

Intertwining algebras of quantum superintegrable systems on the hyperboloid

J A Calzada¹, Ş Kuru², J Negro³ and M A del Olmo³

¹Departamento de Matemática Aplicada, Escuela Superior de Ingenieros Industriales, Universidad de Valladolid, 47011 Valladolid, Spain

²Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey

³Departamento de Física Teórica, Atómica y Óptica, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain

E-mail: juacal@eis.uva.es, kuru@science.ankara.edu.tr, jnegro@fta.uva.es, olmo@fta.uva.es

Abstract. A class of quantum superintegrable Hamiltonians defined on a two-dimensional hyperboloid is considered together with a set of intertwining operators connecting all of them. It is shown that such intertwining operators close a $su(2,1)$ Lie algebra and determine the Hamiltonians through the Casimir operators. The physical states are characterized as unitary representations of $su(2,1)$.

1. Introduction

In this work we will consider a quantum system superintegrable living in a two-dimensional hyperboloid of two-sheets. Although this system is well known in the literature [1]-[6] and can be dealt with standard procedures [7]-[9], it will be studied here under a different point of view based on the properties of intertwining operators (IO), a form of Darboux transformations [10]. We will see how this approach can give a simple explanation of the main features of this physical system. The intertwining operators and integrable Hamiltonians have been studied in previous references [11]-[14], but we will supply here a thorough non-trivial application by means of this example. Besides, there are several points of interest for the specific case here considered.

The intertwining operators are first order differential operators connecting different Hamiltonians in the same class (called hierarchy) and they are associated to separable coordinates of the Hamiltonians. We will obtain just a complete set of such intertwining operators, in the sense that any of the Hamiltonians can be expressed in terms of these operators.

In our case the IO's close an algebraic structure which is the non-compact Lie algebra $su(2,1)$ (see [15] for a compact case). This structure allows us to characterize the discrete spectrum and the corresponding eigenfunctions of the system that constitute a (infinite dimensional) unitary representation. The construction of such representations is not so standard as for compact Lie algebras. We will compute the ground state and characterize the representation space of the wave-functions which share the same energy. Notice that this system includes also a continuum spectrum, but we will not go into this point here.

2. Parametrizations of the two-sheet hyperboloid

Let us consider the two-dimensional two-sheet hyperboloid $s_0^2 + s_1^2 - s_2^2 = -1$, with positive metric $ds^2 = -ds_0^2 - ds_1^2 + ds_2^2$. On this surface, we define the following Hamiltonian

$$H_\ell = J_2^2 - J_1^2 - J_0^2 - \frac{l_2^2 - \frac{1}{4}}{s_2^2} + \frac{l_1^2 - \frac{1}{4}}{s_1^2} + \frac{l_0^2 - \frac{1}{4}}{s_0^2}, \quad (1)$$

where $\ell = (l_0, l_1, l_2) \in R^3$, and the differential operators

$$J_0 = s_1 \partial_2 + s_2 \partial_1, \quad J_1 = s_2 \partial_0 + s_0 \partial_2, \quad J_2 = s_0 \partial_1 - s_1 \partial_0, \quad (2)$$

constitute a realization of the $so(2, 1)$ Lie algebra with Lie commutators

$$[J_0, J_1] = -J_2, \quad [J_2, J_0] = J_1, \quad [J_1, J_2] = J_0.$$

The generator J_2 corresponds to a rotation around the axis s_2 , while the generators J_0 and J_1 give pseudo-rotations (i.e., non-compact rotations) around the axes s_0 and s_1 , respectively. The Casimir operator

$$C = J_0^2 + J_1^2 - J_2^2$$

is the ‘kinetic’ part of the Hamiltonian.

We can parametrize the hyperbolic surface by means of the ‘analogue’ of the spherical coordinates

$$s_0 = \sinh \xi \cos \theta, \quad s_1 = \sinh \xi \sin \theta, \quad s_2 = \cosh \xi, \quad (3)$$

where $0 \leq \theta < 2\pi$, $0 \leq \xi < \infty$ and the invariant measure is $d\mu(\xi, \theta) = \sinh \xi d\xi d\theta$. In this coordinate system, the infinitesimal generators (2) have the following expressions

$$J_0 = \sin \theta \partial_\xi + \cos \theta \coth \xi \partial_\theta, \quad J_1 = \cos \theta \partial_\xi - \sin \theta \coth \xi \partial_\theta, \quad J_2 = \partial_\theta. \quad (4)$$

Also using these coordinates (3), the Hamiltonian (1) is rewritten as

$$H_\ell = -\partial_\xi^2 - \coth \xi \partial_\xi - \frac{l_2^2 - \frac{1}{4}}{\cosh^2 \xi} + \frac{1}{\sinh^2 \xi} \left[-\partial_\theta^2 + \frac{l_1^2 - \frac{1}{4}}{\sin^2 \theta} + \frac{l_0^2 - \frac{1}{4}}{\cos^2 \theta} \right]. \quad (5)$$

Therefore, H_ℓ can be separated in the variables ξ and θ . Choosing its eigenfunctions Φ ($H_\ell \Phi = E \Phi$) in the form

$$\Phi(\theta, \xi) = f(\theta) g(\xi), \quad (6)$$

we get the separated equations

$$H_{l_0, l_1}^\theta f(\theta) \equiv \left[-\partial_\theta^2 + \frac{l_1^2 - \frac{1}{4}}{\sin^2 \theta} + \frac{l_0^2 - \frac{1}{4}}{\cos^2 \theta} \right] f(\theta) = \alpha f(\theta) \quad (7)$$

and

$$\left[-\partial_\xi^2 - \coth \xi \partial_\xi - \frac{l_2^2 - \frac{1}{4}}{\cosh^2 \xi} + \frac{\alpha}{\sinh^2 \xi} \right] g(\xi) = E g(\xi),$$

where $\alpha > 0$ is a separation constant.

3. A complete set of intertwining operators

The second order operator at the l.h.s. of (7) can be factorized in terms of first order operators [16, 17]

$$H_{l_0, l_1}^\theta = A_{l_0, l_1}^+ A_{l_0, l_1}^- + \lambda_{l_0, l_1},$$

being

$$A_{l_0, l_1}^\pm = \pm \partial_\theta - (l_0 + 1/2) \tan \theta + (l_1 + 1/2) \cot \theta, \quad \lambda_{l_0, l_1} = (1 + l_0 + l_1)^2.$$

The Hamiltonian can be also rewritten in terms of the triplet $(A_{l_0-1, l_1-1}^\pm, \lambda_{l_0-1, l_1-1})$

$$H_{l_0, l_1}^\theta = A_{l_0-1, l_1-1}^- A_{l_0-1, l_1-1}^+ + \lambda_{l_0-1, l_1-1} = A_{l_0, l_1}^+ A_{l_0, l_1}^- + \lambda_{l_0, l_1}. \quad (8)$$

In this way we can determine a hierarchy of Hamiltonians

$$\cdots, H_{l_0-n, l_1-n}^\theta, \cdots, H_{l_0-1, l_1-1}^\theta, H_{l_0, l_1}^\theta, H_{l_0+1, l_1+1}^\theta, \cdots, H_{l_0+n, l_1+n}^\theta, \cdots \quad (9)$$

satisfying the following recurrence relations

$$A_{l_0-1, l_1-1}^- H_{l_0-1, l_1-1}^\theta = H_{l_0, l_1}^\theta A_{l_0-1, l_1-1}^-, \quad A_{l_0-1, l_1-1}^+ H_{l_0, l_1}^\theta = H_{l_0-1, l_1-1}^\theta A_{l_0-1, l_1-1}^+.$$

Hence, the operators $\{A_{l_0+n, l_1+n}^\pm\}_{n \in \mathbb{Z}}$ are intertwining operators and act as transformations between the eigenfunctions of the hierarchy of Hamiltonians (9),

$$A_{l_0-1, l_1-1}^- : f_{l_0-1, l_1-1} \rightarrow f_{l_0, l_1}, \quad A_{l_0-1, l_1-1}^+ : f_{l_0, l_1} \rightarrow f_{l_0-1, l_1-1},$$

where the subindex refers to the corresponding Hamiltonian of the hierarchy.

We can define new operators in terms of A_{l_0, l_1}^\pm and a diagonal operator $A_{l_0, l_1} = (l_0 + l_1)\mathbb{I}$ acting in the following way in the space of eigenfunctions

$$\hat{A}^- f_{l_0, l_1} := \frac{1}{2} A_{l_0, l_1}^- f_{l_0, l_1}, \quad \hat{A}^+ f_{l_0, l_1} := \frac{1}{2} A_{l_0, l_1}^+ f_{l_0, l_1}, \quad \hat{A} f_{l_0, l_1} := -\frac{1}{2} (l_0 + l_1) f_{l_0, l_1}.$$

It can be shown from (8) that \hat{A}^- , \hat{A}^+ and \hat{A} satisfy the commutation relations of a $su(2)$ Lie algebra, i.e.

$$[\hat{A}^-, \hat{A}^+] = -2\hat{A}, \quad [\hat{A}, \hat{A}^\pm] = \pm\hat{A}^\pm.$$

The ‘fundamental’ states, f_{l_0, l_1}^0 , of the $su(2)$ representations are determined by the relation $A_{l_0, l_1}^- f_{l_0, l_1}^0(\theta) = 0$. They are

$$f_{l_0, l_1}^0(\theta) = N (\cos \theta)^{l_0+1/2} (\sin \theta)^{l_1+1/2},$$

where N is a normalization constant. These functions are regular and square-integrable when $l_0, l_1 \geq -1/2$. Since $\hat{A} f_{l_0, l_1} \equiv -\frac{1}{2} (l_0 + l_1) f_{l_0, l_1}$, then $j = \frac{1}{2} (l_0 + l_1)$ and the dimension of the unitary representation will be $2j + 1 = l_0 + l_1 + 1$.

Now, observe that because the IO’s A_{l_0, l_1}^\pm depend only on the θ -variable, they can act also as IO’s of the total Hamiltonian H_ℓ (5) and its global eigenfunctions Φ_ℓ (6), leaving the parameter l_2 unchanged (in this framework we will use three-fold indexes)

$$A_{\ell'}^- H_{\ell'} = H_\ell A_{\ell'}^-, \quad A_{\ell'}^+ H_\ell = H_{\ell'} A_{\ell'}^+,$$

where $\ell = (l_0, l_1, l_2)$ and $\ell' = (l_0 - 1, l_1 - 1, l_2)$. In this sense, all the above relations can be extended under this global point of view.

3.1. Pseudo-spherical coordinates around s_1

These coordinates are obtained from the non-compact rotations about the axes s_0 and s_1 , respectively. In this way we obtain the following parametrization of the hyperboloid

$$s_0 = \cosh \psi \sinh \chi, \quad s_1 = \sinh \psi, \quad s_2 = \cosh \psi \cosh \chi. \quad (10)$$

The expressions of the $so(2, 1)$ generators in these coordinates are

$$J_0 = -\tanh \psi \sinh \chi \partial_\chi + \cosh \chi \partial_\psi, \quad J_1 = \partial_\chi, \quad J_2 = \sinh \chi \partial_\psi - \tanh \psi \cosh \chi \partial_\chi.$$

The explicit expression of the Hamiltonian (1) is now

$$H_\ell = -\partial_\psi^2 - \tanh \psi \partial_\psi + \frac{l_1^2 - \frac{1}{4}}{\sinh^2 \psi} + \frac{1}{\cosh^2 \psi} \left[-\partial_\chi^2 + \frac{l_0^2 - \frac{1}{4}}{\sinh^2 \chi} - \frac{l_2^2 - \frac{1}{4}}{\cosh^2 \chi} \right].$$

This Hamiltonian can be separated in the variables ψ and χ considering the eigenfunctions Φ of H_ℓ ($H_\ell \Phi = E \Phi$) as $\Phi(\chi, \psi) = f(\chi) g(\psi)$. We obtain the two following (separated) equations

$$\begin{aligned} H_{l_0, l_2}^\chi f(\chi) &\equiv \left[-\partial_\chi^2 + \frac{l_0^2 - \frac{1}{4}}{\sinh^2 \chi} - \frac{l_2^2 - \frac{1}{4}}{\cosh^2 \chi} \right] f(\chi) = \alpha f(\chi), \\ \left[-\partial_\psi^2 - \tanh \psi \partial_\psi + \frac{l_1^2 - \frac{1}{4}}{\sinh^2 \psi} + \frac{\alpha}{\cosh^2 \psi} \right] g(\psi) &= E g(\psi), \end{aligned} \quad (11)$$

with α a separation constant. The second order operator at the l.h.s. of (11) can be factorized as a product of first order operators

$$H_{l_0, l_2}^\chi = B_{l_0, l_2}^+ B_{l_0, l_2}^- + \lambda_{l_0, l_2} = B_{l_0-1, l_2-1}^- B_{l_0-1, l_2-1}^+ + \lambda_{l_0-1, l_2-1}, \quad (12)$$

being

$$B_{l_0, l_2}^\pm = \pm \partial_\chi + (l_2 + 1/2) \tanh \chi + (l_0 + 1/2) \coth \chi, \quad \lambda_{l_0, l_2} = -(1 + l_0 + l_2)^2. \quad (13)$$

In this case the intertwining relations take the form

$$B_{l_0-1, l_2-1}^- H_{l_0-1, l_2-1}^\chi = H_{l_0, l_2}^\chi B_{l_0-1, l_2-1}^-, \quad B_{l_0-1, l_2-1}^+ H_{l_0, l_2}^\chi = H_{l_0-1, l_2-1}^\chi B_{l_0-1, l_2-1}^+,$$

and imply that these operators B^\pm connect eigenfunctions in the following way

$$B_{l_0-1, l_2-1}^- : f_{l_0-1, l_2-1} \rightarrow f_{l_0, l_2}, \quad B_{l_0-1, l_2-1}^+ : f_{l_0, l_2} \rightarrow f_{l_0-1, l_2-1}.$$

The operators B_{l_0, l_2}^\pm can be expressed in terms of ξ and θ using relations (3) and (10)

$$B_{l_0, l_2}^\pm = \pm J_1 + (l_2 + 1/2) \tanh \xi \cos \theta + (l_0 + 1/2) \coth \xi \sec \theta,$$

where J_1 is given by (4). We define new operators in the following way

$$\hat{B}^- f_{l_0, l_2} := \frac{1}{2} B_{l_0, l_2}^- f_{l_0, l_2}, \quad \hat{B}^+ f_{l_0, l_2} := \frac{1}{2} B_{l_0, l_2}^+ f_{l_0, l_2}, \quad \hat{B} f_{l_0, l_2} := -\frac{1}{2} (l_0 + l_2) f_{l_0, l_2},$$

and, having in mind the expressions (12) and (13), we can prove that they close the $su(1, 1)$ Lie algebra

$$[\hat{B}^-, \hat{B}^+] = 2 \hat{B}, \quad [\hat{B}, \hat{B}^\pm] = \pm \hat{B}^\pm.$$

As the Lie algebra $su(1, 1)$ is non-compact its unitary representations are infinite dimensional. In particular, we will be interested in those having a ground state annihilated by the lowering operator.

The IO's B^\pm can be considered as intertwining operators of the Hamiltonians H_ℓ and their eigenfunctions Φ_ℓ , similarly to the IO's A^\pm described before (in this situation we will also use three-fold indexes but now with l_1 remaining unchanged).

3.2. Pseudo-spherical coordinates around s_0

These coordinates are obtained from the noncompact rotations about the axes s_1 and s_0 , respectively. They give rise to the parametrization

$$s_0 = \sinh \phi, \quad s_1 = \cosh \phi \sinh \beta, \quad s_2 = \cosh \phi \cosh \beta, \quad (14)$$

where the generators take the form

$$J_0 = \partial_\beta, \quad J_1 = \cosh \beta \partial_\phi - \tanh \phi \sinh \beta \partial_\beta, \quad J_2 = -\sinh \beta \partial_\phi + \tanh \phi \cosh \beta \partial_\beta.$$

Now, the Hamiltonian is

$$H_\ell = -\partial_\phi^2 - \tanh \phi \partial_\phi + \frac{l_0^2 - \frac{1}{4}}{\sinh^2 \phi} + \frac{1}{\cosh^2 \phi} \left[-\partial_\beta^2 + \frac{l_1^2 - \frac{1}{4}}{\sinh^2 \beta} - \frac{l_2^2 - \frac{1}{4}}{\cosh^2 \beta} \right],$$

that can be separated in the variables β in terms of the eigenfunctions Φ ($H_\ell \Phi = E \Phi$) such that $\Phi(\beta, \phi) = f(\beta) g(\phi)$ in the following way

$$\begin{aligned} H_{l_1, l_2}^\beta f(\beta) &\equiv \left[-\partial_\beta^2 + \frac{l_1^2 - \frac{1}{4}}{\sinh^2 \beta} - \frac{l_2^2 - \frac{1}{4}}{\cosh^2 \beta} \right] f(\beta) = \alpha f(\beta), \\ \left[-\partial_\phi^2 - \tanh \phi \partial_\phi + \frac{l_0^2 - \frac{1}{4}}{\sinh^2 \phi} + \frac{\alpha}{\cosh^2 \phi} \right] g(\phi) &= E g(\phi), \end{aligned} \quad (15)$$

with α a separation constant. The second order operator at the l.h.s. of expression (15) can be factorized as a product of first order operators

$$H_{l_1, l_2}^\beta = C_{l_1, l_2}^+ C_{l_1, l_2}^- + \lambda_{l_1, l_2} = C_{l_1+1, l_2-1}^- C_{l_1+1, l_2-1}^+ + \lambda_{l_1+1, l_2-1},$$

being

$$C_{l_1, l_2}^\pm = \pm \partial_\beta + (l_2 + 1/2) \tanh \beta + (-l_1 + 1/2) \coth \beta, \quad \lambda_{l_1, l_2} = -(1 - l_1 + l_2)^2.$$

These operators C_{l_1, l_2}^\pm give rise to the intertwining relations

$$C_{l_1+1, l_2-1}^+ H_{l_1, l_2}^\beta = H_{l_1+1, l_2-1}^\beta C_{l_1+1, l_2-1}^+ \quad C_{l_1+1, l_2-1}^- H_{l_1, l_2}^\beta = H_{l_1, l_2}^\beta C_{l_1+1, l_2-1}^-,$$

which imply that

$$C_{l_1+1, l_2-1}^- : f_{l_1+1, l_2-1} \rightarrow f_{l_1, l_2}, \quad C_{l_1+1, l_2-1}^+ : f_{l_1, l_2} \rightarrow f_{l_1+1, l_2-1}.$$

In this case, C_{l_1, l_2}^\pm can also be expressed in terms of ξ and θ using relations (3) and (14)

$$C_{l_1, l_2}^\pm = \pm J_0 + (l_2 + 1/2) \tanh \xi \sin \theta + (-l_1 + 1/2) \coth \xi \csc \theta,$$

where J_0 is given by (4). Now, the new operators are defined as

$$\hat{C}^- f_{l_1, l_2} := \frac{1}{2} C_{l_1, l_2}^- f_{l_1, l_2} \quad \hat{C}^+ f_{l_1, l_2} := \frac{1}{2} C_{l_1, l_2}^+ f_{l_1, l_2} \quad \hat{C} f_{l_1, l_2} := -\frac{1}{2} (l_2 - l_1) f_{l_1, l_2}.$$

They satisfy the commutation relations of the $su(1, 1)$ algebra

$$[\hat{C}^-, \hat{C}^+] = 2\hat{C}, \quad [\hat{C}, \hat{C}^\pm] = \pm \hat{C}^\pm.$$

As in the previous cases, we can consider the IO's C^\pm as connecting global Hamiltonians H_ℓ and their eigenfunctions, having in mind that now the parameter l_0 is unaltered.

4. Algebraic structure of the intertwining operators

If we consider together all the IO's that have appeared in section 3 ($\hat{A}^\pm, \hat{A}, \hat{B}^\pm, \hat{B}, \hat{C}^\pm, \hat{C}$), in addition to the above algebras $su(2)$ and $su(1, 1)$, we find that they close the Lie algebra $su(2, 1)$ since they satisfy the following commutation relations

$$\begin{aligned}
[\hat{A}^+, \hat{B}^+] &= 0 & [\hat{A}^-, \hat{B}^-] &= 0 & [\hat{A}^+, \hat{B}^-] &= -\hat{C}^- & [\hat{A}^-, \hat{B}^+] &= \hat{C}^+ \\
[\hat{C}^+, \hat{B}^+] &= 0 & [\hat{C}^-, \hat{B}^-] &= 0 & [\hat{C}^+, \hat{A}^+] &= -\hat{B}^+ & [\hat{C}^-, \hat{A}^-] &= \hat{B}^- \\
[\hat{C}^+, \hat{B}^-] &= -\hat{A}^- & [\hat{C}^-, \hat{B}^+] &= \hat{A}^+ & [\hat{C}^+, \hat{A}^-] &= 0 & [\hat{C}^-, \hat{A}^+] &= 0 \\
[\hat{A}, \hat{B}^+] &= \frac{1}{2}\hat{B}^+ & [\hat{A}, \hat{B}^-] &= -\frac{1}{2}\hat{B}^- & [\hat{B}, \hat{A}^+] &= \frac{1}{2}\hat{A}^+ & [\hat{B}, \hat{A}^-] &= -\frac{1}{2}\hat{A}^- \\
[\hat{C}, \hat{B}^+] &= \frac{1}{2}\hat{B}^+ & [\hat{C}, \hat{B}^-] &= -\frac{1}{2}\hat{B}^- & [\hat{C}, \hat{A}^+] &= -\frac{1}{2}\hat{A}^+ & [\hat{C}, \hat{A}^-] &= \frac{1}{2}\hat{A}^- \\
[\hat{A}, \hat{C}^-] &= \frac{1}{2}\hat{C}^- & [\hat{A}, \hat{C}^+] &= -\frac{1}{2}\hat{C}^+ & [\hat{B}, \hat{C}^-] &= -\frac{1}{2}\hat{C}^- & [\hat{B}, \hat{C}^+] &= \frac{1}{2}\hat{C}^+ \\
[\hat{A}, \hat{B}] &= 0 & [\hat{A}, \hat{C}] &= 0 & [\hat{B}, \hat{C}] &= 0.
\end{aligned}$$

Obviously $su(2, 1)$ includes as subalgebras the Lie algebras $su(2)$ and $su(1, 1)$ defined in the previous section 3. The second order Casimir operator of $su(2, 1)$ can be written as follows

$$\mathcal{C} = \hat{A}^+ \hat{A}^- - \hat{B}^+ \hat{B}^- - \hat{C}^+ \hat{C}^- + \frac{2}{3} (\hat{A}^2 + \hat{B}^2 + \hat{C}^2) - (\hat{A} + \hat{B} + \hat{C}).$$

It is worthy noticing that in our differential realization we have $\hat{A} - \hat{B} + \hat{C} = 0$, and that there is another generator

$$\mathcal{C}' = l_1 + l_2 - l_0$$

commuting with the rest of generators. Hence, adding this new generator \mathcal{C}' to the other ones we get the Lie algebra $u(2, 1)$.

The eigenfunctions of the Hamiltonians H_ℓ that have the same energy support a unitary representation of $su(2, 1)$ and also they are characterized by a value of \mathcal{C} and other of \mathcal{C}' . In fact, we can show that

$$H_\ell = -4\mathcal{C} + \frac{1}{3}\mathcal{C}'^2 - \frac{15}{4}.$$

These representations can be obtained, as usual, starting with a fundamental state annihilated by the lowering operators A^-, C^- and B^-

$$A_\ell^- \Phi_\ell^0 = C_\ell^- \Phi_\ell^0 = B_\ell^- \Phi_\ell^0 = 0. \quad (16)$$

Solving equations (16) we find

$$\Phi_\ell^0(\xi, \theta) = N(\cos \theta)^{l_0+1/2}(\sin \theta)^{1/2}(\cosh \xi)^{l_2+1/2}(\sinh \xi)^{l_0+1},$$

where $\ell = (l_0, 0, l_2)$. From Φ_ℓ^0 we can get the rest of eigenfunctions in the representation using the raising operators A^+, B^+ and C^+ .

5. Concluding remarks

In this work we have built a set of intertwining operators for a superintegrable system defined on a two-sheet hyperboloid and we have found that they close a non-compact $su(2, 1)$ Lie algebra structure.

We have shown how these IO's can be very helpful in the characterization of the physical system by selecting separable coordinates, determining the eigenvalues and building eigenfunctions.

The IO's can also be used to find the second order integrals of motion for a Hamiltonian H_ℓ and their algebraic relations, which is the usual approach to (super) integrable systems. However, we see that it is much easier to deal directly with the IO's, which are more elementary and simpler, than with constants of motion.

Our program in the near future is the application of this method to wider situations. Besides, in principle, we can also adapt the method to classical versions of such systems. On this aspect we must remark that some symmetry procedures usually considered only for quantum systems can be extended in an appropriate way to classical ones [18].

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