SEMIONICS: A THEORY OF SUPERCONDUCTIVITY BASED ON FRACTIONAL QUANTUM STATISTICS

A. ZEE
Institute for Theoretical Physics
University of California
Santa Barbara, California 93106, USA

ABSTRACT

This talk is intended to give people who are not interested in details a brief conceptual overview of the subject.

We are fortunate enough to meet in Singapore when this nation is celebrating its 25th anniversary. I see everywhere the slogan "One people, one nation, one Singapore." See Figure 1. We are also celebrating a 25th anniversary, and I would like to base my talk on a similar slogan: "One people, one subject, one physics." See Figure 2. (Instead of a lion, I have here what's supposed to be a picture of Bob Marshak.) Theoretical physics is one subject with both conceptual and methodological unity. (The organizers of this conference clearly recognized this fact: I was told that for the first time in the history of the Rochester conference there is a talk on condensed matter physics and an entire session devoted to the interface between field theory and condensed matter physics.)

I am going to talk about a theory of superconductivity based on the idea that the relevant quasi-particles in the superconductors are semions [1]. Certainly one of the most amazing features of quantum physics is the existence of two types of particles, bosons and fermions. This is a totally quantum phenomenon with no classical analog at all. When we interchange two bosons we get the phase factor $+1$, and when we interchange two fermions, we get -1. What is another interesting number besides $+1$ and -1? A very interesting number is i. It would be fun to postulate that there are particles called semions such that when two semions are interchanged the quantum wave function gets a factor of i. The word "semion", being half Latin and half Greek, was coined to describe a particle half way between a boson and a fermion. More generally, as Leinaas and Myrheim and later Wilczek showed [2,3], there can be particles called anyons such that when you interchange two anyons, you get any phase factor you want [3].

Let's emphasize immediately that this very interesting and novel phenomenon is possible only in $2+1$ dimensional spacetimes, but not in the 3+1 dimensional spacetime that we actually live in. 2+1 dimensional spacetime is really different from 3+1 dimensional spacetime. It's not just less than a good thing. In 2 dimensional space a new physical concept emerges: the concept of "going around". A particle in 2 dimensional space, that is to say a plane, can actually go around another particle. This concept of going around does not make sense in 3-dimensional space. In 3 dimensional space, there is no way you can define whether a point particle has gone around another point particle. As a noted architect once said, less is more.

We should be able to do some new physics with this new physical concept and indeed we can. We can postulate an infinite ranged phase interaction between two particles. Consider two particles such that when one particle goes around the other through an angle ϕ the quantum wave function acquires a phase $e^{i\left(\frac{\theta}{\pi}\right)\phi}$. I'm free to postulate such an interaction; whether or not such an interaction actually exists in Nature is, of course, another story. Here θ is just a parameter characteristic of the particle. For different θ's we have different types of particles. Let's understand what θ means. Take the angle $\phi = \pi$ so that one particle has gone half way around the other particle. If you follow this operation by a translation, it is equivalent to interchanging the two particles. According to our definition, the wave function acquires the phase $e^{i\theta}$, and so for $\theta = 0$ we have bosons, for $\theta = \pi$, fermions. In general we can have any phase we want. This infinite ranged phase interaction is known in the literature as fractional statistics, a potentially confusing terminology. First of all, this parameter θ can in fact be any real number, it doesn't have to be a fraction. The word statistics also confuses some people, because we ordinarily think of statistics as the representation of the permutation group.

But as we can see we are dealing here with representations of a group more general than the permutation group, two interchanges is not effectively the same as no interchange.

A very important property of anyons is that they violate time reversal and parity invariances if $\theta \neq 0$ or π. This is very easy to understand. Consider time reversal. Suppose I take a particle around another particle, then according to our formula I will get a phase $e^{i\theta}$. But now if I reverse the direction of time, so that I take the particle going the other way, then I will get $e^{-i\theta}$. These two factors are not the same unless θ is equal to 0 or π. Parity is also very easy to see. Just reflect in the mirror. Unless we have fermions and bosons, we violate T and P.

Another way of understanding the appearance of fractional statistics is to remind ourselves how we learned about quantization of angular momentum in school. We took a course in quantum mechanics and our teachers told us that the rotation group is non-Abelian and from the algebra of the angular momentum generators they derived quantization of angular momentum. But if you live in 2 dimensional space you can only rotate around one axis. The rotation group is $SO(2)$ and abelian, so angular momentum is not quantized and we can have fractional angular momentum. We have more physics because we have less symmetry in lower dimensional space. In a relativistic local field theory, we have spin statistics theorem and hence a new kind of statistics.

A natural question to ask is whether we can incorporate fractional statistics into field theory because what we have done so far is to arbitrarily postulate a new kind of interaction between point particles. The answer is yes. It's at this point that the concept of the gauge field makes its entrance [4].

Here is the recipe for incorporating fractional statistics. Suppose somebody gives you a Lagrangian $\mathcal{L}_0$. It can be any Lagrangian with a conserved current j_μ. This current j_μ can be any conserved current: it can be the current of a point particle, the current carried by a fermionic field, the topological current of solitons, and so on. Then the prescription is as follows: you introduce a gauge potential a_μ, couple it to the current j_μ, and then you add to the Lagrangian a term $\alpha\epsilon^{\mu\nu\lambda}a_\mu f_{\nu\lambda}$ which is known as the Chern-Simons term. It looks rather complicated, but it's in fact a simple object. Here $\epsilon_{\mu\nu\lambda}$ is the totally antisymmetric symbol. Notice that to form this term, we have to contract with a symbol with 3 indices, so this is possible only in 2+1 dimensional space time. And α is any real number. We define $f_{\mu\nu}$ the field strength in terms of the gauge potential a_μ in the usual way. But while $f_{\mu\nu}$ has the same structure as the electromagnetic gauge field would have in (2+1) dimensional spacetime, it should not be confused with the electromagnetic field strength. So we have written down a very fancy Lagrangian. How do we understand the physics of this Lagrangian? Well, it's very simple, we all know what to do: given a Lagrangian we find an equation of motion. Now to find the equation of motion we just vary respect to a_μ. And that's easy to do, I get $2\alpha\epsilon^{\mu\nu\lambda}f_{\nu\lambda} = j^\mu$.

This is the analog of Maxwell's equation of motion, but in fact it's much easier to solve than Maxwell's equations. We can just look at the $\mu = 0$ component here and we see that it says the magnetic field $\epsilon_{ij}f_{ij}$ is proportional to the charged density. We can understand why this describes fractional statistics. Let us consider a particle sitting at rest which carries this charge corresponding to this current that we just talked about. Let's solve for the field around this particle, this is the problem that you solve on the first page of any electrodynamics book. Very far away from the particle, some distance away, the charge density is 0, the current is 0. You don't even have to solve any differential equations, you learn immediately that the electric field is 0 and the magnetic field is 0. So that might be the end of the story, that there is no field at all. But, in fact, as is well known by now the gauge potential can be topologically nontrivial even when it produces no field. We can see this, because if I take the integral of the gauge potential a_i around a circle centered on the particle, then by Stoke's theorem this is just the surface integral of the flux through the area enclosed by the circle. But the magnetic field is proportional to the charge density, so we plug it in and see that we have the surface integral of the charge density, which is not zero by assumption. Therefore, the gauge potential cannot possibly be zero, although the electric field and the magnetic field are zero. The gauge potential, in fact, has to be proportional to the derivative of ϕ, where ϕ is the polar angle, so that we get the right value for the integral of a_i around the circle. Now imagine another particle moving along the circle. If this particle also couples to this gauge potential then we pick up a phase factor as we all learned in a course on quantum electromagnetism, that the phase is We see that this construction precisely incorporates fractional statistics as I promised on the previous transparency. and we can identify θ as $1/\alpha$. Now we have a problem of language here, because this whole subject looks like electromagnetism, we have gauge potentials and gauge fields and it is very tempting to call these things magnetic fields and electric fields, as I have already done. Strictly speaking we should call this the fake magnetic field, the fake charge, just to remind us it's not the ordinary magnetic field and the ordinary electric charge. In the literature people often call this statistical charge, statistical magnetic field, and so on. We will omit this adjective "statistical". Therefore, given this discussion on the previous transparency, we can understand anyons are just particles carrying this statistical charge and flux. So we discovered that after all the physics that we are discussing here is nothing new. It was in some sense known to Dirac; and last year we celebrated the 30th anniversary of the Dirac-Aharonov-Bohm effect.

This is of course all ancient history from 1983, 84 when the subject was developed. Then high temperature superconductivity was discovered and then there was a suggestion that all of this stuff had something to do with high temperature superconductivity.

Before I go on I also have to give you a lightning review of high temperature superconductivity. All you have to know is that the materials that exhibit high temperature superconductivity have a layered structure so that the electrons are largely confined to moving in planes. Most of the theoretical work on the subject assumes as an approximation that the electrons live in a 2 dimensional world. The natural question to ask is whether there is some physics that is generically different in 2+1 dimensional spacetime. Maybe this new phenomenon is associated with the fact that the electrons are forced to move in a 2 dimensional world. Very crudely speaking, theoretical work on high temperature superconductivity follows two main currents. The vast majority of researchers in this field assume that in some sense there is no need for a drastically new physical concept and that more or less the same constructs that have been so successful in condensed matter physics can be applied. This majority view could well be right. But it's sort of a somewhat boring point of view, at least in my opinion and in the opinion of a small minority of theorists who have gone down another path by suggesting that perhaps new theoretical constructs peculiar to 2 dimensional systems are necessary.

Let me emphasize that there is a rather broad spectrum of research in high temperature superconductivity, and today's talk is by no means representative of the research on the subject. On the one end of this spectrum, there are physicists that worry about what happens when you replace some of the copper atoms by zinc and so on. These are the "real physicists". At the other end live "fake physicists" who have only the foggiest notion of what an orbital might be. These people write down topological field theories, talk about fake magnetic fields, and claim to have solved everything. Their philosophical stance may be that they could capture the "global and topological features" of these systems without having to worry about whether it is the p_a+p_b or p_a-p_b orbital on the oxygen that is relevant.

In the brief time available, I will have no way of giving anything other than an overview. Fortunately I have written a very detailed review article with the title of today's talk and it has appeared in a book called *High Temperature Superconductivity* [1] which has been published in March of this year. For those of you who are interested, my original research can be found in a number of publications in various journals [5,6,7]. I don't have time to give proper credit to everyone; suffice it to say that the pioneers in this field include Laughlin, Kalmeyer, Kivelson, Rokshar, Sethna, Dzyloshinski, Polyakov, Wiegmann, and many, many others.

I find it convenient to organize this subject by asking four "big questions". First, under what circumstances does the ground state of a commonly accepted model Hamiltonian spontaneously violate T and P? As I emphasized on one of the first transparencies, fractional statistics particles violate T and P. We know, of course, that if I give you a piece of superconducting material, the underlying Coulomb interaction of the electrons certainly does not violate time reversal and parity. These invariances have to be dynamically and spontaneously violated. So the first question is whether this is possible? Second, how do we describe the dynamics of the excitations about the ground state? What are their quantum numbers and in particular what are their statistics? What is the long distance effective theory? The third question is how does a gas or liquid of fractional statistics particles behave? Is it a superfluid and does it superconduct if these particles are charged? And of course, most important is the question whether there are experimental tests that we can invent to test our scenario.

Most research in this area starts with the Hubbard model, which, although extremely difficult to solve, is a simple model to write down. The Hamiltonian consists of two terms. One term just tells you that electrons can hop on a lattice from site i to a neighboring site j. The other term tells you there's a Coulomb energy if two electrons are on the same site. Of course, by the Pauli exclusion principle two electrons can be on the same site only if they have opposite spin. Let us understand then the physics of the Hubbard model when it's half filled with electrons so that there's one electron per site. Focus on two electrons on neighboring sites. If their spins are opposite then one electron can hop onto the site of the other electron. It can hop back and forth between two sites. We know from the uncertainty principle that if a particle can spread out, it will lower its energy. If you confine a particle to a smaller region of space, that raises its energy. In contrast, if the spins of the two electrons were in the same direction, then they cannot hop back and forth because of the Pauli exclusion principle. Therefore the energy is lower when the spins of neighboring electrons are opposite, that is, antiferromagnetism is favored. Indeed, in the limit of large Coulomb energy, the Hubbard model reduces to the Heisenberg antiferromagnet.

Experimentally, the materials that people have studied are in fact antiferromagnetic and when they are doped with holes up to a certain critical concentration, (doping with holes of course means pulling electrons out of material, leaving empty sites around) they suddenly superconduct. So the homework assignment for theorists is extremely simple: you have to show that the Hubble Hamiltonian, which you can easily write down for a first year quantum mechanics class, when doped with holes describes superconductivity.

On a heuristic level, we can easily understand what holes would do to the system. Holes tend to destroy the antiferromagnetic order. Consider a hole moving in an antiferromagnetic background, so the spins are arranged up-down-up-down, and so on. Suppose this hole decides to move over 3 sites. It means that electrons have to hop over the other way to make room. We no longer have the perfect up-down-up-down arrangement. It raises the energy. A moving hole creates a string of energy behind it. Antiferromagnetic order is not favored in the presence of holes. An even easier way of saying this that a kid can understand is simply that if you have perfectly ordered spins and you pull out some of the electrons and allow things to move around, then of course the ordering is going to be messed up.

Another way of summarizing the effect of holes is to say that effectively the Hamiltonian becomes a so-called frustrated Heisenberg model in which we have a nearest neighbor antiferromagnetic interaction, a next nearest neighbor antiferromagnetic interaction, a next-next neighbor, and so on. How are these terms generated? They are generated by exactly the same argument that I gave earlier, that two neighboring electrons would like to have their spins opposite because then they can hop back and forth. Suppose I introduce holes into this system. Next to a hole, two electrons which are next-nearest neighbors can now start to talk to each other as well, because there is nobody to block them. They also want to be opposite in spin. That generates frustration because the desire of nearest neighbors to have opposite spins and the desire of the next-nearest neighbors to have opposite spin, and so on, impose conflicting demands on the system. Nobody has the foggiest idea what the ground state looks like. This is one of the outstanding problems of theoretical physics today, the nature of the ground state of highly frustrated system. Anderson [8], several years ago, made a guess on what the ground state would look like and called his guess the resonating valence bond state. Other people have their own proposals. Some people believe that the ground state is still essentially antiferromagnetic.

Well, perhaps we can ask a much simpler question, namely, does the ground state violate T and P? This is a binary question with a yes or no answer.

There have been several theoretical approaches towards answering this question which unfortunately I don't have time to explain in detail. All I can give you is a headline service. There are four approaches that I know of. (1) The antiferromagnet can be mapped by mathematical transformation into a problem of hard core boson gas in a magnetic field [9]. (2) We can also study the long-distance physics of an antiferromagnet as represented by a nonlinear sigma model with a topological term added [10]. (3) Wen, Wilczek and I proposed a so-called chiral spin state which is in some sense a new state of matter in which there is a certain order parameter to the state [6]. (4) Finally, we can also use the duality relation between the soliton and particle sectors [11]. Never mind the details of these theoretical approaches. They are however mutually consistent in determining the statistics of the excitations to be $\theta = \frac{\pi}{2n}$ where n is an integer. Take the simplest case $n = 1$: the excitations in the system are semions.

Perhaps I should mention the chiral spin liquid and what the order parameter is. The order parameter is a spin triple cross product $\vec{S}_i \cdot \vec{S}_j \times \vec{S}_k$ where the $\vec{S}$ are these neighboring spin operators. This quantity, if it is nonzero, obviously violates T and P, because if you reverse time all the spins reverse direction and this thing will change sign. So if you do a calculation and this quantity is not equal to zero then it's a signal that T and P are violated. This local order parameter can be extended to a global order parameter, which is in fact a quantity closely related to the Wilson loop [6].

In reality, it is by no means clear what Hamiltonian should be used to describe the actual material. There is also a lot of ifs and buts connected with the word "effective" in the statement "holes lead effectively to frustration." (The presence of holes may be studied directly with the so-called t-J model [12]. There is some indication that the quasiparticles always come out to be semions [13].) Thus, it is perhaps best to think of using the discrete symmetries to divide the space of all plausible Hamiltonians into those that violate T and P spontaneously and those that do not.

Let me go back and summarize what we are discussing. I've told you that the ground state of some condensed matter system can violate T and P under some circumstances and that the quantum numbers of the excitations and quasiparticles obey semion statistics.

We now go on to discuss what happens when you have a gas or liquid of such semions, whether it is a superfluid or a superconductor. The first remark is notice that two semions do not make a fermion. You may think that semions are half way between bosons and fermions and so if I put two semions together they might make a fermion. But in fact, suppose we move a bound state of two semions half way around another bound state of two semions, thus interchanging the two bound states. Then we get a phase. This semion gets a phase of i going around that semion but he also gets a phase from going around the other semion. Now we must do a mathematical calculation: $2 \times 2 = 4$. We get the phase four times. Each time a semion goes around another semion we get a phase of i. Since we get this phase 4 times we get i^4, which is equal to 1. Thus, two semions actually make a boson [14].

We may now give some handwaving arguments We all know that fermions like to stay apart and bosons like to stick together. Semions, being half way between are more likely than fermions to pair. When they do pair, they form bosons, whose condensation can then lead to superfluidity and superconductivity. Just think, if you were a fermion and you want to condense into the ground state, what could you do? You have basically two strategies: (1) Find another fermion and pair with him or her, or (2) turn yourself into a boson. Strategy (2) is an attractive possibility, but in the systems under discussion you can apparently make it only half way and turn yourself into a semion.

Much of the physics of anyons is contained in the following deep mathematical identity $\theta = 0 + \theta = \pi - (\pi - \theta)$. What is the mathematics trying to tell us? The first equality says that an anyon with statistics θ is effectively a boson (statistics 0) with a gauge interaction of "strength" θ; the second equality says that it is also a fermion (statistics π) with a gauge interaction of "strength" $-(\pi - \theta)$. The physics is made completely clear by looking at the 2-anyon problem. In the center of mass, the Schrödinger equation has a centrifugal potential like $(\ell + \theta/\pi)^2/r^2$ with ℓ an integer. The smallest possible value of this potential is thus $0/r^2$ for bosons, $1/r^2$ for fermions, and $1/4r^2$ for semions. Anyons are like bosons with "centrifugal repulsion" between them or fermions with a "centripetal attraction". Thus, theorists have a choice. They can begin either with a fermi gas [15,16,17] — certainly not a superfluid — and show that the attraction gives pairing and superfluidity, or with a boson gas [7,18], - which with a short ranged repulsion is a superfluid as was shown by Bogoliubov ages ago — and show that the repulsion does not destroy superfluidity.

The semion superfluid is a beautifully clean problem in theoretical physics. There is no coupling parameter. We are given a collection of particles such that when two of them are interchanged the wave function gets a phase of i. It may well be that there is an exact solution. What makes the problem difficult is that unlike the case for fermion or for boson, the N-body wave function cannot be written in terms of the 1-body wave function. We have a strongly correlated quantum many-body problem.

People have made some progress using a mean field approximation first proposed in Ref. 19. Consider moving an anyon through a big loop enclosing an area A. Since in the mean, there are nA anyons in the loop, the wave function picks up the phase $e^{i(\theta/\pi)(2\pi)nA}$. But we know that a charged particle moving in a magnetic field b picks up a phase e^{ibA}. Thus, we can consider an approximation: the particles act as if they are moving in a mean (fake) magnetic field given by $b = 2n\theta$ and generated by all the other particles.

As is well known by now, in the quantum Hall effect we have electrons moving in a plane in the presence of external (real) magnetic field. Thus, in the mean field approximation the semion fluid is closely related to the Hall fluid. There is however a crucial difference between the two fluids. The hallmark of the Hall fluid is its incompressibility. The basic physics of this striking phenomenon may be understood heuristically. The electrons are forced by the external fixed magnetic field to move in circles. Because of fermi statistics, the circles can't "overlap." The combination of fermi statistics and the magnetic field makes the system incompressible. Superficially, the semion fluid and the Hall fluid sound similar. We have particles and magnetic fields (real in one case, fake in the other). What's the difference?

The crucial difference is that the magnetic field is external and fixed in one case, but generated by and dynamically tied to the particles in the other. This makes all the difference in the world. (It is sort of like the difference between a fixed political system imposed on the people and a dynamical political system generated by the people.) There is at least no *a priori* reason for the semion fluid to be incompressible and in fact it isn't.

The essential physics behind the compressibility of the semion fluid can be captured, I believe, by a hydrodynamic approximation [7] treating the semions as bosons with gauge repulsion between them. To say a fluid is compressible is to say that it has gapless excitations, that is, excitations that cost infinitesimally little energy. Consider a long wavelength density fluctuation. Where the density is higher, the effective (fake) magnetic field is also higher according to $b = 2n\theta$, and so the (Landau or Larmor) circles the particles move in are correspondingly smaller, just so that they don't bump into each other any more than in the low density regions. As the wavelength becomes longer and longer, the energy of the density wave becomes smaller and smaller.

In the gauge theory language, the massless mode corresponds to the gauge boson.

Another way of approaching the physics is to calculate the effective energy density. Since in a magnetic field the Landau or Larmor energy is $\frac{1}{2}\hbar\omega_c \sim b$, the energy density is $\sim nb \propto n^2$. The quadratic dependence of the energy density on the number density is as if there is a short ranged repulsion between the bosons. We may then invoke Bogoliubov's work to say that we have a superfluid.

We turn finally to our fourth "big question", on whether there are experimental tests., Certainly there are! In contrast to a number of theories of superconductivity which make more or less generic predictions, this theory goes out on a limb and says there should be T and P violation – it's certainly a falsifiable theory.

To look for T and P violation, we can study the polarization of the light reflected from a high-T_c material when the incident light is linearly polarized [20]. Thus far, three experiments have been done with two showing a rotation of the polarization axis and one showing no effect. Of course there are numerous solid state effects to worry about. Also, it is important to see both T and P violation with TP conserved. A rotation of the polarization axis shows only P violation, but not T violation. Another worry is that the statistics parameter θ may alternate in sign as we go from layer to layer, in which case the effect would be washed out. An alternative experiment is to look at the precession of muons injected into the material [16]. In contrast to the polarized light experiment, the muon is a local probe. The experiment has been done and fails to show a signal at the expected level. Thus, we have now two negative experiments and two quasi-positive experiments.

In studying semionic superconductivity I have been struck by the conceptual and methodological unity of theoretical physics. We are playing with the universal concept of gauge fields. In the chiral spin state, due to a remarkable phenomenon, the excitations to which the gauge field couple are effectively relativistic Dirac fermions even though the underlying physics on the lattice is not even rotation invariant, let alone Lorentz invariant. In the nonlinear σ-model, spin-charge separation shows up as a deconfinement transition triggered by spontaneous T and P violation. (Condensed matter physicists have theorized that the spin and charge degrees of freedom carried by an electron in vacuo may separate in a complicated quantum many-body environment.) The chiral spin order parameter formed by the triple spin cross-product is reminiscent of similar operators particle physicists used to study T violation. Indeed, I recently took a semester out from my semionic work to do some particle physics. I ended up doing some work on the electron's electric dipole moment [21], a work discussed by other speakers elsewhere in these proceedings. I was again playing with gauge fields and worrying about T and P violation. Thus is theoretical physics a unified subject.

Besides our fascination with the quantum physics of strongly correlated many-body systems, another motivation in studying these condensed matter systems lies of course in the hope that some ideas and concepts from condensed physics may enrich particle physics. Recently I have made some extremely tentative steps in this direction [22]. It is a glorious historical fact that our understanding of the electroweak interaction owes a great deal to the insight of Nambu and others into the nature of low temperature superconductivity.

To summarize let me list the current answers to the "four questions". (1) Does the ground state violate T and P? Possible (but unlikely?). (2) What are the quantum numbers of the excitations? If the answer to (1) is yes, then we have semions. (3) Is a semion fluid a superfluid? Almost certainly yes. (4) Are there experimental tests? Definitely a falsifiable theory.

In closing, I would like to remind you of an elementary point of logic: Even if the presently known high temperature superconductors are not semion superconductors, it does not follow that semion superconductors do not exist. In particle physics, there is a rule known as Gell-Mann's rule stating that whatever is not forbidden is required. I was told by condensed physicists that they have a priority claim: they have long known this rule as Herring's rule. (And of course the humanists know it as T.H. White's rule.) Unlike particle physicists who have only one darn universe to look at, condensed matter physicists have a wealth of systems to study, particularly in this era of great progress in materials science when you can almost have designer materials made to order. Let us then end with the credo of an extreme optimist. It would certainly be lots of fun if the presently known superconductors turn out to be semionic superconductors, but won't it be even more exciting if the presently known superconductors turn out not to be semionic superconductors? That would mean that there may be a new class of high temperature superconducting materials out there waiting to be discovered!

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