

The effective dark energy in brane universe

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Abstract. We study the dynamical system of braneworld cosmology with the effective cosmological constant as the effective dark energy on brane universe. Firstly we introduce several scenarios of braneworld model then derive the effective EFE on the brane which reduced at low energy limit. Next, we discuss the FRW cosmological model in braneworld scenario considered in static and non-closed system. In the dynamical study of cosmology, we assume the equation of state of dark energy varies with time and can be parameterized as a function of scale factor of the universe. Using the dynamical system approach, we describe the evolution of brane universe with non-vanishing NCC/ BCC term. In this study, we investigate the effects of the presence of NCC/ BCC perturbations in brane universe. Finally, we discuss the NCC/ BCC perturbations in the late-time cosmic acceleration when the dark energy contribution is $\Omega_\Lambda \simeq 0.72$.

1. Introduction

The study of modified Einstein field equation (EFE) in general relativity is recently an active research topic in theoretical physics. One type of modification is braneworld scenario using the extra-dimensional approach. The braneworld model was motivated by M-theory, first proposed by Horava and Witten who showed that the $E_8 \times E_8$ heterotic string theory is equivalent to the realization of M-theory on the eleven-dimensional (11D) orbifold $\mathbf{R}^{10} \times \mathbf{S}^1/Z_2$ [1]. This theory can be compactified into $\mathbf{R}^4 \times \mathbf{S}^1/Z_2 \times [\text{Calabi-Yau}]^6$. As the tiny 6D Calabi-Yau extra space was smaller than other dimensions, the theory can be reduced into the 5D spacetime $\mathbf{R}^4 \times \mathbf{S}^1/Z_2$, where 4D spacetime \mathbf{R}^4 was set as boundary layers [2]. In the braneworld scenario based on the Horava-Witten theory, the 4D spacetime was described as a domain wall called a brane and the 5D full spacetime as the bulk spacetime. The standard model particles are confined on a brane, whereas gravitons propagate in the bulk spacetime [3]. In this case, the matter fields are confined in the 4D spacetime while gravity acts in the full spacetime.

In addition to the formulation by Horava-Witten, the braneworld scenario was also formulated by Randall and Sundrum (RS) in the form of 5D warped spacetime [4,5], and by Dvali, Gabadadze, and Porrati (DGP) with 5D Minkowski spacetime [6]. There are two models in the RS formulation, the first RS model (RS1 model) explains that two identical branes are embedded in a 5D Anti-de-Sitter (AdS) bulk spacetime with Z_2 symmetry and small extra dimension [4]. In the second RS model (RS2 model), a single brane with positive tension (λ) is embedded in a 5D AdS bulk spacetime of infinite extra dimensions [5]. Whereas, in the DGP model, the 3-brane is embedded in a Minkowski bulk spacetime of infinitely large extra dimensions [6].



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Derivation of the EFE in braneworld scenario was proposed by Shiromizu, *et al.* [7], who showed that the effective EFE reduced to the conventional EFE for low energy limit. The effective EFE on brane is obtained by projecting the 5D metric and the 5D EFE onto the brane volume. The effective EFE showed a non-closed system by the presence of 5D Weyl correction term as a bulk nonlocal effect on brane. However, the effective cosmological constant and Λ_5 were not discussed in Ref. [7]. In another study, Maartens, *et al.* [8] derived the effective EFE, generalizing the effective field equation in Ref. [7] with an additional correction term. The derived cosmology in braneworld scenario with a perfect fluid model was introduced in Ref. [8]. However, the effective cosmological constant was negligible for the dynamics of braneworld cosmology in Ref. [8].

In this paper, we report our study on the dynamical system of braneworld cosmology, and try to describe the effects of the presence of nonlocal cosmological constant (NCC) or bulk cosmological constant (BCC) perturbations in brane universe. The evolution of brane universe is described without ignoring the NCC/ BCC term, i.e. Λ_5 as the nonlocal effect on the brane. We also discuss the Friedmann-Robertson-Walker (FRW) brane as a non-closed system and static with the effective cosmological constant contributing at low energy limit. In this case, the matter field terms associated with high energy can be negligible, whereas the bulk nonlocal effects on the brane are non-vanishing for the non-closed brane system. We argue that the dark energy is defined by the effective cosmological constant expressed as the total energy density of the vacuum energy localized on brane and the NCC/ BCC perturbations, where Λ_5 could play a very important role, especially in the late-time cosmic acceleration.

The remaining part of the paper is organized as follows: section 2 describes the effective field equation on the brane for low energy limit. The dynamical study of braneworld cosmology and the effects of the NCC/ BCC perturbations on FRW brane are discussed in section 3 and section 4, respectively. Section 5 is devoted to the concluding remarks.

2. The effective EFE

In the braneworld scenario, the 4D spacetime is described by a domain wall, i.e. 3-brane $(\Sigma, q_{\mu\nu})$ embedded in 5D bulk spacetime (\mathcal{M}, g_{AB}) . The induced metric on Σ in full spacetime is given by $q_{AB} = g_{AB} - n_A n_B$, with n_A being a vector unit normal to Σ and g_{AB} is the 5D metric. The EFE in the bulk spacetime, with negative cosmological constant Λ_5 (see Ref. [7,8]), is given as

$${}^{(5)}G_{AB} = -\Lambda_5 g_{AB} + \kappa_5^2 {}^{(5)}T_{AB}, \quad (1)$$

where ${}^{(5)}G_{AB}$ is the 5D Einstein tensor and κ_5^2 is the 5D gravitational constant. ${}^{(5)}T_{AB}$ is the energy-momentum tensor of the matter fields in the bulk, whereas Λ_5 is the bulk cosmological constant (NCC/ BCC term). Since there are no matter fields other than gravity in the bulk spacetime for braneworld scenario, we can assume that ${}^{(5)}T_{AB} = 0$, so that the 5D EFE becomes ${}^{(5)}G_{AB} = -\Lambda_5 g_{AB}$ in the bulk [8]. Then, by using the 5D metric projections on the brane from Gauss-Codazzi-Mainardi equations [7–10], we have the following effective EFE on the brane:

$${}^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + \kappa^2 T_{\mu\nu} + \frac{6}{\lambda} \kappa^2 \Pi_{\mu\nu} - E_{\mu\nu}, \quad (2)$$

where the extrinsic curvature on Σ is uniquely determined by the Israel junction condition and imposed the Z_2 symmetry with the brane as the fixed point in the full spacetime. Here, $\kappa^2 = 8\pi G_N = \frac{1}{6} \kappa_5^4 \lambda$ is the 4D gravitational constant and $T_{\mu\nu}$ being the energy-momentum tensor of the matter fields on Σ . The effective cosmological constant on the brane [8,10] is written as

$$\Lambda_4 \equiv \frac{1}{2} (\Lambda_5 + \kappa^2 \lambda), \quad (3)$$

where λ being the brane tension or the vacuum energy localized on the brane. Based from RS model [4,5] (see also Ref. [9,10]), Λ_5 is defined as

$$\Lambda_5 \equiv -\frac{6}{\ell^2}. \tag{4}$$

In the effective field equation (2) on the brane, there are two additional correction terms which generalize the conventional EFE. The first correction term is:

$$\Pi_{\mu\nu} \equiv \frac{1}{12}TT_{\mu\nu} - \frac{1}{4}T_{\mu\alpha}T_{\nu}^{\alpha} + \frac{1}{8}q_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{24}q_{\mu\nu}T^2, \tag{5}$$

which arises from the extrinsic curvature terms and quadratic in $T_{\mu\nu}$. This term could play an important role in the early universe when the matter-energy scale is high. The second correction term is the projected 5D Weyl tensor term,

$$E_{\mu\nu} \equiv {}^{(5)}C^A{}_{BRS} n_A n^R q_{\mu}^B q_{\nu}^S, \tag{6}$$

which carries some information about the bulk nonlocal gravitational field on the brane.

By applying the contracted Bianchi identities in Eq. (2) ($D^{\mu} {}^{(4)}G_{\mu\nu} = 0$), we have

$$D^{\mu} E_{\mu\nu} = \frac{6}{\lambda} \kappa^2 D^{\mu} \Pi_{\mu\nu}, \tag{7}$$

in which we imposed the conservation equation $D^{\nu} T_{\mu\nu} = 0$. This equation shows that the $E_{\mu\nu}$ term as the bulk nonlocal gravitational effect on the effective field equation, may strongly be the gravitational field interaction in high energy regime, because its divergence is constrained by the quadratic term relevant with high energy scale of the matter-energy. Following Ref. [7,9], in the low energy limit, the $\Pi_{\mu\nu}$ and the $E_{\mu\nu}$ terms are negligible. Therefore, the effective field equation (2) on the brane can be reduced to the conventional EFE,

$${}^{(4)}G_{\mu\nu} \simeq -\Lambda_4 q_{\mu\nu} + \kappa^2 T_{\mu\nu}, \tag{8}$$

for low energy limit. In this study, Λ_4 is still defined by the effective cosmological constant (3), where Λ_5 is non-vanishing since the brane considered as a non-closed system.

3. The dynamics of braneworld cosmology

3.1. Cosmology of FRW brane

As an alternative approach, the FRW cosmological model in braneworld scenario can be determined by using the Gaussian normal coordinates in the general 5D metric with the cosmological symmetries, given by (see Ref. [8,11])

$$ds^2 = -N^2(t, y)dt^2 + A^2(t, y)\gamma_{ij}dx^i dx^j + dy^2, \tag{9}$$

where $n_{\mu}dx^{\mu} = dy$ is the Gaussian normal coordinate orthogonal to the brane. In the Gaussian normal coordinates, the brane is considered static (not moving but located at a fixed coordinate), which can be set as $y = 0$ on the brane. Therefore, the metric (9) can describe the cosmology of FRW brane, with $A(t, y = 0) \equiv a(t)$ being the scale factor on the FRW brane whereas $N(t, y = 0) \equiv 1$ if t is set as proper time on Σ . For this condition, the metric (9) can properly represent the conventional FRW metric on the static brane.

In the standard cosmological aspects of braneworld, the FRW brane is always considered as a homogeneous and isotropic, where the matter-energy on the brane described as a perfect fluid,

$$T_{\mu\nu} \equiv q_{\mu\nu} T = q_{\mu\nu} \cdot \text{diag}(-\rho, P, P, P), \tag{10}$$

with ρ is the matter-energy density and P is the isotropic pressure. Therefore, the modified Friedmann equation on the brane for Eq. (2) is given by

$$H^2 = \frac{\Lambda_4}{3} + \frac{\kappa^2}{3}\rho \left(1 + \frac{\rho}{2\lambda}\right) - \frac{k}{a^2} + \frac{2\mathcal{U}}{\kappa^2\lambda}, \quad (11)$$

where \mathcal{U} is the dark radiation [8] (see also Ref. [12]). In the low energy limit, as $\rho \ll \lambda$, so that $\rho/\lambda \rightarrow 0$, the last term on Eq. (11) will also vanish. Hence, the Friedmann equation on the brane at low energy limit can be written as

$$H^2 = \frac{\Lambda_4}{3} + \frac{\kappa^2}{3}\rho - \frac{k}{a^2}. \quad (12)$$

Here, we assume that Λ_4 contributes to dark energy on the brane with the equation of state of dark energy $w_\Lambda \neq -1$ and varies with time. In this case, the evolution of Λ_4 [13] is given by

$$\dot{\Lambda}_4 = -3H(1 + w_\Lambda)\Lambda_4. \quad (13)$$

In the cosmological dynamics, we have time derivative of the Friedmann equation (12) on the brane at low energy limit as

$$\dot{H} = \frac{k}{a^2} - \left(\frac{1 + w_\Lambda}{2}\right)\Lambda_4 - \frac{\kappa^2}{2}(\rho + P), \quad (14)$$

where the second term is derived from Eq. (13).

3.2. Dynamical system approach

From Eqs. (12) and (14), the following equations of motion for the flat FRW background ($k = 0$) on the brane are:

$$3H^2 = \Lambda_4 + \kappa^2\rho, \quad (15)$$

$$2\dot{H} = -(1 + w_\Lambda)\Lambda_4 - \kappa^2(\rho + P), \quad (16)$$

where the total energy density ρ and the isotropic pressure P in Eqs. (15) and (16) are given by $\rho = \rho_m + \rho_r$ and $P = \rho_r/3$, respectively. Here ρ_m and ρ_r are the non-relativistic matter density and the radiation density, respectively [14]. In order to study cosmological dynamics, we also introduce the following dimensionless variables:

$$x_1 \equiv \frac{\sqrt{\Lambda_4}}{\sqrt{3}H}, \quad x_2 \equiv \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H}, \quad (17)$$

and the density parameters for dark energy, radiation, and non-relativistic matter are respectively given by,

$$\Omega_\Lambda \equiv x_1^2, \quad \Omega_r \equiv x_2^2, \quad \Omega_m \equiv \frac{\kappa^2\rho_m}{3H^2} = 1 - x_1^2 - x_2^2. \quad (18)$$

These satisfy the relation of total density parameters on the brane, $\Omega_\Lambda + \Omega_r + \Omega_m = 1$ according to Eq. (15). Differentiating the variables x_1 and x_2 with $N = \ln a$ together by using the Eqs. (13) and (15), we obtain the following autonomous equations:

$$\frac{dx_1}{dN} = -\frac{3}{2}(1 + w_\Lambda)x_1 - x_1\frac{\dot{H}}{H^2}, \quad (19)$$

$$\frac{dx_2}{dN} = -2x_2 - x_2 \frac{\dot{H}}{H^2}, \tag{20}$$

where \dot{H}/H^2 is given by

$$\frac{\dot{H}}{H^2} = -\frac{1}{2} (3 + 3w_\Lambda x_1^2 + x_2^2). \tag{21}$$

Here, we have the total effective equation of state on the brane as

$$w_{\text{eff}} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} = w_\Lambda x_1^2 + \frac{x_2^2}{3}. \tag{22}$$

Now, we derive the fixed points of the dynamical system in equations (19)–(20), which satisfy the conditions $dx_1/dN = dx_2/dN = 0$ and set $x_2 = 0$ for dark energy dominated era at low energy limit. In this case, we study the following fixed points:

(i) for $x_1 = 0$, we obtain the saddle point

$$(x_1, x_2) = (0, 0), \quad \Omega_\Lambda = 0, \quad \Omega_m = 1, \quad w_{\text{eff}} = 0. \tag{23}$$

(ii) Λ_4 -dominated point (stable point)

$$(x_1, x_2) = (\pm 1, 0), \quad \Omega_\Lambda = 1, \quad \Omega_m = 0, \quad w_{\text{eff}} = w_\Lambda. \tag{24}$$

(iii) de-Sitter point (by setting $w_\Lambda = -1$)

$$(x_1, x_2) = (\pm 1, 0), \quad \Omega_\Lambda = 1, \quad \Omega_m = 0, \quad w_{\text{eff}} = -1. \tag{25}$$

Point (i) is matter-dominated solution (23), while point (ii) is compatible with dark energy dominated solution $(\Omega_\Lambda, \Omega_M) = (1, 0)$, where the cosmic acceleration is realized when $w_{\text{eff}} < -1/3$ [14,15]. As w_Λ is not specified, then we try to vary the equation of state of Λ_4 for dark energy dominated epoch (ii). The trajectories of phase space solutions for variation of w_Λ are given in Fig. 1.

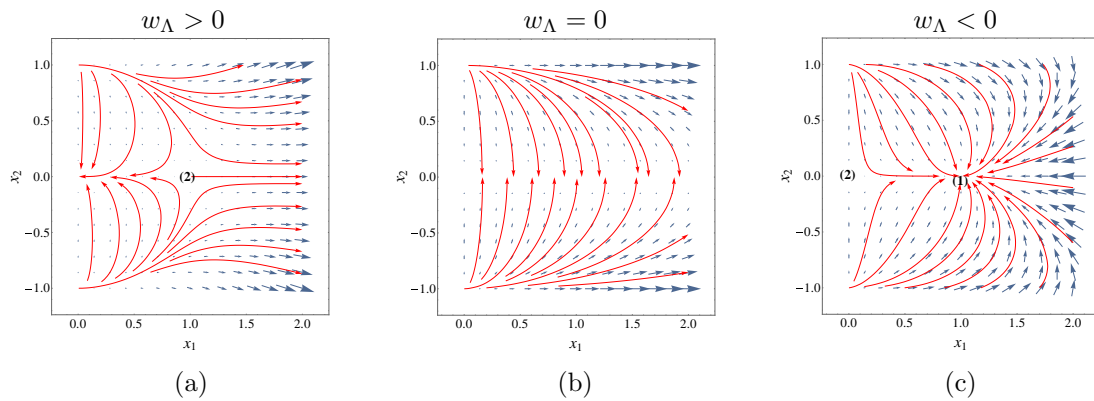


Figure 1. The trajectories of phase space solutions for Eqs. (19)–(20) in the fixed point $(\Omega_\Lambda, \Omega_M) = (1, 0)$ with varying w_Λ . (a) For $w_\Lambda > 0$, critical point (2) at $x_1 = 1, x_2 = 0$ is saddle node, (b) for $w_\Lambda = 0$, the solution becomes singular and does not have a fixed point, and (c) for $w_\Lambda < 0$, critical point (1) at $x_1 = 1, x_2 = 0$ is stable node, while critical point (2) at $x_1 = 0, x_2 = 0$ is saddle node.

Fig. 1 shows that the equation of state of Λ_4 must be smaller than 0 ($w_\Lambda < 0$, $w_\Lambda \in \mathbb{R}^-$) for the current universe, i.e. dark energy dominated epoch (see Fig. 1(c)). In this case, we set w_Λ to satisfy the observational constraints.

Based on the combined data analysis of SN Ia, CMB, and BAO, using the WMAP [14,15], the equation of state for dark energy is given by the bound $-1.098 < w_\Lambda < -0.841$ at the 95% confidence level with dark energy contributing about $\Omega_\Lambda^{(0)} = 0.726 \pm 0.015$ (WMAP 5-year constraint). If $w_\Lambda = -1$ (like in the Λ CDM model), the dark energy density parameter from the CMB shift parameter is constrained to be $0.72 < \Omega_\Lambda^{(0)} < 0.77$ while values of $\Omega_\Lambda^{(0)}$ become smaller for increasing w_Λ [14]. Therefore, we can set the equation of state for dark energy constrained to be $-4/3 < w_\Lambda < -1/3$, for the late-time cosmic acceleration ($w_\Lambda < -1/3$) and the contribution of dark energy compatible with observational data constraints. The trajectories of phase space solutions from the stability point for this case are shown in Fig. 2.

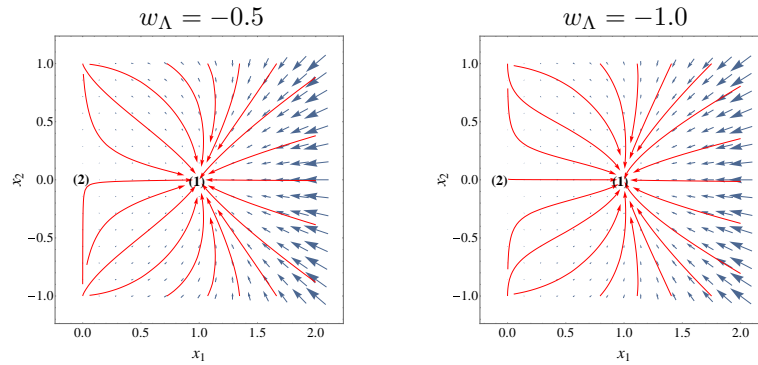


Figure 2. The trajectories of phase space solutions for Eqs. (19)–(20) in the fixed point $(\Omega_\Lambda, \Omega_M) = (1, 0)$ with $-4/3 < w_\Lambda < -1/3$, where critical point (1) at $x_1 = 1$, $x_2 = 0$ is stable node, while critical point (2) at $x_1 = 0$, $x_2 = 0$ is saddle node.

Since the equation of state of Λ_4 varies with time, one needs to parameterize w_Λ to describe its evolution. In this case, we parameterize w_Λ as a function of scale factor of the universe, a [16]. The evolution of w_Λ can be determined by using Eq. (27), with parameterization written in the form (see Ref. [14,16]):

$$w_\Lambda(a) = \sum_{n=0} w_n X_n(a), \quad n \leq 1, \quad (26)$$

where the function of expansion is given by $X_n(a) \equiv \left(1 - \frac{a_0}{a}\right)^n$. Then Eq. (26) can be expanded as

$$w_\Lambda(a) = w_0 + w_1 \left(1 - \frac{a_0}{a}\right), \quad -4/3 < w_\Lambda(a) < -1/3. \quad (27)$$

For $a = a_0 = 1$, we have $w_\Lambda(a) \rightarrow w_0$ and $w_\Lambda(a) = w_0 + w_1$ for $a \rightarrow \infty$. Therefore, we found that $w_0 \lesssim -1/3$ and $w_0 + w_1 \gtrsim -4/3$. In Fig. 3, we plot the evolution of Ω_Λ and w_{eff} versus N for each variation of w_Λ , with $w_\Lambda < 0$ and it is expressed as a constant, including for w_Λ as a function of scale factor of the universe (27). For $w_\Lambda = -0.2$, we found the evolution of Ω_Λ slowly increasing and w_{eff} is too large for the cosmic acceleration today. Otherwise, for $w_\Lambda = -2.0$, we found the evolution of Ω_Λ rapidly increasing, while w_{eff} is very small and incompatible with the observational data constraints.

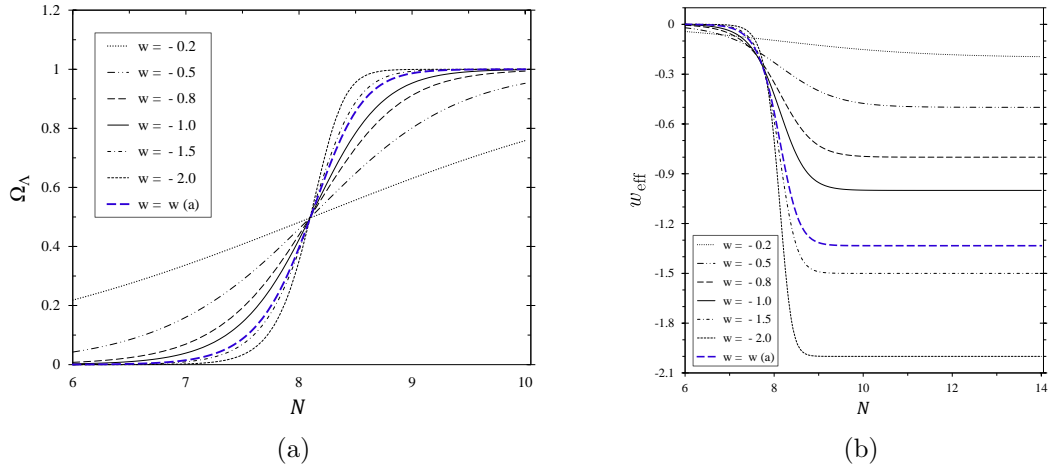


Figure 3. The evolution of: (a) Ω_Λ , and (b) w_{eff} from Eq. (22), for $w_\Lambda = -0.2; -0.5; -0.8; -1.0; -1.5$; and -2.0 , including for $w_\Lambda \equiv w_\Lambda(a) \simeq -4/3 + a_0 e^{-N}$ which represent the equation (27) with $N = \ln a$ and $a_0 = 1$.

4. The NCC/ BCC perturbations

The NCC/ BCC term is a nonlocal effect on the brane, which could play an important role at low energy limit, especially in the late-time cosmic acceleration. If the NCC/ BCC perturbations are non-vanishing on the brane, we have the total effective vacuum energy density called the effective dark energy given by

$$\rho_\Lambda \equiv \rho_\ell + \rho_\lambda, \tag{28}$$

follows from Eqs. (3) and (4), which satisfy the energy density of 4D cosmological constant, i.e. $\rho_\Lambda = \Lambda_4/\kappa^2$. Here ρ_ℓ and ρ_λ are the NCC/ BCC energy density and the vacuum energy density localized on the brane, respectively, which given by

$$^{(5)}\rho_\Lambda = \frac{\Lambda_5}{2\kappa^2} = -\frac{3}{\kappa^2 \ell^2} \equiv \rho_\ell, \tag{29}$$

and

$$\rho_\lambda \equiv \frac{\lambda}{2}. \tag{30}$$

Here, we obtain the total density parameter of the effective dark energy on the brane, follows from Eqs. (28)–(30),

$$\Omega_\Lambda = \frac{\Lambda_4}{3H^2} \equiv -\frac{1}{H^2 \ell^2} + \frac{\kappa^2 \lambda}{6H^2}. \tag{31}$$

From Eq. (31), we introduce a new additional variable:

$$x_3 \equiv \frac{\kappa \sqrt{\rho_\lambda}}{\sqrt{3}H} = \frac{\kappa \sqrt{\lambda}}{\sqrt{6}H}, \tag{32}$$

with the vacuum energy density parameter localized on the brane and the energy density parameter for the NCC/ BCC perturbations are respectively given as

$$\Omega_\lambda \equiv x_3^2, \quad \Omega_\ell \equiv \frac{\Lambda_5}{6H^2} = -(H\ell)^{-2} = x_1^2 - x_3^2, \tag{33}$$

which satisfy the relation $\Omega_\Lambda = \Omega_\ell + \Omega_\lambda = x_1^2$ from Eqs. (18) and (31).

If the brane is a closed system, i.e. $\Lambda_5 = 0$, we found the relation: $\Lambda_4 = \kappa^2 \rho_\lambda$ satisfying the Λ CDM model (clearly discussed in Ref. [17]). In that case, $\dot{\Lambda}_4 = 0 = \dot{\lambda}$ since $w_\Lambda = w_\lambda = -1$ (the equation of state for vacuum energy localized on the brane does not depend on time and has constant value, i.e. $w_\lambda = -1$). However, in this paper, we set the brane to be a non-closed system, i.e. $\Lambda_5 \neq 0$ and assuming that the evolution of Λ_5 is non-zero. For this case, the presence of NCC/ BCC perturbations on the brane could impose $\dot{\Lambda}_4 \neq 0$. In other words, the presence of NCC/ BCC perturbations on the brane leads to $w_\Lambda \neq w_\lambda = -1$ and w_Λ varies with time. Therefore, the derivative of variable x_3 with N is given as

$$\frac{dx_3}{dN} = \frac{x_3}{2} (3 - 3x_3^2 + x_2^2), \quad (34)$$

with initial condition $x_3 = x_1$, and by setting $w_\lambda = -1$ and $\dot{\lambda} = 0$. Completely, the effect of the NCC/ BCC perturbations on the brane is described in the plot of evolution of the universe components given in Fig. 4.

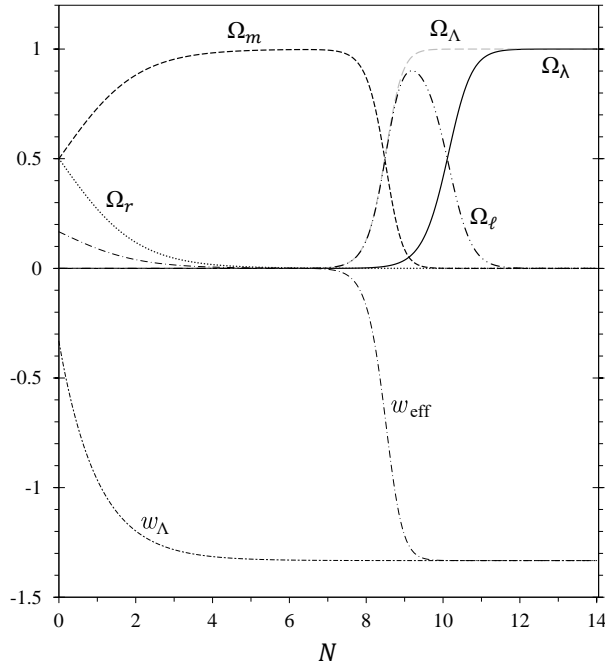


Figure 4. The evolution of Ω_r , Ω_m , Ω_ℓ , and Ω_λ , with $\Omega_\Lambda = \Omega_\ell + \Omega_\lambda$ (the grey dashed-line curve), together with the evolution of $w_\Lambda \equiv w_\Lambda(a)$ and w_{eff} . The initial conditions are given by $x_1 = x_3 = 0.5 \times 10^{-6}$ and $x_2 = 0.707$ at $N = 0$.

It is shown in Fig. 4, the evolution of Ω_r , Ω_m , Ω_Λ , Ω_ℓ , Ω_λ , w_Λ , and w_{eff} versus N with $\Omega_\Lambda = \Omega_\ell + \Omega_\lambda$ is the density parameter of the effective dark energy on the brane, follows from Eq. (31). Starting from the matter-radiation equality epoch, we found the solutions first dwell around the matter-dominated epoch with $w_{\text{eff}} \simeq 0$ and finally approach the dark energy dominated era with $w_{\text{eff}} \simeq w_\Lambda$. Fig. 4 shows that the energy density of the NCC/ BCC term grows with the increase in the density parameter of the effective dark energy Ω_Λ on the brane, but it begins to decrease after the increase of the vacuum energy localized on the brane, Ω_λ . The evolution of Ω_ℓ also has a dynamical solution similar to the evolution of the

Gauss-Bonnet (GB) term which is given in Ref. [14,18]. The GB term has an important role in the accelerated expansion scenario which occurs at the present epoch. Now, if we consider the present universe, based on the observational data constraints [14], the cosmic expansion is accelerated when $w_{\text{eff}} < -1/3$, with $\Omega_M \approx 0.3$ and $\Omega_\Lambda \approx 0.7$. From Fig. 4, at about $\Omega_m \simeq 0.28$ and $\Omega_r \simeq 4.2 \times 10^{-5}$, we obtain:

$$\begin{aligned}\Omega_\Lambda &\simeq 0.7203, \\ \Omega_\ell &\simeq 0.7036, \\ \Omega_\lambda &\simeq 0.0167, \\ w_{\text{eff}} &\simeq -0.96,\end{aligned}\tag{35}$$

for $N \simeq 8.7$. In Eq. (35), we found the values of the density parameter for the effective dark energy, Ω_Λ and w_{eff} corresponding to the combined data analysis using WMAP, SN Ia, CMB, and BAO which are given in Ref. [14,15]. Based on the energy contribution rate, we also found that the energy contribution of the NCC/ BCC term denoted as Ω_ℓ , is more dominant than the contribution of the vacuum energy localized on the brane. Therefore, Λ_5 or the NCC/ BCC term may strongly be the source for dark energy on the brane, which causes the accelerating cosmic expansion at the present epoch.

5. Summary and conclusion

We have presented the dynamical system of braneworld cosmology with the effective cosmological constant as the effective dark energy on the brane. In the dynamical study, we assume that the equation of state of dark energy varies with time ($w_\Lambda \neq -1$). We found the evolution of Ω_Λ slowly increasing for $w_\Lambda = -0.2$ and w_{eff} is too large for the cosmic acceleration today, while w_{eff} is very small for $w_\Lambda = -2.0$. Therefore, for the observational constraints, $w_\Lambda \equiv w_\Lambda(a)$ is the most effective conditions to describing the late-time cosmic acceleration today as seen in Fig. 3. Furthermore, in this FRW cosmological model, we consider the static brane and the non-closed brane system.

In the low energy limit, we found that the present cosmic acceleration occurs at the dark energy dominated epoch, with the NCC/ BCC perturbations energy contributes about 70%, whereas the vacuum energy localized on the brane is only about 2%, based on Eq. (35). We also found that the total effective equation of state, i.e. $w_{\text{eff}} \simeq -0.96$, which satisfies the observational data constraints for the present universe [14,15]. Therefore, dark energy which is observed today, may be due to the NCC/ BCC perturbations. Moreover, since the NCC/ BCC perturbations is non-vanishing on the brane, the energy contribution of Λ_5 can strongly be the source for dark energy on brane universe. Hence, the presence of the NCC/ BCC perturbations play a very important role for evolution of the universe today.

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References

- [1] Horava P and Witten E 1996 *Nucl. Phys. B* **460** 506
Horava P and Witten E 1996 *Nucl. Phys. B* **475** 94
- [2] Lukas A, Ovrut B A, Stelle K S and Waldram D 1999 *Phys. Rev. D* **59** 086001
- [3] Arkani-Hamed N, Dimopoulos S and Dvali G 1998 *Phys. Lett. B* **429** 263
Arkani-Hamed N, Dimopoulos S and Dvali G 1999 *Phys. Rev. D* **59** 086004
- [4] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83**(17) 3370
- [5] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83**(23) 4690
- [6] Dvali G, Gabadadze G and Porrati M 2000 *Phys. Lett. B* **484** 112
Dvali G, Gabadadze G and Porrati M 2000 *Phys. Lett. B* **485** 208

- [7] Shiromizu T, Maeda K and Sasaki M 2000 *Phys. Rev. D* **62** 024012
Sasaki M, Shiromizu T and Maeda K 2000 *Phys. Rev. D* **62** 024008
Maeda K and Wands D 2000 *Phys. Rev. D* **62** 124009
- [8] Maartens R 2004 *Living Rev. Rel.* **7**(7) 2004
Maartens R and Koyama K 2010 *Living Rev. Rel.* **13**(5) 2010
- [9] Maeda K 2002 *Progress of Theoretical Physics Supplement* **148** 59–90
- [10] Durrer R 2005 *AIP Conf. Proc.* **782**(1) 202–240
- [11] Mukohyama S 2000 *Phys. Lett. B* **473** 241
Binétruy P, Deffayet C, Ellwanger U and Langlois D 2000 *Phys. Lett. B* **477** 285
- [12] Maartens R 2000 *Phys. Rev. D* **62** 084023
Maartens R 2001 *Reference Frame and Gravitomagnetism, Proc. of the XXIII Spanish Rel. Meet.* vol 23 p 93–119
- [13] Sorkin R D 2007 *AIP Conf. Proc.* **957** 142–153
- [14] Amendola L and Tsujikawa S 2010 *Dark Energy: Theory and Observation* (Cambridge: Cambridge University Press)
- [15] Kowalski M *et al.* 2008 *Astrophys. J.* **686** 749
- [16] Linder E V 2003 *Phys. Rev. Lett.* **90** 091301
- [17] Gazali Yunus M N *et al.* 2018 Dynamical System Approach of Braneworld Cosmology in Low Energy Limit with Λ CDM Model *Prosiding Seminar Nasional Fisika Makassar* 2018 (Makassar: FMIPA UNHAS) p 42–46 (Translate version)
- [18] Tsujikawa S and Sami M 2007 String-inspired cosmology: Late time transition from scaling matter era to dark energy universe caused by a Gauss-Bonnet coupling *J. Cosmol. Astropart. Phys.* JCAP01(2007)006