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GROUP THEORY OF THE SPONTANEOUSLY BROKEN GAUGE SYMMETRIES*

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ABSTRACT

The patterns of symmetry breaking in the gauge theories are investigated systematically in the general rotation groups and unitary groups, with Higgs scalars in the various representations up to second rank tensors. The occurrences of the fermion mass relations and pseudo-Goldstone bosons are also discussed in various cases.

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I. INTRODUCTION

Recently, it has been shown that it is possible to construct a renormalizable field theory to unify the weak and electromagnetic interactions.[1] In this class of theories, one starts from the usual renormalizable Yang-Mills Lagrangian based on certain symmetry groups, where all the vector gauge bosons are massless and then breaks the symmetry spontaneously to give masses to the gauge bosons in such a way as to preserve the renormalizability of the theory. This solves the long-standing problem in the high-order weak interactions. In the conventional theory of weak interaction, the high-order effects have no meaning due to the uncontrollable divergences present in the non-renormalizable field theory.[2] If a cutoff Λ is introduced in the theory to define these divergent quantities, it turns out that this cutoff Λ is embarrassingly small ($\sim$ 5 GeV) in order to be consistent with the known facts in the weak interactions. The renormalizability of this new type of theory guarantees that the higher order contributions are finite and calculable and presumably small. This opens the possibilities of constructing more realistic models to describe the weak interactions of the leptons and hadrons. Here one has to choose an appropriate gauge group and assigns leptons and hadrons to some representations of the group in such a way that the known facts of the weak interactions are not violated.[3] However due to the limitation of the present available experimental data, there is a large degree of freedom as regards the choice of the group in constructing models. In this paper we attempt to examine systematically the pattern of the symmetry breaking in the general rotation group O(n) and unitary groups SU(n), and various aspects concerning the group structure of the theory. We hope that this approach will provide some useful information as to what to expect in various situations.

First we give a simple example to set up the framework to study this problem. Take the most familiar isospin O(3) group and choose a triplet of scalar bosons, interacting with Yang-Mills fields $\vec{A}_\mu$. The Lagrangian is given by

$$\mathscr{L} = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{2}[(\partial_\mu - g\vec{t}\cdot\vec{A}_\mu)\vec{\phi}]^2 + \frac{1}{2}\mu^2(\vec{\phi}\cdot\vec{\phi}) - \frac{\lambda}{4}(\vec{\phi}\cdot\vec{\phi})^2 \tag{1.1}$$

where

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\vec{A}_\mu\times\vec{A}_\nu$$

and

$$(t^i)_{jk} = i\epsilon_{ijk}$$

This Lagrangian is invariant under the gauge transformation

$$\vec{A}_\mu \rightarrow \vec{A}_\mu + \vec{\epsilon}\times\vec{A}_\mu + \frac{1}{g}\partial_\mu\vec{\epsilon}$$

$$\vec{\phi} \rightarrow \vec{\phi} + \vec{\epsilon}\times\vec{\phi}$$

The spontaneous symmetry breaking is realized by letting the third component of the scalar field have non-zero vacuum expectation value,

$$\langle 0|\phi_i|0\rangle = \delta_{i3}v \tag{1.2}$$

Redefine the fields such that new fields have zero vacuum expectation values

$$\phi_i' \equiv \phi_i - \langle\phi_i\rangle = \phi_i - \delta_{i3}v \qquad \text{and} \qquad \langle\phi_i'\rangle = 0$$

The Lagrangian then becomes

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{int}$$

$$\mathscr{L}_0 = -\frac{1}{4}(\partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu)^2 - \frac{1}{2}gv^2(A_{1\mu}^2 + A_{2\mu}^2) - \frac{1}{2}(\partial_\mu\vec{\phi}')^2 +$$

$$+ [\frac{1}{2}(\mu^2 - \lambda v^2)(\vec{\phi}'\cdot\vec{\phi}') - \lambda v^2\phi_3'^2] + (\mu^2 - \lambda v^2)v\phi_3' \tag{1.3}$$

where $\mathscr{L}_{int}$ contains all the cubic and quartic terms in the Lagrangian. In the tree approximations, we have to eliminate the linear term in $\vec{\phi}'$ in order to ensure the condition $<\phi_i'> = 0$. Hence we can choose

$$v = \sqrt{\frac{\mu^2}{\lambda}} \tag{1.4}$$

From the free Lagrangian in (1.3), we see that the gauge bosons $A_{\mu 1}$, $A_{\mu 2}$ get masses gv^2 and $A_{3\mu}$ remains massless. Hence gauge symmetry of the Lagrangian has been reduced from SU(2) to U(1) with corresponding gauge boson $A_{3\mu}$. With the value (1.4) for v, the quadratic term in (1.3) becomes

$$\left[\frac{1}{2}(\mu^2 - \lambda v^2)(\vec{\phi}' \cdot \vec{\phi}') - \lambda v^2 \phi_3'^2\right] = \lambda v^2 \phi_3'^2$$

Hence ϕ_1' and ϕ_2' are the massless Goldstone bosons. Notice that the number of the Goldstone bosons is the same as the massive gauge bosons. This is due to the fact that these massless Goldstone bosons play the role of providing the extra degrees of freedom needed for the gauge boson to go from massless state with two degrees of freedom to the massive state with three degrees of freedom. So the general feature of this type of symmetry breaking is to have as many zero mass scalar bosons as the massive gauge bosons.

This kind of symmetry breaking has a very simple classical interpretation. We can consider the meson self-interaction and their mass terms as classical potential

$$V(\vec{\phi}) = -\frac{1}{2}\mu^2(\vec{\phi} \cdot \vec{\phi}) + \frac{\lambda}{4}(\vec{\phi} \cdot \vec{\phi})^2 \qquad \lambda > 0 \tag{1.5}$$

If $\mu^2 < 0$, the minimum of $V(\vec{\phi})$, the state of the lowest energy, is at the origin $\vec{\phi} = 0$. However for $\mu^2 > 0$, the minimum is at

$$\vec{\phi} \cdot \vec{\phi} = \frac{\mu^2}{\lambda} \tag{1.6}$$

as shown in Fig. 1. Equation (1.6) contains an infinite number of solutions, related to each other by rotation in O(3) space. Without lose of generality, we can choose $\phi_3 = \sqrt{\frac{\mu^2}{\lambda}}$, $\phi_1 = \phi_2 = 0$, which is invariant under the rotation in (ϕ_1, ϕ_2) space. The symmetry is broken from O(3) to O(2) $\approx$ U(1). Since the minimum is not at the origin we have to shift the origin to the position of the minimum by defining $\phi_i' \equiv \phi_i - \delta_{i3}\sqrt{\frac{\mu^2}{\lambda}}$ such that the perturbation expansion corresponds to Taylor expansion around the ground state.

It is perfectly clear that the symmetry breaking is completely determined by the scalar boson potentials. Therefore to find out the symmetry breaking pattern in any given group, we can follow the procedures;

(a) Choose a particular representation for the scalar boson and write down the most general group invariant potential $V(\phi)$, which is a fourth order polynomial of the scalar fields.

(b) Find the minimum of $V(\phi)$ by solving the equation $\partial V/\partial\phi = 0$.

(c) Calculate the number of the massless gauge bosons which determine the unbroken symmetry.

In section (II) and (III), we discuss the symmetry breaking in the general O(n) and SU(n) groups. For simplicity we consider all the representations up to the second rank tensors. In section (IV) we discuss briefly the situation with products of groups like O(n) $\times$ O(m) or SU(n) $\times$ SU(m).

There are several very interesting phenomena which comes out as a by-product in this class of the renormalizable theories, e.g., zero order fermion mass relations,[4] pseudo-Goldstone bosons.[5] These two kinds of phenomena are purely group theoretical in nature. We discuss them in the section (V) in the context of the groups we are interested in.

Section (VI) summarizes the results obtained and discuss the implications.

II. SYMMETRY BREAKING IN O(n) GROUP

As is well known,[6] in O(n) there are n(n-1)/2 generators which can be represented by

$$L_{ij} = X_i \frac{\partial}{\partial X_j} - X_j \frac{\partial}{\partial X_i} \qquad i, j = 1 \ldots n \tag{2.1}$$

The commutation relation among the generators, the Lie algebra, can be worked out by using the representation (2.1) with the obvious rule $\left[\frac{\partial}{\partial X_i}, X_j\right] = \delta_{ij}$;

$$[L_{ij}, L_{k\ell}] = \delta_{jk} L_{i\ell} + \delta_{i\ell} L_{jk} - \delta_{ik} L_{j\ell} - \delta_{j\ell} L_{ik} \tag{2.2}$$

Hence we have n(n-1)/2 vector gauge bosons, denoted by W^μ_{ij}, with the transformation law

$$W^\mu_{ij} \rightarrow W^\mu_{ij} + \epsilon_{ik} W^\mu_{kj} + \epsilon_{jk} W^\mu_{ik} \qquad \text{and} \qquad W^\mu_{ij} = -W^\mu_{ji} \tag{2.3}$$

where $\epsilon_{ij} = -\epsilon_{ji}$ are the infinitesimal parameters which characterized the infinitesimal rotations in O(n). Under the gauge transformation of second kind,

$$W^\mu_{ij} \rightarrow W^\mu_{ij} + \epsilon_{ij} W^\mu_{kj} + \epsilon_{jk} W^\mu_{ik} + \frac{1}{g} \partial^\mu \epsilon_{ij} \tag{2.4}$$

The Yang-Mills Lagrangian is then

$$\mathscr{L}_W = -\frac{1}{4} F^{ij}_{\mu\nu} F^{\mu\nu}_{ij} \tag{2.5}$$

with

$$F^{\mu\nu}_{ij} = \partial^\mu W^\nu_{ij} - \partial^\nu W^\mu_{ij} + g(W^\mu_{ik} W^\nu_{kj} - W^\nu_{ik} W^\mu_{kj}) \tag{2.6}$$

The irreducible representations in O(n) can be classified into two categories; single-valued and double-valued representations (except O(2)). The single-valued representations have the same transformation properties as the ordinary vectors in the real n-dimensional space and their symmetrized or antisymmetrized tensor products. The double-valued representations, sometime called spinor representations,[7] transform like spinors in n-dimensional coordinate space. These spinors have the property that it returns to its original position under the rotation of angle 4π instead of 2π. They can be constructed along the same line as the familiar Dirac spinors in the Lorentz group. For completeness, we give a very brief descriptions of these spinors in Appendix A. In this section, we discuss the choice of the scalar bosons as vector, second rank tensor, or spinor representations.

First we list their transformation laws and their covariant derivatives, which couple the scalars to the gauge bosons (see Table 1);

(a) Vector Representation

From the transformation law of this representation, it is easy to see that the most general fourth order invariant potential is of the form

$$V(\phi) = -\frac{\mu^2}{2}\, \phi_i \cdot \phi_i + \frac{\lambda}{4}\, (\phi_i \cdot \phi_i)^2 \tag{2.7}$$

where $\lambda > 0$ such that $V(\phi)$ is bounded below. To get the minimum of this potential, we calculate its first derivative,

$$\frac{\partial V}{\partial \phi_i} = (-\mu^2 + \lambda\, \phi_j \phi_j)\, \phi_i = 0 \qquad i = 1 \ldots n \tag{2.8}$$

The spontaneously broken symmetry solution is then given by

$$|\vec{\phi}|^2 = \phi_j \phi_j = \frac{\mu^2}{\lambda} . \tag{2.9}$$

We can choose the solution to be $\vec{\phi} = \left(0, 0, 0, \ldots 0, \sqrt{\frac{\mu^2}{\lambda}}\right)$ or $\phi_i = \delta_{in} \sqrt{\frac{\mu^2}{\lambda}}$.
All the other solutions are related to this one by an O(n) rotation. It is easy to see that this solution is invariant under those rotations which leave the n^{th}-axis

unchanged, which is the subgroup O(n-1). The symmetry is broken from O(n) to O(n-1). The gauge boson mass term is given by

$$\begin{aligned}\mathscr{L}_{w^2} &= -\frac{1}{2} g^2 W^{\mu}_{ij} <\phi_j> W^{\mu}_{ik} <\phi_k> \\ &= -\frac{1}{2} g^2 \sum_{i=1}^{n-1} (W^{\mu}_{in})^2 \left(\frac{\mu^2}{\lambda}\right)\end{aligned} \tag{2.10}$$

So there are (n-1) massive vector bosons W_{ni}, $i=1 \cdots n-1$, and (n-1)(n-2)/2 massless gauge bosons W_{ij}, $i,j=1,\cdots,n-1$, corresponding to the gauge bosons of the unbroken O(n-1) symmetry. The mass matrix of the scalar bosons can be calculated from the formula,

$$\begin{aligned}\mu^2_{ij} = \frac{1}{2} \frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\bigg|_{\phi=<\phi>} &= [-\mu^2 + \lambda <\phi^2>]\ \delta_{ij} + 2\lambda <\phi_i><\phi_j> \\ &= \mu^2 \delta_{in} \delta_{jn}\end{aligned} \tag{2.11}$$

There are (n-1) zero mass Goldstone bosons, the same number as the massive vector bosons as expected.

The pattern of the symmetry breaking in this simple case can be understood as follows. Since the invariant potential (2.7) only depends on the vector through its length $|\vec{\phi}|$, the minimum must be a condition on the length $|\vec{\phi}|$ as in (2.9). Therefore by choosing the solution with all components zero except one, we get the O(n-1) unbroken symmetry. It is then very easy to deduce the results in the case where we have two sets of vector representations, $\vec{\phi}_1$ and $\vec{\phi}_2$. The invariant potential can depend on the length of each vector and the angle between them, $|\vec{\phi}_1|$, $|\vec{\phi}_2|$, and $|\vec{\phi}_1 \cdot \vec{\phi}_2|$. The solutions for the minimum must be conditions on these three variables. We can choose first vector with only first component non-zero, and the second vector with first two components non-zero in order to satisfy these conditions. The symmetry is then reduced from O(n) to O(n-2). We can generalize this argument to any number of vector representations with the result that for m sets of vector representations the symmetry is broken from O(n) to O(n-m), where $m < n$.

In the case of unifying the weak and electromagnetic interactions, we need (n-2) sets of vectors to reduce U(1) symmetry of electromagnetic interaction. If one wants to construct a strong interaction theory from this type of gauge theories, one needs (n-1) set of vectors in order to get rid of all the infrared singularities.

(b) Antisymmetric Second Rank Tensor

For this representation, the most general fourth order invariant potential is of the form

$$V(\phi) = -\frac{\mu^2}{2}\phi'_{ij}\phi'_{ij} + \frac{\lambda_1}{4}(\phi'_{ij}\phi'_{ji})^2 + \frac{\lambda_2}{4}(\phi'_{ij}\phi'_{jk}\phi'_{k\ell}\phi'_{\ell i}) \qquad (2.12)$$

By introducing matrix notation $(\phi')_{ij} = \phi'_{ij}$ with $\phi'^T = -\phi'$, we can write the potential as

$$V(\phi) = \frac{\mu^2}{2}\mathrm{Tr}\phi'^2 + \frac{\lambda_1}{4}(\mathrm{Tr}\phi'^2)^2 + \frac{\lambda_2}{4}\mathrm{Tr}\phi'^4 \qquad (2.13)$$

Notice that the absence of the cubic term $\mathrm{Tr}\phi'^3$ is due to the antisymmetric nature of ϕ'. Since ϕ' is real and antisymmetric, it can be transformed into following standard form by a rotation,

$$\phi' = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_\ell \end{pmatrix} \qquad \text{if } n = 2\ell$$

$$\phi' = \begin{pmatrix} A_1 & & & & \\ & A_2 & & 0 & \\ & & \ddots & & \\ & 0 & & A_\ell & \\ & & & & 0 \end{pmatrix} \quad \text{if } n = 2\ell + 1$$

with

$$A_i = a_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The potential can then be written as

$$V = -\mu^2 \sum_{i=1}^{\ell} a_i^2 + \lambda_1 \left(\sum_{i=1}^{\ell} a_i^2 \right)^2 + \frac{\lambda_2}{2} \left(\sum_{i=1}^{\ell} a_i^4 \right) \tag{2.14}$$

The minimum of this potential is then

$$\frac{\partial V}{\partial a_i} = 2a_i \left[-\mu^2 + 2\lambda_1 \left(\sum_{j=1}^{\ell} a_j^2 \right) + \lambda_2 a_i^2 \right] = 0 \qquad i = 1, \cdots \ell \tag{2.15}$$

From these ~~eq~~uations, we look for solutions where not all a_i's are zero. Suppose that $a_i \neq 0$, for $i=1, \cdots, k$, then from Eq. (2.15), we must have

$$\left[-\mu^2 + 2\lambda_1 \left(\sum_{j=1}^{\ell} a_j^2 \right) + \lambda_2 a_i^2 \right] = 0 \qquad i = 1, \cdots, k$$

which give

$$a_i^2 = \frac{\mu^2}{2\lambda_1 k + \lambda_2} \qquad i = 1, \cdots, k \tag{2.16}$$

With this solution, the potential at minimum is given by

$$V = -\frac{k\mu^4}{2\lambda_1 k + \lambda_2} \tag{2.17}$$

As a function of k, the number of non-zero a_i's, this potential is monotonically increasing when $\lambda_2 > 0$ and monotonically decreasing when $\lambda_2 > 0$, with $\lambda_1 k + \lambda_2 > 0$. Hence for $\lambda_2 > 0$, the solution for ϕ is of the form

$$\phi' = a \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \quad \text{for } n = 2\ell$$

$$= a \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ 0 & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & & & & 0 \end{pmatrix} \quad \text{for } n = 2\ell + 1$$

with

$$a = \sqrt{\frac{\mu^2}{2\ell\,\lambda_1 + \lambda_2}}$$

From the coupling of scalar bosons to the gauge bosons, it can be shown that there are ℓ^2 massless gauge bosons for $n = 2\ell$ or $n = 2\ell + 1$. Therefore the symmetry breaking pattern is

$$O(2\ell) \rightarrow U(\ell)$$

$$O(2\ell + 1) \rightarrow U(\ell)$$

For the case $\lambda_2 < 0$, the solution for ϕ is given by

$$\phi' = b \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} \quad \text{for both even and odd } n$$

with

$$b = \sqrt{\frac{\mu^2}{2\lambda_1 + \lambda_2}}$$

By calculating the masses of the vector gauge bosons, one can see that

$$O(n) \to U(1) \times O(n-2)$$

(c) Symmetric Second Rank Tensor

The invariant potential in this case has the same structure as the anti-symmetric tensor;

$$V(\phi) = -\frac{\mu^2}{2}(\phi_{ij}\phi_{ij}) + \frac{\lambda_1}{4}(\phi_{ij}\phi_{ji})^2 + \frac{\lambda_2}{4}(\phi_{ij}\phi_{jk}\phi_{k\ell}\phi_{\ell i}) \tag{2.18}$$

with

$$\phi_{ij} = \phi_{ji} \qquad\qquad \sum_i \phi_{ii} = 0$$

Here we could have the cubic term $\mathrm{Tr}\,\phi^3$. But to make the discussion simpler, we leave it out by imposing a discrete symmetry $R: \phi \to -\phi$. We will discuss the case with the cubic term in the Appendix B. Again we introduce the matrix notation $(\phi)_{ij} = \phi_{ij}$, with $\phi^T = \phi$ and $\mathrm{Tr}\,\phi = 0$. We can diagonalize this real and symmetric ϕ by an orthogonal transformation to write $(\phi)_{ij} = \phi_{ij} = \delta_{ij}\phi_i$. However, these components are not all independent because of the trace condition $\mathrm{Tr}\,\phi = 0$. We can use the Lagrange multiplier to take this condition into account by writing

$$V(\phi) = -\frac{\mu^2}{2}\sum_{i=1}^{n}\phi_i^2 + \frac{\lambda_1}{4}\left(\sum_{i=1}^{n}\phi_i^2\right)^2 + \frac{\lambda_2}{4}\sum_{i=1}^{n}\phi_i^4 - g\sum_{i=1}^{n}\phi_i \tag{2.19}$$

The condition for the minimum is then

$$\frac{\partial V}{\partial \phi_i} = -\mu^2\phi_i + \lambda_1 \left(\sum_{j=1}^{n} \phi_j^2\right) \phi_i + \lambda_2\phi_i^3 - g = 0 \qquad i=1,\cdots,n \qquad (2.20)$$

with

$$\sum_i \phi_i = 0$$

We leave the detailed calculation of the solutions to this set of equations in Appendix B. Without solving these equations, we can still get some general features. Because of all the ϕ_i's satisfy the same cubic equation (2.20) these ϕ_i's can take at most three different values, say ϕ_1, ϕ_2, ϕ_3. The solution for ϕ can be written in the form

$$\phi = \begin{pmatrix} \overbrace{\phi_1 \cdots \phi_1}^{n_1} & & 0 \\ & \overbrace{\phi_2 \cdots \phi_2}^{n_2} & \\ 0 & & \overbrace{\phi_3 \cdots \phi_3}^{n_3} \end{pmatrix} \qquad \text{with } n_1+n_2+n_3 = n$$

The most general symmetry breaking in this case is

$$O(n) \to O(n_1) \times O(n_2) \times O(n_3)$$

i.e., at most O(n) reduces to products of three smaller rotation groups. However, the detailed calculation shows that it only breaks into two pieces;

$$O(n) \to O(n_1) \times O(n-n_1) \qquad \text{for} \quad \lambda_1 > 0, \quad \lambda_2 > 0$$

where

$$n_1 = \frac{n}{2} \qquad \text{if n is even}$$

$$= \frac{n+1}{2} \qquad \text{if n is odd}$$

$$O(n) \to O(n-1) \qquad \text{if} \quad \lambda_1 > 0, \quad \lambda_2 < 0.$$

(d) Spinor Representation

As explained in Appendix A, this class of representations has dimension 2^ℓ for $n=2\ell$ and $n=2\ell+1$. However due to the existence of the $2^\ell \times 2^\ell$ generalized Dirac matrices, the number of the independent quartic invariants increases with ℓ. The problem of minimization of the invariant potential becomes very hard to do in a general way because the number of terms in the potential increases with the dimension of the space.[8] So far, we have not been able to overcome this difficulty. However, we can still work out the solutions for the rotation group with given dimension. Here we give an example in the case of spinor representation in O(5). It turns out that the invariant potential is of the form

$$V = -\frac{\mu^2}{2}\chi^+\chi + \frac{g_1}{4}(\chi^+\chi)^2 + \frac{g_2}{4}\sum_{i=1}^{5}(\chi^+\gamma_i\chi)(\chi^+\gamma_i\chi) \tag{2.21}$$

where χ is a column vector with 4 components $\chi_1, \chi_2, \chi_3, \chi_4$ and γ_i, $i=1,\cdots 5$, are the generalized Dirac matrices which satisfy the Clifford algebra;

$$\left\{\gamma_i, \gamma_j\right\} = 2\delta_{ij} \tag{2.22}$$

The minimum is given by

$$\frac{\partial V}{\partial \chi_i^*} = -\frac{\mu^2}{2}\chi_i + \frac{g_1}{2}(\chi^+\chi)\chi_i + \frac{g_2}{2}(\chi^+\gamma_j\chi)(\gamma_j\chi)_i = 0 \tag{2.23}$$

and

$$\frac{\partial V}{\partial \chi_i} = -\frac{\mu^2}{2}\chi_i^* + \frac{g_1}{2}(\chi^+\chi)\chi_i^* + \frac{g_2}{2}(\chi^+\gamma_j\chi)(\chi^+\gamma_j)_i = 0 \tag{2.24}$$

If we define $z \equiv \chi^+\chi$, $y_j \equiv \chi^+\gamma_j\chi$, we can rewrite Eq. (2.23) and (2.24) as

$$[-\mu^2 + (g_1 + g_2)z]\, y_j = 0 \tag{2.25}$$

$$-\mu^2 z + g_1 z^2 + g^2 \sum_j y_j y_j = 0 \tag{2.26}$$

The solution for z and y_j are

$$z = \frac{\mu^2}{g_1 + g_2}$$

$$y^2 = \sum_{j=1}^{5} y_j y_j = \frac{\mu^4}{(g_1 + g_2)^2}$$

From the representations of the γ_i's given in the Appendix A, it can be shown that the symmetry breaking is of the form

$$O(5) \rightarrow U(1) \times U(1) \times U(1).$$

III. SYMMETRY BREAKING IN SU(n) GROUP

This class of groups have (n^2-1) generators, U_i^j, $i \cdot j=1, \cdots, n$, with $U_j^i = (U_i^j)^+$,

$$[U_i^j, U_k^\ell] = \delta_j^k U_i^\ell - \delta_i^\ell U_k^j \tag{3.1}$$

There are n^2-1 vector gauge bosons denoted by $W_{\mu i}^{\;j}$, $i, j=1, \cdots, n$ with $W_{\mu j}^{\;i} = (W_{\mu i}^{\;j})^*$ and $W_{\mu i}^{\;i} = 0$. The transformation law for these vector gauge bosons is given by

$$W_{\mu i}^{\;j} \rightarrow W_{\mu i}^{\;j} + i\epsilon_i^\ell W_{\mu\ell}^{\;j} - i\epsilon_k^j W_{\mu i}^{\;k} \tag{3.2}$$

with

$$\epsilon_i^j = (\epsilon_j^i)^*$$

Under the gauge transformation of second kind, we have

$$W_{\mu i}^{\;j} \rightarrow W_{\mu i}^{\;j} + i\epsilon_i^\ell W_{\mu\ell}^{\;j} - i\epsilon_k^j W_{\mu i}^{\;k} + \partial_\mu \epsilon_i^j (x) \tag{3.3}$$

The Yang-Mills Lagrangian is of the form

$$\mathscr{L}_0^W = -\frac{1}{4} F_{\mu\nu i}^{\ \ \ j} F^{\mu\nu i}_{\ \ \ \ j} \tag{3.4}$$

where

$$F_{\mu\nu i}^{\ \ \ j} = \partial_\mu W_{\nu i}^{\ \ j} - \partial_\nu W_{\mu i}^{\ \ j} + g(W_{\mu i}^{\ \ k} W_{\nu k}^{\ \ i} - W_{\nu i}^{\ \ k} W_{\mu k}^{\ \ i}) \tag{3.5}$$

Unlike the case of O(n), all the irreducible representations in SU(n) are single-valued representations and can be obtained by taking appropriate tensor products of the basic vectors in the n-dimensional complex spaces. In this section we consider the vector representations and all the second rank tensor representations. First we list their transformation laws and their couplings to the vector gauge bosons (see Table 2);

(a) Vector Representation

From the transformation law of this representation, it is easy to construct the invariant potential,

$$V(\psi) = -\frac{\mu^2}{2} \psi_i \psi^i + \frac{\lambda}{4} (\psi_i \psi^i)^2 \qquad \text{with } \mu, \lambda \text{ real} \tag{3.6}$$

The minimum is given by

$$\frac{\partial V}{\partial \psi_i} = [-\mu^2 + \lambda(\psi_j \psi^j)]\, \psi_i = 0 \qquad i = 1, \cdots, n \tag{3.7}$$

with the spontaneous broken symmetry solution

$$\psi_j \psi^j = \frac{\mu^2}{\lambda} \tag{3.8}$$

This solution gives rise to (2n-1) massive and $[(n-1)^2-1]$ massless gauge bosons. The symmetry is then reduced from SU(n) to SU(n-1). This is very similar to the case of the vector representation in the O(n) group. By analogy, we can see that the case with two sets of the vector representations will reduce the symmetry from SU(n) $\rightarrow$ SU(n-2). To break the SU(n) symmetry completely we need (n-1) set of vector representations.

(b) Symmetric 2nd Rank Tensor Representation

The invariant potential in this case can be easily written down

$$V(\psi) = -\frac{\mu^2}{2}\psi_{ij}\psi^{ij} + \frac{\lambda_1}{4}(\psi_{ij}\psi^{ij})^2 + \frac{\lambda_2}{4}(\psi_{ij}\psi^{jk}\psi_{k\ell}\psi^{\ell i}) \qquad (3.9)$$

with

$$\psi_{ij} = \psi_{ji} = (\psi^{ij})^*$$

To calculate the minimum, we have

$$\frac{\partial V}{\partial \psi_{ij}} = -\frac{1}{2}\mu^2\psi^{ij} + \frac{\lambda_1}{2}(\psi_{\ell m}\psi^{\ell m})\psi^{ij} + \frac{\lambda_2}{2}(\psi^{jk}\psi_{k\ell}\psi^{\ell i}) = 0 \qquad (3.10)$$

$$i, j = 1, \cdots, n$$

It is very convenient to introduce a hermitian matrix X defined by

$$X_\ell^k \equiv \psi_{\ell m}\psi^{mk} \qquad (3.11)$$

Eq. (3.10) can then be written as

$$-\mu^2\psi^{ij} + \lambda_1(X_\ell^\ell)\psi^{ij} + \lambda_2(X_\ell^j)\psi^{\ell i} = 0 \qquad (3.12)$$

Since the matrix X is hermitian, it can be diagonalized by a unitary transformation, which corresponds to a change of basis vector in the space. Therefore without lose of generality, we can take X to be diagonal to rewrite Eq. (3.12)

$$\left[-\mu^2 + \lambda_1\left(\sum_{k=1}^{n} X_k\right) + \lambda_2 X_i\right]\psi_{ij} = 0 \qquad j=1,\cdots,n \qquad (3.13)$$

Hence if some element $\psi_{ij} \neq 0$, then

$$-\mu^2 + \lambda_1 \left(\sum_{k=1}^{n} X_k \right) + \lambda_2 X_i = 0 \tag{3.14}$$

This equation has the same structure as Eq. (2.15), the tensor representation in the O(n) group. Following the same argument, it is not hard to see that for $\lambda_2 > 0$, the minimum is at $X = c^2 \mathbb{1}$ where $c^2 = \frac{\mu^2}{\lambda_1 n + \lambda_2}$ and $\mathbb{1}$ is the $n \times n$ identity matrix. It is shown in Appendix C that for this solution ψ can be chosen to be

$$\psi_{ij} = c\,\delta_{ij} \qquad \text{or} \qquad \psi = c\,\mathbb{1} = c \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \tag{3.15}$$

This form for ψ exhibits O(n) symmetry, because the group transformation $\psi \to U^T \psi U = U^T U \psi = \psi$ if U is orthogonal. This can also be checked by calculating the masses of the vector gauge bosons. Therefore the symmetry is broken from SU(n) to O(n).

For the case $\lambda_2 < 0$, the minimum is at

$$X = d^2 \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ & & & & 0 \end{pmatrix} \quad \text{with } d^2 = \frac{\mu^2}{\lambda_1 + \lambda_2} \qquad \lambda_1 + \lambda_2 > 0$$

and

$$\psi = d \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

the symmetry is reduced from $SU(n) \to SU(n-1)$.

(c) Antisymmetric 2nd Rank Tensor Representations

The invariant potential in this representation has the same form as the symmetric representation,

$$V(\psi) = -\frac{\mu^2}{2}(\psi_{ij}\psi^{ij}) + \frac{\lambda_1}{4}(\psi_{ij}\psi^{ij})^2 + \frac{\lambda_2}{4}(\psi_{ij}\psi^{jk}\psi_{k\ell}\psi^{\ell i}) \tag{3.16}$$

with $\psi_{ij} = -\psi_{ji} = (\psi^{ij})^*$. Therefore we get the same solution for $X \equiv \psi\psi^*$. However the solutions for ψ are different due to the antisymmetric property of ψ. It is shown in the Appendix that for $\lambda_2 > 0$, we can choose ψ of the form

$$\psi = c\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ 0 & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \qquad \text{for } n = 2\ell \qquad c^2 = \frac{\mu^2}{2\lambda_1\ell+\lambda_2}$$

$$\psi = c\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ 0 & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & & & 0 \end{pmatrix} \qquad \text{for } n = 2\ell + 1$$

the symmetric breaking is

$$U(2\ell) \rightarrow O(2\ell + 1)$$

$$SU(2\ell + 1) \rightarrow O(2\ell + 1)$$

For $\lambda_2 < 0$, we can choose ψ to be of the form

$$\psi = d\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ 0 & & & & 0 \end{pmatrix} \qquad \text{for both even and odd } n$$

with the symmetry breaking

$$SU(n) \rightarrow SU(n-2)$$

(d) Adjoint Representation

For simplicity, we impose the extra symmetry $\psi \rightarrow -\psi$. With this restriction, the invariant potential is given by

$$V = -\frac{\mu^2}{2}\psi_i^j\psi_j^i + \frac{\lambda_1}{4}(\psi_i^j\psi_j^i)^2 + \frac{\lambda_2}{4}(\psi_i^j\psi_j^k\psi_k^\ell\psi_\ell^i) \tag{3.17}$$

with

$$\psi_i^i = 0$$

We can take it to be in the diagonal form

$$\psi_i^j = \delta_i^j\phi_j \qquad \text{with} \quad \phi_i \text{ real}$$

because ψ_i^j is hermitian and can be diagonalized by unitary matrix. The potential (3.17) can be written as

$$V = -\frac{\mu^2}{2}\sum_{i=1}^n \phi_i^2 + \frac{\lambda_1}{4}\left(\sum_{i=1}^n \phi_i^2\right)^2 + \frac{\lambda_2}{4}\left(\sum_{i=1}^n \phi_i^4\right) - g\sum_{i=1}^n{}' \phi_i \tag{3.18}$$

where g is the Lagrange multiplier. This potential has exactly the same structure as the symmetric tensor in the O(n) group. Therefore we can take over what we have learned there, with some obvious substitution. The results are as follows;

if $\lambda_2 > 0$ $\qquad U(n) \rightarrow U(n_1) \times U(n-n_1)$

$n_1 = \frac{n}{2}$ if n is even

$n_1 = \frac{n+1}{2}$ if n is odd

if $\lambda_2 < 0$ $\qquad U(n) \rightarrow U(n-1)$.

IV. PRODUCTS OF SIMPLE GROUPS

The gauge theories based on products of simple groups are very important in constructing models, because of the necessity of including both lepton and hadron symmetries in the theory.

For the product of the simple groups, the generators and irreducible representations can be constructed very easily from those of the groups in the product. If we have $G = G_1 \times G_2$, then the generators in G are simply the direct sum of those generators in G_1 and G_2, and the irreducible representations in G are just the product of the irreducible representations in G_1 and G_2. In this section we study the groups $O(N) \times O(M)$ and $SU(N) \times O(M)$.

(a) $O(N) \times O(M)$

From the properties of the generators, we need two sets of gauge vector mesons, $W^{(1)}_{\mu ij}$, $i, j=1, \cdots, N$, and $W^{(2)}_{\mu\alpha\beta}$, $\alpha, \beta=1, \cdots, M$ with the transformation law

$$
\begin{aligned}
W^{(1)}_{\mu ij} &\rightarrow W^{(1)}_{\mu ij} + (\epsilon_1)_{ik} W^{(1)}_{\mu kj} + (\epsilon_1)_{jk} W^{(1)}_{\mu ik} + (\partial_\mu \epsilon_1)_{ij} \\
W^{(2)}_{\mu\alpha\beta} &\rightarrow W^{(2)}_{\mu\alpha\beta} + (\epsilon_2)_{\alpha\gamma} W^{(2)}_{\mu\gamma\beta} + (\epsilon_2)_{\beta\gamma} W^{(2)}_{\mu\alpha\gamma} + (\partial_\mu \epsilon_2)_{\alpha\beta}
\end{aligned}
\tag{4.1}
$$

For those irreducible representations which transform like a vector or tensor with respect to one group but like a scalar with respect to the other group, the symmetry breaking pattern is the same as those considered in previous sections, e.g., for the representation (N, 1), $O(N) \times O(M) \rightarrow O(N-1) \times O(M)$. The simplest new representation to consider is the representation (N, M) which transforms like N-dimensional vector with respect to O(N) and like M-dimensional vector with respect to O(M), i.e.,

$$
\phi_{i\alpha} \rightarrow \phi_{i\alpha} + (\epsilon_1)_{ij}\, \phi_{j\alpha} + (\epsilon_2)_{\alpha\beta}\, \phi_{i\beta} \tag{4.2}
$$

with

$$i, j = 1, \cdots, N$$

$$\alpha, \beta = 1, \cdots, M$$

Their coupling to the vector gauge bosons is of the form

$$\mathscr{L}_{w\phi} = -\frac{1}{2}(\partial_\mu \phi_{i\alpha} - W^{(1)}_{\mu ik}\phi_{k\alpha} - W^{(2)}_{\alpha\beta}\phi_{i\beta})(\partial^\mu \phi_{i\alpha} - W^{(1)}_{\mu i\ell}\phi_{\ell\alpha} - W^{(2)}_{\mu\alpha\gamma}\phi_{i\gamma}) \tag{4.3}$$

The invariant potential can be easily seen to be

$$V = -\frac{\mu^2}{2}\phi_{i\alpha}\phi_{i\alpha} + \frac{\lambda_1}{4}(\phi_{i\alpha}\phi_{i\alpha})^2 + \frac{\lambda_2}{4}(\phi_{i\alpha}\phi_{i\beta})(\phi_{j\alpha}\phi_{j\beta}) \tag{4.4}$$

The minimum is then given by

$$\frac{\partial V}{\partial \phi_{i\alpha}} = -\mu^2\phi_{i\alpha} + \lambda_1(\phi_{j\beta}\phi_{j\beta})\phi_{i\alpha} + \lambda_2\phi_{i\beta}(\phi_{j\alpha}\phi_{j\beta}) = 0 \tag{4.5}$$

It is convenient to introduce the matrix X, Y defined by

$$X_{ij} = \sum_{\beta=1}^{M} \phi_{i\beta}\phi_{j\beta} = \phi\,\phi^T \tag{4.6}$$

The matrix X defined this way is real and symmetric, which can be diagonalized by an orthogonal transformation. We can choose $X_{ij} = \delta_{ij}X_j$ to rewrite Eq. (4.5) as

$$\left[-\mu^2 + \lambda_1 \sum_{j=1}^{N} X_j + \lambda_2 X_i\right]\phi_{i\alpha} = 0 \tag{4.7}$$

This equation has the same structure as Eq. (2.19) for the O(n) group. Following the same argument, we get

$$X_i = \frac{\mu^2}{\lambda_1 k + \lambda_2} \qquad i = 1, \cdots, k ,$$

$$X_i = 0 \qquad i = k+1, \cdots, N$$

and

$$V = \frac{k\mu^4}{\lambda_1 k + \lambda_2} \tag{4.8}$$

For $\lambda_2 < 0$, the potential is a monotonically increasing function of k. The minimum is at k=1, the X takes the form

$$X = b \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ & & & & 0 \end{pmatrix} \qquad b = \frac{\mu^2}{\lambda_1+\lambda_2} \tag{4.9}$$

For $\lambda_2 > 0$, the potential is a monotonically decreasing function of k. Hence the minimum is at largest value of k allowed, which should be N. However this would imply that X is a multiple of $N \times N$ identity matrix

$$X = c\, \mathbb{1}_N = c \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \tag{4.10}$$

But X is constructed from $N \times M$ matrix ($N \geq M$) by

$$X = \phi\, \phi^T$$

Equation (4.10) implies that if we consider each row as M-component vector, all these N vectors are orthogonal to each other, which is impossible for $N > M$. Therefore the largest value of k allowed is M not N, and the solution for X and

ϕ are

$$X = c^2 \begin{pmatrix} \mathbb{1}_M & & \\ & \ddots & \\ & & 0_{NM} \end{pmatrix} \qquad c^2 = \frac{\mu^2}{N\lambda_1 + \lambda_2}$$

$$\phi = c \begin{pmatrix} \mathbb{1}_M \cdots 0_{MN} \end{pmatrix}$$

By calculating the masses of the vector gauge bosons, we can see that the symmetry is reduced from $O(N) \times O(M) \to O(M) \times O(N-M)$ for $\lambda_2 > 0$. For the case $\lambda_2 < 0$, it is easy to see that

$$\phi = b \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \\ 0 & \cdot & \cdots & 0 \end{pmatrix}$$

the symmetry is reduced from $O(N) \times O(M) \to O(N-1) \times O(M-1)$.

(b) $SU(N) \times SU(M)$

There are also two sets of vector gauge bosons with the transformation laws

$$W^{(1)j}_{\mu i} \to W^{(1)j}_{\mu i} + (\epsilon_1)^k_i W^{(1)j}_{\mu k} + (\epsilon_1)^j_k W^{(1)k}_{\mu i} + (\partial_\mu \epsilon_1)^j_i \qquad i, j = 1, \cdots, N$$

$$W^{(2)\beta}_{\mu\alpha} \to W^{(2)\beta}_{\mu\alpha} + (\epsilon_2)^\gamma_\alpha W^{(2)\beta}_{\mu\gamma} + (\epsilon_2)^\beta_\gamma W^{(2)\gamma}_{\mu\alpha} + (\partial_\mu \epsilon_2)^\beta_\alpha \qquad \alpha, \beta = 1, \cdots, M$$

Here we consider the representation (N, M) which has the transformation property

$$\psi_{i\alpha} \to \psi_{i\alpha} + i(\epsilon_1)^j_i \psi_{j\alpha} + i(\epsilon_2)^\beta_\alpha \psi_{i\beta} \qquad \begin{array}{l} i, j = 1, \cdots, N \\ \alpha, \beta = 1, \cdots, M \end{array}$$

and their complex conjugate transform like $(\overline{N}, \overline{M})$ representation

$$\psi^{i\alpha} \to \psi^{i\alpha} - i(\epsilon_1)^i_j \psi^{j\alpha} - i(\epsilon_2)^\alpha_\beta \psi^{i\beta}$$

Their coupling to the vector gauge bosons is of the form

$$\mathscr{L}_{w\mu} = -\frac{1}{2}\left(\partial_\mu \psi_{i\alpha} - iW^{(1)k}_{\mu\alpha}\,\psi_{k\alpha} - iW^{(2)\beta}_{\mu\alpha}\,\psi_{i\beta}\right)\left(\partial^\mu \psi^{i\alpha} + iW^{(1)i}_{\mu\ell}\,\psi^{\ell\alpha} + iW^{(2)\alpha}_{\mu\gamma}\,\psi^{i\gamma}\right)$$

The invariant potential can be written down as

$$V = -\frac{\mu^2}{2}\left(\psi_{i\alpha}\psi^{i\alpha}\right) + \frac{\lambda_1}{4}\left(\psi_{i\alpha}\psi^{i\alpha}\right)^2 + \frac{\lambda_2}{4}\left(\psi^{i\alpha}\psi_{i\beta}\right)\left(\psi_{j\alpha}\psi^{j\beta}\right)$$

with the minimum given by

$$\frac{\partial V}{\partial \psi_{i\alpha}} = \frac{\mu^2}{2}\,\psi_{i\alpha} + \frac{\lambda_1}{2}\left(\psi_{j\beta}\psi^{j\beta}\right)\psi_{i\alpha} + \frac{\lambda_2}{2}\,\psi_{i\beta}\left(\psi_{j\alpha}\psi^{j\beta}\right) = 0$$

The detailed analysis runs parallel to the case of $O(N) \times O(M)$. The results are

$$\lambda_2 > 0 \qquad SU(N) \times SU(M) \rightarrow SU(M) \qquad \text{for } N > M$$

$$\lambda_2 < 0 \qquad SU(N) \times SU(M) \rightarrow SU(N-1) \times SU(M-1)\,.$$

V. OTHER RELATED TOPICS

It was pointed out by Weinberg[5] that in some cases the restriction of the invariant potential to the fourth order polynomials forces the potential to have symmetry which is higher than the rest of the Lagrangian. Under that circumstance, there are more zero mass Goldstone bosons than the massive vector gauge bosons, because the number of Goldstone bosons are determined by the potential. These extra Goldstone bosons, called pseudo-Goldstone bosons by Weinberg, have vanishing masses in zeroth order and will pick up masses in the higher order correction because the other interactions don't respect the accidental high symmetry. These masses, coming solely from the higher order interaction should be finite and calculable if the theory is renormalizable. This is due to the fact that there is no mass term in the zero order Lagrangian to absorb the divergent masses coming from the higher order corrections. These finite masses are presumably small if the coupling constants are weak. Hopefully these pseudo-Goldstone bosons can be identified as the pions or the whole pseudo-scalar octets. This phenomena provides a very interesting mechanism to explain the approximate symmetries like SU(2) × SU(2), or SU(3) × SU(3), seen in the strong interactions. Hence it is very useful to find all the cases where this phenomena can take place.

For all the representations we have considered in the previous sections, it turns out that only in a very special case can we have pseudo-Goldstone bosons. For the case where there is only one irreducible representation, we have found that the symmetric second rank tensor in O(3) and the adjoint representation in SU(3) can serve the purpose.

Let us illustrate this in the case of O(3). As we have seen in Section II, the most general invariant potential of the second rank symmetric tensor is of the form

$$V(\phi) = -\frac{\mu^2}{2}\,\mathrm{Tr}\phi^2 + \frac{\lambda_1}{4}\,(\mathrm{Tr}\phi^2)^2 + \frac{\lambda_2}{4}\,\mathrm{Tr}\phi^4 \qquad (5.1)$$

However in O(3), it happens that

$$(\mathrm{Tr}\phi^2)^2 = 2\,\mathrm{Tr}\phi^4 \qquad \text{when } \mathrm{Tr}\phi = 0 \qquad (5.2)$$

The potential is then simplified to

$$V(\phi) = -\frac{\mu^2}{2}\,(\mathrm{Tr}\phi^2) + \frac{\lambda'}{4}\,(\mathrm{Tr}\phi^2)^2 \qquad (5.3)$$

$$\text{with } \lambda' = \lambda_1 + \frac{1}{2}\lambda_2$$

This form has the same structure as the vector representation in O(n) with the feature that the quartic term is proportional to the square of the quadratic term.

Since there are five independent components in ϕ, the potential in (5.3) has the O(5) symmetry which is higher than O(3). As we know for the vector representation in O(5) the symmetry is broken to O(4) and there are 4 zero mass Goldstone bosons. But the O(3) symmetry can break into either U(1), or no symmetry, which requires 2 or 3 Goldstone bosons. Therefore there are one or two pseudo-Goldstone bosons. The coupling of ϕ's to the vector gauge bosons is of the form

$$\mathscr{L}_{W\phi} = -\frac{1}{2}(\partial_\mu \phi_{ij} - gW_{\mu ik}\phi_{kj} - gW_{\mu jk}\phi_{ik})^2 \tag{5.4}$$

which has only O(3) symmetry. The higher order will then break the O(5) symmetry of $V(\phi)$ to give these pseudo-Goldstone bosons masses. Notice that it is the relation (5.2) which is responsible for the appearance of the pseudo-Goldstone bosons. It is easy to see that the relation can not hold for values of n, other than 3.

For the unitarity group SU(n), relation (5.2) also holds in the case n=3 for the adjoint representation, octet representation. In this case the higher symmetry is O(8), and the number of pseudo-Goldstone bosons is either 3 or 1. These are the only cases we have found so far if one uses only one irreducible representations. Suppose we have two representations ϕ_1 and ϕ_2 of some group G, such that the potential $V(\phi_1, \phi_2)$ is invariant under G transformations on ϕ_1 or ϕ_2 separately. Then V necessarily has higher symmetry $G \times G$, consisting of independent transformation on ϕ_1 and ϕ_2. This is called unlocking of representations. We have found that in case of O(N) group, if we have spinor representation ϕ_i and vector representation χ_i and if we impose a discrete symmetry $\chi \to -\chi$ then the only coupling between χ and ϕ is of the form $(\phi^+\phi)(\chi_i\chi_i)$. This is invariant under rotation on ϕ or χ separately. We then have unlocking of the spinor representation and the vector representation.

So far we talk only about scalar and vector bosons. The fermions can be included very easily. The most general Lagrangian of the fermion is of the form

$$\mathscr{L}_{\psi} = i \bar{\psi} \gamma^{\mu} D_{\mu} \psi + m \bar{\psi}\psi + f \bar{\psi} \Gamma_i \psi \phi_i$$

where the last term is the Yukawa coupling between fermion and the scalars. This term is the one which is responsible for splitting up the fermion multiplet when the system undergos spontaneous symmetry breaking. If the representation content of ϕ and ψ are such that this term is not present, the fermion multiplet would not know the symmetry breaking in the zero order, and their masses will have higher symmetries. Since this higher symmetry is only special to the fermions, it will be broken by higher order corrections. Again the renormalizability face the mass difference generated from the higher order effect to be finite and calculable.

In the O(n) and SU(n) group, this term is absent if both fermions and the scalars belong to the vector representation except in O(3).

VI. DISCUSSIONS

We have studied all the symmetry breaking patterns in the general O(n) and SU(n) group for all the representations up to the second rank tensors. The results are summerized in Table 3.

Among these results we have obtained so far, the familiar groups O(3) and SU(3) seem to have the special feature in the appearance of the pseudo-Goldstone bosons. For this reason, it seems to be very promising to construct models based on groups which are products of O(3) or SU(3) with some other groups.

This paper which deals with the most general group structure of the gauge theories, can be looked upon as the first step toward building the models. To go further, one has to assign the fermions, both leptons and quarks, to some representations of the groups and study their selection rules. Of course, there must be a large degree of freedom in the choices within the present data. Before the future experiments can nail down the correct group to use, the sensible criterion would be whether the models offer any insight into those mysteries in the weak interactions, like the origin of the Cabbibo angle, the ratio of the muon mass to the electron mass, etc.

In nature, only the U(1) gauge symmetry, corresponding to the electromagnetic interactions, is exact. So one would like to break the symmetry down to U(1). As seen in the table, this situation does not happen very often. However from the work of S. Coleman and E. Weinberg,[9] the symmetry can also be broken spontaneously by the higher order radiative corrections. This gives the possibility of breaking the symmetry in two stages; one starts from a big group G_1 and breaks down to a smaller group G_2 through Higgs mechanism and then breaks further down to the final U(1) symmetry a la Coleman and E. Weinberg. This kind of scheme is very attractive because in first stage, the symmetry breaking effect is usually very large and the second stage symmetry breaking due to the radiative correction, is usually small so that the G_2 group can be used to explain the approximate symmetries, like SU(2) or SU(3) seen in nature. It would be very interesting to see how this scheme can be carried out.

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APPENDIX A

In this appendix, we discuss some general aspects of the spinor representation. This class of representations can be understood most easily in terms of Dirac spinors.

The rotation group O(n) can be considered as those linear transformations on the coordinate $x_1, x_2 \cdots x_n$, such that the quadratic form, $x_1^2 + x_2^2 + \cdots + x_n^2$ is invariant. Now if we want to write this quadratic form as square of a linear form of x_i's

$$x_1^2 + x_2^2 + \cdots + x_n^2 = (\gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n)^2$$

we have to require

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} \tag{A.1}$$

Clearly these γ_i's have to be matrices in order to anticommute with each other. These are the generalized Dirac matrices. The algebra of (A.1) is known as Clifford algebra. First we discuss the case where n is even $n=2\ell$. One particular representation is of the form

$$\gamma_i = \underbrace{\overbrace{\sigma_z \times \sigma_z \cdots \sigma_z \times \sigma_x}^{i} \times 1 \times 1 \cdots \times 1}_{\ell} \qquad \text{for } i=1 \cdots \ell$$

$$\gamma_{i+\ell} = \sigma_z \times \sigma_z \cdots \sigma_z \times \sigma_y \times 1 \times 1 \cdots \times 1 \qquad i=1 \cdots \ell \tag{A.2}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It is easy to check that these expressions for the γ_i's satisfy the anticommutation relations (A.1) by using the rule $(A_1 \times B_1) \cdot (A_2 \times B_2) = (A_1 A_2) \times (B_1 B_2)$. It can be shown that these γ's form a complete matrix algebra in the space of

2^{ℓ} dimension. Now consider a rotation in the coordinate space

$\chi'_i = O_{ik}\,\chi_k$ where O is an orthogonal matrix, i.e., $OO^T = 1$

This rotation induces a transformation on the γ_i matrix,

$$\gamma'_i = O_{ik}\,\gamma_k \tag{A.3}$$

It is easy to see that the anticommutation relations remain unchanged, i.e.,

$$\gamma'_i\gamma'_j + \gamma'_j\gamma'_i = O_{ik}O_{j\ell}(\gamma_i\gamma_j + \gamma_j\gamma_i) = 2\delta_{ij} \tag{A.4}$$

However the original set of γ-matrices form a complete matrix algebra, the new set of γ-matrices must be related to the original set by a similarity transformation,

$$\gamma'_i = S(O)\,\gamma_i\,S^{-1}(O)$$

or

$$O_{ik}\,\gamma_k = S(O)\,\gamma_i\,S^{-1}(O) \tag{A.5}$$

The correspondence $O \rightarrow S(O)$ serves as a representation of the rotation group. This is called the spinor representation of O(n). The quantities ψ_i, which transform like

$$\psi_i \rightarrow \psi'_i = S(O)_{ij}\,\psi_j \tag{A.6}$$

are called covariant spinors. Their complex conjugate ψ^*_i has the transformation property

$$\psi'^*_i = \psi^*_j\,S^{-1}_{ji}(O) \tag{A.7}$$

are called contravariant spinors. From Eqs. (A.5-7), we can construct the following bilinears just like the Dirac spinors,[10] which have the same transformation properties as the tensors seen in Table 4. For an infinitesimal rotation we can parametrize O_{ik} and S(O) by

$$O_{ik} = \delta_{ik} + \epsilon_{ik} \qquad \text{with } \epsilon_{ik} = -\epsilon_{ki}$$

$$S(O) = 1 + i\,S_{ij}\,\epsilon^{ij}$$

Then Eq. (A.5) implies that

$$i\,[S_{k\ell}, \gamma_i] = \frac{1}{2}\,(\delta_{ki}\gamma_\ell - \delta_{\ell i}\gamma_k) \tag{A.8}$$

It is not hard to see that

$$S_{k\ell} = \frac{i}{4}\,\sigma_{k\ell} = \frac{i}{8}\,[\gamma_k, \gamma_\ell] \tag{A.9}$$

and

$$S(\epsilon) = 1 - \frac{1}{8}\,[\gamma_k, \gamma_\ell]\,\epsilon^{k\ell}$$

If one expresses the parameter ϵ_{ik} in terms of rotation angle, one can see that $S(4\pi) = 1$, i.e., S(O) is a double-valued representation of O(n). From these transformation properties, we can work out the covariant derivative as

$$\partial_\mu \psi_i - \frac{i}{4}\,g\,W^\mu_{jk}\,(\sigma_{jk}\psi)_i$$

For the construction of the fourth order invariant potential, we can contract the vector with vector, or second rank tensor with second rank tensor, etc., just like the case of 4-fermion weak interaction Lagrangian. In general we would have (n + 1) quartic terms in the potential. However because all the 4 spinors all identical, not all these (n + 1) terms are independent in contrast to the case of weak interactions. It turns out the number of independent quartic terms increases with the dimension of the space.[8]

APPENDIX B

In this appendix we give the details of the solutions of the minimum in the case of second rank symmetric tensor in the O(n) group and also consider the case with the cubic term $\mathrm{Tr}\phi^3$.

The minimum for the case without the cubic term is given by

$$\frac{\partial V}{\partial \phi_i} = -\mu^2\phi_i + \lambda_1\left(\sum_{j=1}^{n} \phi_j^2\right)\phi_i + \lambda_2\phi_i^3 - g = 0 \qquad i=1\cdots n \tag{B.1}$$

with

$$\sum_{i=1}^{n} \phi_i = 0$$

As we mentioned in the text these ϕ's, which satisfy (B.1), can take at most three different values. This can be seen as follows. Suppose there are three different ϕ's say ϕ_1, ϕ_2, ϕ_3, they have to satisfy Eq. (A.1)

$$\begin{aligned}
&-\mu^2\phi_1 + \lambda_1\left(\sum_{j=1}^{n} \phi_j^2\right)\phi_1 + \lambda_2\phi_1^3 - g = 0 \\
&-\mu^2\phi_2 + \lambda_1\left(\sum_{j=1}^{n} \phi_j^2\right)\phi_2 + \lambda_2\phi_2^3 - g = 0 \\
&-\mu^2\phi_3 + \lambda_1\left(\sum_{j=1}^{n} \phi_j^2\right)\phi_3 + \lambda_2\phi_3^3 - g = 0
\end{aligned} \tag{B.2}$$

By subtracting one equation from the others, we get

$$\begin{aligned}
&\left[-\mu^2 + \lambda_1\left(\sum_{j=1}^{n} \phi_j^2\right) + \lambda_2(\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\right] = 0 \\
&\left[-\mu^2 + \lambda_1\left(\sum_{j=1}^{n} \phi_j^2\right) + \lambda_2(\phi_1^2 + \phi_1\phi_3 + \phi_3^2)\right] = 0
\end{aligned} \tag{B.3}$$

where we have used the fact that $\phi_1 \neq \phi_3$. Substracting again one equation from the other, we get

$$\phi_1 + \phi_2 + \phi_3 = 0 \tag{B.4}$$

If we have another solution ϕ_4, which is different from ϕ_1, ϕ_2 and ϕ_3, by using the same subtraction procedure among ϕ_2, ϕ_3 and ϕ_4, we get

$$\phi_2 + \phi_3 + \phi_4 = 0 \tag{B.5}$$

This implies $\phi_1 = \phi_4$, contradicting the assumption that $\phi_1 \neq \phi_4$. So there can be only three different ϕ_i's which sum to zero. Let us write ϕ matrix in the form

$$\phi = \begin{pmatrix} \overbrace{\phi_1 \cdot \cdot \phi_1}^{n_1} & & 0 \\ & \overbrace{\phi_2 \cdot \cdot \phi_2}^{n_2} & \\ 0 & & \overbrace{\phi_3 \cdot \cdot \phi_3}^{n_3} \end{pmatrix}$$

with

$$n_1 + n_2 + n_3 = n$$

$$\phi_1 + \phi_2 + \phi_3 = 0 \tag{B.6}$$

$$n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0 \tag{B.7}$$

The last equation is due to the trace condition $\mathrm{Tr}\phi = 0$. From (B.6) and (B.7) we can solve ϕ_2 and ϕ_3 in terms of ϕ,

$$\phi_2 = \frac{n_3 - n_1}{n_2 - n_3}\ \phi_1 \tag{B.8}$$

$$\phi_3 = \frac{n_2 - n_1}{n_3 - n_2}\, \phi_1 \tag{B.9}$$

Now we can go back to the potential $V(\phi)$ to express every term in terms of ϕ, and minimize with respect to ϕ_1;

$$V = -a\,\phi_1^2 + b\,\phi_1^4 \tag{B.10}$$

with

$$a = \frac{\mu^2}{2}\,\frac{1}{(n_2-n_3)^2}\left[n_1(n_2 - n_3)^2 + n_2(n_1 - n_3)^2 + n_3(n_1 - n_2)^2\right] \tag{B.11}$$

$$\begin{aligned} b = \frac{1}{4(n_2 - n_3)^4}\Big\{\lambda_1\left[n_1(n_2 - n_3)^2 + n_2(n_1 - n_3)^2 + n_3(n_1 - n_2)^2\right]^2 \\ + \lambda_2\left[n_1(n_2 - n_3)^4 + n_2(n_1 - n_3)^4 + n_3(n_1 - n_2)^4\right]\Big\} \end{aligned} \tag{B.12}$$

It is very easy to see that the potential at the minimum is

$$V_m = -\frac{\mu^4}{4}\left[\frac{1}{\lambda_1+\lambda_2 f(n_1, n_2, n_3)}\right] \tag{B.13}$$

with

$$f(n_1, n_2, n_3) = \frac{n_1(n_2-n_3)^4 + n_2(n_1-n_3)^4 + n_3(n_1-n_2)^4}{[n_1(n_2-n_3)^2 + n_2(n_1-n_3)^2 + n_3(n_1-n_2)^2]^2} \tag{B.14}$$

Now we look for the values of n_1, n_2, n_3, which give the smallest minimum, corresponding to the ground state of the system. From the expression (B.12), we see that if $\lambda_1 > 0$, $\lambda_2 > 0$, the smallest V_m corresponds to the minimum of $f(n_1, n_2, n_3)$, and if $\lambda_1 > 0$, $\lambda_2 < 0$, the smallest V_m corresponds to the maximum of $f(n_1, n_2, n_3)$. By using the identity

$$n_1(n_2-n_3)^4 + n_2(n_1-n_3)^4 + n_3(n_1-n_2)^4 = \frac{1}{2}\left[n_1(n_2-n_3)^2 + n_2(n_1-n_3)^2 + n_3(n_1-n_2)^2\right]\left[(n_2-n_3)^2 + (n_1-n_3)^2 + (n_1-n_2)^2\right]$$

we can reduce $f(n_1, n_2, n_3)$ to a simpler form

$$f(n_1, n_2, n_3) = \frac{(n_2-n_3)^2 + (n_1-n_2)^2 + (n_1-n_3)^2}{2\left[n_1(n_2-n_3)^2 + n_2(n_1-n_3)^2 + n_3(n_1-n_2)^2\right]} \tag{B.15}$$

We introduce the variables $x \equiv n_1 + n_2$, $y \equiv n_1 - n_2$ to rewrite $f(n_1, n_2, n_3)$ as

$$f(x, y) = f(n_1, n_2, n_3) = \frac{3y^2 + (3x-2n)^2}{(8n - 9x)y^2 + x(3x-2n)^2} \tag{B.16}$$

This is an even function of y, we can consider positive y only. The allowed domain for x, y, are

$$0 < x \leq n \qquad 0 < y \leq n \qquad 0 < x - y \leq n$$

the derivative with respect to y is as shown in Fig. 2.

$$\frac{\partial f}{\partial y} = \frac{8y(3x-2n)^3}{\left[(8n-9x)\, y^2 + x(3x-2n)^2\right]^2}$$

Hence for $x \neq \frac{2}{3}n$, f is monotonically increasing or decreasing function of y, the extremum must be the boundaries, y=0 or y=x. Since $\frac{\partial f}{\partial y}$ has the same sign as $(3x-2n)$, the minimum must be on the lines y=x for $x < \frac{2}{3}n$ and y=0 for $\frac{2n}{3} < x < n$. Along the line y=x f is given by

$$f(x, x) = \frac{3x^2 - 3nx + n^2}{xn(n-x)}$$

and has a minimum at $x = \frac{n}{2}$ with value $f = \frac{1}{n}$. Along the other line y=0, we have

$f = \frac{1}{x}$ with minimum at $x=n$ with same value $f=\frac{1}{n}$. Actually these two points $x = y = \frac{n}{2}$ and $x = n$, $y = 0$, which correspond to $n_1 = \frac{n}{2}$, $n_2 = 0$, $n_3 = \frac{n}{2}$ and $n_1 = \frac{n}{2}$, $n_2 = \frac{n}{2}$, $n_3 = 0$ respectively, are equivalent because function $f(n_1, n_2, n_3)$ is symmetric in n_1, n_2 and n_3. Therefore for the case n is even the minimum for $f(n_1, n_2, n_3)$ is at $n_1 = n_2 = \frac{n}{2}$. For the case n is odd since $n_1 = n_2 = \frac{n}{2}$ is not allowed, we have to look at the nearby points. It turns out that the minimum is at $n_1 = \frac{1}{2}(n+1)$, $n_2 = \frac{1}{2}(n-1)$ and $n_3 = 0$. To get the maximum of $f(n_1, n_2, n_3)$, the analysis is very similar. We get the results that for $n_1 = n - 1$, $n_2 = 1$, $n_3 = 0$, f is the maximum, hence V is at minimum.

Now let us consider the invariant potential with the cubic term

$$V = -\frac{\mu^2}{2} \sum_{i=1}^{n} \phi_i^2 + \frac{\lambda_1}{4} \left(\sum_{i=1}^{n} \phi_i^2 \right)^2 + \frac{\lambda_2}{4} \sum_{i=1}^{n} \phi_i^3 - g \sum_{i=1}^{n} \phi_i \tag{B.17}$$

The condition for the minimum is then

$$\frac{\partial V}{\partial \phi_i} = -\mu^2 \phi_i + \lambda_1 \left(\sum_{j=1}^{n} \phi_j^2 \right) \phi_i + \lambda_2 \phi_i^3 + \lambda_3 \phi_i^2 - g = 0 \tag{B.18}$$

We still have only three values for ϕ_i's, ϕ_1, ϕ_2, ϕ_3, which satisfy the condition,

$$\phi_1 + \phi_2 + \phi_3 = -\frac{\lambda_3}{\lambda_2} \tag{B.19}$$

instead of the simple relation (B.4). Combining this equation with the trace condition (B.7), we solve for ϕ_2 and ϕ_3 in terms of ϕ_1,

$$\phi_2 = \frac{1}{(n_3 - n_2)} [\lambda n_3 + (n_1 - n_3)\phi_1] \tag{B.20}$$

$$\phi_3 = \frac{1}{(n_2 - n_3)} [\lambda n_2 + (n_1 - n_2)\phi_1] \tag{B.21}$$

with

$$\lambda \equiv \frac{\lambda_3}{\lambda_2}$$

Because these relations are not homogeneous, the calculation becomes very complicated. Instead of a simple form like (B.10), we have

$$V = a\,\phi_1^4 + b\,\phi_1^3 + c\,\phi_1^2 + d\,\phi_1 + e \tag{B.22}$$

where a, b, c, d and e are functions of n_i's and λ_i's. Its derivative is of the form

$$\frac{\partial V}{\partial \phi_1} = 4a\,\phi_1^3 + 3b\,\phi_1^2 + 2c\,\phi_1 + d = 0 \tag{B.23}$$

We have to solve this cubic equation for ϕ_1 and substitute it in the potential $V(\phi_1)$ to find the value of n_1, n_2, n_3, where $V(\phi)$ is the smallest. This computation is straightforward but very tedious. We only give the results here.

(a) For $\lambda_1 > 0$, $\lambda_2 > 0$, we consider the variation with respect to λ_3. At $\lambda_3 = 0$, we know that O(n) splits into two "almost" even pieces, i.e.,

$$O(n) \to O(n_1) \times O(n-n_1)$$

$$n_1 = \frac{n}{2} \qquad \text{n even}$$

$$n_1 = \frac{n+1}{2} \qquad \text{n odd.}$$

As we increase λ_3, either in positive or negative direction, the minimum starts to shift toward the pattern where O(n) splits into two most uneven pieces, i.e.,

$$O(n) \to O(n-1).$$

(b) For $\lambda_1 > 0$, $\lambda_2 < 0$, the minimum is very stable against the variation of λ_3, i.e.,

$$O(n) \to O(n-1) \qquad \text{for all } \lambda_3.$$

APPENDIX C

In this appendix, we show how to get the solution for the second rank tensor ψ, either symmetric or antisymmetric, if we know that $X = \psi\psi^* = cI$, where I is the $n \times n$ matrix.

(a) Symmetric Tensor

We have $\psi^T = \psi$ and $\psi\psi^* = cI$. Since the matrix ψ is in general complex, we express ψ in terms of its real and imaginary part, $\psi = A + iB$, where A and B are real matrices. Because ψ is symmetric $\psi = \psi^T$. A and B are also symmetric, $A^T = A$, $B^T = B$. In terms of A and B, the condition that $\psi\psi^* = cI$ becomes

$$A^2 + B^2 = cI \tag{C.1}$$

$$AB = BA \tag{C.2}$$

So A and B commute with each other, we can diagonalize these two real and symmetric matrices by a real orthogonal transformation, which is automatically unitary in the complex space. Therefore A and B can be chosen to be of the form

$$A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix} \tag{C.3}$$

with

$$a_j^2 + b_j^2 = c$$

Then the matrix ψ is of the form

$$\psi = A + iB = \begin{pmatrix} a_1 + ib_1 & & & 0 \\ & a_2 + ib_2 & & \\ & & \ddots & \\ 0 & & & a_n + ib_n \end{pmatrix} = \sqrt{c} \begin{pmatrix} e^{i\alpha_1} & & & 0 \\ & e^{i\alpha_2} & & \\ & & \ddots & \\ 0 & & & e^{i\alpha_n} \end{pmatrix} \tag{C.4}$$

where we have defined $a_j + ib_j = \sqrt{c}\, e^{i\alpha_j}$. Remember that under the group transformation of U(n), $\psi \to U\psi U^T$. We can use this property to write

$$\psi = U\psi' U^T$$

with

$$\psi = \sqrt{c}\, I \qquad \text{and} \qquad U = \begin{pmatrix} e^{i\alpha_1/2} & & & 0 \\ & e^{i\alpha_2/2} & & \\ & & \ddots & \\ 0 & & & e^{i\alpha_{n/2}} \end{pmatrix}$$

Hence $\psi' =$ c I is the most general solution.

(b) Antisymmetric Tensor

For even n, n=2ℓ, we have $\psi^T = -\psi$ and $\psi\psi^* = cI$. Again we define two real matrix A and B by $\psi \equiv A + iB$, and A, B are antisymmetric, $A = -A^T$, $B = -B^T$. They satisfy the same conditions as before

$$A^2 + B^2 = cI \tag{C.5}$$

$$AB = BA \tag{C.6}$$

We want to transform A and B to the stand form for the real antisymmetric matrix. Since A and B are antisymmetric, (iA) and (iB) are hermitian and commute with each other because of (C.6). We can diagonalize them simultaneously by unitary transformation,

$$iA = U(iA_d)\, U^+, \qquad iB = U(iA_d)U^+ \tag{C.7}$$

with

$$(iA)_d = \begin{pmatrix} -a_1 & & & & & & 0 \\ & a_1 & & & & & \\ & & -a_2 & & & & \\ & & & a_2 & \ddots & & \\ & & & & & -a_\ell & \\ 0 & & & & & & a_\ell \end{pmatrix} \quad (iB)_d = \begin{pmatrix} -b_1 & & & & & & 0 \\ & b_1 & & & & & \\ & & -b_2 & & & & \\ & & & b_2 & \ddots & & \\ & & & & & -b_\ell & \\ 0 & & & & & & b_\ell \end{pmatrix} \text{ for } n=2\ell \tag{C.8}$$

The eigenvalues, which are real, have to occur in pairs, because det $|\lambda I - iA| = 0$ implies det $|-\lambda I - iA| = 0$ if $A = -A^T$ and n even. To get to the standard antisymmetric form, we can use the matrix $k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ which has the property that

$$k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} k^+ = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{C.9}$$

So we can write A and B in the form

$$A = (UK)\, A_S (UK)^+ \qquad B = (UK)\, B_S (UK)^+ \tag{C.10}$$

with

$$K = \begin{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \end{pmatrix}$$

$$A_S = \begin{pmatrix} a_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & a_\ell \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}, \qquad B_S = \begin{pmatrix} b_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & b_\ell \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \tag{C.11}$$

But we know that any real antisymmetric matrix can be brought into the standard form (C.11) by a real orthogonal transformation. This means that the combination (UK) must be real, i.e., $(UK)^+ = (UK)^T$ and the matrix ψ can be written as

$$\psi = V \psi' V^T$$

where

$$\psi' = \begin{pmatrix} (a_1 + ib_1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & (a_\ell + ib_\ell) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \tag{C.12}$$

Just like the previous case, this matrix ψ' can be written as

$$\psi' = U' \psi'' U'^T$$

where

$$\psi'' = c \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \qquad U' = \begin{pmatrix} e^{i\alpha_1/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \\ & \ddots & \\ & & e^{i\alpha_\ell/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{C.13}$$

So the ψ can be written as the standard form up to a group rotation.

For the case n is odd, $n = 2\ell + 1$, we can use the same analysis with the obvious modification of adding a zero in the diagonal.

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TABLE 1

Properties of various representations in O(n)

Representation	Dimension	Transformation Law	Covariant Derivatives $D_\mu \phi$
vector	n	$\phi_i \rightarrow \phi_i + \epsilon_{ij}\phi_j$	$\partial^\mu \phi_i - gW^\mu_{ij}\phi_j$
2nd rank tensor (sym.)	$\frac{n(n+1)}{2} - 1$	$\phi'_{ij} \rightarrow \phi'_{ij} + \epsilon_{ik}\phi'_{kj} + \epsilon_{jk}\phi'_{ik}$	$\partial^\mu \phi'_{ij} - gW^\mu_{ik}\phi'_{kj} - gW^\mu_{jk}\phi'_{ik}$
2nd rank tensor (antisym.)	$\frac{n(n-1)}{2}$	$\phi_{ij} \rightarrow \phi_{ij} + \epsilon_{ik}\phi_{kj} + \epsilon_{jk}\phi_{ik}$	$\partial^\mu \phi_{ij} - gW^\mu_{ik}\phi_{kj} - gW^\mu_{jk}\phi_{ik}$
spinor	2^ℓ $n=2\ell$ or $n=2\ell+1$	$\chi_i \rightarrow \chi_i - \frac{i}{4}\epsilon_{jk}(\sigma^{jk}\chi)_i$	$\partial^\mu \chi_i - \frac{i}{4} gW^\mu_{jk}(\sigma_{jk}\chi)_i$

TABLE 2

Properties of various representations in SU(n)

Representation	Dimension	Transformation Law	Covariant Derivatives $D_\mu \phi$
vector	n	$\psi_i \rightarrow \psi_i + i\epsilon_i^j \psi_j$	$\partial_\mu \psi_i - igW_{\mu i}^{\ j} \psi_j$
		$\psi^i = (\psi_i)^*, \ \epsilon_i^j = (\epsilon_j^i)^*$	
2nd rank tensor (symmetric)	$\frac{n(n+1)}{2}$	$\psi'_{ij} \rightarrow \psi'_{ij} + i\epsilon_i^k \psi'_{kj} + i\epsilon_j^k \psi'_{ik}$	$\partial_\mu \psi'_{ij} - igW_{\mu i}^{\ \ell} \psi'_{\ell j} - igW_{\mu j}^{\ \ell} \psi'_{i\ell}$
		$\psi'_{ij} = (\psi'^{ij}$, $\psi'_{ij} = \psi'_{ji}$	
2nd rank tensor	$\frac{n(n-1)}{2}$	$\psi_{ij} \rightarrow \psi_{ij} + i\epsilon_i^k \psi_{kj} + i\epsilon_j^k \psi_{ik}$	$\partial_\mu \psi_{ij} - igW_{\mu i}^{\ \ell} \psi_{\ell j} - igW_{\mu j}^{\ \ell} \psi_{i\ell}$
		$\psi_{ij} = -\psi_{ji}$	
adjoint rep.	$n^2 - 1$	$\psi_i^j \rightarrow \psi_i^j + i\epsilon_i^k \psi_k^j - i\epsilon_k^j \psi_i^k$	$\partial_\mu \psi_i^j - igW_{\mu i}^{\ \ell} \psi_\ell^j + igW_{\mu k}^{\ j} \psi_i^k$
		$\psi_i^j = (\psi_j^i)^*$, $\psi_i^i = 0$	

TABLE 3

Summary of the pattern of symmetry breaking

Representation	O(n)	SU(n)
vector	O(n)	SU(n-1)
k vectors	O(n-k)	SU(n-k)
2nd rank symmetry tensor	O(n-1) or $O(\ell) \times O(n-\ell)$ $\ell = \left[\frac{n}{2}\right]$	SU(n-1) or O(n)
2nd rank anti-symmetric tensor	$U(\ell)$ or $U(1) \times O(n-2)$	$O(2\ell+1)$ or SU(n-2) $\ell = \left[\frac{n}{2}\right]$
adjoint representation		$SU(\ell) \times SU(n-\ell) \times U(1)$ or SU(n-1) $\ell = \left[\frac{n}{2}\right]$

TABLE 4

List of the bilinear spinors

scalar	$\Gamma_0 = \psi^+ \psi$	$\Gamma'_0 = \Gamma_0$	
vector	$\Gamma_i = \psi^+ \gamma_i \psi$	$\Gamma'_i = O_{ik} \Gamma_k$	
2nd rank antisymmetric tensor	$\Gamma_{i_1 i_2} = \psi^+ \gamma_{i_1} \gamma_{i_2} \psi$	$\Gamma'_{i_1 i_2} = O_{i_1 k_1} O_{i_2 k_2} \Gamma_{k_1 k_2}$	$i_1 \neq i_2,\ k_1 \neq k_2$
3rd rank antisymmetric tensor	$\Gamma_{i_1 i_2 i_3} = \psi^+ \gamma_{i_1} \gamma_{i_2} \gamma_{i_3} \psi$	$\Gamma'_{i_1 i_2 i_3} = O_{i_1 k_1} O_{i_2 k_2} O_{i_3 k_3} \Gamma_{k_1 k_2 k_3}$	$i_1 \neq i_2 \neq i_3$
	$\Gamma_{i_1, i_2 \cdots in} = \psi^+ \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{in} \psi$	$\Gamma'_{i_1, i_2 \cdots i_n} = O_{i_1 k_1} O_{i_2 k_2} \cdots O_{i_n k_n} \Gamma_{k_1 \cdots k_n}$	

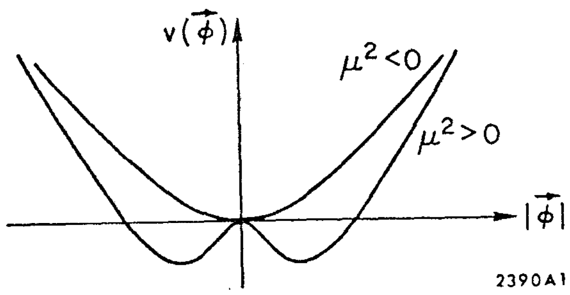

Fig. 1

The classical potential for the spontaneously broken symmetry.

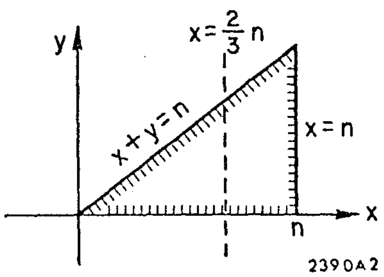

Fig. 2

The domain for the function f(x, y) in Eq. B.16).