A matrix model for a noncommutative formulation of the $D = 11$ supermembrane compactified torus

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We introduce a $SU(N)$ regularized matrix model of the noncommutative formulation of the $D = 11$ supermembrane compactified on a torus with non-trivial winding. In this model we show that the sector characterized by the winding number $n \neq 0$ has no local string-like spikes, and the bosonic sector of the theory has discrete spectrum. We also discuss the spectrum for the complete supersymmetric model.
Keywords: Supermembrane; matrix models; SU(N) regularization; noncommutative geometry.

Presentamos un modelo de regularización $SU(N)$ de la formulación no conmutativa de la supermembrana compactada en un toro en el caso de enrollamiento no trivial. En este modelo demostramos que el sector caracterizado por el número de enrollamiento $n \neq 0$ no tiene configuraciones tipo cuerdas y el sector bosónico de la teoría tiene espectro discreto. También discutimos el espectro para el modelo supersimétrico completo.
Descriptores: Supermembrana; modelos matriciales; regularización SU(N); geometría no conmutativa.

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1. Introduction

There has been several attempts to quantize the supermembrane $D = 11$ in the Light Cone gauge. One of these attempts is a non perturbative regularizing model found by De Wit, Hoppe and Nicolai [1]. In this approximation, De Wit, Lüsher and Nicolai [2], found that the spectrum of the supermembrane in $D = 11$ is continuous. This instability of the theory is due to the existence of singular configurations, which are string-like configurations, and to the supersymmetry of the model. De Wit Peeters and Plefka tried to find a matrix model of the supermembrane in the case of compactified target spaces in order to see if this instabilities remained, but they could not obtain a regularized model although by heuristical reasoning they were able to see that the spectrum properties remain unchanged [3]. We will show that is possible to obtain for the sector of the theory describing non trivial winding, a matrix model regularization which exhibit stability properties. This result does not contradict the one in [3] since the instability arises from the sector with non trivial winding.

We start with the Hamiltonian of the compactified D=11 supermembrane on $(R^9 \times S^1 \times S^1)$. The Hamiltonian for the double compactified D=11 supermembrane was obtained in Ref. 4 and 5 starting from the lagrangian formulation of the D=11 supermembrane. It was important to follow step by step the dualization procedure in order to show that the nontrivial winding of the supermembrane was indeed described by the nontrivial bundle over which the gauge field, dual to the compactified coordinates, is defined. Having that geometrical structure one may introduce in an intrinsic way a symplectic structure on the world volume. One finally may formulate the double compactified D=11 supermembrane as a symplectic noncommutative gauge theory [4, 5]. The final form of the Hamiltonian is

$$\begin{aligned} H = \int_\Sigma \frac{1}{2\sqrt{W}}[(P^m)^2 + (\Pi_r)^2 + 1/2W\{X^m, X^n\}^2 \\ + W(\mathcal{D}_r X^m)^2 + 1/2W(\mathcal{F}_{rs})^2] \\ + \int_\Sigma [1/8\sqrt{W}n^2 - \Lambda(\mathcal{D}_r\Pi^r + \{X^m, P_m\})] \\ - \frac{1}{4}\int_\Sigma \sqrt{W}n^*\mathcal{F} + \int_\Sigma \sqrt{W}[-\bar{\theta}\Gamma_-\Gamma_r\mathcal{D}_r\theta \\ + \bar{\theta}\Gamma_-\Gamma_m\{X^m, \theta\} + \Lambda\{\bar{\theta}\Gamma_-, \theta\}] \end{aligned} \quad (1)$$

in terms of the original Majorana spinors of the D=11 formulation, which may be decomposed in terms of a complex 8-component spinor of $SO(7) \times U(1)$. $m = 1, ...7$ are the indices denoting the scalar fields once the supermembrane is formulated in the light cone gauge. $r, s = 1, 2$ are the indices related to the two compactified directions of the target space. Σ is the spatial part of the world volume which is assumed to be closed Riemann surface of topology g. P_M and Π_r are the conjugate momenta to X^M and the connection 1-form $\mathcal{A}_r$ respectively. Where, $\mathcal{D}_r = D_r + \{\mathcal{A}_r, \}$ and $\mathcal{F}_{rs} = D_r\mathcal{A}_s - D_s\mathcal{A}_r + \{\mathcal{A}_r, \mathcal{A}_s\}$. The bracket $\{,\}$ is defined as $\{*, \diamond\} = 2\epsilon^{sr}/n(D_r*)(D_s\diamond)$ where n denotes the integer which characterizes the non trivial principle bundle under consideration. D_r is a tangent space derivative

$D_r\diamond = \widehat{\Pi}^a_r \partial_a \diamond /\sqrt{W} = \{\widehat{\Pi}_r, \diamond\} \quad r,s = 1,2 : a = 1,2.$ where ∂_a denotes derivatives with respect to the local coordinates of the world volume while $\widehat{\Pi}^a_r = \epsilon^{au}\partial_u \Pi_r$ is a zweibein defined from the minimal solution of the Hamiltonian of the theory. It satisfies $\epsilon^{rs}\widehat{\Pi}^a_r\widehat{\Pi}^b_s\epsilon_{ab} = n\sqrt{W}$.

Our model consists in the following: Multivalued fields are decomposed on a fixed but multivalued background which is a minimum of the hamiltonian $\widehat{\Pi}_r$ and a single-valued field A_r without transitions over the surface,Σ, as it follows, $\widetilde{\mathcal{A}}_r = \widehat{\Pi}_r + A_r$. We expand on an ortonormal complete basis Y_A which is an eigenfunction of the operator D_r. We only make the expansion over the single-valued fields. Multivalued background $\widehat{\Pi}_r$ is absorved in the definition of covariant derivatives. We can define, the structure constants associated to the infinite group of the area preserving diffeomorphisms,(DPA) as $g^C_{AB} = \int d^2\sigma\sqrt{\omega}\,\{Y_A, Y_B\}\,Y^C$. They also verify that $D_r Y_A = \lambda^B_{rA} Y_B$ and $\lambda^B_{rA} = \lambda_{rA}\delta_{AB}$, where Y_A is chosen the Fourier basis for the torus, with $\widehat{\Pi}_r$ the local coordinates over Σ. In this case the new elements $\lambda^c_{rA} = -ig^C_{V_r,A-V_r}$ are specific elements of the full group of structure constants, where $V_r = (0,1),(1,0)$.

2. Results

This model can be regularized by truncation of the indices $A = 1,..N$ and aproximating the structure constants of the infinite group of diffeomorphisms by the structure constants of SU(N) group. The new elements λ^c_{rA} which appear as a consequence of the process of target space compactification can be also regularized as they are particular structure constants of the full group, DPA. The SU(N) regularized hamiltonian for a dual supermembrane compactified on a torus with non-trivial winding is (see for details Ref. 6),

$$\begin{aligned} H =& tr\Big(\frac{1}{2N^3}(P^{0m}T_0P^0_mT_0 + \Pi^{r0}T_0\Pi^{-0}_rT_0 \\ &+ (P^m)^2 + (\Pi)^2_r) + \frac{n^2}{16\pi^2N^3}[X^m, X^n]^2 \\ &+ \frac{n^2}{8\pi^2N^3}(\frac{i}{N}[T_{V_r}, X^m]T_{-V_r} - [\mathcal{A}_r, X^m])^2 \\ &+ \frac{n^2}{16\pi^2N^3}([\mathcal{A}_r, \mathcal{A}_s] + \frac{1}{8}n^2 \\ &\frac{i}{N}([T_{V_s}, \mathcal{A}_r]T_{-V_s} - [T_{V_r}, \mathcal{A}_s]T_{-V_r}))^2 + \frac{n}{4\pi N^3} \\ &\Lambda([X^m, P_m] - \frac{i}{N}[T_{V_r}, \Pi_r]T_{-V_r} + [\mathcal{A}_r, \Pi_r]) \\ &+ \frac{in}{4\pi N^3}(\overline{\Psi}\gamma_-\gamma_m[X^m, \Psi] - \overline{\Psi}\gamma_-\gamma_r[\mathcal{A}_r, \Psi] \\ &+ \Lambda[\overline{\Psi}\gamma_-, \Psi] - \frac{i}{N}\overline{\Psi}\gamma_-\gamma_r[T_{V_r}, \Psi]T_{-V_r})\Big), \end{aligned} \tag{2}$$

restricted to the gauge fixing condition

$$\mathcal{A}_1 = \mathcal{A}^{(a_1,0)}_1 T_{(a_1,0)}, \quad \mathcal{A}_2 = \mathcal{A}^{(a_1,a_2)}_2 T_{(a_1,a_2)}, \tag{3}$$

with $a_2 \neq 0$. Where were used the following definitions:

$$X^m = X^{mA}T_A, \qquad \mathcal{A}_r = \mathcal{A}^A_r T_A, \tag{4}$$

respectively to their conjugate momenta $P^m = P^{mA}T_A$ and $\Pi^r = \Pi^{rA}T_A$ using $[T_B, T_C] = f^A_{BC}T_A$ with $lim_{N\to\infty} f^A_{BC} = g^A_{BC}$. In order to see the existence of local singular configurations, it is important to come back to the global condition which was imposed in order to obtain the Hamiltonian of the model under consideration . It was

$$\int_\Sigma \sqrt{W}^* \mathcal{F} = 0. \tag{5}$$

The annihilation of that term (5) which is perfectly valid when we formulate our model over a fixed non trivial line bundle, has important consequences with respect to the non-existence of the local string-like configurations with zero energy density. To analyze this point let us see first what occurs for the compactified membrane without that assumption. Without the assumption (5), there are local string-like configurations arising from following configurations:

$$X^m{=}X^m(X(\sigma_1,\sigma_2)), \;\; \mathcal{A}_r{=} - \widehat{\Pi}_r + f_r(X(\sigma_1,\sigma_2)). \tag{6}$$

These configurations depend on an arbitrary uniform map $X(\sigma_1,\sigma_2)$. After some calculations one can show that the hamiltonian density of (1) over those configurations becomes zero. Hence the compactified supermembrane allows local string like spikes with zero energy. Let us now discuss the sector of the theory arising from the imposition of the global condition (5). In the SU(N) model, the singular configurations (6) do not arise because $\mathcal{A}_r$ is single valued in distinction to $\widehat{\Pi}_r$ which is necessarily multivalued over Σ. String-like configurations appear as zeros of the bosonic potential $V(X,\mathcal{A})$ which can be rewritten as:

$$\begin{aligned} &\widetilde{\lambda}_{rA}X^{mA} = 0 \quad \text{with} \quad r = 1,2 \\ &\text{and} \quad k^{-1/2(V_r\times A)}\widetilde{\lambda}_{rA}\mathcal{A}^A_s - k^{-1/2(V_s\times A)}\widetilde{\lambda}_{sA}\mathcal{A}^A_r \\ &\qquad + f^A_{BC}\mathcal{A}^B_r\mathcal{A}^C_s = 0, \end{aligned}$$

which is only satisfied for

$$\mathcal{A}^A_r = 0; \quad X^{mA} = 0; \quad \Pi^A_r = 0; \quad P^A_m = 0. \tag{7}$$

Hence there are not string-like configurations of zero energy density for this model, [6]. The constraint which is included in the hamiltonian determines $\Pi^{(a,b)}_1 b \neq 0$ and $\Pi^{(a,0)}_2$ which, together the gauge condition, allow a canonical reduction H_R of the hamiltonian. After this reduction the canonical momenta contribution becomes non-trivial, however, as they are positive, we can bound the mass operator

$$\mu_R = H_R - Tr(\frac{1}{2N^3}P^0_mT_0P^0_mT_0 + \Pi^0_rT_0\Pi^0_rT_0) \tag{8}$$

by an operator μ without the constraint. If the resulting μ which is defined in the whole space $\mathbb{R}^M$, is bounded from below and has a compact resolvent, the same properties are valid for μ_R, which is valid only in the interior of an open cone $K \subset \mathbb{R}^M$. If we denote the hamiltonian by

$$\mu = -\Delta_X - \Delta_{\mathcal{A}} + V(X,\mathcal{A}) \tag{9}$$

acting on $L^2(X,\mathcal{A}) \in \mathbb{R}^M$, and define rigorously μ as the self-adjoint Friedrichs extension of $(\mu, C_c^\infty(\mathbb{R}^M))$, in order to show that μ has compact resolvent we must see the properties of $V(X,\mathcal{A})$, (see Ref. 7 for details): Firstly we can note that $V(X,\mathcal{A})$ is positive. Secondly the potential $V(X,\mathcal{A}) = 0$ if and only if $X^{Bm} = 0 \quad and \quad \mathcal{A}_r^B = 0$ for all indices B, m, r. This was shown previously when we discussed that there are no local string-like configurations. The third property that fill the requeriments for μ to be compact is that the potential $V(X,A) \to \infty$ when $(X,A) \to \infty$ in every direction. Let see this third requeriment in more detail. We write

$$X^{Bm} = R\phi^{Bm}; \mathcal{A}_r^A = R\psi_r^A, \tag{10}$$

with ϕ^{Bm}, ψ_r^A angular variables and $R > 0$. The proof consists in showing that

$$\inf_{(\phi,\psi)\in T} V(R\phi, R\psi) \to \infty \quad as \quad R \to \infty. \tag{11}$$

As a consequence of this third property the resolvent of μ is compact so the spectrum of this sector consists of pure isolated positive eigenvalues of finite multiplicity. This ensures that the resolvent of the bosonic sector of the theory H_R is also compact so the spectrum is discrete [7]. We now consider the supersymmetric extension of the hamiltonian.

3. The supersymmetric model

If we write the fermionic potential without considering the constraint

$$\begin{aligned} V_{Fermionic} =& \overline{\Psi}^{(-A)}[-f_{BC}^A \gamma_- \gamma_m X^{mB} \Psi^C \\ &+ f_{AB}^C \mathcal{A}_r^B \gamma_- \gamma_r \Psi^C + \lambda_{rA} \gamma_- \gamma_r \Psi^A] \end{aligned} \tag{12}$$

Each spinor can be expressed in terms of an eight complex components spinor χ^A:

$$\Psi^A = \begin{pmatrix} -i\chi^A \\ 0 \\ \chi^{\dagger A} \\ 0 \end{pmatrix} \tag{13}$$

and its complex conjugate is

$$\overline{\Psi}^A = \Psi^t C = \begin{pmatrix} 0, & -i\chi^{\dagger A}, & 0, & -\chi^A \end{pmatrix} \tag{14}$$

Written in terms of χ^A the total hamiltonian of the system is

$$\begin{aligned} H =& (\frac{1}{4}[X^m, X^n]^2 + \frac{1}{2}(\widetilde{\mathcal{D}}_r X^{mA})^2 + \frac{1}{8}n^2 + \frac{1}{4}(\mathcal{F}_{rs}^A)^2)I \\ &- 2(C_C^A)_{\alpha'\beta'}(\chi_{\alpha'}^{\dagger A}\chi_{\beta'}^C) + i(A_{1C}^A)_{\alpha\alpha}\chi_\alpha^{\dagger A}\chi_\alpha^{\dagger C} \\ &- i(A_{1C}^{*A})_{\alpha\alpha}\chi_\alpha^A\chi_\alpha^C, \end{aligned} \tag{15}$$

where we have made the following definitions,

$$\begin{aligned} C_C^A &= \frac{in}{4\pi\sqrt{2}N^3} f_{BC}^A \Gamma_- \Gamma_m X^{mB} \\ A_C^A &= \frac{in}{4\pi\sqrt{2}N^3}(f_{BC}^A(i\mathcal{A}_1 + \mathcal{A}_2)^B \\ &\quad + (i\lambda_{1A} + \lambda_{2A})^B \delta_C^A I_{8\times 8}). \end{aligned} \tag{16}$$

We have explicitly evaluated the spectrum of this hamiltonian on a subspace of the whole physical configuration space, $2^{8(N^2-1)} \times 2^{8(N^2-1)}$, with $2^8 \times 2^8$ associated to the matrix representation of spinors, and $N^2 - 1$ to the $SU(N)$ group. It turns out that the spectrum of the superymmetric model is discrete. We will present a general proof in a forthcoming paper. [8]

4. Conclusions

We showed that it is possible to obtain a $SU(N)$ matrix model formulation of the supermembrane dual in compactified spaces for non trivial winding. For this sector of the theory there are not string-like configurations of zero energy density which cause instabilities in the theory. We showed that the bosonic sector of the spectrum of the theory is discrete. We also discussed the spectrum of the supersymmetric model. Explicit calculations have been performed over a subspace of the physical configuration space. Our results point out that the spectrum of the supersymmetric model is discrete.

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