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6 Involution Requirement on a Boundary Makes Massless Fermions Compactified on a Finite Flat Disk Mass Protected

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Abstract. The genuine Kaluza-Klein-like theories—with no fields in addition to gravity—have difficulties with the existence of massless spinors after the compactification of some space dimensions [1]. We proposed in ref. [2] a boundary condition which allows massless spinors compactified on a flat disk to be of only one handedness. Massless spinors then chirally couple to the corresponding background gauge gravitational field (which solves equations of motion for a free field linear in the Riemann curvature). In this paper we study the same toy model: $M^{(1+3)} \times M^{(2)}$, looking this time for an involution which transforms a space of solutions of Weyl equations in $d = 1 + 5$ from the outside of the flat disk in x^5 and x^6 into its inside (or conversely). The natural boundary condition that on the wall an outside solution must coincide with the corresponding inside one leads to massless spinors of only one handedness (and accordingly mass protected), chirally coupled to the corresponding background gauge gravitational field. We introduce the Hermitean operators of momenta and discuss the orthogonality of solutions, ensuring that to each mass only one solution of equations of motion corresponds.

6.1 Introduction

The major problem of the compactification procedure in all Kaluza-Klein-like theories with only gravity and no additional gauge fields is how to ensure that massless spinors be mass protected after the compactification. Namely, even if we start with only one Weyl spinor in some even dimensional space of $d = 2$ modulo 4 dimensions (i.e. in $d = 2(2n+1)$, $n = 0, 1, 2, \cdots$) so that there appear no Majorana mass if no conserved charges exist and families are allowed, as we have proven in ref. [3], and accordingly with the mass protection from the very beginning, a compactification of m dimensions gives rise to a spinor of one handedness in d with both handedness in $d-m$ and is accordingly not mass protected any longer.

And in addition, since a spin (or the total conserved angular momentum) in the compactified part of space will in $d - m$ space appear as a charge and will manifest both values (positive and negative ones) and since in the second quantization procedure anti particles of opposite charges appear anyhow, doubling the number of massless spinors of both—positive and negative—charges when coming from $d(=2(2n+1))$-dimensional space down to $d=4$ and after a second quantized procedure is not in agreement with what we observe. Accordingly there must be some requirements, some boundary conditions, which ensure in a compactification procedure that only spinors of one handedness survive, if Kaluza-Klein-like theories have some meaning. However, the idea of Kaluza and Klein of having only gravity as a gauge field seems too beautiful not to have the realization in Nature.

One of us[4,5,6,7,8] has for long tried to unify the spin and all the charges to only the spin, so that spinors would in $d \geq 4$ carry nothing but a spin and interact accordingly with only the gauge fields of the Poincaré group, that is with vielbeins $f^{\alpha}{}_{a}$ [1] and spin connections $\omega_{ab\alpha}$, which are the gauge fields of the Poincaré group.

In this paper we take (as we did in the ref. [2]) the covariant momentum of a spinor, when applied on a spinor function ψ, to be

$$p_{0a} = f^{\alpha}{}_{a} p_{0\alpha}, \quad p_{0\alpha}\psi = p_{\alpha} - \frac{1}{2} S^{cd} \omega_{cd\alpha}. \tag{6.1}$$

A kind of a total covariant derivative of $e^{a}{}_{\alpha}$ (a vector with both—Einstein and flat index) will be taken to be $p_{0\alpha} e^{a}{}_{\beta} = i e^{a}{}_{\beta;\alpha} = i(e^{a}{}_{\beta,\alpha} + \omega^{a}{}_{d\alpha} e^{d}{}_{\beta} - \Gamma^{\gamma}{}_{\beta\alpha} e^{a}{}_{\gamma})$, with the require that this derivative of a vielbein is zero: $e^{a}{}_{\beta;\alpha} = 0$.

The corresponding Lagrange density $\mathcal{L}$ for a Weyl spinor has the form $\mathcal{L} = E\frac{1}{2}[(\psi^{\dagger}\gamma^{0}\gamma^{a} p_{0a}\psi) + (\psi^{\dagger}\gamma^{0}\gamma^{a} p_{0a}\psi)^{\dagger}]$ and leads to

$$\mathcal{L} = E\psi^{\dagger}\gamma^{0}\gamma^{a}(p_{a} - \frac{1}{2} S^{cd}\omega_{cda})\psi, \tag{6.2}$$

with $E = \det(e^{a}{}_{\alpha})$ [2].

The authors of this paper have tried to find a way out of this "Witten's no go theorem" for a toy model of $M^{(1+3)} \times$ a flat finite disk in $(1+5)$-dimensional space [2] by postulating a particular boundary condition, which allows a spinor to carry after the compactification only one handedness. Massless spinors then chirally couple to the corresponding background gauge gravitational field, which

[1] $f^{\alpha}{}_{a}$ are inverted vielbeins to $e^{a}{}_{\alpha}$ with the properties $e^{a}{}_{\alpha} f^{\alpha}{}_{b} = \delta^{a}{}_{b}, \; e^{a}{}_{\alpha} f^{\beta}{}_{a} = \delta^{\beta}_{\alpha}$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indices from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

[2] To generate more than one family, we actually observe up to now three families, a second kind of the Clifford algebra objects has also been introduced[5,6,9,10], which anti commute with the ordinary Dirac γ^{a} matrices ($\{\gamma^{a}, \tilde{\gamma^{b}}\}_{+} = 0$) and generate equivalent representations with respect to the generators $S^{ab} = \frac{i}{4}(\gamma^{a}\gamma^{b} - \gamma^{b}\gamma^{a})$ and are used accordingly to generate families[5,6,9,10]. In this work we shall not take families into account.

solves equations of motion for a free field, linear in the Riemann curvature, while the current through the wall is for a massless and massive solutions equal to zero.

In the ref. [2] the boundary condition was written in a covariant way as

$$\hat{\mathcal{R}}\psi|_{\rm wall} = 0,$$
$$\hat{\mathcal{R}} = \frac{1}{2}(1 - i n^{(\rho)}{}_a\, n^{(\phi)}{}_b\, \gamma^a\gamma^b), \quad \hat{\mathcal{R}}^2 = \hat{\mathcal{R}}$$

with $n^{(\rho)} = (0,0,0,0,\cos\phi,\sin\phi)$, $n^{(\phi)} = (0,0,0,0,-\sin\phi,\cos\phi)$, which are the two unit vectors perpendicular and tangential to the boundary of the disk (at ρ_0), respectively. The projector $\hat{\mathcal{R}}$ can for the above choice of the two vectors $n^{(\rho)}$ and $n^{(\phi)}$ be written as

$$\hat{\mathcal{R}} = \overset{56}{[-]} = \frac{1}{2}(1 - i\gamma^5\gamma^6). \tag{6.3}$$

The reader can find more about the Clifford algebra objects $\overset{ab}{(\pm)}, \overset{ab}{[\pm]}$ in the Appendix (section 6.8).

The boundary condition requires that only massless states (determined by Eq.(6.2)) of one (let us say right) handedness with respect to the compactified disk degrees of freedom are allowed. Accordingly also massless states of only one handedness are allowed also in $d = 1 + 3$.

In this paper we reformulate the above boundary condition as an *involution*, which transforms solutions of equations of motion from outside the boundary of the disk into its inside. We do this by the intention that the limitation of M2 on a finite disk would have a natural explanation, originated in a symmetry relation. We also define the Hermitean momentum p^s and comment on the orthogonality relations of solutions of equations of motion, which fulfill the boundary conditions.

We make use of the technique presented in ref. [9,10] when writing the equations of motion and their solutions. It turns out that all the derivations and discussions appear to be very transparent when using this technique. We briefly repeat this technique in Appendix 6.8.

6.2 Equations of motion and solutions

We assume that a two dimensional space, spanned by x^5 and x^6, is an Euclidean manifold $M^{(2)}$ (with no gravity)

$$f^{\sigma}{}_s = \delta^{\sigma}{}_s,\ \omega_{56s} = 0. \tag{6.4}$$

and accordingly with the rotational symmetry around an origin.

Wave functions describing spinors in $(1+5)$-dimensional space demonstrating $M^{(1+3)} \times M^{(2)}$ symmetry are required to obey the equations of motion

$$\gamma^0\gamma^a p_a \psi^{(6)} = 0,\ a = m, s;\ m = 0,1,2,3;\ s = 5,6. \tag{6.5}$$

The most general solution for a free particle in $d = 1 + 5$ should be written as a superposition of all four $(2^{6/2-1})$ states of one Weyl representation. We ask the reader to see Appendix 6.8 for the technical details how to write one Weyl representation in terms of the Clifford algebra objects after making a choice of the Cartan sub algebra set, for which we make a choice: S^{03}, S^{12}, S^{56}. In our technique [9] the four states, which all are the eigenstates of the Cartan sub algebra set, are expressed with the following four products of projections ($\stackrel{ab}{[k]}$) and nilpotents ($\stackrel{ab}{(k)}$):

$$\begin{aligned}
\varphi^1_1 &= \stackrel{56}{(+)}\stackrel{03}{(+i)}\stackrel{12}{(+)}\,\psi_0,\\
\varphi^1_2 &= \stackrel{56}{(+)}\stackrel{03}{[-i]}\stackrel{12}{[-]}\,\psi_0,\\
\varphi^2_1 &= \stackrel{56}{[-]}\stackrel{03}{[-i]})\stackrel{12}{(+)}\,\psi_0,\\
\varphi^2_2 &= \stackrel{56}{[-]}\stackrel{03}{(+i)}\stackrel{12}{[-]}\,\psi_0,
\end{aligned} \tag{6.6}$$

where ψ_0 is a vacuum state. If we write the operators of handedness in $d = 1+5$ as $\Gamma^{(1+5)} = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\gamma^6$ $(= 23iS^{03}S^{12}S^{56})$, in $d = 1{+}3$ as $\Gamma^{(1+3)} = -i\gamma^0\gamma^1\gamma^2\gamma^3$ $(= 22iS^{03}S^{12})$ and in the two dimensional space as $\Gamma^{(2)} = i\gamma^5\gamma^6$ $(= 2S^{56})$, we find that all four states are left handed with respect to $\Gamma^{(1+5)}$, with the value -1, the first two are right handed and the second two left handed with respect to $\Gamma^{(2)}$, with the values 1 and -1, respectively, while the first two are left handed and the second two right handed with respect to $\Gamma^{(1+3)}$ with the values -1 and 1, respectively.

Taking into account Eq.(6.6) we may write a wave function $\psi^{(6)}$ in $d = 1+5$ as

$$\psi^{(6)} = (\mathcal{A}\stackrel{56}{(+)} + \mathcal{B}\stackrel{56}{[-]})\psi^{(4)}, \tag{6.7}$$

where $\mathcal{A}$ and $\mathcal{B}$ depend on x^5 and x^6, while $\psi^{(4)}$ determines the spin and the coordinate dependent part of the wave function $\psi^{(6)}$ in $d = 1+3$.

Spinors, which manifest masslessness in $d = 1+3$, must obey the equation

$$\gamma^0\gamma^s p_s \psi^{(6)} = 0, \quad s = 5, 6, \tag{6.8}$$

since what will demonstrate as an effective action in $d = 1+3$ is

$$\begin{aligned}
&\int \prod_m dx^m \mathrm{Tr}_{0123}\Big(\int dx^5 dx^6 \mathrm{Tr}_{56}(\psi^{(6)\dagger}\gamma^0(\gamma^m p_m + \gamma^s p_s)\psi^{(6)})\Big) = \\
&\int \prod_m dx^m \mathrm{Tr}_{0123}(\psi^{(4)\dagger}\gamma^0\gamma^m p_m \psi^{(4)}) - \int \prod_m dx^m \mathrm{Tr}_{0123}(\psi^{(4)\dagger}\gamma^0 m \psi^{(4)}),
\end{aligned} \tag{6.9}$$

where integrals go over all the space on which the solutions are defined. $\psi^{(6)}$ and $\psi^{(4)}$ are the solutions in $d = 1+5$ and $d = 1+3$, respectively. Tr_{0123} and Tr_{56} mean the trace over the spin degrees of freedom in $x0, x1, x2, x3$ and in x^5, x^6, respectively. (One finds, for example, that $\mathrm{Tr}(\stackrel{56}{[\pm]}) = 1$.) For massless spinors it must be that $\int dx^5 dx^6 \mathrm{Tr}_{56}(\psi^{(6)\dagger}\gamma^0\gamma^s p_s\psi^{(6)}) = \psi^{(4)\dagger}\gamma^0(-m)\psi^{(4)} = 0$.

To find the effective action in $1+3$ for massive spinors, we recognize that for the mass term we have

$$\begin{aligned}
&\psi^{(4)\dagger}\gamma^0(-\mathcal{A}^* \stackrel{56}{(+)}^\dagger + \mathcal{B}^* \stackrel{56}{[-]}^\dagger)\gamma^s p_s(\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]})\psi^{(4)} = \\
&\psi^{(4)\dagger}\gamma^0(-\mathcal{A}^* \stackrel{56}{(+)}^\dagger + \mathcal{B}^* \stackrel{56}{[-]}^\dagger)(-m)(-\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]})\psi^{(4)},
\end{aligned} \tag{6.10}$$

with $s = 5, 6$, $\stackrel{56}{(\pm)}^\dagger = - \stackrel{56}{(\mp)}$ and $\stackrel{56}{[\pm]}^\dagger = \stackrel{56}{[\mp]}$, while $(^*)$ means complex conjugation. We took into account that $\gamma^0 \stackrel{56}{(+)} = - \stackrel{56}{(+)} \gamma^0$, while $\gamma^0 \stackrel{56}{[-]} = \stackrel{56}{[-]} \gamma^0$. We find that $Tr_{56}(\stackrel{56}{(+)}^\dagger \stackrel{56}{(+)}) = Tr_{56}(\stackrel{56}{[-]}) = 1$ and $Tr_{56}(\stackrel{56}{[-]}^\dagger \stackrel{56}{[-]}) = Tr_{56}(\stackrel{56}{[-]}) = 1$. In order that $\int dx^5 dx^6 Tr_{56}(\psi^{(4)\dagger}\gamma^0(-\mathcal{A}^* \stackrel{56}{(+)}^\dagger + \mathcal{B}^* \stackrel{56}{[-]}^\dagger)\gamma^s p_s(\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]})\psi^{(4)}$ will appear in $d = 1+3$ as a mass term $\psi^{(4)\dagger}\gamma^0(-m)\psi^{(4)}$, we must solve the equation $\gamma^s p_s(\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]}) = (-m)(-\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]})$.

We can rewrite equations of motion in terms of the two complex superposition of x^5 and x^6: $z := x^5 + i x^6$ and $\bar{z} := x^5 - i x^6$ and their derivatives, defined as $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x^5} - i\frac{\partial}{\partial x^6})$, $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x^5} + i\frac{\partial}{\partial x^6})$ and in terms of the two projectors $\stackrel{56}{[\pm]} := \frac{1}{2}(1 \pm i\gamma^5\gamma^6)$ as follows

$$2i\gamma^5\{\frac{\partial}{\partial z} \stackrel{56}{[-]} + \frac{\partial}{\partial \bar{z}} \stackrel{56}{[+]}\}(\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]}) = -m(-\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]}). \tag{6.11}$$

Since in Eq.(6.11) $\psi^{(4)}$ would be just a spectator, we skipped it.

In the massless case the superposition of the first two states ($\psi^{(6)m=0}_{+} = \stackrel{56}{(+)} \psi^{(4)m=0}_{+}$, with $\psi^{(4)m=0}_{+} = (\alpha \stackrel{03}{(+i)}\stackrel{12}{(+)} + \beta \stackrel{03}{[-i]}\stackrel{12}{[-]})\psi_0$) or the second two states ($\psi^{(6)m=0}_{-} = \stackrel{56}{[-]} \psi^{(4)m=0}_{-}$, with $\psi^{(4)m=0}_{-} = (\alpha \stackrel{03}{[-i]}\stackrel{12}{(+)} + \beta \stackrel{03}{(+i)}\stackrel{12}{[-]})\psi_0$) of the left handed Weyl representation presented in Eq.(6.6) must be taken, with the ratio of the two parameters α and β determined by the dynamics in x^m space. In the massive case $\psi^{(6)m}$ is the superposition of all the states to which γ^5 and γ^0 separately transform the starting state: $\psi^{(6)m} = (\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]})\psi^{(4)m}_{\pm}$, with $\psi^{(4)m}_{\pm} = \{\alpha[\stackrel{03}{(+i)}\stackrel{12}{(+)} \pm \stackrel{03}{[-i]}\stackrel{12}{(+)}] + \beta[\stackrel{03}{[-i]}\stackrel{12}{[-]} \pm \stackrel{03}{(+i)}\stackrel{12}{[-]}]\}\psi_0$. The sign $\pm$ denotes the eigenvalue of γ^0 on these states.

We shall therefore simply write (as suggested in Eq.(6.7)) $\psi^{(6)} = (\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]})\psi^{(4)}$ in the massless and the massive case, taking into account that in the massless case either $\mathcal{A}$ or $\mathcal{B}$ is nonzero, while in the massive case both are nonzero. Accordingly also $\psi^{(4)}$ differs in the massless and the massive case.

We want our states to be eigenstates of the total angular momentum operator M^{56} around a chosen origin in the flat two dimensional manifold ($M^{(2)}$)

$$M^{56} = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} + S^{56}. \tag{6.12}$$

Taking into account that $\gamma^5 \stackrel{56}{(+)} = -\stackrel{56}{[-]}$, $\gamma^0 \stackrel{56}{(+)} = -\stackrel{56}{(+)} \gamma^0$ and $\gamma^5 \stackrel{56}{[-]} = \stackrel{56}{(+)}$, $\gamma^0 \stackrel{56}{[-]} = \stackrel{56}{[-]} \gamma^0$ (see Appendix6.8) we end up with equations for $\mathcal{A}$ and $\mathcal{B}$

$$\frac{\partial \mathcal{B}}{\partial z} + \frac{im}{2}\mathcal{A} = 0,$$
$$\frac{\partial \mathcal{A}}{\partial \bar{z}} + \frac{im}{2}\mathcal{B} = 0. \tag{6.13}$$

For $m = 0$ we get as solutions

$$\psi^{(6)m=0}_{n+1/2} = a_n z^n \stackrel{56}{(+)} \psi^{(4)}_{+},$$
$$\psi^{(6)m=0}_{-(n+1/2)} = b_n \bar{z}^n \stackrel{56}{[-]} \psi^{(4)}_{-}, \; n \geq 0. \tag{6.14}$$

We required $n \geq 0$ to ensure the integrability of solutions at the origin. The solutions have the eigenvalues of M^{56} equal to $(n+1/2)$ and $-(n+1/2)$, respectively.

Since in the massless case the contribution from $(p^5)2$ compensates the one from $(p^6)2$ for all the solutions from Eq. (6.14) with $n \geq 1$ and has therefore obviously one of the two contributions to the zero $m2$ a negative real value unless $n = 0$, it seems natural to expect that the only massless solutions are the two solutions with the eigenvalues $M^{(56)}$ equal to $1/2$ for the right handed spinor ($\psi^{(6)m=0}_{1/2} = a_0 \stackrel{56}{(+)} \psi^{(4)}_{+}$) and to $-1/2$ for the left handed spinor ($\psi^{(6)m=0}_{-1/2} = b_0 \stackrel{56}{[-]} \psi^{(4)}_{-}$), and accordingly with the corresponding $\psi^{(4)m=0}_{+}$ and $\psi^{(4)m=0}_{-}$ of the left and right handedness in $d = 1+3$, respectively. We shall reformulate the operator of momentum to be Hermitean on the vector space of solutions fulfilling the involution boundary condition in sect. 6.5. Having solutions of both handedness we must conclude that in such cases there is no mass protection.

For $m \neq 0$ we get

$$\psi^{(6)m}_{n+1/2} = a_n (J_n \stackrel{56}{(+)} - i J_{n+1} e^{i\phi} \stackrel{56}{[-]}) e^{\pm i n\phi} \psi^{(4)m}, \text{ for } n \geq 0, \tag{6.15}$$

where J_n is the Bessel's functions of the first order. The easiest way to see that J_n and J_{n+1} determine the massive solution is to use Eq.(6.13), take into account that $z = \rho e^{i\phi}$, define $r = m\rho, \rho = \sqrt{(x^5)2 + (x^6)2}$, recognize that $\frac{\partial}{\partial z} = \frac{1}{2} e^{-i\phi}(\frac{\partial}{\partial \rho} - \frac{i}{\rho}\frac{\partial}{\partial \phi})$ and we find $\mathcal{B} = -\frac{2}{im}\frac{\partial \mathcal{A}}{\partial \bar{z}}$. Then for the choice $\mathcal{A} = J_n e^{in\phi}$ it follows that $\mathcal{B} = -i e^{i(n+1)\phi}(\frac{n}{r} J_n - \frac{\partial J_n}{\partial r})$, which tells that $\mathcal{B} = -i J_{n+1} e^{i(n+1)\phi}$.

6.3 Boundary conditions and involution

In the ref. [2] we make a choice of particular solutions of the equations of motion by requiring that $\hat{\mathcal{R}}\psi|_{\text{wall}} = 0$, where the wall were put on the circle of the radius ρ_0 of the finite disk (Eq.(6.3)).

This boundary condition requires that in the massless case (since $\stackrel{56}{[-]}\stackrel{56}{(+)} = 0$ while $\stackrel{56}{[-]}\stackrel{56}{[-]} = \stackrel{56}{[-]}$) only the right handed solution (Eq.6.14) $\psi^{(6)m=0}_{1/2} = a_0 \stackrel{56}{(+)}$

$\psi_{+}^{(4)m=0}$ (that is the left handed with respect to $SO(1,3)$) is allowed, while the left handed solution must be zero ($b_n = 0$) making the mass protection mechanism work in $d = 1 + 3$.

In the massive case the boundary condition determines masses of solutions, since only the solutions with $J_{n+1}|_{\rho=\rho_0} = 0$ are allowed from the same reason as discussed for the massless case. This boundary condition determines masses of spinors through the relation $m_{n+1/2}\rho_0$ is equal to a zero of J_{n+1}:

$$J_{n+1}(m_{n+1/2}\rho_0) = 0.$$

This time we look for the *involution boundary conditions*.

First we recognize that for a flat M2 - {0} manifold, with the origin $x^5 = 0 = x^6$ excluded, the Z_2 or involution symmetry can be recognized: *The transformation* $\rho/\rho_0 \to \frac{\rho_0}{\rho}$ (which can be written also as $z/\rho_0 \to \frac{\rho_0}{\bar{z}}$) *transforms the exterior of the disk into the interior of the disk* and conversely.

Then we extend the involution operator to operate also on the space of solutions

$$\begin{aligned} \hat{\mathcal{O}} &= (I - 2\hat{\mathcal{R}}')|_{z/\rho_0 \to \rho_0/\bar{z}}, \\ \hat{\mathcal{O}}^2 &= I. \end{aligned} \tag{6.16}$$

The involution condition $\hat{\mathcal{O}}^2 = I$ requires, that $\hat{\mathcal{R}}'$ is a projector

$$(\hat{\mathcal{R}}')^2 = \hat{\mathcal{R}}' \tag{6.17}$$

and can be written as $\hat{\mathcal{R}}' = \hat{\mathcal{R}} + \hat{\mathcal{R}}_{add}$, where $\hat{\mathcal{R}}_{add}$ must be a nilpotent operator fulfilling the conditions

$$(\hat{\mathcal{R}}_{add})^2 = 0, \;\; \hat{\mathcal{R}}_{add}\hat{\mathcal{R}} = 0, \;\; \hat{\mathcal{R}}\hat{\mathcal{R}}_{add} = \hat{\mathcal{R}}_{add}, \tag{6.18}$$

We had $\hat{\mathcal{R}} = \overset{56}{[-]}$, which is the projector. Since we find that $\overset{56}{[-]}\overset{56}{(-)}=\overset{56}{(-)}$ (see Appendix 6.8), while $\overset{56}{(-)}\overset{56}{[-]}= 0$, we can choose $\hat{\mathcal{R}}_{add} = \alpha \; \overset{56}{(-)}$, where α is any function of z and $\frac{\partial}{\partial z}$. Let us point out that $\hat{\mathcal{R}}_{add}$ is not a Hermitean operator, since $\overset{56}{(-)}^{\dagger}= - \overset{56}{(+)}$ and $z^{\dagger} = \bar{z}$, $(\frac{\partial}{\partial z})^{\dagger} = \frac{\partial}{\partial \bar{z}}$. Accordingly also neither $\hat{\mathcal{R}}'$ nor $\hat{\mathcal{O}}$ is a Hermitean operator.

We now make a choice of a natural boundary conditions on the wall $\rho = \rho_0$

$$\{\hat{\mathcal{O}}\psi = \psi\}|_{\text{wall}}, \tag{6.19}$$

saying that what ever the involution operator is, the state ψ and its involution $\hat{\mathcal{O}}\psi$ must be the same on the wall, that is at $\rho = \rho_0$.

It is worthwhile to write the involution operator $\hat{\mathcal{O}}$ and correspondingly the projector $\hat{\mathcal{R}}'$ in a covariant way. Recognizing that $n^{(\rho)}{}_a\, \gamma^a n^{(\rho)}{}_b\, p^b = [e^{2i\phi}\frac{1}{2}(p^5 - ip^6) + \frac{1}{2}(p^5 + ip^6)] \; \overset{56}{(-)} \; +[\frac{1}{2}(p^5 + ip^6) + e^{-2i\phi}\frac{1}{2}(p^5 + ip^6)] \; \overset{56}{(+)}$, we may write $\frac{1}{2}(1 - in^{(\rho)}{}_a\, n^{(\phi)}{}_b\, \gamma^a\gamma^b)(1 - \beta n^{(\rho)}{}_a\, \gamma^a\, n^{(\rho)}{}_b\, p^b) = \overset{56}{[-]} \; (I + \beta i[e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}] \; \overset{56}{(-)})$.

This is just our generalized projector $\hat{\mathcal{R}}'$, if we make a choice for α from Eq.(6.18) as follows: $\alpha = \beta i[e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}]$ (since $\overset{56}{[-]}\overset{56}{(-)}=\overset{56}{(-)}$). We then have

$$\hat{\mathcal{R}}' = \overset{56}{[-]}\ (I + \beta i[e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}]\ \overset{56}{(-)}), \tag{6.20}$$

where β is any complex number.

6.4 Current through the wall

The current perpendicular to the wall can be written as

$$\begin{aligned} n^{(\rho)s} j_s &= \psi^{\dagger}\gamma^0\gamma^s n^{(\rho)}_s \psi = \psi^{\dagger}\gamma^0(-)\{e^{-i\phi}\ \overset{56}{(+)}\ + e^{i\phi}\ \overset{56}{(-)}\}\psi = \psi^{\dagger}\hat{j}_{\perp}\psi, \\ \hat{j}_{\perp} &= -\gamma^0\{e^{-i\phi}\ \overset{56}{(+)}\ + e^{i\phi}\ \overset{56}{(-)}\}. \end{aligned} \tag{6.21}$$

We need to know the current through the wall, which for physically acceptable cases when spinors are localized inside the disk (involution transforms outside the disk into its inside, or equivalently, it transforms inside the disk into its outside) must be zero. We find for the current

$$\{\psi^{\dagger}\hat{j}_{\perp}\psi\}|_{\text{wall}} = \{\psi^{\dagger}\hat{\mathcal{O}}^{\dagger}\hat{j}_{\perp}\hat{\mathcal{O}}\psi\}|_{\text{wall}}. \tag{6.22}$$

Since $\hat{\mathcal{O}}^{\dagger} = I - 2(\hat{\mathcal{R}} + \hat{\mathcal{R}}^{\dagger}_{add})$ and $\hat{\mathcal{R}}^{\dagger}_{add} = (\alpha\ \overset{56}{(-)})^{\dagger} = -\alpha^*\ \overset{56}{(+)}$, it follows that $\hat{\mathcal{O}}^{\dagger}\hat{j}_{\perp}\hat{\mathcal{O}} = -\hat{j}_{\perp} - 2\alpha^*\gamma^0 e^{i\phi}\ \overset{56}{[+]}\ - 2\alpha\gamma^0 e^{-i\phi}\ \overset{56}{[+]}$.

It must then be

$$\{\psi^{\dagger}\hat{j}_{\perp}\psi\}|_{\text{wall}} = (-\psi^{\dagger}\{\hat{j}_{\perp} + 2\gamma^0(\alpha^* e^{i\phi} + \alpha e^{-i\phi})\ \overset{56}{[+]}\}\psi)|_{\text{wall}}. \tag{6.23}$$

First we check the current on the wall for the "old" case, when $\alpha = 0$ and $\hat{\mathcal{O}} = I - 2\hat{\mathcal{R}}, \hat{\mathcal{R}} = \overset{56}{[-]}$. Not to be in contradiction with Eq.(6.23) the current on the wall must for either massless or massive case be zero. In the case of massless solutions (Eq.(6.14)) only $\psi^{(6)m=0}_{n+1/2}$ can fulfill this boundary condition ($\psi^{(4)m=0\dagger}\bar{z}^n(-)\ \overset{56}{(-)}$ $\{-\gamma^0(e^{-i\phi}\ \overset{56}{(+)}\ + e^{i\phi}\ \overset{56}{(-)})\}z^n\ \overset{56}{(+)}\ \psi^{(4)m=0}_{+})|_{\text{wall}} = 0$, for each nonnegative n. The chosen boundary condition accordingly allows only the right handed solutions. We shall conclude when discussing Hermiticity of the operators that only $n = 0$ is the physically acceptable solution.

In the massive case the solutions of equations of motion (Eq.(6.15)) contribute no current through the wall, if $J_{n+1}|_{\text{wall}} = 0$, which is exactly what the boundary condition (Eq.(6.19)) $\hat{\mathcal{O}}\psi|_{wall} = \psi|_{wall}$ required.

Then we check the general case with $\hat{\mathcal{O}} = I - 2\hat{\mathcal{R}}'$, where $\hat{\mathcal{R}}' = \hat{\mathcal{R}} + \hat{\mathcal{R}}_{add} = \overset{56}{[-]}$ $+\beta i[e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}]\ \overset{56}{(-)}$. For massless solutions it is not difficult to see that *for any* *nonzero choice of* β *only one handedness - the right handed one - survives and that only* $n = 0$ *is allowed*. In the massive case we find

$$\{-i\beta[e^{2i\phi}\frac{\partial \mathcal{A}}{\partial z} + \frac{\partial \mathcal{A}}{\partial \bar{z}}] + \mathcal{B}\}|_{\text{wall}} = 0. \tag{6.24}$$

Since equations of motion require that $\mathcal{B} = -\frac{2}{im}\frac{\partial \mathcal{A}}{\partial \bar{z}}$ and since $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}e^{i\phi}(\frac{i}{\rho}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial \rho})$, we fulfill the involution condition on the wall for $\mathcal{A} = J_n e^{in\phi}$ only if $\mathcal{B} = -iJ_{n+1}e^{i(n+1)\phi}$, with the requirement that $J_n|_{\text{wall}} = 0$ and $\beta = \frac{1}{m}$. While $J_n|_{\text{wall}} = 0$ can always be fulfilled, the second requirement $\beta = \frac{1}{m}$ means, since β can not be an arbitrary number, that our generalized condition is not written in an covariant form, and is accordingly not the acceptable boundary condition.

6.5 Hermiticity of operators and the orthogonality of solutions

In this section we comment on the Hermiticity properties of the operators, in particular of p_s and on the orthogonality properties of those solutions of the equations of motion which fulfill the involution boundary conditions. We expect the solutions
i) to be orthogonal ($\int d^2x\psi_i{}^{\dagger}((p^5)2 + (p^6)2)\psi_j = \int d^2x\psi_i{}^{\dagger}\psi_j m2\delta_{ij}$) and that
ii) on the space of these solutions the operators p_s are Hermitean and have accordingly expectation values of the operators $(p^s)2$ positive contribution to $m2$ for each s.

Let us first check the orthogonality relations of the massive and massless solutions. We immediately see that the massive solutions $\psi^{(6)m}_{n+1/2}$ belonging to different n are all orthogonal due to the orhogonality of the functions $e^{in\phi}$. We find $\int d^2x \mathrm{Tr}_{56}(\psi^{(6)m\dagger}_{n+1/2}\psi^{(6)m}_{k+1/2}) = \delta_{nk}\; a^*_n a_n \frac{1}{2}\psi^{(4)m\dagger}\psi^{(4)m}\int_0^{\rho_0}(J^*_n J_n + J^*_{n+1}J_{n+1})\rho d\rho$.

It also turns out that the massless solutions ($\psi^{(6)m=0}_{n+1/2}$ (Eq.(6.14)) are orthogonal to all the massive states (Eq.(6.15)) due to the properties of the J_n Bessel's function. Namely,

$$\int d^2x \mathrm{Tr}_{56}(\psi^{(6)m=0\dagger}_{n+1/2}\psi^{(6)m}_{k+1/2}) = \delta_{nk}a^{0*}_{n+1/2}a_k\frac{1}{\sqrt{2}}\psi^{(4)m=0\dagger}\psi^{(4)m}\int_0^{\rho_0}\rho^n J_n\rho d\rho = 0,$$

since $\int_0^{\rho_0}\rho^n J_n \rho d\rho = \rho_0^{n+1}J_{n+1}(\rho_0)$, but $J_{n+1}(\rho_0)$ must be zero in order that the massive state with $n + 1/2$ obeys the involution boundary condition. Massless solutions are again due to the $e^{in\phi}$ part orthogonal among themselves.

So we conclude that all the states, which obey the equations of motion and the involution boundary condition, are orthogonal.

Are p_s Hermitean on the space of these solutions?

We know that $p_s = -i\frac{\partial}{\partial x^s}$ is Hermitean on the vectors space ψ_i if for any two functions ψ_i and ψ_j from the vector space of solutions $\mathrm{Tr}_{56}(\int d^2x p_s(\psi_i^{\dagger}\psi_j)) = 0$ (since then $\int d^2x\psi_i{}^{\dagger}p_s\psi_j + \int d^2x(-p_s\psi_i)^{\dagger}\psi_j = 0$).

We find that

$$p_s = i\frac{\partial}{\partial x^s} = i\begin{pmatrix}\cos\phi\frac{\partial}{\partial \rho} - \sin\phi\frac{1}{\rho}\frac{\partial}{\partial \phi}\\ \sin\phi\frac{\partial}{\partial \rho} + \cos\phi\frac{1}{\rho}\frac{\partial}{\partial \phi}\end{pmatrix}, \tag{6.25}$$

for $s = 5$ (first row) and 6(second row). Since either messless ($\psi^{(6)m=0}_{n+1/2}$, Eq.(6.14)) or massive ($\psi^{(6)m}_{n+1/2}$, Eq.(6.15)) states can be written as a product of $e^{in\phi}$ and the rest, say ψ_n, we see that $\int d^2x p_s(\psi^\dagger_n \psi_k)$ is nonzero only if $|n-k| = 1$.

In this case we get that the integral $\mathrm{Tr}_{56}(\int d^2x p_s(\psi_n{}^\dagger \psi_{n\pm 1}))$, $s = x^5, x^6$, proportional to $i\pi(^1_i)|\rho\psi^\dagger_n\psi_{n+1}|_{\rho_0}$, with $|\rho\psi^\dagger_n\psi_{n+1}|^{\rho_0}_0$ equal to
i)$a^{m=0*}_n a^{m=0}_{n+1}(\rho_0)^{2(n+1)}$ in the case that two massless states are concerned,
ii) $a^{m=0*}_n a^m_{n+1}\rho^{n+1}_0 J_{n+1}(\rho_0)$ in the case that one massless and one massive state are concerned,
iii) $a^{m*}_n a^m_{n+1}\rho_0(J_n J_{n+1} + J_{n+1}J_{n+2})|_{\rho_0}$ in the case that two massive states are concerned. None of these integral is zero, since the two J_n and J_{n+1} are not correlated (J_n and J_{n+1} are correlated, if both belong to the solution with the same mass, determined by $J_{n+1}(m\rho = m\rho_0) = 0$). We conclude that for none of the solutions p_s are Hermitean operators.

One can check, however, that $\hat{p}_s$

$$\hat{p}_s = i\{\frac{\partial}{\partial x^s} - \frac{1}{2}\frac{x^s}{\rho}\delta(\rho-\rho_0)\stackrel{56}{[+]}\}, \tag{6.26}$$

are Hermitean operators on the space of massive and massless solutions, fulfilling the involution boundary conditions. It contains the part with the δ function which corrects those parts of solutions, which are nonzero on the wall—the radial parts which appear with $\stackrel{56}{(+)}$. It can be shown that the integral over the part with the $\delta(\rho-\rho_0)$ function contributes just the terms which compensate the nonzero contribution in each of the three cases i)-iii).

What we must check now is, what appears in this new definition of the operator of the momenta (Eq.(6.25)) as $\gamma^s p_s \gamma^t p_t$ and whether now the integral $\mathrm{Tr}_{56}(\int d^2x\, \psi^\dagger\, \gamma^s p_s \gamma^t p_t\, \psi)$, $s = x^5, x^6$, is still manifesting as just the mass term for those ψ which we accept as solutions of equations of motion (Eq.(6.14,6.15)).

One finds

$$\begin{aligned}\gamma^s\hat{p}_s\gamma^t\hat{p}_t &= p_s p^s \\ &+ \frac{1}{2}\{[\frac{\partial}{\partial\rho}\delta(\rho-\rho_0) + \frac{1}{\rho}\delta(\rho-\rho_0) + \delta(\rho-\rho_0)(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi})]\stackrel{56}{[+]} \\ &+ \delta(\rho-\rho_0)(\frac{\partial}{\partial\rho} + \frac{i}{\rho}\frac{\partial}{\partial\phi})\stackrel{56}{[-]}\}.\end{aligned} \tag{6.27}$$

One notices that the first row of Eq.(6.27) represent the usual momentum squared. The last two terms are zero everywhere except on the wall. What we must check is the integral of the last two terms for all solutions fulfilling our involution boundary condition.

We find that the integral $\mathrm{Tr}_{56}(\int d^2x\,(z\stackrel{56}{(+)})^\dagger\,\gamma^s p_s\gamma^t p_t\, z^n\stackrel{56}{(+)})$ is for massless solutions (Eq.(6.14)) obeying the involution boundary condition proportional to n and it is zero only for $n = 0$.

The integral $\mathrm{Tr}_{56}(\int d^2x\,(z\stackrel{56}{(+)})^\dagger\,\gamma^s p_s\gamma^t p_t\, z^n\stackrel{56}{(+)})$ demonstrates for massive solutions (Eq.(6.15)) the mass term squared originating in the first row of Eq.(6.27), while the rest contributes zero.

The requirement that the integral $\mathrm{Tr}_{56}(\int d^2x\, \psi^\dagger \gamma^s p_s \gamma^t p_t\, \psi), s = x^5, x^6$, *must be zero for massless solutions, makes a choice of only one among all possible massless solutions: the one with* $n = 0$.

Our the only possible solution is in the massless case $\psi^{(6)m=0}_{1/2}$. For the massive solutions we have $\psi^{(6)m}_{1/2} = a_{1/2}\frac{1}{\sqrt{2}}(J_0 \overset{56}{(+)} - iJ_1 e^{i\phi} \overset{56}{[-]})$, with $m_{1/2}\rho_0$ as a zero of J_1, $\psi^m_{3/2} = a_{3/2}\frac{1}{\sqrt{2}}(J_1 - iJ_2 e^{i\phi})e^{i\phi})$, with $m_{3/2}\rho_0$ as a zero of J_2, $\psi^{(6)m}_{-1/2} = a_{-1/2}\frac{1}{\sqrt{2}}(J_1 \overset{56}{(+)} - iJ_0 \overset{56}{[-]} e^{-i\phi}))e^{-i\phi})$, with $m_{-1/2}\rho_0$ equal to a zero of J_0 and so on.

6.6 Properties of spinors in $d = 1 + 3$

To study how do spinors couple to the Kaluza-Klein gauge fields in the case of $M^{(1+5)}$, "broken" to $M^{(1+3)}\times$ a flat disk with ρ_0 and with the involution boundary condition, which allows only right handed spinors at ρ_0, we first look for (background) gauge gravitational fields, which preserve the rotational symmetry on the disk. Following ref. [2] we find for the background vielbein field

$$e^a{}_\alpha = \begin{pmatrix} \delta^m{}_\mu & e^m{}_\sigma = 0 \\ e^s{}_\mu & e^s{}_\sigma \end{pmatrix}, f^\alpha{}_a = \begin{pmatrix} \delta^\mu{}_m & f^\sigma{}_m \\ 0 = f^\mu{}_s & f^\sigma{}_s \end{pmatrix}, \tag{6.28}$$

with $f^\sigma{}_m = A_\mu \delta^\mu{}_m \varepsilon^\sigma{}_\tau x^\tau$ and the spin connection field

$$\omega_{st\mu} = -\varepsilon_{st} A_\mu, \quad \omega_{sm\mu} = -\frac{1}{2} F_{\mu\nu} \delta^\nu{}_m \varepsilon_{s\sigma} x^\sigma. \tag{6.29}$$

The $U(1)$ gauge field A_μ depends only on x^μ. All the other components of the spin connection fields are zero, since for simplicity we allow no gravity in $(1+3)$ dimensional space.

To determine the current, coupled to the Kaluza-Klein gauge fields A_μ, we analyze the spinor action

$$\begin{aligned}\mathcal{S} = \int d^dx E \bar\psi^{(6)} \gamma^a p_{0a} \psi^{(6)} = \int d^dx \bar\psi^{(6)} \gamma^m \delta^\mu{}_m p_\mu \psi^{(6)} + \\ \int d^dx \bar\psi^{(6)} \gamma^m (-) S^{sm} \omega_{sm\mu} \psi^{(6)} + \int d^dx \bar\psi^{(6)} \gamma^s \delta^\sigma{}_s p_\sigma \psi^{(6)} + \\ \int d^dx \bar\psi^{(6)} \gamma^m \delta^\mu{}_m A_\mu (\varepsilon^\sigma{}_\tau x^\tau p_\sigma + S^{56}) \psi^{(6)}.\end{aligned} \tag{6.30}$$

$\psi^{(6)}$ are solutions of the Weyl equation in $d = 1+3$. E is for $f^\alpha{}_a$ from (6.28) equal to 1. The first term on the right hand side of Eq.(6.30) is the kinetic term (together with the last term defines the covariant derivative $p_{0\mu}$ in $d = 1+3$). The second term on the right hand side contributes nothing when integration over the disk is performed, since it is proportional to x^σ ($\omega_{sm\mu} = -\frac{1}{2} F_{\mu\nu} \delta^\nu{}_m \varepsilon_{s\sigma} x^\sigma$).

We end up with

$$j^\mu = \int d^2x \bar\psi^{(6)} \gamma^m \delta^\mu{}_m M^{56} \psi^{(6)} \tag{6.31}$$

as the current in $d = 1 + 3$. The charge in $d = 1 + 3$ is proportional to the total angular momentum $M^{56} = L^{56} + S^{56}$ on a disk, for either massless or massive spinors.

6.7 Conclusions

In this paper we were looking for what we call a "natural boundary condition"—a condition which would, due to some symmetry relations, make massless spinors which live in M^{1+5} and carry nothing but the charge to live in $M^{(1+3)} \times$ a flat disk, manifesting in $M^{(1+3)}$, if massless, as a left handed spinor (with no right handed partner) and would accordingly be mass protected. The spin in x^5 and x^6 of the left handed massless spinor should in $M^{(1+3)}$ manifest as the charge and should chirally couple with the Kaluza-Klein charge of only one value to the corresponding gauge field, in order that after the second quantization procedure a particle and an antiparticle would not appear each of both charges.

We found the involution boundary condition

$$\{\hat{\mathcal{O}}\psi = \psi\}|_{wall}, \quad \mathcal{O} = (I - (I - i n^{(\rho)}{}_a n^{(\phi)}{}_b \gamma^a \gamma^b))_{\frac{\rho}{\rho_0} \to \frac{\rho_0}{\rho}}, \quad \mathcal{O}^2 = I, \tag{6.32}$$

which transforms solutions of the Weyl equations inside the flat disk into outside of it (or conversely) and allows in the massless case only the right handed spinor to live on the disk and accordingly manifests left handedness in $M^{(1+3)}$. The massless solution carries in the fifth and sixth dimension (only) the spin $1/2$, which then manifests as the charge in $d = 1 + 3$.

We defined a generalized momentum p_s

$$\hat{p}_s = i\{\frac{\partial}{\partial x^s} - \frac{1}{2}\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix} \delta(\rho - \rho_0) \overset{56}{[+]}\}, \tag{6.33}$$

which is the Hermitean operator in the case of our involution boundary condition.

The requirement that $\gamma^s \hat{p}_s \gamma^t \hat{p}_t$ manifests as the square of the mass leads in the massless case to to the solution with the total angular momentum $1/2$ as the only solution, while the massive solutions carry all half integer angular momenta: $\pm 1/2, \pm 3/2, \cdots$. The angular momenta in the fifth and sixth dimensions then manifests as the charge in the $1 + 3$ dimension. The massless solution with the spin $1/2$ is mass protected and chirally coupled to the corresponding Kaluza-Klein field.

The negative charge of the massless $1/2$ charge state appears only after the second quantization procedure in agreement with what we observe.

All the solutions fulfilling the involution boundary conditions are orthogonal and in this vector space and the generalized operators are Hermitean.

The involution boundary condition of Eq.(6.32) are equivalent to the boundary condition, which we present in the ref. [2]. Both take care that massless solutions of one handedness appear in $d = 1 + 3$.

We were looking for generalized boundary conditions with

$$
\begin{aligned}
\hat{\mathcal{O}} &= I - 2\hat{\mathcal{R}}', \\
\hat{\mathcal{R}}' &= \overset{56}{[-]}\,(I + \beta i[e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}]\,\overset{56}{(-)}),
\end{aligned} \tag{6.34}
$$

where β is any complex number. This generalized boundary $\hat{\mathcal{R}}'$ can be written in a covariant way as

$$
\begin{aligned}
\hat{\mathcal{R}}' &= \frac{1}{2}(1 - i n^{(\rho)}{}_a\, n^{(\phi)}{}_b\, \gamma^a\gamma^b)(1 - \beta n^{(\rho)}{}_a\, \gamma^a\, n^{(\rho)}{}_b\, p^b) \\
&= \overset{56}{[-]}\,(I + \beta i[e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}]\,\overset{56}{(-)}).
\end{aligned} \tag{6.35}
$$

But while in the massless case the generalized boundary condition $\{\hat{\mathcal{O}}\psi = \psi\}|_{wall}$ forbids all but $s = 1/2$ solution, it fails in the massive case to demonstrate the covariance and is accordingly not an acceptable boundary condition.

6.8 Appendix: Spinor representation technique in terms of Clifford algebra objects

We define[9] spinor representations as superposition of products of the Clifford algebra objects γ^a so that they are eigen states of the chosen Cartan sub algebra of the Lorentz algebra $SO(d)$, determined by the generators $S^{ab} = i/4(\gamma^a\gamma^b - \gamma^b\gamma^a)$. By introducing the notation

$$
\begin{aligned}
\overset{ab}{(\pm i)} &: = \frac{1}{2}(\gamma^a \mp \gamma^b), \quad \overset{ab}{[\pm i]} := \frac{1}{2}(1 \pm \gamma^a\gamma^b), \text{ for } \eta^{aa}\eta^{bb} = -1, \\
\overset{ab}{(\pm)} &: = \frac{1}{2}(\gamma^a \pm i\gamma^b), \quad \overset{ab}{[\pm]} := \frac{1}{2}(1 \pm i\gamma^a\gamma^b), \text{ for } \eta^{aa}\eta^{bb} = 1,
\end{aligned} \tag{6.36}
$$

it can be checked that the above binomials are really "eigenvectors" of the generators S^{ab}

$$
S^{ab}\,\overset{ab}{(k)} : = \frac{k}{2}\,\overset{ab}{(k)}, \quad S^{ab}\,\overset{ab}{[k]} := \frac{k}{2}\,\overset{ab}{[k]}\,. \tag{6.37}
$$

Accordingly we have

$$
\begin{aligned}
\overset{03}{(\pm i)} &: = \frac{1}{2}(\gamma^0 \mp \gamma^3), \quad \overset{03}{[\pm i]} := \frac{1}{2}(1 \pm \gamma^0\gamma^3), \\
\overset{12}{(\pm)} &: = \frac{1}{2}(\gamma^1 \pm i\gamma^2), \quad \overset{12}{[\pm]} := \frac{1}{2}(1 \pm i\gamma^1\gamma^2), \\
\overset{56}{(\pm)} &: = \frac{1}{2}(\gamma^5 \pm i\gamma^6), \quad \overset{56}{[\pm]} := \frac{1}{2}(1 \pm i\gamma^5\gamma^6),
\end{aligned} \tag{6.38}
$$

with eigenvalues of S^{03} equal to $\pm\frac{i}{2}$ for $\overset{03}{(\pm i)}$ and $\overset{03}{[\pm i]}$, and to $\pm\frac{1}{2}$ for $\overset{12}{(\pm)}$ and $\overset{12}{[\pm]}$, as well as for for $\overset{56}{(\pm)}$ and $\overset{56}{[\pm]}$.

We further find

$$\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]},$$
$$\gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)} . \qquad (6.39)$$

We also find

$$\stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0, \quad \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa} \stackrel{ab}{[k]}, \quad \stackrel{ab}{[k]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0,$$
$$\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0, \quad \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}, \quad \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}, \quad \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0. \qquad (6.40)$$

To represent one Weyl spinor in $d = 1 + 5$, one must make a choice of the operators belonging to the Cartan sub algebra of 3 elements of the group $SO(1, 5)$

$$S^{03}, S^{12}, S^{56}. \qquad (6.41)$$

Any eigenstate of the Cartan sub algebra (Eq.(6.41)) must be a product of three binomials, each of which is an eigenstate of one of the three elements. A left handed spinor ($\Gamma^{(1+5)} = -1$) representation with $2^{6/2-1}$ basic states is presented in Eq.(6.6). for example, the state $\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} \psi_0$, where ψ_0 is a vacuum state (any, which is not annihilated by the operator in front of the state) has the eigenvalues of S^{03}, S^{12} and S^{56} equal to $\frac{i}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$, correspondingly. All the other states of one representation of $SO(1, 5)$ follow from this one by just the application of all possible $S^(ab)$, which do not belong to Cartan sub algebra.

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