On Stellar Structure and Evolution

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Abstract
In this paper, an exposition has been provided on the structure and evolution of stars based on the major functionality on their initial mass. First, we deal with the role of the radiation pressure and its contribution in the overall equilibrium of the stellar stability, then we discuss about the formation of white dwarfs and Newtonian polytropes, then to Neutron stars and pulsars and finally we touch about the black-hole singularity.

Key Words: Radiative transfer; Newtonian Polytropes; White Dwarf; Neutron Star; Black-holes.

1. Introduction

As to a known fact, like humans, stars have a similar life cycle except for the fact that their life time is in the scale of millions and their end is majestically spectacular. Now, skipping the basic notion like how a star forms i.e. proto-star, the Jean's instability we will directly jump to their evolution and stability-structure much of which is controlled by their initial mass. The spectral and mass-based classification can be elaborately studied on the basis of Hertzsprung-Russell (H-R) diagram (*annot.)* given below:

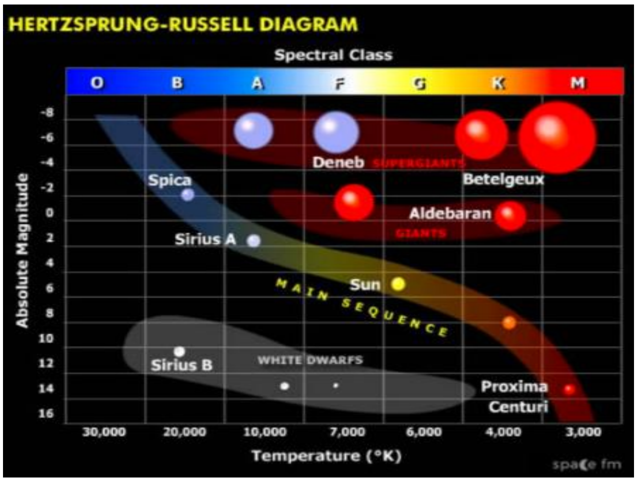

Fig-1

So, based on the classification; a star can be broadly grouped as dwarf, main sequence, giant and super-giant *(annot.)*. Most of our equation of states will be dwindling around main-sequence stars and can also be applied to dwarf and we will eventually be considering the fate of a giant to super-giant which is much more interesting.

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However, to understand the final stages of a giant star we need to consider general relativity. Now, of the current theories of stellar evolution derive its success largely from the fact the following combination of dimensions of a mass provides a correct measure of stellar masses:

$$\left(\frac{hc}{G}\right)^{\frac{3}{2}}\left(\frac{1}{H^2}\right) \approx 29.2 \odot \tag{1}$$

where $\odot$ is the Solar Mass.

2. The Role of Radiation Pressure

The central fact concerning about a normal star is the radiation pressure which plays an important role in the hydrostatic equilibrium of a star *(annot.)*. The equation governing this equilibrium can be written as

$$\frac{dp}{dr} = \frac{GM(r)}{r^2}\rho \tag{2}$$

Where P stands for total pressure, ρ is the density, and M(r) is the mass interior to a sphere of radius r. There are two factors contributing to P. First is due to the material and the other is due to radiation because of thermo-nuclear radiation deep inside stellar core *(annot.)*. On the assumption the matter at such high temp. acts perfectly as Maxwellian , the material or the gas pressure is given as

$$p_{gas} = \frac{k}{\mu H}\rho T, \tag{3}$$

The radiation pressure is given according to the equation

$$P_{rad} = \frac{1}{3}\alpha T^4 \tag{4}$$

Where α is the Stephen's constant. If radiation contribution by a fraction of $(1-\beta)$ then the total pressure is given by

$$P = \frac{1}{\beta}\frac{K}{\mu H}\rho T. \tag{5}$$

The importance of $(1-\beta)$ fraction on stellar pressure was first indicated by Eddington *(annot.)*. We may now express the absolute temperature T in terms of P ,ρ and β as

$T = \left(\frac{K}{\mu H}\frac{3}{\alpha}\frac{1-\beta}{\beta}\right)^{\frac{1}{3}}\rho^{\frac{1}{3}}$ and pressure can be given as

$$P = \left[\left(\frac{K}{\mu H}\right)^4\left(\frac{3}{\alpha}\right)\frac{1-\beta}{\beta}\right]^{\frac{1}{3}}\rho^{\frac{4}{3}} \equiv C(\beta)\rho^{\frac{4}{3}} \text{ (say)} \tag{6}$$

A more rational version of Eddington's argument can in the following sense: There is a general theorem *(annot.)*. which states that the pressure P_C at the centre of the star of mass M is hydrostatic equilibrium in which the density $\rho(r)$ at a point of radial distance r from the centre does not exceed the mean density $\bar{\rho}(r)$,interior to the same point r must satisfy the following inequality :

$$\frac{1}{2}G\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}\bar{\rho}^{\frac{4}{3}}M^{\frac{2}{3}} \leq \rho_C \leq \frac{1}{2}G\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}\rho_C^{\frac{4}{3}}M^{\frac{2}{3}} \tag{7}$$

The right-hand side of (7) along with P from (6) yields for the stable existence of stars, the condition is

$$\left[\frac{\left(\frac{k}{\mu H}\right)^4\left(\frac{3}{\alpha}\right)(1-\beta_C)}{\beta_C}\right]^{\frac{1}{3}} \leq \left(\frac{\pi}{6}\right)^{\frac{1}{3}}GM^{\frac{2}{3}}$$

More evidently

$$M \geq \left(\frac{6}{\pi}\right)^{\frac{1}{2}}\left[\left(\frac{k}{\mu H}\right)^4\left(\frac{3}{\alpha}\right)\frac{1-\beta_C}{\beta_C^4}\right]^{\frac{1}{2}}\left(\frac{1}{G^{\frac{3}{2}}}\right) \tag{8}$$

Where in the foregoing equations $\beta_C \rightarrow \beta$ at the centre of the star. The Stephen's constant can be given accord. To Plank's law

$$\alpha = \frac{8\pi^5k^4}{15h^3c^3}. \tag{9}$$

Inserting this value of α in equation (8) we obtain

$$\mu^2M\left(\frac{\beta_C^4}{1-\beta_C}\right)^{\frac{1}{2}} \geq \frac{(135)^{\frac{1}{2}}}{2\pi^3}\left(\frac{hc}{G}\right)^{\frac{3}{2}}\left(\frac{1}{H^2}\right) = 0.1873\left(\frac{hc}{G}\right)^{\left(\frac{3}{2}\right)}\left(\frac{1}{H^2}\right) \tag{10}$$

We observe that the inequality (10) has isolated the combination (1) of natural const. of the dimensions of a mass; by inserting it's numerical values in equation (1) we obtain the inequality

$$\mu^2M\left(\frac{\beta_C^4}{1-\beta_C}\right)^{\frac{1}{2}} \geq 5.48 \odot. \tag{11}$$

This inequality (11) provides an upper limit to

$$(1-\beta_C) \leq 1-\beta_* . \tag{12}$$

Where $(1-\beta_*)$ is uniquely determined by the stellar mass M and the mean molecular weight μ, by the quadratic equation

$$\mu^2 M = 5.48\left(\frac{\beta_C^4}{1-\beta_C}\right)^{\left(-\frac{1}{2}\right)} \times\odot. \quad (13).$$

Table-1 has been listed for several values of $(1-\beta_*)$ and corresponding $\mu^2 M$ *(annot.)*. From this table we find that for a star of solar masses with mean molecular weight to 1, the radiation pressure at the centre cannot exceed 3 percent of the total pressure.

TABLE-1.

$1-\beta_*$	$\mu^2 M/\odot$	$1-\beta_*$	$\mu^2 M/\odot$
0.01	0.56	0.50	15.49
.03	1.01	.60	26.52
.10	2.14	.70	50.92
.20	3.84	.80	122.5
.03	6.12	.85	224.4

So, from the above discussed details we conclude that the equation (13) is the base equilibrium of an actual stars, to that extent the combination of natural constants (1), providing a mass of proper magnitude for the measurements of stellar masses, is at base of a physical theory of stellar structure.

3. Cooling of Stars

The same combination of natural const.(1) emerged soon afterward in a much more fundamental context of resolving a paradox raised by Eddington in the following aphorism: *"a star needs energy to cool".(annot.)*. The paradox arose while considering the ultimate of a star in the light of then new knowledge of white-dwarfs stars, such as Sirius-B, which have a mean densities in the range $10^5 - 10^7 gm\ cm^{-3}$. R.H Fowler restated the paradox in the following statement: An estimate of the electrostatic energy, Ev ,per unit volume of an assembly of atoms of atomic no. Z ionized down to bare nuclei, is given by

$$E_V = 1.32 * 10^{11} Z^2 \rho^{\frac{4}{3}} \quad (14)$$

While the kinetic energy of the thermal motions, E_{kin} per unit volume of free particles in the form of a perfect gas of density ρ and temperature T, is given by

$$E_{kin} = \left(\frac{3}{2}\right)\left(\frac{k}{\mu H}\right)\rho T = \frac{1.24*10^8}{\mu}\rho T. \quad (15)$$

Now, if such matter were released of the pressure to which it is subjected, it can resume a state of ordinary normal atoms only if

$$E_{kin} > E_v\ , \quad (16)$$

Or, according to equations (14) and (15) only if

$$\rho < \left(0.94 * 10^{-3}\frac{T}{\mu Z^2}\right)^3 \quad (17)$$

This inequality will be clearly be violated if the density is sufficiently high. This is the essence of Eddington's paradox as formulated by R.H Fowler. And the resolution of this paradox also given by Fowler himself in a landmark paper *"On Dense Matter"* in 1926; where for the first time in stellar structure Fermi-Dirac statistics was used.

a. Fowler's Resolution of Eddington's Paradox

In a completely degenerate electron gas all available parts of the phase-space, with momenta less than a certain 'threshold' value P_0- the Fermi 'threshold' are occupied consistently with the Pauli exclusion principle i.e. with two electrons per 'cell' of volume h^3 of six-dimensional phase-space. So, if $n(p)dp$ denotes the no. of electrons per unit volume between p and p+dp, then the assumption of complete degeneracy is equivalent to the assertion

$$n(p) = \begin{cases} \frac{8\pi}{h^3}p^2 & (p \le p_0) \\ 0 & (p > p_0) \end{cases} \quad (18)$$

The threshold momentum p_0, is determined by the normalization condition

$$n = \int_0^{p_0} n(p)dp = \frac{8\pi}{3h^3}p_0^3 \quad (19)$$

Where n denotes the total no. of electrons per unit volume. For the dist. given by (18) and the pressure p and the kinetic energy E_{kin} of the electrons are given by

$$P = \frac{8\pi}{3h^3}\int_0^{p_0} p^3 v_p dp \quad (20)$$

And

$$E_{kin} = \frac{8\pi}{3h^3}\int_0^{p_0} p^2 T_P dp \quad (21)$$

Where v_p and T_P are the velocity and temperature of the electrons having momentum p. If we set $v_p = \frac{p}{m}$ and $T_P = \frac{p^2}{2m}$; appropriate for non-relativistic mechanics in equations (20) and (21) ,we find that

$$P = \frac{8\pi}{15h^3 m} p_0^5 = \left(\frac{3}{40}\right)\left(\frac{3}{\pi}\right)^{\frac{2}{3}}\left(\frac{h^2}{m}\right) n^{\frac{5}{3}}. \quad (22)$$

And

$$E_{KIN} = \frac{8\pi}{10h^3 m} p_0^5 = \frac{3}{40}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}\frac{h^2}{m} n^{\frac{5}{3}}. \quad (23).$$

Fowler's resolution of Eddington's paradox in this : at the temperatures and densities that may be expected to prevail in the interiors of white-dwarf stars, the electrons will be highly degenerate and kinetic energy must be calculated accord. to equation (23) not accord.(15) and equation (23) gives

$$E_{kin} = 1.39 * 10^{13}\left(\frac{\rho}{\mu}\right)^{\frac{5}{3}}. \quad (24)$$

Now comparing equation (24) and (14) we see that for a matter of the density of white-dwarfs namely $\rho \approx 10^5 \text{gm cm}^{-3}$, the total kinetic energy is about two to four times the negative potential-energy ;and Eddington's paradox does not arise *(annot.)*.

4. Theory of White Dwarf and Newtonian Polytropes

On this account, finite equilibrium configurations are predicted for all masses. And it came to be accepted that the white-dwarfs represents the end fate of all stars irrespective of their initial mass. But soon it was realized by Chandrasekhar that at such a high degenerate-compressed core the electrons must have to travel close to the speed of light and special-relativity become important to consider which was also bragged by Pauli's exclusion principle. Inserting the relativistic transformation for velocity and temperature we have

$$V_P = P/\left[m\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}}\right] \text{ and}$$

$$T_P = mc^2\left[\left(1+\frac{p^2}{m^2c^2}\right)^{\frac{1}{2}} - 1\right] \quad (25)$$

Now based on the relativistic transformation one could easily predict the pressure as

$$P = \frac{8\pi c}{3h^3}\int_0^{p_0} p^3 dp = \frac{2\pi c}{3h^3} p_0^4. \quad (26)$$

The relation between P and ρ corresponding to the limiting form (26) is

$$P = K_2 \rho^{\frac{4}{3}} \text{ where } K_2 = \frac{1}{8}\left(\frac{\pi}{3}\right)^{\frac{1}{3}}\frac{hc}{(\mu_e H)^{\frac{4}{3}}} \quad (27).$$

In this limit, the configuration is an Emden polytrope *(annot.)* of index 3. And it is well known that when the polytropic index is 3, the mass of the resulting equilibrium configuration is uniquely determined by the constant of proportionalityK_2. We have accordingly :

$$M_{lm} = 4\pi\left(\frac{K_2}{\pi G}\right)^{\frac{3}{2}}(2.018) = 0.197\left(\frac{hc}{G}\right)^{\frac{3}{2}}\frac{1}{(\mu_e H)^2} = 5.76\mu_e^{-2} \odot \quad (28)$$

In equation (28), 2.018 is a numerical constant derived from the explicit solution of the Lane-Emden equation for $n = 3$.
Thus ; The important conclusions which follow from the foregoing considerations are: *first,* there is an upper limit, to the mass of stars which can become degenerate configurations, as the last stage in their evolution; and *second,* that Stars with $M > M_{lm}$ must have end states which cannot be predicted from the considerations we have presented so far.

b. *Degenerate Stellar Core*

For our present purposes, the principal content is the criterion that for a star to develop degeneracy, it is necessary that the radiation pressure be less than 9.2 percent of total pressure. This last inference is so central to all current schemes of stellar evolution that the directness and the simplicity of the early arguments are worth repeating. The two principal elements of the early arguments were these: *first*, that radiation pressure becomes increasingly dominant as the mass of the star increases; and *second,* that the degeneracy of electrons is possible only so long as the radiation pressure is not a significant fraction of the total pressure - indeed, as we have seen, it must not exceed 9.2 percent of total pressure. The second of these elements in the arguments is a direct and an elementary consequence of the physics of degeneracy. While the evolution of the massive stars was thus left uncertain, there was no such uncertainty regarding the final states of stars of sufficiently low mass." The reason is that by virtue, again, of the inequality (7), the maximum central pressure attainable in a star must be less than that provided by the degenerate equation of state, so long as

$$\frac{1}{2}G\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}M^{\frac{1}{3}} < K_2 = \frac{1}{8}\left(\frac{3}{\pi}\right)^{\frac{1}{3}}\frac{hc}{(\mu_e H)^{\frac{4}{3}}}$$

(29)

Or equivalently

$$M < \frac{3}{16\pi}\left(\frac{hc}{G}\right)^{\frac{3}{2}}\frac{1}{(\mu_e H)^2} = 1.74\mu_e^{-2} \odot \qquad (30)$$

We conclude that there can be no surprises in the evolution of stars of mass less than $\mathbf{1.44}\odot$ for $\mu_e = 1.099$; which is called the ***Chandrasekhar's Limit***. *(annot.).*

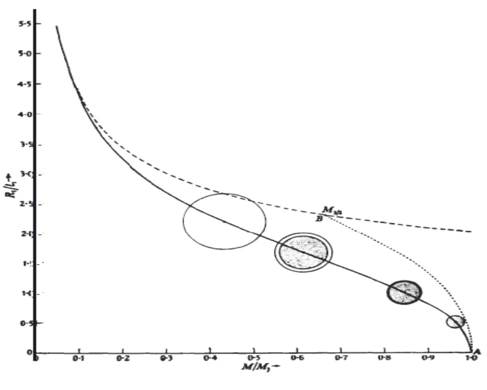

Fig-2 *(annot.)*

The inability of the massive stars to become white dwarfs must result in the development of much more extreme conditions in their interiors and, eventually, in the onset of gravitational collapse attended by the super-nova phenomenon *(annot.).* In the case of less massive stars the degenerate cores, which are initially formed, are not highly relativistic. But the mass of core increases with the further burning of the nuclear fuel at the interface of the core and the mantle; and when the core reaches the limiting mass, an explosion occurs following instability, and it is believed that this is the cause underlying super-nova phenomenon of type-I.

5. Neutron Stars

The first theoretical predictions of neutron stars and their equation of states was carried out by Landau with the application of then modified neutron dynamics but violating Heisenberg's uncertainty principle and letter developed in a fully-fledged quantum-mechanical concept by Baade & Zwicky . The equation of states were first established by Oppenheimer and Volkoff and the limiting mass for a star to become a neutron star is thus came to known as Oppenheimer - Volkoff limit which is close to $2.17\odot$.

5.1 Interior and EOS for a neutron star

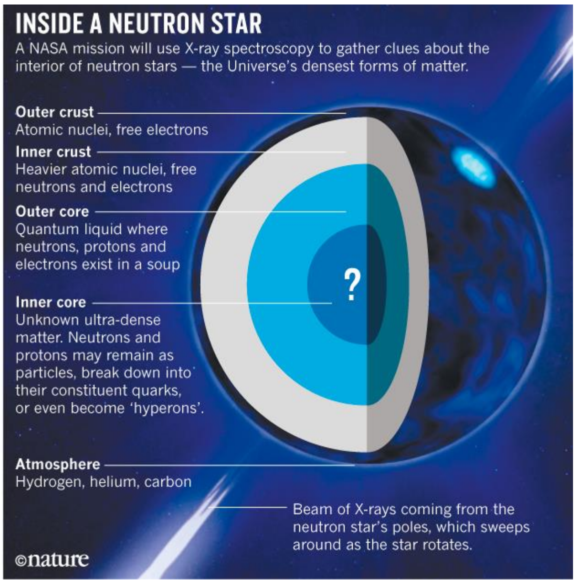

Fig-3

The outer crust:
(the outer envelope) extends from the atmosphere bottom to the layer of the density $\rho = \rho_{ND} \approx 4 * 10^{11}$gm cm^{-3}. Its thickness is some hundred meters. Its matter consists of ions Z and electrons e .A very thin surface layer (up to few meters in a hot star) contains a non-degenerate electron gas. In deeper layers the electrons

constitute a strongly degenerate, almost ideal gas, which becomes ultra-relativistic at $\rho \gg 10^6 \text{gm cm}^{-3}$. The pressure is mainly provided by electrons. In the outer atmosphere layers the ions may constitute a Boltzmann gas, but in deeper layers they form a strongly coupled Coulomb system (liquid or solid). A larger fraction of the envelope is usually solidified; hence, the envelope is often called *the crust*. The electron Fermi energy grows with increasing ρ. This induces beta captures in atomic nuclei and enriches the nuclei with neutrons. At the base of the outer crust the neutrons start to drip out from the nuclei producing a free neutron gas.

The inner crust (the inner envelope):
May be about one kilometre thick. The density ρ in the inner crust varies from ρ_{ND} at the upper boundary to $\sim 0.5\rho_0$ at the base. Here, ρ_0 is the saturation nuclear matter density .The matter of the inner crust consists of electrons, free neutrons n, and neutron-rich atomic nuclei. The fraction of free neutrons increases with growing ρ. The neutronization at $\rho \approx \rho$ND greatly softens the EOS, but at the crust bottom the repulsive short-range component of the neutron-neutron interaction comes into play and introduces a considerable stiffness. In the bottom layers of the crust, the nuclei may become essentially nonspherical and form a “mantle”, but this result is model dependent. The nuclei disappear at the crust-core interface. Free neutrons in the inner crust and nucleons confined in the atomic nuclei can be in superfluid state.

The outer core:
occupies the density range $0.5\rho_0 \leq \rho \leq 2\rho_0$ and is several kilometres thick . Its matter consists of neutrons with several per cent admixture of protons p, electrons, and possibly muons μ (the so called $npe\mu$ composition). The state of this matter is determined by the conditions of electric neutrality and beta equilibrium, supplemented by a microscopic model of many-body nucleon interaction. The beta equilibrium implies the equilibrium with respect to the beta (muon) decay of neutrons and inverse processes. All $npe\mu$-plasma components are strongly degenerate. The electrons and muons form almost ideal Fermi gases. The neutrons and protons, which interact via nuclear forces, constitute a strongly interacting Fermi liquid and can be in superfluid state.

The inner core:
where $\rho \geq 2\rho_0$,, occupies the central regions of massive neutron stars (and does not occur in low-mass stars whose outer core extends to the very centre). Its radius can reach several kilometres, and its central density can be as high as (10−15) ρ_0. Its composition and the EOS are very model dependent. Several hypotheses have been put forward, predicting the appearance of new fermions and/or boson condensates.*(annot.)*.

5.2 Equation of Structure

Since neutron stars are the remnants of massive-super massive stars; the space-time structure around the neutron star is sufficiently deformed so as to apply G.T.R; i.e. the for a locally symmetric field there exist an unique curvature tensor $R^{\lambda}_{\mu\nu\kappa}$ which can be constructed form the metric tensor $g_{\nu\mu}$ and it's first and second derivative. So, for that case the locally inertial gravitational field can be described by Toloman-Oppenhimer-Volkoff (TOV) equation which also tends to give the maximum mass for a neutron star.

Now for a slowly rotating neutron star the metric is given by Hartle-Throne metric *(annot.)*. As Considering a spherically symmetric fluid the metric components are given as

$$c\, d\tau^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2. \qquad (31)$$

now by the perfect-fluid assumption the diagonal of the stress-energy tensor is given as $T^0_0 = \rho c^2$ for energy-density eigenvalue and

$$T^i_j = -P\delta^i_j \qquad (32)$$

for eigenvalue for pressure. To proceed further, we solve Einstein's field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} g - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{c^4} G T_{\mu\nu} \ \ \textit{(annot)} \qquad (33).$$

Now considering the first G_{00} component (33) is modified to

$$\frac{8\pi}{c^4} G\, \rho c^2 e^{\nu} = \frac{e^{\nu}}{r^2}\left(1 - \frac{d}{dr} r e^{-\lambda}\right) \qquad (34)$$

where the value of $T_{\mu\nu}$ is derived from (32); then integrating equation (34) from 0 to r we have

$$e^{-\lambda} = 1 - \frac{2Gm}{rc^2} \qquad (35)$$

where *m(r)* satisfies the $\frac{dm}{dr} = 4\pi r^2 \rho$ condition. Now, considering the G_{11} component we have

$$-\frac{8\pi G}{c^4} P e^{\lambda} = \frac{-r\nu' + e^{\lambda} - 1}{r^2} \qquad (36).$$

Now using the value of e^{λ} from equation (35), equation (36) simplified to

$$\frac{d\nu}{dr} = \frac{1}{r}\left(1 - \frac{2Gm}{c^2 r}\right)^{-1}\left(\frac{2Gm}{c^2 r} + \frac{8\pi G}{c^4} r^2 P\right) \qquad (37).$$

We obtain a second equation by continuity of the stress-energy tensor: $\nabla_\mu T^\nu_\mu = 0$. Putting $\partial_t \rho = \partial_t P = 0$ (static/slowly rotating) and $\partial_\theta P = \partial_\phi P = 0$ (for isotropy) we obtain a particular solution as $0 = \nabla_\mu T^\nu_1 = -\frac{dp}{dr} - \frac{\frac{1}{2}(P+\rho c^2)d\nu}{dr}$; rearranging the terms we have

$$\frac{dP}{dr} = -\left(\frac{\rho c^2 + P}{2}\right)\left(\frac{d\nu}{dr}\right) \qquad (38)$$

Thus, from equations (38) and (37) eliminating $\frac{d\nu}{dr}$, we obtain

$$\frac{dP}{dr} = -\frac{1}{r}\left(\frac{\rho c^2 + P}{2}\right)\left(\frac{2Gm}{c^2 r} + \frac{8\pi G}{c^4} r^2 P\right)\left(1 - \frac{2Gm}{c^2 r}\right)^{-1} \qquad (39)$$

Finally, simplifying equation (39) we have the TOV equation written as

$$\frac{dP}{dr} = -\frac{G}{r^2}\left(\rho + \frac{P}{c^2}\right)\left(m + 4\pi r^3 \frac{P}{c^2}\right)\left(1 - \frac{2Gm}{c^2 r}\right)^{-1} \qquad (40)$$

5.3 Equation of States:

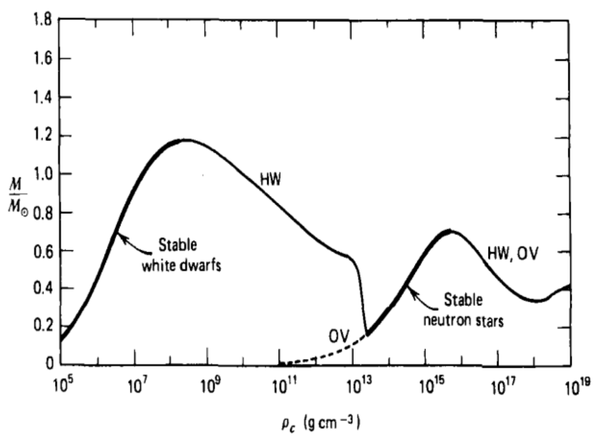

Fig-4

Beta-Equilibrium between Relativistic Electrons and Nuclei: The Harrison-Wheeler (HW) Equation of State-

For a quantitative treatment, we start by writing the energy density of a mixture of nuclei, free electrons, and free neutrons in the form

$$\varepsilon = n_N M(A,Z) + \varepsilon_e'(n_e) + \varepsilon_n(n_n) \qquad (41).$$

The baryon density n and electron density n_e are then given by

$$n = n_N A + n_A \quad, n_e = n_Z Z ;\quad 1 = A\, Y_N + Y_n \;, \quad Y_e = Y_N\, Z. \qquad (42).$$

They used the semi-empirical mass formula of *Green*

$$M(A,Z) = \left[(Z-A)m_n c^2 + Z\left(m_p + m_e\right)c^2 - A\bar{E}_b\right] = m_u c^2 [b_1 A + b_2 A^{\frac{2}{3}} - b_3 Z + b_4 A\left(\frac{1}{2} + \frac{Z}{A}\right)^2 + \frac{b_5 Z^2}{A^{\frac{1}{3}}}] \qquad (43)$$

$b_1, = 0.991749$, $b_3, = 0.000840$, $b_5 = 0.000763$, $b_2, = 0.01911$, $b_4 = 0.10175$.

This expression is based on the "liquid-drop" nuclear model, and the terms have the following interpretation: The dominant contribution to $\bar{E}_b$ is proportional to the volume of the nucleus, as in the case of a liquid drop. The difference of b_1, from unity is due largely to this volume binding energy. Now from equations (41) and (43) we have

$$\varepsilon = n(1 - Y_n)\left(\frac{M(A,Z)}{A}\right) + \varepsilon_e'(n_e) + \varepsilon_n(n_n) \qquad (44)$$

Where, $n_e = \frac{n(1-Y_n)Z}{A}$ and $n_n = nY_n$.

Next; by Wheeler-Harrison approximation we have

$$\frac{\partial M}{\partial Z} = -(E_{Fe} - m_e c^2) \qquad (45).$$

the continuum limit of the P-stability condition in equilibrium with the free electron being at the Fermi sea,

we have

$$A^2 \frac{\partial}{\partial A}\left(\frac{M}{A}\right) = Z(E_{Fe} - m_e c^2) \quad \text{which gives}$$

$$Z\frac{\partial M}{\partial Z} + A\frac{\partial M}{\partial A} - M = 0 \qquad (46).$$

Taking $\frac{\partial \varepsilon}{\partial r_n} = 0$, we have

$$\frac{\partial M}{\partial A} = E_{Fe} \quad (47).$$

Thus, for a continuum version of the condition for M to be in equilibrium with $M(Z, A-1)$, and free electrons we have from (45) as

$$b_3 + b_4\left(1 - \frac{2Z}{A}\right) - 2b_5 \frac{Z}{A^{\frac{1}{3}}} = \left[(1 + x_e^2)^{\frac{1}{2}} - 1\right]\frac{m_e}{m_u} \quad (48)$$

Where $x_e = \frac{P_F^e}{m_e c}$. Thus equation (46) gives

$$Z = \left(\frac{b_2}{2b_5}\right)^{\frac{1}{2}} A^{\frac{1}{2}} = 3.54\, A^{\frac{1}{2}} \quad (49).$$

And from equation (47) we have

$$b_1 + \frac{2b_2 A^{-\frac{1}{3}}}{3} + b_4\left(\frac{1}{4} - \frac{Z^2}{A^2}\right) - \frac{b_5 Z^2}{3A^{\frac{4}{3}}} = (1 + x_n^2)^{\frac{1}{2}}\frac{m_n}{m_u} \quad (50),$$

where $x_n = \frac{P_F^n}{m_n c}$.

Thus, under the restricted conditions of 'neutron drip' we have

$$\rho = \frac{\varepsilon}{c^2} \equiv (n_e M(A,Z)/Z + \varepsilon_e' + \varepsilon_n)/c^2$$

$$P = P_e + P_n \quad \text{and} \quad n = n_e \frac{A}{Z} + n_e \quad (51).$$

Thus equation (51) is the ***ESO for a normal neutron star*** with $P = P(\rho)$ *(annot.* GW170817*).*

6. Pulsars

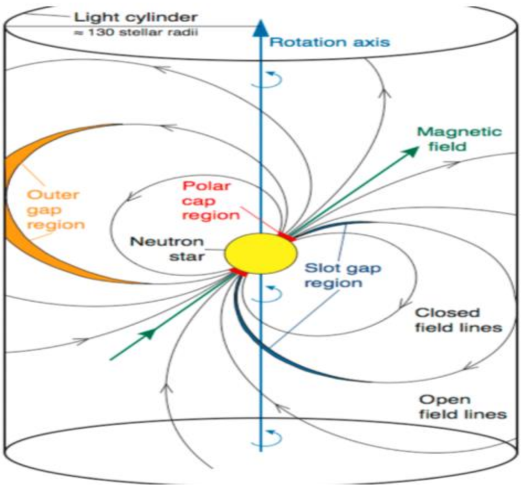

Fig-5

a. Gold's Model of Pulsars (key features)

1. In the intense magnetic field of a pulsar, any charged particle that may escape from it's surface will be constrained to move only along the magnetic field lines as in the fig.
2. The tangential velocity of these whirling particles gradually increases as they move further from the stellar surface.*(velocity of light circle as in the fig-5.).*
3. The relativistic beam near this circle will radiate radio-waves perpendicular to the beam but the velocity of the light circle where the particles moves at nearly the speed of light gradually separates and spreads into the surrounding region creating periodic flashes.

According to this model, any asymmetry in the emission region may be responsible for the observed fine structure in the pulsar.

7. Black-Holes

black hole partitions the four-dimensional space into two regions: an inner region which is bounded by a smooth two-dimensional surface called the *event horizon;* and an outer region, external to the event horizon, which is asymptotically flat; and it is required (as a part of the definition) that no point. A in the inner region can communicate with any point of the outer region. This incommunicability is guaranteed by the impossibility of any light signal, originating in the inner region, crossing the event horizon. The requirement of asymptotic flatness of the outer region is equivalent to the requirement that the black hole is isolated in space and that far from the event horizon the space-time approaches the customary space-time of terrestrial physics. To the extent they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described *exactly* by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. *(annot.).*

Result and Discussion

While considering the structure and evolution of stars, the initial mass of the stellar object plays a very major faction in determining the final state of the star. After reaching the Jean's stability in proto stellar gas, consisting enough mass contracting into itself, igniting sufficient thermo-nuclear reaction to counter balance inward gravitational contraction, the star reaches its hydrostatic equilibrium. Where, the pressure plays a very major role in the balancing the overall geometrical structure of the star. This pressure consist of both from radiation and the matter itself. Exceeding the middle ages, the star exhaust all it's fuel to expand itself into a red-giant, from where, depending on its initial mass, will either shed its outer atmosphere in a planetary nebulae (Chandrasekhar limit) to become a white dwarf, if it exceeds that mass to then within the Oppenheimer-Volkoff limit it converts into a neutron star or pulsar. If it exceeds that mass limit, the inward gravitational contraction become so strong, that the star collapses into itself causing a space-time singularity in the local geometry of the space-time.

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No new data is (are) been analysed or generated in this paper.

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