf{r}}(U,V)$ is the sum of a finite number of conformal blocks, we expect that

$$\mathcal{S}^{(\mathcal{M})}_{\mathbf{r}}(U,V)=r^{\Delta}\left(q_{\mathbf{r}}(\eta)+O(r^2)\right) \tag{2.44}$$

for some polynomial $q_{\mathbf{r}}(\eta)$.

Let us translate (2.39) into radial coordinates:

$$\left[r^2(r^2-1)\partial_r^2+2r^3\partial_r-(r^2-1)(\eta^2-1)\partial_\eta^2-2(r^2-1)\eta\partial_\eta\right]\mathcal{S}^{(\mathcal{M})}_{\mathbf{r}}(r,\eta)=0\,. \tag{2.45}$$

Substituting (2.44) into this equation we find that $q_{\mathbf{r}}(\eta)$ satisfies Legendre's equation

$$(1-\eta^2)q_{\mathbf{r}}''(\eta) - 2\eta q_{\mathbf{r}}'(\eta) + \Delta(\Delta-1)q_{\mathbf{r}}(\eta) = 0\,. \tag{2.46}$$

Hence, $q_{\mathbf{r}}(\eta)$ is a polynomial if and only if $\Delta \in \mathbb{Z}$, in which case $q_{\mathbf{r}}(\eta) = aP_{\Delta+1}(\eta)$ for some arbitrary constant a. Since unitarity implies that $\Delta \geq 0$, we conclude that $\mathcal{S}_{\mathbf{r}}^{(\mathcal{M})}(r,\eta)$ includes a contribution from either an operator with twist $\Delta - \ell = 1$, or else from the identity operator $\Delta = \ell = 0$.

All operators in a superconformal multiplet have twist greater than or equal to the twist of the superconformal primary. Thus, if $\mathcal{M}$ is not the trivial supermultiplet, then its superconformal primary must have twist one. Examining Table 1.2, we see that aside from the stress tensor multiplet the only other such multiplets are conserved currents: A-type multiplets whose superprimary is an R-symmetry singlet with conformal dimension $\Delta = \ell + 1$. We conclude that any superblock in which the only exchanged operators transform in the **1** or **15** must correspond to the exchange of the trivial, stress tensor, or a conserved current multiplet.

2.3 Superconformal Casimir Equation

Just as the s-channel conformal blocks are eigenfunctions of the quadratic conformal Casimir when the Casimir acts only on the first two operators in a four-point function, superconformal blocks are eigenfunctions of the quadratic superconformal Casimir (see for instance [70, 71] for similar discussions with less supersymmetry). In the conformal case, this fact implies that the conformal blocks obey a second order differential equation. In the superconformal case, the equation obeyed is more complicated because it mixes together four-point functions of operators with different spins. In the case we are interested in, namely for the four-point function of the stress tensor multiplet superconformal primary, the superconformal Casimir equation involves both the $\langle SSSS\rangle$ four-point function as well as four-point functions of two scalar and two fermionic operators.

Using the conformal generators $M_\alpha{}^\beta$, $P_{\alpha\beta}$, $K_{\alpha\beta}$, and D introduced in Section 1.1.2, we can write the quadratic conformal Casimir as

$$C_C = \frac{1}{2}M_\alpha{}^\beta M_\beta{}^\alpha + D(D-3) - \frac{1}{2}P_{\alpha\beta}K^{\alpha\beta}\,, \tag{2.47}$$

which commutes with all conformal generators. The normalization of $M_\alpha{}^\beta$ and D is such that, when acting on any conformal primary $\mathcal{O}_{\Delta,\ell}(0)$,

$$\frac{1}{2}M_\alpha{}^\beta M_\beta{}^\alpha \mathcal{O}_{\Delta,\ell}(0) = \ell(\ell+1)\mathcal{O}_{\Delta,\ell}(0)\,, \qquad D\mathcal{O}_{\Delta,\ell}(0) = \Delta\mathcal{O}_{\Delta,\ell}(0)\,. \tag{2.48}$$

Further, recall that the special conformal generators $K_{\alpha\beta}$ annihilate all conformal primaries, and so it follows that $\mathcal{O}_{\Delta,\ell}(0)$ is an eigenstate of the Casimir

$$C_C\mathcal{O}_{\Delta,\ell}(0) = \lambda_C(\Delta,\ell)\mathcal{O}_{\Delta,\ell}(0)\,, \qquad \lambda_C(\Delta,\ell) \equiv \ell(\ell+1) + \Delta(\Delta-3)\,. \tag{2.49}$$

Conformal symmetry then implies that this continues to hold away from $\vec{x} = 0$, and so

$$C_C\mathcal{O}(\vec{x}) = \lambda_C(\Delta,\ell)\mathcal{O}(\vec{x})\,. \tag{2.50}$$

The discussion in the previous paragraph can be generalized to the superconformal case for a theory with $\mathcal{N}$-extended superconformal symmetry. (We will of course set $\mathcal{N} = 6$ shortly, but let us keep $\mathcal{N}$ arbitrary for now.) The superalgebra now also includes the Poincaré supercharges $Q_{\alpha I}$, the superconformal charges $S^{\alpha I}$, and the R-symmetry generators R_{IJ}. Using the $\mathfrak{osp}(\mathcal{N}|4)$ commutation relations given in Section 1.1.2, it is straightforward to check that the quadratic superconformal Casimir

$$C_S = C_C + \mathcal{N}D - \frac{1}{2}C_R + \frac{i}{2}Q_{\alpha I}S_I^\alpha\,, \qquad \text{where} \qquad C_R \equiv \frac{1}{2}R_{IJ}R_{IJ}\,, \tag{2.51}$$

commutes with all the conformal generators. Here, the R-symmetry generators are such that when acting on an operator in a representation $\mathbf{r}$ of $\mathfrak{so}(\mathcal{N})$, we have $C_R = \lambda_R(\mathbf{r})$, where $\lambda_R(\mathbf{r})$ is the eigenvalue of the quadratic Casimir of $\mathfrak{so}(\mathcal{N})$ normalized so that $\lambda_R(\mathcal{N}) = \mathcal{N} - 1$. For the case of $\mathfrak{so}(6)$ and the various representations we will encounter, we list the quadratic Casimir eigenvalues in Table 2.4. Equation (2.51) implies that when acting on the superconformal primary operator $\mathcal{O}_{\Delta,\ell,\mathbf{r}}$

irrep $\mathbf{r}$ of $\mathfrak{so}(6)$	$\lambda_R(\mathbf{r})$
$\mathbf{1}$	0
$\mathbf{6}$	5
$\mathbf{15}$	8
$\mathbf{20'}$	12
$\mathbf{45}$, $\overline{\mathbf{45}}$	16
$\mathbf{84}$	20

Table 2.4: Eigenvalues of C_R in the $\mathcal{N} = 6$ case where the R-symmetry algebra is $\mathfrak{so}(6)_R$.

of spin ℓ, dimension Δ, and R-symmetry representation $\mathbf{r}$, placed at $\vec{x}=0$,

$$C_S\mathcal{O}(\vec{x})=\lambda(\Delta,\ell,\mathbf{r})\mathcal{O}(\vec{x})\,,\qquad \lambda_S(\Delta,\ell,\mathbf{r})\equiv\lambda_C(\Delta,\ell)+\mathcal{N}\Delta-\frac{1}{2}\lambda_R(\mathbf{r}) \tag{2.52}$$

Superconformal symmetry then implies that if $\mathcal{O}$ is any operator in a superconformal multiplet whose superconformal primary has dimension Δ, spin ℓ and R-symmetry irrep $\mathbf{r}$, we have

$$C_S\mathcal{O}=\lambda(\Delta,\ell,\mathbf{r})\mathcal{O}\,. \tag{2.53}$$

Let us now use the Casimirs above to obtain an equation for the superconformal blocks. Suppose we have four superconformal primary scalar operators ϕ_i, $i=1,\ldots,4$, of dimension Δ_ϕ and R-symmetry representation $\mathbf{r}_\phi$. The four-point function has a conformal block decomposition

$$\langle\phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4)\rangle=\frac{1}{|\vec{x}_{12}|^{2\Delta_\phi}|\vec{x}_{34}|^{2\Delta_\phi}}\sum_{\substack{\text{conf primaries}\\ \mathcal{O}_{\Delta,\ell,\mathbf{r}}}}c_{\Delta,\ell,\mathbf{r}}\,g_{\Delta,\ell}(U,V)\,. \tag{2.54}$$

A superconformal block corresponding to the supermultiplet $\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}$ whose superconformal primary has quantum numbers $(\Delta_0,\ell_0,\mathbf{r}_0)$ consists of the conformal primary operators in the sum on the right-hand side of (2.54) that belong to the same supermultiplet as $\mathcal{O}_{\Delta_0,\ell_0,\mathbf{r}_0}$:

$$\langle\phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4)\rangle\Big|_{\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}=\frac{1}{|\vec{x}_{12}|^{2\Delta_\phi}|\vec{x}_{34}|^{2\Delta_\phi}}\sum_{\substack{\text{conf primaries}\\ \mathcal{O}_{\Delta,\ell,\mathbf{r}}\,\in\,\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}}c_{\Delta,\ell,\mathbf{r}}\,g_{\Delta,\ell}(U,V)\,. \tag{2.55}$$

Let us now applying the superconformal Casimir operator (2.51) on the first two operators only. To specify which of the four ϕ's an operator is acting on, let us use a subscript "(12)" if the operator is acting on ϕ_1 and ϕ_2 and a superscript "(i)" if the operator acts only on ϕ_i. From (2.53), we see that

$$C_S^{(12)}-C_C^{(12)}+\frac{1}{2}C_R^{(12)}-\sum_{i=1}^{2}\left(C_S^{(i)}-C_C^{(i)}+\frac{1}{2}C_R^{(i)}\right)=\frac{i}{2}\left(Q^{(1)}_{\alpha I}S_I^{(2)\alpha}+Q^{(2)}_{\alpha I}S_I^{(1)\alpha}\right)\,. \tag{2.56}$$

When we apply this expression to (2.55), we act with the Casimirs with upper index (i) on the left-hand side of the equation, and with the ones with upper index (12) on the right-hand side of the equation—for instance $C_C^{(1)}$ simply gives $\lambda_C(\Delta_\phi,0)$, while $C_R^{(12)}$ gives $\lambda_R(\mathbf{r})$. Thus, we obtain the following relation:

$$
\begin{aligned}
&\frac{i}{2}\left(\langle (Q_{\alpha I}\phi_1)(\vec{x}_1)(S_I^\alpha\phi_2)(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4)\rangle - \langle (S_I^\alpha\phi_1)(\vec{x}_1)(Q_{I\alpha}\phi_2)(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4)\rangle\right)\Big|_{\mathcal{M}_{\Delta_0,\ell_0,\mathbf{r}_0}}\\
&= \frac{1}{|\vec{x}_{12}|^{2\Delta_\phi}\,|\vec{x}_{34}|^{2\Delta_\phi}} \sum_{\substack{\text{conf primaries}\\ \mathcal{O}_{\Delta,\ell,\mathbf{r}}\,\in\,\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}} \alpha_{\Delta,\ell,\mathbf{r}} c_{\Delta,\ell,\mathbf{r}}\, g_{\Delta,\ell}(U,V)
\end{aligned}
\tag{2.57}
$$

where

$$
\alpha_{\Delta,\ell,\mathbf{r}} \equiv \lambda_S(\Delta_0,\ell_0,\mathbf{r}_0) - \lambda_C(\Delta,\ell) + \frac{1}{2}\lambda_R(\mathbf{r}) - 2\mathcal{N}\Delta_\phi\,. \tag{2.58}
$$

The right-hand side of equation (2.57) can be easily evaluated provided we know all the conformal primaries occurring in the multiplet $\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}$. To evaluate the left-hand side, note that

$$
(S_I^\alpha\phi)(\vec{x}) = x^\mu\gamma_\mu^{\alpha\beta}(Q_{\beta I}\phi)(\vec{x})\,, \tag{2.59}
$$

and so equation (2.57) becomes

$$
\begin{aligned}
&\frac{i}{2}x_{21}^\mu\gamma_\mu^{\alpha\beta}\langle (Q_{\alpha I}\phi_1)(\vec{x}_1)(Q_{\beta I}\phi_2)(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4)\rangle\Big|_{\mathcal{M}_{\Delta_0,\ell_0,\mathbf{r}_0}}\\
&\qquad= \frac{1}{|\vec{x}_{12}|^{2\Delta_\phi}\,|\vec{x}_{34}|^{2\Delta_\phi}} \sum_{\substack{\text{conf primaries}\\ \mathcal{O}_{\Delta,\ell,\mathbf{r}}\,\in\,\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}} \alpha_{\Delta,\ell,\mathbf{r}} c_{\Delta,\ell,\mathbf{r}}\, g_{\Delta,\ell}(U,V)\,.
\end{aligned}
\tag{2.60}
$$

In general, Ward identities relate the left-hand side of (2.60) to $\langle\phi_1\phi_2\phi_3\phi_4\rangle$, but these relations may not be sufficient to completely determine the left-hand side of (2.60) in terms of $\langle\phi_1\phi_2\phi_3\phi_4\rangle$.

This general discussion can be applied to the case of interest to us, namely the $\langle SSSS\rangle$ correlator in 3d $\mathcal{N}=6$ SCFTs. If we replace $\phi_i(\vec{x}_i)$ by $S(\vec{x}_i,X_i)$, then $\langle SSSS\rangle$ can be expanded in R-symmetry channels as in (2.5), and so can all the equations above. In particular, we replace $c_{\Delta,\ell,\mathbf{r}} \to c^i_{\Delta,\ell,\mathbf{r}}\mathcal{B}_i$ in all these equations, with $c^i_{\Delta,\ell,\mathbf{r}}$, placed in a row vector, determined in terms of the coefficients $a_{\Delta,\ell,\mathbf{r}}$ defined in (2.18) via

$$
c^i_{\Delta,\ell,\mathbf{r}} = \begin{pmatrix} a_{\Delta,\ell,\mathbf{1}_s} & a_{\Delta,\ell,\mathbf{15}_a} & a_{\Delta,\ell,\mathbf{15}_s} & a_{\Delta,\ell,\mathbf{20}'_s} & a_{\Delta,\ell,\mathbf{45}_a\oplus\overline{\mathbf{45}}_a} & a_{\Delta,\ell,\mathbf{84}_s}\end{pmatrix}\mathbf{B}^{-1}\,, \tag{2.61}
$$

with $\mathbf{B}$ defined in (2.17). Thus (2.60) becomes

$$
\begin{aligned}
&\frac{i}{2}x_{21}^{\mu}\gamma_{\mu}^{\alpha\beta}\langle Q_{\alpha I}S(\vec{x}_1,X_1)Q_{\beta I}S(\vec{x}_2,X_2)S(\vec{x}_3,X_3)S(\vec{x}_4,X_4)\rangle\Big|_{\mathcal{M}_{\Delta_0,\ell_0,\mathbf{r}_0}} \\
&\qquad = \frac{1}{|\vec{x}_{12}|^2\,|\vec{x}_{34}|^2}\sum_{\substack{\text{conf primaries}\\ \mathcal{O}_{\Delta,\ell,\mathbf{r}}\in\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}}\alpha_{\Delta,\ell,\mathbf{r}}c^{i}_{\Delta,\ell,\mathbf{r}}\,g_{\Delta,\ell}(U,V)\mathcal{B}_i\,,
\end{aligned}
\tag{2.62}
$$

with $\alpha_{\Delta,\ell,\mathbf{r}}$ evaluated in this particular case to

$$
\alpha_{\Delta,\ell,\mathbf{r}}\equiv\lambda_S(\Delta_0,\ell_0,\mathbf{r}_0)-\lambda_C(\Delta,\ell)+\frac{1}{2}\lambda_R(\mathbf{r})-12\,. \tag{2.63}
$$

To determine what operators appear a given supermultiplet and can contribute to $S\times S$, we use the $\mathfrak{osp}(6|4)$ characters as explained in Appendix B.

The remaining challenge is to evaluate the left-hand side of (2.62). This can be done by noting that $Q_{\alpha I}S(\vec{x},X)$ is a linear combination of the fermions χ, F, and $\overline{F}$ in the stress tensor multiplet, as given in (2.10). Consequently, the left-hand side of (2.62) can be written in terms of the functions $\mathcal{C}^{i,a}$, $\mathcal{E}^{i,a}$, $\mathcal{F}^{i,a}$, and $\mathcal{G}^{i,a}$ introduced in Appendix A to describe the correlators $\langle SS\chi\chi\rangle$, $\langle SS\chi F\rangle$, $\langle SSFF\rangle$ and $\langle SSF\bar{F}\rangle$. Here, the index i runs over the R-symmetry structures and the index $a=1,2$ runs over the two spacetime structures of a fermion-fermion-scalar-scalar correlator. Denoting

$$
\mathcal{X}^{n,a}=(\mathcal{F}^{1,a},\mathcal{F}^{1,a},\mathcal{G}^{1,a},\mathcal{G}^{2,a},\mathcal{G}^{3,a},\mathcal{G}^{4,a},\mathcal{E}^{1,a},\mathcal{E}^{2,a},\mathcal{E}^{3,a},\mathcal{C}^{1,a},\mathcal{C}^{2,a},\mathcal{C}^{3,a})\,, \tag{2.64}
$$

where $n=1,\ldots,12$, we find

$$
\begin{aligned}
&\frac{i}{2}x_{21}^{\mu}\gamma_{\mu}^{\alpha\beta}\langle Q_{\alpha I}S(\vec{x}_1,X_1)Q_{\beta I}S(\vec{x}_2,X_2)S(\vec{x}_3,X_3)S(\vec{x}_4,X_4)\rangle \\
&\qquad = \frac{\sum_{i,n}\beta_{i,n}(\mathcal{X}^{n,1}-\frac{V-U-1}{2U}\mathcal{X}^{n,2})\mathcal{B}_i}{|\vec{x}_{12}|^2\,|\vec{x}_{34}|^2}\,,
\end{aligned}
\tag{2.65}
$$

with the coefficients $\beta_{i,n}$ given by

$$
\beta_{i,n}=\begin{pmatrix}
4 & 4 & -32 & -4 & -4 & 0 & 16 & 16 & 0 & -16 & -128 & -128\\
20 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & -16 & 0 & -128 & -128\\
4 & 20 & 0 & 0 & 0 & 4 & 0 & 0 & 16 & 0 & -128 & -128\\
-12 & -12 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 128 & 128\\
-4 & -8 & 0 & 2 & -10 & -2 & -8 & -24 & -8 & 0 & 128 & 0\\
-8 & -4 & 0 & -10 & 2 & -2 & -24 & -8 & 8 & 0 & 0 & 128
\end{pmatrix}. \tag{2.66}
$$

Thus, equation (2.60) reduces to the 6 equations (one for each i):

$$\sum_{n=1}^{12} \beta_{i,n} \left[\mathcal{X}^{n,1}(U,V) - \frac{V-U-1}{2U} \mathcal{X}^{n,2}(U,V) \right] \Big|_{\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}} = \sum_{\substack{\text{conf primaries} \\ \mathcal{O}_{\Delta,\ell,\mathbf{r}} \in \mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}} \alpha_{\Delta,\ell,\mathbf{r}} c^i_{\Delta,\ell,\mathbf{r}} \, g_{\Delta,\ell}(U,V) \,. \tag{2.67}$$

To use this equation to find the coefficients $c^i_{\Delta,\ell,\mathbf{r}}$ of a given superconformal block, we will also need to expand the fermion-fermion-scalar-scalar correlators on the left-hand side in conformal blocks corresponding to operators belonging to the supermultiplet $\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}$. Fortunately, we do not have to do this for all 24 functions $\mathcal{X}^{n,a}$ because, using the superconformal Ward identities given in Appendix A, we can completely determine $\mathcal{C}^{i,a}$, $\mathcal{E}^{i,a}$, $\mathcal{F}^{i,a}$, and $\mathcal{G}^{i,a}$ from $\mathcal{S}^i$ and $\mathcal{F}^{1,a}$. Since we have already expanded the $\mathcal{S}^i$ in conformal blocks,

$$\mathcal{S}^i(U,V) \Big|_{\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}} = \sum_{\substack{\text{conf primaries} \\ \mathcal{O}_{\Delta,\ell,\mathbf{r}} \in \mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}} c^i_{\Delta,\ell,\mathbf{r}} \, g_{\Delta,\ell}(U,V) \,, \tag{2.68}$$

all that is left to do is to also expand $\mathcal{F}^{1,a}$.

The s-channel conformal block decomposition of a fermion-fermion-scalar-scalar four-point function was derived in [67]. For each conformal primary being exchanged, there are two possible blocks appearing with independent coefficients. For $\mathcal{F}^{1,a}$, if we denote the corresponding coefficients by $d_{\Delta,\ell,\mathbf{r}}$ for the first block and $e_{\Delta,\ell,\mathbf{r}}$ for the second block, we can then write:

$$\begin{pmatrix} \mathcal{F}^{1,1} \\ \mathcal{F}^{1,2} \end{pmatrix} \Big|_{\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}} = \sum_{\substack{\text{conf primaries} \\ \mathcal{O}_{\Delta,\ell,\mathbf{r}} \in \mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}} d_{\Delta,\ell,\mathbf{r}} \begin{pmatrix} g_{\Delta,\ell} \\ 0 \end{pmatrix} + e_{\Delta,\ell,\mathbf{r}} \begin{pmatrix} \mathcal{D}_1 g_{\Delta,\ell} \\ \mathcal{D}_2 g_{\Delta,\ell} \end{pmatrix} , \tag{2.69}$$

where $g_{\Delta,\ell}$ are the scalar conformal blocks appearing above and $\mathcal{D}_{1,2}$ are differential operators:

$$\begin{aligned} \mathcal{D}_1 &= 2 + 2U \left[-2\partial_V - 2V\partial_V^2 - \partial_U + 2U\partial_U^2 \right] \,, \\ \mathcal{D}_2 &= 4U \left[(V-1)(\partial_V + V\partial_V^2) + U(\partial_U + 2V\partial_U\partial_V + U\partial_U^2) \right] \,. \end{aligned} \tag{2.70}$$

(Each doublet of functions $(\mathcal{X}^{n,1}, \mathcal{X}^{n,2})$ appearing on the LHS of (2.67) has a similar block decomposition, but as mentioned above, we only need this decomposition for $(\mathcal{F}^{1,1}, \mathcal{F}^{1,2})$.)

Using the relations between $\mathcal{X}^{n,a}$, $\mathcal{S}^i$ and $\mathcal{F}^{1,a}$ given in Appendix A together with the decompositions (2.68) and (2.69), we obtain a system of linear equations for $c^i_{\Delta,\ell,\mathbf{r}}$, $d_{\Delta,\ell,\mathbf{r}}$, and $e_{\Delta,\ell,\mathbf{r}}$ that has to be obeyed for all values of (U,V). Expanding $g_{\Delta,\ell}$ to sufficiently high orders in U is then enough to determine the linearly-independent solutions of this system of equations, and thus determine the coefficients $c^i_{\Delta,\ell,\mathbf{r}}$ of the superconformal block corresponding to the supermultiplet $\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}$.

We performed this analysis for all the multiplets described in Table 2.3. The coefficients $c^i_{\Delta,\ell,\mathbf{r}}$ for each multiplet are included in the `Mathematica` attached to the paper [24] which this section is based on. The multiplets marked in red in Table 2.3 did not give solutions to the system of equations that determines the $c^i_{\Delta,\ell,\mathbf{r}}$. For each of the remaining multiplets we found between one and three solutions. Since any linear combination of superconformal blocks is a superconformal block, we are free to choose a basis of blocks with specific normalizations. In other words, the coefficients $a_{\Delta,\ell,\mathbf{r}}$ in (2.18) can be written as

$$a_{\Delta,\ell,\mathbf{r}} = \sum_I \lambda_I^2 a^I_{\Delta,\ell,\mathbf{r}}\,, \tag{2.71}$$

where I ranges over all superconformal blocks, λ_I^2 are theory-dependent coefficients, and $a^I_{\Delta,\ell,\mathbf{r}}$ represent the solution to the super-Casimir equation for superconformal block I, normalized according to our conventions. In Table 2.5, we list all the superconformal blocks as well as enough values for $a^I_{\Delta,\ell,\mathbf{r}}$ in order to determine the normalization of the blocks.[4] A superconformal block $\mathfrak{G}_I$ is simply

$$\mathfrak{G}^{\mathbf{r}}_I(U,V) = \sum_{\substack{\text{conf primaries}\\ \mathcal{O}_{\Delta,\ell,\mathbf{r}}\,\in\,\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}}} a^I_{\Delta,\ell,\mathbf{r}}\, g_{\Delta,\ell}(U,V)\,, \qquad I \equiv \mathcal{M}^{\mathbf{r}_0,n}_{\Delta_0,\ell_0}\,, \tag{2.72}$$

where the index $I = \mathcal{M}^{\mathbf{r}_0,n}_{\Delta_0,\ell_0}$ of the block encodes both the supermultiplet $\mathcal{M}^{\mathbf{r}_0}_{\Delta_0,\ell_0}$ as well as an integer $n = 1,2,\ldots$ denoting which block this is according to Table 2.5. In the cases where there is a single superconformal block per multiplet, we omit the index n.

Table 2.5 also includes the $\mathcal{P}$ and $\mathcal{Z}$ charges relative to that of the superconformal primary, which are relevant for $\mathcal{N}=6$ theories that are invariant under these discrete symmetries. We derive $\mathcal{P}$ charges for each superblock by noting that any two primaries $\mathcal{O}_{\Delta,\ell}$ and $\mathcal{O}_{\Delta',\ell'}$ in a supermultiplet have the same parity if and only if $\Delta'-\Delta \equiv \ell'-\ell \mod 2$. To derive the $\mathcal{Z}$ charges we use the $O(6)$ tensor product of two pseudo-tensors:

$$\mathbf{15}^- \otimes \mathbf{15}^- = \mathbf{1}^+_s \oplus \mathbf{15}^+_a \oplus \mathbf{15}^-_s \oplus \mathbf{20'}^+_s \oplus \mathbf{90}_a \oplus \mathbf{84}^+_s\,. \tag{2.73}$$

[4]The $(A,+)$ and $(A,-)$ multiplets are each other's complex conjugates and must appear together in the $S\times S$ OPE.

Superconformal block	normalization	$\mathcal{P}$	$\mathcal{Z}$	Isolated?
$\text{Long}_{\Delta,0}^{[000],n}$	$n=1:\quad (a_{\Delta,0,\mathbf{1}}, a_{\Delta+1,0,\mathbf{20'}}) = (1,0)$	$+$	$+$	
	$n=2:\quad (a_{\Delta,0,\mathbf{1}}, a_{\Delta+1,0,\mathbf{20'}}) = (0,1)$	$-$	$+$	
$\text{Long}_{\Delta,\ell}^{[000]}$, $\ell \geq 1$ odd	$a_{\Delta+1,\ell+1,\mathbf{15}_s} = 1$	$+$	$+$	
$\text{Long}_{\Delta,\ell}^{[000],n}$, $\ell \geq 2$ even	$n=1:\quad (a_{\Delta,\ell,\mathbf{1}}, a_{\Delta+1,\ell,\mathbf{1}}, a_{\Delta+1,\ell,\mathbf{15}_s}) = (1,0,0)$	$+$	$+$	
	$n=2:\quad (a_{\Delta,\ell,\mathbf{1}}, a_{\Delta+1,\ell,\mathbf{1}}, a_{\Delta+1,\ell,\mathbf{15}_s}) = (0,1,0)$	$-$	$+$	
	$n=3:\quad (a_{\Delta,\ell,\mathbf{1}}, a_{\Delta+1,\ell,\mathbf{1}}, a_{\Delta+1,\ell,\mathbf{15}_s}) = (0,0,1)$	$-$	$-$	
$(A,1)_{\ell+2,\ell}^{[100],n}$, $\ell - \frac{1}{2} \geq 1$ odd	$n=1:\quad (a_{\ell+\frac{5}{2},\ell+\frac{1}{2},\mathbf{1}}, a_{\ell+\frac{5}{2},\ell+\frac{1}{2},\mathbf{15}_s}) = (1,0)$	$+$	$+$	
	$n=2:\quad (a_{\ell+\frac{5}{2},\ell+\frac{1}{2},\mathbf{1}}, a_{\ell+\frac{5}{2},\ell+\frac{1}{2},\mathbf{15}_s}) = (0,1)$	$+$	$-$	
$(A,2)_{\ell+2,\ell}^{[011]}$, $\ell \geq 0$ even	$a_{\ell+2,\ell,\mathbf{15}_s} = 1$	$+$	$-$	✓
$(A,2)_{\ell+2,\ell}^{[011]}$, $\ell \geq 0$ odd	$a_{\ell+2,\ell,\mathbf{15}_a} = 1$	$+$	$+$	✓
$(A,+)_{\ell+2,\ell}^{[020]}$, $\ell - \frac{1}{2} \geq 0$ even	$a_{\ell+\frac{5}{2},\ell+\frac{1}{2},\mathbf{15}_a} = 1$	$+$		✓
$(A,-)_{\ell+2,\ell}^{[002]}$, $\ell - \frac{1}{2} \geq 0$ even	$a_{\ell+\frac{5}{2},\ell+\frac{1}{2},\mathbf{15}_a} = 1$	$+$		✓
$(A,\text{cons})_{\ell+1,\ell}^{[000]}$, $\ell \geq 0$ even	$a_{\ell+1,\ell,\mathbf{1}} = 1$	$+$	$+$	
$(A,\text{cons})_{\ell+1,\ell}^{[000]}$, $\ell \geq 1$ odd	$a_{\ell+2,\ell+1,\mathbf{15}_s} = 1$	$+$	$-$	
$(B,1)_{2,0}^{[200]}$	$a_{2,0,\mathbf{20'}} = 1$	$+$	$+$	
$(B,2)_{2,0}^{[022]}$	$a_{2,0,\mathbf{84}} = 1$	$+$	$+$	✓
$(B,2)_{1,0}^{[011]}$	$a_{1,0,\mathbf{15}_s} = 1$	$+$	$-$	✓

Table 2.5: A summary of the superconformal blocks and their normalizations in terms of a few OPE coefficients. The values $a_{\Delta,\ell,\mathbf{r}}$ in this table correspond to $a^I_{\Delta,\ell,\mathbf{r}}$ in Eq. (2.71)—we omitted the index I for clarity. The right-hand column lists whether the superconformal blocks are isolated, as described in the main text. Note that the $(A,\pm)$ are complex conjugates and do not by themselves have well-defined $\mathcal{Z}$ parity, but together they can be combined into a $\mathcal{Z}$-even and a $\mathcal{Z}$-odd structure.

Reflection positivity implies that the coefficients $a_{\Delta,\ell,\mathbf{r}}$ in (2.18) are non-negative for all $\mathbf{r}$. Because for each superconformal block in Table 2.5 there exists an operator that receives contributions only from that block, it follows that the coefficients λ_I^2 in (2.71) are non-negative. This is the reason why we wrote these coefficients in (2.71) manifestly as perfect squares. They are the squares of real OPE coefficients.[5]

Let us end this section by describing the unitarity limits of the long blocks obtained by taking $\Delta \to \ell + 1$. For the scalar blocks, we obtain (up to normalization) either a spin-0 conserved block for the parity-even structure or a $(B,1)^{[200]}_{2,0}$ block for the parity odd structure:

$$\text{Long}^{[000],1}_{\Delta,0} \to (A,\text{cons})^{[000]}_{1,0}\,, \qquad \text{Long}^{[000],2}_{\Delta,0} \to (B,1)^{[200]}_{2,0}\,. \tag{2.74}$$

For odd $\ell \geq 1$ there is a single block, which it approaches a spin-ℓ conserved block:

$$\ell \geq 1 \text{ odd:} \qquad \text{Long}^{[000]}_{\Delta,\ell} \to (A,\text{cons})^{[000]}_{\ell+1,\ell}\,. \tag{2.75}$$

Lastly, for even $\ell \geq 2$ we have three superconformal blocks. The parity even one approaches a spin-ℓ conserved block, while the parity odd ones approach the two superconformal blocks for the $(A,1)^{[100]}_{\ell+3/2,\ell-1/2}$ multiplet:

$$\begin{aligned} \ell \geq 2 \text{ even:} \qquad & \text{Long}^{[000],1}_{\Delta,\ell} \to (A,\text{cons})^{[000]}_{\ell+1,\ell}\,, \\ & \text{Long}^{[000],2}_{\Delta,\ell} \to (A,1)^{[100],1}_{\ell+3/2,\ell-1/2}\,, \\ & \text{Long}^{[000],3}_{\Delta,\ell} \to (A,1)^{[100],2}_{\ell+3/2,\ell-1/2}\,. \end{aligned} \tag{2.76}$$

Even though the blocks on the RHS of (2.74)–(2.76) involve short or semishort superconformal multiplets, they sit at the bottom of the continuum of long superconformal blocks. All other short and semishort superconformal blocks are isolated, as they cannot recombine into a long superconformal block. In particular, if the correlator $\langle SSSS \rangle$ contains one of these isolated superconformal blocks, any sufficiently small deformation of $\langle SSSS \rangle$ also must, while the other blocks can instead disappear by recombining into a long block. This distinction will be important when we consider the numerical bootstrap.

[5]In other words, for each multiplet for which there are several superconformal blocks, the number of superconformal 3-point structures equals the number of superconformal blocks. This is so because each superconformal 3-point structure contains different operators from the exchanged multiplet.

2.4 Examples: GFFT and Free Field Theory

Let us begin by computing $\langle SSSS \rangle$ in free field theory, which, as we recall from Section 1.2 consists of four complex scalars $\Phi^a(x)$ and four complex fermions Ψ_a^α, with two-point functions given in (1.33). Stress tensor multiplet operators are bilinears of the free Φ and Ψ fields, and in particular

$$S(\vec{x}, X) = 4\pi \Phi^a(\vec{x}) \bar{\Phi}_b(\vec{x}) X_a{}^b \,, \qquad P(\vec{x}, X) = -2\sqrt{2}\pi i \bar{\Psi}^{\alpha a}(\vec{x}) \Psi_{\alpha b}(\vec{x}) X_a{}^b \,. \tag{2.77}$$

The $\langle SSSS \rangle$ correlator can then be computed using Wick contractions of Φ and $\bar{\Phi}$, and so we find that

$$\begin{aligned} \mathcal{S}^i_{\text{free-1}}(U,V) &= \mathcal{S}^i_{\text{disc}}(U,V) + \mathcal{S}^i_{\text{free}}(U,V) \,, \\ \text{where} \quad \mathcal{S}^i(U,V)_{\text{disc}} &= \begin{pmatrix} 1 & U & \frac{U}{V} & \frac{U}{\sqrt{V}} & 0 & 0 & 0 \end{pmatrix} , \\ \mathcal{S}^i(U,V)_{\text{free}} &= \begin{pmatrix} 0 & 0 & 0 & \frac{U}{\sqrt{V}} & \frac{\sqrt{U}}{\sqrt{V}} & \sqrt{U} \end{pmatrix} . \end{aligned} \tag{2.78}$$

Here we have separated out the disconnected correlator $\mathcal{S}^i_{\text{disc}}(U,V)$ from the free connected correlator $\mathcal{S}^i_{\text{free}}(U,V)$, both of which play important roles in this thesis.

We can also consider a more general free field theory with N hypermultiplets $(\Phi_i^a, \Psi_{ia}^\alpha)$, which has $c_T = 16N$. We then define

$$S(\vec{x}, X) = \frac{4\pi}{N} X_a{}^b \sum_{i=1}^N \Phi_i^a(\vec{x}) \bar{\Phi}_{ib}(\vec{x}) \,, \tag{2.79}$$

and so find that for this theory

$$\mathcal{S}^i_{\text{free-}N}(U,V) = \mathcal{S}^i_{\text{disc}}(U,V) + \frac{1}{N} \mathcal{S}^i_{\text{free}}(U,V) \,. \tag{2.80}$$

In the limit where we take $N \to \infty$, all connected correlators vanish and we have what is known as generalized free field theory (GFFT). All correlation functions can be computed by simply taking Wick contractions of S using (2.4). It is not a true superconformal field theory because $c_T \to \infty$ in this limit, so that the "stress tensor" decouples from all other operators in the theory. Nevertheless, GFFT is important as it represents the leading term in any $1/c_T$ expansion.

With our superconformal blocks in hand, we can now determine the superconformal decomposition of both free field theory and generalized free field theory. We will begin with the latter theory. If we think of the S operators as single-trace, then in the superconformal block decomposition of

$\langle SSSS\rangle$ only double-trace operators appear. These schematically take the form $S\partial_{\mu_1}\cdots\partial_{\mu_\ell}\Box^p S$, with spin ℓ and conformal dimension $2+\ell+2p$. We can expand the disconnected correlator $\mathcal{S}^i_{\text{disc}}(U,V)$ in superconformal blocks, and so produce the CFT data given in Table 2.6.

Now consider the free field theory with $N=1$, where the four-point function is given in (2.78). This four-point function can be expanded into superconformal blocks to give the CFT data listed in Table 2.6. Note that the same spectrum of supermultiplets contribute to free theory and GFFT, except that free theory also contains conserved current multiplets for each spin, has a stress tensor multiplet, and does not have a $(B,1)^{[200]}_{2,0}$ multiplet.

	$\mathcal{S}^i_{\text{disc}}$	$\mathcal{S}^i_{\text{free-1}}$
$\lambda^2_{(B,2)^{[011]}_{1,0}}$	0	4
$\lambda^2_{(B,2)^{[022]}_{2,0}}$	2	4
$\lambda^2_{(B,1)^{[200]}_{2,0}}$	$\frac{4}{3}$	0
$\lambda^2_{(A,\text{cons.})^{[000]}_{\ell+1,\ell}}$ for $\ell=0,1,2,\ldots$	0	4
$\lambda^2_{(A,2)^{[011]}_{\ell+2,\ell}}$ for $\ell=0,1,2,\ldots$	$\frac{16}{3},\ \frac{256}{45},\ \frac{4096}{525},\ \frac{32768}{3675},\ \ldots$	$\frac{8}{3},\ \frac{32}{5},\ \frac{3712}{525},\ \frac{34304}{3675},\ \ldots$
$\lambda^2_{(A,+)^{[002]}_{\ell+5/2,\ell+1/2}}$ for $\ell=0,2,\ldots$	$\frac{16}{9},\ \frac{6144}{1225},\ \ldots$	$\frac{8}{3},\ \frac{7872}{1225},\ \ldots$
$\lambda^2_{(A,1)^{[100],1}_{\ell+7/2,\ell+3/2}}$ for $\ell=0,2,\ldots$	$\frac{512}{315},\ \ldots$	$\frac{64}{63},\ \ldots$
$\lambda^2_{(A,1)^{[100],2}_{\ell+7/2,\ell+3/2}}$ for $\ell=0,2,\ldots$	$\frac{1024}{105},\ \ldots$	$\frac{128}{21},\ \ldots$
$\Delta_{(0,1)}$	$2,\ 4,\ \ldots$	$2,\ 4,\ \ldots$
$\Delta_{(0,2)}$	$3,\ 5,\ \ldots$	$3,\ 5,\ \ldots$
$\Delta_{\ell\geq 1}$, ℓ odd	$\ell+2,\ \ell+4,\ \ldots$	$\ell+2,\ \ell+4,\ \ldots$
$\Delta_{(\ell\geq 2,1)}$, ℓ even	$\ell+2,\ \ell+4,\ \ldots$	$\ell+2,\ \ell+4,\ \ldots$
$\Delta_{(\ell\geq 2,2)}$, ℓ even	$\ell+3,\ \ell+5,\ \ldots$	$\ell+3,\ \ell+5,\ \ldots$
$\Delta_{(\ell\geq 2,3)}$, ℓ even	$\ell+3,\ \ell+5,\ \ldots$	$\ell+3,\ \ell+5,\ \ldots$

Table 2.6: Low-lying CFT data for the generalized free field theory (GFFT) $\mathcal{S}^i_{\text{disc}}$ and the free theory $\mathcal{S}^i_{\text{free}}$. We write $\Delta_{(\ell,n)}$ to denote the scaling dimension of the superblock corresponding to the structure $\text{Long}^{[000],n}_{\Delta,\ell}$.

For both the GFFT theory and the free theory of an $\mathcal{N}=6$ hypermultiplet, one can alternatively obtain the CFT data listed in Table 2.6 by performing a decomposition of the correlators in the analogous $\mathcal{N}=8$ SCFTs, as described in Appendix C. Indeed, the $\mathcal{N}=6$ GFFT is a subsector of the $\mathcal{N}=8$ GFFT, where the $\mathcal{N}=6$ stress tensor multiplet is embedded into the $\mathcal{N}=8$ stress tensor multiplet. Similarly, as noted in Section 1.2, the $\mathcal{N}=6$ free field theory has $\mathcal{N}=8$ supersymmetry.

As a final task, we shall use the free field conformal block decomposition to relate $\lambda^2_{(B,2)^{[011]}_{1,0}}$ to

c_T. Recall that $\lambda_{(B,2)^{[011]}_{1,0}}$ is the OPE coefficient for the three-point function

$$\langle S(x_1, X_1) S(x_2, X_2) S(x_3, X_3) \rangle = \frac{\lambda_{(B,2)^{[011]}_{1,0}}}{2} \frac{\operatorname{tr}(X_1 \{X_2, X_3\})}{\sqrt{x_{12} x_{13} x_{23}}} \,, \tag{2.81}$$

which is related by supersymmetry to the three-point function $\langle SST^{\mu\nu} \rangle$. This latter three-point function is completely fixed by the conformal Ward identity (1.32), and so must be proportional to $c_T^{-1/2}$. From this we can conclude that the quantity $\lambda_{(B,2)^{[011]}_{1,0}} c_T^{1/2}$ is completely fixed by supersymmetry, and so must be the same in all $\mathcal{N} = 6$ theories. Because in free field theory $c_T = 16$ and $\lambda_{(B,2)^{[011]}_{1,0}} = 2$, we conclude that

$$\lambda_{(B,2)^{[011]}_{1,0}} = \frac{8}{\sqrt{c_T}} \tag{2.82}$$

in any $\mathcal{N} = 6$ superconformal field theory.

Chapter 3

Exact Results from Supersymmetric Localization

In this chapter, we discuss the constraints that supersymmetric localization places on scalar four-point functions in $\mathcal{N}=6$ theories. Supersymmetric localization is a technique for computing observables in a supersymmetric field theory which are closed but not exact under certain $\mathcal{Q}$ supercharges. By deforming the path-integral with a term which is $\mathcal{Q}$-exact, in certain favorable conditions the path-integral will localize onto specific BPS field configurations, reducing the calculation to an ordinary integral.

In this thesis we focus on the mass-deformed S^3 partition function. Given any conformal field theory defined on $\mathbb{R}^3$, we can define correlators on the sphere S^3 by performing a Weyl transform

$$\mathcal{O}_{\text{sphere}}(\vec{x}) = \Omega(\vec{x})^{\Delta}\mathcal{O}_{\text{flat}}(\vec{x})\,, \qquad \text{with} \qquad \Omega(\vec{x}) = \frac{1}{1+\frac{x^2}{4r^2}}\,, \tag{3.1}$$

where r is the radius of the three-sphere with metric

$$ds^2 = \Omega(\vec{x})^2 d\vec{x}^2\,. \tag{3.2}$$

If we place an $\mathcal{N}=2$ superconformal field theory with a $U(1)_F$ flavor current on S^3, the theory admits a real mass deformation

$$m\int_{S^3} d^3\vec{x}\left(\frac{iJ}{r}+K\right)+O(m^2)\,, \tag{3.3}$$

where the operators $J(\vec{x})$ and $K(\vec{x})$ are scalars with conformal dimension $\Delta = 1$ and 2 respectively, belonging to the same superconformal multiplet as the flavor current. This deformation breaks conformal symmetry, but preserves the S^3 isometries. It is furthermore closed, but not exact, under certain linear combinations of the $Q_{\alpha I}$ and $S_{\alpha I}$ supercharges. In the presence of this mass deformation, supersymmetric localization can be used to compute the S^3 partition function $Z(m)$ exactly.

Now consider $\mathcal{N} = 6$ superconformal theories, which possess both an $SO(6)_R$ R-symmetry and a $U(1)_F$ flavor symmetry. When viewed as an $\mathcal{N} = 2$ theory, we can choose a $U(1)_R$ subalgebra to form the $\mathcal{N} = 2$ R-symmetry. There are then three other $U(1)$ generators which commute both with each other and with the $U(1)_R$, which, from the $\mathcal{N} = 2$ perspective, are the Cartans of a $SO(4) \times U(1)$ flavor symmetry. We can associate to each of these commuting $U(1)$'s a real mass parameter, giving us a total of three distinct real mass parameters for any $\mathcal{N} = 6$ theory.

We will focus on just two of these three mass parameters,[1]

$$m_+ \int_{S^3} (iJ_+ + K_+) + m_- \int_{S^3} (iJ_- + K_-) + O(m_\pm^2)\,, \tag{3.4}$$

where for simplicity we have set the sphere radius $r = 1$, and where we define

$$\begin{aligned} J_\pm(\vec{x}) &= \frac{\sqrt{c_T}}{2^5\pi} S(\vec{x}, X_\pm)\,, \qquad K_\pm(\vec{x}) = \frac{\sqrt{2c_T}}{2^5\pi} P(\vec{x}, X_\mp)\,, \\ \text{with } X_+ &= \operatorname{diag}\{1,-1,0,0\}\,, \qquad X_- = \operatorname{diag}\{0,0,1,-1\}\,. \end{aligned} \tag{3.5}$$

The reason for the peculiar normalization of the operators $J_\pm$ and $K_\pm$, and indeed the term "mass deformation", becomes apparent if we consider the precise expression for (3.4) in free field theory. Recall from Section 1.2 that the $\mathcal{N} = 6$ free field theory consists of an $\mathcal{N} = 6$ hypermultiplet (Φ^a, Ψ_a^α) and its complex conjugate $(\bar{\Phi}_a, \bar{\Psi}^{\alpha a})$. Using the equations (2.77) which relate S and P to these fields, we can rewrite the mass deformation as

$$\begin{aligned} i\int d^3\vec{x}\,\sqrt{g(\vec{x})}\,\bigg(&\frac{m_+}{2}|\Phi^1|^2 - \frac{m_+}{2}|\Phi^2|^2 - \frac{m_-}{2}|\Phi^3|^2 + \frac{m_-}{2}|\Phi^4|^2 \\ &+\frac{m_-}{2}|\Psi_1|^2 - \frac{m_-}{2}|\Psi_2|^2 - \frac{m_+}{2}|\Psi_3|^2 + \frac{m_+}{2}|\Psi_4|^2 + O(m_\pm^2)\bigg)\,. \end{aligned} \tag{3.6}$$

Hence, for free field theories, and more generally for the Lagrangian theories built from free field theories, the mass deformations quite literally gives masses to the hypermultiplets.

[1] In terms of symmetries, the two mass parameters that we consider correspond to linear combinations of $U(1)_F$ and one of the Cartans of an $SU(2)$ factor inside $SO(4) \cong SU(2) \times SU(2)$.

Our plan for the rest of the chapter is as follows. In the next section, we will derive expressions for various derivatives of $Z(m_+, m_-)$. These expressions rely only on superconformal symmetry and so hold in any $\mathcal{N}=6$ superconformal field theory. In Sections 3.2 and 3.3 we then compute explicit localization results for the $U(N)_k \times U(N+M)_{-k}$ ABJ and $SO(2)_{2k} \times USp(2+2M)_{-k}$ OSp families of theories, respectively. Finally, in Section 3.4 we show that additional $U(1)$ factors do not change $Z(m_+, m_-)$, up to an overall constant. As a result, the localization calculations performed in this chapter cover all known families of $\mathcal{N}=6$ Lagrangian theories.

3.1 Integrated Correlators on S^3

Our aim in this section is to derive simplified expressions for derivatives of the S^3 sphere partition function $Z(m_+, m_-)$. Let us begin with the second derivatives of $Z(m_+, m_-)$:

$$\left.\frac{\partial^2 \log Z}{\partial m_+^2}\right|_{m_\pm=0}, \qquad \left.\frac{\partial^2 \log Z}{\partial m_+ \partial m_-}\right|_{m_\pm=0}, \qquad \left.\frac{\partial^2 \log Z}{\partial m_-^2}\right|_{m_\pm=0}. \tag{3.7}$$

Combining (3.5) with the S and P two-point functions (2.4) and then explicitly evaluating the integrals, we find that [72]

$$\left.\frac{\partial^2 \log Z}{\partial m_\pm^2}\right|_{m_\pm=0} = \left\langle \int_{S^3} (iJ_\pm + K_\pm)^2 \right\rangle = -\frac{\pi^2 c_T}{64}. \tag{3.8}$$

Naively, we might expect that the mixed mass derivative should vanish, as

$$\begin{aligned}\left.\frac{\partial^2 \log Z}{\partial m_+ \partial m_-}\right|_{m_\pm=0} &= \left\langle \left(\int_{S^3} iJ_+ + K_+\right)\left(\int_{S^3} iJ_- + K_-\right)\right\rangle \\ &= \frac{ic_T}{128\sqrt{2}\pi^2} \int d^3x_1 d^3x_2 \sqrt{g(x_1)}\sqrt{g(x_2)} \langle S(x_1, X_+) P(x_2, X_+)\rangle.\end{aligned} \tag{3.9}$$

Conformal invariance however only requires that $\langle S(x_1, X_1) P(x_2, X_2)\rangle$ vanishes at separated points, and allows the possibility of a delta function [72, 73]

$$\langle S(x_1, X_1) P(x_2, X_2)\rangle = \kappa \operatorname{tr}(X_1 X_2) \delta^{(3)}(x_1 - x_2). \tag{3.10}$$

We hence find that

$$\left.\frac{\partial^2 \log Z}{\partial m_+ \partial m_-}\right|_{m_\pm=0} = \frac{ic_T \kappa}{64\sqrt{2}}, \tag{3.11}$$

and as we shall see κ does not generically vanish in ABJ theory.

Having derived expressions for the second derivatives of $Z(m_+, m_-)$, we next move onto third derivatives. It is straightforward to check that these all vanish

$$\left.\frac{\partial^3 \log Z}{\partial m_+^3}\right|_{m_\pm=0} = \left.\frac{\partial^3 \log Z}{\partial m_+^2 \partial m_-}\right|_{m_\pm=0} = \left.\frac{\partial^3 \log Z}{\partial m_+ \partial m_-^2}\right|_{m_\pm=0} = \left.\frac{\partial^3 \log Z}{\partial m_-^3}\right|_{m_\pm=0} = 0\,, \tag{3.12}$$

because the trace of any three $X_\pm$ matrices always vanish. Finally, we move to fourth derivatives

$$\left.\frac{\partial^4 \log Z}{\partial m_+^4}\right|_{m_\pm=0}, \qquad \left.\frac{\partial^4 \log Z}{\partial m_+^3 \partial m_-}\right|_{m_\pm=0}, \qquad \left.\frac{\partial^4 \log Z}{\partial m_+^2 \partial m_-^2}\right|_{m_\pm=0}. \tag{3.13}$$

Note that because X_+ and X_- are related by an $O(6)$ transformation and all four-point scalar correlators are $\mathcal{Z}$-invariant,

$$\left.\frac{\partial^4 \log Z}{\partial m_+^4}\right|_{m_\pm=0} = \left.\frac{\partial^4 \log Z}{\partial m_-^4}\right|_{m_\pm=0}, \qquad \left.\frac{\partial^4 \log Z}{\partial m_+^3 \partial m_-}\right|_{m_\pm=0} = \left.\frac{\partial^4 \log Z}{\partial m_+ \partial m_-^3}\right|_{m_\pm=0}, \tag{3.14}$$

so that we need only consider the three expressions given in (3.13). We can now directly compute

$$\begin{aligned}
\frac{\partial^4 \log Z}{\partial m_+^4} &= \left\langle \left(\int (iJ_+ + K_+)\right)^4 \right\rangle_{\text{conn}} + (\text{2- and 3-pt functions})\,,\\
\frac{\partial^3 \log Z}{\partial m_+^3 \partial m_-} &= \left\langle \left(\int (iJ_+ + K_+)\right)^3 \left(\int (iJ_- + K_-)\right) \right\rangle_{\text{conn}} + (\text{2- and 3-pt functions})\,,\\
\frac{\partial^4 \log Z}{\partial m_+^2 \partial m_-^2} &= \left\langle \left(\int (iJ_+ + K_+)\right)^2 \left(\int (iJ_- + K_-)\right)^2 \right\rangle_{\text{conn}} + (\text{2- and 3-pt functions})\,.
\end{aligned} \tag{3.15}$$

where the 2- and 3-point function terms not written in (3.15) come from the $O(m^2)$ terms not written in (3.4). We will not write down these 2- and 3-point function contributions because they will be automatically taken into account in the final formulas, by analogy with the similar situation encountered in [74].

While in principle we can evaluate each of these expressions by expanding out (3.15), it is possible to obtain simpler formulas by making use of the fact that all $\mathcal{N} \geq 4$ SCFTs in 3d have a 1d topological sector [75–80]. In general, a 3d $\mathcal{N} = 4$ SCFT has $SU(2)_H \times SU(2)_C$ R-symmetry, and one can consider 1/2-BPS operators that have scaling dimension $\Delta = j_H$, where j_H is the $SU(2)_H$ spin, and are invariant under $SU(2)_C$. Such operators can be written as rank-$2j_H$ symmetric tensors $\mathcal{O}_{a_1 a_2 \ldots a_{2j_H}}(\vec{x})$ where $a_i = 1, 2$ are $SU(2)_H$ spinor indices. From these operators, we can construct 1d topological operators by inserting them on a line, say the line $(0, 0, x)$, and contracting the $SU(2)_H$

indices with position-dependent polarizations:

$$\tilde{\mathcal{O}}_{\mathbb{R}^3}(x) = \mathcal{O}_{a_1 a_2 \ldots a_{2j_H}}(0,0,x) u^{a_1}(x) \cdots u^{a_{2j_H}}(x)\,, \tag{3.16}$$

where we can take[2]

$$u^a(x) = \begin{pmatrix} 1 + \frac{ix}{2} \\ 1 - \frac{ix}{2} \end{pmatrix} . \tag{3.17}$$

If we want to express the topological operator in terms of the operator $\mathcal{O}_{a_1 a_2 \ldots a_{2j_H}}$ when the theory is placed on S^3, we have

$$\tilde{\mathcal{O}}(x) = \frac{1}{\left(1 + \frac{x^2}{4}\right)^{j_H}} \mathcal{O}_{a_1 a_2 \ldots a_{2j_H}}(0,0,x) u^{a_1}(x) \cdots u^{a_{2j_H}}(x)\,, \tag{3.18}$$

where the extra factor accounts for the fact that the operators on $\mathbb{R}^3$ and those on S^3 differ by a Weyl factor. In this case, the 1d topological theory lives on a circle parameterized by x, with the point at $x = +\infty$ being identified with the point at $x = -\infty$.

To connect this discussion to the $\mathcal{N} = 6$ case, let us embed the $\mathcal{N} = 4$ $SU(2)_H \times SU(2)_C$ R-symmetry into $SU(4)_R$ such that $SU(2)_H$ corresponds to the top left 2×2 block of an $SU(4)_R$ matrix written in the fundamental representation and $SU(2)_C$ corresponds to the bottom right 2×2 block. Raising and lowering indices with the epsilon symbol, Eqs. (3.16) and (3.18) applied to S give

$$\tilde{S}(x) = \frac{\left(1 + \frac{ix}{2}\right)^2}{1 + \frac{x^2}{4}} {S_1}^2(0,0,x) - \frac{\left(1 - \frac{ix}{2}\right)^2}{1 + \frac{x^2}{4}} {S_2}^1(0,0,x) + {S_1}^1(0,0,x) - {S_2}^2(0,0,x) \tag{3.19}$$

on S^3 and $\tilde{S}_{\mathbb{R}^3}(x) = \left(1 + \frac{x^2}{4}\right) S(x)$ on $\mathbb{R}^3$. It is straightforward to check that the superconformal Ward identities (2.12) imply that the four-point function of $\tilde{S}_{\mathbb{R}^3}$, namely

$$\begin{aligned} \langle \tilde{S}_{\mathbb{R}^3}(x_1) \tilde{S}_{\mathbb{R}^3}(x_2) \tilde{S}_{\mathbb{R}^3}(x_3) \tilde{S}_{\mathbb{R}^3}(x_4) \rangle &= \mathcal{S}^1 + \frac{\mathcal{S}^2}{z^2} + \frac{(1-z)^2 \mathcal{S}^3}{z^2} \\ &\quad + \frac{2(1-z)\mathcal{S}^4}{z^2} - \frac{2(1-z)\mathcal{S}^5}{z} + \frac{2\mathcal{S}^6}{z}\Big|_{\substack{U = z^2 \\ V = (1-z)^2}} \\ \text{with } z \equiv \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}\,, \end{aligned} \tag{3.20}$$

[2]In the notation of [78] this choice corresponds to ${h_a}^b = {(\sigma_3)_a}^b$.

is piece-wise constant.

The advantage of the topological sector is that we can replace the integrated operator $\int_{S^3} d^3\vec{x}\,\sqrt{g}(iJ_+ + K_+)$ by a different operator that is integrated only along the circle. Such a replacement can be rigorously justified in the class of $\mathcal{N}=4$ theories studied in [78–80] where it was shown how one can obtain a 1d action for the topological sector by using supersymmetric localization in the 3d $\mathcal{N}=4$ theory. Unfortunately, the theories considered in this thesis fall outside the range of theories studied in [78–80]. Nevertheless, as explained in Section 3.1 of [81], we expect that such a replacement should be possible in these theories as well, and in particular that

$$4\pi \int \frac{dx}{1+\frac{x^2}{4}} i\tilde{J}(x) = \int d^3\vec{x}\,\sqrt{g}(iJ_+ + K_+) + Q\text{-exact terms}\,, \tag{3.21}$$

where we define

$$\tilde{J}(x) = \frac{\sqrt{c_T}}{64\pi}\tilde{S}(x)\,. \tag{3.22}$$

Thus, instead of (3.15), we may write

$$\begin{aligned}
\frac{\partial \log Z}{\partial m_+^4} &= (4\pi)^4 \left\langle \left(\int \frac{dx}{1+\frac{x^2}{4}} i\tilde{J}(x) \right)^4 \right\rangle_{\text{conn}}\,, \\
\frac{\partial \log Z}{\partial m_+^3 \partial m_-} &= (4\pi)^3 \left\langle \left(\int d^3\vec{x}\,\sqrt{g}\,(iJ_-(\vec{x}) + K_-(\vec{x})) \right), \left(\int \frac{dx}{1+\frac{x^2}{4}} i\tilde{J}(x) \right)^3 \right\rangle_{\text{conn}} \\
\frac{\partial \log Z}{\partial m_+^2 \partial m_-^2} &= (4\pi)^2 \left\langle \left(\int d^3\vec{x}\,\sqrt{g}\,(iJ_-(\vec{x}) + K_-(\vec{x})) \right)^2 \left(\int \frac{dx}{1+\frac{x^2}{4}} i\tilde{J}(x) \right)^2 \right\rangle_{\text{conn}}\,.
\end{aligned} \tag{3.23}$$

Let us begin with the first equation in (3.23). Because the correlation function $\langle \tilde{J}\tilde{J}\tilde{J}\tilde{J}\rangle$ is topological, we can place the four operators at any four locations of our choosing and multiply the answer by $(2\pi)^4$. Using (3.19), we have

$$\frac{\partial \log Z}{\partial m_+^4} = \frac{\pi^4 c_T^2}{2^{13}} I_{++}[\mathcal{S}^i]\,, \tag{3.24}$$

where

$$I_{++}[\mathcal{S}^i] = 2\left[\mathcal{S}^1 + \frac{\mathcal{S}^2}{z^2} + \frac{(1-z)^2\mathcal{S}^3}{z^2} + \frac{2(1-z)\mathcal{S}^4}{z^2} - \frac{2(1-z)\mathcal{S}^5}{z} + \frac{2\mathcal{S}^6}{z} \bigg|_{\substack{U=z^2 \\ V=(1-z)^2}} \right] - 6\,, \tag{3.25}$$

and where the -6 comes from subtracting the disconnected part. The quantity $I_{++}[\mathcal{S}^i]$ is independent of z. It can be simplified significantly using the conformal block expansion introduced in Equation (2.18). Indeed, (3.25) can be written as

$$I_{++}[\mathcal{S}^i] = 2\bigg[\mathcal{S}_{\mathbf{1}} + \mathcal{S}_{\mathbf{15}_a}\frac{2(z-2)}{z} + \mathcal{S}_{\mathbf{15}_s} + 2\mathcal{S}_{\mathbf{20}'} + \mathcal{S}_{\mathbf{45}\oplus\overline{\mathbf{45}}}\frac{4-2z}{z} \\ + \mathcal{S}_{\mathbf{84}}\left(\frac{16}{z^2} - \frac{16}{z} + \frac{44}{15}\right)\bigg]\bigg|_{\substack{U=z^2 \\ V=(1-z)^2}} - 6\,. \tag{3.26}$$

Each $\mathcal{S}_{\mathbf{r}}$ must be expanded in conformal blocks $g_{\Delta,\ell}\left(z^2,(1-z)^2\right)$, and as $z\to 0$ these behave as $(z/4)^{\Delta}$ where Δ is the scaling dimension of the corresponding conformal primary. Since I_{++} is independent of z, it follows that the only conformal primaries that can contribute must have either $\Delta = 0$ in the $\mathbf{1}$, $\mathbf{15}_s$, $\mathbf{20}'$ channels, $\Delta = 1$ in the $\mathbf{15}_a$ and $\mathbf{45}\oplus\overline{\mathbf{45}}$ channels, or $\Delta = 2$ in the $\mathbf{84}$ channel. The only $\Delta = 0$ operator is the identity operator, which appears in the $\mathbf{1}$ channel with squared OPE coefficient $\lambda^2_{0,0,\mathbf{1}} = 1$ by convention. The $\mathbf{15}_a$ and $\mathbf{45}\oplus\overline{\mathbf{45}}$ channels contain only odd spin operators, and for them $\Delta = 1$ would violate the unitarity bound. Thus, there are no $\Delta = 1$ operators contributing to (3.26). Consequently, the only operators that can contribute to (3.26) are the identity operator and any $\Delta = 2$ operators in the $\mathbf{84}$. The only such operator that can appear in an $\mathcal{N} = 6$ theory is the superconformal primary of the $(B,2)^{[022]}_{2,0}$. Because the conformal block $g_{2,0}(U,V)\approx U/16$ at small U, we find that

$$I_{++}[\mathcal{S}^i] = -4 + 2\lambda^2_{(B,2)^{[022]}_{2,0}}\,. \tag{3.27}$$

where $\lambda_{(B,2)^{[022]}_{2,0}}$ is the OPE coefficient between two S operators and the $(B,2)^{[022]}_{2,0}$ supermultiplet. Combining this with the expression (3.8) for c_T, we conclude that

$$\lambda^2_{(B,2)^{[022]}_{2,0}} = 2 + \frac{\partial^4_{m_\pm}\log Z}{(\partial^2_{m_\pm}\log Z)^2}\bigg|_{m=0}\,. \tag{3.28}$$

Now let us move on to simplify the second equation in (3.23). Because correlators of $\tilde{J}(x)$ are topological, we can place the three $\tilde{J}$ operators at 0, 1, and infinity, so that

$$\frac{\partial^4\log Z}{\partial^3 m_+\partial m_-} = 2^9\pi^6\left\langle \tilde{J}(0)\tilde{J}(1)\tilde{J}(\infty)\left(\int d^3x\sqrt{g}(iJ_-(\vec{x}) + K_-(\vec{x}))\right)\right\rangle\,. \tag{3.29}$$

Next we expand the right-hand correlator using (3.5), (2.5) and (2.13). The $\langle\tilde{J}\tilde{J}\tilde{J}J_-\rangle$ correlator automatically vanish, and so

$$\frac{\partial^4 \log Z}{\partial^3 m_+ \partial m_-} = -\frac{i c_T^2 \pi^2}{2^{12}\sqrt{2}} \int \frac{d^3\vec{x}}{(4+|\vec{x}|^2)|\vec{x}|} \times \left(\mathcal{T}^1(U,V) + 5\mathcal{T}^2(U,V) + \mathcal{T}^3(U,V) + 8\mathcal{T}^4(U,V) + 2\mathcal{T}^6(U,V) \right)\Bigg|_{\substack{U=\frac{1}{|\vec{x}-\hat{e}_3|^2} \\ V=\frac{|\vec{x}|^2}{|\vec{x}-\hat{e}_3|^2}}} , \tag{3.30}$$

where $\hat{e}_3 = (0,0,1)$. We can then use the superconformal Ward identity (2.15) to eliminate $\mathcal{T}^6$, and so find that

$$\begin{aligned} \frac{\partial^4 \log Z}{\partial^3 m_+ \partial m_-} = &- \frac{i c_T^2 \pi^2}{2^{13}\sqrt{2}} \int d^3\vec{x} \left(2\mathcal{T}^2\left(\frac{1}{|\vec{x}-\hat{e}_3|^2}, \frac{|\vec{x}|^2}{|\vec{x}-\hat{e}_3|^2}\right) \right. \\ &\left. + 2\mathcal{T}^3\left(\frac{1}{|\vec{x}-\hat{e}_3|^2}, \frac{|\vec{x}|^2}{|\vec{x}-\hat{e}_3|^2}\right) + 4\mathcal{T}^4\left(\frac{1}{|\vec{x}-\hat{e}_3|^2}, \frac{|\vec{x}|^2}{|\vec{x}-\hat{e}_3|^2}\right) \right)\Bigg| . \end{aligned} \tag{3.31}$$

Switching to spherical coordinates $\vec{x} = r(\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), \cos(\theta))$ and then integrating over ϕ, we arrive at our final expression

$$\frac{\partial^4 \log Z}{\partial^3 m_+ \partial m_-} = -\frac{i c_T^2 \pi^2}{2^{13}\sqrt{2}} I_{\text{odd}}\left[\mathcal{T}^i\right] , \tag{3.32}$$

where we define the linear functional

$$\begin{aligned} I_{\text{odd}}\left[\mathcal{T}^i\right] &= \tilde{I}_{\text{odd}}[\mathcal{T}^2] + \tilde{I}_{\text{odd}}[\mathcal{T}^3] + 2\tilde{I}_{\text{odd}}[\mathcal{T}^4] , \\ \tilde{I}_{\text{odd}}[\mathcal{T}^i] &= \int_0^\infty dr \int_0^\pi d\theta \, 4\pi r \sin\theta \, \mathcal{T}^i\left(\frac{1}{1+r^2-2r\cos\theta}, \frac{r^2}{1+r^2-2r\cos\theta}\right) . \end{aligned} \tag{3.33}$$

Finally, we turn to the last equation in (3.23). Again we use the fact that $\tilde{J}$ is topological to place the first $\tilde{J}$ at $x_3 = 0$ and the second at $x_4 = \infty$ and multiply by $(2\pi)^2$. Then, relating all the operators in the second line of (3.23) to $\mathcal{S}^i$ and $\mathcal{R}^i$ and computing the required traces of M matrices, we obtain

$$\frac{\partial \log Z}{\partial m_+^2 \partial m_-^2} = \frac{c_T^2}{2^{16}} \left[\tilde{I}_1[2\mathcal{S}^1] - 4\tilde{I}_2[2\mathcal{R}^1 + \mathcal{R}^2 + \mathcal{R}^3 + 2\mathcal{R}^5 + 2\mathcal{R}^6] \right] , \tag{3.34}$$

where

$$\tilde{I}_\Delta[\mathcal{G}] \equiv \int d^3\vec{x}_1 \, d^3\vec{x}_2 \, \frac{[\Omega(\vec{x}_1)\Omega(\vec{x}_2)]^{3-\Delta}}{\vec{x}_{12}^{2\Delta}} \mathcal{G}\left(\frac{\vec{x}_{12}^2}{\vec{x}_1^2}, \frac{\vec{x}_2^2}{\vec{x}_1^2}\right) , \qquad \Omega(x) = \frac{1}{1+\frac{x^2}{4}} . \tag{3.35}$$

We can evaluate (3.35) as follows. Using rotational symmetry, we can set $\vec{x}_1 = (r_1, 0, 0)$ and $\vec{x}_2 = (r_2 \cos\theta, r_2 \sin\theta, 0)$ and perform the angular integrals which give $4\pi \times 2\pi = 8\pi^2$. Thus

$$\tilde{I}_\Delta[\mathcal{G}] \equiv 8\pi^2 \int dr_1\, dr_2\, d\theta\, r_1^2 r_2^2 \sin\theta \frac{\left[\left(1+\frac{r_1^2}{4}\right)\left(1+\frac{r_2^2}{4}\right)\right]^{\Delta-3}}{(r_1^2+r_2^2-2r_1r_2\cos\theta)^\Delta} \mathcal{G}\left(\frac{r_1^2+r_2^2-2r_1r_2\cos\theta}{r_1^2}, \frac{r_2^2}{r_1^2}\right) . \tag{3.36}$$

Let us now change variables by setting $r_1 = 2\rho$ and $r_2 = 2r\rho$. Then (3.36) becomes

$$\tilde{I}_\Delta[\mathcal{G}] \equiv 2^{9-2\Delta}\pi^2 \int d\rho\, dr\, d\theta\, \rho^{5-2\Delta} r^2 \sin\theta \, \left[\left(1+\rho^2\right)\left(1+r^2\rho^2\right)\right]^{\Delta-3} \frac{\mathcal{G}\left(1+r^2-2r\cos\theta, r^2\right)}{(1+r^2-2r\cos\theta)^\Delta} . \tag{3.37}$$

The ρ integral can be done analytically. For the cases of interest, namely $\Delta = 1$ and 2, the result is

$$\begin{aligned} \tilde{I}_1[\mathcal{G}] &= 2^7\pi^2 \int dr\, d\theta\, r^2 \sin\theta \frac{1-r^2+(1+r^2)\log r}{(r^2-1)^3} \frac{\mathcal{G}\left(1+r^2-2r\cos\theta, r^2\right)}{1+r^2-2r\cos\theta} , \\ \tilde{I}_2[\mathcal{G}] &= 2^5\pi^2 \int dr\, d\theta\, r^2 \sin\theta \frac{\log r}{r^2-1} \frac{\mathcal{G}\left(1+r^2-2r\cos\theta, r^2\right)}{(1+r^2-2r\cos\theta)^2} . \end{aligned} \tag{3.38}$$

The expression (3.34) can be simplified further after using the Ward identity relating $\mathcal{R}^i$ to $\mathcal{S}^i$ in equations (A.23)–(A.26), and integrating by parts. We find

$$\begin{aligned} \tilde{I}_2[2\mathcal{R}^1+\mathcal{R}^2+\mathcal{R}^3+2\mathcal{R}^5+2\mathcal{R}^6] = &\int dr\, d\theta\, \mathcal{S}^1\left(1+r^2-2r\cos\theta, r^2\right) \\ &\times \left(-16\pi^2 \sin\theta \frac{-1-5r^2+5r^4+r^6-8(r^2+r^4)\log r}{(r^2-1)^3(1+r^2-2r\cos\theta)}\right) . \end{aligned} \tag{3.39}$$

Combining with (3.34), we obtain

$$\frac{\partial \log Z}{\partial m_+^2 \partial m_-^2} = \frac{c_T^2\pi^2}{2^{11}} \int dr\, d\theta\, \sin\theta \frac{\mathcal{S}^1\left(1+r^2-2r\cos\theta, r^2\right)}{1+r^2-2r\cos\theta} . \tag{3.40}$$

Once again we can view the right-hand side as a linear functional defined on $\mathcal{S}$, defining

$$I_{+-}[\mathcal{S}^i] = \int dr\, d\theta\, \sin\theta \frac{\mathcal{S}^1\left(1+r^2-2r\cos\theta, r^2\right)}{1+r^2-2r\cos\theta} \tag{3.41}$$

so that

$$\frac{\partial \log Z}{\partial m_+^2 \partial m_-^2} = \frac{\pi^2 c_T^2}{2^{11}} I_{+-}[\mathcal{S}^i]. \tag{3.42}$$

3.2 $U(N)_k \times U(N+M)_{-k}$ Theory

3.2.1 Simplifying the Partition Function

Using supersymmetric localization, the mass-deformed $U(N)_k \times U(N+M)_{-k}$ partition function can be reduced to $M+2N$ integrals [82, 83]:

$$Z_{M,N,k}(m_+, m_-) \\ = \int d^{M+N}\mu \, d^N\nu \; \frac{e^{-i\pi k(\sum_i \mu_i^2 - \sum_a \nu_a^2)} \prod_{i<j} 4\sinh^2[\pi(\mu_i-\mu_j)] \prod_{a<b} 4\sinh^2[\pi(\nu_a-\nu_b)]}{\prod_{i,a} 4\cosh\left[\pi(\mu_i-\nu_a)+\frac{\pi m_+}{2}\right]\cosh[\pi(\mu_i-\nu_a)+\frac{\pi m_-}{2}]}\,, \tag{3.43}$$

up to an overall $m_\pm$-independent normalization factor. Our first task will be to write (3.43) as an N-dimensional integral which prove simpler to evaluate. To achieve this we generalize the methods of [84], which studied the special case $m_+ = m_- = 0$. We are ultimately only interested in computing Z up to an overall normalization constant Z_0 which is independent of $m_\pm$, and so will ignore any overall factors.

Our first step is to use the determinant formula:

$$\frac{\prod_{i<j} 2\sinh\frac{x_i-x_j}{2} \prod_{a<b} 2\sinh\frac{y_a-y_b}{2}}{\prod_{i,a} 2\cosh\frac{x_i-y_a}{2}} = \prod_{i=1}^{N+M} e^{-\frac{1}{2}Mx_i} \prod_{a=1}^{N} e^{\frac{1}{2}My_a} \det\left(A(x,y)\right) \tag{3.44}$$

where $A(x,y)$ is the matrix

$$A_{ij}(x,y) = \frac{\theta_{N,i}}{2\cosh\frac{x_i-y_j}{2}} + e^{(N+M+1/2-j)y_i}\theta_{j,N+1}\,, \quad \text{where} \quad \theta_{i,j} = \begin{cases} 1 & i \geq j \\ 0 & \text{otherwise} \end{cases}\,, \tag{3.45}$$

which is proven in [84] using a generalization of the Cauchy determinant formula. Applying this formula with $x_i = 2\pi\mu_i$ and $y_i = 2\pi\nu_i + \pi m_-$, we can rewrite (3.43) as

$$Z_{M,N,k}(m_+, m_-) = (N+M)!\, e^{-\frac{\pi}{2}MN(m_+ + m_-)} \int d^{M+N}\mu\, d^N\nu \prod_{j=1}^{N+M} e^{-\pi(ik\mu_j^2 + 2M\mu_j)} \\ \times \prod_{a=1}^{N} \frac{e^{\pi(ik\nu_a^2 + 2M\nu_a)}}{2\cosh\left[\pi(\mu_a-\nu_a)+\frac{\pi m_+}{2}\right]} \prod_{j=N+1}^{N+M} e^{(2(N+M-j)+1)\pi\mu_j} \\ \times \sum_{\text{perms } \sigma} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{N+M} \left(\frac{\theta_{N,\sigma(i)}}{2\cosh\left[\pi(\mu_i - \nu_{\sigma(i)}) + \frac{\pi m_-}{2}\right]} + e^{(2(N+M-\sigma(i))+1)\pi\mu_i}\theta_{\sigma(i),N+1} \right)\,, \tag{3.46}$$

where the sum over σ is a sum over all permutations $\sigma(i)$ of $N+M$ elements.

We next take the Fourier transform of the coshines

$$\frac{1}{2\cosh(\pi p)} = \frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{e^{2ipx}}{2\cosh(x)}\,. \tag{3.47}$$

The μ and ν integrals then become Gaussian and can be easily performed. We thus find that

$$\begin{aligned} Z_{M,N,k}(m_+,m_-) &\propto e^{-\frac{\pi}{2}MN(m_++m_-)} \\ &\times \sum_\sigma (-1)^\sigma \Bigg(\int d^N x\, d^{N+M} y \prod_{a=1}^{N} \frac{e^{-\frac{2i}{k\pi}x_a(y_a-y_{\sigma(a)})+\frac{2}{k}M(y_a-y_{\sigma(a)})+i(x_a m_+ + y_a m_-)}}{4\cosh(x_a)\cosh(y_a)} \\ &\times \prod_{l=N+1}^{N+M} \left[\pi e^{-\frac{i\pi}{k}(N+\frac{1}{2}-l)^2}\delta\left(y_l + i\pi(N+M+1/2-l)\right) e^{\frac{i}{k\pi}y_l^2+\frac{2}{k}(N+1/2-l)y_{\sigma(l)}}\right]\Bigg)\,. \end{aligned} \tag{3.48}$$

So long as $2M < |k|+1$, we can integrate over x_i, leaving

$$\begin{aligned} Z_{M,N,k}(m_+,m_-) \propto e^{-\frac{\pi}{2}MN(m_++m_-)} &\sum_\sigma (-1)^\sigma \int d^{N+M}y \prod_{a=1}^{N} \frac{e^{\frac{2}{k}M(y_a-y_{\sigma(a)})+iy_a m_-}}{4\cosh\left[\frac{y_a-y_{\sigma(a)}}{k} - \frac{\pi m_+}{2}\right]\cosh(y_a)} \\ \times \prod_{l=N+1}^{N+M} &\left[\pi e^{-\frac{i\pi}{k}(N+\frac{1}{2}-l)^2}\delta\left(y_l + i\pi(N+M+1/2-l)\right) e^{\frac{i}{k\pi}y_l^2+\frac{2}{k}(N+1/2-l)y_{\sigma(l)}}\right]\,. \end{aligned} \tag{3.49}$$

After a change of variable $y_a \to y_a/2$ and judicious use of the equation $\sum_a y_a = \sum_a y_{\sigma(a)}$ we find that

$$\begin{aligned} Z_{M,N,k}(m_+,m_-) \propto e^{-\frac{\pi}{2}MN(m_++m_-)} &\sum_\sigma (-1)^\sigma \int d^{N+M}y \prod_{a=1}^{N} \frac{e^{\frac{i}{2}y_a m_-}}{2\cosh\left[\frac{y_a}{2}\right]} \\ \times \prod_{l=N+1}^{N+M} &\left[\pi e^{-\frac{i\pi}{k}(N+\frac{1}{2}-l)^2}\delta\left(y_l + i\pi(N+M+1/2-l)\right) e^{\frac{i}{4k\pi}y_l^2-\frac{M}{k}y_l}\right] \\ \times \det &\left(\frac{\theta_{N,l}}{2\cosh\frac{y_j-y_l+k\pi m_+}{2k}} + e^{\frac{1}{k}(N+M+1/2-l)y_j}\theta_{l,N+1}\right)\,. \end{aligned} \tag{3.50}$$

Applying (3.44) again, integrating over $y_{N+1},\ldots,y_{N+M}$, and then performing a final change of variables $y_a \to 2\pi y_a$, we arrive at our final expression for the mass-deformed S^3 partition function,

$$
\begin{aligned}
&Z_{M,N,k}(m_+, m_-) \\
&\quad = \frac{e^{-\frac{\pi}{2}MNm_-} Z_0}{\cosh^N \frac{\pi m_+}{2}} \int d^N y \prod_{a<b} \frac{\sinh^2 \frac{\pi(y_a - y_b)}{k}}{\cosh\left[\frac{\pi(y_a-y_b)}{k} + \frac{\pi m_+}{2}\right] \cosh\left[\frac{\pi(y_a-y_b)}{k} - \frac{\pi m_+}{2}\right]} \\
&\qquad \times \prod_{a=1}^{N} \left(\frac{e^{i\pi y_a m_-}}{2\cosh(\pi y_a)} \prod_{l=0}^{M-1} \frac{\sinh\left[\frac{\pi\left(y_a + i(l+1/2)\right)}{k}\right]}{\cosh\left[\frac{\pi\left(y_a+i(l+1/2)\right)}{k} - \frac{\pi m_+}{2}\right]} \right) .
\end{aligned}
\tag{3.51}
$$

3.2.2 Finite M, N, k Calculations

In this section, we will evaluate localization results for small M, N and k. For simplicity we focus on the single mass case, taking $m_- = m$ and $m_+ = 0$, so that

$$
\begin{aligned}
Z_{M,N,k}(m) = Z_0 e^{-\frac{\pi}{2}MNm} &\int d^N y \prod_{a<b} \tanh^2 \frac{\pi(y_a - y_b)}{k} \\
&\times \prod_{a=1}^{N} \left[\frac{e^{i\pi y_a m}}{2\cosh(\pi y_a)} \prod_{l=0}^{M-1} \tanh \frac{\pi\left(y_a + i(l+1/2)\right)}{k} \right] .
\end{aligned}
\tag{3.52}
$$

Let us begin with the case $N = 1$. We must compute

$$
\hat{Z}_{M,1,k}(m) \equiv \frac{Z_{M,1,k}(m)}{Z_0} = e^{-\frac{\pi}{2}Mm} \int_{-\infty}^{\infty} dx\; e^{i\pi x m} F_{M,k}(x) \,, \tag{3.53}
$$

where we define

$$
F_{M,k}(x) = \frac{1}{2\cosh(\pi x)} \prod_{l=0}^{M-1} \tanh \frac{\pi(x + i(l+1/2))}{k} \,. \tag{3.54}
$$

All poles of $F_{M,k}(x)$ are located at $x = \frac{i}{2} + iK$ for $K \in \mathbb{Z}$. Furthermore, $F_{M,k}(x)$ is periodic in the complex plane, with

$$
F_{M,k}(x + ik) = (-1)^k F_{M,k}(x) \,. \tag{3.55}
$$

By closing the integral (3.53) in the upper-half of the complex, we may therefore reduce it to a finite sum of poles

$$
\hat{Z}_{M,1,k}(m) = \frac{2\pi i e^{-\frac{\pi}{2}Mm}}{1 - (-1)^k e^{-k\pi m}} \sum_{K=1}^{k-1} \operatorname*{Res}_{x=\frac{i}{2}+iK} \left[e^{i\pi m x} F_{M,k}(x) \right] . \tag{3.56}
$$

We can evaluate the residues and derive analytic expressions for $\hat{Z}_{M,1,k}(m)$ for any M and k, and then compute c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ using (3.8) and (3.28). Table 3.1 lists these quantities for various

M	k	$\frac{16}{c_T}$	$\lambda^2_{(B,2)^{[022]}_{2,0}}$
1	2	$\frac{3}{4} = 0.75$	$\frac{16}{5} = 3.2$
	3	$\frac{5+2\sqrt{3}}{13} \approx 0.6511$	$\frac{612-62\sqrt{3}}{169} \approx 2.986$
	4	$\frac{3(\pi-2)}{4(3\pi-8)} \approx 0.6009$	$\frac{2(512-441\pi+90\pi^2)}{5(8-3\pi)^2} \approx 2.921$
2	4	$\frac{3(\pi-4)}{15\pi-52} \approx 0.5281$	$\frac{4(12544-6936\pi+945\pi^2)}{5(52-15\pi)^2} \approx 2.715$
	5	0.4667	2.618
	6	0.4309	2.582
3	6	0.4005	2.498
4	8	0.3211	2.381
5	10	0.2674	2.307
6	12	0.2290	2.258

Table 3.1: OPE coefficients $\frac{16}{c_T}$ and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ in various $U(1)_k \times U(1+M)_{-k}$ ABJ theories.

values of M and k. Note that the analytic results become increasingly elaborate as M and k become larger, and so we include analytic expressions in Table 3.1 only if a concise expression exists.

The above analysis can be generalized to the $N > 1$ case by repeatedly integrating over z_k. When $N = 2$, for instance, we must compute

$$Z_{M,2,k}(m) = e^{-\frac{\pi}{2}MNm} \int dz_1\, dz_2\, e^{i\pi(z_1+z_2)m} \tanh^2 \frac{\pi(z_1 - z_2)}{k} \prod_{a=1}^{2} F_{M,k}(z_a)\,. \tag{3.57}$$

We evaluate this by first integrating over z_1 while fixing $|\mathrm{Im}(z_2)| < \frac{k}{2}$. We can perform this integral by closing the contour in the upper half complex plane and then summing over the poles, which occur at

$$z_1 = \frac{iN}{2} \quad \text{and} \quad z_1 = z_2 + ik(K + 1/2)\,,$$

where K is a positive integer. Because both $K(z)$ and $\tanh z$ are periodic in the complex plane, we need only sum the poles with imaginary part less than k; the rest can be resummed as a geometric series. Having integrated over z_1, we perform the z_2 integral in a similar fashion. For general N we must repeat this process for each of the N integration variables. We list results in Table 3.2.

3.2.3 Supergravity Limit

We will now study localization in the large N limit. Using the Fermi gas method [85], the localization formula (3.43) for the mass deformed partition function with $M = 0$ was computed to all orders in

M	k	$\frac{16}{c_T}$	$\lambda^2_{(B,2)^{[022]}_{2,0}}$
1	2	0.3177	2.479
	3	0.2697	2.384
	4	0.2425	2.339
2	4	0.2242	2.302
	5	0.1986	2.262
	6	0.1822	2.239
3	6	0.1736	2.221
4	8	0.1419	2.175
5	10	0.1201	2.144
6	12	0.1041	2.122

N	k	$\frac{16}{c_T}$	$\lambda^2_{(B,2)^{[022]}_{2,0}}$
2	2	$\frac{3}{8} = 0.375$	$\frac{13}{5} = 2.6$
3	2	$\frac{3(\pi^2-10)}{45\pi^2-446} \approx 0.2095$	2.309
	3	0.1838	2.258
4	2	0.1381	2.195
	3	0.1191	2.161
	4	0.1071	2.143

Table 3.2: OPE coefficients $\frac{16}{c_T}$ and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ in various $U(2)_k \times U(M+2)_{-k}$ ABJ theories (left) and $U(N)_k \times U(N)_{-k}$ ABJM theories (right)

$1/N$ in [81, 86]. The answer, up to overall factors which are independent of $m_\pm$, is

$$Z_{M,N,k}(m_+, m_-) = \frac{e^{A_k(m_+,m_-)}}{C_k(m_+,m_-)^{1/3}} \text{Ai}\left[\frac{\tilde{N}_k(N,M) - B_k(m_+,m_-)}{C_k(m_+,m_-)^{1/3}}\right] , \tag{3.58}$$

where we define the functions

$$\begin{aligned} \tilde{N}_k(N,M) &= N + \frac{M}{2} - \frac{M^2}{2k} , \\ C_k(m_+,m_-) &= \frac{2}{\pi^2 k(1+m_+^2)(1+m_-^2)} , \\ B_k(m_+,m_-) &= \frac{\pi^2 C_k(m_+,m_-)}{3} - \frac{1}{6k}\left(\frac{1}{1+m_+^2} + \frac{1}{1+m_-^2}\right) - \frac{k}{24} , \\ A_k(m_+,m_-) &= \frac{\mathcal{A}[k(1+im_+)] + \mathcal{A}[k(1-im_+)] + \mathcal{A}[k(1+im_-)] + \mathcal{A}[k(1-im_-)]}{4} , \end{aligned} \tag{3.59}$$

and where the constant map function $\mathcal{A}(k)$ is given by

$$\mathcal{A}(k) = \frac{2\zeta(3)}{\pi^2 k}\left(1 - \frac{k^3}{16}\right) + \frac{k^2}{\pi^2}\int_0^\infty dx \frac{x}{e^{kx}-1} \log\left(1 - e^{-2x}\right) . \tag{3.60}$$

We will be interested in derivatives of $Z_{M,N,K}(m_+, m_-)$ at $m_\pm = 0$, in which case we expect the non-perturbative corrections to take the form $e^{-\sqrt{Nk}}$ and $e^{-\sqrt{N/k}}$. This is the case for $Z_{0,N,K}(0,0)$, which has been computed exactly for all N and k in [85, 87–93], and we expect this to continue to hold true for derivatives of $Z_{M,N,k}(m_+, m_-)$. We can then apply the large N expansion to the stringy regimes described in Section 1.3.2: the finite k limit, the strong 't Hooft coupling limit $\lambda \gg 1$, and the finite μ limit interpolating between the two. Here, as in Section 1.3.2, λ and μ are defined by the equations

$$\lambda = \frac{\tilde{N}_k(N,M)}{k} - \frac{1}{3k^2} - \frac{1}{24}\,, \qquad \mu = \frac{\tilde{N}_k(N,K)}{k^5}\,. \tag{3.61}$$

We begin by computing c_T using (3.8), and find that to all orders in $1/N$,

$$c_T = \frac{32k^2\mathcal{A}''(k)}{\pi^2} - \frac{64}{3\pi^2} - \frac{32\left(3+2k^2\lambda\right)\mathrm{Ai}'\left(\lambda\left(\frac{\pi^2k^4}{2}\right)^{1/3}\right)}{3\mathrm{Ai}\left(\lambda\left(\frac{\pi^2k^4}{2}\right)^{1/3}\right)}\left(\frac{2}{\pi k^2}\right)^{2/3}. \tag{3.62}$$

Expanding this expression at large N and finite k, we find that

$$\text{finite } k: \quad c_T = \frac{64\sqrt{2k}N^{3/2}}{3\pi} - \frac{8(12M^2-12kM+k^2-16)N^{1/2}}{3\pi\sqrt{2k}} + \frac{32(k^2\mathcal{A}''(k)-1)}{\pi^2} + O(N^{-1/2})\,. \tag{3.63}$$

We can likewise expand both $\frac{\partial^4 \log Z}{\partial m_\pm^4}$, and $\frac{\partial^4 \log Z}{\partial m_+^2 \partial m_-^2}$ to all orders in $1/N$, although the results are more complicated. Expanding at large N and then systematically eliminating N in favor of c_T, we find that

$$\begin{aligned} \text{finite } k: \quad \frac{1}{c_T^2}\frac{\partial^4 \log Z}{\partial m_\pm^4} &= \frac{3\pi^2}{64}\frac{1}{c_T} + \frac{3^{\frac{4}{3}}\pi^{4/3}}{2^{\frac{8}{3}}k^{2/3}}\frac{1}{c_T^{\frac{5}{3}}} + \frac{k^4\mathcal{A}^{(4)}(k) - 3k^2\mathcal{A}''(k) - 3}{2{c_T}^2} + O(c_T^{-\frac{7}{3}})\,, \\ \frac{1}{c_T^2}\frac{\partial^4 \log Z}{\partial m_+^2\partial m_-^2} &= -\frac{\pi^2}{64}\frac{1}{c_T} + \frac{5\pi^{4/3}}{4\;6^{2/3}k^{2/3}}\frac{1}{c_T^{\frac{5}{3}}} + \frac{k^2\mathcal{A}''(k)-1}{2{c_T}^2} + O(c_T^{-\frac{7}{3}})\,, \end{aligned} \tag{3.64}$$

where we have only shown the lowest couple terms in $1/c_T$ for simplicity. We can evaluate $\mathcal{A}^{(4)}(k)$ and $\mathcal{A}''(k)$ exactly using the definition of (3.60).[3] Note that neither expression in (3.64) depend on M when written as a series in $1/c_T$. This is because both M and N only enter into the large N partition function (3.58) through the combined quantity $\tilde{N}(N,M)$, and so when we eliminate N in favor of c_T this also eliminates the M dependence.

Next we consider the strong coupling 't Hooft limit. Using the large k expansion

$$\mathcal{A}(k) = -\frac{\zeta(3)}{8\pi^2}k^2 + 2\zeta'(-1) + \frac{\log\left[\frac{4\pi}{k}\right]}{6} + \sum_{g=2}^{\infty}\left(\frac{2\pi i}{k}\right)^{2g-2}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\,, \tag{3.66}$$

derived in [94], where B_n denote the Bernoulli numbers, we can expand (3.62) at large N with $\lambda \gg 1$

[3] For instance, for $k=1,2$ these values are [81]

$$\begin{aligned} \mathcal{A}''(1) &= \frac{1}{6} + \frac{\pi^2}{32}\,, & \mathcal{A}''(2) &= \frac{1}{24}\,, \\ \mathcal{A}''''(1) &= 1 + \frac{4\pi^2}{5} - \frac{\pi^4}{32}\,, & \mathcal{A}''''(2) &= \frac{1}{16} + \frac{\pi^2}{80}\,. \end{aligned} \tag{3.65}$$

to find that

$$\text{'t Hooft:}\quad c_T = \left(\frac{128}{3\pi(2\lambda)^{1/2}} - \frac{64}{9\pi(2\lambda)^{3/2}} - \frac{32\zeta(3)}{\pi^4\lambda^2} + O(\lambda^{5/2})\right) N(N+M) + \\ + \left(\frac{32(5-6M^2)\lambda^{1/2}}{9\pi} - \frac{80}{3\pi^2} + \frac{16(21M^2+2)}{27\pi} + O(\lambda^{-1})\right) + O(N^{-1})\,. \tag{3.67}$$

We can now expand the fourth-derivatives at large N, and then eliminate N in favor of c_T to find

$$\begin{aligned} \text{'t Hooft:}\quad \frac{1}{c_T^2}\frac{\partial^4 \log Z}{\partial m_\pm^4} &= \left[\frac{3\pi^2}{64} + \frac{9\zeta(3)}{512\sqrt{2}\pi}\frac{1}{\lambda^{\frac{3}{2}}} + \frac{27\zeta(3)^2}{8192\pi^4}\frac{1}{\lambda^3} + O(\lambda^{-\frac{9}{2}})\right]\frac{1}{c_T} \\ &+ \left[\frac{3}{2}\pi\sqrt{2\lambda} - \frac{5}{4} - \frac{9\zeta(3)}{16\pi^2}\frac{1}{\lambda} + \frac{15\zeta(3)}{32\sqrt{2}\pi^3}\frac{1}{\lambda^{\frac{3}{2}}} + O(\lambda^{-\frac{5}{2}})\right]\frac{1}{c_T^2} + O(c_T^{-3})\,, \\ \frac{1}{c_T^2}\frac{\partial^4 \log Z}{\partial m_+^2 \partial m_-^2} &= \left[-\frac{\pi^2}{64} - \frac{3\zeta(3)}{512\sqrt{2}\pi}\frac{1}{\lambda^{\frac{3}{2}}} - \frac{9\zeta(3)^2}{8192\pi^4}\frac{1}{\lambda^3} + O(\lambda^{-\frac{9}{2}})\right]\frac{1}{c_T} \\ &+ \left[\frac{5}{6}\pi\sqrt{2\lambda} - \frac{5}{12} + \frac{3\zeta(3)}{16\pi^2\lambda} - \frac{5\zeta(3)}{32\sqrt{2}\pi^3}\frac{1}{\lambda^{\frac{3}{2}}} + O(\lambda^{-\frac{5}{2}})\right]\frac{1}{c_T^2} + O(c_T^{-3})\,. \end{aligned} \tag{3.68}$$

Once again, the final result does not depend on M to any order in $1/c_T$. Furthermore note that $\zeta(3)$ and π are the only transcendental numbers that appear to any order in $1/\lambda$ and $1/c_T$ expansion.

Finally, for the finite μ limit we can again use the large k expansion of $\mathcal{A}(k)$ to compute

$$\text{finite } \mu:\quad c_T = \frac{64\sqrt{2}N^{8/5}}{3\pi\mu^{1/10}} - \frac{4\sqrt{2}N^{4/5}}{3\pi\mu^{3/10}} + \frac{256\sqrt{2}MN^{3/5}}{15\pi\mu^{1/10}} + O(N^{2/5})\,, \tag{3.69}$$

and then, upon eliminating N in favor of c_T,

$$\begin{aligned} \text{finite } \mu:\quad \frac{1}{c_T^2}\frac{\partial^4 \log Z}{\partial m_\pm^4} &= \frac{3\pi^2}{64}\frac{1}{c_T} + \frac{3^{\frac{5}{4}}\left(4\sqrt{2}\pi^3\sqrt{\mu} + \zeta(3)\right)}{16\; 2^{5/8}\pi^{7/4}\mu^{3/8}}\frac{1}{c_T^{\frac{7}{4}}} - \frac{5}{4}\frac{1}{c_T^2} + O(c_T^{-\frac{9}{4}})\,, \\ \frac{1}{c_T^2}\frac{\partial^4 \log Z}{\partial m_+^2 \partial m_-^2} &= -\frac{\pi^2}{64}\frac{1}{c_T} + \frac{20\sqrt{2}\pi^3\sqrt{\mu} - 3\zeta(3)}{16\; 2^{5/8}3^{3/4}\pi^{7/4}\mu^{3/8}}\frac{1}{c_T^{\frac{7}{4}}} - \frac{5}{12}\frac{1}{c_T^2} + O(c_T^{-\frac{9}{4}})\,. \end{aligned} \tag{3.70}$$

Once again, the results do not depend explicitly on M.

From the finite μ limit we can derive both the 't Hooft limit and the finite k limit by taking $\mu \to 0$ and $\mu \to \infty$ respectively. To reproduce the 't Hooft limit (3.68) we first solve for μ in terms of λ and c_T using (3.8) and (3.61), which at leading order in $1/c_T$ gives

$$\mu = \frac{8192\lambda^4}{9c_T^2\pi^2} + \dots\,. \tag{3.71}$$

We then take the large c_T limit followed by the large λ limit. The $\zeta(3)\mu^{-\frac{3}{8}}c_T^{-\frac{7}{4}}$ and $\mu^{\frac{1}{8}}c_T^{-\frac{7}{4}}$ terms give rise to the $\zeta(3)\lambda^{-\frac{3}{2}}c_T^{-1}$ and $\sqrt{\lambda}c_T^{-2}$ terms in (3.68), respectively.

k	M	$\frac{16}{c_T}$	$\lambda^2_{(B,2)^{[022]}_{2,0}}$
2	0	0.0361459	2.04699
3	0	0.0301815	2.03842
4	0	0.0265295	2.03342
	1	0.0250946	2.03158
6	0	0.0221553	2.02766
	1	0.0208109	2.02595
	2	0.0200682	2.02501
10	0	0.0178216	2.02218
	1	0.0166285	2.02067
	2	0.0157899	2.01961
	3	0.0152331	2.01891
	4	0.0149146	2.01851

Table 3.3: OPE coefficients $\frac{16}{c_T}$ and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ in various $U(10)_k \times U(10+M)_{-k}$ ABJ theories, as computed from the all orders in $1/N$ formula (3.58).

To extract the finite k limit (3.64) from (3.70) we solve for μ in terms of c_T and k using (3.8), which at leading order in $1/c_T$ gives

$$\mu = \frac{(3\pi)^{2/3} {c_T}^{2/3}}{2^{\frac{13}{3}} k^{\frac{16}{3}}} + \dots . \tag{3.72}$$

We then take the large c_T limit. In this limit, the ratio $c_T^2 \mu^{-3}$ is finite, so we must sum infinitely many terms in the finite μ limit to recover the finite k limit. This infinite sum cancels all the $\zeta(3)$ terms which appear at finite μ. The $\mu^{\frac{1}{8}} c_T^{-\frac{7}{4}}$ term becomes a $c_T^{-\frac{5}{3}}$ term at finite k.

While we have so far focused on computing the leading large N corrections from (3.58), we can also use the all orders $1/N$ expansion as a tool to calculate localization results at finite N. For low $k = 1, 2$ and $M = 1, 2$, some explicit examples were given in [81], where it was shown that the large N expansion compares well even down to the exact $N = 2$ result. In Chapter 6, we will study the $U(10)_k \times U(10+M)_{-k}$ ABJ theories for various values of M and k. Using (3.58), we can calculate $\frac{16}{c_T}$ and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ for these theories for a range of M, k, and we summarize our results in Table 3.3.

3.2.4 Higher-Spin Limit

We now compute $Z_{M,N,k}(m_+, m_-)$ at large M and fixed $\lambda = \frac{M}{k}$, which is the higher-spin limit of the $U(N)_k \times U(N+M)_{-k}$ theory. The special case where $m_\pm = 0$ has already been considered in [95], so our task is to generalize their results to non-zero masses.

To begin, let us define

$$F_1(x) = \sum_{l=-\frac{M-1}{2}}^{\frac{M-1}{2}} \log\tanh\left[\frac{\pi\left(x+il\right)}{k}\right] - \log R(x)\,,$$
$$F_2(x) = \sum_{l=-\frac{M-1}{2}}^{\frac{M-1}{2}} \log\cosh\left[\frac{\pi\left(x+il\right)}{k}\right] \tag{3.73}$$

where $R(x) = \cosh(\pi x)$ if M is even and $R(x) = \sinh(\pi x)$ if M is odd, and

$$G(x, \hat{m}_+) = \log\left(\frac{k\sinh^2\frac{\pi x}{\sqrt{k}}}{\pi^2 x^2}\,\mathrm{sech}\left[\frac{2\pi x + \pi\hat{m}_+}{2\sqrt{k}}\right]\mathrm{sech}\left[\frac{2\pi x - \pi\hat{m}_+}{2\sqrt{k}}\right]\right)\,, \tag{3.74}$$

where $\hat{m}_\pm = k^{-1/2}m_\pm$. After a change of variables $y_a \to \sqrt{k}\left(x_a - \frac{iM}{2}\right)$, we find that

$$Z_{M,N,k}(\hat{m}_+, \hat{m}_-) \propto \frac{1}{\cosh^N\frac{\pi\hat{m}_+}{2\sqrt{k}}}\int d^N x \prod_{a<b}(x_a - x_b)^2 \exp\left(G(x_a - x_b, \hat{m}_+)\right)$$
$$\times\exp\left(\sum_a i\pi x_a \hat{m}_- + F_1\left(x_a\sqrt{k}\right) + F_2\left(x_a\sqrt{k}\right) - F_2\left(\frac{\sqrt{k}}{2}(2x - \hat{m}_+)\right)\right)\,. \tag{3.75}$$

We now expand $F_1(x)$, $F_2(x)$ and $G(x)$ at large M and k, holding x, $\hat{m}_\pm$ and λ fixed. The large M expansion of $F_1(x)$ has already been computed in [95], where it was shown that

$$F_1(x) \equiv \sum_{l=-\frac{M-1}{2}}^{\frac{M-1}{2}} \log\tanh\frac{\pi(x+il)}{k} - R(x) \sim \frac{\cos\frac{2x\partial_\lambda}{k}}{\sinh\frac{\partial_\lambda}{k}}\log\tan\frac{\pi\lambda}{2} \tag{3.76}$$

The right-hand expression should be understood as a formal series expansion, which can be written more verbosely as

$$F_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n f_{2n}(k,\lambda)}{(2n)!}\frac{x^{2n}}{k^{2n-1}}\,,$$
$$\text{where }\; f_{2n}(k,\lambda) = \sum_{p=0}^{\infty}\frac{4^n(2-4^p)B_{2p}}{(2p)!k^{2p}}\partial_\lambda^{2p+2n-1}\log\tan\frac{\pi\lambda}{2}\,, \tag{3.77}$$

and so we find that

$$F_1\left(x\sqrt{k}\right) = \text{cons.} - 2\pi\csc(\pi\lambda)x^2 + \frac{1}{3}\pi^3(\cos(2\lambda\pi)+3)\csc^3(\lambda\pi)\frac{x^4}{k} + O(k^{-2})\,, \tag{3.78}$$

Next we expand $F_2(x)$ using the Euler-MacLaurin expansion, finding that

$$\begin{aligned} F_2\left(x\right) &= \sum_{l=-\frac{M-1}{2}}^{\frac{M-1}{2}} \log\cosh\left[\frac{\pi\left(x+il\right)}{k}\right] \\ &= \frac{\pi x^2 \tan\frac{\pi\lambda}{2}}{k} - \frac{2\pi^3 x^2(2x^2+1)\sin^4\frac{\pi\lambda}{2}}{k^3\cos\pi\lambda} + O(k^{-5})\,. \end{aligned} \tag{3.79}$$

Finally, we can expand $G(x)$ by simply using the Taylor series expansion around $k^{-1/2}=0$, so that

$$G(x,\hat{m}_+) = -\frac{\pi^2(8x^2+3\hat{m}_+^2)}{12k} + \frac{\pi^4(224x^4+360x^2\hat{m}_+^2+15\hat{m}_+^4)}{1440k^2} + O(k^{-3})\,. \tag{3.80}$$

Putting everything together, we have

$$\begin{aligned} &Z_{M,N,k}(\hat{m}_+,\hat{m}_-) \\ &\quad\propto \frac{1}{\cosh^N\frac{\pi\hat{m}_+}{2\sqrt{k}}}\int d^N x\prod_{a<b}(x_a-x_b)^2\exp\left(-2\pi\csc(\pi\lambda)\sum_a x_a^2 + O(k^{-1})\right)\,. \end{aligned} \tag{3.81}$$

where all higher order terms are polynomial in x and $\hat{m}_\pm$. To compute

$$\left.\frac{\partial^{n_1+n_2}Z_{M,N,k}(\hat{m}_+,\hat{m}_-)}{\partial^{n_1}\hat{m}_+\partial^{n_2}\hat{m}_-}\right|_{\hat{m}_\pm=0} \tag{3.82}$$

at each order in k^{-1}, all we must do now is evaluate Gaussian integrals of the form

$$\int d^N x\, p(x_a)\prod_{a<b}(x_a-x_b)^2\exp\left(-2\pi\csc(\pi\lambda)\sum_a x_a^2\right)\,, \tag{3.83}$$

where $p(x_a)$ is a polynomial in x_a. These are just polynomial expectation values in a Gaussian matrix model. They can be computed at finite N as sums of $U(N)$ Young tableux [96], as described in detail in Appendix B of [97].

After a little work, we find that

$$\begin{aligned} c_T &= \frac{16Nk\sin(\pi\lambda)}{\pi} + 4N^2(3+\cos(2\pi\lambda)) \\ &\quad - \frac{\pi N\left(16-18N^2+(1-14N^2)\cos(2\lambda\pi)\right)}{3k} + O(k^{-2})\,. \end{aligned} \tag{3.84}$$

We invert this series to eliminate k in favor of c_T, and so find that

$$\begin{aligned}\lambda^2_{(B,2)^{[022]}_{2,0}} &= 2 + \frac{8(3+\cos(2\pi\lambda))}{c_T} - \frac{64N^2\sin^2(\pi\lambda)(3+5\cos(2\pi\lambda))}{c_T^2} \\ &- \frac{512N^3\left(29-\frac{121}{4N^2}+\left(44-\frac{19}{N^2}\right)\cos(2\pi\lambda)+\left(23+\frac{5}{4N^2}\right)\cos(4\pi\lambda)\right)\sin^2(\pi\lambda)}{3c_T^3} + O\left(c_T^{-4}\right) ,\end{aligned} \tag{3.85}$$

from which we see that in the limit of large c_T, the theory with the lowest value of $\lambda^2_{(B,2)^{[022]}_{2,0}}$ at fixed c_T is that with $\lambda = 1/2$ and $N = 1$, corresponding to the $U(1)_{2M} \times U(1+M)_{-2M}$ theories. This family will prove of special interest to us when considering numeric bootstrap bounds. Specializing to this case, we can easily expand $\lambda^2_{(B,2)^{[022]}_{2,0}}$ to much higher order in $1/c_T$:

$$\begin{aligned}\lambda^2_{(B,2)^{[022]}_{2,0}} = 2 &+ \frac{16}{c_T} + \frac{1}{2}\left(\frac{16}{c_T}\right)^2 + \frac{1}{12}\left(\frac{16}{c_T}\right)^3 + \frac{2}{3}\left(\frac{16}{c_T}\right)^4 - \frac{217}{240}\left(\frac{16}{c_T}\right)^5 \\ &+ \frac{979}{480}\left(\frac{16}{c_T}\right)^6 - \frac{71291}{15120}\left(\frac{16}{c_T}\right)^7 + \cdots .\end{aligned} \tag{3.86}$$

Comparing to the exact values computed in Table 3.1, we find that (3.86) gives answers to within 1% of the exact results already for $M = 4$.

For the mixed derivatives of $\log Z$, we find that

$$\begin{aligned}\frac{1}{c_T^2}\frac{\partial^4 \log Z_{M,N,k}}{\partial^2 m_+ \partial^2 m_-}\bigg|_{m_\pm=0} &= -\frac{\pi^4\sin^2(\pi\lambda)}{256 c_T} + \frac{\pi^4 N^2(1-5\cos(2\pi\lambda))\sin^2(\pi\lambda)}{64 c_T^2} + O(c_T^{-3}) , \\ \frac{1}{c_T^2}\frac{\partial^4 \log Z_{M,N,k}}{\partial^3 m_+ \partial m_-}\bigg|_{m_\pm=0} &= -\frac{i\pi^4\sin(2\pi\lambda)}{512 c_T} + \frac{5i\pi^4 N^2\cos(\pi\lambda)\sin^3(\pi\lambda)}{32 c_T^2} + O(c_T^{-3}) .\end{aligned} \tag{3.87}$$

Note that for each of these quantities, the $O(c_T^{-1})$ term is independent of N, the $O(c_T^{-2})$ term is proportional to N^2, and further subleading terms have more complicated N dependence. This overall behavior is expected for the following reason. The $U(N)_k$ gauge factor is very weakly coupled in the higher-spin limit at finite N, so we can construct N^2 different "single-trace" operators of the $U(M+N)$ factor (which are a combined adjoint and singlet of $SU(N)$, where the $SU(N)$-adjoint is not a gauge-invariant operator in the full theory), and because of the weak $U(N)$ coupling the "double-trace" operators constructed from pairs of each of these N^2 "single-trace" operators contribute the same, so we get a factor of N^2. Note that it is important to distinguish the single trace operators in scare quotes from single-trace operators in the usual sense, which are gauge-invariant.

Finally, we give the coefficient of the delta function in $\langle SP\rangle$, which we can compute using (3.11):

$$\kappa = \sqrt{2}\tan\left(\frac{\pi\lambda}{2}\right) - \frac{32\sqrt{2}}{c_T}N^2\csc(\pi\lambda)\sin^4\left(\frac{\pi\lambda}{2}\right) + O(c_T^{-2})\,. \tag{3.88}$$

As promised, this expression does not vanish, and, in fact, is not even invariant under Seiberg duality $\lambda \to 1-\lambda$.

3.3 $SO(2)_{2k} \times USp(2+2M)_{-k}$ Theory

3.3.1 Simplifying the Partition Function

We now discuss the mass-deformed sphere partition function for the $SO(2)_{2k} \times USp(2+2M)_{-k}$ theory. Using supersymmetric localization, this quantity can be written as an $(M+1)$-dimensional integral [82, 98]:

$$\begin{aligned} Z_{M,k}(m_+, m_-) \propto &\int d\mu\, d^M\nu\, e^{2\pi i k(\mu^2 - \sum_a \nu_a^2)} \\ &\times \frac{\prod_a \sinh^2(2\pi\nu_a)\prod_{a<b}\sinh^2[\pi(\nu_a+\nu_b)]\sinh^2[\pi(\nu_a-\nu_b)]}{\prod_b \cosh\frac{2\pi(\mu-\nu_b)+\pi m_+}{2}\cosh\frac{2\pi(\mu+\nu_b)+\pi m_+}{2}\cosh\frac{2\pi(\mu-\nu_b)+\pi m_-}{2}\cosh\frac{2\pi(\mu+\nu_b)+\pi m_-}{2}}\,, \end{aligned} \tag{3.89}$$

up to an overall $m_\pm$-independent factor. In this section we shall reduce this expression down to a single integral. We follow the derivation in [99], which considered the special case $m_+ = m_- = 0$. They however consider the general $SO(2N) \times USp(2N+2M)$ theory, while here we only focus on the $N = 1$ case, for which the manifest $\mathcal{N} = 5$ SUSY is enhanced to $\mathcal{N} = 6$.

To ease comparison with [99], we rewrite (3.89):

$$\begin{aligned} Z_{M,k}(m_+, m_-) \propto &\int d\mu\, d^{M+1}\nu\, e^{\frac{i}{4\tilde{k}\pi}(\mu^2 - \sum_a \nu_a^2)} \\ &\times \frac{\prod_a \sinh^2\frac{\nu_a}{\tilde{k}}\prod_{a<b}\sinh^2\frac{\nu_a+\nu_b}{2\tilde{k}}\sinh^2\frac{\nu_a-\nu_b}{2\tilde{k}}}{\prod_b \cosh\left(\frac{\mu-\nu_b}{2\tilde{k}}+\frac{\pi m_+}{2}\right)\cosh\left(\frac{\mu+\nu_b}{2\tilde{k}}+\frac{\pi m_+}{2}\right)\cosh\left(\frac{\mu-\nu_b}{2\tilde{k}}+\frac{\pi m_-}{2}\right)\cosh\left(\frac{\mu+\nu_b}{2\tilde{k}}+\frac{\pi m_-}{2}\right)} \end{aligned} \tag{3.90}$$

with $\tilde{k} = 2k$. Next we use the Cauchy-Vandermonde determinant, given in (2.6) of [99]:

$$\begin{aligned} &\frac{\prod_a \sinh \frac{\nu_a}{\tilde{k}} \prod_{a<b} \sinh \frac{\nu_a+\nu_b}{2\tilde{k}} \sinh \frac{\nu_a-\nu_b}{2\tilde{k}}}{\prod_b \cosh \frac{\mu-\nu_b+\tilde{k}\pi m_+}{2\tilde{k}} \cosh \frac{\mu+\nu_b+\tilde{k}\pi m_+}{2\tilde{k}}} \\ &= \det \left(\begin{matrix} \left[\frac{\sinh \frac{\nu_a}{\tilde{k}}}{\cosh \frac{\mu+\tilde{k}\pi m_+ - \nu_a}{2\tilde{k}} \cosh \frac{\mu+\tilde{k}\pi m_+ + \nu_a}{2\tilde{k}}} \right] \\ \left[\sinh \frac{b\nu_a}{\tilde{k}} \right]_{b=1,\ldots,M} \end{matrix} \right) . \end{aligned} \tag{3.91}$$

We can simplify this expression by introducing canonical position and momentum operators $\hat{q}$ and $\hat{p}$ which satisfy $[\hat{q}, \hat{p}] = 2\pi i\tilde{k}$. We denote the $\hat{q}$ eigenstates by $|\nu\rangle$, and introduce states $|b]]$ such that

$$2 \sinh \frac{b\nu_a}{\tilde{k}} = [[b\,|\nu_a\rangle \ , \tag{3.92}$$

allowing us to simplify the lower block of (3.91). The upper block can be simplified using the Fourier transform, giving us (2.9) of [99]:

$$\frac{\sinh \frac{\nu_a}{\tilde{k}}}{\cosh \frac{\mu+\tilde{k}\pi m_+ - \nu_a}{2\tilde{k}} \cosh \frac{\mu+\tilde{k}\pi m_+ + \nu_a}{2\tilde{k}}} \propto \langle\mu| e^{\frac{i\hat{p}m_+}{4}} \frac{1}{\sinh \frac{\hat{p}}{2}} \hat{\Pi}_- \,|\nu_a\rangle \ . \tag{3.93}$$

We then follow [99] in performing similarity transforms

$$|\mu\rangle \to e^{-\frac{i}{4\pi\tilde{k}}\hat{p}^2}\,|\mu\rangle \ , \qquad |\nu_a\rangle \to e^{-\frac{i}{4\pi\tilde{k}}\hat{p}^2}\,|\nu_a\rangle \ , \tag{3.94}$$

so that (3.89) becomes

$$\begin{aligned} Z_{M,k}(m_+, m_-) &\propto \int d\mu\, d^{M+1}\nu \det \left(\begin{matrix} \left[\langle\mu| e^{\frac{i\hat{p}m_+}{4}} \frac{1}{\sinh \frac{\hat{p}}{2}} \hat{\Pi}_- \,|\nu_a\rangle \right] \\ \left[[[b| e^{\frac{i}{4\pi\tilde{k}}\hat{p}^2} \,|\nu_a\rangle \right]_{b=1,\ldots,M} \end{matrix} \right) \\ &\times \langle\mu| e^{\frac{i}{4\pi\tilde{k}}\hat{p}^2} e^{\frac{i}{4\pi\tilde{k}}\hat{q}^2} e^{\frac{i\hat{p}m_-}{4}} \frac{1}{2\sinh\frac{\hat{p}}{2}} \hat{\Pi}_- e^{-\frac{i}{4\pi\tilde{k}}\hat{q}^2} e^{-\frac{i}{4\pi\tilde{k}}\hat{p}^2} |\nu_1\rangle \prod_{a=1}^{M} [[a| e^{-\frac{i}{4\pi\tilde{k}}\hat{q}^2} e^{-\frac{i}{4\pi\tilde{k}}} |\nu_{a+1}\rangle \end{aligned} \tag{3.95}$$

We can now apply (2.12-5) of [99] to simplify the matrix elements, but with the modification

$$\begin{aligned} &\langle\mu| e^{\frac{i}{4\pi\tilde{k}}\hat{p}^2} e^{\frac{i}{4\pi\tilde{k}}\hat{q}^2} e^{\frac{i\hat{p}m_-}{4}} \frac{1}{2\sinh\frac{\hat{p}}{2}} \hat{\Pi}_- e^{-\frac{i}{4\pi\tilde{k}}\hat{q}^2} e^{-\frac{i}{4\pi\tilde{k}}\hat{p}^2} |\nu_1\rangle \\ &= \frac{\pi\tilde{k} e^{\frac{im_-\mu}{2}}}{i\sinh\frac{\mu}{2}} \left(\delta(\mu-\nu_1) - \delta(\mu+\nu_1) \right) , \end{aligned} \tag{3.96}$$

and so find that

$$
\begin{aligned}
Z_{M,k}(m_+,m_-) &\propto \int d\mu\, d^{M+1}\nu\ \frac{\prod_a \sinh\frac{\nu_a}{\tilde{k}} \prod_{a<b}\sinh\frac{\nu_a+\nu_b}{2\tilde{k}}\sinh\frac{\nu_a-\nu_b}{2\tilde{k}}}{\prod_b \cosh\frac{\mu-\nu_b+\tilde{k}\pi m_+}{2\tilde{k}}\cosh\frac{\mu+\nu_b+\tilde{k}\pi m_+}{2\tilde{k}}} \\
&\quad\times \frac{e^{\frac{im_-\mu}{2}}\delta(\mu-\nu_1)}{\sinh\frac{\mu}{2}}\prod_{m=1}^{M}\delta(\nu_{m+1}-2\pi i m) \\
&\propto \frac{1}{\cosh\frac{\pi m_+}{2}}\int d\mu \frac{e^{\frac{im_-\mu}{2}}\sinh\left[\frac{\mu}{2k}\right]}{\sinh\left[\frac{\mu}{2}\right]\cosh\left[\frac{\mu}{2k}+\frac{\pi m_+}{2}\right]} \\
&\quad\times \frac{\prod_{l=1}^{M}\sinh\left[\frac{\mu+2\pi il}{4k}\right]\sinh\left[\frac{\mu-2\pi il}{4k}\right]}{\prod_{l=1}^{M}\cosh\left[\frac{\mu+2\pi il}{4k}+\frac{\pi m_+}{2}\right]\cosh\left[\frac{\mu-2\pi il}{4k}+\frac{\pi m_+}{2}\right]}\,.
\end{aligned}
\tag{3.97}
$$

After a change of variables $\mu \to 2x$ and some further simplifications, we arrive at our final expression for the partition function

$$
\begin{aligned}
&Z_{M,k}(m_+,m_-) \\
&= \frac{Z_0}{\cosh\frac{\pi m_+}{2}}\int dx\ \frac{e^{i\pi m_- x}\cosh\left[\frac{\pi x}{2k}\right]\cosh\left[\frac{\pi x}{2k}+\frac{\pi m_+}{2}\right]}{\sinh\left[\pi x\right]\cosh\left[\frac{\pi x}{k}+\frac{\pi m_+}{2}\right]}\prod_{l=-M}^{M}\frac{\sinh\left[\frac{\pi(x+il)}{2k}\right]}{\cosh\left[\frac{\pi(x+il)}{2k}+\frac{\pi m_+}{2}\right]}\,.
\end{aligned}
\tag{3.98}
$$

3.3.2 Finite M, k Calculations

We now compute c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ for finite values of M and k. Using (3.98) for the case $m_+ = 0$ and $m_- = m$, we find that

$$
\hat{Z}_{M,k}(0,m) \equiv \frac{Z_{M,k}(0,m)}{Z_0} = \int_{-\infty}^{\infty} dx\, e^{i\pi m x} G_{M,k}(x) \tag{3.99}
$$

where we define

$$
G_{M,k}(x) = \frac{\cosh^2\frac{\pi x}{2k}}{\sinh\pi x\cosh\frac{\pi x}{k}}\prod_{l=-M}^{M}\tanh\frac{\pi(x+il)}{2k}\,. \tag{3.100}
$$

Similarly to $F_{M,k}(x)$ in Section 3.2.2, all poles of $G_{M,k}(x)$ are located at $x = \frac{iK}{2}$ for $K \in \mathbb{Z}$, and furthermore $G_{M,k}(x)$ is periodic in the complex plane, with

$$
G_{M,k}(x+2ik) = G_{M,k}(x)\,. \tag{3.101}
$$

M	k	$\frac{16}{c_T}$	$\lambda^2_{(B,2)^{[022]}_{2,0}}$
0	1	1	4
	2	$\frac{3}{4}=0.75$	$\frac{16}{5}=3.2$
	3	0.6511	2.986
	4	0.6009	2.921
1	2	$\frac{3}{4}=0.75$	$\frac{16}{5}=3.2$
	3	0.4778	2.648
	4	0.3879	2.503
2	4	0.3879	2.503
	5	0.3021	2.366
	6	0.2603	2.312
3	6	0.2603	2.312
4	8	0.1957	2.225
5	10	0.1568	2.175
6	12	0.1307	2.144

Table 3.4: OPE coefficients $\frac{16}{c_T}$ and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ in various $SO(2)_{2k}\times USp(2+2M)_{-k}$ theories

By closing the integral (3.99) in the upper-half of the complex, we may therefore reduce it to a finite sum of poles

$$\hat{Z}_{M,k}(m)=\frac{2\pi i}{1-e^{-2k\pi m}}\sum_{K=1}^{4k-1}\operatorname*{Res}_{x=\frac{iK}{2}}\left[e^{i\pi m x}G_{M,k}(x)\right]\,. \tag{3.102}$$

For small values of M and k we can easily sum over poles, and then compute c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ using (3.8) and (3.28). We list results for various M and k in Table 3.4.

3.3.3 Higher-Spin Limit

We now study the large M expansion of $SO(2)_{2k}\times USp(2+2M)_{-k}$ sphere partition function, holding $\lambda=\frac{2M+1}{2k}$ fixed. Defining

$$\begin{aligned}
\tilde{F}_1(x)&=\sum_{l=-M}^{M}\log\tanh\left[\frac{\pi\big(x+il\big)}{2k}\right]-\log\sinh\left(\pi x\right)\,,\\
\tilde{F}_2(x)&=\sum_{l=-M}^{M}\log\cosh\left[\frac{\pi\big(x+il\big)}{2k}\right]\,,\\
\hat{G}(x,\hat{m}_+)&=\log\left(\frac{\cosh\left[\frac{\pi x}{2\sqrt{k}}\right]\cosh\left[\frac{\pi(x+\hat{m}_+)}{2\sqrt{k}}\right]}{\cosh\frac{\pi\hat{m}_+}{2\sqrt{k}}\cosh\left[\frac{\pi(2x+\hat{m}_+)}{2\sqrt{k}}\right]}\right)\,,
\end{aligned} \tag{3.103}$$

and then performing a change of variables $x = k^{-1/2}y$, we find that

$$Z_{M,k}(m_+, m_-) \propto \int dx \ \exp\left(i\pi\hat{m}_- x + \tilde{G}(x, \hat{m}_+) + \tilde{F}_1(x) + \tilde{F}_2(x) - \tilde{F}_2(x + \hat{m}_+)\right) . \qquad (3.104)$$

Each of $\tilde{F}_1(x)$, $\tilde{F}_2(x)$ and $\tilde{G}(x, \hat{m}_-)$ can be expanded at large k with x and $\hat{m}_\pm$ fixed in a completely analogous fashion to $F_1(x)$, $F_2(x)$ and $G(x, \hat{m}_-)$ respectively, as derived in Section 3.2.4 . We then find that

$$Z_{M,k}(m_+, m_-) \propto \int dx \ \exp\left(-\pi \csc(\pi\lambda) x^2 + \dots\right) , \qquad (3.105)$$

where at each order in k^{-1} and $\hat{m}_\pm$ the terms in the exponent are polynomial in x. Derivatives of $Z_{M,k}(m_+, m_-)$ at $m_\pm = 0$ now reduce to a number of Gaussian integrals at each order in k^{-1}.

After a little work, we find that

$$\begin{aligned} c_T &= \frac{32k\sin(\pi\lambda)}{\pi} + 16\cos^2(\pi\lambda) - \frac{\pi\sin(\pi\lambda)[15 + 29\cos(2\pi\lambda)]}{3k} + O(k^{-2}) \\ \lambda^2_{(B,2)^{[022]}_{2,0}} &= 2 + \frac{\pi[3 + \cos(2\pi\lambda)]\csc(\pi\lambda)}{4k} \\ &\quad - \frac{\pi^2[39 + 44\cos(2\pi\lambda) - 19\cos(4\pi\lambda)]\csc^2(\lambda\pi)}{128k^2} + O(k^{-3}) . \end{aligned} \qquad (3.106)$$

Comparing to the exact results in Table 3.4, we see that already for $k = 2$ the approximations (3.106) are within a couple percent of the exact answers. Solving for $\lambda^2_{(B,2)^{[022]}_{2,0}}$ in terms of c_T, we find

$$\lambda^2_{(B,2)^{[022]}_{2,0}} = 2 + \frac{8(\cos(2\lambda\pi) + 3)}{c_T} - \frac{32\sin^2(\pi\lambda)(17 + 23\cos(2\pi\lambda))}{c_T^2} + \cdots , \qquad (3.107)$$

from which we see that, at least at large c_T, the theory with $\lambda = 1/2$ has the smallest value of $\lambda^2_{(B,2)^{[022]}_{2,0}}$. This value of $\lambda^2_{(B,2)^{[022]}_{2,0}}$ is still larger than that of the $U(1)_k \times U(1+M)_{-k}$ theory, however. By specializing to the $\lambda = 1/2$ theory, we can easily compute the large M expansion to higher orders in $1/c_T$, finding that

$$\begin{aligned} \lambda^2_{(B,2)^{[022]}_{2,0}} &= 2 + \frac{16}{c_T} + \frac{3}{4}\left(\frac{16}{c_T}\right)^2 - \frac{25}{24}\left(\frac{16}{c_T}\right)^3 + \frac{437}{64}\left(\frac{16}{c_T}\right)^4 - \frac{20997}{640}\left(\frac{16}{c_T}\right)^5 \\ &\quad + \frac{259523}{1536}\left(\frac{16}{c_T}\right)^6 - \frac{897994667}{967680}\left(\frac{16}{c_T}\right)^7 + \cdots . \end{aligned} \qquad (3.108)$$

Finally, for the mixed mass derivatives, we find that

$$\begin{aligned}
\frac{1}{c_T^2}\frac{\partial^4 \log Z}{\partial^2 m_+ \partial^2 m_-} &= -\frac{\pi^4 \sin^2(\pi\lambda)}{256 c_T} - \frac{\pi^4(5+23\cos(2\pi\lambda))\sin^2(\pi\lambda)}{128 c_T^2} + O(c_T^{-3})\,,\\
\frac{1}{c_T^2}\frac{\partial^4 \log Z}{\partial^3 m_+ \partial m_-} &= \frac{i\pi^4 \sin(2\pi\lambda)}{512 c_T} - \frac{i\pi^4(30\sin(2\pi\lambda)-23\sin(4\pi\lambda))}{512 c_T^2} + O(c_T^{-3})\,.
\end{aligned} \tag{3.109}$$

3.4 Additional $U(1)$ Factors

To close this chapter, we will show that given an $\mathcal{N}=6$ Chern-Simon gauge theory with gauge group $G = SU(N)\times SU(N+M)$ or $G = USp(2+2M)$, the S^3 partition function for the theory $G\times U(1)^L$ is equivalent to that of the theory $G\times U(1)$, up to an overall constant. This provides evidence for our conjecture in Section 1.2 that additional $U(1)$ factors do not affect the $\langle SSSS\rangle$ correlator.

Note that the mass-deformed S^3 partition function for the $G\times U(1)^L$ can be generically written as

$$Z_{G\times U(1)^L}(m_+, m_-) = \int d\chi_1 \dots d\chi_N \, e^{i\pi\sum_{ab} K_{ab}\chi_a\chi_b} Z_G(m_+ + 2q\cdot\chi, m_- + 2q\cdot\chi)\,, \tag{3.110}$$

where K_{ab} is the matrix of Chern-Simons levels for the $U(1)$s $q = (q_1, \dots, q_L)$ are the charges of the (bi)fundamentals under each $U(1)$, and $Z_G(m_+, m_-)$ is the S^3 partition function for the theory without any $U(1)$ factors. In order for $G\times U(1)^N$ to have $\mathcal{N}=6$ supersymmetry, K_{ab} and q_a must satisfy the condition

$$\sum_{a,b} K^{ab} q_a q_b = \frac{1}{k_G} \tag{3.111}$$

for some G dependent constant k_G, where K^{ab} is the inverse of K_{ab} [19].

To simplify (3.110), we first perform a change of basis of χ_a such that $q_a = (1, 0, \dots, 0)$. Because K_{ab} is symmetric, we can then perform a second change of basis to $\chi_2, \dots, \chi_L$, so that K_{ab} take the form

$$K_{ab} = \begin{pmatrix} K_{11} & K_{12} & 0 & \dots \\ K_{12} & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.112}$$

We can now integrate over $\chi_2\,, \chi_3\,, \ldots$ leaving us with

$$Z_{G\times U(1)^L}(m_+, m_-) \propto \int d\chi_1 \, e^{i\pi(K_{11}-K_{12}^2)\chi^2} Z_G(m_+ + 2\chi_1, m_- + 2\chi_1)\,, \tag{3.113}$$

We then note that, in this basis, the condition (3.111) becomes:

$$K_{11} - K_{12}^2 = k_G\,, \tag{3.114}$$

and so

$$Z_{G\times U(1)^L}(m_+, m_-) \propto \int d\chi_1 \, e^{i\pi k_G \chi^2} Z_G(m_+ + 2\chi_1, m_- + 2\chi_1)\,. \tag{3.115}$$

We now simply recognize the right-hand side of this equation is the partition function for the $G\times U(1)$ theory, and have hence shown what we set out to prove.

Chapter 4

String and M-Theory Limits

In this chapter, we will study the string and M-theory limits of the $U(N)_k \times U(N+M)_{-k}$ ABJ theory. In these limits the bulk can be described semiclassical, and so CFT correlators can be computed using bulk Witten diagrams. Though such holographic correlators have been a subject of study since the early days of the AdS/CFT correspondence [20, 56, 57] (see for example [100–108] for early work on four-point functions), they are in many cases hard or even impossible to compute directly. For instance, for higher derivative contact interactions in string theory or M-theory the full supersymmetric completion of the first correction to the supergravity action is not completely known (see however [109–112]), and so one cannot even write down the full set of relevant Witten diagrams. In the past few years, however, it has become clear that in certain cases one can essentially 'bootstrap' the answer using various consistency conditions [65, 74, 113–119]. These consistency conditions include crossing symmetry, the analytic properties of correlators in Mellin space, and supersymmetry. In particular, for tree-level Witten diagrams with supergravity and/or higher derivative vertices in 2d [120–122], 3d [65,117,118], 4d [74,115,116], 5d [123], and 6d [114,119] maximally supersymmetric theories, these consistency conditions determine the Witten diagrams contributing to the 4-point functions[1] of 1/2-BPS operators up to a finite number of coefficients. For low orders in the derivative expansion, one can further fix these coefficients using other methods, such as supersymmetric localization [15, 82] or the relation between the Mellin amplitudes and flat space scattering amplitudes in 10d or 11d [22,125–129]. In particular, Refs. [65, 74, 119] showed that the tree-level Witten diagram corresponding to an R^4 contact interaction, which is the first correction to supergravity in both 10d and 11d, can be completely determined using either supersymmetric

[1]See also [124] for recent work on holographic five-point functions in the 4d $\mathcal{N}=4$ super-Yang-Mills theory in the supergravity approximation.

localization or the flat space scattering amplitudes. The agreement between the two methods of fixing the undetermined coefficients in this case provides a precision test of AdS/CFT beyond supergravity.

Our goal in this chapter is to apply analytic bootstrap techniques to the $\langle SSSS \rangle$ correlator in $U(N)_k \times U(N+M)_{-k}$ ABJ theory. Unlike previous studies, the $U(N)_k \times U(N+M)_{-k}$ theories have only $\mathcal{N} = 6$, rather than the maximal $\mathcal{N} = 8$, supersymmetry, and so are more challenging to study. The reason for pursuing this generalization is that it offers the possibility of an unprecedented test of AdS/CFT at finite string coupling g_s. Indeed, if in ABJ theory we take N to be large and of the same order as k^5, then the holographic dual is a weakly curved $AdS_4 \times \mathbb{CP}^3$ background of type IIA string theory with finite g_s [16]. Using the consistency conditions mentioned above supplemented by supersymmetric localization results, we will be able to fully determine the contribution of the R^4 contact diagrams to the four-point function of the lowest dimension operator in the same super-multiplet as the stress tensor. The flat space limit of the Mellin amplitude then reproduces precisely the R^4 contribution to the four-point scattering of super-gravitons in type IIA string theory as a function of g_s. This function receives contributions from genus zero and genus one string worldsheets [130]. The reason why such a finite g_s test of AdS/CFT is not available in the maximally supersymmetric cases is that in 3d and 6d the bulk dual is an M-theory as opposed to string theory background, while in the 4d case, whose dual is type IIB string theory on $AdS_5 \times S^5$, the required supersymmetric localization result in the limit of large N and finite $g_s \propto g_{\text{YM}}^2$ is hard to evaluate due to the contribution of instantons in the localized S^4 partition function [15, 131–134].

Our primary challenge is to determine the first few tree-level corrections to the correlator $\langle SSSS \rangle$. As we shall see in Section 4.1, this task is simplified in Mellin space. Tree-level Mellin amplitudes have a simple analytic structure, and can be related to flat space scattering amplitudes via the Penedones formula. In Section 4.2 we study these flat space amplitudes, and solve the problem of computing tree-level correlators in this simpler setting. Using both the flat space limit and the superconformal Ward identities derived in Chapter 2, we are then able to compute tree-level corrections up to the R^4 term, which we do in Section 4.3.

With this task solved, we can apply our results to the string and M-theory limits of ABJ. In Section 4.4 we use the known IIA string theory and M-theory flat space scattering amplitudes to organize the large c_T expansion of $\langle SSSS \rangle$, and can fix the contributions of certain Mellin amplitudes by considering their flat space limits. We finish in Section 4.5 by combining certain CFT constraints and the supersymmetric localization constraints derived in Chapter 3 to completely determine the first few corrections to $\langle SSSS \rangle$ at large c_T in each of the three stringy limits studied in this chapter. Certain coefficients can be computed independently from both supersymmetric localization and the flat space limit, allowing us to test AdS/CFT beyond supergravity.

4.1 Mellin Space

Holographic correlators are simpler in Mellin space. To compute the Mellin transform of $\mathcal{S}^i(U,V)$, we first compute the connected correlator by subtracting the disconnected part, defined in (2.78),

$$\mathcal{S}^i_{\text{conn}}(U,V) \equiv \mathcal{S}^i(U,V) - \mathcal{S}^i_{\text{disc}}(U,V)\,, \tag{4.1}$$

and then define $M^i(s,t)$ through the equation

$$\mathcal{S}^i_{\text{conn}}(U,V) = \int_{-i\infty}^{i\infty} \frac{ds\,dt}{(4\pi i)^2}\, U^{\frac{s}{2}} V^{\frac{u}{2}-1} \Gamma^2\left[1-\frac{s}{2}\right]\Gamma^2\left[1-\frac{t}{2}\right]\Gamma^2\left[1-\frac{u}{2}\right] M^i(s,t)\,, \tag{4.2}$$

where $u = 4-s-t$. The Mellin transform (4.2) is defined such that a bulk contact Witten diagrams coming from a vertex with $n = 2m$ derivatives gives rise to a polynomial $M^i(s,t)$ of degree m [22]. The two integration contours in (4.2) are chosen such that[2]

$$\text{Re}(s) < 2\,, \quad \text{Re}(t) < 2\,, \quad \text{Re}(u) = 4 - \text{Re}(s) - \text{Re}(t) < 2\,, \tag{4.3}$$

which include all poles of the Gamma functions on one side or the other of the contour. These poles naturally incorporate the effect of double trace operators [135].

In this chapter we focus on contact Witten diagrams, and in particular aim to find a basis of Mellin amplitudes that can be used to write the contribution from contact Witten diagrams with small numbers of derivatives. These Mellin amplitudes must satisfy three constraints:

1. They obey the crossing symmetry requirements

$$\begin{aligned} &M^1(s,t) = M^1(s,u)\,, \qquad M^2(s,t) = M^1(t,s)\,, \qquad M^3(s,t) = M^1(u,t)\,,\\ &M^4(s,t) = M^4(s,u)\,, \qquad M^5(s,t) = M^4(t,s)\,, \qquad M^6(s,t) = M^4(u,t) \end{aligned} \tag{4.4}$$

   coming from the crossing symmetry of the full $\langle SSSS \rangle$ correlator.

2. They obey the supersymmetric Ward identities derived in Chapter 2. The SUSY Ward identities not only constrain $M^i(s,t)$, but they also allow us to determine the Mellin amplitudes

[2]This is the correct choice of contour provided that $M^i(s,t)$ does not have any poles with $\text{Re}(s) < 2$ or $\text{Re}(t) < 2$ or $\text{Re}(u) < 2$. If this is not the case (such as for the supergravity Mellin amplitude), the integration contour will have to be modified in such a way that the extra poles are on the same side of the contour as the other poles in s, t, u, respectively.

corresponding to correlators of other operators in the stress tensor multiplet.

3. The $M^i(s,t)$ and all other Mellin amplitudes related to them by SUSY are polynomials in s, t. We call the collection of Mellin amplitudes corresponding to four-point functions of operators in the same super-multiplet a super-Mellin amplitude, and we define the degree of a polynomial super-Mellin amplitude n to be the highest degree of any component Mellin amplitude.

For fixed m, we will label the Mellin amplitudes obeying these requirements as $M^i_m(s,t)$ in cases where there is a unique such amplitude for a given m or by $M^i_{m,k}(s,t)$ in the cases where there are multiple such amplitudes indexed by k. These Mellin amplitudes represent a basis for contact Witten diagrams, with the number of derivatives in the interaction vertex being bounded from below by $2m$.

Note that, in general, supersymmetry relates the contact interactions for bulk fields with various spins, and in flat space SUSY preserves the number of derivatives of the interaction vertices it relates. In AdS however, the number of derivatives within a given super-vertex may vary, with the change in the number of derivatives being compensated by an appropriate power of the AdS radius L. Thus, it may happen that a four-scalar vertex with a given number of derivatives is part of a supervertex containing other vertices with more derivatives. The corresponding Mellin amplitudes $M^i(s,t)$ will then have lower degree than those of some four-point function of superconformal descendants of S, and so $M^i_n(s,t)$ may have degree less than n. This fact will be very important in the analysis that follows.

4.1.1 The Flat-Space Limit

Finding the Mellin amplitudes $M^i_n(s,t)$ that obey the conditions listed above is a difficult task, as satisfying the third condition requires us to calculate Ward identities for many different correlators and then examine the locality properties of their Mellin amplitudes. We can simplify matters by first solving an analogous problem for flat space scattering amplitudes.

At large AdS radius, we can recover flat space scattering amplitudes for scalars using the Penedones formula [128]. Applied to the superconformal primary S the relationship is (up to an overall normalization $\mathcal{N}(L)$)

$$\mathcal{A}^i(s,t) = \lim_{L\to\infty} \mathcal{N}(L)\sqrt{\pi} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{d\alpha}{2\pi i}\, e^{\alpha}\alpha^{-\frac{1}{2}} M^i\left(\frac{L^2}{2\alpha}s, \frac{L^2}{2\alpha}t\right) . \tag{4.5}$$

Here, $\kappa > 0$, and $\mathcal{A}^i(s,t)$ is the corresponding 4d flat space scattering amplitude of graviscalars (or more precisely a scattering amplitude in 10d string theory or 11d M-theory with the momenta restricted to lie within 4d and polarizations transverse to this 4d space), computed in the limit where the AdS radius L is taken to infinity while keeping some other dimensionful length scale ℓ_{UV} fixed. For string or M-theory duals we can take ℓ_{UV} to be either the 10d string length or 11d Planck length, as we will do in Section 4.4.

From (4.5) we expect that each Mellin amplitude $M^i_{m,k}(s,t)$ gives rise to a local $\mathcal{N}=6$ scattering amplitude $\mathcal{A}^i_{m,k}(s,t)$. This mapping should furthermore be one-to-one, since if two amplitudes M^i_{m,k_1} and M^i_{m,k_2} have the same large s,t limit, then their difference $M^i_{m,k_1}-M^i_{m,k_2}$ will be a local Mellin amplitude with degree at most $m-1$. Thus, if we can find all of the number of local scattering amplitudes of a given degree in s,t, then this will also tell us the number of Mellin amplitudes which occur at this degree:[3]

$$\begin{aligned}&\#\text{ of degree } m \text{ scattering amplitudes in 4d SUGRA}\\ &\qquad = \quad \#\text{ of degree } m \text{ Mellin amplitudes in 3d SCFT}\,.\end{aligned} \tag{4.6}$$

Because the flat space scattering amplitudes are obtained as the large s, t limits of Mellin amplitudes, finding all crossing-invariant, supersymmetric, and local $\mathcal{N}=6$ flat space scattering amplitudes is a strictly simpler problem than finding all Mellin amplitudes with the same properties.

4.2 Scattering Amplitudes with $\mathcal{N}=6$ Supersymmetry

The toy problem described in the previous section is that of finding four-point scattering amplitudes corresponding to counterterms in 4d $\mathcal{N}=6$ supergravity. Spinor helicity and on-shell supersymmetric methods provide an efficient means to classify allowed counterterms in a theory. They were first applied to 4d $\mathcal{N}=8$ in [136, 137], and have subsequently been generalized to other maximally supersymmetric theories in [138, 139]. In the context of $\mathcal{N}=6$ supergravity these methods have been applied to study amplitudes involving bulk graviton exchange [140, 141].

We will begin this section with a lightning review of spinor-helicity variables, including their discrete symmetries, before discussing the on-shell superspace formalism as applied to $\mathcal{N}=6$ theories. For a much more detailed treatment of spinor-helicity and on-shell methods we recommend [142]. In Section 4.2.3 we apply these methods to classify counterterms in $\mathcal{N}=6$ supergravity which con-

[3] At a more abstract level, we can justify the correspondence (4.6) as follows. Local Mellin amplitudes correspond to bulk contact Witten diagrams, which are themselves in one-to-one correspondence with local counterterms in AdS. But since AdS is maximally symmetric, local counterterms in AdS are equivalent to local counterterms in flat space. Since local counterterms in flat space correspond exactly to scattering amplitudes, we find that Mellin amplitudes and scattering amplitudes are in one-to-one correspondence.

tribute to four-particle scattering, and discuss their implications for $\mathcal{N}=6$ SCFTs in Section 4.2.4. We then close with a discussion of four-particle exchange diagrams in Section 4.2.5.

4.2.1 Spinor-Helicity Review

For massless fermions, the Dirac equation for the wavefunction of 4-component spinors implies

$$\not{p}v_{\pm}(p) = 0\,, \qquad \overline{u}_{\pm}(p)\not{p} = 0\,. \tag{4.7}$$

Here $\pm$ indicated the helicity $h = \pm\frac{1}{2}$ of the wavefunction. If we take our Dirac matrices to be in the Weyl basis, namely

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.8}$$

where 1 stands for the 2×2 identity matrix and σ^i, $i = 1,2,3$ are the standard Pauli matrices, then the top two components of the Dirac spinor transform in the $(1/2,0)$ and bottom two in the $(0,1/2)$ of $SO(3,1)$. For a given momentum $p^\mu = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta)$, we can then define the angle and square brackets as

$$\begin{aligned} |p\rangle^{\dot{a}} &= \sqrt{2E}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \qquad |p]_a = \sqrt{2E}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix}, \\ [p|^a &= \sqrt{2E}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{-i\phi} \end{pmatrix}, \qquad \langle p|_{\dot{a}} = \sqrt{2E}\begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{-i\phi} \end{pmatrix}, \end{aligned} \tag{4.9}$$

such that

$$\begin{aligned} v_+(p) &= \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \qquad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}, \\ \overline{u}_+(p) &= \begin{pmatrix} [p|^a & 0 \end{pmatrix}, \qquad \overline{u}_-(p) = \begin{pmatrix} 0 & \langle p|_{\dot{a}} \end{pmatrix}. \end{aligned} \tag{4.10}$$

are solutions to (4.7). Now consider the scattering of massless particles $b_i^{\pm}$ with helicity $\pm h_i$ for $i = 1, 2, \ldots$. We define the scattering amplitude to be:

$$A[b_1^{\pm}b_2^{\pm}\ldots]\delta^{(4)}(p_1+p_2+\ldots) = \langle a_1^{\pm}(p_1)a_2^{\pm}(p_2)\ldots\rangle \tag{4.11}$$

where $a_i^{\pm}(p)$ is the annihilation operator of the i^{th} particle, annihilating a particle of helicity $\pm h_i$ and momentum p_i. The amplitude $A[b_1^{\pm} b_2^{\pm} \ldots]$ must be a Lorentz scalar which transforms covariantly under the little group transformations.

We will finish by reviewing the discrete symmetries $\mathcal{P}$ and $\mathcal{CT}$. Under parity $\mathcal{P}$, we reverse the spatial components of the momentum of a particle, while leaving the spin unchanged. Flipping the direction of $\vec{p}$ is equivalent to sending $\theta \to \pi - \theta$ and $\phi \to \phi \pm \pi$ in (4.9). Under this transformation, the spinors in the first line of (4.9) get interchanged and so do the spinors on the bottom line. Thus, parity acts[4] as either $\mathcal{P}_{a\dot{a}}$ or $\mathcal{P}^{\dot{a}a}$

$$\mathcal{P}_{a\dot{a}}|p\rangle^{\dot{a}} = |p]_a\,, \qquad \mathcal{P}^{\dot{a}a}|p]_a = |p\rangle^a\,, \qquad [p|^a \mathcal{P}_{a\dot{a}} = \langle p|_{\dot{a}}\,, \qquad \langle p|_{\dot{a}} \mathcal{P}^{\dot{a}a} = [p|^a\,. \tag{4.12}$$

The effect of parity is hence to swap all angle brackets with square brackets and vice versa, while leaving all coefficients unchanged. For instance, $\mathcal{P}(c\langle 12\rangle) = c[12]$ for any constant c.

The second discrete symmetry we consider is $\mathcal{CT}$, which is the product of charge conjugation and time-reversal. Under $\mathcal{CT}$, the spatial components of momentum also flip sign, just like for $\mathcal{P}$, but in addition $\mathcal{CT}$ also implements complex conjugation. Thus, from (4.9), we see that $\mathcal{CT}$ acts as either $(\mathcal{CT})_{\dot{a}\dot{b}}$ or $(\mathcal{CT})^{ab}$ as follows:

$$(\mathcal{CT})_{\dot{a}\dot{b}}|p\rangle^{\dot{b}} = \langle p|_{\dot{a}}\,, \qquad (\mathcal{CT})^{ab}|p]_b = [p|^a\,, \qquad \langle p|_{\dot{a}}(\mathcal{CT})_{\dot{a}\dot{b}} = |p\rangle^{\dot{b}}\,, \qquad [p|^a(\mathcal{CT})^{ab} = |p]_b\,. \tag{4.13}$$

Thus, the effect of $\mathcal{CT}$ is to flip all the brackets and perform complex conjugation on the coefficients—for instance $\mathcal{CT}(c\langle 12\rangle) = c^*\langle 21\rangle$ for any constant c.

The combined transformation of the two symmetries above, $\mathcal{CPT}$, is a symmetry of all unitary QFTs. On amplitudes, it acts by exchanging angle brackets with flipped square brackets and vice versa, and it complex conjugates the coefficients. For instance, $\mathcal{CPT}(c\langle 12\rangle) = c^*[21]$. Using $\mathcal{CPT}$, we can relate a given amplitude to the amplitude of the $\mathcal{CPT}$ conjugate particles. For particles b_1, b_2, etc. with $\mathcal{CPT}$ conjugate particles $\bar{b}_1$, $\bar{b}_2$, etc., we have

$$\mathcal{CPT}\left(A[b_1^{\pm} b_2^{\pm} \ldots]\right) = A[\bar{b}_1^{\mp} \bar{b}_2^{\mp} \ldots]\,. \tag{4.14}$$

[4] In terms of the four-component spinors (4.10), the action of parity takes the usual form:

$$v_{\pm}(p^0, -\vec{p}) = \gamma^0 v_{\pm}(p^0, \vec{p}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v_{\pm}(p^0, \vec{p})\,.$$

Φ Particles	h^+	ψ^+	g^+	F^+	ϕ	χ^-	a^-
Helicity	$+2$	$+3/2$	$+1$	$+1/2$	0	$-1/2$	-1
$SU(6)_R$	**1**	**6**	**15**	**20**	$\overline{\mathbf{15}}$	$\overline{\mathbf{6}}$	**1**
Ψ Particles	a^+	χ^+	$\overline{\phi}$	F^-	g^-	ψ^-	h^-
Helicity	$+1$	$+1/2$	0	$-1/2$	-1	$-3/2$	-2
$SU(6)_R$	**1**	**6**	**15**	**20**	$\overline{\mathbf{15}}$	$\overline{\mathbf{6}}$	**1**

Table 4.1: Massless particles in $\mathcal{N} = 6$ supergravity.

4.2.2 $\mathcal{N} = 6$ On-Shell Formalism

In $\mathcal{N} = 6$ supergravity, the massless particles split into two supermultiplets: a multiplet we denote by Φ that contains the positive helicity graviton h^+, and its $\mathcal{CPT}$ conjugate multiplet we denote by Ψ that contains the negative helicity graviton h^-. In addition to the graviton $h^\pm$, these multiplets also contain the gravitino $\psi^\pm$, the gauginos $g^\pm$, fermions $F^\pm$ and $\chi^\pm$, scalars ϕ, and the graviphoton $a^\pm$. Table 4.1 lists the particles in these multiplets, along with their transformation properties under the $SU(6)$ R-symmetry of $\mathcal{N} = 6$ supergravity. In the on-shell superspace formalism, the Φ and Ψ superfields are polynomials in the Grassmann variables η^I, with $I = 1, \ldots 6$ transforming in the $\overline{\mathbf{6}}$ of $SU(6)$:[5]

$$\begin{aligned}
\Phi \equiv\ & h^+ + \eta^I \psi_I^+ + \frac{1}{2!}\eta^I\eta^J g_{IJ}^+ + \frac{1}{3!}\eta^I\eta^J\eta^K F_{IJK}^+ + \frac{1}{4!2}\eta^I\eta^J\eta^K\eta^L \epsilon_{IJKLMN}\phi^{MN} \\
& + \frac{1}{5!}\eta^I\eta^J\eta^K\eta^L\eta^M \epsilon_{IJKLMN}\chi^{N-} + \frac{1}{6!}\eta^I\eta^J\eta^K\eta^L\eta^M\eta^N \epsilon_{IJKLMN} a^- \\
\Psi \equiv\ & a^+ + \eta^I \chi_I^+ + \frac{1}{2!}\eta^I\eta^J \overline{\phi}_{IJ} + \frac{1}{3!}\eta^I\eta^J\eta^K F_{IJK}^+ + \frac{1}{4!2}\eta^I\eta^J\eta^K\eta^L \epsilon_{IJKLMN} g^{MN} \\
& + \frac{1}{5!}\eta^I\eta^J\eta^K\eta^L\eta^M \epsilon_{IJKLMN}\psi^{N-} + \frac{1}{6!}\eta^I\eta^J\eta^K\eta^L\eta^M\eta^N \epsilon_{IJKLMN} h^- \,.
\end{aligned} \tag{4.15}$$

In a four-point superamplitude, such as $\mathcal{A}[\Phi\Phi\Psi\Psi]$, each particle $i = 1, \ldots, 4$ is associated to some Grassmannian variable η_i^I. To compute a component scattering amplitude we simply differentiate with respect to some of the Grassmannian variables while setting all others to zero. For instance:

$$\begin{aligned}
A[h^+h^+h^+h^+] &= \mathcal{A}[\Phi\Phi\Psi\Psi]\Big|_{\eta_i^I=0}\,, \\
A[h^+h^+h^-h^-] &= \left(\prod_{J=1}^{6}\frac{\partial}{\partial\eta_3^J}\right)\left(\prod_{K=1}^{6}\frac{\partial}{\partial\eta_4^K}\right)\mathcal{A}[\Phi\Phi\Psi\Psi]\Big|_{\eta_i^I=0}\,, \\
A[\phi^{56}\phi^{56}\overline{\phi}_{12}\overline{\phi}_{12}] &= \left(\prod_{J=1}^{4}\frac{\partial}{\partial\eta_1^J}\right)\left(\prod_{K=1}^{4}\frac{\partial}{\partial\eta_2^K}\right)\left(\prod_{L=1}^{2}\frac{\partial}{\partial\eta_3^L}\right)\left(\prod_{M=1}^{2}\frac{\partial}{\partial\eta_4^M}\right)\mathcal{A}[\Phi\Phi\Psi\Psi]\Big|_{\eta_i^I=0}\,.
\end{aligned} \tag{4.16}$$

[5] Upper $I, J, K, \ldots$ indices transform in the $\overline{\mathbf{6}}$ of $SU(6)$ while lower $I, J, K, \ldots$ indices transform in the $\mathbf{6}$ of $SU(6)$.

In this way a superamplitude $\mathcal{A}$ encodes all the amplitudes of its component particles.

Up to crossing, there are five possible 4 particle superamplitudes we can construct from Φ and Ψ. However, under $\mathcal{CPT}$ the two supermultiplets Φ and Ψ are conjugates, and their scattering amplitudes are related by complex conjugation

$$\mathcal{A}[\Psi\Psi\Psi\Psi] = (\mathcal{A}[\Phi\Phi\Phi\Phi])^* \,, \qquad \mathcal{A}[\Psi\Psi\Psi\Phi] = (\mathcal{A}[\Phi\Phi\Phi\Psi])^* \,. \tag{4.17}$$

This leaves us only three independent superamplitudes, $\mathcal{A}[\Phi\Phi\Psi\Psi]$, $\mathcal{A}[\Phi\Phi\Phi\Psi]$, and $\mathcal{A}[\Phi\Phi\Phi\Phi]$. Our task now is to constrain the forms of these superamplitudes, beginning with invariance under supersymmetry.

As explained in [142], for a given particle i the supermomentum is defined to be

$$q_i^I = |i\rangle \eta_i^I \,, \qquad \tilde{q}_{Ii} = |i] \frac{\partial}{\partial \eta_i^I} \,, \tag{4.18}$$

and it satisfies the on-shell SUSY algebra by construction. For a given amplitude the total supermomentum is thus:

$$Q^I = \sum_i q_i^I \,, \qquad \tilde{Q}_I = \sum_i \tilde{q}_{Ii} \,. \tag{4.19}$$

Superamplitudes must be annihiliated by these supercharges. For a four-point amplitude such as $\mathcal{A}[\Phi\Phi\Psi\Psi]$ this implies that

$$Q^I \mathcal{A}[\Phi\Phi\Psi\Psi] = 0 \,, \qquad \tilde{Q}_I \mathcal{A}[\Phi\Phi\Psi\Psi] = 0 \,. \tag{4.20}$$

Imposing these conditions uniquely fixes any four-point superamplitudes up to an arbitrary function of s and t:

$$\begin{aligned}
\mathcal{A}[\Phi\Phi\Psi\Psi] &= \delta^{12}(Q) \frac{[12]^4}{\langle 34\rangle^2} f_1(s,t) \,, \\
\mathcal{A}[\Phi\Phi\Phi\Psi] &= \delta^{12}(Q) \frac{[12]^5 \langle 14\rangle\langle 24\rangle}{\langle 34\rangle^4} f_2(s,t) \,, \\
\mathcal{A}[\Phi\Phi\Phi\Phi] &= \delta^{12}(Q) \frac{[12]^4}{\langle 34\rangle^4} f_3(s,t) \,,
\end{aligned} \tag{4.21}$$

where the first factor is the Grassmann delta function

$$\delta^{12}(Q) = \frac{1}{2^4} \prod_{I=1}^{6} \sum_{i,j=1}^{4} \langle ij\rangle \eta_i^I \eta_j^I \,, \tag{4.22}$$

$\sum_i \lvert h_i\rvert$	$\mathcal{A}[\Phi\Phi\Psi\Psi]$	$\mathcal{A}[\Phi\Phi\Phi\Psi]$	$\mathcal{A}[\Phi\Phi\Phi\Phi]$
0	$A[\phi\phi\overline{\phi}\overline{\phi}]$	None	None
1	$A[F^+\chi^-\overline{\phi}\overline{\phi}]$ $A[\phi\phi\chi^+F^-]$ $A[\phi\chi^-\chi^+\overline{\phi}]$ $A[\phi F^+F^-\overline{\phi}]$	$A[\phi\phi\phi a^+]$ $A[\phi\phi F^+\chi^+]$ $A[\phi\phi g^+\overline{\phi}]$	None
2	$A[F^+F^+F^-F^-]$ $A[\chi^-\chi^-\chi^+\chi^+]$ $A[\phi g^+g^-\overline{\phi}]$ $A[\phi a^-a^+\overline{\phi}]$ $A[\phi g^+F^-F^-]$ ...	$A[F^+F^+F^+F^-]$ $A[F^+F^+\psi^-\chi^+]$ $A[\phi\phi\psi^+F^-]$ $A[\psi^+\chi^-\phi\overline{\phi}]$ $A[g^+F^+\chi^-\overline{\phi}]$ ...	$A[F^+F^+F^+F^+]$ $A[g^+F^+F^+\phi]$ $A[g^+g^+\phi\phi]$ $A[\psi^+F^+\phi\phi]$ $A[h^+\phi\phi\phi]$
...	...	...	...
6	...	...	$A[h^+h^+a^-a^-]$
7	...	$A[h^+h^+a^-h^-]$	None
8	$A[h^+h^+h^-h^-]$	None	None

Table 4.2: Component amplitudes of each superamplitude, organized by total helicity $\sum_i |h_i|$. Here h_i is the helicity of the i^{th} particle. We have not included amplitudes equivalent to the ones listed here under crossing.

which is annihilated by both Q^I and $\tilde{Q}_I$, and $f_i(s,t)$ are functions of s and t. The delta function $\delta^{12}(Q)$ is automatically invariant under $SU(6)_R$, even if the full theory does not preserve $SU(6)_R$ [137].[6] Note that every term in each superamplitude contains exactly 12 Grassmannian variables, and, as a result, many component amplitudes vanish, including $A[h^+h^+h^+h^+] = A[\phi\phi\phi\phi] = 0$. See Table 4.2 for a list of component amplitudes that do not vanish. The angle and square brackets in (4.21) are required so that the Φ and Ψ components have the correct helicity, which for instance can be fixed by considering

$$\begin{aligned} A[h^+h^+h^-h^-] &= [12]^4\langle 34\rangle^4 f_1(s,t)\,, \\ A[h^+h^+h^-a^-] &= [12]^5\langle 34\rangle^2\langle 14\rangle\langle 24\rangle f_2(s,t)\,, \\ A[h^+h^+a^-a^-] &= [12]^4\langle 34\rangle^2 f_3(s,t)\,. \end{aligned} \tag{4.23}$$

Due to $\mathcal{CPT}$ invariance, the function $f_1(s,t)$ must always be real. We can see this by considering

[6] In flat space $\mathcal{N}=6$ the supersymmetry algebra does not require there to be an R-symmetry; it is an accidental symmetry of the supergravity action. On the other hand, the superconformal algebra does require that at least an $SO(6)_R$ symmetry be present in order for an AdS solution to preserve all supersymmetries of the theory.

the first equation in (4.23), and then crossing both $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ to find that

$$A[h^- h^- h^+ h^+] = \langle 12\rangle^4 [34]^4 f_1(s,t)\,, \tag{4.24}$$

But the amplitude $A[h^- h^- h^+ h^+]$ is also related to $A[h^+ h^+ h^- h^-]$ by $\mathcal{CPT}$,

$$\mathcal{CPT}\left(A[h^+ h^+ h^- h^-]\right) = A[h^- h^- h^+ h^+] = \langle 12\rangle^4 [34]^4 f_1^*(s,t)\,, \tag{4.25}$$

and from comparing this expression with (4.24) we conclude that $f_1(s,t)$ must be real. As a consequence of this, we find that

$$\mathcal{CT}\left(A[h^+ h^+ h^- h^-]\right) = [12]^4 \langle 34\rangle^4 f_1^*(s,t) = A[h^+ h^+ h^- h^-]\,, \tag{4.26}$$

and so $A[h^+ h^+ h^- h^-]$ is always $\mathcal{CT}$-even. This relation extends to the full multiplet thus showing that $A[\Phi\Phi\Psi\Psi]$ is $\mathcal{CT}$-even.

The functions $f_{2,3}(s,t)$, on the other hand, are in general complex, with their real and imaginary parts corresponding to $\mathcal{CT}$ even and odd amplitudes respectively. For instance, if we consider the third amplitude in (4.23) we see that

$$\mathcal{CT}\left(A[h^+ h^+ a^- a^-]\right) = [12]^4 \langle 34\rangle^2 f_3^*(s,t) \tag{4.27}$$

and so the amplitude $A[h^+ h^+ a^- a^-]$ is $\mathcal{CT}$ even / odd if $f_3(s,t)$ is real / pure imaginary. From this we conclude that $\mathcal{A}[\Phi\Phi\Phi\Phi]$ can be thought of as containing two distinct superstructures, one of which is $\mathcal{CT}$ even and the other $\mathcal{CT}$ odd. Similar manipulations show that $\mathcal{A}[\Phi\Phi\Phi\Psi]$ also contains a $\mathcal{CT}$ even and $\mathcal{CT}$ odd structure, corresponding to $f_2(s,t)$ purely real and purely imaginary respectively.

So far we have focused on the discrete spacetime symmetries. However, $\mathcal{N}=6$ superconformal symmetry also allows a discrete R-symmetry $\mathcal{Z}$, and so it is natural to extend this to $\mathcal{N}=6$ flat space scattering amplitudes . Recall that the various particles in the Φ and Ψ multiplets transform under an $SU(6)_R$ R-symmetry

$$\eta^I \to M^I{}_J \eta^J\,, \tag{4.28}$$

where $M^I{}_J$ is a unitary matrix with determinant 1. Let us now relax the determinant condition, and instead consider a more general element of $U(6)$. If we demand that the gravitons h^+ and h^-

are preserved under $\mathcal{Z}$, then without loss of generality we can take $\mathcal{Z}$ to act as

$$\mathcal{Z} : \Phi \to \Phi\,, \qquad \Psi \to -\Psi\,, \qquad \eta^I \to i\eta^I\,. \tag{4.29}$$

Under both $\mathcal{Z}$ and $-\mathcal{Z}$ the gauge fields flip sign

$$\mathcal{Z}:\ a^\pm \to -a^\mp\,, \qquad g^\pm \to -g^\mp \tag{4.30}$$

while the gravitons $h^\pm$ and the graviscalar ϕ are left invariant. The fermions will transform with additional factors of i:

$$\mathcal{Z}:\ \Psi^\pm \to \pm i\Psi^\mp\,, \qquad F^\pm \to \mp iF^\mp\,, \qquad \chi^\pm \to \pm i\chi^\mp\,. \tag{4.31}$$

The full symmetry group is now $(\mathbb{Z}_4 \times SU(6))/\mathbb{Z}_2$, the subgroup of $U(6)$ of matrices with determinant ± 1. Note however that only fermion bilinears are physical. As a result, the transformation $\eta^I \to -\eta^I$ acts trivially on all amplitudes. After quotienting the $SU(6)$ by this $\mathbb{Z}_2$ symmetry, we find that the symmetry group acting on the amplitudes is $\mathbb{Z}_2 \times (SU(6)/\mathbb{Z}_2)$, with $\mathcal{Z}^2 = I$.

While $\mathcal{Z}$ is a discrete R-symmetry of pure supergravity, it may or may not be a symmetry of the corrections to supergravity, so we can classify the various amplitude structures as $\mathcal{Z}$-even or $\mathcal{Z}$-odd. Since $\delta^{(12)}(Q)$ contains twelve η's, it is even under $\mathcal{Z}$, and so we conclude that $\mathcal{A}[\Phi\Phi\Psi\Psi]$ and $\mathcal{A}[\Phi\Phi\Phi\Phi]$ are even under $\mathcal{Z}$ and that $\mathcal{A}[\Phi\Phi\Phi\Psi]$ is odd. We can alternatively deduce this from (4.23), since $\mathcal{A}[\Phi\Phi\Phi\Psi]$ contains an amplitude with an odd number of gauge fields, while the other two amplitudes contain an even number.

To summarize, we have found that there are five linearly independent superamplitudes which contribute to the scattering of four supergravitons. Two of these structures are both $\mathcal{CT}$ and $\mathcal{Z}$ even, and there is a unique superamplitude for each of the other three possible $\mathcal{CT}$ and $\mathcal{Z}$ parity combinations.

4.2.3 Counterterms in $\mathcal{N} = 6$ Supergravity

We are now left to constrain the forms of $f_i(s,t)$ using locality and crossing symmetry. A tree-level scattering amplitude is local if and only if it can be written as a polynomial in the spinor helicity variables $[ij]$ and $\langle ij\rangle$; note that

$$s = [12]\langle 12\rangle = [34]\langle 34\rangle\,, \qquad t = [13]\langle 13\rangle = [24]\langle 24\rangle\,, \qquad u = [14]\langle 14\rangle = [23]\langle 23\rangle\,. \tag{4.32}$$

From (4.23) we immediately see that it is not possible for $f_i(s,t)$ to contain poles in s, t or u, or else the amplitudes in (4.23) would lead to non-polynomial expressions. Hence $f_i(s,t)$ are necessarily polynomials for tree-level amplitudes. This is also sufficient, as when $f_i(s,t) = 1$ one can check that all amplitudes in the superamplitude are local.

Crossing symmetry imposes a series of further constraints. For instance, in (4.23) the amplitudes must be invariant under interchanging the first and second particles. This gives us the relations

$$f_{1,3}(s,t) = f_{1,3}(s,u)\,, \qquad f_2(s,t) = -f_2(s,u)\,, \tag{4.33}$$

where $u = -s - t$ is the third Mandelstam variable. The superamplitudes $\mathcal{A}[\Phi\Phi\Phi\Phi]$ and $\mathcal{A}[\Phi\Phi\Phi\Psi]$ are also invariant under crossing which exchange the first and third particles, giving rise to the further conditions:

$$f_2(s,t) = -f_2(u,t)\,, \qquad f_3(s,t) = f_3(u,t)\,. \tag{4.34}$$

Together, Eqs. (4.33) and (4.34) suffice to guarantee crossing under all possible permutations.

Having determined the allowed forms of $f_i(s,t)$, we can now determine the number of derivatives in each interaction vertex. To this count each angle and square bracket contribute 1, $\delta^{12}(Q)$ contributes 6, and each power of s, t, u contributes 2. For instance, if we set $f_2(s,t) = s^k$ and consider the amplitude $\mathcal{A}[\Phi\Phi\Psi\Psi] = s^k \delta^{12}(Q) \frac{[12]^4}{\langle 34\rangle^2}$, it follows that this amplitude comes from an interaction vertex with $8 + 2k$ derivatives, namely from an $D^{2k}R^4$ term.

With this in mind, we can now systematically find all local counterterms up to a certain number of derivatives. In Table 4.3 we list all local counterterms up to 15 derivatives, corresponding to Mellin amplitudes up to degree 7.5.[7] In particular, the first local counterterm has 6 derivatives, is unique, and contributes only to $\mathcal{A}[\Phi\Phi\Phi\Phi]$. The next local counterterm has 8 derivatives and is also unique and contributes only to $\mathcal{A}[\Phi\Phi\Psi\Psi]$. There are two 10 derivative counterterms, one contributing to $\mathcal{A}[\Phi\Phi\Phi\Phi]$ and one to $\mathcal{A}[\Phi\Phi\Phi\Phi]$, and so on. The counterterm with the lowest number of derivatives that contributes to $\mathcal{A}[\Phi\Phi\Phi\Psi]$ has 15 derivatives and will not be important in this work.

4.2.4 Implications of Flat-Space Amplitudes for $\mathcal{N} = 6$ SCFTs

Having systematically computed the local amplitudes in $\mathcal{N} = 6$ supergravity, we will now discuss the implications for holographic $\mathcal{N} = 6$ SCFTs. First, we can deduce that there are five independent

[7] A Mellin amplitude of degree 7.5 would seem to require non-polynomial contributions to $M^i(s,t)$. Because $\mathcal{A}[\Phi\Phi\Phi\Psi]$ always violates $\mathcal{Z}$ while $\langle SSSS\rangle$ is $\mathcal{Z}$ preserving, the Mellin amplitudes corresponding to $\mathcal{A}[\Phi\Phi\Phi\Psi]$ never contribute to $\langle SSSS\rangle$ and so $M^i(s,t)$ remains a polynomial in s and t.

deg.	$f_1(s,t)$	$f_2(s,t)$	$f_3(s,t)$	Counterterms	# sols.	even sols.
3	—	—	1	F^2R^2	2	1
4	1	—	—	R^4	1	1
5	s	—	$s^2+t^2+u^2$	$D^4F^2R^2$, D^2R^4	3	2
6	s^2, t^2+u^2	—	stu	$D^6F^2R^2$, D^4R^4	4	3
7	s^3, $s(t^2+u^2)$	—	$(s^2+t^2+u^2)^2$	$D^8F^2R^2$, D^6R^4	4	3
7.5	—	$(s-t)(t-u)(u-s)$	—	D^8FR^3	2	0

Table 4.3: Counterterms in $\mathcal{N}=6$ supergravity, up to 15 derivatives. The last column lists the number solutions which are $\mathcal{CT}$ and $\mathcal{Z}$ even.

superconformal invariants in the four-point function of four stress tensor multiplets. This counting follows from the number of unknown real functions needed to fully determine the scattering amplitudes of supergravitons, one for $f_1(s,t)$ and two each for $f_2(s,t)$ and $f_3(s,t)$, as these latter two functions are in general complex.

Second, from Table 4.3 we can immediately deduce how many polynomial Mellin super-amplitudes exist for a given degree in s,t. For instance, at third degree we have a single polynomial super-Mellin amplitude with scalar component $M_3^i(s,t)$, and at fourth degree we additionally have another polynomial super-Mellin amplitude with scalar component $M_4^i(s,t)$. Here, by third and fourth degree we mean that the super-amplitudes that $M_3^i(s,t)$ and $M_4^i(s,t)$ have degree 3 or 4 for some of the components of the amplitude, but not necessarily for the scalar components $M_3^i(s,t)$ and $M_4^i(s,t)$ themselves. These scalar components may be of less than third and fourth degree, respectively.

In fact, it can be argued that while the scalar component $M_4^i(s,t)$ is of degree 4 in s,t, the scalar component $M_3^i(s,t)$ is actually at most quadratic. This is because the leading order behavior of the super-Mellin amplitudes that $M_3^i(s,t)$ and $M_4^i(s,t)$ are part of at large s and t must match the corresponding super-scattering amplitude. Since the $M_3^i(s,t)$ amplitude contributes only to the superamplitude $\mathcal{A}[\Phi\Phi\Phi\Phi]$ (as can be seen from Table 4.2), it does not give rise to a scalar scattering amplitude. Therefore $M_3^i(s,t)$ must be at most quadratic, rather than cubic, in s and t. On the other had, $M_4^i(s,t)$ contributes to the superamplitude $\mathcal{A}[\Phi\Phi\Psi\Psi]$, and this superamplitude does include a scalar scattering amplitude, $A[\phi\phi\overline{\phi\phi}]$. Thus, $M_4^i(s,t)$ must have degree 4.

We can be more precise and also find the leading large s, t behavior of all $\langle SSSS\rangle$ Mellin amplitudes $M^i(s,t)$ for which $M^i(s,t)$ is of highest degree in the super-Mellin amplitude. (This means we will be able to find the leading large s,t behavior of $M_4^i(s,t)$ but not of $M_3^i(s,t)$.) As per (4.5), the leading large s,t behavior of $M^i(s,t)$ comes from the flat space amplitude $\mathcal{A}^i(s,t)$. The only scattering amplitude with a scalar component is $\mathcal{A}[\Phi\Phi\Psi\Psi]$ which is fixed in terms of $f_1(s,t)$, and so the leading large s,t behavior of $M^i(s,t)$ depends only on $f_1(s,t)$. The calculation proceeds in two steps. First we compute the amplitude $A[\phi_{ABCD}\phi_{EFGH}\overline{\phi}_{IJ}\overline{\phi}_{KL}]$, where we have made explicit the $SU(6)$ indices on ϕ and $\overline{\phi}$. We then relate ϕ and $\overline{\phi}$ to the CFT operator ${S_b}^a$, requiring us to convert the $SU(6)$ structures to $SO(6)$ structures.

To compute $A[\phi_{ABCD}\phi_{EFGH}\overline{\phi}_{IJ}\overline{\phi}_{KL}]$, we must differentiate $\mathcal{A}[\Phi\Phi\Psi\Psi]$ with respect to the Grassmannian variables:

$$\begin{aligned} A[\phi_{ABCD}\phi_{EFGH}\overline{\phi}_{IJ}\overline{\phi}_{KL}] &= \frac{\partial}{\partial\eta_1^A}\cdots\frac{\partial}{\partial\eta_4^L}\mathcal{A}[\Phi\Phi\Psi\Psi] \\ &= \frac{\partial}{\partial\eta_1^A}\cdots\frac{\partial}{\partial\eta_4^L}\delta^{12}(Q)\frac{[12]^4}{\langle 34\rangle^2}f_1(s,t) \\ &= \frac{[12]^4}{2^4\langle 34\rangle^2}f_1(s,t)\frac{\partial}{\partial\eta_1^A}\cdots\frac{\partial}{\partial\eta_4^L}\prod_{M=1}^{6}\sum_{i,j=1}^{4}\langle ij\rangle\eta_i^M\eta_j^M\,. \end{aligned} \tag{4.35}$$

To simplify the process of differentiating $\delta^{(12)}(Q)$, we can use $SU(6)$ invariance to expand

$$\begin{aligned} &A[\phi_{ABCD}\phi_{EFGH}\overline{\phi}_{IJ}\overline{\phi}_{KL}] \\ &\quad= \epsilon_{ABCDIJ}\epsilon_{EFGHKL}F_1(s,t)+\epsilon_{ABCDKL}\epsilon_{EFGHIJ}F_2(s,t)+\epsilon_{ABEFIK}\epsilon_{CDGHJL}F_3(s,t). \end{aligned} \tag{4.36}$$

for some functions $F_i(s,t)$. We can then choose specific numbers for each index A through L to isolate each structure, and hence find that

$$F_1(s,t)=2s^2u(4t-u)f_1(s,t)\,,\quad F_2(s,t)=2s^2t(4u-t)f_1(s,t)\quad F_3(s,t)=-s^2tuf_1(s,t)\,. \tag{4.37}$$

Now we must relate $A[\phi\phi\overline{\phi\phi}]$ to $\langle SSSS\rangle$. To do so, we can rewrite ${S_a}^b$ as an antisymmetric 6×6 matrix:

$$\check{S}^{IJ}={S_a}^bC^{[I}_{bc}\overline{C}^{J]ac}\,, \tag{4.38}$$

where recall that C^I_{ac} are the $SO(6)$ gamma matrices defined in Section 2.1. Up to normalization, we then find that

$$\check{S}^{IJ}\underset{\text{flat space}}{\longrightarrow}\phi_{ABCD}\epsilon^{ABCDIJ}+\delta^{IA}\delta^{JB}\overline{\phi}_{AB}. \tag{4.39}$$

This expression for $\check{S}^{IJ}$ breaks the $SU(6)$ symmetry down to $SO(6)$ due to the presence of the δ^{IA} symbol. Applying this to the four-point function, we find that

$$\langle\check{S}^{I_1J_1}\ldots\check{S}^{I_4J_4}\rangle\underset{\text{flat space}}{\longrightarrow}\text{sum of contracted permutations of }A[\phi\phi\overline{\phi\phi}]. \tag{4.40}$$

We must now expand our final answer in terms of the $SO(6)$ structures appearing in (2.5). To do so we choose a series of polarization matrices $(X_i)^a{}_b$ and then define

$$\check{X}_i^{IJ} = (X_i)^a{}_b C_{ac}^{[I} \overline{C}^{J]bc} \,. \tag{4.41}$$

Contracting both sides of (4.40) with matrices X_i^{IJ}, on the left-hand side we find that

$$\langle \check{S}^{I_1 J_1}(\vec{x}_1) \dots \check{S}^{I_4 J_4}(\vec{x}_4) \rangle \check{X}_1^{I_1 J_1} \cdots \check{X}_4^{I_4 J_4} \propto \langle S(\vec{x}_1, X_1) \cdots S(\vec{x}_4, X_4) \rangle = \frac{1}{x_{12}^2 x_{34}^2} \sum_{i=1}^{6} \mathcal{S}^i(U,V) \mathcal{B}_i \,. \tag{4.42}$$

We then Mellin transform and take the flat space limit (4.5) to find that

$$\langle \check{S}^{I_1 J_1}(\vec{x}_1) \dots \check{S}^{I_4 J_4}(\vec{x}_4) \rangle \check{X}_1^{I_1 J_1} \cdots \check{X}_4^{I_4 J_4} \underset{\text{flat space}}{\longrightarrow} \frac{\mathcal{N}}{x_{12}^2 x_{34}^2} \left[\mathcal{A}^1(s,t) A_{12} A_{34} + \cdots + \mathcal{A}^6(s,t) B_{1342} \right] \tag{4.43}$$

for some overall normalization constant $\mathcal{N}$. Computing the right-hand side of (4.40) is more straightforward; we simply contract the $X_i^{I_i J_i}$ matrices with the various permutations of $A[\phi\phi\overline{\phi\phi}]$. By imposing (4.40) for many different matrices $(X_i)^a{}_b$ we can completely determine $\mathcal{A}^i(s,t)$ in terms of $f_1(s,t)$, and upon choosing a suitable value for $\mathcal{N}$, we find that

$$\begin{aligned}
\mathcal{A}^1(s,t) &= -\frac{1}{2} tu \left(-s^2 f_1(s,t) + u^2 f_1(u,s) + t^2 f_1(t,s) \right) \,, \\
\mathcal{A}^2(s,t) &= -\frac{1}{2} su \left(s^2 f_1(s,t) + u^2 f_1(u,s) - t^2 f_1(t,s) \right) \,, \\
\mathcal{A}^3(s,t) &= -\frac{1}{2} ts \left(s^2 f_1(s,t) - u^2 f_1(u,s) + t^2 f_1(t,s) \right) \,, \\
\mathcal{A}^4(s,t) &= -\frac{1}{2} stu \left(u f_1(u,s) + t f_1(t,s) \right) \,, \\
\mathcal{A}^5(s,t) &= -\frac{1}{2} stu \left(u f_1(u,s) + s f_1(s,t) \right) \,, \\
\mathcal{A}^6(s,t) &= -\frac{1}{2} stu \left(s f_1(s,t) + t f_1(t,s) \right) \,.
\end{aligned} \tag{4.44}$$

From (4.44), we can also determine $f_1(s,t)$ in terms of $\mathcal{A}^i(s,t)$:

$$f_1(s,t) = -\frac{1}{s^3} \left(\frac{\mathcal{A}^2(s,t)}{u} + \frac{\mathcal{A}^3(s,t)}{t} \right) \,. \tag{4.45}$$

We can then apply (4.44) to $M_4^i(s,t)$, which at large s,t should asymptote to $\mathcal{A}^i(s,t)$ with $f_1(s,t) = 1$ (see Table 4.3). We hence find

$$M_4^i(s,t) = \begin{pmatrix} t^2 u^2 & s^2 u^2 & s^2 t^2 & \frac{s^2 tu}{2} & \frac{st^2 u}{2} & \frac{stu^2}{2} \end{pmatrix} + \text{subleading in } s,t \,. \tag{4.46}$$

4.2.5 Exchange Amplitudes

So far we have considered local contact amplitudes. The only other tree-level diagrams which appear in four-point functions are exchange diagrams. These can be built up from the on-shell three point amplitudes using on-shell recursion relations (see for instance chapter 3 of [142]), and so our first task is to find the allowed three point amplitudes.

Three point amplitudes are subtle due to special kinematics; conservation of momentum implies that either

$$[12] = [13] = [23] = 0 \text{ or } \langle 12\rangle = \langle 13\rangle = \langle 23\rangle = 0\,. \tag{4.47}$$

For real momenta $[ij]^* = \langle ji\rangle$ so this would seem to rule out any interesting amplitudes. This issue is resolved by analytically continuing to complex momenta. Locality and little-group scaling then uniquely fix three-point functions to take the form:

$$A[1^{h_1}2^{h_2}3^{h_3}] = \begin{cases} c[12]^{h_1+h_2-h_3}[13]^{h_1+h_3-h_2}[23]^{h_2+h_3-h_1} & \text{if } h_1+h_2+h_3>0 \\ c\langle 12\rangle^{h_3-h_1-h_2}\langle 13\rangle^{h_2-h_1-h_3}\langle 23\rangle^{h_1-h_2-h_3} & \text{if } h_1+h_2+h_3<0 \\ c & \text{if } h_1=h_2=h_3=0 \\ 0 & \text{otherwise} \end{cases} \tag{4.48}$$

where c is an arbitrary constant [142, 143]. Superamplitudes must furthermore satisfy the supersymmetric Ward identities, and this uniquely fixes them to take the form:

$$\begin{aligned} \mathcal{A}[\Phi\Phi\Psi] &= \frac{g_1}{[13]^2[23]^2}\delta^{(6)}([12]\eta_3+[23]\eta_1+[31]\eta_2) + \frac{g_2\langle 12\rangle^3}{\langle 13\rangle^7\langle 23\rangle^7}\delta^{(12)}(\langle 12\rangle\eta_3+\langle 23\rangle\eta_1+\langle 31\rangle\eta_2)\,, \\ \mathcal{A}[\Phi\Phi\Phi] &= \frac{g_3}{[12][13][23]}\delta^{(6)}([12]\eta_3+[23]\eta_1+[31]\eta_2)\,, \end{aligned} \tag{4.49}$$

where

$$\begin{aligned} \delta^{(6)}([12]\eta_3+[23]\eta_1+[31]\eta_2) &= \prod_{I=1}^{6}([12]\eta_{3I}+[23]\eta_{1I}+[31]\eta_{2I})\,, \\ \delta^{(12)}(\langle 12\rangle\eta_3+\langle 23\rangle\eta_1+\langle 31\rangle\eta_2) &= \prod_{I=1}^{6}(\langle 12\rangle\eta_{3I}+\langle 23\rangle\eta_{1I}+\langle 31\rangle\eta_{2I})^2\,. \end{aligned} \tag{4.50}$$

The g_1 term in the $\mathcal{A}[\Phi\Phi\Psi]$ superamplitude corresponds to the usual supergravity three-point function, and in particular gives rise to a graviton scattering amplitude

$$\mathcal{A}[h^+h^+h^-] = g_1 \frac{[12]^6}{[13]^2[23]^2} \,. \tag{4.51}$$

The g_2 and g_3 terms both vanish due to crossing symmetry; if we exchange $1 \leftrightarrow 2$ then $\mathcal{A}[\Phi\Phi\Phi]$ and $\mathcal{A}[\Phi\Phi\Psi]$ must be even, but this is only possible if $g_2 = g_3 = 0$.

Since there is only one supergravity three-point function, we can now determine the corresponding unique four-point exchange amplitude. Because the tree-level graviton amplitudes in pure supergravity are identical to those in pure gravity [142], we can simply use the pure gravity result to deduce that

$$f_1^{\text{SG}}(s,t) = \frac{g_1^2}{stu} \,, \qquad f_2^{\text{SG}}(s,t) = f_3^{\text{SG}}(s,t) = 0 \,. \tag{4.52}$$

We can then substitute this into (4.44) to find that the $A[\phi\phi\overline{\phi\phi}]$ amplitude at large s,t is expected to be

$$M^i_{\text{SG}}(s,t) = g_1^2 \begin{pmatrix} \frac{tu}{s} & \frac{su}{t} & \frac{st}{u} & \frac{s}{2} & \frac{t}{2} & \frac{u}{2} \end{pmatrix} + \text{subleading in } s,t \,. \tag{4.53}$$

4.3 Holographic Correlators with $\mathcal{N} = 6$ Supersymmetry

We will now determine the full form of the first few Mellin amplitudes contributing to $\langle SSSS \rangle$. We will begin by fixing $M_3(s,t)$, by using the supersymmetric Ward identities and requiring that all four-scalar and two-scalar, two-fermion amplitudes are polynomial. We then compute the degree four amplitude $M_4^i(s,t)$ and the supergravity amplitude $M_{SG}^i(s,t)$ by reducing the known $\mathcal{N} = 8$ results to $\mathcal{N} = 6$.

4.3.1 Computing $M_3(s,t)$

We can translate the $\langle SSSS \rangle$ Ward identities (2.12) into Mellin space using the definition of the Mellin transform $M^i(s,t)$ of $\langle SSSS \rangle$ in (4.2). We then find that the effect of multiplication by $U^m V^n$ and of differentiating with respect to U and V corresponds in Mellin space to the operators

$$\begin{aligned} \widehat{U^mV^n} M(s,t) =& M(s-2m, t+2m+2n) \left(1 - \frac{s}{2}\right)_m^2 \left(1 - \frac{t}{2}\right)_{-m-n}^2 \left(1 - \frac{u}{2}\right)_n^2 \,, \\ \widehat{\partial_U^m} M(s,t) =& \left(\frac{s}{2} + 1 - m\right)_m \widehat{U}^{-m} M(s,t) \,, \\ \widehat{\partial_V^m} M(s,t) =& \left(\frac{u}{2} - m\right)_m \widehat{V}^{-m} M(s,t) \,. \end{aligned} \tag{4.54}$$

Applying these rules to the position space Ward identities (2.12) yields a pair of finite difference equations for $M^i(s,t)$. We now impose the following constraints to find M_3:

1. M_3^i must satisfy crossing symmetry (4.4).

2. M_3^i must be a degree 2 polynomial solution of the $\langle SSSS\rangle$ Ward identity. The ansatz for M_3 is only degree 2, since in the previous section we showed that $\mathcal{A}_3$ does not appear in the scattering of four scalars, so M_3 must vanish in the flat space limit.

3. M_3 must remain a polynomial when expressed as correlator of other operators in the stress tensor multiplet using the Ward identities in the previous section.[8] The degree of these polynomials is at most 2 if the corresponding flat space amplitude vanishes, and 3 otherwise.

Condition 3 was trivially satisfied in the maximally supersymmetric cases previously considered in various dimensions [74,118], where polynomial Mellin amplitudes for $\langle SSSS\rangle$ remained polynomials for all other stress tensor multiplets correlators. In our case though, we find that just imposing conditions 1 and 2 leads to five linearly independent solutions: a degree 0, a degree 1, and three degree 2:

$$\begin{aligned}
\text{degree 0:}\qquad & M^1 = 1\,, \qquad M^4 = 1\,,\\
\text{degree 1:}\qquad & M^1 = s\,, \qquad M^4 = \frac{s-4}{2}\,,\\
\text{1st degree 2:}\qquad & M^1 = (t-2)(u-2)\,, \qquad M^4 = \left(s-\frac{4}{3}\right)(s-2)\,,\\
\text{2nd degree 2:}\qquad & M^1 = tu\,, \qquad M^4 = \frac{s(s-4)}{2}\,,\\
\text{3rd degree 2:}\qquad & M^1 = s^2\,, \qquad M^4 = s^2+tu-3s\,.
\end{aligned} \tag{4.55}$$

To reduce these to a unique amplitude, we must consider the other Ward identities for $\langle SS\chi\chi\rangle$, $\langle SS\chi F\rangle$, $\langle SSFF\rangle$, and $\langle SSF\overline{F}\rangle$ given in Appendix A. To translate these into Mellin space, we first note that any four-point correlator between two dimension 1 scalars ϕ and two dimension 3/2 fermions $\psi^\alpha(x)$ takes the form

$$\langle\phi(\vec{x}_1)\phi(\vec{x}_2)\psi^\alpha(\vec{x}_3)\psi^\beta(\vec{x}_4)\rangle = \frac{i\not{x}_{34}^{\alpha\beta}}{x_{12}^2 x_{34}^4}\mathcal{H}_1(U,V) + \frac{i(\not{x}_{13}\not{x}_{24}\not{x}_{12})^{\alpha\beta}}{2x_{12}^4 x_{34}^4}\mathcal{H}_2(U,V)\,. \tag{4.56}$$

[8]Instead of imposing this requirement, we could alternatively impose the condition that certain operators in the $S\times S$ OPE do not acquire anomalous dimensions. For instance, we can uniquely determine M_3 if we impose this requirement for the spin 0 operators of dimension 2 in the **84**, **20′**, and $\mathbf{15}_s$ irreps of $SO(6)_R$, as well as for the spin 1 operator of dimension 3 in the $\mathbf{45}\oplus\overline{\mathbf{45}}$, all of which belong to protected multiplets and do not mix with unprotected operators.

We can then define Mellin transforms $M_i^{\phi\phi\psi\psi}(s,t)$ of the connected parts of the correlators $\mathcal{H}^{\phi\phi\psi\psi}_{\text{conn},i}$ by the equations

$$\begin{aligned}
\mathcal{H}^{\phi\phi\psi\psi}_{\text{conn},1}(U,V) &= \int_{-i\infty}^{i\infty} \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{u}{2}-1} \Gamma\left[1-\frac{s}{2}\right]\Gamma\left[2-\frac{s}{2}\right]\Gamma^2\left[1-\frac{t}{2}\right]\Gamma^2\left[1-\frac{u}{2}\right] M_1^{\phi\phi\psi\psi}(s,t)\,,\\
\mathcal{H}^{\phi\phi\psi\psi}_{\text{conn},2}(U,V) &= \int_{-i\infty}^{i\infty} \frac{ds\,dt}{(4\pi i)^2} U^{\frac{s}{2}} V^{\frac{u}{2}-1} \Gamma^2\left[2-\frac{s}{2}\right]\Gamma^2\left[1-\frac{t}{2}\right]\Gamma^2\left[1-\frac{u}{2}\right] M_2^{\phi\phi\psi\psi}(s,t)\,,
\end{aligned} \tag{4.57}$$

where, as previously, we define $u = 4 - s - t$. These expressions were derived in [144] using AdS_4 Witten diagram calculations. The arguments of the Gamma functions were chosen so that bulk contact Witten diagrams correspond to polynomial Mellin amplitudes. Using the definition (4.57), we find that drivatives of U and V and powers of U and V in position space act on $M_i^{\phi\phi\psi\psi}(s,t)$ as

$$\begin{aligned}
\widehat{\partial_U^m} M_i^{\phi\phi\psi\psi}(s,t) &= \left(\frac{s}{2}+1-m\right)_m \widehat{U}^{-m} M_i^{\phi\phi\psi\psi}(s,t)\,,\\
\widehat{\partial_V^m} M_i^{\phi\phi\psi\psi}(s,t) &= \left(\frac{u}{2}-m\right)_m \widehat{V}^{-m} M_i^{\phi\phi\psi\psi}(s,t)\,,\\
\widehat{U^m V^n} M_1^{\phi\phi\psi\psi}(s,t) &= M_1^{\phi\phi\psi\psi}(s-2m,t+2m+2n)\left(1-\frac{s}{2}\right)_m\left(2-\frac{s}{2}\right)_m\left(1-\frac{t}{2}\right)^2_{-m-n}\left(1-\frac{u}{2}\right)^2_n\,,\\
\widehat{U^m V^n} M_2^{\phi\phi\psi\psi}(s,t) &= M_2^{\phi\phi\psi\psi}(s-2m,t+2m+2n)\left(2-\frac{s}{2}\right)^2_m\left(1-\frac{t}{2}\right)^2_{-m-n}\left(1-\frac{u}{2}\right)^2_n\,.
\end{aligned} \tag{4.58}$$

For the correlators $\langle SS\chi\chi\rangle$, $\langle SS\chi F\rangle$, $\langle SSFF\rangle$, and $\langle SSF\overline{F}\rangle$, we define their Mellin amplitudes by Mellin transforming the individual functions of U and V given in Appendix A. For instance, we can write $\langle SS\chi\chi\rangle$ in terms of the structures $\mathcal{C}^{a,I}(U,V)$ defined in (A.4), where the indices $a = 1,2,3$ and $I = 1,2$ refer to the various R-symmetry and conformal structures, respectively. The Mellin transform $M_{a,I}^{SS\chi\chi}(s,t)$ of these $\mathcal{C}^{a,I}(U,V)$ is then defined by (4.57). We can relate $M_{a,I}^{SS\chi\chi}(s,t)$ to $M^i(s,t)$ as

$$\begin{aligned}
M_{a,1}^{SS\chi\chi} &= \left(1-\frac{s}{2}\right)^{-1} \widehat{\mathcal{D}^C_{ai,1}}(U,V,\partial_U,\partial_V) M^i(s,t)\,,\\
M_{a,2}^{SS\chi\chi} &= \left(1-\frac{s}{2}\right)^{-2} \widehat{\mathcal{D}^C_{ai,2}}(U,V,\partial_U,\partial_V) M^i(s,t)\,,
\end{aligned} \tag{4.59}$$

where the $\langle SS\chi\chi\rangle$ Ward identity $\mathcal{D}^C_{ai,1}$ is given in position space in (A.5), we express derivatives and powers of U and V in Mellin space using the rules (4.54), and s-dependent prefactors come from the difference in the definition of the scalar and fermion Mellin amplitudes in (4.2) and (4.57). We find that degree 0 amplitude in (4.55) gives

$$\begin{aligned} \text{degree 0:}\quad & M_{1,1}^{SS\chi\chi}(s,t)=0\,,\quad M_{2,1}^{SS\chi\chi}(s,t)=\frac{1}{16}\,,\quad M_{3,1}^{SS\chi\chi}(s,t)=\frac{2-t}{16u}\,,\\ & M_{1,2}^{SS\chi\chi}(s,t)=0\,,\quad M_{2,2}^{SS\chi\chi}(s,t)=\frac{1}{8t}\,,\quad M_{3,2}^{SS\chi\chi}(s,t)=\frac{1}{8u}\,, \end{aligned} \tag{4.60}$$

which contain poles, and so must be discarded.

When we apply this method to the Ward identities for $\langle SSFF\rangle$ and $\langle SSF\overline{F}\rangle$, a new subtlety emerges. These Ward identities (A.13), (A.15), and (A.17) depend on both $\langle SSSS\rangle$ and $\langle SSFF\rangle$, and in particular can be written in terms of not only $\mathcal{S}^1(U,V)$ and $\mathcal{S}^4(U,V)$, but also the functions $\mathcal{F}^{a,1}(U,V)$ for $\langle SSFF\rangle$ defined in (A.3), where $a=1,2$ labels the two R-symmetry structures. To derive the constraints from these Ward identities up to degree 2, we must consider a degree 2 polynomial ansatz for the Mellin transform $M_{a,1}^{SSFF}(s,t)$ of $\mathcal{F}^{a,1}(U,V)$, which satisfies the crossing relations

$$\begin{aligned} M_{1,1}^{SSFF}(s,t) &= M_{2,1}^{SSFF}(s,u)+\left(1-\frac{s}{2}\right)M_{2,2}^{SSFF}(s,u)\,,\\ M_{2,1}^{SSFF} &= M_{1,1}^{SSFF}(s,u)+\left(1-\frac{s}{2}\right)M_{1,2}^{SSFF}(s,u)\,,\\ M_{1,2}^{SSFF}(s,t) &= -\left(1-\frac{s}{2}\right)M_{2,2}^{SSFF}(s,u)\,,\\ M_{2,2}^{SSFF}(s,t) &= -\left(1-\frac{s}{2}\right)M_{1,2}^{SSFF}(s,u)\,, \end{aligned} \tag{4.61}$$

where the s-dependent prefactors come from the difference in the definition of the fermion Mellin amplitudes in (4.57) for the two different conformal structures. After imposing the $\langle SS\chi F\rangle$, $\langle SSFF\rangle$, and $\langle SSF\overline{F}\rangle$ Ward identities, just as we did for $\langle SS\chi\chi\rangle$ above, and demanding that all poles vanish, we find that $M_{a,1}^{SSFF}(s,t)$ is completely fixed in terms of $M^i(s,t)$ up to degree 2, and that only a single degree 2 solution for $M^i(s,t)$ survives:

$$M_3:\qquad M_3^1=(t-2)(u-2)\,,\qquad M_3^4=\left(s-\frac{4}{3}\right)(s-2)\,. \tag{4.62}$$

Thus we have found the unique degree 3 Mellin amplitude $M_3(s,t)$ which contributes to $\langle SSSS\rangle$.

4.3.2 Computing $M_4(s,t)$ and Supergravity Exchange

To compute both $M_4^i(s,t)$ and $M_{\text{SG}}^i(s,t)$, we can make use of previous results in the literature for $\mathcal{N}=8$ SCFTs. Recall that $\mathcal{N}=8$ SCFTs are special cases of $\mathcal{N}=6$ SCFTs. The $\mathcal{N}=8$ stress tensor multiplet is a superset of the $\mathcal{N}=6$ stress tensor multiplet. In particular, the superconformal primary is a $\Delta = 1$scalar operator $\overline{S}_{AB}(\vec{x})$ transforming in the $\mathbf{35}_c$ irrep of the $\mathfrak{so}(8)_R$ R-symmetry.[9] (Here $\overline{S}_{AB}(\vec{x})$, with $A, B = 1, \ldots, 8$ being $\mathbf{8}_c$ indices, is a traceless symmetric tensor.) Like in the $\mathcal{N} = 6$ case, we can use an index-free notation by contracting $\overline{S}_{AB}(\vec{x})$ with a symmetric traceless 8×8 matrix $\overline{X}_{AB}$. The four-point function of the $\mathbf{35}_c$ scalar operator is restricted by conformal invariance and $\mathfrak{so}(8)_R$ to take the form

$$\langle \overline{S}(\vec{x}_1, \overline{X}_1) \cdots \overline{S}(\vec{x}_4, \overline{X}_4) \rangle = \frac{1}{x_{12}^2 x_{34}^2} \Big[\overline{\mathcal{S}}^1(U,V) \overline{A}_{12} \overline{A}_{34} + \overline{\mathcal{S}}^2(U,V) \overline{A}_{13} \overline{A}_{24} + \overline{\mathcal{S}}^3(U,V) \overline{A}_{14} \overline{A}_{23} + \overline{\mathcal{S}}^4(U,V) \overline{B}_{1423} + \overline{\mathcal{S}}^5(U,V) \overline{B}_{1234} + \overline{\mathcal{S}}^6(U,V) \overline{B}_{1342} \Big] , \tag{4.63}$$

where we define[10]

$$\overline{A}_{ij} \equiv \mathrm{tr}(\overline{X}_i \overline{X}_j) \,, \qquad \overline{B}_{ijkl} \equiv \mathrm{tr}(\overline{X}_i \overline{X}_j \overline{X}_k \overline{X}_l) \,. \tag{4.64}$$

The Mellin amplitudes for $\overline{S}^i$ which correspond to contact interactions were found in [65]. With our definition (4.2) (with $\mathcal{S}^i_{\text{conn}}$ replaced by $\overline{\mathcal{S}}^i_{\text{conn}}$ and M^i replaced by $\overline{M}^i$), the result in [65] for the quartic amplitude is

$$\begin{aligned} \overline{M}_4^1 &= \frac{1}{35}(t-2)(u-2)(35tu + 100s - 112) \,, \\ \overline{M}_4^4 &= \frac{2}{35}(s-2)(35stu - 90(t^2+u^2) - 280tu - 324s + 1072) \,. \end{aligned} \tag{4.65}$$

To relate (4.65) to $M_4^i(s,t)$ we must relate the $\mathfrak{so}(8)_R$ structures (4.64) to the $\mathfrak{su}(4)_R$ ones defined in (2.6). Under the decomposition $\mathfrak{so}(8) \to \mathfrak{su}(4)$, we have $\mathbf{8}_c \to \mathbf{4} + \overline{\mathbf{4}}$, which implies $\mathbf{35}_c \to \mathbf{10} + \overline{\mathbf{10}} + \mathbf{15}$. To select the $\mathbf{15}$, we restrict the 8×8 matrices $\overline{X}$ to take the form

$$\overline{X} = \frac{1}{\sqrt{2}} \Big[(\mathrm{Re}X) \otimes I_2 + (\mathrm{Im}X) \otimes (i\sigma_2) \Big] \,, \tag{4.66}$$

where X is a 4×4 traceless hermitian matrix, I_2 is the 2×2 identity matrix, and σ_2 is the second Pauli matrix. (See equation (3.16) of [65].[11]) It is straightforward to check that

$$\overline{A}_{12} = \mathcal{B}_1 \,, \qquad \overline{B}_{1423} = \frac{1}{4} \mathcal{B}_4 \,, \tag{4.67}$$

[9]The fact that this representation is the $\mathbf{35}_c$ as opposed to one of the other two 35-dimensional irreducible representations of $\mathfrak{so}(8)_R$ assumes a choice of the triality frame.

[10]Despite the use of matrix $\mathfrak{so}(8)$ polarizations here, the $\overline{\mathcal{S}}^i(U,V)$ here are equal to the $\mathcal{S}_i(U,V)$ in [65].

[11]The factor of $1/\sqrt{2}$ is just a choice of normalization.

with analogous expressions for the rest of the R-symmetry structures in $\langle SSSS\rangle$. This implies that $\mathcal{S}^i = \overline{\mathcal{S}}^i$ for $i = 1, 2, 3$ and $\mathcal{S}^i = \frac{1}{4}\overline{\mathcal{S}}^i$ for $i = 4, 5, 6$ and analogously for the Mellin amplitudes. Thus,

$$\begin{aligned} M_4: \qquad & M_4^1 = \frac{1}{35}(t-2)(u-2)(35tu + 100s - 112)\,, \\ & M_4^4 = \frac{1}{70}(s-2)(35stu - 90(t^2+u^2) - 280tu - 324s + 1072)\,, \end{aligned} \tag{4.68}$$

where the other M_4^i are given by crossing (4.4). The Mellin amplitudes M_4^i are normalized so that at large s, t they obey (4.46).

We can fix the supergravity amplitude $M_{\rm SG}^i(s,t)$ in an identical fashion. The $\mathcal{N} = 8$ supergravity amplitude was derived in [117]. Using equations (E.1) and (4.8) of [118], and converting to $\mathcal{N} = 6$ notation, we find that

$$\begin{aligned} M_{\rm SG}^1 &= -\frac{(t-2)(u-2)}{s(s+2)}\left(\frac{4\Gamma\left(\frac{1-s}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{s}{2}\right)} - (4+s)\right), \\ M_{\rm SG}^4 &= -\frac{s-2}{2tu}\left(\frac{2u\Gamma\left(\frac{1-t}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{t}{2}\right)} + \frac{2t\Gamma\left(\frac{1-u}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{u}{2}\right)} + 2s - tu - 8\right), \end{aligned} \tag{4.69}$$

where the other $M_{\rm SG}^i$ are given by crossing (4.4). We normalize $M_{\rm SG}^i$ so that at large s, t they obey (4.53) with $g_1 = 1$. Note that, as an exchange diagram, $M_{\rm SG}^i(s,t)$ contains an infinite series of poles that correspond to the exchange of stress tensor multiplet operators (or the exchange of the graviton multiplet in the bulk) and their conformal descendants.

4.4 ABJ Correlators at Large c_T

We now fix the first few corrections to $\langle SSSS\rangle$ in each of the stringy regimes of ABJ described in Section 1.3.2. In each of these limits, we can use the Penedones formula (4.5) to relate the $\langle SSSS\rangle$ Mellin amplitude to the four-point scattering amplitudes of gravitons and their superpartners in 11d (in the M-theory case) or 10d (in the type IIA case) flat space, with momenta restricted to lie within a four-dimensional subspace. Of course, the flat space limit of the $\langle SSSS\rangle$ correlator in ABJ theory cannot give the four-point scattering amplitude of *all* massless particles in 11d or 10d. Indeed, in either 11d M-theory or in 10d type IIA string theory, the massless particle spectrum consists of 128 bosons and 128 fermions that are related by maximal SUSY. The flat space limit of the $\langle SSSS\rangle$ correlator must match the four-point scattering amplitude of only 15 of the 128 bosons, which all have the property that after restricting their momenta to lie within 4d, they can be thought of as scalars from the 4d point of view.[12] Note that when using Eq. (4.5), we should keep either the 11d Planck length ℓ_{11} or the 10d string length ℓ_s fixed as we send $L \to \infty$. In other words, we should more precisely send L/ℓ_{11} or L/ℓ_s to infinity.

As we have seen in the previous sections, the ingredients we will use to construct the first few terms in the large N expansion of the $\langle SSSS \rangle$ correlator are the Mellin amplitudes

$$M^i_{\rm SG}(s,t)\,, \quad M^i_3(s,t)\,, \quad M^i_4(s,t)\,, \tag{4.70}$$

given in (4.69), (4.62), and (4.68), respectively. $M^i_{\rm SG}$ is the Mellin amplitude corresponding to an exchange Witten diagram with supergravity vertices. M^i_3 is a polynomial Mellin amplitude that represents the $\langle SSSS \rangle$ component of a degree 3 super-Mellin amplitude corresponding to a contact Witten diagram with an F^2R^2 contact interaction vertex. Likewise, M^i_4 is part of a degree 4 super-Mellin amplitude corresponding to a contact Witten diagram with an R^4 super-vertex. As explained in Section 4.2, if we apply the Penedones formula (4.5) to each of the Mellin amplitudes (4.70), we find that

$$\begin{aligned} \frac{1}{L^2\mathcal{N}(L)} M^i_{\rm SG}(s,t) &\underset{\text{flat space}}{\longrightarrow} \quad \mathcal{A}^i_{\rm SG}(s,t) = \begin{pmatrix} \frac{tu}{s} & \frac{su}{t} & \frac{st}{u} & \frac{s}{2} & \frac{t}{2} & \frac{u}{2} \end{pmatrix}\,, \\ \frac{1}{L^6\mathcal{N}(L)} M^i_3(s,t) &\underset{\text{flat space}}{\longrightarrow} \quad \mathcal{A}^i_3(s,t) = 0\,, \\ \frac{1}{L^8\mathcal{N}(L)} M^i_4(s,t) &\underset{\text{flat space}}{\longrightarrow} \quad \mathcal{A}^i_4(s,t) = \frac{stu}{105} \mathcal{A}^i_{\rm SG}(s,t)\,. \end{aligned} \tag{4.71}$$

Here, the normalization constant $\mathcal{N}(L)$ appearing in (4.5) depends on our precise choice of normalization for the $\langle SSSS \rangle$ correlator. If we normalize this correlator such that the disconnected piece scales as c_T^0, then we should take $\mathcal{N}(L) = \mathcal{N}_0 L^D$, where $D = 7$ for the case of an 11d dual and $D = 6$ for the case of a 10d dual.

In addition to (4.70), we will also consider the contact Mellin amplitudes

$$M^i_{5,1}(s,t)\,, \qquad M^i_{5,2}(s,t)\,, \tag{4.72}$$

which are part of degree-5 super-Mellin amplitudes corresponding to D^2R^4 and $D^4F^2R^2$ interaction vertices, respectively. While in Section 4.3 we did not determine the forms of $M^i_{5,1}$ and $M^i_{5,2}$, we know that such Mellin amplitudes must exist because they must reproduce the scattering amplitudes in the 3rd line of Table 4.3 in the flat space limit. Upon a convenient choice of normalization, the

[12] More generally, from all the 4-point CFT correlators of the $\mathcal{N} = 6$ stress tensor multiplet, we would be able to determine the 4-point scattering amplitudes of precisely half (64 bosons + 64 fermions) of the massless particles of both 11d M-theory and 10d type IIA string theory.

flat space limits of the Mellin amplitudes can be taken to be

$$\begin{aligned}
\frac{1}{L^{10}\mathcal{N}(L)} M^i_{5,1}(s,t) &\underset{\text{flat space}}{\longrightarrow} \quad \mathcal{A}^i_{5,1}(s,t) = \frac{1}{945} stu\left(s^2+3t^2+3u^2 \quad \cdots\right) , \\
\frac{1}{L^{10}\mathcal{N}(L)} M^i_{5,2}(s,t) &\underset{\text{flat space}}{\longrightarrow} \quad \mathcal{A}^i_{5,2}(s,t) = 0 \,.
\end{aligned} \tag{4.73}$$

The Mellin amplitudes M^i_{SG}, M^i_3, M^i_4, $M^i_{5,1}$, and $M^i_{5,2}$ are the only crossing-invariant Mellin amplitudes that obey the SUSY Ward identities and that grow at most as the fifth power of s, t at large s and t.

We now analyze the (Mellin transform of) the $\langle SSSS \rangle$ correlator in each of the three large N stringy limits of ABJ.

4.4.1 Large c_T, finite k

Recall from Section 1.3.2 that at large c_T limit with k fixed, ABJ theory is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, with

$$\frac{L^9}{\ell_{11}^9} = \frac{3\pi k}{2^{11}} c_T + O(c_T^0) \,, \tag{4.74}$$

From this relation, the flat space limits (4.71) and (4.73), as well as the requirement that in the flat space limit the scattering amplitude should have an expansion in ℓ_{11} times momentum, we infer that $M^i(s,t)$ has the large c_T expansion

$$\begin{aligned}
M^i(s,t) =& \frac{1}{c_T} A^1_{\text{SG}} M^i_{\text{SG}} + \frac{1}{c_T^{\frac{13}{9}}}\left[A^3_{\text{SG}} M^i_{\text{SG}} + A^3_3 M^i_3\right] + \frac{1}{c_T^{\frac{5}{3}}}\left[A^4_{\text{SG}} M^i_{\text{SG}} + A^4_3 M^i_3 + A^4_4 M^i_4\right] \\
&+ \frac{1}{c_T^{\frac{17}{9}}}\left[A^5_{\text{SG}} M^i_{\text{SG}} + A^5_3 M^i_3 + A^5_4 M^i_4 + A^5_{5,1} M^i_{5,1} + A^5_{5,2} M^i_{5,2}\right] + O(c_T^{-2}) \,,
\end{aligned} \tag{4.75}$$

where $A^l_{i,j}$ are k-dependent numerical coefficients. In the flat space limit only the maximal degree Mellin amplitudes contribute at each order in $1/c_T$, and so from (4.71) and (4.73) we find that

$$\mathcal{A}^i(s,t) = \ell_{11}^9 \left(A^1_{\text{SG}} \mathcal{A}^i_{\text{SG}} + \left(\frac{3k\pi}{2^{11}}\right)^{2/3} \ell_{11}^6 A^4_4 \mathcal{A}^i_4 + \left(\frac{3k\pi}{2^{11}}\right)^{8/9} \ell_{11}^8 A^5_{5,1} \mathcal{A}^i_{5,1} + \cdots \right) . \tag{4.76}$$

Note that neither $\mathcal{A}^i_3$ nor $\mathcal{A}^i_{5,2}$ give rise to scalar scattering amplitudes in flat space, which is why they do not appear in (4.76). Comparing (4.76) to the known M-theory four-point scattering amplitude [145]

$$\mathcal{A}^{11} = \mathcal{A}^{11}_{\rm SG} \left[1 + \ell_{11}^6 \frac{1}{3 \cdot 2^7} stu + O(\ell_{11}^9) \right] , \tag{4.77}$$

where $\mathcal{A}^{11}_{\rm SG}$ is the 11d supergravity scattering amplitude, we can immediately deduce that

$$\frac{A_4^4}{A_{\rm SG}^1} = 35 \left(\frac{2}{9\pi^2 k^2} \right)^{1/3} , \quad A_{5,1}^5 = 0 \, . \tag{4.78}$$

Although M_3^i and $M_{5,2}^i$ do not give rise to scattering amplitudes for the 11d super-gravitons that are scalars from the 4d point of view, they do contribute to the scattering of other particles in the same multiplet. The M-theory amplitude (4.77) however encodes the scattering amplitudes for all such particles, and it does not contain any terms of order ℓ_{11}^{13} or ℓ_{11}^{17}. From this we conclude that

$$A_3^3 = A_{5,2}^5 = 0 \, . \tag{4.79}$$

As a final aside, note that the $O(c_T^{-2})$ term (4.75) is not a local Mellin amplitude. It instead corresponds to the one-loop supergravity term, which is not analytic in s and t. We will not study this term any further.

4.4.2 't Hooft strong coupling limit

We next consider the strong coupling 't Hooft limit of ABJ theory, whereby we first take $N \to \infty$ with fixed λ (see (1.56) for the definition of λ), and then take $\lambda \to \infty$, holding M finite. As discussed in Section 1.3.2, in this double limit, ABJ theory is dual to weakly coupled type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ with

$$\frac{L^8}{\ell_s^8} = 4\pi^4 \lambda^2 + \dots , \qquad g_s^2 = \frac{512 \lambda^2}{3 c_T} + \dots , \tag{4.80}$$

Similarly to the M-theory limit discussed above, we can expand $M^i(s,t)$ in powers of ℓ_s/L, with the appropriate powers of ℓ_s/L chosen such that after taking the flat space limit, the string theory scattering amplitude has an expansion in ℓ_s times momentum. Unlike M-theory however, type IIA string theory has an additional dimensionless parameter, the string coupling constant g_s, that governs the strength of string interactions. Simultaneously expanding in both, we find that

$$\begin{aligned} M(s,t) = &\frac{1}{c_T}\Big[B^1_{\rm SG}M_{\rm SG} + \frac{1}{\lambda}\left(B^3_{\rm SG}M_{\rm SG} + B^3_3 M_3\right) + \frac{1}{\lambda^{\frac{3}{2}}}\left(B^4_{\rm SG}M_{\rm SG} + B^4_3 M_3 + B^4_4 M_4\right) \\ &+\frac{1}{\lambda^2}\left(B^5_{\rm SG}M_{\rm SG} + B^5_3 M_3 + B^5_4 M_4 + B^5_{5,1}M_{5,1} + B^5_{5,2}M_{5,2}\right) + O(\lambda^{-\frac{5}{2}})\Big] \\ &+\frac{1}{c_T^2}\Big[\lambda^2 \widetilde{B}^1_{\rm SG}M_{\rm SG} + \lambda\left(\widetilde{B}^3_{\rm SG}M_{\rm SG} + \widetilde{B}^3_3 M_3\right) \\ &+\sqrt{\lambda}\left(\widetilde{B}^4_{\rm SG}M_{\rm SG} + \widetilde{B}^4_3 M_3 + \widetilde{B}^4_4 M_4\right) + O(\lambda^0)\Big] + O(c_T^{-3})\,, \end{aligned} \tag{4.81}$$

where $B^l_{i,j}$ and $\widetilde{B}^l_{i,j}$ are numerical coefficients. The leading order $1/c_T$ behavior corresponds to tree-level string theory, and the higher order terms are loop corrections. At fixed order in $1/c_T$ and $1/\lambda$ only the maximal degree Mellin amplitudes contribute in the flat space limit, and so we find that

$$\begin{aligned} \mathcal{A}^i(s,t) = &\frac{3\pi^4}{128}g_s^2\ell_s^8\left(B^1_{\rm SG}\mathcal{A}^i_{\rm SG} + 2\sqrt{2}\pi^3\ell_s^6 B^4_4\mathcal{A}^i_4 + 4\pi^4\ell_s^8 B^5_{5,1}\mathcal{A}^i_{5,1} + \cdots\right) \\ &+\frac{9\pi^4}{2^{16}}g_s^4\ell_s^8\left(\tilde{B}^1_{\rm SG}\mathcal{A}^i_{\rm SG} + 2\sqrt{2}\pi^3\ell_s^6\tilde{B}^4_4\mathcal{A}^i_4 + \cdots\right)\,. \end{aligned} \tag{4.82}$$

Although the $1/c_T^2$ terms are one-loop corrections, non-analytic Mellin amplitudes will occur first at λ^0/c_T^2 corresponding to the one-loop correction in supergravity. Comparing this to the IIA S-matrix at weak coupling [146]

$$\mathcal{A}^{10}_{\rm IIA} = \mathcal{A}^{10}_{\rm SG}\left[\left(1 + \ell_s^6\frac{\zeta(3)}{32}stu + O(\ell_s^{10})\right) + g_s^2\left(\ell_s^6\frac{\pi^2}{96}stu + O(\ell_s^8)\right) + O(g_s^4)\right]\,, \tag{4.83}$$

we find that

$$\frac{B^4_4}{B^1_{\rm SG}} = \frac{105\zeta(3)}{64\sqrt{2}\pi^3}\,,\qquad \frac{\widetilde{B}^4_4}{B^1_{\rm SG}} = \frac{140\sqrt{2}}{3\pi}\,,\qquad B^5_{5,1} = \widetilde{B}^1_{\rm SG} = 0\,. \tag{4.84}$$

Like the M-theory amplitude, the type IIA super-amplitude does not contain any terms which could correspond to M^i_3 or $M^i_{5,2}$, which in 10d contribute at ℓ_s^{12} and ℓ_s^{16}. We hence conclude that these terms do not contribute at leading order:

$$B^3_3 = \widetilde{B}^3_3 = B^5_{5,2} = 0\,. \tag{4.85}$$

4.4.3 Large c_T, finite μ

Finally, consider the large N expansion of ABJ at finite μ. This regime interpolates between the two limit considered previously. The bulk and boundary quantities are related by the equation

$$\frac{L^8}{\ell_s^8} = \frac{3c_T\pi^5\sqrt{\mu}}{16\sqrt{2}} + \cdots\,, \qquad g_s^4 = 32\pi^2\mu + \cdots\,, \tag{4.86}$$

with corrections suppressed at large c_T. They imply that $M^i(s,t)$ can be expanded at large c_T in terms of $M_n^i(s,t)$ as

$$M^i(s,t) = \frac{1}{c_T}C^1_{\text{SG}}M^i_{\text{SG}} + \frac{1}{c_T^{\frac{3}{2}}}\left[C^3_{\text{SG}}M^i_{\text{SG}} + C^3_3M^i_3\right] + \frac{1}{c_T^{\frac{7}{4}}}\left[C^4_{\text{SG}}M^i_{\text{SG}} + C^4_3M^i_3 + C^4_4M^i_4\right] + O(c_T^{-2})\,, \tag{4.87}$$

where now $C^l_{i,j}$ are μ-dependent numerical coefficients. This expansion is nothing but a reorganized version of the double expansion (4.81). Unlike in the previous limits, we do not include the two amplitudes $M^i_{5,1}$ and $M^i_{5,2}$ because in this case they contribute at the same order in $1/c_T$ as the one-loop supergravity Mellin amplitude. Taking the flat space limit of (4.87) we find that

$$\mathcal{A}^i(s,t) = \frac{3\pi^4}{128}g_s^2\ell_s^8\left(C^1_{\text{SG}}\mathcal{A}^i_{\text{SG}} + \ell_s^6\left(\frac{9\pi^8g_s^4}{2^{14}}\right)^{3/8}C^4_4\mathcal{A}^i_4 + O(\ell_s^8)\right) \tag{4.88}$$

This expression can be compared with the type IIA scattering amplitude at fixed g_s, computed in [147]:

$$\mathcal{A}^{10}_{\text{IIA}} = \mathcal{A}^{10}_{\text{SG}}\left[1 + \ell_s^6 stu\left(\frac{\zeta(3)}{32} + g_s^2\frac{\pi^2}{96}\right) + O(\ell_s^8)\right]\,. \tag{4.89}$$

Note that the ℓ_s^6 term receives contributions at tree-level and one-loop, and has no further perturbative or non-perturbative corrections.

By comparing (4.89) and (4.88), we conclude that

$$\frac{C^4_4}{C^1_{\text{SG}}} = \frac{35}{2\pi^4}\left(\frac{9\pi^2}{32\mu^3}\right)^{1/8}\left(\zeta(3) + \frac{4}{3}\sqrt{2\mu\pi^3}\right)\,. \tag{4.90}$$

We can recover both the finite k and strong coupling 't Hooft limit expansions from (4.87) by taking the $\mu \to \infty$ and $\mu \to 0$ limits respectively, as we explain at the end of Section 3.2.3. Using the relations (3.71) and (3.72), we find that the $c_T^{-\frac{7}{4}}$ term becomes the $c_T^{-\frac{5}{3}}$ term at finite k, and gives rise to both the $c_T^{-1}\lambda^{-\frac{3}{2}}$ and $c_T^{-2}\lambda^{\frac{1}{2}}$ terms in the strong coupling 't Hooft limit.

4.5 Constraints from CFT Data

In the previous section, we derived general ansaetze (4.75), (4.81) and (4.87) for the Mellin amplitudes $M^i(s,t)$ at large c_T, and fixed terms that were leading in the flat space limit using the known 10d IIA string theory and 11d M-theory flat space scattering amplitudes. Now we shall use properties of the $U(N)_k \times U(N+M)_{-k}$ ABJ theories in order to compute the rest of the unknown coefficients. First we fix the contribution of the supergravity term using the OPE coefficient $\lambda^2_{(B,2)^{[011]}_{1,0}}$. We will then make use of the large N supersymmetric localization results derived in Chapter 3 to compute the contributions of the contact terms. In certain cases we shall find that the same coefficients can be fixed from localization and flat space independently, giving us precision checks of the AdS/CFT duality beyond supergravity.

4.5.1 Fixing the Supergravity Terms

We begin by noting that, as derived at the end of Section 2.4, c_T is related to $\lambda_{(B,2)^{[011]}_{1,0}}$ by the equation

$$\lambda^2_{(B,2)^{[011]}_{1,0}} = \frac{64}{c_T} \,. \tag{4.91}$$

The OPE coefficient $\lambda^2_{(B,2)^{[011]}_{1,0}}$ controls the exchange of the scalar S itself, which is the lowest twist operator transforming in the $\mathbf{15}_s$. Using (1.18), we then deduce that as we take $U \to 0$ while setting $V = 1$,

$$\mathcal{S}_{\mathbf{15}_s}(U,1) = \lambda^2_{(B,2)^{[011]}_{1,0}} g_{1,0}(U,1) + \cdots = \frac{\lambda^2_{(B,2)^{[011]}_{1,0}}}{4}\sqrt{U} + \cdots \,, \tag{4.92}$$

Thus, in order to extract $\lambda^2_{(B,2)^{[011]}_{1,0}}$, all we need to do is extract the coefficient of $\sqrt{U}$ in the small U expansion of $\mathcal{S}_{\mathbf{15}_s}(U,1)$. Note that the disconnected piece $\mathcal{S}_{\text{disc},\mathbf{15}_s}(U,1) = O(U)$ in this limit. The $\sqrt{U}$ term in the small U expansion of $\mathcal{S}_{\mathbf{15}_s}(U,1)$ must hence come from a pole at $s = 1$ in the Mellin amplitude $M_{\mathbf{15}_s}(s,t)$

$$M_{\mathbf{15}_s} \equiv \frac{1}{6}\left(M^2 + M^3 - M^4\right) + \frac{1}{2}(M^5 + M^6) \,, \tag{4.93}$$

corresponding to $\mathcal{S}_{\mathbf{15}_s}(U,V)$, which is defined in (2.17). Performing the s integral in (4.2) and picking up the residue at $s = 1$, we obtain

$$\mathcal{S}_{\mathbf{15}_s}(U,1) = -\frac{\sqrt{U}}{8i}\int_{-i\infty}^{i\infty} dt\, \Gamma^2\left(1-\frac{t}{2}\right)\Gamma^2\left(\frac{t-1}{2}\right)\lim_{s\to 1}\Big[(s-1)M_{\mathbf{15}_s}(s,t)\Big] + \cdots \,, \tag{4.94}$$

where the integration contour can be chosen such that Re$t < 2$. Comparing with (4.92), we have

$$\lambda^2_{(B,2)^{[011]}_{1,0}} = -\frac{1}{2i}\int_{-i\infty}^{i\infty} dt\, \Gamma^2\left(1-\frac{t}{2}\right)\Gamma^2\left(\frac{t-1}{2}\right)\lim_{s\to 1}\Big[(s-1)M_{\mathbf{15}_s}(s,t)\Big]\,. \tag{4.95}$$

As can be seen from (4.95), only the pole as $s \to 1$ in $M_{\mathbf{15}_s}$ contributes to $\lambda^2_{(B,2)^{[011]}_{1,0}}$. Therefore local Mellin amplitudes cannot contribute to $\lambda^2_{(B,2)^{[011]}_{1,0}}$, so the only contribution will come from the supergravity exchange Mellin amplitude. Indeed, the supergravity exchange amplitude $M^i_{\text{SG}}(s,t)$ does have a pole at $s = 1$ with a residue independent of t:

$$\lim_{s\to 1}\Big[(s-1)M_{\text{SG},\mathbf{15}_s}(s,t)\Big] = -\frac{1}{\pi}\,. \tag{4.96}$$

and thus M_{SG} in each of the expansions presented above contributes to $\lambda^2_{(B,2)^{[011]}_{1,0}}$ an amount equal to

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dt\, \Gamma^2\left(1-\frac{t}{2}\right)\Gamma^2\left(\frac{t-1}{2}\right) = 2\pi^2\,. \tag{4.97}$$

Note that although we have not discussed Mellin amplitudes for loop corrections, by suitably adding to them an appropriate multiple of M_{SG} we can always define them such that they do not contribute to the $\sqrt{U}$ term, so that $\lambda^2_{(B,2)^{[011]}_{1,0}}$ is purely fixed by the coefficient of M_{SG}. Since this OPE coefficient is related to c_T^{-1} via equation (4.91), we conclude that

$$\begin{aligned} &A^1_{\text{SG}} = B^1_{\text{SG}} = C^1_{\text{SG}} = \frac{32}{\pi^2}\,, \qquad \widetilde{B}^1_{\text{SG}} = 0\,, \\ &A^l_{\text{SG}} = B^l_{\text{SG}} = \widetilde{B}^l_{\text{SG}} = C^l_{\text{SG}} = 0\,, \qquad \text{for } l > 1\,. \end{aligned} \tag{4.98}$$

The same result was previously for the $\mathcal{N} = 8$ case in [117]. Indeed, the supergravity term does not depend on k when written in terms of c_T, as c_T is proportional to the effective 4d Newton constant G_N via the equation (1.52) in any theory where the bulk gravity is semiclassical.

4.5.2 Integrating Holographic Correlators

We now turn to the constraints on $\langle SSSS\rangle$ from supersymmetric localization. To implement these, we must compute the linear functionals $I_{++}[\mathcal{S}]$ and $I_{+-}[\mathcal{S}]$ for each of the Mellin amplitudes M^i_{SG}, M^i_3 and M^i_4. Our first step is to rewrite both linear functionals in Mellin space.

Let us begin with $I_{++}[\mathcal{S}]$, which, recall, is defined by equation (3.27) in terms of the OPE coefficient $\lambda^2_{(B,2)^{[022]}_{2,0}}$. This OPE coefficient controls the exchange of an 1/3-BPS operator whose superprimary is a $\Delta = 2$ scalar transforming in the $\mathbf{84}$. The operator provides the lowest twist contribution to $\mathcal{S}_{\mathbf{84}}(U,V)$, and so at small U we find that

$$\mathcal{S}_{\mathbf{84}}(U,1) = \lambda^2_{(B,2)^{[022]}_{2,0}} g_{2,0}(U,1) + \cdots = \frac{\lambda^2_{(B,2)^{[022]}_{2,0}}}{16} U + \cdots . \tag{4.99}$$

The coefficient receives contributions from both the disconnected piece,

$$\mathcal{S}_{\text{disc},\mathbf{84}}(U,1) = \frac{1}{16}\left(U + \frac{U}{V}\right)\Big|_{V=1} = \frac{U}{8}, \tag{4.100}$$

as well as from the connected piece. To extract the latter contribution we must consider $s = 2$ pole in the Mellin integral. The Gamma functions in the definition (4.2) of the Mellin transform have a double pole at $s = 2$, so

$$M_{\mathbf{84}}(s,t) = \frac{1}{16}\big(M^2(s,t) + M^3(s,t) + 2M^4(s,t)\big) \tag{4.101}$$

must vanish at least linearly as $s \to 2$. Combining the contribution of this pole with (4.100), we have

$$\mathcal{S}_{\mathbf{84}}(U,1) = U\left[\frac{1}{8} + \frac{i}{2\pi}\int_{-i\infty}^{i\infty} dt\, \Gamma^2\left(1 - \frac{t}{2}\right)\Gamma^2\left(\frac{t}{2}\right)\lim_{s\to 2}\frac{M_{\mathbf{84}}(s,t)}{s-2}\right] + \cdots . \tag{4.102}$$

We must be careful with the integration contours. As we take $s \to 2$, the the poles in t and $u = 4 - s - t$ may potentially overlap. For polynomial Mellin amplitudes no problem occurs. But for the supergravity amplitude, which has a pole both at $t = 1$ and $u = 1$, we must be careful to keep the t-channel poll to the right of the t-contour but the u-channel pole to the left. If we take the contour $0 < \text{Re}(t) < 1$, then we have to subtract of the $u = 1$ pole by hand. Thus, the correct formula is

$$\begin{aligned}\mathcal{S}_{\mathbf{84}}(U,1) = U\Bigg[&\frac{1}{8} + \pi^2 \lim_{s\to 2}\lim_{t\to 3-s}\frac{(u-1)M_{\mathbf{84}}(s,t)}{s-2} \\ &+ \frac{i}{2\pi}\int_{-i\infty}^{i\infty} dt\, \Gamma^2\left(1 - \frac{t}{2}\right)\Gamma^2\left(\frac{t}{2}\right)\lim_{s\to 2}\frac{M_{\mathbf{84}}(s,t)}{s-2}\Bigg] + \cdots .\end{aligned} \tag{4.103}$$

Comparing with (4.99), we extract

$$
\begin{aligned}
\lambda^2_{(B,2)^{[022]}_{2,0}} &= 2 + 16\pi^2 \lim_{s\to 2}\lim_{t\to 3-s} \frac{(u-1)M_{\mathbf{84}}(s,t)}{s-2} \\
&\quad + \frac{8i}{\pi}\int_{-i\infty}^{i\infty} dt\, \Gamma^2\left(1-\frac{t}{2}\right)\Gamma^2\left(\frac{t}{2}\right) \lim_{s\to 2}\frac{M_{\mathbf{84}}(s,t)}{s-2}\,,
\end{aligned}
\tag{4.104}
$$

with the t contour obeying $0 < \mathrm{Re}(t) < 1$, and so

$$
\begin{aligned}
I_{++}[M^i] &= 32\pi^2 \lim_{s\to 2}\lim_{t\to 3-s} \frac{(u-1)M_{\mathbf{84}}(s,t)}{s-2} \\
&\quad + \frac{16i}{\pi}\int_{-i\infty}^{i\infty} dt\, \Gamma^2\left(1-\frac{t}{2}\right)\Gamma^2\left(\frac{t}{2}\right) \lim_{s\to 2}\frac{M_{\mathbf{84}}(s,t)}{s-2}
\end{aligned}
\tag{4.105}
$$

where again the t contour obeys $0 < \mathrm{Re}(t) < 1$.

Let us now convert $I_{+-}[\mathcal{S}^i]$ to Mellin space, using (3.41) and (4.2) to write

$$
\begin{aligned}
I_{+-}[\mathcal{S}^i] &= \int_0^\infty dr \int_0^\pi d\theta\, \sin\theta\, \frac{\mathcal{S}^1\left(1+r^2-2r\cos\theta, r^2\right)}{1+r^2-2r\cos\theta} \\
&= \int \frac{ds\; dt}{(4\pi i)^2}\Bigg(\Gamma^2\left[1-\frac{s}{2}\right]\Gamma^2\left[1-\frac{t}{2}\right]\Gamma^2\left[\frac{s+t-2}{2}\right] M^1(s,t) \\
&\qquad \times \int_0^\infty dr \int_0^\pi d\theta\, \sin\theta\, \left(1+r^2-2r\cos\theta\right)^{s/2-1} r^{2-s-t}\Bigg).
\end{aligned}
\tag{4.106}
$$

The integral of r and θ can now be performed explicitly using

$$
\int_0^\infty dr \int_0^\pi d\theta\, \sin\theta\, \left(1+r^2-2r\cos\theta\right)^{s/2-1} r^{u-2} = \frac{\pi^{1/2}\Gamma\left[\frac{s+1}{2}\right]\Gamma\left[\frac{3-s-t}{2}\right]\Gamma\left[\frac{t-1}{2}\right]}{2\Gamma\left[1-\frac{s}{2}\right]\Gamma\left[2-\frac{t}{2}\right]\Gamma\left[\frac{s+t}{2}\right]}, \tag{4.107}
$$

and so we find that

$$
\begin{aligned}
I_{+-}[M^i] &= \int \frac{ds\; dt}{(4\pi i)^2}\frac{2\sqrt{\pi}}{(2-t)(s+t-2)} M^1(s,t) \\
&\quad \times \Gamma\left[1-\frac{s}{2}\right]\Gamma\left[\frac{s+1}{2}\right]\Gamma\left[1-\frac{t}{2}\right]\Gamma\left[\frac{t-1}{2}\right]\Gamma\left[\frac{s+t-2}{2}\right]\Gamma\left[\frac{3-s-t}{2}\right].
\end{aligned}
\tag{4.108}
$$

Having derived Mellin space expressions for $I_{++}[\mathcal{S}^i]$ and $I_{+-}[\mathcal{S}^i]$, we now compute these linear functionals acting on the Mellin amplitudes M^i_{SG}, M_3, and M_4. In each case, the integrals are performed by closing the contours at infinity and then summing over all poles enclosed by the contours. In some cases, the pole summation is easily done using the Barnes lemma

$$
\int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s) = \frac{\Gamma(a+c)\Gamma(b+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\,, \tag{4.109}
$$

which holds for contours for which the poles of each Gamma function lie either entirely to the left or to the right of the contour.

Let us begin with $I_{+-}[M^i]$ for the polynomial Mellin amplitudes $M_3^1(s,t)$ and $M_4^1(s,t)$. In both cases the amplitudes take the form $(t-2)(2-s-t)$ times a polynomial in s,t, so $I_{+-}[\mathcal{S}_n^i]$ can be evaluated for $n=3,4$ by writing the integrand as a sum of products of six Gamma functions in s,t and then applying the Barnes lemma twice. For example, for $M_3^1(s,t)=(t-2)(2-s-t)$ we compute

$$\begin{aligned}I_{+-}[M_3^i] &= \int \frac{dsdt}{(4\pi i)^2} 2\sqrt{\pi} \\ &\times \Gamma\left[1-\frac{s}{2}\right]\Gamma\left[\frac{s+1}{2}\right]\Gamma\left[1-\frac{t}{2}\right]\Gamma\left[\frac{t-1}{2}\right]\Gamma\left[\frac{2-s-t}{2}\right]\Gamma\left[\frac{3-s-t}{2}\right] \\ &= \int \frac{dt}{4\pi i}\pi^{3/2}\Gamma\left[1-\frac{t}{2}\right]\Gamma\left[2-\frac{t}{2}\right]\Gamma\left[\frac{t-1}{2}\right]\Gamma\left[\frac{t}{2}\right] \\ &= \frac{2\pi^2}{3}\,,\end{aligned} \tag{4.110}$$

where the last two equalities followed from the Barnes lemma. We can evaluate $I_{+-}[\mathcal{S}_4^i]$ in an identical fashion, finding that

$$I_{+-}[M_4^i] = \frac{8}{7}\pi^2\,. \tag{4.111}$$

The supergravity Mellin amplitude $M_{\text{SG}}^1(s,t)$ (4.69) is also proportional to $(t-2)(2-s-t)$, but the remaining function is not a polynomial in s,t and so we must work harder. We compute

$$\begin{aligned}I_{+-}[M_{\text{SG}}^i] &= \int \frac{dsdt}{(4\pi i)^2}\frac{1}{4\pi^2 s(2+s)}\left[\sqrt{\pi}(4+s)\Gamma\left[1-\frac{s}{2}\right]-4\Gamma\left[\frac{1-s}{2}\right]\right] \\ &\times \Gamma\left[\frac{s+1}{2}\right]\Gamma\left[1-\frac{t}{2}\right]\Gamma\left[\frac{t-1}{2}\right]\Gamma\left[\frac{s+t-2}{2}\right]\Gamma\left[\frac{3-s-t}{2}\right] \\ &= \int \frac{ds}{4\pi i}\frac{\Gamma\left[1-\frac{s}{2}\right]\Gamma\left[\frac{s}{2}\right]\Gamma\left[\frac{s+1}{2}\right]}{4\pi s(2+s)}\left[\sqrt{\pi}(4+s)\Gamma\left[1-\frac{s}{2}\right]-4\Gamma\left[\frac{1-s}{2}\right]\right] \\ &= -\pi^2\,,\end{aligned} \tag{4.112}$$

where in the first equality we used the Barnes lemma, and in the second equality we summed over poles with the contour $0<\text{Re}(s)<1$. Note that this contour is different from the range $0<\text{Re}(s)<2$ that would follow from (4.3). This is because the supergravity term includes the stress tensor multiplet superblock, which contribute extra poles that require a more constraining contour [148].

Finally, we evaluate $I_{++}[M^i]$ for M_3^i, M_4^i and M_{SG}^i using (4.105). For the polynomial Mellin amplitudes M_3^i and M_4^i, the first term in (4.105) vanishes, and in the second term we have

$$\begin{aligned}\lim_{s\to 2}\frac{M_{3,\mathbf{84}}}{s-2} &= -\frac{1}{24}\,,\\ \lim_{s\to 2}\frac{M_{3,\mathbf{84}}}{s-2} &= -\frac{1}{5}-\frac{3t(t-2)}{56}\,.\end{aligned} \tag{4.113}$$

We now integrate over t using the Barnes lemma, which yields

$$I_{++}[M_3^i] = \frac{8}{3}\,,\qquad I_{++}[M_4^i] = \frac{288}{35}\,. \tag{4.114}$$

For the supergravity amplitude, the first term in (4.105) gives $8\pi/3$ and in the integrand of the second term we have

$$\begin{aligned}\lim_{s\to 2}\frac{M_{\text{SG},\mathbf{84}}}{s-2} = &\frac{(t-2)\Gamma\left(\frac{1-t}{2}\right)}{8\sqrt{\pi}t(t+2)\Gamma\left(1-\frac{t}{2}\right)} - \frac{t^2\Gamma\left(\frac{t-1}{2}\right)}{16\sqrt{\pi}(t-2)(t-4)\Gamma\left(1+\frac{t}{2}\right)}\\ &-\frac{t^4-4t^3-12t^2+32t-32}{16t(t-4)(t^2-4)}\,.\end{aligned} \tag{4.115}$$

We can then compute the integral over t by summing over all poles which lie to the right of the t contour, and so find that

$$I_{++}[M_{\text{SG}}^i] = 12\,. \tag{4.116}$$

4.5.3 Fixing $\langle SSSS\rangle$ with Localization

In the previous section, we computed $I_{++}[M]$ and $I_{+-}[M]$ for each of the amplitudes $M_{\text{SG}}(s,t)$, $M_3(s,t)$ and $M_4(s,t)$ contributing to $\langle SSSS\rangle$ in the first few orders of the large c_T expansion. We found that

$$\begin{aligned} I_{++}[M_{\text{SG}}^i] &= 12\,, & I_{+-}[M_{\text{SG}}^i] &= -\pi^2\,,\\ I_{++}[M_3^i] &= \frac{8}{3}\,, & I_{+-}[M_3^i] &= \frac{2}{3}\pi^2\,,\\ I_{++}[M_4^i] &= \frac{288}{35}\,, & I_{+-}[M_4^i] &= \frac{8}{7}\pi^2\,.\end{aligned} \tag{4.117}$$

We can now combine these calculations with the supersymmetric localization results derived in Chapter 3, using them to constrain the coefficients $A_{i,j}^l$, $B_{i,j}^l$, $\widetilde{B}_{i,j}^l$, and $C_{i,j}^l$. Plugging (4.117) into (3.24) and (3.42) and using the large N localization results (3.64), (3.68), and (3.70) derived in Section 3.2.3, we can obtain the following results. First, without using the constraints from the flat space limit or the constraints (4.98) coming from the superconformal block expansion, the supersymmetric localization constraints (3.24) and (3.42) reproduce the coefficients in the first line of (4.98). This is a stringent consistency check on the accuracy of our computations.

Second, using the constraints (4.98) coming from the superconformal block expansion as an input, the supersymmetric localization constraints allow us to fix the coefficients at the next two orders in each of the expansions (4.75), (4.81), and (4.87). The result is

$$\begin{aligned}
&\text{finite } k: && A_4^4 = \frac{2240}{(6\pi^4 k)^{2/3}}\,, \qquad A_3^3 = A_3^4 = 0\,,\\
&\text{'t Hooft:} && B_3^4 = -\frac{54\sqrt{2}\zeta(3)}{\pi^5}\,, \qquad B_4^4 = \frac{105\zeta(3)}{2\sqrt{2}\pi^5}\,, \qquad \widetilde{B}_4^4 = \frac{4480\sqrt{2}}{3\pi^3}\,,\\
& && B_3^3 = \widetilde{B}_3^3 = \widetilde{B}_3^4 = 0\,,\\
&\text{finite } \mu: && C_3^4 = -\frac{576\ 2^{3/8}3^{\frac{1}{4}}\zeta(3)}{\pi^{23/4}\mu^{3/8}}\,, \qquad C_4^4 = \frac{2^{\frac{3}{8}}280}{3^{3/4}\pi^{23/4}}\left(4\sqrt{2}\pi^3\mu^{\frac{1}{8}} + 3\zeta(3)\mu^{-\frac{3}{8}}\right)\,,\\
& && C_3^3 = 0\,.
\end{aligned} \tag{4.118}$$

These equations agree with the constraints from the flat space limit, thus providing a very non-trivial precision test of AdS/CFT.

Third, using both the constraints (4.98) and the constraints coming from the flat space limit as input, the constraints from supersymmetric localization allow us to conclude that

$$A_3^5 = A_4^5 = B_3^5 = B_4^5 = 0\,. \tag{4.119}$$

We can then plug these values back into (4.75), (4.81), and (4.87) to get our final results

$$\begin{aligned}
\text{M-theory}: \quad M(s,t) &= \frac{1}{c_T}\frac{32}{\pi^2}M_{\text{SG}}(s,t) + \frac{1}{c_T^{\frac{5}{3}}}\frac{1120}{3\pi^3}\left(\frac{6\pi}{k^2}\right)^{1/3} M_4(s,t) + O\left(c_T^{-2}\right)\,,\\
\text{'t Hooft}: \quad M(s,t) &= \frac{1}{c_T}\left(\frac{32}{\pi^2}M_{\text{SG}}(s,t) + \frac{3\sqrt{2}\zeta(3)}{4\pi^5}\big[35M_4(s,t) - 72M_3(s,t)\big]\lambda^{-\frac{3}{2}} + O(\lambda^{-\frac{5}{2}})\right)\\
&\quad + \frac{1}{c_T^2}\left(\frac{4480\sqrt{2}}{3\pi^3}M_4(s,t)\lambda^{\frac{1}{2}} + O(\lambda^0)\right) + O(c_T^{-3})\,,\\
\text{fixed } \mu: \quad M(s,t) &= \frac{1}{c_T}\frac{32}{\pi^2}M_{\text{SG}}(s,t)\\
&\quad + \frac{1}{c_T^{\frac{7}{4}}}\left(-\frac{576\ 2^{3/8}3^{\frac{1}{4}}\zeta(3)}{\pi^{23/4}\mu^{3/8}}M_3(s,t) + \frac{2^{\frac{3}{8}}280}{3^{3/4}\pi^{23/4}}\left(4\sqrt{2}\pi^3\mu^{\frac{1}{8}} + 3\zeta(3)\mu^{-\frac{3}{8}}\right)M_4(s,t)\right)\\
&\quad + O\left(c_T^{-2}\right)\,,
\end{aligned} \tag{4.120}$$

for the leading large c_T corrections to the $\langle SSSS\rangle$ Mellin amplitude in each of three regimes considered in this chapter.

Chapter 5

The Higher-Spin Limit

In this chapter we study $\mathcal{N} = 6$ theories with weakly broken higher-spin symmetry. As discussed in the Introduction, such theories include the $U(N)_k \times U(N+M)_{-k}$ and $SO(2)_{2k} \times USp(2+2M)_{-k}$ theories at large M. By combining the constraints of weakly broken higher-spin symmetry with the supersymmetric localization results for these theories computed in Chapter 3, we derive the leading $1/c_T$ correction to $\langle SSSS \rangle$ in the higher-spin limits of both theories. Our strategy parallels that of the previous chapter, where we studied the string and M-theory limits of ABJ theory. First we fix an ansatz for $\langle SSSS \rangle$ at large c_T involving only finitely many undetermined coefficients. We then determine these coefficients for the specific theories of interest using supersymmetric localization.

While the overall strategy may be similar, the structure of the $1/c_T$ expansion in the higher-spin limit is very different from that of the string and M-theory limits. In the latter case, the bulk theory contains only single trace operators of spin at most two. If we focus on $\langle SSSS \rangle$, then at tree-level the only single trace operators we need to consider are those in the stress tensor multiplet. Holographic tree-level correlators then consist of a tree-level supergravity exchange diagram whose form is completely fixed by superconformal symmetry, along with an infinite number of contact Witten diagrams which appear at increasingly higher powers of $1/c_T$.

Unlike the supergravity limit, the higher-spin limit has single trace particles of every spin, their exchange diagrams are not completely fixed by superconformal symmetry, and the contact terms can no longer be fixed using the flat space limit which does not exist for higher-spin gravity [149]. We will resolve these problems by combining slightly broken higher-spin Ward identities with the Lorentzian inversion formula [150], as in the recent calculations of the analogous non-supersymmetric correlator in [151, 152].[1] In particular, we will first compute tree-level three-point functions of

[1] See also [153] for a similar calculation of a spinning non-supersymmetric correlator, as well as [154] for a more

single trace operators in terms of c_T and another free parameter using weakly broken higher-spin symmetry, which generalizes the non-supersymmetric analysis of [155] to $\mathcal{N}=6$ theories.[2] We then use these three-point functions to fix the infinite single trace exchange diagrams that appear in $\langle SSSS\rangle$. Finally, we use the Lorentzian inversion formula to argue that only contact diagrams with six derivatives or fewer can appear, of which only a single linear combination is allowed by $\mathcal{N}=6$ superconformal symmetry. In sum, we find that $\langle SSSS\rangle$ is fixed at $O(1/c_T)$ in the higher-spin limit in terms of two free parameters.

We then try to fix these two free parameters using the supersymmetric localization constraints for $\frac{\partial^4 \log Z}{\partial m_+^4}\Big|_{m_\pm=0}$ and $\frac{\partial^4 \log Z}{\partial m_+^2 \partial m_-^2}\Big|_{m_\pm=0}$ derived in Chapter 3. When we do so however, we find that the constraints are redundant. To resolve this issue we use the weakly broken higher-spin Ward identities to compute $\langle SSSP\rangle$, which we can constrain using $\frac{\partial^4 \log Z}{\partial m_+^3 \partial m_-}\Big|_{m_\pm=0}$. With this additional constraint, we will be able to fix both free parameters for the $U(N)_k\times U(N+M)_{-k}$ ABJ and $SO(2)_{2k}\times USp(2+2M)_{-k}$ OSp theories in terms of the 't Hooft coupling $\lambda\sim\frac{M}{k}$, whose precise definition is given in (1.60) and (1.61).[3]

Our plan for the chapter is as follows. We begin in Section 5.1 by describing the higher-spin conserved currents, and then study the pseudocharge $\tilde{\delta}(X)$ associated to the conserved vector $H_1^\mu(x,X)$ in the scalar conserved current multiplet. In Sections 5.2 and 5.3 we use the pseudoconservation of the pseudocharge $\tilde{\delta}(X)$ to derive the general form of $\langle SSSS\rangle$ and $\langle SSSP\rangle$ respectively, and then in Section 5.4 use supersymmetric localization constraints to completely fix $\langle SSSS\rangle$. We close in Section 5.5 with a discussion of the similarities between our results and those for non-supersymmetric higher-spin theories.

5.1 Weakly Broken Higher-Spin Symmetry

5.1.1 $\mathcal{N}=6$ Conserved Currents

The $\mathfrak{osp}(6|4)$ superalgebra allows two kinds of unitary conserved current multiplets: the $(B,2)^{[011]}_{1,0}$ stress tensor multiplet, and the $(A,\text{cons})^{[000]}_{\ell+1,\ell}$ higher-spin multiplets. We are hopefully by now quite familiar with the stress tensor multiplet, which we first introduced all the way back in Section 1.1.3. Unlike the stress tensor multiplet, the $(A,\text{cons})^{[000]}_{\ell+1,\ell}$ multiplets are semishort rather than short, and

direct diagrammatic approach.

[2]Note that [155] applies to higher-spin theories with only one single trace operator of each spin. This excludes the $\mathcal{N}=6$ higher-spin theories we consider, whose single trace spectrum includes one higher-spin multiplet of each spin plus the stress tensor multiplet, whose component operators includes multiple operators of each spin.

[3]We will use λ rather than λ_{HS} throughout this Chapter to refer to the 't Hooft coupling in the higher-spin limit. It should not be confused with the 't Hooft coupling in the large N regime!

contain conserved currents with spin greater than two. When $\ell > 0$ superconformal primary for the multiplet is a spin-ℓ conserved current $B_\ell(\vec{x})$, and the bosonic descendants of $B_\ell(\vec{x})$ are conserved currents $H_{\ell+1}(\vec{x}, X)$, $J_{\ell+2}(\vec{x}, X)$, and $T_{\ell+3}(\vec{x})$ with spins $\ell+1$, $\ell+2$ and $\ell+3$ respectively. The bottom and top components B_ℓ and $T_{\ell+3}$ are R-symmetry singlets, while the middle two components $H_{\ell+1}$ and $J_{\ell+2}$ transform in the **15**. There is also a scalar higher-spin multiplet $(A, \text{cons})^{[000]}_{1,0}$ whose primary $B_0(\vec{x})$ is a dimension 1 scalar. This multiplet has the same structure as the $\ell > 0$ higher-spin multiplets, except that it also contains an additional scalar $C_0(\vec{x})$ with dimension 2. We normalize all of these operators so that

$$\begin{aligned}
\langle \mathcal{J}_\ell^{\mu_1\ldots\mu_\ell}(\vec{x}_1)\mathcal{J}_\ell^{\nu_1\ldots\nu_\ell}(\vec{x}_2)\rangle &= \left(\frac{I^{(\mu_1\ldots\mu_n)(\nu_1\ldots\nu_n)}(x_{12})}{x_{12}^{2\ell-1}} - \text{traces}\right),\\
\langle \mathcal{K}_\ell^{\mu_1\ldots\mu_\ell}(\vec{x}_1, X_1)\mathcal{K}_\ell^{\nu_1\ldots\nu_\ell}(\vec{x}_2, X_2)\rangle &= \operatorname{tr}(X_1X_2)\left(\frac{I^{(\mu_1\ldots\mu_n)(\nu_1\ldots\nu_n)}(x_{12})}{x_{12}^{2\ell-2}} - \text{traces}\right),\\
\text{where } I^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}(x_{12}) &= \left(\delta^{\mu_1\nu_1} - \frac{x_{12}^{\mu_1}x_{12}^{\nu_1}}{x_{12}^2}\right)\ldots\left(\delta^{\mu_n\nu_n} - \frac{x_{12}^{\mu_n}x_{12}^{\nu_n}}{x_{12}^2}\right),
\end{aligned} \tag{5.1}$$

for operators $\mathcal{J}_\ell$ and $\mathcal{K}_\ell$ transforming in the **1** and **15** of the $\mathfrak{so}(6)_R$ R-symmetry respectively.

We will sometimes find it convenient to think of the stress tensor multiplet as being the $\ell = -1$ conserved current multiplet. To this end, we relabel the R-symmetry current $J_1(\vec{x}, X)$ and the stress tensor $T_2(\vec{x})$, as they are the natural continuations of the $J_\ell(\vec{x}, X)$ and $T_\ell(\vec{x})$ families of conserved currents.

We restrict our attention to theories where the single-trace operators consist of a stress tensor multiplet, along with a single higher-spin multiplet $(A, \text{cons})^{[000]}_{\ell+1,\ell}$ for each $\ell = 0, 1, 2, \ldots$. This, in particular, is the spectrum of free field theory, and also of ABJ theory at large M. We list the single-trace operator content of such theories in Table 5.1. Observe that for each spin $\ell \geq 2$ the bosonic conserved currents come in pairs, so that for each $B_\ell(\vec{x})$ and $H_\ell(\vec{x})$ there is a $T_\ell(\vec{x})$ and $J_\ell(\vec{x})$ respectively with the same quantum numbers but belonging to different SUSY multiplets. As we shall see, these pairs of operators are mixed by the higher-spin conserved currents.

Let us now consider three-point functions between the scalars S, P and a conserved current $\mathcal{J}$.

	Higher-Spin Multiplet					
Spin	Stress-tensor	Spin 0	Spin 1	Spin 2	Spin 3	...
0	$\mathbf{15}+\mathbf{15}$	$\mathbf{1}+\mathbf{1}$				
1/2	$\mathbf{6}+\mathbf{10}+\overline{\mathbf{10}}$	$\mathbf{6}$				
1	$\mathbf{1}+\mathbf{15}$	$\mathbf{15}$	$\mathbf{1}$			
3/2	$\mathbf{6}$	$\mathbf{10}+\overline{\mathbf{10}}$	$\mathbf{6}$			
2	$\mathbf{1}$	$\mathbf{15}$	$\mathbf{15}$	$\mathbf{1}$		
5/2		$\mathbf{6}$	$\mathbf{10}+\overline{\mathbf{10}}$	$\mathbf{6}$		
3		$\mathbf{1}$	$\mathbf{15}$	$\mathbf{15}$	$\mathbf{1}$	
7/2			$\mathbf{6}$	$\mathbf{10}+\overline{\mathbf{10}}$	$\mathbf{6}$	
4			$\mathbf{1}$	$\mathbf{15}$	$\mathbf{15}$	...
$\vdots$				$\vdots$	$\vdots$	$\ddots$

Table 5.1: Single trace operators for higher-spin $\mathcal{N}=6$ CFTs.

Conformal invariance, R-symmetry, and crossing symmetry together imply that

$$\begin{aligned}
\langle\phi(\vec{x}_1,X_1)\phi(\vec{x}_2,X_2)\mathcal{J}_\ell^{\mu_1\dots\mu_\ell}(\vec{x}_3)\rangle &= \begin{cases}\lambda_{\phi\phi\mathcal{J}}\,\mathrm{tr}\,(X_1X_2)\,\mathcal{C}^{\mu_1\dots\mu_\ell}_{\phi\phi\ell}(x_i) & \text{even } \ell\\ 0 & \text{odd } \ell\end{cases}\\
\langle\phi(\vec{x}_1,X_1)\phi(\vec{x}_2,X_2)\mathcal{K}_\ell^{\mu_1\dots\mu_\ell}(\vec{x}_3,X_3)\rangle &= \begin{cases}\lambda_{\phi\phi\mathcal{K}}\,\mathrm{tr}\,(\{X_1,X_2\}X_3)\,\mathcal{C}^{\mu_1\dots\mu_\ell}_{\Delta_\phi\Delta_\phi\ell}(x_i) & \text{even } \ell\\ \lambda_{\phi\phi\mathcal{K}}\,\mathrm{tr}\,([X_1,X_2]X_3)\,\mathcal{C}^{\mu_1\dots\mu_\ell}_{\Delta_\phi\Delta_\phi\ell}(x_i) & \text{odd } \ell\end{cases}
\end{aligned}\tag{5.2}$$

where we define the conformally covariant structure[4]

$$\mathcal{C}^{\mu_1\dots\mu_\ell}_{\Delta_1\Delta_2\ell}(x_i)=\sqrt{\frac{(1/2)_\ell}{2^{\ell+2}\ell!}}\left(\frac{x_{13}^{\mu_1}}{x_{13}^2}-\frac{x_{23}^{\mu_1}}{x_{23}^2}\right)\cdots\left(\frac{x_{13}^{\mu_\ell}}{x_{13}^2}-\frac{x_{23}^{\mu_\ell}}{x_{23}^2}\right)\frac{1}{x_{12}^{\Delta_1+\Delta_2-1}x_{23}^{\Delta_2-\Delta_1+1}x_{31}^{\Delta_1-\Delta_2+1}}\,. \tag{5.3}$$

Note that $\langle SP\mathcal{J}\rangle$ automatically vanishes when $\mathcal{J}$ is a conserved current, as $\mathcal{C}_{\Delta_1\Delta_2\ell}$ is not conserved unless $\Delta_1=\Delta_2$.

In Chapter 2 we computed the superconformal blocks for $\langle SSSS\rangle$, which relate the OPE coefficients of operators in the same supermultiplet. We also derived superconformal Ward identities relating $\langle SSSS\rangle$ to $\langle SSPP\rangle$, and so can use these to derive the superconformal block expansion for $\langle SSPP\rangle$. Using the results of that chapter, we find every integer ℓ there is a unique superconformal structure between two S operators and the supermultiplet $(A,\text{cons})^{[000]}_{\ell+1,\ell}$. For even ℓ the OPE

[4]Our choice of prefactors multiplying $\mathcal{C}_{\phi_1\phi_2\ell}$ is such that the three-point coefficients $\lambda_{\phi_1\phi_2\mathcal{O}}$ match the OPE coefficients multiplying the conformal blocks in Chapter 2.

coefficients are all related to λ_{SSB_ℓ} via the equations[5]

$$\begin{aligned}
\lambda_{SSB_\ell} &= \lambda_{SSH_{\ell+1}} = \lambda_{SSJ_{\ell+2}} = \lambda_{(A,\text{cons})^{[000]}_{\ell+1,\ell}}\,, & \lambda_{SST_{\ell+3}} &= 0\,,\\
\lambda_{PPB_\ell} &= -\ell\lambda_{SSB_\ell}\,, & \lambda_{PPH_{\ell+1}} &= -(\ell+1)\lambda_{SSB_\ell}\,,\\
\lambda_{PPJ_{\ell+2}} &= -(\ell+2)\lambda_{SSB_\ell}\,, & \lambda_{PPT_{\ell+3}} &= 0\,,
\end{aligned} \tag{5.4}$$

while for odd ℓ the OPE coefficients are related to $\lambda_{SST_{\ell+3}}$:

$$\begin{aligned}
\lambda_{SSH_{\ell+1}} &= \lambda_{SSJ_{\ell+2}} = \lambda_{SST_{\ell+3}} = \lambda_{(A,\text{cons})^{[000]}_{\ell+1,\ell}}\,, & \lambda_{SSB_\ell} &= 0\,,\\
\lambda_{PPB_\ell} &= 0\,, & \lambda_{PPH_{\ell+1}} &= (\ell+1)\lambda_{SST_{\ell+3}}\,,\\
\lambda_{PPJ_{\ell+2}} &= (\ell+2)\lambda_{SST_{\ell+3}}\,, & \lambda_{PPT_{\ell+3}} &= (\ell+3)\lambda_{SST_{\ell+3}}\,.
\end{aligned} \tag{5.5}$$

Note that $\lambda_{SST_{\ell+3}}$ vanishes for even ℓ, and λ_{SSB_ℓ} for odd ℓ, simply as a consequence of $1 \leftrightarrow 2$ crossing symmetry. The superconformal blocks for the stress tensor and the scalar conserved current have the same structure (where we treat the stress tensor block as having spin -1), with the additional equations

$$\begin{aligned}
\lambda_{SSS} &= \lambda_{SST_2} = \lambda_{(B,2)^{[011]}_{1,0}} = \frac{8}{\sqrt{c_T}}\,, \qquad \lambda_{SSP} = \lambda_{PPS} = \lambda_{PPP} = 0\,,\\
\lambda_{SSC_0} &= \lambda_{PPC_0} = 0\,,
\end{aligned} \tag{5.6}$$

for the scalars S and P, and dimension 2 scalar C_0, in the stress tensor and the scalar conserved current multiplet respectively.

5.1.2 The $\mathfrak{so}(6)$ Pseudocharge

Having reviewed the properties of conserved current multiplets in $\mathcal{N} = 6$ theories, we now consider weakly breaking the higher-spin symmetries. We will follow the strategy employed in [155] and use the weakly broken higher-spin symmetries to constrain three-point functions. Unlike that paper however, which studies the non-supersymmetric case and so considers the symmetries generated by a spin 4 operator, we will instead focus on the spin 1 operator $H_1^\mu(\vec{x}, X)$. While itself not a higher-spin conserved current, it is related to the spin 3 current $T_3(\vec{x})$ by supersymmetry.

[5]The superconformal blocks themselves relate $\lambda^2_{SSH_{\ell+1}}$ or $\lambda^2_{SST_{\ell+3}}$ to $\lambda^2_{SS\mathcal{O}}$ and $\lambda_{SS\mathcal{O}}\lambda_{PP\mathcal{O}}$ for all superdescendants $\mathcal{O}$ of B_ℓ. Although the superconformal blocks do not fix the sign of $\lambda_{SS\mathcal{O}}$, we can always redefine $\mathcal{O} \to -\mathcal{O}$ so that $\lambda_{SS\mathcal{O}}/\lambda_{SSB_\ell}$ or $\lambda_{SS\mathcal{O}}/\lambda_{SST_{\ell+3}}$ is positive.

We begin by using $H_1(\vec{x})$ to define a pseudocharge:

$$\tilde{\delta}(X)\mathcal{O}(0) = \frac{1}{4\pi}\int_{|x|=r} dS\cdot H_1(\vec{x},X)\,\mathcal{O}(0)\Big|_{\text{finite as } r\to 0}\,. \tag{5.7}$$

The action of $\tilde{\delta}(X)$ is fixed by the 3-points functions $\langle H_1\mathcal{O}\mathcal{O}'\rangle$. Because H_1^μ has spin 1, it must act in the same way on conformal primaries as would any other spin 1 conserved current. In particular, it relates conformal primaries to other conformal primaries with the same spin and conformal dimension.

Now consider the action of $\tilde{\delta}(X)$ on an arbitrary three-point function. We can use the divergence theorem to write:

$$\begin{aligned}\tilde{\delta}(X)&\big\langle\mathcal{O}_1(\vec{y}_1)\mathcal{O}_2(\vec{y}_2)\mathcal{O}_3(\vec{y}_3)\big\rangle\\ &= -\frac{1}{4\pi}\int_{\mathcal{R}_r} d^3x\,\langle\nabla\cdot H_1(\vec{x},X)\mathcal{O}_1(\vec{y}_1)\mathcal{O}_2(\vec{y}_2)\mathcal{O}_3(\vec{y}_3)\rangle\Big|_{\text{finite } r\to 0}\,,\end{aligned} \tag{5.8}$$

where $\mathcal{R}_r$ is the set of $\vec{x}\in\mathbb{R}^3$ for which $|\vec{x}-\vec{y}_i|>r$ for each y_i. If the operator $H_1^\mu(x,X)$ were conserved, the right-hand side of this expression would vanish, and we would find that correlators were invariant under $\tilde{\delta}(X)$. When the higher-spin symmetries are broken, however, $\nabla\cdot H_1$ will no longer vanish and so (5.8) will give us a non-trivial identity: a pseudoconservation rule for the pseudocharge.

In the infinite c_T limit, $\nabla\cdot H_1$ is a conformal primary distinct from H_1^μ. In order to work out what this primary is, we can use the $\mathcal{N}=6$ multiplet recombination rules [13]:[6]

$$\begin{aligned}\text{Long}_{\Delta,0}^{[000]} &\underset{\Delta\to 1}{\longrightarrow} (A,\text{cons})_{1,0}^{[100]}\oplus(B,1)_{2,0}^{[200]}\,,\\ \text{Long}_{\Delta,\ell}^{[000]} &\underset{\Delta\to \ell+1}{\longrightarrow} (A,\text{cons})_{\ell+1,\ell}^{[100]}\oplus(A,1)_{\ell+3/2,\ell-1/2}^{[100]}\qquad\text{for } \ell>0\,.\end{aligned} \tag{5.9}$$

From this we see that, unlike the other conserved current multiplets, the scalar conserved current multiplet recombines with a B-type multiplet, the $(B,1)_{2,0}^{[200]}$. The only such multiplet available in higher-spin $\mathcal{N}=6$ SCFTs at infinite c_T is the double-trace operator $S^{[a}{}_{[b}S^{c]}{}_{d]}$, whose descendants are also double-traces of stress tensor operators. From this we deduce that

$$\begin{aligned}\nabla\cdot H_1(\vec{x},X) &= -\frac{\alpha}{\sqrt{c_T}}\Phi(\vec{x},X) + \text{fermion bilinears} + O(c_T^{-1})\\ \text{with } \Phi(\vec{x},X) &= X^a{}_b\left(S^b{}_c(\vec{x})P^c{}_a(\vec{x}) - P^b{}_c(\vec{x})S^c{}_a(\vec{x})\right)\,,\end{aligned} \tag{5.10}$$

[6] For the decomposition of the superconformal blocks associated to these multiplets, see equations (2.74) through (2.76).

where α is some as yet undetermined coefficient. We then conclude that

$$\tilde{\delta}(X)\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = \frac{\alpha}{4\pi\sqrt{c_T}}\int d^3x\,\langle\Phi(\vec{x},X)\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle + \text{fermion bilinears} + O({c_T}^{-3/2})\,, \tag{5.11}$$

where we have left the regularization of the right-hand integral implicit.

Consider the case where $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ are any three bosonic conserved currents. In this case, $\langle\Phi\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle\sim c_T^{-3/2}$ and so

$$\tilde{\delta}(X)\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = O(c_T^{-3/2})\,. \tag{5.12}$$

We thus find that, at leading order in the $1/c_T$ expansion, these three-point functions are invariant under $\tilde{\delta}(X)$. This is a strong statement, allowing us to import statements about conserved currents and apply them to H_1.

We will now consider the R-symmetry current J_1, which has the same quantum numbers as H_1, and let us define

$$\delta(X)\mathcal{O}(0) = \frac{1}{4\pi}\int_{|x|=r} dS\cdot J_1(\vec{x})\mathcal{O}(0)\Big|_{\text{finite as } r\to 0}\,, \tag{5.13}$$

which generates the $\mathfrak{so}(6)_R$ symmetry. Because any correlator of both J_1 and H_1 is conserved under $\delta(X)$ and $\tilde{\delta}(X)$ at leading order, the (pseudo)charges $\delta(X)$ and $\tilde{\delta}(X)$ form a semisimple Lie algebra.[7] The $\mathfrak{so}(6)_R$ symmetry implies the commutator relations

$$[\delta(X),\delta(Y)] = \zeta\delta([X,Y])\,,\qquad [\delta(X),\tilde{\delta}(Y)] = \zeta\tilde{\delta}([X,Y])\,, \tag{5.14}$$

for some non-zero constant ζ, while

$$[\tilde{\delta}(X),\tilde{\delta}(Y)] = \zeta\delta([X,Y]) + 2\gamma\tilde{\delta}([X,Y])\,, \tag{5.15}$$

for some additional γ. Note that both the second equation in (5.14) and the first term in (5.15) are fixed by the same conformal structure in the three-point function $\langle H_1H_1J_1\rangle$, which is why they are both proportional to ζ. We can now define charges $\delta_L(X)$ and $\delta_R(X)$ by the equations

$$\begin{aligned} \delta(X) &= \zeta\left(\delta_L(X)+\delta_R(X)\right)\,, \qquad & \tilde{\delta}(X) &= \zeta_L\delta_L(X)+\zeta_R\delta_R(X)\,,\\ \text{with}\quad \zeta_L &= \gamma+\sqrt{\zeta^2+\gamma^2}\,, & \zeta_R &= \gamma-\sqrt{\zeta^2+\gamma^2}\,, \end{aligned} \tag{5.16}$$

[7]Note that this Lie algebra structure only holds when $\delta(X)$ and $\tilde{\delta}(X)$ act on spinning single trace operators, so that (5.12) holds. In particular, equation (5.14) and (5.15) are true when $\delta(X)$ and $\tilde{\delta}(X)$ act on such operators.

which satisfy the commutator relations

$$\begin{aligned} &[\delta_L(X), \delta_L(Y)] = \delta_L([X,Y])\,, \qquad [\delta_L(X), \delta_R(Y)] = 0 \\ &[\delta_R(X), \delta_R(Y)] = \delta_R([X,Y])\,. \end{aligned} \tag{5.17}$$

These are precisely the commutation relations of an $\mathfrak{so}(6) \times \mathfrak{so}(6)$ Lie algebra, where the $\delta_L(X)$ generates the left-hand and $\delta_R(X)$ the right-hand $\mathfrak{so}(6)$ respectively.

As we have showed previously, three-point functions of bosonic conserved currents are $\tilde{\delta}(X)$ invariant at leading order in the large c_T expansion. As a consequence, the higher-spin operators $H_\ell(\vec{x}, X)$ and $J_\ell(\vec{x}, X)$ will together form representations of $\mathfrak{so}(6)\times\mathfrak{so}(6)$. There are two possibilities. Either both operators transform in the adjoint of the same $\mathfrak{so}(6)$, or instead the operators split into left and right-handed operators

$$\begin{aligned} \mathcal{J}_\ell^L(\vec{x}, X) &= \quad \cos(\theta_\ell) H_\ell(\vec{x}, X) + \sin(\theta_\ell) J_\ell(\vec{x}, X)\,, \\ \mathcal{J}_\ell^R(\vec{x}, X) &= -\sin(\theta_\ell) H_\ell(\vec{x}, X) + \cos(\theta_\ell) J_\ell(\vec{x}, X)\,, \end{aligned} \tag{5.18}$$

with some mixing angle θ_ℓ, such that

$$\tilde{\delta}(X)\mathcal{J}_\ell^L(\vec{y}, Y) = \zeta_L \mathcal{J}_\ell^L(\vec{y}, [X,Y])\,, \qquad \tilde{\delta}(X)\mathcal{J}_\ell^R(\vec{y}, Y) = \zeta_R \mathcal{J}_\ell^R(\vec{y}, [X,Y])\,. \tag{5.19}$$

As we shall see in the next section, it is this latter possibility which is actually realized in all theories for which $\lambda_{SSB_0} \neq 0$.

5.1.3 Pseudocharge Action on Scalars

So far we have been avoiding the scalars S and P. Because the $H_1^\mu(\vec{x}, X)$ eats a bilinear of S and P, correlators involving these scalars are not automatically conserved at leading order, and so we can not assign these operators well-defined $\mathfrak{so}(6) \times \mathfrak{so}(6)$ transformation properties. The action of $\tilde{\delta}(X)$ is, however, still fixed by the delta function appearing in the three-point functions

$$\langle S \mathcal{O} \nabla \cdot H_1 \rangle \qquad \text{and} \qquad \langle S \hat{\mathcal{O}} \nabla \cdot H_1 \rangle$$

when $\mathcal{O}$ and $\hat{\mathcal{O}}$ are scalars of dimension 1 and 2 respectively. Let us now work through the possibilities, beginning with $\tilde{\delta}(X) S(\vec{y}, Y)$.

To fix $\tilde{\delta}(X)S(\vec{y},Y)$, we must evaluate

$$\langle\tilde{\delta}(X)S(0,Y)\mathcal{O}(\hat{e}_3)\rangle = \frac{1}{4\pi}\int_{|x|=r} dS\cdot\langle H_1(\vec{x},X)S(0,Y)\mathcal{O}(\hat{e}_3)\rangle\Big|_{\text{finite as } r\to\infty} \tag{5.20}$$

for general operators $\mathcal{O}(\hat{e}_3)$ located at $\hat{e}_3 = (0,0,1)$. We first note that the right-hand side of (5.20) is only non-zero if $\mathcal{O}$ is a scalar with conformal dimension 1. For this special case, conformal invariance implies that

$$\begin{aligned}\langle H_1(\vec{x},X)S(0,Y)\mathcal{O}(\hat{e}_3)\rangle &= f_{S\mathcal{O}H_1}(X,Y)\mathcal{C}^{\mu}_{1,1,1}(0,\hat{e}_3,\vec{x})\\ &= \frac{f_{S\mathcal{O}H_1}(X,Y)}{4}\left(\frac{(\hat{e}_3-x)^\mu}{|\hat{e}_3-x|^2}-\frac{x^\mu}{|x|^2}\right)\frac{1}{|x||x-\hat{e}_3|}\,,\end{aligned} \tag{5.21}$$

where $f_{S\mathcal{O}H_1}(X,Y)$ is a function of X and Y whose exact form depends on the $\mathfrak{so}(6)_R$ properties of $\mathcal{O}$. Substituting this into (5.20), we find that

$$\langle\tilde{\delta}(X)S(0,Y)\mathcal{O}(\hat{e}_3)\rangle = -\frac{1}{4}f_{S\mathcal{O}H_1}(X,Y)\,. \tag{5.22}$$

The only two dimension 1 scalars in higher-spin $\mathcal{N}=6$ theories are $S(\vec{y},Y)$ itself and $B_0(\vec{x})$. For S, we apply (5.22) with $\mathcal{O}(\hat{e}_3) = S(\hat{e}_3,Z)$ to find that[8]

$$\langle\tilde{\delta}(X)S(0,Y)S(\hat{e}_3,Z)\rangle = -\frac{\lambda_{SSH_1}\text{tr}([Y,Z]X)}{4} = -\frac{\lambda_{SSH_1}}{4}\langle S(0,[X,Y])S(\hat{e}_3,Z)\rangle\,, \tag{5.23}$$

while for B_0 we find that

$$\langle\tilde{\delta}(X)S(0,Y)B_0(\hat{e}_3)\rangle = -\frac{\lambda_{SB_0H_1}\text{tr}(XY)}{4} = -\frac{\lambda_{SB_0H_1}\text{tr}(XY)}{4}\langle B_0(0)B_0(\hat{e}_3)\rangle\,. \tag{5.24}$$

However, as we will now show, $\lambda_{SB_0H_1} = 0$. To see this, we compute:

$$\tilde{\delta}(X)\langle S(0,Y)SH_1\rangle = \langle\tilde{\delta}(X)S(0,Y)SH_1\rangle + \cdots = -\frac{\lambda_{SB_0H_1}\text{tr}(XY)}{4}\langle B_0(0)S_0H_1\rangle + \dots\,, \tag{5.25}$$

where the additional terms come from the variations of the second S and H_1, and from the multiplet recombination. Note that $\tilde{\delta}(X)\langle S(0,Y)SH_1\rangle$ contains a term proportional to $\lambda^2_{SB_0H_1}\text{tr}(XY)$. But it is straightforward to check that no additional term appears in either $\left\langle S(0,Y)\tilde{\delta}(X)\,(SH_1)\right\rangle$ or in $\langle S\tilde{P}H_1\rangle$ with the right R-symmetry structure needed to cancel such a contribution, and so conclude

[8]Throughout this section and the next, we will abuse notation slightly and use $\lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ to refer to the leading large c_T behavior of the OPE coefficient, which for three single trace operators scales as $c_T^{-1/2}$.

that $\lambda_{SB_0H_1} = 0$. Having exhausted the possible operators that could appear in $\tilde{\delta}(X)S(y,Y)$, we conclude that

$$\tilde{\delta}(X)S(\vec{y},Y) = -\frac{\lambda_{SSH_1}}{4}S(y,[X,Y]) = -\frac{\lambda_{SSB_0}}{4}S(y,[X,Y])\,. \tag{5.26}$$

We can constrain $\tilde{\delta}(X)P(\vec{y},Y)$ in much the same way, except now we have to consider not only the single trace operators P and C_0, but also double-trace operators built from S and B_0, and in particular

$$\begin{aligned} S^2(\vec{y},Y) &\equiv Y^a{}_b\left(S^b{}_c(\vec{y})S^c{}_a(\vec{y}) - \frac{1}{4}\delta^a{}_bS^c{}_d(\vec{y})S^d{}_c(\vec{y})\right)\,, \\ SB_0(\vec{y},Y) &\equiv Y^a{}_bS^b{}_a(\vec{y})B_0(\vec{y})\,. \end{aligned} \tag{5.27}$$

The most general expression we can write is

$$\begin{aligned} \tilde{\delta}(X)P(\vec{y},Y) = \kappa_0P(\vec{y},[X,Y]) + \kappa_1S^2(\vec{y},[X,Y]) + \kappa_2SB_0(\vec{y},[X,Y]) \\ + \mu_1\mathrm{tr}(XY)\mathcal{O}_1(\vec{y}) + \mu_2\mathcal{O}_2(\vec{y},\{X,Y\}) + \mu_3S(\vec{y},X)S(y,Y)\,, \end{aligned} \tag{5.28}$$

where $\mathcal{O}_1(y)$ is some linear combination of $S^a{}_bS^b{}_a$ and B_0^2, and $\mathcal{O}_2$ is some linear combination of P, S^2 and SB_0. By computing $\langle\tilde{\delta}PP\rangle$ we find that

$$\kappa_0 = -\frac{1}{4}\lambda_{PPH_1} = \frac{1}{4}\lambda_{SSH_1}\,. \tag{5.29}$$

If we instead consider $\langle\tilde{\delta}P\mathcal{O}_i\rangle$, we find that μ_i are proportional to OPE coefficients $\lambda_{PP\mathcal{O}_i}$, but can then check that the $\tilde{\delta}\langle PPH_1\rangle$ Ward identity is satisfied if and only if $\mu_i = 0$.

Computing κ_1 and κ_2 will prove somewhat more involved. Let us begin with $\tilde{\delta}(X)\langle SSP\rangle$. As listed in equation (5.6), supersymmetry forces both $\langle SSP\rangle$ and $\langle SPP\rangle$ to vanish. Expanding the left-hand side of the higher-spin Ward identity, we thus find that

$$\tilde{\delta}(X)\langle SSP(y_3,Y_3)\rangle = \kappa_1\langle SSS^2(\vec{y}_3,[X,Y_3])\rangle\,, \tag{5.30}$$

while expanding the right-hand side we instead find that

$$\tilde{\delta}(X)\langle SSP(y_3,Y_3)\rangle = -\frac{\alpha}{\sqrt{c_T}}\langle SS\tilde{S}(\vec{y}_3,[X,Y_3])\rangle\,. \tag{5.31}$$

Equating these two expressions, we conclude that

$$\kappa_1 = \frac{\alpha \lambda_{SSS}}{4\sqrt{c_T}} \,. \tag{5.32}$$

The variation $\tilde{\delta}(X)\langle SB_0 P\rangle$ is a little trickier, as $\langle SPB_0\rangle$ does not vanish at $O(c_T^{-1/2})$. It is instead related by supersymmetry to the three-point function $\langle SPH_1\rangle$, so that

$$\frac{\lambda_{SPH_1}}{\lambda_{SPB_0}} = \sqrt{\frac{2(\Delta+1)}{\Delta}} \underset{\Delta\to 1}{\longrightarrow} 2 \,, \tag{5.33}$$

where Δ is the conformal dimension of B_0. We can then in turn relate λ_{SPH_1} to α using the multiplet recombination formula (5.10), and so find that

$$\lambda_{SPB_0} = -\frac{1}{2}\lambda_{SPH_1} = \frac{2\alpha}{\sqrt{c_T}} \,. \tag{5.34}$$

Now that we have computed λ_{SPB_0}, let us turn to $\tilde{\delta}\langle SPB_0\rangle$. Expanding this using (5.38), we find that

$$\begin{aligned} &\tilde{\delta}(X)\langle S(\vec{y}_1, Y_1)P(\vec{y}_2, Y_2)B_0(\vec{y}_3)\rangle \\ &\qquad = \frac{1}{2}\lambda_{SSB_0}\langle S(\vec{y}_1, Y_1)P(\vec{y}_2, [X, Y_2])B_0(\vec{y}_3)\rangle + \kappa_2\langle S(\vec{y}_1, Y_1)SB(\vec{y}_2, [X, Y_2])B_0(\vec{y}_3)\rangle \,. \end{aligned} \tag{5.35}$$

But if we instead use the multiplet recombination rule, we find that

$$\tilde{\delta}(X)\langle S(\vec{y}_1, Y_1)P(\vec{y}_2, Y_2)B_0(\vec{y}_3)\rangle = \frac{\alpha}{\sqrt{c_T}}\left\langle S(\vec{y}_1, Y_1)\tilde{S}(\vec{y}_2, [X, Y_2])B_0(\vec{y}_3)\right\rangle \,, \tag{5.36}$$

where we dropped $\langle \tilde{P}PB_0\rangle$ as it vanishes due to supersymmetry. Equating the two expressions and solving for κ_2, we conclude that

$$\kappa_2 = -\frac{\alpha\lambda_{SSB_0}}{4\sqrt{c_T}} \,. \tag{5.37}$$

Putting everything together, we conclude

$$\begin{aligned} \tilde{\delta}(X)S(\vec{y}, Y) &= -\frac{1}{4}\lambda_{SSB_0}S(\vec{y}, [X, Y]) \,, \\ \tilde{\delta}(X)P(\vec{y}, Y) &= \frac{1}{4}\lambda_{SSB_0}P(\vec{y}, [X, Y]) + \frac{\alpha\lambda_{SSS}}{4\sqrt{c_T}}S^2(\vec{y}, [X, Y]) + \frac{\alpha\lambda_{SSB_0}}{4\sqrt{c_T}}SB_0(\vec{y}, [X, Y]) \,, \end{aligned} \tag{5.38}$$

5.2 $\langle SSSS\rangle$ in the Higher-Spin Limit

We now derive the four-point function $\langle SSSS\rangle$, proceeding in two steps. First we use pseudocharge pseudoconservation to compute the three-point functions $\langle SS\mathcal{J}\rangle$ between two scalars S and a higher-spin conserved current $\mathcal{J}$. Combining these three-point functions with the Lorentzian inversion formula, we can then fix $\langle SSSS\rangle$ up to two free parameters. With this achieved, we then use the superconformal Ward identities to compute $\langle SSPP\rangle$ and $\langle PPPP\rangle$ from $\langle SSSS\rangle$.

5.2.1 Three-Point Functions

Let us begin by considering the three-point function of two scalars with a spin ℓ conserved current $\mathcal{O}^L_\ell(\vec{y},Y)$ transforming in the left-handed **15**, so that

$$\tilde{\delta}(X)\mathcal{O}^L_\ell(\vec{y},Y) = \zeta_L \mathcal{O}^L_\ell(\vec{y},[X,Y])\,, \tag{5.39}$$

and consider the weakly broken $\tilde{\delta}(X)$ Ward identity:

$$\begin{aligned}
&\tilde{\delta}(X)\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)\mathcal{O}^L_\ell(\vec{y}_3,Y_3)\rangle \\
&\quad= \frac{\alpha}{4\pi\sqrt{c_T}}\int d^3x\, X^a{}_b \langle S(\vec{y}_1,Y_1)S^b{}_c(\vec{x})\rangle\langle P^c{}_a(\vec{x})S(\vec{y}_2,Y_2)\mathcal{O}^L_\ell(\vec{y}_3,Y_3)\rangle \\
&\quad- \frac{\alpha}{4\pi\sqrt{c_T}}\int d^3x\, X^a{}_b \langle S(\vec{y}_1,Y_1)S^c{}_a(\vec{x})\rangle\langle P^b{}_c(\vec{x})S(\vec{y}_2,Y_2)\mathcal{O}^L_\ell(\vec{y}_3,Y_3)\rangle + 1\leftrightarrow 2\,.
\end{aligned} \tag{5.40}$$

Defining the operators $\tilde{S}(\vec{x},X)$ and $\tilde{P}(\vec{x},X)$ to be the "shadow transforms" [156] of $S(\vec{x},X)$ and $P(\vec{x},X)$ respectively:

$$\tilde{S}(\vec{x},X) = \frac{1}{4\pi}\int \frac{d^3z}{|\vec{x}-\vec{z}|^4}S(\vec{z},X)\,, \qquad \tilde{P}(\vec{x},X) = \frac{1}{4\pi}\int \frac{d^3z}{|\vec{x}-\vec{z}|^2}P(\vec{z},X)\,, \tag{5.41}$$

we can then rewrite the Ward identity as:

$$\begin{aligned}
&\tilde{\delta}(X)\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)\mathcal{O}^L_\ell(\vec{y}_3,Y_3)\rangle \\
&= \frac{\alpha}{\sqrt{c_T}}\Big(\langle \tilde{P}(\vec{y}_1,[X,Y_1])S(\vec{y}_2,Y_2)\mathcal{O}^L_\ell(\vec{y}_3,Y_3)\rangle + \langle S(\vec{y}_1,Y_1)\tilde{P}(\vec{y}_2,[X,Y_2])\mathcal{O}^L_\ell(\vec{y}_3,Y_3)\rangle\Big)\,.
\end{aligned} \tag{5.42}$$

Our task now is to expand correlators in this Ward identity in terms of conformal and R-symmetry covariant structures. Using (5.2), the left-hand side of (5.42) becomes

$$\begin{aligned}&\tilde{\delta}(X)\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)\mathcal{O}^L_\ell(\vec{y}_3,Y_3)\rangle\\&\qquad\qquad=\left(\zeta_L+\frac{\lambda_{SSH_1}}{4}\right)\lambda_{SS\mathcal{O}^L_\ell}\operatorname{tr}\left([Y_1,Y_2]_\pm[X,Y_3]\right)\mathcal{C}_{11\ell}(\vec{y}_i)\,,\end{aligned}\tag{5.43}$$

where $[Y_i,Y_j]_\pm$ is a commutator when ℓ is even, and anticommutator when ℓ is odd. To evaluate the right-hand side we first note that, using both conformal and R-symmetry invariance,

$$\begin{aligned}&\langle S(\vec{y}_1,Y_1)P(\vec{y}_2,Y_2)\mathcal{O}^L_\ell(\vec{y}_3,Y_3)\rangle\\&\qquad=\left(\lambda^+_{SP\mathcal{O}^L_\ell}\operatorname{tr}\left([Y_1,Y_2]_\pm Y_3\right)+\lambda^-_{SP\mathcal{O}^L_\ell}\operatorname{tr}\left([Y_1,Y_2]_\mp Y_3\right)\right)\mathcal{C}_{12\ell}(\vec{y}_i)\,,\end{aligned}\tag{5.44}$$

for some OPE coefficients $\lambda^\pm_{SP\mathcal{O}^L_\ell}$. Because the operator $\mathcal{O}^L_\ell(\vec{y},Y)$ is not conserved at finite c_T, this three-point function does not necessarily vanish at $O(c_T^{-1/2})$. We can then compute the shadow transform using the identity [157]

$$\begin{aligned}&\int\frac{d^3z}{|\vec{z}-\vec{x}_1|^{2\Delta_1-6}}\mathcal{C}^{\mu_1\dots\mu_\ell}_{\Delta_1,\Delta_2,\ell}(\vec{z}_1,\vec{x}_2,\vec{x}_3)\\&\qquad=\frac{\pi^{3/2}\Gamma\left(\Delta_1-\frac{3}{2}\right)\Gamma\left(\frac{\Delta_2-\Delta_1+2}{2}\right)\Gamma\left(\frac{4+2\ell-\Delta_1-\Delta_2}{2}\right)}{\Gamma\left(3-\Delta_1\right)\Gamma\left(\frac{\Delta_1+\Delta_2-1}{2}\right)\Gamma\left(\frac{2\ell+\Delta_1-\Delta_2+1}{2}\right)}\mathcal{C}^{\mu_1\dots\mu_\ell}_{3-\Delta_1,\Delta_2\ell}(\vec{x}_1,\vec{x}_2,\vec{x}_3)\,.\end{aligned}\tag{5.45}$$

Putting everything together, we conclude that

$$\lambda^+_{SP\mathcal{O}^L_\ell}=-\frac{(\lambda_{SSB_0}+4\zeta_L)\ell!}{\pi^{3/2}\Gamma(\ell+1/2)}\frac{\lambda_{SS\mathcal{O}^L_\ell}\sqrt{c_T}}{\alpha}\,,\qquad\lambda^-_{SP\mathcal{O}^L_\ell}=0\,.\tag{5.46}$$

So far we have considered the weakly broken Ward identity for the three-point function $\langle SS\mathcal{O}^L_\ell\rangle$, but it is straightforward to repeat this exercise with the variations

$$\begin{aligned}\tilde{\delta}\langle SP\mathcal{O}^L_\ell\rangle&=\frac{\alpha}{\sqrt{c_T}}\left(\langle\tilde{P}P\mathcal{O}^L_\ell\rangle-\langle S\tilde{S}\mathcal{O}^L_\ell\rangle\right)\,,\\\tilde{\delta}\langle PP\mathcal{O}^L_\ell\rangle&=-\frac{\alpha}{\sqrt{c_T}}\left(\langle\tilde{S}P\mathcal{O}^L_\ell\rangle+\langle P\tilde{S}\mathcal{O}^L_\ell\rangle\right)\,.\end{aligned}\tag{5.47}$$

Expanding each of these correlators and using (5.45), we find that

$$\lambda_{PP\mathcal{O}^L_\ell}=\left(\frac{4\zeta_L+\lambda_{SSB_0}}{4\zeta_L-\lambda_{SSB_0}}\right)\ell\lambda_{SS\mathcal{O}^L_\ell}\,,\qquad16\zeta_L^2-\lambda_{SSB_0}^2=\frac{2\alpha^2\pi^2}{c_T}\,.\tag{5.48}$$

Applying the same logic to a right-handed operator $\mathcal{O}_\ell^R(\vec{x}, X)$, we immediately see that

$$\lambda_{PP\mathcal{O}_\ell^R} = \left(\frac{4\zeta_R + \lambda_{SSB_0}}{4\zeta_R - \lambda_{SSB_0}}\right)\ell\lambda_{SS\mathcal{O}_\ell^R}\,, \qquad 16\zeta_R^2 - \lambda_{SSB_0}^2 = \frac{2\alpha^2\pi^2}{c_T}\,. \tag{5.49}$$

In particular, taking the last equations of (5.48) and (5.49) and combining them with (5.16), we find that

$$\zeta_L = -\zeta_R = \zeta\,. \tag{5.50}$$

As we saw in the previous section, ζ fixes the action of the R-symmetry charge $\delta(X)$, which, unlike $\tilde{\delta}(X)$, is exactly conserved in any $\mathcal{N} = 6$ theory. We can therefore relate it to the three-point function $\langle SSJ_1\rangle$, and thus to the OPE coefficient λ_{SSS}:

$$\zeta = -\frac{1}{4}\lambda_{SSJ_1} = -\frac{1}{4}\lambda_{SSS}\,. \tag{5.51}$$

We now apply (5.48) and (5.49) to the operators H_ℓ and J_ℓ. Recall that these operators either transform identically under $\mathfrak{so}(6)$, or they split into left-handed and right-handed operators. Let us begin with the possibility that they transform identically under $\mathfrak{so}(6) \times \mathfrak{so}(6)$, and assume without loss of generality that both are left-handed. Combining (5.48) with the superblocks (5.4) and (5.5), we find that

$$(-1)^\ell\lambda_{SSH_\ell} = \left(\frac{\lambda_{SSS} - \lambda_{SSB_0}}{\lambda_{SSS} + \lambda_{SSB_0}}\right)\lambda_{SSH_\ell}\,, \quad -(-1)^\ell\lambda_{SSJ_\ell} = \left(\frac{\lambda_{SSS} - \lambda_{SSB_0}}{\lambda_{SSS} + \lambda_{SSB_0}}\right)\lambda_{SSJ_\ell}\,. \tag{5.52}$$

Because $\lambda_{SSS} \neq 0$, the only way to satisfy these equations is if $\lambda_{SSB_0} = 0$. We know however that λ_{SSB_0} is nonzero for generic higher-spin $\mathcal{N} = 6$ CFTs, such as for ABJ theories, and in particular does not vanish in free field theory. We therefore conclude that is not possible for H_ℓ and J_ℓ to transform identically under $\mathfrak{so}(6) \times \mathfrak{so}(6)$.

We now turn to the second possibility, that H_ℓ and J_ℓ recombine into left and right-handed multiplets $\mathcal{J}_\ell^L$ and $\mathcal{J}_\ell^R$ under $\mathfrak{so}(6) \times \mathfrak{so}(6)$, satisfying

$$\lambda_{PP\mathcal{J}_\ell^L} = \left(\frac{\lambda_{SSS} - \lambda_{SSB_0}}{\lambda_{SSS} + \lambda_{SSB_0}}\right)\ell\lambda_{SS\mathcal{J}_\ell^L}\,, \qquad \lambda_{PP\mathcal{J}_\ell^R} = \left(\frac{\lambda_{SSS} + \lambda_{SSB_0}}{\lambda_{SSS} - \lambda_{SSB_0}}\right)\ell\lambda_{SS\mathcal{J}_\ell^R}\,. \tag{5.53}$$

We can then use the superconformal blocks (5.4) and (5.5) to find that

$$\lambda_{PPH_\ell} = (-1)^\ell\ell\lambda_{SSH_\ell}\,, \qquad \lambda_{PPJ_\ell} = (-1)^{\ell+1}\ell\lambda_{SSJ_\ell}\,, \tag{5.54}$$

and, from (5.18), we see that

$$\begin{aligned}\lambda_{SS\mathcal{J}_\ell^L} &= \lambda_{SSH_\ell}\cos\theta_\ell + \lambda_{SSJ_\ell}\sin\theta_\ell\,, & \lambda_{SS\mathcal{J}_\ell^R} &= -\lambda_{SSH_\ell}\sin\theta_\ell + \lambda_{SSJ_\ell}\cos\theta_\ell \\ \lambda_{PP\mathcal{J}_\ell^L} &= \lambda_{PPH_\ell}\cos\theta_\ell + \lambda_{PPJ_\ell}\sin\theta_\ell\,, & \lambda_{PP\mathcal{J}_\ell^R} &= -\lambda_{PPH_\ell}\sin\theta_\ell + \lambda_{PPJ_\ell}\cos\theta_\ell\,.\end{aligned} \tag{5.55}$$

Combining (5.53), (5.54), and (5.55) together, we have 8 equations which are linear in the 8 OPE coefficients. Generically, the only solution to these equations will be the trivial one where all of OPE coefficients vanish. However, if we set

$$\theta_\ell = \frac{\pi}{4} + \frac{n\pi}{2} \text{ for } n \in \mathbb{Z}\,, \tag{5.56}$$

then we find that the equations become degenerate, allowing non-trivial solutions. By suitably redefining the conserved currents $H_\ell \to -H_\ell$ we can always fix $n = 0$ so that $\lambda_{SSH_\ell} \geq 0$, and can then solve the equations to find that

$$\lambda_{SSH_\ell} = \begin{cases} \frac{\lambda_{SSS}}{\lambda_{SSB_0}}\lambda_{SSJ_\ell} & \ell \text{ is even} \\ \frac{\lambda_{SSB_0}}{\lambda_{SSS}}\lambda_{SSJ_\ell} & \ell \text{ is odd}\,. \end{cases} \tag{5.57}$$

To complete our derivation, we simply note that from the superblocks (5.4), (5.5) and (5.6) that

$$\lambda_{SSJ_{\ell+2}} = \lambda_{SSH_{\ell+1}}\,, \qquad \lambda_{SSJ_1} = \lambda_{SSS}\,, \qquad \lambda_{SSH_1} = \lambda_{SSB_0}\,, \tag{5.58}$$

so that

$$\lambda_{SSH_{\ell+1}} = \begin{cases} \lambda_{SSS} & \ell \text{ is even} \\ \lambda_{SSB_0} & \ell \text{ is odd}\,. \end{cases} \tag{5.59}$$

Let us now apply (5.59) to two special cases: free field theory and parity preserving theories. In free field theory the higher-spin currents remain conserved, so that $\alpha = 0$ and hence, using (5.48), we find that $\lambda_{SSB_0} = \lambda_{SSS}$. We conclude that each conserved current supermultiplet must contribute equally to $\langle SSSS\rangle$, which is precisely what we found in Section 2.4.

For parity preserving theories, supersymmetry requires that S is a scalar but that P is a pseudoscalar. As we see from (5.10), the operator H_1^μ eats a pseudoscalar, and so is also a pseudovector rather than a vector. Parity preservation then requires that $\lambda_{SSB_0} = 0$, and so we conclude that for parity preserving theories only conserved current supermultiplets with odd spin contribute to $\langle SSSS\rangle$. Note that this does not apply to free field theory (which is parity preserving), because H_1^μ

remains short.

5.2.2 Ansatz for $\langle SSSS\rangle$

In the previous section we showed that the OPE coefficient between two S operators and a conserved current is completely fixed by H_1^μ pseudo-conservation in terms of λ_{SSS} and λ_{SSB_0}. Our task now is to work out the implications of this for the $\langle SSSS\rangle$ four-point function.

As shown in [151] using the Lorentzian inversion formula [150], $\mathcal{S}^i(U,V)$ is fully fixed by its double discontinuity up to a finite number of contact interactions in AdS. More precisely, we can write:

$$\mathcal{S}^i(U,V) = \mathcal{S}^i_{\text{disc}}(U,V) + \frac{1}{c_T}\Big(\mathcal{S}^i_{\text{exchange}}(U,V) + \mathcal{S}^i_{\text{contact}}(U,V)\Big) + O(c_T^{-2})\,. \tag{5.60}$$

Here the disconnected correlator is defined as in (2.78). The $\mathcal{S}^i_{\text{exchange}}(U,V)$ term is any CFT correlator with the same single trace exchanges as $\mathcal{S}^i(U,V)$, and with good Regge limit behavior so that the Lorentzian inversion formula holds. Finally, $\mathcal{S}^i_{\text{contact}}(U,V)$ is a sum of contact interactions in AdS with at most six derivatives, which contribute to CFT data with spin two or less. We will focus on each of these last two contributions in turn.

Let us begin with the exchange term. In higher-spin $\mathcal{N}=6$ theories the only single trace operators are conserved currents, and their contributions to $\langle SSSS\rangle$ are fixed by the OPE coefficients computed in the previous section. We can write the superconformal blocks for the conserved currents in the original $\mathcal{B}^i$ basis of R-symmetry structures as

$$\begin{aligned}\mathfrak{G}^i_{\text{stress tensor}}(U,V) &= \mathfrak{g}^i_S(U,V) + \mathfrak{g}^i_{J_1}(U,V) + \mathfrak{g}^i_{T_2}(U,V)\,,\\ \mathfrak{G}^i_{\text{cons},\ell}(U,V) &= \mathfrak{g}^i_{B_\ell}(U,V) + \mathfrak{g}^i_{H_{\ell+1}}(U,V) + \mathfrak{g}^i_{J_{\ell+2}}(U,V) + \mathfrak{g}^i_{T_{\ell+3}}(U,V)\,,\end{aligned} \tag{5.61}$$

where we define

$$\begin{aligned}\mathfrak{g}^i_{B_\ell}(U,V) = \mathfrak{g}^i_{T_\ell}(U,V) &= g_{\ell+1,\ell}(U,V)\times\begin{cases}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\end{pmatrix} & \text{even } \ell\,,\\ 0 & \text{odd } \ell\,,\end{cases}\\ \mathfrak{g}^i_{B_\ell}(U,V) = \mathfrak{g}^i_{T_\ell}(U,V) &= g_{\ell+1,\ell}(U,V)\times\begin{cases}\begin{pmatrix}-1 & 0 & 0 & 0 & 1 & 1\end{pmatrix} & \text{even } \ell\,,\\ \begin{pmatrix}0 & 0 & 0 & 0 & 1 & -1\end{pmatrix} & \text{odd } \ell\,.\end{cases}\end{aligned} \tag{5.62}$$

These superconformal blocks can be derived by expanding each $\mathcal{S}_{\mathbf{r}}(U,V)$ as a sum of conformal blocks using the OPE coefficients (5.4), (5.5) and (5.6), and then using (2.17) to convert back to the basis $\mathcal{S}^i(U,V)$ of R-symmetry structures. We can now write

$$\begin{aligned} \frac{1}{c_T}\mathcal{S}^i_{\text{exchange}}(U,V) &= \lambda^2_{SSS}\left(\mathfrak{G}^i_{\text{stress tensor}}(U,V) + \sum_{\text{odd } \ell}\mathfrak{G}^i_{\text{cons},\ell}(U,V)\right) \\ &\quad + \lambda^2_{SSB_0}\sum_{\text{even } \ell}\mathfrak{G}^i_{\text{cons},\ell}(U,V) + \text{crossing} + \text{double trace terms}\,, \end{aligned} \tag{5.63}$$

where the double trace terms are some combination of contact terms required so that $\mathcal{S}^i_{\text{exchange}}(U,V)$ has good Regge behavior.

To make further progress, we note that

$$\frac{1}{4}\left(\mathfrak{G}^i_{\text{stress tensor}}(U,V) + \sum_{\ell}\mathfrak{G}^i_{\text{cons},\ell}(U,V) + \text{crossing}\right) = \mathcal{S}^i_{\text{free}}(U,V)\,, \tag{5.64}$$

where $\mathcal{S}^i_{\text{free}}(U,V)$ is the connected correlator in the $\mathcal{N}=6$ free field theory defined in (2.78). This equality can be verified using the superconformal block decomposition of $\langle SSSS\rangle$ in free field theory computed in Section 2.4. Because $\mathcal{S}^i_{\text{free}}(U,V)$ is a correlator in a unitary CFT, it is guaranteed to have the necessary Regge behavior required for the Lorentzian inversion formula.

Having derived an expression for the sum of odd and even conserved current superblocks, let us turn to the difference. Note that for $\ell > 0$, each contribution from B_ℓ, $J_{\ell+1}$, H_ℓ and $T_{\ell+1}$ appearing in an even superblock comes matched with contributions from T_ℓ, $J_{\ell+1}$, H_ℓ, and $B_{\ell+1}$ from an odd superblock. We thus find that if we take the difference between the odd and even blocks, the contributions from spinning operators will cancel, leaving us only with the scalar conformal blocks

$$\mathfrak{G}^i_{\text{stress tensor}}(U,V) + \sum_{\text{odd } \ell}\mathfrak{G}^i_{\text{cons},\ell}(U,V) - \sum_{\text{even } \ell}\mathfrak{G}^i_{\text{cons},\ell}(U,V) = \mathfrak{g}^i_S(U,V) - \mathfrak{g}^i_{B_0}(U,V)\,. \tag{5.65}$$

On their own, the difference of two conformal blocks does not have good Regge behavior. We can however replace these conformal blocks with scalar exchange diagrams in AdS. Such exchange diagrams do have good Regge behavior, and the only single trace operators that appears in their OPE have the same quantum numbers as the exchanged particle. Using the general scalar exchange diagram computed in [22], and inverting (2.17) to convert from the s-channel R -symmetry basis to $\mathcal{S}^i(U,V)$, we find that[9]

$$\begin{aligned} \mathcal{S}^1_{\text{scal}}(U,V) &= -\frac{2U}{\pi^{5/2}}\bar{D}_{1,1,\frac{1}{2},\frac{1}{2}}(U,V)\,, \\ \mathcal{S}^4_{\text{scal}}(U,V) &= \frac{U}{\pi^{5/2}}\left[\bar{D}_{\frac{1}{2},1,1,\frac{1}{2}}(U,V) + \bar{D}_{1,\frac{1}{2},1,\frac{1}{2}}(U,V)\right]\,, \end{aligned} \tag{5.66}$$

[9]Our conventions for $\bar{D}$-functions can be found in Appendix D .

which has been normalized so that the exchange of S itself contributes equally to $\mathcal{S}^i_{\text{free}}(U,V)$ and $\mathcal{S}^1_{\text{scal}}(U,V)$. Using (2.82) to eliminate λ^2_{SSS} in favor of c_T^{-1}, we arrive at our ansatz for the exchange contribution:

$$\frac{1}{c_T}\mathcal{S}^i_{\text{exchange}}(U,V) = \frac{1}{c_T}\left((16-a_1(\lambda))\mathcal{S}^i_{\text{free}}(U,V) + a_1(\lambda)\mathcal{S}^i_{\text{scal}}(U,V)\right)\,, \tag{5.67}$$

where $a_1(\lambda)$ is related to $\lambda^2_{SSB_0}$ by the equation

$$a_1(\lambda) = 8 - \frac{c_T\lambda^2_{SSB_0}}{8}\,. \tag{5.68}$$

Because $\lambda^2_{SSB_0}$ is always positive in unitary theories, $a_1(\lambda) \leq 8$.

Now that we have an expression for the exchange terms, let us now turn to the contact terms. As already noted, $\mathcal{S}^i_{\text{contact}}(U,V)$ must be a sum of contact Witten diagrams that contribute to CFT data of spin two or less, which in Mellin space requires the correlator to be a polynomial of degree at most two. Furthermore, because our theory is supersymmetric these contact Witten diagrams must also preserve $\mathcal{N}=6$ supersymmetry. In Chapter 4 we solved the task of computing all such Mellin amplitudes, and found that there was a unique solution, $M_3(s,t)$. Converting this Mellin amplitude to position space using equation (D.4), we find that

$$\begin{aligned}
\mathcal{S}^1_{\text{cont}}(U,V) &= 4UV\bar{D}_{2,2,3,1}(U,V)\,,\\
\mathcal{S}^4_{\text{cont}}(U,V) &= 4U\left(\bar{D}_{1,1,1,3}(U,V) - \frac{4}{3}\bar{D}_{1,1,2,2}(U,V)\right)\,.
\end{aligned} \tag{5.69}$$

Putting everything together, we arrive at our ansatz for $\langle SSSS\rangle$ in higher-spin $\mathcal{N}=6$ theories:

$$\begin{aligned}
\mathcal{S}^i(U,V) &= \mathcal{S}^i_{\text{disc}}(U,V)\\
&\quad + \frac{1}{c_T}\left((16-a_1(\lambda))\mathcal{S}^i_{\text{free}}(U,V) + a_1(\lambda)\mathcal{S}^i_{\text{scal}}(U,V) + a_2(\lambda)\mathcal{S}^i_{\text{cont}}(U,V)\right) + O(c_T^{-2})\,.
\end{aligned} \tag{5.70}$$

5.2.3 $\langle SSPP\rangle$ and $\langle PPPP\rangle$

Given our ansatz for $\langle SSSS\rangle$, we will now derive expressions for $\langle SSPP\rangle$ and $\langle PPPP\rangle$ using the superconformal Ward identities derived in Chapter 2. Applying these identities to the various terms in our ansatz (5.70), we find that for the disconnected term

$$\begin{aligned}
&\mathcal{R}^1_{\text{disc}}(U,V) = 1\,, \qquad \mathcal{R}^i_{\text{disc}}(U,V) = 0 \text{ for } i = 2\,,\ldots\,,6\,,\\
&\mathcal{P}^1_{\text{disc}}(U,V) = 1\,, \qquad \mathcal{P}^4_{\text{disc}}(U,V) = 0\,,
\end{aligned} \tag{5.71}$$

and for the free connected term

$$\mathcal{R}^i_{\text{free}}(U,V)=0\,,\qquad \mathcal{P}^1_{\text{free}}(U,V)=0\,,\qquad \mathcal{P}^4_{\text{free}}(U,V)=\frac{U^2(U-V-1)}{4V^{3/2}}\,, \tag{5.72}$$

both of which can be checked against their expressions in free field theory. For the scalar exchange term, we find

$$\begin{aligned}
&\mathcal{R}^1_{\text{scal}}(U,V)=0\,,\qquad \mathcal{R}^2_{\text{scal}}(U,V)=\frac{4U}{\pi^2}\,,\qquad \mathcal{R}^4_{\text{scal}}(U,V)=\frac{2U(U-V-1)}{\pi^2 V}\,,\\
&\mathcal{R}^5_{\text{scal}}(U,V)=-\frac{2U}{\pi^2 V}+\frac{U^2}{2\pi^{5/2}}\bar{D}_{1,2,\frac{1}{2},-\frac{1}{2}}(U,V)\,,\qquad \mathcal{P}^i_{\text{scal}}(U,V)=0\,,
\end{aligned} \tag{5.73}$$

where we use the $\bar{D}$ function relations in (D.5) to relate derivatives of $\bar{D}$ functions to each other. Finally, for the degree 2 contact term $\langle SSPP\rangle$ is given by

$$\begin{aligned}
\mathcal{R}^1_{\text{cont}}(U,V)&=-\frac{4U^2}{3}\left(4\bar{D}_{2,2,1,1}-6\bar{D}_{2,2,2,2}+15\bar{D}_{4,2,1,3}+15\bar{D}_{3,2,2,3}-30\bar{D}_{3,2,1,2}\right)\\
\mathcal{R}^2_{\text{cont}}(U,V)&=4U^2\left(\bar{D}_{3,2,2,3}-2\bar{D}_{2,2,2,2}\right)\,,\qquad \mathcal{R}^4_{\text{cont}}(U,V)=4U^2\left(5\bar{D}_{3,2,2,3}-2\bar{D}_{2,2,2,2}\right)\,,\\
\mathcal{R}^5_{\text{cont}}(U,V)&=8U^2\left(\frac{4}{3}\bar{D}_{2,2,1,1}-6(\bar{D}_{2,2,2,2}+\bar{D}_{3,2,1,2})+5\bar{D}_{3,2,2,3}+\frac{5}{2}(\bar{D}_{2,2,3,3}+\bar{D}_{4,2,1,3})\right)\,.
\end{aligned} \tag{5.74}$$

We will not need the equivalent expression for $\langle PPPP\rangle$.

5.3 $\langle SSSP\rangle$ in the Higher-Spin Limit

Our task in this section is to use the weakly broken higher-spin Ward identities to compute $\langle SSSP\rangle$. Recall from Section 2.1.4 that the correlator $\langle SSSP\rangle$ can be written in terms of six functions $\mathcal{T}^i(U,V)$, but, due to the crossing relations (2.14), we only need to specify $\mathcal{T}^1(U,V)$ and $\mathcal{T}^4(U,V)$ in order to complete fix $\langle SSSP\rangle$.

Acting with the pseudocharge $\tilde{\delta}(X)$ on $\langle SSSP\rangle$, we find that

$$\begin{aligned}
&\tilde{\delta}(X)\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)S(\vec{y}_3,Y_3)P(\vec{y}_4,Y_4)\rangle\\
&\quad=\frac{\alpha}{\sqrt{N}}\Bigg(\Big\langle\tilde{P}(\vec{y}_1,[X,Y_1])S(\vec{y}_2,Y_2)S(\vec{y}_3,Y_3)P(\vec{y}_4,Y_4)\Big\rangle+1\leftrightarrow 2+1\leftrightarrow 3\\
&\qquad\qquad-\Big\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)S(\vec{y}_3,Y_3)\tilde{S}(\vec{y}_4,[X,Y_4])\Big\rangle\Bigg)\,.
\end{aligned} \tag{5.75}$$

To expand the right-hand side of this identity, we define

$$\begin{aligned}\langle S(\vec{x}_1,X_1)S(\vec{x}_2,X_2)S(\vec{x}_3,X_3)\tilde{S}(\vec{x}_4,X_4)\rangle &= \frac{x_{13}}{x_{12}^2x_{34}^3x_{14}}\sum_{i=1}^{6}\tilde{\mathcal{S}}^i(U,V)\mathcal{B}_i\,,\\ \langle S(\vec{x}_1,X_1)S(\vec{x}_2,X_2)\tilde{P}(\vec{x}_3,X_3)P(\vec{x}_4,X_4)\rangle &= \frac{x_{13}}{x_{12}^2x_{34}^3x_{14}}\sum_{i=1}^{6}\tilde{\mathcal{R}}^i(U,V)\mathcal{B}_i\,,\end{aligned}\tag{5.76}$$

where $\tilde{\mathcal{S}}^i(U,V)$ and $\tilde{\mathcal{R}}^i(U,V)$ can be computed by taking the shadow transform of $\langle SSSS\rangle$ and $\langle SSPP\rangle$. To expand the left-hand side, we use (5.38) and $SO(6)_R$ invariance to write

$$\begin{aligned}&\tilde{\delta}(X)\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)S(\vec{y}_3,Y_3)P(\vec{y}_4,Y_4)\rangle\\ &\qquad= \frac{1}{2}\lambda_{SSB_0}\,\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)S(\vec{y}_3,Y_3)P(\vec{y}_4,[X,Y_4])\rangle\\ &\qquad+\kappa_1\left\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)S(\vec{y}_3,Y_3)S^2(\vec{y}_4,[X,Y_4])\right\rangle\\ &\qquad+\kappa_2\,\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)S(\vec{y}_3,Y_3)SB_0(\vec{y}_4,[X,Y_4])\rangle\ .\end{aligned}\tag{5.77}$$

The two double-trace terms can each be expanded at $O(c_T^{-3/2})$ as a product of a two-point and a three-point function, so that for instance

$$\begin{aligned}&\left\langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)S(\vec{y}_3,Y_3)SB_0(\vec{y}_4,Y_4)\right\rangle\\ &\quad= \langle S(\vec{y}_1,Y_1)S(\vec{y}_2,Y_2)B_0(\vec{y}_4)\rangle\,\langle S(\vec{y}_3,Y_3)S(\vec{y}_4,Y_4)\rangle + \text{ permutations } + O(c_T^{-3/2})\,.\end{aligned}\tag{5.78}$$

We can then solve (5.75) to find that it fully fixes $\langle SSSP\rangle$ in terms of $\langle SSSS\rangle$ and $\langle SSPP\rangle$:

$$\begin{aligned}\mathcal{T}^1(U,V) &= -\frac{2\alpha}{\lambda_{SSB_0}\sqrt{c_T}}\left(\tilde{\mathcal{R}}^1(U,V)+\tilde{\mathcal{S}}^1(U,V)-\frac{\lambda_{SSB_0}^2}{8}\sqrt{U}\right)\,,\\ \mathcal{T}^4(U,V) &= -\frac{2\alpha}{\lambda_{SSB_0}\sqrt{c_T}}\left(\tilde{\mathcal{R}}^4(U,V)+\tilde{\mathcal{S}}^4(U,V)+\frac{\lambda_{SSS}^2}{8}U\left(1+\frac{1}{\sqrt{V}}\right)\right)\,.\end{aligned}\tag{5.79}$$

To calculate $\mathcal{T}^i(U,V)$ for the various contributions to our $\langle SSSS\rangle$ ansatz, we must compute the shadow transforms of both $\langle SSSS\rangle$ and $\langle SSPP\rangle$, which, using (5.76) we can express in terms of functions

$$\begin{aligned}\tilde{\mathcal{S}}^i(U,V) &= \frac{x_{34}^3x_{14}}{x_{13}}\int\frac{d^3z}{4\pi|\vec{z}-\vec{x}_4|^4}\frac{\mathcal{S}^i\left(\frac{x_{12}^2|\vec{x}_3-z|^2}{x_{13}^2|\vec{x}_2-\vec{z}|^2},\frac{|\vec{x}_1-\vec{z}|^2x_{23}^2}{x_{13}^2|\vec{x}_2-\vec{z}|^2}\right)}{|\vec{x}_3-\vec{z}|^2}\,,\\ \tilde{\mathcal{R}}^i(U,V) &= \frac{x_{34}^3x_{14}}{x_{13}}\int\frac{d^3z}{4\pi|\vec{z}-\vec{x}_3|^2}\frac{\mathcal{P}^i\left(\frac{x_{12}^2|x_4-\vec{z}|^2}{|\vec{x}_1-\vec{z}|^2x_{24}^2},\frac{x_{14}^2|\vec{x}_2-\vec{z}|^2}{|\vec{x}_1-\vec{z}|^2x_{24}^2}\right)}{|\vec{x}_4-\vec{z}|^4}\,.\end{aligned}\tag{5.80}$$

Let us begin with the free connected term, for which

$$\mathcal{S}^1_{\text{free}}(U,V) = \mathcal{R}^i_{\text{free}}(U,V) = 0\,, \tag{5.81}$$

so that the only non-trivial computation is

$$\tilde{\mathcal{S}}^4_{\text{free}}(U,V) = \frac{x_{12}^2 x_{34}^3 x_{14}}{x_{13}^2 x_{23}} \int \frac{d^3 z}{4\pi|\vec{z}-\vec{x}_4|^4} \frac{1}{|\vec{x}_1-\vec{z}||\vec{x}_2-\vec{z}|}\,. \tag{5.82}$$

We can evaluate this integral using the star-triangle relation

$$\int \frac{d^3 z}{|\vec{x}_1-\vec{z}|^{2\Delta_1}|\vec{x}_2-\vec{z}|^{2\Delta_2}|\vec{x}_3-\vec{z}|^{2\Delta_3}} = \left(\prod_{i=1}^{3} \frac{\Gamma\left(\frac{3}{2}-\Delta_i\right)}{\Gamma(\Delta_i)}\right) \frac{\pi^{3/2}}{x_{12}^{d-2\Delta_3} x_{13}^{d-2\Delta_2} x_{23}^{d-2\Delta_1}}\,, \tag{5.83}$$

and so find that

$$\tilde{\mathcal{S}}^4_{\text{free}}(U,V) = -\frac{1}{2}\sqrt{\frac{U^3}{V}}\,. \tag{5.84}$$

Next we turn to the contact term. As shown in Appendix D, the shadow transform of a D-function is another D-function:

$$\int \frac{d^3 y}{|\vec{x}_4-\vec{y}|^{6-2r_4}} D_{r_1,r_2,r_3,r_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{y}) = \frac{\pi^{3/2}\Gamma\left(r_4-\frac{3}{2}\right)}{\Gamma(r_4)} D_{r_1,r_2,r_3,3-r_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4)\,. \tag{5.85}$$

When we write

$$\frac{\mathcal{S}^i_{\text{cont}}(U,V)}{x_{12}^2 x_{34}^2} \text{ and } \frac{\mathcal{R}^i_{\text{cont}}(U,V)}{x_{12}^2 x_{34}^4}$$

in terms of D-functions, the result is a sum of D-functions multiplied by rational functions of x_{ij}^2. Using the identity [158]

$$\begin{aligned} &4r_1 r_2 x_{12}^2 D_{r_1+1,r_2+1,r_3,r_4} - 4r_3 r_4 x_{34}^2 D_{r_1,r_2,r_3+1,r_4+1} \\ &\qquad = (r_1+r_2-r_3-r_4)(3-r_1-r_2-r_3-r_4) D_{r_1,r_2,r_3,r_4}\,. \end{aligned} \tag{5.86}$$

along with its crossings, we can always rearrange the integrands in (5.80) into a form such that we can apply (5.85) term by term.

Finally, we turn to the exchange term. To compute the shadow transform for this term, we use equation (D.3) to rewrite the $\bar{D}$ functions as conformal integrals

$$\begin{aligned}\tilde{\mathcal{S}}^1_{\text{scal}}(U,V) &= -\frac{2}{\pi^{5/2}}\frac{x_{34}^3 x_{14} x_{12}^2}{x_{13}^3}\int \frac{d^3z}{4\pi|\vec{z}-\vec{x}_4|^4}\frac{\bar{D}_{1,1,\frac{1}{2},\frac{1}{2}}\left(\frac{x_{12}^2|\vec{x}_3-\vec{z}|^2}{x_{13}^2|\vec{x}_2-\vec{z}|^2},\frac{|\vec{x}_1-\vec{z}|^2x_{23}^2}{x_{13}^2|\vec{x}_2-\vec{z}|^2}\right)}{|\vec{x}_2-\vec{z}|^2} \\ &= -\frac{1}{2\pi^4}\frac{x_{12}^2 x_{14} x_{34}^3}{x_{13}}\int\frac{d^3z\,d^3w}{|\vec{z}-\vec{x}_4|^4|\vec{w}-\vec{x}_1|^2|\vec{w}-\vec{x}_2|^2|\vec{w}-\vec{x}_3||\vec{w}-\vec{z}||\vec{z}-\vec{x}_3|}\,.\end{aligned} \tag{5.87}$$

Performing the integral over z using the star-triangle relation (5.83), we then find that the integral over w can also be performed using the star-triangle relation, and so

$$\tilde{\mathcal{S}}^1_{\text{scal}}(U,V) = \sqrt{U}\,. \tag{5.88}$$

We can evaluate $\tilde{\mathcal{S}}^4_{\text{scal}}$ in a similar fashion, finding that

$$\tilde{\mathcal{S}}^4_{\text{scal}}(U,V) = -\frac{1}{2}U\left(1+\frac{1}{\sqrt{V}}\right)\,. \tag{5.89}$$

Now we turn to computing $\tilde{\mathcal{R}}^i_{\text{scal}}(U,V)$. Because ultimately our goal is to compute $\langle SSSP\rangle$, we only need $\tilde{\mathcal{R}}^1_{\text{scal}}(U,V)$ and $\tilde{\mathcal{R}}^4_{\text{scal}}(U,V)$, as these suffice to compute $\tilde{\mathcal{T}}^1(U,V)$ and $\tilde{\mathcal{T}}^4(U,V)$. But $\mathcal{R}^1_{\text{scal}}(U,V)=0$, and $\tilde{\mathcal{R}}^4_{\text{scal}}(U,V)$ can be computed by using the star-triangle relation term by term, so that

$$\tilde{\mathcal{R}}^1(U,V)=0 \qquad \tilde{\mathcal{R}}^4 = \frac{(\sqrt{U}-\sqrt{V}-1)}{2\sqrt{V}}\,. \tag{5.90}$$

Having computed all of the needed shadow transforms, we can now write our final expression for $\langle SSSP\rangle$. Substituting our results into (5.79) and the using (5.48) and (5.68) to simplify the prefactor on the right-hand side of [10]

$$\frac{2\alpha}{\lambda_{SSB_0}\sqrt{c_T}} = \frac{1}{\pi}\sqrt{\frac{2a_1(\lambda)}{8-a_1(\lambda)}}\,, \tag{5.91}$$

we find that

$$\begin{aligned}\mathcal{T}^i(U,V) &= -\frac{1}{\pi}\sqrt{\frac{2a_1(\lambda)}{8-a_1(\lambda)}} \\ &\times\left((16-a_1(\lambda))\mathcal{T}^i_{\text{free}}(U,V)+a_1(\lambda)\mathcal{T}^i_{\text{scal}}(U,V)+a_2(\lambda)\mathcal{T}^i_{\text{cont}}(U,V)\right)+O(c_T^{-2})\,,\end{aligned} \tag{5.92}$$

[10] Because equation (5.48) gives an expression for α^2, it only fixes α up to an overall sign. Note that this sign is determined by the sign convention for P, such that by redefining $P\to -P$ we can always fix $\alpha\geq 0$. This choice turns out to also be consistent with our conventions in Chapter 3.

where we define

$$
\begin{aligned}
\mathcal{T}^1_{\text{free}}(U,V) &= -\sqrt{U}\,, \qquad \mathcal{T}^4_{\text{free}}(U,V) = -\frac{1}{2}\left(\sqrt{\frac{U^3}{V}} - U - \frac{U}{\sqrt{V}}\right)\,,\\
\mathcal{T}^1_{\text{scal}}(U,V) &= +\sqrt{U}\,, \qquad \mathcal{T}^4_{\text{scal}}(U,V) = \frac{1}{2}\left(\sqrt{\frac{U^3}{V}} - U - \frac{U}{\sqrt{V}}\right)\,,\\
\mathcal{T}^1_{\text{cont}}(U,V) &= \frac{8\pi^{1/2}}{3}UV\Big(2U\bar{D}_{2,3,2,2} + 2V\bar{D}_{1,3,3,2} - 2\bar{D}_{2,2,3,2} - 3\bar{D}_{1,2,2,2}\Big)\,,\\
\mathcal{T}^4_{\text{cont}}(U,V) &= -\frac{32\pi^{1/2}}{3}U^2\left(U\bar{D}_{3,3,1,2} - \bar{D}_{2,2,1,2}\right)\,.
\end{aligned}
\tag{5.93}
$$

It is not hard to check that each of these contributions individually satisfies the $\langle SSSP\rangle$ superconformal Ward identity (2.15).

We conclude by applying (5.93) to parity preserving theories, where $\langle SSSP\rangle$ must vanish. We see that this is possible only if $a_1 = 8$ and $a_2 = 0$, and so conclude that

$$
\mathcal{S}^i(U,V) = \mathcal{S}^i_{\text{disc}}(U,V) + \frac{8}{c_T}\left(\mathcal{S}^i_{\text{free}}(U,V) + \mathcal{S}^i_{\text{scal}}(U,V)\right) + O(c_T^{-2})\,, \tag{5.94}
$$

in such theories. In particular, we see that

$$
\lambda^2_{SSB_0} = \frac{8(8-a_1)}{c_T} = 0\,, \tag{5.95}
$$

just as argued at the end of Section 5.2.1.

5.4 Constraints from Localization

We will now fix the two unknown coefficients $a_1(\lambda)$ and $a_2(\lambda)$ in $\langle SSSS\rangle$ using supersymmetric localization. To do so we will first need to compute the integrated constraints for the functions $\mathcal{S}^i_{\text{free}}$, $\mathcal{S}^i_{\text{scal}}$, $\mathcal{S}^i_{\text{cont}}$ and their $\langle SSSP\rangle$ equivalents. We can then use the localization results of Chapter 3 to fix both parameters in the $U(N)_k \times U(N+M)_{-k}$ ABJ and $SO(2)_{2k} \times USp(2_2M)_{-k}$ OSp theories. Having fully determines the leading large c_T correction to $\langle SSSS\rangle$ in both theories, we decompose the result into superconformal blocks, allowing us to compute the leading corrections to certain OPE coefficients and conformal dimensions.

5.4.1 Integrating Higher-Spin Correlators

Let us begin by computing I_{++} and I_{+-} for the free connected scalar correlator $\mathcal{S}^i_{\text{free}}(U,V)$. We can immediately read of $\lambda^2_{(B,2)^{[022]}_{2,0}}$ from Table 2.6, and so

$$I_{++}[\mathcal{S}^i_{\text{free}}] = 2\lambda^2_{(B,2)^{[022]}_{2,0}}[\mathcal{S}^i_{\text{free}}] = 4\,, \tag{5.96}$$

while because $\mathcal{S}^1_{\text{free}}(U,V)=0$, using (3.42) we find

$$I_{+-}[\mathcal{S}^i_{\text{free}}] = 0\,. \tag{5.97}$$

Computing I_{odd} for $\mathcal{T}^i_{\text{free}}(U,V) = -\mathcal{T}^i_{\text{scal}}(U,V)$ is also straightforward. Combining (5.93) and (3.33), we can just directly compute

$$I_{\text{odd}}\left[\mathcal{T}^i_{\text{free}}\right] = -I_{\text{odd}}\left[\mathcal{T}^i_{\text{scal}}\right] = -4\pi\int dr\,d\theta\,\frac{\sin\theta}{r^2-2r\cos\theta+1} = -2\pi^3\,. \tag{5.98}$$

The rest of the integrated correlator computations are more tractable in Mellin space. We already computed I_{++} and I_{+-} for $\mathcal{S}^i_{\text{cont}}(U,V)$ in Section 4.5.2 by using its Mellin transform $M^i_3(s,t)$,

$$I_{++}[\mathcal{S}^i_{\text{cont}}] = I_{++}[M^i_3] = \frac{8}{3}\,,\qquad I_{+-}[\mathcal{S}^i_{\text{cont}}] = I_{+-}[M^i_3] = \frac{2}{3}\pi^2\,. \tag{5.99}$$

We will likewise find it convenient to convert $\mathcal{S}^i_{\text{scal}}(U,V)$ to Mellin space, where

$$M^1_{\text{scal}}(s,t) = -\frac{2\Gamma\left(\frac{1-s}{2}\right)}{\pi^{5/2}\Gamma\left(\frac{2-s}{2}\right)}\,,\qquad M^4_{\text{scal}}(s,t) = \frac{\Gamma\left(\frac{1-t}{2}\right)}{\pi^{5/2}\Gamma\left(\frac{2-t}{2}\right)} + \frac{\Gamma\left(\frac{1-u}{2}\right)}{\pi^{5/2}\Gamma\left(\frac{2-u}{2}\right)}\,. \tag{5.100}$$

To compute $I_{++}[M^i_{\text{scal}}]$ we simply note that

$$M_{\text{scal},\mathbf{84}}(s,t) = \frac{1}{8}(M^2_{\text{scal}}(s,t) + M^3_{\text{scal}}(s,t) + 2M^4_{\text{scal}}(s,t)) = 0\,, \tag{5.101}$$

and so from (4.105) we find that

$$I_{++}[\mathcal{S}^i_{\text{scal}}] = 0\,. \tag{5.102}$$

To compute $I_{+-}[\mathcal{S}^i_{\text{scal}}]$, we use (4.108) to compute

$$I_{+-}[M^i_{\text{scal}}] = -\frac{4}{\pi}\int\frac{dt\,du}{(4\pi i)^2}\frac{\sec\frac{\pi(t+u)}{2}}{(t-2)(u-2)}\Gamma\left(\frac{2-t}{2}\right)\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{2-u}{2}\right)\Gamma\left(\frac{u-1}{2}\right)\,. \tag{5.103}$$

Using `Mathematica`, we can evaluate this integral numerically to arbitrary high precision, and find that, to within the numerical error,

$$I_{+-}[\mathcal{S}^i_{\text{scal}}] = -\pi^2\,. \tag{5.104}$$

Finally, let us turn to $I_{\text{odd}}[\mathcal{T}^i_{\text{cont}}]$, which we also evaluate in Mellin space. Let us begin by defining Mellin amplitudes for $\langle SSSP\rangle$ through the equation

$$\begin{aligned}\mathcal{T}^i(U,V) = \int \frac{ds\,dt}{(4\pi i)^2} N^i(s,t) U^{s/2} V^{u/2-1}\\ \times \Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(\frac{3-t}{2}\right)\Gamma\left(1-\frac{u}{2}\right)\Gamma\left(\frac{3-u}{2}\right),\end{aligned} \tag{5.105}$$

where $u = 5 - s - t$. Just like for $\langle SSSS\rangle$, the s and t contours are defined to satisfy

$$\text{Re}(s) < 2\,, \quad \text{Re}(t) < 2\,, \quad \text{Re}(u) < 2\,, \tag{5.106}$$

which include all poles of the Gamma functions on one side or the other of the contour [135]. The crossing equations (2.14) imply that

$$\begin{aligned}&N^1(s,t) = N^1(s,u)\,, \qquad N^2(s,t) = N^1(t,s)\,, \qquad N^3(s,t) = N^1(u,t)\,,\\ &N^4(s,t) = N^4(s,u)\,, \qquad N^5(s,t) = N^4(t,s)\,, \qquad N^6(s,t) = N^4(u,t)\,.\end{aligned} \tag{5.107}$$

Using (5.105), we can rewrite $I_{\text{odd}}[\mathcal{T}^i]$ in terms of its Mellin transform $N^i(s,t)$. The integrals over r and θ become tractable, and so we find that

$$I_{\text{odd}}[N^i] = -8\pi^{9/2}\int \frac{ds\,dt}{(4\pi i)^2}\frac{N^i(s,t)\csc(\pi s)\csc(\pi t)\csc(\pi u)\left(\sin(\pi s)+\sin(\pi t)+\sin(\pi u)\right)}{(s-2)(s-3)}\,. \tag{5.108}$$

We can convert $\mathcal{T}^i_{\text{cont}}$ into Mellin space using (D.4):

$$N^1_{\text{cont}}(s,t) = -\frac{2\pi^{1/2}}{3}(t-2)(u-2)\,, \qquad N^4_{\text{cont}}(s,t) = -\frac{2\pi^{1/2}}{3}(s-2)^2\,. \tag{5.109}$$

and then compute

$$\begin{aligned}I_{\text{odd}}[N^i_{\text{cont}}] &= -\frac{64\pi^5}{3}\int \frac{ds\,dt}{(4\pi i)^2}\csc(\pi s)\csc(\pi t)\csc(\pi u)\left(\sin(\pi s)+\sin(\pi t)+\sin(\pi u)\right)\\ &= -64\pi^5\int\frac{ds\,dt}{(4\pi i)^2}\csc(\pi s)\csc(\pi t)\\ &= -4\pi^3\,.\end{aligned} \tag{5.110}$$

To summarize the results of this section, we have found that

$$\begin{aligned}
&I_{++}\left[\mathcal{S}^i_{\text{free}}\right]=4\,, && I_{+-}\left[\mathcal{S}^i_{\text{free}}\right]=0\,, && I_{\text{odd}}\left[\mathcal{T}^i_{\text{free}}\right]=-2\pi^3\\
&I_{++}\left[\mathcal{S}^i_{\text{scal}}\right]=0\,, && I_{+-}\left[\mathcal{S}^i_{\text{scal}}\right]=-\pi^2\,, && I_{\text{odd}}\left[\mathcal{T}^i_{\text{scal}}\right]=2\pi^3\\
&I_{++}\left[\mathcal{S}^i_{\text{cont}}\right]=\frac{8}{3}\,, && I_{+-}\left[\mathcal{S}^i_{\text{cont}}\right]=\frac{2}{3}\pi^2\,, && I_{\text{odd}}\left[\mathcal{T}^i_{\text{cont}}\right]=-4\pi^3
\end{aligned} \tag{5.111}$$

5.4.2 Applying the Constraints

We are now finally in a position to fully fix the coefficients $a_i(\lambda)$ in higher-spin ABJ theory. Let us begin with the parity even constraints. Combining the expressions in Section 3.1 with our ansatz for $\langle SSSS\rangle$ and the integrated correlators computed in the previous section, we find that

$$\begin{aligned}
\frac{1}{c_T}\left(32-2a_1(\lambda)+\frac{4}{3}a_2(\lambda)\right)&=\lambda^2_{(B,2)^{[022]}_{2,0}}\Big|_{1/c_T}\,,\\
-a_1(\lambda)+\frac{2}{3}a_2(\lambda)&=\lim_{c_T\to 0}\frac{2^{11}}{\pi^4 c_T}\left.\frac{\partial^4\log Z}{\partial m_+^2\partial m_-^2}\right|_{m_\pm=0}\,.
\end{aligned} \tag{5.112}$$

Note however that these equations are redundant, and in particular they imply that

$$\left.\frac{\partial^4\log Z}{\partial m_+^2\partial m_-^2}\right|_{m_\pm=0}=\frac{c_T\pi^4}{2^{11}}\left(16+c_T\left(2-\lambda^2_{(B,2)^{[022]}_{2,0}}\right)\right)+O(c_T^0)\,, \tag{5.113}$$

regardless of the values of $a_i(\lambda)$. Solving the parity even constraints (5.112) with either the localization results (3.85) and (3.87) for the $U(N)_k\times U(N+M)_{-k}$ theory or (3.106) and (3.109) for the $SO(2)_{2k}\times USp(2+2M)_{-k}$ theory (the localization results are identical at leading order in c_T^{-1}), we find that

$$a_2(\lambda)=\frac{3}{2}a_1(\lambda)+6\cos(2\lambda\pi)-6\,. \tag{5.114}$$

To solve for $a_1(\lambda)$, we now turn to the parity odd constraint

$$\sqrt{\frac{a_1(\lambda)}{8-a_1(\lambda)}}\,(8-a_1(\lambda)+a_2(\lambda))=\frac{2^{11}i}{\pi^4 c_T}\frac{\partial^4\log Z}{\partial^3 m_+\partial m_-}+O(c_T^{-1})\,, \tag{5.115}$$

which also follows from combining the results of Section 3.1 with the integrated correlators computed in the previous section. Substituting (5.114) into (5.115) and squaring both sides, we find that

$$\frac{a_1(\lambda)(a_1(\lambda)+12\cos(2\pi\lambda)+4)^2}{2(8-a_1(\lambda))}=32\sin^2(2\pi\lambda)\,, \tag{5.116}$$

which upon further rearrangement becomes the cubic equation

$$\big(a_1(\lambda) - 8\sin^2(\pi\lambda)\big)\left(a_1(\lambda)^2 + 4(5\cos(2\pi\lambda)+3)a_1(\lambda) + 256\cos^2(\pi\lambda)\right) . \tag{5.117}$$

This has three solutions for $a_1(\lambda)$. However, two of these solutions are not real for all $\lambda \in [0, \frac{1}{2}]$ and so we discard them as non-physically. We therefore conclude that

$$a_1(\lambda) = 8\sin^2(\pi\lambda)\,, \tag{5.118}$$

which in turn implies that

$$a_2(\lambda) = 0\,. \tag{5.119}$$

Substituting these values into our ansatz for $\langle SSSS\rangle$, we arrive at the expression

$$\mathcal{S}^i(U,V) = \mathcal{S}^i_{\text{disc}}(U,V) + \frac{8}{c_T}\Big[(2-\sin^2(\pi\lambda))\mathcal{S}^i_{\text{free}}(U,V) + \sin^2(\pi\lambda)\mathcal{S}^i_{\text{scal}}(U,V)\Big] + O(c_T^{-2}) \tag{5.120}$$

We can then also use our expressions for $\langle SSSP\rangle$, $\langle SSPP\rangle$ and $\langle PPPP\rangle$ computed in Sections 5.2.3 and 5.3 to find that

$$\begin{aligned}
\mathcal{T}^i(U,V) &= -\frac{8\sqrt{2}}{c_T\pi}\sin(2\pi\lambda)\mathcal{T}^i_{\text{free}}(U,V) + O(c_T^{-2})\,,\\
\mathcal{R}^i(U,V) &= \mathcal{R}^i_{\text{disc}}(U,V) + \frac{8}{c_T}\sin^2(\pi\lambda)\mathcal{R}^i_{\text{scal}}(U,V) + O(c_T^{-2})\,,\\
\mathcal{P}^i(U,V) &= \mathcal{P}^i_{\text{disc}}(U,V) + \frac{8}{c_T}(2-\sin^2(\pi\lambda))\mathcal{P}^i_{\text{free}}(U,V) + O(c_T^{-2})\,.
\end{aligned} \tag{5.121}$$

and so, as desired, we have computed the leading $1/c_T$ correction to each of these correlators in the higher-spin limit. Note that, when expressed in terms of λ and c_T our final results are identical for the ABJ and OSp theories, and are independent of N.

So far in this chapter we have focused on the large M expansion. We can, however, safely take $\lambda \sim \frac{M}{k} \to 0$ and so rearrange the large M expansion into the semiclassical large k expansion. Using our final expression (5.120) for $\langle SSSS\rangle$ at large M, along with expressions for c_T in terms of M, N and k computed in Chapter 3, we find that in the $U(N)_k \times U(N+M)_{-k}$ ABJ theory,

$$\begin{aligned}
\mathcal{S}^i(U,V) &= \mathcal{S}^i_{\text{disc}}(U,V) + \frac{1}{N(M+N)}\mathcal{S}^i_{\text{free}}(U,V) + O(k^{-2})\,,\\
\mathcal{T}^i(U,V) &= \frac{\sqrt{2}M}{N(M+N)k}\mathcal{T}^i_{\text{free}}(U,V) + O(k^{-2})\,.
\end{aligned} \tag{5.122}$$

and for the $SO(2)_{2k} \times USp(2+2M)_{-k}$ theory,

$$\begin{aligned} \mathcal{S}^i(U,V) &= \mathcal{S}^i_{\text{disc}}(U,V) + \frac{1}{2M+2}\mathcal{S}^i_{\text{free}}(U,V) + O(k^{-2})\,, \\ \mathcal{T}^i(U,V) &= \frac{1}{k\sqrt{2}}\mathcal{T}^i_{\text{free}}(U,V) + O(k^{-2})\,. \end{aligned} \tag{5.123}$$

5.4.3 Extracting CFT Data

Having computed the leading correction to $\langle SSSS\rangle$ at large M, we will now expand our answer in superblocks

$$\mathcal{S}_{\mathbf{r}}(U,V) = \sum_{I\in S\times S} \lambda_I^2 \mathfrak{G}_I^{\mathbf{r}}(U,V) \tag{5.124}$$

as defined in Section 2.3. At large c_T the CFT data takes the form

$$\lambda_I^2 = \lambda^2_{I,\text{disc}} + \frac{1}{c_T}\lambda^2_{I,\text{tree}} + O(c_T^{-2})\,, \qquad \Delta_I = \Delta_{I,\text{disc}} + \frac{1}{c_T}\Delta_{I,\text{tree}} + O(c_T^{-2})\,, \tag{5.125}$$

and so using (5.124) we find that

$$\begin{aligned} &\mathcal{S}_{\mathbf{r}}(U,V) \\ &= \sum_{I\in S\times S}\left[\lambda^2_{I,\text{disc}} + \frac{1}{c_T}\left(\lambda^2_{I,\text{tree}} + \lambda^2_{I,\text{disc}}\Delta_{I,\text{tree}}\partial_\Delta\right) + O(c_T^{-2})\right]\mathfrak{G}_I^{\mathbf{r}}(U,V)\Big|_{\Delta=\Delta_{\text{disc}}}\,. \end{aligned} \tag{5.126}$$

Comparing this general superblock expansion to the explicit correlator in (5.120), we can extract the CFT data at GFFT and tree-level by expanding both sides around $U\sim 0$ and $V\sim 1$. Expressions for the $U\sim 0$ and $V\sim 1$ expansion of the $\bar{D}$ functions in $\mathcal{S}_{\text{scal}}$ are given in Appendix D.

Note that there are two cases where we cannot extract tree-level CFT data from the tree-level correlator. If operators are degenerate at GFFT we cannot distinguish them at tree-level, and the coefficient of the logarithm will contain contributions from the anomalous dimensions of all the degenerate operators. We can lift this degeneracy either by computing other correlators at tree-level, or by computing $\langle SSSS\rangle$ at higher order in $1/c_T$.

The second case where the anomalous dimension cannot be computed occurs when an operator first appears at tree-level. In this case its tree-level anomalous dimension cannot be extracted from tree-level $\langle SSSS\rangle$ because $\lambda^2_{I,\text{GFFT}} = 0$, and so we would need to compute $\langle SSSS\rangle$ at 1-loop in order to extract the tree-level anomalous dimension.

We will now show the results of the CFT data extraction. For the semishort multiplets,[11] we find the squared OPE coefficients:

$$\begin{aligned}
\ell \geq 0 \text{ even}: \quad \lambda^2_{(A,+)^{[002]}_{\ell+5/2,\ell+1/2}} &= \frac{\pi(\ell+1)(\ell+2)\Gamma(\ell+2)^2}{2\Gamma\left(\ell+\frac{5}{2}\right)^2} \\
&+ \frac{8}{c_T}\left[(2-\sin^2(\pi\lambda))\frac{4^\ell(\ell+1)^5\Gamma\left(\frac{\ell+1}{2}\right)^4}{\pi(\ell+2)\Gamma\left(\ell+\frac{5}{2}\right)^2} + \sin^2(\pi\lambda)\mathbb{S}_{(A,+)^{[002]}_{\ell+5/2,\ell+1/2}}\right] + O(c_T^{-2})\,, \\
\ell \geq 0 \text{ even}: \quad \lambda^2_{(A,2)^{[011]}_{\ell+2,\ell}} &= \frac{\pi(\ell+2)\Gamma(\ell+1)\Gamma(\ell+3)}{(2\ell+3)\Gamma\left(\ell+\frac{3}{2}\right)^2} \\
&+ \frac{8}{c_T}\left[-(2-\sin^2(\pi\lambda))\frac{2^{2\ell+1}(2\ell+3)\Gamma\left(\frac{\ell+1}{2}\right)^2\Gamma\left(\frac{\ell+3}{2}\right)^2}{\pi\Gamma\left(\ell+\frac{5}{2}\right)^2} + \sin^2(\pi\lambda)\mathbb{S}_{(A,2)^{[011]}_{\ell+2,\ell}}\right] + O(c_T^{-2})\,, \\
\ell \geq 0 \text{ odd}: \quad \lambda^2_{(A,2)^{[011]}_{\ell+2,\ell}} &= \frac{\pi\Gamma(\ell+2)\Gamma(\ell+4)}{(2\ell^2+7\ell+6)\,\Gamma\left(\ell+\frac{3}{2}\right)^2} \\
&+ \frac{8}{c_T}\left[(2-\sin^2(\pi\lambda))\frac{2^{2\ell+1}(2\ell+3)\Gamma\left(\frac{\ell}{2}+1\right)^4}{\pi\Gamma\left(\ell+\frac{5}{2}\right)^2} + \sin^2(\pi\lambda)\mathbb{S}_{(A,2)^{[011]}_{\ell+2,\ell}}\right] + O(c_T^{-2})\,, \\
\ell \geq 0 \text{ even}: \quad \lambda^2_{(A,1)^{[100],2}_{\ell+7/2,\ell+3/2}} &= \frac{\pi\Gamma(\ell+3)\Gamma(\ell+5)}{\Gamma\left(\ell+\frac{5}{2}\right)\Gamma\left(\ell+\frac{9}{2}\right)} \\
&+ \frac{8}{c_T}\left[-(2-\sin^2(\pi\lambda))\frac{2^{2\ell+7}\Gamma\left(\frac{\ell+3}{2}\right)^2\Gamma\left(\frac{\ell+5}{2}\right)^2}{\pi\Gamma\left(\ell+\frac{5}{2}\right)\Gamma\left(\ell+\frac{9}{2}\right)} + \sin^2(\pi\lambda)\mathbb{S}_{(A,1)^{[100],2}_{\ell+7/2,\ell+3/2}}\right] + O(c_T^{-2})\,,
\end{aligned} \tag{5.127}$$

where the contributions $\mathbb{S}_I$ from the scalar exchange term $\mathcal{S}^i_{\text{scal}}$ are given in Table 5.2. Note that we did not include the result for $\lambda^2_{(A,1)^{[100],1}_{\ell+7/2,\ell+3/2}}$, since it cannot be unambiguously extracted from $\langle SSSS\rangle$ at $O(c_T^{-1})$ due to mixing with the single trace operators, as we discuss next.

For the long multiplets, we first consider the single trace approximately conserved current multiplets with superprimary B_ℓ, starting with $\ell = 0$. For generic λ when parity is not a symmetry, we expect this multiplet at $c_T \to \infty$ to contribute to both $n = 1, 2$ structures of the $\mathfrak{G}_{\text{Long}^{[000],n}_{\Delta,0}}$ superblock at unitarity $\Delta = 1$, where recall from (2.74) that we can formally identify $\mathfrak{G}_{\text{Long}^{[000],2}_{1,0}} = \mathfrak{G}_{(B,1)^{[200]}_{2,0}}$ and $\mathfrak{G}_{\text{Long}^{[000],1}_{1,0}} = \mathfrak{G}_{(A,\text{cons})^{[000]}_{1,0}}$. For each structure, we find the OPE coefficients

$$\begin{aligned}
\lambda^2_{\text{Long}^{[000],1}_{1,0}} &= \frac{64}{c_T}(1-\sin^2(\pi\lambda)) + O(c_T^{-2})\,, \\
\lambda^2_{\text{Long}^{[000],2}_{1,0}} &= \frac{4}{3} + O(c_T^{-1})\,,
\end{aligned} \tag{5.128}$$

where the $n = 2$ structure starts at $O(c_T^0)$ since $\mathfrak{G}_{\text{Long}^{[000],2}_{1,0}} = \mathfrak{G}_{(B,1)^{[200]}_{2,0}}$ appears in the GFFT, while the $n = 1$ starts at $O(c_T^{-1})$ since $\mathfrak{G}_{\text{Long}^{[000],1}_{1,0}} = \mathfrak{G}_{(A,\text{cons})^{[000]}_{1,0}}$ does not appear at GFFT. Note that

[11] We already computed the short multiplet $\lambda^2_{(B,2)^{[022]}_{2,0}}$ in Chapter 3 using supersymmetric localization.

$\ell =$	0	2	4	6
$\mathbb{S}_{(A,+)^{[002]}_{\ell+5/2,\ell+1/2}}$	$\frac{160}{27\pi^2}$	$\frac{11264}{1225\pi^2}$	$\frac{104071168}{10085229\pi^2}$	$\frac{60842573824}{5590245375\pi^2}$
$\mathbb{S}_{(A,2)^{[011]}_{\ell+2,\ell}}$	$-\frac{32}{\pi^2}$	$-\frac{2048}{175\pi^2}$	$-\frac{262144}{31185\pi^2}$	$-\frac{2046820352}{289864575\pi^2}$
$\mathbb{S}_{(A,2)^{[011]}_{\ell+3,\ell+1}}$	$-\frac{2048}{135\pi^2}$	$-\frac{65536}{3675\pi^2}$	$-\frac{1375731712}{72837765\pi^2}$	$-\frac{517543559168}{26609567985\pi^2}$
$\mathbb{S}_{(A,1)^{[100],2}_{\ell+7/2,\ell+3/2}}$	$-\frac{63488}{1575\pi^2}$	$-\frac{29229056}{694575\pi^2}$	$-\frac{318465114112}{7439857425\pi^2}$	$-\frac{49162474150166528}{1138027162647375\pi^2}$
$\mathbb{S}_{\mathrm{Long}^{[000],1}_{\ell+4,\ell+2}}$	$-\frac{2048(560\log 2-613)}{7875\pi^2}$	$-\frac{4194304(45045\log 2-46819)}{1158782625\pi^2}$	$-\frac{33554432(24504480\log 2-24789439)}{4780852381305\pi^2}$	$-\frac{17179869184(1629547920\log 2-1620985787)}{157730564742926175\pi^2}$
$\mathbb{S}_{\mathrm{Long}^{[000]}_{\ell+3,\ell+1}}$	$\frac{2048}{315\pi^2}$	$\frac{2490368}{848925\pi^2}$	$\frac{6677331968}{4463914455\pi^2}$	$\frac{820340901019648}{1138027162647375\pi^2}$
$\mathbb{S}_{\mathrm{Long}^{[000],2}_{\ell+5,\ell+2}}$	$\frac{16384(290521\log 2-166093)}{196101675\pi^2}$	$\frac{262144(8305440\log 2-4857781)}{60091156125\pi^2}$	$\frac{268435456(32580565875\log 2-19860644894)}{203408194351737375\pi^2}$	$\frac{34359738368(16345498091104\log 2-10295061309185)}{118661038436089179780075\pi^2}$
$\mathbb{S}_{\mathrm{Long}^{[000],3}_{\ell+5,\ell+2}}$	$\frac{827392}{56595\pi^2}$	$\frac{82168512512}{5150670525\pi^2}$	$\frac{597016157618176}{36046109224125\pi^2}$	$\frac{1017159024789149450 24}{6013025156384371125\pi^2}$

Table 5.2: Contribution $\mathbb{S}_I$ of the scalar exchange diagram $\mathcal{S}^i_{\mathrm{scal}}$ to various OPE coefficients squared. Note that the $\log 2$ terms for CFT data with nonzero tree-level anomalous dimensions come from the 4^Δ factor in the definition of our conformal block [159].

$\lambda^2_{\text{Long}^{[000],1}_{1,0}}$ is what we called λ_{SSB_0} from Section 5.1, and vanishes for the parity preserving $\lambda = 1/2$ theory as discussed before. For $N > 1$ we cannot unambiguously determine the $O(c_T^{-1})$ correction to $\lambda^2_{\text{Long}^{[000],2}_{1,0}}$ using just tree-level $\langle SSSS\rangle$,[12] since we cannot distinguish it from the correction to $\lambda^2_{(B,1)^{[200]}_{2,0}}$, which can be explicitly constructed in any $U(N)_{-k} \times U(N+M)_k$ theory with $N > 1$.[13] To unmix these degenerate operators, we would need to compute $\langle SSSS\rangle$ at $O(c_T^{-2})$, in which case the $O(c_T^{-1})$ correction to $\lambda^2_{\text{Long}^{[000],2}_{1,0}}$ will multiply the anomalous dimension, so that it can be unambiguously read off. The $O(c_T^{-1})$ anomalous dimension should be the same for either structure, but in practice we can only extract it from tree-level $\langle SSSS\rangle$ using the $\mathfrak{G}_{\text{Long}^{[000],2}_{1,0}}$ structure, because that is the only structure whose OPE coefficient is $O(c_T^0)$. From this structure we find

$$\Delta_{(0,2)} = 1 + \frac{128}{\pi^2 c_T} \sin^2(\pi\lambda) + O(c_T^{-2})\,. \tag{5.130}$$

For the $\mathfrak{G}_{\text{Long}^{[000],1}_{1,0}}$ structure, the tree-level anomalous dimension would first appear in $\langle SSSS\rangle$ at $O(c_T^{-2})$, since the leading order OPE coefficient starts at $O(c_T^{-1})$.

Next, we consider the single trace approximately conserved currents with superprimary B_ℓ and even $\ell > 0$. For generic λ parity is not a symmetry, and so we expect this multiplet at $c_T \to \infty$ to contribute to both $n = 1, 2$ structures of the $\mathfrak{G}_{\text{Long}^{[000],n}_{\Delta,\ell}}$ superblock at unitarity $\Delta = \ell + 1$, where recall from (2.74) that we can formally identify $\mathfrak{G}_{\text{Long}^{[000],2}_{\ell+1,\ell}} = \mathfrak{G}_{(A,1)^{[100],1}_{\ell+3/2,\ell-1/2}}$ and $\mathfrak{G}_{\text{Long}^{[000],1}_{\ell+1,\ell}} = \mathfrak{G}_{(A,\text{cons})^{[000]}_{\ell+1,\ell}}$. For each structure, we find the OPE coefficients

$$\begin{aligned}
\ell > 0 \text{ even}: &\quad \lambda^2_{\text{Long}^{[000],1}_{\ell+1,\ell}} = \frac{64}{c_T}(1 - \sin^2(\pi\lambda)) + O(c_T^{-2})\,,\\
\ell > 0 \text{ even}: &\quad \lambda^2_{\text{Long}^{[000],2}_{\ell+1,\ell}} = \frac{\pi(\ell+2)\Gamma(\ell+3)\Gamma(\ell+4)}{3\Gamma\left(\ell+\frac{5}{2}\right)\Gamma\left(\ell+\frac{9}{2}\right)} + O(c_T^{-1})\,,
\end{aligned} \tag{5.131}$$

where the $n = 2$ structure start at $O(c_T^0)$ since $\mathfrak{G}_{\text{Long}^{[000],2}_{\ell+1,\ell}} = \mathfrak{G}_{(A,1)^{[100],1}_{\ell+3/2,\ell-1/2}}$ appears in the GFFT, while the $n = 1$ starts at $O(c_T^{-1})$ since $\mathfrak{G}_{\text{Long}^{[000],1}_{\ell+1,\ell}} = \mathfrak{G}_{(A,\text{cons})^{[000]}_{\ell+1,\ell}}$ does not appear at GFFT. Note that $\lambda^2_{\text{Long}^{[000],1}_{\ell+1,\ell}}$ is what we called λ_{SSB_ℓ} from Section 5.1, and vanishes for the parity preserving $\lambda = 1/2$ theory as discussed before. We did not write the $O(c_T^{-1})$ correction to $\lambda^2_{\text{Long}^{[000],2}_{\ell+1,\ell}}$, since we cannot distinguish it from the correction to $\lambda^2_{(A,1)^{[100],1}_{\ell+7/2,\ell+3/2}}$ using just tree-level $\langle SSSS\rangle$. To unmix

[12] For $N = 1$, the unambiguous tree-level correction will then be

$$\lambda^2_{\text{Long}^{[000],2}_{1,0}} = \frac{4}{3} + \frac{8}{c_T}\left[-\frac{4}{3}(2 - \sin^2(\pi\lambda)) + \frac{32}{3\pi^2}\sin^2(\pi\lambda)\right] + O(c_T^{-2})\,. \tag{5.129}$$

[13] In particular, at GFFT one can construct two $(B,1)^{[200]}_{2,0}$ operators, one using adjoints of the $SU(N)$ gauge group factor and one using singlets. The latter $(B,1)^{[200]}_{2,0}$ is what is eaten by the conserved current at tree-level, while the former remains. For $N = 1$, there is of course no adjoint, which is why the extra $(B,1)^{[200]}_{2,0}$ does not exist.

these degenerate operators, we would need to compute $\langle SSSS\rangle$ at $O(c_T^{-2})$, in which case the $O(c_T^{-1})$ correction to $\lambda^2_{\text{Long}^{[000],2}_{\ell+1,\ell}}$ will multiply the anomalous dimension, so that it can be unambiguously read off. The $O(c_T^{-1})$ anomalous dimension should be the same for either structure, but in practice we can only extract it from tree-level $\langle SSSS\rangle$ using the $\mathfrak{G}_{\text{Long}^{[000],2}_{\ell+1,\ell}}$ structure, because that is the only structure which contributes at $O(c_T^0)$. From this structure we find

$$\ell > 0 \text{ even}: \quad \Delta_{(\ell,2)} = \ell + 1 + \frac{8\ell(2\ell+1)(2\ell+3)^2(2\ell+5)(2\ell+7)}{\pi^2(\ell+1)^2(\ell+2)^3(\ell+3)^2 c_T}\sin^2(\pi\lambda) + O(c_T^{-2})\,. \qquad (5.132)$$

For the $\text{Long}^{[000],1}_{\ell+1,\ell}$ structure, the tree-level anomalous dimension first contributes to $\langle SSSS\rangle$ at $O(c_T^{-2})$, as discussed above.

Finally, we consider the single trace approximately conserved current multiplets with superprimary B_ℓ for odd $\ell > 0$. In this case there is just a single structure, which from (2.74) is identified at unitarity with $\mathfrak{G}_{\text{Long}^{[000]}_{\ell+1,\ell}} = \mathfrak{G}_{(A,\text{cons})^{[000]}_{\ell+1,\ell}}$. We find the tree-level OPE coefficient

$$\ell > 0 \text{ odd}: \quad \lambda^2_{\text{Long}^{[000]}_{\ell+1,\ell}} = \frac{64}{c_T} + O(c_T^{-2})\,, \qquad (5.133)$$

which is what we called $\lambda_{SST_{\ell+3}}$ from Section 5.1, and does not depend on λ as discussed before. We would need to compute $\langle SSSS\rangle$ at $O(c_T^{-2})$ in order to extract the tree-level anomalous dimension.

We now move on to the double trace long multiplets. We will only consider the lowest twist in each sector, since higher twist double trace long multiplets are expected to be degenerate, so we cannot extract them from just $\langle SSSS\rangle$. For twist two, we find that only $\mathfrak{G}_{\text{Long}^{[000],1}_{\ell+2,\ell}}$ receives contributions for all even ℓ:

$$\begin{aligned}
\ell \geq 0 \text{ even}: \quad \Delta_{(\ell,1)} &= \ell + 2 - \frac{128(2\ell+3)(2\ell+5)}{\pi^2(\ell+1)(\ell+3)(\ell+4)c_T}\sin^2(\pi\lambda) + O(c_T^{-2})\,,\\
\lambda^2_{\text{Long}^{[000],1}_{\ell+2,\ell}} &= \frac{\pi(\ell+4)\Gamma(\ell+1)\Gamma(\ell+3)}{2\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+\frac{7}{2}\right)}\\
&\quad + \frac{8}{c_T}\left[-(2-\sin^2(\pi\lambda))\frac{4^{\ell+1}\ell\Gamma\left(\frac{\ell+1}{2}\right)^2\Gamma\left(\frac{\ell+3}{2}\right)^2}{\pi\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+\frac{7}{2}\right)} + \sin^2(\pi\lambda)\mathbb{S}_{\text{Long}^{[000],1}_{\ell+2,\ell}}\right] + O(c_T^{-2})\,.
\end{aligned} \qquad (5.134)$$

For odd ℓ at twist two, only $\mathfrak{G}_{\text{Long}^{[000]}_{\ell+2,\ell}}$ receives contributions:

$$\begin{aligned}
\ell > 0 \text{ odd}: \quad \Delta_{(\ell,1)} &= \ell + 2 + O(c_T^{-2})\,,\\
\lambda^2_{\text{Long}^{[000]}_{\ell+2,\ell}} &= \frac{\pi\ell\Gamma(\ell+2)\Gamma(\ell+4)}{2\Gamma\left(\ell+\frac{5}{2}\right)\Gamma\left(\ell+\frac{7}{2}\right)}\\
&\quad + \frac{8}{c_T}\left[(2-\sin^2(\pi\lambda))\frac{4^{\ell+2}\Gamma\left(\frac{\ell}{2}+1\right)^2\Gamma\left(\frac{\ell}{2}+2\right)^2}{\pi\Gamma\left(\ell+\frac{5}{2}\right)\Gamma\left(\ell+\frac{7}{2}\right)} + \sin^2(\pi\lambda)\mathbb{S}_{\text{Long}^{[000]}_{\ell+2,\ell}}\right] + O(c_T^{-2})\,,
\end{aligned} \tag{5.135}$$

where note that the tree-level corrections to the anomalous dimension vanish. For twist three, we find that both $\mathfrak{G}_{\text{Long}^{[000],2}_{\ell+3,\ell}}$ and $\mathfrak{G}_{\text{Long}^{[000],3}_{\ell+3,\ell}}$ receive contributions for all even ℓ, though only the former receives an anomalous dimension:

$$\begin{aligned}
\ell \geq 0 \text{ even}: \quad \Delta'_{(\ell,2)} &= \ell + 3 + \frac{128(2\ell+5)(2\ell(\ell+4)+5)}{\pi^2(\ell+1)(\ell+3)(\ell+4)(\ell+5)c_T}\sin^2(\pi\lambda) + O(c_T^{-2})\,,\\
\lambda^2_{\text{Long}^{[000],2}_{\ell+3,\ell}} &= \frac{\pi\Gamma(\ell+3)\Gamma(\ell+4)}{3(2\ell+3)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+\frac{9}{2}\right)}\\
&\quad + \frac{8}{c_T}\left[(2-\sin^2(\pi\lambda))\frac{4^{\ell+3}(\ell+2)\Gamma\left(\frac{\ell+1}{2}\right)\Gamma\left(\frac{\ell+3}{2}\right)\Gamma\left(\frac{\ell+5}{2}\right)^2}{3\pi(\ell+4)(2\ell+3)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+\frac{9}{2}\right)} + \sin^2(\pi\lambda)\mathbb{S}_{\text{Long}^{[000],2}_{\ell+3,\ell}}\right] + O(c_T^{-2})\,,\\
\lambda^2_{\text{Long}^{[000],3}_{\ell+3,\ell}} &= \frac{\pi\Gamma(\ell+2)\Gamma(\ell+5)}{(2\ell+3)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+\frac{9}{2}\right)}\\
&\quad + \frac{8}{c_T}\left[(2-\sin^2(\pi\lambda))\frac{4^{\ell+3}\Gamma\left(\frac{\ell+1}{2}\right)\Gamma\left(\frac{\ell+3}{2}\right)\Gamma\left(\frac{\ell+5}{2}\right)^2}{\pi(2\ell+3)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+\frac{9}{2}\right)} + \sin^2(\pi\lambda)\mathbb{S}_{\text{Long}^{[000],3}_{\ell+3,\ell}}\right] + O(c_T^{-2})\,,
\end{aligned} \tag{5.136}$$

where $\Delta'_{(\ell,2)}$ denotes that these are the second lowest dimension operators in their sector that we consider, after the single trace operators with twist one.

5.5 Discussion

Our main result in this chapter is the tree-level expression for $\langle SSSS\rangle$ in both $U(N)_k\times U(N+M)_{-k}$ and $SO(2)_{2k}\times USp(2+2M)_{-k}$ ABJ theory in the higher-spin large M, k limit as a function of finite λ. It is instructive to compare our $\mathcal{N}=6$ correlators $\langle SSSS\rangle$ in (5.120) and $\langle PPPP\rangle$ in (5.121) to the tree-level correlator of the scalar single trace quasibosonic $\mathcal{O}_{qb}$ and quasifermionic $\mathcal{O}_{qf}$ operators for non-supersymmetric vector models in [151]:

$$\langle \mathcal{O}_{qb}(\vec{x}_1)\mathcal{O}_{qb}(\vec{x}_2)\mathcal{O}_{qb}(\vec{x}_3)\mathcal{O}_{qb}(\vec{x}_4)\rangle = \frac{1}{x_{12}^2 x_{34}^2}\frac{8}{c_T}\Bigg[\sqrt{U}+\sqrt{\frac{U}{V}}+\frac{U}{\sqrt{V}} - \frac{2}{\pi^{\frac{5}{2}}}\sin^2\left(\frac{\pi\lambda_{qb}}{2}\right)\left(U\bar{D}_{1,1,\frac{1}{2},\frac{1}{2}}(U,V)+U\bar{D}_{1,1,\frac{1}{2},\frac{1}{2}}(V,U)+\bar{D}_{1,1,\frac{1}{2},\frac{1}{2}}(\frac{1}{U},\frac{V}{U})\right)\Bigg], \tag{5.137}$$

and

$$\langle \mathcal{O}_{qf}(\vec{x}_1)\mathcal{O}_{qf}(\vec{x}_2)\mathcal{O}_{qf}(\vec{x}_3)\mathcal{O}_{qf}(\vec{x}_4)\rangle = \frac{1}{x_{12}^4 x_{34}^4}\frac{2}{c_T}\left[\frac{U^2(U-V-1)}{V^{3/2}} - \sqrt{U}(U-V+1) - \frac{\sqrt{U}(U+V-1)}{V^{\frac{3}{2}}}\right]. \tag{5.138}$$

To facilitate comparison we converted the parameters in [151] to the notation of [160]:

$$\frac{1}{\tilde{N}} = \frac{2}{c_T}\,, \qquad \tilde{\lambda}_{qb} = \tan(\frac{\pi\lambda_{qb}}{2})\,. \tag{5.139}$$

The four-point functions are then those of a $U(N_{qb})_{k_{qb}}$ Chern-Simons matter theory with 't Hooft coupling

$$\lambda_{qb} \equiv \frac{N_{qb}}{k_{qb}} \tag{5.140}$$

and either a complex scalar for the quasibosonic case, or a complex fermion for the quasifermionic case. We should compare the quasibosonic case correlator to $\langle SSSS\rangle$, as both S and $\mathcal{O}_{qb}$ are scalars with $\Delta = 1$ at tree-level, and the quasifermionic correlator to $\langle PPPP\rangle$, as both P and $\mathcal{O}_{qf}$ are pseudoscalars with $\Delta = 2$ at tree-level.

The $\mathcal{N} = 6$ correlators are structurally similar to those of the quasibosonic and quasifermionic theories. In all of these cases, the contact terms allowed by the Lorentzian inversion formula vanish. For both the quasiboson and $\langle SSSS\rangle$, the tree-level correlator includes a free connected term and a scalar exchange term, while for the quasifermion and $\langle PPPP\rangle$ only a free connected term appears. For the $\mathcal{N} = 6$ theory both $\langle SSSS\rangle$ and $\langle PPPP\rangle$ depend on λ through $\sin^2(\pi\lambda)$, while in the nonsupersymetric case only the quasiboson depends on λ_{qb}, and with slightly different periodicity $\sin^2(\frac{\pi\lambda_{qb}}{2})$.[14]

Although both the quasiboson and $\langle SSSS\rangle$ correlators contain a scalar exchange term, their physical origin is quite different in each case. In the quasibosonic case it was shown in [155] that for spin ℓ single trace operators J_ℓ, all tree-level $\langle \mathcal{O}_{qb}\mathcal{O}_{qb}J_\ell\rangle$ were the same as the free theory except

[14] The factor of two discrepancy in the periodicity between the ABJ case and the non-supersymmetric case is discussed in Section 6.2 of [21].

for $J_0 \equiv \mathcal{O}_{qb}$, which depends on λ_{qb}. The scalar exchange then appears so as to compensate for the fact that tree-level $\langle \mathcal{O}_{qb}\mathcal{O}_{qb}\mathcal{O}_{qb}\rangle$ is not given by the free theory result. In the $\mathcal{N}=6$ case, we found that the tree-level three-point functions between two S's and a higher-spin multiplet are given by the free theory result only for odd ℓ, while for even ℓ they are all proportional to the same λ dependent coefficient. The contribution of the exchange diagrams for the even and odd spin single trace long multiplets, which at tree-level coincide with conserved supermultiplets, exactly canceled so that only the scalar exchange diagrams remains.

We showed that the contact terms allowed by the Lorentzian inversion formula for $\langle SSSS\rangle$ vanished by combining localization results with the $\langle SSSP\rangle$ four-point function computed using the weakly broken Ward identity. There is in fact a possible alternative argument that only uses $\mathcal{N}=6$ superconformal symmetry, and so would apply to any $\mathcal{N}=6$ higher-spin theory. $\mathcal{N}=6$ superconformal symmetry only allows a single contact term with four derivatives or fewer, which thus contributes to spin two or less as allowed by the large M Lorentzian inversion formula [150]. But as we saw in the previous chapter, $\mathcal{S}^i_{\text{cont}}$, which corresponds to the Mellin amplitude $M_3(s,t)$, should really be thought of as a six derivative contact term. The six derivatives contributions vanish for $\langle SSSS\rangle$ but would appear in correlators such as $\langle SSJJ\rangle$. Since six derivative contact terms generically contribute to spin three CFT data in correlators of non-identical operators [161], they would be disallowed by the Lorentzian inversion formula for correlators with spin [162], which would then disallow the putative four derivative $\langle SSSS\rangle$ contact term.

As further evidence for this, $\mathcal{S}^i_{\text{cont}}(U,V)$ contributes to a scalar long multiplet which contains a spin three descendant. This spin three descendant happens to not contribute to the $\langle SSSS\rangle$ superblock, but could well appear in the $\langle SSJJ\rangle$ superblock. It would be interesting to derive the superconformal Ward identity that explicitly relates $\langle SSJJ\rangle$ to $\langle SSSS\rangle$, so that we could verify this alternative argument for the vanishing of the contact term. Our tree-level result would then just be fixed in terms of a single free parameter, analogous to a recent argument in [151, 152] showing that the contact terms must vanish in non-supersymmetric theories due to the higher-spin Ward identities.

This chapter relied almost exclusively on the CFT side of the higher-spin AdS/CFT duality. This is mostly because supersymmetric higher-spin gravity is still poorly understood. The only known formulation so far is in terms of Vasiliev theory [21, 163–169], which is just a classical equation of motion with no known action, and so cannot be used to compute loops. Even on the classical level, it has been difficult to regularize the calculation of various correlation functions [170, 171]. Recently, a higher-spin action has been derived in [172] for the $O(N)$ free and critical vector models, which manifestly reproduces the correct CFT results to all orders in $1/N$. If this construction could be extended to $\mathcal{N} = 6$, then it is possible that the bulk dual of $\langle SSSS \rangle$ could be computed and the absence of contact terms understood from the bulk perspective.

Chapter 6

Numeric Conformal Bootstrap

Numerical bootstrap techniques provide one of the few general tools applicable to non-perturbative conformal field theories; for recent reviews see [8, 26, 173]. Bootstrap studies with extended supersymmetry have so far however only been performed for operators that belong to half-BPS supermultiplets [70, 76, 81, 174–191].[1] Consequently, general constraints on the space of such SCFTs have so far been explored only when these SCFTs preserve the maximal amount of supersymmetry in their respective dimensions, because only then does the stress tensor sit in a half-BPS multiplet. Our aim in this chapter is to perform a general study of $\mathcal{N} = 6$ SCFTs, where the stress tensor multiplet only a third-BPS.[2]

We begin in Section 6.1 by deriving the crossing equations for $\langle SSSS \rangle$, using the superconformal blocks computed in Chapter 2. We then study general numerical bounds on OPE coefficients and conformal dimensions in Section 6.2. In particular, we find that the $U(1)_{2M} \times U(1+2M)_{-2M}$ ABJ theory is very close to saturating the lower bounds on the short $\lambda^2_{(B,2)^{[022]}_{2,0}}$ OPE coefficient, and so we can derive a conjectural spectrum for this theory using the extremal functional method. In Section 6.3 we restrict our attention to specific ABJ theories by using supersymmetric localization results as input to the bootstrap, allowing us to find small islands in OPE space for these theories. We close in Section 6.4 with a short discussion.

[1] See, however, [192, 193].

[2] Half-BPS multiplets in 3d $\mathcal{N} = 6$ SCFTs have been studied in [194].

6.1 Crossing Equations

The crossing equations for $\langle SSSS\rangle$ are written in (2.8). For the s-channel superblock expansion the nontrivial constraint is the one given by crossing $(x_1, X_1) \leftrightarrow (x_3, X_3)$. In terms of the $\mathcal{S}_{\mathbf{r}}(U,V)$ basis in (2.17), we can write the crossing equations (2.8) using a 6-component vector

$$d^i(U,V) = \begin{pmatrix} 15F_{-,\mathbf{1}_s} + 80F_{-,\mathbf{45}_a\oplus\overline{\mathbf{45}}_a} + 64F_{-,\mathbf{84}_s} \\ F_{-,\mathbf{15}_a} + F_{-,\mathbf{45}_a\oplus\overline{\mathbf{45}}_a} - 4F_{-,\mathbf{84}_s} \\ 3F_{-,\mathbf{15}_s} - 12F_{-,\mathbf{45}_a\oplus\overline{\mathbf{45}}_a} + 16F_{-,\mathbf{84}_s} \\ 3F_{-,\mathbf{20}'_s} - 2F_{-,\mathbf{45}_a\oplus\overline{\mathbf{45}}_a} + 2F_{-,\mathbf{84}_s} \\ 15F_{+,\mathbf{1}_s} - 15F_{+,\mathbf{15}_s} - 60F_{+,\mathbf{20}'_s} - 60F_{+,\mathbf{45}_a\oplus\overline{\mathbf{45}}_a} - 56F_{+,\mathbf{84}_s} \\ 3F_{+,\mathbf{15}_a} - 3F_{+,\mathbf{15}_s} - 9F_{+,\mathbf{20}'_s} - 3F_{+,\mathbf{45}_a\oplus\overline{\mathbf{45}}_a} + 14F_{+,\mathbf{84}_s} \end{pmatrix}, \tag{6.1}$$

where we define

$$F_{\pm,\mathbf{r}}(U,V) \equiv V^2\mathcal{S}_{\mathbf{r}}(U,V) \pm U^2\mathcal{S}_{\mathbf{r}}(V,U)\,. \tag{6.2}$$

Combining the crossing equations with the superconformal block decomposition derived in Chapter 2, we can then define a function $d^i_I(U,V)$ for each superconformal block I listed in Table 2.5 by replacing each $\mathcal{S}_{\mathbf{r}}(U,V)$ in $d^i(U,V)$ by $\mathfrak{G}^{\mathbf{r}}_I(U,V)$ as defined in (2.72). In terms of these $d^i_I(U,V)$, the crossing equations can now be written as

$$0 = d^i_{\text{Id}} + \frac{64}{c_T} d^i_{(B,2)^{[011]}_{1,0}} + \sum_{I\neq \text{Id},(B,2)^{[011]}_{1,0}} \lambda_I^2\, d^i_I\,, \tag{6.3}$$

where we normalize the squared OPE coefficient of the identity multiplet to $\lambda^2_{\text{Id}} = 1$, and parameterize our theories by the value of

$$\lambda^2_{(B,2)^{[011]}_{1,0}} = \frac{64}{c_T}$$

(see (2.82)). The sum in (6.3) should then be understood as running over all other superconformal blocks for multiplets appearing in the $S\times S$ OPE.

These six crossing equations are in fact redundant due to $\mathcal{N}=6$ superconformal symmetry, akin to the $\mathcal{N}=8$ case studied in [178, 179]. It is important to remove these redundancies, as otherwise they lead to numerical instabilities in the bootstrap algorithm. As in [179], we eliminate redundancies by rewriting the functions $F_{\pm,\mathbf{r}}(U,V)$ as sums of superconformal blocks. We then expand in $z, \bar{z}$ derivatives as

$$
\begin{aligned}
F_{+,\mathbf{r}}(U,V) &= \sum_{\substack{p+q=\text{even} \\ \text{s.t. } p\leq q}} \frac{2}{p!q!}\left(z-\frac{1}{2}\right)^p\left(\bar{z}-\frac{1}{2}\right)^q \partial_z^p\partial_{\bar{z}}^q F_{+,\mathbf{r}}(U,V)|_{z=\bar{z}=\frac{1}{2}}\,, \\
F_{-,\mathbf{r}}(U,V) &= \sum_{\substack{p+q=\text{odd} \\ \text{s.t. } p<q}} \frac{2}{p!q!}\left(z-\frac{1}{2}\right)^p\left(\bar{z}-\frac{1}{2}\right)^q \partial_z^p\partial_{\bar{z}}^q F_{-,\mathbf{r}}(U,V)|_{z=\bar{z}=\frac{1}{2}}\,,
\end{aligned}
\tag{6.4}
$$

where $z, \bar{z}$ are written in terms of U, V as

$$
U = z\bar{z}\,, \qquad V = (1-z)(1-\bar{z})\,. \tag{6.5}
$$

We then truncate these sums to a finite number of terms by imposing that

$$
p + q \leq \Lambda\,, \tag{6.6}
$$

and then consider the finite dimensional matrix $\widetilde{d}_i^{(p,q)}$ whose rows as labeled by $i = 1, \ldots 6$ are those of d^i, and whose columns as labeled by (p, q) are the coefficients of the $\partial_z^p\partial_{\bar{z}}^q \mathcal{S}_{\mathbf{r}}(U,V)|_{z=\bar{z}=\frac{1}{2}}$ that appear in each entry of d^i after expanding like (6.4) using the definition (6.2) of $F_{\pm,\mathbf{r}}(U,V)$ in terms of $\mathcal{S}_{\mathbf{r}}(U,V)$. Finally, we check numerically to see which crossing equations are linearly independent for each value of Λ, and find that a linearly independent subspace for any Λ is given by

$$
\{d^3\,, d^4\,, d^5\,, d^6\}\,, \tag{6.7}
$$

where we include all nonzero $z, \bar{z}$ derivatives for the crossing equations listed.[3]

We now have all the ingredients to perform the numerical bootstrap using the crossing equations (6.3), where we restrict to the linearly independent set of crossing equations (6.7). We can derive numerical bounds on both OPE coefficients and conformal dimensions using numerical algorithms that are by now standard (see for instance [179,195]) and can be implemented using `SDPB` [196,197]. In each case, the numerical algorithms involve finding functionals α that act on the vector of functions $d^i(U,V)$ and return a linear combination of derivatives of these functions evaluated at the crossing-symmetric point $U = V = 1/4$. In all the numerical studies presented below, we will restrict the total derivative order Λ defined in (6.6) to be $\Lambda = 39$, and we will only consider acting with α on blocks that have spin up to $\ell_{\max} = 50$.

[3] In the analogous $\mathcal{N} = 8$ case studied in [178], the linearly independent set consisted of just one crossing equation with all of its derivatives, as well as a second crossing equation with only derivatives in z.

6.2 Numeric Bootstrap Bounds

6.2.1 Short OPE Coefficients

We begin by deriving numerical bootstrap bounds on the squared OPE coefficients $\lambda^2_{(B,2)^{[011]}_{1,0}} = \frac{64}{c_T}$ and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ that were computed using supersymmetric localization in specific $\mathcal{N}=6$ SCFTs from the ABJ family in the Chapter 3.

We first derive a lower bound on c_T that applies to all $\mathcal{N}=6$ SCFTs. To do so, we consider linear functionals α satisfying

$$\begin{aligned} &\alpha(d^i_{(B,2)^{[011]}_{1,0}}) = 1\,, \\ &\alpha(d^i_I) \geq 0\,, \qquad\qquad \text{for all superconf. blocks } I \notin \{\text{Id}, (B,2)^{[011]}_{1,0}\}\,. \end{aligned} \tag{6.8}$$

From (6.3), the existence of such an α implies

$$\frac{64}{c_T} \leq -\alpha(d^i_{\text{Id}})\,. \tag{6.9}$$

We performed such a numerical study, and we found

$$c_T \geq 15.5\,, \tag{6.10}$$

where recall that $c_T = 16$ corresponds to the theory of a free $\mathcal{N}=6$ massless hypermultiplet, which also has $\mathcal{N}=8$ SUSY. The bound (6.10) can be compared to the analogous $\mathcal{N}=8$ bound $c_T \geq 15.9$ computed in [81] with $\Lambda = 43$. In both cases, we expect the numerics should converge to $c_T \geq 16$ in the infinite Λ limit, as there are no known $\mathcal{N}=6$ SCFTs with c_T smaller than 16. The fact that the $\mathcal{N}=6$ bound (6.10) is weaker than the $\mathcal{N}=8$ one suggests that the $\mathcal{N}=6$ numerics are slightly less converged than the $\mathcal{N}=8$ numerics. In the remainder of this thesis we will only show results for $c_T \geq 16$.

Let us now compute bounds on the squared OPE coefficient $\lambda^2_{(B,2)^{[022]}_{2,0}}$ as a function of c_T. In general, to computer upper and lower bounds on the OPE coefficient of an isolated superblock I^*

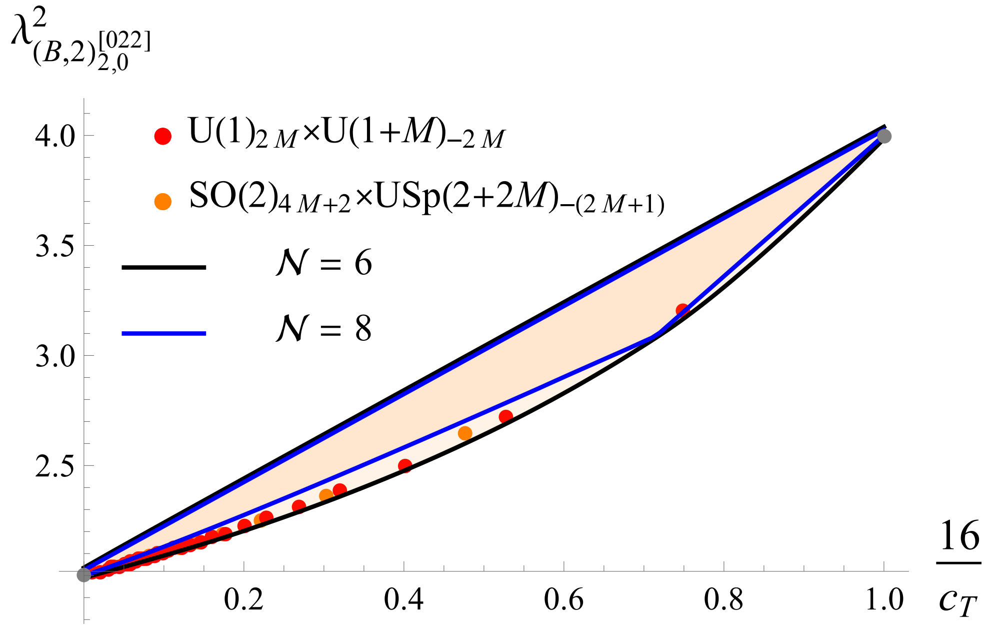

Figure 6.1: Upper and lower bounds on the $\lambda^2_{(B,2)^{[022]}_{2,0}}$ OPE coefficient in terms of the stress tensor coefficient c_T, where the orange shaded region is allowed, and the plot ranges from the generalized free field theory (GFFT) limit $c_T \to \infty$ to the free theory $c_T = 16$. The **black** lines denote the $\mathcal{N} = 6$ upper/lower bounds computed in this work with $\Lambda = 39$, the **blue** lines denotes the $\mathcal{N} = 8$ upper/lower bounds computed in [81] with $\Lambda = 43$. The **red** and **orange** dots denote the exact values in Tables 3.1 and 3.4 for the $U(1)_{2M} \times U(1+M)_{-2M}$ and $SO(2)_{4M+2} \times USp(2+2M)_{-(2M+1)}$ theories, respectively, for $M = 1, 2, \ldots$, while the gray dots denote the GFFT and free theory values from Table 2.6.

appearing in (6.3), we consider linear functionals α satisfying

$$
\begin{aligned}
&\alpha(d^i_{I^*}) = s\,, && s = 1 \text{ for upper bounds, } s = -1 \text{ for lower bounds}\,,\\
&\alpha(d^i_I) \geq 0\,, && \text{for all short and semi-short } I \notin \{\text{Id}, (B,2)^{[011]}_{1,0}, I^*\}\,,\\
&\alpha(d^i_I) \geq 0\,, && \text{for all long } I \text{ with } \Delta_I \geq \ell + 1\,.
\end{aligned}
\tag{6.11}
$$

The existence of such an α implies that

$$
\begin{aligned}
&\text{if } s = 1, \text{ then} && \lambda^2_{I^*} \leq -\alpha(d^i_{\text{Id}}) - \frac{64}{c_T}\alpha(d^i_{(B,2)^{[011]}_{1,0}})\,,\\
&\text{if } s = -1, \text{ then} && \lambda^2_{I^*} \geq \alpha(d^i_{\text{Id}}) + \frac{64}{c_T}\alpha(d^i_{(B,2)^{[011]}_{1,0}})\,,
\end{aligned}
\tag{6.12}
$$

thus giving us both an upper and a lower bound on $\lambda^2_{I^*}$. Using this procedure, our numerical study gives the upper and lower bounds shown in black in Figure 6.1. On the same plot, we indicated in blue the bounds obtained with $\Lambda = 43$ in the $\mathcal{N} = 8$ case, as derived in [81]. While the upper bounds for the $\mathcal{N} = 6$ and $\mathcal{N} = 8$ cases are very similar and likely differ only because of the different value of Λ that was used, the lower bounds are qualitatively different. Indeed, the $\mathcal{N}=6$ and $\mathcal{N}=8$ lower bounds meet at $\frac{16}{c_T}=0$, 1, and at around .71, where the $\mathcal{N}=8$ bound has a kink.[4] At other values of $\frac{16}{c_T}$, the $\mathcal{N}=6$ lower bound is significantly weaker than the $\mathcal{N}=8$ one.

In Chapter 3, we noted that at large c_T, the $U(1)_{2M}\times U(1+M)_{-2M}$ theory had the minimal value of $\lambda^2_{(B,2)^{[022]}_{2,0}}$ of all theories studied. In Figure 6.1 we hence plot the analytic value of $\lambda^2_{(B,2)^{[022]}_{2,0}}$ for various $U(1)_{2M}\times U(1+M)_{-2M}$ theories in red, and note that they appear to come very close to saturating the lower bound in Figure 6.1. For comparison, we also show the $SO(2)_{4M+2}\times USp(2M+2)_{-(2M+1)}$ theories in orange, which lie slightly above the $U(1)_{2M}\times U(1+M)_{-2M}$ dots. We hence conjecture that, in the infinite Λ limit, the $U(1)_{2M}\times U(1+M)_{-2M}$ theory saturates the numerical lower bound on $\lambda^2_{(B,2)^{[022]}_{2,0}}$.

At the boundary of the allowed and forbidden region, it is believed that there is a unique solution to the crossing equation and the CFT data can be extracted using the extremal functional method [198–200].[5] One application of this method has been to the 3d Ising model, which was argued to saturate the lower bound on the coefficient c_T which appears in the stress tensor two-point function [200, 202]. As we have just seen, it appears that the $U(1)_{2M}\times U(1+M)_{-2M}$ ABJ theory saturates the lower bound on $\lambda^2_{(B,2)^{[022]}_{2,0}}$, and so should be amenable to a precision bootstrap study. This is reminiscent of the $\mathcal{N}=8$ case in [81], where the $U(N)_2\times U(N+1)_{-2}$ theory was found to saturate the corresponding lower bound.

At large M, the $U(1)_{2M}\times U(1+M)_{-2M}$ ABJ theory has weakly broken higher-spin symmetry. We computed $\langle SSSS\rangle$ in the higher-spin limit in Chapter 5; the $U(1)_{2M}\times U(1+M)_{-2M}$ theory has $\lambda=\frac{1}{2}$ and so is parity preserving. Using the results of Section 5.4.3, we can thus compute the tree-level corrections to the $U(1)_{2M}\times U(1+M)_{-2M}$ theory, and will compare them to the extremal functional results in the next section.

6.2.2 Semishort OPE Coefficients

Let us now discuss upper and lower bounds on OPE coefficients for isolated superconformal blocks that appear in $\langle SSSS\rangle$. The isolated superconformal blocks are listed in Table 2.5. They consist of those superblocks which do not appear on the RHS of (2.74)–(2.76), and so are unable to recombine with other short multiplets to become long. This includes all semishort multiplets in Table 2.5 except for $(A,1)^{[100],n}_{\ell+2,\ell}$.

Using the algorithm presented in (6.11)–(6.12), we determined such bounds as shows in Figure 6.2.

[4] This $\mathcal{N}=8$ kink was previously observed in [76, 178].

[5] Ref. [201] showed that it is sometimes possible that there could be several extremal functionals, but in all cases that were studied they produced the same CFT spectrum.

In these plots, our $\Lambda = 39$ $\mathcal{N} = 6$ upper/lower bounds are shown in black, and they can be compared to the $\Lambda = 43$ $\mathcal{N} = 8$ bounds computed in [81], which in these figures are shown in blue. As in Figure 6.1 discussed above, in all these plots, the $\mathcal{N} = 6$ and $\mathcal{N} = 8$ lower bounds meet at around $\frac{16}{c_T} \sim .71$. Note that the $\mathcal{N} = 6$ upper/lower bounds do not converge at the GFFT and free theory points yet, whose exactly known values were listed in Table 2.6 and are denoted by gray dots, which is evidence that they are not fully converged. The exception is the bound on the OPE coefficient for $(A,+)^{[002]}_{\ell+2,\ell}$, which is our most constraining plot.

In addition to the upper and lower bounds, in Figure 6.2 we also plot in dashed red the values of the OPE coefficients as extracted from the extremal functional under the assumption that the lower bound of Figure 6.1 is saturated. As we can see, the extremal functional values for the OPE coefficients come close to saturating several of the bounds in this figure, but not all.

We further include on these plots the tree-level results for $U(1)_{2M} \times U(1+M)_{-2M}$ computed in Section 5.4.3, shown in green. For comparison, we also include the tree-level results in the supergravity limit as computed in [117, 203] are shown in orange. Recall that the supergravity results apply to the leading large c_T correction to both the M-theory and string theory limits. As first noted in [203] and visible in these plots, they match the large c_T regime of the $\mathcal{N} = 8$ lower bounds.[6] For $\mathcal{N} = 6$, we see in all these plots that the tree-level results approximately match the conjectured $U(1)_{2M} \times U(1+M)_{-2M}$ spectrum in the large c_T regime. Curiously, the conjectured spectrum approximately coincides with the $\mathcal{N} = 6$ lower bounds for $\lambda^2_{(A,+)^{[020]}_{\ell+5/2,\ell+1/2}}$ and $\lambda^2_{(A,2)^{[011]}_{\ell,\ell+2}}$ with odd ℓ, but not for $\lambda^2_{(A,2)^{[011]}_{\ell,\ell+2}}$ with even ℓ.[7]

The $\mathcal{N} = 6$ numerics are not completely converged yet, which can be seen from the fact that at $c_T \to \infty$ the numerics do not exactly match the GFFT value shown as a grey dot. On the other hand, it has been observed in many previous numerical bootstrap studies [76, 81, 174–176, 178] that the bounds change uniformly as precision is increased, so that the large c_T slope is still expected to be reasonably accurate, even if the intercept is slightly off. In Table 6.1, we compare the coefficient of the $1/c_T$ term as read off from the numerics at large c_T to the tree-level results, and find a good match for between the extremal functional and the analytic results. The match is especially good for the most protected quantities, which are the 1/4-BPS $\lambda^2_{(A,+)^{[020]}_{\ell+5/2,\ell+1/2}}$. In fact, this quantity is so constrained that it is difficult to distinguish by eye between the $\mathcal{N} = 8$ and $\mathcal{N} = 6$ numerical and analytical results in Figure 6.2. Nevertheless, the exact tree correction for supergravity and

[6]We have converted the $\mathcal{N} = 8$ results in [203] to $\mathcal{N} = 6$ using the superblock decomposition Appendix C.

[7]Recall that, as described in Table 2.5, the superblocks for $\lambda^2_{(A,2)^{[011]}_{\ell,\ell+2}}$ have completely different structures for even/odd values of ℓ.

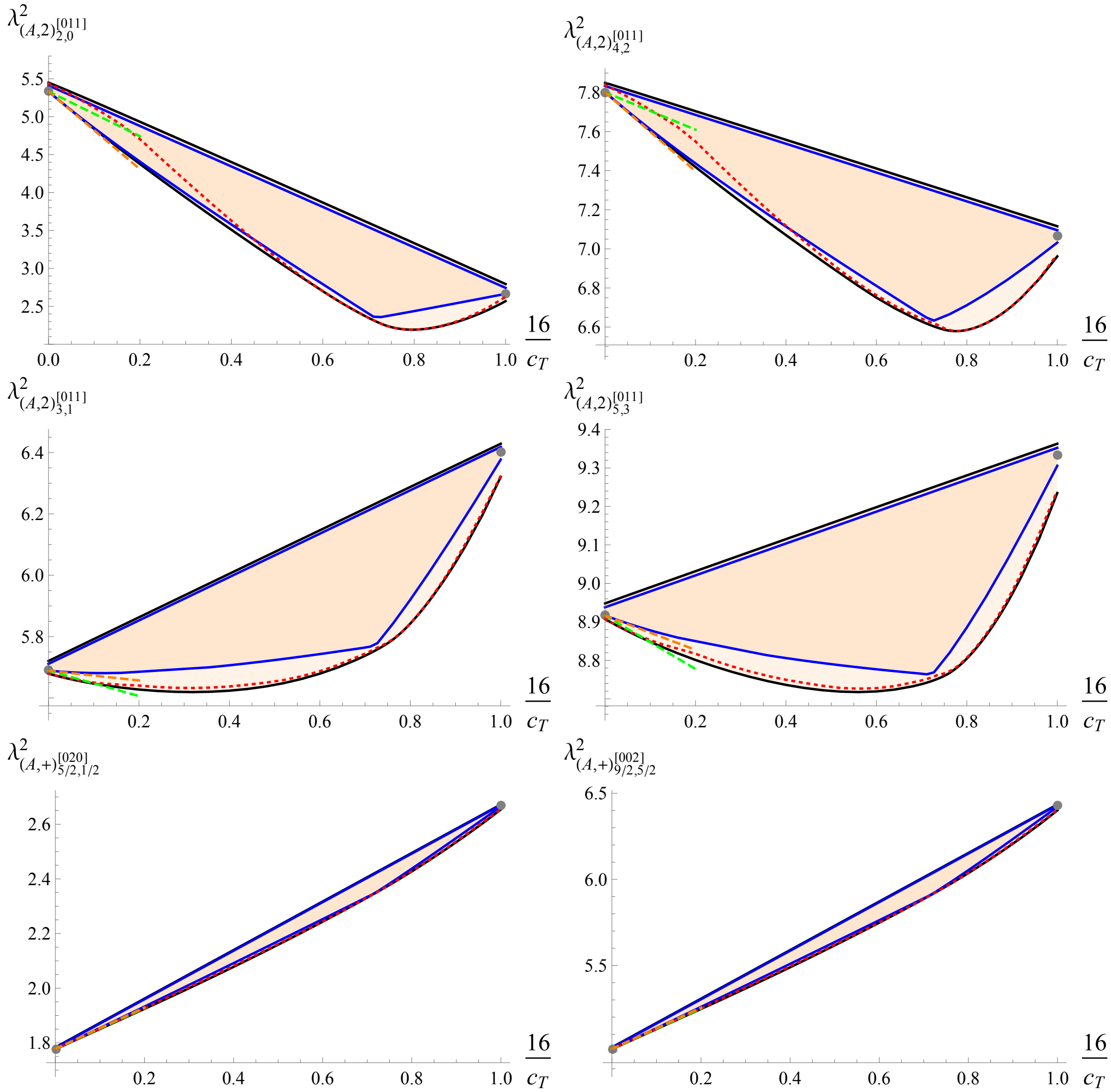

Figure 6.2: Upper and lower bounds on various semishort OPE coefficients squared in terms of c_T, where the orange shaded regions are allowed, and the plots ranges from the GFFT limit $c_T \to \infty$ to the free theory $c_T = 16$. The **black** lines denote the $\mathcal{N} = 6$ upper/lower bounds computed in this thesis with $\Lambda = 39$, the **blue** lines denote the $\mathcal{N} = 8$ upper/lower bounds computed in [81] with $\Lambda = 43$. The **red** dotted lines denotes the spectrum read off from the functional saturating the lower bound on $\lambda^2_{(B,2)^{[022]}_{2,0}}$, which we conjecture is the $U(1)_{2M} \times U(1+M)_{-2M}$ theory. The **green** dashed lines denote the $O(c_T^{-1})$ correction for the $U(1)_{2M} \times U(1+M)_{-2M}$ theory computed in this work, while the **orange** dashed lines denote the $O(c_T^{-1})$ correction for the supergravity limit of ABJM theory as computed in [117, 203]. The **gray** dots denote the GFFT and free theory values.

CFT data	Extremal Functional	Tree level HS	Tree level SUGRA
$\lambda^2_{(A,+)^{[002]}_{5/2,1/2}}$	11.9	$\frac{64}{9}+\frac{1280}{27\pi^2}\approx 11.91$	$-\frac{5120}{27}+\frac{17920}{9\pi^2}\approx 12.11$
$\lambda^2_{(A,+)^{[002]}_{9/2,5/2}}$	18.7	$\frac{13824}{1225}+\frac{90112}{1225\pi^2}\approx 18.74$	$-\frac{2490368}{1225}+\frac{10633216}{525\pi^2}\approx 19.18$
$\lambda^2_{(A,2)^{[011]}_{2,0}}$	-45	$-\frac{64}{3}-\frac{256}{\pi^2}\approx -47.27$	$-\frac{1024}{3}+\frac{2560}{\pi^2}\approx -81.96$
$\lambda^2_{(A,2)^{[011]}_{4,2}}$	-18	$-\frac{1024}{175}-\frac{16384}{175\pi^2}\approx -15.34$	$-\frac{262144}{105}+\frac{5472256}{225\pi^2}\approx -32.36$
$\lambda^2_{(A,2)^{[011]}_{3,1}}$	-6	$\frac{256}{45}-\frac{16384}{135\pi^2}\approx -6.61$	$-\frac{131072}{135}+\frac{28672}{3\pi^2}\approx -2.54$
$\lambda^2_{(A,2)^{[011]}_{5,3}}$	-10	$\frac{4096}{1225}-\frac{524288}{3675\pi^2}\approx -11.11$	$-\frac{16777216}{3675}+\frac{495976448}{11025\pi^2}\approx -7.14$
$\Delta_{(0,1)}$	-16	$-\frac{160}{\pi^2}\approx -16.21$	$-\frac{1120}{\pi^2}\approx -113.48$
$\Delta_{(0,2)}$	16	$\frac{128}{\pi^2}\approx 12.97$	$-\frac{1120}{\pi^2}\approx -113.48$

Table 6.1: The $1/c_T$ correction to the conformal dimensions $\Delta_{0,1}$ and $\Delta_{0,2}$ for the lowest dimension $\text{Long}^{[000],1}_{\Delta,0}$ and $\text{Long}^{[000],2}_{\Delta,0}$ operators, respectively, as well as the OPE coefficients squared of various semishort operators. The extremal functional results come from a large c_T fit to the functional that we conjecture applies to the $U(1)_{2M}\times U(1+M)_{-2M}$ theory, and corresponds to the dashed red lines in Figure 6.2. The analytic tree-level results of the higher-spin theory are those for $U(1)_{2M}\times U(1+M)_{-2M}$, computed in Chapter 5 of this thesis, and the analytic results for supergravity were computed in [117, 203].

tree-level $U(1)_{2M}\times U(1+M)_{-2M}$ are different, as we can see from Table 6.1.

6.2.3 Bounds on Long Scaling Dimensions

Lastly, we will study bounds on the conformal dimensions of long multiplets. To find upper bounds on the scaling dimension Δ^* of the lowest dimension operator in a long supermultiplet with spin ℓ^* that appears in (6.3), we consider linear functionals α satisfying

$$\begin{aligned}
&\alpha(d^i_{\text{Id}})+\frac{64}{c_T}\alpha(d^i_{(B,2)^{[011]}_{1,0}})=1\,,\\
&\alpha(d^i_I)\geq 0\,, \qquad \text{for all short and semi-short } I\notin\{\text{Id},(B,2)^{[011]}_{1,0}\}\,,\\
&\alpha(d^i_I)\geq 0\,, \qquad \text{for all long } I \text{ with } \Delta_I\geq\Delta'_I\,,
\end{aligned} \tag{6.13}$$

where we set all Δ'_I to their unitarity values except for Δ'_{I*}. If such a functional α exists, then this α applied to (6.3) along with the reality of λ_I would lead to a contradiction. By running this algorithm for many values of (c_T,Δ'_{I*}) we can find an upper bound on Δ'_{I*} in this plane.

Since for the long multiplets $\text{Long}^{[000]}_{\Delta,\ell}$ of even spin ℓ there are several superconformal blocks (two for $\ell=0$ and three for $\ell\geq 2$), we can ask what the upper bound on Δ is independently for each superconformal structure $\text{Long}^{[000],n}_{\Delta,\ell}$. To be explicit, we denote by $\Delta_{(\ell,n)}$ the bound obtained from the structure $\text{Long}^{[000],n}_{\Delta,\ell}$. (For odd ℓ, we simply denote the bound by Δ_ℓ.).

For general $\mathcal{N}=6$ SCFTs, the bounds for different n need not be the same, but we do expect that a long multiplet $\text{Long}_{\Delta,\ell}^{[000]}$ in a generic $\mathcal{N}=6$ SCFTs will contribute to all superconformal structures and, if this is the case, the lowest dimension long multiplet must obey all the bounds obtained separately from each superconformal structure. Since the superconformal structures are distinguished by they parity $\mathcal{P}$ and $\mathcal{Z}$ charges (see Table 2.5), in an SCFT that preserves these symmetries, $\Delta_{(\ell,n)}$ represents the upper bound on the lowest long multiplet with the $\mathcal{P}$ and $\mathcal{Z}$ charges that correspond to the structure $\text{Long}_{\Delta,\ell}^{[000],n}$ as given in Table 2.5.

Let us begin with bounds on scalar long multiplets. We show the bounds for the parity even structure $\text{Long}_{\Delta,0}^{[000],1}$ in Figure 6.3. The bound appears to smoothly interpolate between the generalized free field value of 2 at $\frac{16}{c_T}=0$ and the free field value of 1 at $\frac{16}{c_T}=1$. This can be compared to the $\mathcal{N}=8$ bounds, plotted in blue, which is of course always lower than the $\mathcal{N}=6$ bound, but exhibits a kink at $\frac{16}{c_T}\sim .71$ where the two bounds appear to meet. We also show the extremal functional, conjectured to be the $U(1)_{2M}\times U(1+2M)_{-2M}$ theory, and see that it coincides with the $\mathcal{N}=6$ upper bound. The green and orange dashed lines plot the tree-level result for higher-spin theory and SUGRA respectively, which as we can see approximately match the $\mathcal{N}=6$ and $\mathcal{N}=8$ bounds respectively.

Now let us consider the bounds for the parity odd scalar structure $\text{Long}_{\Delta,0}^{[000],1}$. Recall that, as per (2.74), the unitarity limit of the $\text{Long}_{\Delta,0}^{[000],2}$ superconformal block is the $(B,1)_{2,0}^{[200]}$ superconformal block, so our bound on $\Delta_{(0,2)}$ depends on whether we assume that a $(B,1)_{2,0}^{[200]}$ multiplet appears in the $S\times S$ OPE. If we assume that there are no $(B,1)_{2,0}^{[200]}$ operators that appear in the $S\times S$ OPE, then we obtain the bound in bottom plot of Figure 6.3. As we can see from this figure, the bound $\Delta_{0,2}$ smoothly goes from the GFFT value 1 at $\frac{16}{c_T}=0$ to the free theory value 3 at $\frac{16}{c_T}=1$. The extremal functional, shown in red, comes extremely close to the upper bound, and both match match reasonably well with the tree-level result for the $U(1)_{2M}\times U(1+2M)_{-2M}$ theory.

For comparison we also show the $\mathcal{N}=8$ upper bound computed in [81], computed with no assumptions about the spectrum. For this reason, the $\mathcal{N}=6$ bounds need not be above the $\mathcal{N}=8$ bounds. Indeed, it was shown in [76] that all $\mathcal{N}=8$ SCFTs with $\frac{16}{c_T}<.71$ contain a short multiplet (namely the $(B,2)_{2,0}^{[0200]}$) that upon reduction to $\mathcal{N}=6$ includes a $(B,1)_{2,0}^{[200]}$ multiplet.[8] It may also seem curious that in bottom plot of Figure 6.3, at $c_T=16$, where the free theory has in fact $\mathcal{N}=8$ SUSY, the lowest operator (marked by a gray dot) that contributes to the $\text{Long}_{\Delta,0}^{[000],2}$ block does not obey the $\mathcal{N}=8$ bound in blue. This is because in that case the $\text{Long}_{\Delta,0}^{[0000]}$ multiplet that gives the $\mathcal{N}=8$ bound is replaced by an $\mathcal{N}=8$ conserved current multiplet $(A,\text{cons.})_{1,0}^{[0000]}$ which no longer

[8]See (C.4) for the reduction of $\mathcal{N}=8$ superconformal blocks to $\mathcal{N}=6$ ones.

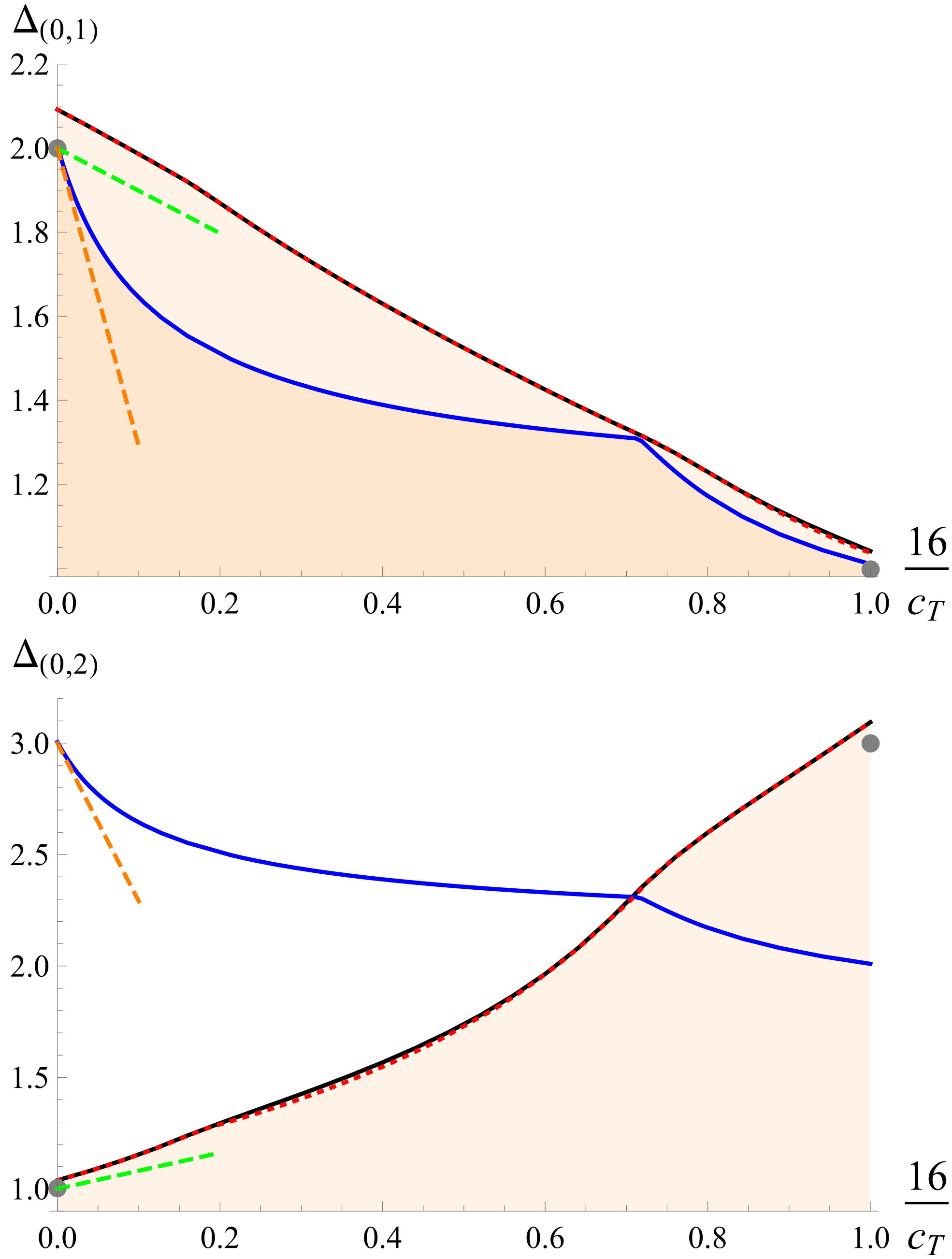

Figure 6.3: Upper bounds on the scaling dimension of the lowest dimension $\ell = 0$ long multiplet as a function of c_T, for the $\text{Long}_{\Delta,0}^{[000],1}$ (**top**) and $\text{Long}_{\Delta,0}^{[000],2}$ (**bottom**) superconformal structures. The **black** lines denote the $\mathcal{N} = 6$ upper bounds computed here with $\Lambda = 39$, the **blue** lines denote the $\mathcal{N} = 8$ upper bounds computed in [81] with $\Lambda = 43$, and we shade the allowed region **orange**. The **red** dotted lines denote the extremal functional spectrum saturating the $\lambda^2_{(B,2)_{2,0}^{[022]}}$ lower bound, which we conjecturally identify with the $U(1)_{2M} \times U(1+M)_{-2M}$ theory. The **green** dashed lines denote the $O(c_T^{-1})$ correction for the $U(1)_{2M} \times U(1+M)_{-2M}$ theory computed in Chapter 5, while the **orange** dashed lines denote the $O(c_T^{-1})$ correction for the supergravity limit of ABJM theory as computed in [117, 203]. The **gray** dots denote the GFFT and free theory values.

contributes to the $\text{Long}_{\Delta,0}^{[000],2}$. The gray dot in Figure 6.3 instead comes from a $\text{Long}_{2,0}^{[0000]}$ multiplet in $\mathcal{N}=8$.

Next we study bounds on long multiplets of spin $\ell=1$ and $\ell=2$. Let us begin with Figure 6.4, which shows the bounds on $\mathcal{P}$ even, $\mathcal{Z}$ even, superblock for $\ell=1$ and $\ell=2$. These bounds smoothly interpolate between the values of the corresponding conformal dimensions at the free $\mathcal{N}=6$ hypermultiplet theory at $\frac{16}{c_T}=1$ and the GFFT at $\frac{16}{c_T}=0$. This behavior is distinct from the $\mathcal{N}=8$ bounds, which exhibit a kink at $\frac{16}{c_T}\sim .71$. We do not show extremal functional results for these plots because our numerics are not yet sufficiently accurate. In particular, we expect that the $\ell=1$ multiplet should become approximately conserved current at large c_T, but this will be particularly hard to see because, as a single trace operator, its OPE coefficient starts at $O(c_T^{-1})$.

The bounds on the other $\ell=2$ long superconformal blocks, the $\text{Long}_{\Delta,2}^{[000],2}$ and $\text{Long}_{\Delta,2}^{[000],3}$, are shown in Figure 6.5. At unitarity these superblocks become the $(A,1)_{\ell+5/2,\ell+1/2}^{[100],1}$ and $(A,1)_{\ell+5/2,\ell+1/2}^{[100],2}$, as per (2.76), and so our bounds depend on what assumptions we make about the presence of these operators. Our first results is that if we assume the $(A,1)_{7/2,3/2}^{[100],2}$ and $(A,1)_{7/2,3/2}^{[100],3}$ do not appear in the $S\times S$ OPE, then the long multiplet bounds are at the unitarity bound. This in turn implies that our assumption was false, and so we conclude all $\mathcal{N}=6$ SCFTs must contain $(A,1)_{7/2,3/2}^{[100]}$ multiplets! This is consistent with the result in [76] that all $\mathcal{N}=8$ SCFTs must contain an $\mathcal{N}=8$ $(A,2)_{3,1}^{[0020]}$ multiplet, which reduces to the $(A,1)_{7/2,3/2}^{[100]}$ $\mathcal{N}=6$ multiplet as per (C.4).

Finally, we can derive revised bounds on these superblocks under the assumption that the $S\times S$ OPE contains the $(A,1)_{7/2,3/2}^{[100],2}$ and $(A,1)_{7/2,3/2}^{[100],3}$ superblocks. As we can see from Figure 6.5, we found that the bounds $\Delta_{(2,n)}$ are slightly above 5 for all c_T, with little dependence on c_T. This is consistent with the value at both GFFT and free theory. For comparison, we also show the second lowest operator for $\mathcal{N}=8$ theories, which corresponds to the lowest long spin 2 $\mathcal{N}=8$ operator.[9] We do not show any extremal functional results for these plots, because we do not yet have sufficient numerical precision.

6.3 Islands for Semishort OPE Coefficients

In the previous section we discussed numerical bounds that apply to all 3d $\mathcal{N}=6$ SCFTs. In particular, we noticed that the upper/lower bounds on $(A,+)_{\ell+2,\ell}^{[002]}$ for $\ell=1/2,5/2$ were extremely

[9] It may again seem curious that at $c_T=16$, where the free theory has $\mathcal{N}=8$ SUSY, the gray dot does not obey the $\mathcal{N}=8$ bound. At exactly the free theory point, this $\mathcal{N}=8$ operator becomes a conserved current which no longer decomposes to a parity odd $\mathcal{N}=6$ long multiplet, which is why the $\mathcal{N}=6$ free theory value of the second lowest operator, denoted by the second lowest gray dot, does not coincide with the $\frac{16}{c_T}\to 1$ limit of the $\mathcal{N}=8$ upper bound.

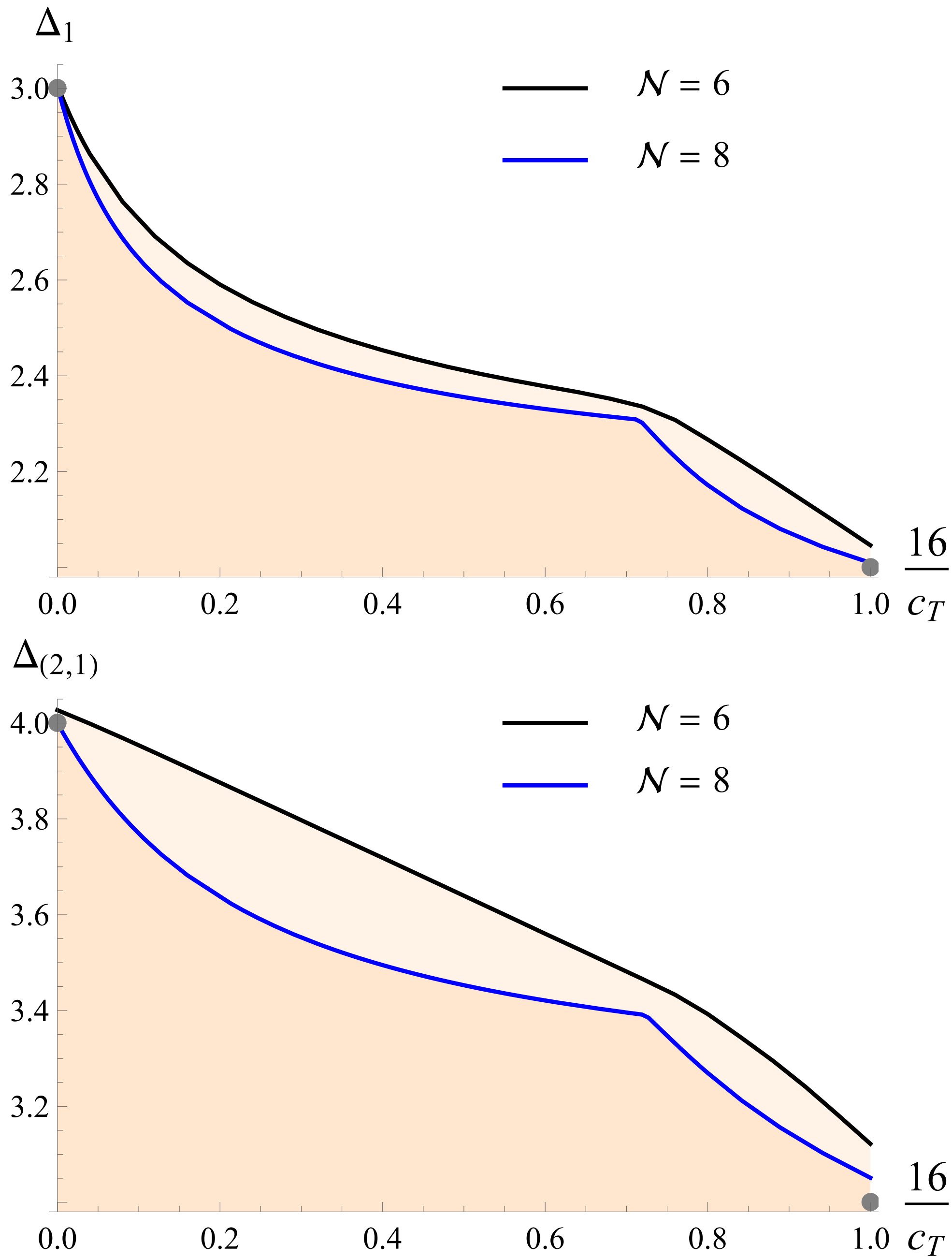

Figure 6.4: Upper bounds on the conformal dimensions as a function of c_T. The **top** plot shows the bounds on the lowest $\ell = 1$ long multiplet (for which there is a unique conformal structure). The **bottom** plot shows the bounds for the parity even $\ell = 2$ long multiplet, corresponding to the $\text{Long}_{\Delta,2}^{[000],1}$ superconformal block. The **black** line denotes the $\mathcal{N} = 6$ upper bound computed in here with $\Lambda = 39$, the **blue** line denote the $\mathcal{N} = 8$ upper bound computed in [81] with $\Lambda = 43$, and we shade the allowed region orange. The **gray** dots denote the GFFT and free theory values from Table 2.6.

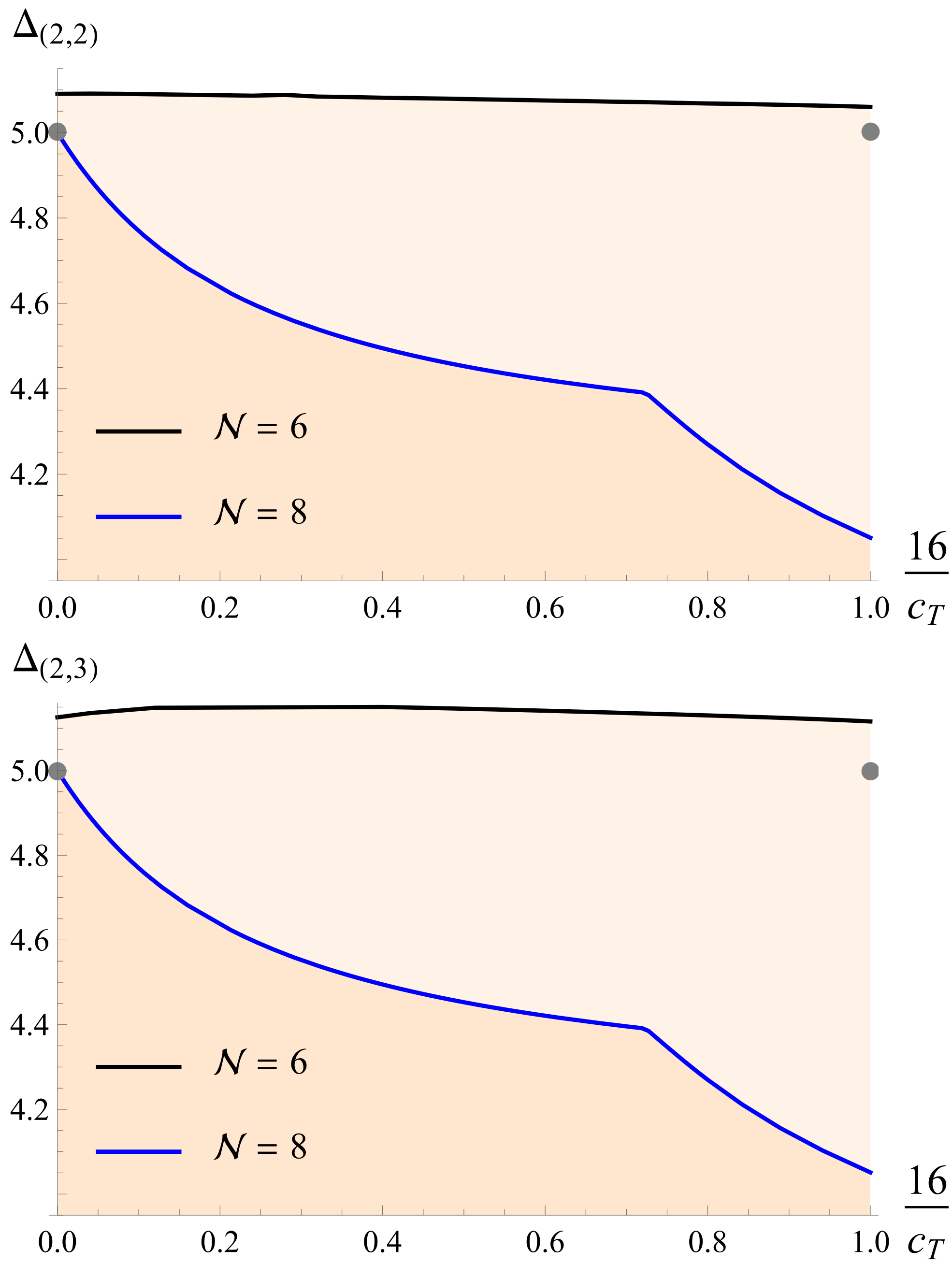

Figure 6.5: Upper bounds on the scaling dimension of the lowest dimension $\ell = 2$ long multiplet in terms of c_T for the $\text{Long}_{\Delta,2}^{[000],2}$ (**top**) and $\text{Long}_{\Delta,2}^{[000],3}$ (**bottom**) superconformal structures, which for parity preserving theories have the opposite parity as the superprimary, and for $\mathcal{Z}$ preserving theories has the same charge for $\text{Long}_{\Delta,2}^{[000],2}$ and the opposite charge for $\text{Long}_{\Delta,2}^{[000],3}$. The **black** line denotes the $\mathcal{N} = 6$ upper bound computed in here with $\Lambda = 39$, the **blue** line denote the $\mathcal{N} = 8$ upper bound computed in [81] with $\Lambda = 43$, and we shade the allowed region orange. The gray dots denote the GFFT and free theory values from Table 2.6.

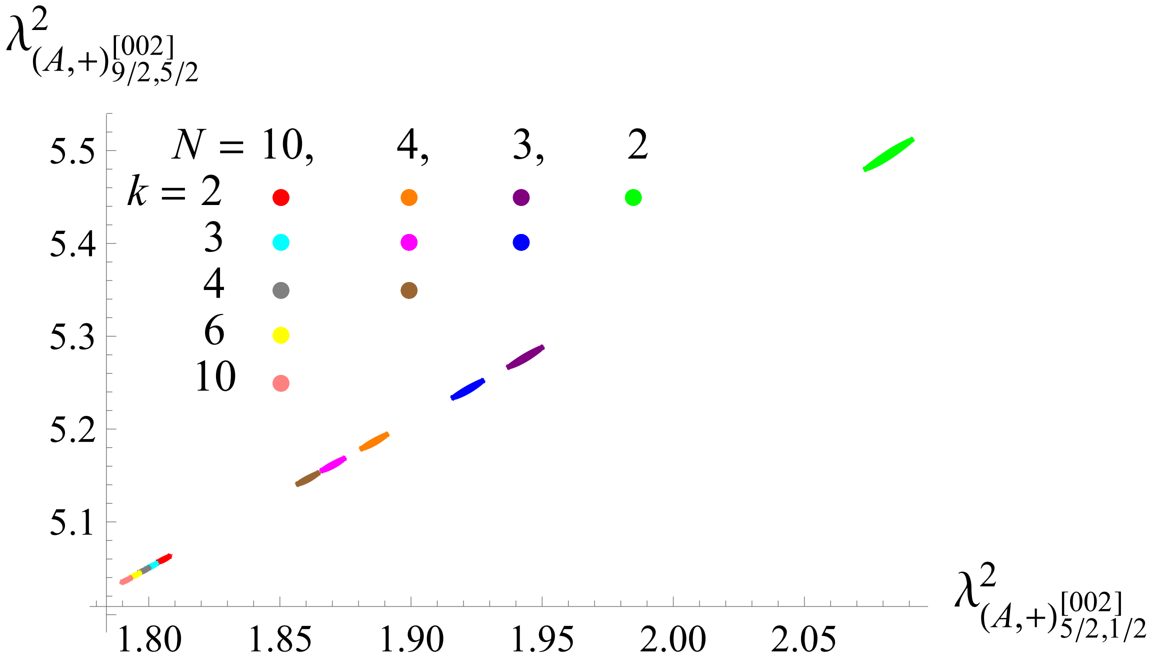

Figure 6.6: Islands in the space of the semi-short OPE coefficients $\lambda^2_{(A,+)^{[002]}_{5/2,1/2}}$, $\lambda^2_{(A,+)^{[002]}_{9/2,5/2}}$ (to be defined precisely later) for $U(N)_k \times U(N)_{-k}$ ABJM theory for various N, k. These bounds are derived from the $\mathcal{N} = 6$ bootstrap with $\Lambda = 39$ derivatives, and with the short OPE coefficients (i.e. c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$) fixed to their values in each theory using the exact localization results of [76] for $N = 2, 3, 4$ as shown in Table 3.2 and the all orders in $1/N$ formulae in [81] for $N = 10$ as shown in Table 3.3.

constraining. This implies that for a given value of c_T, we could find a small island in the space of OPE coefficients $(\lambda^2_{(A,+)^{[002]}_{5/2,1/2}}, \lambda^2_{(A,+)^{[002]}_{9/2,5/2}})$ using the OPE island algorithm described in the previous subsection.

To make these islands even smaller and correlate them to specific physical theories, we can impose values of c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ computed using supersymmetric localization in Chapter 3. Such islands were found for $\mathcal{N} = 8$ SCFTs in [76,179], and we ow find similar islands for $\mathcal{N} = 6$ theories. We show our results for $U(N)_k \times U(N)_{-k}$ for a variety of N, k in the Figure 6.6. Note that the islands are small enough that we can distinguish each value of N and k, which allows us to non-perturbatively interpolate between M-theory at small k and Type IIA at large k.

One difficulty with trying to fix a physical theory by imposing two exactly computed quantities, c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$, is that the most general $\mathcal{N} = 6$ ABJ theory has gauge group $U(N)_k \times U(N+M)_{-k}$ and so is described by 3 parameters M, N, and k. While for physical theories these parameters should be integers, we expect that the numerical bootstrap should find theories with any real value of these parameters, so we are effectively trying to parameterize a 3-dimensional space of theories. Since

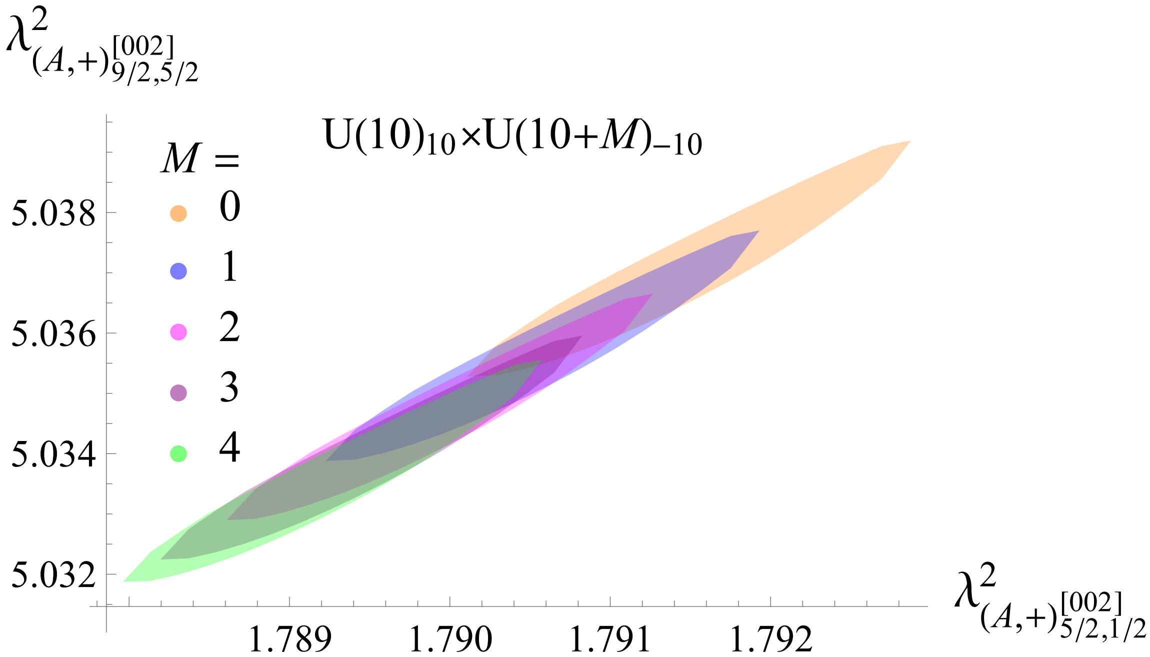

Figure 6.7: Islands in the space of the semi-short OPE coefficients $\lambda^2_{(A,+)^{[002]}_{5/2,1/2}}$, $\lambda^2_{(A,+)^{[002]}_{9/2,5/2}}$ for $U(10)_{10} \times U(10+M)_{-10}$ ABJ theory for $M < k/2 = 5$. These bounds are derived from the $\mathcal{N} = 6$ bootstrap with $\Lambda = 39$ derivatives, and with c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ fixed to their values in each theory using the all orders in $1/N$ localization formulae in [81] for $N = 10$. Note that the axes describe a very narrow range in parameter space.

we are only imposing two quantities, these islands are expected to have a finite area even at high numerical precision corresponding to the third direction in "theory space". Thankfully, this third direction appears to be very small. We can quantify this by fixing $N = k = 10$ and computing islands for several different values of $M \leq k/2 = 5$. As shown in Figure 6.7, the island is not very sensitive to the value of $M < N$, which explains why we were able to get such small islands in a 3-dimensional space by just imposing two values of the parameters.

6.4 Discussion

In this chapter we studied $\mathcal{N} = 6$ theories non-perturbatively using the numerical conformal bootstrap. In particular, by inputting the exact values of c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ for a given ABJ theory, we found precise rigorous islands in the space of semishort OPE coefficients that interpolate between M-theory at small k and type IIA string theory at $k \sim N$. We also conjectured that in the infinite precision limit, the numerical lower bound on $\lambda^2_{(B,2)^{[022]}_{2,0}}$ is saturated by the family of $U(1)_{2M} \times U(1+M)_{-2M}$ theories, which allowed us to non-rigorously read off all CFT data in $\langle SSSS \rangle$

using the extremal functional method. Interestingly, in the regime of large c_T we found a spin zero long multiplet whose scaling dimension approaches a zero spin conserved current multiplet at large c_T, as expected from weakly broken higher-spin symmetry.

There are several ways we can improve upon our 3d $\mathcal{N}=6$ bootstrap study. From the numerical perspective, it will be useful to improve the precision of our study. This is parameterized by the parameter Λ defined previously. While we used $\Lambda = 39$ in this work, which is close to the $\Lambda = 43$ values used in the analogous $\mathcal{N}=8$ studies [81, 179], for $\mathcal{N}=6$ this value has not led to complete convergence. For instance, we found the lower bound $c_T \geq 15.5$, compared to the $\mathcal{N}=8$ result $c_T \geq 15.9$; both are expected to converge to the free theory $c_T = 16$. More physically, we expect that approximately conserved currents should appear in the extremal functional that conjecturally describes the $U(1)_{2M} \times U(1+M)_{-2M}$ theory. We found such an operator in the zero spin sector as shown in Figure 6.3, but do not yet have sufficient precision to see them for higher-spin. The main obstacle to increasing Λ at the moment is not SDPB, which due to the recent upgrade [197] can easily handle four crossing equations at very high Λ, but simply the difficulty in computing numerical approximations to the superblocks at large Λ. In particular it would be extremely useful to have an efficient code for approximations of linear combinations of conformal blocks with Δ dependent coefficients around the crossing symmetric point. Currently the code `scalar_blocks` code, found on the bootstrap collaboration website,[10] is only able to efficiently compute single conformal blocks.

We could also make further use of localization to improve our results. In this chapter we only considered constraints from c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$, but ABJ is parameterized by three parameters M, N, and k. For this reason there are not enough constraints to uniquely pick out a single ABJ theory and so we should not expect our islands to shrink indefinitely as we increase Λ. We think this is the reason why the islands shown in Figure 6.6, while small, are still much bigger than the $\mathcal{N}=8$ islands computed in [179]. In Chapter 3 we studied a third quantity $\frac{\partial^4 \log Z}{\partial m_+^2 \partial m_-^2}$ which constrains $\langle SSSS \rangle$, however, it is not yet known how to use this to constrain the numerical bootstrap in our case. Perhaps the method used in [204], where a similar integrated constraint was successfully imposed on the numerical bootstrap of a certain supersymmetric 2d theory, could be applied to our case. Another option would be to look at a larger system of correlators, such as those involving fermions or $\langle SSSP \rangle$. This would allow us to impose parity, which would restrict the set of known $\mathcal{N}=6$ SCFTs to a few families such as $U(N)_k \times U(N)_{-k}$ parameterized by only two parameters each.

[10]This code can be found at `https://gitlab.com/bootstrapcollaboration/scalar_blocks/blob/master/Install.md`.

Chapter 7

Conclusion

In this thesis we studied $\mathcal{N} = 6$ superconformal field theories. Our results can be roughly divided into two classes: those which apply to all $\mathcal{N} = 6$ SCFTs, and those which apply to specific holographic regimes. In the former class we have the supersymmetric Ward identities and superconformal block expansion for $\langle SSSS \rangle$ derived in Chapter 2, the supersymmetric localization constraints derived in Chapter 3, and the numerical conformal bootstrap results in Chapter 6. The latter class of results includes the holographic expressions for $\langle SSSS \rangle$ derived in Chapter 4 for the string and M-theory regimes and in Chapter 5 for higher-spin theories.

Throughout this thesis we made use of the fact that derivatives of the mass-deformed sphere partition function can be computed exactly using supersymmetric localization. We have not yet exhausted all of the constraints these provide. As explained in Chapter 3, in addition to the two masses $m_\pm$ considered in this thesis, $\mathcal{N} = 6$ theories admit a third mass deformation, $\widetilde{m}$, and on top of this we can also consider placing the theory on a squashed sphere parameterized by squashing parameter b [83] (with $b = 1$ corresponding to the round case). This leads to a much larger number of four derivative constraints to consider, although not all of these are necessarily independent. In [97] it was shown that in ABJ theory at large N, the only independent four derivative quantities are

$$\partial_{m_\pm}^4 \log Z\,, \quad \partial_{m_+}^3 \partial_{m_-} \log Z\,, \quad \partial_{m_+}^2 \partial_{m_-}^2 \log Z\,, \quad \partial_{m_+}^2 \partial_{\widetilde{m}}^2 \log Z\,,$$

all evaluated at $m_\pm = \widetilde{m} = 0$ and $b = 1$. Of these, only the last quantity, which is complex, was not studied in this thesis. It would be interesting to work out whether these redundancies are specific to ABJ at large N, or whether they hold for more general $\mathcal{N} = 6$ theories as a consequence of the superconformal Ward identities.

In either case, it should be possible to derive the integrated constraint imposed by $\partial_{m_+}^2 \partial_{\tilde{m}}^2 \log Z$, and to derive expressions for $\partial_{m_+}^2 \partial_{\tilde{m}}^2 \log Z$ at both large N and large M. With these results, we could then have enough constraints to extend our calculation in Chapter 4 to degree six. As shown in Table 4.3, we have one degree 3, one degree 4, two degree 5, and three degree 6 local Mellin amplitudes. With the flat space limit we can fix all three degree 6 terms, while we have just enough parity-even localization constraints to then fix the other four Mellin amplitudes. Unfortunately, however, this would not provide us with any additional checks of AdS/CFT beyond the ones already considered in this thesis. For $\mathcal{N} = 8$ ABJM theory we can make further progress due to the enhanced supersymmetry. In [118] $\langle SSSS \rangle$ was computed up to the D^4R^4 term, corresponding to a degree 6 Mellin amplitude, and it seems likely that this could be extended up to the degree 8 D^6R^4 term.

Apart from supersymmetric localization, another potential source of exactly computable quantities for $\mathcal{N} = 6$ theories is from integrability. It would be interesting to try to match integrability results for the lowest dimension singlet scaling dimension in the leading large N 't Hooft limit at fixed $\lambda_{\text{'t Hooft}} = N/k$ and $M = 0$, computed in [205, 206], to numerical bootstrap results, where we use inputs from supersymmetric localization to constrain ourselves to the relevant ABJM theories. For this to work we would need to compute the derivatives of the mass deformed free energy in the $1/N$ expansion at finite $\lambda_{\text{'t Hooft}}$. In fact, the zero mass free energy has already been computed in this limit in [91] by applying topological recursion to the Lens space $L(2,1)$ matrix model, so computing c_T and $\lambda^2_{(B,2)^{[022]}_{2,0}}$ should correspond to just computing two- and four-body operators in this matrix model. This could potentially lead to the first precise comparison between integrability and the numerical conformal bootstrap.

In both the large N limits studied in Chapter 4 and the large M limit studied in Chapter 5, we focused our attention on tree-level correlators. A logical next step would be to extend these results to 1-loop. In [207] it was shown that 1-loop corrections to holographically correlators can in general be computed from the "square" of the tree-level anomalous dimensions. The challenge here is that these double traces are generally degenerate, and to compute the 1-loop correction they must first be unmixed. One-loop corrections for supergravity and for the R^4 term have been computed in both 4d $\mathcal{N} = 4$ theories [208–212] and 6d $(2,0)$ theories [213], and it should be possible to extend these calculations to both 3d $\mathcal{N} = 6$ and $\mathcal{N} = 8$ theories in the supergravity limit. No similar calculations have so far been performed for higher-spin theories, although the mixing problem in non-supersymmetric case was considered in [214].

While in this thesis we focused on $\mathcal{N} = 6$ SCFTs, many of the tools we have developed should generalize to theories with less supersymmetry. The next logical step is to study $\mathcal{N} = 5$ superconformal theories, and in particular the $\mathcal{N}=5$ $O(N_1)_{2k} \times USp(N_2)_{-k}$ Chern-Simons matter theories, of which the $SO(2)_{2k} \times USp(2+2M)_{-k}$ theories considered in this thesis are merely a special case. The ABJ quadrality of [66] extends the ABJ triality by relating the bulk duals of these theories to those of ABJ with an additional orientifold. From the string theory perspective, these theories are obtained by orientifolding the brane construction of the $U(N)_k \times U(N+M)_{-k}$ theory, so that the $O(N_1)_{2k} \times USp(N_2)_{-k}$ theories are dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3/\mathbb{Z}_2$. In the string or M-theory limit, orientifolding changes the single trace spectrum, such that certain tree-level correlators vanish, and the 1-loop corrections are suitably modified [213]. The $O(N_1)_{2k} \times USp(N_2)_{-k}$ also have two distinct higher-spin limits, where $N_1 \gg N_2$ or $N_2 \gg N_1$. The orientifold does not affect the single trace spectrum aside from reducing the supersymmetry when $N_1 \neq 2$ from $\mathcal{N}=6$ to $\mathcal{N}=5$, so we expect that the general structure of the $\mathcal{N}=5$ tree-level correlator should be very similar to our $\mathcal{N}=6$ result. The precise dependence on λ could still be different, as that depends on the Lagrangian of the specific theory, as well as the specific form of the $\mathcal{N}=5$ version of the $\mathcal{N}=6$ integrated constraints discussed in this thesis. To fully fix the correlator, one might also need to consider integrated constraints involving the squashed sphere, as computed in [83, 97, 215, 216].

We could also study less supersymmetric theories in 3d with the numeric conformal bootstrap. For both $\mathcal{N}=4$ and $\mathcal{N}=5$ superconformal field theories the superprimary of the stress tensor multiplet remains a scalar. It should hence be feasible to generalize the strategy employed in Chapter 2 to derive the superconformal blocks for these theories, and then use these superblocks to perform a numeric bootstrap study. We could similarly study the superprimary for a conserved current multiplet in $\mathcal{N}=2$ or $\mathcal{N}=3$; for $\mathcal{N}=4$ theories the multiplet is half-BPS and have been studied in [184]. Many localization results exist which could be applied to each of these cases.

Let us finish by considering the broader picture. As we discussed in the introduction, our motivation for studying $\mathcal{N}=6$ SCFTs was two-fold: the theories provide highly symmetric examples of quantum field theories, and, through holography, they provide highly symmetric examples of AdS/CFT duality. By studying $\mathcal{N}=6$ SCFTs, we may hope to gain a more general insight into the space of possible quantum field theories, and into the behavior of quantum gravity.

On the former front, we already have a classification of all Lagrangian $\mathcal{N}=6$ SCFTs, and it would be extremely interesting to understand whether other $\mathcal{N}=6$ exist. Perhaps insight into the classification problem could be gained through a deeper understanding of the structure of observables that can be computed through supersymmetric localization. Or perhaps a better understanding of the superconformal crossing equations could yield insight. The conformal bootstrap itself is still in its infancy, and a precise understanding of why certain theories saturate bootstrap bounds, or which theories can be isolated using the bootstrap, is still lacking. $\mathcal{N} = 6$ theories provide a constrained setting in which one could try to explore these more general questions.

On the holographic front, an ambitious goal would be to derive string and M-theory scattering amplitudes beyond those quantities protected by supersymmetry. While in string theory we can systematically compute scattering amplitudes in perturbation theory, no similar method is known for M-theory. Studying ABJ theory, whether through the numeric bootstrap or with some other method, provides one possible avenue to fully compute the M-theory S-matrix.

Another goal would be to understand how to relate the stringy limits of ABJ theory with the higher-spin limit. The higher-spin limit is in many ways more tractable than the stringy limits, and so one could hope that understanding AdS/CFT in the former case may give insight into AdS/CFT in the latter. Conversely, stringy theories of quantum gravity are in many ways much better understood than their higher-spin cousins, and so by studying how ABJ theory interpolates between these two theories we may hope to gain insight into both types of quantum gravity.

Appendix A

Supersymmetric Ward Identities

We begin by listing general expressions for two-scalar, two-fermion correlators which are invariant under both conformal and $\mathfrak{so}(6)_R$ symmetry. As explained in Section 2.1 we restrict to those structures which are $\mathcal{P}$ and $\mathcal{Z}$ invariant:

$$\begin{aligned}
&\langle S(\vec{x}_1, X_1) S(\vec{x}_2, X_2) \chi^\alpha(\vec{x}_3, Z_3) \chi^\beta(\vec{x}_4, Z_4)\rangle \\
&= \frac{i \not{x}_{34}^{\alpha\beta}}{x_{12}^2 x_{34}^4}\Big[\mathrm{Tr}(X_1 X_2)(Z_3 \cdot Z_4)\mathcal{C}^{1,1} + (Z_3 \check{X}_1 \check{X}_2 Z_4)\mathcal{C}^{2,1} + (Z_3 \check{X}_1 \check{X}_2 Z_4)\mathcal{C}^{3,1}\Big] \\
&+ \frac{i(\not{x}_{13}\not{x}_{24}\not{x}_{12})^{\alpha\beta}}{2x_{12}^4 x_{34}^4}\Big[\mathrm{Tr}(X_1 X_2)(Z_3 \cdot Z_4)\mathcal{C}^{1,2} + (Z_3 \check{X}_1 \check{X}_2 Z_4)\mathcal{C}^{2,2} + (Z_3 \check{X}_1 \check{X}_2 Z_4)\mathcal{C}^{3,2}\Big],
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&\langle S(\vec{x}_1, X_1) S(\vec{x}_2, X_2) \chi^\alpha(\vec{x}_3, Z_3) F^\beta(\vec{x}_4, Y_4)\rangle \\
&= \frac{i \not{x}_{34}^{\alpha\beta}}{x_{12}^2 x_{34}^4}\Big[\mathrm{Tr}(X_1 X_2 Y_4 \not{Z}_3)\mathcal{E}^{1,1} + \mathrm{Tr}(X_2 X_1 Y_4 \not{Z}_3)\mathcal{E}^{2,1} + \mathrm{Tr}(X_2 Y_4 X_1^T \not{Z}_3)\mathcal{E}^{3,1}\Big] \\
&+ \frac{i(\not{x}_{13}\not{x}_{24}\not{x}_{12})^{\alpha\beta}}{2x_{12}^4 x_{34}^4}\Big[\mathrm{Tr}(X_1 X_2 Y_4 \not{Z}_3)\mathcal{E}^{1,2} + \mathrm{Tr}(X_2 X_1 Y_4 \not{Z}_3)\mathcal{E}^{2,2} + \mathrm{Tr}(X_2 Y_4 X_1^T \not{Z}_3)\mathcal{E}^{3,2}\Big],
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\langle S(\vec{x}_1, X_1) S(\vec{x}_2, X_2) F^\alpha(\vec{x}_3, Z_3) F^\beta(\vec{x}_4, Y_4)\rangle \\
&= \frac{i \not{x}_{34}^{\alpha\beta}}{x_{12}^2 x_{34}^4}\Big[\big(\epsilon_{abcd}(X_1)^a{}_e Y_1^{eb}(X_2)^c{}_f Y_2^{fd}\big)\mathcal{F}^{1,1} + \big(\epsilon_{abcd}(X_1)^a{}_e Y_2^{eb}(X_2)^c{}_f Y_1^{fd}\big)\mathcal{F}^{2,1}\Big] \\
&+ \frac{i(\not{x}_{13}\not{x}_{24}\not{x}_{12})^{\alpha\beta}}{2x_{12}^4 x_{34}^4}\Big[\big(\epsilon_{abcd}(X_1)^a{}_e Y_1^{eb}(X_2)^c{}_f Y_2^{fd}\big)\mathcal{F}^{1,2} + \big(\epsilon_{abcd}(X_1)^a{}_e Y_2^{eb}(X_2)^c{}_f Y_1^{fd}\big)\mathcal{F}^{2,2}\Big],
\end{aligned} \tag{A.3}$$

$$
\begin{aligned}
&\langle S(\vec{x}_1, X_1) S(\vec{x}_2, X_2) \overline{F}^{\beta}(\vec{x}_3, \overline{Y}_3) F^{\alpha}(\vec{x}_4, Y_4)\rangle \\
&= \frac{i \not{x}_{34}^{\alpha\beta}}{x_{12}^2 x_{34}^4} \Big[\mathrm{Tr}(X_1 X_2)\mathrm{Tr}(Y_4 \overline{Y}_3)\mathcal{G}^{1,1} + \mathrm{Tr}(Y_4 \overline{Y}_3 X_2 X_1)\mathcal{G}^{2,1} \\
&\qquad\qquad + \mathrm{Tr}(Y_4 \overline{Y}_3 X_2 X_1)\mathcal{G}^{3,1} + \mathrm{Tr}(Y_4 X_2^T \overline{Y}_3 X_1)\mathcal{G}^{4,1} \Big] \\
&+ \frac{i(\not{x}_{13}\not{x}_{24}\not{x}_{12})^{\alpha\beta}}{2 x_{12}^4 x_{34}^4} \Big[\mathrm{Tr}(X_1 X_2)\mathrm{Tr}(Y_4 \overline{Y}_3)\mathcal{G}^{1,2} + \mathrm{Tr}(Y_4 \overline{Y}_3 X_2 X_1)\mathcal{G}^{2,2} \\
&\qquad\qquad + \mathrm{Tr}(Y_4 \overline{Y}_3 X_2 X_1)\mathcal{G}^{3,2} + \mathrm{Tr}(Y_4 X_2^T \overline{Y}_3 X_1)\mathcal{G}^{4,2} \Big] .
\end{aligned} \tag{A.4}
$$

We will now give the Ward identities for two-scalar, two-fermion correlators. We will begin with $\langle SS\chi\chi\rangle$ and $\langle SS\chi F\rangle$, which can be derived from $\delta\langle SSS\chi\rangle$. We will omit those functions of the cross-ratios that are related to these under crossing. The expressions for $\langle SS\chi\chi\rangle$ are:

$$
\begin{aligned}
\mathcal{C}^{1,1} = &-\frac{1}{2U}\Big(U^2\partial_V\mathcal{S}^1(U,V) + 4U^2\partial_U\mathcal{S}^1(U,V) + 4U^2\partial_U\mathcal{S}^5(U,V) + U(V-U)\partial_V\mathcal{S}^2(U,V) \\
&+ U(-U+V-1)\partial_U\mathcal{S}^2(U,V) + UV\partial_V\mathcal{S}^3(U,V) + U(U+V-1)\partial_U\mathcal{S}^3(U,V) \\
&+ 2UV\partial_V\mathcal{S}^4(U,V) + 2U(V-1)\partial_U\mathcal{S}^4(U,V) - 4U\mathcal{S}^1(U,V) - 3U\mathcal{S}^5(U,V) - U\mathcal{S}^6(U,V) \\
&+ (U-V+1)\mathcal{S}^2(U,V) - (V-1)\mathcal{S}^3(U,V) + (U-2V+2)\mathcal{S}^4(U,V)\Big) ,
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
\mathcal{C}^{2,1} = &-\frac{1}{32U}\Big(U^2\partial_U\mathcal{S}^2(U,V) + U^2\partial_U\mathcal{S}^3(U,V) - U^2\partial_V\mathcal{S}^1(U,V) + U(U+V)\partial_V\mathcal{S}^2(U,V) \\
&+ UV\partial_U\mathcal{S}^2(U,V) - U\partial_U\mathcal{S}^2(U,V) + UV\partial_V\mathcal{S}^3(U,V) + UV\partial_U\mathcal{S}^3(U,V) - U\partial_U\mathcal{S}^3(U,V) \\
&+ 2UV\partial_V\mathcal{S}^4(U,V) + 2UV\partial_U\mathcal{S}^4(U,V) - 2U\partial_U\mathcal{S}^4(U,V) - U\mathcal{S}^2(U,V) + U\mathcal{S}^4(U,V) \\
&+ U\mathcal{S}^5(U,V) - U\mathcal{S}^6(U,V) - V\mathcal{S}^2(U,V) + \mathcal{S}^2(U,V) - V\mathcal{S}^3(U,V) + \mathcal{S}^3(U,V) - 2V\mathcal{S}^4(U,V) \\
&+ 2\mathcal{S}^4(U,V)\Big) ,
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
\mathcal{C}^{1,2} = &\frac{1}{2}(U((3V+1)\partial_V\mathcal{S}^1(U,V) + 3U\partial_U\mathcal{S}^1(U,V) - \partial_V\mathcal{S}^2(U,V) - \partial_U\mathcal{S}^2(U,V) \\
&+ (U-1)\partial_U\mathcal{S}^3(U,V) - 2\partial_U\mathcal{S}^4(U,V) + V(\partial_V\mathcal{S}^3(U,V) + 4\partial_V\mathcal{S}^5(U,V)) + 4U\partial_U\mathcal{S}^5(U,V)) \\
&+ \mathcal{S}^2(U,V) + \mathcal{S}^3(U,V) + 2\mathcal{S}^4(U,V)) ,
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
\mathcal{C}^{2,2} = &\frac{1}{32}(U((V-1)\partial_V\mathcal{S}^1(U,V) + U\partial_U\mathcal{S}^1(U,V) + \partial_V\mathcal{S}^2(U,V) - \partial_U\mathcal{S}^2(U,V) + V\partial_V\mathcal{S}^3(U,V) \\
&+ (U-1)\partial_U\mathcal{S}^3(U,V) - 2\partial_U\mathcal{S}^4(U,V)) + \mathcal{S}^2(U,V) + \mathcal{S}^3(U,V) + 2\mathcal{S}^4(U,V)) .
\end{aligned} \tag{A.8}
$$

The expressions for $\langle SSF\chi\rangle$ are:

$$\begin{aligned}\mathcal{E}^{1,1} =& -V\partial_V\mathcal{S}^2(U,V)-(V-1)\partial_U\mathcal{S}^2(U,V)-V\partial_V\mathcal{S}^3(U,V)-(U+V-1)\partial_U\mathcal{S}^3(U,V)\\&-2V\partial_V\mathcal{S}^4(U,V)-2(V-1)\partial_U\mathcal{S}^4(U,V)-2U\partial_U\mathcal{S}^5(U,V)\\&+\frac{(V-1)\mathcal{S}^2(U,V)}{U}+\frac{(V-1)\mathcal{S}^3(U,V)}{U}+\mathcal{S}^5(U,V)+\mathcal{S}^6(U,V)-\frac{(U-2V+2)\mathcal{S}^4(U,V)}{U}\,,\end{aligned} \tag{A.9}$$

$$\mathcal{E}^{3,1} = -U(\partial_V\mathcal{S}^2(U,V)+\partial_U\mathcal{S}^2(U,V)-\partial_U\mathcal{S}^3(U,V))+\mathcal{S}^2(U,V)-\mathcal{S}^3(U,V)\,, \tag{A.10}$$

$$\begin{aligned}\mathcal{E}^{1,2} =& U(-\partial_U\mathcal{S}^2(U,V)+V\partial_V\mathcal{S}^3(U,V)+(U-1)\partial_U\mathcal{S}^3(U,V)-2\partial_U\mathcal{S}^4(U,V)+2V\partial_V\mathcal{S}^5(U,V)\\&+2U\partial_U\mathcal{S}^5(U,V))+\mathcal{S}^2(U,V)+\mathcal{S}^3(U,V)+2\mathcal{S}^4(U,V)\,,\end{aligned} \tag{A.11}$$

$$\mathcal{E}^{3,2} = U(\partial_V\mathcal{S}^2(U,V)-V\partial_V\mathcal{S}^3(U,V)-U\partial_U\mathcal{S}^3(U,V))\,. \tag{A.12}$$

Next we shall give expressions for $\langle SSFF\rangle$ and $\langle SS\overline{F}F\rangle$, which can be computed from $\delta\langle SSSF\rangle$. Unlike the previous correlators, we cannot completely fix these in terms of $\langle SSSS\rangle$. We will instead also leave $\mathcal{F}^{1,1}(U,V)$ and $\mathcal{F}^{2,1}(U,V)$ undetermined. We then find that the other components of $\langle SSFF\rangle$ are:

$$\begin{aligned}\mathcal{F}^{2,1}(U,V) =& \frac{1}{V}\Big(-4UV\partial_V\mathcal{S}^4(U,V)-4UV\partial_U\mathcal{S}^4(U,V)-2(U-2V)\mathcal{S}^4(U,V)\\&+(U-V)\mathcal{F}^{1,1}(U,V)+\mathcal{F}^{1,2}(U,V)\Big)\,,\end{aligned} \tag{A.13}$$

$$\mathcal{F}^{2,2}(U,V) = -\frac{1}{V}\Big(U\big(-4V\partial_V\mathcal{S}^4(U,V)-2\mathcal{S}^4(U,V)+\mathcal{F}^{1,1}(U,V)\big)+\mathcal{F}^{1,2}(U,V)\Big)\,. \tag{A.14}$$

Furthermore, by imposing conservation on $\langle SSSJ\rangle$, we find that $\mathcal{F}^{1,1}(U,V)$ and $\mathcal{F}^{2,1}(U,V)$ are constrained by the Ward identities:

$$
\begin{aligned}
\mathcal{F}^{1,1}(U,V) = \frac{1}{3U} & 2U^3(U+2V-2)\partial_U^2\mathcal{S}^1(U,V) + 2U^2V(U+2V-2)\partial_V^2\mathcal{S}^1(U,V) \\
& + U^2(U+2V-2)\partial_U\mathcal{S}^1(U,V) + 2U^2(U+V-1)(U+2V-2)\partial_U\partial_V\mathcal{S}^1(U,V) \\
& - 2U^2(U+2V-2)\partial_U^2\mathcal{S}^2(U,V) - 2U^2V(U+2V-2)\partial_U^2\mathcal{S}^3(U,V) \\
& + 8U^2V(U-V+1)\partial_U^2\mathcal{S}^4(U,V) - 2UV^2(U+2V-2)\partial_V^2\mathcal{S}^3(U,V) \\
& + 8UV^2(U-V+1)\partial_V^2\mathcal{S}^4(U,V) + U(2U-V+1)(U+2V-2)\partial_V\mathcal{S}^1(U,V) \\
& - 2UV(U+2V-2)\partial_V^2\mathcal{S}^2(U,V) + U(U+2V-2)\partial_U\mathcal{S}^2(U,V) \\
& - 2U(U+V-1)(U+2V-2)\partial_U\partial_V\mathcal{S}^2(U,V) - (U-1)U(U+2V-2)\partial_U\mathcal{S}^3(U,V) \\
& - 2UV(U+V-1)(U+2V-2)\partial_U\partial_V\mathcal{S}^3(U,V) + 4(U-1)U(U-V+1)\partial_U\mathcal{S}^4(U,V) \\
& + 8UV(U-V+1)(U+V-1)\partial_U\partial_V\mathcal{S}^4(U,V) \\
& - (U-2V+2)(U+2V-2)\partial_V\mathcal{S}^2(U,V) \\
& + V(U+2V-2)(-3U+2V-2)\partial_V\mathcal{S}^3(U,V) \\
& + 4V(U-V+1)(3U-2V+2)\partial_V\mathcal{S}^4(U,V) - 2U\Big(U^2 - U(2V+1) \\
& + (V-1)^2\Big)\partial_U\mathcal{F}^{1,1}(U,V) + \Big(U^2(1-2V) + U(4V+3)(V-1) \\
& - 2(V-1)^3\Big)\partial_V\mathcal{F}^{1,1}(U,V) + \Big(U^2 - 3U(V+1) + 2(V-1)^2\Big)\partial_V\mathcal{F}^{1,2}(U,V) \\
& + U(-U+V-1)\partial_U\mathcal{F}^{1,2}(U,V) - (U+2V-2)\mathcal{S}^2(U,V) - (U+2V-2)\mathcal{S}^3(U,V) \\
& + 4(U-V+1)\mathcal{S}^4(U,V)\,,
\end{aligned} \tag{A.15}
$$

$$
\begin{aligned}
\mathcal{F}^{1,2}(U,V) = \frac{1}{3} & \Big(2U^3\partial_U^2\mathcal{S}^1(U,V) + 2U^2V\partial_V^2\mathcal{S}^1(U,V) + U^2\partial_U\mathcal{S}^1(U,V) \\
& + 2U^2(U+V-1)\partial_U\partial_V\mathcal{S}^1(U,V) - 2U^2\partial_U^2\mathcal{S}^2(U,V) - 2U^2V\partial_U^2\mathcal{S}^3(U,V) \\
& - 4U^2V\partial_U^2\mathcal{S}^4(U,V) - 2UV^2\partial_V^2\mathcal{S}^3(U,V) - 4UV^2\partial_V^2\mathcal{S}^4(U,V) \\
& + U(2U-V+1)\partial_V\mathcal{S}^1(U,V) - 2UV\partial_V^2\mathcal{S}^2(U,V) + U\partial_U\mathcal{S}^2(U,V) \\
& - 2U(U+V-1)\partial_U\partial_V\mathcal{S}^2(U,V) - (U-1)U\partial_U\mathcal{S}^3(U,V) \\
& - 2UV(U+V-1)\partial_U\partial_V\mathcal{S}^3(U,V) - 2(U-1)U\partial_U\mathcal{S}^4(U,V) \\
& - 4UV(U+V-1)\partial_U\partial_V\mathcal{S}^4(U,V) - (U-2V+2)\partial_V\mathcal{S}^2(U,V) \\
& + V(-3U+2V-2)\partial_V\mathcal{S}^3(U,V) + 2V(-3U+2V-2)\partial_V\mathcal{S}^4(U,V) \\
& + U(U-V+1)\partial_U\mathcal{F}^{1,1}(U,V) + 2U\partial_U\mathcal{F}^{1,2}(U,V) + \Big(U(V+1) \\
& - (V-1)^2\Big)\partial_V\mathcal{F}^{1,1}(U,V) + (U+V-1)\partial_V\mathcal{F}^{1,2}(U,V) - \mathcal{S}^2(U,V) - \mathcal{S}^3(U,V) \\
& - 2\mathcal{S}^4(U,V)\Big)\,.
\end{aligned} \tag{A.16}
$$

We also find the following expressions for $\langle SS\overline{F}F\rangle$:

$$\begin{aligned}
\mathcal{G}^{1,1}(U,V) = \frac{1}{U}\Big(&- 2U^2\partial_U\mathcal{S}^1(U,V) - 4U^2\partial_U\mathcal{S}^5(U,V) - 2UV\partial_V\mathcal{S}^3(U,V) \\
&- 2U(U+V-1)\partial_U\mathcal{S}^3(U,V) - 4UV\partial_V\mathcal{S}^4(U,V) - 4U(V-1)\partial_U\mathcal{S}^4(U,V) \\
&+ 2U\mathcal{S}^1(U,V) + 2U\mathcal{S}^5(U,V) + 2(V-1)\mathcal{S}^3(U,V) - 2(U-2V+2)\mathcal{S}^4(U,V) \\
&+ (U-V+1)\mathcal{F}^{1,1}(U,V) + \mathcal{F}^{1,2}(U,V)\Big)\,, \qquad \text{(A.17)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}^{2,1}(U,V) = \frac{1}{U}\Big(&4U^2\partial_U\mathcal{S}^5(U,V) - 2UV\partial_V\mathcal{S}^2(U,V) - 2U(V-1)\partial_U\mathcal{S}^2(U,V) + 2UV\partial_V\mathcal{S}^3(U,V) \\
&+ 2U(U+V-1)\partial_U\mathcal{S}^3(U,V) + 4UV\partial_V\mathcal{S}^4(U,V) + 4U(V-1)\partial_U\mathcal{S}^4(U,V) \\
&- 2U\mathcal{S}^5(U,V) + 2U\mathcal{S}^6(U,V) + 2(V-1)\mathcal{S}^2(U,V) - 2(V-1)\mathcal{S}^3(U,V) \\
&+ 2(U-2V+2)\mathcal{S}^4(U,V) - (U-2V+2)\mathcal{F}^{1,1}(U,V) - 2\mathcal{F}^{1,2}(U,V)\Big)\,, \qquad \text{(A.18)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}^{4,1}(U,V) = \frac{1}{V}U\Big(&2V\Big(\partial_V\mathcal{S}^2(U,V) + \partial_U\mathcal{S}^2(U,V) + \partial_U\mathcal{S}^3(U,V) + 2(\partial_V\mathcal{S}^4(U,V) + \partial_U\mathcal{S}^4(U,V))) \\
&+ 2\mathcal{S}^4(U,V) - \mathcal{F}^{1,1}(U,V)\Big) - \mathcal{F}^{1,2}(U,V)\Big) - 2\Big(\mathcal{S}^2(U,V) + \mathcal{S}^3(U,V) + 2\mathcal{S}^4(U,V)\Big)\,, \\
&\qquad \text{(A.19)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}^{1,2}(U,V) = 2U\Big(&U\partial_U\mathcal{S}^1(U,V) + (U-1)\partial_U\mathcal{S}^3(U,V) - 2\partial_U\mathcal{S}^4(U,V) + V(\partial_V\mathcal{S}^1(U,V) \\
&+ \partial_V\mathcal{S}^3(U,V) + 2\partial_V\mathcal{S}^5(U,V)) + 2U\partial_U\mathcal{S}^5(U,V)\Big) + 2\mathcal{S}^3(U,V) + 4\mathcal{S}^4(U,V) \\
&- \mathcal{F}^{1,1}(U,V)\,, \qquad \text{(A.20)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}^{2,2}(U,V) = -2U\Big(&\partial_U\mathcal{S}^2(U,V) + V\partial_V\mathcal{S}^3(U,V) + (U-1)\partial_U\mathcal{S}^3(U,V) - 2\partial_U\mathcal{S}^4(U,V) \\
&+ 2V\partial_V\mathcal{S}^5(U,V) + 2U\partial_U\mathcal{S}^5(U,V)\Big) + 2\mathcal{S}^2(U,V) - 2\mathcal{S}^3(U,V) - 4\mathcal{S}^4(U,V) \\
&+ 2\mathcal{F}^{1,1}(U,V) + \mathcal{F}^{1,2}(U,V)\,, \qquad \text{(A.21)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}^{4,2}(U,V) = \frac{1}{V}\Big(&U(-2V(\partial_V\mathcal{S}^2(U,V) + V\partial_V\mathcal{S}^3(U,V) + U\partial_U\mathcal{S}^3(U,V) + 2\partial_V\mathcal{S}^4(U,V)) - 2\mathcal{S}^4(U,V) \\
&+ \mathcal{F}^{1,1}(U,V)) + \mathcal{F}^{1,2}(U,V)\Big) - \mathcal{F}^{1,2}(U,V)\,. \qquad \text{(A.22)}
\end{aligned}$$

Next we give expressions for $\langle SSPP\rangle$, which can be derived by considering the supersymmetric variation $\delta\langle SSP\chi\rangle$:

$$\begin{aligned}
\mathcal{R}^1(U,V) = {} & 2V^2\partial_V^2\mathcal{S}^2(U,V) + 2V^2\partial_V^2\mathcal{S}^3(U,V) + 4V^2\partial_V^2\mathcal{S}^4(U,V) + 2V(U+V-1)\partial_U\partial_V\mathcal{S}^2(U,V) \\
& + 2UV\partial_U^2\mathcal{S}^2(U,V) - \frac{V(-3U+2V-2)\partial_V\mathcal{S}^3(U,V)}{U} + 2V(U+V-1)\partial_U\partial_V\mathcal{S}^3(U,V) \\
& + 2UV\partial_U^2\mathcal{S}^3(U,V) + 4V(U+V-1)\partial_U\partial_V\mathcal{S}^4(U,V) + 4UV\partial_U^2\mathcal{S}^4(U,V) \\
& + 2V\partial_V\mathcal{S}^5(U,V) - 2V\partial_V\mathcal{S}^6(U,V) + \frac{V(U-2V+2)\partial_V\mathcal{S}^2(U,V)}{U} \\
& + \frac{4V(U-V+1)\partial_V\mathcal{S}^4(U,V)}{U} - U\partial_U\mathcal{S}^1(U,V) - (V+1)\partial_U\mathcal{S}^2(U,V) \\
& - (-U+V+1)\partial_U\mathcal{S}^3(U,V) - 2(-U+V+1)\partial_U\mathcal{S}^4(U,V) - 2U\partial_U\mathcal{S}^6(U,V) + \mathcal{S}^1(U,V) \\
& - \frac{(U-2(V+1))\mathcal{S}^4(U,V)}{U} + \mathcal{S}^5(U,V) + \mathcal{S}^6(U,V) + \frac{(V+1)\mathcal{S}^2(U,V)}{U} \\
& + \frac{(V+1)\mathcal{S}^3(U,V)}{U}\,, \qquad \text{(A.23)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^2(U,V) = {} & -U^2\partial_U\mathcal{S}^1(U,V) - 2U^2V\partial_U^2\mathcal{S}^1(U,V) - 4U^2V\partial_U^2\mathcal{S}^5(U,V) - 2UV^2\partial_V^2\mathcal{S}^1(U,V) \\
& - 4UV^2\partial_V^2\mathcal{S}^5(U,V) + 2V^2\partial_V^2\mathcal{S}^2(U,V) + 2V^2\partial_V^2\mathcal{S}^3(U,V) + 4V^2\partial_V^2\mathcal{S}^4(U,V) \\
& - 2UV(U+V-1)\partial_U\partial_V\mathcal{S}^1(U,V) + 2UV\partial_U^2\mathcal{S}^2(U,V) + 2UV\partial_U^2\mathcal{S}^3(U,V) \\
& + 4UV\partial_U^2\mathcal{S}^4(U,V) - 2(U-1)U\partial_U\mathcal{S}^5(U,V) - 4UV(U+V-1)\partial_U\partial_V\mathcal{S}^5(U,V) \\
& - 2U\partial_U\mathcal{S}^6(U,V) - V(3U-2V+2)\partial_V\mathcal{S}^1(U,V) - (V+1)\partial_U\mathcal{S}^2(U,V) \\
& + 2V(U+V-1)\partial_U\partial_V\mathcal{S}^2(U,V) - (-U+V+1)\partial_U\mathcal{S}^3(U,V) \\
& + 2V(U+V-1)\partial_U\partial_V\mathcal{S}^3(U,V) - 2(-U+V+1)\partial_U\mathcal{S}^4(U,V) \\
& + 4V(U+V-1)\partial_U\partial_V\mathcal{S}^4(U,V) - 2V(3U-2V+1)\partial_V\mathcal{S}^5(U,V) - 2V\partial_V\mathcal{S}^6(U,V) \\
& + \frac{V(U-2V+2)\partial_V\mathcal{S}^2(U,V)}{U} - \frac{V(-3U+2V-2)\partial_V\mathcal{S}^3(U,V)}{U} \\
& + \frac{4V(U-V+1)\partial_V\mathcal{S}^4(U,V)}{U} - \mathcal{S}^5(U,V) + \mathcal{S}^6(U,V) + \frac{(V+1)\mathcal{S}^2(U,V)}{U} \\
& + \frac{(V+1)\mathcal{S}^3(U,V)}{U} - \frac{(U-2(V+1))\mathcal{S}^4(U,V)}{U}\,, \qquad \text{(A.24)}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{R}^4(U,V) = & -\frac{1}{2}\Big(2\Big(-2U^2-(U+3)V+U+2V^2+1\Big)\partial_V\mathcal{S}^5(U,V)+2U^2\partial_U\mathcal{S}^1(U,V) \\
& +2U^2(2U+V-1)\partial_U^2\mathcal{S}^1(U,V)+4U^2(U+V-1)\partial_U^2\mathcal{S}^5(U,V) \\
& +\Big(4U^2+U-2(V-1)^2\Big)\partial_V\mathcal{S}^1(U,V)-\frac{\Big(U^2+U(4-3V)+2(V-1)^2\Big)\partial_V\mathcal{S}^2(U,V)}{U} \\
& -\frac{2\Big(U^2+U-2(V-1)^2\Big)\partial_V\mathcal{S}^4(U,V)}{U} \\
& -\frac{\Big((U(2U-1)-4)V-2(U+1)(U-1)^2-2V^2\Big)\partial_V\mathcal{S}^3(U,V)}{U} \\
& +2UV(2U+V-1)\partial_V^2\mathcal{S}^1(U,V)+2U(U+V-1)(2U+V-1)\partial_U\partial_V\mathcal{S}^1(U,V) \\
& -2U(2U+V-1)\partial_U^2\mathcal{S}^2(U,V)-2U\Big(V-(U-1)^2\Big)\partial_U^2\mathcal{S}^3(U,V) \\
& -4U(U+V-1)\partial_U^2\mathcal{S}^4(U,V)+4UV(U+V-1)\partial_V^2\mathcal{S}^5(U,V)+2(U-1)U\partial_U\mathcal{S}^5(U,V) \\
& +4U(U+V-1)^2\partial_U\partial_V\mathcal{S}^5(U,V)+2U\partial_U\mathcal{S}^6(U,V)-2V(2U+V-1)\partial_V^2\mathcal{S}^2(U,V) \\
& +(3U+V-1)\partial_U\mathcal{S}^2(U,V)-2(U+V-1)(2U+V-1)\partial_U\partial_V\mathcal{S}^2(U,V) \\
& -2V\Big(V-(U-1)^2\Big)\partial_V^2\mathcal{S}^3(U,V)+(U+V-1)\partial_U\mathcal{S}^3(U,V) \\
& +2\Big((U-1)^2-V\Big)(U+V-1)\partial_U\partial_V\mathcal{S}^3(U,V)-4V(U+V-1)\partial_V^2\mathcal{S}^4(U,V) \\
& +2(U+V-1)\partial_U\mathcal{S}^4(U,V)-4(U+V-1)^2\partial_U\partial_V\mathcal{S}^4(U,V)+2(U+V-1)\partial_V\mathcal{S}^6(U,V) \\
& +\mathcal{S}^5(U,V)-\mathcal{S}^6(U,V)-\frac{(3U+V-1)\mathcal{S}^2(U,V)}{U}-\frac{(2U+V-1)\mathcal{S}^3(U,V)}{U} \\
& -\frac{(3U+2V-2)\mathcal{S}^4(U,V)}{U}\Big), \qquad \text{(A.25)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}^5(U,V) = & -\frac{1}{2}\Big(-2U^2\partial_U^2\mathcal{S}^1(U,V)-2V^2\partial_V^2\mathcal{S}^3(U,V)-4V^2\partial_V^2\mathcal{S}^4(U,V)+2UV\partial_V^2\mathcal{S}^1(U,V) \\
& +2U(U+V-1)\partial_U\partial_V\mathcal{S}^1(U,V)-2U(U+V)\partial_U^2\mathcal{S}^2(U,V)-2UV\partial_U^2\mathcal{S}^3(U,V) \\
& -4UV\partial_U^2\mathcal{S}^4(U,V)+2U\partial_U\mathcal{S}^6(U,V)+(U-2V+2)\partial_V\mathcal{S}^1(U,V) \\
& -2V(U+V)\partial_V^2\mathcal{S}^2(U,V)+(U+V+1)\partial_U\mathcal{S}^2(U,V) \\
& -2(U+V-1)(U+V)\partial_U\partial_V\mathcal{S}^2(U,V)-(U-V-1)\partial_U\mathcal{S}^3(U,V) \\
& -2V(U+V-1)\partial_U\partial_V\mathcal{S}^3(U,V)-2(U-V-1)\partial_U\mathcal{S}^4(U,V) \\
& -4V(U+V-1)\partial_U\partial_V\mathcal{S}^4(U,V)-2V\partial_V\mathcal{S}^5(U,V)+2V\partial_V\mathcal{S}^6(U,V) \\
& -\frac{(U+V)(U-2V+2)\partial_V\mathcal{S}^2(U,V)}{U}-\frac{V(3U-2V+2)\partial_V\mathcal{S}^3(U,V)}{U} \\
& -\frac{4V(U-V+1)\partial_V\mathcal{S}^4(U,V)}{U}-\mathcal{S}^5(U,V)-\mathcal{S}^6(U,V)-\frac{(U+V+1)\mathcal{S}^2(U,V)}{U} \\
& -\frac{(V+1)\mathcal{S}^3(U,V)}{U}-\frac{(-U+2V+2)\mathcal{S}^4(U,V)}{U}\Big). \qquad \text{(A.26)}
\end{aligned}
$$

Appendix B

Characters of $\mathfrak{osp}(6|4)$

In this appendix we review the character formulas of $\mathfrak{osp}(6|4)$, which were computed in [31], as well as their decomposition under $\mathfrak{osp}(6|4) \to \mathfrak{so}(3,2) \oplus \mathfrak{so}(6)$. This decomposition was used in Chapter 2 to determine which conformal primaries reside in each supermultiplet appearing in the $S \times S$ OPE.

The $\mathfrak{osp}(6|4)$ characters are defined in terms of the quantum numbers and generators given in Section 2.2 as

$$\chi_{(\Delta;j;\mathrm{r})}(s,x,y) \equiv \mathrm{Tr}_{\mathcal{R}_{(\Delta;j;\mathrm{r})}} \left(s^{2D} x^{2J_3} y_1^{H_1} y_2^{H_2} y_3^{H_3} \right) . \tag{B.1}$$

Their explicit form for the multiplets we consider are

$$\begin{aligned} \chi^{(A,\pm)}_{(\Delta;j;r,r,r)}(s,x,y) = s^{2\Delta} P(s,x) & \sum_{a_1,a_2,a_3=0}^{2} \sum_{\bar{a}_1,\bar{a}_2,\bar{a}_3=0}^{1} s^{a_1+a_2+a_3+\bar{a}_1+\bar{a}_2+\bar{a}_3} \chi_{2j+\bar{a}_1+\bar{a}_2+\bar{a}_3}(x) \\ & \times \left(\prod_{i=1}^{3} \chi_{j_{a_i}}(x) \right) \chi_{(r+\bar{a}_1-a_1,\, r+\bar{a}_2-a_2,\, \pm r \pm \bar{a}_3 \mp a_3)}(y) , \end{aligned} \tag{B.2}$$

$$\chi^{(B,+)}_{(\Delta;0;r,r,r)}(s,x,y) = s^{2\Delta} P(s,x) \sum_{a_1,a_2,a_3=0}^{2} s^{a_1+a_2+a_3} \left(\prod_{i=1}^{3} \chi_{j_{a_i}}(x) \right) \chi_{(r-a_1,\, r-a_2,\, r-a_3)}(y) , \tag{B.3}$$

$$\chi^{(A,n)}_{(\Delta;j;r_1,\ldots,r_1,r_{n+1},\ldots,r_3)}(s,x,y) = s^{2\Delta}P(s,x)\sum_{a_1,a_2,a_3=0}^{2}\;\sum_{\bar{a}_{n+1},\ldots,\bar{a}_3=0}^{2}\;\sum_{\bar{a}_1,\ldots,\bar{a}_n=0}^{1} s^{a_1+a_2+a_3+\bar{a}_1+\bar{a}_2+\bar{a}_3}\chi_{2j+\bar{a}_1+\cdots+\bar{a}_n}(x)$$
$$\times\left(\prod_{i=n+1}^{3}\chi_{j_{\bar{a}_i}}(x)\right)\left(\prod_{i=1}^{3}\chi_{j_{a_i}}(x)\right)\chi_{(r_1+\bar{a}_1-a_1,r_2+\bar{a}_2-a_2,r_3+\bar{a}_3-a_3)}(y)\,, \tag{B.4}$$

$$\chi^{(B,n)}_{(\Delta;0;r_1,\ldots,r_1,r_{n+1},\ldots,r_3)}(s,x,y) = s^{2\Delta}P(s,x)\sum_{a_1,a_2,a_3,\bar{a}_{n+1},\ldots,\bar{a}_3=0}^{2} s^{a_1+a_2+a_3+\bar{a}_{n+1}+\cdots+\bar{a}_3}\left(\prod_{i=n+1}^{3}\chi_{j_{\bar{a}_i}}(x)\right)$$
$$\times\left(\prod_{i=1}^{3}\chi_{j_{a_i}}(x)\right)\chi_{(r_1-a_1,\ldots,r_1-a_n,r_{n+1}+\bar{a}_{n+1}-a_{n+1},\ldots,r_3+\bar{a}_3-a_3)}(y)\,, \tag{B.5}$$

where the long multiplet corresponds to $(A,0)$, we define $j_a \equiv a \pmod 2$, the $\mathfrak{su}(2)$ and $\mathfrak{so}(6)$ characters are

$$\chi_j(x) = \frac{x^{j+1}-x^{-j-1}}{x-x^{-1}}\,, \tag{B.6}$$
$$\chi_{\mathrm{r}}(y) = \frac{\det\left[y_i^{r_j+3-j}+y_i^{-r_j-3+j}\right]+\det\left[y_i^{r_j+3-j}-y_i^{-r_j-3+j}\right]}{2\prod_{1\le i<j\le 3}(y_i+y_i^{-1}-y_j-y_j^{-1})}\,, \tag{B.7}$$

and the function $P(s,x)$ is related to the $\mathfrak{so}(3,2)$ character and takes the form

$$P(s,x) = \frac{1}{1-s^4}\sum_{n=0}^{\infty}s^{2n}\chi_{2n}(x)\,. \tag{B.8}$$

The products of the $\mathfrak{su}(2)$ characters in (B.2)–(B.5) are easily transformed into sums of such characters by decomposing $\mathfrak{su}(2)$ tensor products. After doing so, we see that (B.2)–(B.5) become sums over $\mathfrak{so}(3,2)\oplus\mathfrak{so}(6)$ characters, as desired.[1]

[1]Sometimes the $\mathfrak{so}(6)$ characters in (B.2)–(B.5) appear with negative Dynkin labels. One can then try to use the identity

$$\chi_{\mathrm{r}^\omega}(y) = (-)^{\ell(\omega)}\chi_{\mathrm{r}}(y)\,,$$

to obtain a character with non-negative Dynkin labels. In this identity ω is an element of the $\mathfrak{so}(6)$ Weyl group S_4, $\mathrm{r}^\omega = \omega(\mathrm{r}+\rho)-\rho$ is a Weyl reflection, $\rho=(2,1,0)$ is the Weyl vector, and $(-)^{\ell(\omega)}$ is the signature of the Weyl transformation. If there is no Weyl transformation such that r^ω correspond to non-negative integer Dynkin labels, then $\chi_{\mathrm{r}}=0$.

Appendix C

Decomposing $\mathcal{N} = 8$ Superblocks to $\mathcal{N} = 6$

In this appendix we discuss how the superblocks that appeared in the four-point function of the $\mathcal{N} = 8$ stress tensor superprimary $\overline{S}$ decompose into the $\mathcal{N} = 6$ superblocks discussed for $\langle SSSS \rangle$ in the main text. This serves as both a consistency check of our $\mathcal{N} = 6$ superblocks, and also allows us to translate the $\mathcal{N} = 8$ numerical bootstrap results of [76, 81, 178, 179] into $\mathcal{N} = 6$ language, which we use to compare to the $\mathcal{N} = 6$ results in Chapter 6. $\overline{S}$ transforms in the $\mathbf{35_c}$ of the $\mathcal{N} = 8$ R-symmetry group $SO(8)$, which decomposes to $SO(6) \times U(1)$ as

$$\mathbf{35} \to \mathbf{15}_0 \oplus \mathbf{10}_2 \oplus \overline{\mathbf{10}}_{-2} \,, \tag{C.1}$$

so $\overline{S}$ decomposes to S as well as the superprimaries of the multiplets $(B, +)^{020}_{1,0}$ and $(B, -)^{002}_{1,0}$ that are charged under $U(1)$. Since we are only interested in correlators of S, we will always restrict to $U(1)$ singlets when decomposing from $\mathcal{N} = 8$ to $\mathcal{N} = 6$ in this appendix, which we will denote using an arrow instead of an equality.

The $\mathcal{N} = 8$ multiplets that appear in $\overline{S} \times \overline{S}$ are listed in Table C.1.[1] We can decompose the characters for these superblocks, as computed in [178], into the characters of the $\mathcal{N} = 6$ superblocks

[1] In [178], the long multiplet was denoted as $(A, 0)$.

Type	(Δ, ℓ)	$\mathfrak{so}(8)_R$ irrep	spin ℓ	BPS
$(B,+)$	$(1,0)$	$\mathbf{35}_c = [0020]$	0	1/2
$(B,+)$	$(2,0)$	$\mathbf{294}_c = [0040]$	0	1/2
$(B,2)$	$(2,0)$	$\mathbf{300} = [0200]$	0	1/4
$(A,+)$	$(\ell+2,\ell)$	$\mathbf{35}_c = [0020]$	even	1/4
$(A,2)$	$(\ell+2,\ell)$	$\mathbf{28} = [0100]$	odd	1/8
$(A,\text{cons.})$	$(\ell+1,\ell)$	$\mathbf{1} = [0000]$	even	5/16
Long	$\Delta \geq \ell+1$	$\mathbf{1} = [0000]$	even	0

Table C.1: The possible superconformal multiplets in the $\overline{S}\times\overline{S}$ OPE. The $\mathfrak{so}(3,2)\oplus\mathfrak{so}(8)_R$ quantum numbers are those of the superconformal primary in each multiplet.

as computed from the previous Appendix to get the following decomposition of multiplets:

$$\begin{aligned}
(B,+)^{[0020]}_{1,0} &\quad\to\quad (B,2)^{[011]}_{1,0}\,,\\
(B,+)^{[0040]}_{2,0} &\quad\to\quad (B,2)^{[022]}_{2,0}\,,\\
(B,2)^{[0200]}_{2,0} &\quad\to\quad (B,2)^{[022]}_{2,0}\oplus(B,1)^{[200]}_{2,0}\oplus(A,2)^{[011]}_{2,0}\oplus(A,0)^{[000]}_{2,0}\,,\\
(A,+)^{[0020]}_{\ell+2,\ell} &\quad\to\quad (A,+)^{[020]}_{\ell+5/2,\ell+1/2}\oplus(A,-)^{[002]}_{\ell+5/2,\ell+1/2}\oplus(A,2)^{[011]}_{\ell+2,\ell}\oplus(A,2)^{[011]}_{\ell+3,\ell+1}\,,\\
(A,2)^{[0100]}_{\ell+2,\ell} &\quad\to\quad (A,2)^{[011]}_{\ell+2,\ell}\oplus(A,2)^{[011]}_{\ell+3,\ell+1}\oplus 2\times(A,1)^{[100]}_{\ell+5/2,\ell+1/2}\oplus(A,0)^{[000]}_{\ell+3,\ell+1}\oplus(A,0)^{[000]}_{\ell+2,\ell}\,,\\
\text{Long}^{[0000]}_{\Delta,\ell} &\quad\to\quad \text{Long}^{[000]}_{\Delta,\ell}\oplus\text{Long}^{[000]}_{\Delta+1,\ell-1}\oplus 2\times\text{Long}^{[000]}_{\Delta+1,\ell}\oplus\text{Long}^{[000]}_{\Delta+1,\ell+1}\oplus\text{Long}^{[000]}_{\Delta+2,\ell}\,,\\
(A,\text{cons.})^{[0000]}_{\ell+1,\ell} &\quad\to\quad (A,\text{cons.})^{[000]}_{\ell+1,\ell}\oplus(A,\text{cons.})^{[000]}_{\ell+2,\ell+1}\,,
\end{aligned}\tag{C.2}$$

where $2\times$ denotes that the multiplet appears twice.

The $\mathcal{N}=8$ stress tensor correlator was written in [178] in the basis

$$\begin{aligned}
\langle\overline{S}(\vec{x}_1,Y_1)\overline{S}(\vec{x}_2,Y_2)\overline{S}(\vec{x}_3,Y_3)\overline{S}(\vec{x}_4,Y_4)\rangle = \frac{1}{x_{12}^2x_{34}^2}\Big[&\overline{\mathcal{S}^1}(U,V)Y_{12}^2Y_{34}^2+\overline{\mathcal{S}^2}(U,V)Y_{13}^2Y_{24}^2+\overline{\mathcal{S}^3}(U,V)Y_{14}^2Y_{23}^2\\
&+\overline{\mathcal{S}^4}(U,V)Y_{13}Y_{14}Y_{23}Y_{24}+\overline{\mathcal{S}^5}(U,V)Y_{12}Y_{14}Y_{23}Y_{34}+\overline{\mathcal{S}^6}(U,V)Y_{12}Y_{13}Y_{24}Y_{34}\Big]\,,
\end{aligned}\tag{C.3}$$

where Y are $\mathfrak{so}(8)$ null vectors. As shown in Section 4.3.2,[2] this decomposes to the $\mathcal{N}=6$ basis in

[2]This was for a basis of $SO(8)$ matrices $\overline{X}$, but as noted there its the exact same decomposition for the basis of Y's.

(2.5) as

$$\{\overline{\mathcal{S}^1}\,,\overline{\mathcal{S}^2}\,,\overline{\mathcal{S}^3}\,,\overline{\mathcal{S}^4}\,,\overline{\mathcal{S}^5}\,,\overline{\mathcal{S}^6}\}\to\{\mathcal{S}^1\,,\mathcal{S}^2\,,\mathcal{S}^3\,,4\mathcal{S}^4\,,4\mathcal{S}^5\,,4\mathcal{S}^6\}\,. \tag{C.4}$$

Finally, we can decompose the explicit $\mathcal{N}=8$ superblocks $\mathfrak{G}_{\mathcal{M}^{[d_1d_2d_3d_4]}_{\Delta,\ell}}(U,V)$ given in [178] into the $\mathcal{N}=6$ superblocks $\mathfrak{G}_{\mathcal{M}^{[d_1d_2d_3]}_{\Delta,\ell}}(U,V)$ given in the attached `Mathematica` file to get

$$\begin{aligned}
\mathfrak{G}_{(B,+)^{[0020]}_{1,0}} &\to \frac{1}{4}\mathfrak{G}_{(B,2)^{[011]}_{1,0}}\,,\\
\mathfrak{G}_{(B,+)^{[0040]}_{2,0}} &\to \frac{1}{4}\mathfrak{G}_{(B,2)^{[022]}_{2,0}}\,,\\
\mathfrak{G}_{(B,2)^{[0200]}_{2,0}} &\to \frac{1}{16}\mathfrak{G}_{(B,2)^{[022]}_{2,0}}+\frac{1}{3}\mathfrak{G}_{(A,2)^{[011]}_{2,0}}+\frac{4}{35}\mathfrak{G}_{\text{Long}^{[000],1}_{2,0}}+\frac{1}{8}\mathfrak{G}_{(B,1)^{[200]}_{2,0}}\,,\\
\mathfrak{G}_{(A,+)^{[0020]}_{\ell+2,\ell}} &\to \frac{1}{4}\mathfrak{G}_{(A,2)^{[011]}_{\ell+2,\ell}}+\frac{(4+\ell)^2}{(5+2\ell)(7+2\ell)}\mathfrak{G}_{(A,2)^{[011]}_{\ell+3,\ell+1}}+\frac{1+\ell}{4+2\ell}\mathfrak{G}_{(A,+)^{[020]}_{\ell+5/2,\ell+1/2}}\,,\\
\mathfrak{G}_{(A,2)^{[0020]}_{\ell+2,\ell}} &\to -\frac{1}{4}\mathfrak{G}_{(A,2)^{[011]}_{\ell+2,\ell}}-\frac{(3+\ell)^2}{(3+2\ell)(5+2\ell)}\mathfrak{G}_{(A,2)^{[011]}_{\ell+3,\ell+1}}-\frac{\ell^2}{3+4\ell(2+\ell)}\mathfrak{G}_{\text{Long}^{[000]}_{\ell+2,\ell}}\\
&\quad-\frac{(5+\ell)^2}{(7+2\ell)(9+2\ell)}\mathfrak{G}_{\text{Long}^{[000],1}_{\ell+3,\ell+1}}-\frac{1+\ell}{9+3\ell}\mathfrak{G}_{(A,1)^{[000],1}_{\ell+5/2,\ell+1/2}}-\mathfrak{G}_{(A,1)^{[000],2}_{\ell+5/2,\ell+1/2}}\,,\\
\mathfrak{G}_{\text{Long}^{[0000]}_{\Delta,\ell}} &\to \mathfrak{G}_{\text{Long}^{[000],1}_{\Delta,\ell}}+\frac{4(\ell-1)^2(-\Delta+\ell+1)(\Delta+\ell)}{(2\ell-1)(2\ell+1)(\ell-\Delta)(\Delta+\ell+1)}\mathfrak{G}_{\text{Long}^{[000]}_{\Delta+1,\ell-1}}\\
&\quad+\frac{4\Delta(-\Delta+\ell+1)(\Delta+\ell)}{3(\Delta+2)(\ell-\Delta)(\Delta+\ell+1)}\mathfrak{G}_{\text{Long}^{[000],2}_{\Delta+1,\ell}}+\frac{4(-\Delta+\ell+1)(\Delta+\ell)}{(\ell-\Delta)(\Delta+\ell+1)}\mathfrak{G}_{\text{Long}^{[000],3}_{\Delta+1,\ell}}\\
&\quad+\frac{4(\ell+1)(\ell+2)(\Delta+\ell)(\Delta+\ell+2)}{(2\ell+1)(2\ell+3)(\Delta+\ell+1)(\Delta+\ell+3)}\mathfrak{G}_{\text{Long}^{[000]}_{\Delta+1,\ell+1}}\\
&\quad+\frac{4(\Delta+4)^2(-\Delta+\ell+1)(\Delta+\ell)}{(2\Delta+5)(2\Delta+7)(\ell-\Delta)(\Delta+\ell+1)}\mathfrak{G}_{\text{Long}^{[000],1}_{\Delta+2,\ell}}\,,\\
\mathfrak{G}_{(A,\text{cons.})^{[0000]}_{\ell+1,\ell}} &\to \mathfrak{G}_{(A,\text{cons.})^{[000]}_{\ell+1,\ell}}+\mathfrak{G}_{(A,\text{cons.})^{[000]}_{\ell+2,\ell+1}}\,,
\end{aligned} \tag{C.5}$$

where for $\mathfrak{G}_{\text{Long}^{[0000]}_{\Delta,0}}$ we should ignore the $\mathfrak{G}_{\text{Long}^{[000]}_{\Delta+1,-1}}$ and $\mathfrak{G}_{\text{Long}^{[000],3}_{\Delta+1,0}}$ terms on the RHS, and rescale $\mathfrak{G}_{\text{Long}^{[000],2}_{\Delta+1,0}}$ by $\frac{\Delta-1}{\Delta+2}$.

Appendix D

$\bar{D}$ Functions

In this appendix, we list useful properties of D and $\bar{D}$ functions. By definition, the D function is the quartic contact Witten diagram

$$D_{r_1,r_2,r_3,r_4}(x_i) = \int_{AdS_4} dz \prod_{i=1}^{4} G_{B\partial}^{r_i}(z,\vec{x}_i)\,, \qquad G_{B\partial}^{r}(z,\vec{x}) = \left(\frac{z_0}{z_0^2 + (\vec{z}-\vec{x})^2}\right)^r \tag{D.1}$$

and the $\bar{D}$ function is defined in terms of the D function as

$$\bar{D}_{r_1,r_2,r_3,r_4}(U,V) = \frac{x_{13}^{\frac{1}{2}\sum_{i=1}^4 r_i - r_4} x_{24}^{r_2}}{x_{14}^{\frac{1}{2}\sum_{i=1}^4 r_i - r_1 - r_4} x_{34}^{\frac{1}{2}\sum_{i=1}^4 r_i - r_3 - r_4}} \frac{2\prod_{i=1}^4 \Gamma(r_i)}{\pi^{\frac{3}{2}} \Gamma\left(\frac{-3+\sum_{i=1}^4 r_i}{2}\right)} D_{r_1,r_2,r_3,r_4}(x_i)\,. \tag{D.2}$$

When $\sum_i r_i = 3$, this definition of $\bar{D}$ becomes singular, however, for that special case we can alternatively define:

$$\bar{D}_{r_1,r_2,r_3,r_4}(U,V) = \frac{\prod_{i=1}^4 \Gamma(r_i)}{\pi^{3/2}} \frac{x_{13}^{3-2r_4} x_{24}^{2r_2}}{x_{14}^{3-2r_1-2r_4} x_{34}^{3-2r_3-2r_4}} \int d^3x \prod_{i=1}^{4} \frac{1}{|\vec{x}-\vec{x}_i|^{2r_i}}\,. \tag{D.3}$$

$\bar{D}$ functions take a particularly simple form in Mellin space:

$$\begin{aligned} \bar{D}_{r_1,r_2,r_3,r_4}(U,V) = \int_{-i\infty}^{\infty} \frac{ds\,dt}{(4\pi i)^2} U^{\frac{1}{2}(s+r_1-r_2)} V^{\frac{1}{2}(u-r_1-r_2)} \Gamma\left(\frac{r_1+r_2-s}{2}\right) \Gamma\left(\frac{r_3+r_4-s}{2}\right) \\ \times \Gamma\left(\frac{r_1+r_3-t}{2}\right) \Gamma\left(\frac{r_2+r_4-t}{2}\right) \Gamma\left(\frac{r_1+r_4-u}{2}\right) \Gamma\left(\frac{r_2+r_3-u}{2}\right)\,. \end{aligned} \tag{D.4}$$

Using (D.4), polynomial Mellin amplitudes can be converted into sums of $\bar{D}$ functions multiplied by powers of U and V.

We can relate $\bar{D}_{r_1,r_2,r_3,r_4}(U,V)$ to each other using the relations [158, 217]

$$
\begin{aligned}
\bar{D}_{r_1+1,r_2+1,r_3,r_4} &= -\partial_U \bar{D}_{r_1,r_2,r_3,r_4}\,, \\
\bar{D}_{r_1,r_2,r_3+1,r_4+1} &= \left(\frac{r_3+r_4-r_1-r_2}{2} - U\partial_U\right)\bar{D}_{r_1,r_2,r_3,r_4}\,, \\
\bar{D}_{r_1,r_2+1,r_3+1,r_4} &= -\partial_V \bar{D}_{r_1,r_2,r_3,r_4}\,, \\
\bar{D}_{r_1+1,r_2,r_3,r_4+1} &= \left(\frac{r_1+r_4-r_2-r_3}{2} - V\partial_V\right)\bar{D}_{r_1,r_2,r_3,r_4}\,, \\
\bar{D}_{r_1,r_2+1,r_3,r_4+1} &= \left(r_2 + U\partial_U + V\partial_V\right)\bar{D}_{r_1,r_2,r_3,r_4}\,, \\
\bar{D}_{r_1+1,r_2,r_3+1,r_4} &= \left(\frac{r_1+r_2+r_3-r_4}{2} + V\partial_V + U\partial_U\right)\bar{D}_{r_1,r_2,r_3,r_4}\,,
\end{aligned}
\tag{D.5}
$$

which can easily be checked using the Mellin space expression (D.4). In particular, we can combine the first and second equations of (D.5) to derive D function relation

$$
\begin{aligned}
&4r_1r_2x_{12}^2 D_{r_1+1,r_2+1,r_3,r_4} - 4r_3r_4x_{34}^2 D_{r_1,r_2,r_3+1,r_4+1} \\
&\qquad = (r_1+r_2-r_3-r_4)(3-r_1-r_2-r_3-r_4)D_{r_1,r_2,r_3,r_4}\,,
\end{aligned}
\tag{D.6}
$$

which, along with its crossings, will prove useful when computing shadow transforms Chapter 5.

The shadow transform of a D-function is another D-function:

$$
\int \frac{d^3y}{|\vec{x}_4-\vec{y}|^{6-2r_4}} D_{r_1,r_2,r_3,r_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{y}) = \frac{\pi^{3/2}\Gamma\left(r_4-\frac{3}{2}\right)}{\Gamma(r_4)} D_{r_1,r_2,r_3,3-r_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4)\,. \tag{D.7}
$$

This identity is a straightforward consequence of the fact that the shadow transform of a bulk-boundary propagator is another bulk-boundary propagator [218]:

$$
\int \frac{d^3y}{|\vec{x}-\vec{y}|^{6-2r}} G^r_{B\partial}(z,\vec{y}) = \frac{\pi^{3/2}\Gamma\left(r-\frac{3}{2}\right)}{\Gamma(r)} G^{3-r}_{B\partial}(z,\vec{x})\,, \tag{D.8}
$$

Finally, when computing the superconformal expansion of $\mathcal{S}^i(U,V)$ in the higher-spin limit in Chapter 3, we will need to compute the $U \sim 0$, $V \sim 1$ expansion of certain $\bar{D}$ functions. General expressions are given [219], and applying these to the cases of interest to us, we find that

$$
\begin{aligned}
\bar{D}_{1,1,\frac{1}{2},\frac{1}{2}}(U,V) &= \sum_{m,n=0}^{\infty} \frac{\pi U^m (1-V)^n \left(\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+n+\frac{1}{2}\right)^2}{\sqrt{U}\Gamma(2m+n+1)} - \frac{\Gamma(m+1)^2\Gamma(m+n+1)^2}{\Gamma\left(m+\frac{3}{2}\right)\Gamma(2m+n+2)} \right)}{m!n!}\,, \\
\bar{D}_{\frac{1}{2},1,1,\frac{1}{2}}(U,V) &= -\sum_{m,n=0}^{\infty} \left[\frac{2U^m(1-V)^n \Gamma\left(m+\frac{1}{2}\right)^2 (m+n)!^2}{m!^2 n! \Gamma\left(2m+n+\frac{3}{2}\right)} \Big(\psi(m+n+1) - \psi\left(2m+n+\frac{3}{2}\right) \right. \\
&\qquad \left. + \psi\left(m+\frac{1}{2}\right) - \psi^{(0)}(m+1) \Big) + \frac{1}{2}\log U \right], \\
\bar{D}_{1,\frac{1}{2},1,\frac{1}{2}}(U,V) &= \sum_{m,n=0}^{\infty} \left[\frac{\sqrt{\pi} U^m (1-V)^n \Gamma\left(m+\frac{1}{2}\right)\Gamma(2m+2n+1)}{4^{m+n} m! n! \Gamma\left(2m+n+\frac{3}{2}\right)} \Big(2\psi\left(2m+n+\frac{3}{2}\right) \right. \\
&\qquad \left. - 2\psi(2m+2n+1) - \psi\left(m+\frac{1}{2}\right) + \psi(m+1) + \log(4) \Big) + \frac{1}{2}\log U \right],
\end{aligned}
\tag{D.9}
$$

where $\psi(x)$ is the Digamma function. Note in particular the $\log U$ dependence in the last two expressions.

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