POST-NEWTONIAN HIGHER ORDER ORDER SPIN EFFECTS IN INSPIRALING COMPACT BINARIES

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We report recent results for the effects of the spins in binary systems of compact objects (black holes and/or neutron stars), obtained in the post-Newtonian framework at quadratic and cubic order in the spins. The new contributions enter at the third and third-and-a-half post-Newtonian order respectively, and further complete existing theoretical predictions for the graviational wave signals expected from these binaries. The treatment of higher orders in spins required an extension of the Lagrangian formalism for spinning point particles to the quadrupolar and octupolar orders.

1 Introduction

With the advent of a new generation of ground-based gravitational waves detectors such as LIGO, VIRGO and KAGRA, gravitational waves astronomy is expected to enter a new observational era. The most promising sources for these detectors are binary systems of compact objects, neutron stars and/or black holes. Among the different approaches to the joint problems of the dynamics and gravitational waves emission by such systems, the post-Newtonian theory provides analytical predictions, in the form of formal series, covering the long inspiraling phase of the system. Extending this framework to include the effects of the angular momentum (or spin) of the compact bodies has driven a lot of effort in the past few years, as they are expected to be significant for binaries containing black holes.

Beyond the linear order in spin (or spin-orbit, SO), the terms quadratic in the spins (SS) enter at the second post-Newtonian order (2PN), and the cubic terms at the 3.5PN order. Recent results for the dynamics, derived by other authors, cover the 3PN (and partially 4PN) SS contributions [2,3,4,5,6,7] and the 3.5PN SSS [8] (as well as 4PN SSSS) terms. In the work we are reporting [9,10], using the multipolar post-Newtonian method [1], which provides a comprehensive treatment of both the dynamics and the gravitational waves generation, we confirmed these results for the dynamics and extended them to compute the energy flux emitted in gravitational waves, thus predicting the phasing of the binary for circular orbits. These results will be useful to further improve post-Newtonian waveform templates used in the data analysis of the detectors.

2 Lagrangian formalism for spinning point particles

2.1 Definitions

The representation of the effects of the spins requires an extension of the point particle approximation. The approach we use here is based on a Lagrangian formalism, first introduced by Hanson&Regge [11] and Bailey&Israel [12]. To represent the rotational degrees of freedom, an orthonormal tetrad $\epsilon_A{}^\mu$ is introduced, and the antisymmetric rotation coefficients are defined as (with[a] τ the proper time, u^μ the 4-velocity, and $\mathrm{D}/\mathrm{d}\tau = u^\mu \nabla_\mu$)

$$\Omega^{\mu\nu} \equiv \epsilon^{A\mu} \frac{\mathrm{D}\epsilon_A{}^\nu}{\mathrm{d}\tau} . \tag{1}$$

We make then the following ansatz for the action describing the particle's dynamics:

$$S = \int \mathrm{d}\tau \, L \left[u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma} \right] . \tag{2}$$

The couplings to the Riemann tensor and its derivative are included here to represent spin-induced finite-size effects up to the octupolar order. From this general form of the Lagrangian, the linear momentum p_μ and the spin tensor $S_{\mu\nu}$ are defined as conjugate momenta for the positional and rotational degrees of freedom, and the quadrupolar and octupolar moments $J^{\mu\nu\rho\sigma}$ and $J^{\lambda\mu\nu\rho\sigma}$ as partial derivatives with respect to the curvature tensor, according to

$$p_\mu \equiv \frac{\partial L}{\partial u^\mu} , \quad S_{\mu\nu} \equiv 2 \frac{\partial L}{\partial \Omega^{\mu\nu}} , \quad J^{\mu\nu\rho\sigma} \equiv -6 \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} , \quad J^{\lambda\mu\nu\rho\sigma} \equiv -12 \frac{\partial L}{\partial \nabla_\lambda R_{\mu\nu\rho\sigma}} . \tag{3}$$

2.2 Equations of motion and stress-energy tensor

The equations of motion governing the dynamics are derived by varying the action with respect to the worldline and to the rotational degrees of freedom, and read

$$\begin{aligned} \frac{\mathrm{D}p_\mu}{\mathrm{d}\tau} &= -\frac{1}{2} R_{\mu\nu\rho\sigma} u^\nu S^{\rho\sigma} - \frac{1}{6} J^{\lambda\nu\rho\sigma} \nabla_\mu R_{\lambda\nu\rho\sigma} - \frac{1}{12} J^{\tau\lambda\nu\rho\sigma} \nabla_\mu \nabla_\tau R_{\lambda\nu\rho\sigma} , \\ \frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} &= 2 p^{[\mu} u^{\nu]} + \frac{4}{3} R^{[\mu}{}_{\lambda\rho\sigma} J^{\nu]\lambda\rho\sigma} + \frac{2}{3} \nabla^\lambda R^{[\mu}{}_{\tau\rho\sigma} J_\lambda{}^{\nu]\tau\rho\sigma} + \frac{1}{6} \nabla^{[\mu} R_{\lambda\tau\rho\sigma} J^{\nu]\lambda\tau\rho\sigma} . \end{aligned} \tag{4}$$

Introducing the number density $n(x) = \int \mathrm{d}\tau \delta^4(x-z)/\sqrt{-g}$, with z^μ the trajectory of the particle, the stress-energy tensor is obtained with a variation with respect to the metric as

$$\begin{aligned} T^{\mu\nu}_{\text{pole-dipole}} &= p^{(\mu} u^{\nu)} n - \nabla_\rho \left[S^{\rho(\mu} u^{\nu)} n \right] , \\ T^{\mu\nu}_{\text{quad}} &= \frac{1}{3} R^{(\mu}{}_{\lambda\rho\sigma} J^{\nu)\lambda\rho\sigma} n - \nabla_\rho \nabla_\sigma \left[\frac{2}{3} J^{\rho(\mu\nu)\sigma} n \right] , \\ T^{\mu\nu}_{\text{oct}} &= \left(\frac{1}{6} \nabla^\lambda R^{(\mu}{}_{\xi\rho\sigma} J_\lambda{}^{\nu)\xi\rho\sigma} + \frac{1}{12} \nabla^{(\mu} R_{\xi\tau\rho\sigma} J^{\nu)\xi\tau\rho\sigma} \right) n \\ &\quad + \nabla_\rho \left[\left(-\frac{1}{6} R^{(\mu}{}_{\xi\lambda\sigma} J^{|\rho|\nu)\xi\lambda\sigma} - \frac{1}{3} R^{(\mu}{}_{\xi\lambda\sigma} J^{\nu)\rho\xi\lambda\sigma} + \frac{1}{3} R^\rho{}_{\xi\lambda\sigma} J^{(\mu\nu)\xi\lambda\sigma} \right) n \right] \\ &\quad + \nabla_\lambda \nabla_\rho \nabla_\sigma \left[\frac{1}{3} J^{\sigma\rho(\mu\nu)\lambda} n \right] , \end{aligned} \tag{5}$$

where we separated the pole-dipole, quadrupolar and octupolar contributions.

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[a] See the original papers [9,10] for more details on the conventions.

2.3 Spin-induced multipolar moments and conserved norm spin vector

In the case of interest for us, where the extended-size structure of the compact bodies is only induced by their spin, it is possible to derive a unique structure for the quadrupolar and octupolar moments, given by

$$
\begin{aligned}
J^{\mu\nu\rho\sigma} &= \frac{3\kappa}{m} u^{[\mu} S^{\nu]\lambda} S_{\lambda}{}^{[\rho} u^{\sigma]} \\
J^{\lambda\mu\nu\rho\sigma} &= \frac{\lambda}{4m^2} \Big[\Theta^{\lambda[\mu} u^{\nu]} S^{\rho\sigma} + \Theta^{\lambda[\rho} u^{\sigma]} S^{\mu\nu} - \Theta^{\lambda[\mu} S^{\nu][\rho} u^{\sigma]} - \Theta^{\lambda[\rho} S^{\sigma][\mu} u^{\nu]} \\
&\quad - S^{\lambda[\mu} \Theta^{\nu][\rho} u^{\sigma]} - S^{\lambda[\rho} \Theta^{\sigma][\mu} u^{\nu]} \Big] \, . \qquad (6)
\end{aligned}
$$

Here $\Theta^{\mu\nu} = S^{\mu}{}_{\rho} S^{\nu\rho}$, and κ, λ are polarizability constants describing the structure of the compact object. Their value is 1 for black holes and must be determined numerically for neutron stars.

An important feature of the formalism is the requirement of a supplementary spin condition, corresponding to fixing the worldline inside the rotating body. We choose the covariant condition $p_\nu S^{\mu\nu} = 0$, which allows the definition of a spin covector as

$$\tilde{S}_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \frac{p^\nu}{m} S^{\rho\sigma} \, . \qquad (7)$$

By introducing a tetrad $(u^\mu, e_a{}^\mu)$, one can then define a spin vector of conserved Euclidean norm $S_a = e_a{}^\mu \tilde{S}_\mu$, which will obey a precession equation of the form $\dot{\boldsymbol{S}} = \boldsymbol{\Omega} \times \boldsymbol{S}$.

3 Post-Newtonian dynamics and emission of gravitational waves

3.1 Multipolar post-Newtonian formalism

Equipped with the multipolar point particle respresentation of spinning extended objects described above, we were able to use the general framework of the multipolar post-Newtonian formalism (MPN) [1] to derive new results at the SS 3PN and SSS 3.5PN orders. Introducing the metric perturbation $h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$, with $\eta^{\mu\nu}$ the background Minkowski metric, in the harmonic gauge $\partial_\nu h^{\mu\nu} = 0$, the gravitational field equations take the form

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} |g| T^{\mu\nu} + \Lambda^{\mu\nu}[h] \equiv \frac{16\pi G}{c^4} \tau^{\mu\nu} \, , \qquad (8)$$

where $\Lambda^{\mu\nu}$ contains the non-linearities in h, and $T^{\mu\nu}$ is the multipolar stress-energy tensor (5).

The MPN formalism contains (i) a near-zone iteration of the Einstein equation (8), where the full metric is parametrized by a set of potentials which are iteratively solved for, and then plugged in the equations of motion (4); (ii) a vacuum iteration of (8) outside the source, allowing the calculation of the radiative moments U_L, V_L, parametrizing the waveform at infinity, in terms of source moments I_L, J_L containing both the matter source and the gravitational field.

When iterating the near-zone metric, we use the Hadamard regularization to give sense to distributional sources such as (5). The conserved energy can then be deduced from the results for the dynamics obtained in (i), and the emitted energy flux is computed from the radiative moments obtained in (ii) as (where L stands for a multi-index $i_1 \ldots i_\ell$)

$$\mathcal{F} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left[\frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\,\ell!(2\ell+1)!!} \dot{U}_L \dot{U}_L + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \dot{V}_L \dot{V}_L \right] . \qquad (9)$$

3.2 Results

After reduction in the center-of-mass frame, and specialization to the case of circular orbits with no eccentricity nor precession, the results for the conserved energy E and the flux $\mathcal{F}$ can be written as series in the PN expansion parameter $x = (Gm\omega/c^3)^{2/3}$, with ω the orbital frequency:

$$E = -\frac{1}{2}m\nu c^2 x\left[1 + xe_{\rm NS} + x^{3/2}\frac{e_{\rm SO}}{Gm^2} + x^2\frac{e_{\rm SS}}{G^2m^4} + x^{7/2}\frac{e_{\rm SSS}}{G^3m^6} + \mathcal{O}(8)\right],$$
$$\mathcal{F} = \frac{32\nu^2}{5G}c^5x^5\left[1 + xf_{\rm NS} + x^{3/2}\frac{f_{\rm SO}}{Gm^2} + x^2\frac{f_{\rm SS}}{G^2m^4} + x^{7/2}\frac{f_{\rm SSS}}{G^3m^6} + \mathcal{O}(8)\right]. \quad (10)$$

The complete expressions for the new coefficients are too long to be displayed here and can be found for $e_{\rm SSS}$ and $f_{\rm SSS}$ in eqs. (6.17) and (6.19) of[9], and for $e_{\rm SS}$ and $f_{\rm SS}$ in eqs. (3.33) and (4.14) of[10]. The balance equation $\mathcal{F} = -{\rm d}E/{\rm d}t$ can then be used to derive the frequency evolution of the binary with time, and ultimately the expected phasing of the gravitational wave signal. Table 1 gives an illustration of the contribution of each PN term in this phasing.

LIGO/Virgo	$10M_\odot + 1.4M_\odot$	$10M_\odot + 10M_\odot$
Newtonian	3558.9	598.8
1PN	212.4	59.1
1.5PN	$-180.9 + 114.0\chi_1 + 11.7\chi_2$	$-51.2 + 16.0\chi_1 + 16.0\chi_2$
2PN	$9.8 - 10.5\chi_1^2 - 2.9\chi_1\chi_2$	$4.0 - 1.1\chi_1^2 - 2.2\chi_1\chi_2 - 1.1\chi_2^2$
2.5PN	$-20.0 + 33.8\chi_1 + 2.9\chi_2$	$-7.1 + 5.7\chi_1 + 5.7\chi_2$
3PN	$2.3 - 13.2\chi_1 - 1.3\chi_2$ $-1.2\chi_1^2 - 0.2\chi_1\chi_2$	$2.2 - 2.6\chi_1 - 2.6\chi_2$ $-0.1\chi_1^2 - 0.2\chi_1\chi_2 - 0.1\chi_2^2$
3.5PN	$-1.8 + 11.1\chi_1 + 0.8\chi_2 +$ (SS) $-0.7\chi_1^3 - 0.3\chi_1^2\chi_2$	$-0.8 + 1.7\chi_1 + 1.7\chi_2 +$ (SS) $-0.05\chi_1^3 - 0.2\chi_1^2\chi_2 - 0.2\chi_1\chi_2^2 - 0.05\chi_2^3$
4PN	(NS) $- 8.0\chi_1 - 0.7\chi_2 +$ (SS)	(NS) $- 1.5\chi_1 - 1.5\chi_2 +$ (SS)

Table 1: Contribution of each PN order to the number of cycles of a gravitational wave signal for typical neutron star/black hole and black hole/black hole systems, between an entry frequency in the dectector band of 10Hz and the Schwarzschild ISCO $r = 6M$. We assume circular orbits and aligned spins, and χ_A stands for the dimensionless Kerr parameter. All contributions known to date are included (except absorption terms accross the horizon).

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