

Spectral geometry of nuts and bolts

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Abstract

We study the spectrum of Laplace operators on a one-parameter family of gravitational instantons of bi-axial Bianchi IX type coupled to an abelian connection with self-dual curvature. The family of geometries includes the Taub-NUT (TN), Taub-bolt and Euclidean Schwarzschild geometries and interpolates between them. The interpolating geometries have conical singularities along a submanifold of co-dimension two, but we prove that the associated Laplace operators have natural selfadjoint extensions and study their spectra. In particular, we determine the essential spectrum and prove that its complement, the discrete spectrum, is infinite. We compute some of these eigenvalues numerically and compare the numerical results with an analytical approximation derived from the asymptotic TN form of each of the geometries in our family.

Keywords: gravitational instanton, Laplace operator, spectrum, nuts and bolts, Taub-NUT space, Taub-bolt space, Euclidean Schwarzschild space

(Some figures may appear in colour only in the online journal)

1. Introduction

Laplacians on manifolds with non-trivial Riemannian metrics arise in rather diverse contexts in mathematical physics. The simplest and most familiar example is the Laplacian on the Lie group $SO(3)$ equipped with a left-invariant metric, whose spectrum describes the quantum excitations of a rigid body. More involved examples arise generally when collective coordinates are used to model a physical system where kinetic effects are important. Mathematically, such

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effects are captured by an induced Riemannian metric on the space of collective coordinates, and in the corresponding quantum theory the associated Laplace operator is a natural candidate for the kinetic part of the Hamiltonian. This point of view is systematically explored in the study of collective coordinates for solitons, often called moduli spaces in this context [1, 2].

The spectrum of Laplacians associated to four-dimensional Riemannian geometries which satisfy the Einstein equations has received particular attention in the literature. Such spaces were first studied in the context of the path-integral approach to quantum gravity and are called gravitational instantons [3]¹. The spectrum of the Laplace operator associated to a gravitational instanton is important in understanding the fluctuations around the instanton, and in determining its semiclassical contribution to the path integral. Gravitational instantons also occur as moduli spaces of magnetic monopole [1], and here the Laplace operator can be used to study quantum properties of monopoles [2].

It is often natural to couple the Laplace operator to a connection on the Riemannian manifold. In the context of collective coordinates, such connections may be induced as Berry connections when additional linear collective coordinates are included in the model. In the context of gravitational instantons, the relevant connections are often themselves Yang–Mills instantons, i.e., their curvatures obey equations which are mathematically natural on four-dimensional Riemannian manifolds.

In this paper we examine the spectral properties of a family of non-compact four-dimensional Riemannian geometries which obey the Einstein equations and which permit isometric actions of $(SU(2) \times U(1))/\mathbb{Z}_2$. The family includes three much studied spaces—the self-dual Taub-NUT geometry (TN), the Taub-bolt space (TB) and the Euclidean Schwarzschild space (ES)—but also interpolating geometries which have so-called edge-cone singularities [4]. Each geometry comes with an abelian connection whose curvature two-form is self-dual (an abelian instanton), and in each case we consider the Laplace operator minimally coupled to that connection.

The essential difference between TN, TB and ES lies in the geometry of the $U(1)$ orbits. Each space has an interior region where the $U(1)$ -action has fixed points. There is a single fixed point, called a nut, for TN, but a two-sphere of fixed points, called a bolt, for ES and TB [5]. Away from the fixed points, the spaces are principal $U(1)$ bundles over a three-dimensional space which is conformal to flat Euclidean space. The radius of the $U(1)$ -orbits tends to a constant value as the proper distance from the fixed points tends to infinity. At a fixed positive distance, the orbits are fibres of a non-trivial fibre bundle over a two-sphere in the case of TN and TB, but fibres of a trivial fibration for ES.

Our motivation for studying the spectral properties of this family of geometries is two-fold. The family interpolates between the TN and ES geometries whose spectral properties were studied in [6, 7], with several interesting results. The spectrum of TB has not been studied, so our first motivation is to fill that gap and to provide a unified understanding of the spectra associated to three superficially rather different geometries. Also, with a bolt in its interior and a non-trivial bundle structure asymptotically, TB is the most interesting of the three spaces.

Our second motivation stems from the conical singularities in the interpolating geometries. Geometries with conical singularities on submanifolds of real co-dimension two are of interest in differential geometry and analysis, see [4] and references therein. In particular, there are two famous one-parameter families of such edge-cone geometries, one due to Hitchin [8] and one due to Abreu [9], which involve spaces closely related to the ones we study here.

¹ In differential geometry the term gravitational instanton is often reserved for the more restricted class of hyperkähler metrics, but we will use the original, wider definition here.

Unlike our manifolds, the manifolds considered by Abreu and Hitchin are compact except for limiting cases, which are, respectively, the TN and Atiyah–Hitchin spaces. Edge-cone singularities may be problematic for the classical geometry, but here we show that, at least in our examples, they do not pose any difficulties for the spectral theory. The Laplacians we consider have natural selfadjoint extensions and infinitely many bound states even when the corresponding classical geometry has conical singularities. This echoes observations about Laplacians on two-dimensional cones in [10].

The smoothing out of the conical singularities in the spectral theory is reminiscent of a similar effect for singular potentials in standard Hamiltonians studied in atomic and nuclear physics, where the quantum version (the spectral theory) is well-behaved, even when the classical theory (trajectories obeying suitable differential equations) is not. Since Laplacians associated to Riemannian geometries are increasingly used to model quantum effects, it is natural to explore the robustness of these operators when singularities are introduced in the classical picture.

Our paper is organised as follows. In section 2 we discuss the geometry of gravitational instantons of the so-called Bianchi IX form which include and interpolate between the TN, TB and ES geometries, paying particular attention to the interpolating geometries with conical singularities. We also introduce a family of self-dual two-forms on these geometries, and interpret them as curvatures of a family of $U(1)$ connections. In section 3 we define the Laplace operators on the family of geometries, minimally coupled to the family of $U(1)$ connections. Exploiting the symmetry and paying careful attention to the topology of the $U(1)$ bundle over the bolt for the TB and ES spaces, we separate variables and reduce the study of gauged Laplace operators to the study of radial Hamiltonians, defined on the half-line. The selfadjointness and spectral properties of the radial Hamiltonians are the subject of section 4, which contains our main analytical results. The radial Hamiltonians depend on geometric parameters and on parameters characterising angular momenta of the functions used in separating variables. The intricate dependence of spectral properties on these two sets of parameters, which one may call classical and quantum, and on their interplay is the focus of this section. The TN geometry is special among the geometries studied here in that both its geodesics and its spectrum can be determined analytically, even when the coupling to the self-dual connection is included [6]. In section 5 we exploit the fact that all Hamiltonians studied in this paper have an asymptotic form of the TN type to compute analytical approximations to the discrete spectrum of the TB family of metrics. These analytical results are then compared with a numerical study of the discrete spectrum for illustrative choices of the parameters, using a Riesz–Galerkin approximation. We end the paper with a brief discussion of the results, and with our conclusions.

2. Geometry of nuts and bolts in bi-axial Bianchi IX geometries

2.1. A family of Taub-bolt spaces and its limits

The gravitational instantons we want to consider have sufficient symmetry to reduce both the Einstein equations and the eigenvalue problem for the Laplace operator to problems involving ordinary differential equations. Technically, this means that we consider four dimensional Riemannian manifolds with isometric group actions whose orbits are generically three dimensional, i.e. group actions of co-homogeneity one. If the isometry group is $SU(2)$ and the action has generically three-dimensional orbits, the Einstein condition implies that the metric can be expressed in the Bianchi IX form [11]

$$ds^2 = f^2 dr^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2, \quad (2.1)$$

where σ_1, σ_2 and σ_3 are left-invariant one-forms on $SU(2)$ and a, b, c and f functions of a transverse or radial coordinate which will denote as r in the following.

We adopt the conventions of [6] for our parametrisation of $SU(2)$ in terms of Euler angles, but name the angles (ϕ, θ, ψ) instead of the (α, β, γ) used in [6]. With the ranges $0 \leq \phi < 2\pi, 0 \leq \theta \leq \pi, 0 \leq \psi < 4\pi$, an element $g \in SU(2)$ is parametrised in terms of Lie algebra generators $t_i, i = 1, 2, 3$, with commutators $[t_i, t_j] = \epsilon_{ijk} t_k$ as

$$g = \exp(\phi t_3) \exp(\theta t_2) \exp(\psi t_3). \tag{2.2}$$

These coordinates need to be used with care when $\theta = 0$ or $\theta = \pi$, where neither ϕ nor ψ is individually well-defined. However, for us it will be important that $\phi + \psi$ is well-defined when $\theta = 0$ and $\phi - \psi$ is well-defined when $\theta = \pi$. Written in terms of these angles, the left-invariant one-forms on $SU(2)$ are

$$\begin{aligned} \sigma_1 &= \sin \psi d\theta - \cos \psi \sin \theta d\phi, \\ \sigma_2 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\ \sigma_3 &= d\psi + \cos \theta d\phi. \end{aligned}$$

They satisfy $d\sigma_3 = -\sigma_1 \wedge \sigma_2$ and its cyclic permutations. For our study of the Laplace–Beltrami operator we also require the basis of vector fields dual to left-invariant one-forms, i.e. satisfying $\sigma_i(\xi_j) = \delta_{ij}$. In terms the Euler angles, they are

$$\begin{aligned} \xi_1 &= \cot \theta \cos \psi \partial_\psi + \sin \psi \partial_\theta - \frac{\cos \psi}{\sin \theta} \partial_\phi, \\ \xi_2 &= -\cot \theta \sin \psi \partial_\psi + \cos \psi \partial_\theta + \frac{\sin \psi}{\sin \theta} \partial_\phi, \\ \xi_3 &= \partial_\psi. \end{aligned}$$

The function f can be chosen to fix the radial coordinate, but a, b and c satisfy a system of second order ordinary differential equations by virtue of the Einstein condition. In general it is a difficult problem to determine and classify all solutions of these equations which result in smooth Riemannian geometries. However, with the assumption of an additional $U(1)$ symmetry, this problem simplifies and is easily solved. This is the situation we study in the present paper. The additional symmetry requires that $a = b$, leading to the so-called bi-axial Bianchi IX type, and the Einstein equations are equivalent to a system of coupled ordinary differential equations [12, 13]. Writing a dot for the derivative $\frac{1}{f} \frac{d}{dr}$ with respect to proper radial distance, the system consists of two second order evolution equations

$$\ddot{a} - \frac{\dot{a} \dot{c}}{a c} + \frac{c^2}{4a^4} = 0, \quad \ddot{c} + 2 \frac{\dot{a} \dot{c}}{a c} - \frac{c^2}{2a^4} = 0,$$

and the conservation law

$$2 \frac{\dot{a} \dot{c}}{a c} + \left(\frac{\dot{a}}{a} \right)^2 - \frac{1}{a^2} + \frac{c^2}{4a^4} = 0.$$

The general solution of this system depends on three constants, one of which can be viewed as an overall additive shift in the proper radial distance. The most widely used form of the metric is based on the choice $f = 2N/c$ and depends on two real parameters N and M which can be chosen to be positive without loss of generality [14] and which are ordered $M > N$ for

a complete metric without boundary [12]. The solution is conveniently expressed in terms of the function

$$U = \frac{r^2 - 2Mr + N^2}{r^2 - N^2}, \tag{2.3}$$

and takes the form

$$a = b = \sqrt{r^2 - N^2}, \quad c = 2N\sqrt{U}, \quad f = \frac{1}{\sqrt{U}},$$

so that

$$ds^2 = \frac{1}{U}dr^2 + (r^2 - N^2)(\sigma_1^2 + \sigma_2^2) + 4N^2U\sigma_3^2. \tag{2.4}$$

This is the general family of metrics with which we work in this paper. We will use the orientation determined by the volume form

$$dV = 2N(r^2 - N^2)\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge dr. \tag{2.5}$$

Factorising

$$r^2 - 2Mr + N^2 = (r - r_+)(r - r_-), \quad r_{\pm} = M \pm \sqrt{M^2 - N^2},$$

and defining a parameter $k \geq 1$ via

$$r_+ = kN, \quad r_- = \frac{N}{k},$$

it is necessary (but not sufficient) to take the range of r as $[kN, \infty]$ for a complete and non-singular metric. This range avoids the curvature singularities at the poles $r = \pm N$ of U [14], and the zero of U at $r = kN$ ensures that this is not a boundary of the manifold but, in the terminology introduced in [5], either a nut or a bolt. The former occurs when $k = 1$ and means that the $SU(2)$ orbit collapses to a point. The latter occurs when $k > 1$ and means that the $SU(2)$ orbit collapses to the two-sphere parametrised by the polar angles (θ, ϕ) . In this paper we think of the metrics (2.4) as a two-parameter family, parametrised by $N > 0$ and $k \geq 1$. We are particularly interested in the dependence on k and only comment on the role of N briefly in our conclusions.

To avoid conical singularities at the bolt one in fact requires $k = 2$, as can be seen by switching to the coordinate $\varrho = \sqrt{r - r_+}$. Near the bolt (at $r = r_+$) one checks that $U \approx \varrho^2/(kN)$ so that, keeping leading terms in ϱ , the metric is approximately

$$ds^2 \approx 4Nk \left[(d\varrho)^2 + \varrho^2 \frac{\sigma_3^2}{k^2} \right] + (k^2 - 1)N^2(\sigma_1^2 + \sigma_2^2).$$

This is a metric on the branched cover of cotangent bundle to the two-sphere. The metric on the two-dimensional cotangent plane at fixed θ and ϕ is approximately

$$ds^2 \approx 4Nk \left[(d\varrho)^2 + \varrho^2 \frac{d\psi^2}{k^2} \right].$$

With ψ in the interval $[0, 4\pi]$, this is the flat metric on the plane precisely if $k = 2$ but when $k \neq 2$ it is the geometry of a singular cone. The smooth geometry obtained when $k = 2$, first discovered by Page in [14], corresponds to TB. In this case, $r_+ = 2N, r_- = \frac{1}{2}N, M = \frac{5}{4}N$ and so

$$U = \frac{r^2 - \frac{5}{2}Nr + N^2}{r^2 - N^2}.$$

The case $k = 1$ has to be treated separately since $r_+ = r_- = N = M$ in this case, leading to a cancellation and reduction in the order of the rational function U , which is now $U = \frac{r-N}{r+N}$. The resulting space is the Euclidean TN gravitational instanton [3], so TN in our notation. Its metric

$$ds^2 = \frac{r+N}{r-N}dr^2 + (r^2 - N^2)(\sigma_1^2 + \sigma_2^2) + 4N^2\frac{r-N}{r+N}\sigma_3^2, \quad r \geq N,$$

is frequently expressed in terms of the alternative radial parameter $\rho = r - N$. Then, with $\Lambda = 2N$, it takes the isotropic form

$$ds^2 = \left(1 + \frac{\Lambda}{\rho}\right) (d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2)) + \Lambda^2\frac{\rho}{\rho + \Lambda}\sigma_3^2, \quad \rho \geq 0, \quad (2.6)$$

used in [15]². The apparent singularity at $r = N$ or $\rho = 0$ is only a coordinate singularity. It can be removed by working with $\varrho = \sqrt{r - N}$. Keeping leading terms only, one finds for small ϱ that

$$ds^2 \approx 8N \left[d\varrho^2 + \frac{\varrho^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \right].$$

This shows that the geometry near $r = N$ is that of Euclidean \mathbb{R}^4 , with the being the origin.

Even though the classical geometry is only smooth for $k = 1$ or $k = 2$, we will treat k as a real parameter in the range $k \in [1, \infty)$ in this paper. This means that we allow conical singularities at the bolt. We refer to the metrics with $1 < k < \infty$ as the TB family, and sometimes simply as TB spaces.

Relative to the flat angular range 2π , the cones have a surplus angle for $1 < k < 2$ and deficit angle for $k > 2$ given by

$$\Delta\psi = \left(\frac{2-k}{k}\right) 2\pi.$$

It is remarkable that in the limit $k \rightarrow 1$, as the surplus angle approaches 2π and the bolt collapses to a nut, the Riemannian geometry near the bolt jumps from that of the cotangent bundle of S^2 to that of \mathbb{R}^4 .

The discussion so far assumes that the range of the angle ψ is $[0, 4\pi)$. However, it is geometrically natural to consider Bianchi IX manifolds whose generic $SU(2)$ orbit is the Lens space S^3/\mathbb{Z}_n , where the generator of \mathbb{Z}_n acts by sending $\psi \mapsto \psi + \frac{4\pi}{n}$ for some positive integer n . In our parametrisation this simply leads to the shorter range $\psi \in [0, \frac{4\pi}{n})$. Repeating the above analysis with this range, the requirement of a smooth geometry at the bolt when $k > 1$ now requires $kn = 2$, which is only possible if $n = 1$ and again $k = 2$, while in the case $k = 1$ smoothness immediately yields $n = 1$ [14]. However, if one permits conical singularities as we do here, we can allow any positive integer n in the case where $k > 1$, i.e. when there is a

²The constant L in that paper is denoted Λ here to avoid a notational clash later in our discussion.

bolt at $r = r_+$. The conical geometry always has a deficit angle when $n \geq 2$, with deficit angle $4\pi/(kn) - 2\pi$.

All the geometries we have considered so far are asymptotically locally flat, or ALF in the terminology used for classifying gravitational instantons. In our examples this means that for large r (but in fact $r > r_+$ is sufficient), the spaces are topologically circle bundles over $\mathbb{R}^3 \setminus \{0\}$, with the circle (parametrised by ψ here) having a finite radius which tends to $4N/n$ for $r \rightarrow \infty$. At fixed $r > r_+$ they are the non-trivial circle bundles over S^2 whose total space is the Lens space S^3/\mathbb{Z}_n .

To end this review of the classical geometry, we observe that two further rotationally symmetric gravitational instantons with bolts can be obtained from the general solution (2.4) by taking limits, namely the Eguchi–Hanson space and the ES space. The details are summarised in [14] but we note that the Eguchi–Hanson space is obtained for $n = 2$ by taking the limit $k \rightarrow 1$ while keeping $4(k^2 - 1)N^2$ constant (so that necessarily $N \rightarrow \infty$). It has a self-dual Riemann curvature and is an example of an asymptotically locally Euclidean space: for fixed radial distance from the bolt, the geometry is that of a non-trivial circle bundle over S^2 and for large radial distances the radius of both the circle and the sphere grow linearly with that distance.

The ES space is obtained by picking a positive constant L , and taking the limit

$$N = \frac{L}{k}, \quad k \rightarrow \infty. \tag{2.7}$$

This implies that $r_+ = L$, but $r_- \rightarrow 0$. We also change angular coordinates according to

$$\chi = \frac{\psi}{k} = \frac{N}{L}\psi,$$

and observe that in the ES limit

$$2N\sigma_3 = 2N \cos \theta d\phi + 2L d\chi \rightarrow 2L d\chi.$$

Then

$$U \rightarrow V = 1 - \frac{L}{r}, \quad 4N^2\sigma_3^2 \rightarrow 4L^2 d\chi^2,$$

and the metric becomes

$$ds^2 = \frac{1}{V} dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + 4L^2 V d\chi^2, \quad r \geq L,$$

which is the standard form of the ES metric, as studied in [6]. For a fixed positive distance from the bolt, the topology is that of a trivial circle bundle over S^2 .

2.2. Magnetic fields

The study of harmonic two-forms on Riemannian four-manifolds is interesting for a number of apparently different reasons. Physically, such forms represent solutions of the Maxwell equations, albeit in a Riemannian rather than the Lorentzian setting [16]. Geometrically, they arise as the curvature of abelian connections on the manifold. In differential topology, the existence of harmonic two-forms is related to the homology of the four-manifold. The situation is simplest in the compact setting, where harmonic forms are the Hodge representatives of cohomology classes. In the non-compact setting, the relationship between harmonic forms and homology of the underlying manifold depends on asymptotic behaviour, but happily this has been studied in detail for the TB and TN manifolds recently [17]. The final reason why

harmonic forms are interesting is particularly relevant here. Coupling linear differential operators naturally associated to a Riemannian manifold—such as Dirac or Laplace operators—to abelian connections enriches the associated spectral theory. For example, on the TN space, the Laplace operator has no discrete spectrum, but the same operator coupled minimally to an abelian connection with self-dual and square-integrable curvature has infinitely many bound states [6].

Here we are interested in a family of connections of the TN–TB family of metrics (2.4) with $k \geq 1$ which generalises the self-dual connection on the TN space. As explained in [17], the TB space ($k = 2$) has a two-dimensional space of harmonic two-forms, which contains a one-dimensional subspace of self-dual two-forms. These forms can easily be generalised for any value $k > 1$ in such a way that they become a self-dual form on TN in the limit $k \rightarrow 1$. The resulting family of forms, self-dual with respect to the volume element (2.5), can be written as

$$F = \frac{pk + 1}{2k - 1} \left(\frac{2N}{(r + N)^2} dr \wedge \sigma_3 + \frac{r - N}{r + N} \sigma_2 \wedge \sigma_1 \right).$$

Our choice of normalisation for F and the interpretation of the parameter p follow from our wish to treat iF as the curvature of a connection on the TB manifold. At the bolt,

$$iF|_{r=kN} = -i \frac{p}{2} \sin \theta d\theta \wedge d\phi,$$

showing that our normalisation implies that, for $k > 1$, p is the first Chern class of the connection iA over the bolt:

$$\frac{i}{2\pi} \int_{\text{bolt}} (iF) = p.$$

The integrality of the first Chern number is therefore equivalent to

$$p \in \mathbb{Z}.$$

The curvature form is not globally exact, but away from the bolt, $iF = i dA$ with

$$iA = i \frac{pk + 1}{2k - 1} \frac{r - N}{r + N} \sigma_3. \tag{2.8}$$

This formula fails at the bolt since the angle ψ and hence σ_3 is not well-defined there. We will use iA as a gauge potential in our gauged version of the Laplace operator, but to obtain a connection over TB we also need gauge potentials on the bolt. They can be constructed in analogy with the gauge potentials used in the discussion of the Dirac monopole. Defining local gauge transformation $g_{\pm} = \exp(i \frac{p}{2} (\psi \mp \phi))$, and monopole gauge potentials via

$$iA_{\pm} = iA - g_{\pm}^{-1} dg_{\pm}, \tag{2.9}$$

one checks that, at the bolt,

$$iA_{\pm}|_{r=kN} = i \frac{p}{2} (\cos \theta \pm 1) d\phi,$$

so that A_+ is regular in the southern ($\theta > 0$) and A_- in the northern ($\theta < \pi$) coordinate patches covering the bolt. In practice, one can work in the smooth monopole gauge, with two gauge potentials iA_+ and iA_- , smooth in the open sets $\theta > 0$ and $\theta < \pi$ and related on their overlap by the smooth gauge transformation $g = \exp(ip\phi)$, or one can apply the gauge transformation g_{\pm} , singular at the bolt, to work with the gauge potential iA , also singular at the bolt.

The necessity of local gauge potentials, related by local gauge transformations, suggests that in studying the spectrum of the gauged Laplace operator one may have to work with local sections, also related by gauge transformations. Explicitly, writing Φ for a section in the singular gauge, and Φ_{\pm} for local sections in the monopole gauges, we need to combine (2.9) with

$$\Phi_{\pm} = g_{\pm} \Phi \tag{2.10}$$

to satisfy the covariance condition $(d + iA_{\pm})\Phi_{\pm} = g_{\pm}(d + iA)\Phi$. Happily, this is not necessary in practice. As in the analogous treatment of the gauged Laplace operator on the ES space in [7], one can exploit the covariance of the gauged Laplace operator to work with the singular gauge potential iA , even at the bolt, but then determine the allowed solutions Φ by requiring that the gauge transformed sections (2.10) are smooth at the bolt. We will see in the next section that this requirement can be implemented very naturally in practice.

A gauge potential for the TN space can formally be obtained by taking the limit $k \rightarrow 1$ and simultaneously re-scaling A by taking $p \rightarrow 0$ in such a way that $\frac{k+1}{k-1}p \rightarrow \tilde{p}$ for some $\tilde{p} \in \mathbb{R}$. The resulting one-form is³

$$A_{\text{TN}} = \frac{\tilde{p} r - N}{2 r + N} \sigma_3. \tag{2.11}$$

Note that this is globally defined on TN since the radial function multiplying σ_3 vanishes at the nut. As discussed in [15], the coefficient \tilde{p} is not quantised in the TN case, essentially because there is no non-contractible two-sphere in TN.

3. Gauged Laplace operator on TN, TB and ES geometries

The Laplace operator associated to metrics of the Bianchi IX form (2.1) is [2]

$$\Delta = \frac{1}{abc f} \partial_r \frac{abc}{f} \partial_r + \frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2} + \frac{\xi_3^2}{c^2},$$

where we write ∂_r for $\partial/\partial r$. Gauging this operator means the minimal substitution of ordinary derivatives by covariant derivatives with respect to the connection iA discussed above, i.e.

$$\partial_r \mapsto \partial_r + iA(\partial_r), \quad \xi_j \mapsto \xi_j + iA(\xi_j), \quad j = 1, 2, 3.$$

Focussing on the TB family, the particular form of the connection (2.8) and the duality relation $\sigma_i(\xi_j) = \delta_{ij}$ mean that we only need to substitute

$$\xi_3 \mapsto \xi_3 + i \frac{p(k+1)r - N}{2(k-1)r + N}.$$

The Laplace operator on the metric (2.4) for the TB family minimally coupled to the self-dual connection (2.8) is therefore

$$\begin{aligned} \Delta_k &= \frac{1}{r^2 - N^2} \partial_r (r - kN) \left(r - \frac{N}{k} \right) \partial_r + \frac{1}{r^2 - N^2} (\xi_1^2 + \xi_2^2) \\ &\quad + \frac{r^2 - N^2}{4N^2 (r - kN) \left(r - \frac{N}{k} \right)} \left(\xi_3 + i \frac{p k + 1}{2} \frac{r - N}{k - 1} \frac{r - N}{r + N} \right)^2, \quad 1 < k < \infty. \end{aligned} \tag{3.1}$$

³ The parameter \tilde{p} is denoted p in [6, 15].

Treating the TN case analogously, and including the resulting gauged Laplace operator with connection (2.11) in the TB family as the limiting case $k = 1$, we define

$$\Delta_1 = \frac{1}{r^2 - N^2} \partial_r (r - N)^2 \partial_r + \frac{1}{r^2 - N^2} (\xi_1^2 + \xi_2^2) + \frac{r + N}{4N^2(r - N)} \left(\xi_3 + i \frac{\tilde{p} r - N}{2r + N} \right)^2. \quad (3.2)$$

To analyse the spectrum of the family of Laplace operators (3.1) and of (3.2) we exploit the symmetry of the TB and TN spaces and separate variables into a function of the radial variable and Wigner functions D_{ms}^j on $SU(2)$. With the conventions of [18], the latter have the form

$$D_{ms}^j(\phi, \theta, \psi) = e^{-im\phi} d_{ms}^j(\theta) e^{-is\psi},$$

where the functions d_{ms}^j are often called elements of the small d -matrix; their explicit form is given in [18]. Here we only require that the functions D_{ms}^j are eigenfunctions of the total angular momentum, and the left and right actions of rotations about the three-axis:

$$(\xi_1^2 + \xi_2^2 + \xi_3^2) D_{ms}^j = -j(j + 1) D_{ms}^j, \quad i\xi_3 D_{ms}^j = s D_{ms}^j, \quad i\partial_\phi D_{ms}^j = m D_{ms}^j.$$

The range of the labels is

$$j \in \frac{1}{2} \mathbb{N}^0, \quad s, m \in \{-j, -j + 1, \dots, j - 1, j\}, \quad (3.3)$$

and for later use we note that

$$(\xi_1^2 + \xi_2^2) D_{ms}^j = (-j(j + 1) + s^2) D_{ms}^j.$$

We recall that in the TB case we need to check if the angular dependence is non-singular in the non-singular monopole gauge. If we use the Wigner function D_{ms}^j in the singular gauge, this amounts to checking, according to (2.10), if

$$D_{msp}^{j\pm}(\phi, \theta, \psi) := e^{i\frac{p}{2}(\psi \mp \phi)} D_{ms}^j(\phi, \theta, \psi) = e^{-i(m \pm \frac{p}{2})\phi} d_{ms}^j(\theta) e^{-i(s - \frac{p}{2})\psi} \quad (3.4)$$

is smooth on the bolt (where ψ is not defined) for $\theta > 0$ (+) and $\theta < \pi$ (-). However, this happens precisely if $s = p/2$, in which case (3.4) is independent of ψ and

$$D_{m\frac{p}{2}p}^{j\pm}(\phi, \theta, \psi) = D_{m\frac{p}{2}}^j(\phi, \theta, \pm\phi)$$

which are a standard expressions for local sections, valid in the open sets $\theta > 0$ (+) and $\theta < \pi$ (-), of the line bundle of Chern class p over the two-sphere. We conclude that we can use Wigner functions when separating variables on the TB and TN spaces equipped with the connections (2.8) and (2.11), but that in the TB case we need to impose $s = p/2$ for any function which is non-zero at the bolt. As we shall see, this is implemented rather naturally in our spectral analysis.

Inserting the ansatz

$$\Phi(r, \phi, \theta, \psi) = u_{jms}(r) D_{ms}^j(\phi, \theta, \psi),$$

(no sum) into the equation

$$-\Delta_k \Phi = \epsilon \Phi$$

leads to the radial problem

$$\tilde{H}_k u_{j\ell m} = \epsilon u_{j\ell m},$$

with radial Hamiltonians, labelled by $k > 1$, defined initially on $C^\infty((kN, \infty))$ and given by

$$\begin{aligned} \tilde{H}_k = & -\frac{1}{r^2 - N^2} \frac{d}{dr} (r - kN) \left(r - \frac{N}{k} \right) \frac{d}{dr} + \frac{j(j+1) - s^2}{r^2 - N^2} \\ & + \frac{r^2 - N^2}{4N^2(r - kN) \left(r - \frac{N}{k} \right)} \left(s - \frac{pk + 1}{2} \frac{r - N}{r + N} \right)^2. \end{aligned} \tag{3.5}$$

The condition $s = p/2$ which characterises a special situation in the angular dependence also leads to a special feature in the radial Hamiltonian. A cancellation in the last term in this case leads to the simplified form

$$\begin{aligned} \tilde{H}_k^{sp} = & -\frac{1}{r^2 - N^2} \frac{d}{dr} (r - kN) \left(r - \frac{N}{k} \right) \frac{d}{dr} + \frac{j(j+1) - s^2}{r^2 - N^2} \\ & + \frac{s^2(r - kN)(r - N)}{N^2 \left(r - \frac{N}{k} \right) (k - 1)^2 (r + N)}. \end{aligned} \tag{3.6}$$

The radial TN Hamiltonian, defined initially on $C^\infty((N, \infty))$, is

$$\tilde{H}_1 = -\frac{1}{r^2 - N^2} \frac{d}{dr} (r - N)^2 \frac{d}{dr} + \frac{j(j+1) - s^2}{r^2 - N^2} + \frac{r + N}{4N^2(r - N)} \left(s - \frac{\tilde{p}r - N}{2r + N} \right)^2.$$

For completeness, we also include the radial Hamiltonian which arises in the study of the ES space. Using the conventions of [7], it has the form

$$\begin{aligned} \tilde{H}_\infty = & -\frac{1}{r^2} \frac{d}{dr} (r^2 - Lr) \frac{d}{dr} + \frac{1}{r^2} \left(j(j+1) - \frac{p^2}{4} \right) \\ & + \frac{r}{r - L} \left(\frac{n}{2L} + \frac{p}{2} \left(\frac{1}{L} - \frac{1}{r} \right) \right)^2, \quad r \geq L. \end{aligned}$$

Here $L > 0$ is a real parameter, $p, n \in \mathbb{Z}$ and $j = \frac{|p|}{2}, \frac{|p|}{2} + 1, \frac{|p|}{2} + 2, \dots$. We have denoted the Hamiltonian \tilde{H}_∞ since it can formally be obtained from \tilde{H}_k of (3.5) by identifying L with the product kN in the ES limit $k \rightarrow \infty$ (2.7) together with the identification

$$\left(s - \frac{p}{2} \right) = -\frac{n}{k} \tag{3.7}$$

Table 1. Parameters of the models considered in this paper.

Model	k	N	p	j	s	n
TN	1	$(0, \infty)$	$\tilde{p} \in \mathbb{R}$	$\frac{1}{2}\mathbb{N}_0$	$\{-j, -j+1, \dots, j-1, j\}$	—
TB	$(1, \infty)$	$(0, \infty)$	\mathbb{Z}	$\frac{1}{2}\mathbb{N}_0$	$\{-j, -j+1, \dots, j-1, j\}$	—
ES	∞	0	\mathbb{Z}	$\frac{ p }{2} + \mathbb{N}_0$	—	\mathbb{Z}

for fixed $n \in \mathbb{Z}$. For ease of reference, we summarise the range of the different parameters in the different models we consider in table 1.

4. Spectral properties of the radial Hamiltonian

4.1. Standard form

In order to apply the results and techniques of [19] to the study of the radial Hamiltonian \tilde{H}_k with $1 \leq k \leq \infty$, we write it in standard Schrödinger form via a unitary transformation. This requires, as a first step, expressing it in terms of the proper radial distance from the bolt or nut. Using the notation introduced in our discussion of the metric in (2.4), we define the proper radial distance

$$R(r) = \int_{kN}^r f(x)dx, \quad f(x) = \sqrt{\frac{x^2 - N^2}{(x - kN)(x - \frac{N}{k})}}. \tag{4.1}$$

We then rescale by the function

$$\nu(r) = \sqrt{a^2c} = \sqrt{2N} \left((r - kN) \left(r - \frac{N}{k} \right) (r^2 - N^2) \right)^{\frac{1}{4}}.$$

Noting that the radial derivative term in the Hamiltonian is

$$-\frac{1}{r^2 - N^2} \frac{d}{dr} (r - kN) \left(r - \frac{N}{k} \right) \frac{d}{dr} = -\frac{1}{\nu^2} \frac{d}{dR} \nu^2 \frac{d}{dR}$$

and that

$$-\frac{1}{\nu^2} \frac{d}{dR} \nu^2 \frac{d}{dR} \left(\frac{\eta}{\nu} \right) = \frac{1}{\nu} \left(-\frac{d^2 \eta}{dR^2} + \frac{1}{\nu} \frac{d^2 \nu}{dR^2} \eta \right), \tag{4.2}$$

the operators \tilde{H}_k defined on $[kN, \infty)$ with measure $\nu^2 f dr$ are unitarily equivalent to Hamiltonians of the standard form

$$\hat{H}_k = -\frac{d^2}{dR^2} + V_{jsp}^{\text{eff}}, \tag{4.3}$$

defined on the half-line $[0, \infty)$ with measure dR and effective potential

$$V_{jsp}^{\text{eff}} = \frac{1}{\nu} \frac{d^2 \nu}{dR^2} + \frac{j(j+1) - s^2}{r^2 - N^2} + Q. \tag{4.4}$$

Here Q combines the remaining non-derivative terms in \hat{H}_k into a radial potential. It depends on the parameters k, s and p (respectively \tilde{p}) summarised in table 1. Written in terms of r , we distinguish the following cases, making it explicit that there are important cancellations for some value of those parameters:

$$Q(r) = \begin{cases} \frac{r+N}{4N^2(r-N)} \left(s - \frac{\tilde{p}r-N}{2r+N} \right)^2 & \text{if } k = 1, s \neq 0, \\ \frac{\tilde{p}^2}{16N^2} \frac{r-N}{r+N} & \text{if } k = 1, s = 0, \\ \frac{(r^2 - N^2)}{4N^2(r - kN) \left(r - \frac{N}{k} \right)} \left(s - \frac{pk + 1}{2k - 1} \frac{r - N}{r + N} \right)^2 & \text{if } k > 1, s \neq \frac{p}{2}, \\ \frac{p^2(r - kN)(r - N)}{4N^2 \left(r - \frac{N}{k} \right) (k - 1)^2(r + N)} & \text{if } k > 1, s = \frac{p}{2}. \end{cases} \quad (4.5)$$

We now describe the behaviour of the effective potential (4.4) for small r and large R . In practice, we used

$$\frac{1}{\nu} \frac{d^2\nu}{dR^2} = \frac{1}{f\nu} \frac{d}{dr} \left(\frac{1}{f} \frac{d\nu}{dr} \right), \quad (4.6)$$

to derive the asymptotic expansions in the following sections.

4.2. Short range asymptotics

Defining

$$\rho = r - kN,$$

we find the following expansions for small ρ of the right-hand side of (4.6),

$$\frac{1}{f\nu} \frac{d}{dr} \left(\frac{1}{f} \frac{d\nu}{dr} \right) = \begin{cases} \frac{3}{32N\rho} - \frac{1}{64N^2} + o(\rho) & \text{if } k = 1, \\ -\frac{1}{16kN\rho} + \frac{11}{16(k^2 - 1)N^2} + o(\rho) & \text{if } k > 1. \end{cases}$$

Now, still for small ρ , the definition (4.1) implies that

$$R = \begin{cases} 2\sqrt{2N\rho} + o(\rho^{\frac{3}{2}}) & \text{if } k = 1, \\ 2\sqrt{kN\rho} + o(\rho^{\frac{3}{2}}) & \text{if } k > 1, \end{cases}$$

or

$$\rho = \begin{cases} \frac{R^2}{8N} + o(R^4) & \text{if } k = 1, \\ \frac{R^2}{4kN} + o(R^4) & \text{if } k > 1. \end{cases}$$

These expressions also hold for the ES limit (2.7). Hence

$$\frac{1}{\nu} \frac{d^2\nu}{dR^2} = \begin{cases} \frac{3}{4R^2} - \frac{1}{64N^2} + o(R^2) & \text{if } k = 1, \\ -\frac{1}{4R^2} + \frac{11}{16(k^2 - 1)N^2} + o(R^2) & \text{if } k > 1. \end{cases} \quad (4.7)$$

Table 2. Asymptotic behaviour of the effective potentials. The ranges of the various parameters are given in table 1.

Model	V_{jsp}^{eff} for $R \rightarrow 0$	V_{jsp}^{eff} for $R \rightarrow \infty$
TN ($k = 1$)	$[(2j + 1)^2 - \frac{1}{4}] \frac{1}{R^2} + o(1)$	$\frac{(s - \frac{\tilde{p}}{2})^2}{4N^2} + \frac{(s - \frac{\tilde{p}}{2})(s + \frac{\tilde{p}}{2})}{2N} \frac{1}{R} + o(\frac{1}{R})$
TB ($1 < k < \infty$)	$[k^2(s - \frac{p}{2})^2 - \frac{1}{4}] \frac{1}{R^2} + o(1)$	$\frac{(s - \frac{p}{2} \frac{k+1}{k-1})^2}{4N^2} + \frac{(k + \frac{1}{k})(s - \frac{p}{2} \frac{k+1}{k-1}) [s - \frac{p}{2} \frac{k+1}{k-1} (1 - \frac{4k}{k^2+1})]}{4N} \frac{1}{R} + o(\frac{1}{R})$
ES ($k = \infty$)	$(n^2 - \frac{1}{4}) \frac{1}{R^2} + o(1)$	$\frac{(n+p)^2}{4L^2} + \frac{1}{4L(n+p)(n-p)} \frac{1}{R} + o(\frac{1}{R})$

We also have the expansion of the j -dependent term

$$\frac{j(j+1) - s^2}{r^2 - N^2} = \begin{cases} \frac{j(j+1) - s^2}{2N\rho} + o(1) & \text{if } k = 1, \\ \frac{j(j+1) - s^2}{(k^2 - 1)N^2} + o(\rho) & \text{if } k > 1, \end{cases}$$

which turns into

$$\frac{j(j+1) - s^2}{r^2 - N^2} = \begin{cases} 4 \frac{j(j+1) - s^2}{R^2} + o(1) & \text{if } k = 1, \\ \frac{j(j+1) - s^2}{(k^2 - 1)N^2} + o(R^2) & \text{if } k > 1. \end{cases} \tag{4.8}$$

To obtain the corresponding results for the ES case we need to combine the limits (2.7) and (3.7), so replace $(k^2 - 1)N^2 \mapsto L^2$ and $s \mapsto \frac{p}{2}$ in the above expressions.

Finally turning to Q , for $k > 1$ we obtain

$$\frac{(r^2 - N^2)}{4N^2(r - kN)(r - \frac{N}{k})} \left(s - \frac{p}{2} \frac{k+1}{k-1} \frac{r-N}{r+N} \right)^2 = \frac{k^2}{R^2} \left(s - \frac{p}{2} \right)^2 + o(1),$$

with the ES limit given by the replacement $k^2(s - p/2)^2 \mapsto n^2$. For $k = 1$ we find

$$\frac{r+N}{4N^2(r-N)} \left(s - \frac{\tilde{p}}{2} \frac{r-N}{r+N} \right)^2 = \frac{4s^2}{R^2} + o(1).$$

We can combine this term with the terms (4.7) and (4.8) to write the centrifugal potential in the $k = 1$ case as

$$\frac{\frac{3}{4} + 4j(j+1)}{R^2} = \frac{(2j+1)^2 - \frac{1}{4}}{R^2}.$$

Our results are summarised in the second column of table 2.

4.3. Long range asymptotics

The computation of the long range asymptotic behaviour is a little messy, but straightforward in principle. Looking at the form of Q when $k > 1$ and $s \neq \frac{p}{2}$ in (4.5), for example, one finds

$$\begin{aligned} & \frac{(r^2 - N^2)}{4N^2(r - kN) \left(r - \frac{N}{k}\right)} \left(s - \frac{pk + 1}{2k - 1} \frac{r - N}{r + N}\right)^2 \\ &= \frac{1}{4N^2} \left(s - \frac{pk + 1}{2k - 1}\right)^2 + \frac{\left(k + \frac{1}{k}\right)}{4N} \left(s - \frac{pk + 1}{2k - 1}\right) \left(s - \frac{k + 1}{k - 1} \left(1 - \frac{4k}{k^2 + 1}\right) \frac{p}{2}\right) \frac{1}{r} \\ &+ o\left(\frac{1}{r^2}\right). \end{aligned}$$

Using the integral defining R to determine the asymptotic relation between R and r ,

$$R = r + \frac{1}{2} \left(k + \frac{1}{k}\right) N \ln r + o(1),$$

we deduce

$$\begin{aligned} & \frac{(r^2 - N^2)}{4N^2(r - kN) \left(r - \frac{N}{k}\right)} \left(s - \frac{pk + 1}{2k - 1} \frac{r - N}{r + N}\right)^2 \\ &= \frac{1}{4N^2} \left(s - \frac{pk + 1}{2k - 1}\right)^2 \\ &+ \frac{\left(k + \frac{1}{k}\right)}{4N} \left(s - \frac{pk + 1}{2k - 1}\right) \left(s - \frac{k + 1}{k - 1} \left(1 - \frac{4}{k^2 + 1}\right) \frac{p}{2}\right) \frac{1}{R} + o\left(\frac{1}{R}\right). \end{aligned}$$

One checks that the case $s = \frac{p}{2}$ has the same large- R asymptotics. Expanding Q similarly for $k = 1$ and using

$$\nu^2 = 2N \sqrt{(r - kN) \left(r - \frac{N}{k}\right) (r^2 - N^2)} = 2Nr^2 \left(1 - \frac{1}{2} \left(k + \frac{1}{k}\right) \frac{N}{r}\right) + o(1)$$

to evaluate (4.6) finally leads to the asymptotic expansions for the effective potential summarised in the third column of table 2.

The results collected in table 2 show that the effective potential typically has a $1/R^2$ singularity at $R = 0$ with a coefficient $\geq -\frac{1}{4}$ while the leading large R terms in table 2 are attractive Coulomb ($1/R$) potentials provided the following conditions are satisfied:

$$\begin{aligned} k = 1 : & -\frac{|\tilde{p}|}{2} < s < \frac{|\tilde{p}|}{2}, \\ 1 < k < \infty, p \geq 0 : & \frac{k + 1}{k - 1} \left(1 - \frac{4k}{k^2 + 1}\right) \frac{p}{2} < s < \frac{pk + 1}{2k - 1}, \\ 1 < k < \infty, p < 0 : & \frac{pk + 1}{2k - 1} < s < \frac{k + 1}{k - 1} \left(1 - \frac{4k}{k^2 + 1}\right) \frac{p}{2}, \\ k = \infty : & -|p| < n < |p|. \end{aligned} \tag{4.9}$$

When $k > 1$, we have $\frac{k+1}{k-1} \left(1 - \frac{4k}{k^2+1}\right) < 1$, so that the allowed interval for an attrac-

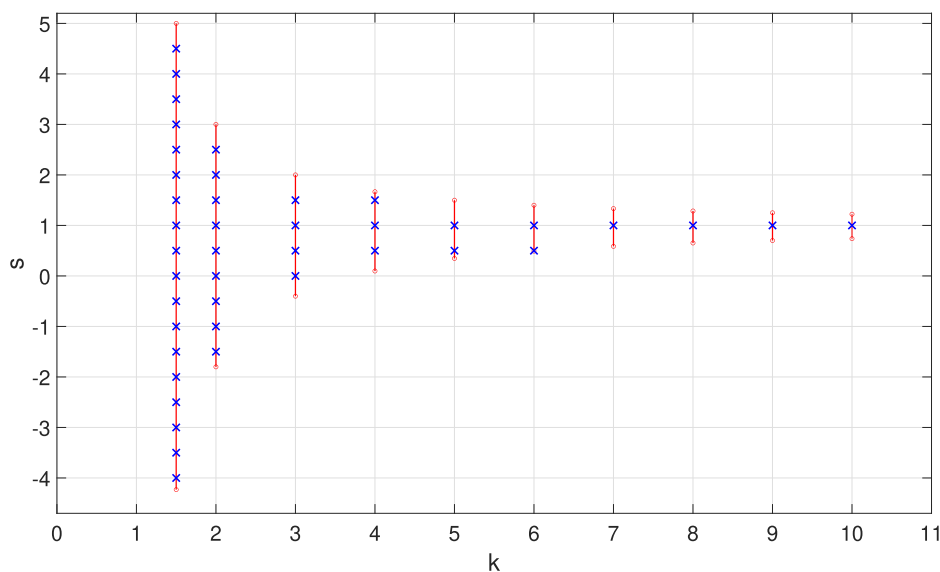


Figure 1. Values of s that allow for bound states according to (4.9) when $p = 2$.

tive Coulomb potential always includes $s = \frac{p}{2}$. In particular, it is never empty. We illustrate the values of s that lead to Coulomb attraction for given p as a function of k in figure 1.

The conditions for attractive Coulomb tails in the TN and ES cases were previously derived in [6, 7] and also interpreted geometrically. As pointed out there, the existence of bound orbits and bound states can be understood qualitatively in terms of the motion of a charged particle on certain two-dimensional submanifolds of the four-geometries. In terms of the coordinates introduced after (2.1), these submanifolds are the disks with angular coordinate ψ and radial coordinate c at fixed values of (θ, ϕ) . The dynamics on these disks is that of a charged particle moving on a disk under the influence of a transverse magnetic field (stemming from the $U(1)$ connection), much studied in the context of Landau levels. The particle moves in circular orbits whose radius is proportional to the angular momentum, but in order to fit onto the disk the angular momentum has to satisfy a bound set by the strength of the magnetic field. For the motion on TN and ES, the magnetic field strength is proportional to $\frac{p}{2}$ and p , and the relevant angular momenta are s and n , resulting in the conditions (4.9). For details and further references see [6], in particular the discussion of a toy model in section 2 of that paper. The condition for bound states in the TB family depends on the geometrical parameter k and is more intricate, but the qualitative picture is as for TN and ES.

4.4. Domain, selfadjointness and spectrum

We now state and prove the central results of this paper. Their validity is a consequence of the fact that, irrespective of the model, the effective potentials (4.4) in the radial reduction of the Hamiltonian turns into a one-dimensional Schrödinger operator which has a Calogero-type singularity at $R = 0$ and a Coulomb-type behaviour at $R = \infty$.

We begin by describing the domain of a selfadjoint realisation of the radial Hamiltonian (4.3). Let

$$m_k = \begin{cases} 2j + 1 & k = 1 \\ k \left(s - \frac{p}{2} \right) & k \in (1, \infty), \\ n & k = \infty \end{cases}$$

so that the Calogero contribution of V^{eff} at $R = 0$ has coefficient $m_k^2 - \frac{1}{4}$. Since $m_k^2 \geq 0$, the differential expression \widehat{H}_k is symmetric and semi-bounded below on the domain $C_0^\infty((0, \infty))$ of compactly supported functions, see [19, lemma 1]. In fact we know that it is always bounded below by zero, from the positivity of the Laplacian (2.1) associated to a Riemannian metric. The operator \widehat{H}_k is limit point at $R = \infty$ (so no boundary condition is required there) but the value of m_k determines the nature of the operator at $R = 0$ and hence its domain of selfadjointness. To see this, consider

$$H = -\frac{d^2}{dR^2} + \frac{m_k^2 - \frac{1}{4}}{R^2}$$

on its minimal domain $\mathcal{D}_{\min} \subset L^2(\mathbb{R}^+)$, which is found from the closure of the operator defined on $C_0^\infty((0, \infty))$. Let $\chi \in C^\infty([0, \infty))$ be a mollifier identically equal to 1 in $[0, 1]$ and vanishing in $[2, \infty)$ and set $\eta_{m_k}(R) = \chi(R)R^{\frac{1}{2}+m_k}$. Defining

$$\mathcal{D} = \mathcal{D}_{\min} + \eta_{m_k}\mathbb{C},$$

we see that $\mathcal{D} = \mathcal{D}_{\min}$ for $m_k \geq 1$. According to [20, proposition 4.17], it is known that the operator H with domain \mathcal{D} is selfadjoint. The following lemma shows that the operator \widehat{H}_k extended to \mathcal{D} also has this property.

Lemma 4.1. *The operator $\widehat{H}_k : C_0^\infty((0, \infty)) \rightarrow L^2(\mathbb{R}^+)$ is essentially selfadjoint, when*

- (a) $k = 1$ (TN space), or
- (b) $1 < k < \infty$ (TB family) and $m_k \geq 1$, or
- (c) $k = \infty$ (ES space) and $n \neq 0$.

Otherwise, it has deficiency indices (1, 1). Moreover, the extension H_k of \widehat{H}_k to \mathcal{D} is a selfadjoint operator.

Proof. It is classically known that $H \upharpoonright_{C_0^\infty((0, \infty))}$ is essentially selfadjoint if and only if $m_k \geq 1$. Moreover, for $0 \leq m_k < 1$, it has deficiency indices (1, 1). According to table 2 the condition for $m_k \geq 1$ holds always for TN and whenever $|n| \geq 1$ for ES.

Now, according to [19, lemma 4], and with the following notation for the constant term in the long range asymptotic (cf table 2),

$$C_0 = \begin{cases} \frac{1}{4N^2} \left(s - \frac{\tilde{p}}{2} \right)^2 & \text{for } k = 1, \\ \frac{1}{4N^2} \left(s - \frac{p}{2} \frac{k+1}{k-1} \right)^2 & \text{for } 1 < k < \infty, \\ \frac{(n+p)^2}{4L^2} & \text{for } k = \infty, \end{cases} \tag{4.10}$$

the potential $V^{\text{eff}}(R) - \frac{m_k^2 - \frac{1}{4}}{R^2} - C_0$ is relatively compact with respect to H . Hence it is relatively bounded with bound 0 with respect to $H \upharpoonright_{C_0^\infty((0, \infty))}$. Thus the Kato–Rellich theorem ensures the conclusions stated in the lemma. \square

We now characterise the spectrum of H_k .

Theorem 4.2. *Let C_0 be defined as in (4.10). Then*

$$\text{Spec}_{\text{ess}}(H_k) = [C_0, \infty).$$

Moreover, H_k has infinitely many eigenvalues below C_0 provided the conditions (4.9) hold.

Proof. The claim about the essential spectrum is a consequence of [19, lemma 4]. Moreover, under the hypotheses (4.9), the coefficient of the long range Coulombic part of the potential is negative and so the claim about the discrete spectrum follows from [19, theorem 6]. This ensures the validity of the theorem. \square

The condition $s = p/2$ plays a special role in the spectral analysis of the TB family and deserves a comment. In our discussion of sections of the line bundle over the bolt used in separating variables, this emerged as a regularity condition for the angular dependence on the bolt, see the paragraph following equation (3.4). With the selfadjoint extension H_k we now find that eigenfunctions of the gauged Laplace operator of the TB family are non-vanishing at the bolt precisely when the angular dependence is regular, i.e. when $s = p/2$. To see this, recall from section 4.1 that η/ν is an eigenfunction of the radial Hamiltonian \hat{H}_k if η is an eigenfunction of \hat{H}_k . Near $R = 0$, the leading term of the latter is $O(R^{\frac{1}{2} + m_k})$. With $\nu = O(\sqrt{R})$ near $R = 0$, the eigenfunction η/ν has leading term R^{m_k} , which, in the TB case, is non-vanishing at the bolt precisely if $s = p/2$. This is a more involved version of what happens when studying the Laplace operator in the plane in terms of polar coordinates (ρ, φ) , see also the discussion of lemma 4 in [19]. The angular functions $e^{in\varphi}$ for integer n are ill-defined at the origin except when $n = 0$. With the right choice of selfadjoint extension of the radial Laplace operator, briefly reviewed above, eigenfunctions are non-vanishing (but finite) at the origin if and only if $n = 0$, thus ensuring that eigenfunctions obtained via separating variables are in the domain of the planar Laplacian.

It is natural to wonder what happens when the coefficient of the Coulomb contribution in the large R regime is non-negative, i.e. in the complement of the intervals given by (4.9). When the Coulomb term is repulsive, we necessarily have $C_0 > 0$ and we cannot rule out a discrete spectrum below C_0 . However, the neutral case (vanishing Coulomb term) is more subtle. For $k = 1, s = \frac{p}{2}$ implies $C_0 = 0$ while $s = -\frac{p}{2}$ implies $C_0 > 0$. For $1 < k < \infty, s = \frac{p}{2} \frac{k+1}{k-1}$ implies $C_0 = 0$ while $s = \frac{p}{2} \frac{k+1}{k-1} \left(1 - \frac{4k}{k^2+1}\right)$ implies $C_0 > 0$. Finally, for $k = \infty, n = -p$ implies $C_0 = 0$ while $n = p$ implies $C_0 > 0$. For $C_0 = 0$, we know from $\text{Spec}(H_k) = [0, \infty)$ that there is no discrete spectrum in this case. However, when the Coulomb contribution vanishes while $C_0 > 0$ we are once again not able to draw any definite conclusion.

5. Approximating the eigenvalues

5.1. Exact and numerical eigenvalues

Having established the existence of infinitely many eigenvalues for the gauged Laplacians of the TN, TB and ES geometries, we now turn to their numerical computation. The TN metric is special compared with the other two geometries in that both the associated classical problem (geodesics) and the associated quantum problem (spectrum of the Laplacian) can be solved

analytically, with simple algebraic expressions for the eigenvalues. This remains true in the presence of the magnetic field (2.11) and is explored in detail in [6]. There is no known method for obtaining exact eigenvalues of the TB or ES Laplacians, so numerical schemes are required to investigate these two cases. The ES case was examined in [7], via an implementation of the shooting method. The main purpose of this section is the approximation of the eigenvalues for the TB family by means of a projection method, but we begin by presenting an analytical approximation of the eigenvalues in terms of TN eigenvalues.

5.2. The Taub-NUT approximation

The starting point of this approximation is the asymptotic form of the TB Laplacian. This turns out to be of the TN form and its spectral properties can therefore be determined exactly. In this way we obtain an analytic approximation to the TB spectrum which we call the TN approximation. Although we have not been able to justify the validity of this approximation rigorously, we show evidence that it is in close agreement with our numerical results.

Following [6], the radial eigenvalue equation for the TN Laplacian coupled to (2.11), expressed in the isotropic coordinates (2.6) is

$$-\frac{d^2\eta}{d\rho^2} + \left[\frac{\tilde{j}(\tilde{j} + 1)}{\rho^2} + \frac{2\tilde{s}(\tilde{s} - \frac{\tilde{p}}{2})}{\Lambda\rho} + \left(\frac{\tilde{s} - \frac{\tilde{p}}{2}}{\Lambda} \right)^2 \right] u = E \left(1 + \frac{\Lambda}{\rho} \right) \eta, \quad \rho \geq 0. \quad (5.1)$$

We have added tildes to the angular momentum eigenvalues j and s to avoid a clash in the notation. The eigenvalues E exist when the condition (4.9) for $k = 1$ holds. The entire spectrum is given by [6]

$$E(\tilde{p}, \tilde{s}, \tilde{n}) = \frac{2}{\Lambda^2} \left(\tilde{s} \left(\tilde{s} - \frac{\tilde{p}}{2} \right) + \tilde{n} \sqrt{\tilde{n}^2 - \left(\tilde{s} - \frac{\tilde{p}}{2} \right) \left(\tilde{s} + \frac{\tilde{p}}{2} \right) - \tilde{n}^2} \right), \quad \tilde{n} = |\tilde{s}| + 1, |\tilde{s}| + 2, \dots \quad (5.2)$$

An important aspect of this formula is that j only restricts the range of s according to (3.3) but does not enter explicitly. This is consequence of a dynamical symmetry of the spectral problem which manifests itself through additional operators of Runge–Lenz type which commute with the Hamiltonian [6]. It matters in the current context because it implies that the spectrum is determined by the coefficients of the terms in the eigenvalue equation (5.1) which are $O(1)$, $O(1/\rho)$, $O(E)$ and $O(E/\rho)$.

We now expand the radial eigenvalue problem for the TB Hamiltonian \tilde{H}_k (3.5) to extract the terms of this order. For this purpose, we introduce abbreviations

$$F(r) = \frac{r^2 - N^2}{(r - kN) \left(r - \frac{N}{k} \right)}, \quad G(r) = s - \frac{pk + 1}{2k - 1} \frac{r - N}{r + N},$$

for the functions which appear in (3.5), and multiply the radial eigenvalue problem by F . The resulting equation

$$F\tilde{H}_k u = EFu$$

is equivalent to

$$-\frac{1}{\mu^2} \frac{d}{dr} \left(\mu^2 \frac{du}{dr} \right) + \frac{j(j+1) - s^2}{r^2 - N^2} u + \frac{F^2 G^2}{4N^2} u = EFu, \tag{5.3}$$

where $\mu^2 = (r - kN) \left(r - \frac{N}{k} \right)$. We can use the identity (4.2) to deduce that (5.3) can be written as

$$-\frac{d^2 \eta}{dr^2} + \frac{1}{\mu} \frac{d^2 \mu}{dr^2} \eta + \frac{j(j+1) - s^2}{r^2 - N^2} \eta + \frac{F^2 G^2}{4N^2} \eta = EF\eta, \tag{5.4}$$

for $\eta = u/\mu$. In this formulation, the TB eigenvalue equation can be compared with the TN eigenvalue equation (5.1). To match the range of the radial variable in the TN problem, we set $\rho = r - kN$, and think of F and G as functions of ρ . Then we expand

$$F(\rho) = \frac{(\rho + (k+1)N)(\rho + (k-1)N)}{\rho(\rho + (k - \frac{1}{k})N)} = 1 + \frac{N(k + \frac{1}{k})}{\rho} + O\left(\frac{1}{\rho^2}\right),$$

and

$$G(\rho) = \left(s - \frac{pk+1}{2k-1} \frac{\rho + (k-1)N}{\rho + (k+1)N} \right) = \left(s - \frac{pk+1}{2k-1} \right) + p \frac{k+1}{k-1} \frac{N}{\rho} + O\left(\frac{1}{\rho^2}\right),$$

to deduce

$$\begin{aligned} \frac{1}{4N^2} F^2(\rho) G^2(\rho) &= \frac{1}{4N^2} \left(s - \frac{pk+1}{2k-1} \right)^2 \\ &+ \frac{1}{2N} \left(s - \frac{pk+1}{2k-1} \right) \left(\left(k + \frac{1}{k} \right) s + \frac{pk+1}{2k-1} \left(2 - k - \frac{1}{k} \right) \right) \frac{1}{\rho} + O\left(\frac{1}{\rho^2}\right). \end{aligned}$$

The remaining terms in (5.4) are $O(1/\rho^2)$. Although we do not need the coefficient of the $1/\rho^2$ term for the computation of the TN approximation, we include it for completeness. Setting

$$J^2 = j(j+1) + \frac{(k+1)(k-1) \left(s - \frac{pk+1}{2k-1} \right) \left((k^2 - 3)s - (k^2 - 2k + 3) \frac{p}{2} \right)}{4k^2},$$

the TN approximation to the TB radial equation is

$$\begin{aligned} -\frac{d^2 \eta}{d\rho^2} + \frac{J^2}{\rho^2} \eta + \frac{1}{2N} \left(s - \frac{pk+1}{2k-1} \right) \left(\left(k + \frac{1}{k} \right) s + \frac{pk+1}{2k-1} \left(2 - k - \frac{1}{k} \right) \right) \frac{1}{\rho} \eta \\ + \frac{1}{4N^2} \left(s - \frac{pk+1}{2k-1} \right)^2 \eta = E \left(1 + \frac{N(k + \frac{1}{k})}{\rho} \right) \eta. \end{aligned}$$

Comparing with (5.1) we deduce that by setting

$$\Lambda = \left(k + \frac{1}{k} \right) N,$$

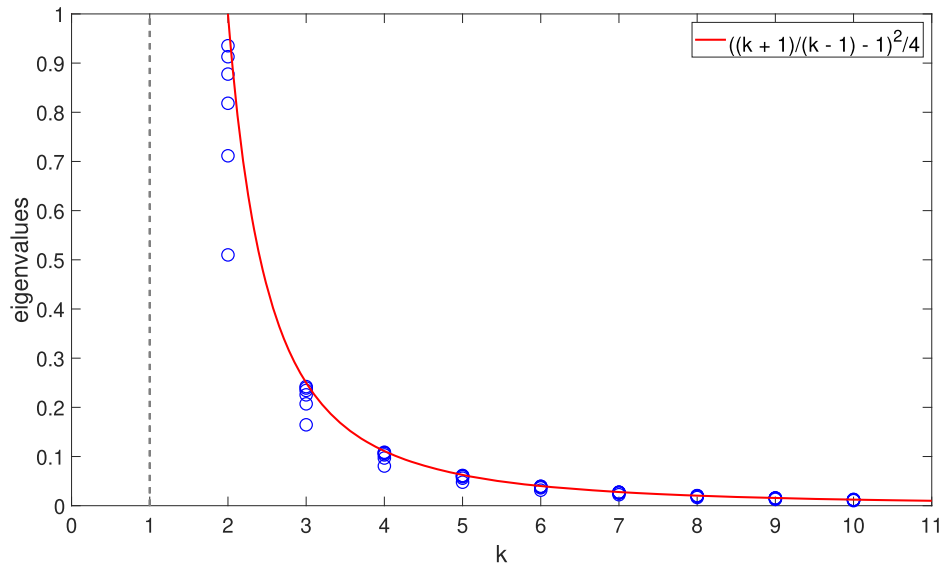


Figure 2. The numerically computed eigenvalues as a function of k when $(s, j, p) = (1, 1, 2)$, with the bottom of the essential spectrum C_0 shown as a continuous curve.

as well as

$$\tilde{p} = p \frac{k+1}{k-1}, \quad \tilde{s} = \frac{\tilde{p}}{2} + \frac{1}{2} \left(k + \frac{1}{k} \right) \left(s - \frac{pk+1}{2k-1} \right), \quad (5.5)$$

we obtain the TN approximation to the energy levels from (5.2). It is convenient to set $\sigma = \tilde{s} - \frac{\tilde{p}}{2}$ and to compute the TN approximation from

$$\varepsilon(p, s, \tilde{n}) = \frac{1}{N^2 \left(k + \frac{1}{k} \right)^2} \left(2\sigma^2 + \tilde{p}\sigma + 2\tilde{n} \sqrt{\tilde{n}^2 - (\sigma^2 + \tilde{p}\sigma) - 2\tilde{n}^2} \right), \quad \tilde{n} = |\tilde{s}| + 1, |\tilde{s}| + 2, \dots \quad (5.6)$$

Substituting the parameters N, k, s, p of a given TB radial Hamiltonian into (5.5) and (5.6) gives the TN approximation to the eigenvalues of that Hamiltonian which we compare with numerically computed eigenvalues below. As a consistency check we compared the conditions for bound states in the TN problem derived in [6], and found that it agrees with the condition derived from (4.9) for the case $1 < k < \infty$.

It is instructive to determine the dependence of the eigenvalues on \tilde{n} as $\tilde{n} \rightarrow \infty$, and to compare with the standard Coulomb problem. Firstly, note that

$$\sigma + \tilde{p} = \frac{1}{2} \left(k + \frac{1}{k} \right) \left(s - \frac{k+1}{k-1} \left(1 - \frac{4k}{k^2+1} \right) \frac{p}{2} \right).$$

Hence,

$$\begin{aligned} \varepsilon(p, s, \tilde{n}) &= \frac{1}{4N^2} \left(s - \frac{pk+1}{2k-1} \right)^2 - \frac{(\sigma(\sigma + \tilde{p}))^2}{4N^2(k + \frac{1}{k})^2} \frac{1}{\tilde{n}^2} + O\left(\frac{1}{\tilde{n}^4}\right) \\ &= \frac{1}{4N^2} \left(s - \frac{pk+1}{2k-1} \right)^2 \\ &\quad - \frac{(k + \frac{1}{k})^2}{64N^2} \left(s - \frac{pk+1}{2k-1} \right)^2 \left(s - \frac{k+1}{k-1} \left(1 - \frac{4k}{k^2+1} \right) \frac{p}{2} \right)^2 \frac{1}{\tilde{n}^2} + O\left(\frac{1}{\tilde{n}^4}\right). \end{aligned}$$

Thus, the leading terms agrees with the lower bound on the essential spectrum for the TB family given in (4.10), and the coefficient of the eigenvalue spacing term $1/\tilde{n}^2$ is precisely a quarter of the square of the coefficient of the Coulomb potential in table 2, which is a familiar feature of the standard Coulomb problem in three dimensions.

5.3. Numerical results

For the computation of eigenvalues of the radial Hamiltonians (3.5) (when $s \neq p/2$) and (3.6) (when $s = p/2$) in the TB family, we have set the parameter N in (2.3) to $N = 1$, but consider a range of benchmark values for k . We numerically compute upper bounds on the eigenvalues using the Riesz–Galerkin method via the Matlab toolbox Chebfun which implements a Chebyshev basis, see [21] and references therein. This requires a truncation of the interval $[r_+, \infty)$ to a finite interval which avoids singularities at r_+ . In practice we work on $[r_+ + h, m]$ for suitable small h and suitably large m . For a detailed discussion of the method in a similar context see [19].

We illustrate the dependence of the numerically computed eigenvalues on the geometrical parameter k in figure 2, where we plot their numerical approximation together with the lower end of the essential spectrum C_0 for the case $(s, j, p) = (1, 1, 2)$ (still keeping $N = 1$).

In table 3 we show the numerically computed six lowest eigenvalues $\lambda_1, \dots, \lambda_6$ for the four cases $(s, j, p) = (1, 1, 2), (1, 2, 2), (0, 2, 2), (0, 0, 2)$ respectively. We compare the answer with the TN approximation formula, and compute the relative error according to

$$RE_l = \frac{|\lambda_l - \lambda_l^{\text{TN}}|}{C_0 - \lambda_l},$$

where λ_l is the numerically calculated l th eigenvalue, λ_l^{TN} is the TN approximation for the l th eigenvalue computed according to (5.6), and C_0 is the lower bound of the essential spectrum given in (4.10). Some of our numerically computed eigenvalues lie above C_0 . By construction, the Riesz–Galerkin method does not give any information about eigenvalues embedded in the essential spectrum [22, section 4.5]. Where our numerical results lie above the threshold of the essential spectrum we have therefore discounted them.

The data collected in table 3 shows several trends regarding the agreement between the TN approximation and the numerically computed eigenvalues.

- As already seen in [7] for ES, the agreement is best when $j = s$. In two such cases, namely $k = 2$ with $(j, s, p) = (1, 1, 2)$ and $k = 3$ with $(j, s, p) = (0, 0, 2)$ it is consistently very useful, with an accuracy better than 8%. When $|s| < j$, the TN approximation is not useful.
- The agreement improves with the order of the eigenvalue. This is expected since eigenfunctions for higher eigenvalues have more of their support in the asymptotic region of TB where, by construction, the Hamiltonian is close to its TN approximation. There is

Table 3. The numerically computed first six eigenvalues for $N = 1$ and a range of values for the parameter k , for the four cases $(s, j, p) = (1, 1, 2), (1, 2, 2), (0, 2, 2), (0, 0, 2)$ from top to bottom in each cell. For the latter two cases, bound states only occur for the first two values of k according to the condition (4.9), illustrated in figure 1. The relative errors of the TN approximation (5.3) are shown as percentages.

k	λ_1	Rel error	λ_2	Rel error	λ_3	Rel error	λ_4	Rel error	λ_5	Rel error	λ_6	Rel error
2	0.509 9813	7.76%	0.711 5164	4.55%	0.818 3046	3.2%	0.877 6668	2.48%	0.912 9560	2.02%	0.935 2771	1.71%
	0.736 107 42	100.09%	0.830 7225	78.18%	0.884 4950	62.34%	0.917 0277	51.09%	0.937 8735	42.94%	0.951 9037	36.87%
	2.052 49738	118.20%	2.120 307 36	92.76%	2.159 462 05	74.40%	2.183 666 72	61.30%	2.199 504 90	51.76%	2.210 367 87	44.60%
	1.884 3085	17.84%	2.033 4290	15.44%	2.110 3984	13.11%	2.153 7739	11.19%	2.180 1354	9.68%	2.197 1772	8.49%
3	0.164 4953	15.42%	0.207 1437	9.39%	0.225 4277	6.66%	0.234 3488	5.15%	0.239 2419	4.19%	0.242 1806	3.53%
	0.213 1688	168%	0.228 2401	115.4%	0.235 8353	85%	0.240 1095	66.4%	0.242 7268	54%	0.244 4367	45.5%
	0.991 7801	107.70%	0.994 8492	76.86%	0.996 4864	59.12%	0.997 4558	47.82%	0.998 0752	40.06%	0.998 5059	35.50%
4	0.983 5014	3.48%	0.991 1283	2.68%	0.994 5269	2.15%	0.996 3053	1.79%	0.997 3446	1.52%	0.998 0021	1.33%
	0.080 6650	21.76%	0.096 9355	13.41%	0.103 2707	9.54%	0.106 2113	7.37%	0.107 7803	5.99%	0.108 7071	5.05%
5	0.099 2625	212.9%	0.104 3036	136.2%	0.106 7431	96.6%	0.108 0860	73.9%	0.108 8977	59.5%	0.109 4250	49.8%
	0.047 7721	26.79%	0.055 9210	16.49%	0.058 9276	11.71%	0.060 2881	9.02%	0.061 0043	7.32%	0.061 4286	6.61%
6	0.057 0821	244.6%	0.059 4288	149.5%	0.060 5425	103.8%	0.061 1494	78.5%	0.061 5242	64.5%	0.061 8603	78.5%
	0.031 5512	30.73%	0.036 3237	18.82%	0.038 0257	13.30%	0.038 7841	10.22%	0.039 1840	8.77%	0.039 4826	21.67%
7	0.036 9997	268.1%	0.038 3124	158.8%	0.038 9285	108.8%	0.039 2705	83.7%	0.039 5636	103.4%	0.039 9422	988.6%
	0.022 3782	33.93%	0.025 4703	20.66%	0.026 5475	14.55%	0.027 0234	11.25%	0.027 3029	16.65%	0.027 6131	138.03%
8	0.025 9057	286.3%	0.026 7301	165.8%	0.027 1160	113.0%	0.027 3732	107.4%	0.027 6875	513.4%	—	—
	0.016 6916	36.56%	0.018 8406	22.15%	0.019 5767	15.55%	0.019 9055	13.32%	0.020 1593	50.66%	—	—
9	0.019 1417	300.7%	0.019 7019	171.1%	0.019 9741	121.3%	0.020 2229	207.4%	—	—	—	—
	0.012 9257	38.75%	0.014 4977	23.36%	0.015 0295	16.40%	0.015 2837	19.98%	0.015 5447	234.83%	—	—
10	0.014 7169	312.5%	0.015 1206	175.7%	0.015 3413	144.3%	0.015 6005	1570.4%	—	—	—	—
	0.010 3028	40.60%	0.011 4992	24.37%	0.011 9010	17.37%	0.012 1221	37.56%	—	—	—	—
	0.011 6655	322.2%	0.011 9712	181.1%	0.012 1731	202.3%	—	—	—	—	—	—

an increase in the relative error in the cases directly before where our numerical approximation become unreliable and lies above the threshold of the essential spectrum. This indicates that they are numerical artefacts.

- In the case $s = \frac{p}{2}$ the accuracy of the TN approximation gets worse with increasing k , while for $s \neq \frac{p}{2}$ it improves (for a smaller data set).

6. Conclusions

In this paper we generalised and extended the study of spectral properties of the gauged TN and ES geometries initiated in [6, 7], to a parameter-dependant family of gauged TB geometries which interpolates between them and combined various non-trivial features of both. We showed that the gauged Laplace operators of all members of the family have qualitatively similar spectra, including an infinite tower of Coulombic bound states when the fibre angular momentum s satisfies a bound set by the Chern class of the connection. Our findings are underpinned by a rigorous analysis of selfadjoint extensions which also applies to the previous studies of the radial TN and ES Laplacians. Remarkably, the edge-cone singularities in the classical geometry of the TB family cause no particular difficulty for the spectral analysis.

One surprising aspect of our study is the accuracy of the analytical approximation to the spectrum provided by the exactly solvable gauged TN Laplacian, and its dependence on the total angular momentum j and the fibre angular momentum s . The gauged TN Laplacian has a dynamical symmetry of Runge–Lenz type [6], and as a result its eigenvalues do not directly depend on the value of the total angular momentum. This symmetry is not present in the TB family: its spectrum shows no unexpected additional degeneracy and does depend on j . It is therefore not surprising that the TN approximation is better for some choices of j than for others. However, it is not clear why the choice $j = s$ is particularly advantageous, and this should be investigated in future research.

A further direction for future work is the consideration of negative values of the parameter N which was kept fixed and positive here. The TN metric for negative N —the ‘negative mass TN space’—has a well-defined Laplacian even though the classical geometry has a singularity, and its spectrum can be determined analytically [2]. Remarkably, this Laplacian has infinitely many Coulombic bound states even without the presence of a connection. It would therefore be of interest to consider the case of negative N for all of the geometries discussed here, to see if the Laplacian can be given a satisfactory definition and to study its spectrum. The interplay of the parameters k and N and their interaction with a self-dual connection on these spaces provide a rich field for further study.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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