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Cosmological particle production in time-dependent backgrounds

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Abstract

One of the most interesting and at the same time complex problems of modern physics is to determine how does the past and the future of our Universe look like. Especially, we are interested in the first moments of its history as they are beyond our experimental reach at present but they influence later stages of the evolution of the Universe we are able to observe. Generally accepted cosmological scenario starts with the Big Bang, we speculate that the Universe was homogeneous and isotropic with very high temperature and pressure at the beginning, after which our Universe starts to gradually cool down and expand ending up as the Universe we live in. We assume that very early in the evolution of the Universe the process of exponential expansion occurred which we call inflation. During this period density fluctuations were amplified giving the origin to the seeds that would form all the large scale structure in the Universe – stars, galaxies etc. After inflation we distinguish two periods important for the presented research – preheating and reheating, during which out of the inflaton – the field that drives inflation, elementary particles that fill the Universe now were produced.

The thesis describes the processes of cosmological particle production in the time-dependent theories and it focuses on three main subjects: gravitational reheating with an instant period of particle creation, multi-stages non-perturbative production in both adiabatic approximation and interacting theory. All of them are based on the fact that the vacuum state changes in time and that in the parameter space there exist a region where particle production is energetically favourable and efficient enough to be observed.

In the chapter concerning gravitational reheating particles are produced solely due to the change in the evolution of the observed Universe in time, which is described by the scale factor that depends on time. In our research we assume this kind of particle production right after the end of inflation in a very general way. From our analysis we can draw generic conclusions about available observables involving the features of inflation, observed spectrum of gravitational waves and even characteristics of dark matter or dark energy.

The main part of the thesis concentrates on the general description of the multiphase non-perturbative production of particles, especially in case of inflation. The essence of my research in this matter lies in the fact that production of particles can repeat itself until it is energetically possible and the previous stage can affect the next one. We investigate the role of the masses of particles and values of couplings in various scenarios motivated by the usual cosmological considerations with and without supersymmetry. In addition, we focus on the role of light states in the theory proving that in general even massless states not coupled to inflaton can be efficiently produced due to effects of quantum physics and also that additional light sector present in the theory can quench the production after inflation completely due to backreaction.

Streszczenie

Jednym z najciekawszych i jednocześnie najbardziej złożonych problemów współczesnej fizyki jest ustalenie, jak wygląda przeszłość i przyszłość Wszechświata. Szczególnie interesuje nas jego wczesna ewolucja, ponieważ znajduje się ona obecnie poza naszym zasięgiem eksperymentalnym, ale wpływa na późniejszą historię Wszechświata, którą już możemy obserwować. Ogólnie przyjęty model kosmologiczny zaczyna się od Wielkiego Wybuchu i zakłada, że Wszechświat na początku był jednorodny i izotropowy z bardzo wysoką temperaturą i ciśnieniem, po czym zaczął stopniowo się ochładzać i rozszerzać, kończąc jako Wszechświat, w którym żyjemy. Poza tym zakłada się, że bardzo wcześnie w ewolucji Wszechświata nastąpił proces eksponencjalnego rozszerzania się, który nazywamy inflacją i w którym fluktuacje gęstości zostały wzmocnione, dając początek wszystkim strukturom wielkoskalowym - gwiazdom, galaktykom itp. Po zakończeniu inflacji miały miejsce dwa procesy bardzo istotne dla przedstawionych badań – reheating i preheating, podczas których z inflatonu, czyli pola napędzającego inflację, powstały cząstki elementarne obecnie wypełniające Wszechświat.

Rozprawa opisuje procesy kosmologicznej produkcji cząstek w teoriach zależnych od czasu i skupia się na trzech głównych tematach: grawitacyjny reheating z błyskawicznym procesem tworzenia cząstek oraz wieloetapowa i nieperturbacyjna produkcja zarówno w przybliżeniu adiabatycznym, jak i w teorii oddziałującej. Wszystkie opierają się na fakcie, że stan próżni zmienia się w czasie i że w przestrzeni parametrów istnieje obszar, w którym wytwarzanie cząstek jest energetycznie korzystne i wystarczająco wydajne, aby mogło nastąpić.

W rozdziale dotyczącym reheatingu grawitacyjnego cząstki są wytwarzane wyłącznie ze względu na zmianę ewolucji obserwowanego Wszechświata w czasie opisywaną przez zależący od czasu czynnik skali. W naszych badaniach rozważamy w bardzo ogólny sposób tego rodzaju produkcję cząstek tuż po zakończeniu inflacji, co pozwala wyciągnąć ogólne wnioski na temat dostępnych obserwabli dotyczących inflacji, obserwowanego spektrum fal grawitacyjnych, a nawet własności ciemnej materii lub ciemnej energii.

Główna część pracy koncentruje się na ogólnym opisie wielofazowej nieperturbacyjnej produkcji cząstek, w szczególności w przypadku inflacji. Istota tych badań polega na tym, że produkcja cząstek może się powtarzać, dopóki jest dozwolona energetycznie, a poprzedni jej etap może wpłynąć na następny. Badamy rolę mas cząstek i wartości sprzężeń w różnych scenariuszach motywowanych standardowymi rozważaniami kosmologicznymi z uwzględnieniem i bez supersymetrii. Ponadto koncentrujemy się na roli stanów bezmasowych w teorii, dowodząc, że nawet bezmasowe stany niesprzężone bezpośrednio z inflatonem mogą być produkowane ze względu na poprawki kwantowe, a także, że dodatkowy bezmasowy sektor może zatrzymać całkowicie produkcję po inflacji.

Contents

Chapter 1

Introduction

Quantum field theory can describe creation of particles in presence of an external perturbation. The perturbation can have various character with the key examples of Schwinger effect coming from adding an external electric field to the quantum electrodynamics (QED) [1, 2, 3, 4], Hawking radiation from the blackholes originating in gravitational horizon effects [5, 6, 7] or the Unruh effect involving an accelerating observer [8, 9].

Particles are created during the transition between the initial and final free configurations in equilibrium that comes from the perturbing and usually time-dependent background. Free configuration allows to define positive and negative energy states unambiguously, respectively particles and antiparticles, which in turn compared for the asymptotic vacua can be related with the final particle number of produced states.

Description of the intermediate period entails a severely compound problem, both analytical and numerical, with a proper definition of the time-dependent particle number involving the whole non-equilibrium dynamics and the backreaction. Distinction between particles and antiparticles is not so clear then and one has to make use of some assumptions about the considered system allowing to approximate the intermediate basis of states in an useful way, for instance one assumes slow variation of the perturbation and then uses standard adiabatic expansion. There are several equivalent approaches to describe the process of particle creation such as diagonalization of the instantaneous Hamiltonian [10], the Unruh-de Witt method [11] or the Bogoliubov transformation between two different basis states representing vacua [12] to name a few and this dissertation is based on the last one. Particular choice of the method should be insignificant as they should give the same results in the end[1].

Perhaps the most prominent realisation of the process of particle creation in time-dependent background is post-inflationary particle production. It is a very complex and multi-stage process that mixes perturbative and non-perturbative production, which is schematically depicted in the Figure 1.1.

This dissertation describes cosmological particle production in time-dependent

[1]These three methods have been compared for the de Sitter space in [13].

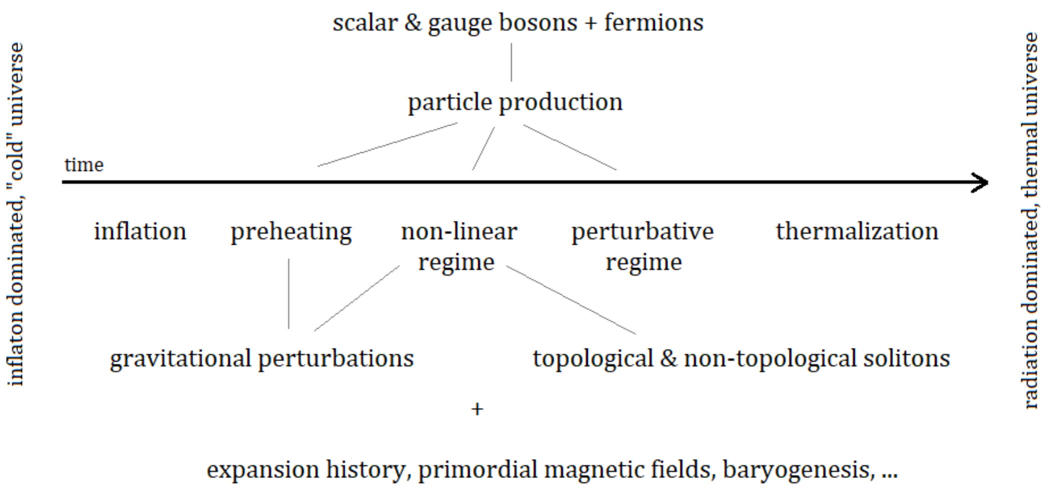

Figure 1.1: Post-inflationary particle production is a very complex process consisting of several separate phases. Scheme copied from [14].

backgrounds and consists of seven chapters with the current introduction being the first of them. Based on the classical literature [6, 15] in Chapter 2 we introduce all the necessary information about the physical foundations of the presented analysis - features of the curved spacetime (the issue of quantization, definition of the vacuum, adiabaticity), Bogoliubov transformation and its application to some simple illustrative forms of the gravitational background or description of particle production in a theory with time-dependent mass term. A brief thermal history of the Universe and the cosmological description of the homogeneous Universe has been included there. In Chapter 3 we analyse particle production in adiabatic approximation for massless background for two supersymmetric models including quantum corrections in detail, partially based on [16], while in Chapter 4 we use the same approximation to address the production in massive backgrounds and compare it with the classical post-inflationary production in a form of parametric resonance. Moreover, we provide the useful description of this classical approach based on [17]. In Chapter 5 we introduce the interacting theory to capture the intermediate period of production in a more precise way [18] and then in Chapter 6 we focus on gravitational realisation of reheating with an instant period of particle creation [19]. Finally, Chapter 7 summarizes the whole dissertation. The details of some chosen calculations, which we find useful are presented in five appendices.

Chapter 2

Foundations

2.1 Quantization of the scalar field in curved spacetime

Investigating cosmological production of particles one often considers simple scalar field in a flat or curved spacetime. The simplest action for a massive scalar field for Minkowski metric is of the form

$$S = \frac{1}{2}\int d^4x \left(\eta_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - m^2\phi^2\right), \tag{2.1.1}$$

which for more general homogeneous and isotropic FRW metric in flat spacetime, $ds^2 = dt^2 - a^2(t)d\vec{x}^2$, changes to:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - m^2\phi^2\right). \tag{2.1.2}$$

Euler-Lagrange equation

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial_\mu\phi} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{2.1.3}$$

for the action above reads:

$$g^{\mu\nu}\phi_{\mu\nu} + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial_\mu}\left(g^{\mu\nu}\sqrt{-g}\right)\phi_\nu + m^2\phi = 0. \tag{2.1.4}$$

For Minkowski metric it simplifies to:

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0, \tag{2.1.5}$$

which is the usual harmonic oscillator equation of motion, while for FRW metric it reads:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{\nabla^2}{a^2}\phi + m^2\phi = 0 \tag{2.1.6}$$

for comoving coordinates (t, x^μ) and:

$$\phi'' + 2\frac{a'}{a}\phi' - \Delta\phi + m^2a^2(\eta)\phi = 0 \tag{2.1.7}$$

for conformal coordinates (η, x^μ), where $d\eta = \frac{dt}{a}$. Both of them are again harmonic oscillator equations but with additional external force. In order to eliminate the first derivative terms in the above equations and simplify them χ field defined as $\chi = a\phi$ is introduced. Then

$$\chi'' - \Delta\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0, \tag{2.1.8}$$

which is the harmonic oscillator equation again but with the time-dependent frequency, which makes the whole analysis much more complicated than in the Minkowski case. In particular it is not always possible to find its explicit solution - unambiguous set of modes. Due to the coupling between the scale factor a and scalar field ϕ production of particles associated with ϕ is possible as there is energy transferred between the gravitational background and the field.

Action for the new field reads:

$$S = \frac{1}{2}\int d^3x d\eta \left(\chi'^2 - (\partial\chi)^2 - m_{\text{eff}}^2(\eta)\chi^2\right), \tag{2.1.9}$$

where $m_{\text{eff}}^2 = m^2a^2 - \dfrac{a''}{a}$. It is a matter of convention to choose the following mode decomposition:

$$\chi(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^3}\left(e^{i\vec{k}\cdot\vec{x}}v_{\vec{k}}^*(\eta)a_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x}}v_{\vec{k}}(\eta)a_{\vec{k}}^\dagger\right), \tag{2.1.10}$$

where $\vec{k}$ denotes momentum and $v_{\vec{k}}$ mode function, which results in the following equation of motion for the modes

$$v_{\vec{k}}'' + \omega_{\vec{k}}^2(\eta)v_{\vec{k}} = 0 \tag{2.1.11}$$

with $\omega_{\vec{k}}^2(\eta) = |\vec{k}|^2 + m_{\text{eff}}^2(\eta)$.[1] Solving it means finding some particular set of modes. Vacuum state is then defined by the condition: $\forall_{\vec{k}}\ a_{\vec{k}}|0\rangle = 0$ with the commutation relations:

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta(\vec{k} - \vec{k}') \tag{2.1.15}$$

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0 \tag{2.1.16}$$

[1]Mode functions $v_{\vec{k}}$ are orthonormal, which means that:

$$\left(v_{\vec{k}}, v_{\vec{k}'}\right) = \delta(\vec{k} - \vec{k}'), \tag{2.1.12}$$

$$\left(v_{\vec{k}}, v_{\vec{k}'}^*\right) = 0 \tag{2.1.13}$$

with scalar product

$$\left(f_1, f_2\right) = i\int d^3x|g|^{1/2}\Big[f_1^*(\vec{x},t)\cdot\partial_0 f_2(\vec{x},t) - \partial_0 f_1^*(\vec{x},t)\cdot f_2(\vec{x},t)\Big]. \tag{2.1.14}$$

2.2 Notion of vacuum in curved spacetime

In Minkowski space vacuum is the unique energy-eigenstate of the Hamiltonian corresponding to the lowest energy and it is the same for all inertial observers. In curved spacetime situation is quite different - in a general situation no set of mode functions is distinguished as the Hamiltonian is explicitly time-dependent and there are no time-independent eigenvectors that can serve as vacuum. It is a matter of choice how to define vacuum in such a setup.

In this thesis we shall use two types of vacuum in curved spacetime: instantaneous and adiabatic vacuum.

Hamiltonian in curved spacetime reads

$$H(\eta) = \frac{1}{2}\int d^3x\Big(\dot{\chi}^2 + (\nabla\chi)^2 + m_{\text{eff}}^2\chi^2\Big) \tag{2.2.1}$$

and instantaneous vacuum at η_0 is just the lowest energy-state of the instantaneous Hamiltonian $\hat{H}(\eta_0)$. We can compute the expectation value of the instantaneous Hamiltonian

$$\langle 0_v|\hat{H}(\eta_0)|0_v\rangle = \frac{1}{4}\delta^3(0)\int d^3k\Big(|v_k'|^2 + \omega_k^2(\eta)|v_k|^2\Big) \tag{2.2.2}$$

and minimize it with respect to some chosen set of modes $v_k(\eta)$. Equivalently we may minimize the energy density

$$\epsilon(\eta_0) = \frac{1}{4}\int d^3k\Big(|v_k'(\eta_0)|^2 + \omega_k^2(\eta_0)|v_k(\eta_0)|^2\Big). \tag{2.2.3}$$

Provided the normalization condition coming from the time-independence of the Wronskian: $\text{Im}(v'v^*) = 1$, we find the initial conditions that determine the set of modes defining instantaneous vacuum

$$\begin{cases} v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}e^{i\gamma_k(\eta_0)} \\ v_k'(\eta_0) = i\omega_k(\eta_0)v_k(\eta_0) \end{cases} \tag{2.2.4}$$

with arbitrary phase γ_k, assuming that $\omega_k^2 > 0$.

The procedure of finding adiabatic vacuum is based on the WKB approximation for the solution of equation of motion for the modes

$$\ddot{v}_k(\eta) + \omega^2(\eta)v_k(\eta) = 0 \tag{2.2.5}$$

assuming slowly changing background. Ansatz for the vacuum is of the form

$$v_k(\eta) = \frac{1}{\sqrt{W_k(\eta)}}e^{i\int_{\eta_0}^{\eta} W_k(\eta)d\eta} \tag{2.2.6}$$

and it determines the equation for W_k

$$W_k^2 = \omega_k^2 - \frac{1}{2}\Big[\frac{W_k''}{W_k} - \frac{3}{2}\Big(\frac{W_k'}{W_k}\Big)^2\Big]. \tag{2.2.7}$$

For a slowly changing spacetime: $W_k'^2/W_k^2$, $W_k''/W_k \ll \omega_k^2$, the 0^{th} order approximation states that

$$W_k^{(0)}(\eta) = \omega_k(\eta), \tag{2.2.8}$$

while the higher orders can be estimated by iteration. WKB method does not work for small values of the couplings, therefore it cannot include the particle production coming entirely from the expansion of the spacetime [20] as in Chapter 6.

For 0^{th} order we can link both kinds of vacuum by the relation

$$\frac{1}{4}E_k^{\text{adiabatic}} = \frac{1}{4}E_k^{\text{instantaneous}} + \frac{\omega_k}{16}\epsilon^2, \tag{2.2.9}$$

where $\epsilon = \dfrac{\dot{\omega}_k}{\omega_k^2}$ is the adiabaticity parameter.

Moreover, for asymptotically flat geometry

$$a(t) \sim \begin{cases} a_1 \text{ for } t \to -\infty \\ a_2 \text{ for } t \to +\infty \end{cases} \tag{2.2.10}$$

vacuum can be described in analogy to the Minkowski case with well defined "in" and "out" asymptotic states.

2.2.1 Adiabaticity

As the name suggests concept of adiabaticity is connected with the adiabatic vacuum. Taking the 0^{th} order of the mode function

$$v_k \sim \frac{1}{\sqrt{\omega_k}} e^{\pm i \int \omega(t')dt'} \tag{2.2.11}$$

we can see that the equation of motion (2.2.5) can be solved in two regimes:

- for slowly varying background, when $\dot{\omega_k}/\omega_k^2 < 1$: adiabatic region,
- for rapidly varying background, when $\dot{\omega_k}/\omega_k^2 > 1$: non-adiabatic region.

In the first case occupation number reads:

$$n_k(t) \approx \frac{\rho_k}{\omega_k} \approx \frac{|\dot{v}_k|^2}{\omega_k} \approx \frac{1}{\omega_k}\left(\sqrt{\omega e^{\pm i \int \omega}}\right)^2 \approx \text{const} \tag{2.2.12}$$

and therefore there are no states produced there, while in the non-adiabatic region $n_k(t) \neq \text{const}$, which corresponds to the production of particles.

2.3 Bogoliubov transformation

Particle production occurs when the background of the considered theory changes e.g. the mass of the background field or the scale factor are not constant. This change influences also the vacuum structure as the vacua before and after production differ, $|0_{\rm in}\rangle \neq |0_{\rm out}\rangle$. Moreover, following the procedure of second quantization we can notice the change of the set of creation and annihilation operators and the mode functions:

$$\left(a_k^{\rm in}, a_k^{\rm in\,\dagger}\right) \neq \left(a_k^{\rm out}, a_k^{\rm out\,\dagger}\right), \tag{2.3.1}$$

$$v_k^{\rm in} \neq v_k^{\rm out}, \tag{2.3.2}$$

keeping: $a_k^{\rm in}|0_{\rm in}\rangle = 0$ and $a_k^{\rm out}|0_{\rm out}\rangle = 0$.

These two sets of operators act in the same Hilbert space so we can express one by the other

$$a_k^{\rm out} = \alpha_k a_k^{\rm in} + \beta_k a_k^{\rm in\,\dagger}, \tag{2.3.3}$$

$$a_k^{\rm out\,\dagger} = \alpha_k^* a_k^{\rm in\,\dagger} + \beta_k^* a_k^{\rm in}, \tag{2.3.4}$$

where α_k and β_k are some coefficients. Commutation relation for the scalar field in the new basis reads

$$[a_k^{\rm out}, a_k^{\rm out\,\dagger}] = [\alpha_k a_k^{\rm in} + \beta_k a_k^{\rm in\,\dagger}, \alpha_k^* a_k^{\rm in\,\dagger} + \beta_k^* a_k^{\rm in}] = \left(|\alpha_k|^2 - |\beta_k|^2\right)[a_k^{\rm in}, a_k^{\rm in\,\dagger}], \tag{2.3.5}$$

which implies the normalization condition for newly introduced coefficients

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{2.3.6}$$

as the commutation relation is fixed[2]. Transformation (2.3.3)-(2.3.4) is called Bogoliubov transformation [15, 21] and parameters α_k and β_k are the Bogoliubov coefficients. For the mode functions it translates into the relation

$$v_k^{\rm in} = \alpha_k v_k^{\rm out} + \beta_k^* v_k^{\rm out\,*}. \tag{2.3.7}$$

Occupation number of produced particles can be described by only one of the Bogoliubov coefficients

$$n_k \equiv \langle 0^{\rm in}|N_k|0^{\rm in}\rangle = \langle 0^{\rm in}|a_{\vec{k}}^{\rm out\,\dagger} a_{\vec{k}}^{\rm out}|0^{\rm in}\rangle = V|\beta_k|^2. \tag{2.3.8}$$

2.3.1 Simple examples of gravitational particle production

There are two classical and very pedagogical examples illustrating the idea of purely gravitational particle production in terms of the Bogoliubov transformation: due to the sudden jump of the scale factor and its well shape.

The first case, a sudden jump, can be described by the scale factor of the form

$$a(\eta) = a_1 - a_2\Theta(\eta) = \begin{cases} a_1 & \eta < \eta_0 = 0 \\ a_2 & \eta > \eta_0 \end{cases}, \tag{2.3.9}$$

[2]For fermions $|\alpha_k|^2 + |\beta_k|^2 = 1$ because of the different form of commutation relation.

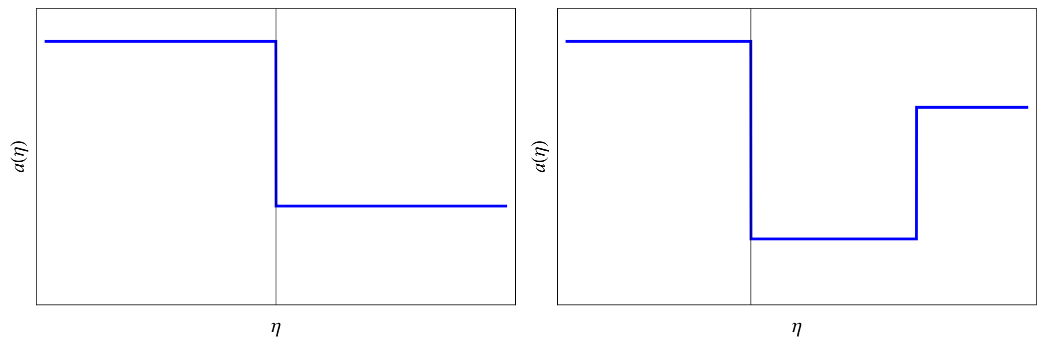

Figure 2.1: Simple scale factor profiles that are followed by with the particle production: sudden jump (*left*) and well (*right*).

which leads to the occupation number of the form

$$|\beta|^2 = \frac{\left(\sqrt{k^2 + m^2(a_1 - a_2)^2} - \sqrt{k^2 + m^2 a_1^2}\right)^2}{4\sqrt{k^2 + m^2(a_1 - a_2)^2}\sqrt{k^2 + m^2 a_1^2}} \tag{2.3.10}$$

relying on the Bogoliubov transformation between the two regions in η.

The second case, a well, can be described by

$$a(\eta) = \begin{cases} a_1 & \eta < \eta_0 = 0 \\ a_2 & \eta_0 < \eta < \eta_1 \\ a_3 & \eta > \eta_1 \end{cases} \tag{2.3.11}$$

and, thus, results in the occupation number that reads

$$|\beta|^2 = \frac{D^2(\omega_2 - \omega_3)^2 + E^2(\omega_2 + \omega_3)^2 - 2DE(\omega_2^2 - \omega_3^2)\cos(2\omega_2\eta_1)}{4\omega_3^2} \tag{2.3.12}$$

with $D = \dfrac{\omega_2 - \omega_1}{2\omega_2}$, $E = \dfrac{\omega_1 + \omega_2}{2\omega_2}$ and $\omega_i^2 = k^2 + m^2 a_i^2$.

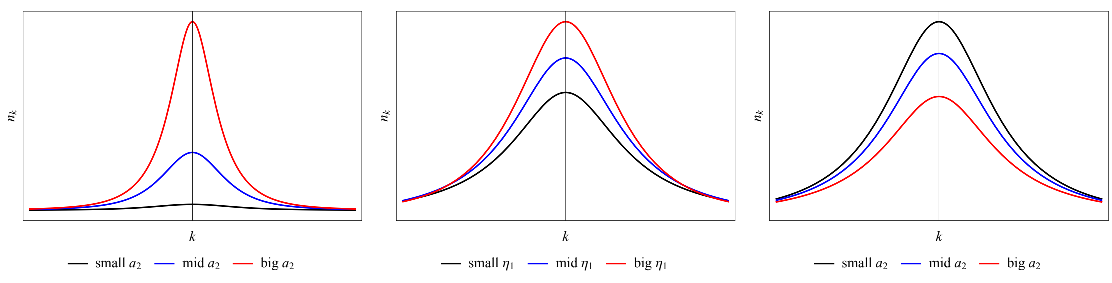

Figure 2.2: Occupation number of produced particles as a function of momentum for the two profiles: sudden jump depending on its height (*left*) and well depending on its width (*middle*) and depth (*right*).

From Figure 2.2 we can infer that the quantity of produced particles relies on the features of the scale factor profile - it gets bigger as the height of the jump decreases, the width of the well increases and the depth of the well increases.

2.4 Particle production in a theory with time-varying mass terms

Gathering all the information we can describe the procedure of obtaining number density of produced particles in a theory with time-varying mass terms for both scalars and fermions.

The simplest Lagrangian describing such theory for a real scalar field reads

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2(t)\phi^2 \tag{2.4.1}$$

with corresponding Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right). \tag{2.4.2}$$

We start the whole procedure with expanding ϕ in plane waves

$$\phi = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\phi_k(x^0)a_{\mathbf{k}} + \phi_k^*(x^0)a_{-\mathbf{k}}^\dagger\right) \tag{2.4.3}$$

with the usual equation of motion for the modes

$$0 = \ddot{\phi}_k + \omega_k^2\phi_k, \tag{2.4.4}$$

where $\omega_k \equiv \sqrt{k^2+m^2}$. Inner product relation is of the form

$$1 = i(\phi_k^*\dot{\phi}_k - \dot{\phi}_k^*\phi_k) \equiv (\phi_k, \phi_k), \tag{2.4.5}$$

which is equivalent to the following set of the commutation relations

$$[\phi(t,\mathbf{x}), \dot{\phi}(t,\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}'), \tag{2.4.6}$$
$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] = [\dot{\phi}(t,\mathbf{x}), \dot{\phi}(t,\mathbf{x}')] = 0, \tag{2.4.7}$$
$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}'), \tag{2.4.8}$$
$$[a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger] = 0. \tag{2.4.9}$$

Substituting (2.4.3) into the Hamiltonian, we obtain

$$H = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[\Omega_k(t)\left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}a_{-\mathbf{k}}^\dagger\right) + \Lambda_k(t)a_{-\mathbf{k}}a_{\mathbf{k}} + \Lambda_k^*(t)a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger\right], \tag{2.4.10}$$

where[3]

$$\Omega_k(t) \equiv |\dot{\phi}_k(t)|^2 + \omega_k^2(t)|\phi_k(t)|^2, \tag{2.4.11}$$
$$\Lambda_k(t) \equiv \dot{\phi}_k^2(t) + \omega_k^2(t)\phi_k^2(t). \tag{2.4.12}$$

[3]These two functions $\Omega_k(t)$ and $\Lambda_k(t)$ connect via frequency

$$\Omega_k^2(t) - |\Lambda_k(t)|^2 = \omega_k^2(t).$$

Vacuum state $|0\rangle$ defined by the relation $a_{\mathbf{k}}|0\rangle \equiv 0$ turns out not to be the eigenstate of our Hamiltonian due to the terms proportional to $\Lambda_k(t)$. But if we use Bogoliubov transformation [22]

$$\bar{a}_{\mathbf{k}} = \alpha_k a_{\mathbf{k}} + \beta_k a^{\dagger}_{-\mathbf{k}} \tag{2.4.13}$$

in order to diagonalize our Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2}\left(\bar{a}^{\dagger}_{\mathbf{k}}\bar{a}_{\mathbf{k}} + \bar{a}_{-\mathbf{k}}\bar{a}^{\dagger}_{-\mathbf{k}}\right), \tag{2.4.14}$$

we end up with the proper vacuum condition $\bar{a}_{\mathbf{k}}\left|\bar{0}\right\rangle \equiv 0$. Comparing (2.4.10) and (2.4.14) we get two conditions:

$$\Omega_k = \omega_k(|\alpha_k|^2 + |\beta_k|^2), \tag{2.4.15}$$
$$\Lambda_k = 2\omega_k\alpha_k\beta_k^*, \tag{2.4.16}$$

which combined determine the Bogoliubov coefficients that are explicitly time-dependent

$$\alpha_k(t) = \sqrt{\frac{\Omega_k}{2\omega_k} + \frac{1}{2}}, \tag{2.4.17}$$
$$\beta_k(t) = \frac{\Lambda_k^*}{|\Lambda_k|}\sqrt{\frac{\Omega_k}{2\omega_k} - \frac{1}{2}}, \tag{2.4.18}$$

but they sustain a proper normalization

$$|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1. \tag{2.4.19}$$

Finally, the occupation number of produced scalar states reads

$$N_k(t) = \langle 0|\,\bar{a}^{\dagger}_{\mathbf{k}}\bar{a}_{\mathbf{k}}\,|0\rangle = |\beta_k(t)|^2 \cdot \int d^3x = \left(\frac{\Omega_k}{2\omega_k} - \frac{1}{2}\right)\cdot \int d^3x, \tag{2.4.20}$$

where d^3x is the usual volume of the system.

We can apply this procedure to the time-dependent fermionic case basing on the simple Lagrangian of the form

$$\mathcal{L} = \xi^{\dagger}i\bar{\sigma}^{\mu}\partial_{\mu}\xi + \eta i\sigma^{\mu}\partial_{\mu}\eta^{\dagger} - m(t)\eta\xi - m^*(t)\xi^{\dagger}\eta^{\dagger} \tag{2.4.21}$$

with the corresponding Hamiltonian

$$H = \int d^3x\left(-\xi^{\dagger}i\bar{\sigma}^{i}\partial_{i}\xi - \eta i\sigma^{i}\partial_{i}\eta^{\dagger} + m(t)\eta\xi + m^*(t)\xi^{\dagger}\eta^{\dagger}\right). \tag{2.4.22}$$

We start again with the expansion into modes

$$\xi(x) = \int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{s=\pm} e^s_{\mathbf{k}}\left(u^s_k(x^0)a^s_{\mathbf{k}} + v^{s*}_k(x^0)b^{s\dagger}_{-\mathbf{k}}\right), \tag{2.4.23}$$

$$\eta^{\dagger}(x) = \int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{s=\pm} \bar{\sigma}^0 e^s_{\mathbf{k}}\left(v^s_k(x^0)a^s_{\mathbf{k}} - u^{s*}_k(x^0)b^{s\dagger}_{-\mathbf{k}}\right), \tag{2.4.24}$$

where $e^s_{\mathbf{k}}$ is a helicity operator and equations of motion read

$$0 = \dot{u}^s_k - is|\mathbf{k}|u^s_k + im^* v^s_k, \quad (2.4.25)$$
$$0 = \dot{v}^s_k + is|\mathbf{k}|v^s_k + imu^s_k. \quad (2.4.26)$$

Wave functions are normalized as usual

$$|u^s_k(t)|^2 + |v^s_k(t)|^2 = 1. \quad (2.4.27)$$

Side note: Helicity operator

Physical meaning of helicity is that it is the projection of the total angular momentum on the direction of the linear momentum of the particle. Helicity operator $e^s_{\mathbf{k}}$ is defined by

$$-k^i \bar{\sigma}^i e^s_{\mathbf{k}} = s|\mathbf{k}|\bar{\sigma}^0 e^s_{\mathbf{k}} \quad (2.4.28)$$

with $s = \pm$.

We choose the following representation of this operator

$$e^s_{\mathbf{k}\alpha} = \frac{1}{\sqrt{2}} e^{i\rho^s_{\mathbf{k}}} \begin{pmatrix} \sqrt{1 + sk^3/|\mathbf{k}|} \\ se^{i\theta_{\mathbf{k}}}\sqrt{1 - sk^3/|\mathbf{k}|} \end{pmatrix}_\alpha , \quad (2.4.29)$$

where $\rho^s_{\mathbf{k}}$ and $\theta_{\mathbf{k}}$ are two phases. The first one is arbitrary and we choose $\rho^s_{\mathbf{k}} = 0$, while the second reads

$$e^{i\theta_{\mathbf{k}}} \equiv \frac{k^1 + ik^2}{\sqrt{(k^1)^2 + (k^2)^2}} \quad (2.4.30)$$

and it is completely defined by x- and y-components of the given momentum.

Some useful properties of the helicity operator:

$$e^{s\dagger}_{\mathbf{k}} \bar{\sigma}^0 e^r_{\mathbf{k}} = e^r_{\mathbf{k}} \sigma^0 e^{s\dagger}_{\mathbf{k}} = \delta^{sr}, \quad (2.4.31)$$
$$e^s_{\mathbf{k}} e^r_{-\mathbf{k}} = se^{i\theta_{\mathbf{k}}} e^{i\rho^s_{\mathbf{k}} + i\rho^s_{-\mathbf{k}}} \cdot \delta^{sr}, \quad (2.4.32)$$
$$e^s_{\mathbf{k}\alpha} e^{s\dagger}_{\mathbf{k}\dot{\alpha}} = \frac{1}{2}\left(\sigma^0 + \frac{s\sigma^i k^i}{|\mathbf{k}|}\right)_{\alpha\dot{\alpha}} . \quad (2.4.33)$$

In the last equation there is no summation over s.

Substituting (2.4.23) and (2.4.24) into the Hamiltonian, we obtain

$$H = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \left[E^s_k(t)\left(a^{s\dagger}_{\mathbf{k}} a^s_{\mathbf{k}} - b^s_{-\mathbf{k}} b^{s\dagger}_{-\mathbf{k}}\right) + F^s_k(t) b^s_{-\mathbf{k}} a^s_{\mathbf{k}} + F^{s*}_k(t) a^{s\dagger}_{\mathbf{k}} b^{s\dagger}_{-\mathbf{k}}\right], \quad (2.4.34)$$

where[4]

$$E_k^s(t) \equiv -sk(|u_k^s|^2 - |v_k^s|^2) + mu_k^s v_k^{s*} + m^* u_k^{s*} v_k^s, \tag{2.4.35}$$
$$F_k^s(t) \equiv -2sku_k^s v_k^s - mu_k^{s\,2} + m^* v_k^{s\,2}. \tag{2.4.36}$$

Diagonalizing the Hamiltonian (2.4.34) using the Bogoliubov transformation

$$\begin{pmatrix} \bar{a}_{\mathbf{k}}^s \\ \bar{b}_{-\mathbf{k}}^{s\dagger} \end{pmatrix} = \begin{pmatrix} \alpha_k^s & \beta_k^s \\ -\beta_k^{s*} & \alpha_k^{s*} \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}}^s \\ b_{-\mathbf{k}}^{s\dagger} \end{pmatrix} = \begin{pmatrix} \alpha_k^s a_{\mathbf{k}}^s + \beta_k^s b_{-\mathbf{k}}^{s\dagger} \\ -\beta_k^{s*} a_{\mathbf{k}}^s + \alpha_k^{s*} b_{-\mathbf{k}}^{s\dagger} \end{pmatrix} \tag{2.4.37}$$

leads to the Hamiltonian of the form

$$H = \int d^3x \sum_{s=\pm} \omega_k \left(\bar{a}_{\mathbf{k}}^{s\dagger} \bar{a}_{\mathbf{k}}^s - \bar{b}_{-\mathbf{k}}^s \bar{b}_{-\mathbf{k}}^{s\dagger} \right) \tag{2.4.38}$$

and two constraints:

$$E_k^s = \omega_k (|\alpha_k^s|^2 - |\beta_k^s|^2), \tag{2.4.39}$$
$$F_k^s = 2\omega_k \alpha_k^s \beta_k^{s*}. \tag{2.4.40}$$

Combining them we obtain the time-dependent Bogoliubov coefficients

$$\alpha_k^s(t) = \sqrt{\frac{1}{2} + \frac{E_k^s}{2\omega_k}}, \tag{2.4.41}$$
$$\beta_k(t) = \frac{F_k^{s*}}{|F_k^s|} \sqrt{\frac{1}{2} - \frac{E_k^s}{2\omega_k}} \tag{2.4.42}$$

with a proper normalization

$$|\alpha_k^s(t)|^2 + |\beta_k^s(t)|^2 = 1. \tag{2.4.43}$$

Once again we can express the occupation number in terms of the modes

$$N_k(t) = \langle 0| \bar{a}_{\mathbf{k}}^{s\dagger} \bar{a}_{\mathbf{k}}^s |0\rangle = \langle 0| \bar{b}_{\mathbf{k}}^{s\dagger} \bar{b}_{\mathbf{k}}^s |0\rangle = |\beta_k^s(t)|^2 \cdot \int d^3x = \left(\frac{1}{2} - \frac{E_k^s}{2\omega_k} \right) \cdot \int d^3x. \tag{2.4.44}$$

2.5 Brief thermal history of the Universe

Observational data is consistent with the expansion of the Universe as for instance the light from distant galaxies is redshifted and relic abundance of light elements is in agreement with the theory of Big Bang Nucleosynthesis (BBN). Moreover, three independent sources: supernovae Ia, temperature fluctuations of the Cosmic Microwave Background (CMB) and the distribution of galaxies suggest that ΛCDM[5] is a reliable model of our Universe.

[4]These two functions $E_k^{s\,2}(t)$ and $F_k^s(t)$ connect via frequency

$$E_k^{s\,2}(t) + |F_k^s(t)|^2 = \omega_k^2(t)$$

with $\omega_k(t) \equiv \sqrt{|\mathbf{k}|^2 + |m(t)|^2}$.

[5]ΛCDM (Lambda cold dark matter) model is a parametrization of cosmological model with the Big Bang and the Universe containing a cosmological constant (Λ) related to dark energy and cold dark matter (CDM).

	time	energy	
Planck epoch?	$< 10^{-43}$ s	10^{18} GeV	
string scale?	$\gtrsim 10^{-43}$ s	$\lesssim 10^{18}$ GeV	
Grand Unification ?	$\sim 10^{-36}$ s	10^{15} GeV	
inflation?	$\gtrsim 10^{-34}$ s	$\lesssim 10^{15}$ GeV	
SUSY breaking?	$< 10^{-10}$ s	> 1 TeV	
baryogenesis?	$< 10^{-10}$ s	> 1 TeV	
electroweak unification	10^{-10} s	1 TeV	
quark-hadron transition	10^{-4} s	100 MeV	
nucleon freeze-out	0.01 s	10 MeV	
neutrino decoupling	1 s	1 MeV	
BBN	3 min	0.1 MeV	
			redshift
matter-radiation equality	10^4 yrs	1 eV	10^4
recombination	10^5 yrs	0.1 eV	1 100
Dark Ages	$10^5 - 10^8$ yrs		> 25
reionization	10^8 yrs		25-6
galaxy formation	$\sim 6 \times 10^8$ yrs		~ 10
dark energy	$\sim 10^9$ yrs		~ 2
solar system	8×10^9 yrs		0.5
us	14×10^9 yrs	1 meV	0

Table 2.1: Major events in the history of the Universe, the table is taken from [23]. Question marks denote the events corresponding to the energy scales currently unavailable in the observations.

The Universe was homogeneous and isotropic right after Big Bang filled with the energy of tremendous density, temperature and pressure. Due to expansion it cooled down going through the phase transitions related to the breaking of symmetries restored at high energies. Table 2.1 presents the most important events in the history of the Universe - its thermal history, with some speculative processes and ideas marked with question marks. This table has been excerpted from [23], which may result in some discrepancies between the energies contained there and currently valid ones.

Based on our understanding and experimental observations of particle physics, nuclear physics and gravity we can have some credible picture of the evolution of the Universe from 10^{-10} seconds to today. We assume that before this time it went through the processes important for this dissertation, namely inflation.

Inflation is a period of exponential expansion of the Universe, so far hypothetical but being consistent with observations for some of its scenarios. Particular particle physics realization of inflation is not set yet but its paradigm helps with solving a lot of cosmological problems - isotropy and homogeneity of the Universe, its flatness, lack of magnetic monopoles and the origin of the large-scale structure. For the detailed description of inflation and associated particle production see Section 4.1.

2.6 The Homogeneous Universe

The main goal of cosmology is to describe the structure and evolution of the Universe at the largest scales, which is usually limited by the conditions of its homogeneity and isotropy. Homogeneous space is translationally invariant - there is no special point in it, while isotropic one is rotationally invariant - there is no special direction there. The most common metric describing the Universe that fulfils both these conditions is the Friedmann-Robertson-Walker (FRW) metric of the form

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \tag{2.6.1}$$

where $a(t)$ is the scale factor and the curvature parameter k is $+1$ for positively curved space, 0 for flat one and -1 for negatively curved space. For the FRW metric all the information about the evolution of the Universe is encoded exactly in the scale factor, which is determined by the Einstein equations and the matter content of the Universe. It characterizes the FRW spacetime expansion rate through the Hubble parameter

$$H = \frac{\dot{a}}{a}, \tag{2.6.2}$$

which is positive for an expanding universe and negative for a collapsing one. Its importance lies also in the fact that it defines two important cosmological scales - Hubble time $t \sim H^{-1}$, which is the scale of the age of the Universe, and Hubble length $d \sim H^{-1}$, which corresponds to the size of the observable Universe.

The dynamics of the universe is described by the Einstein equations[6]

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.6.3}$$

which determines the time evolution of the scale factor. Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{2.6.4}$$

is defined by the Ricci tensor, $R_{\mu\nu}$, and the Ricci scalar, R, which are of the form

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}, \tag{2.6.5}$$

$$R \equiv g^{\mu\nu}R_{\mu\nu}, \tag{2.6.6}$$

where

$$\Gamma^{\mu}_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\nu}[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}] \tag{2.6.7}$$

are the Christoffel symbols.

Right hand side of the Einstein equation includes the energy-momentum tensor of the universe, $T_{\mu\nu}$. To describe it, it is essential to introduce a set of observers with worldlines tangent to the timelike velocity

$$u^{\mu} \equiv \frac{dx^{\mu}}{d\tau}, \tag{2.6.8}$$

[6] There are various conventions of their form, setting $8\pi G \equiv 1$ is one of them.

where τ is the proper time ($g_{\mu\nu}u^\mu u^\nu = -1$). We can also define the metric of the 3-dimensional spatial sections orthogonal to u_μ: the tensor $\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. Then, for some general fluid $T_{\mu\nu}$ is of the form

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\gamma_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \Sigma_{\mu\nu}, \tag{2.6.9}$$

where $\rho = T_{\mu\nu}u^\mu u^\nu$ is the matter energy density, $p = \frac{1}{3}T_{\mu\nu}\gamma^{\mu\nu}$ is the pressure, $q_\mu = -\gamma^\alpha_\mu T_{\alpha\beta}u^\beta$ is the energy-flux vector and $\Sigma_{\mu\nu} = \gamma^\alpha_{\langle\mu}\gamma^\beta_{\nu\rangle}T_{\alpha\beta}$ is the anisotropic stress tensor.[7] For a perfect fluid it simplifies to

$$T^\mu_\nu = g^{\mu\alpha}T_{\alpha\nu} = (\rho + p)u^\mu u_\nu - p\delta^\mu_\nu \tag{2.6.10}$$

as there exists some special velocity for which $q_\mu = \Sigma_{\mu\nu} = 0$ and then ρ and p are the proper energy density and pressure in the fluid rest frame, while u^μ is its 4-velocity.

Energy-momentum tensor can simplify even more if we choose a frame comoving with the fluid ($u^\mu = (1, 0, 0, 0)$)

$$T^\mu_\nu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}, \tag{2.6.11}$$

which also simplifies the Einstein equations called the Friedmann equations then

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2}, \tag{2.6.12}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho(t) + 3p(t)\right). \tag{2.6.13}$$

They can be combined resulting in the continuity equation of the form

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0 \tag{2.6.14}$$

or

$$\frac{d\ln\rho}{d\ln a} = -3(1 + w), \tag{2.6.15}$$

where w is the equation of state (or barotropic) parameter

$$w \equiv \frac{p}{\rho}. \tag{2.6.16}$$

Continuity equation can be integrated into

$$\rho \propto a^{-3(1+w)}, \tag{2.6.17}$$

which combined with the Friedmann equation determines the evolution of the universe

$$a(t) \propto \begin{cases} t^{2/3(1+w)} & w \neq -1 \\ e^{Ht} & w = -1 \end{cases}. \tag{2.6.18}$$

Solutions for different values of w are presented in the Table 2.2.

[7] We use the notation $t_{\langle\mu\nu\rangle} = \gamma^\alpha_{(\mu}\gamma^\beta_{\nu)}t_{\alpha\beta} - \frac{1}{3}\gamma^{\alpha\beta}t_{\alpha\beta}\gamma_{\mu\nu}$ and $t_{(\mu\nu)} = \frac{1}{2}(t_{\mu\nu} + t_{\nu\mu})$.

	w	$\rho(a)$	$a(t)$	$a(\tau)$	τ_i
MD	0	a^{-3}	$t^{2/3}$	τ^2	0
RD	$\frac{1}{3}$	a^{-4}	$t^{1/2}$	τ	0
ΛD	-1	a^0	e^{Ht}	$-\tau^{-1}$	$-\infty$

Table 2.2: FRW solutions for a flat universe dominated by relativistic matter (MD), radiation (RD) or a cosmological constant (ΛD).

Side note: Conformal time

Conformal time η, which simplifies the study of the null geodesics, is connected with the cosmic time t through the relation:

$$a(\eta)d\eta = dt. \tag{2.6.19}$$

For conformal time for flat case ($k = 0$) the Friedmann and continuity equations change their form to:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2(\eta) = \left(\frac{a'(\eta)}{a^2(\eta)}\right)^2 = \mathcal{H}^2(\eta)\frac{1}{a^2} = \frac{8\pi G}{3}\rho(\eta), \tag{2.6.20}$$

$$\mathcal{H}'(\eta) + \mathcal{H}^2(\eta) = \frac{a''(\eta)}{a(\eta)} = \frac{4\pi G}{3}(\rho(\eta) - 3p(\eta))a^2, \tag{2.6.21}$$

$$\rho'(\eta) = -3a(\eta)\mathcal{H}(\eta)(\rho(\eta) + p(\eta)), \tag{2.6.22}$$

where $\mathcal{H}(\eta) = \dfrac{a'}{a^2}$.

Pressure and energy density for homogeneous scalar field $\phi(t)$ are equal to

$$\rho(t) = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{2.6.23}$$

$$p(t) = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{2.6.24}$$

which means that these equations transform into

$$\mathcal{H}^2(\eta) = \frac{8\pi G}{3}\left(\frac{1}{2a^2}(\phi')^2 + V(\phi)\right)a^2, \tag{2.6.25}$$

$$\mathcal{H}'(\eta) + \mathcal{H}^2(\eta) = \frac{a''(\eta)}{a(\eta)} = \frac{4\pi G}{3}(4a^2V(\phi) - (\phi')^2), \tag{2.6.26}$$

$$\rho'(\eta) = -3a(\eta)\mathcal{H}3(\eta)(\rho(\eta) + p(\eta)). \tag{2.6.27}$$

From the continuity equation we can also derive the equation of motion for the field ϕ in terms of conformal time:

$$\phi'' + 2\frac{a'}{a}\phi' + a^2\frac{dV}{d\phi} = 0. \tag{2.6.28}$$

Chapter 3

Non-adiabatic particle production for massless background field

In a free quantum field theory the total number of particles is conserved, which is no longer true as soon as interactions, which induce dynamical production of particles, come into play. Another source of new states can be the coupling between the quantum theory and a classical source, which breaks the invariance under space and time translations and violates the energy conservation.

Following the method developed in [24, 25] we want to analyse non-perturbative production of particles in the time-dependent background, namely with the vev of the background field linear in time, in a more general way including backreaction and said quantum corrections. Our approach is based on the asymptotic approximation of the wave functions and analytical continuation of the time variable, which makes the calculation of the number density simpler and well-grounded in the theory of penetration through an inverted parabolic potential.

Our starting potential is the supersymmetric version of the potential considered in [24], which allows us to introduce the method including the influence of quantum corrections based on the Yang-Feldman formalism. Such a potential has been investigated thoroughly before, for instance analytically in the context of massless preheating in [26] and numerically with lattice calculations for the production during inflation in [27].

3.1 Method

The bases of the method describing the process of particle production in time-dependent backgrounds, generalizing the reasoning from [24, 25] to the case of a complex field and including backreaction, can be introduced based on the simple Lagrangian [24]:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \bar{\phi} + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \frac{1}{2} g^2 |\phi|^2 \chi^2, \tag{3.1.1}$$

where a complex scalar field $\phi = \phi_1 + i\phi_2$ interacts with a real scalar field χ via the coupling g. For simplicity we neglect expansion of the universe now, leaving it for the Section 3.6. The above Lagrangian is invariant under phase rotations $\phi \to \phi e^{i\theta}$

and thus the angular momentum is conserved - it is important for constraining the amplitude of ϕ's oscillations that are described further in this section.

In this model point $\phi = 0$ in the phase space is distinguished as the other field χ becomes massless there. It is called an enhanced symmetry point (ESP). We aim to consider time-dependent background and its influence on particle creation. It can be realized in this framework keeping $\langle\chi\rangle = 0$ and $\langle\phi\rangle = vt + i\mu$ asymptotically, when quantum effects are negligible - it is a pure classical solution. Both parameters, v and μ, are real and they represent a velocity in the ϕ-space and an impact parameter, respectively, see Figure 3.1. In consequence, χ acquires a time-dependent mass $m_\chi^2(t) = g^2|\phi(t)|^2$, which leads to its production.

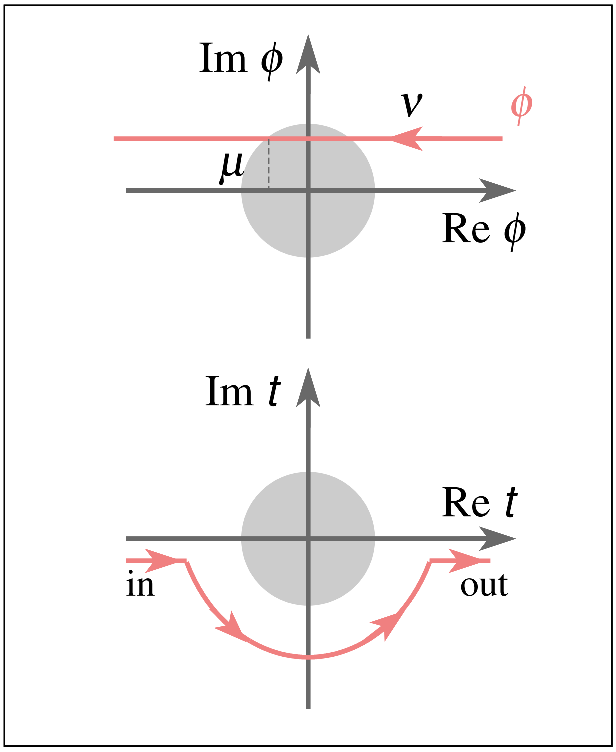

Figure 3.1: Illustration of the parameters v and μ in the phase space (*top*) and contour of integration for a complex t (*bottom*).

At first we investigate the production of χ particles without the effects of back-reaction on the trajectory of ϕ field. Particular modes of χ field with time-varying frequency $\omega(t) = \sqrt{k^2 + g^2|\phi(t)|^2}$ and momentum fixed by

$$\frac{k^2 + g^2\mu^2}{gv} \lesssim 1 \tag{3.1.2}$$

become excited, when they enter the non-adiabatic region $|\phi| \lesssim \Delta\phi$ with

$$\Delta\phi = \sqrt{\frac{v}{g}}. \tag{3.1.3}$$

This condition is fulfilled only for a time

$$\Delta t \sim \frac{\sqrt{v/g}}{v} \sim (gv)^{-1/2}, \tag{3.1.4}$$

which agrees with the condition (3.1.2) due to the uncertainty principle - produced states carry the energy of the order $E \sim (\Delta t)^{-1}$. In this region mass of χ field becomes tiny, which makes it energetically favourable to transfer kinetic energy of ϕ to χ initiating χ's particle production.

Side note: Vacuum expectation value and Lorentz invariance

Assuming that only a scalar field carries a vacuum expectation value is encouraged by the fact that if anything else than a Lorentz scalar has a vev, Lorentz invariance is spontaneously broken.

If a fermion carries a vev: $\psi \to v + \psi$, its mass term

$$\mathcal{L} \supset m\psi\bar{\psi} \tag{3.1.5}$$

changes to

$$\mathcal{L} \supset m|v|^2 + \underline{mv\bar{\psi}} + \underline{mv\psi} + m\psi\bar{\psi} \tag{3.1.6}$$

with the 2nd and 3rd terms breaking Lorentz invariance.

If a gauge boson carries a vev: $A_\mu \to v + A_\mu$, its mass term

$$\mathcal{L} \supset m^2 A_\mu A^\mu \tag{3.1.7}$$

changes to

$$\mathcal{L} \supset \underline{m^2 v(A_\mu + A^\mu)} + m^2 v^2 + m^2 A_\mu A^\mu \tag{3.1.8}$$

with the 1st term breaking Lorentz invariance.

First, we have to find the equation of motion for the modes χ_k that couples the classical evolution of ϕ with the time-dependent quantum field χ, which in this case reads

$$\left(\partial_t^2 + k^2 + g^2|\phi(t)|^2\right) u_k = 0. \tag{3.1.9}$$

For each k there are two linearly-independent solution to this equation: u_k^{in} with vacuum state with no particles in the far past and u_k^{out} with no particles in the far future, connected with each other through a Bogoliubov transformation. We want to start with the state u_k^{in} and see what is the number density of particles in the far future, which is given by $n_k = |\beta_k|^2$ for the k^{th} mode. We can choose modes u_k^{in} of the quite simple form in the far past using the WKB approximation

$$u_k^{in} \to \frac{1}{\sqrt{2\sqrt{k^2 + g^2|\phi|^2}}} e^{-i\int^t \sqrt{k^2+g^2|\phi(t')|^2}dt'} \quad \text{as} \quad t \to -\infty, \tag{3.1.10}$$

which defines our adiabatic vacuum $|\text{in}\rangle$ with no particles in the far past.

We can find the solution for u_k solving equation (3.1.9) respecting the boundary conditions in terms of hypergeometric functions but there exist a more general method using some physical arguments. We can treat (3.1.9) as a one dimensional Schrödinger equation for particle scattering/penetration through an inverted parabolic potential. A wave sent from the far right of the potential partially penetrates to the far left with an asymptotic amplitude $T_k\psi_k^{out}$ and is partially reflected back to the right with an asymptotic amplitude $R_k\psi_k^{in*}$. Parameters T_k and R_k are the transmission and reflection amplitudes, respectively. These two sets of modes are linked by the relation $u_k^{in}(t \to -\infty) = T_k^*\psi_k^{out*}$ and the Bogoliubov coefficient β_k can be expressed in terms of these amplitudes as

$$\beta_k = \frac{R_k^*}{T_k^*}. \tag{3.1.11}$$

Using WKB method we can find R and T introducing a simple trick - analytical continuation for t. When we move along the real time coordinate the solution (3.1.10) is spoiled for small t as we enter the non-adiabatic region. But if we assume t to be complex we can move from $t = -\infty$ to $t = +\infty$ along a complex contour in such a way that the WKB approximation is not destroyed whatsoever, see Figure 3.1. Integral $\int^t dt'$ becomes then a contour integral along a semicircle of large radius in the lower complex t plane.

For large values of $|t|$ we can expand the integral in (3.1.10) to obtain

$$\sqrt{k^2 + g^2|\phi|^2} \sim gvt + \frac{k^2 + g^2\mu^2}{2gvt}, \tag{3.1.12}$$

which induces a factor

$$\left(e^{-i\pi}\right)^{-i(k^2+g^2\mu^2)/2gv-1/2} = ie^{-\pi(k^2+g^2\mu^2)/2gv} \tag{3.1.13}$$

as we go around half of the circle in the phase space. This factor corresponds precisely to the ratio R^* and T^* and thus

$$n_k = |\beta_k|^2 = e^{-\pi(k^2+g^2\mu^2)/gv}. \tag{3.1.14}$$

This result is nonperturbative in coupling g since there is a factor g in the denominator of the exponent not g^2.

Interaction terms in our potential generate radiative corrections to the effective potential, which results in the Coleman-Weinberg effective potential and three UV-divergent terms:

$$V_{eff}(\phi) = \Lambda_{\text{eff}} + g^2 m_{\text{eff}}^2\phi^2 + g^4\lambda_{\text{eff}}\phi^4, \tag{3.1.15}$$

which can be eliminated by the proper counterterms[1]. To obtain an appropriate picture of particle production at the considered order of these corrections we remove

[1] In the supersymmetric models these divergences do not exist.

the whole Coleman-Weinberg effective potential for ϕ induced by loops of χ particles by hand. It means substituting χ mass-squared with $g^2|\phi(t)|^2$ to delete loop contributions from χ in $\langle\chi^2\rangle$, which reads

$$\langle\chi^2(t)\rangle \equiv \langle \text{in}|\chi^2(t)|\text{in}\rangle = \int \frac{d^3k}{(2\pi)^3}|u_k^{in}(t)|^2 \tag{3.1.16}$$

at time t. Thus, subtracted correction δ_M, which should be included in the classical equation of motion for ϕ

$$\left(\partial^2 + g^2(\langle\chi^2\rangle - \delta_M)\right)\phi = 0, \tag{3.1.17}$$

is of the form

$$\delta_M \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\sqrt{k^2+g^2|\phi(t)|^2}}. \tag{3.1.18}$$

It can be now explicitly seen that large values of μ correspond to negligible $(\langle\chi^2\rangle - \delta_M)$ term and the trajectory of the background cannot be affected by the presence of the non-adiabatic region, therefore avoiding particle creation completely then.

All above equations of motion can be translated into the energy transfer between the two systems: χ and ϕ. Energy conservation in this time-dependent scenario reads

$$\frac{d}{dt}H_\phi = -\frac{d}{dt}\langle \text{in}|H_\chi|\text{in}\rangle, \tag{3.1.19}$$

where the left hand side describes the classical energy of the rolling $\phi(t)$ field, while the right hand side concerns the vacuum expectation value of the time-dependent χ Hamiltonian.

Once χ particles are produced and trajectory of $\langle\phi\rangle$ leaves the non-adiabatic region, we can observe the effects of backreaction of the created states on the evolution of the background. In general, description of the backreaction is very complicated but in this approach a simple one is accessible. The crucial fact is that we can treat the initial production of χ states as instant, since it takes place in the vicinity of $\phi = 0$, and that the induced potential $V \sim |\phi|$ is linear making the evolution of ϕ under its impact quite simple.

Newly produced χ particles are nonrelativistic since $k \lesssim \sqrt{gv} \ll g|\phi|$ and their total density is given by

$$\rho_\chi(t) = \int \frac{d^3k}{(2\pi)^3} n_k \sqrt{k^2 + g^2|\phi(t)|^2} \approx g|\phi(t)|n_\chi, \tag{3.1.20}$$

where n_χ is defined by (3.1.14). It looks like a new linear potential inducing an attractive force acting in the field space on the trajectory of $\langle\phi\rangle$.

As ϕ evolves in the phase space some of its energy is inherited by χ field, resulting in some new excitations. When ϕ moves away from the point $\phi = 0$, the mass of excited χ states grows increasing the energy cumulated in this sector and backreacting on the evolution of the background via the potential (3.1.20). Induced attractive force slows it down and at some point ϕ_*, where the initial kinetic energy density $\frac{1}{2}\dot{\phi}^2 \equiv \frac{1}{2}v^2$ is comparable to the energy density ρ_χ contained in χ sector, turns it back towards the non-adiabatic region inducing another production of χ particles. We can estimate this bending point as[2]

$$\phi_* = \frac{4\pi^3}{g^{5/2}} v^{1/2} e^{\pi g \mu^2/v}. \tag{3.1.21}$$

Then again linear potential is established, which is steeper this time as it corresponds to the stronger attractive force, and the process repeats itself again and again until it is energetically favourable, see Figure 3.2. The oscillating nature of the evolution of the background is sustained with the trajectory agreeing with the effective potential and the angular momentum conservation. In some cases, for example when taking into account the evolution of the scale factor, the amplitude of these oscillation may significantly drop, see Section 3.6.

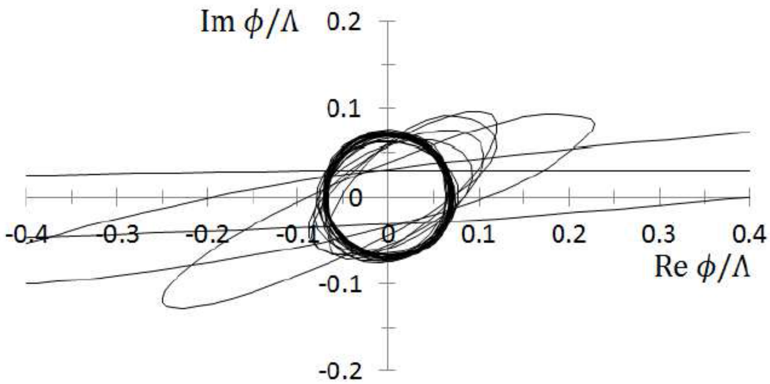

Figure 3.2: Oscillating nature of the background. Figure from [25].

Apart from that we must take under control all other quantum effects to build a full picture of particle production process here. These are loop corrections to the effective action including kinetic corrections, which can be expressed as a series expanded in v^2/ϕ^4, and Coleman-Weinberg potential energy, which is removed by hand, but also interaction effects, which are the subject of the next section, all of them being important in the regime of dominating kinetic energy. The first two effects are governed by the parameters, nonadiabaticity and kinetic factor, diverging near the origin. Nevertheless, for a weak coupling the nonadiabaticity parameter gets enhanced in comparison to the kinetic corrections, $v^2/g^2\phi^4 \gg v^2/\phi^4$, and it is justified to pay attention only to the first effect and accompanying particle production. In particular, sufficiently large impact parameter μ prevents kinetic correction from domination at all.

[2]We can notice that for a very weak coupling g the length of the first pass through the non-adiabatic region is much greater than the impact parameter μ, which in consequence makes the evolution of the background one-dimensional after the first pass.

So far in our analysis we also neglected the impact of scattering and decay of the χ particles, which is the subject of Section 3.4.4.

Note that this method is much more general than only applying to the precise $\phi = i\mu + vt$ evolution of the background. We can also use it each time the trajectory of the background can be approximately linear near the origin of the phase space, no matter what is its behaviour away from the origin.

3.2 Production of fermions

In the case of fermionic production we have to differentiate between various representations of the fermionic field. Again, we are interested in the theories with time-dependent backgrounds, which drive particle production.

3.2.1 Weyl fermions

The simplest Lagrangian for the Weyl representation of the fermionic field reads:

$$\mathcal{L} = \xi^\dagger i\bar{\partial\!\!\!/}\xi + \eta^\dagger i\bar{\partial\!\!\!/}\eta - m\left(\xi\eta + \xi^\dagger\eta^\dagger\right), \tag{3.2.1}$$

where $m = m(t) \in \mathbb{R}$. Equations of motions for these four fields are of the form:

$$0 = i\bar{\partial\!\!\!/}^{\dot{\alpha}\alpha}\xi_\alpha - m\eta^{\dagger\dot{\alpha}}, \tag{3.2.2}$$
$$0 = i\partial\!\!\!/_{\alpha\dot{\alpha}}\xi^{\dagger\dot{\alpha}} - m\eta_\alpha, \tag{3.2.3}$$
$$0 = i\bar{\partial\!\!\!/}^{\dot{\alpha}\alpha}\eta_\alpha - m\xi^{\dagger\dot{\alpha}}, \tag{3.2.4}$$
$$0 = i\partial\!\!\!/_{\alpha\dot{\alpha}}\eta^{\dagger\dot{\alpha}} - m\xi_\alpha, \tag{3.2.5}$$

which can be translated into two second order differential equations

$$0 = \left(\partial^2 + m^2\right)\xi_\alpha + i\dot{m}\sigma^0_{\alpha\dot{\alpha}}\eta^{\dagger\dot{\alpha}}, \tag{3.2.6}$$
$$0 = \left(\partial^2 + m^2\right)\eta^{\dagger\dot{\alpha}} + i\dot{m}\bar{\sigma}^{0\dot{\alpha}\alpha}\xi_\alpha \tag{3.2.7}$$

and then diagonalized to

$$0 = \left(\partial^2 + m^2 \pm i\dot{m}\right)\psi_{\pm\alpha}, \tag{3.2.8}$$

introducing the notation $\psi_{\pm\alpha} \equiv \xi_\alpha \pm \sigma^0_{\alpha\dot{\alpha}}\eta^{\dagger\dot{\alpha}}$. It can be decomposed into mode functions

$$\psi_\pm(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}\psi_{\mathbf{k}\pm}(t) \tag{3.2.9}$$

with the equation of motion of the form

$$0 = \ddot{\psi}_{\mathbf{k}\pm} + \left(\omega_k^2 \pm i\dot{m}\right)\psi_{\mathbf{k}\pm}, \tag{3.2.10}$$

where $\omega_k \equiv \sqrt{\mathbf{k}^2 + m^2}$. In the adiabatic region, $\left|\dot{\omega}_k/2\omega_k^2\right| \ll 1$, it possesses two independent solutions

$$\psi_{\mathbf{k}\pm} \sim \sqrt{1 \pm \frac{m}{\omega_k}}e^{-i\int^t dt'\omega_k(t')}, \sqrt{1 \mp \frac{m}{\omega_k}}e^{+i\int^t dt'\omega_k(t')}, \tag{3.2.11}$$

which for our choice of the eigenspinor of the helicity operators turns into

$$\psi_{\pm\alpha} \sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^s_{\mathbf{k}\alpha} \times \left(\alpha^s_{\mathbf{k}\pm} \sqrt{1 \pm \frac{m}{\omega_k}} e^{-i\int^t dt'\omega_k(t')} + \beta^s_{\mathbf{k}\pm} \sqrt{1 \mp \frac{m}{\omega_k}} e^{+i\int^t dt'\omega_k(t')} \right). \tag{3.2.12}$$

where $\alpha^s_{\mathbf{k}\pm}$ and $\beta^s_{\mathbf{k}\pm}$ are some constants. Combining (3.2.2) and (3.2.5) we can represent one set of these coefficients with the other as:

$$\alpha^s_{\mathbf{k}-} = +s\alpha^s_{\mathbf{k}+}, \tag{3.2.13}$$

$$\beta^s_{\mathbf{k}-} = -s\beta^s_{\mathbf{k}+}. \tag{3.2.14}$$

Side note: Helicity operator, Weyl fermions

Eigenspinor of the helicity operator $e^s_{\mathbf{k}}$ satisfies the relation

$$\frac{k^i}{|\mathbf{k}|} \left(\sigma^0 \bar{\sigma}^i\right)_\alpha^{\ \beta} e^s_{\mathbf{k}\beta} = s e^s_{\mathbf{k}\alpha} \tag{3.2.15}$$

with $s = \pm$.

We choose the following representation of this spinor

$$e^s_{\mathbf{k}\alpha} = e^{i\theta^s_{\mathbf{k}}} \times \sqrt{1 + \frac{sk^3}{|\mathbf{k}|}} \begin{pmatrix} \frac{-k^1+ik^2}{s|\mathbf{k}|+k^3} \\ 1 \end{pmatrix} = e^{i\theta^s_{\mathbf{k}}} \times \begin{pmatrix} s e^{i\alpha_{\mathbf{k}}} \sqrt{1 - sk^3/|\mathbf{k}|} \\ \sqrt{1 + sk^3/|\mathbf{k}|} \end{pmatrix}, \tag{3.2.16}$$

where $\theta^s_{\mathbf{k}}$ is some indefinite phase and

$$e^{i\alpha_{\mathbf{k}}} \equiv \frac{-k^1 + ik^2}{\sqrt{(k^1)^2 + (k^2)^2}}. \tag{3.2.17}$$

Some useful properties of the helicity operator:

$$e^{s\dagger}_{\mathbf{k}} \bar{\sigma}^0 e^r_{\mathbf{k}} = e^r_{\mathbf{k}} \sigma^0 e^{s\dagger}_{\mathbf{k}} = 2\delta^{sr}, \tag{3.2.18}$$

$$\epsilon^{\alpha\beta} e^s_{\mathbf{k}\beta} = e^{i\left(\theta^s_{\mathbf{k}} + \theta^s_{-\mathbf{k}}\right)} \times s e^{i\alpha_{-\mathbf{k}}} e^{s\dagger}_{-\mathbf{k}\dot{\alpha}} \bar{\sigma}^{0\dot{\alpha}\alpha}, \tag{3.2.19}$$

$$e^s_{-\mathbf{k}} e^r_{\mathbf{k}} = e^{i\left(\theta^s_{-\mathbf{k}} + \theta^r_{\mathbf{k}}\right)} \times 2s e^{i\alpha_{\mathbf{k}}} \delta^{sr}, \tag{3.2.20}$$

$$e^s_{\mathbf{k}\alpha} e^{s\dagger}_{\mathbf{k}\dot{\alpha}} = \left(\sigma^0 - \frac{s\sigma^i k^i}{|\mathbf{k}|}\right)_{\alpha\dot{\alpha}}, \quad e^{s\dagger\dot{\alpha}}_{\mathbf{k}} e^{s\alpha}_{\mathbf{k}} = \left(\bar{\sigma}^0 - \frac{s\bar{\sigma}^i k^i}{|\mathbf{k}|}\right)^{\dot{\alpha}\alpha}. \tag{3.2.21}$$

In the last line there is no summation over s.

Finally, wave functions of ξ and η are of the form:

$$\xi_\alpha = \frac{1}{2}\left(\psi_{+\alpha} + \psi_{-\alpha}\right) \tag{3.2.22}$$
$$\sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^s_{\mathbf{k}\alpha} \times \left[\alpha^s_{\mathbf{k}+} u^s_{k+} e^{-i\int^t dt'\omega_k(t')} + \beta^s_{\mathbf{k}+} u^s_{k-} e^{+i\int^t dt'\omega_k(t')}\right],$$
$$\eta^{\dagger\dot\alpha} = \frac{1}{2}\bar\sigma^{0\dot\alpha\alpha}\left(\psi_{+\alpha} - \psi_{-\alpha}\right) \tag{3.2.23}$$
$$\sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s \bar\sigma^{0\dot\alpha\alpha} e^s_{\mathbf{k}\alpha} \left[\alpha^s_{\mathbf{k}+} u^{-s}_{k+} e^{-i\int^t dt'\omega_k(t')} + \beta^s_{\mathbf{k}+} u^{-s}_{k-} e^{+i\int^t dt'\omega_k(t')}\right],$$

where

$$u^s_{k\pm} \equiv \frac{1}{2}\left[\sqrt{1 \pm \frac{m}{\omega_k}} \pm s\sqrt{1 \mp \frac{m}{\omega_k}}\right]. \tag{3.2.24}$$

Quantization can be performed by replacing

$$\alpha^s_{\mathbf{k}+} \longrightarrow \frac{1}{\sqrt{2}} a^s_{\mathbf{k}}, \tag{3.2.25}$$
$$\beta^s_{\mathbf{k}+} \longrightarrow \frac{1}{\sqrt{2}} b^s_{-\mathbf{k}}, \tag{3.2.26}$$

which comes from the canonical anti-commutation relation.

3.2.2 Dirac fermions

The simplest Lagrangian for the Dirac representation of the fermionic field reads

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi, \tag{3.2.27}$$

where again we assume $m = m(t)$. Corresponding equation of motion is of the form

$$0 = (i\gamma^\mu\partial_\mu - m)\psi, \tag{3.2.28}$$

which for the mode functions, $\psi(t,\mathbf{x}) \sim \psi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$, becomes

$$0 = i\gamma^0\dot\psi_{\mathbf{k}} - (\mathbf{k}\cdot\gamma + m)\psi_{\mathbf{k}}. \tag{3.2.29}$$

For Dirac representation γ matrices are equal to

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{3.2.30}$$

while mode functions create a doublet $\psi_{\mathbf{k}} = \begin{pmatrix} \psi_{\mathbf{k}\uparrow} \\ \psi_{\mathbf{k}\downarrow} \end{pmatrix}$, which put together leads to the following equations of motion:

$$0 = i\dot\psi_{\mathbf{k}\uparrow} - \mathbf{k}\cdot\sigma\psi_{\mathbf{k}\downarrow} - m\psi_{\mathbf{k}\uparrow}, \tag{3.2.31}$$
$$0 = i\dot\psi_{\mathbf{k}\downarrow} - \mathbf{k}\cdot\sigma\psi_{\mathbf{k}\uparrow} + m\psi_{\mathbf{k}\downarrow}. \tag{3.2.32}$$

Note that also $\frac{\mathbf{k}\cdot\sigma}{k}\psi_{\mathbf{k}\uparrow(\downarrow)}$ satisfies the same set of the equations of motion. Thus, we can choose the basis of the wave functions $\psi_{\mathbf{k}\uparrow(\downarrow)}$ as the eigenstates of the helicity operator denoted as $\psi^s_{\mathbf{k}\uparrow(\downarrow)}$ $(s=\pm)$ satisfying

$$\frac{\mathbf{k}\cdot\sigma}{k}\psi^s_{\mathbf{k}\uparrow(\downarrow)} = s\psi^s_{\mathbf{k}\uparrow(\downarrow)}. \tag{3.2.33}$$

Second order equations of motion for them read

$$0 = \ddot{\psi}^s_{\mathbf{k}\uparrow} + \left(\omega_k^2 + i\dot{m}\right)\psi^s_{\mathbf{k}\uparrow}, \tag{3.2.34}$$

$$0 = \ddot{\psi}^s_{\mathbf{k}\downarrow} + \left(\omega_k^2 - i\dot{m}\right)\psi^s_{\mathbf{k}\downarrow}, \tag{3.2.35}$$

where $\omega_k \equiv \sqrt{\mathbf{k}^2+m^2}$ denotes the frequency.

WKB solutions for this set of equations read

$$\psi^s_{\mathbf{k}\uparrow} = \alpha^s_{\mathbf{k}}\xi^s_{\mathbf{k}}\sqrt{1+\frac{m}{\omega_k}}e^{-i\int^{t'}dt'\omega_k(t')} + \beta^s_{\mathbf{k}}\eta^s_{\mathbf{k}}\sqrt{1-\frac{m}{\omega_k}}e^{+i\int^{t'}dt'\omega_k(t')}, \tag{3.2.36}$$

$$\psi^s_{\mathbf{k}\downarrow} = s\alpha^s_{\mathbf{k}}\xi^s_{\mathbf{k}}\sqrt{1-\frac{m}{\omega_k}}e^{-i\int^{t'}dt'\omega_k(t')} - s\beta^s_{\mathbf{k}}\eta^s_{\mathbf{k}}\sqrt{1+\frac{m}{\omega_k}}e^{+i\int^{t'}dt'\omega_k(t')}, \tag{3.2.37}$$

where $\alpha^s_{\mathbf{k}}$ and $\beta^s_{\mathbf{k}}$ correspond to the Bogoliubov coefficients with the normalization condition $|\alpha^s_{\mathbf{k}}| + |\beta^s_{\mathbf{k}}| = 1$, while $\xi^s_{\mathbf{k}}$ and $\eta^s_{\mathbf{k}}$ are the normalized eigenspinors for the helicity operator.

Side note: Helicity operator, Dirac fermions

Eigenspinor $\xi^s_{\mathbf{k}}$ for the helicity operator $\frac{\mathbf{k}\cdot\sigma}{k}$ with the momentum $\mathbf{k} = \{k^1, k^2, k^3\}$ is given by

$$\xi^s_{\mathbf{k}} = \frac{1}{2}e^{i\theta^s_{\mathbf{k}}}\begin{pmatrix} \frac{sk^-}{\sqrt{|\mathbf{k}|(|\mathbf{k}|-sk^3)}} \\ \sqrt{\frac{|\mathbf{k}|-sk^3}{|\mathbf{k}|}} \end{pmatrix}, \tag{3.2.38}$$

where $\theta^s_{\mathbf{k}} \in \mathbb{R}$, $k^\pm \equiv k^1 \pm ik^2$ and $s=\pm$ corresponds to the eigenvalue. It is orthogonal in a sense that

$$\xi^{r\dagger}_{\mathbf{k}}\xi^s_{\mathbf{k}} = \frac{1}{2}\delta^{rs}. \tag{3.2.39}$$

$\mathcal{C}$-transformed state

$$\eta^s_{\mathbf{k}} \equiv -s\epsilon\xi^{s*}_{-\mathbf{k}} \tag{3.2.40}$$

possesses the same properties and its explicit form reads

$$\eta^s_{\mathbf{k}} = -s\epsilon\xi^{s*}_{-\mathbf{k}} = -\frac{1}{2}e^{-i\theta^s_{-\mathbf{k}}}\frac{k^+}{\sqrt{\mathbf{k}^2-(k^3)^2}}\begin{pmatrix} \frac{sk^-}{\sqrt{|\mathbf{k}|(|\mathbf{k}|-sk^3)}} \\ \sqrt{\frac{|\mathbf{k}|-sk^3}{|\mathbf{k}|}} \end{pmatrix}. \tag{3.2.41}$$

The difference between $\xi^s_{\mathbf{k}}$ and $\eta^s_{\mathbf{k}}$ is just a phase because $|k^+|^2 = \mathbf{k}^2-(k^3)^2$.

Finally, we can obtain the WKB solution for $\psi^s_{\mathbf{k}}$ as

$$\psi^s_{\mathbf{k}} = \alpha^s_{\mathbf{k}} \begin{pmatrix} \xi^s_{\mathbf{k}}\sqrt{1+\frac{m}{\omega_k}} \\ s\xi^s_{\mathbf{k}}\sqrt{1-\frac{m}{\omega_k}} \end{pmatrix} e^{-i\int^{t'} dt'\omega_k(t')} + \beta^s_{\mathbf{k}} \begin{pmatrix} \eta^s_{\mathbf{k}}\sqrt{1-\frac{m}{\omega_k}} \\ -s\eta^s_{\mathbf{k}}\sqrt{1+\frac{m}{\omega_k}} \end{pmatrix} e^{+i\int^{t'} dt'\omega_k(t')}. \tag{3.2.42}$$

As in the bosonic case we want to extend this solution to the complex values of t assuming that mass is linear in time

$$m(t) = gvt. \tag{3.2.43}$$

Even for complex t WKB solution is valid for large $|t|$ and then

$$\omega_k \sim gvt. \tag{3.2.44}$$

It means that for the far past the positive frequency is equal to $-\omega_{\mathbf{k}}(t)$, while for the far future $+\omega_{\mathbf{k}}(t)$ and therefore the initial and final WKB solutions read[3]

$$u^s_{\mathbf{k}in} \sim \begin{pmatrix} \xi^s_{\mathbf{k}}\sqrt{1-\frac{m}{\omega_k}} \\ -s\xi^s_{\mathbf{k}}\sqrt{1+\frac{m}{\omega_k}} \end{pmatrix} e^{+i\int^{t} dt'\omega_k(t')}, \tag{3.2.45}$$

$$u^s_{\mathbf{k}out} \sim \alpha^s_{\mathbf{k}} \begin{pmatrix} \xi^s_{\mathbf{k}}\sqrt{1+\frac{m}{\omega_k}} \\ s\xi^s_{\mathbf{k}}\sqrt{1-\frac{m}{\omega_k}} \end{pmatrix} e^{-i\int^{t} dt'\omega_k(t')} + \beta^s_{\mathbf{k}} \begin{pmatrix} \eta^s_{\mathbf{k}}\sqrt{1-\frac{m}{\omega_k}} \\ -s\eta^s_{\mathbf{k}}\sqrt{1+\frac{m}{\omega_k}} \end{pmatrix} e^{+i\int^{t} dt'\omega_k(t')}. \tag{3.2.46}$$

The asymptotic form of the integral appearing in the exponential functions above reads

$$\int^t dt'\omega(t') = \int^t dt' gvt' \left(1 + \frac{k^2}{2g^2v^2t^2} + \dots\right) = \tag{3.2.47}$$

$$= \frac{1}{2}gvt^2 + \frac{k^2}{2gv}\left(\int^{t|_{\theta=-\pi}} dt'\frac{1}{t'} + \int_{\theta=-\pi}^{\theta} dt'\frac{1}{t'}\right) + \dots = \tag{3.2.48}$$

$$= \tfrac{1}{2}gv|t|^2 e^{2i\theta} + \tfrac{k^2}{2gv}\left(\ln|t| + i(\theta+\pi)\right) + \dots, \tag{3.2.49}$$

where we assume $t = |t|e^{i\theta}$ and $\theta \to -\pi$ corresponds to the in-state frequency ($t \to -\infty$). Therefore, we can express $u^s_{\mathbf{k}in}$ on the complex t-plane as

$$u^s_{\mathbf{k}in} \sim \begin{pmatrix} \xi^s_{\mathbf{k}}\sqrt{1-\frac{m(t)}{\omega_k(t)}} \\ -s\xi^s_{\mathbf{k}}\sqrt{1+\frac{m(t)}{\omega_k(t)}} \end{pmatrix} e^{+\frac{1}{2}gv|t|^2(i\cos 2\theta - \sin 2\theta) + \frac{k^2}{2gv}(i\ln|t| - \theta - \pi) + \dots} \tag{3.2.50}$$

and going to $\theta \to 0$ through the lower half plane we obtain

$$u^s_{\mathbf{k}in}\big|_{\theta\to 0} \sim \begin{pmatrix} \xi^s_{\mathbf{k}}\sqrt{1-\frac{m(|t|)}{\omega_k(|t|)}} \\ -s\xi^s_{\mathbf{k}}\sqrt{1+\frac{m(|t|)}{\omega_k(|t|)}} \end{pmatrix} e^{+i\frac{1}{2}gv|t|^2 + \frac{k^2}{2gv}(i\ln|t| - \pi) + \dots}. \tag{3.2.51}$$

[3]We choose $\xi^s_{\mathbf{k}}$ not $\eta^s_{\mathbf{k}}$ for the helicity basis but the difference between them is just a phase, see (3.2.38) and (3.2.41).

This expression should be consistent with the second term of (3.2.46), which means that

$$\beta^s_{\mathbf{k}} = \text{phase} \times e^{-\pi \frac{k^2}{2gv}} \tag{3.2.52}$$

and finally number density of produced fermions is proportional to

$$|\beta^s_{\mathbf{k}}|^2 = e^{-\pi \frac{k^2}{gv}}. \tag{3.2.53}$$

3.2.3 Majorana fermions

For the Majorana representation of fermions the simplest Lagrangian is of the form

$$\mathcal{L} = \xi^\dagger i\bar{\partial\!\!\!/}\xi - \frac{1}{2}m\xi\xi - \frac{1}{2}m^*\xi^\dagger\xi^\dagger, \tag{3.2.54}$$

where we assume $m = m(t) \in \mathbb{C}$ and $\dot{m} = \dot{m}^*$, with the equations of motion that read

$$0 = i\bar{\partial\!\!\!/}^{\dot{\alpha}\alpha}\xi_\alpha - m^*\xi^{\dagger\dot{\alpha}}, \tag{3.2.55}$$

$$0 = i\partial\!\!\!/_{\alpha\dot{\alpha}}\xi^{\dagger\dot{\alpha}} - m\xi_\alpha. \tag{3.2.56}$$

As in the Dirac case we can obtain second order differential equations

$$0 = \left(\partial^2 + |m|^2\right)\xi_\alpha + i\dot{m}^*\sigma^0_{\alpha\dot{\alpha}}\xi^{\dagger\dot{\alpha}}, \tag{3.2.57}$$

$$0 = \left(\partial^2 + |m|^2\right)\xi^{\dagger\dot{\alpha}} + i\dot{m}\bar{\sigma}^{0\dot{\alpha}\alpha}\xi_\alpha, \tag{3.2.58}$$

that can be diagonalized. It is difficult to obtain a general solution of these equations so in our analysis we consider two cases:

- Case A: $\mathrm{Arg}(m)$ is constant
  Diagonalized equation of motion is given by

$$0 = \left(\partial^2 + |m|^2 \pm i|m|^{\cdot}\right)\Xi^\theta_{\pm\alpha}, \tag{3.2.59}$$

  where $m = |m|e^{i\theta}$, $\theta \in \mathbb{R}$ and $\Xi^\theta_{\pm\alpha} \equiv \left(\xi \pm e^{-i\theta}\sigma^0\xi^\dagger\right)_\alpha$.

- Case B: $\dot{m}$ is constant
  Time-dependent mass can be written in the form: $m(t) = \dot{m}t + m_0$, where $m_0 = \text{const} \in \mathbb{C}$. Choosing the notation $\dot{m} = |\dot{m}|e^{i\rho}$ ($\rho \in \mathbb{R}$) it can be further rewritten as

$$m(t) = \left[|\dot{m}|\left(t + \frac{\mathrm{Re}(m_0 e^{-i\rho})}{|\dot{m}|}\right) + i\,\mathrm{Im}(m_0 e^{-i\rho})\right]e^{i\rho} \equiv \left[|\dot{m}|\,(t - t_0) + i\mu\right]e^{i\rho}, \tag{3.2.60}$$

  where $t_0, \mu \in \mathbb{R}$. Diagonalized equation of motion is given by

$$0 = \left(\partial^2 + |\dot{m}|^2(t - t_0)^2 + \mu^2 \pm i|\dot{m}|\right)\Xi^\rho_{\pm\alpha}. \tag{3.2.61}$$

Case A is similar to the theory of Dirac fermions with one difference: m should be replaced with $|m|$. Equation governing Ξ^θ_+ and Ξ^θ_-,

$$0 = \dot{\Xi}^\theta_+ + i|m|\Xi^\theta_+ + \sigma^0\bar{\sigma}^i\partial_i\Xi^\theta_-, \tag{3.2.62}$$

is also similar to the corresponding Dirac one (3.2.31), which provides with the solutions of the form

$$\Xi^\theta_{+\alpha} \sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^s_{\mathbf{k}\alpha} \left(\alpha^s_{\mathbf{k}} \sqrt{1+\frac{|m|}{\omega_k}} e^{-i\int^t dt'\omega_k(t')} + \beta^s_{\mathbf{k}} \sqrt{1-\frac{|m|}{\omega_k}} e^{+i\int^t dt'\omega_k(t')} \right), \tag{3.2.63}$$

$$\Xi^\theta_{-\alpha} \sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^s_{\mathbf{k}\alpha} \left(s\alpha^s_{\mathbf{k}} \sqrt{1-\frac{|m|}{\omega_k}} e^{-i\int^t dt'\omega_k(t')} - s\beta^s_{\mathbf{k}} \sqrt{1+\frac{|m|}{\omega_k}} e^{+i\int^t dt'\omega_k(t')} \right), \tag{3.2.64}$$

where $\alpha^s_{\mathbf{k}}$ and $\beta^s_{\mathbf{k}}$ are some constants. For ξ and $\xi^\dagger$ it means that

$$\xi_\alpha = \frac{1}{2}\left(\Xi^\theta_{+\alpha} + \Xi^\theta_{-\alpha}\right) \tag{3.2.65}$$

$$\sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^s_{\mathbf{k}\alpha} \left[\alpha^s_{\mathbf{k}} v^s_{k+} e^{-i\int^t dt'\omega_k(t')} + \beta^s_{\mathbf{k}} v^s_{k-} e^{+i\int^t dt'\omega_k(t')} \right],$$

$$\xi^{\dagger\dot{\alpha}} = \frac{1}{2} e^{+i\theta}\bar{\sigma}^{0\dot{\alpha}\alpha}\left(\Xi^\theta_{+\alpha} - \Xi^\theta_{-\alpha}\right) \tag{3.2.66}$$

$$\sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^{+i\theta}\bar{\sigma}^{0\dot{\alpha}\alpha} e^s_{\mathbf{k}\alpha} \left[\alpha^s_{\mathbf{k}} v^{-s}_{k+} e^{-i\int^t dt'\omega_k(t')} + \beta^s_{\mathbf{k}} v^{-s}_{k-} e^{+i\int^t dt'\omega_k(t')} \right],$$

where

$$v^s_{k\pm} \equiv \frac{1}{2}\left[\sqrt{1 \pm \frac{|m|}{\omega_k}} \pm s\sqrt{1 \mp \frac{|m|}{\omega_k}} \right]. \tag{3.2.67}$$

Considering the consistency between ξ and $\xi^\dagger$ we are provided with the relation linking $\alpha^s_{\mathbf{k}}$ and $\beta^s_{\mathbf{k}}$

$$\beta^s_{\mathbf{k}} = e^{-i\left(\theta^s_{\mathbf{k}} + \theta^s_{-\mathbf{k}}\right)} \times e^{-i\alpha_{\mathbf{k}}} e^{-i\theta} \cdot \alpha^{s*}_{-\mathbf{k}}, \tag{3.2.68}$$

where both phases are coming from the definition of the helicity operator. Quantization can be again performed by substituting $\alpha^s_{\mathbf{k}}$ with $\frac{1}{\sqrt{2}} a^s_{\mathbf{k}}$ again.

Case B demands WKB solution for Ξ^ρ_+ of the form

$$\Xi^\rho_{+\alpha} \sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^s_{\mathbf{k}\alpha} \times \tag{3.2.69}$$

$$\times \left(\alpha^s_{\mathbf{k}} \sqrt{1 + \frac{|\dot{m}|(t-t_0)}{\omega_k}} e^{-i\int^t dt'\omega_k(t')} + \beta^s_{\mathbf{k}} \sqrt{1 - \frac{|\dot{m}|(t-t_0)}{\omega_k}} e^{+i\int^t dt'\omega_k(t')} \right),$$

where again $\alpha^s_{\mathbf{k}}$ and $\beta^s_{\mathbf{k}}$ are some coefficients. Relation between Ξ^ρ_+ an Ξ^ρ_-,

$$0 = \dot{\Xi}^\rho_+ + i|\dot{m}|(t-t_0)\Xi^\rho_+ + \sigma^0\bar{\sigma}^i\partial_i\Xi^\rho_- - \mu\Xi^\rho_-, \tag{3.2.70}$$

coming from (3.2.55) and (3.2.56) determines the WKB solution for Ξ_-

$$\Xi^{\rho}_{-\alpha} \sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^s_{\mathbf{k}\alpha} \frac{s|\mathbf{k}| - i\mu}{\sqrt{\mathbf{k}^2+\mu^2}} \times \tag{3.2.71}$$
$$\times \left(\alpha^s_{\mathbf{k}} \sqrt{1 - \frac{|\dot{m}|\,(t-t_0)}{\omega_k}} e^{-i\int^t dt'\omega_k(t')} - \beta^s_{\mathbf{k}} \sqrt{1 + \frac{|\dot{m}|\,(t-t_0)}{\omega_k}} e^{+i\int^t dt'\omega_k(t')} \right).$$

For ξ and $\xi^\dagger$ it means that

$$\xi_\alpha = \frac{1}{2}\left(\Xi^\rho_{+\alpha} + \Xi^\rho_{-\alpha}\right) \tag{3.2.72}$$
$$\sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^s_{\mathbf{k}\alpha} \left[\alpha^s_{\mathbf{k}} w^s_{k+} e^{-i\int^t dt'\omega_k(t')} + s e^{-i\alpha_{\mathbf{k}}} e^{-i\rho} \cdot \alpha^{s*}_{-\mathbf{k}} w^s_{k-} e^{+i\int^t dt'\omega_k(t')} \right],$$
$$\xi^{\dagger\dot{\alpha}} = \frac{1}{2} e^{+i\theta} \bar{\sigma}^{0\dot{\alpha}\alpha} \left(\Xi^\rho_{+\alpha} - \Xi^\rho_{-\alpha}\right) \tag{3.2.73}$$
$$\sim \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_s e^{+i\theta} \bar{\sigma}^{0\dot{\alpha}\alpha} e^s_{\mathbf{k}\alpha} \left[\alpha^s_{\mathbf{k}} w^{-s*}_{k+} e^{-i\int^t dt'\omega_k(t')} - s e^{-i\alpha_{\mathbf{k}}} e^{-i\rho} \cdot \alpha^{s*}_{-\mathbf{k}} w^{-s*}_{k-} e^{+i\int^t dt'\omega_k(t')} \right],$$

where

$$w^s_{k\pm} \equiv \frac{1}{2} \left[\pm\sqrt{1 + \frac{|\dot{m}|(t-t_0)}{\omega_k}} + \frac{s|\mathbf{k}| \mp i\mu}{\sqrt{|\mathbf{k}|^2+\mu^2}} \sqrt{1 - \frac{|\dot{m}|(t-t_0)}{\omega_k}} \right]. \tag{3.2.74}$$

Considering the consistency between ξ and $\xi^\dagger$ we are once again provided with the relation linking $\alpha^s_{\mathbf{k}}$ and $\beta^s_{\mathbf{k}}$:

$$\beta^s_{\mathbf{k}} = e^{-i\left(\theta^s_{\mathbf{k}} + \theta^s_{-\mathbf{k}}\right)} \times e^{-i\alpha_{\mathbf{k}}} e^{-i\rho} \frac{|\mathbf{k}| + is\mu}{\sqrt{\mathbf{k}^2+\mu^2}} \cdot \alpha^{s*}_{-\mathbf{k}}, \tag{3.2.75}$$

where the indefinite phases $e^{-i\left(\theta^s_{\mathbf{k}} + \theta^s_{-\mathbf{k}}\right)}$ can be included in $\alpha_{\mathbf{k}}$ coefficient by redefinition.

Quantization can be performed by substituting $\alpha^s_{\mathbf{k}}$ with $\frac{1}{\sqrt{2}} a^s_{\mathbf{k}}$.

3.3 Influence of interactions

We can extend our method to include quantum corrections, what enables us to investigate the influence of the interactions on the particle production process.

To obtain the proper occupation number of the produced states we follow the Bogoliubov transformation method described in Section 2.3, which in the theories with large interaction terms becomes complicated since the out-state gets affected by all other fields through the interaction terms. Thus, operator $a^{\text{out}}_{\mathbf{k}}$ should differ then from the non-interacting case.

Side note: Production of gauge bosons

Method presented in this Chapter can be also applied to the production of gauge bosons easily. Starting from the Lagrangian [28]

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \left(D^{\mu}\phi\right)^{*} D_{\mu}\phi - \lambda\left(|\phi|^2 - v\right)^2 \tag{3.3.1}$$

with $D_\mu\phi = \partial_\mu + ieA_\mu\phi$, we can expand the kinetic term for the scalar field ϕ

$$\left(D^{\mu}\phi\right)^{*} D_{\mu}\phi = \partial^{\mu}\phi^{*}\partial_{\mu}\phi - ieA^{\mu}\phi^{*}\partial_{\mu}\phi + ieA_{\mu}\phi\partial^{\mu}\phi^{*} + e^2 A_{\mu}A^{\mu}|\phi|^2 \tag{3.3.2}$$

obtaining the mass of our gauge boson

$$m_A^2 = 2e^2|\phi|^2. \tag{3.3.3}$$

Its equation of motion is given by [17]

$$A_k'' + \omega_k^2 A_k = -\sigma_k a A_k' \approx 0 \tag{3.3.4}$$

with $\omega_k^2 = k^2 + 2e^2(v^2t^2 + \mu^2)$, when we assume linear time evolution of the background: $\langle\phi\rangle = vt + i\mu$.

Let us consider some general real scalar field Ψ satisfying the following commutation relation

$$\left[\Psi(t,\mathbf{x}), \dot{\Psi}(t,\mathbf{y})\right] = i\delta^3(\mathbf{x}-\mathbf{y}) \tag{3.3.5}$$

with the equation of motion that reads

$$0 = \left(\partial^2 + M^2(x)\right)\Psi(x) + J(x). \tag{3.3.6}$$

Here M denotes a mass of Ψ, which in general can be coordinate-dependent and we focus on its time-dependence, and J a source term, which is an operator consisting of the fields present in the model.

The solution to (3.3.6) is known as the Yang-Feldman equation [29]

$$\Psi(x) = \sqrt{Z}\Psi^{\mathrm{as}}(x) - iZ\int_{t^{\mathrm{as}}}^{x^0} dy^0 \int d^3y \left[\Psi^{\mathrm{as}}(x), \Psi^{\mathrm{as}}(y)\right] J(y) \tag{3.3.7}$$

with Ψ^{as} being the free asymptotic field fulfilling

$$0 = \left(\partial^2 + M^2\right)\Psi^{\mathrm{as}} \tag{3.3.8}$$

and Z the field renormalization constant. For the details of the derivation of the Yang-Feldman equation see Appendix A.

For $x^0 = t^{\mathrm{as}}$ (3.3.7) transforms into

$$\Psi(t^{\mathrm{as}},\mathbf{x}) = \sqrt{Z}\Psi^{\mathrm{as}}(t^{\mathrm{as}},\mathbf{x}), \tag{3.3.9}$$

so it corresponds to the time, when our interacting field Ψ behaves like a free field. Thus, we can relate the in-state defined at $t^{\text{as}} = -\infty \equiv t^{\text{in}}$ with the out-state defined at $t^{\text{as}} = +\infty \equiv t^{\text{out}}$ through the expression

$$\Psi^{\text{out}}(t^{\text{out}}, \mathbf{x}) = \Psi^{\text{in}}(t^{\text{out}}, \mathbf{x}) - i\sqrt{Z} \int d^4y \left[\Psi^{\text{in}}(t^{\text{out}}, \mathbf{x}), \Psi^{\text{in}}(y)\right] J(y). \tag{3.3.10}$$

As asymptotic fields are free, they can be expanded into modes

$$\Psi^{\text{as}}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[\Psi_k^{\text{as}}(x^0) a_{\mathbf{k}}^{\text{as}} + \Psi_k^{\text{as}*}(x^0) a_{-\mathbf{k}}^{\text{as}\dagger}\right] \tag{3.3.11}$$

with the equation of motion that reads

$$0 = \ddot{\Psi}_k^{\text{as}} + (\mathbf{k}^2 + M^2)\Psi_k^{\text{as}} \tag{3.3.12}$$

and the commutation relations of the form

$$\left[a_{\mathbf{k}}^{\text{as}}, a_{\mathbf{k}'}^{\text{as}\dagger}\right] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}'), \qquad \text{(other relations)} = 0. \tag{3.3.13}$$

Moreover, mode functions satisfy time-independent inner product relation

$$\dot{\Psi}_k^{\text{as}*} \Psi_k^{\text{as}} - \Psi_k^{\text{as}*} \dot{\Psi}_k^{\text{as}} = i/Z, \tag{3.3.14}$$

which allows to derive $a_{\mathbf{k}}^{\text{as}}$ in terms of Ψ^{as}

$$a_{\mathbf{k}}^{\text{as}} = -iZ \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \left(\dot{\Psi}_k^{\text{as}*} \Psi^{\text{as}} - \Psi_k^{\text{as}*} \dot{\Psi}^{\text{as}}\right). \tag{3.3.15}$$

The above relation for out-states reads

$$\begin{aligned} a_{\mathbf{k}}^{\text{out}} &= \alpha_k a_{\mathbf{k}}^{\text{in}} + \beta_k a_{-\mathbf{k}}^{\text{in}\dagger} + \qquad (3.3.16)\\ -Z^{3/2} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} &\int d^4y \left[\dot{\Psi}_k^{\text{out}*}(x)\Psi^{\text{in}}(x) - \Psi_k^{\text{out}*}(x)\dot{\Psi}^{\text{in}}(x), \Psi^{\text{in}}(y)\right] J(y) = \\ &= \alpha_k a_{\mathbf{k}}^{\text{in}} + \beta_k a_{-\mathbf{k}}^{\text{in}\dagger} + \qquad (3.3.17)\\ -i\sqrt{Z} \int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}} &\left(-\beta_k \Psi_k^{\text{in}}(x^0) + \alpha_k \Psi_k^{\text{in}*}(x^0)\right) J(x), \end{aligned}$$

where

$$\alpha_k \equiv -iZ \left(\dot{\Psi}_k^{\text{out}*} \Psi_k^{\text{in}} - \Psi_k^{\text{out}*} \dot{\Psi}_k^{\text{in}}\right), \tag{3.3.18}$$

$$\beta_k \equiv -iZ \left(\dot{\Psi}_k^{\text{out}*} \Psi_k^{\text{in}*} - \Psi_k^{\text{out}*} \dot{\Psi}_k^{\text{in}*}\right) \tag{3.3.19}$$

are some time-independent coefficients. They satisfy the relations

$$\Psi_k^{\text{out}} = \alpha_k^* \Psi_k^{\text{in}} - \beta_k^* \Psi_k^{\text{in}*}, \tag{3.3.20}$$

$$\Psi_k^{\text{in}} = \alpha_k \Psi_k^{\text{out}} + \beta_k^* \Psi_k^{\text{out}*} \tag{3.3.21}$$

and have a proper normalization

$$|\alpha_k|^2 - |\beta_k|^2 = 1, \tag{3.3.22}$$

so we can identify them as the Bogoliubov coefficients including quantum corrections. So relations (3.3.16) and (3.3.17) describe the generalized Bogoliubov transformation law with the first two terms reproducing the usual transformation connected with the time-dependence of the mass term and the third one purely describing the effect of interactions.

Finally, the occupation number of produced states including quantum corrections is of the form

$$\begin{aligned} n_k \equiv \left\langle 0^{\text{in}} \left| a_{\mathbf{k}}^{\text{out}\dagger} a_{\mathbf{k}}^{\text{out}} \right| 0^{\text{in}} \right\rangle = \\ = \left| \left(\beta_k a_{-\mathbf{k}}^{\text{in}\dagger} - i\sqrt{Z} \int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}} \left(-\beta_k \Psi_k^{\text{in}} + \alpha_k \Psi_k^{\text{in}*} \right) J \right) \left| 0^{\text{in}} \right\rangle \right|^2 = \\ = \begin{cases} V \cdot |\beta_k|^2 & + \quad \cdots & (\beta_k \neq 0) \\ 0 & + \quad Z \left| \int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}} \Psi_k^{\text{in}*} J \left| 0^{\text{in}} \right\rangle \right|^2 & (\beta_k = 0) \end{cases}, \end{aligned} \tag{3.3.23}$$

where V denotes the volume of the system. Although at the leading order occupation number can vanish here for $\beta_k = 0$, we can see that the particle production process can be also initiated due to quantum corrections then.

For the fermionic case, the results are analogous with one difference: proper commutation relation.

3.4 SUSY model with a single coupling

It is interesting to investigate the details of our method describing particle production in some supersymmetric model. This way we introduce fermions in a very natural way also cancelling the possible divergences. We can choose a simple but non-trivial superpotential

$$W = \frac{1}{2} g \Phi X^2, \tag{3.4.1}$$

where Φ and X are chiral superfields which interact via the coupling g. For simplicity we assume the supersymmetry is unbroken for now. Interactions in this system are described by the Lagrangian

$$\mathcal{L}_{\text{int}} = -g^2 |\phi|^2 |\chi|^2 - \frac{1}{4} g^2 |\chi|^4 - \left(\frac{1}{2} g \phi \psi_\chi \psi_\chi + g \chi \psi_\phi \psi_\chi + h.c. \right), \tag{3.4.2}$$

where ϕ/ψ_ϕ and χ/ψ_χ are the scalar/fermion components of Φ and X supermultiplets, respectively. Again, we assume that ϕ carries a vev $\langle \phi \rangle \equiv \langle 0^{\text{in}} | \phi | 0^{\text{in}} \rangle$, which induces a mass $g \langle \phi \rangle$ for χ and ψ_χ, keeping ψ_ϕ and $\tilde{\phi} \equiv \phi - \langle \phi \rangle$ massless. ESP is placed in the phase space at $\chi = \psi_\psi = \psi_\chi = 0$.

Important feature of this system is the fact that it is the minimal supersymmetric model containing not only the fields with masses varying in time but also massless fields. It allows us to compare cases with $\beta_k = 0$ $(\tilde{\phi}, \psi_\phi)$ and $\beta_k \neq 0$ (χ, ψ_χ) within a single model.

3.4.1 One-loop corrections

Potential corresponding to the considered superpotential reads

$$V = g^2|\phi|^2|X|^2 + \frac{1}{4}g^2|X|^2 + \frac{1}{2}g\phi\psi_X\psi_X + \frac{1}{2}g\phi^*\psi_X^+\psi_X^+ + \\ +gX\psi_\phi\psi_X + gX^*\psi_\phi^+\psi_X^+ + m_S^2|X|^2. \tag{3.4.3}$$

The particle content of our theory is: one massive fermion ψ_X (m_ψ), one massive scalar X (m_X), one massles fermion ψ_ϕ and one massless scalar ϕ, and both fermions are Weyl spinors. Two-component notation follows [30].

We are interested in one-loop corrections to χ's and ϕ's propagators at low energies, so we can use on-shell renormalization scheme:

$$\Pi^2(p^2 = m_R^2) = 0, \tag{3.4.4}$$

$$\frac{d}{dp^2}\Pi^2(p^2 = m_R^2) = 0, \tag{3.4.5}$$

where m_R denotes the renormalized mass and Π^2 - one-loop correction. In this scheme[4]

$$m_{phys}^2 = m_R^2 = m^2 - \delta m^2, \tag{3.4.7}$$

where

$$\delta m^2 = i\Big(m^2\frac{d}{dp^2}L^1_{p^2=m_R^2} - L^1_{p^2=m_R^2}\Big) \tag{3.4.8}$$

and L^1 stands for all the diagrams contributing to the one-loop mass correction, while m^2 is the bare mass. We use dimensional regularization with $D = 4 - \epsilon$.

If the background field carries the vacuum expectation value:

$$\phi \to \langle\phi\rangle + \phi \tag{3.4.9}$$

our potential changes its form to

$$V = g^2|\phi|^2|X|^2 + g^2|\langle\phi\rangle|^2|X|^2 + g^2\langle\phi\rangle\phi^*|X|^2 + g^2\langle\phi\rangle^*\phi|X|^2 + \frac{1}{4}g^2|X|^2 + \\ +\frac{1}{2}g\phi\psi_X\psi_X + \frac{1}{2}g\langle\phi\rangle\psi_X\psi_X + \frac{1}{2}g\phi^*\psi_X^+\psi_X^+ + \frac{1}{2}g\langle\phi\rangle^*\psi_X^+\psi_X^+ + \\ +gX\psi_\phi\psi_X + gX^*\psi_\phi^+\psi_X^+ + m_S^2|X|^2, \tag{3.4.10}$$

where m_S denotes possible soft mass.

[4]This relation follows the expression

$$\Pi^2(p^2) = -p^2\delta Z + \delta m^2 + iL^1. \tag{3.4.6}$$

One-loop corrections for χ and ϕ read then

$$\Pi_\chi^2(p^2) = \frac{2g^4|\langle\phi\rangle|^2}{(4\pi)^2}\left((m_{\chi R}^2 - p^2)\left[\frac{m_\psi^2}{m_{\chi R}^4} + \frac{m_\psi^2}{m_{\chi R}^4}\left(\frac{m_\psi^2}{m_{\chi R}^2} - 1\right)\log\frac{m_\psi^2 - m_{\chi R}^2}{m_\psi^2} - \frac{1}{2m_{\chi R}^2}\right] + \right.$$
$$+\log\frac{m_\psi^2 - m_{\chi R}^2}{m_\psi^2 - p^2} + \frac{m_\psi^2}{p^2}\log\frac{m_\psi^2 - p^2}{m_\psi^2} - \frac{m_\psi^2}{m_{\chi R}^2}\log\frac{m_\psi^2 - m_{\chi R}^2}{m_\psi^2} - \frac{1}{2}\frac{p^2}{m_{\chi R}^4}\left[-m_{\chi R}^2 + \right.$$
$$\left.\left. +(m_\chi^2 + m_S^2)\log\frac{m_\chi^2 + m_S^2}{m_\chi^2 - m_{\chi R}^2 + m_S^2}\right] - \frac{1}{2} + \frac{1}{2}\frac{m_\chi^2 + m_S^2}{p^2}\log\frac{m_\chi^2 + m_S^2}{m_\chi^2 - p^2 + m_S^2}\right), \quad (3.4.11)$$

$$\Pi_\phi^2(p^2) = \frac{2g^4|\langle\phi\rangle|^2}{(4\pi)^2}\left(\left(\frac{1}{2}p^2 - m_{\phi_R}^2\right)\left[\frac{1}{m_{\phi_R}^2} - \frac{4m_\psi^2}{m_{\phi_R}^2\sqrt{4m_\psi^2 - m_{\phi_R}^2}}\arctan\frac{m_{\phi_R}}{\sqrt{4m_\psi^2 - m_{\phi_R}^2}}\right] + \right.$$
$$+2\left[\frac{\sqrt{4m_\psi^2 - p^2}}{p}\arctan\frac{p}{\sqrt{4m_\psi^2 - p^2}} + \frac{\sqrt{4m_\psi^2 - m_{\phi_R}^2}}{m_{\phi_R}}\arctan\frac{m_{\phi_R}}{\sqrt{4m_\psi^2 - m_{\phi_R}^2}}\right] - \frac{p^2}{m_{\phi_R}^2} +$$
$$+\frac{4p^2(m_\chi^2 + m_S^2)}{m_{\phi_R}^3\sqrt{4m_\chi^2 - m_{\phi_R}^2 + 4m_S^2}}\arctan\frac{m_{\phi_R}}{\sqrt{4m_\chi^2 - m_{\phi_R}^2 + 4m_S^2}} + \frac{3}{2} + \log(m_\chi^2 + m_S^2) +$$
$$+\frac{1}{m_{\phi_R}}\frac{2m_S^2 - m_{\phi_R}^2 - m_\chi^2}{\sqrt{4m_\chi^2 - m_{\phi_R}^2 - m_S^2}}\arctan\frac{m_{\phi_R}}{\sqrt{m_\chi^2 - m_{\phi_R}^2 - m_S^2}} +$$
$$\left.+\frac{\sqrt{4m_S^2 - m_{\phi_R}^2 - m_\chi^2}}{p}\arctan\frac{m_{\phi_R}}{\sqrt{4m_\chi^2 - m_{\phi_R}^2 + 4m_S^2}}\right), \quad (3.4.12)$$

while the physical masses are given by these implicit expressions

$$m_{\chi R}^2 = g^2|\langle\phi\rangle|^2 + m_S^2 - \frac{2g^2}{(4\pi)^2}\left[m_\psi^2\left(\frac{m_\psi^2}{m_{\chi R}^2} + \frac{m_\psi^2}{m_{\chi R}^4}(m_\psi^2 - m_{\chi R}^2)\log\frac{m_\psi^2 - m_{\chi R}^2}{m_\psi^2} - \frac{1}{2}\right) + \right.$$
$$\left. -(g^2|\langle\phi\rangle|^2 + m_S^2)\log(g^2|\langle\phi\rangle|^2 + m_S^2) - m_\psi^2\left(\frac{m_\psi^2}{m_{\chi R}^2}\log\frac{m_\psi^2 - m_{\chi R}^2}{m_\psi^2} + 2\right) + \frac{1}{2}g^2|\langle\phi\rangle|^2\right], \quad (3.4.13)$$

$$m_{\phi_R}^2 = \frac{2g^2}{(4\pi)^2}\left(m_\psi^2 + (m_\chi^2 + m_S^2)\log(m_\chi^2 + m_S^2) + \frac{m_\psi^2 m_{\phi_R}}{\sqrt{4m_\psi^2 - m_{\phi_R}^2}}\arctan\frac{m_{\phi_R}}{\sqrt{4m_\psi^2 - m_{\phi_R}^2}} + \right.$$
$$-2m_\psi^2\log(m_\psi^2) + \frac{1}{2}g^2|\langle\phi\rangle|^2 - \frac{1}{2}g^2|\langle\phi\rangle|^2\left[\log(m_\chi^2 + m_S^2) + \right.$$
$$\left.\left. + \frac{2}{m_{\phi_R}}\frac{2m_S^2 - m_{\phi_R}^2 - m_\chi^2}{\sqrt{4m_\chi^2 - m_{\phi_R}^2 + 4m_S^2}}\arctan\frac{m_{\phi_R}}{\sqrt{4m_\chi^2 - m_{\phi_R}^2 + 4m_S^2}}\right]\right) \quad (3.4.14)$$

and are presented in the Figures 3.3 and 3.4. There can exist several solutions for a physical mass for a given set of parameters but some of them can be excluded due to its real value: $m_{\text{phys}}^2 > 0$.

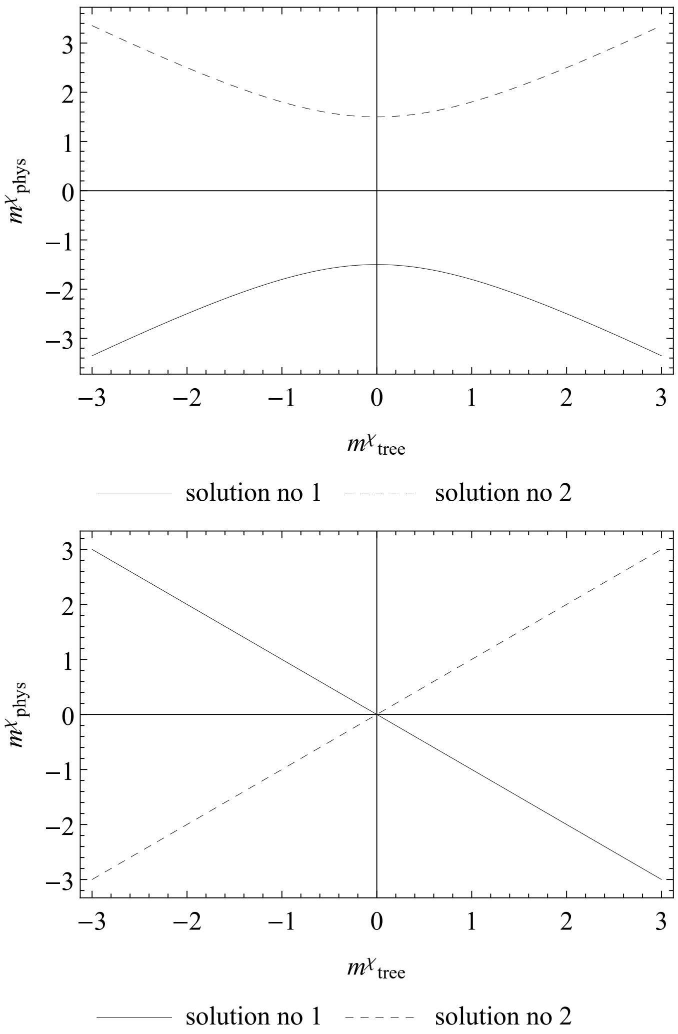

Figure 3.3: Physical mass of χ as a function of a tree-level mass with non-zero (*top*) and vanishing (*bottom*) soft mass.

3.4.2 Influence of interactions

Equations of motion for all the fields are given by

$$0 = \partial^2\phi + J_\phi^\dagger, \tag{3.4.15}$$

$$0 = \left(\partial^2 + g^2\left|\langle\phi\rangle\right|^2\right)\chi + J_\chi^\dagger, \tag{3.4.16}$$

$$0 = i\bar{\sigma}^\mu\partial_\mu\psi_\phi - J_{\psi_\phi}^\dagger, \tag{3.4.17}$$

$$0 = i\bar{\sigma}^\mu\partial_\mu\psi_\chi - g\left\langle\phi^\dagger\right\rangle\psi_\chi^\dagger - J_{\psi_\chi}^\dagger \tag{3.4.18}$$

with source terms of the form

$$J_\phi^\dagger \equiv g^2|\chi|^2\phi + \frac{1}{2}g\psi_\chi^\dagger\psi_\chi^\dagger, \tag{3.4.19}$$

$$J_\chi^\dagger \equiv g^2\left(|\phi|^2 - |\langle\phi\rangle|^2\right)\chi + \frac{1}{2}g^2|\chi|^2\chi + g\psi_\chi^\dagger\psi_\phi^\dagger, \tag{3.4.20}$$

$$J_{\psi_\phi}^\dagger \equiv g\chi^\dagger\psi_\chi^\dagger, \tag{3.4.21}$$

$$J_{\psi_\chi}^\dagger \equiv g\left(\phi^\dagger - \left\langle\phi^\dagger\right\rangle\right)\psi_\chi^\dagger + g\chi^\dagger\psi_\phi^\dagger. \tag{3.4.22}$$

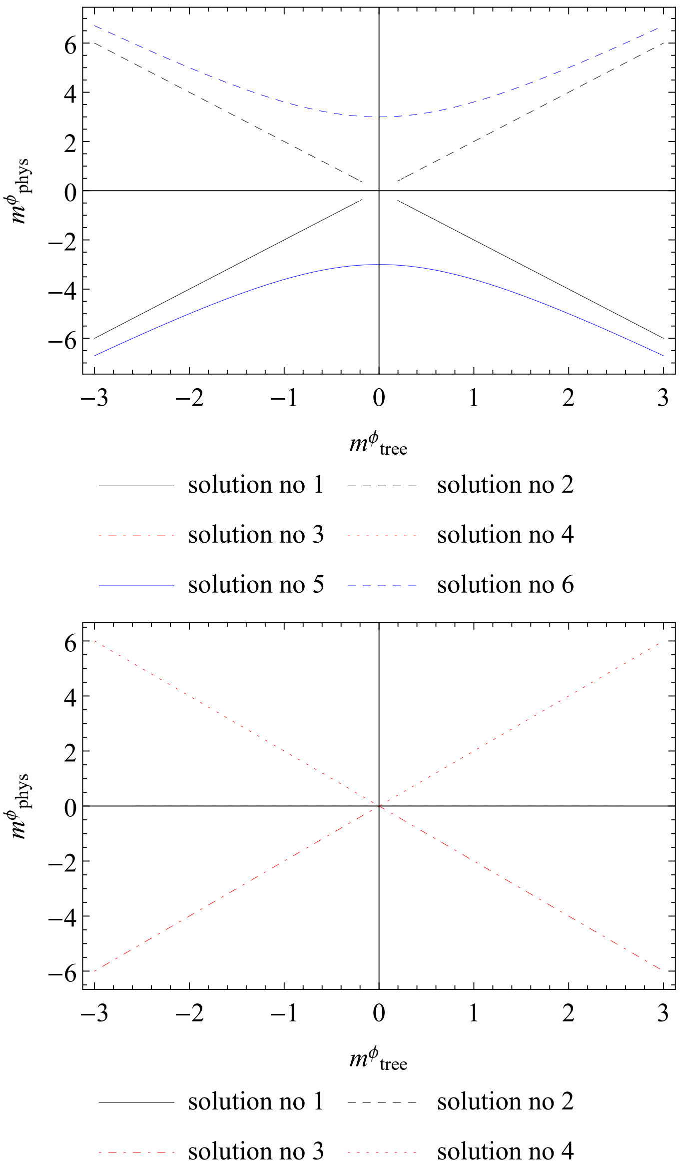

Figure 3.4: Physical mass of ϕ as a function of a tree-level mass with non-zero (*top*) and vanishing (*bottom*) soft mass.

Yang-Feldman equations for all the fields read

$$\phi(x) = \sqrt{Z_\phi}\phi^{\rm as}(x) - iZ_\phi \int_{t^{\rm as}}^{x^0} dy^0 \int d^3y \left[\phi^{\rm as}(x), \phi^{\rm as*}(y)\right] J_\phi^\dagger(y), \tag{3.4.23}$$

$$\chi(x) = \sqrt{Z_\chi}\chi^{\rm as}(x) - iZ_\chi \int_{t^{\rm as}}^{x^0} dy^0 \int d^3y \left[\chi^{\rm as}(x), \chi^{\rm as*}(y)\right] J_\chi^\dagger(y), \tag{3.4.24}$$

$$\psi_{\phi\alpha}(x) = \sqrt{Z_{\psi_\phi}}\psi_{\phi\alpha}^{\rm as}(x) - iZ_{\psi_\phi} \int_{t^{\rm as}}^{x^0} dy^0 \int d^3y \left\{\psi_{\phi\alpha}^{\rm as}(x), \psi_{\phi\dot\beta}^{\rm as\dagger}(y)\right\} J_{\psi_\phi}^{\dagger\dot\beta}(y), \tag{3.4.25}$$

$$\psi_{\chi\alpha}(x) = \sqrt{Z_{\psi_\chi}}\psi_{\chi\alpha}^{\rm as}(x) + \tag{3.4.26}$$

$$-iZ_{\psi_\chi} \int_{t^{\rm as}}^{x^0} dy^0 \int d^3y \left(\left\{\psi_{\chi\alpha}^{\rm as}(x), \psi_{\chi\dot\beta}^{\rm as\dagger}(y)\right\} J_{\psi_\chi}^{\dagger\dot\beta}(y) + \left\{\psi_{\chi\alpha}^{\rm as}(x), \psi_{\chi}^{\rm as\beta}(y)\right\} J_{\psi_\chi\beta}(y)\right).$$

Asymptotic fields can be expanded into modes

$$\phi^{\rm as} = \langle 0^{\rm as}\left|\phi^{\rm as}\right| 0^{\rm as}\rangle + \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\phi_k^{\rm as} a_{\phi\mathbf{k}}^{+,{\rm as}} + \phi_k^{{\rm as}*} a_{\phi-\mathbf{k}}^{-,{\rm as}\dagger}\right), \tag{3.4.27}$$

$$\chi^{\rm as} = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\chi_k^{\rm as} a_{\chi\mathbf{k}}^{+,{\rm as}} + \chi_k^{{\rm as}*} a_{\chi-\mathbf{k}}^{-,{\rm as}\dagger}\right), \tag{3.4.28}$$

$$\psi_\phi^{\rm as} = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(e_{\mathbf{k}}^{+} \psi_{\phi k}^{\rm as} a_{\psi_\phi\mathbf{k}}^{+,{\rm as}} + e_{\mathbf{k}}^{-} \psi_{\phi k}^{{\rm as}*} a_{\psi_\phi-\mathbf{k}}^{-,{\rm as}\dagger}\right), \tag{3.4.29}$$

$$\psi_\chi^{\rm as} = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{s=\pm} e_{\mathbf{k}}^{s} \left(\psi_{\chi k}^{(+)s,{\rm as}} a_{\psi_\chi\mathbf{k}}^{s,{\rm as}} - s e^{-i\theta_{\mathbf{k}}} \psi_{\chi k}^{(-)s,{\rm as}*} a_{\psi_\chi-\mathbf{k}}^{s,{\rm as}\dagger}\right), \tag{3.4.30}$$

satisfying the equations of motion

$$0 = \ddot{\phi}_k^{\rm as} + \mathbf{k}^2 \phi_k^{\rm as}, \tag{3.4.31}$$

$$0 = \ddot{\chi}_k^{\rm as} + \left(\mathbf{k}^2 + g^2 \left|\langle\phi\rangle\right|^2\right) \chi_k^{\rm as}, \tag{3.4.32}$$

$$0 = \dot{\psi}_{\phi k}^{\rm as} + i|\mathbf{k}|\psi_{\phi k}^{\rm as}, \tag{3.4.33}$$

$$0 = \dot{\psi}_{\chi k}^{(+)s,{\rm as}} + is|\mathbf{k}|\psi_{\chi k}^{(+)s,{\rm as}} + ig\left\langle\phi^\dagger\right\rangle \psi_{\chi k}^{(-)h,{\rm as}}, \tag{3.4.34}$$

$$0 = \dot{\psi}_{\chi k}^{(-)s,{\rm as}} - is|\mathbf{k}|\psi_{\chi k}^{(-)s,{\rm as}} + ig\left\langle\phi\right\rangle \psi_{\chi k}^{(+)h,{\rm as}}, \tag{3.4.35}$$

where the helicity of fermions is taken into account. In all the expression $\pm$ distinguishes between the particle and antiparticle for scalars and different helicities for fermions.

Commutation relations given by

$$\left[a_{\phi\mathbf{k}}^{s,{\rm as}}, a_{\phi\mathbf{k}'}^{r,{\rm as}\dagger}\right] = \left[a_{\chi\mathbf{k}}^{s,{\rm as}}, a_{\chi\mathbf{k}'}^{r,{\rm as}\dagger}\right] = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta^{sr}, \tag{3.4.36}$$

$$\left\{a_{\psi_\phi\mathbf{k}}^{s,{\rm as}}, a_{\psi_\phi\mathbf{k}'}^{r,{\rm as}\dagger}\right\} = \left\{a_{\psi_\chi\mathbf{k}}^{s,{\rm as}}, a_{\psi_\chi\mathbf{k}'}^{r,{\rm as}\dagger}\right\} = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\delta^{sr} \tag{3.4.37}$$

entail the following inner product relations

$$i = Z_\phi \left(\dot{\phi}_k^{{\rm as}*} \phi_k^{\rm as} - \phi_k^{{\rm as}*} \dot{\phi}_k^{\rm as}\right), \tag{3.4.38}$$

$$i = Z_\chi \left(\dot{\chi}_k^{{\rm as}*} \chi_k^{\rm as} - \chi_k^{{\rm as}*} \dot{\chi}_k^{\rm as}\right), \tag{3.4.39}$$

$$1 = Z_{\psi_\phi} \left|\psi_{\phi k}^{\rm as}\right|^2, \tag{3.4.40}$$

$$1 = Z_{\psi_\chi} \left(\left|\psi_{\chi k}^{(+)s,{\rm as}}\right|^2 + \left|\psi_{\chi k}^{(-)s,{\rm as}}\right|^2\right), \tag{3.4.41}$$

which in turn enable us to formulate all the out-state annihilation operators

$$a^{+,\mathrm{out}}_{\phi\mathbf{k}} = -iZ\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\dot{\phi}^{\mathrm{out}*}_k\left(\phi^{\mathrm{out}} - \left\langle 0^{\mathrm{out}}\left|\phi^{\mathrm{out}}\right| 0^{\mathrm{out}}\right\rangle\right)\right.$$
$$\left.-\phi^{\mathrm{out}*}_k\left(\dot{\phi}^{\mathrm{out}} - \left\langle 0^{\mathrm{out}}\left|\dot{\phi}^{\mathrm{out}}\right| 0^{\mathrm{out}}\right\rangle\right)\right], \tag{3.4.42}$$

$$a^{-,\mathrm{out}\dagger}_{\phi-\mathbf{k}} = +iZ\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\dot{\phi}^{\mathrm{out}}_k\left(\phi^{\mathrm{out}} - \left\langle 0^{\mathrm{out}}\left|\phi^{\mathrm{out}}\right| 0^{\mathrm{out}}\right\rangle\right)\right.$$
$$\left.-\phi^{\mathrm{out}}_k\left(\dot{\phi}^{\mathrm{out}} - \left\langle 0^{\mathrm{out}}\left|\dot{\phi}^{\mathrm{out}}\right| 0^{\mathrm{out}}\right\rangle\right)\right], \tag{3.4.43}$$

$$a^{+,\mathrm{out}}_{\chi\mathbf{k}} = -iZ_\chi\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\dot{\chi}^{\mathrm{out}*}_k\chi^{\mathrm{out}} - \chi^{\mathrm{out}*}_k\dot{\chi}^{\mathrm{out}}\right), \tag{3.4.44}$$

$$a^{-,\mathrm{out}\dagger}_{\chi-\mathbf{k}} = +iZ_\chi\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\dot{\chi}^{\mathrm{out}}_k\chi^{\mathrm{out}} - \chi^{\mathrm{out}}_k\dot{\chi}^{\mathrm{out}}\right), \tag{3.4.45}$$

$$a^{+,\mathrm{out}}_{\psi_\phi\mathbf{k}} = Z_{\psi_\phi}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\psi^{\mathrm{out}*}_{\phi k}\cdot e^{+\dagger}_{\mathbf{k}}\bar{\sigma}^0\psi^{\mathrm{out}}_\phi, \tag{3.4.46}$$

$$a^{-,\mathrm{out}\dagger}_{\psi_\phi-\mathbf{k}} = Z_{\psi_\phi}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\psi^{\mathrm{out}}_{\phi k}\cdot e^{-\dagger}_{\mathbf{k}}\bar{\sigma}^0\psi^{\mathrm{out}}_\phi, \tag{3.4.47}$$

$$a^{s,\mathrm{out}}_{\psi_\chi\mathbf{k}} = Z_{\psi_\phi}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\psi^{(+)s,\mathrm{out}*}_{\chi k}\cdot e^{s\dagger}_{\mathbf{k}}\bar{\sigma}^0\psi^{\mathrm{out}}_\chi + se^{-i\theta_{\mathbf{k}}}\psi^{(-)s,\mathrm{out}}_{\chi k}\cdot\psi^{\mathrm{out}\dagger}_\chi\bar{\sigma}^0 e^s_{-\mathbf{k}}\right), \tag{3.4.48}$$

using proper Yang-Feldman equations.

Moreover, the relation between in and out-states constituting the Bogoliubov transformation can be obtained

$$\phi^{\mathrm{out}}(t^{\mathrm{out}},\mathbf{x}) = \phi^{\mathrm{in}}(t^{\mathrm{out}},\mathbf{x}) - i\sqrt{Z_\phi}\int d^4y\left[\phi^{\mathrm{in}}(t^{\mathrm{out}},\mathbf{x}),\phi^{\mathrm{in}\dagger}(y)\right]J^\dagger_\phi(y), \tag{3.4.49}$$

$$\chi^{\mathrm{out}}(t^{\mathrm{out}},\mathbf{x}) = \chi^{\mathrm{in}}(t^{\mathrm{out}},\mathbf{x}) - i\sqrt{Z_\chi}\int d^4y\left[\chi^{\mathrm{in}}(t^{\mathrm{out}},\mathbf{x}),\chi^{\mathrm{in}\dagger}(y)\right]J^\dagger_\chi(y), \tag{3.4.50}$$

$$\psi^{\mathrm{out}}_{\phi\alpha}(t^{\mathrm{out}},\mathbf{x}) = \psi^{\mathrm{in}}_{\phi\alpha}(t^{\mathrm{out}},\mathbf{x}) - i\sqrt{Z_{\psi_\phi}}\int d^4y\left\{\psi^{\mathrm{in}}_{\phi\alpha}(t^{\mathrm{out}},\mathbf{x}),\psi^{\mathrm{in}\dagger}_{\phi\dot{\beta}}(y)\right\}J^{\dagger\dot{\beta}}_{\psi_\phi}(y), \tag{3.4.51}$$

$$\psi^{\mathrm{out}}_{\chi\alpha}(t^{\mathrm{out}},\mathbf{x}) = \psi^{\mathrm{in}}_{\chi\alpha}(t^{\mathrm{out}},\mathbf{x}) - i\sqrt{Z_{\psi_\chi}}\int d^4y\left(\left\{\psi^{\mathrm{in}}_{\chi\alpha}(t^{\mathrm{out}},\mathbf{x}),\psi^{\mathrm{in}\dagger}_{\chi\dot{\beta}}(y)\right\}J^{\dagger\dot{\beta}}_{\psi_\chi}(y)+\right.$$
$$\left.+\left\{\psi^{\mathrm{in}}_{\chi\alpha}(t^{\mathrm{out}},\mathbf{x}),\psi^{\mathrm{in}\beta}_\chi(y)\right\}J_{\psi_\chi\beta}(y)\right), \tag{3.4.52}$$

corresponding to the Bogoliubov transformation for creation/annihilation operators given by

$$a^{+,\mathrm{out}}_{\phi\mathbf{k}} = \left\langle a^{+,\mathrm{out}}_{\phi\mathbf{k}}\right\rangle + a^{+,\mathrm{in}}_{\phi\mathbf{k}} - i\sqrt{Z_\phi}\int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}}\phi^{\mathrm{out}*}_k\left(J^\dagger_\phi - \left\langle J^\dagger_\phi\right\rangle\right), \tag{3.4.53}$$

$$a^{-,\mathrm{out}\dagger}_{\phi-\mathbf{k}} = \left\langle a^{-,\mathrm{out}\dagger}_{\phi-\mathbf{k}}\right\rangle + a^{-,\mathrm{in}\dagger}_{\phi-\mathbf{k}} + i\sqrt{Z_\phi}\int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}}\phi^{\mathrm{out}}_k\left(J^\dagger_\phi - \left\langle J^\dagger_\phi\right\rangle\right), \tag{3.4.54}$$

$$a^{+,\mathrm{out}}_{\chi\mathbf{k}} = \alpha_{\chi k}a^{+,\mathrm{in}}_{\chi\mathbf{k}} + \beta_{\chi k}a^{-,\mathrm{in}\dagger}_{\chi-\mathbf{k}} - i\sqrt{Z_\chi}\int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}}\chi^{\mathrm{out}*}_k J^\dagger_\chi, \tag{3.4.55}$$

$$a^{-,\mathrm{out}\dagger}_{\chi-\mathbf{k}} = \beta^*_{\chi k}a^{+,\mathrm{in}}_{\chi\mathbf{k}} + \alpha^*_{\chi k}a^{-,\mathrm{in}\dagger}_{\chi-\mathbf{k}} + i\sqrt{Z_\chi}\int d^4x e^{-i\mathbf{k}\cdot\mathbf{y}}\chi^{\mathrm{out}}_k J^\dagger_\chi, \tag{3.4.56}$$

$$a^{+,\mathrm{out}}_{\psi_\phi\mathbf{k}} = a^{+,\mathrm{in}}_{\psi_\phi\mathbf{k}} - i\sqrt{Z_{\psi_\phi}}\int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}}\psi^{\mathrm{out}*}_{\phi k}\cdot e^{+\dagger}_{\mathbf{k}}J^\dagger_{\psi_\phi}, \tag{3.4.57}$$

$$a^{-,\mathrm{out}\dagger}_{\psi_\phi-\mathbf{k}} = a^{-,\mathrm{in}\dagger}_{\psi_\phi-\mathbf{k}} - i\sqrt{Z_{\psi_\phi}}\int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}}\psi^{\mathrm{out}}_{\phi k}\cdot e^{-\dagger}_{\mathbf{k}}J^\dagger_{\psi_\phi}, \tag{3.4.58}$$

$$a^{s,\mathrm{out}}_{\psi_\chi\mathbf{k}} = \alpha_{\chi k}a^{s,\mathrm{in}}_{\psi_\chi\mathbf{k}} + \beta_{\chi\mathbf{k}}a^{s,\mathrm{in}\dagger}_{\psi_\chi-\mathbf{k}} + \tag{3.4.59}$$
$$-i\sqrt{Z_{\psi_\chi}}\int d^4x e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\psi^{(+)s,\mathrm{out}*}_{\chi k}\cdot e^{s\dagger}_{\mathbf{k}}J^\dagger_{\psi_\chi} + se^{-i\theta_{\mathbf{k}}}\psi^{(-)s,\mathrm{out}*}_{\chi k}\cdot e^s_{-\mathbf{k}}J_{\psi_\chi}\right).$$

Finally, Bogoliubov coefficients for massive fields are of the form

$$\alpha_{\chi k} \equiv -iZ_\chi \left(\dot{\chi}_k^{\text{out}*}\chi_k^{\text{in}} - \chi_k^{\text{out}*}\dot{\chi}_k^{\text{in}}\right), \tag{3.4.60}$$

$$\beta_{\chi k} \equiv -iZ_\chi \left(\dot{\chi}_k^{\text{out}*}\chi_k^{\text{in}*} - \chi_k^{\text{out}*}\dot{\chi}_k^{\text{in}*}\right), \tag{3.4.61}$$

$$\alpha^s_{\psi_\chi k} \equiv Z_{\psi_\chi} \left(\psi_{\chi k}^{(+)s,\text{out}*}\psi_{\chi k}^{(+)s,\text{in}} + \psi_{\chi k}^{(-)s,\text{out}*}\psi_{\chi k}^{(-)s,\text{in}}\right), \tag{3.4.62}$$

$$\beta^s_{\psi_\chi \mathbf{k}} \equiv -Z_{\psi_\chi} s e^{-i\theta_{\mathbf{k}}} \left(\psi_{\chi k}^{(+)s,\text{out}*}\psi_{\chi k}^{(-)s,\text{in}*} - \psi_{\chi k}^{(-)s,\text{out}*}\psi_{\chi k}^{(+)s,\text{in}*}\right). \tag{3.4.63}$$

For massless fields ϕ and ψ_ϕ the usual Bogoliubov coefficients vanish as their mass does not depend on time and thus their in-states equal to the out-states, $\phi_k^{\text{in}}(t) = \phi_k^{\text{out}}(t)$ and $\psi_{\phi k}^{\text{in}}(t) = \psi_{\phi k}^{\text{out}}(t)$.

In the interacting theories it is difficult to obtain precise analytical results in an easy way, usually it is even impossible, and we are left with the numerical analysis. In order to find the evolution of number density we need to solve equations of motion for $\langle\phi\rangle$ and all the in-modes.

For massive particles occupation number at the leading order is given by

$$n_{\chi k} \equiv \sum_{s=\pm} \left\langle 0^{\text{in}} \left| a^{s,\text{out}\dagger}_{\chi\mathbf{k}} a^{s,\text{out}}_{\chi\mathbf{k}} \right| 0^{\text{in}} \right\rangle = V \cdot 2 \left|\beta_{\chi k}\right|^2 + \cdots, \tag{3.4.64}$$

$$n_{\psi_\chi k} \equiv \sum_{s=\pm} \left\langle 0^{\text{in}} \left| a^{s,\text{out}\dagger}_{\psi_\chi\mathbf{k}} a^{s,\text{out}}_{\psi_\chi\mathbf{k}} \right| 0^{\text{in}} \right\rangle = V \cdot \sum_{s=\pm} \left|\beta^s_{\psi_\chi \mathbf{k}}\right|^2 + \cdots, \tag{3.4.65}$$

where factor 2 counts degrees of freedom of a complex scalar field. For the linear evolution of the background, $\langle\phi\rangle = vt + i\mu$, it translates into

$$\left|\beta_{\chi\, k}\right|^2 = \left|\beta^s_{\psi_\chi\, \mathbf{k}}\right|^2 \sim e^{-\pi\frac{k^2+g^2\mu^2}{g|v|}}. \tag{3.4.66}$$

Quasi-classically number densities of produced massive scalars and fermions are equal and read

$$n_\chi = n_{\psi_\chi} = \int \frac{d^3k}{(2\pi)^3}\frac{n_{\chi k}}{V} \sim 2 \times \frac{(g|v|)^{3/2}}{(2\pi)^3} e^{-\pi g\mu^2/|v|}. \tag{3.4.67}$$

For the massless fields, $\tilde{\phi}$ and ψ_ϕ, occupation number at the lowest order in g is given by

$$\begin{aligned} n_{\phi k} &\equiv \sum_{s=\pm} \left\langle 0^{\text{in}} \left| \left(a^{s,\text{out}\dagger}_{\phi\mathbf{k}} - \left\langle a^{s,\text{out}\dagger}_{\phi\mathbf{k}}\right\rangle\right)\left(a^{s,\text{out}}_{\phi\mathbf{k}} - \left\langle a^{s,\text{out}}_{\phi\mathbf{k}}\right\rangle\right)\right| 0^{\text{in}} \right\rangle \approx \\ &\approx V \cdot g^2 \int \frac{d^3p}{(2\pi)^3}\left[Z_\phi Z_\chi^2 \left| \int dt\, \phi_k^{\text{out}} \chi^{\text{in}}_{|\mathbf{k}+\mathbf{p}|}\chi^{\text{in}}_p \cdot g\left\langle\phi^*\right\rangle\right|^2 + Z_\phi Z_\chi^2 \left| \int dt\, \phi_k^{\text{out}} \chi^{\text{in}}_{|\mathbf{k}+\mathbf{p}|}\chi^{\text{in}}_p \cdot g\left\langle\phi\right\rangle\right|^2 + \right. \\ &\left. + \frac{1}{4} Z_\phi Z_{\psi_\chi}^2 \sum_{s,r,q}\left(1 + rq\frac{\mathbf{p}\cdot(\mathbf{k}+\mathbf{p})}{p|\mathbf{k}+\mathbf{p}|}\right) \times \left|\int dt\, \phi_k^{\text{out}} \psi^{(s)r,\text{in}}_{\chi|\mathbf{k}+\mathbf{p}|}\psi^{(s)q,\text{in}}_{\chi p}\right|^2\right], \end{aligned} \tag{3.4.68}$$

$$\begin{aligned} n_{\psi_\phi k} &\equiv \sum_{s=\pm} \left\langle 0^{\text{in}} \left| a^{s,\text{out}\dagger}_{\psi_\phi\mathbf{k}} a^{s,\text{out}}_{\psi_\phi\mathbf{k}} \right| 0^{\text{in}} \right\rangle \approx \\ &\approx V \cdot g^2 Z_\chi Z_{\psi_\phi} Z_{\psi_\chi} \int \frac{d^3p}{(2\pi)^3}\sum_{s,r}\frac{1}{2}\left(1 - sr\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\right) \times \left|\int dt\, \psi^{\text{out}}_{\phi k}\chi^{\text{in}}_{|\mathbf{k}+\mathbf{p}|}\psi^{(s)r,\text{in}}_{\chi p}\right|^2. \end{aligned} \tag{3.4.69}$$

These expressions simplify when we investigate their physical meaning looking at the corresponding diagrams, which are presented in the Figure 3.5. It seems like they violate the energy and momentum conservation laws but nothing more wrong - time-dependence of the background balances their amount in the system. These diagrams correspond to the "inverse decay" processes in varying external background, which turns out to be the main channel in the production of massless states.

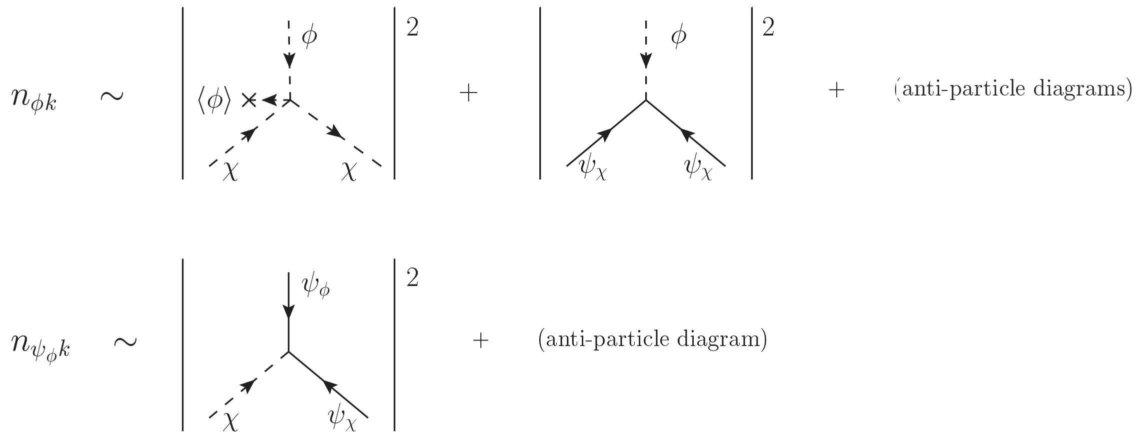

Figure 3.5: Diagrams corresponding to the production of massless states.

Analytical calculation of the equations (3.4.68) and (3.4.69) is highly limited in case of the time-dependent theories. Although, it is possible to simplify them to the form

$$n_{\phi k}/V \sim g^2 \int \frac{d^3p}{(2\pi)^3} \left| \int dt \frac{1}{\sqrt{2k}} \right|^2 e^{-\pi p^2/g|v|} \propto g^2 \cdot (g|v|)^{3/2} |t|^2/k, \tag{3.4.70}$$

$$n_{\psi_\phi k}/V \sim g^2 \int \frac{d^3p}{(2\pi)^3} \left| \int dt \frac{1}{\sqrt{2g|v||t|}} \right|^2 e^{-\pi p^2/g|v|} \propto g^2 \cdot \sqrt{g|v|}|t|, \tag{3.4.71}$$

using some crude approximations for small momentum or equivalently large value of the background's vev:

$$\chi_k^{\text{in}} \sim \frac{1}{\sqrt{2\omega_k}} e^{-\pi k^2/2g|v|}, \tag{3.4.72}$$

$$\psi_{\chi k}^{(s)r,\text{in}} \sim e^{-\pi k^2/2g|v|}, \tag{3.4.73}$$

$$\omega_k \sim g|v||t|. \tag{3.4.74}$$

These approximate results correspond to the exponential suppression of (3.4.66).

We can find analytical expressions for the in-modes for the massless states:

$$\sqrt{Z_\phi}\phi_k^{\text{in}}(t) = \frac{1}{\sqrt{2|\mathbf{k}|}} e^{-i|\mathbf{k}|t}, \tag{3.4.75}$$

$$\sqrt{Z_{\psi_\phi}}\psi_{\phi k}^{\text{in}}(t) = e^{-i|\mathbf{k}|t}, \tag{3.4.76}$$

while for the massive ones it is only possible to presume WKB approximation. In our model

$$\sqrt{Z_\chi}\chi_k^{\rm in}(t) \sim \frac{1}{\sqrt{2\omega_k(t)}} e^{-i\int^t dt\omega_k(t')}, \tag{3.4.77}$$

$$\sqrt{Z_{\psi_\chi}}\psi_{\chi k}^{(+)s,{\rm in}}(t) \sim \frac{1}{\sqrt{2}} \cdot \sqrt{1+\frac{s|\mathbf{k}|}{\omega_k(t)}} e^{-i\int^t dt\omega_k(t')}, \tag{3.4.78}$$

$$\sqrt{Z_{\psi_\chi}}\psi_{\chi k}^{(-)s,{\rm in}}(t) \sim \frac{\langle\phi(t)\rangle}{\sqrt{2}\cdot|\langle\phi(t)\rangle|}\sqrt{1-\frac{s|\mathbf{k}|}{\omega_k(t)}} e^{-i\int^t dt\omega_k(t')}, \tag{3.4.79}$$

where $\omega_k \equiv \sqrt{\mathbf{k}^2 + g^2 |\langle\phi\rangle|^2}$. This approximation is valid as long as $|\langle\phi\rangle| \gg \sqrt{v/g}$, which means that if we start with $\langle\phi\rangle$ big enough, it is safe to choose initial conditions given by

$$\sqrt{Z_\chi}\chi_k^{\rm in}(0) = \frac{1}{\sqrt{2\omega_k(0)}}, \tag{3.4.80}$$

$$\dot{\chi}_k^{\rm in}(0) = -i\sqrt{\frac{\omega_k(0)}{2}}, \tag{3.4.81}$$

$$\sqrt{Z_{\psi_\chi}}\psi_{\chi k}^{(+)s,{\rm in}}(0) = \frac{1}{\sqrt{2}} \cdot \sqrt{1+\frac{s|\mathbf{k}|}{\omega_k(0)}}, \tag{3.4.82}$$

$$\sqrt{Z_{\psi_\chi}}\psi_{\chi k}^{(-)s,{\rm in}}(0) = \frac{\phi(0)}{\sqrt{2}\cdot|\phi(0)|}\sqrt{1-\frac{s|\mathbf{k}|}{\omega_k(0)}}. \tag{3.4.83}$$

For the out-modes we are again deep outside the non-adiabatic region and thus we can utilize the WKB-type solutions of the form

$$\sqrt{Z_\chi}\chi_k^{\rm out} = \frac{1}{\sqrt{2\omega_k}} e^{-i\int^t dt'\omega_k(t')}, \tag{3.4.84}$$

$$\sqrt{Z_{\psi_\chi}}\psi_{\chi k}^{(+)s,{\rm out}} = \frac{1}{\sqrt{2}} \cdot \sqrt{1+\frac{sk}{\omega_k}} e^{-i\int^t dt'\omega_k(t')}, \tag{3.4.85}$$

$$\sqrt{Z_{\psi_\chi}}\psi_{\chi k}^{(-)s,{\rm out}} = \frac{\langle\phi\rangle}{\sqrt{2}\,|\langle\phi\rangle|} \cdot \sqrt{1-\frac{sk}{\omega_k}} e^{-i\int^t dt'\omega_k(t')}. \tag{3.4.86}$$

Taking into account the asymptotic field expansion we can also describe the impact of backreaction on the evolution of ϕ's vev. At the 1-loop level in momentum integration equation governing $\langle\phi\rangle$ reads

$$0 = \left\langle 0^{\rm in}\right| \left(\partial^2\phi + g^2 |\chi|^2 \phi + \frac{1}{2} g \psi_\chi^\dagger \psi_\chi^\dagger \right) \left|0^{\rm in}\right\rangle \sim$$
$$\sim \left\langle \ddot{\phi} \right\rangle + g \int \frac{d^3p}{(2\pi)^3} \left(Z_\chi \left|\chi_p^{\rm in}\right|^2 \cdot g\langle\phi\rangle - \frac{1}{2} Z_{\psi_\chi} \sum_s \psi_{\chi p}^{(-)s,{\rm in}} \psi_{\chi p}^{(+)s,{\rm in}*} \right). \tag{3.4.87}$$

Finally, combining all the information we can end up with the number densities for massive fields

$$\left|\beta_{\chi k}\right|^2 \sim \frac{Z_\chi\left(|\dot{\chi}_k^{\text{in}}|^2+\omega_k^2\,|\chi_k^{\text{in}}|^2\right)}{2\omega_k}-\frac{1}{2}, \tag{3.4.88}$$

$$\left|\beta^s_{\psi_\chi k}\right|^2 \sim \frac{1}{2}+\frac{sk}{2\omega_k}Z_{\psi_\chi}\left(\left|\psi_{\chi k}^{(-)s,\text{in}}\right|^2-\left|\psi_{\chi k}^{(+)s,\text{in}}\right|^2\right)+ \tag{3.4.89}$$
$$-\frac{1}{\omega_k}Z_{\psi_\chi}\text{Re}\left(g\left\langle\phi\right\rangle\psi_{\chi k}^{(-)s,\text{in}*}\psi_{\chi k}^{(+)s,\text{in}}\right).$$

Equations needed for determining number densities of produced states, namely (3.4.68), (3.4.69), (3.4.88) and (3.4.89), are quite difficult to solve analytically but of course there are numerical solutions available. Figure 3.6 shows our results for some specific choice of the parameters - our first conclusion is that all the species are produced, not only the massive ones. Moreover, numerical results for massive states are in good agreement with the analytical ones, which claim $n_\chi \sim n_{\psi_\chi} \sim 2.81\times10^{-3}$. For this choice of parameters the ratios of final densities equal approximately to $n_\phi/n_\chi \sim 28\%$ for bosons and $n_{\psi_\phi}/n_{\psi_\chi} \sim 1.5\%$ for fermions. To see some general picture there is a comparison between the number densities of all the species for different values of coupling g presented in the Table 3.1.

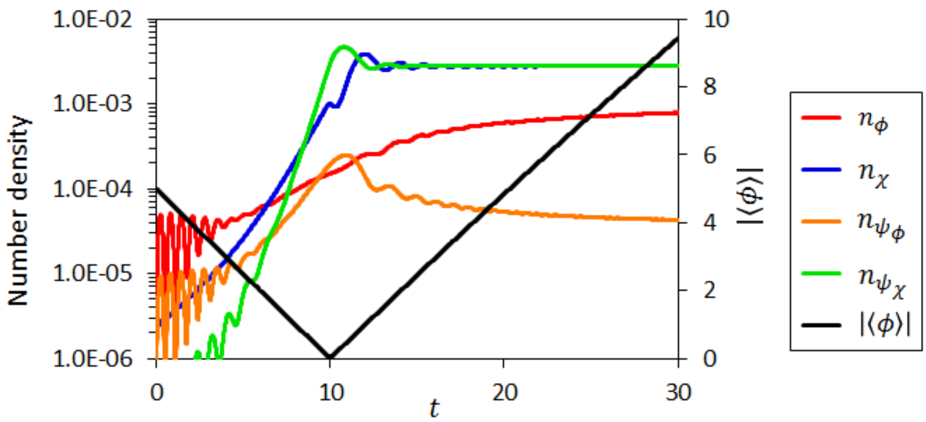

Figure 3.6: Time evolution of $|\langle\phi\rangle|$ (*right axis*) and number densities for all the species (*left axis*) for $\phi(t=0)=5.0+0.05i$, $\dot{\phi}(t=0)=-0.5$ and $g=1$. Approximate final values of number densities are equal to: $n_\phi=7.82\times10^{-4}$, $n_\chi=2.77\times10^{-3}$, $n_{\psi_\phi}=4.26\times10^{-5}$, $n_{\psi_\chi}=2.78\times10^{-3}$.

We can see in the Table 3.1 that for the bigger values of g number densities of produced massless states rise significantly, which is illustrated by the Figure 3.7. Crucial difference lies in the fact that for stronger coupling we can observe the trapping effect leading to the oscillations of the background, which is absent in the case of the weak g. For stronger coupling massless bosons ϕ can be produced as abundantly as massive bosons χ and it seems that their number density exceeds the latter in the end. We believe it is an artificial effect coming from the limited accuracy of

g	n_χ	n_{ψ_χ}	n_ϕ	n_{ψ_ϕ}
0.1	45.85	50.66	1.83	1.66
0.5	47.33	47.74	4.26	0.66
0.8	45.26	45.36	8.72	0.66
1.5	36.8	37.04	25.03	1.13
1.6	35.67	35.94	27.16	1.24
1.8	32.85	33.14	32.59	1.41
2	43.45	43.61	12.27	0.67

Table 3.1: Number densities of produced species as a part of the whole production (in %).

our numerical calculations - only up to terms of the order of g^2, that can be solved using interacting field theory as in Chapter 5. Nonetheless, the main conclusion that the massless species can be produced as efficiently as the massive ones holds.

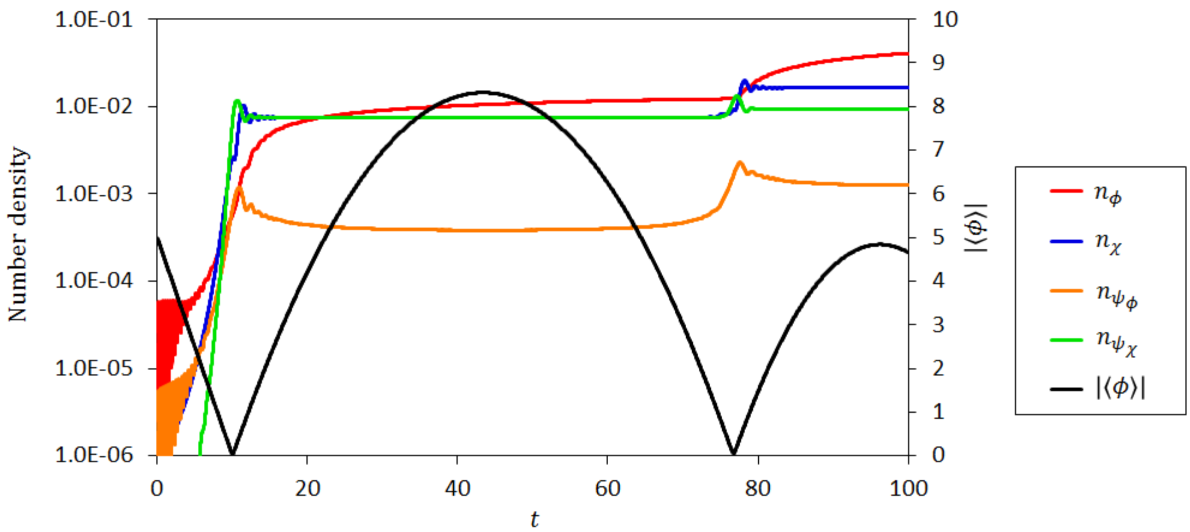

Figure 3.7: Time evolution of $|\langle\phi\rangle|$ (*right axis*) and all number densities (*left axis*) for the stronger coupling $g = 2$. Other parameters are the same as in the Figure 3.6.

Moreover, we are able to find some functions reproducing time-evolution of the number densities for the massless species with good accuracy, which is helpful with no analytical results available. Fitted functions are presented in the Figure 3.8.

Our approximate functions are given by

$$n_{\phi k}/V \sim 0.16 \cdot \frac{g^2}{4\pi} \frac{1}{e^{\sqrt{\pi k^2/g|v|}} - 1} \cdot g|v|(t-t_*)^2 \left[\frac{\sin 0.52k(t-t_*)}{0.52k(t-t_*)}\right]^2, \quad (3.4.90)$$

$$n_{\psi_\phi k}/V \sim 0.40 \cdot \frac{g^2}{4\pi} \frac{1}{e^{\sqrt{\pi k^2/g|v|}} + 1} \cdot \sqrt{g|v|}(t-t_*) \left[\frac{\sin 0.59k(t-t_*)}{0.59k(t-t_*)}\right]^2, \quad (3.4.91)$$

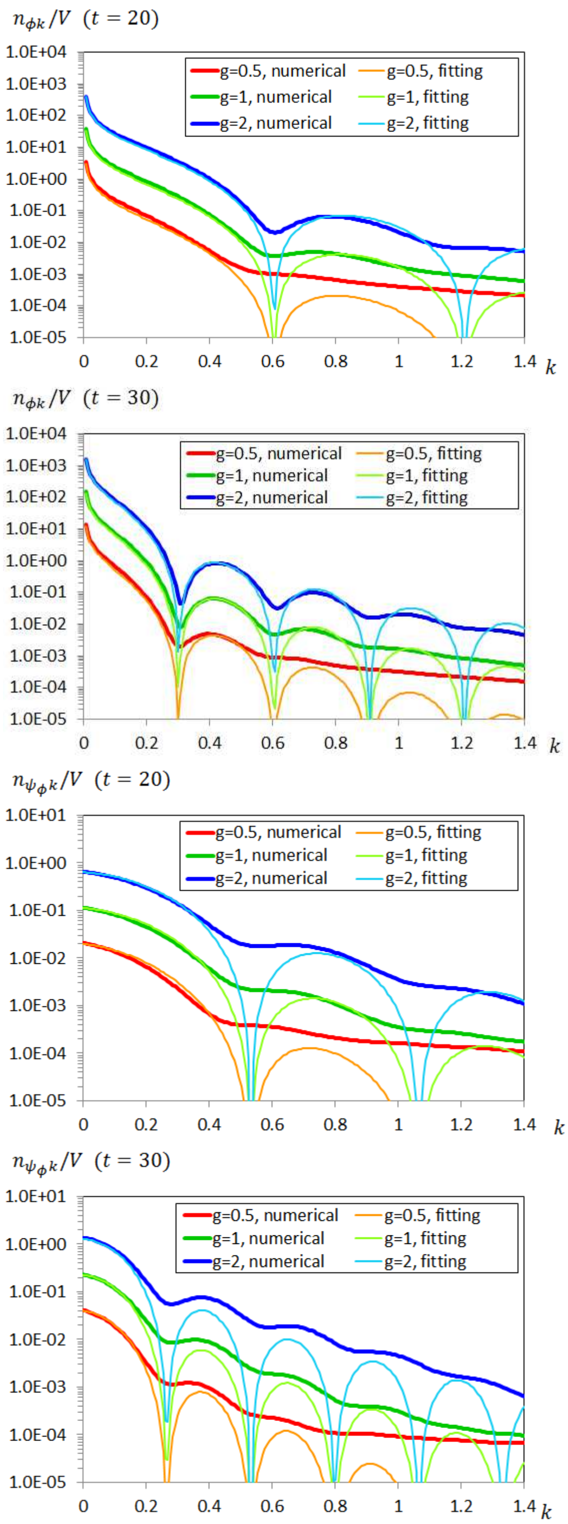

Figure 3.8: Numerically found distribution functions for massless species $n_{\phi k}, n_{\psi_\phi k}$ (thick lines) and corresponding approximate functions for various couplings $g = 0.5, 1, 2$ (thin lines). The values of all the parameters are consistent with the previous plots with numerical results: $\phi(t = 0) = 5.0 + 0.05i$, $\dot{\phi}(t = 0) = -0.5$ and $|v| = 0.5, t_* = 10$ for these functions.

where t_* denotes the time when ϕ's trajectory is closest to the point $\phi = 0$. They possess a very interesting structure as they consist of four factors with some numerical constant - perturbative suppression $g^2/4\pi$, usual bosonic/fermionic distribution, time-dependent factor identified as (3.4.70) or (3.4.71) and oscillating part (sine). For low momentum they mimic the behaviour of (3.4.70) and (3.4.71), but they are not only limited to this case - they fit numerical results over the whole range of momenta very well due to the oscillating factor.

3.4.3 SUSY-breaking

We can include SUSY-breaking in our model in two ways:

a) SUSY can be broken from the beginning, before the non-perturbative production occurs,

b) SUSY can be broken after the non-perturbative production and before rescattering.

In both cases SUSY is broken during rescattering and decays of the products and it should be taken into account to have a complete analysis.

We consider two possible soft SUSY-breaking terms:

a) $\delta\mathcal{L}_{soft} = m_S^2|\chi|^2$,

b) $\delta\mathcal{L}_{soft} = m_S^2|\chi|^2 + A\phi\chi^2 + h.c.$,

where m_S denotes the soft mass. Second possibility is crucial in gravity-mediation scenarios of SUSY-breaking, when $A \sim m$, while in other scenarios it just comes down to the first one ($A \ll m$).

For the scenario with soft mass term distribution of produced states after one "oscillation" reads

$$n_k^\chi = \frac{(gv)^{3/2}}{(2\pi)^3} e^{-\frac{\pi(g^2\mu^2 + m_S^2 + k^2)}{gv}} \tag{3.4.92}$$

for SUSY broken before the production and

$$n_k^\chi = \frac{(gv)^{3/2}}{(2\pi)^3} e^{-\frac{\pi}{gv}(g^2\mu^2 + k^2)} \tag{3.4.93}$$

for SUSY broken after the production. In the Figure 3.9 there is the ratio of number distributions of produced states for broken and unbroken SUSY as a function of v for different values of soft mass depicted. Soft mass does not depend on the coupling so in general lighter particles are produced more efficiently.

3.4.4 Comparison between different sources of production

Time-varying vacuum expectation value of the background field induces the process of particle production that may have a very miscellaneous origin. We have

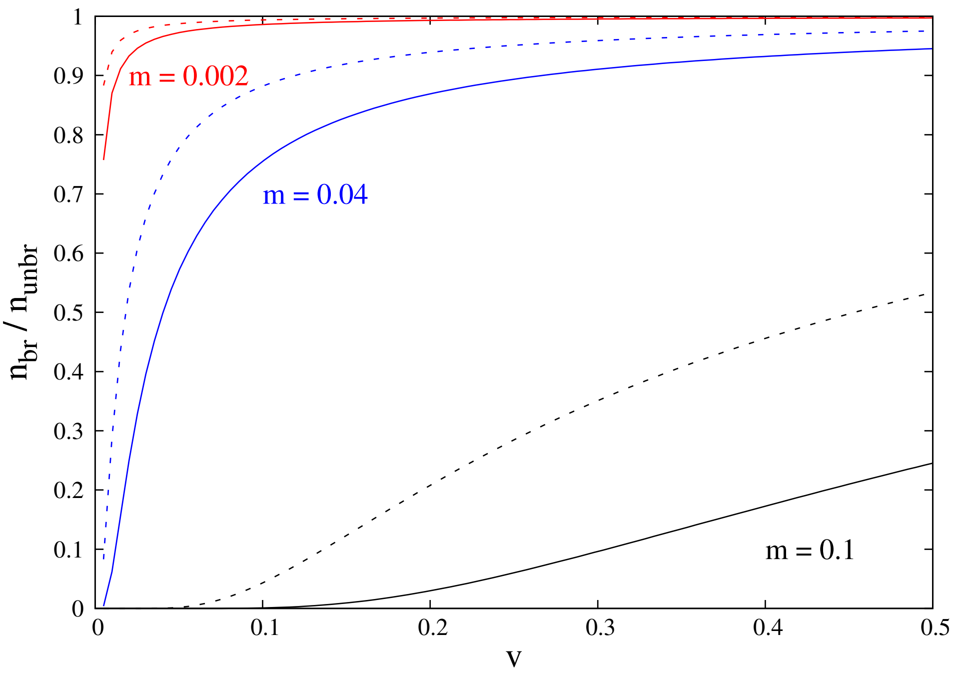

Figure 3.9: Ratio of number distributions of produced states for broken and unbroken SUSY as a function of v for different values of soft mass ($m_S = 0.1, 0.04, 0.002$). Solid lines correspond to the smaller values of the coupling g than for the dashed lines.

already investigated the ones deriving from the vacuum change and quantum corrections, but apart from that it is also important to explore the influence of the rescattering, the perturbative production or the possible rotation of the basis during the whole process.

In our analysis we consider $\langle\chi\rangle$ to vanish all the time but it may not be generally true. Vev of χ can be affected by the production in the non-adiabatic area, keeping the condition $\langle\chi\rangle = 0$ only asymptotic. When $\langle\chi\rangle \neq 0$ we can observe rotation of the basis for fermionic mass eigenstates, which we can now describe in more details.

For the considered superpotential Lagrangian of interactions for fermions can be written as

$$\mathcal{L}_{\psi_{\rm int}} = -gX\psi_X\psi_\phi - \frac{1}{2}g\phi\psi_X\psi_X - gX^*\bar{\psi}_X\bar{\psi}_\phi - \frac{1}{2}g\phi^*\bar{\psi}_X\bar{\psi}_X = \\ = \Psi^T M\Psi + \bar{\Psi}^T M^*\bar{\Psi}, \tag{3.4.94}$$

where

$$\Psi = \begin{pmatrix} \psi_X \\ \psi_\phi \end{pmatrix}, \bar{\Psi} = \begin{pmatrix} \bar{\psi}_X \\ \bar{\psi}_\phi \end{pmatrix}, \tag{3.4.95}$$

$$M = \begin{pmatrix} -\frac{1}{2}g\phi & -\frac{1}{2}gX \\ -\frac{1}{2}gX & 0 \end{pmatrix}. \tag{3.4.96}$$

Matrix M is explicitly not diagonal, so in order to obtain mass eigenstates we need to diagonalize it using the formula

$$M = U^T \tilde{M} U, \tag{3.4.97}$$

where U is some unitary matrix ($U^{-1} = U^\dagger$) and $\tilde{M}$ is diagonal. In our case these two matrices are given by

$$\tilde{M} = \begin{pmatrix} \frac{g}{4}(\sqrt{\phi^2+4X^2}-\phi) & 0 \\ 0 & -\frac{g}{4}(\sqrt{\phi^2+4X^2}+\phi) \end{pmatrix}, \tag{3.4.98}$$

$$U = \begin{pmatrix} -\frac{2X}{\sqrt{4X^2+(\phi+\sqrt{\phi^2+4X^2})^2}} & \frac{2X}{\sqrt{4X^2+(\sqrt{\phi^2+4X^2}-\phi)^2}} \\ \frac{\phi+\sqrt{\phi^2+4X^2}}{\sqrt{4X^2+(\phi+\sqrt{\phi^2+4X^2})^2}} & \frac{\sqrt{\phi^2+4X^2}-\phi}{\sqrt{4X^2+(\sqrt{\phi^2+4X^2}-\phi)^2}} \end{pmatrix}. \tag{3.4.99}$$

Mass eigenstates, denoted with a tilde, are simply the product of the following rotation

$$\begin{cases} \Psi \to \tilde{\Psi} = U\Psi \\ \bar{\Psi} \to \tilde{\bar{\Psi}} = U^*\bar{\Psi} \\ \Psi^T \to \tilde{\Psi}^T = \Psi^T U^T \\ \bar{\Psi}^T \to \tilde{\bar{\Psi}}^T = \bar{\Psi}^T U^\dagger \end{cases} \tag{3.4.100}$$

and their masses read

$$\tilde{m}_X^2 = \frac{1}{4}g^2\left(\sqrt{\langle\phi\rangle^2+4\langle X\rangle^2}-\langle\phi\rangle\right)^2, \tag{3.4.101}$$

$$\tilde{m}_\phi^2 = \frac{1}{4}g^2\left(\sqrt{\langle\phi\rangle^2+4\langle X\rangle^2}+\langle\phi\rangle\right)^2. \tag{3.4.102}$$

Kinetic part of the Lagrangian for fermions reads

$$\begin{aligned} \mathcal{L}_{\psi_{\text{kin}}} &= \frac{1}{2}\bar{\psi}_\phi i\bar{\partial\!\!\!/}\psi_\phi + \frac{1}{2}\psi_\phi i\bar{\partial\!\!\!/}\bar{\psi}_\phi + \frac{1}{2}\bar{\psi}_X i\bar{\partial\!\!\!/}\psi_X + \frac{1}{2}\psi_X i\bar{\partial\!\!\!/}\bar{\psi}_X = \\ &= \frac{i}{2}\bar{\Psi}^T\bar{\partial\!\!\!/}_{4x4}\Psi + \frac{i}{2}\Psi^T\bar{\partial\!\!\!/}_{4x4}\bar{\Psi}, \end{aligned} \tag{3.4.103}$$

which after the rotation (3.4.100) transforms into

$$\begin{aligned} \mathcal{L}_\psi = {} & \tilde{\Psi}^T\tilde{M}\tilde{\Psi} + \tilde{\bar{\Psi}}^T\tilde{M}^*\tilde{\bar{\Psi}} + \frac{i}{2}\tilde{\bar{\Psi}}^T U^{\dagger^{-1}}\bar{\partial\!\!\!/}_{4x4}U^{-1}\tilde{\Psi} + \frac{i}{2}\tilde{\bar{\Psi}}^T U^{\dagger^{-1}}U^{-1}\bar{\partial\!\!\!/}_{4x4}\tilde{\Psi} + \\ & + \frac{i}{2}\tilde{\Psi}^T U^{T^{-1}}\bar{\partial\!\!\!/}_{4x4}U^{*-1}\tilde{\bar{\Psi}} + \frac{i}{2}\tilde{\Psi}^T U^{T^{-1}}U^{*-1}\bar{\partial\!\!\!/}_{4x4}\tilde{\bar{\Psi}}. \end{aligned} \tag{3.4.104}$$

We recover canonically normalized kinetic terms as the matrix U is unitary and together with introducing the matrix V defined as

$$V = U^{\dagger^{-1}}\bar{\partial\!\!\!/}_{4x4}U^{-1}, \tag{3.4.105}$$

$$V^* = (U^T)^{-1}\bar{\partial\!\!\!/}_{4x4}(U^*)^{-1} \tag{3.4.106}$$

the whole Lagrangian simplifies to

$$\mathcal{L}_\psi = \tilde{\Psi}^T\tilde{M}\tilde{\Psi} + \tilde{\bar{\Psi}}^T\tilde{M}^*\tilde{\bar{\Psi}} + \frac{i}{2}\tilde{\bar{\Psi}}^T V\tilde{\Psi} + \frac{i}{2}\tilde{\bar{\Psi}}^T\bar{\partial\!\!\!/}_{4x4}\tilde{\Psi} + \frac{i}{2}\tilde{\Psi}^T V^*\tilde{\bar{\Psi}} + \frac{i}{2}\tilde{\Psi}^T\bar{\partial\!\!\!/}_{4x4}\tilde{\bar{\Psi}}. \tag{3.4.107}$$

For our time-dependent theory interaction Lagrangian becomes

$$\tilde{\mathcal{L}}_{\psi_{int}} = \frac{i}{2}\tilde{\bar{\Psi}}^T U\partial_0 U^\dagger \tilde{\Psi} + \frac{i}{2}\tilde{\Psi}^T U^* \partial_0 U^T \tilde{\bar{\Psi}}. \tag{3.4.108}$$

It means that rotation of the basis introduces some new interactions to the theory given by

$$\tilde{\mathcal{L}}_{\psi_{int}} = iA_{\bar{X}X}\tilde{\bar{\psi}}_X\tilde{\psi}_X + iA_{\bar{\phi}\phi}\tilde{\bar{\psi}}_\phi\tilde{\psi}_\phi + iA_{\bar{\phi}X}\tilde{\psi}_X\tilde{\bar{\psi}}_\phi + iA_{\bar{X}\phi}\tilde{\psi}_\phi\tilde{\bar{\psi}}_X, \tag{3.4.109}$$

where

$$\begin{aligned} A_{\bar{X}X} = 2\Re\Bigg(&-2X\frac{(X^*\dot{\phi}^* - \dot{X}^*\phi^*)}{\sqrt{\phi^{*2}+4X^{*2}}}\Bigg[\frac{1}{\sqrt{4X^2+(\phi+\sqrt{\phi^2+4X^2})^2}}\times \\ &\times\frac{(\phi^*+\sqrt{\phi^{*2}+4X^{*2}})^2}{\sqrt{4X^{*2}+(\phi^{*2}+\sqrt{\phi^{*2}+4X^{*2}})^2}} + \\ &-\frac{1}{\sqrt{4X^2+(\sqrt{\phi^2+4X^2}-\phi)^2}}\frac{(\sqrt{\phi^{*2}+4X^{*2}}-\phi^*)^2}{\sqrt{4X^{*2}+(\sqrt{\phi^{*2}+4X^{*2}}-\phi^{*2})^2}}\Bigg]\Bigg), \end{aligned} \tag{3.4.110}$$

$$\begin{aligned} A_{\bar{\phi}\phi} = 2\Re\Bigg(&2X^*\frac{(X^*\dot{\phi}^* - \dot{X}^*\phi^*)}{\sqrt{\phi^{*2}+4X^{*2}}}\Bigg[\frac{\phi+\sqrt{\phi^2+4X^2}}{\sqrt{4X^2+(\phi+\sqrt{\phi^2+4X^2})^2}}\times \\ &\times\frac{\phi^*+\sqrt{\phi^{*2}+4X^{*2}}}{\sqrt{4X^{*2}+(\phi^{*2}+\sqrt{\phi^{*2}+4X^{*2}})^2}} + \\ &+\frac{\sqrt{\phi^2+4X^2}-\phi}{\sqrt{4X^2+(\sqrt{\phi^2+4X^2}-\phi)^2}}\frac{\sqrt{\phi^{*2}+4X^{*2}}-\phi^*}{\sqrt{4X^{*2}+(\sqrt{\phi^{*2}+4X^{*2}}-\phi^{*2})^2}}\Bigg]\Bigg), \end{aligned} \tag{3.4.111}$$

$$A_{\bar{\phi}X} = A^*_{\bar{X}\phi}, \tag{3.4.112}$$

$$\begin{aligned} A_{\bar{X}\phi} = &\frac{(X\dot{\phi}-\dot{X}\phi)}{\sqrt{\phi^2+4X^2}}\Bigg(\frac{\phi^*+\sqrt{\phi^{*2}+4X^{*2}}}{\sqrt{4X^{*2}+(\phi^*+\sqrt{\phi^{*2}+4X^{*2}})^2}}\frac{(\phi+\sqrt{\phi^2+4X^2})^2}{\sqrt{4X^2+(\phi^2+\sqrt{\phi^2+4X^2})^2}} + \\ &+\frac{\sqrt{\phi^{*2}+4X^{*2}}-\phi^*}{\sqrt{4X^{*2}+(\sqrt{\phi^{*2}+4X^{*2}}-\phi^*)^2}}\frac{(\sqrt{\phi^2+4X^2}-\phi)^2}{\sqrt{4X^2+(\sqrt{\phi^2+4X^2}-\phi^2)^2}}\Bigg) + \\ &-4|X|^2\frac{(X^*\dot{\phi}^*-\dot{X}^*\phi^*)}{\sqrt{\phi^{*2}+4X^{*2}}}\Bigg(\frac{1}{\sqrt{4X^2+(\phi+\sqrt{\phi^2+4X^2})^2}}\frac{\phi^*+\sqrt{\phi^{*2}+4X^{*2}}}{\sqrt{4X^{*2}+(\phi^{*2}+\sqrt{\phi^{*2}+4X^{*2}})^2}} + \\ &-\frac{1}{\sqrt{4X^2+(\sqrt{\phi^2+4X^2}-\phi)^2}}\frac{\sqrt{\phi^{*2}+4X^{*2}}-\phi^*}{\sqrt{4X^{*2}+(\sqrt{\phi^{*2}+4X^{*2}}-\phi^{*2})^2}}\Bigg) \end{aligned} \tag{3.4.113}$$

and we use notation that $\langle X\rangle := X$ and $\langle\phi\rangle := \phi$. All the As are complex or even purely imaginary but it is acceptable as fermions can carry phases. If we assume linear dependence on time for both vevs:

$$X = v_X t + i\mu_X, \tag{3.4.114}$$

$$\phi = v_\phi t + i\mu_\phi, \tag{3.4.115}$$

there is no additional interactions when all the parameters have similar values ($v_X \approx \mu_X \approx v_\phi \approx \mu_\phi$) and when both vevs are real or purely imaginary at the same time ($X \in \mathbb{R}$, $\phi \in \mathbb{R}$ or $X \in i\mathbb{R}$, $\phi \in i\mathbb{R}$).

For the choice of parameters we use throughout the whole Section, $g = 1$, $v = -0.5$, $\mu = 0.05$, number densities of produced states due to vacuum change combined with quantum effects are of the order $n_\phi, n_\chi, n_{\psi_\chi} \sim 10^{-3}$ and $n_{\psi_\phi} \sim 10^{-5}$. For the rotation of the basis we face more parameters and we cannot choose all of them in a symmetric way for both fields because there is no new effect left then ($\tilde{\mathcal{L}}_{\psi_{int}} = 0$). Instead, we can choose them to be just of the same order, for example: $g = 1$, $v_\chi = 0.5$, $v_\phi = 0.3$, $\mu_\chi = 0.01$, $\mu_\phi = 0.02$, and then, if we approximate number densities using the numerical factors from the Lagrangian, final number densities coming from the rotation of the basis are of the order of $10^{-8} \div 10^{-9}$. It is the mixed term that gives the strongest effect. For the opposite choice of parameters we get approximately the same result, which has been also checked for different combinations of these parameters' values. It turns out that the overall effect is a few orders of magnitude smaller than the production connected with the change of the vacuum combined with quantum corrections.

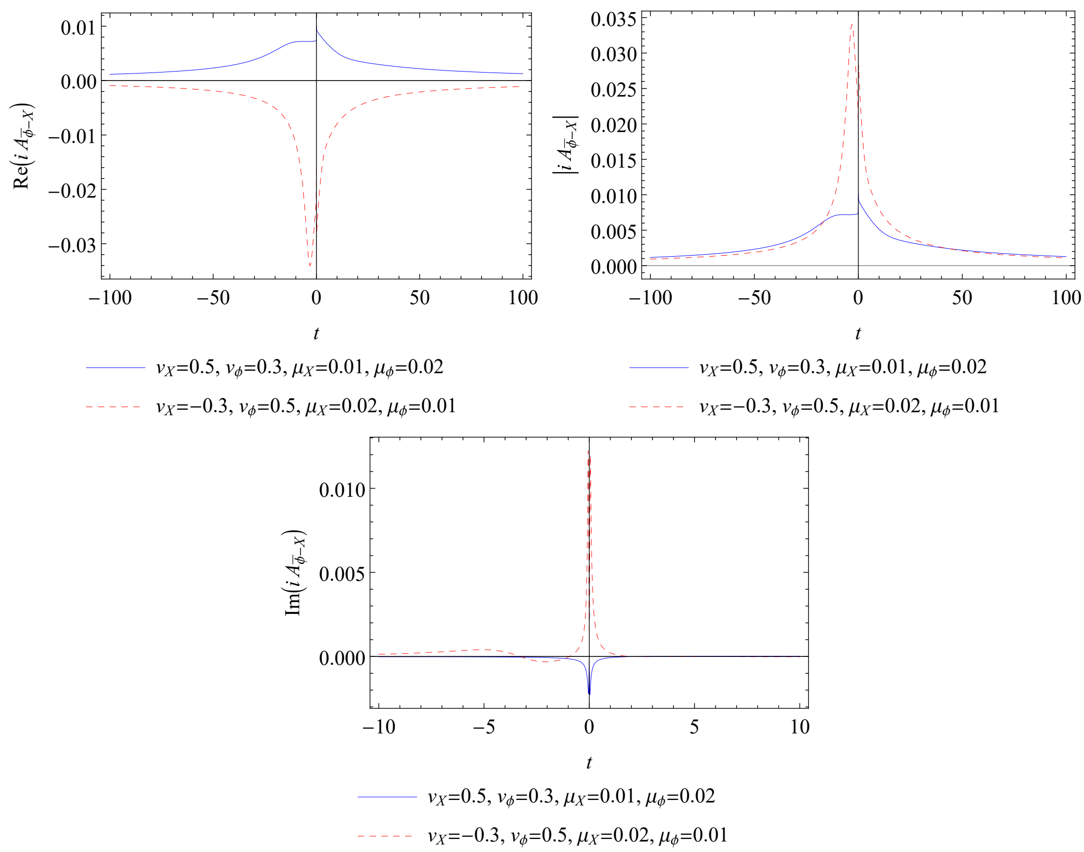

Figure 3.10: Time-dependence of the real part, imaginary part and absolute value of $iA_{\bar{\phi}-X}$ for two different choices of free parameters.

Figure 3.10 presents the time-dependence of one of the new A terms for different choices of our free parameters. Crucial for our analysis is the fact that their influence on the overall production is localized only in the vicinity of the bottom of the potential - all of them possess a symmetric or asymmetric δ shape, which means they do not influence the asymptotic states determining the final number density of produced states in this approximation.

To decide whether scatterings of particles during the process of non-perturbative production is important one needs to compare the mean path $\frac{1}{\Gamma}$ with the time spent in the non-adiabatic area $\Delta t = \frac{v}{gn_\chi}$; when $\frac{1}{\Gamma} \ll \Delta t$ their effect is sizeable. For the supersymmetric model with a single coupling possible processes of such scatterings are

a) $\psi_\chi\psi_\chi \to \psi_\phi\psi_\phi$, $\chi\chi \to \psi_\phi\psi_\phi$, $\psi_\chi\psi_\chi \to \phi\phi$, $\chi\chi \to \phi\phi$ (in both SUSY-breaking scenarios, see Section 3.4.3),

b) $\chi\chi \to \psi_\chi\psi_\chi$ (additional process in gravity-mediation scenario).

Their affect is negligible since

$$\Gamma^{-1}_{\psi_\chi\psi_\chi\to\psi_\phi\psi_\phi} \propto \frac{v^4}{n_\chi^3} \gg \frac{v}{gn_\chi}, \tag{3.4.116}$$

$$\Gamma^{-1}_{\chi\chi\to\psi_\phi\psi_\phi} \propto \frac{v^4}{n_\chi^3} \gg \frac{v}{gn_\chi}, \tag{3.4.117}$$

$$\Gamma^{-1}_{\psi_\chi\psi_\chi\to\phi\phi} \propto \frac{v^4}{n_\chi^3} \gg \frac{v}{gn_\chi}, \tag{3.4.118}$$

$$\Gamma^{-1}_{\chi\chi\to\phi\phi} \propto \frac{v^4}{g^4 n_\chi^3} \gg \frac{v}{gn_\chi}, \tag{3.4.119}$$

$$\Gamma^{-1}_{\chi\chi\to\psi_\chi\psi_\chi} \propto \frac{v^8}{g^4 n_\chi^5} \gg \frac{v}{gn_\chi} \tag{3.4.120}$$

and n_χ is exponential.

Another possible channel of creating particles is perturbative production outside the non-adiabatic region in the phase space. Direct decays could in principle spoil the non-perturbative production as too many χ decays could decrease the energy stored in their sector so much that the trajectory of $\langle\phi\rangle$ does not bend and just roll away to infinity.

Usually we investigate perturbative production using Boltzmann equation [31] but for our purpose it is enough to discuss some simple qualitative estimation without it. If we choose $t = 0$ to correspond to the beginning of some stage of the non-perturbative production, it finishes at $t \sim 1/\sqrt{gv}$ and momentum of produced ϕs and χs is of the order

$$|\mathbf{k}| \lesssim \sqrt{gv} \equiv k_{\max}. \tag{3.4.121}$$

At the same time χ's mass is then of the form

$$m_\chi = g\phi \sim gvt \gtrsim \sqrt{gv}, \tag{3.4.122}$$

which means that the scattering $\chi\chi \to \phi\phi$ (2 massive to 2 massless) is allowed, while $\phi\phi \to \chi\chi$ (2 massless to 2 massive) is not in the perturbative limit. The cross-section of this allowed process is given by

$$\sigma(\chi_1\chi_2 \to \phi_3\phi_4) =$$
$$= \frac{1}{4v_{12}E_1E_2}\int \frac{d^3k_3}{(2\pi)^3}\frac{d^3k_4}{(2\pi)^3}\frac{|\mathcal{M}(\chi_1\chi_2\to\phi_3\phi_4)|^2}{4E_3E_4}(2\pi)^4\delta^4(k_1+k_2-k_3-k_4) =$$
$$= \frac{1}{128\pi^2}\frac{1}{\sqrt{s^2-4m_\chi^2 s}}\int d\Omega|\mathcal{M}|^2 \sim \frac{g^4}{32\pi}\frac{1}{\sqrt{s^2-4m_\chi^2 s}}, \qquad (3.4.123)$$

where $s \equiv (k_1+k_2)^2 = (E_1+E_2)^2 - (\mathbf{k}_1+\mathbf{k}_2)^2$. Then, the actual number of particles produced perturbatively between the two subsequent non-perturbative productions can be estimated as

$$\int_{1/\sqrt{gv}}^{2v/n_\chi - 1/\sqrt{gv}} dt\, n_\chi \sigma v_{\chi\chi} = \frac{g^4}{64\pi}n_\chi \int_{1/\sqrt{gv}}^{2v/n_\chi - 1/\sqrt{gv}} dt\, \frac{1}{k_{\max}^2 + m_\chi^2} \sim$$
$$\sim \frac{g^4}{64\pi}n_\chi \int_{1/\sqrt{gv}}^{2v/n_\chi - 1/\sqrt{gv}} dt\, \frac{1}{m_\chi^2} \sim \frac{g^3}{32\pi}\frac{n_\chi}{\sqrt{gv^3}}\left(1+\mathcal{O}\left(\frac{n_\chi}{\sqrt{gv^3}}\right)\right) \sim \qquad (3.4.124)$$
$$\sim \frac{g^4}{(4\pi)^4}\left(1+\mathcal{O}\left(\frac{g}{(2\pi)^3}\right)\right) \ll 1,$$

assuming that $\mathbf{k}_1 = -\mathbf{k}_2, |\mathbf{k}_1| \sim k_{\max}$ and $n_\chi \sim (gv)^{3/2}/(2\pi)^3$. As the above value is much smaller than 1, we can conclude that perturbative effects can be safely neglected in our analysis provided that the parameters of the theory are not chosen in a very peculiar way, which agrees with [24]. The notation used in this paragraph is presented in the Figure 3.11.

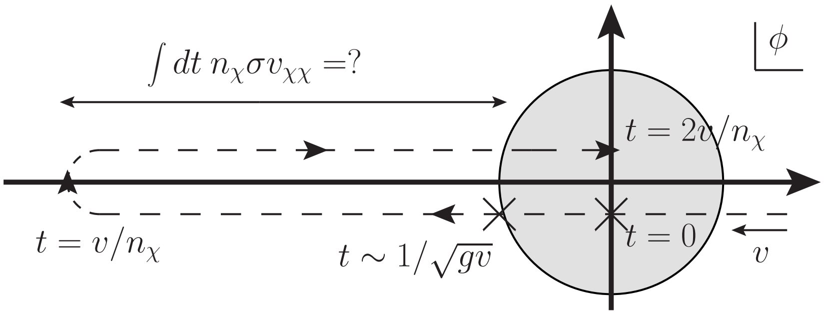

Figure 3.11: Illustration of the parameters used to describe the perturbative production. The gray circle represents the non-perturbative region in the phase space.

3.5 SUSY model with two couplings

In the model with a single coupling only g determines both the strength of the interactions and the mass of produced states. Therefore it is difficult to settle, which effect influences the final production in what way. We can extend it by adding another term with the second coupling h and the third supermultiplet Ψ, which reads

$$W = \frac{g}{2}\Phi X^2 + h\Phi X \Psi^2. \qquad (3.5.1)$$

Even though there is some correction to the previous case added our vacuum choice is still the same: $\langle\phi\rangle \neq 0$ ($\langle\phi\rangle^{\text{as}} = vt + i\mu$)) and $\langle\chi\rangle = \langle\psi\rangle = 0$, since the scalar potential reads:

$$V_{\text{scalar}} = |g\chi + h\psi|^2|\phi|^2 + \left|\frac{g}{2}\chi + h\psi\right| |\chi|^2 + h^2|\phi|^2|\chi|^2. \tag{3.5.2}$$

Equations of motion for this model are given by:

$$\begin{gathered}
0 = \partial^2\phi + \left[\left(g^2 + h^2\right)|\chi|^2 + h^2|\psi|^2 + gh\left(\chi^*\psi + \chi\psi^*\right)\right]\phi + \frac{1}{2}g\tilde{\chi}^\dagger\tilde{\chi}^\dagger, \\
0 = \partial^2\chi + \left[\left(g^2 + h^2\right)|\phi|^2 + \frac{1}{2}g^2|\chi|^2 + h^2|\psi|^2 + \frac{1}{2}gh\left(\chi^*\psi + \chi\psi^*\right)\right]\chi + \\
+gh\left(|\phi|^2 + \frac{1}{2}|\chi|^2\right)\psi + g\tilde{\phi}^\dagger\tilde{\chi}^\dagger + h\tilde{\phi}^\dagger\tilde{\psi}^\dagger, \\
0 = \partial^2\psi + gh\left(\frac{1}{2}|\chi|^2 + |\phi|^2\right)\chi + h^2\left(|\chi|^2 + |\phi|^2\right)\psi + \frac{1}{2}h\tilde{\phi}^\dagger\tilde{\chi}^\dagger, \\
0 = i\bar{\sigma}^\mu\partial_\mu\tilde{\phi} - (g\chi^* + h\psi^*)\tilde{\chi}^\dagger - h\chi^*\tilde{\psi}^\dagger, \\
0 = i\bar{\sigma}^\mu\partial_\mu\tilde{\psi} - h\chi^*\tilde{\phi}^\dagger - h\phi^*\tilde{\chi}^\dagger, \\
0 = i\bar{\sigma}^\mu\partial_\mu\tilde{\chi} - (g\chi^* + h\psi^*)\tilde{\phi}^\dagger - g\phi^*\tilde{\chi}^\dagger - h\phi^*\tilde{\psi}^\dagger,
\end{gathered} \tag{3.5.3}$$

where we can observe explicit mixing of the states χ and ψ. In order to analyse particle production here we need to diagonalize the mass matrix and to construct the basis of mass eigenstates.

Introducing new notation in the above formulae:

$$\begin{gathered}
J_\chi^\dagger = \left[\frac{1}{2}g^2|\chi|^2 + h^2|\psi|^2 + \frac{1}{2}gh\left(\chi^*\psi + \chi\psi^*\right)\right]\chi + \frac{1}{2}gh|\chi|^2\psi + g\tilde{\phi}^\dagger\tilde{\chi}^\dagger + h\tilde{\phi}^\dagger\tilde{\psi}^\dagger, \\
J_\psi^\dagger = \frac{1}{2}gh|\chi|^2\chi + h^2|\chi|^2\psi + \frac{1}{2}h\tilde{\phi}^\dagger\tilde{\chi}^\dagger, \\
J_{\tilde{\chi}}^\dagger = (g\chi^* + h\psi^*)\tilde{\phi}^\dagger, \\
J_{\tilde{\psi}}^\dagger = h\chi^*\tilde{\phi}^\dagger
\end{gathered} \tag{3.5.4}$$

and diagonalizing the mass matrix, we obtain equations of motion for mass eigenstates (bosons: χ', ψ', fermions: $\tilde{\chi}', \tilde{\psi}'$) of the form:

$$\begin{gathered}
\partial^2\chi' + m_{\chi'}^2\chi' + J_{\chi'}^\dagger = 0, \\
\partial^2\psi' + m_{\psi'}^2\chi' + J_{\psi'}^\dagger = 0, \\
0 = i\bar{\sigma}^\mu\partial_\mu\tilde{\chi}' - m_{\tilde{\chi}'}\tilde{\chi}' - J_{\tilde{\chi}'}^\dagger, \\
0 = i\bar{\sigma}^\mu\partial_\mu\tilde{\psi}' - m_{\tilde{\psi}'}\tilde{\psi}' - J_{\tilde{\psi}'}^\dagger.
\end{gathered} \tag{3.5.5}$$

Rotation matrices defined as

$$\begin{gathered}
\begin{pmatrix}\chi' \\ \psi'\end{pmatrix} = U\begin{pmatrix}\chi \\ \psi\end{pmatrix}, \quad \begin{pmatrix}J_{\chi'}^\dagger \\ J_{\psi'}^\dagger\end{pmatrix} = U\begin{pmatrix}J_\chi^\dagger \\ J_\psi^\dagger\end{pmatrix}, \\
\begin{pmatrix}\tilde{\chi}' \\ \tilde{\psi}'\end{pmatrix} = V\begin{pmatrix}\tilde{\chi} \\ \tilde{\psi}\end{pmatrix}, \quad \begin{pmatrix}J_{\tilde{\chi}'}^\dagger \\ J_{\tilde{\psi}'}^\dagger\end{pmatrix} = V\begin{pmatrix}J_{\tilde{\chi}}^\dagger \\ J_{\tilde{\psi}}^\dagger\end{pmatrix}
\end{gathered} \tag{3.5.6}$$

are of the form:

$$U = \begin{pmatrix} -\frac{2h}{\sqrt{4h^2+(g-\sqrt{g^2+4h^2})^2}} & \frac{\sqrt{g^2+4h^2}-g}{\sqrt{4h^2+(g-\sqrt{g^2+4h^2})^2}} \\ -\frac{2h}{\sqrt{4h^2+(g+\sqrt{g^2+4h^2})^2}} & \frac{\sqrt{g^2+4h^2}+g}{\sqrt{4h^2+(g+\sqrt{g^2+4h^2})^2}} \end{pmatrix}, \tag{3.5.7}$$

$$V = \begin{pmatrix} \frac{2h}{\sqrt{4h^2+(g-\sqrt{g^2+4h^2})^2}} & \frac{\sqrt{g^2+4h^2}-g}{\sqrt{4h^2+(g-\sqrt{g^2+4h^2})^2}} \\ -\frac{2h}{\sqrt{4h^2+(g+\sqrt{g^2+4h^2})^2}} & \frac{\sqrt{g^2+4h^2}+g}{\sqrt{4h^2+(g+\sqrt{g^2+4h^2})^2}} \end{pmatrix}. \tag{3.5.8}$$

Mass eigenstates are free since there are no interaction terms coming from the derivative of diagonalizing matrices which are constant, with masses

$$m^2_{\chi'} = m^2_{\tilde{\chi}'} = \frac{1}{2}|\langle\phi\rangle|^2\Big(2h^2 + g^2 + g\sqrt{g^2+4h^2}\Big), \tag{3.5.9}$$

$$m^2_{\psi'} = m_{\tilde{\psi}'} = \frac{1}{2}|\langle\phi\rangle|^2\Big(2h^2 + g^2 - g\sqrt{g^2+4h^2}\Big). \tag{3.5.10}$$

After one "oscillation" number densities of produced species are then given by

$$n_{\chi'} = n_{\tilde{\chi}'} = 2\frac{(\tilde{g}v_\phi)^{3/2}}{(2\pi)^3}e^{-\pi\tilde{g}\mu^2/v}, \tag{3.5.11}$$

$$n_{\psi'} = n_{\tilde{\psi}'} = 2\frac{(\tilde{h}v_\phi)^{3/2}}{(2\pi)^3}e^{-\pi\tilde{h}\mu^2/v}, \tag{3.5.12}$$

where

$$\tilde{g} \equiv \frac{1}{2}\Big(g + \sqrt{g^2+4h^2}\Big), \tag{3.5.13}$$

$$\tilde{h} \equiv \frac{1}{2}\left|g - \sqrt{g^2+4h^2}\right|. \tag{3.5.14}$$

$$\tag{3.5.15}$$

Details of their evolution are presented in the Figures 3.12 and3.13. We can infer that the influence of g is stronger than h as it changes the behaviour of n_k for different species more significantly. Moreover, impact parameter μ differentiates between the two states only for its small values staying indistinguishable asymptotically. Apart from that we can see that heavier states are produced more abundantly no matter what is the choice of the parameters g and h.

3.5.1 Influence of interactions

Again, we are interested in investigating the role of quantum corrections in the particle production in the extended model. Equations of motion for asymptotic

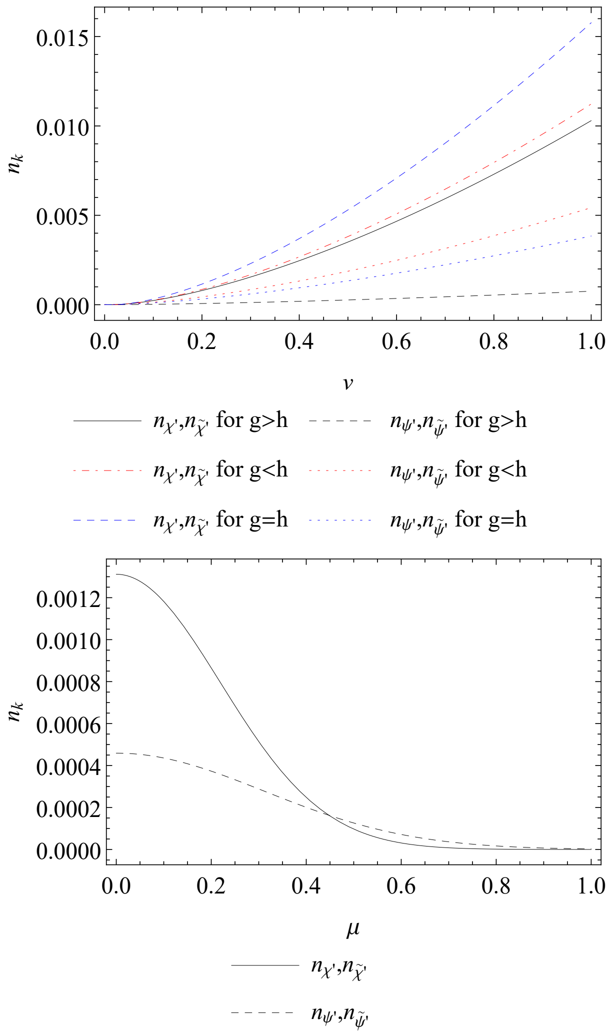

Figure 3.12: Occupation number n_k for the model with two couplings as a function of v (*top*) and μ (*bottom*).

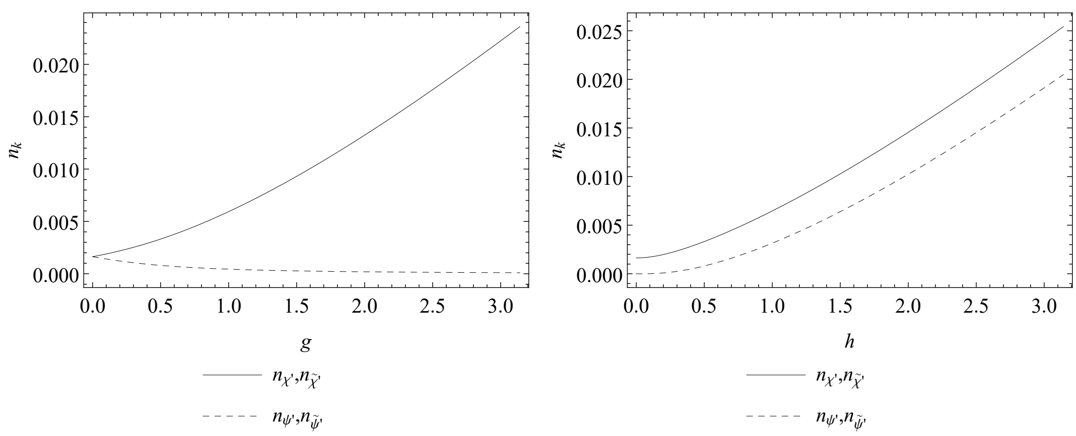

Figure 3.13: Occupation number n_k for the model with two couplings as a function of g (*left*) and h (*right*).

fields read now

$$0 = \ddot{\phi}_k^{\rm as} + k^2\phi_k^{\rm as}, \tag{3.5.16}$$

$$0 = \ddot{\chi}_k'^{\rm as} + \left(k^2 + m^2_{\chi'_k}\right)\chi_k'^{\rm as}, \tag{3.5.17}$$

$$0 = \ddot{\psi}_k'^{\rm as} + \left(k^2 + m^2_{\psi'_k}\right)\psi_k'^{\rm as}, \tag{3.5.18}$$

$$0 = \dot{\tilde{\phi}}_k^{\rm as} + i|k|\tilde{\phi}_k^{\rm as}, \tag{3.5.19}$$

$$0 = \dot{\tilde{\psi}}_k'^{{\rm as}(+),s} + is|k|\tilde{\psi}_k'^{{\rm as}(+),s} + im_{\tilde{\psi}'_k}\tilde{\psi}_k'^{{\rm as}(-),s}, \tag{3.5.20}$$

$$0 = \dot{\tilde{\psi}}_k'^{{\rm as}(-),s} - is|k|\tilde{\psi}_k'^{{\rm as}(-),s} + im^*_{\tilde{\psi}'_k}\tilde{\psi}_k'^{{\rm as}(+),s}, \tag{3.5.21}$$

$$0 = \dot{\tilde{\chi}}_k'^{{\rm as}(+),s} + is|k|\tilde{\chi}_k'^{{\rm as}(+),s} + im_{\tilde{\chi}'_k}\tilde{\chi}_k'^{{\rm as}(-),s}, \tag{3.5.22}$$

$$0 = \dot{\tilde{\chi}}_k'^{{\rm as}(-),s} - is|k|\tilde{\chi}_k'^{{\rm as}(-),s} + im^*_{\tilde{\chi}'_k}\tilde{\chi}_k'^{{\rm as}(+),s}. \tag{3.5.23}$$

Having analogous Yang-Feldman equations and inner product relations as for the case with a single coupling in Section 3.4, we obtain the following occupation numbers for massive fields

$$n_k^{\chi'} = 2V|\beta_k^{\chi'}|^2, \tag{3.5.24}$$

$$n_k^{\tilde{\chi}'} = V\sum_{s=\pm}|\beta_k^{s\tilde{\chi}'}|^2, \tag{3.5.25}$$

$$n_k^{\psi'} = 2V|\beta_k^{\psi'}|^2, \tag{3.5.26}$$

$$n_k^{\tilde{\psi}'} = V\sum_{s=\pm}|\beta_k^{s\tilde{\psi}'}|^2, \tag{3.5.27}$$

where

$$\beta_k^{\chi'} = -iZ_{\chi'}\left(\dot{\chi}_k'^{\rm out*}\chi_k'^{\rm in*} - \chi_k'^{\rm out*}\dot{\chi}_k'^{\rm in*}\right), \tag{3.5.28}$$

$$\beta_k^{s\tilde{\chi}'} = -Z_{\tilde{\chi}'}se^{-i\theta_k^{\tilde{\chi}'}}\left(\tilde{\chi}_k'^{(+)s,{\rm out}*}\tilde{\chi}_k'^{(-)s,{\rm in}*} - \tilde{\chi}_k'^{(-)s,{\rm out}*}\tilde{\chi}_k'^{(+)s,{\rm in}*}\right), \tag{3.5.29}$$

$$\beta_k^{\psi'} = -iZ_{\psi'}\left(\dot{\psi}_k'^{\rm out*}\psi_k'^{\rm in*} - \psi_k'^{\rm out*}\dot{\psi}_k'^{\rm in*}\right), \tag{3.5.30}$$

$$\beta_k^{s\tilde{\psi}'} = -Z_{\tilde{\psi}'}se^{-i\theta_k^{\tilde{\psi}'}}\left(\tilde{\psi}_k'^{(+)s,{\rm out}*}\tilde{\psi}_k'^{(-)s,{\rm in}*} - \tilde{\psi}_k'^{(-)s,{\rm out}*}\tilde{\psi}_k'^{(+)s,{\rm in}*}\right). \tag{3.5.31}$$

For massless fields it reads

$$\begin{aligned}
n_k^{\phi'} &= \sum_{s=\pm}\langle 0^{\rm in}|\left(a_{\phi'_k}^{s,{\rm out}\,\dagger} - \langle a_{\phi'_k}^{s,{\rm out}\,\dagger}\rangle\right)\left(a_{\phi'_k}^{s,{\rm out}} - \langle a_{\phi'_k}^{s,{\rm out}}\rangle\right)|0^{\rm in}\rangle \approx \\
&\approx Z_{\phi'}V\int\frac{d^3p}{(2\pi)^3}\Bigg[\Bigg|\int dt\phi_k'^{\rm out}\Big(\chi'(g^2+h^2)\chi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}\chi_p'^{\rm in} + h^2 Z_{\psi'}\psi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}\psi_p'^{\rm in} + \\
&\quad + ghZ_{\chi'}^{1/2}Z_{\psi'}^{1/2}(\chi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}\psi_p'^{\rm in} + \psi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}\chi_p'^{\rm in})\Big)\langle\phi'^*\rangle\Bigg|^2 + \\
&\quad + \Bigg|\int dt\phi_k'^{\rm out}\Big(Z_{\chi'}(g^2+h^2)\chi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}\chi_p'^{\rm in} + h^2 Z_{\psi'}\psi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}\psi_p'^{\rm in} + \\
&\quad + ghZ_{\chi'}^{1/2}Z_{\psi'}^{1/2}(\chi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}\psi_p'^{\rm in} + \psi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}\chi_p'^{\rm in})\Big)\langle\phi'\rangle\Bigg|^2 + \\
&\quad + \frac{1}{4}Z^2_{\tilde{\chi}'}g^2\sum_{s,q,r}\left(1 + rq\frac{\mathbf{p}\cdot|\mathbf{k}+\mathbf{p}|}{p|\mathbf{k}+\mathbf{p}|}\right)\left|\int dt\phi_k'^{\rm out}\tilde{\chi}'^{(s)r,{\rm in}}_{|\mathbf{k}+\mathbf{p}|}\tilde{\chi}_p'^{(s)q,{\rm in}}\right|^2\Bigg]
\end{aligned} \tag{3.5.32}$$

for boson and

$$n_k^{\tilde{\phi}'} = \sum_{s=\pm} \langle 0^{\rm in} | a^{s,{\rm out}\,\dagger}_{\tilde{\phi}'_k} a^{s,{\rm out}}_{\tilde{\phi}'_k} | 0^{\rm in} \rangle \approx$$

$$\approx \frac{1}{2} Z_{\tilde{\phi}'} V \int \frac{d^3p}{(2\pi)^3} \sum_{s,r} \left(1 - sr\frac{\mathbf{k}\cdot\mathbf{p}}{kp}\right) \times \tag{3.5.33}$$

$$\left| \int dt \tilde{\phi}'^{\rm out}_k \left((g Z_{\chi'}^{1/2} \chi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|} + h Z_{\psi'}^{1/2} \psi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|}) Z_{\tilde{\chi}'}^{1/2} \tilde{\chi}'^{(s)r,{\rm in}}_p + h Z_{\chi'}^{1/2} Z_{\tilde{\psi}'}^{1/2} \chi'^{\rm in}_{|\mathbf{k}+\mathbf{p}|} \tilde{\psi}'^{(s)r,{\rm in}}_p \right) \right|^2$$

for fermion, where we can explicitly see the presence of the new diagrams introduced by adding a term ”$h\Phi X \Psi^2$” to the superpotential.

Side note: Scale factor in the universe filled with perfect fluid in terms of cosmic time

Perfect fluid can be described unambiguously by the barotropic parameter w combining its pressure p and energy density ρ defined as:

$$w = \frac{p}{\rho}. \tag{3.5.34}$$

We can derive the evolution of the scale factor in the universe filled with some perfect fluid depending solely on this parameter and initial conditions using continuity and Friedmann equations. For general $w \neq -1$ we get

$$\rho(t) = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \tag{3.5.35}$$

and thus the scale factor in terms of cosmic time reads

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}, \tag{3.5.36}$$

which for a flat space, $H^2 = \dfrac{8\pi}{3M_{PL}^2}\rho$, transforms into

$$a = a_0 \left(1 + \frac{3}{2}(w+1) H_0 (t - t_0)\right)^{\frac{2}{3(w+1)}}. \tag{3.5.37}$$

If $t \gg t_0$: $a \propto t^{\frac{2}{3(1+w)}}$.

Value of w determines the way the universe evolves

- $w = -1/3$: $\dot{a} = a_0 H_0 = \text{const}$, expansion at a constant rate,
- $w > -1/3$: $\dot{a} \propto t^\alpha$, accelerating expansion,
- $w < -1/3$: $\dot{a} \propto 1/t^{|\alpha|}$, decelerating expansion.

3.6 Expanding universe

Due to the lack of energy conservation in curved spacetime, not the one connected with the explicit time-dependence of the background field but with the expansion of the universe, transformation of our analysis from the flat to curved spacetimes needs to be done very carefully. It may even introduce some new processes such as decays that are forbidden in the flat spacetime because the energy is conserved there [32].

3.6.1 Production without quantum corrections

Equations of motion for the scalar sector in our model in curved spacetime without quantum corrections are given by

$$0 = \ddot{\phi} + 3H\dot{\phi}, \tag{3.6.1}$$

$$0 = \ddot{\chi} + 3H\dot{\chi} - \frac{1}{a^2}\partial_i^2\chi + g^2|\phi|^2\chi. \tag{3.6.2}$$

The first one corresponds to the conservation of $\dot{\phi}$ in the comoving volume

$$0 = \frac{d}{dt}\left(\dot{\phi}a^3\right) \tag{3.6.3}$$

and thus the time evolution of ϕ is given by

$$\phi - \phi_0 = \frac{2}{3(w-1)}\frac{\dot{\phi}_0}{H_0}\left[\left(1 + \frac{3}{2}(1+w)H_0(t-t_0)\right)^{\frac{w-1}{w+1}} - 1\right]. \tag{3.6.4}$$

For different values of the barotropic parameter it means that

- $w = 0$ (matter domination): $\phi = \phi_0 + \dot{\phi}_0(t - t_0)$ (as in the flat case),
- $w = \dfrac{1}{3}$ (radiation domination): $\phi = \phi_0 + \dot{\phi}_0(t - t_0)$ (as in the flat case),
- $w = -1$ (scalar field domination, e.g. inflation): $\phi - \phi_0 = \dfrac{\dot{\phi}_0}{3H_0}\left(1 - e^{-3H_0(t-t_0)}\right)$.

In order to solve (3.6.2) we need to use the plane wave expansion first

$$0 = \ddot{\chi}_k + 3H\dot{\chi}_k + \frac{k}{a^2}\chi_k + g^2|\phi|^2\chi_k, \tag{3.6.5}$$

where k denotes the momentum not the curvature parameter, and then introduce some new variable $\zeta_k := a^{3/2}\chi_k$ depending on the cosmic time. It is defined in such a way to eliminate the term with a single time derivative in (3.6.5) obtaining a very simple equation of motion instead

$$0 = \ddot{\zeta}_k + \omega_k^2\zeta_k, \tag{3.6.6}$$

where

$$\omega_k^2 = -\frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a} + \frac{k^2}{a^2} + g^2|\phi|^2. \tag{3.6.7}$$

In our model this frequency simplifies to

$$\omega_k^2 \approx \frac{k^2}{a_0^2}+g^2\mu^2+\frac{9w}{4}H_0^2-H_0\left(\frac{27}{4}H_0^2 w(1+w)+2\frac{k^2}{a_0^2}\right)(t-t_0)+g^2v^2(t-t_0)^2 \quad (3.6.8)$$

for $H_0(t-t_0) \ll 1$, where $H_0 = \sqrt{\frac{1}{3}}|v|$. This condition assures that the mean time spent by the $\langle\phi\rangle$'s trajectory in the non-adiabatic region is smaller than the Hubble time and it is safe to neglect the expansion of the universe during a single "oscillation", see Figure 3.14. It can be described by the condition

$$\frac{1}{\sqrt{gv}} < \frac{2}{3H(w+1)}, \quad (3.6.9)$$

where H is a Hubble and w a barotropic parameter. Specific conditions valid in the periods dominated by different components are presented in the Table 3.2.

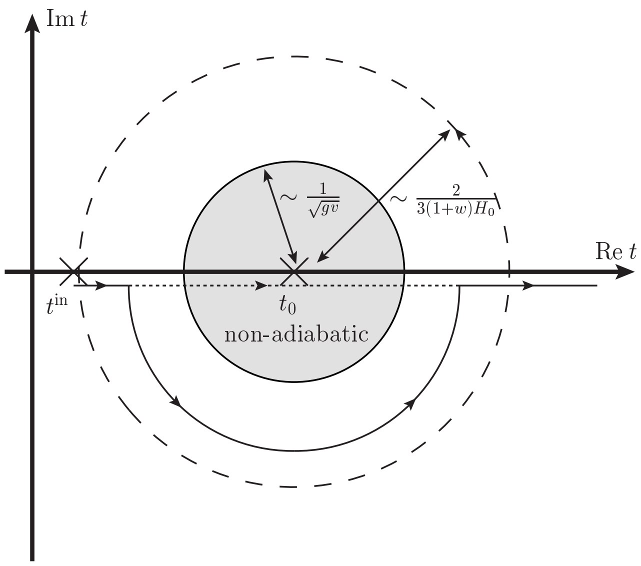

Figure 3.14: Mean time spent by the trajectory of the background in the adiabatic region compared with the Hubble time.

main component	w	condition
matter	0	$\sqrt{gv} > \frac{3}{2}H$
radiation	$\frac{1}{3}$	$\sqrt{gv} > 2H$
kinetic term	1	$\sqrt{gv} > 3H$

Table 3.2: Conditions allowing for neglecting the expansion of the universe in our analysis for different dominating components in its energy density budget.

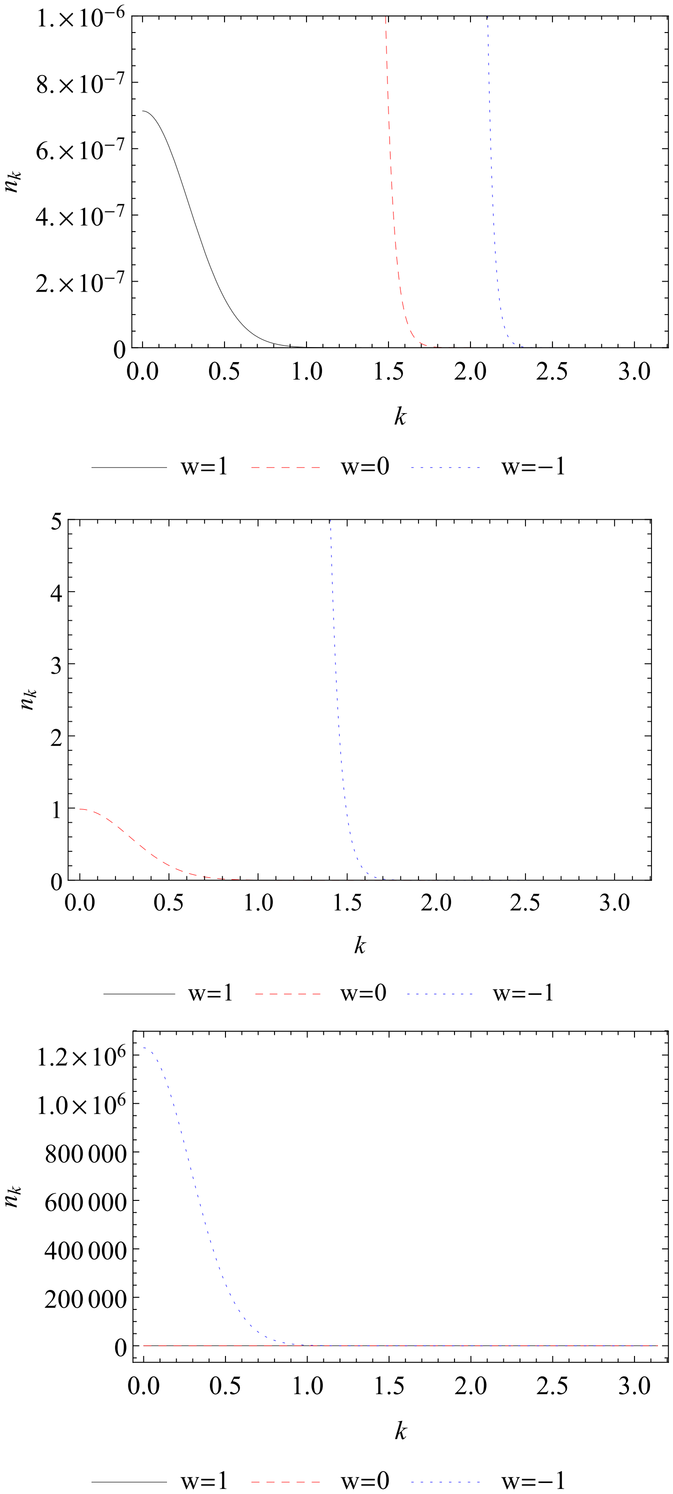

Figure 3.15: Occupation number n_k^ζ for different values of barotropic parameter, $w \in \{-1, 0, 1\}$ for different ranges on y-axis.

Occupation number presented in the Figure 3.15 is given by

$$n_k^\zeta = \exp\left[-\frac{\pi}{gv}\left(\frac{k^2}{a_0^2} + g^2\mu^2 + \frac{9w}{4}H_0^2\right)\right], \tag{3.6.10}$$

$$n_k^\chi = \frac{1}{a^3}\exp\left[-\frac{\pi}{gv}\left(\frac{k^2}{a_0^2} + g^2\mu^2 + \frac{9w}{4}H_0^2\right)\right], \tag{3.6.11}$$

while the number density reads

$$n^{\zeta} = \frac{(gv)^{3/2}}{(2\pi)^3} a_0^3 \exp\left[-\frac{\pi}{gv}\left(g^2\mu^2 + \frac{9w}{4}H_0^2\right)\right], \tag{3.6.12}$$

$$n^{\chi} = \frac{(gv)^{3/2}}{(2\pi)^3}\left(\frac{a_0}{a}\right)^3 \exp\left[-\frac{\pi}{gv}\left(g^2\mu^2 + \frac{9w}{4}H_0^2\right)\right]. \tag{3.6.13}$$

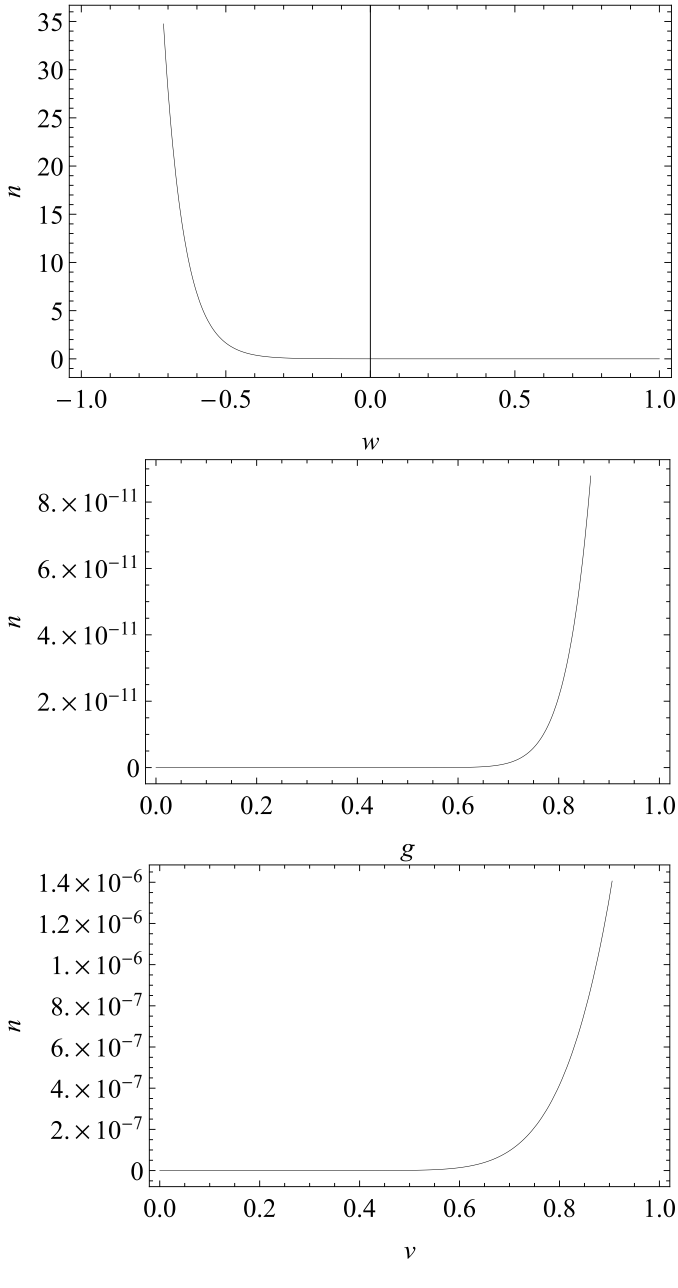

Figure 3.16: The dependence of number density n^{ζ} on w, g and v from top to bottom.

The above formulae hold also for massive fermions as the whole analysis is quasi-classical, although in this case we observe the cut-off momentum $k_{\max}$

$$k_{\max}^2/a_0^2 = \frac{gv}{\pi}\ln(2) - g^2\mu^2 - \frac{9w}{4}H_0^2. \tag{3.6.14}$$

In this approximation massless states are not produced yet.

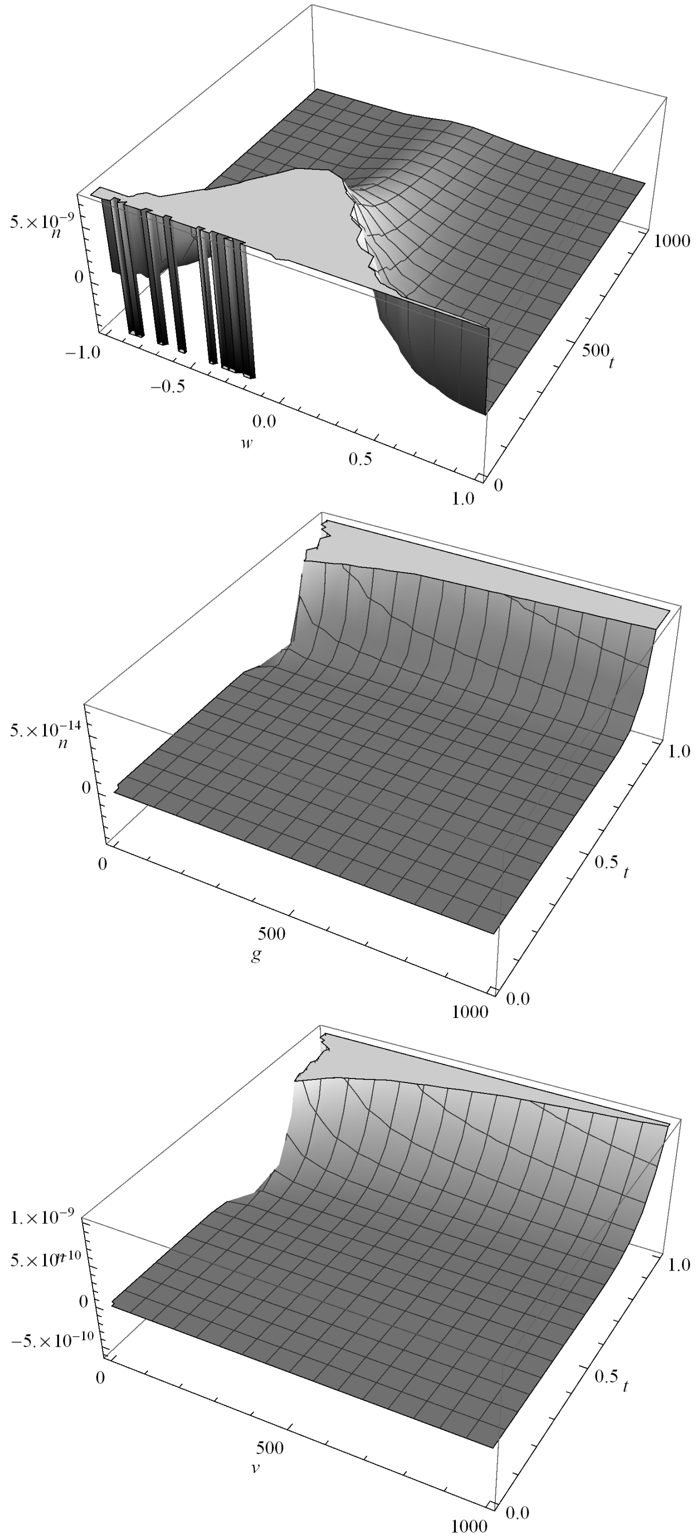

Figure 3.17: The dependence of number density n^{χ} on w, g and v from top to bottom as a function of time.

3.6.2 Production including quantum corrections

In general, commutation relation for some real scalar field Ψ in curved spacetime changes to

$$[\Psi(t,\mathbf{x}),\dot{\Psi}(t,\mathbf{y})]=\frac{i}{a^3}\delta^3(\mathbf{x}-\mathbf{y}), \tag{3.6.15}$$

while the equation of motion reads

$$\Big(\partial_0^2-\frac{1}{a^2}\partial_i^2+3\frac{\dot{a}}{a}\partial_0+M^2(x)\Big)\Psi(x)+J_\Psi(x)=0. \tag{3.6.16}$$

Introducing a new field $\psi=a^{3/2}\Psi$ with the commutation relation

$$[\psi(t,\mathbf{x}),\dot{\psi}(t,\mathbf{y})]=i\delta^3(\mathbf{x}-\mathbf{y}), \tag{3.6.17}$$

once again removes the first derivative term in the above equation transforming it into

$$\Big(\partial_0^2-\frac{1}{a^2}\partial_i^2-\frac{3}{2}\frac{\ddot{a}}{a}-\frac{3}{4}H^2+M^2(x)\Big)\psi(x)+J_\psi(x)=0 \tag{3.6.18}$$

with the solution in a form of the Yang-Feldman equation

$$\psi(x)=\sqrt{Z}\psi^{\rm as}(x)-iZ\int_{t^{\rm as}}^{x^0}dy^0\sqrt{-g}\int d^3y[\psi^{\rm as}(x),\psi^{\rm as}(y)]J_\psi(y). \tag{3.6.19}$$

Again $\psi^{\rm as}$ denotes the asymptotic field

$$\psi(t^{\rm as},\mathbf{x})=\sqrt{Z}\psi^{\rm as}(t^{\rm as},\mathbf{x}), \tag{3.6.20}$$

satisfying the free field equation of motion

$$\Big(\partial_0^2-\frac{1}{a^2}\partial_i^2-\frac{3}{2}\frac{\ddot{a}}{a}-\frac{3}{4}H^2+M^2(x)\Big)\psi^{\rm as}(x)=0. \tag{3.6.21}$$

Following the same procedure as for the flat spacetime, we can finally obtain the expression describing occupation number of produced ψ states including quantum corrections

$$n_k^\psi(t)=\Big|\Big(\beta_k(t)a_{-\mathbf{k}}^{\rm in\dagger}-i\sqrt{Z}\int d^4x\sqrt{-g}J_\psi(x)\Big[-\beta_k\psi_k^{\rm in}(x^0)+\alpha_k\psi_k^{\rm in*}(x^0)\Big]\Big)|0^{\rm in}\rangle\Big|^2, \tag{3.6.22}$$

which reads

$$n_k^\psi(t)=V|\beta_k(t)|^2+... \tag{3.6.23}$$

for $\beta_k\neq 0$ and

$$n_k^\psi(t)=0+Z\Big|\int d^4x\sqrt{-g}J_\psi(x)\psi_k^{\rm in*}(x^0)|0^{\rm in}\rangle\Big|^2 \tag{3.6.24}$$

for $\beta_k=0$. The presence of the scale factor is encoded in parametrization connecting Ψ and ψ, the volume of the system V and the new definition of the source term J_ψ.

For comoving coordinates Dirac equation in curved spacetime is given by

$$i\Big(\gamma^0\partial_0+\frac{1}{a}\gamma^i\partial_i-\frac{\dot{a}}{a}\gamma^i\Sigma^{0i}\Big)\Psi=m\Psi, \tag{3.6.25}$$

where $\Sigma^{\alpha\beta} = \frac{1}{4}[\gamma^\alpha, \gamma^\beta]$[5]. For FRW metric it reads

$$i\Big(\gamma^0\partial_0 + \frac{1}{a}\gamma^i\partial_i - \frac{3}{2}H\gamma^0\Big)\Psi = m\Psi. \tag{3.6.26}$$

Thus, equations of motion for the considered model with the reparametrization

$$\begin{cases} \zeta_k = a^{3/2}\chi_k \\ \theta_k = a^{3/2}\phi_k \\ \tilde{\zeta}_k = a^{3/2}\psi_{\chi_k} \\ \tilde{\theta}_k = a^{3/2}\psi_{\phi_k} \end{cases} \tag{3.6.27}$$

are given by

$$0 = \left[\partial_0^2 - \frac{1}{a^2}\partial_i^2 - \frac{3}{2}\frac{\ddot{a}}{a} - \frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2\right]\theta + J_\theta^\dagger, \tag{3.6.28}$$

$$0 = \left[\partial_0^2 - \frac{1}{a^2}\partial_i^2 - \frac{3}{2}\frac{\ddot{a}}{a} - \frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2 + g^2|\langle\phi\rangle|^2\right]\zeta + J_\zeta^\dagger, \tag{3.6.29}$$

$$0 = i\left[\bar{\sigma}^0\partial_0 + \frac{k}{a}\bar{\sigma}^i\right]\tilde{\theta}_k - J_{\tilde{\theta}}^\dagger, \tag{3.6.30}$$

$$0 = i\left[\bar{\sigma}^0\partial_0 + \frac{k}{a}\bar{\sigma}^i\right]\tilde{\zeta}_k - g\langle\phi^\dagger\rangle\tilde{\zeta}_k^\dagger - J_{\tilde{\zeta}}^\dagger \tag{3.6.31}$$

with the source terms of the form

$$J_\theta^\dagger = a^{-3}\left[g^2|\zeta|^2\theta + \frac{1}{2}a^{3/2}g\tilde{\zeta}^\dagger\tilde{\zeta}^\dagger\right], \tag{3.6.32}$$

$$J_\zeta^\dagger = a^{-3}\left[g^2\left(\frac{1}{2}|\zeta|^2 + |\theta|^2 + a^3|\langle\phi\rangle|^2\right) + ga^{3/2}\tilde{\zeta}^\dagger\tilde{\theta}^\dagger\right], \tag{3.6.33}$$

$$J_{\tilde{\theta}}^\dagger = a^{-3/2}g\zeta_k^\dagger\tilde{\zeta}_k^\dagger, \tag{3.6.34}$$

$$J_{\tilde{\zeta}}^\dagger = a^{-3/2}g\left(\zeta_k^\dagger\tilde{\theta}_k^\dagger + \theta_k^\dagger\tilde{\zeta}_k - a^{3/2}\langle\phi^\dagger\rangle\tilde{\zeta}_k\right). \tag{3.6.35}$$

Equations of motion for the free asymptotic fields read

$$0 = \ddot{\theta}_k^{\rm as} + \left[\frac{k^2}{a^2} - \frac{3}{2}\frac{\ddot{a}}{a} - \frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2\right]\theta_k^{\rm as}, \tag{3.6.36}$$

$$0 = \ddot{\zeta}_k^{\rm as} + \left[\frac{k^2}{a^2}\partial_i^2 - \frac{3}{2}\frac{\ddot{a}}{a} - \frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2 + g^2\left|\langle\phi\rangle\right|^2\right]\zeta_k^{\rm as}, \tag{3.6.37}$$

$$0 = \dot{\tilde{\theta}}_k^{\rm as} + i\frac{k}{a}\tilde{\theta}_k^{\rm as}, \tag{3.6.38}$$

$$0 = \dot{\tilde{\zeta}}_k^{(+)s,{\rm as}} + is\frac{k}{a}\tilde{\zeta}_k^{(+)s,{\rm as}} + ig\langle\phi^\dagger\rangle\tilde{\zeta}_k^{(-)h,{\rm as}}, \tag{3.6.39}$$

$$0 = \dot{\tilde{\zeta}}_k^{(-)s,{\rm as}} - is\frac{k}{a}\tilde{\zeta}_k^{(-)s,{\rm as}} + ig\langle\phi^\dagger\rangle\tilde{\zeta}_k^{(+)h,{\rm as}}, \tag{3.6.40}$$

where $\pm$ labels distinguish between a particle and antiparticle for scalars and indicate the helicity of fermions.

[5] $\Sigma^{0i} = \frac{1}{4}[\gamma^0, \gamma^i]$.

Occupation numbers of massive particles ζ and $\tilde{\zeta}$ agree with the case of production without quantum corrections at the leading order, while for massless states they are given by

$$\begin{aligned} n_{\theta k} \approx g^2 V \int \frac{d^3p}{(2\pi)^3} \Bigg[& Z_\theta Z_\zeta^2 \left| \int dt\, a^{-3} \theta_k^{\text{out}} \zeta_{|\mathbf{k}+\mathbf{p}|}^{\text{in}} \zeta_p^{\text{in}} \cdot g \left\langle \theta^* \right\rangle \right|^2 + \\ & + Z_\theta Z_\zeta^2 \left| \int dt\, a^{-3} \theta_k^{\text{out}} \zeta_{|\mathbf{k}+\mathbf{p}|}^{\text{in}} \zeta_p^{\text{in}} \cdot g \left\langle \theta \right\rangle \right|^2 + \\ & + \frac{1}{4} Z_\theta Z_{\tilde{\zeta}}^2 \sum_{s,q,r} \left(1 + rq \frac{\mathbf{p} \cdot (\mathbf{k}+\mathbf{p})}{p|\mathbf{k}+\mathbf{p}|} \right) \left| \int dt\, a^{-3/2} \theta_k^{\text{out}} \tilde{\zeta}_{|\mathbf{k}+\mathbf{p}|}^{(s)r,\text{in}} \tilde{\zeta}_p^{(s)q,\text{in}} \right|^2 \Bigg] \end{aligned} \tag{3.6.41}$$

for boson and

$$n_{\tilde{\theta} k} \approx g^2 V Z_\zeta Z_{\tilde{\theta}} Z_{\tilde{\zeta}} \int \frac{d^3p}{(2\pi)^3} \sum_{s,r} \frac{1}{2} \left(1 - sr \frac{\mathbf{k} \cdot \mathbf{p}}{kp} \right) \left| \int dt\, a^{-3/2} \tilde{\theta}_k^{\text{out}} \zeta_{|\mathbf{k}+\mathbf{p}|}^{\text{in}} \tilde{\zeta}_p^{(s)r,\text{in}} \right|^2 \tag{3.6.42}$$

for fermion, where we can explicitly observe the destructive influence of the scale factor on the final occupation number for both states.

Summary of the chapter

- We describe the method of calculating number density of the states produced due to the time-dependence of the background field based on the asymptotic approximation of the wave functions and analytic continuation of the time coordinate. We prove that massless states can be produced as abundantly as the massive ones due to quantum corrections. We extend our method to include fermions of different types (Weyl, Majorana and Dirac) and obtain the general formulae needed for calculating their final number densities. We introduce the influence of interactions (quantum corrections) on the process of fermionic production based on the Yang-Feldman formalism.

- We apply the whole method to the supersymmetric model with a single coupling, bosonic and fermionic sector containing both massive and massless fields. We also compare different possible sources of production (quantum corrections, perturbative production and rotation of the basis), investigate the one-loop corrections leading to the physical mass and address the issue of the SUSY-breaking.

- We extend our considerations to the supersymmetric model with two couplings.

- We show how the expansion of the universe changes our analysis.

Chapter 4

Particle production in adiabatic approximation for massive background field - parametric resonance

Cosmic inflation [33, 34, 35] is a well adopted theory assuming a phase of the accelerated expansion of the scale factor in the early evolution of the Universe, which solves many problems of the classical cosmological model with the Big Bang [36] and is compatible with the recent experimental data [37, 38].

Post-inflationary particle production aiming at gaining the radiation dominance in the Universe is a very complex process that mixes perturbative and non-perturbative processes [24, 39, 40, 41]. Usually we can distinguish two main stages there:

a) preheating - when exponentially and non-perturbatively produced states typically correspond to the fields directly interacting with the inflaton, they do affect the mass term of the inflaton through backreaction effects,

b) reheating (thermalization) [42, 43, 44, 45] - when the inflaton decays perturbatively and produced particles end up in thermal equilibrium with a well-defined temperature.

For recent reviews of post-inflationary particle production see [14] or [43].

This Chapter presents this notion following the classical literature and the Mathieu approach, which is in some sense limited to the case when the amplitude of oscillations is almost constant $\Phi \approx \Phi_0$ - apart from some particular cases it is impossible to obtain closed-form formulae for the Floquet exponents [14]. Moreover, we address the issue of parametric resonance following the method from Section 3.1 used previously for the massless background neglecting the influence of quantum corrections for now. The case including them is the subject of the next chapter.

4.1 Inflation and post-inflationary particle production

Cosmological inflation is a well-grounded hypothesis proposed in 1981 by A. Guth [36] during his research on magnetic monopoles. It predicts the existence of a period of accelerated expansion in the early Universe that must end before the beginning of nucleosynthesis. Field responsible for inflation is called inflaton and it can realize accelerated expansion of the Universe, $\frac{d^2a}{dt^2} > 0$, by carrying negative pressure[1]. There is a plethora of particle physics inflationary scenarios and, while some of them are consistent with observations, we are unable to chose the "true" one. Nevertheless, usually it is assumed that after inflation inflaton decays to radiation and some SM (Standard Model) fields or mediators, which is followed by the usual thermal history of the Universe with radiation domination, matter domination etc.

Accelerated exponential expansion associated with inflation means that two inertial observers recede with growing velocity, which corresponds to the metric for one of them that reads

$$ds^2 = -(1 - \Lambda r^2)dt^2 + \frac{1}{1 - \Lambda r^2}dr^2 + r^2 d\Omega^2,$$

which is analogous to the black hole metric and describes the de Sitter spacetime. It requires cosmological constant, Λ, domination and the equation of state $p = -\rho$, which dictates almost constant Hubble parameter, $H = \frac{\dot{a}}{a} \propto \sqrt{\Lambda}$, and scale factor exponential in time, e^{Ht}.

It has been soon realized that inflationary hypothesis copes with several severe problems of the present cosmological model dynamically without fine-tuning of the initial conditions:

a) flatness problem - how to explain the flatness of the Universe?

   Present energy density of the Universe is very close to the critical value, ρ_{cr}, which corresponds to the vanishing curvature of space (Euclidean space). Actual limit reads: $|\Omega_0 - 1| < 0.02$, where $\Omega = \frac{\rho}{\rho_{cr}} = \frac{k}{a^2H^2} + 1$ and "0" corresponds to the present time. Function "$\Omega_0 - 1$" grows in time[2], which requires significant fine-tuning. It can be estimated that if right after Big Bang $\frac{\rho_{BB}}{\rho_{cr,BB}} = 99,99\%$, it would be observed at present that $\frac{\rho_0}{\rho_{cr,0}} = 10^{-11}\%$, which corresponds to the considerable curvature.

   <u>Inflationary solution</u>:
   During inflation energy density of the inflaton is more or less constant, while for other components (inhomogeneities, curvature, SM particles etc.) it strongly

[1]It follows from the Friedmann equation that $\frac{d^2a}{dt^2} > 0$ corresponds to $\rho + 3p < 0$
[2]For RD $a \propto t^{1/2}$ and $|\Omega - 1| \propto a^2$, while for MD $a \propto t^{2/3}$ and $|\Omega - 1| \propto a$.

depends on time making them negligible during inflation. It makes the Universe flat, symmetric and empty[3] after inflation.

b) horizon problem - why the Universe is isotropic and CMB so homogeneous?

It follows from the observations of CMB (Cosmic Microwave Background) that its temperature fluctuations are of the order $\frac{\Delta T}{T} \sim 10^{-5}$ on large scales. Regions connected causally at the moment of CMB formation include sections around $\Delta\theta \approx 2^o$ now. So there were about 10^5 separated regions at the moment of recombination in which observed inhomogeneities are $\sim 10^{-5}$ at present.

Inflationary solution:
In inflationary theory expansion of the Universe was initially slow enough to enable communication between the regions within our horizon and equalize the temperature there as well.

c) magnetic monopole problem - why no magnetic monopoles are observed?

According to the GUT theory in the early Universe lots of heavy, non-relativistic and stable magnetic monopoles should be created but they have not been observed so far. They are hypothetically produced before radiation domination, $\rho_{RD} \propto a^{-4}$, and their density behaves like $\rho_{mono} \propto \frac{1}{a^3}$, which means that even if their initial abundance is very low they should dominate energy density of the Universe right away. Apart from the monopoles it concerns also moduli and gravitinos.

Inflationary solution:
It is assumed that inflation occurs after the hypothetical production of the monopoles. Rapid expansion of the Universe leads to the decrease of the observed monopoles' density by many orders of magnitude.

4.1.1 Early models of inflation

Observed problem of magnetic monopoles lead Guth to its solution by the decay of false vacuum in the early Universe followed by the inflation driven by the scalar field. At the same time Starobinsky developed his model of inflation from the perspective of modified gravity. Both scenarios predict the de Sitter epoch but differ in details.

Starobinsky noticed that quantum corrections to general relativity should be significant for the early Universe, which resulted in corrections to Einstein-Hilbert action quadratic in the Ricci scalar R and $f(R)$ modification of gravity [46]. For large curvature it leads to the existence of the effective cosmological constant, which Starobinsky identified with the de Sitter epoch in the early Universe. It dealt with

[3]It is only filled with quantum fluctuations.

the problems of the hitherto cosmological model and introduced some specific corrections to the CMB spectrum that can be calculated. Starobinsky used the action

$$S = \frac{1}{2}\int d^4x \left(R + \frac{R^2}{6M^2}\right), \tag{4.1.1}$$

which corresponds to the potential $V(\phi) = \Lambda^4 \left(1 - e^{-\sqrt{2/3}\phi/M_{PL}}\right)^2$ in the Einstein frame. Inflationary observables for this model are equal to: $n_s = 1 - \frac{2}{N}$ and $r = \frac{12}{N^2}$, which for $50 < N < 60$ are in agreement with the experimental data from PLANCK.

Guth based his inflationary hypothesis on the metastable false vacuum with huge positive energy in the early Universe that behaves like cosmological constant. Positive energy corresponds to negative pressure, which causes accelerated expansion of the Universe with fixed energy density. The fate of this false vacuum is to decay via quantum tunnelling into the true vacuum. Guth realized that there is a problem with reheating after inflation in his model - there was no radiation produced during nucleation of the bubbles. The only possibility of producing radiation there would be during the collisions of the bubbles but there are too rare, when inflation lasts long enough to solve cosmological problems.

4.1.2 Slow-roll inflation

Problem with bubble collisions was solved at the same time by Linde [47], Albrecht and Steinhardt [48] with so-called new inflation scenario. Their idea was to drive inflation by the slow roll of the scalar field on the slope of the potential (slow-roll inflation), which ends with the reheating phase described by the oscillations of the field around the minimum of the potential, see Figure 4.1.

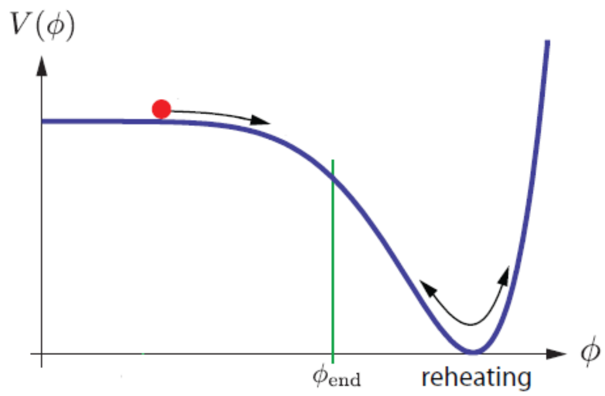

Figure 4.1: Scheme of the slow-roll inflation from [23].

Assuming that inflaton ϕ is minimally coupled to gravity and that its interactions with other fields become significant after inflation, its action reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{4.1.2}$$

Then energy-stress tensor defined as

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \mathcal{L} g^{\mu\nu} \tag{4.1.3}$$

corresponds to energy density and pressure of the scalar field of the form

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{4.1.4}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{4.1.5}$$

assuming FRW metric and vanishing spatial derivatives $\partial_i \phi \to 0$. From $T^{\mu\nu}_{;\nu} = 0$ we obtain the equation of motion for the inflaton

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} + V'(\phi) = 0. \tag{4.1.6}$$

Slow-roll means that during inflation kinetic energy of inflaton is much smaller than its potential energy, $E_{kin} \ll E_{pot}$, which corresponds to the approximate de Sitter: $\rho_\phi \approx V(\phi)$ and $p_\phi \approx -V(\phi)$. Friedmann equation translates then into

$$H^2 = \frac{8\pi}{3M_{PL}^2} V(\phi) \tag{4.1.7}$$

and the equation of motion simplifies to

$$3H\dot{\phi} + V'(\phi) = 0. \tag{4.1.8}$$

We can define two slow-roll parameters

$$\epsilon = \frac{M_{PL}^2}{16\pi}\left(\frac{V'}{V}\right)^2, \tag{4.1.9}$$

$$\eta \equiv \frac{M_{PL}^2}{8\pi}\frac{|V''|}{V}, \tag{4.1.10}$$

that are very small during inflation, $\frac{1}{3}\epsilon \ll 1$ and $\frac{1}{3}|\eta| \ll 1$, and their growth constrains its end. They originate from the need that during slow-roll inflation kinetic energy has to be much smaller than potential energy (ϵ) and that second time derivative of inflaton is much smaller than the first one (η). Inflation ends when one of these parameters is of the order of one.

Duration of inflation can be characterized by the number of e-folds

$$N \equiv \log\left(\frac{a_{fin}}{a_{ini}}\right) \tag{4.1.11}$$

describing exponential growth of the Universe. For the slow-roll inflation it is equal to

$$N = \int_{\phi_{ini}}^{\phi_{fin}} \frac{H}{\dot{\phi}} d\phi = \frac{2\sqrt{\pi}}{M_{PL}} \int_{\phi_{ini}}^{\phi_{fin}} \frac{d\phi}{\sqrt{\epsilon}}. \tag{4.1.12}$$

Depending on the inflationary scenario and the details of reheating we need $N \sim 50 \div 70$.

4.1.3 Reheating

Temperature of the Universe decreases significantly during inflation but afterwards the Universe is dominated by inflaton and gets reheated in the reheating process. Huge potential energy of the inflaton is then transferred onto the SM particles or the mediators, including electromagnetic radiation, starting radiation domination epoch this way. Reheating is a very complicated multi-stage process, which incorporates the end of inflation in a very natural way.

For simplicity we can describe reheating perturbatively as oscillations of the inflaton around the minimum of the potential ϕ_0 with the amplitude decreasing in time right after the breakdown of the slow-roll regime. Equation of motion for the inflaton reads then

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} + V'(\phi) = 0 \tag{4.1.13}$$

with[4] $V \approx \frac{1}{2}\omega^2(\phi - \phi_0)^2$. At the beginning of reheating energy density is accumulated in the inflaton but it gets attenuated by two effects - expansion of the Universe (H) and the decay of the inflaton (Γ_ϕ).

Averaging over time we obtain the equation of state of dust with $p = 0$. Asymptotically for $mt \gg 1$ it corresponds to the solution of the form

$$\phi(t) \to \frac{\phi_0}{a^{3/2}(t)} \sin(mt) = \frac{m_{PL}}{\sqrt{3\pi} mt} \sin(mt), \tag{4.1.14}$$

where $a(t) \approx t^{2/3}$.

Equation of motion for quadratic potential reads

$$\ddot{\phi}\dot{\phi} + 3H\dot{\phi}^2 + \Gamma_\phi \dot{\phi}^2 + \omega^2(\phi - \phi_0)\dot{\phi} = \tag{4.1.15}$$

$$= \left(\frac{1}{2}\dot{\phi}^2\right)^{\cdot} + \frac{1}{2}\omega^2 \left((\phi - \phi_0)^2\right)^{\cdot} + (3H + \Gamma_\phi)\, \dot{\phi}^2 = 0. \tag{4.1.16}$$

We can rewrite it using

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) = \langle \frac{1}{2}\dot{\phi}^2 \rangle + \langle \frac{1}{2}\omega^2 (\phi - \phi_0)^2 \rangle = \langle \dot{\phi}^2 \rangle \tag{4.1.17}$$

to finally get

$$\dot{\rho}_\phi + (3H + \Gamma_\phi)\, \rho_\phi = 0. \tag{4.1.18}$$

For small coupling Γ_ϕ is typically smaller than H at the end of inflation, which means that at the beginning of reheating decays are negligible in comparison with the expansion of the Universe. As H decreases production of particles becomes more effective and at time when $H = \Gamma_\phi$ produced states gain thermal distribution with so-called reheating temperature:

$$T_{RH} \sim (\Gamma_\phi M_{PL})^{1/2}. \tag{4.1.19}$$

Since Γ_ϕ is proportional to some power of the coupling, which in general is small, perturbative reheating is very ineffective and T_{RH} can be surprisingly much smaller

[4]Even if the potential is not quadratic, we can approximate it by this expression around its minimum.

that the scale of inflation. Nevertheless, T_{RH} is not the maximal temperature reached during reheating, this one is denoted as T_{MAX} and is proportional to $T_{MAX} \sim (T_{RH}M)^{1/2}$, where M is the scale of inflation.

4.1.4 Preheating

Perturbative approach is unfortunately incomplete as it ignores the fact that rapid oscillations of inflaton can take place far beyond the equilibrium and lead to the parametric resonance (preheating). Apart from that it does not take into consideration the nature of inflaton at the beginning of the oscillation phase - it is rather a coherent oscillating homogeneous field, not a superposition of asymptotic states. Large amplitude of oscillations indicates that we can treat inflaton as a classical background in which χ quantum states are produced.

Preheating involves production of particles under extreme conditions - high energy, instabilities, non-linearities. It concerns the non-adiabatic and non-perturbative stage of coherent oscillations during which the masses of produced states can change rapidly in time and even become larger than the mass of inflaton.

The idea of preheating can be illustrated based on very simple model of interactions

$$\mathcal{L}_{int} = -\frac{1}{2}g^2\chi^2\phi^2, \tag{4.1.20}$$

where χ is some SM field or the mediator. For simplicity we can neglect expansion of the Universe as it is reasonable to assume that the period of oscillations is shorter that the Hubble time H^{-1}, see Section 3.6.

Quantum modes of the field $\hat{\chi}$

$$\hat{\chi}(t, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k (\chi_k^* \hat{a}_k e^{i\mathbf{kx}} + \chi_k \hat{a}_k^\dagger e^{-i\mathbf{kx}}) \tag{4.1.21}$$

fulfil equation of motion

$$\ddot{\chi}_k + (k^2 + m_\chi^2 + g^2\Phi^2 \sin^2(mt))\chi_k = 0, \tag{4.1.22}$$

where Φ is the amplitude of oscillations. This equation is called Mathieu equation

$$\chi_k'' + (A_k - 2q\cos(2z))\chi_k = 0 \tag{4.1.23}$$

with

$$z = mt, \tag{4.1.24}$$

$$A_k = \frac{k^2 + m_\chi^2}{m^2} + 2q, \tag{4.1.25}$$

$$q = \frac{g^2\Phi^2}{4m^2} \tag{4.1.26}$$

and prime denoting the derivative over z. We can consider its solutions mode after mode remembering that exponential growth of the mode corresponds to particle production. Mathieu equation is unstable for some values of k and leads to exponential growth of $\chi_k \propto e^{\mu_k z}$, where μ_k are Floquet exponents.

For small values of q ($q \ll 1$) resonance occurs in the narrow area around $k = m$, thus it is called *narrow resonance*. It is much more effective for $q \gg 1$, when also modes with $k \to 0$ are produced, and it is called *broad resonance* then. Production has to be non-adiabatic, which is equivalent to breaking the WKB approximation for the evolution of our field $\chi_k \propto e^{\pm i \int \omega_k dt}$ holding for

$$\frac{d\omega_k^2}{dt} \leqslant 2\omega_k^3, \tag{4.1.27}$$

where

$$\omega_k = \sqrt{k^2 + m_\chi^2 + g^2\Phi^2(t)\sin^2(mt)}. \tag{4.1.28}$$

It means that modes with momentum

$$k^2 \leqslant \frac{2}{3\sqrt{3}} gm\Phi - m_\chi^2 \tag{4.1.29}$$

are produced each time ϕ approaches zero.

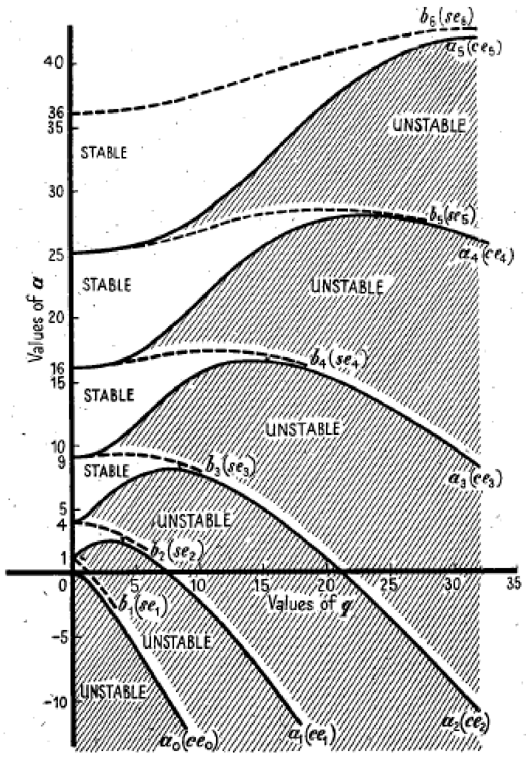

Figure 4.2: Instability bands of Mathieu equation on the A-q plane. Figure copied from [40].

4.1.5 Observables

Theory of inflation predicts that all the visible structures of the Universe have their origin in the quantum fluctuations from the inflationary epoch. If their distribution is Gaussian, all the statistical information about primordial fluctuation is specified by the power spectrum (two-point correlation function). Such a spectrum depends on two parameters: spectral index n_s that measures the deviation of the spectrum from the exact scale invariance (de Sitter universe) and tensor-to-scalar ratio r. Hypothetical primordial non-gaussianity may be encoded in higher order correlation functions - for a single field inflation should be small, while for multifield inflation or single field with non-trivial kinetic term and broken slow-roll conditions could be non negligible.

The most important measure of primordial fluctuations is power spectrum for $\mathcal{R}$ (gauge-invariant comoving curvature scalar), $\mathcal{P}_{\mathcal{R}}(k)$, that is directly connected with density fluctuations:

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \mathcal{P}_{\mathcal{R}}(k), \tag{4.1.30}$$

$$\Delta_s^2 \equiv \Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2} \mathcal{P}_{\mathcal{R}}(k) \tag{4.1.31}$$

with the normalization condition for the dimensionless spectrum $\Delta_{\mathcal{R}}^2(k)$: $\langle \mathcal{R}\mathcal{R} \rangle = \int_0^\infty \Delta_{\mathcal{R}}^2(k) d \ln k$, where $\langle ... \rangle$ denotes the average over all possible parameters. Scale dependence of the spectrum is determined by the spectral index:

$$n_s \equiv \frac{d \ln \Delta_s^2}{d \ln k}, \tag{4.1.32}$$

which for $n_s = 1$ corresponds to the exact scale-invariance.

We also define power spectrum of both polarizations of tensor metric fluctuations h_{ij}, $h \in \{h^\times, h^+\}$, as

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \mathcal{P}_h(k), \tag{4.1.33}$$

$$\Delta_h^2 = \frac{k^3}{2\pi^2} \mathcal{P}_h(k). \tag{4.1.34}$$

Sumarically, tensor power spectrum is equal to $\Delta_t^2 = 2\Delta_h^2$ and can be normalized in comparison to the scalar spectrum via tensor-to-scalar ratio

$$r \equiv \frac{\Delta_t^2(k)}{\Delta_s^2(k)}. \tag{4.1.35}$$

Since $\Delta_s^2(k)$ is fixed and $\Delta_t^2(k) \propto H^2 \approx V$, r translates into the inflationary energy scale

$$H = 2.54 \times 10^{13} \text{GeV} \left(\frac{r}{0.01} \right)^{1/2}. \tag{4.1.36}$$

4.2 Results for scalars

Our analysis based on the method presented in Section 3.1 starts with the simplest scalar model of post-inflationary production with inflaton ϕ and coupled field χ, both real scalars,

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (\partial \chi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} g^2 \phi^2 \chi^2. \tag{4.2.1}$$

Time evolution of the inflaton can be described as

$$\phi(t) = \phi_0 \cos[m(t - t_0)] = (-1)^j \phi_0 \cos[m(t - t_j)] = (-1)^j \phi_0 \sin[m(t - t_j - t_*)], \tag{4.2.2}$$

where

$$t_j \equiv t_0 + j\frac{\pi}{m}, \quad j = 0, 1, 2, ... \tag{4.2.3}$$

$$t_* \equiv \frac{\pi}{2m} \tag{4.2.4}$$

and j stands for the number of oscillations. For t_j inflaton reaches the maximal amplitude and t_* corresponds to the transition between the minimum and maximum of the oscillations, see Figure 4.3.

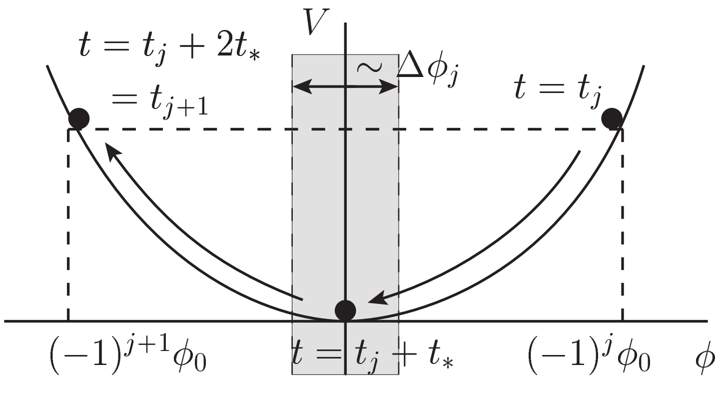

Figure 4.3: Oscillations of the inflaton and our notation.

Effective production of particles occurs when the non-adiabatic condition is satisfied, namely

$$\left|\frac{\dot{\omega}_k}{\omega_k^2}\right|_{k\sim 0} \gtrsim 1 \tag{4.2.5}$$

with $\omega_k \equiv \sqrt{\mathbf{k}^2 + g^2\phi^2(t)}$. As long as $m \ll g|\phi_0|$ it means

$$|\phi| \lesssim \sqrt{\frac{m|\phi_0|}{g}} \sim \sqrt{\frac{v_*^j}{g}} \equiv \Delta\phi_j, \tag{4.2.6}$$

where $v_*^j \equiv |\dot{\phi}(t = t_j + t_*)|$. So each time the production process lasts for

$$\Delta t_j \sim \frac{\Delta\phi_j}{v_*^j} \sim \frac{1}{\sqrt{g v_*^j}} \tag{4.2.7}$$

for $t \sim t_j \pm (t_* - \Delta t_j)$ we cannot use the usual approximate WKB solution for the modes of our coupled field χ

$$\chi_k(t) \sim \frac{1}{\sqrt{2\omega_k(t)}} \left(A_k e^{-i\int^t dt'\omega_k(t')} + B_k e^{+i\int^t dt'\omega_k(t')} \right), \tag{4.2.8}$$

where A_k and B_k are constants satisfying the normalization condition $|A_k|^2 - |B_k|^2 = 1$. We label the mode valid around $t \sim t_j$ as χ_k^j. Mode for the subsequent oscillation $\chi_k^{j'}(t)$, valid around $t \sim t_{j'} = t_j + 2(j' - j)t_*$, can be connected with $\chi_k^j(t)$ via linear combination

$$\chi_k^{j'}(t) = \alpha_k^{j'j*}\chi_k^j(t) - \beta_k^{j'j*}\chi_k^{j*}(t),^5 \tag{4.2.9}$$

where $\alpha_k^{j'j}$ and $\beta_k^{j'j}$ are Bogoliubov coefficients

$$\alpha_k^{j'j} = \left(\chi_k^{j'}, \chi_k^j\right), \tag{4.2.10}$$

$$\beta_k^{j'j} = \left(\chi_k^{j'}, \chi_k^{j*}\right), \tag{4.2.11}$$

satisfying $\left|\alpha_k^{j'j}\right|^2 - \left|\beta_k^{j'j}\right|^2 = 1$.

Number density of produced states can be expressed as

$$n_k^j = \left|\beta_k^{j0}\right|^2, \tag{4.2.12}$$

what establishes the recurrence equation for the occupation number of produced states

$$n_k^j = \left|\beta_k^{j,j-1}\right|^2 + \left(1 + 2\left|\beta_k^{j,j-1}\right|^2\right) n_k^{j-1} + 2\left|\beta_k^{j,j-1}\right| \sqrt{1 + \left|\beta_k^{j,j-1}\right|^2}\sqrt{n_k^{j-1}\left(1 + n_k^{j-1}\right)} \cos\theta_k^{j,j-1}, \tag{4.2.13}$$

where $\theta_k^{j,j-1} \equiv \mathrm{Arg}\,\alpha_k^{j,j-1}\beta_k^{j,j-1*}\alpha_k^{j-1,0}\beta_k^{j-1,0}$ is some phase. According to [40], the last term can be neglected because the stochastic contribution can be averaged to zero. Following the method from [49], we can express β coefficient between the subsequent oscillations as

$$\left|\beta_k^{l+1,l}\right|^2 \sim \exp\left[-\pi\frac{k^2}{gv_*^l}\right] = \exp\left[-\pi\frac{k^2}{gm|\phi_0|}\right] \equiv \Delta_k^l, \tag{4.2.14}$$

so (4.2.13) can be expressed as

$$n_k^j \sim \Delta_k^{j-1} + \left(1 + 2\Delta_k^{j-1}\right) n_k^{j-1}, \tag{4.2.15}$$

which is equivalent to

$$n_k^j + \frac{1}{2} \sim \left(n_k^{j-1} + \frac{1}{2}\right)\left(1 + 2\Delta_k^{j-1}\right). \tag{4.2.16}$$

It indicates that since we can think of Δ_k^l as

$$(1 \text{ d.o.f.}) \times (\text{Boltzmann suppression factor}), \tag{4.2.17}$$

the first factor in (4.2.16) means that each time after χ field exits the non-adiabatic area 1 d.o.f. is added. The second term states that 2 d.o.f.s are added per original 1 d.o.f. after the non-adiabatic period, see also Figure 4.4.

[5] There is no summation over j.

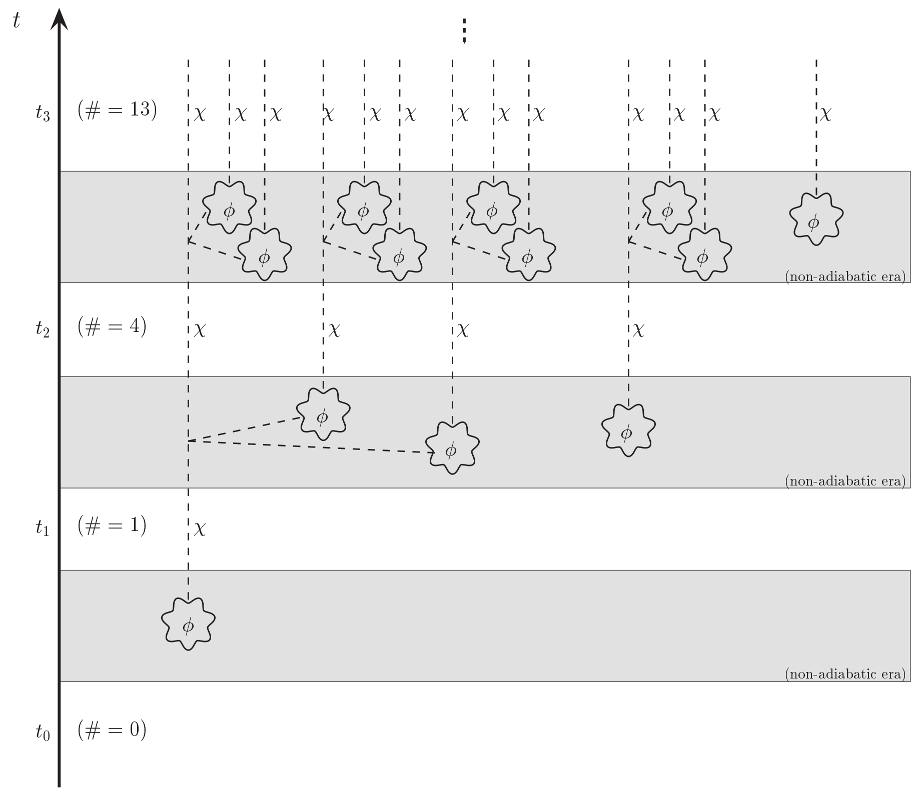

Figure 4.4: The scheme of the resonant production.

For $n_k^l \gg 1$ relation (4.2.16) can be approximated as [49]

$$n_k^j \sim 3^{j-1} \exp\left[-\frac{\pi k^2}{g v_*^0} F_j\right] \tag{4.2.18}$$

with $F_j \equiv 1 + \frac{2}{3}\sum_{l=1}^{j-1} \frac{v_*^0}{v_*^l} \sim \frac{2j+1}{3}$, which for late times ($j \gg 1$) is proportional to

$$n^j \sim n^1 \cdot 3^{j-1} \left(\frac{3}{2j}\right)^{3/2}, \tag{4.2.19}$$

see Appendix B.

4.3 Results for fermions

It has been already proved that not only bosons but also fermions undergo preheating and one of the main differences between them is the possibility of creating very heavy fermions from much lighter inflatons, if they are oscillating coherently [50, 51].

If we consider Majorana fermions ψ coupled to the background scalar field ϕ, Lagrangian changes to

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \psi^\dagger \bar{\sigma}^\mu i \partial_\mu \psi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}y\phi\psi\psi - \frac{1}{2}y\phi\psi^\dagger\psi^\dagger. \tag{4.3.1}$$

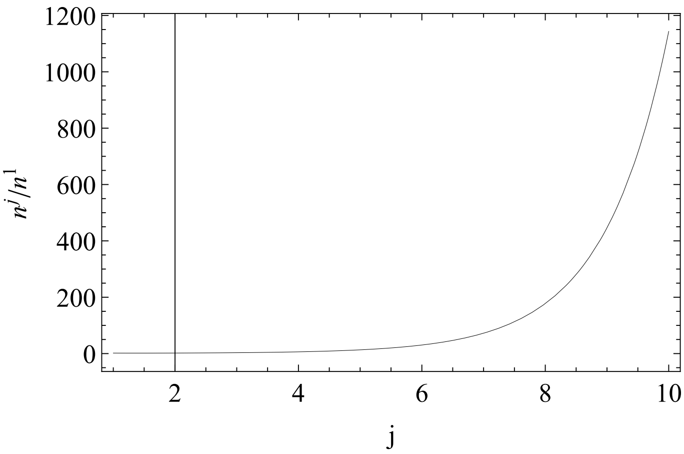

Figure 4.5: Ratio n^j/n^1 as a function of j for a massive background field according to (4.2.19).

We can expand the field operator ψ as[6]

$$\psi(t,\mathbf{x}) = \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} e^s_{\mathbf{k}} \left(\psi_k^{(+)s}(t) a^s_{\mathbf{k}} - s e^{-i\theta_{\mathbf{k}}} \psi_k^{(-)s*}(t) a^{s\dagger}_{-\mathbf{k}} \right), \tag{4.3.4}$$

where $\psi_k^{(\pm)s}$ and $a^s_{\mathbf{k}}$ are wave functions and creation/annihilation operators respectively.

Fermionic character of produced fields changes Bogoliubov coefficients [16] to

$$\alpha_k^{s,j'j} = \psi_k^{j'(+)s*} \psi_k^{j(+)s} + \psi_k^{j'(-)s*} \psi_k^{j(-)s}, \tag{4.3.5}$$
$$\beta_{\mathbf{k}}^{s,j'j} = -s e^{-i\theta_{\mathbf{k}}} \left(\psi_k^{j'(+)s*} \psi_k^{j(-)s*} - \psi_k^{j'(-)s*} \psi_k^{j(+)s*} \right), \tag{4.3.6}$$

satisfying normalization condition $\left|\alpha_k^{s,j'j}\right|^2 + \left|\beta_{\mathbf{k}}^{s,j'j}\right|^2 = 1$. Bogoliubov transformation for the wave functions acquires the following form

$$\psi_k^{j'(+)s} = \alpha_k^{s,j'j*} \psi_k^{j(+)s} - s e^{-i\theta_{\mathbf{k}}} \beta_{\mathbf{k}}^{s,j'j*} \psi_k^{j(-)s*}, \tag{4.3.7}$$
$$\psi_k^{j'(-)s} = s e^{-i\theta_{\mathbf{k}}} \beta_{\mathbf{k}}^{s,j'j*} \psi_k^{j(+)s*} + \alpha_k^{s,j'j*} \psi_k^{j(-)s}, \tag{4.3.8}$$

with $e^{i\theta_{\mathbf{k}}} \equiv (k^1 + ik^2)/\sqrt{(k^1)^2 + (k^2)^2}$. The wave function satisfies a following equation of motion

$$0 = \dot{\psi}_k^{(\pm)s} \pm isk\psi_k^{(\pm)s} + iy\phi\psi_k^{(\mp)s}. \tag{4.3.9}$$

[6]Inner product is defined as

$$\left|\psi_k^{(+)s}\right|^2 + \left|\psi_k^{(-)s}\right|^2 = 1 \tag{4.3.2}$$

and we have the following commutation relations

$$\{a^s_{\mathbf{k}}, a^r_{\mathbf{k}'}\} = \left\{a^{s\dagger}_{\mathbf{k}}, a^{r\dagger}_{\mathbf{k}'}\right\} = 0, \quad \left\{a^s_{\mathbf{k}}, a^{r\dagger}_{\mathbf{k}'}\right\} = (2\pi)^3 \delta^{sr} \delta^3(\mathbf{k} - \mathbf{k}'), \tag{4.3.3}$$

where $e^s_{\mathbf{k}}$ is the helicity eigenspinor.

In analogy to the scalar case, we can find the recurrence equation

$$\beta_{\mathbf{k}}^{s,j0} = \alpha_k^{s,j,j-1}\beta_{\mathbf{k}}^{s,j-1,0} + \beta_{\mathbf{k}}^{s,j,j-1}\alpha_k^{s,j-1,0*} \tag{4.3.10}$$

and represent the distribution of produced ψ particles as

$$\begin{aligned} n_k^{s,j} = {} & \left|\beta_{\mathbf{k}}^{s,j,j-1}\right|^2 + \left(1 - 2\left|\beta_{\mathbf{k}}^{j,j-1}\right|^2\right) n_k^{s,j-1} + \\ & + 2\left|\beta_{\mathbf{k}}^{s,j,j-1}\right| \sqrt{1 - \left|\beta_{\mathbf{k}}^{s,j,j-1}\right|^2} \sqrt{n_k^{s,j-1}\left(1 - n_k^{s,j-1}\right)} \cos\theta_{\mathbf{k}}^{s,j,j-1}, \end{aligned} \tag{4.3.11}$$

where $\theta_{\mathbf{k}}^{s,j,j-1} \equiv \text{Arg}\, \alpha_k^{s,j,j-1}\beta_{\mathbf{k}}^{s,j,j-1*}\alpha_k^{s,j-1,0}\beta_{\mathbf{k}}^{s,j-1,0}$. Once again we finally approximate the distribution as

$$n_k^{s,j} \sim \frac{1}{2} - \frac{1}{2}\left(1 - 2e^{-\pi k^2/ym|\phi_0|}\right)^j, \tag{4.3.12}$$

which is consistent with Pauli exclusion principle: $0 < n_k^{s,j} < 1$ for any j, because $0 < \tilde{\Delta}_k^l < 1$ for any l.

Summary of the chapter

- We introduce the concept of inflation and post-inflationary particle production following the classical literature [24, 39, 40, 41]. We describe general advantages of the inflationary models and the idea of slow-roll inflation. Moreover, we look more carefully at the main stages of post-inflationary production: reheating and preheating.
- We develop the method used in the case of massless background to include massive background neglecting quantum corrections (the framework including them is the subject of Chapter 5). Obtained results are consistent with the literature.

Chapter 5

Particle production in interacting theory for massive background field

Usually preheating is considered to proceed because of the direct coupling between the inflaton and other fields, whose production affects inflaton's mass term but it is not always the case - it is possible to observe successful particle production even without the direct coupling, see Chapter 3. In our analysis there we prove that production of such light states can be abundant, even if they are massless, because of the quantum corrections.

In general, it is important to investigate carefully the role of light fields present in the model in the course of preheating. It has been proven that their influence on the process of particle production and energy transfer is of high importance during [52, 53, 54] and after inflation for instance for multi-field inflation models and for curvaton scenarios [55, 56, 57].

Our goal in this Chapter is to develop the results from Chapter 3 in the preheating scenarios with additional light sector. There is one fundamental difference between these two analyses - our previous results possess some artificial infinite growth coming from the approximation by asymptotic field, which we want to avoid now using the theory of interacting field.

5.1 Theory of interacting field

Usually defined occupation number operator $N_{\mathbf{k}} \equiv a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}$ assumes that produced states can be treated as free fields, see Section 2.4, so their equations of motion are linear. But what is interesting from our point of view happens when this linearity gets spoiled, which corresponds to the interactions between particles, and non-perturbative effects become important. In such a case we need to define number operator in a proper way using the theory of interacting fields, which takes into account all these non-linear effects. In this Section we focus on the case of the scalar field as it is useful for our further analysis, for the fermionic case see Appendix C.

5.1.1 Real scalar field

Our starting point is the Lagrangian (2.4.1) with the general potential V describing all the interactions

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi^2 - V[\phi, \text{(other fields)}], \tag{5.1.1}$$

where m_0 is a bare mass of ϕ. Equation of motion reads then

$$0 = (\partial^2 + m_0^2)\phi + \frac{\partial V}{\partial \phi} = (\partial^2 + M^2)\phi + J, \tag{5.1.2}$$

where M denotes the physical mass that in general can be time-dependent[1] and should be a complex number, while

$$J \equiv (m_0^2 - M^2)\phi + \frac{\partial V}{\partial \phi} \tag{5.1.3}$$

stands for the source term that could be an operator. Its solution is the Yang-Feldman equation (see Appendix A)

$$\phi(x) = \phi^{(t)}(x) - \int_t^{x^0} d^4y\, i[\phi^{(t)}(x), \phi^{(t)}(y)]J(y), \tag{5.1.4}$$

where $\phi^{(t)}(x)$ is an asymptotic free field defined at $x^0 = t$. When ϕ does not carry a vev, $\langle 0^{\text{as}}|\,\phi\,|0^{\text{as}}\rangle \equiv \langle\phi\rangle = 0$, this asymptotic field can be decomposed into modes

$$\phi^{(t)}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\phi_k^{(t)} a_{\mathbf{k}}^{(t)} + \phi_k^{(t)*} a_{-\mathbf{k}}^{(t)\dagger}\right) \tag{5.1.5}$$

fulfilling harmonic oscillator equation with $\omega_k^2 = k^2 + M^2$.

From (5.1.4) we can infer the relation between two asymptotic fields defined at times $x^0 = t$ and $x^0 = t^{\text{in}}$

$$\phi^{(t)}(x) = \phi^{\text{in}}(x) - \int_{t^{\text{in}}}^{t} d^4y\, i[\phi^{\text{in}}(x), \phi^{\text{in}}(y)]J(y), \tag{5.1.6}$$

where we introduce the notation $\phi^{\text{in}}(x) \equiv \phi^{(t^{\text{in}})}(x)$. Its inner product with the mode function, $\left(\phi^{(t)}, \phi_k^{(t)}\right)$, results in the Bogoliubov transformation that reads [16]

$$a_{\mathbf{k}}^{(t)} = \alpha_k a_{\mathbf{k}}^{\text{in}} + \beta_k a_{-\mathbf{k}}^{\text{in}\dagger} - \int_{t^{\text{in}}}^{t} d^4y\, i[\alpha_k a_{\mathbf{k}}^{\text{in}} + \beta_k a_{-\mathbf{k}}^{\text{in}\dagger}, \phi(y)]J(y), \tag{5.1.7}$$

with the Bogoliubov coefficients defined as

$$\alpha_k = \alpha_k(t, t^{\text{in}}) \equiv (\phi_k^{(t)}, \phi_k^{\text{in}}), \tag{5.1.8}$$
$$\beta_k = \beta_k(t, t^{\text{in}}) \equiv (\phi_k^{(t)}, \phi_k^{\text{in}*}). \tag{5.1.9}$$

[1] In principle physical mass M can depend both on time and space coordinates but we focus on its time-dependence as it is a more common case in cosmological considerations and it is simpler at the same time.

For the wave function it means that

$$\phi_k^{(t)} = \alpha_k^* \phi_k^{\rm as} - \beta_k^* \phi_k^{\rm as*}. \tag{5.1.10}$$

Corresponding Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_0^2\phi^2 + V\right] \tag{5.1.11}$$

in terms of the field $\phi^{(t)}(x)$ defined by (5.1.6) gains a very complicated structure, see Appendix D, which simplifies when the Bogoliubov coefficients are of the form:

$$\alpha_k(t) = \sqrt{\frac{\Omega_k^{\rm in}}{2\omega_k} + \frac{1}{2}}, \tag{5.1.12}$$

$$\beta_k(t) = \frac{\Lambda_k^{\rm in*}}{|\Lambda_k^{\rm in}|}\sqrt{\frac{\Omega_k^{\rm in}}{2\omega_k} - \frac{1}{2}}, \tag{5.1.13}$$

where[2]

$$\Omega_k^{\rm in} \equiv |\dot{\phi}_k^{\rm in}|^2 + \omega_k^2|\phi_k^{\rm in}|^2, \tag{5.1.14}$$

$$\Lambda_k^{\rm in} \equiv (\dot{\phi}_k^{\rm in})^2 + \omega_k^2(\phi_k^{\rm in})^2. \tag{5.1.15}$$

Hamiltonian can be diagonalized in the kinetic terms then and reads

$$H = \int \frac{d^3k}{(2\pi)^3}\,\omega_k \left(a_{\mathbf{k}}^{(t)\dagger} a_{\mathbf{k}}^{(t)} + \frac{1}{2}(2\pi)^3\delta^3(\mathbf{k}=0)\right) + \int d^3x \left[\frac{1}{2}(m_0^2 - M^2)\phi^2 + V\right]. \tag{5.1.16}$$

Second integral describes the effective potential and the term with Dirac delta corresponds to the zero-point energy in our theory, which means that the operator

$$N_{\mathbf{k}}(t) \equiv a_{\mathbf{k}}^{(t)\dagger} a_{\mathbf{k}}^{(t)} \tag{5.1.17}$$

is actually the number operator. Note that these are not the same creation and annihilation operators as in the free field theory but the ones defined by (5.1.7), which implies that

$$N_{\mathbf{k}}(t) = \frac{1}{2\omega_k}\left(\dot{\hat{\phi}}_{\mathbf{k}}^\dagger \dot{\hat{\phi}}_{\mathbf{k}} + \omega_k^2 \hat{\phi}_{\mathbf{k}}^\dagger \hat{\phi}_{\mathbf{k}}\right) + \frac{i}{2}\left(\hat{\phi}_{\mathbf{k}}^\dagger \dot{\hat{\phi}}_{\mathbf{k}} - \dot{\hat{\phi}}_{\mathbf{k}}^\dagger \hat{\phi}_{\mathbf{k}}\right) \equiv \frac{1}{2}\left[N_{\mathbf{k}}^+(t) + N_{\mathbf{k}}^-(t)\right]. \tag{5.1.18}$$

Here $N_{\mathbf{k}}^{\pm}(t) = a_{\mathbf{k}}^{(t)\dagger} a_{\mathbf{k}}^{(t)} \pm a_{-\mathbf{k}}^{(t)\dagger} a_{-\mathbf{k}}^{(t)}$, $N_{\mathbf{k}}^+$ denotes a total and $N_{\mathbf{k}}^-$ a net number of particles with momentum between **k** and **-k** and hat denotes the Fourier transformation

$$\hat{\phi}_{\mathbf{k}}(t) \equiv \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(t,\mathbf{x}). \tag{5.1.19}$$

[2]Functions $\Omega_k^{\rm in}$ and $\Lambda_k^{\rm in}$ are related by the condition:

$$|\Omega_k^{\rm in}|^2 - |\Lambda_k^{\rm in}|^2 = \omega_k^2.$$

Side note: Energy budget in a classical system

Classically the total Hamiltonian of the system, H, can be described as

$$H = NE + V_{\text{eff}} + V_0, \tag{5.1.20}$$

where V_{eff} an effective potential, V_0 a zero-point energy, E is a one-particle energy and N number of particles.

Actual expressions for $N^{\pm}_{\mathbf{k}}(t)$ read

$$N^{+}_{\mathbf{k}}(t) = \frac{1}{\omega_k}\left(\dot{\hat{\phi}}^{\dagger}_{\mathbf{k}}\dot{\hat{\phi}}_{\mathbf{k}} + \omega_k^2 \hat{\phi}^{\dagger}_{\mathbf{k}}\hat{\phi}_{\mathbf{k}}\right) - (2\pi)^3\delta^3(\mathbf{k}=0), \tag{5.1.21}$$

$$N^{-}_{\mathbf{k}}(t) = i\left(\hat{\phi}^{\dagger}_{\mathbf{k}}\dot{\hat{\phi}}_{\mathbf{k}} - \dot{\hat{\phi}}^{\dagger}_{\mathbf{k}}\hat{\phi}_{\mathbf{k}}\right) + (2\pi)^3\delta^3(\mathbf{k}=0), \tag{5.1.22}$$

where the zero point term represents the volume of the system

$$(2\pi)^3\delta^3(\mathbf{k}=0) = \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}|_{\mathbf{k}=0} = \int d^3x = V. \tag{5.1.23}$$

Thus, final occupation numbers are of the form

$$n^{+}_{\mathbf{k}} = \frac{N^{+}_{\mathbf{k}}}{V} = \frac{1}{\omega_k}\left(\frac{1}{V}\dot{\hat{\phi}}^{\dagger}_{\mathbf{k}}\dot{\hat{\phi}}_{\mathbf{k}} + \omega_k^2\cdot\frac{1}{V}\hat{\phi}^{\dagger}_{\mathbf{k}}\hat{\phi}_{\mathbf{k}}\right) - 1, \tag{5.1.24}$$

$$n^{-}_{\mathbf{k}} = \frac{N^{-}_{\mathbf{k}}}{V} = i\left(\frac{1}{V}\hat{\phi}^{\dagger}_{\mathbf{k}}\dot{\hat{\phi}}_{\mathbf{k}} - \frac{1}{V}\dot{\hat{\phi}}^{\dagger}_{\mathbf{k}}\hat{\phi}_{\mathbf{k}}\right) + 1, \tag{5.1.25}$$

$$n_{\mathbf{k}} = \frac{1}{2\omega_k}\left(\frac{1}{V}\dot{\hat{\phi}}^{\dagger}_{\mathbf{k}}\dot{\hat{\phi}}_{\mathbf{k}} + \omega_k^2\cdot\frac{1}{V}\hat{\phi}^{\dagger}_{\mathbf{k}}\hat{\phi}_{\mathbf{k}}\right) + \frac{i}{2}\left(\frac{1}{V}\hat{\phi}^{\dagger}_{\mathbf{k}}\dot{\hat{\phi}}_{\mathbf{k}} - \frac{1}{V}\dot{\hat{\phi}}^{\dagger}_{\mathbf{k}}\hat{\phi}_{\mathbf{k}}\right). \tag{5.1.26}$$

Thus far we assumed that ϕ does not have a vev. If it is not a case and $\langle\phi\rangle \equiv \langle 0^{\text{in}}|\,\phi\,|0^{\text{in}}\rangle \neq 0$ the Yang-Feldman equation for a real scalar changes to

$$\phi(x) = \langle\,\phi^{\text{as}}(x)\,\rangle + \tilde{\phi}^{\text{as}}(x) - \int_{t^{\text{as}}}^{x^0} d^4y\, i[\tilde{\phi}^{\text{as}}(x), \tilde{\phi}^{\text{as}}(y)]J(y) =$$

$$= \langle\,\phi(x)\,\rangle + \tilde{\phi}^{\text{as}}(x) - \int_{t^{\text{as}}}^{x^0} d^4y\, i[\tilde{\phi}^{\text{as}}(x), \tilde{\phi}^{\text{as}}(y)]\left(J(y) - \langle J(y)\rangle\right), \tag{5.1.27}$$

where we divide the asymptotic field ϕ^{as} into the background $\langle\,\phi^{\text{as}}\,\rangle$ and the quantum fluctuation $\tilde{\phi}^{\text{as}}$ ($\langle\,\tilde{\phi}^{\text{as}}\,\rangle = 0$), which in turn can be decomposed into modes

$$\tilde{\phi}^{\text{as}}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}\left(\phi^{\text{as}}_k a^{\text{as}}_{\mathbf{k}} + \phi^{\text{as}*}_k a^{\text{as}\dagger}_{-\mathbf{k}}\right). \tag{5.1.28}$$

Note that (5.1.27) can be rewritten as

$$\phi(x) = \langle\, \phi(x)\,\rangle + \left[\tilde{\phi}^{\rm as}(x) - \int_{t^{\rm as}}^{t} d^4y\, i[\tilde{\phi}^{\rm as}(x), \tilde{\phi}^{\rm as}(y)]\,(J(y) - \langle J(y)\rangle)\right] + \\ - \int_{t}^{x^0} d^4y\, i[\tilde{\phi}^{\rm as}(x), \tilde{\phi}^{\rm as}(y)]\,(J(y) - \langle J(y)\rangle)\,, \quad (5.1.29)$$

which implies that

$$\tilde{\phi}^{(t)}(x) - \langle\, \tilde{\phi}^{(t)}(x)\,\rangle = \tilde{\phi}^{\rm as}(x) - \int_{t^{\rm as}}^{t} d^4y\, i[\tilde{\phi}^{\rm as}(x), \tilde{\phi}^{\rm as}(y)]\,(J(y) - \langle J(y)\rangle) \quad (5.1.30)$$

and results in the Bogoliubov transformation of the form

$$a^{(t)}_{\mathbf{k}} - \langle\, a^{(t)}_{\mathbf{k}}\,\rangle = \alpha_k(t) a^{\rm as}_{\mathbf{k}} + \beta_k(t) a^{\rm as\dagger}_{-\mathbf{k}} \quad (5.1.31) \\ - \int_{t^{\rm as}}^{t} d^4y\, i[\alpha(t) a^{\rm as}_{\mathbf{k}} + \beta_k(t) a^{\rm as\dagger}_{-\mathbf{k}}, \phi^{\rm as}(y)]\,(J(y) - \langle J(y)\rangle)\,.$$

So the overall difference between the cases with and without the vev is the following shift: $a^{(t)}_{\mathbf{k}} \to a^{(t)}_{\mathbf{k}} - \langle\, a^{(t)}_{\mathbf{k}}\,\rangle$ and $J \to J - \langle J\rangle$, while the final results are the same. For details see Appendix D.

5.1.2 Complex scalar field

When ϕ is a complex scalar there are a few changes in the previous reasoning but the overall result for (5.1.21) and (5.1.22) stays the same. We start with the Lagrangian

$$\mathcal{L} = |\partial\phi|^2 - m_0^2|\phi|^2 - V[\phi, \phi^\dagger] \quad (5.1.32)$$

with the corresponding equation of motion

$$0 = (\partial^2 + m_0^2)\phi + \frac{\partial V}{\partial \phi^\dagger} = (\partial^2 + M^2)\phi + J^\dagger, \quad (5.1.33)$$

where again M can depend on time and

$$J^\dagger \equiv (m_0^2 - M^2)\phi + \frac{\partial V}{\partial \phi^\dagger}. \quad (5.1.34)$$

Yang-Feldman equation slightly changes its form

$$\phi(x) = \phi^{(t)}(x) - \int_t^{x^0} d^4y\, i[\phi^{(t)}(x), \phi^{(t)\dagger}(y)] J^\dagger(y), \quad (5.1.35)$$

where

$$\phi^{(t)}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\phi^{(t)}_k a^{(t)}_{\mathbf{k}} + \phi^{(t)*}_k b^{(t)\dagger}_{-\mathbf{k}}\right) \quad (5.1.36)$$

and $b^{(t)\dagger}_{-\mathbf{k}}$ denotes a creation operator for an anti-state.

Bogoliubov transformation reads now

$$a^{(t)}_{\mathbf{k}} = \alpha_k(t)a^{\rm in}_{\mathbf{k}} + \beta_k(t)b^{\rm in\dagger}_{-\mathbf{k}} - \int_{t^{\rm in}}^t d^4y\, i[\alpha(t)a^{\rm in}_{\mathbf{k}} + \beta_k(t)b^{\rm in\dagger}_{-\mathbf{k}}, \phi^{\rm in}(y)]J^\dagger(y), \quad (5.1.37)$$

$$b^{(t)\dagger}_{-\mathbf{k}} = \beta^*_k(t)a^{\rm in}_{\mathbf{k}} + \alpha^*_k(t)b^{\rm in\dagger}_{-\mathbf{k}} - \int_{t^{\rm in}}^t d^4y\, i[\beta^*(t)a^{\rm in}_{\mathbf{k}} + \alpha^*_k(t)b^{\rm in\dagger}_{-\mathbf{k}}, \phi^{\rm in}(y)]J^\dagger(y), \quad (5.1.38)$$

where the Bogoliubov coefficients are still defined by (5.1.8) and (5.1.9) and can be chosen as before, (5.1.12) and (5.1.13), to diagonalize the Hamiltonian

$$H(t) = \int d^3x \left(|\dot\phi(t,\mathbf{x})|^2 + |\nabla\phi(t,\mathbf{x})|^2 + m_0^2|\phi(t,\mathbf{x})|^2 + V(t,\mathbf{x})\right) = \quad (5.1.39)$$

$$= \int \frac{d^3k}{(2\pi)^3}\omega_k(t)\left(a^{(t)\dagger}_{\mathbf{k}}a^{(t)}_{\mathbf{k}} + b^{(t)}_{-\mathbf{k}}b^{(t)\dagger}_{-\mathbf{k}}\right) + \int d^3x\left[(m_0^2 - M^2)\phi(t,\mathbf{x}) + V(t,\mathbf{x})\right].$$

Previous $N^\pm_{\mathbf{k}}(t)$ becomes $N^\pm_{\mathbf{k}}(t) = a^{(t)\dagger}_{\mathbf{k}}a^{(t)}_{\mathbf{k}} \pm b^{(t)\dagger}_{-\mathbf{k}}b^{(t)}_{-\mathbf{k}}$ now but it does not change the final number of produced states in comparison with the case of real scalar.

5.2 Numerical results for multi-scalar systems

In each particular model we are in the end interested in time-dependence of the background (inflaton) and evolution of particle number density for all considered species

$$n(t) = \int \frac{d^3k}{(2\pi)^3}\frac{\langle N_{\mathbf{k}}\rangle}{V} = \int \frac{d^3k}{(2\pi)^3}n_{\mathbf{k}}, \quad (5.2.1)$$

where V is the volume of the system and $n_{\mathbf{k}}$ the distribution.

In our numerical analysis we follow the procedure described in the previous Section - we consider some time range and solve equations of motion for all the states for the initial time calculating also their number density. We repeat this step for slightly later time but now taking into account the backreaction of previously produced states on the evolution of the background and all the species coming from the induced potential connected with the non-zero energy density. Iteratively we reach the final time.

However, there is an important subtlety in this calculation, which can be presented based on the real scalar case. Time evolution of all the distributions there is determined by time evolution of bilinear products of field operators: $\langle\hat\phi^\dagger_{\mathbf{k}}\hat\phi_{\mathbf{k}}\rangle$, $\langle\hat\phi^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle$ and $\langle\dot{\hat\phi}^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle$, whose equations of motion read

$$\langle\hat\phi^\dagger_{\mathbf{k}}\hat\phi_{\mathbf{k}}\rangle^\cdot = \langle\dot{\hat\phi}^\dagger_{\mathbf{k}}\hat\phi_{\mathbf{k}}\rangle + \langle\hat\phi^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle, \quad (5.2.2)$$

$$\langle\hat\phi^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle^\cdot = \langle\dot{\hat\phi}^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle + \langle\hat\phi^\dagger_{\mathbf{k}}\ddot{\hat\phi}_{\mathbf{k}}\rangle = \langle\dot{\hat\phi}^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle - \omega_k^2\langle\hat\phi^\dagger_{\mathbf{k}}\hat\phi_{\mathbf{k}}\rangle - \langle\hat\phi^\dagger_{\mathbf{k}}\hat J_{\mathbf{k}}\rangle, \quad (5.2.3)$$

$$\langle\dot{\hat\phi}^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle^\cdot = \langle\ddot{\hat\phi}^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle + \langle\dot{\hat\phi}^\dagger_{\mathbf{k}}\ddot{\hat\phi}_{\mathbf{k}}\rangle = -\omega_k^2(\langle\dot{\hat\phi}^\dagger_{\mathbf{k}}\hat\phi_{\mathbf{k}}\rangle + \langle\hat\phi^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle) - \langle\dot{\hat\phi}^\dagger_{\mathbf{k}}\hat J_{\mathbf{k}}\rangle - \langle\hat J^\dagger_{\mathbf{k}}\dot{\hat\phi}_{\mathbf{k}}\rangle \quad (5.2.4)$$

Physical mass of ϕ is of the form

$$0 = \langle\hat\phi^\dagger_{\mathbf{k}}\hat J_{\mathbf{k}}\rangle = (m^2 - M^2)\langle\hat\phi^\dagger_{\mathbf{k}}\hat\phi_{\mathbf{k}}\rangle + \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\left\langle\hat\phi^\dagger_{\mathbf{k}}\frac{dV(x)}{d\phi}\right\rangle \quad (5.2.5)$$

in order to compensate the infinite part of mass correction.

In the following Subsections we present our numerical results for some specific models.

5.2.1 Two-scalar system

As the first application we consider a simple theory of two scalar fields ϕ and χ

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{2}m_\chi^2\chi^2 - \frac{1}{4}g^2\phi^2\chi^2, \tag{5.2.6}$$

which is important because it can be compared with other methods present in the literature. In this system we have the field ϕ being the inflaton with time-varying vev $\langle 0^{\text{in}}|\,\phi\,|0^{\text{in}}\rangle = \langle\phi(t)\rangle$ and another field χ with vanishing vev, for instance some mediator between the inflaton and the SM. We assume that inflaton is much heavier than the other field, $m_\phi \gg m_\chi$.

Asymptotically quantum effects can be neglected and the vacuum solution for (5.2.6) is linear in time

$$\langle\phi\rangle = vt + i\mu, \tag{5.2.7}$$

where v is a velocity in the ϕ-space and μ an impact parameter. For the description of particle production in such a case with asymptotic approximation see Chapter 3.

In the two-scalar system number operators are of the form:

$$N^{(+)}_{\phi\mathbf{k}}(t) = \frac{1}{\omega_{\phi k}}\left(\dot{\tilde{\phi}}^\dagger_{\mathbf{k}}\dot{\tilde{\phi}}_{\mathbf{k}} + \omega^2_{\phi k}\tilde{\phi}^\dagger_{\mathbf{k}}\tilde{\phi}_{\mathbf{k}}\right) - V, \tag{5.2.8}$$

$$N^{(-)}_{\phi\mathbf{k}}(t) = i\left(\tilde{\phi}^\dagger_{\mathbf{k}}\dot{\tilde{\phi}}_{\mathbf{k}} - \dot{\tilde{\phi}}^\dagger_{\mathbf{k}}\tilde{\phi}_{\mathbf{k}}\right) + V \tag{5.2.9}$$

$$\tag{5.2.10}$$

for ϕ and

$$N^{(+)}_{\chi\mathbf{k}}(t) = \frac{1}{\omega_{\chi k}}\left(\dot{\chi}^\dagger_{\mathbf{k}}\dot{\chi}_{\mathbf{k}} + \omega^2_{\phi k}\chi^\dagger_{\mathbf{k}}\chi_{\mathbf{k}}\right) - V, \tag{5.2.11}$$

$$N^{(-)}_{\chi\mathbf{k}}(t) = i\left(\chi^\dagger_{\mathbf{k}}\dot{\chi}_{\mathbf{k}} - \dot{\chi}^\dagger_{\mathbf{k}}\chi_{\mathbf{k}}\right) + V \tag{5.2.12}$$

for χ, where $\tilde{\phi} \equiv \phi - \langle\phi\rangle$. We can rewrite their form to

$$\frac{1}{V}N^{(+)}_{\phi\mathbf{k}}(t) = A_{\phi\mathbf{k}} + B_{\phi\mathbf{k}}, \tag{5.2.13}$$

$$\frac{1}{V}N^{(-)}_{\phi\mathbf{k}}(t) = C_{\phi\mathbf{k}} + C^\dagger_{\phi\mathbf{k}}, \tag{5.2.14}$$

$$\frac{1}{V}N^{(+)}_{\chi\mathbf{k}}(t) = A_{\chi\mathbf{k}} + B_{\chi\mathbf{k}}, \tag{5.2.15}$$

$$\frac{1}{V}N^{(-)}_{\chi\mathbf{k}}(t) = C_{\chi\mathbf{k}} + C^\dagger_{\chi\mathbf{k}}, \tag{5.2.16}$$

introducing a new set of operators[3]

$$A_{\phi\mathbf{k}} \equiv \frac{1}{V} \cdot \frac{1}{\omega_{\phi k}} \dot{\tilde{\phi}}^{\dagger}_{\mathbf{k}} \dot{\tilde{\phi}}_{\mathbf{k}} - \frac{1}{2}, \tag{5.2.21}$$

$$B_{\phi\mathbf{k}} \equiv \frac{1}{V} \cdot \omega_{\phi k} \tilde{\phi}^{\dagger}_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}} - \frac{1}{2}, \tag{5.2.22}$$

$$C_{\phi\mathbf{k}} \equiv \frac{1}{V} \cdot i \tilde{\phi}^{\dagger}_{\mathbf{k}} \dot{\tilde{\phi}}_{\mathbf{k}} + \frac{1}{2}, \tag{5.2.23}$$

$$A_{\chi\mathbf{k}} \equiv \frac{1}{V} \cdot \frac{1}{\omega_{\chi k}} \dot{\chi}^{\dagger}_{\mathbf{k}} \dot{\chi}_{\mathbf{k}} - \frac{1}{2}, \tag{5.2.24}$$

$$B_{\chi\mathbf{k}} \equiv \frac{1}{V} \cdot \omega_{\chi k} \chi^{\dagger}_{\mathbf{k}} \chi_{\mathbf{k}} - \frac{1}{2}, \tag{5.2.25}$$

$$C_{\chi\mathbf{k}} \equiv \frac{1}{V} \cdot i \chi^{\dagger}_{\mathbf{k}} \dot{\chi}_{\mathbf{k}} + \frac{1}{2}. \tag{5.2.26}$$

Using the approximation

$$\langle\, \dot{\tilde{\phi}}^{\dagger}_{\mathbf{k}} \chi_{\mathbf{p}_1} \chi_{\mathbf{p}_2} \,\rangle = \langle\, \dot{\tilde{\phi}}^{\dagger}_{\mathbf{k}} \,\rangle \, \langle \chi_{\mathbf{p}_1} \chi_{\mathbf{p}_2} \rangle + \mathcal{O}(g^2) = 0 + \mathcal{O}(g^2), \tag{5.2.27}$$

$$\langle\, \dot{\tilde{\phi}}^{\dagger}_{\mathbf{k}} \tilde{\phi}_{\mathbf{p}_1} \chi_{\mathbf{p}_2} \chi_{\mathbf{p}_3} \,\rangle = \langle\, \dot{\tilde{\phi}}^{\dagger}_{\mathbf{k}} \tilde{\phi}_{\mathbf{p}_1} \,\rangle \, \langle \chi_{\mathbf{p}_2} \chi_{\mathbf{p}_3} \rangle + \mathcal{O}(g^2) \tag{5.2.28}$$

and momentum conservation[4]

$$\langle\, \dot{\tilde{\phi}}^{\dagger}_{\mathbf{k}} \tilde{\phi}_{\mathbf{p}_1} \,\rangle = \frac{1}{V} \cdot (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{p}_1) \, \langle\, \dot{\tilde{\phi}}^{\dagger}_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}} \,\rangle \,, \tag{5.2.32}$$

$$\langle \chi_{\mathbf{p}_2} \chi_{\mathbf{p}_3} \rangle = \frac{1}{V} \cdot (2\pi)^3 \delta^3(\mathbf{p}_2 + \mathbf{p}_3) \, \langle\, \chi^{\dagger}_{\mathbf{p}_3} \chi_{\mathbf{p}_3} \,\rangle \,, \tag{5.2.33}$$

we can obtain time derivatives of new operators, see Appendix E, and equation of

[3] Their vevs vanish for WKB type solutions

$$\tilde{\phi}_{\mathbf{k}}(x^0 = t^{\text{in}}) = \tilde{\phi}^{\text{in}}_{\mathbf{k}}(x^0 = t^{\text{in}}) = \phi^{\text{in}}_k a^{\text{in}}_{\mathbf{k}} + \phi^{\text{in}*}_k a^{\text{in}\dagger}_{-\mathbf{k}}, \tag{5.2.17}$$

$$\chi_{\mathbf{k}}(x^0 = t^{\text{in}}) = \chi^{\text{in}}_{\mathbf{k}}(x^0 = t^{\text{in}}) = \chi^{\text{in}}_k a^{\text{in}}_{\mathbf{k}} + \chi^{\text{in}*}_k a^{\text{in}\dagger}_{-\mathbf{k}}, \tag{5.2.18}$$

where

$$\phi^{\text{in}}_k(x^0) \sim \frac{1}{\sqrt{2\omega_{\phi k}(x^0)}} \exp\left[-i \int_{t^{\text{in}}}^{x^0} dt' \omega_{\phi k}(t')\right], \tag{5.2.19}$$

$$\chi^{\text{in}}_k(x^0) \sim \frac{1}{\sqrt{2\omega_{\chi k}(x^0)}} \exp\left[-i \int_{t^{\text{in}}}^{x^0} dt' \omega_{\chi k}(t')\right]. \tag{5.2.20}$$

[4] Momentum conservation indicates that for instance

$$\langle \hat{X}^{\dagger}_{\mathbf{p}} \hat{X}_{\mathbf{p}'} \rangle = C_{\mathbf{p}} (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'), \tag{5.2.29}$$

where $C_{\mathbf{p}}$ is some numerical factor. For $\mathbf{p}' = \mathbf{p}$:

$$\langle \hat{X}^{\dagger}_{\mathbf{p}} \hat{X}_{\mathbf{p}} \rangle = V \cdot C_{\mathbf{p}} \tag{5.2.30}$$

and thus

$$C_{\mathbf{p}} = \frac{1}{V} \langle \hat{X}^{\dagger}_{\mathbf{p}} \hat{X}_{\mathbf{p}} \rangle. \tag{5.2.31}$$

motion for the background

$$0 = \langle \ddot{\phi} \rangle + M_\phi^2 \langle \phi \rangle + \langle J_\phi \rangle \approx \langle \ddot{\phi} \rangle + \left(m_\phi^2 + \frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_{\chi k}} \left(\langle B_{\chi \mathbf{k}} \rangle + \frac{1}{2} \right) \right) \langle \phi \rangle \tag{5.2.34}$$

both up to the order $\mathcal{O}(g^4)$.

Choosing the physical masses as

$$M_\phi^2 = m_\phi^2 + \frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} \frac{\langle B_{\chi \mathbf{p}} \rangle}{\omega_{\chi k}}, \tag{5.2.35}$$

$$M_\chi^2 = m_\chi^2 + \frac{1}{2} g^2 \langle \phi \rangle^2 + \frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} \frac{\langle B_{\phi \mathbf{p}} \rangle}{\omega_{\phi k}} \tag{5.2.36}$$

simplifies the final differential equation we need to solve deleting the source terms, see Appendix E.

Finally, we obtain the formulas for the physical masses

$$M_\phi^2 = m_\phi^2 + \frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{V} \langle \hat{\chi}_{\mathbf{p}}^\dagger \hat{\chi}_{\mathbf{p}} \rangle - \frac{1}{2\omega_{\chi p}} \right], \tag{5.2.37}$$

$$M_\chi^2 = m_\phi^2 + \frac{1}{2} g^2 \langle \phi \rangle^2 + \frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{V} \langle \hat{\phi}_{\mathbf{p}}^\dagger \hat{\phi}_{\mathbf{p}} \rangle - \frac{1}{2\omega_{\phi p}} \right] \tag{5.2.38}$$

and the set of differential equations

$$0 = \langle \ddot{\phi} \rangle + M_\phi^2 \langle \phi \rangle, \tag{5.2.39}$$

$$\langle \hat{\phi}_{\mathbf{k}}^\dagger \hat{\phi}_{\mathbf{k}} \rangle^{\cdot} = \langle \dot{\hat{\phi}}_{\mathbf{k}}^\dagger \hat{\phi}_{\mathbf{k}} \rangle + \langle \hat{\phi}_{\mathbf{k}}^\dagger \dot{\hat{\phi}}_{\mathbf{k}} \rangle, \tag{5.2.40}$$

$$\langle \hat{\phi}_{\mathbf{k}}^\dagger \dot{\hat{\phi}}_{\mathbf{k}} \rangle^{\cdot} = \langle \dot{\hat{\phi}}_{\mathbf{k}}^\dagger \dot{\hat{\phi}}_{\mathbf{k}} \rangle - \omega_{\phi k}^2 \langle \hat{\phi}_{\mathbf{k}}^\dagger \hat{\phi}_{\mathbf{k}} \rangle, \tag{5.2.41}$$

$$\langle \dot{\hat{\phi}}_{\mathbf{k}}^\dagger \dot{\hat{\phi}}_{\mathbf{k}} \rangle^{\cdot} = -\omega_{\phi k}^2 (\langle \dot{\hat{\phi}}_{\mathbf{k}}^\dagger \hat{\phi}_{\mathbf{k}} \rangle + \langle \hat{\phi}_{\mathbf{k}}^\dagger \dot{\hat{\phi}}_{\mathbf{k}} \rangle), \tag{5.2.42}$$

$$\langle \hat{\chi}_{\mathbf{k}}^\dagger \hat{\chi}_{\mathbf{k}} \rangle^{\cdot} = \langle \dot{\hat{\chi}}_{\mathbf{k}}^\dagger \hat{\chi}_{\mathbf{k}} \rangle + \langle \hat{\chi}_{\mathbf{k}}^\dagger \dot{\hat{\chi}}_{\mathbf{k}} \rangle, \tag{5.2.43}$$

$$\langle \hat{\chi}_{\mathbf{k}}^\dagger \dot{\hat{\chi}}_{\mathbf{k}} \rangle^{\cdot} = \langle \dot{\hat{\chi}}_{\mathbf{k}}^\dagger \dot{\hat{\chi}}_{\mathbf{k}} \rangle - \omega_{\chi k}^2 \langle \hat{\chi}_{\mathbf{k}}^\dagger \hat{\chi}_{\mathbf{k}} \rangle, \tag{5.2.44}$$

$$\langle \dot{\hat{\chi}}_{\mathbf{k}}^\dagger \dot{\hat{\chi}}_{\mathbf{k}} \rangle^{\cdot} = -\omega_{\chi k}^2 (\langle \dot{\hat{\chi}}_{\mathbf{k}}^\dagger \hat{\chi}_{\mathbf{k}} \rangle + \langle \hat{\chi}_{\mathbf{k}}^\dagger \dot{\hat{\chi}}_{\mathbf{k}} \rangle), \tag{5.2.45}$$

whose solutions put into (5.2.8)-(5.2.12) allow to compute number densities of produced states.

Figure 5.1 presents our numerical results for the two-scalar system for some chosen values of free parameters. We can see that both species, ϕ and χ, are produced and abundance of the lighter states is greater as it is easier to produce them. Moreover, for some specific choices of parameters energy transfer between the background and the particles associated with inflaton, $\tilde{\phi}$, is that small they can be even neglected in the general picture.

Our numerical results presented in Figure 5.1 are consistent with the analytical predictions from [24]. From their estimation of the number density of produced χ particles we can infer

$$n_\chi^{(1)} \sim \frac{1}{(2\pi)^3} \left(g m_\phi \langle \phi(0) \rangle \right)^{3/2} \sim 4 \times 10^{-9}. \tag{5.2.46}$$

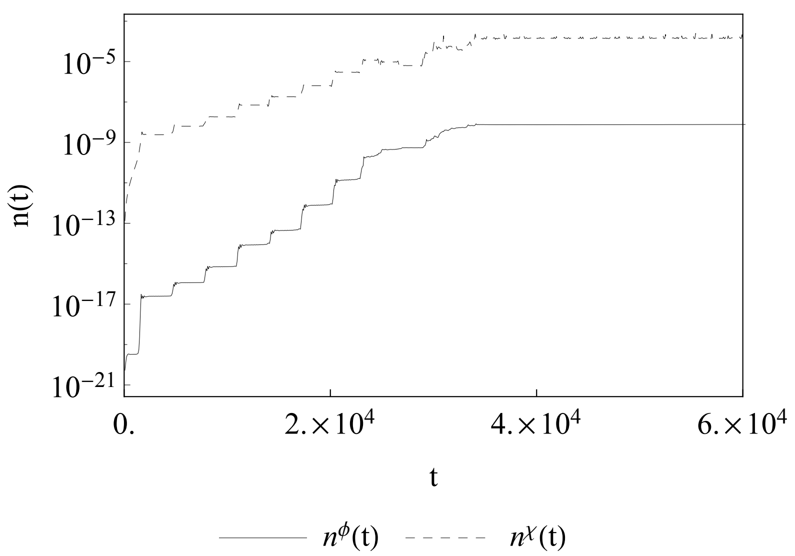

Figure 5.1: Time evolution of number density of produced states for $g = 0.1$, $m_\phi = 0.001$, $\phi\ (t = 0) = 1$, $\dot{\phi}(t = 0) = 0$ in the system with two scalars. Chosen values of parameters correspond to $m_\phi \sim 5 \times 10^{14}$ GeV and $\phi(t = 0) = 0.04 M_{PL}$, where M_{PL} denotes the Planck mass.

Besides, it is difficult to describe analytically the indirect production of states like $\tilde{\phi}$.

We are also in agreement with [49] and their expression for the number density of produced states

$$n_\chi^{(j)} \sim n_\chi^{(1)} \cdot 3^{j-1} \left(\frac{5}{2}\right)^{3/2} \frac{1}{j^{5/2}}, \tag{5.2.47}$$

where j denotes the number of oscillations. Based on the Figure 5.1 we can take $j \sim 10$, which according to the above formula corresponds to

$$n_\chi^{(10)} \sim n_\chi^{(1)} \cdot \frac{3^9}{2^3} \sim 2.5 \cdot 10^3 \tag{5.2.48}$$

and is consistent with our results. We can also notice that oscillations cease around the time when $\frac{1}{2} m_\phi^2 \langle \phi_j \rangle^2 \sim \rho_\chi^{(j)} \sim g \langle \phi_j \rangle n_\chi^{(j)}$.

Produced states are not in thermal equilibrium but we can estimate some maximal reheating temperature

$$T_R^{\max} \sim \left(\frac{30 \rho_R}{g_* \pi^2}\right)^{1/4} \tag{5.2.49}$$

under the assumption that all the energy from the background is transferred to the light sector. These light states interact then with each other, and maybe with some other particles neglected in our simple Lagrangian, gaining thermal distribution in the end. In the formula above ρ_R denotes energy density of the relativistic particles and $g_* \sim \mathcal{O}(10^2)$ the number of relativistic degrees of freedom. The values of the couplings we consider are big enough to use the relation $\rho = mn$ for the energy density, while the mass of χ is chosen as in Table 5.1. Thus, our final results for $T_R^{\max}$ are also presented in Table 5.1.

m_χ	ρ_χ	$T_R^{\max}$
125	10^{-6}	$1.3 \cdot 10^{-2}$
700	$5.7 \cdot 10^{-6}$	$2 \cdot 10^{-2}$

Table 5.1: Maximal reheating temperature and energy density of produced states for two reasonable choices of χ mass, both in GeV. Number densities correspond to Figure 5.1: $n_\phi \approx 3.96 \cdot 10^{-2}$ GeV and $n_\chi \approx 8.2 \cdot 10^{-9}$ GeV, and we assume that $m_\phi = 10^{16}$ GeV.

5.2.2 System with the additional light sector

It is often in the analysis of preheating that the light fields not coupled directly to the inflaton are not taken into account. But it turns out that their presence can affect the features of preheating as they may be produced abundantly via interactions with some field directly coupled to the background, which is produced through parametric resonance. We can realize this scenario by adding to (5.2.6) a sector of very light fields ξ_n ($m_\phi \gg m_\chi, m_\xi$) not coupled to ϕ and the time-varying background $\langle\phi\rangle$ at the tree-level

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{2}m_\chi^2\chi^2 - \frac{1}{4}g^2\phi^2\chi^2 \\ &+\sum_n \frac{1}{2}(\partial\xi_n)^2 - \sum_n \frac{1}{2}m_\xi^2\xi_n^2 - \sum_n \frac{1}{4}y^2\chi^2\xi_n^2. \end{aligned} \tag{5.2.50}$$

Other fields, χ and ξ_n, do not have a vev and they are produced in two ways - χ resonantly and ξ_n via the interactions with χ.

Up to the fourth order in couplings the physical mass of ξ_n reads

$$M_\xi^2 = m_\xi^2 + \frac{1}{2}y^2 \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{V}\langle\hat{\chi}^\dagger_{\mathbf{p}}\hat{\chi}_{\mathbf{p}}\rangle - \frac{1}{2\omega_{\chi p}}\right) + \mathcal{O}(y^4, y^2g^2, g^4). \tag{5.2.51}$$

Indeed, we notice that ξ_n can affect the background through the operator $\langle\chi^\dagger_{\mathbf{p}}\chi_{\mathbf{p}}\rangle$ in their mass term.

Results for only one additional field ξ are presented in Figure 5.2. In fact, we notice that all the states are produced effectively here, even ξ due to enhanced back-reaction effects related to the strong coupling. However, parameters are chosen in this figure in such a way we can observe quenching of the preheating - energy transfer stops when the final number density of ξ is comparable to the one for χ, $n_\xi \sim n_\chi$.

Our expectation is that the energy transfer would be the most effective towards very light ξ_n fields and the more ξ_n we have, the more energy is transferred there. Nothing could be more wrong. We can infer from the Figure 5.3 that the more light species are present in the system, the larger the final value of $|\langle\phi\rangle|$ we obtain, which means smaller energy transfer that could be even terminated. The reason is the

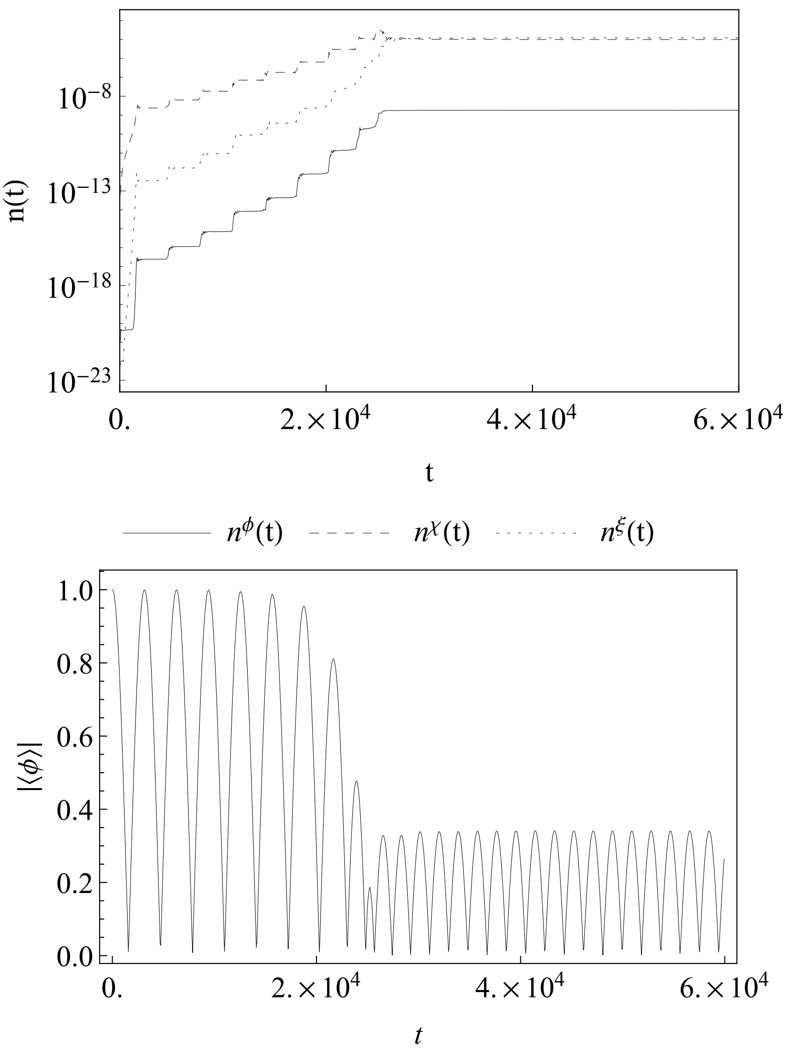

Figure 5.2: Behaviour of the number density of produced states (*top*) and the background (*bottom*) in the system with additional light sector for $g = 0.1$, $y = 1$, $n = 1$, $m_\phi = 0.001$, $\phi(t = 0) = 1$, $\dot{\phi}(t = 0) = 0$.

form of χ's physical mass

$$M_\chi^2 = m_\chi^2 + \frac{1}{2}g^2\langle\phi\rangle^2 + \frac{1}{2}g^2 \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{V}\langle\hat{\phi}^\dagger_{\mathbf{p}}\hat{\phi}_{\mathbf{p}}\rangle - \frac{1}{2\omega_{\phi p}}\right) + \\ +\frac{1}{2}y^2\sum_n\left(\frac{1}{V}\langle\hat{\xi}^\dagger_{n\mathbf{p}}\hat{\xi}_{n\mathbf{p}}\rangle - \frac{1}{2\omega_{\xi p}}\right) + \mathcal{O}(y^4, y^2g^2, g^4), \tag{5.2.52}$$

which translates into

$$M_\chi^2 \approx m_\chi^2 + \frac{1}{2}g^2\langle\phi\rangle^2 + \frac{1}{2}g^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\phi_p}}\frac{1}{V}\left\langle N^{(+)}_{\phi_{\mathbf{p}}}\right\rangle + \frac{1}{2}y^2\sum_n\left(\frac{1}{2\omega_{\xi_{n_p}}}\frac{1}{V}\left\langle N^{(+)}_{\xi_{n\mathbf{p}}}\right\rangle\right) + \\ +\mathcal{O}(y^4, y^2g^2, g^4), \tag{5.2.53}$$

using the approximation $\dot{\hat{X}}_{\mathbf{p}} \sim -i\omega_{Xp}\hat{X}_{\mathbf{p}}$ for $X = \phi, \xi_n$. Once ϕ and ξ_n are produced they automatically affect χ's effective mass making it bigger and as a result χ particle production area becomes smaller. Production of χ is then suppressed, which means that too big production of states not coupled to the background, ξ_n, damps the production of particles directly coupled to the background, χ.

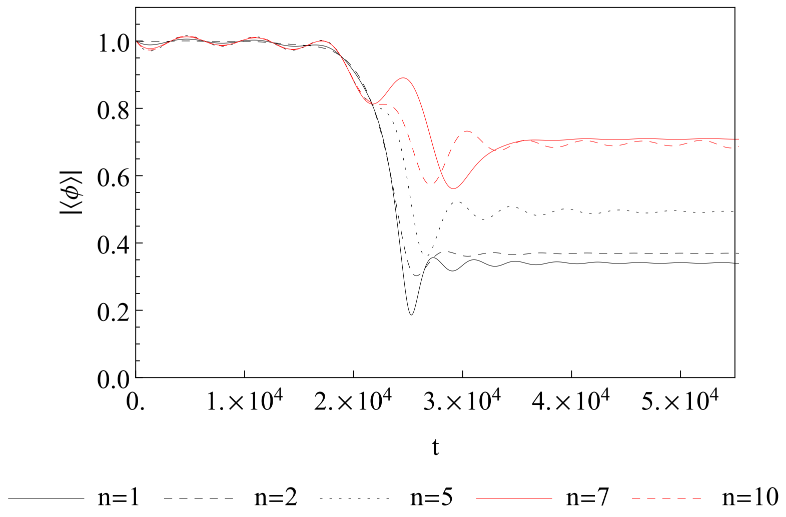

Figure 5.3: Envelope function of the time-dependence of the background $\langle\phi\rangle$ for $g = 0.1$, $y = 1$, $m_\phi = 0.001$, $\phi(t = 0) = 1$, $\dot{\phi}(t = 0) = 0$ for different numbers of light fields ξ_n: $n \in \{1, 2, 5, 7\}$.

Above mass corrections correspond to the square of so-called plasma frequency, which is a critical value settling if the wave of χ particles can enter X's plasma or not. If X particles possess masses big enough,

$$\int \frac{d^3p}{(2\pi)^3} \frac{\langle N_{\mathbf{k}}^{(+)}\rangle}{2\omega_{Xp}V} \propto \frac{n_X}{M_X}. \tag{5.2.54}$$

Furthermore, for the thermal distribution for massless states[5]

$$\frac{1}{V}\langle N_{X\mathbf{p}}^{(+)}\rangle = 2\frac{1}{e^{p/T}-1} \tag{5.2.55}$$

it corresponds to the thermal mass that reads

$$\int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{V}\langle \hat{X}_{\mathbf{p}}^\dagger \hat{X}_{\mathbf{p}}\rangle - \frac{1}{2\omega_{X_p}}\right) \sim \frac{T^2}{6}.$$

Assuming that χ or ξ is the Higgs field means that Higgs is the mediator or the light field in the scenario at hand. Table 5.2 gathers the maximal reheating temperature and energy densities for each state as for the two-scalar system. Comparing it with the Table 5.1 proves that adding the light sector rises $T_R^{\max}$ lowering the number density of ϕ particles.

[5]Factor 2 results from the existence of states with $\mathbf{k}$ and $-\mathbf{k}$ momenta.

m_χ	m_ξ	n_ξ	ρ_χ	ρ_ξ	$T_R^{\max}$
125	100	$1.21 \cdot 10^{-5}$	$1.24 \cdot 10^{-3}$	$1.21 \cdot 10^{-3}$	$0.93 \cdot 10^{-1}$
700	125	$1.21 \cdot 10^{-5}$	$6.94 \cdot 10^{-3}$	$1.51 \cdot 10^{-3}$	$1.26 \cdot 10^{-1}$

Table 5.2: Maximal reheating temperature and energy density of produced states for two reasonable choices of χ and ξ mass, both in GeV. Number densities correspond to Figure 5.2: $n_\phi \approx 1.82 \cdot 10^{-9}$ GeV and $n_\chi \approx 9.91 \cdot 10^{-6}$ GeV, and we assume that $m_\phi = 10^{16}$ GeV.

5.3 Discussion

5.3.1 Role of the couplings

Both couplings, g and y, affect the details of particle production in different way, because their role in the model is different - g couples χ to ϕ and the background $\langle\phi\rangle$, while y couples the light sector ξ_n to χ.

Figure 5.4 presents what happens with the features of preheating if we change the coupling y with fixed g. We can see that the initial stage of particle production of χ and ϕ states is not sensitive to the change of y, only the final abundance of these states responses to its variation - the bigger y is, the smaller abundance we observe. Behaviour of ξ_n's production differs as both initial and final stages are affected by the change of y. In Figure 5.5 we can see that the bigger the value of y is, the higher abundance of ξ we end up with and that the energy transfer is more effective as the value of y decreases.

produced states	varying g	varying y
χ, ϕ	both initial and final stages are strongly influenced: $g\uparrow \;\Leftrightarrow\; n_\phi\uparrow, n_\chi\uparrow$	does not influence the initial stage of preheating but influences the final n_χ and n_ϕ: $y\uparrow \;\Leftrightarrow\; n_\chi^{\text{final}}\downarrow, n_\phi^{\text{final}}\downarrow$
ξ_n	both initial and final stages are strongly influenced $g\uparrow \;\Leftrightarrow\; n_\xi\uparrow$ $g\uparrow \;\Leftrightarrow\; n_\xi^{\text{final}}\downarrow$	both initial and final stages are strongly influenced $y\uparrow \;\Leftrightarrow\; n_\xi^{\text{initial}}\uparrow$
energy transfer	$g\uparrow\Leftrightarrow$ energy transfer from $\langle\phi\rangle\uparrow$	$y\uparrow\Leftrightarrow$ energy transfer from $\langle\phi\rangle\downarrow$

Table 5.3: Influence of the value of couplings g and y on the details of the production of all the species and the energy transfer.

Greater g corresponds to the bigger production of χ and ϕ on both stages, which is in agreement with intuition - g controls ϕ's mass term and thus influence the production of χ via parametric resonance. In the case of ξ initial growth of number density is stronger when g is bigger, but the final value of its number density, n_ξ^{final}, behaves in an opposite way. Also the energy transfer from the background is more effective as g gets bigger.

Particular values of these two parameters also play significant role in obtaining $n_\xi^{\rm final} \sim n_\chi^{\rm final}$, which states the condition for quenching preheating.

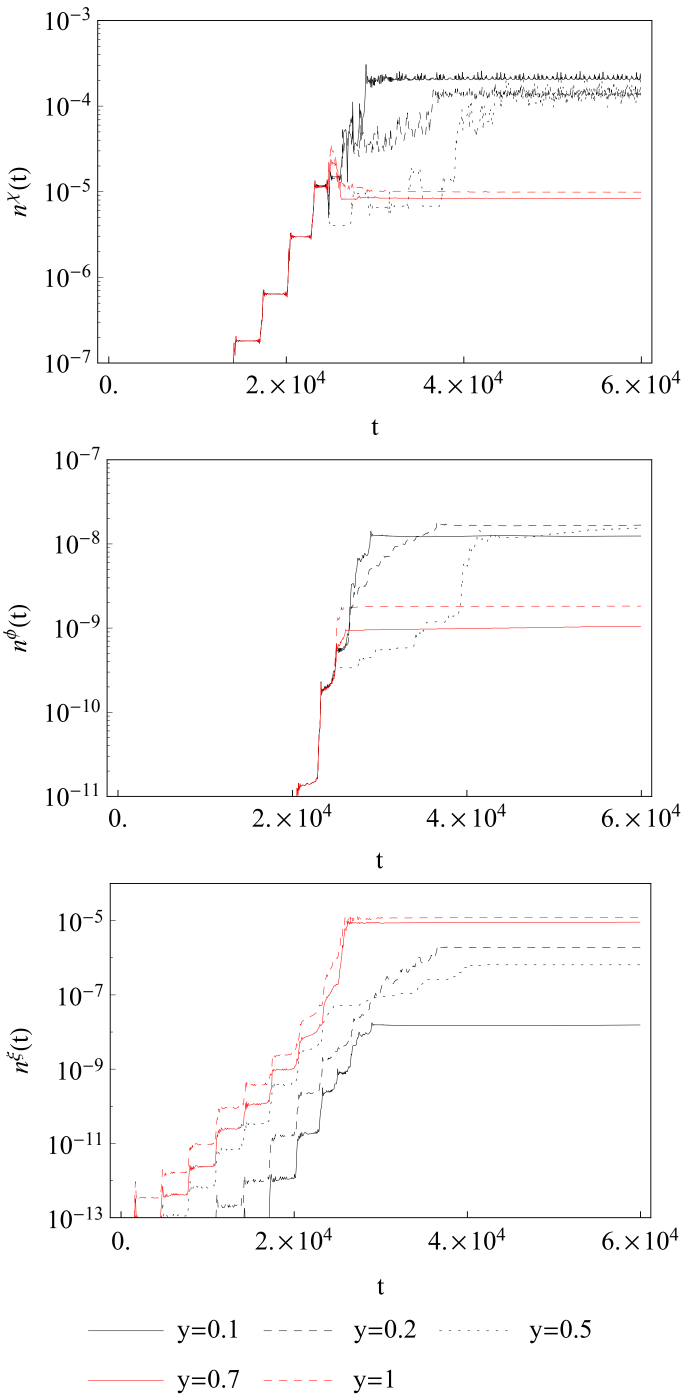

Figure 5.4: Number density of produced states χ, ϕ and ξ for $g = 0.1$, $n = 1$, $m_\phi = 0.001$, $\phi(t=0) = 1$, $\dot{\phi}(t=0) = 0$ and different values of y coupling. For the values $y = 0.7$ and $y = 1$ we observe quenching of the energy transfer.

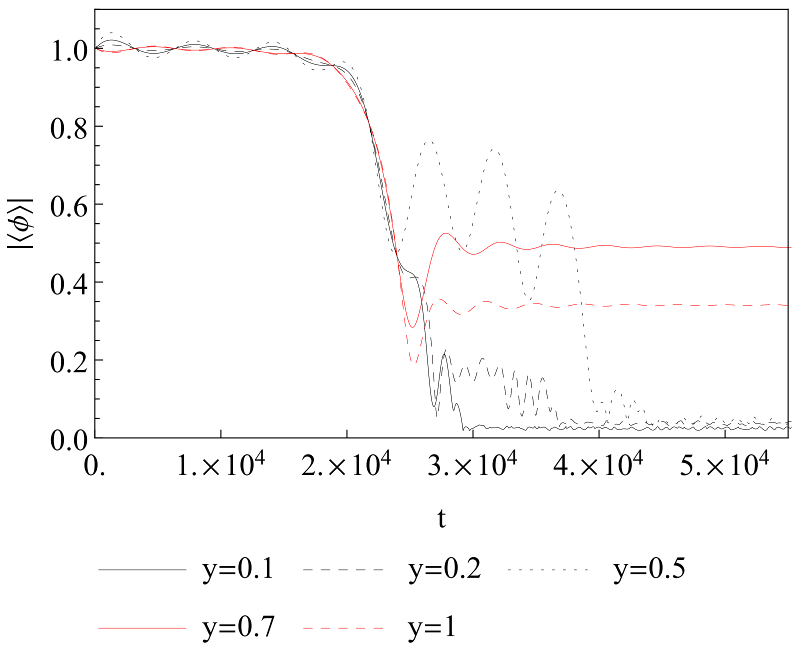

Figure 5.5: Envelope of the time evolution of the background $\langle\phi\rangle$ for $g = 0.1$, $n = 1$, $m_\phi = 0.001$, $\phi(t = 0) = 1$, $\dot{\phi}(t = 0) = 0$ and different values of y coupling. For the values $y = 0.7$ and $y = 1$ we observe quenching of the energy transfer.

old method S. Enomoto, O. Fuksińska, Z. Lalak: 1412.7442	*new* method O. Czerwińska, S. Enomoto, Z. Lalak: 1701.00015
massless background asymptotic approximation secularity for massless states	massive background interacting field theory no secularity

Table 5.4: Comparison between the *old* and *new* methods of describing particle production used in this dissertation. *New* denotes the interacting theory described here and *old* - asymptotic approximation presented in [16] and Chapter 3.

5.3.2 Secularity - comparison with our previous work

Our results from Chapter 3 show that the production of particles not directly connected to the source can be abundant due to quantum corrections (secondary effect). However, it seems to include some artificial behaviour at the late stages of the resonant particle production as we observe linear or even quadratic growth of number densities of massless particles resulting in their infinite production. This effect is known as secular growth and comes from an inappropriate perturbative expansion, even in presence of weak coupling. In general it is not very difficult to overcome this minor inaccuracy and there are a lot of ways to improve one's calculation as long as the condition of universality is fulfilled - late-time behaviour has to be insensitive to the chosen initial conditions.

Our previous approach assumed that to obtain number densities of produced states solutions should be expanded around asymptotic free fields. In order to avoid such secularity perturbative expansion has to be applied not to solutions but to equations of motion [58] and non-linear effects without any significant approximations have to be included for the sake of universality. This secularity is caused by time integral of the interaction effects with the Green functions - at the late stages particle production is overestimated because it complies "inverse decay" processes. To improve our framework we need to represent the number densities by the original interacting fields, not by the asymptotic ones, and take into account mass corrections as we do in this chapter. Notwithstanding, the results possessing secularity are still applicable for the early stages of particle production.

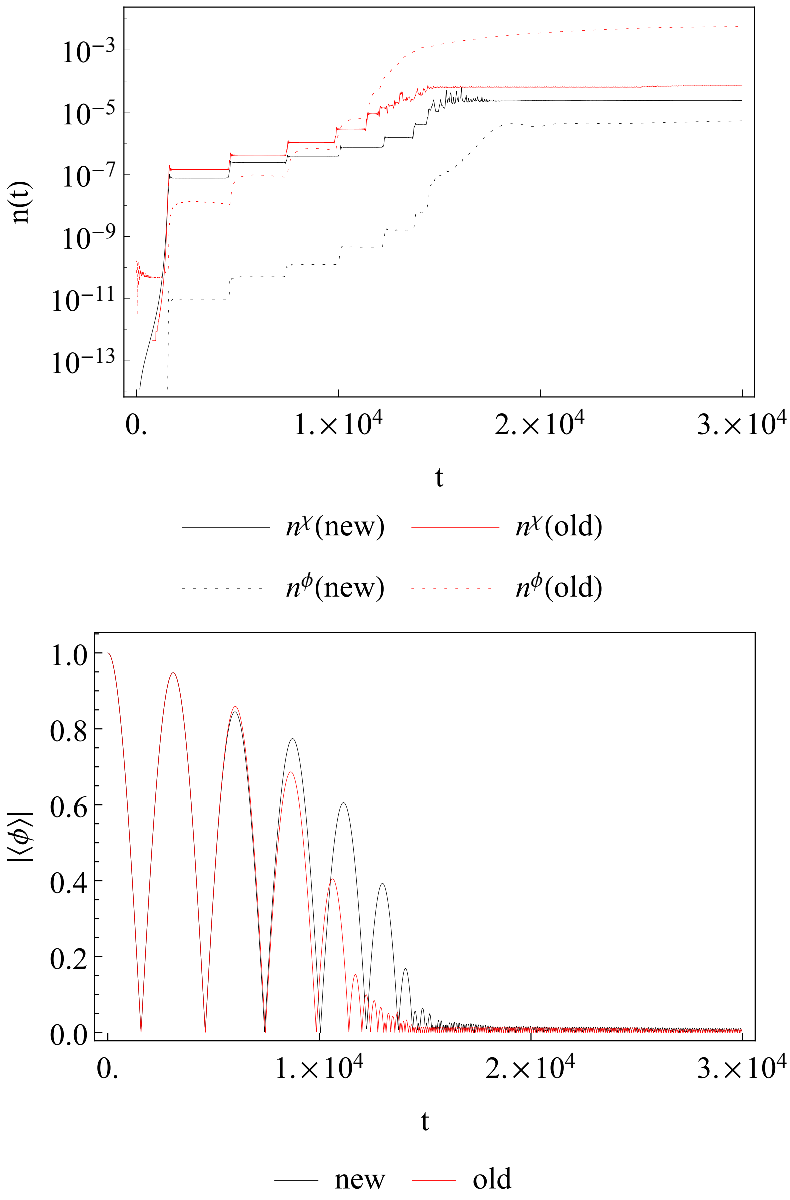

Figure 5.6: Time evolution of number density of produced states (*top*) and the background $\langle\phi\rangle$ (*bottom*) for a *new* and *old* methods for $g = 1$, $m_\phi = 0.001$, $\phi(t = 0) = 1$, $\dot{\phi}(t = 0) = 0$. *New* denotes the interacting theory described here and *old* - asymptotic approximation presented in [16] and Chapter 3.

In the Figure 5.6 we compare these two methods for the Lagrangian (5.2.6).

5.3.3 Instant preheating

On first sight it seems that our considerations may be similar to the process of instant preheating [42] but it is not the case. In the instant preheating scenario the system consists of three fields - background ϕ, χ interacting with the background and some other field ψ not coupled to ϕ. It is based on the assumption that χ particles produced within one-time oscillation of ϕ field decay instantly to ψ before the next oscillation of ϕ begins. So as in our case ψ particles can be also produced even though there is no direct interaction between ϕ and ψ, but the mechanism of production is different - decay instead of quantum corrections, and quenching of preheating originates in the rapid decay not in a plasma gas effect.

Table 5.5 compares our work with instant preheating in a very general way.

	our work	instant preheating
inflaton's behaviour	oscillations	no oscillations
mechanism of production	quantum corrections	decay
origin of the quenching	backreaction	rapid decay

Table 5.5: Brief comparison between instant preheating and our considerations.

Summary of the chapter

- We develop the results from Chapter 3 in two preheating scenarios with fields coupled indirectly to the background, two-scalar system and system with additional light sector, using the theory of interacting field.
- We investigate the role of the couplings in these models discussing the conditions for obtaining successful quenching of the energy transfer between the background and the dynamical sector.
- We discuss the profits of using theory of interacting field, which allows to avoid artificial infinite growth coming from the approximation by asymptotic field seen before.
- We compare our analysis with instant preheating, which artificially seems to be related.

Chapter 6

Gravitational reheating and its cosmological consequences

Cosmological inflation is usually realized by assuming some couplings between the inflaton and matter fields to obtain a proper reheating scenario. However, when these couplings are strong, it may lead to non-trivial loop corrections to inflationary potential, which in principle may spoil its flatness ruining the very basis of the inflationary concept. In consequence, this could even spoil the predictions of inflation [59].

This motivates the studies of alternative models of reheating with gravitational reheating, which is another example of particle production in the time-dependent background, as a prime example. In this scenario post-inflationary production of particles proceeds only due to gravitational interactions [6, 60, 61, 62, 63, 64].

In particular it is interesting to consider instant transition between de-Sitter evolution and a decelerating universe, which produces quantum modes of scalar fields capable of dominating the Universe and increasing its temperature sufficiently at some point. Moreover, in this scenario the inflaton does not need to be coupled to any Standard Model degree of freedom and therefore it may be a part of a dark sector, which can combine several cosmological open questions - inflation, dark matter (DM) and dark energy (DE) - at once. In our study we exploit this scenario to investigate its possible phenomenological consequences regardless of the particular structure of the dark and inflationary sector. We introduce a dark inflaton as it couples to SM only gravitationally and is a component of the dark sector. Our dark inflation is followed by the domination of a perfect fluid with a barotropic parameter w and then by the usual radiation and matter domination.

There is one other important reason to study gravitational particle production - the uncertainty in the thermal history of the Universe excludes the precise calculation of the exact moment of the horizon crossing of the pivot scale [65, 66]. The number of e-folds before the end of inflation at the pivot scale horizon crossing, $N_\star$, depends on the energy scale at which radiation starts to dominate, ρ_{th}, which can vary from $\rho_{th} \sim \rho_{end}$ (an instant reheating scenario) to $\rho_{th} \sim \mathrm{MeV}^4$ (radiation has to dominate before the Big Bang Nucleosynthesis[1]). This ambiguity significantly

[1]This condition is the most stringent current experimental constraint on the temperature of

affects $N_\star$ and therefore also the predictions of inflationary models. In the dark inflationary scenario reheating temperature is a precise function of inflationary parameters, such as scale of inflation and the post-inflationary equation of state, which makes the results more precise.

Moreover, an attractive feature of the dark inflationary scenario is its simplicity. Most reheating mechanisms require the existence of additional couplings between the inflaton and matter fields, while gravitational particle production always occurs, with or without them, at the end of inflation, regardless of the form of the inflationary potential. Therefore dark inflation can decrease the amount of new physics needed in order to explain the matter content of the present Universe.

In this chapter we use the convention $8\pi G = M_p^{-2}$, where $M_p = 2.435 \times 10^{18}$ GeV is the reduced Planck mass.

6.1 Dark inflation with gravitational reheating

In order to investigate the gravitational particle production during the transition era between the de-Sitter expansion, which is a good approximation of the cosmic inflation era, and a decelerating universe we consider the evolution of the inflaton and a scale factor as a function of a conformal time η.

As shown in [6], the energy density of radiation generated by gravitational particle production for the inflaton minimally coupled to gravity is equal to

$$\rho_r = \frac{H_{\rm inf}^4}{128\pi^2}\left(\frac{a_{end}}{a}\right)^4 I\,, \tag{6.1.1}$$

where $H_{\rm inf}$ is the value of the Hubble parameter at the plateau and a_{end} is the value of the scale factor at the and of inflation. Integral I is defined by

$$I = -\int_{-\infty}^{x} dx_1 \int_{-\infty}^{x} dx_2 \log(|x_1 - x_2|)\frac{d\tilde{V}(x_1)}{dx_1}\frac{d\tilde{V}(x_2)}{dx_2}\,, \tag{6.1.2}$$

$$\tilde{V}(x) = \frac{f_{xx}f - \frac{1}{2}f_x^2}{f^2}\,, \tag{6.1.3}$$

$$f(H_{\rm inf}\eta) = a^2(\eta)\,, \tag{6.1.4}$$

where $\tilde{V}$ is a rescaled Ricci scalar, $x = H_{\rm inf}a$ and $f_x = \dfrac{df}{dx}$. The upper limit for the integration of I corresponds to the moment when $\tilde{V} < 1$ and the notion of particle is well-defined at all times.

Gravitational particle production rate can be estimated analytically by investigating the instant transition between de-Sitter universe and a decelerating solution, for instance radiation domination [6] or domination of the kinetic energy of a free

reheating. BBN occurs at the MeV scale [67] and the exact temperature varies from 0.7MeV to 5MeV[68, 69].

scalar field [60]. Our analysis is more general as we assume that the decelerating universe is filled with any perfect fluid with a constant barotropic parameter w and thus we can describe the evolution of $f(x)$ in the following way

$$f(x) = \begin{cases} \frac{1}{x^2} & x < -1, \text{ de Sitter} \\ a_0 + a_1 x + a_2 x^2 + a_3 x^3 & -1 < x < x_0 - 1, \text{ transition} \\ b_0 (b_1 + x)^{\frac{4}{3w+1}} & x_0 - 1 < x, \text{ general } w \neq -1/3 \end{cases}, \tag{6.1.5}$$

where x_0 is the transition time between the de-Sitter and decelerating solutions.

Side note: Scale factor in the universe filled with perfect fluid in terms of conformal time

The scale factor in terms of conformal time reads:

$$\begin{cases} a(\eta) = a_0 \left(1 + \frac{3w+1}{3(w+1)} \left(\frac{\eta}{\eta_0} - 1\right)\right)^{\frac{2}{3w+1}} & \text{for } w \neq -\frac{1}{3} \\ a(\eta) = a_0 \exp\left(\frac{\eta}{\eta_0} - 1\right) & \text{for } w = -\frac{1}{3} \end{cases}, \tag{6.1.6}$$

with a_0, t_0 and η_0 describing chosen initial conditions.

For the de Sitter phase w is equal to -1 and the Hubble parameter is constant $H = H_{inf}$, so the evolution of the scale factor in terms of cosmic and conformal time can be expressed as:

$$a(t) = a_0 e^{H_{\text{inf}}(t - t_0)}, \tag{6.1.7}$$

$$a(\eta) = \frac{a_0}{1 - a_0 \mathcal{H}_{\text{inf}}(\eta - \eta_0)} \tag{6.1.8}$$

with t_0, η_0 and a_0 being the constants of integration again.

Our a_i and b_i coefficients can be calculated using continuity conditions for $f(x)$, $f'(x)$ and $f''(x)$ at $x = -1$ and $x = x_0 - 1$, which makes the Ricci scalar continuous throughout the whole period. We assume that the transition under consideration occurs within the Hubble time, which means $x_0 < 1$ and it follows that:

$$a_0 = \frac{1}{4}\left(29 - 8w + 3w^2\right) - \frac{1+w}{2x_0}, \tag{6.1.9}$$

$$a_1 = \frac{3}{4}\left(\frac{47}{3} - 8w + 3w^2\right) - 3\frac{1+w}{2x_0}, \tag{6.1.10}$$

$$a_2 = \frac{3}{4}\left(9 - 8w + 3w^2\right) - 3\frac{1+w}{2x_0}, \tag{6.1.11}$$

$$a_3 = \frac{1}{4}\left(5 - 8w + 3w^2\right) - \frac{1+w}{2x_0}, \tag{6.1.12}$$

$$b_0 = \left(\frac{2}{1+3w}\right)^{-\frac{4}{1+3w}}, \tag{6.1.13}$$

$$b_1 = \frac{3(1+w)}{1+3w}. \tag{6.1.14}$$

For the $w = -1/3$ case after the transition ($x > x_0 - 1$) $f(x) = b_0 \exp(b_1 x - 1)$ for and the continuity conditions imply that $b_0 = 1/2$ and $b_1 = 2$ up to $\mathcal{O}(x_0)$ terms, while a_i coefficients stay the same as for $w \neq -1/3$.

The biggest contribution to I comes from integration around the transition time, for $x \in (x_0 - 1, -1)$, so it can be estimated by

$$I \simeq 9(w+1)^2 \log\left(\frac{1}{x_0}\right) . \tag{6.1.15}$$

The hitherto results are fully consistent with [6, 60].

Interesting is also to consider the transition between two de Sitter space-times with different values of $H_{\rm inf}$ (two subsequent inflationary events at different energy scales) for which $f = b_0/(x+b_1)^2$ for $x > x_0 - 1$. It turns out that the a_i coefficients satisfy (6.1.9)-(6.1.12), which means there is no x_0^{-1} term corresponding to $I \propto x_0^2$ for such a case resulting in a strong suppression of particle production, since we assume $x_0 \ll 1$.

Following (6.1.1) the energy density of radiation produced at the end of the dark inflation can be estimated as

$$\rho_r \simeq H_{\rm inf}^4 \frac{9N_{\rm eff}(1+w)^2}{128\pi^2} \left(\frac{a_{end}}{a}\right)^4 \log\left(\frac{1}{x_0}\right) , \tag{6.1.16}$$

where $N_{\rm eff}$ denotes number of scalar species produced gravitationall for an inflaton minimally coupled to gravity[2]. Non-minimal coupling results in an additional numerical coefficient in (6.1.16) that is absorbed into $N_{\rm eff}$. We want to keep our considerations as model-independent as possible so we simply include a wide range of possible values of $N_{\rm eff}$ and show the consequences of each choice. For example, in a specific case of a scalar-tensor theory with a non-minimal coupling of the form $\xi\phi^2 R$ we would have $N_{\rm eff} = N(1-6\xi)^2$. Thus, the gravitational particle production can be strongly suppressed by the non-minimal coupling close to the conformal value. The $\log(1/x_0)$ term should be of order of unity [6] and it is neglected in the further part of this analysis.

As gravitational particle production is highly inefficient in comparison with most of the other mechanisms of reheating [60], dark inflation leads to the universe initially dominated by the inflaton field which needs to be taken care of in order to end up with the viable thermal history of the Universe. In such a case to obtain a proper radiation-domination (RD), which is necessary at least during the BBN era, we need the energy density of the inflaton redshifting faster than radiation after inflation. This means that dark inflation changes the thermal history of the Universe by introducing a long period of inflaton domination in the post-inflationary era but before RD. We assume that the inflaton field may be treated as a perfect fluid with constant barotropic parameter w and we limit ourselves to $w \in]1/3, 1]$ due to the presumption about its redshift.

[2]Both fermions and vectors may be also produced during the transition between two gravitational vacua but energy densities related to them are too small to significantly contribute to the reheating of the Universe.

6.1.1 Inflationary parameters

The universe does not reheat at once when radiation is produced but when it stars to dominate the energy density of the universe, which means that the reheating temperature T_R fulfils the condition $\rho_r = \rho_\phi$. It gives us the expression for the energy density of inflaton

$$\rho_\phi = 3H_{\rm inf}^2 M_p^2 \left(\frac{a_{end}}{a}\right)^{3(1+w)}, \tag{6.1.17}$$

where a_{end} is the scale factor at the end of inflation. Combining (6.1.16) and (6.1.17) we obtain the scale factor at the moment of reheating

$$a_R = a_{end} \left(\frac{128\pi^2 M_p^2}{3N_{\rm eff}(1+w)^2 H_{\rm inf}^2}\right)^{\frac{1}{3w-1}} \tag{6.1.18}$$

and the radiation energy density at that time

$$\rho_R \equiv \rho_r(a_R) = 3H_{\rm inf}^2 M_p^2 \left(\frac{128\pi^2 M_p^2}{3N_{\rm eff}(1+w)^2 H_{\rm inf}^2}\right)^{-\frac{3(w+1)}{3w-1}}. \tag{6.1.19}$$

Finally it gives us the reheating temperature

$$\frac{T_R}{M_p} = \left(\frac{90}{\pi^2 g_*(T_R)}\right)^{1/4} \left(\frac{128\pi^2}{3N_{\rm eff}(1+w)^2}\right)^{-\frac{3(1+w)}{4(3w-1)}} \left(\frac{H_{\rm inf}}{M_p}\right)^{\frac{3w+1}{3w-1}}, \tag{6.1.20}$$

which is presented as a function of the barotropic parameter w for some particular values of $H_{\rm inf}$ and $N_{\rm eff}$ in Figure 6.1. We can infer from it that reheating temperature grows with the barotropic parameter for fixed values of $H_{\rm inf}$ and $N_{\rm eff}$ - it corresponds to radiation starting to dominate earlier as the inflaton redshifts away faster.

Successful BBN requires that radiation must dominate the Universe before its beginning around $T_{\rm BBN} \approx 1$MeV [70, 71], which sets a constraint on the values of T_R and $H_{\rm inf}$. However, radiation domination does not mean that there are no inflaton remnants present at the MeV scale, so the overall and the usual radiation domination Hubble parameters read

$$H^2 = \frac{1}{3M_p^2}(\rho_r + \rho_\phi), \qquad H_r^2 = \frac{1}{3M_p^2}\rho_r = \frac{1}{3M_p^2}\frac{\pi^2}{30} g_* T^4. \tag{6.1.21}$$

Then the upper bound on H from BBN can be expressed as [72, 73]

$$\left(\frac{H}{H_r}\right)^2\Bigg|_{T=T_{\rm BBN}} \leq 1 + \frac{7}{43}\Delta N_{\nu_{\rm eff}} \equiv \alpha \approx 1.038, \tag{6.1.22}$$

where $\Delta N_{\nu_{\rm eff}} = 3.28 - 3.046$ denotes the difference between the measured value and SM prediction for the effective number of neutrinos. In terms of the scale factor it means that

$$\alpha - 1 \geq \frac{\rho_\phi}{\rho_r}\Bigg|_{T=T_{\rm BBN}} = \frac{\rho_\phi}{\rho_r}\Bigg|_{\rm T=T_R} \left(\frac{a_{\rm BBN}}{a_R}\right)^{4-3(1+w)} = \left(\frac{a_{\rm BBN}}{a_R}\right)^{1-3w}. \tag{6.1.23}$$

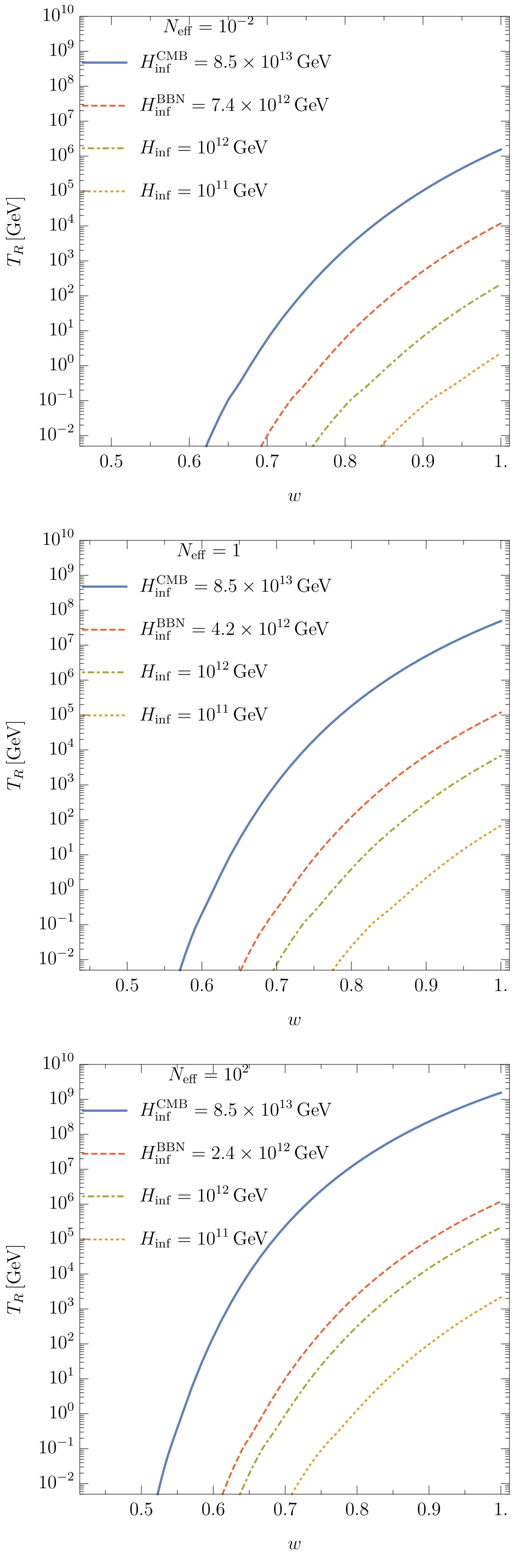

Figure 6.1: Temperature of reheating, T_R, as a function of w for some fixed values of inflationary scale $H_{\rm inf}$ with $N_{\rm eff} \in \{10^{-2}, 1, 100\}$ from top to bottom. $N_{\rm eff} = 10^{-2}$ corresponds to the nearly conformal non-minimal coupling to gravity ($\xi \simeq 1/6$), $N_{\rm eff} = 1$ to $\xi = 0$ and $N = 1$ (SM scenario), while $N = 100$ matches to the supersymmetric models with $\xi = 0$.

We can approximate the energy density of the universe at the moment of BBN by

$$\rho_{\rm BBN} \approx \rho_r(a_{\rm BBN}) = \frac{\pi^2}{30} g_*\left(T_{\rm BBN}\right) T_{\rm BBN}^4\,, \tag{6.1.24}$$

which combined with (6.1.16) specifies the value of the scale factor then

$$a_{\rm BBN} = a_{end}\frac{H_{\rm inf}}{\rho_{\rm BBN}^{1/4}}\left(\frac{9N_{\rm eff}(1+w)^2}{128\pi^2}\right)^{\frac{1}{4}}\,. \tag{6.1.25}$$

All of the above finally results in the limit on the scale of inflation

$$\frac{H_{\rm inf}}{M_p} \geq \left[(\alpha-1)^{-\frac{1}{3w-1}}\,\frac{\left(\frac{1}{3}\rho_{\rm BBN}\right)^{1/4}}{M_p}\left(\frac{3N_{\rm eff}(1+w)^2}{128\pi^2}\right)^{-\frac{3}{4}\frac{1+w}{3w-1}}\right]^{\frac{3w-1}{3w+1}}\,, \tag{6.1.26}$$

preserving the proper BBN, which translates into the analogous constraint on the temperature of reheating

$$T_R \geq \left(\frac{30}{\pi^2 g_\star(T_R)}\right)^{1/4}(\alpha-1)^{-\frac{1}{3w-1}}\,\rho_{\rm BBN}^{\frac{1}{4}}\,. \tag{6.1.27}$$

The above limit turns out to be $N_{\rm eff}$-independent.

Our lower limit on the scale of inflation corresponds to the usual radiation domination starting slightly above the BBN temperature, see Figure 6.2. In this figure we can also see that the minimal reheating temperature is a decreasing function of w as expected - inflaton remaining after the end of inflation needs more time to redshift away enough for small w. Moreover, there are two constraints included in this figure. The first one is that the minimal $H_{\rm inf}$ allowed is already higher than the upper bound from the CMB polarization ($H_{\rm inf} \lesssim H_{\rm inf}^{\rm CMB} = 8.5\times10^{13}$ [74]) for lower values of w, which automatically excludes them. The second one $H_{\rm inf} \lesssim H_{\rm inf}^{\rm BBN}$, also present in Figure 6.3, arises from the necessity not to overproduce gravitational waves (see Section 6.2) as it would contribute to radiation and eventually spoil BBN predictions.

We can make an interesting observation in the context of the Higgs instability [75, 76] analyzing Figure 6.3. For some part of the parameter space obtained scales of inflation can be smaller than the GUT scale but if they are greater than the barrier between the electroweak vacuum and the true high energy minimum of the Higgs potential ($H_{\rm inf} \gtrsim 10^{10}-10^{12}$ GeV), the field would be pushed into the deeper minimum during inflation [77, 78]. This is true assuming that SM is valid at least up to the instability scale because any physics beyond SM below it would allow a higher scale of inflation.

Another parameter characterizing the inflationary scenarios that we can limit in our considerations about the dark inflation is the number of e-folds before the end of inflation at the pivot scale horizon crossing [65, 66]

$$N_\star \simeq 67 - \log\left(\frac{k_\star}{a_0H_0}\right) + \frac{1}{4}\log\left(\frac{V_{hor}}{M_p^4}\right) + \frac{1}{4}\log\left(\frac{V_{hor}}{\rho_{end}}\right) + \frac{1-3w}{12(1+w)}\log\left(\frac{\rho_{th}}{\rho_{end}}\right)\,, \tag{6.1.28}$$

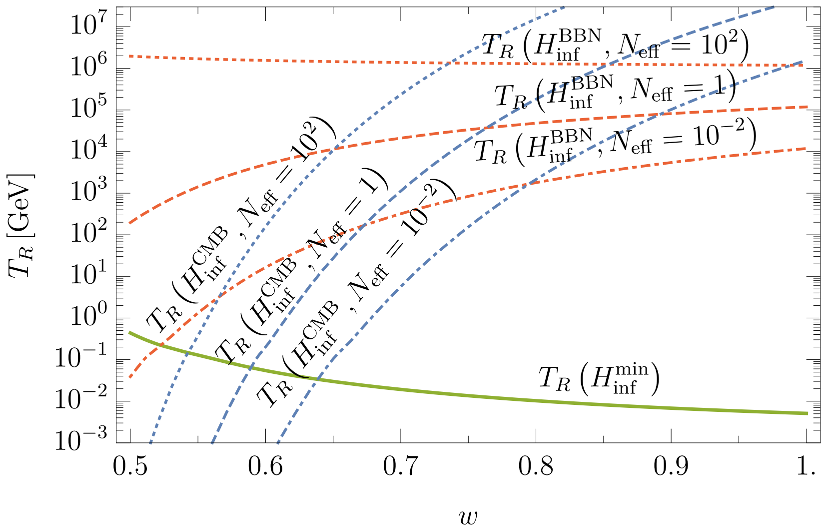

Figure 6.2: The minimal reheating temperature T_R not spoiling the proper BBN (solid green line) together with T_R corresponding to maximal inflationary scale allowed by CMB polarization data for different values of $N_{\rm eff}$ ($N_{\rm eff} = 10^{-2}$ dot-dashed, $N_{\rm eff} = 1$ dashed and $N_{\rm eff} = 10^2$ dotted blue lines) and maximal T_R not spoiling the BBN with overproduction of GWs ($N_{\rm eff} = 10^{-2}$ dot-dashed, $N_{\rm eff} = 1$ dashed and $N_{\rm eff} = 10^2$ dotted red lines). The allowed range of reheating temperatures lies between $T_R(H_{\rm inf}^{\rm CMB/BBN})$ and $T_R(H_{\rm inf}^{\rm min})$.

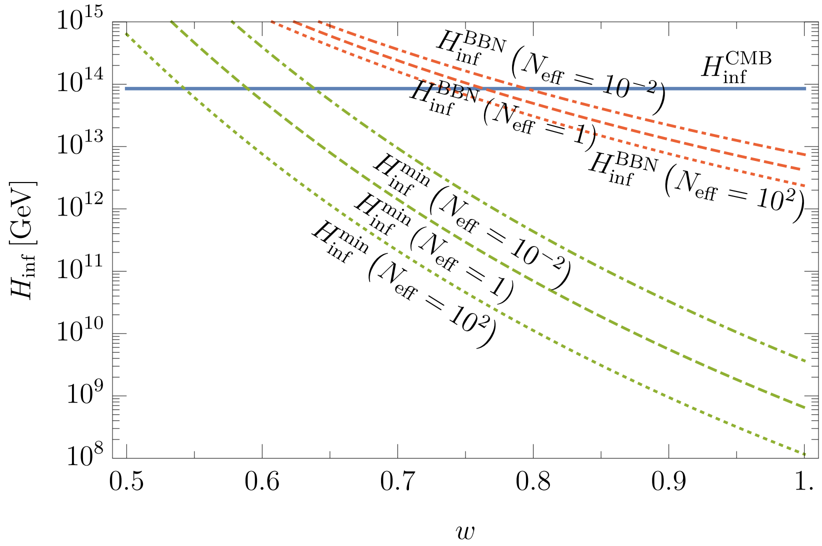

Figure 6.3: The minimal inflationary scale $H_{\rm inf}^{\rm min}$ not spoiling the proper BBN for different values of $N_{\rm eff}$ ($N_{\rm eff} = 10^{-2}$ dot-dashed, $N_{\rm eff} = 1$ dashed and $N_{\rm eff} = 10^2$ dotted green lines) together with the maximal $H_{\rm inf}^{\rm CMB}$ allowed by current CMB polarization data (solid blue line) and the maximal $H_{\rm inf}^{\rm BBN}$ not spoiling the BBN with overproduction of GWs ($N_{\rm eff} = 10^{-2}$ dot-dashed, $N_{\rm eff} = 1$ dashed and $N_{\rm eff} = 10^2$ dotted red lines). The allowed range of inflationary Hubble scales lies between $H_{\rm inf}^{\rm CMB/BBN}$ and $H_{\rm inf}^{\rm min}$.

where $k_\star$ denotes the pivot scale, $a_0 H_0$ is the inverse of the comoving Hubble radius at present, V_{hor} is the value of the inflaton potential at the horizon crossing, ρ_{end} is the energy density (energy scale) at the end of inflation, ρ_{th} describes the energy scale at which radiation starts to dominate, while w defines the equation of state after inflation but before the radiation domination era.

For the dark inflation scenario $V_{hor} \sim \rho_{end} \sim 3H^2_{\rm inf}$ and $\rho_{th} = \rho_R$, which together with $k_\star = 0.002{\rm Mpc}^{-1}$ and ρ_r defined by (6.1.19) results in

$$N_\star \simeq 64.82 + \frac{1}{4}\ln\left(\frac{128\pi^2}{N_{\rm eff}(1+w)^2}\right). \tag{6.1.29}$$

This value does not directly depend on $H_{\rm inf}$, which reduces its uncertainty as $H_{\rm inf}$ varies from the GUT scale to the MeV scale in different models of inflation. For dark inflation $N_\star$ depends only on two free parameters of the model, namely w and $N_{\rm eff}$, and this dependence is logarithmic so it does not influence the change of $N_\star$ much, see Figure 6.4. As for the range of these two - w has to be larger than 1/3 in order to provide sufficient redshift of the inflaton after inflation, while $N_{\rm eff}$ may vary from $N_{\rm eff} = 1$ for SM minimally coupled gravity to $N_{\rm eff} \approx 100$ for MSSM. In the case of non-minimal coupling for the inflaton $N_{\rm eff}$ can acquire even greater values.

There is one subtlety in the reasoning above as the scale of inflation does not necessarily need to be of the same order of magnitude as ρ_{end}, for example for the big field models. In case of $m^2\phi^2$ inflation the value of V_{hor} may be two orders of magnitude bigger than $\rho_{\rm end}$. Even though the large field models are disfavoured by the data it is interesting to investigate the influence of the hierarchy between V_{hor} and ρ_{end} on (6.1.29) and assume the relation $\rho_{end} = \zeta V_{hor}$. Then

$$N_\star \simeq 64.82 + \frac{1}{4}\ln\left(\frac{128\pi^2}{N_{\rm eff}(1+w)^2}\right) + \frac{1}{6}\frac{1+3w}{1+w}\log(\zeta). \tag{6.1.30}$$

Logarithmic dependence on ζ means that the deviation from (6.1.29) is not very strong - for the extreme case with $w = 1$ and $\zeta = 10^{-2}$ the difference is equal to $-2/3$, which is not a very altering correction. Apart from that it seems that $\zeta \lesssim 1$ is a grounded assumption for a huge class of inflationary models consistent with observations, for example Starobinsky inflation, α-attractors or ξ-attractors.

Another parameter testing the viability of the inflationary model is the tensor-to-scalar ratio r characterizing the ratio between tensor and scalar power spectrum, see Section 4.1, which according to the data should be rather small with the present upper bound $r < 0.09$ [38]. There is a correspondence between r and the inflationary scale described by the power spectrum normalization condition

$$\frac{r}{0.01} = \left(\frac{H_{\rm inf}}{\Lambda_{\rm COBE}}\right)^2 \tag{6.1.31}$$

with $\Lambda_{\rm COBE} = 2.54\times 10^{13}{\rm GeV}$. Therefore, we can translate our bound on inflationary scale (6.1.26) to the one on the tensor-to-scalar ratio

$$\frac{r}{0.01} \geqslant \left(\frac{M_p}{\Lambda_{\rm COBE}}\right)^2 \left[(\alpha-1)^{-\frac{1}{3w-1}} \frac{\left(\frac{1}{3}\rho_{\rm BBN}\right)^{1/4}}{M_p} \left(\frac{3N_{\rm eff}(1+w)^2}{128\pi^2}\right)^{-\frac{3}{4}\frac{1+w}{3w-1}}\right]^{2\frac{3w-1}{3w+1}} \tag{6.1.32}$$

Figure 6.4 presents the values of $r_{\min}$ as a function of N and w and shows that dark inflation is compatible with the experimental data as long as the value of w is not very small.

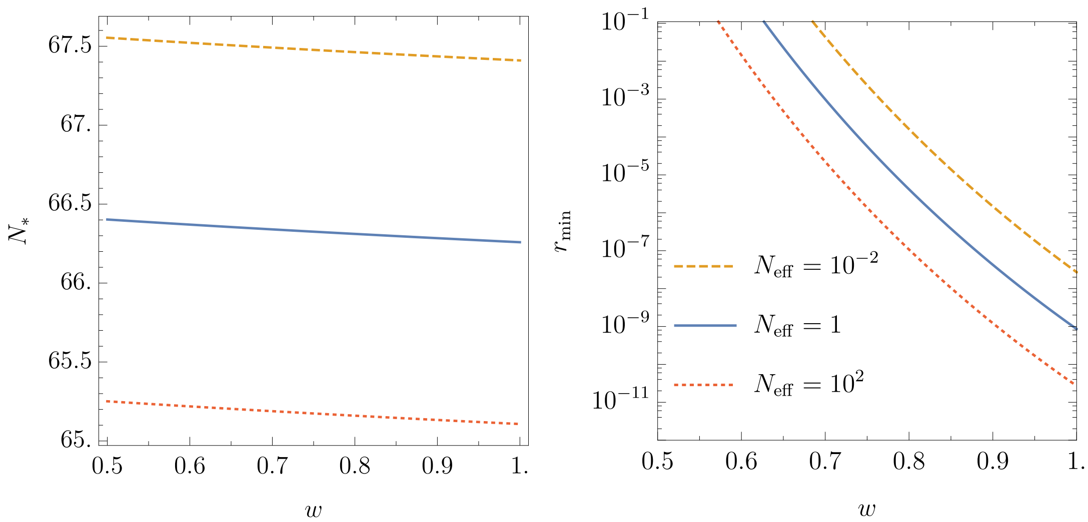

Figure 6.4: The values of $N_\star$ (*left*) and $r_{\min}$ (*right*) as a function of w for several values on N_{eff} listed in the right panel. We can infer from the behaviour of tensor-to-scalar ratio that most of specific realizations of the inflationary sector can be made consistent with these constraints as it is natural to obtain $r \lesssim 0.1$ here.

6.2 Gravitational waves

Inflationary scenarios predict also the features of produced background of primordial gravitational waves (GW) represented by the tensor fluctuations of the metric tensor. They could be observed in the polarization of the CMB and could exclude or confirm different inflationary models.

For simplicity we assume a scale invariant primordial power spectrum that can be approximated as in [79] for the dark inflation scenario

$$P_{\text{GW}}(k) = \frac{2H_{\text{inf}}^2}{\pi^2 M_p^2}, \tag{6.2.1}$$

but we need its present shape in order to compare it with observations. The present GW spectrum can be described by [80]

$$\Omega_{\text{GW}} h^2(k,\tau_0) = \begin{cases} \frac{k^2}{12a_0^2H_0^2} P_{\text{GW}}(k) T_{\text{T}}^2(k,\tau_0) & \text{for } k \le k_R \\ \frac{k^2}{12a_0^2H_0^2} P_{\text{GW}}(k) T_{\text{T}}^2(k,\tau_0) \left(\frac{k}{k_R}\right)^{\frac{6w-2}{3w+1}} & \text{for } k_R < k \le k_{end} \end{cases}, \tag{6.2.2}$$

where $T_{\text{T}}^2(k,\tau_0)$ is the transfer function [81, 82], k_{end} is the highest reachable scale and $k_R = a_R H(a_R)$, which during radiation domination era can be computed using (6.1.18)-(6.1.21). Behaviour for $k_R < k \le k_{end}$ comes from the period of

w-domination after inflation.

The role of the transfer function is to describe the evolution of GWs in the late history of the Universe starting from matter domination era. These waves reenter the horizon at late times and very small scale which corresponds to a very low frequency $f \approx 10^{-17} - 10^{-18}$ Hz. This sets an upper limit on the inflationary scale with the current value $H_{\text{inf}} \lesssim 8.5 \times 10^{-13}$. However, we are focused in GWs that reeenter the horizon much earlier - during the domination of inflaton after inflation, because GWs redshift much slower than the background then, which results in an enhanced abundance at present. The transfer function reads

$$T_{\mathrm{T}}^2(k, \tau_0) = \frac{3\Omega_m j_1(k\tau_0)}{k\tau_0} \sqrt{1 + 1.36\left(\frac{k}{k_{\text{eq}}}\right) + 2.5\left(\frac{k}{k_{\text{eq}}}\right)^2}, \tag{6.2.3}$$

where Ω_m is the present matter abundance, k_{eq} corresponds to matter-radiation equality and $j_1(x) \approx (\sqrt{2}x)^{-1}$ is the spherical Bessel function. Scale k_{end} may be obtained using (6.1.18)-(6.1.21) again

$$\frac{k_{end}}{k_R} = \frac{a_{end} H(a_{end})}{a_R H(a_R)} = \frac{1}{2}\left(\frac{\rho_{end}}{\rho_R}\right)^{\frac{1}{2} - \frac{1}{3(1+w)}} = \frac{1}{2}\left(\frac{128\pi^2 M_p^2}{3N_{\text{eff}}(1+w)^2 H_{\text{inf}}^2}\right)^{\frac{1+3w}{2(1-3w)}}. \tag{6.2.4}$$

For dark inflation we observe the increase of GW density $\Omega_{\text{GW}}h^2$ for high frequencies, which may have observational consequences in the future. Unfortunately, if inflaton is minimally coupled to gravity ($N_{\text{eff}} \geq 1$), it would be impossible to observe this additional increase in the near future, see Figure 6.5. However, in the case of non-minimal coupling close to its conformal value ($N_{\text{eff}} \ll 1$) this new feature can be reached by LIGO or future spaced based experiments in coming years, see Figure 6.6.

In our analysis for illustrative purpose we focus on the case with $w = 1$ as the effect of the amplification of Ω_{GW} due to w-domination period is the strongest here.

As $N_{\text{eff}} \ll 1$ can be realized by inflaton non-minimally coupled to gravity $\xi\phi^2 R$ for ξ very close to the conformal value features of GWs signal can not only probe thermal history of the Universe but also the presence of non-minimal coupling.

In Figures 6.2 and 6.3 we include additional upper bound preserving standard BBN from the overproduction of GWs. It could be caused by the smaller redshift of GWs during inflaton dominated period than during radiation domination [61], which ends up with GWs effectively contributing as extra radiation at BBN. Its present abundance is limited by observations and, as we can see n Figure 6.2 and 6.3, for big values of w it constraints the parameter space even more than the usual CMB bound.

Our assumption in this section is that the spectrum of primordial gravitational waves is completely scale-independent, which seems to be consistent with observations. Scale dependence is described by the spectral index n_T and strongly depends on the considered inflationary scenario. For the small field models currently favored by the data n_T is small, which corresponds to quasiscale independence. In fact,

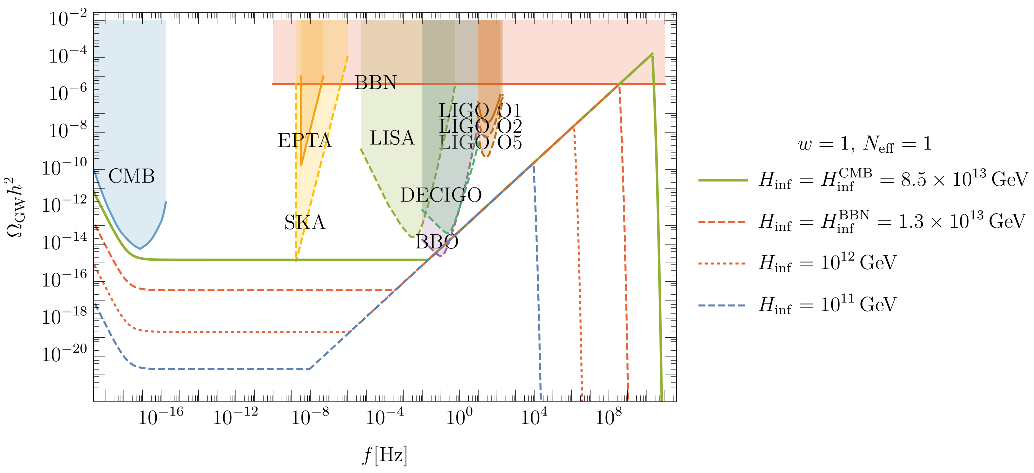

Figure 6.5: Gravitational wave energy density, $\Omega_{\rm GW}h^2$, for $w = N_{\rm eff} = 1$ and several values of $H_{\rm inf}$. The reach of current experiments using BBN data [83, 84], present and planned LIGO capabilities [85, 86, 87] and European Pulsar Timing Array [88] is marked. Also planned reach of future experiments LISA (with the most optimistic configuration A5M5) [82], BBO and DECIGO [89], SKA [90] and CMB polarization [91, 92] is shown.

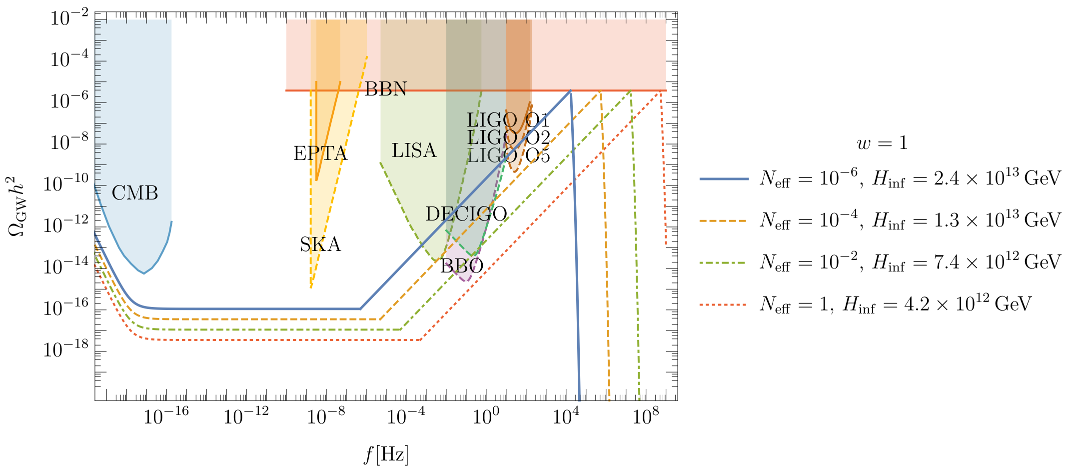

Figure 6.6: Gravitational wave energy density, $\Omega_{\rm GW}h^2$, for fixed $w = 1$, several values of $N_{\rm eff}$ and maximal $H_{\rm inf}$ allowed.

non-zero index would not influence the above results much and our analysis stays as model-independent as it can be.

Summary of chapter

- We introduce the concept of dark inflation driven by a field coupled to the SM only gravitationally and investigate its cosmological consequences. Such a field does not fully decay after inflation and can dominate the energy density of the universe for some time after inflation resulting in the presence of additional era before RD in the thermal history of the Universe.

- We discuss gravitational reheating in this scenario in a very general way regardless of the precise dynamics of the inflationary sector. It is realized by purely gravitational particle production during the transition between a de-Sitter inflationary era and a post-inflationary era dominated by a perfect fluid with some barotropic parameter w. In order to obtain the standard thermal history at later times energy density of the remaining inflaton has to redshift faster than radiation, which results in $w > 1/3$.

- We calculate several inflationary parameters - scale of inflation, temperature of reheating, tensor-to-scalar ratio and number of e-folds - for our model and describe their meaning for the cosmological predictions. For example, we argue that $N_\star$ does not depend strongly on $H_{\rm inf}$ and w being almost constant for all considered examples, which defines the thermal history of the Universe more accurately.

- We also present experimental bounds on the inflationary scale. There are two competing upper bounds - the usual one originating in the upper limit on tensor-to-scalar ratio and the one coming from the demand not to overproduce gravitational waves during inflaton domination. As for the lower limit on $H_{\rm inf}$, it corresponds to the remaining inflaton redshifting fast enough in order not to spoil BBN.

- We obtain the evolution of the primordial gravitational waves generated during inflation with an amplification of the signal during the w-domination period. We discuss the possibility of measuring this effect by the present and forthcoming experiments focusing on the $w = 1$ case as it produces the strongest signal - only in the case of $N_{\rm eff} \ll 1$ their reach includes the region of amplification, otherwise signal is strengthened for too high frequencies. $N_{\rm eff} \ll 1$ can be realized by nearly conformal non-minimal coupling to gravity.

Chapter 7

Summary

The aim of this dissertation is to study cosmological particle production in time-dependent backgrounds. We consider various realisations of this problem starting from the adiabatic approximation for massless and then massive backgrounds, followed by the interacting theory describing the intermediate period of production in a more accurate way and gravitational reheating with an instant period of particle creation in the end.

In Chapter 3 we describe the method of calculating number density of the states produced due to the time-dependence of the background field based on the asymptotic approximation of the wave functions and analytic continuation of the time coordinate. We prove that massless states can be produced as abundantly as the massive ones due to quantum corrections. We extend our method to include fermions of different types (Weyl, Majorana and Dirac) and obtain the general formulae needed for calculating their final number densities. We introduce the influence of interactions (quantum corrections) on the process of fermionic production based on the Yang-Feldman formalism. We apply the whole method to the supersymmetric model with a single coupling, bosonic and fermionic sector containing both massive and massless fields. We also compare different possible sources of production (quantum corrections, perturbative production and rotation of the basis), investigate the one-loop corrections leading to the physical mass and address the issue of the supersymmetry-breaking terms. We extend our considerations to the supersymmetric model with two couplings. We show how the expansion of the universe changes our analysis.

In Chapter 4 we introduce the concept of inflation and post-inflationary particle production following the classical literature [24, 39, 40, 41]. We describe general advantages of the inflationary models and the idea of slow-roll inflation. Moreover, we look more carefully at the main stages of post-inflationary production: reheating and preheating. We develop the method used in the case of massless background to include massive background neglecting quantum corrections. Obtained results are consistent with the literature.

In Chapter 5 we develop the results from Chapter 3 in two preheating scenarios with fields coupled indirectly to the background, two-scalar system and system with additional light sector, using the theory of interacting fields. We investigate the role of the couplings in these models discussing the conditions for obtaining successful quenching of the energy transfer between the background and the dynamical sector. We discuss the profits of using theory of interacting field, which allows to avoid artificial infinite growth coming from the approximation by asymptotic field seen before. Moreover, we compare our analysis with instant preheating, which turns out to be related in a sense.

In Chapter 6 we introduce the concept of dark inflation driven by a field coupled to the SM only gravitationally and investigate its cosmological consequences. Such a field does not fully decay after inflation and can dominate the energy density of the universe for some time after inflation resulting in the presence of an additional era before RD in the thermal history of the Universe. We discuss gravitational reheating in this scenario in a very general way regardless of the precise dynamics of the inflationary sector. It is realized by purely gravitational particle production during the transition between a de-Sitter inflationary era and a post-inflationary era dominated by a perfect fluid with certain barotropic parameter w. In order to obtain the standard thermal history at later times the energy density of the remaining inflaton has to redshift faster than radiation, which results in $w > 1/3$. We calculate several inflationary parameters - scale of inflation, temperature of reheating, tensor-to-scalar ratio and number of e-folds - in our model and describe their meaning for the cosmological predictions. For example, we argue that $N_\star$ does not depend strongly on $H_{\rm inf}$ and w being almost constant for all considered examples, which defines the thermal history of the Universe more accurately. We also present experimental bounds on the inflationary scale. There are two competing upper bounds - the usual one originating in the upper limit on tensor-to-scalar ratio and the one coming from the demand not to overproduce gravitational waves during inflaton domination. As for the lower limit on $H_{\rm inf}$, it corresponds to the remaining inflaton redshifting fast enough in order not to spoil BBN. Moreover, we obtain the evolution of the primordial gravitational waves generated during inflation with an amplification of the signal during the w-domination period. We discuss the possibility of measuring this effect by the present and forthcoming experiments focusing on the $w = 1$ case as it produces the strongest signal - only in the case of $N_{\rm eff} \ll 1$ their reach includes the region of amplification, otherwise signal is strengthened for too high frequencies. $N_{\rm eff} \ll 1$ can be realized by nearly conformal non-minimal coupling to gravity.

There are several ways of completing our analysis in the future. One idea we consider is to add the non-minimal coupling between the scalar field and Ricci scalar and analyse the production of particles accompanying Higgs inflation using our method. Apart from that one could investigate for instance higher order corrections and temperature corrections in the considered models.

What seems to be the most interesting future prospect at the moment is to compare our strictly time-dependent analysis with position-dependent one, such as 1D scattering, which has been investigated in [93] using Bogoliubov transformation. Both [93] and [94] present general upper and lower limits on the Bogoliubov coefficients in such a case. Particular cosmological realisation of the simple position dependent system could be given by a cosmic string interacting with the Higgs field, which is the subject of our ongoing investigation.

To conclude, the main novel results of the dissertation are:

I. We prove that the massless states, for which the standard Bogoliubov coefficient β is equal to zero, can be produced as abundantly as the massive ones with $\beta \neq 0$ due to quantum corrections [Chapter 3].

II. We discuss the conditions for obtaining successful quenching of the energy transfer between the background and the dynamical sector in the theories including some light sector in the framework of the theory of interacting field [Chapter 5].

III. We investigate instant gravitational reheating in the dark inflation scenario in a very general way regardless of the precise dynamics of the inflationary sector obtaining the inflationary observables and gravitational waves signals that can fit the experimental data for the reasonable choice of the parameters [Chapter 6].

Appendix A

Yang-Feldman equation

We can express any function $f(x)$ as

$$f(x) = \int d^4y \delta^{(4)}(x-y) f(y), \tag{A.1}$$

where δ-function is a product of two other δ-functions:

$$\delta(x^0 - y^0) = \partial_{x^0} \left[\theta(x^0 - y^0) - g(y^0)\right] \equiv \Theta^{x^0}_{t^{\mathrm{as}}}(y^0), \tag{A.2}$$

$$\delta^{(3)}(\vec{x} - \vec{y}) = \tfrac{1}{i} \left[\phi(y^0, \vec{y}), \dot{\phi}(x^0, \vec{x})\right]_{x^0 = y^0} \tag{A.3}$$

with $g(y^0)$ being some general function - without loss of generality we can choose $g(y^0) = \theta(t^{\mathrm{as}} - y^0)$. After integrating by parts $f(x)$ reads

$$f(x) = \partial^x_0 \int d^4y \Theta^{x^0}_{t^{\mathrm{as}}}(y^0) i \langle * | [\dot{\phi}(x), \phi(y)] | * \rangle f(y) - \int d^4y \Theta^{x^0}_{t^{\mathrm{as}}} i \langle * | [\ddot{\phi}(x), \phi(y)] | * \rangle f(y). \tag{A.4}$$

Following ϕ's equation of motion:

$$\ddot{\phi} - \nabla^2 \phi + M^2 \phi + J = 0, \tag{A.5}$$

we get

$$f(x) = (\partial^x_0)^2 \int d^4y \Theta^{x^0}_{t^{\mathrm{as}}}(y^0) i \langle * | [\phi(x), \phi(y)] | * \rangle f(y) + \tag{A.6}$$
$$+ \int d^4y \Theta^{x^0}_{t^{\mathrm{as}}} i \langle * | [(-\nabla^2 + M^2)\phi(x) + J(x), \phi(y)] | * \rangle f(y).$$

Finally:

$$f(x) = \left((\partial^x)^2 + M^2\right) \int d^4y G^{(t^{\mathrm{as}})}[\phi, \phi]_{xy} f(y) + \int d^4y G^{(t^{\mathrm{as}})}[J, \phi]_{xy} f(y), \tag{A.7}$$

where $\langle * | * \rangle = 1$ and $G^{(t^{\mathrm{as}})}[X, \phi]_{xy} \equiv \Theta^{x^0}_{t^{\mathrm{as}}}(y^0) i \langle * | [X(x), \phi(y)] | * \rangle$.

On the other hand equation of motion can be expressed as

$$0 = \left(\partial^2 + M^2\right) \phi(x) + J(x) = \left(\partial^2 + M^2\right) \phi(x) + J(x) + f(x) - f(x), \tag{A.8}$$

which using (A.7) turns into:

$$0 = \left((\partial^x)^2 + M^2\right) \left(\phi(x) + \int d^4y G^{(t^{\mathrm{as}})}[\phi, \phi]_{xy} f(y)\right) +$$
$$\int d^4y G^{(t^{\mathrm{as}})}[J, \phi]_{xy} f(y) + J(x) - f(x), \tag{A.9}$$

where $[A, B]_{xy} \equiv [A(x), B(y)]$. In order to keep it identically zero, we should choose:

$$f(x) = \int d^4y \left(1 - G^{(t^{\mathrm{as}})}[J, \phi]\right)^{-1}_{xy} J(y) = \tag{A.10}$$
$$= J(x) + \int d^4y G^{(t^{\mathrm{as}})}[J, \phi]_{xy} J(y) + \int d^4y d^4z G^{(t^{\mathrm{as}})}[J, \phi]_{xy} G^{(t^{\mathrm{as}})}[J, \phi]_{yz} J(z) + ...$$

and $\phi(x)$ in the form of Yang-Feldman equation:

$$\phi(x) = \sqrt{Z}\phi^{\mathrm{as}} - \int d^4y d^4z G^{(t^{\mathrm{as}})}[\phi, \phi]_{xy} \left(1 - G^{(t^{\mathrm{as}})}[J, \phi]\right)^{-1}_{yz} J(z). \tag{A.11}$$

$\phi^{\mathrm{as}}(x)$ is an asymptotic free field,

$$\left(\partial^2 + M^2\right) \phi^{\mathrm{as}}(x) = 0, \tag{A.12}$$

defined by the relation

$$\phi(x^0 = t^{\mathrm{as}}) = \sqrt{Z}\phi^{\mathrm{as}}(x^0 = t^{\mathrm{as}}). \tag{A.13}$$

Keeping only the terms with up to one commutator under the integral, we get a simplified version of Yang-Feldman equation:

$$\phi(x) \approx \sqrt{Z}\phi^{\mathrm{as}}(x) - iZ \int\limits_{t^{\mathrm{as}}}^{x^0} dy^0 \int d^3y \left[\phi^{\mathrm{as}}, \phi^{\mathrm{as}}\right]_{xy} J(y). \tag{A.14}$$

Commutation relation for the field:

$$\left[\dot{\phi}(t, \vec{x}), \phi(t, \vec{y})\right] = -i\delta^{(3)}(\vec{x} - \vec{y}) \tag{A.15}$$

for $t = t^{\mathrm{as}}$ translates into:

$$\left[\phi^{\mathrm{as}}(x), \dot{\phi}^{\mathrm{as}}(y)\right] = \frac{i}{Z}\delta^{(3)}(\vec{x} - \vec{y}). \tag{A.16}$$

Appendix B

Number density for the j-th resonant particle production using the saddle point method

Equation (4.2.16) for the distribution of produced particles can be expressed as

$$n_k^j + \frac{1}{2} \sim \left(n_k^0 + \frac{1}{2}\right) \prod_{l=0}^{j-1} \left(1 + 2\Delta_k^l\right), \tag{B.1}$$

which for $n_k^0 = 0$ and $\Delta_k^l \sim \Delta_k^0 = e^{-\pi k^2/gm|\phi_0|}$ transforms into

$$n_k^j \sim \frac{1}{2}\left(1 + 2\Delta_k^0\right)^j - \frac{1}{2} \sim \frac{1}{2}\sum_{l=1}^{j} \frac{j!}{l!(j-l)!} \cdot 2^l e^{-l\pi k^2/gm|\phi_0|}. \tag{B.2}$$

Thus, the number density is of the form

$$\begin{aligned} n^j &\sim \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sum_{l=1}^{j} \frac{j!}{l!(j-l)!} \cdot 2^l e^{-l\pi k^2/gm|\phi_0|} = \\ &= \frac{(gm|\phi_0|)^{3/2}}{(2\pi)^3} \cdot \sum_{l=1}^{j} \frac{j!}{l!(j-l)!} \frac{2^{l-1}}{l^{3/2}} \equiv \frac{(gm|\phi_0|)^{3/2}}{(2\pi)^3} \cdot R_j. \end{aligned} \tag{B.3}$$

Factor multiplying R_j is just n^1, so we can interpret R_j as the deviation from the first production.

For $j \gg 1$ we can use the following approximations

$$\sum_{l=1}^{j} \sim \int_1^j dl, \tag{B.4}$$

$$l! \sim \sqrt{2\pi l}(l/e)^l, \tag{B.5}$$

which results in

$$\begin{aligned} R_j &\sim \int_1^j dl \frac{1}{\sqrt{2\pi}} \frac{j^{j+1/2}}{l^{l+1/2}(j-l)^{j-l+1/2}} \frac{2^{l-1}}{l^{3/2}} \equiv \frac{j^{j+1/2}}{\sqrt{2\pi}} \cdot \int_1^j dl\, e^{f(l)} = \\ &= \frac{j^{j+1/2}}{\sqrt{2\pi}} \cdot \int_1^j dl\, \exp\left[f(\bar{l}) + f'(\bar{l})(l-\bar{l}) + \frac{1}{2}f''(\bar{l})(l-\bar{l})^2 + \cdots\right]. \end{aligned} \tag{B.6}$$

If we choose $\bar{l} = \dfrac{2j-1}{3} + \mathcal{O}(1/j)$: $f'(\bar{l}) \sim 0$ and

$$R_j \sim \frac{j^{j+1/2}}{\sqrt{2\pi}} \cdot \int_1^j dl \, \exp\left[f(\bar{l}) + \frac{1}{2}f''(\bar{l})(l-\bar{l})^2 + \cdots\right] \sim \frac{j^{j+1/2}}{\sqrt{2\pi}} \cdot e^{f(\bar{l})}\sqrt{\frac{2\pi}{-f''(\bar{l})}} =$$

$$= \frac{j^{j+1/2}}{\bar{l}^{\bar{l}+1/2}(j-\bar{l})^{j-\bar{l}+1/2}} \frac{2^{\bar{l}-1}}{\bar{l}^{3/2}} \frac{1}{\sqrt{\frac{1}{\bar{l}} - \frac{2}{\bar{l}^2} + \frac{1}{j-\bar{l}} - \frac{1}{2(j-\bar{l})^2}}} \sim \frac{3^j}{2}\left(\frac{3}{2j}\right)^{3/2} \qquad \text{(B.7)}$$

we finally obtain

$$n^j \sim n^1 \cdot \frac{3^j}{2}\left(\frac{3}{2j}\right)^{3/2}. \qquad \text{(B.8)}$$

Appendix C

Fermions in the theory of interacting field

Apart from bosons we can also consider the production of fermions in the framework of interacting fields. The simplest Lagrangian for Dirac fermion useful in this case reads

$$\mathcal{L} = \xi^\dagger i\bar{\sigma}^\mu \partial_\mu \xi + \eta i\sigma^\mu \partial_\mu \eta^\dagger - m_0 \eta\xi - m_0^* \xi^\dagger \eta^\dagger - V[\xi, \xi^\dagger, \eta, \eta^\dagger]. \tag{C.1}$$

Corresponding equations of motion are of the form

$$0 = i\bar{\sigma}^\mu \partial_\mu \xi - m_0^* \eta^\dagger - \frac{\partial}{\partial \xi^\dagger} V = i\bar{\sigma}^\mu \partial_\mu \xi - M^* \eta^\dagger - J_\xi^\dagger, \tag{C.2}$$

$$0 = i\sigma^\mu \partial_\mu \eta^\dagger - m_0 \xi - \frac{\partial}{\partial \eta} V = i\sigma^\mu \partial_\mu \eta^\dagger - M\xi - J_\eta, \tag{C.3}$$

where $M = M(t)$ is a time-dependent physical mass and

$$J_\xi^\dagger \equiv (m_0^* - M^*)\eta^\dagger + \frac{\partial V}{\partial \xi^\dagger}, \tag{C.4}$$

$$J_\eta \equiv (m_0 - M)\xi + \frac{\partial V}{\partial \eta} \tag{C.5}$$

are the source terms.

Yang-Feldman equation for the fermionic sector can be written in a condensed way

$$\begin{pmatrix} \xi(x) \\ \eta^\dagger(x) \end{pmatrix} = \begin{pmatrix} \xi^{\text{as}}(x) \\ \eta^{\text{as}\dagger}(x) \end{pmatrix} - \int_{t^{\text{as}}}^{x^0} d^4y i \begin{pmatrix} \{\xi^{\text{as}}(x), \eta^{\text{as}}(y)\} & \{\xi^{\text{as}}(x), \xi^{\text{as}\dagger}(y)\} \\ \{\eta^{\text{as}\dagger}(x), \eta^{\text{as}}(y)\} & \{\eta^{\text{as}\dagger}(x), \xi^{\text{as}\dagger}(y)\} \end{pmatrix} \begin{pmatrix} J_\eta(y) \\ J_\xi^\dagger(y) \end{pmatrix} \tag{C.6}$$

with the following decomposition of the asymptotic fields into modes

$$\xi^{\text{as}}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{s=\pm} e_{\mathbf{k}}^s \left(u_k^{\text{as},s} a_{\mathbf{k}}^{\text{as},s} + v_k^{\text{as},s*} b_{-\mathbf{k}}^{\text{as},s\dagger} \right), \tag{C.7}$$

$$\eta^{\text{as}\dagger}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{s=\pm} \bar{\sigma}^0 e_{\mathbf{k}}^s \left(v_k^{\text{as},s} a_{\mathbf{k}}^{\text{as},s} - u_k^{\text{as},s*} b_{-\mathbf{k}}^{\text{as},s\dagger} \right). \tag{C.8}$$

Equations of motion for the modes read

$$0 = \dot{u}_k^{\mathrm{as},s} - is|\mathbf{k}|u_k^{\mathrm{as},s} + iM^* v_k^{\mathrm{as},s}, \tag{C.9}$$

$$0 = \dot{v}_k^{\mathrm{as},s} + is|\mathbf{k}|v_k^{\mathrm{as},s} + iM u_k^{\mathrm{as},s} \tag{C.10}$$

and they are normalized by

$$|u_k^{\mathrm{as},s}(t)|^2 + |v_k^{\mathrm{as},s}(t)|^2 = 1. \tag{C.11}$$

Bogoliubov transformation for creation/annihilation operators becomes[1]

$$\begin{pmatrix} a_{\mathbf{k}}^{(t),s} \\ b_{-\mathbf{k}}^{(t),s\dagger} \end{pmatrix} = \begin{pmatrix} \alpha_k^s(t) & \beta_k^s(t) \\ -\beta_k^{s*}(t) & \alpha_k^{s*}(t) \end{pmatrix} \Bigg[\begin{pmatrix} a_{\mathbf{k}}^{\mathrm{as},s} \\ b_{-\mathbf{k}}^{\mathrm{as},s\dagger} \end{pmatrix}$$
$$-i \int\limits_{t^{\mathrm{as}}}^{t} d^4y \begin{pmatrix} \{a_{\mathbf{k}}^{\mathrm{as},s}, \eta^{\mathrm{as}}(y)\} & \{a_{\mathbf{k}}^{\mathrm{as},s}, \xi^{\mathrm{as}\dagger}(y)\} \\ \{b_{-\mathbf{k}}^{\mathrm{as},s\dagger}, \eta^{\mathrm{as}}(y)\} & \{b_{-\mathbf{k}}^{\mathrm{as},s\dagger}, \xi^{\mathrm{as}\dagger}(y)\} \end{pmatrix} \begin{pmatrix} J_\eta(y) \\ J_\xi^\dagger(y) \end{pmatrix} \Bigg], \tag{C.13}$$

where

$$\alpha_k^s(t) \equiv u_k^{(t),s*} u_k^{\mathrm{as},s} + v_k^{(t),s*} v_k^{\mathrm{as},s}, \tag{C.14}$$

$$\beta_k^s(t) \equiv u_k^{(t),s*} v_k^{\mathrm{as},s*} - v_k^{(t),s*} u_k^{\mathrm{as},s*} \tag{C.15}$$

with normalization condition changing its sign to

$$|\alpha_k^s(t)|^2 + |\beta_k^s(t)|^2 = 1. \tag{C.16}$$

Relation (C.13) can be again transformed to

$$\begin{pmatrix} a_{\mathbf{k}}^{(t),s} \\ b_{-\mathbf{k}}^{(t),s\dagger} \end{pmatrix} = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \begin{pmatrix} \alpha_k^s(t) & \beta_k^s(t) \\ -\beta_k^{s*}(t) & \alpha_k^{s*}(t) \end{pmatrix} \begin{pmatrix} u_k^{\mathrm{as},s*}(t) & v_k^{\mathrm{as},s*}(t) \\ v_k^{\mathrm{as},s}(t) & -u_k^{\mathrm{as},s}(t) \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} e_{\mathbf{k}}^{s\dagger}\bar{\sigma}^0 & 0 \\ 0 & e_{\mathbf{k}}^{s\dagger} \end{pmatrix} \begin{pmatrix} \xi(t,\mathbf{x}) \\ \eta^\dagger(t,\mathbf{x}) \end{pmatrix}, \tag{C.17}$$

which is useful for evaluating the number of produced states.

Corresponding Hamiltonian

$$H(t) = \int d^3x \left(-\xi^\dagger i\bar{\sigma}^i \partial_i \xi - \eta i \sigma^i \partial_i \eta^\dagger + m_0 \eta\xi + m_0^* \xi^\dagger \eta^\dagger + V \right) = \tag{C.18}$$

$$= \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \omega_k(t) \left(a_{\mathbf{k}}^{(t)\dagger} a_{\mathbf{k}}^{(t)} - b_{-\mathbf{k}}^{(t)} b_{-\mathbf{k}}^{(t)\dagger} \right) + \int d^3x \left[(m_0 - M)\eta\xi + (m_0^* - M^*)\xi^\dagger\eta^\dagger + V \right] \tag{C.19}$$

[1]This is equivalent to the following Bogoliubov transformation for the wave functions

$$\begin{pmatrix} u_k^{(t),s} \\ v_k^{(t),s*} \end{pmatrix} = \begin{pmatrix} \alpha_k^{s*}(t) & \beta_k^{s*}(t) \\ -\beta_k^s(t) & \alpha_k^s(t) \end{pmatrix} \begin{pmatrix} u_k^{\mathrm{as},s} \\ v_k^{\mathrm{as},s*} \end{pmatrix}. \tag{C.12}$$

can be diagonalized provided the Bogoliubov coefficients are of the form

$$\alpha_k^s(t) = \sqrt{\frac{1}{2} + \frac{E_k^{\text{in},s}}{2\omega_k}}, \tag{C.20}$$

$$\beta_k(t) = \frac{F_k^{\text{in},s*}}{|F_k^{\text{in},s}|}\sqrt{\frac{1}{2} - \frac{E_k^{\text{in},s}}{2\omega_k}}, \tag{C.21}$$

where

$$E_k^{\text{in},s}(t) \equiv -sk(|u_k^{\text{in},s}|^2 - |v_k^{\text{in},s}|^2) + m u_k^{\text{in},s} v_k^{\text{in},s*} + m^* u_k^{\text{in},s*} v_k^{\text{in},s}, \tag{C.22}$$

$$F_k^{\text{in},s}(t) \equiv -2sk u_k^{\text{in},s} v_k^{\text{in},s} - m u_k^{\text{in},s\,2} + m^* v_k^{\text{in},s\,2}. \tag{C.23}$$

Finally, total and net numbers of particles with helicity s in terms of the interacting fields read

$$\begin{aligned} N_{\mathbf{k}}^{+(s)}(t) &= a_{\mathbf{k}}^{(t),s\dagger} a_{\mathbf{k}}^{(t),s} + b_{-\mathbf{k}}^{(t),s} b_{-\mathbf{k}}^{(t),s\dagger} = \\ &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\omega_k}\Big[-sk\xi^\dagger(x)\bar{\sigma}^0 e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\bar{\sigma}^0\xi(y) + sk\eta(x) e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\eta^\dagger(y) + \\ &\quad + M\eta(x) e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\bar{\sigma}^0\xi(y) + M^*\xi^\dagger(x)\bar{\sigma}^0 e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\eta^\dagger(y)\Big] = \\ &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{2\omega_k}\left[-sk\xi^\dagger(x)\left(\bar{\sigma}^0 - \frac{sk^i}{|\mathbf{k}|}\bar{\sigma}^i\right)\xi(y) + sk\eta(x)\left(\sigma^0 + \frac{sk^i}{|\mathbf{k}|}\sigma^i\right)\eta^\dagger(y) + \right. \\ &\quad \left. + M\eta(x)\left(1 - \frac{sk^i}{|\mathbf{k}|}\sigma^0\bar{\sigma}^i\right)\xi(y) + M^*\xi^\dagger(x)\left(1 + \frac{sk^i}{|\mathbf{k}|}\bar{\sigma}^0\sigma^i\right)\eta^\dagger(y)\right], \end{aligned} \tag{C.24}$$

$$\begin{aligned} N_{\mathbf{k}}^{-(s)}(t) &= a_{\mathbf{k}}^{(t),s\dagger} a_{\mathbf{k}}^{(t),s} - b_{-\mathbf{k}}^{(t),s} b_{-\mathbf{k}}^{(t),s\dagger} = \\ &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \Big[\xi^\dagger(x)\bar{\sigma}^0 e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\bar{\sigma}^0\xi(y) + \eta(x) e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\eta^\dagger(y)\Big] = \\ &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cdot \frac{1}{2}\left[\xi^\dagger(x)\left(\bar{\sigma}^0 - \frac{sk^i}{|\mathbf{k}|}\bar{\sigma}^i\right)\xi(y) + \eta(x)\left(\sigma^0 + \frac{sk^i}{|\mathbf{k}|}\sigma^i\right)\eta^\dagger(y)\right], \end{aligned} \tag{C.25}$$

which summed over helicities become

$$\begin{aligned} N_{\mathbf{k}}^{+}(t) &= \sum_{s=\pm}\left(a_{\mathbf{k}}^{(t),s\dagger} a_{\mathbf{k}}^{(t),s} + b_{-\mathbf{k}}^{(t),s} b_{-\mathbf{k}}^{(t),s\dagger}\right) = \\ &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\omega_k}\Big[\xi^\dagger(x) k^i\bar{\sigma}^i\xi(y) + \eta(x) k^i\sigma^i\eta^\dagger(y) + M\eta(x)\xi(y) + M^*\xi^\dagger(x)\eta^\dagger(y)\Big], \end{aligned} \tag{C.26}$$

$$N_{\mathbf{k}}^{-}(t) = \sum_{s=\pm}\left(a_{\mathbf{k}}^{(t),s\dagger} a_{\mathbf{k}}^{(t),s} - b_{-\mathbf{k}}^{(t),s} b_{-\mathbf{k}}^{(t),s\dagger}\right) = \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\Big[\xi^\dagger(x)\bar{\sigma}^0\xi(y) + \eta(x)\sigma^0\eta^\dagger(y)\Big] \tag{C.27}$$

and $N_{\mathbf{k}}(t) = \frac{1}{2}\left(N_{\mathbf{k}}^{+}(t) + N_{\mathbf{k}}^{-}(t)\right)$ still holds.

In the case of Majorana fermion ξ Lagrangian simplifies to

$$\mathcal{L} = \xi^\dagger i\bar{\sigma}^\mu\partial_\mu\xi - \frac{1}{2}m_0\xi\xi - \frac{1}{2}m_0^*\xi^\dagger\xi^\dagger - V[\xi, \xi^\dagger] \tag{C.28}$$

with the equation of motion

$$0 = i\bar{\sigma}^{\mu}\partial_{\mu}\xi - m_0^*\xi^{\dagger} - \frac{\partial}{\partial\xi^{\dagger}}V = i\bar{\sigma}^{\mu}\partial_{\mu}\xi - M^*\xi^{\dagger} - J_{\xi}^{\dagger}, \tag{C.29}$$

where $M = M(t)$ denotes a physical mass and

$$J_{\xi}^{\dagger} = (m_0^* - M^*)\xi^{\dagger} + \frac{\partial V}{\partial \xi^{\dagger}}. \tag{C.30}$$

Yang-Feldman equation reads now

$$\begin{pmatrix} \xi(x) \\ \xi^{\dagger}(x) \end{pmatrix} = \begin{pmatrix} \xi^{\rm as}(x) \\ \xi^{\rm as\dagger}(x) \end{pmatrix} - \int_{t^{\rm as}}^{x^0} d^4y i \begin{pmatrix} \{\xi^{\rm as}(x), \xi^{\rm as}(y)\} & \{\xi^{\rm as}(x), \xi^{\rm as\dagger}(y)\} \\ \{\xi^{\rm as\dagger}(x), \xi^{\rm as}(y)\} & \{\xi^{\rm as\dagger}(x), \xi^{\rm as\dagger}(y)\} \end{pmatrix} \begin{pmatrix} J_{\xi}(y) \\ J_{\xi}^{\dagger}(y) \end{pmatrix}, \tag{C.31}$$

where

$$\xi^{\rm as}(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{s=\pm} e_{\mathbf{k}}^s \left(u_k^{{\rm as},s} a_{\mathbf{k}}^{{\rm as},s} + s e^{-i\theta_{\mathbf{k}}} v_k^{{\rm as},s*} a_{-\mathbf{k}}^{{\rm as},s\dagger} \right) \tag{C.32}$$

and

$$0 = \dot{u}_k^{{\rm as},s} - is|\mathbf{k}|u_k^{{\rm as},s} + iM^* v_k^{{\rm as},s}, \tag{C.33}$$
$$0 = \dot{v}_k^{{\rm as},s} + is|\mathbf{k}|v_k^{{\rm as},s} + iM u_k^{{\rm as},s}. \tag{C.34}$$

Finally, total and net numbers of particles with helicity s in terms of the interacting fields read

$$\begin{aligned} N_{\mathbf{k}}^{+(s)}(t) &= a_{\mathbf{k}}^{(t),s\dagger} a_{\mathbf{k}}^{(t),s} + a_{-\mathbf{k}}^{(t),s} a_{-\mathbf{k}}^{(t),s\dagger} = \\ &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cdot \frac{1}{\omega_k} \Big[-sk\xi^{\dagger}(x)\bar{\sigma}^0 e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\bar{\sigma}^0\xi(y) + \\ &+ \frac{1}{2}M\xi(x)e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\bar{\sigma}^0\xi(y)] + \frac{1}{2}M^*\xi^{\dagger}(x)\bar{\sigma}^0 e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\xi^{\dagger}(y) \Big] = \\ &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cdot \frac{1}{2\omega_k} \left[-sk\xi^{\dagger}(x) \left(\bar{\sigma}^0 - \frac{sk^i}{|\mathbf{k}|}\bar{\sigma}^i \right) \xi(y) + \right. \\ &\left. + \frac{1}{2}M\xi(x)\left(1 - \frac{sk^i}{|\mathbf{k}|}\sigma^0\bar{\sigma}^i\right)\xi(y) + \frac{1}{2}M^*\xi^{\dagger}(x)\left(1 + \frac{sk^i}{|\mathbf{k}|}\bar{\sigma}^0\sigma^i\right)\xi^{\dagger}(y) \right], \end{aligned} \tag{C.35}$$

$$\begin{aligned} N_{\mathbf{k}}^{-(s)}(t) &= a_{\mathbf{k}}^{(t),s\dagger} a_{\mathbf{k}}^{(t),s} - a_{-\mathbf{k}}^{(t),s} a_{-\mathbf{k}}^{(t),s\dagger} = \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cdot \xi^{\dagger}(x)\bar{\sigma}^0 e_{\mathbf{k}}^s \cdot e_{\mathbf{k}}^{s\dagger}\bar{\sigma}^0\xi(y) = \\ &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cdot \frac{1}{2}\xi^{\dagger}(x)\left(\bar{\sigma}^0 - \frac{sk^i}{|\mathbf{k}|}\bar{\sigma}^i\right)\xi(y), \end{aligned} \tag{C.36}$$

which summed over helicities become

$$N_{\mathbf{k}}^{+}(t) = \sum_{s=\pm} \left(a_{\mathbf{k}}^{(t),s\dagger} a_{\mathbf{k}}^{(t),s} + a_{-\mathbf{k}}^{(t),s} a_{-\mathbf{k}}^{(t),s\dagger} \right) = \tag{C.37}$$

$$\begin{aligned} &= \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\omega_k} \left[\xi^{\dagger}(x)k^i\bar{\sigma}^i\xi(y) + \frac{1}{2}M\xi(x)\xi(y) + \frac{1}{2}M^*\xi^{\dagger}(x)\xi^{\dagger}(y) \right], \\ N_{\mathbf{k}}^{-}(t) &= \sum_{s=\pm} \left(a_{\mathbf{k}}^{(t),s\dagger} a_{\mathbf{k}}^{(t),s} - a_{-\mathbf{k}}^{(t),s} a_{-\mathbf{k}}^{(t),s\dagger} \right) = \int d^3x d^3y e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \cdot \xi^{\dagger}(x)\bar{\sigma}^0\xi(y). \end{aligned} \tag{C.38}$$

So the transformation that links Dirac and Majorana cases is of the form:

$$\eta \to \xi, \tag{C.39}$$

$$b_{\mathbf{k}}^{\mathrm{as},s} \to s e^{-i\theta_{\mathbf{k}}} a_{\mathbf{k}}^{\mathrm{as},s}. \tag{C.40}$$

Appendix D

Diagonalized Hamiltonian in the theory of interacting field

The actual form of the Hamiltonian (5.1.11) for a real scalar field in the interacting field theory reads:

$$
\begin{aligned}
H(t) &= \int d^3x \left[\frac{1}{2}\dot{\phi}^2(t,\mathbf{x}) + \frac{1}{2}(\nabla\phi(t,\mathbf{x}))^2 + \frac{1}{2}m_0^2\phi^2(t,\mathbf{x}) + V(t,\mathbf{x})\right] = \qquad \text{(D.1)}\\
&= \int d^3x \left[\frac{1}{2}\left(\dot{\phi}^{\rm in}(t,\mathbf{x}) - i\int_{t^{\rm in}}^{t} d^4y[\dot{\phi}^{\rm in}(t,\mathbf{x}),\phi^{\rm in}(y)]J(y)\right)^2 + \right.\\
&\qquad + \frac{1}{2}\left(\nabla\phi^{\rm in}(t,\mathbf{x}) - i\int_{t^{\rm in}}^{t} d^4y[\nabla\phi^{\rm in}(t,\mathbf{x}),\phi^{\rm in}(y)]J(y)\right)^2 + \\
&\qquad + \frac{1}{2}M^2\left(\phi^{\rm in}(t,\mathbf{x}) - i\int_{t^{\rm in}}^{t} d^4y[\phi^{\rm in}(t,\mathbf{x}),\phi^{\rm in}(y)]J(y)\right)^2 + \\
&\qquad \left. + \frac{1}{2}(m_0^2 - M^2)\phi^2(t,\mathbf{x}) + V(t,\mathbf{x})\right] = \qquad \text{(D.2)}\\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[\Omega_k^{\rm in}(t)(a_{\mathbf{k}}^{\rm in\dagger}a_{\mathbf{k}}^{\rm in} + a_{-\mathbf{k}}^{\rm in}a_{-\mathbf{k}}^{\rm in\dagger}) + \Lambda_k^{\rm in}(t)a_{-\mathbf{k}}^{\rm in}a_{\mathbf{k}}^{\rm in} + \Lambda_k^{\rm in*}(t)a_{\mathbf{k}}^{\rm in\dagger}a_{-\mathbf{k}}^{\rm in\dagger} + \right.\\
&\qquad + i\int_{t^{\rm in}}^{t} d^4y e^{i\mathbf{k}\cdot\mathbf{y}}\left\{\left(\Omega_k^{\rm in}(t)\phi_k^{\rm in}(y^0) - \Lambda_k^{\rm in}(t)\phi_k^{\rm in*}(y^0)\right)a_{\mathbf{k}}^{\rm in} + \right.\\
&\qquad\qquad \left. + \left(\Lambda_k^{\rm in*}(t)\phi_k^{\rm in}(y^0) - \Omega_k^{\rm in}(t)\phi_k^{\rm in*}(y^0)\right)a_{-\mathbf{k}}^{\rm in\dagger}\right\}J(y) + \\
&\qquad + i\int_{t^{\rm in}}^{t} d^4y e^{i\mathbf{k}\cdot\mathbf{y}}J(y)\left\{\left(\Omega_k^{\rm in}(t)\phi_k^{\rm in}(y^0) - \Lambda_k^{\rm in}(t)\phi_k^{\rm in*}(y^0)\right)a_{\mathbf{k}}^{\rm in} + \right.\\
&\qquad\qquad \left. + \left(\Lambda_k^{\rm in*}(t)\phi_k^{\rm in}(y^0) - \Omega_k^{\rm in}(t)\phi_k^{\rm in*}(y^0)\right)a_{-\mathbf{k}}^{\rm in\dagger}\right\} + \\
&\qquad + \int_{t^{\rm in}}^{t} d^4y d^4z e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{z})}J(y)\left\{\Omega_k^{\rm in}(t)\left(\phi_k^{\rm in}(y^0)\phi_k^{\rm in*}(z^0) + \phi_k^{\rm in*}(y^0)\phi_k^{\rm in}(z^0)\right) + \right.\\
&\qquad\qquad \left.\left. - \Lambda_k^{\rm in}(t)\phi_k^{\rm in*}(y^0)\phi_k^{\rm in*}(z^0) - \Lambda_k^{\rm in*}(t)\phi_k^{\rm in}(y^0)\phi_k^{\rm in}(z^0)\right\}J(z)\right] + \\
&\quad + \int d^3x\left[\frac{1}{2}(m_0^2 - M^2)\phi^2(t,\mathbf{x}) + V(t,\mathbf{x})\right]. \qquad \text{(D.3)}
\end{aligned}
$$

For a real scalar with non-vanishing vev it is convenient to formulate this calculation in terms of "vectors"

$$A_{\mathbf{k}}^{\text{as}} = \begin{pmatrix} a_{\mathbf{k}}^{\text{as}} \\ a_{-\mathbf{k}}^{\text{as}\dagger} \end{pmatrix}, \quad A_{\mathbf{k}}^{(t)} = \begin{pmatrix} a_{\mathbf{k}}^{(t)} \\ a_{-\mathbf{k}}^{(t)\dagger} \end{pmatrix} \tag{D.4}$$

and

$$\Phi_k^{\text{as}} = \begin{pmatrix} \phi_k^{\text{as}} \\ \phi_k^{\text{as}*} \end{pmatrix}. \tag{D.5}$$

Bogoliubov transformation can be simplified then to

$$A_{\mathbf{k}}^{(t)} - \langle A_{\mathbf{k}}^{(t)} \rangle = \mathcal{B}_k \left[A_{\mathbf{k}}^{\text{as}} - \int_{t^{\text{as}}}^{t} d^4y \, i[A_{\mathbf{k}}^{\text{as}}, \phi^{\text{as}}(y)] \left(J(y) - \langle J(y) \rangle\right) \right], \tag{D.6}$$

where

$$\mathcal{B}_k = \begin{pmatrix} \alpha_k & \beta_k \\ \beta_k^* & \alpha_k^* \end{pmatrix} \tag{D.7}$$

is a matrix composed of the Bogoliubov coefficients.

Relation between the interacting field defined at time t and t^{in} can be expressed as

$$\begin{aligned} \phi(t,\mathbf{x}) = \langle \phi(t,\mathbf{x}) \rangle + \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} (\Phi_k^{\text{in}})^T \left[A_{\mathbf{k}}^{\text{in}} - \int_{t^{\text{in}}}^{t} d^4y \, i[A_{\mathbf{k}}^{\text{in}}, \phi^{\text{in}}(y)] \left(J(y) - \langle J(y) \rangle\right) \right] = \\ = \langle \phi(t,\mathbf{x}) \rangle + \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} (\Phi_k^{\text{in}})^T \mathcal{B}_k^{-1} \left[A_{\mathbf{k}}^{(t)} - \langle A_{\mathbf{k}}^{(t)} \rangle \right], \end{aligned} \tag{D.8}$$

while its derivative reads

$$\dot{\phi}(x)\Big|_{x^0 \to t} = \langle \dot{\phi}(t,\mathbf{x}) \rangle + \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} (\dot{\Phi}_k^{\text{in}})^T \mathcal{B}_k^{-1} \left[A_{\mathbf{k}}^{(t)} - \langle A_{\mathbf{k}}^{(t)} \rangle \right]. \tag{D.9}$$

Therefore the Hamiltonian

$$H(t) = \int d^3x \left[\frac{1}{2}\dot{\phi}^2(t,\mathbf{x}) + \frac{1}{2}(\nabla\phi(t,\mathbf{x}))^2 + \frac{1}{2}m_0^2\phi^2(t,\mathbf{x}) + V(t,\mathbf{x}) \right] \tag{D.10}$$

in the framework of interacting fields reads

$$\begin{aligned} H(t) = \int d^3x \left(\frac{1}{2} \langle \dot{\phi} \rangle^2 + \frac{1}{2} M^2 \langle \phi \rangle^2 + \frac{1}{2}(m_0^2 - M^2)\phi^2 + V \right) + \\ + \int \frac{d^3k}{(2\pi)^3} (2\pi)^3 \delta^3(\mathbf{k}) \frac{1}{2} \left[\left(\langle \dot{\phi} \rangle (\dot{\Phi}_k^{\text{in}})^T + \omega_k^2 \langle \phi \rangle (\Phi_k^{\text{in}})^T \right) \mathcal{B}_k^{-1} \left(A_{\mathbf{k}}^{(t)} - \langle A_{\mathbf{k}}^{(t)} \rangle \right) + (h.c.) \right] + \\ + \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left(A_{\mathbf{k}}^{(t)\dagger} - \langle A_{\mathbf{k}}^{(t)\dagger} \rangle \right) \mathcal{B}_k^{\dagger -1} \mathcal{E}_k \mathcal{B}_k^{-1} \left(A_{\mathbf{k}}^{(t)} - \langle A_{\mathbf{k}}^{(t)} \rangle \right). \end{aligned} \tag{D.11}$$

It is useful to rewrite it as

$$
\begin{aligned}
H(t) = &\int d^3x \left(\frac{1}{2}\left\langle \dot{\phi} \right\rangle^2 + \frac{1}{2}M^2 \left\langle \phi \right\rangle^2 + \frac{1}{2}(m_0^2 - M^2)\phi^2 + V\right) + \\
&- \int \frac{d^3k}{(2\pi)^3}(2\pi)^3\delta^3(\mathbf{k})\frac{1}{2}\left[\left(\langle \dot{\phi} \rangle\, (\dot{\Phi}_k^{\text{in}})^T + \omega_k^2 \,\langle \phi \rangle\, (\Phi_k^{\text{in}})^T\right)\mathcal{B}_k^{-1}\,\langle A_{\mathbf{k}}^{(t)} \rangle + (h.c.)\right] + \\
&+ \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\,\langle A_{\mathbf{k}}^{(t)\dagger} \rangle\, \mathcal{B}_k^{\dagger -1}\mathcal{E}_k\mathcal{B}_k^{-1}\,\langle A_{\mathbf{k}}^{(t)} \rangle + \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}A_{\mathbf{k}}^{(t)\dagger}\mathcal{B}_k^{\dagger -1}\mathcal{E}_k\mathcal{B}_k^{-1}A_{\mathbf{k}}^{(t)} + \\
&+ \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[\left\{(2\pi)^3\delta^3(\mathbf{k})\left(\langle \dot{\phi} \rangle\, (\dot{\Phi}_k^{\text{in}})^T + \omega_k^2\,\langle \phi \rangle\, (\Phi_k^{\text{in}})^T\right)\right.\right. \\
&\left.\left. - \langle A_{\mathbf{k}}^{(t)\dagger} \rangle\, \mathcal{B}_k^{\dagger -1}\mathcal{E}_k\right\}\mathcal{B}_k^{-1}A_{\mathbf{k}}^{(t)} + (h.c.)\right],
\end{aligned}
\tag{D.12}
$$

where

$$
\mathcal{E}_k \equiv (\dot{\Phi}_k^{\text{in}})^*(\dot{\Phi}_k^{\text{in}})^T + \omega_k^2(\Phi_k^{\text{in}})^*(\Phi_k^{\text{in}})^T = \begin{pmatrix} \Omega_k^{\text{in}} & \Lambda_k^{\text{in}*} \\ \Lambda_k^{\text{in}} & \Omega_k^{\text{in}} \end{pmatrix}, \tag{D.13}
$$

$$
\Omega_k^{\text{in}} \equiv |\dot{\phi}_k^{\text{in}}|^2 + \omega_k^2|\phi_k^{\text{in}}|^2, \tag{D.14}
$$

$$
\Lambda_k^{\text{in}} \equiv (\dot{\phi}_k^{\text{in}})^2 + \omega_k^2(\phi_k^{\text{in}})^2. \tag{D.15}
$$

In order to diagonalize the Hamiltonian (D.12) these two conditions need to be fulfilled

$$
(2\pi)^3\delta^3(\mathbf{k})\left(\langle \dot{\phi} \rangle\, (\dot{\Phi}_k^{\text{in}})^T + \omega_k^2\,\langle \phi \rangle\, (\Phi_k^{\text{in}})^T\right) - \langle A_{\mathbf{k}}^{(t)\dagger} \rangle\, \mathcal{B}_k^{-1}\mathcal{E}_k = 0, \tag{D.16}
$$

$$
2\Omega_k^{\text{in}}\alpha_k\beta_k - \Lambda_k^{\text{in}}\beta_k^2 - \Lambda_k^{\text{in}*}\alpha_k^2 = 0. \tag{D.17}
$$

The second one means that the off-diagonal part of the term $\mathcal{B}_k^{\dagger -1}\mathcal{E}_k\mathcal{B}_k^{-1}$ needs to be zero. Combining these two conditions with proper normalization $|\alpha_k|^2 - |\beta_k|^2 = 1$, we obtain the Bogoliubov coefficients of the form

$$
|\beta_k|^2 = \frac{\Omega_k^{\text{in}}}{2\omega_k} - \frac{1}{2}, \tag{D.18}
$$

$$
\alpha_k = \frac{\Omega_k^{\text{in}} + \omega_k}{\Lambda_k^{\text{in}*}}\beta_k \tag{D.19}
$$

and

$$
\langle A_{\mathbf{k}}^{(t)} \rangle = (2\pi)^3\delta^3(\mathbf{k}) \cdot \mathcal{B}_k\mathcal{E}_k^{-1}\left((\dot{\Phi}_k^{\text{in}})^*\,\langle \dot{\phi} \rangle + (\Phi_k^{\text{in}})^*\omega_k^2\,\langle \phi \rangle\right). \tag{D.20}
$$

Diagonalized Hamiltonian reads then

$$
H(t) = \int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2}\left(a_{\mathbf{k}}^{(t)\dagger}a_{\mathbf{k}}^{(t)} + a_{-\mathbf{k}}^{(t)}a_{\mathbf{k}}^{(t)\dagger}\right) + \int d^3x\left(\frac{1}{2}(m_0^2 - M^2)\phi^2 + V\right), \tag{D.21}
$$

which suggests that as in the non-vev case number of produced particles is of the form

$$
N_{\mathbf{k}}(t) \equiv a_{\mathbf{k}}^{(t)\dagger}a_{\mathbf{k}}^{(t)}. \tag{D.22}
$$

Note that all terms with the vev disappear here due to the condition (D.16).

Using the inner product relation for the wave function

$$
i\left((\Phi_k^{\text{in}})^*(\dot{\Phi}_k^{\text{in}})^T - (\dot{\Phi}_k^{\text{in}})^*(\Phi_k^{\text{in}})^T\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{D.23}
$$

(D.8) transforms into

$$A^{(t)}_{\mathbf{k}} = \langle\, A^{(t)}_{\mathbf{k}}\,\rangle + \mathcal{B}_k \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} i \left[(\Phi^{\text{in}}_k)^* (\dot\phi - \langle\, \dot\phi\,\rangle) - (\dot\Phi^{\text{in}}_k)^* (\phi - \langle\, \phi\,\rangle) \right]$$
$$= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \mathcal{B}_k \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} i \left[(\Phi^{\text{in}}_k)^* \dot\phi - (\dot\Phi^{\text{in}}_k)^* \phi \right]. \tag{D.24}$$

Final expressions for total and net numbers of particles are of the form

$$N^+_{\mathbf{k}}(t) = A^{(t)\dagger}_{\mathbf{k}} A^{(t)}_{\mathbf{k}} = a^{(t)\dagger}_{\mathbf{k}} a^{(t)}_{\mathbf{k}} + a^{(t)}_{-\mathbf{k}} a^{(t)\dagger}_{-\mathbf{k}} =$$
$$= \int d^3x e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1}{\omega_k} \left(\dot\phi(t,\mathbf{x}) \dot\phi(t,\mathbf{y}) + \omega_k^2 \phi(t,\mathbf{x}) \phi(t,\mathbf{y}) \right), \tag{D.25}$$

$$N^-_{\mathbf{k}}(t) = A^{(t)\dagger}_{\mathbf{k}} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} A^{(t)}_{\mathbf{k}} = a^{(t)\dagger}_{\mathbf{k}} a^{(t)}_{\mathbf{k}} - a^{(t)}_{-\mathbf{k}} a^{(t)\dagger}_{-\mathbf{k}} =$$
$$= \int d^3x e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} i \left(\phi(t,\mathbf{x}) \dot\phi(t,\mathbf{y}) - \dot\phi(t,\mathbf{x}) \phi(t,\mathbf{y}) \right), \tag{D.26}$$

which are the same as in the non-vev case.

Appendix E

Details of the calculation for the two-scalar system in the theory of interacting field

Time derivatives of the operators introduced to describe number operator for all the species read

$$\langle A_{\phi\mathbf{k}}\rangle^{\cdot} = -\frac{\dot{\omega}_{\phi k}}{\omega_{\phi k}}\left(\langle A_{\phi\mathbf{k}}\rangle + \frac{1}{2}\right) + \frac{i}{\omega_{\phi k}}\Big(\omega_{\phi k}^2 + m_\phi^2 - M_\phi^2 + \tag{E.1}$$

$$+\frac{1}{2}g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\chi p}}\left(\langle B_{\chi\mathbf{p}}\rangle + \frac{1}{2}\right)\Big)\left(\langle\, C_{\phi\mathbf{k}}\,\rangle - \langle\, C^\dagger_{\phi\mathbf{k}}\,\rangle\right),$$

$$\langle B_{\phi\mathbf{k}}\rangle^{\cdot} = \frac{\dot{\omega}_{\phi k}}{\omega_{\phi k}}\left(\langle B_{\phi\mathbf{k}}\rangle + \frac{1}{2}\right) - i\omega_{\phi k}\left(\langle\, C_{\phi\mathbf{k}}\,\rangle - \langle\, C^\dagger_{\phi\mathbf{k}}\,\rangle\right), \tag{E.2}$$

$$\langle C_{\phi\mathbf{k}}\rangle^{\cdot} = i\omega_{\phi k}\left(\langle A_{\phi\mathbf{k}}\rangle - \langle B_{\phi\mathbf{k}}\rangle\right) - \frac{i}{\omega_{\phi k}}\Big(m_\phi^2 - M_\phi^2 +$$

$$+\frac{1}{2}g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\chi p}}\left(\langle B_{\chi\mathbf{p}}\rangle + \frac{1}{2}\right)\Big)\left(\langle B_{\phi\mathbf{k}}\rangle + \frac{1}{2}\right), \tag{E.3}$$

$$\langle A_{\chi\mathbf{k}}\rangle^{\cdot} = -\frac{\dot{\omega}_{\chi k}}{\omega_{\chi k}}\left(\langle A_{\chi\mathbf{k}}\rangle + \frac{1}{2}\right) + \frac{i}{\omega_{\chi k}}\Big(\omega_{\chi k}^2 + m_\chi^2 - M_\chi^2 + \frac{1}{2}g^2\langle\phi\rangle^2 +$$

$$+\frac{1}{2}g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\phi p}}\left(\langle B_{\phi\mathbf{p}}\rangle + \frac{1}{2}\right)\Big)\left(\langle\, C_{\chi\mathbf{k}}\,\rangle - \langle\, C^\dagger_{\chi\mathbf{k}}\,\rangle\right), \tag{E.4}$$

$$\langle B_{\chi\mathbf{k}}\rangle^{\cdot} = \frac{\dot{\omega}_{\chi k}}{\omega_{\chi k}}\left(\langle B_{\chi\mathbf{k}}\rangle + \frac{1}{2}\right) - i\omega_{\chi k}\left(\langle\, C_{\chi\mathbf{k}}\,\rangle - \langle\, C^\dagger_{\chi\mathbf{k}}\,\rangle\right), \tag{E.5}$$

$$\langle C_{\chi\mathbf{k}}\rangle^{\cdot} = i\omega_{\phi k}\left(\langle A_{\chi\mathbf{k}}\rangle - \langle B_{\chi\mathbf{k}}\rangle\right) - \frac{i}{\omega_{\chi k}}\Big(m_\chi^2 - M_\chi^2 +$$

$$+\frac{1}{2}g^2\langle\phi\rangle^2 + \frac{1}{2}g^2\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\phi p}}\left(\langle B_{\phi\mathbf{p}}\rangle + \frac{1}{2}\right)\Big)\left(\langle B_{\chi\mathbf{k}}\rangle + \frac{1}{2}\right). \tag{E.6}$$

Choosing physical masses of ϕ and χ as (5.2.35) and (5.2.36) we can obtain the set of simplified equations of motion

$$\langle \ddot{\phi} \rangle = -M_\phi^2 \langle \phi \rangle + \mathcal{O}(g^4), \tag{E.7}$$

$$\langle A_{\phi\mathbf{k}} \rangle^{\cdot} = -\frac{\dot{\omega}_{\phi k}}{\omega_{\phi k}} \left(\langle A_{\phi\mathbf{k}} \rangle + \frac{1}{2} \right) + i\omega_{\phi k} \left(\langle C_{\phi\mathbf{k}} \rangle - \langle C_{\phi\mathbf{k}}^\dagger \rangle \right) + \mathcal{O}(g^4), \tag{E.8}$$

$$\langle B_{\phi\mathbf{k}} \rangle^{\cdot} = \frac{\dot{\omega}_{\phi k}}{\omega_{\phi k}} \left(\langle B_{\phi\mathbf{k}} \rangle + \frac{1}{2} \right) - i\omega_{\phi k} \left(\langle C_{\phi\mathbf{k}} \rangle - \langle C_{\phi\mathbf{k}}^\dagger \rangle \right) + \mathcal{O}(g^4), \tag{E.9}$$

$$\langle C_{\phi\mathbf{k}} \rangle^{\cdot} = i\omega_{\phi k} \left(\langle A_{\phi\mathbf{k}} \rangle - \langle B_{\phi\mathbf{k}} \rangle \right) + \mathcal{O}(g^4), \tag{E.10}$$

$$\langle A_{\chi\mathbf{k}} \rangle^{\cdot} = -\frac{\dot{\omega}_{\chi k}}{\omega_{\chi k}} \left(\langle A_{\chi\mathbf{k}} \rangle + \frac{1}{2} \right) + i\omega_{\chi k} \left(\langle C_{\chi\mathbf{k}} \rangle - \langle C_{\chi\mathbf{k}}^\dagger \rangle \right) + \mathcal{O}(g^4), \tag{E.11}$$

$$\langle B_{\chi\mathbf{k}} \rangle^{\cdot} = \frac{\dot{\omega}_{\chi k}}{\omega_{\chi k}} \left(\langle B_{\chi\mathbf{k}} \rangle + \frac{1}{2} \right) - i\omega_{\chi k} \left(\langle C_{\chi\mathbf{k}} \rangle - \langle C_{\chi\mathbf{k}}^\dagger \rangle \right) + \mathcal{O}(g^4), \tag{E.12}$$

$$\langle C_{\chi\mathbf{k}} \rangle^{\cdot} = i\omega_{\phi k} \left(\langle A_{\chi\mathbf{k}} \rangle - \langle B_{\chi\mathbf{k}} \rangle \right) + \mathcal{O}(g^4), \tag{E.13}$$

where

$$\frac{\dot{\omega}_{\phi k}}{\omega_{\phi k}} = \frac{1}{2\omega_{\phi k}^2} \cdot \frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} \left(\frac{g^2 \langle \phi \rangle \langle \dot{\phi} \rangle}{2\omega_{\chi p}^3} - i \left(\langle C_{\chi\mathbf{p}} \rangle - \langle C_{\chi\mathbf{p}}^\dagger \rangle \right) \right) + \mathcal{O}(g^4), \tag{E.14}$$

$$\frac{\dot{\omega}_{\chi p}}{\omega_{\chi p}} = \frac{1}{2\omega_{\chi k}^2} \cdot \left[2g^2 \langle \phi \rangle \langle \dot{\phi} \rangle - \frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} i \left(\langle C_{\phi\mathbf{p}} \rangle - \langle C_{\phi\mathbf{p}}^\dagger \rangle \right) \right] + \mathcal{O}(g^4). \tag{E.15}$$

Putting it together an evaluating up to the fourth order in coupling results in the final form of differential equations we need to solve (5.2.39)-(5.2.45).

In the above analysis we neglect the divergent terms

$$\frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_k} \tag{E.16}$$

due to the chosen convention. Usually these terms are regularized and included in the bare masses that are constant in time. However, in this case the divergent part is time-dependent, so our regularization corresponds to the time-dependent bare masses. If we take $m_\phi = m_\chi = 0$,

$$\frac{1}{2} g^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_k} \propto g^2 M^2 \propto g^4, \tag{E.17}$$

so we may also justify neglecting these terms by perturbation theory and our order of approximation.

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