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Algebraic models of many-body systems and their dynamic symmetries and supersymmetries

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Abstract. An overview of the method of spectrum generating algebras (SGA) and dynamical symmetries (DS) is given and applied to a class of models (the so-called s-b boson models) with SGA≡u(n). Quantum phase transitions (QPT) in these systems are discussed by introducing the coset spaces U(n)/U(n-1)⊗U(1). Finally, spectrum generating superalgebras (SGSA) and dynamical supersymmetries (SUSY) of a class of Bose-Fermi systems with SGA≡u(n/m) are introduced and applied to s-b Bose-Fermi models.

1. Introduction

Algebraic methods have been used extensively in physics since their introduction by Heisenberg (1932), Wigner (1936) and Racah (1942). Since 1974, a general formulation of algebraic methods has emerged. (See, for example, [1]). In this formulation, a quantum mechanical many-body system is mapped onto an algebraic structure. The logic of the method is:

Many-body system
↓
Quantization in terms of boson and/or fermion operators
↓
Algebraic structure
(Lie algebra, Graded Lie algebra, Kac-Moody algebra, q-deformed algebra,...)
↓
Computation of observables
(Spectra, Transitions, ...)
↓
Comparison with experiments

2. Algebraic models

An algebraic model [2] is an expansion of the Hamiltonian and other operators in terms of elements, $G_{\alpha\beta}$, of an algebra (often a Lie algebra g, or a contraction of it). The algebra $G_{\alpha\beta} \in g$ is called the *spectrum generating algebra* (SGA). In most applications, the expansion is a polynomial in the elements,

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$$H = E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} G_{\alpha\beta} + \sum_{\alpha\beta\gamma\delta} u_{\alpha\beta\gamma\delta} G_{\alpha\beta} G_{\gamma\delta} + ... \quad , \quad T = t_0 + \sum_{\alpha\beta} t_{\alpha\beta} G_{\alpha\beta} + ... \tag{1}$$

but more complicated forms have been considered

$$H = f(G_{\alpha\beta}) . \tag{2}$$

An interesting situation occurs when H does not contain all elements of g, but only the invariant (Casimir) operators of a chain of algebras $g \supset g' \supset g'' \supset ...$

$$H = E_0 + \alpha C(g) + \alpha' C(g') + \alpha'' C(g'') + ... \quad , \tag{3}$$

called a *dynamical symmetry* (DS). (For the concept of DS, see also [3]). In this case, the energy eigenvalues can be written explicitly in terms of quantum numbers labeling the representations of g

$$E = \langle H \rangle = E_0 + \alpha \langle C(g) \rangle + \alpha' \langle C(g') \rangle + \alpha'' \langle C(g'') \rangle + ... \tag{4}$$

Also here more complicated functionals of invariant operators have been considered

$$H = f(C_i) \quad . \tag{5}$$

When a dynamic symmetry occurs, one can also calculate matrix elements of all operators in explicit analytic form, in terms of Clebsch-Gordan coefficients (isoscalar factors) of the algebras $g \supset g' \supset g'' \supset ...$

Dynamic symmetries were introduced implicitly by Pauli in 1926 [4] and later by Fock [5] for the hydrogen atom. The Hamiltonian with Coulomb interaction is invariant under a set of transformations, G, larger than rotations (Runge-Lenz transformations, SO(4)). (In this article, algebras will be denoted by lowercase letters, g, and groups by capital letters, G). It can be written in terms of Casimir operators of G,

$$H = \frac{p^2}{2m} - \frac{e^2}{r} = -\frac{A}{C_2(SO(4)) + 1} \quad , \tag{6}$$

with eigenvalues given by the Bohr formula

$$E(n, \ell, m) = -\frac{A}{n^2} \quad . \tag{7}$$

States can be classified by representations of $so(4) \supset so(3) \supset so(2)$ with quantum number n, ℓ, m respectively. The corresponding spectrum is shown in figure 1 where it is compared with experiment. Apart from small relativistic corrections non included in Eq.(7), the agreement is excellent.

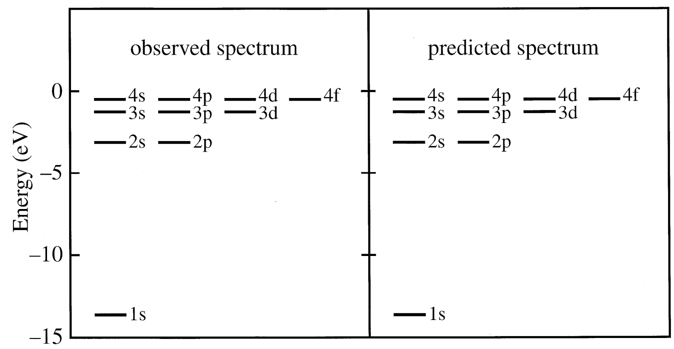

Figure 1. The spectrum of the hydrogen atom is shown as evidence of SO(4) symmetry.

In this case, the expansion is $1/H = f(C_i)$. Pauli's construction has been generalized to any number of dimensions $\nu \geq 2$ with DS so(ν+1) and to scattering states with DS so(ν,1). In this case, only the Casimir operator of g appears, and therefore the algebra is often called *degeneracy algebra*, g_d. The embedding of g_d into a larger algebra (the so-called dynamical algebra) which contains all the states of the system, is discussed in several books [2, 6].

DS assumed an important role in physics with the introduction of flavor symmetry by Gell'mann and Ne'eman [7-8], $SU_f(3)$. States can be classified by representations of $su_f(3) \supset su_I(2) \oplus u_Y(1) \supset spin_I(2) \oplus u_Y(1)$ with mass formula

$$\begin{aligned} M &= a + b\left[C_1(U(1))\right] + c\left[C_2(SU(2)) - \frac{1}{4}C_1^2(U(1))\right] \\ M(Y,I,I_3) &= a + bY + c\left[I(I+1) - \frac{1}{4}Y^2\right] \end{aligned} \quad , \tag{8}$$

where Y, I, I_3 are the hypercharge, isotopic spin and third component of the isotopic spin labeling the representations of $u_Y(1), su_I(2), spin_I(2)$ respectively.

The corresponding spectrum is shown in figure 2 where is compared with experiment.

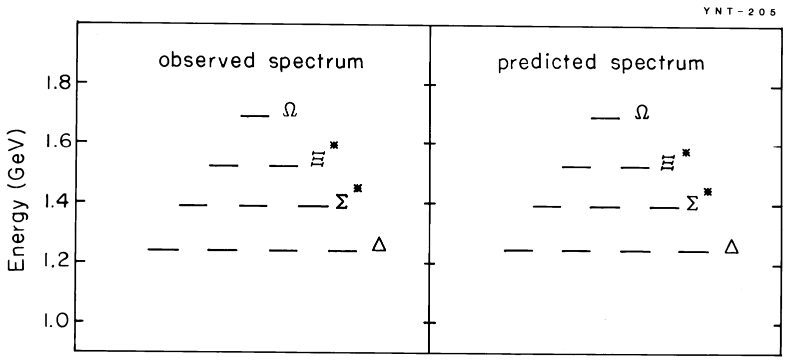

Figure 2. The spectrum of the baryon decuplet is shown as an example of dynamic $SU_f(3)$ symmetry in baryons.

2.1. Bosonic systems

It is convenient to write the elements $G_{\alpha\beta} \in g$ as bilinear products of creation and annihilation operators (Jordan-Schwinger realization). For bosonic systems

$$G_{\alpha\beta} = b_\alpha^\dagger b_\beta \qquad \alpha, \beta = 1, \ldots, n \quad . \tag{9}$$

From $\left[b_\alpha, b_\beta^\dagger\right] = \delta_{\alpha\beta}$, $\left[b_\alpha, b_\beta\right] = \left[b_\alpha^\dagger, b_\beta^\dagger\right] = 0$, one obtains the commutation relations

$$\left[G_{\alpha\beta}, G_{\gamma\delta}\right] = \delta_{\beta\gamma} G_{\alpha\delta} - \delta_{\alpha\delta} G_{\gamma\beta} \tag{10}$$

which define the real form of *g=u(n)* [or *gl(n)*]. The basis upon which the elements act is the totally symmetric representation, with one-row Young tableaux,

$$\left|N\right\rangle \equiv [N,0,0,\ldots,0] \qquad \left|N\right\rangle = \frac{1}{\sqrt{N!}}(b_\alpha^\dagger)^{n_\alpha}(b_{\alpha'}^\dagger)^{n_{\alpha'}} \ldots \left|0\right\rangle \tag{11}$$

characterized by the total number of bosons *N*.

2.2. Fermionic systems

Fermionic systems can also be treated algebraically in terms of bilinear products of anti-commuting operators

$$G_{ij} = a_i^\dagger a_j \qquad i, j = 1, \ldots, m \quad . \tag{12}$$

From $\left\{a_i, a_j^\dagger\right\} = \delta_{ij}$, $\left\{a_i, a_j\right\} = \left\{a_i^\dagger, a_j^\dagger\right\} = 0$, one obtains

$$\left[G_{ij},G_{kl}\right]=\delta_{jk}G_{il}-\delta_{il}G_{jk} \quad , \qquad (13)$$

spanning the Lie algebra *u(m)* as before. The basis upon which these elements act is, however, the totally anti-symmetric representation, with one-column Young tableaux

$$\left|N_F\right\rangle \equiv [1,1,...,1,0,...,0] \qquad \left|N_F\right\rangle = \frac{1}{\sqrt{N_F\,!}}a_i^\dagger a_{i'}^\dagger ...\left|0\right\rangle \qquad (14)$$

characterized by the total number of fermions N_F.

2.3. Mixed Bose-Fermi systems

Mixtures of bosonic and fermionic systems can be treated in terms of bilinear products

$$\begin{aligned} G_{\alpha\beta} &= b_\alpha^\dagger b_\beta \\ G_{ik} &= a_i^\dagger a_k \qquad && \alpha,\beta=1,...,n \\ F_{\alpha i}^\dagger &= b_\alpha^\dagger a_i && i,j=1,...,m \\ F_{i\alpha} &= a_i^\dagger b_\alpha \end{aligned} \qquad (15)$$

often placed in matrix form

$$\begin{pmatrix} b_\alpha^\dagger b_\beta & b_\alpha^\dagger a_i \\ a_i^\dagger b_\alpha & a_i^\dagger a_j \end{pmatrix} . \qquad (16)$$

The bilinear products above generate the graded Lie algebra (also called superalgebra) *U(n/m)*. (For a classification of superalgebras see [9-10]. The superalgebra *su(n/m)* is *A(n-1,m-1)* in Kac's classification). In a single index notation, the commutation relations of the graded algebra are

$$\left[X_\alpha,X_\beta\right]=\sum_\gamma c_{\alpha\beta}^\gamma X_\gamma \;;\; \left[X_\alpha,Y_i\right]=\sum_j c_{\alpha i}^j Y_j \;;\; \left\{Y_i,Y_k\right\}=\sum_\gamma f_{ik}^\gamma X_\gamma \qquad (17)$$

where X are bosonic operators and Y are fermionic operators, together with the Jacobi identities. The basis upon which the elements act is the totally supersymmetric representation, with supertableux

$$\left|N=N_B+N_F\right\rangle = [N,0,...,0\} \qquad (18)$$

where N is the total number of particles bosons+fermions. A symbol (square-curly bracket) different from Eq.(11) has been used in Eq.(18) to indicate that these representations are totally symmetric under the interchange of bosons and totally anti-symmetric under the interchange of fermions [11].

3. Geometry of algebraic models

Geometry can be associated to algebraic models with algebra g by constructing appropriate coset spaces, obtained by splitting g (Cartan decomposition) into

$$g=h\oplus p \qquad (19)$$

where h, a subalgebra of g, is called the stability algebra and p the remainder (not closed with respect to commutation). For models with *u(n)* structure, the appropriate coset space (*g/h*) is

$$u(n)/u(n-1)\oplus u(1) \quad . \qquad (20)$$

This space is a globally symmetric Riemannian space [2, chapter 5] with dimension *2(n-1)*. For bosonic models, the algebra h can be constructed by selecting one boson, b_1, and choosing

$$\begin{aligned} h &\doteq b_1^\dagger b_1, b_\alpha^\dagger b_\beta \qquad && \alpha,\beta=2,...,n \\ p &\doteq b_1^\dagger b_\alpha, b_\alpha^\dagger b_1 && \alpha=2,...,n \end{aligned} \quad . \qquad (21)$$

Associated with the Cartan decomposition, there are geometric variables η_i defined by

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$$\left|\eta_i\right\rangle = \exp\left[\sum_i \eta_i p_i\right]\left|\Lambda_{ext}\right\rangle \quad , \quad p_i \in p \quad . \tag{22}$$

For bosonic systems

$$\left|N;\eta_\alpha\right\rangle = \left[\exp\left(\eta_\alpha b_\alpha^\dagger b_1 - \eta_\alpha^* b_1^\dagger b_\alpha\right)\right]\frac{1}{\sqrt{N!}}\left(b_1^\dagger\right)^N\left|0\right\rangle \ . \tag{23}$$

For systems with fixed value of N it is convenient to introduce projective coherent states in terms of projective variables [12]

$$\left|N;\vartheta_\alpha\right\rangle = \frac{1}{\sqrt{N!}}\left[b_1^\dagger + \vartheta_\alpha b_\alpha^\dagger\right]^N\left|0\right\rangle \quad . \tag{24}$$

For algebraic models written in terms of boson operators $b_1 \equiv s$ and $b_m (m = 2,...,n)$, it is convenient to rewrite the coherent state as

$$\left|N;\alpha\right\rangle = \frac{1}{\sqrt{N!}}\left(s^\dagger + \sum_m \alpha_m b_m^\dagger\right)^N\left|0\right\rangle \tag{25}$$

with normalization

$$\left\langle N;\alpha\middle|N;\alpha\right\rangle = \left(1 + \sum_m \left|\alpha_m\right|^2\right)^N . \tag{26}$$

The semiclassical energy functional associated with the quantum algebraic Hamiltonian is

$$E(N;\alpha) = \frac{\left\langle N;\alpha\middle|H\middle|N;\alpha\right\rangle}{\left\langle N;\alpha\middle|N;\alpha\right\rangle} . \tag{27}$$

This energy functional depends on the complex coordinates α_m which can be split into real coordinates, q_m and momenta, p_m. (For details, see [13]).

Geometry can also be associated to algebraic Bose-Fermi models with graded Lie algebra g^* [14]. However, the construction involves Grassmann variables and will not be discussed here.

4. Algebraic models in physics

In the last 50+ years, many algebraic models of many-body systems have been constructed, for bosonic, fermionic and mixed Bose-Fermi systems. Here, only a selected class of algebraic models of bosonic systems and their associated Bose-Fermi systems will be discussed.

4.1. Bosonic models: The Interacting Boson Model

A list of algebraic bosonic models with SGA *u(n)*, called s-b models, and their associated dynamic symmetries extensively investigated is: u(2) [15,16]; u(3) [16]; u(4) [16]; u(6) [17]. These provide a description of many-body problems with $f=n-1$ degrees of freedom [13]. A convenient realization of these models is with a scalar boson, called s, and another boson, b_m, with $f = 2\ell + 1$ components. The integer or half-integer number $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ...$ is called quasi-spin. The use of both integer and half-integer values allows one to treat problems in both odd and even dimensions.

One of the best examples of algebraic bosonic models is the Interacting Boson Model (IBM) [17]. This model describes even-even nuclei as a collection of correlated pairs of nucleons with angular momentum J=0 and J=2. The pairs are then treated as bosons, called s (J=0) and d (J=2). Introducing boson creation and annihilation operators

$$\begin{array}{l} s^\dagger, d_\mu^\dagger (\mu = 0, \pm 1, \pm 2) \\ s, d_\mu (\mu = 0, \pm 1, \pm 2) \end{array} \quad , \tag{28}$$

generically denoted by $b_\alpha^\dagger, b_\alpha (\alpha = 1,...,6)$, one can write the Hamiltonian, *H*, and the transition operators, *T*, as

$$\begin{aligned} H &= E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} b_\alpha^\dagger b_\beta + \sum_{\alpha\beta\gamma\delta} u_{\alpha\beta\gamma\delta} b_\alpha^\dagger b_\beta^\dagger b_\gamma b_\delta + ... \\ T &= t_0 + \sum_{\alpha\beta} t_{\alpha\beta} b_\alpha^\dagger b_\beta + ... \end{aligned} \tag{29}$$

The bilinear products of the six creation and six annihilation operators

$$g \doteq G_{\alpha\beta} = b_\alpha^\dagger b_\beta \qquad \alpha, \beta = 1,...,6 \tag{30}$$

span the Lie algebra *u(6)* which is then the SGA of the IBM. The basis *B* upon which the elements act is the totally symmetric representation

$$|N\rangle \equiv [N,0,...,0] \qquad |N\rangle = \frac{1}{\sqrt{N!}} b_\alpha^\dagger b_{\alpha'}^\dagger ... |0\rangle \quad . \tag{31}$$

For rotationally invariant problems, like the one described here, it is convenient to introduce the Racah form of the algebra [18]. To this end, one constructs operators that transform as irreducible representations of the rotation algebra, $so(3)$, called spherical tensor boson operators $b_{\ell,m}^\dagger, b_{\ell,m} (\ell = 0,2; m = 0, \pm 1, \pm 2)$ with commutation relations

$$\left[b_{\ell,m}, b_{\ell',m'}^\dagger \right] = \delta_{\ell\ell'} \delta_{mm'} \quad \left[b_{\ell,m}, b_{\ell',m'} \right] = \left[b_{\ell,m}^\dagger, b_{\ell',m'}^\dagger \right] = 0 \tag{32}$$

It should be noted that if $b_{\ell,m}^\dagger$ transforms as a spherical tensor, $b_{\ell,m}$ does not. An operator that transforms as a spherical tensor is $\tilde{b}_{\ell,m} = (-)^{\ell-m} b_{\ell,-m}$. The Lie algebra *g* in Racah form is obtained by taking tensor products

$$G_\kappa^{(k)}(\ell, \ell') = \left[b_\ell^\dagger \times \tilde{b}_{\ell'} \right]_\kappa^{(k)} \qquad (\ell, \ell' = 0,2) \ . \tag{33}$$

Tensor products of two operators with respect to $so(3)$ are defined as

$$\left[T^{(k_1)} \times U^{(k_2)} \right]_\kappa^{(k)} = \sum_{\kappa_1,\kappa_2} \langle k_1\kappa_1 k_2\kappa_2 | k\kappa \rangle T_{\kappa_1}^{(k_1)} U_{\kappa_2}^{(k_2)} \quad . \tag{34}$$

4.1.1. Dynamic symmetries of the Interacting Boson Model

The algebra *u(6)* can be broken, with the constraint that the angular momentum algebra *so(3)* be contained in it, into three subalgebra chains:

$$\begin{aligned} &(I): u(6) \supset u(5) \supset so(5) \supset so(3) \supset so(2) \\ &(II): u(6) \supset su(3) \supset so(3) \supset so(2) \\ &(III): u(6) \supset so(6) \supset so(5) \supset so(3) \supset so(2) \end{aligned} \quad . \tag{35}$$

For each of these three cases, DS, it is possible to construct an energy formula, which gives the energies in terms of quantum numbers labeling the representations of $g \supset g' \supset g'' \supset ...$

Dynamic symmetry (I): *u(5)*.

The Hamiltonian, *H*, and energy formula, *E*, are

$$\begin{aligned} H^{(I)} &= E_0 + \varepsilon C_1(u(5)) + \alpha C_2(u(5)) + \beta C_2(so(5)) + \gamma C_2(so(3)) \\ E^{(I)}(N, n_d, v, n_\Delta, L, M_L) &= E_0 + \varepsilon n_d + \alpha n_d(n_d + 4) + \beta v(v+3) + \gamma L(L+1) \end{aligned} \quad , \tag{36}$$

where N, n_d, v, L, M_L denote the representations of $u(6), u(5), so(5), so(3), so(2)$ respectively, while v_Δ is an additional quantum number (missing label) that takes into account the fact that the breaking $u(5) \supset so(5) \supset so(3)$ is non-canonical [2, chapter 6]. An example is given in figure 3, where the energy formula (36) is compared with experiment.

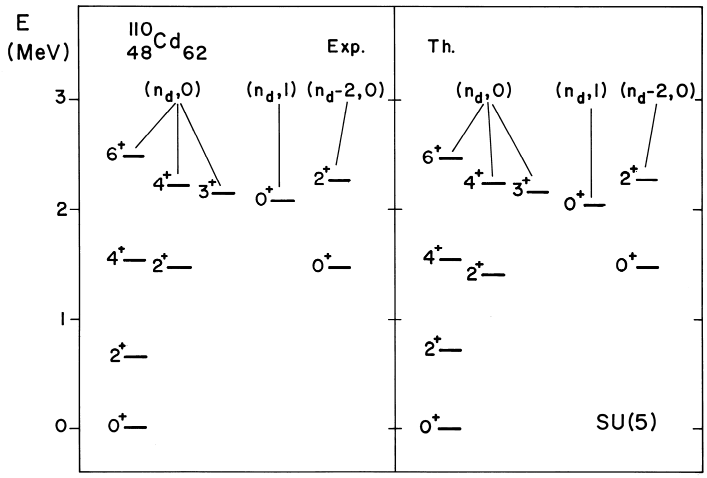

Figure 3. The spectrum of the nucleus ^{110}Cd is shown as an example of u(5) symmetry.

Dynamic symmetry (II): *su(3)*
The Hamiltonian and energy formula for this case are

$$H^{(II)} = E_0 + \kappa C_2(su(3)) + \kappa' C_2(so(3))$$
$$E^{(II)}(N,\lambda,\mu,K,L,M_L) = E_0 + \kappa\left(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu\right) + \kappa' L(L+1) \quad , (37)$$

where N,λ,μ,K,L,M_L label the states of Chain II. Again here K is a missing label.
An example is given in figure 4, where the energy formula (37) is compared with experiment.

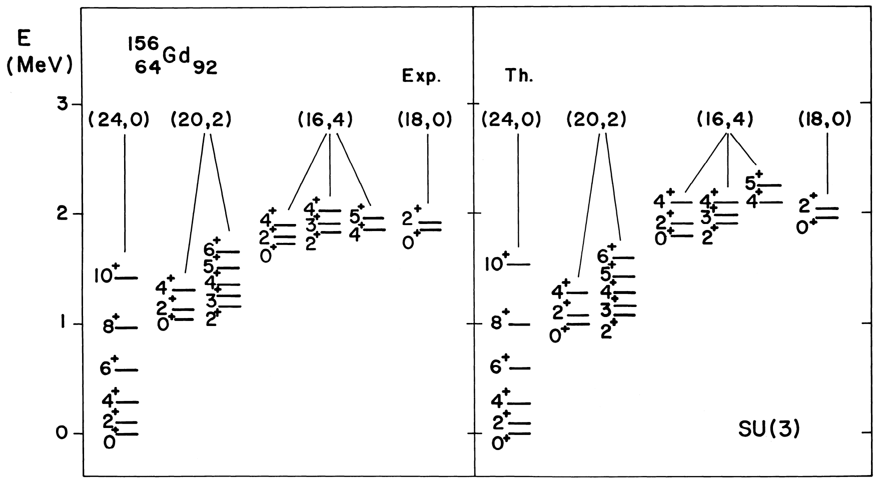

Figure 4. The spectrum of the nucleus ^{156}Gd is shown as an example of su(3) symmetry.

Dynamic symmetry (III): *so(6)*
The Hamiltonian and energy levels for this symmetry are

$$H^{(III)} = E_0 + AC_2(so(6)) + BC_2(so(5)) + CC_2(so(3))$$
$$E^{(III)}(N,\sigma,\tau,\nu_\Delta,L,M_L) = E_0 + A\sigma(\sigma+4) + B\tau(\tau+3) + CL(L+1) \quad , \quad (38)$$

where $N,\sigma,\tau,\nu_\Delta,L,M_L$ label the states of Chain III with ν_Δ the missing label.
An example is given in figure 5, where the energy formula (38) is compared with experiment.

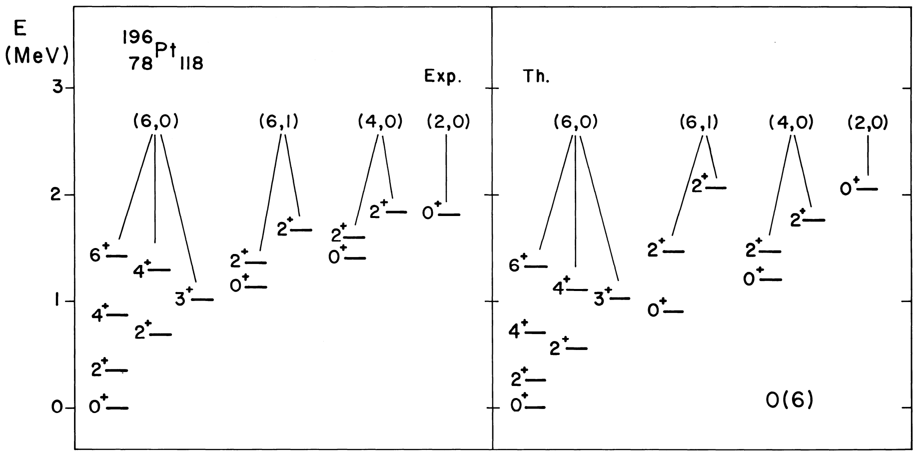

Figure 5. The spectrum of the nucleus ^{196}Pt is shown as an example of so(6) symmetry.

4.1.2. Numerical studies

In many cases, spectra of nuclei cannot be described by a dynamic symmetry. For these cases, H must be diagonalized numerically. Lie algebras are useful here to construct the basis B in which the diagonalization is done. For the Interacting Boson Model, the basis used in the diagonalization is usually the basis of Chain (I) $u(6) \supset u(5) \supset so(5) \supset so(3) \supset so(2)$, with quantum numbers given in Eq.(36). Note that for $u(n), n \geq 5$ the construction of the basis containing the angular momentum algebra *so(3)* is not simple, since hidden quantum numbers (missing labels) appear [17].

An Hamiltonian often used in numerical studies is

$$\begin{aligned} H &= E_0 + \varepsilon \hat{n}_d + \kappa\left(\hat{Q}^\chi \bullet \hat{Q}^\chi\right) \\ \hat{n}_d &= \left(d^\dagger \bullet \tilde{d}\right) \\ \hat{Q}^\chi &= \left(d^\dagger \times \tilde{s} + s^\dagger \times \tilde{d}\right)^{(2)} + \chi\left(d^\dagger \times \tilde{d}\right)^{(2)} \end{aligned} \quad . \qquad (39)$$

This Hamiltonian provides a good description of nuclei in terms of three parameters, $\varepsilon, \kappa, \chi$. Using this Hamiltonian it has been possible to provide a classification of all nuclei in terms of symmetry groups, a portion of which is shown in figure 6.

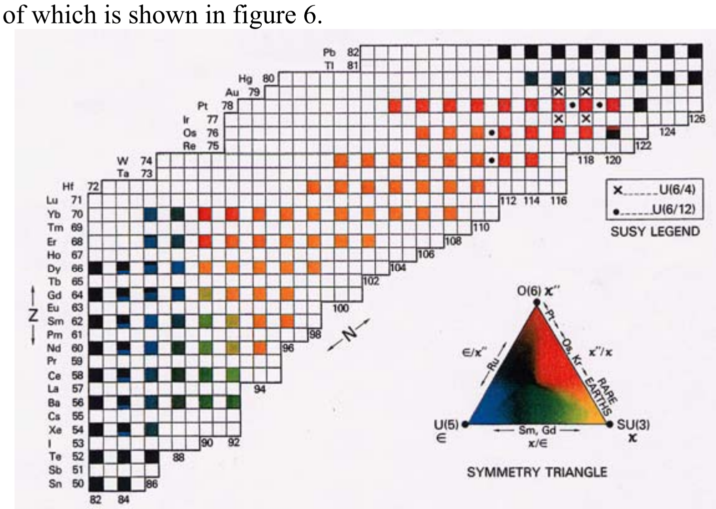

Figure 6. Symmetry classification of nuclei with Z=50-82 and N=82-126.

In this figure a color code has been used to denote each of the three symmetries, as indicated in the symmetry triangle in the bottom right.

4.2. Bose-Fermi models: The Interacting Boson-Fermion Model

For each of the bosonic models of Sect. 4.1, one can construct a corresponding Bose-Fermi model by adding fermions to the bosons. One of the best examples of Bose-Fermi models is the Interacting Boson-Fermion Model (IBFM) [19]. In odd-even nuclei at least one particle is unpaired, and at higher excitation energies, pairs may break. A more accurate description of nuclei is then in terms of correlated pairs with *J=0* and *J=2*, treated as bosons, *s* and *d*, plus unpaired fermions with angular momentum *j*. The Hamiltonian is now

$$H = H_B + H_F + V_{BF} \tag{40}$$

with

$$\begin{aligned} H_B &= E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} b_\alpha^\dagger b_\beta + \sum_{\alpha\alpha'\beta\beta'} v_{\alpha\alpha'\beta\beta'} b_\alpha^\dagger b_{\alpha'}^\dagger b_\beta b_{\beta'} \\ H_F &= E'_0 + \sum_{ik} \eta_{ik} a_i^\dagger a_k + \sum_{ii'kk'} u_{ii'kk'} a_i^\dagger a_{i'}^\dagger a_k a_{k'} \\ V_{BF} &= \sum_{\alpha\beta ik} w_{\alpha\beta ik} b_\alpha^\dagger b_\beta a_i^\dagger a_k \end{aligned} \quad . \tag{41}$$

4.2.1. Dynamic supersymmetries of the Interacting Boson-Fermion Model

Dynamic supersymmetries of Bose-Fermi models require more stringent conditions than dynamic symmetries of models of bosons or of fermions. Two of these conditions are:

(i) To each boson α there exists a fermion *j* related to it by a supersymmetry transformation.

(ii) In the Hamiltonian, *H*, of Eq.(40), all couplings must be related by a supersymmetry transformation. If g_B , g_F and g_{BF} are the coupling constants in the first, second and third term in the r.h.s. of Eq.(40), then $2g_B = 2g_F = g_{BF}$.

In the case of the IBFM, for each dynamic symmetry of the bosons, *U(5)*, *SU(3)* and *SO(6)*, several classes of supersymmetries are possible, described by the graded Lie algebra $u(6/\Omega)$ and its subalgebras, where Ω is the dimension of the fermionic space,

$$\Omega = \sum_i (2j_i + 1) \quad . \tag{42}$$

Here j_i are the values of the angular momenta of the fermions. A class of supersymmetry extensively studied is that in which, in addition to bosons with *J=0,2*, one has fermions with *j=3/2*. The supersymmetry is then described by the chain

$$u(6/4) \supset so(6) \otimes su(4) \supset spin(6) \supset spin(5) \supset spin(3) \supset spin(2) \quad . \tag{43}$$

A consequence of supersymmetry is that if bosonic states are known one can predict fermionic states and vice versa. Both are given by the same formula, which for the case mentioned above is

$$\begin{aligned} & E(N,(\sigma_1,\sigma_2,\sigma_3),(\tau_1,\tau_2),\nu_\Delta,J,M) = E_0 \\ & +A\left[\sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2\right] + B\left[\tau_1(\tau_1+3)+\tau_2(\tau_2+1)\right] + CJ(J+1) \end{aligned} \quad . \tag{44}$$

Here $N=N_B+N_F$ is the total number of bosons plus fermions, and states are identified by the quantum numbers of the representations of the algebras appearing in the chain (43). As a result, if one knows the spectra of even-even nuclei (bosonic), one can predict those of odd-even nuclei (fermionic). In the 1980’s, several case of spectra of nuclei with supersymmetric properties were found [20,21]. An example is shown in figure 7.

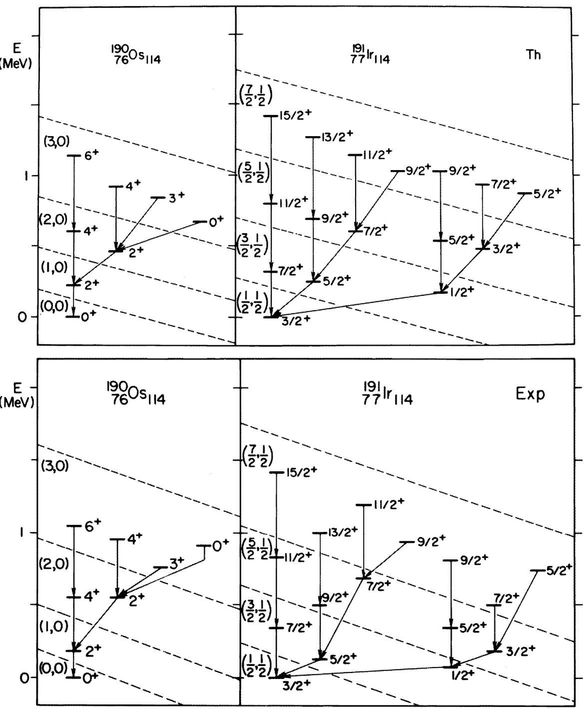

Figure 7. An example of *u(6/4)* supersymmetry in nuclei: the pair of nuclei ^{190}Os-^{191}Ir.

Recently, supersymmetry in nuclei has been confirmed by a series of new experiments. These experiments deal with an even more complex situation than that described above, wherein a distinction is made between protons and neutrons with the introduction of proton and neutron pairs with *J=0,2* treated as bosons, and denoted by $s_\pi, d_\pi; s_\nu, d_\nu$. The corresponding model is known as Interacting Boson Model-2 (IBM-2) and has an algebraic structure $u_\pi(6) \oplus u_\nu(6)$. Consequently, when fermions are added one has a model with two types of bosons (protons and neutrons) and two types of fermions (protons and neutrons), called Interacting Boson Fermion Model-2 (IBFM-2), with algebraic structure $u_\pi(6/\Omega_\pi) \oplus u_\nu(6/\Omega_\nu)$. If supersymmetry occurs for this very complex structure, one expects to have supersymmetric partners composed of a quartet of nuclei, even-even, even-odd, odd-even and odd-odd. In a major effort that has involved several laboratories worldwide, it has been possible to measure spectra of odd-odd nuclei and thus test the occurrence of super symmetry in nuclei [22]. An example is shown in figure 8.

Dynamic supersymmetry in nuclei is the only experimentally verified case of supersymmetry in Physics (as of 2018). The supersymmetric constituents are nucleons (fermions) and nucleon pairs (bosons). Bosons are thus composite objects. A major part of the program at CERN-LHC will be devoted in the next few years to a search for supersymmetry in particle physics. The expected constituents are fundamental particles, quarks (fermions) and squarks (bosons), or gluons (bosons) and gluinos (fermions). Several authors, in particular Y. Nambu, have suggested that the only supersymmetries that can occur in Nature are of the “composite” type. Indeed, Nambu [23] has constructed an effective supersymmetry in Type-II superconductors, where supersymmetric partners are electrons and Cooper pairs. Also, Catto and Gürsey [24] and others have applied the same idea to hadronic spectroscopy where the supersymmetric partners are diquarks (D) and quarks (q). This scheme is an enlargement of Gürsey-Radicati SU(6) [25]. Since the diquarks transform as the representation 21 and the quarks as the representation 6 of SU(6), the appropriate superalgebra here is u(6/21). This scheme is thus similar to the u(6/Ω) scheme of nuclear physics described above, except that the role of u(6) is interchanged, since in nuclear physics u(6) refers to the Bose sector and u(Ω) to the Fermi sector, while in Catto-Gürsey supersymmetry u(6) refers to the quarks (Fermi part) and u(21) to the diquark (Bose part). Gürsey's scheme implies the occurrence of diquark-antidiquark states, for which there was little evidence at the time when u(6/21) was introduced [24], but for which now substantial evidence has been accumulated. It remains to be seen whether or not Nambu's suggestion is substantiated by future experiments.

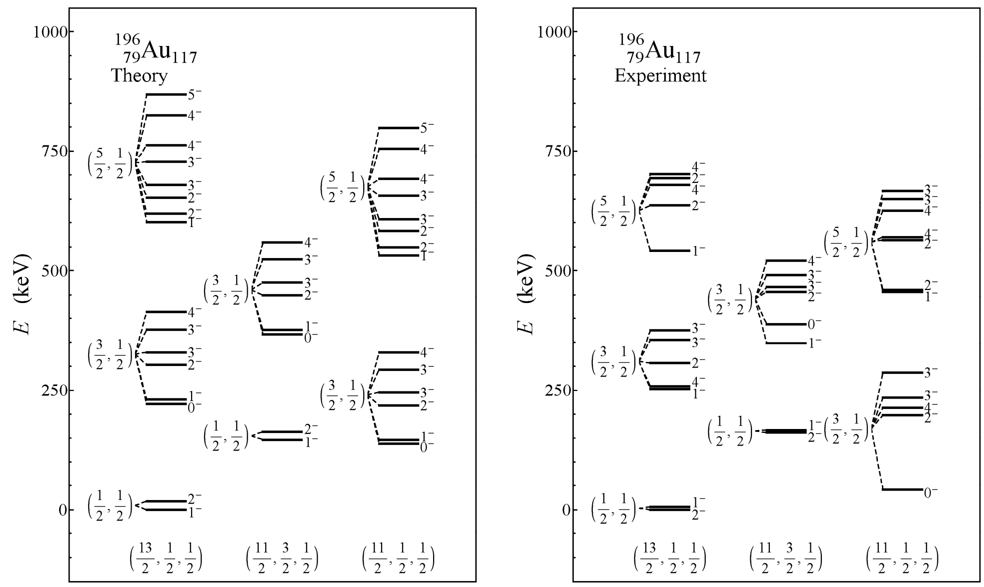

Figure 8. An example of supersymmetry in odd-odd nuclei: the spectrum of ^{196}Au, supersymmetric partner of ^{194}Pt, ^{195}Pt, ^{195}Au [22].

5. Quantum Phase Transitions in algebraic models

Quantum phase transitions (QPT) are phase transitions that occur as a function of a coupling constant, ξ, that appears in the quantum Hamiltonian, *H*, that describes the system

$$H = \varepsilon\left[\left(1-\xi\right)H_1 + \xi H_2\right] \quad , \; 0 \le \xi \le 1 \quad . \tag{45}$$

As ξ changes from 0 to 1, the eigenstates of *H* change from those of H_1 to those of H_2. Associated with phase transitions there are order parameters, the expectation values of some suitable chosen operators that describe the state of the system, $\langle O \rangle$. Introduced in the 1970's [26], they have become in recent years of great importance in a variety of systems. QPT are also called ground state phase transitions and/or zero temperature phase transitions.

QPT can be conveniently studied within the framework of algebraic models. For these models, one can do both the semi-classical and the quantal analysis. Also, in many-body systems, finite size scaling (1/N expansion) can be easily investigated. The latter point is particularly important in applications to finite systems: nuclei, molecules, finite polymers, photonic crystals, optical lattices, etc. In this article, particular emphasis will be given to the semi-classical analysis of algebraic models.

5.1. Semi-classsical analysis of QPT in algebraic models

An algorithm to study QPT in algebraic models was developed by Gilmore [26].

(i) Consider and Hamiltonian *H* which is a mixture of Casimir invariants of two (or more) algebras as in Eq.(45),

$$H=\varepsilon\left[\left(1-\xi\right)C(g_1)+\xi C(g_2)\right] \quad . \tag{46}$$

(ii) Construct the energy per particle, $E(N;\alpha)/N$, where the energy functional E is given in Eq.(27).

(iii) Minimize E as a function of the variables α, thus determining E_{min}.

(iv) Study the behavior of E_{min} and its derivatives as a function of the coupling constants ξ (control parameters) which appear in front of the Casimir operators. The phase transition is said to be of zeroth order if E is discontinuous at the critical point, of first order if $\partial E_{\min}/\partial\xi$ is discontinuous, of second order if $\partial^2 E_{\min}/\partial\xi^2$ is discontinuous, etc. (Erhenfest classification).

5.1.1. QPT in the Interacting Boson Model

A convenient Hamiltonian to study QPT in the Interacting Boson Model is given in Eq.(39), rewritten as

$$H=\varepsilon\left[\left(1-\xi\right)\hat{n}_d+\frac{\xi}{N}\hat{Q}^{\chi}\cdot\hat{Q}^{\chi}\right] \quad . \tag{47}$$

In this Hamiltonian there are two control parameters ξ and χ. The number of classical variables α_m associated with $u(6)$ is five (see Sect.3). However, by exploiting the rotational invariance of the problem, it is possible to reduce the number of variables to two, called intrinsic or Bohr variables [17, Ch.3] β,γ and to write the coherent state as

$$\left|N;\beta,\gamma\right\rangle=\frac{1}{\sqrt{N!}}\left(s^\dagger+\beta\left[\cos\gamma d_0^\dagger+\frac{1}{\sqrt{2}}\sin\gamma\left(d_{+2}^\dagger+d_{-2}^\dagger\right)\right]\right)^N\left|0\right\rangle \quad . \tag{48}$$

The energy surface for the Hamiltonian H of Eq.(46) is

$$E(N;\beta,\gamma;\xi,\chi)=\left(\varepsilon N\right)\left\{\begin{array}{l}\dfrac{\beta^2}{1+\beta^2}\left[\left(1-\xi\right)-\left(\chi^2+1\right)\dfrac{\xi}{4N}\right]-\dfrac{5\xi}{4N(1+\beta^2)^2}\\[2ex] -\dfrac{\xi(N-1)}{4N(1+\beta^2)^2}\left[4\beta^2-4\sqrt{\dfrac{7}{2}}\chi\beta^3\cos3\gamma+\dfrac{2}{7}\chi^2\beta^4\right]\end{array}\right\} \quad . \tag{49}$$

Using the algorithm described above, one can construct the phase diagram of the interacting boson model given in figure 9 [27]. The three phases (dynamic symmetries) are at the vertices of the triangle. There is a line of 1st order transitions ending in a point of 2nd order transition.

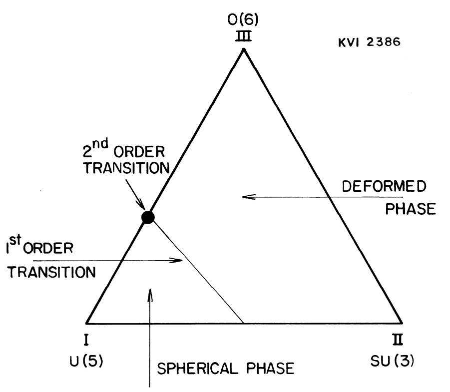

Figure 9. Phase diagram of the Interacting Boson Model [27].

Experimental evidence for these phase transitions has been found [17].

6. Summary of algebraic models in physics

Since its introduction in the 1970's, algebraic modelling of many-body systems has seen many applications, encompassing many aspects of physics and chemistry. Here are summarized some of these applications and their associated algebras.

6.1. Nuclear Physics

The Interacting Boson Model, IBM-1 [17] with algebra $u(6)$.

The Proton-Neutron Interacting Boson Model, IBM-2 [17] with algebra $u_\pi(6)\oplus u_\nu(6)$.

The Interacting Boson-Fermion Model, IBFM-1 [19] with superalgebra $u(6/\Omega)$.

The Proton-Neutron Interacting Boson-Fermion Model, IBFM-2 [19] with superalgebra $u_\pi(6/\Omega_\pi)\oplus u_\nu(6/\Omega_\nu)$.

These models have been briefly described in Sect. 4.

6.2. Molecular Physics

The Vibron Model, VM [16] with algebra $u(4)$.

The Electron-Vibron Model, EVM [28] with superalgebra $u(4/\Omega)$.

6.3. Hadronic Physics

The Algebraic Model of Mesons [29] with algebra $u(4)$

The Algebraic Model of Baryons [30,31] with algebra $u(7)$

6.4. Polymer Physics

The Algebraic Model of Anharmonic Polymer chains [32, 33], with algebra $\sum_i \oplus u_i(2)$

6.5. Cluster Physics

The Algebraic Cluster Model (ACM): Two-body problems [34] with algebra $u(4)$

The Algebraic Cluster Model (ACM): Three-body problems [35] with algebra $u(7)$

The Algebraic Cluster Model (ACM): Four-body problems [36] with algebra $u(10)$

For applications to cluster physics one needs also to include the discrete symmetry of the cluster. Symmetries that have been considered are: two-body Z_2, three-body D_{3h} and four-body T_d.

6.6. Crystal Physics

The Algebraic Lattice Model: one-dimensional systems [37] with algebra $\sum_i \oplus_i u_i(2)$

The Algebraic Lattice Model: two-dimensional systems [37] with algebra $\sum_{ii'} \oplus_{ii'} u_{ii'}(2)$.

Also here one needs to include the discrete symmetry of the lattice. Symmetries that have been considered are: D_{4h} (square lattice), D_{6h} (hexagonal lattice).

7. Conclusions

Symmetry in its various forms (space-time, gauge, dynamic, …) has become a guiding principle in the description of Physics and Chemistry. Algebraic modelling is an ideal tool for discovering symmetries in complex many-body systems. Through the symmetries and supersymmetries of algebraic models it has been possible to uncover order and regularities in the spectra of complex quantum systems, molecules, atoms, nuclei and hadrons. This discovery is part of the *simplicity in complexity* program. As Herman Weyl wrote: *Nature loves symmetry*! In the 21st Century, as the complexity of the phenomena that we are studying increases, dynamic symmetry and supersymmetry, and their underlying algebraic models, may play an equally important role. In fact, one of the lessons we have learned is that the more complex is the structure, as in figure 10, the more useful is the concept of symmetry.

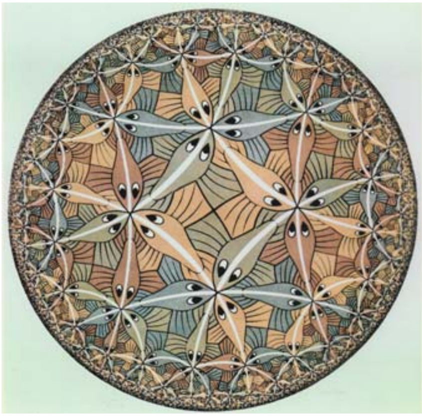

Figure 10. Tessellation of the hyperbolic Poincare' plane. (From M.C. Escher, *Circle Limit III*, 1959).

I am therefore looking forward to many more years of Group Theory Colloquia.

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