2020

# Theoretical Studies of C and CP Violation in $\eta \to \pi^+ \pi^- \pi^0$ Decay

Jun Shi
*University of Kentucky*, shijunlj@hotmail.com
Author ORCID Identifier:
https://orcid.org/0000-0001-9484-616X
Digital Object Identifier: https://doi.org/10.13023/etd.2020.388

STUDENT AGREEMENT:

I represent that my thesis or dissertation and abstract are my original work. Proper attribution has been given to all outside sources. I understand that I am solely responsible for obtaining any needed copyright permissions. I have obtained needed written permission statement(s) from the owner(s) of each third-party copyrighted matter to be included in my work, allowing electronic distribution (if such use is not permitted by the fair use doctrine) which will be submitted to UKnowledge as Additional File.

I hereby grant to The University of Kentucky and its agents the irrevocable, non-exclusive, and royalty-free license to archive and make accessible my work in whole or in part in all forms of media, now or hereafter known. I agree that the document mentioned above may be made available immediately for worldwide access unless an embargo applies.

I retain all other ownership rights to the copyright of my work. I also retain the right to use in future works (such as articles or books) all or part of my work. I understand that I am free to register the copyright to my work.

REVIEW, APPROVAL AND ACCEPTANCE

The document mentioned above has been reviewed and accepted by the student's advisor, on behalf of the advisory committee, and by the Director of Graduate Studies (DGS), on behalf of the program; we verify that this is the final, approved version of the student's thesis including all changes required by the advisory committee. The undersigned agree to abide by the statements above.

Jun Shi, Student

Dr. Susan Gardner, Major Professor

Dr. Christopher Crawford, Director of Graduate Studies

# THEORETICAL STUDIES OF C AND CP VIOLATION IN $\eta \to \pi^+\pi^-\pi^0$ DECAY

---

DISSERTATION

---

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Jun Shi
Lexington, Kentucky

Director: Dr. Susan Gardner, Professor of Physics
Lexington, Kentucky
2020


https://orcid.org/0000-0001-9484-616X

ABSTRACT OF DISSERTATION

THEORETICAL STUDIES OF C AND CP VIOLATION IN $\eta \to \pi^+\pi^-\pi^0$ DECAY

A violation of mirror symmetry in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot has long been recognized as a signal of C and CP violation. In this thesis, we show how the isospin of the underlying C- and CP-violating structures can be reconstructed from their kinematic representation in the Dalitz plot. Our analysis of the most recent experimental data reveals, for the first time, that the C- and CP-violating amplitude with total isospin I = 2 is much more severely suppressed than that with total isospin I = 0.

In searching for C- and CP-violating sources beyond the SM, we enumerate the leading-dimension, CP-violating effective operators that share the gauge symmetry and particle content of the Standard Model (SM), carefully separating the operators that are P-odd from those that are C-odd below the electroweak scale. The P-odd and CP-odd effective operators that generate permanent electric dipole moments have been the subject of much investigation in the literature; we now focus on C-odd and CP-odd operators and study their effects systematically. We emphasize that while for flavor-changing interactions the C-odd and CP-odd operators appear in mass dimension 6, for flavor-conserving interactions the C-odd and CP-odd operators appear in mass dimension 8, though some operators can be of mass dimension 6 in numerical effect. In the flavor-changing case, the C-odd and CP-odd operators and P-odd and CP-odd operators probe different linear combinations of common low-energy coefficients in SM effective field theory. Remarkably, however, in the flavor-conserving case, we find that low-energy coefficients probed by a P-odd and CP-odd observable, such as by the permanent electric dipole moment of the neutron, and by a C-odd and CP-odd observable, as probed by mirror symmetry breaking in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot, can be completely different.

Finally, from the C-odd and CP-odd flavor-conserving operators, we determine the operators with definite isospin that can contribute to $\eta \to \pi^+\pi^-\pi^0$ decay. We illustrate how these operators can be represented in chiral perturbation theory, for the eventual determination of their contribution to mirror-symmetry breaking in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot.

KEYWORDS: CP violation, beyond the SM, effective field theory, $\eta \rightarrow \pi^{+}\pi^{-}\pi^{0}$ decay, effective operators

Jun Shi

August 5, 2020

THEORETICAL STUDIES OF C AND CP VIOLATION IN $\eta \to \pi^+\pi^-\pi^0$ DECAY

By
Jun Shi

Dr. Susan Gardner
Director of Dissertation

Dr. Christopher Crawford
Director of Graduate Studies

August 5, 2020
Date

# ACKNOWLEDGMENTS

I would like to first show my deepest gratitude to my supervisor Prof. Susan Gardner. I appreciate the opportunity she kindly offered for continuing my study in University of Kentucky which will leave a indelible and special memory in the rest of my life. Susan is remarkably knowledgeable and shows fantastic enthusiasm in discovering the mysteries of high energy physics and astronomy. It is my great honor to do my research under her direction. It is wholeheartedly appreciated that her great advice for my study proved monumental towards the accomplishment of this project. Her virtuous character, unswerving enthusiasm and fighting spirit will affect my whole life and guide me through my future career.

I also want to show my great thankfulness to other members of my PhD Advisory Committee - Prof. Keh-Fei Liu, Prof. Brad-Plaster and Prof. Anne-Frances Miller. Thank you for all your help and being so nice to me. I appreciate Prof. Jeffery Todd Hastings being the outside examiner of my defense.

I am indebted to all the help from the former DGS Prof. Tim Gorringe, the present DGS Prof. Christopher Crawford in my graduate study. Plenty thanks are given to Diane Riddell, Libby Weir, Suann Reese, Maxwell Ankrah and Max Brown for their kind help. I also want to show my gratitude to the lecture professors that I have assisted, Prof. Michael Eides, Prof. Susan Gardner, Prof. Terrence Draper, Prof. Wolfgang Korsch, Prof. Tim Gorringe and Dr. Ivan Horvath. Thanks to my friends Qing Chen, Sinong Liu, Xinshuai Yan, Gen Wang, Yi-Bo Yang, Xuguang Ai, Tongxi Hu, Austin Hinkel, Nick Floyd, Parisa Alimohamadi and so many others, it has been a wonderful life in University of Kentucky.

Last but not least, I wish to acknowledge the support and great love of my parents, my parents in law, my husband Jian Liang and my son Xiaokuan Liang. They kept me going on and this work would not have been possible without them.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

# INTRODUCTION

Particle physics, also known as high energy physics, is a branch of physics that explores what nature is made of and how it evolves with time at both the smallest microscopic scale and the largest cosmological scale. How a system behaves under certain kinds of transformations plays important roles in particle physics. If a system remains invariant under a certain transformation, we say that the system has a symmetry under that transformation. Symmetries can substantially constrain how a system evolves and interacts with other systems. However, the breaking of a symmetry can be more essential sometimes, mainly due to the fact that it can indicate new physics beyond our knowledge.

## 1.1 Histories of C, P, and T in Particle Physics

Charge conjugation (C), parity (P), and time reversal (T) are three discrete transformations of wide interest in particle physics. A charge conjugation transformation interchanges a particle with its antiparticle, parity relates to space inversion, and time reversal is just as its name implies.

Until the 1950s, all three discrete symmetries were believed to be valid throughout the macroscopic and microscopic world. In 1956, in order to explain the $\tau - \theta$ puzzle, Lee and Yang [1] proposed the idea that parity might not be conserved in the weak interactions, even though it had been verified to be conserved in processes mediated by strong and electromagnetic interactions. They also suggested possible experimental tests [1]. Inspired by Lee and Yang [1], parity breaking in the weak interaction was first experimentally verified in beta decay by Wu et al. in 1957 [2]. Soon after, Garwin, Lederman, and Weinrich [3] verified that both parity and charge conjugation are not conserved in the successive decays $\pi^+ \to \mu^+ + \nu_\mu$ and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$. The Nobel Prize in Physics in 1957 was jointly awarded to Yang and Lee "for their penetrating investigation of the so-called parity laws which has led to important discoveries regarding the elementary particles". After that, people supposed that although charge conjugation and parity could be violated separately, the combined symmetry CP should still be conserved. However in 1964, Cronin and Fitch [4] first discovered CP violation in the decay of neutral K mesons, and their discovery earned them the Nobel Prize in Physics in 1980.

Then we can wonder how about the fate of time reversal symmetry? In 1951, Schwinger [5] first proved the CPT theorem, which states the combined operation of charge conjugation, parity, and time reversal is an exact symmetry of any interaction and that under the three transformations all physical laws must be invariant. Shortly after that, Lüders [6], Pauli [7], and Bell [8] independently gave a more explicit proof, which is based on Lorentz invariance in local quantum field theory with a Hermitian Hamiltonian. We refer to Streater and Wightman [9], pp. 175-176 for further discussion of this early work. As a basic principle in particle physics, the CPT theorem indicates that when CP violation appears, then time reversal is also violated.

These symmetry violating discoveries drastically changed people's perspective on discrete symmetry transformations and aroused great interest in studying violations of CP symmetry.

## 1.2 Baryon Asymmetry in Universe

The Big Bang theory [10, 11, 12, 13] is a cosmological model of the universe describing its origin and subsequent large-scale evolution. It is the leading explanation of how our universe began. According to the Big Bang theory, the beginning of the universe was extremely hot and matter and anti-matter were created and annihilated in pairs in thermal equilibrium [14]. But today, everything we see from our life on Earth to stellar objects in space is made mostly of matter. After the big bang, but before big-bang nucleosynthesis (BBN), it is believed that some yet to be determined mechanism involving the interactions between particle physics and cosmology [15] induced a small but crucial asymmetry of matter and antimatter in the universe at that time [16]. The baryon asymmetry of the universe (BAU) $\eta_B$ is a parameter that can reflect the excess of matter. Since once all the nucleon-antinucleon annihilation and $e^+ - e^-$ annihilation have occurred in the early universe, the number of nucleons and the number of photons in a comoving volume stay conserved [16], thus the ratio of the number of baryons to that of photons stays unchanged. Therefore we can conveniently estimate the BAU by the baryon to photon ratio observed nowadays [14]

$$\eta_B = \frac{N_B}{N_\gamma}\Big|_{T=3K} = \frac{N_B - N_{\overline{B}}}{N_\gamma}\Big|_{T=3K}. \tag{1.1}$$

BBN constrains the possibility of new, light degrees of freedom (dark radiation) expressed as the number of equivalent neutrinos $\Delta N_\nu$, the baryon density $\eta_{10}$, where $\eta_{10} \equiv 10^{10}\eta_B$, and a possible lepton asymmetry $\xi$. From the BBN-predicted primordial abundances, $\eta_B = (5.96 \pm 0.28) \times 10^{-10}$ when taking $\Delta N_\nu = 0 = \xi$, $\eta_B = (6.27 \pm 0.28) \times 10^{-10}$ when only $\xi = 0$, and $\eta_B = (6.01 \pm 0.28) \times 10^{-10}$ when only $\Delta N_\nu = 0$ [16]. The latest Planck experiment [17] determined that $\eta_B$ is about $(6.14 \pm 0.25) \times 10^{-10}$, which is calculated from $\eta_{10} = 10^{10}\eta_B = 273.9\Omega_b h^2$ [18], where $\Omega_b h^2 = 0.0224 \pm 0.0001$ [17] denotes the combination of the critical mass density and the Hubble parameter [18].

The nonzero BAU indicates that before BBN, physical laws for matter and antimatter should act differently. In 1967, Sakharov [15] formulated three necessary conditions to produce a BAU from particle physics interactions:

i) baryon number violation,

ii) C and CP violation,

iii) deviation from thermal equilibrium.

The three Sakharov conditions must be met in order for a mechanism of baryogenesis to be successful. Baryon number violation is obviously a necessary condition to produce an unequal number of baryons over antibaryons. But C violation is also needed so that interactions producing more baryons will not be counterbalanced by interactions with more production of antibaryons. CP violation is also required because otherwise equal numbers of left-handed baryons and right-handed antibaryons would be produced, as well as equal productions would occur for right-handed baryons and left-handed antibaryons. Finally, out of thermal equilibrium is an essential role because otherwise CPT symmetry would assure equal compensations between processes decreasing and increasing the baryon number [19]. There have been some competing hypotheses explaining matter and antimatter through electroweak baryogenesis, but there are not as yet widely agreed explanations on the observed BAU. As said in a research paper [14],"the origin of matter remains one of the great mysteries in physics".

### 1.3 BAU effects from the Standard Model

The Standard Model (SM) of particle physics encapsulates our best understanding of how fundamental particles and three of the four fundamental interactions in the universe – the strong interaction, electromagnetic interaction, and weak interaction – are related to each other. Developed in the 1970s, it has successfully explained almost all experimental observables and given precise predictions to a variety of phenomena.

It is very natural that we would first wonder whether the SM can meet the Sakharov conditions and explain the large BAU. For the first Sakharov condition, baryon number can be violated through anomalies in the SM electroweak interaction [20, 21, 22, 23, 24, 25]. For the second Sakharov condition, the phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix [26] breaks CP and the structure of the charged weak currents show that C and P are broken "maximally". The third Sakharov condition in the SM could also be large enough [27, 24] if a first order weak phase transition in the Coleman-Weinberg scenario [28, 29] with a lighter Higgs took place in the evolution of the universe [30]. So at first glance, it would seem that the SM could fulfill all three Sakharov conditions. However, the weak baryogenesis from the CP violation in the SM gives rise to an upper bound on the magnitude of $\eta_B$ as $|\eta_B| < 10^{-26}$ [31, 32, 33, 19, 34], which is negligibly small compared with the observed BAU value. It has been thought for a long time that the too small values of $\eta_B$ showed that the mechanism of CP violation in the SM was much too restrictive, so that new sources of CP violation should exist. Meanwhile, since the discovery of the Higgs and the determination of its mass it has become clear that the first order phase transition does not exist and the SM does not satisfy the third Sakharov condition [35, 36, 37, 38]. Therefore the result for $\eta_B$ in the SM under the Sakharov conditions is zero.

The failure to explain the observed BAU is one of the primary reasons why people search for new physics beyond the SM, other reasons include no explanation of dark matter or dark energy, not involving gravity, and no neutrino oscillations or origins of neutrino masses. In this thesis we explore the possibility of new sources of C and CP violation.

## 1.4 Why search for CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay

As for the CP-violating effects studied thus far, much effort has been put into flavor-changing meson decays and into searches for permanent electric dipole moments (EDMs), the latter being promising ways to test for CP violation beyond the SM. Flavor-changing meson decays, like $B$, $D$, and $K$ decays, are loosely consistent with the SM [39, 40, 41]. An EDM breaks P and T, which breaks CP if CPT symmetry is assumed. But the existing experimental measurements continue to find null results, for example the latest neutron EDM is tested to be $d_n = (0.0 \pm 1.1_{\text{stat}} \pm 0.2_{\text{sys}}) \times 10^{-26}\text{e} \cdot \text{cm}$ [42]. Theoretically, new P and CP violating sources [43, 44, 45] and flavor diagonal meson decays such as $\eta \to \pi\pi$ [46, 47, 48, 49] and $\eta \to 4\pi^0$ [50, 51] that are P and CP violating have also been extensively studied, but they appear to be strongly connected with or constrained by the existing EDM limits. However, C and CP violating processes are far from adequately studied.

The charge asymmetry in $\eta \to \pi^+\pi^-\pi^0$ decay is a little-studied CP violating process that has special features. The quantum numbers of $\eta$ and $\pi^0$ are[1] $I^G J^{PC} = 0^+0^{-+}$ and $I^G J^{PC} = 1^-0^{-+}$ separately, for $\pi^\pm$, $I^G J^P = 1^-0^-$. This process breaks G parity, and the charge conjugation of $\pi^+\pi^-\pi^0$ is related to their total isospin $C = -(-1)^I$ [52], so $\eta \to \pi^+\pi^-\pi^0$ can happen either through isospin breaking or C violating mechanisms. For example, if we assume isospin is conserved, then the charge conjugation of the final state is $C = -1$, compared with $C = +1$ for the eta, so that charge conjugation symmetry is violated. To see the parity of the final state, working in the rest frame of two pions coupled to angular momentum $l$, the parity of the three pions is [53] $P|[\pi_1(\boldsymbol{p})\pi_2(-\boldsymbol{p})]_l\pi_3(\boldsymbol{p}')_l\rangle = -1 = P|\eta\rangle$, so that parity is conserved. Thus if there exists CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay, it would be C and CP violating. If we plot the Dalitz distribution in terms of the Mandelstam variables $t \equiv (p_{\pi^-} + p\pi^0)^2$ and $u \equiv (p_{\pi^+} + p_{\pi^0)^2}$, the asymmetry of the momentum distribution of $\pi^+$ and $\pi^-$, that is the charge asymmetry, corresponds to a failure of mirror symmetry, i.e., of $t \leftrightarrow u$ exchange, in the Dalitz plot. An observation of the mirror symmetry breaking in the Dalitz plot in terms of $t$ and $u$ would provide definite evidence of C and CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay. More details are explained in Chapter 3.

In order for a CP-violating observable to be nonzero in the SM, all three generations of quarks have to contribute. So CP violation is invariably associated with the appearance of loop effects in charged current processes involving hadrons. Since $\eta \to \pi^+\pi^-\pi^0$ decay is a flavor-diagonal process, the contribution of SM should be negligibly small since only flavor-changing elements of the CKM matrix contain the nonzero phase that leads to CP violation. Moreover, since the CP-violation in $\eta \to 3\pi$ decay is also C violating, there should be no strong constraints from EDMs which are P and CP violating. Thus the study of mirror-symmetry breaking in $\eta \to \pi^+\pi^-\pi^0$ decay is an ideal process to search for new CP violation.

CP violation in $\eta \to \pi^+\pi^-\pi^0$ first attracted people's attention after the discovery of CP violation in the observation of $K_L \to \pi^+\pi^-$ decay in 1964 [4], because it can

[1] $I$, $G$, $J$, $P$, and $C$ stand for isospin, $G$ parity, spin, parity, and charge conjugation quantum numbers, respectively.

be used to test whether CP violation in $K_L \to \pi^+\pi^-$ could be generated by an interaction much stronger than the usual weak interaction [54, 55]. Under this alternative viewpoint, e.g., CP violation in $K_L \to \pi^+\pi^-$ decay can occur from the interference of the CP-conserving weak interaction with a new "strong" interaction that breaks C and CP, which can be identified through the presence of a charge asymmetry in the momentum distribution of $\pi^+$ and $\pi^-$ [54, 52, 56]. Such a charge asymmetry could arise through the interference of a C-conserving, but isospin-breaking amplitude with a isospin-conserving, but C-violating one in $\eta \to \pi^+\pi^-\pi^0$ [52]. Numerical estimates were made by assuming that the isospin-violating contributions were driven by the electromagnetic interaction [52, 56, 57]. Since those early works, our understanding of meson decays within the SM has changed completely: the weak interaction does break CP in flavor-changing transitions driven by the phase in CKM matrix. Moreover, it is known that the dominant effect in $\eta \to 3\pi$ is provided by isospin breaking in the strong interaction mediated by the up-down quark mass difference [58, 59, 60, 61, 62, 63, 64].

Modern theoretical studies of $\eta \to 3\pi$ focus on a complete description of the final-state interactions within the SM, in order to precisely extract the isospin-breaking, light-quark mass ratio $Q \equiv \sqrt{m_s^2 - \hat{m}^2}/\sqrt{m_d^2 - m_u^2}$, with $\hat{m} = (m_d + m_u)/2$. There has been no further theoretical study of CP violation in $\eta \to \pi^+\pi^-\pi^0$ since 1966.

There has been a number of experiments conducted over the years to test for a charge asymmetry in $\eta \to \pi^+\pi^-\pi^0$. Some early experiments found evidence for a nonzero asymmetry [65, 66, 67], but there could be some possible systematic problems which have become apparent only later, such as those discussed in Ref. [68]. Other experiments [69, 70, 71, 72, 73, 74] find no evidence for a charge asymmetry or C violation. However, some new, high statistics experiments [75, 76, 77] are planned. Our study of C and CP violation in $\eta \to \pi^+\pi^-\pi^0$ will give important insights into the analysis of these upcoming experiments [75, 76, 77].

### 1.5 Organization of this thesis

Here we will give an the outline of the following chapters.

Chapter 2 touches on some of the basic definitions and fundamental properties of C, P, and T transformations in classical dynamics, quantum mechanics, and quantum field theory, respectively. This theoretical background can give us some conceptual feeling for the effects of the three discrete transformations and help us to understand the later chapters better.

In Chapter 3, CP violation within and beyond the SM is explained, with a brief introduction to general CP-violating observables there as well. In the part of CP violation within the SM, the mechanism of how the CKM matrix gives rise to CP violation is carefully investigated, and another CP violating source that does not seem to operate – the $\theta$ term, which is natural in the QCD Lagrangian, but with a coefficient forced to be extremely small because of experimental constraints on the neutron EDM, a mismatch termed the "strong CP problem" – is discussed. We do not understand the resolution to the strong CP problem, but that one solution, axions, which are the Goldstone bosons associated with spontaneously broken Peccei-Quinn symmetry [78], could also help to solve the dark matter problem. For CP violation beyond the SM, we note several popular new-physics models to give us an impression as to how new physics can be formulated.

We can also describe the appearance of new physics in a minimally model-independent way. In this approach, termed SM effective field theory (SMEFT), assumes new physics enters at very high energy scale, so that new physics effects can be encapsulated in higher-dimensional operators composed of the SM fields and sharing SM gauge symmetries. SMEFT is introduced more specifically in Chapter 4. The SMEFT shares the same gauge symmetries and fundamental fields as the SM, with new physics effects embedded in the appearance of new, low-energy operators. It is the basis for a significant portion of this thesis..

In Chapter 5, we talk about meson decays at low energy, where chiral perturbation theory (ChPT) plays an essential role and therefore it is also concisely introduced. The theoretical analysis of $\eta \to 3\pi$ and formulation of isospin-breaking sources is also presented.

Based on the content of the previous chapters, in Chapter 6, patterns of C and CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay are studied through its Dalitz plot. There we show how the isospin of the underlying C- and CP-violating structures can be reconstructed from their kinematic representation in the Dalitz plot. Our analysis of the most recent experimental data reveals important features of C- and CP-violating amplitudes with different total isospin.

In searching for C- and CP-violating sources beyond the SM, in Chapter 7, leading-dimension CP violating effective operators from the SMEFT are enumerated and subsequently P-odd and C-odd operators below the electroweak scale are carefully separated. The C- and CP-odd operators and P- and CP-odd operators for flavor-changing and flavor-conserving interactions are both studied explicitly. We show that the relations between the constraints from P- and CP-odd observables and C-odd and CP-odd observables for flavor-changing and flavor-conserving cases are remarkably distinct.

From the C-odd and CP-odd flavor-conserving operators derived in the last chapter, in Chapter 8, the operators with definite isospin that can contribute to $\eta \to \pi^+\pi^-\pi^0$ decay are also given. We also illustrate how these operators can be represented in ChPT for the eventual determination of their contributions to mirror-symmetry breaking in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot.

Finally, Chapter 9 provides a summary of the thesis as well as an outlook on other work that can be completed.

CHAPTER 2

# C, P, AND T IN PHYSICS

In this chapter, we give some introduction to the parity, charge conjugation, and time reversal transformations in classical physics, quantum mechanics, and quantum field theory in particular. More detailed information can be found in Bigi and Sanda [79], Branco, Lavoura, and Silva. [80], and Peskin and Schroeder [81].

## 2.1 Parity and time reversal in classical physics

Let us first look into the discrete transformations in classical physics, which is closely related to our daily life. This part mainly follows the structure and content of Branco, Lavoura, and Silva [80]. Since a charge conjugation transformation associates the existence of an antiparticle with that of any particle, and there is no concept of antiparticles in classical physics, we mainly discuss parity and time reversal transformations here.

### 2.1.1 Parity

The transformation of parity, denoted by P, sends the 3-dimensional space coordinate $\boldsymbol{r}$ to $-\boldsymbol{r}$. This is equivalent to the inversion of the three coordinate axes through the origin. Such a transformation changes the handedness of a system of axes, for example the left-handed system becomes right-handed under parity transformation. The inversion of the coordinate system can also be achieved through a mirror reflection on a coordinate plane followed by a $\pm 180$ degree rotation around the axis perpendicular to that plane, so that parity symmetry is sometimes called mirror symmetry.

The general physical quantities - velocity $\boldsymbol{v}$, momentum $\boldsymbol{p}$, force $\boldsymbol{F}$, and angular momentum $\boldsymbol{l}$ are defined as

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}, \quad \boldsymbol{p} = m\boldsymbol{v}, \quad \boldsymbol{F} = m\boldsymbol{a} = m\frac{d^2\boldsymbol{r}}{dt^2}, \quad \boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}. \tag{2.1}$$

Since under P, $\boldsymbol{r} \xrightarrow{P} -\boldsymbol{r}$, we have

$$\boldsymbol{v} \xrightarrow{P} -\boldsymbol{v}, \quad \boldsymbol{p} \xrightarrow{P} -\boldsymbol{p}, \quad \boldsymbol{F} \xrightarrow{P} -\boldsymbol{F}, \quad \boldsymbol{l} \xrightarrow{P} \boldsymbol{l}. \tag{2.2}$$

Using the International System of Units (SI) (SI is used throughout this chapter), the Lorentz force acting on a particle with charge $q$ is defined as

$$\boldsymbol{F}_L = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{2.3}$$

where $\boldsymbol{E}$ is the electric field strength and $\boldsymbol{B}$ is the magnetic field strength. Since $\boldsymbol{F}_L \xrightarrow{P} -\boldsymbol{F}_L$ and $\boldsymbol{v} \xrightarrow{P} -\boldsymbol{v}$, the transformation properties of $\boldsymbol{E}$ and $\boldsymbol{B}$ are

$$\boldsymbol{E} \xrightarrow{P} -\boldsymbol{E}, \quad \boldsymbol{B} \xrightarrow{P} \boldsymbol{B}. \tag{2.4}$$

The scalar potential $\Phi$ and vector potential $\boldsymbol{A}$ are defined as

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t}, \tag{2.5}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{2.6}$$

since $\nabla = \partial/\partial \boldsymbol{x}$ changes sign under P, to fulfill Eq. (2.4), $\boldsymbol{A}$ should change sign under P, whereas $\Phi$ stays unchanged.

We can classify vectors and scalar products according to their properties under P. Vectors that change sign under P, or equivalently that are P odd, are called polar vectors, for example $\boldsymbol{E}$. Vectors that do not change sign under P, or that are P even, such as $\boldsymbol{B}$, are called axial vectors or pseudovectors. P-even vector products such as $\boldsymbol{p}_1 \cdot \boldsymbol{p}_2$ are called scalars, whereas P-odd ones such as $\boldsymbol{E} \cdot \boldsymbol{B}$ are called pseudoscalars.

### 2.1.2 Time reversal

The time reversal transformation, denoted as T, sends $t$ to $-t$, leaving the space coordinate $\boldsymbol{x}$ unchanged.

Under T, the quantities defined in Eq. (2.1) become

$$\boldsymbol{v} \xrightarrow{T} -\boldsymbol{v}, \quad \boldsymbol{p} \xrightarrow{T} -\boldsymbol{p}, \quad \boldsymbol{F} \xrightarrow{T} \boldsymbol{F}, \quad \boldsymbol{l} \xrightarrow{T} -\boldsymbol{l}. \tag{2.7}$$

Meanwhile, from the general definition of any force in Eq. (2.1, Lorentz force defined in Eq. (2.3) is invariant under T, and it constrains the electric field and magnetic field to transform as

$$\boldsymbol{E} \xrightarrow{T} \boldsymbol{E}, \quad \boldsymbol{B} \xrightarrow{T} -\boldsymbol{B}. \tag{2.8}$$

Accordingly, the dependance of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ on the scalar potential $\Phi$ and vector potential $\boldsymbol{A}$ shown in Eq. (2.6) indicates

$$\Phi \xrightarrow{T} \Phi, \quad \boldsymbol{A} \xrightarrow{T} -\boldsymbol{A}. \tag{2.9}$$

The fundamental time reversal transformation $T$ goes beyond the above discussed mathematical transformations with entities changing sign or not, since it also interchanges the initial and final states.

### 2.1.3 Maxwell equations

The Maxwell equations govern electrodynamics. We will investigate the properties of Maxwell equations under P and T discrete transformations. The differential Maxwell equations are

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{2.10}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.11}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.12}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left( \boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right), \tag{2.13}$$

where $\rho$ is the charge density and $\boldsymbol{J}$ denotes the electric current vector, $\epsilon_0$ and $\mu_0$ are the permittivity and permeability of free space, respectively. Note that $\rho$ is expected to stay the same under P as well as T. Since the electric current vector $\boldsymbol{J}$ is parallel to $\boldsymbol{v}$ of a moving charged particle, it changes sign either under P or T, i.e. $\boldsymbol{J} \xrightarrow{P} -\boldsymbol{J}$ and $\boldsymbol{J} \xrightarrow{T} -\boldsymbol{J}$. Recalling $\nabla \xrightarrow{P} -\nabla$, $\boldsymbol{E} \xrightarrow{P} -\boldsymbol{E}$, $\boldsymbol{B} \xrightarrow{P} \boldsymbol{B}$, $\partial/\partial t \xrightarrow{P} \partial/\partial t$ , and $\nabla \xrightarrow{T} \nabla$, $\boldsymbol{E} \xrightarrow{T} \boldsymbol{E}$, $\boldsymbol{B} \xrightarrow{T} -\boldsymbol{B}$, $\partial/\partial t \xrightarrow{T} -\partial/\partial t$ , the Maxwell equations remain invariant under P and T. Thus we can conclude that Maxwell equations are invariant under P and T just as the Lorentz force law is.

The charge conjugation transformation, denoted by C, has no analogue in classical physics, since it relates the existence of an antiparticle to that of a particle. However, upon noting that the effect of C on a charged particle is to make its charge opposite, it is natural to consider $\rho \xrightarrow{C} -\rho$ , $\boldsymbol{J} \xrightarrow{C} -\boldsymbol{J}$, while $\nabla$ and $\partial/\partial t$ remain unchanged. From the general definition of any force in Eq. (2.1, the Lorentz force should be invariant under C. From the definition of Lorentz force in Eq. (2.3), under C, $\boldsymbol{E} \xrightarrow{C} -\boldsymbol{E}$, $\boldsymbol{B} \xrightarrow{C} -\boldsymbol{B}$ should hold. Thus the Maxwell equations as Eq. (2.10) stay invariant under C.

### 2.1.4 Spin, dipole moments, and helicity

Spin $\boldsymbol{s}$ is an intrinsic angular momentum carried by an elementary particle. It does not appear in classical physics. But we can treat it as a special angular momentum with the same properties as a classical angular momentum. Thus we have $\boldsymbol{s} \xrightarrow{P} \boldsymbol{s}$ and $\boldsymbol{s} \xrightarrow{T} -\boldsymbol{s}$.

If a particle with spin $\boldsymbol{s}$ moves in an electric field with field strength $\boldsymbol{E}$ or magnetic field with field strength $\boldsymbol{B}$, there may arise terms in Hamiltonian of the form

$$-d_e \frac{\boldsymbol{s}}{|\boldsymbol{s}|} \cdot \boldsymbol{E}, \tag{2.14}$$

$$-d_m \frac{\boldsymbol{s}}{|\boldsymbol{s}|} \cdot \boldsymbol{B}. \tag{2.15}$$

If the interaction in Eq. (2.14) exists, we say the particle possesses a permanent electric dipole moment $d_e$. If the interaction in Eq. (2.15) emerges, the particle is said to have an magnetic dipole moment $d_m$. Note that for a particle with a spin and electrically charged constituents, even if it has no net electric charge, it will have a nonzero magnetic moment. But whether it also has a permanent electric dipole moment needs to be established. According to the transformation properties of $\boldsymbol{E}$ and $\boldsymbol{B}$ in Eq. (2.4) and Eq. (2.8), we conclude that $d_m$ stays unchanged under either P or T, while $d_e \to -d_e$ under any of them. Thus, the electric dipole moment breaks both P and T, which is equivalent with breaking P and CP when the CPT theorem is assumed.

Another important quantity that depends on the spin of a particle is the helicity $h$, which is defined as

$$h = \frac{\boldsymbol{s} \cdot \boldsymbol{p}}{|\boldsymbol{s}||\boldsymbol{p}|}. \tag{2.16}$$

Helicity $h$ describes the handedness of a particle. If the helicity of a particle is positive (negative), we call the particle right-handed (left-handed). Helicity is the same as chirality in the case of a massless particle. According to the transformation properties of $\boldsymbol{s}$ and $\boldsymbol{p}$, under $P$ and $T$, we have $h \xrightarrow{P} -h$ and $h \xrightarrow{T} h$.

We summarize the P and T transformation properties of the quantities discussed above in Table 2.1 [80].

Table 2.1: Transformations properties under P and T in classical physics.

| name | symbol | $P$ | $T$ |
|---|---|---|---|
| velocity | $\boldsymbol{v}$ | $-$ | $-$ |
| momentum | $\boldsymbol{p}$ | $-$ | $-$ |
| force | $\boldsymbol{F}$ | $-$ | $+$ |
| angular momentum | $\boldsymbol{l}$ | $+$ | $-$ |
| electric field | $\boldsymbol{E}$ | $-$ | $+$ |
| magnetic field | $\boldsymbol{B}$ | $+$ | $-$ |
| spin | $\boldsymbol{s}$ | $+$ | $-$ |
| helicity | $\boldsymbol{h}$ | $-$ | $+$ |

## 2.2 C, P, and T in quantum mechanics

Quantum mechanics describes nature at microscopic scales and serves as the foundation of modern physics. The state of a system in the quantum world is usually represented by the wave function $|\psi_i\rangle$. The superposition principle is one of the basic concepts in quantum mechanics, which can be stated that if $|\psi_i\rangle$ are vectors in Hilbert space, so are their linear combinations. An operator $\mathcal{O}$ acting on the states can be expressed as

$$|\psi_i\rangle \xrightarrow{\mathcal{O}} \mathcal{O}|\psi_i\rangle, \quad \langle\psi_i| \xrightarrow{\mathcal{O}} \langle\psi_i|\mathcal{O}^\dagger. \tag{2.17}$$

If such an operation represents a symmetry, we must have

$$|\langle\psi'|\mathcal{O}^\dagger\mathcal{O}|\psi\rangle|^2 = |\langle\psi'|\psi\rangle|^2 \tag{2.18}$$

so that the quantum-mechanical probabilities associated with the measurement remain uninfluenced. This can be satisfied either by

$$\langle\psi'|\mathcal{O}^\dagger\mathcal{O}|\psi\rangle = \langle\psi'|\psi\rangle \Rightarrow \mathcal{O}^\dagger\mathcal{O} = I, \tag{2.19}$$

or

$$\langle\psi'|\mathcal{O}^\dagger\mathcal{O}|\psi\rangle = \langle\psi'|\psi\rangle^*. \tag{2.20}$$

An operator $\mathcal{O}$ that operates as in Eq. (2.19) is called a linear or unitary operator, whereas if the operator satisfies Eq. (2.20), it is called an anti-linear or anti-unitary operator. In the later sections we will see that P and C are unitary operators and T is an anti-unitary operator.

In quantum theory, the fundamental operator is the Hamiltonian $H$, which is Hermitian and generates the time evolution operator $U(t,\ t_i) = \exp(-i/\hbar\ H\Delta t)$ where $\Delta t = t - t_i$. With the definition $|\psi(t)\rangle \equiv U(t,\ t_0)|\psi(t_0)\rangle$, we immediately have the Schrödinger equation of the state vector $|\psi(t)\rangle$ as

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle \quad = \quad H|\psi(t)\rangle, \tag{2.21}$$

with

$$H \quad = \quad \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\hat{\boldsymbol{x}}). \tag{2.22}$$

where $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{x}}$ are the momentum and position operators, respectively. A symmetry requires the invariance of the Hamiltonian under the corresponding transformation $\mathcal{O}$, i.e. $\mathcal{O}$ should commute with $H$: $[\mathcal{O}, H] = 0$. This implies that if the operator is not explicitly time dependent, $\langle\mathcal{O}\rangle$ is a constant of motion.

### 2.2.1 Parity and charge conjugation

The transformation T requires a mathematical formalism distinct from the one used for P and C transformations. We will first treat P and C jointly, and then discuss the properties of T later.

The relation $[\mathcal{P}, H] = 0$ would imply that P is a good symmetry in nature. This is not true since there exists P violation in weak interactions, so that we cannot write a general form for all the states which adequately represents the parity transformation in quantum mechanics [82]. C is also not a good symmetry in nature if we are to account for that our description of the weak interaction has only left-handed neutrinos and right-handed anti-neutrinos. Therefore, there is no general expression which represents correctly the C transformation for every state. We will follow the procedure of Branco et al. [80], that we first restrict ourselves to an appropriate part of the complete Hamiltonian that is C and P invariant, then define operators $\mathcal{P}$ and $\mathcal{C}$ suitable to this part, and if some of the other parts of the Hamiltonian are not invariant under $\mathcal{P}$ or $\mathcal{C}$, we say that there is P or C violation.

The simple way to obtain the behavior of observables under P is to apply the correspondence principle passing from classical mechanics to quantum mechanics [83] and to require the expectation value of the position operator $\hat{\boldsymbol{x}}$ to change sign under P [79]:

$$\langle\psi(t)|\hat{\boldsymbol{x}}|\psi(t)\rangle \xrightarrow{P} \langle\psi(t)|\mathcal{P}\hat{\boldsymbol{x}}\mathcal{P}^\dagger|\psi(t)\rangle = -\langle\psi(t)|\hat{\boldsymbol{x}}|\psi(t)\rangle, \tag{2.23}$$

which is guaranteed to happen if

$$\mathcal{P}\hat{\boldsymbol{x}}\mathcal{P}^\dagger = -\hat{\boldsymbol{x}} \quad \text{or} \quad \{\hat{\boldsymbol{x}}, \mathcal{P}\} = 0, \tag{2.24}$$

where $\{\hat{\boldsymbol{x}}, \mathcal{P}\} \equiv \hat{\boldsymbol{x}}\mathcal{P} + \mathcal{P}\hat{\boldsymbol{x}}$ is an anti-commutator. In the position representation, the momentum operator can be expressed as

$$\hat{\boldsymbol{p}} = -i\hbar\frac{\partial}{\partial\boldsymbol{x}}. \tag{2.25}$$

Similarly we have

$$\mathcal{P}\hat{\boldsymbol{p}}\mathcal{P}^\dagger = -\hat{\boldsymbol{p}} \quad \text{or} \quad \{\hat{\boldsymbol{p}}, \mathcal{P}\} = 0. \tag{2.26}$$

Assuming $[\mathcal{P}, H] = 0$ is satisfied with $H$ defined in Eq. (2.22) with an even potential, i.e. $V(\hat{\boldsymbol{x}}) = V(-\hat{\boldsymbol{x}})$, in order for the Schrödinger equation to hold, we must have

$$\mathcal{P}i\mathcal{P}^\dagger = i, \tag{2.27}$$

which guarantees that $\mathcal{P}$ is an unitary operator, i.e. $\mathcal{P}^{-1} = \mathcal{P}^\dagger$. Applying the transformation properties of $i$, $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ under P, we see that the basic quantum commutator

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \tag{2.28}$$

still holds under P.

For the charge conjugation transformation, we introduce the minimal electromagnetic coupling to the Hamiltonian

$$H = \frac{1}{2m}\left(\hat{\boldsymbol{p}} - e\boldsymbol{A}\right)^2 + e\phi, \tag{2.29}$$

where $e$ is the charge of the particle, $\boldsymbol{A}$ is the vector potential and $\phi$ is the electric potential. Note that the potential $V = H - \hat{\boldsymbol{p}}^2/(2m)$ is momentum dependent here. The Hamiltonian is invariant under C when we have the following transformation properties

$$e \xrightarrow{C} -e, \quad \boldsymbol{A} \xrightarrow{C} -\boldsymbol{A}, \quad \phi \xrightarrow{C} -\phi, \tag{2.30}$$

with $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{x}}$ remaining unchanged. Thus to make the Schrödinger equation, Eq(2.21) invariant under C, we must require $\mathcal{C}$ to be a unitary operator, i.e. $\mathcal{C}i\mathcal{C}^\dagger = i$ and $\mathcal{C}^{-1} = \mathcal{C}^\dagger$.

### 2.2.2 Time reversal

Now let us turn to the case of time reversal. Similarly as in the derivation of Eq. (2.24), we deduce the following properties

$$\mathcal{T}\hat{\boldsymbol{x}}\mathcal{T}^\dagger = \hat{\boldsymbol{x}}, \tag{2.31}$$

$$\mathcal{T}\hat{\boldsymbol{p}}\mathcal{T}^\dagger = -\hat{\boldsymbol{p}}. \tag{2.32}$$

A straightforward way to realize that the time reversal transformation must be anti-unitary is to consider Eq. (2.28), which is the most important foundation of quantum mechanics. Under time reversal, $\hat{p}_j$ changes sign but $\hat{x}_i$ does not. So Eq. (2.28) can stay undisturbed if and only if

$$\mathcal{T}i\mathcal{T}^\dagger = -i. \tag{2.33}$$

Using the T transformation properties of $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{p}}$, and $i$ together, the Schrödinger equation 2.21 is invariant under T.

## 2.3 C, P, and T in quantum field theory

Quantum field theory (QFT) is a set of notions and mathematical tools that combines three of the major themes of modern physics: the quantum theory, the field concept, and the principle of relativity. The theory underlies modern elementary particle physics, and supplies comprehensive applications to nuclear physics, atomic physics, condensed matter physics, and astrophysics.

In this section, we discuss the transformation properties of some basic and essential fields under parity, charge conjugation and time reversal in QFT, which is closely connected to my research projects. This part mainly follows Bigi and Sanda [79].

### 2.3.1 Conventions

Some notations and conventions need to be addressed for later convenience. We will use the natural unit system from now on, i.e.,

$$c = \hbar = 1. \tag{2.34}$$

Following Peskin and Schroeder [81], we use the metric tensor

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with Greek indices $\mu,\ \nu$ running over $0,\ 1,\ 2,\ 3$ or $t,\ x\ ,y\ ,\ z$. Roman indices $i,\ j$, etc. denote only the three spatial components. The 4-vector and the product of two 4-vectors take the general following form

$$v^\mu = (v^0,\ \boldsymbol{v}), \quad v_\mu = g_{\mu\nu} v^\nu = (v^0,\ -\boldsymbol{v}), \tag{2.35}$$

$$v \cdot u = v^\mu u_\mu = g_{\mu\nu} v^\mu u^\nu = v^0 u^0 - \boldsymbol{v} \cdot \boldsymbol{u} \tag{2.36}$$

For the position and momentum 4-vectors, we have

$$x^\mu = (t,\ \boldsymbol{x}), \quad p^\mu = (E,\ \boldsymbol{p}), \tag{2.37}$$

and $p^2 = p^\mu p_\mu = E^2 - |\boldsymbol{p}|^2 = m^2$ for a particle with mass $m$. The derivative operator is defined as follows

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t},\ \boldsymbol{\nabla}) \tag{2.38}$$

### 2.3.2 The Photon Field

The Lagrangian for the photon field is [80, 81]

$$\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.39}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ being the field strength tensor, and $A^\mu = (\Phi, \boldsymbol{A})$ where $\Phi$ denotes the electric scalar potential and $\boldsymbol{A}$ the vector potential. The relativistic form of Maxwell's equations can be expressed in a compact and manifestly Lorentz invariant form [81]

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \quad \partial_\mu F_{\mu\nu} = eJ^\nu, \tag{2.40}$$

where $J^\mu$ is the current density vector.

The electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ can be related with the photon field as

$$E_i = F_{0i} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\Phi. \tag{2.41}$$

$$B_i = -\epsilon_{ijk}\partial_j A_k = -\frac{1}{2}\epsilon_{ijk}F_{jk} \tag{2.42}$$

From our experience with the classical field, we can postulate the transformation properties for the quantized fields

$$\mathcal{P}A_\mu(t,\boldsymbol{x})\mathcal{P}^\dagger = (-1)^\mu A_\mu(t,-\boldsymbol{x}), \quad \mathcal{P}J_\mu(t,\boldsymbol{x})\mathcal{P}^\dagger = (-1)^\mu J_\mu(t,-\boldsymbol{x}) \tag{2.43}$$

$$\mathcal{C}A_\mu(t,\boldsymbol{x})\mathcal{C}^\dagger = -A_\mu(t,\boldsymbol{x}), \quad \mathcal{C}J_\mu(t,\boldsymbol{x})\mathcal{C}^\dagger = -J_\mu(t,\boldsymbol{x}) \tag{2.44}$$

$$\mathcal{T}A_\mu(t,\boldsymbol{x})\mathcal{T}^\dagger = (-1)^\mu A_\mu(-t,\boldsymbol{x}), \quad \mathcal{T}J_\mu(t,\boldsymbol{x})\mathcal{T}^\dagger = (-1)^\mu J_\mu(-t,\boldsymbol{x}), \tag{2.45}$$

with the convention [81]

$$(-1)^\mu \equiv +1 \text{ for } \mu = 0 \text{ and } (-1)^\mu \equiv -1 \text{ for } \mu = 1,\ 2,\ 3. \tag{2.46}$$

For the differential operator, we have

$$\mathcal{P}\partial_\mu\mathcal{P}^\dagger = (-1)^\mu\partial_\mu, \quad \mathcal{C}\partial_\mu\mathcal{C}^\dagger = \partial_\mu, \quad \mathcal{T}\partial_\mu\mathcal{T}^\dagger = -(-1)^\mu\partial_\mu. \tag{2.47}$$

Using the transformation properties above, we can check that the form of the Lagrangian remains invariant under P, C, and T

$$\mathcal{P}\mathcal{L}_\gamma(t,\boldsymbol{x})\mathcal{P}^\dagger = \mathcal{L}_\gamma(t,-\boldsymbol{x}), \tag{2.48}$$

$$\mathcal{C}\mathcal{L}_\gamma(t,\boldsymbol{x})\mathcal{C}^\dagger = \mathcal{L}_\gamma(t,\boldsymbol{x}), \tag{2.49}$$

$$\mathcal{T}\mathcal{L}_\gamma(t,\boldsymbol{x})\mathcal{T}^\dagger = \mathcal{L}_\gamma(-t,\boldsymbol{x}). \tag{2.50}$$

### 2.3.3 Klein-Gordon field

The Klein-Gordon Lagrangian for a spin-0 field $\phi$ with mass $m$ and electric charge $q$, moving in the electromagnetic field given by the potential $A^\mu$, is

$$\mathcal{L}_\phi = (\partial_\mu - iqA_\mu)\phi^\dagger(\partial^\mu + iqA^\mu)\phi - m^2\phi^\dagger\phi. \tag{2.51}$$

After second quantization, $\phi$ is a field with the conditions

$$[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})] = [\phi^\dagger(t,\boldsymbol{x}),\phi^\dagger(t,\boldsymbol{y})] = 0 \tag{2.52}$$

$$[\phi(t,\boldsymbol{x}),\partial_t\phi^\dagger(t,\boldsymbol{y})] = [\phi^\dagger(t,\boldsymbol{x}),\partial_t\phi(t,\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}). \tag{2.53}$$

The field can be expanded as

$$\phi(t,\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(b_{\boldsymbol{p}}e^{-ipx} + d^\dagger_{\boldsymbol{p}}e^{ipx}), \tag{2.54}$$

where $b_{\boldsymbol{p}}$ is the annihilation operator for particle and $d_{\boldsymbol{p}}^{\dagger}$ is the creation operator for antiparticle. Note that for a scalar that does not carry a charge, there is no difference between $b_{\boldsymbol{p}}$ and $d_{\boldsymbol{p}}^{\dagger}$.

Under parity, we can postulate that [80]

$$\mathcal{P}\phi(t,\boldsymbol{x})\mathcal{P}^{\dagger} = e^{i\alpha_p}\phi(t,-\boldsymbol{x}), \tag{2.55}$$

with an arbitrary phase $\alpha_p$ which is real. We use the phase convention of Branco [80]. Note that as discussed by Gardner and Yan [84], where they discuss Majorana fermions only if baryon number or lepton number is broken., the associated phase of the transformation is not arbitrary if it is associated with Majorana fermions. When the phase is taken to be zero, the transformation properties are consistent with Peskin and Schroeder [81]. Taking the hermitian conjugate of Eq. 2.55. we get

$$\mathcal{P}\phi^{\dagger}(t,\boldsymbol{x})\mathcal{P}^{\dagger} = e^{-i\alpha_p}\phi^{\dagger}(t,-\boldsymbol{x}). \tag{2.56}$$

With these properties together with those of $\partial_\mu$ and $A_\mu$, we see that the Lagrangian and the quantization conditions Eq. 2.52 stay invariant. Moreover,

$$\mathcal{P}^2\phi(t,\boldsymbol{x}){\mathcal{P}^{\dagger}}^2 = e^{2i\alpha_p}\phi(t,\boldsymbol{x}). \tag{2.57}$$

Note that the phase choice that is appropriate to the Majorana case requires $\exp(i\alpha) \propto i$, so that the phase on the right-hand-side of Eq. (2.57) is -1 [84].

Under charge conjugation,

$$\mathcal{C}b_{\boldsymbol{p}}\mathcal{C}^{\dagger} = d_{\boldsymbol{p}}, \quad \mathcal{C}b_{\boldsymbol{p}}^{\dagger}\mathcal{C}^{\dagger} = d_{\boldsymbol{p}}^{\dagger}, \tag{2.58}$$

then we have [80]

$$\mathcal{C}\phi(t,\ \boldsymbol{x})\mathcal{C}^{\dagger} = e^{i\alpha_c}\phi^{\dagger}(t,\ \boldsymbol{x}), \tag{2.59}$$
$$\mathcal{C}\phi^{\dagger}(t,\ \boldsymbol{x})\mathcal{C}^{\dagger} = e^{-i\alpha_c}\phi(t,\ \boldsymbol{x}). \tag{2.60}$$

The Lagrangian and the quantization conditions Eq. 2.52 remain invariant under C upon applying the transformation properties. Notice that

$$\mathcal{C}^2\phi(t,\ \boldsymbol{x}){\mathcal{C}^{\dagger}}^2 = \phi(t,\ \boldsymbol{x}), \tag{2.61}$$

which is different for the case of P as Eq. (2.57).

For time reversal, we can define

$$\mathcal{T}\phi(t,\ \boldsymbol{x})\mathcal{T}^{\dagger} = e^{i\alpha_t}\phi^{\dagger}(-t,\boldsymbol{x}), \tag{2.62}$$
$$\mathcal{T}\phi^{\dagger}(t,\ \boldsymbol{x})\mathcal{T}^{\dagger} = e^{-i\alpha_t}\phi(-t,\boldsymbol{x}), \tag{2.63}$$

meanwhile using $\mathcal{T}i\mathcal{T}^{\dagger} = -i$, $\mathcal{T}\partial_\mu\mathcal{T}^{\dagger} = -\partial^\mu$, and $\mathcal{T}A_\mu\mathcal{T}^{\dagger} = A^\mu$, the Lagrangian and the quantization conditions Eq. 2.52 are invariant under T. Notice that

$$\mathcal{T}^2\phi(t,\ \boldsymbol{x}){\mathcal{T}^{\dagger}}^2 = \phi(t,\ \boldsymbol{x}). \tag{2.64}$$

### 2.3.4 Dirac Field

The Dirac Lagrangian for a free spin 1/2 particle is

$$\mathcal{L}_\psi = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x), \tag{2.65}$$

where $\psi(x)$ satisfies the anti-commutation relations

$$\{\psi_a(t,\boldsymbol{x}), \psi_b^\dagger(t,\boldsymbol{y})\} = \delta^3(\boldsymbol{x}-\boldsymbol{y})\delta_{ab} \tag{2.66}$$

$$\{\psi_a(t,\boldsymbol{x}), \psi_b(t,\boldsymbol{y})\} = \{\psi_a^\dagger(t,\boldsymbol{x}), \psi_b^\dagger(t,\boldsymbol{y})\} = 0. \tag{2.67}$$

The spinor fields can be conveniently expressed as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s (a_{\boldsymbol{p}}^s u^s(p)e^{-ipx} + b_{\boldsymbol{p}}^{s\dagger} v^s(p)e^{ipx}), \tag{2.68}$$

where $a_{\boldsymbol{p}}^s$ annihilate the particle and $b_{\boldsymbol{p}}^{s\dagger}$ create the antiparticle with momentum $\boldsymbol{p}$ and spin index $s$, respectively. The Dirac spinors $u^s(p)$ and $v^s(p)$ represent solutions to the Dirac equation in momentum space

$$(\not{p} - m)u^s(p) = 0 \tag{2.69}$$

$$(\not{p} + m)v^s(p) = 0. \tag{2.70}$$

The Dirac matrices $\gamma^\mu$ are four $4\times 4$ matrices which obey the anti-commutation algebra (Clifford algebra)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{2.71}$$

We use the Weyl or chiral representation to write $\gamma^\mu$ in $2\times 2$ block form as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

where $\sigma^i$ are Pauli matrices expressed as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then spinors $u^s(p)$ and $v^s(p)$ are given by [81]

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^{-s} \\ -\sqrt{p\cdot\bar{\sigma}}\xi^{-s} \end{pmatrix}. \tag{2.72}$$

where $s = 1, 2$ (spin "up" or "down"), $\sigma^\mu = (1, \ \boldsymbol{\sigma})$, $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, and

$$\xi^{-s} = \left(\xi^2, \ -\xi^1\right). \tag{2.73}$$

Associating $s$ with the physical spin component of the fermion along a specific axis with polar coordinate $\theta$ and $\phi$, the two-component spinors with spin up and down are

$$\xi^1 = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \xi^2 = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}. \end{pmatrix} \tag{2.74}$$

There is another important matrix $\gamma_5$ defined as

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.75}$$

which satisfies

$$\{\gamma_5, \gamma^\mu\} = 0, \quad (\gamma_5)^2 = 1. \tag{2.76}$$

Using the Weyl or chiral representation $\gamma_5$ can be expressed in $2 \times 2$ block form as

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Following the discussion from Peskin and Schroeder [81] and phase convention from Branco, Lavoura, and Silva [80], we give the transformation properties of Dirac fields under P, C, and T, respectively as follows. The resulting transformation properties are consistent with Peskin and Schroeder [81] when choosing the phases to be 1.

### Parity

The parity transformation should reverse the momentum of a particle without flipping its spin, which can be expressed mathematically as

$$Pa^s_{\boldsymbol{p}}P^\dagger = \eta_a a^s_{-\boldsymbol{p}}, \qquad Pb^s_{\boldsymbol{p}}P^\dagger = \eta_b b^s_{-\boldsymbol{p}} \tag{2.77}$$

where $\eta_a$ and $\eta_b$ are possible phases. Using the relation [81]

$$u^s(p) = \gamma^0 u^s(\tilde{p}), \quad v^s(p) = -\gamma^0 v^s(\tilde{p}), \tag{2.78}$$

where $\tilde{p} = (p^0, -\boldsymbol{p})$, we have

$$\begin{aligned} \mathcal{P}\psi(t,\boldsymbol{x})\mathcal{P}^\dagger &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s(\eta_a a^s_{-\boldsymbol{p}}u^s(p)e^{-ipx} + \eta_b^* b^{s\dagger}_{-\boldsymbol{p}}v^s(p)e^{ipx}) \\ &= \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s(\eta_a a^s_{\tilde{p}}\gamma^0 u^s(\tilde{p})e^{-i\tilde{p}(t,-\boldsymbol{x})} - \eta_b^* b^{s\dagger}_{\tilde{p}}\gamma^0 v^s(\tilde{p})e^{i\tilde{p}(t,-\boldsymbol{x})}). \end{aligned} \tag{2.79}$$

This should equal some constant matrix times $\psi(t, -\boldsymbol{x})$, then we need

$$e^{i\beta_p} \equiv \eta_a = -\eta_b^*, \tag{2.80}$$

and as a result we have

$$\mathcal{P}\psi(t,\boldsymbol{x})\mathcal{P}^\dagger = e^{i\beta_p}\gamma^0\psi(t,-\boldsymbol{x}), \tag{2.81}$$

Then

$$\mathcal{P}\overline{\psi}(t,\boldsymbol{x})\mathcal{P}^\dagger = e^{-i\beta_p}\overline{\psi}(t,-\boldsymbol{x})\gamma^0, \tag{2.82}$$

Notice that

$$\mathcal{P}^2\psi(t,\boldsymbol{x})\mathcal{P}^{\dagger 2} = e^{2i\beta_p}\gamma^0\psi(t,-\boldsymbol{x}). \tag{2.83}$$

Note that for Majorana case $e^{i\beta_p} \propto i$, thus the phase on the right-hand side of the above equation is $-1$ [84].

The quark bilinears we list below are all the ones that could appear in a Lagrangian, and it is convenient to write down their behaviors under certain discrete transformation.

$$\overline{\psi}\psi, \quad \overline{\psi}\gamma_5\psi, \quad \overline{\psi}\gamma^\mu\psi, \quad \overline{\psi}\gamma^\mu\gamma_5\psi, \quad \overline{\psi}\sigma^{\mu\nu}\psi, \quad \overline{\psi}\sigma^{\mu\nu}\gamma_5\psi. \tag{2.84}$$

Note that the above quark bilinears are independent of each other except for $\overline{\psi}\sigma^{\mu\nu}\gamma_5\psi$, which can be written as a linear combinations of the other ones. Using the anti-commutation relations of the gamma matrices and the transformation properties of $\psi$ under P, we have

$$\mathcal{P}\overline{\psi}(t,\boldsymbol{x})\psi(t,\boldsymbol{x})\mathcal{P}^\dagger = +\overline{\psi}(t,-\boldsymbol{x})\psi(t,-\boldsymbol{x}) \tag{2.85}$$
$$\mathcal{P}\overline{\psi}(t,\boldsymbol{x})\gamma_5\psi(t,\boldsymbol{x})\mathcal{P}^\dagger = -\overline{\psi}(t,-\boldsymbol{x})\gamma_5\psi(t,-\boldsymbol{x}) \tag{2.86}$$
$$\mathcal{P}\overline{\psi}(t,\boldsymbol{x})\gamma^\mu\psi(t,\boldsymbol{x})\mathcal{P}^\dagger = (-1)^\mu\overline{\psi}(t,-\boldsymbol{x})\gamma^\mu\psi(t,-\boldsymbol{x}) \tag{2.87}$$
$$\mathcal{P}\overline{\psi}(t,\boldsymbol{x})\gamma^\mu\gamma_5\psi(t,\boldsymbol{x})\mathcal{P}^\dagger = -(-1)^\mu\overline{\psi}(t,-\boldsymbol{x})\gamma^\mu\gamma_5\psi(t,-\boldsymbol{x}) \tag{2.88}$$
$$\mathcal{P}\overline{\psi}(t,\boldsymbol{x})\sigma^{\mu\nu}\psi(t,\boldsymbol{x})\mathcal{P}^\dagger = (-1)^\mu(-1)^\nu\overline{\psi}(t,-\boldsymbol{x})\sigma^{\mu\nu}\psi(t,-\boldsymbol{x}) \tag{2.89}$$
$$\mathcal{P}\overline{\psi}(t,\boldsymbol{x})\sigma^{\mu\nu}\gamma_5\psi(t,\boldsymbol{x})\mathcal{P}^\dagger = -(-1)^\mu(-1)^\nu\overline{\psi}(t,-\boldsymbol{x})\sigma^{\mu\nu}\gamma_5\psi(t,-\boldsymbol{x}). \tag{2.90}$$

### Time reversal

The time reversal operator reverses the 3-momentum of the particle as well as flips the spin, which can be expressed as

$$\mathcal{T}a^s_{\boldsymbol{p}}\mathcal{T}^\dagger = e^{i\beta_t}a^{-s}_{-\boldsymbol{p}}, \quad \mathcal{T}b^s_{\boldsymbol{p}}\mathcal{T}^\dagger = e^{i\beta_t}b^{-s}_{-\boldsymbol{p}}. \tag{2.91}$$

We take the phase factor in the transformation of $a^s_{\boldsymbol{p}}$ and $b^s_{\boldsymbol{p}}$ to be the same, which is required to make the $\psi(t,\boldsymbol{x})$ transforms to some constant matrix times $\psi(-t,\boldsymbol{x})$ under T, which can be seen upon further derivation. Accounting for the anti-unitary property of T, we have $\mathcal{T}i\mathcal{T}^\dagger = -i$. The Dirac spinors of the particle and antiparticle have the following properties [81]

$$u^{-s}(\tilde{p}) = -\gamma^1\gamma^3[u^s(p)]^*, \quad v^{-s}(\tilde{p}) = -\gamma^1\gamma^3[v^s(p)]^*. \tag{2.92}$$

Then we have

$$\begin{aligned}
\mathcal{T}\psi(t,\boldsymbol{x})\mathcal{T}^\dagger &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s e^{i\beta_t}(a^{-s}_{-\boldsymbol{p}}[u^s(p)]^*e^{ipx} + b^{-s\dagger}_{-\boldsymbol{p}}[v^s(p)]^*e^{-ipx}) \\
&= e^{i\beta_t}(\gamma^1\gamma^3)\int\frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s(a^{-s}_{\tilde{p}}u^{-s}(\tilde{p})e^{-i\tilde{p}(-t,\boldsymbol{x})} + b^{-s\dagger}_{\tilde{p}}v^{-s}(\tilde{p})e^{i\tilde{p}(-t,\boldsymbol{x})}), \\
&= e^{i\beta_t}(\gamma^1\gamma^3)\psi(-t,\boldsymbol{x}).
\end{aligned} \tag{2.93}$$

Notice that

$$\mathcal{T}^2\psi(t,\boldsymbol{x})\mathcal{T}^{\dagger 2} = e^{i\beta_t}(\gamma^1\gamma^3)^*\mathcal{T}\psi(-t,\boldsymbol{x})\mathcal{T}^\dagger = e^{-i\beta_t}e^{i\beta_t}(\gamma^1\gamma^3)(\gamma^1\gamma^3)\psi(t,\boldsymbol{x}) = -\psi(t,\boldsymbol{x}). \tag{2.94}$$

So $T^2$ acting on the Dirac field equals $-1$, whereas it results in $+1$ for the Klein-Gordon field as shown in the previous section. $T^2 = -1$ and $T^2 = +1$ can generate very different physical impacts [79]. Suppose there is a system invariant under T and $|E\rangle$ and $|E^{(T)}\rangle = T|E\rangle$ are eigenstates of T with the same eigenvalue. If $T^2 = -1$ which implies $(T^\dagger)^2 = -1$ as well, we have

$$\langle E|E^{(T)}\rangle = \langle E^{(T)}|T^\dagger T|E\rangle = \langle E|(T^\dagger)^2 T|E\rangle = -\langle E|T|E\rangle = -\langle E|E^{(T)}\rangle, \tag{2.95}$$

which means $|E\rangle$ and $|E^{(T)}\rangle\rangle$ are orthogonal and they describe different physical states. This is called Kramers' degeneracy [79]. For $\overline{\psi}$, we obtain

$$\mathcal{T}\overline{\psi}(t,\boldsymbol{x})\mathcal{T}^\dagger = \mathcal{T}\psi^\dagger\mathcal{T}^\dagger(\gamma^0)^* = e^{-i\beta_t}\psi^\dagger(-t,\boldsymbol{x})(\gamma^1\gamma^3)^\dagger\gamma^0 = -e^{-i\beta_t}\overline{\psi}(-t,\boldsymbol{x})(\gamma^1\gamma^3). \tag{2.96}$$

Then using

$$(\gamma^1\gamma^3)\gamma^{\mu *}(\gamma^1\gamma^3) = \gamma_\mu = (-1)^\mu\gamma^\mu, \tag{2.97}$$

we have the transformation properties of fermion bilinears

$$\mathcal{T}\overline{\psi}(t,\boldsymbol{x})\psi(t,\boldsymbol{x})\mathcal{T}^\dagger = +\overline{\psi}(-t,\boldsymbol{x})\psi(-t,\boldsymbol{x}) \tag{2.98}$$

$$\mathcal{T}i\overline{\psi}(t,\boldsymbol{x})\gamma_5\psi(t,\boldsymbol{x})\mathcal{T}^\dagger = -i\overline{\psi}(-t,\boldsymbol{x})\gamma_5\psi(-t,\boldsymbol{x}) \tag{2.99}$$

$$\mathcal{T}\overline{\psi}(t,\boldsymbol{x})\gamma^\mu\psi(t,\boldsymbol{x})\mathcal{T}^\dagger = (-1)^\mu\overline{\psi}(-t,\boldsymbol{x})\gamma^\mu\psi(-t,\boldsymbol{x}) \tag{2.100}$$

$$\mathcal{T}\overline{\psi}(t,\boldsymbol{x})\gamma^\mu\gamma_5\psi(t,\boldsymbol{x})\mathcal{T}^\dagger = (-1)^\mu\overline{\psi}(-t,\boldsymbol{x})\gamma^\mu\gamma_5\psi(-t,\boldsymbol{x}) \tag{2.101}$$

$$\mathcal{T}\overline{\psi}(t,\boldsymbol{x})\sigma^{\mu\nu}\psi\mathcal{T}^\dagger = -(-1)^\mu(-1)^\nu\overline{\psi}(-t,\boldsymbol{x})\sigma^{\mu\nu}\psi(-t,\boldsymbol{x}), \tag{2.102}$$

$$\mathcal{T}\overline{\psi}(t,\boldsymbol{x})\sigma^{\mu\nu}\gamma_5\psi(t,\boldsymbol{x})\mathcal{T}^\dagger = -(-1)^\mu(-1)^\nu\overline{\psi}(-t,\boldsymbol{x})\sigma^{\mu\nu}\gamma_5\psi(t,-\boldsymbol{x}), \tag{2.103}$$

### 2.3.5 Charge conjugation

Charge conjugation is defined to turn a fermion with a given spin orientation into an anti-fermion with the same spin orientation. This can be conveniently expressed as

$$\mathcal{C}a^s_{\boldsymbol{p}}\mathcal{C}^\dagger = e^{i\beta_c}b^s_{\boldsymbol{p}}, \quad \mathcal{C}b^s_{\boldsymbol{p}}\mathcal{C}^\dagger = e^{i\beta_c}a^s_{\boldsymbol{p}}, \tag{2.104}$$

where we take the phase factor of $a^s_{\boldsymbol{p}}$ and $b^s_{\boldsymbol{p}}$ to be equal for the same reason stated in the time reversal section 2.3.4. The Dirac spinors of the particle and its antiparticle have the relation [81]

$$u^s(p) = -i\gamma^2[v^s(p)]^*, \quad v^s(p) = -i\gamma^2[u^s(p)]^*. \tag{2.105}$$

Then we have

$$\begin{aligned}\mathcal{C}\psi(x)\mathcal{C}^\dagger &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\eta_c\left(b^s_{\boldsymbol{p}}[-i\gamma^2(v^s(p))^*]e^{-ipx} + a^{s\dagger}_{\boldsymbol{p}}[-i\gamma^2(u^s(p))^*]e^{ipx}\right)\\ &= -ie^{i\beta_c}\gamma^2\psi^*(x) = -ie^{i\beta_c}\gamma^2[\psi^\dagger(x)]^T = -ie^{i\beta_c}[\overline{\psi}(x)\gamma^0\gamma^2]^T,\end{aligned} \tag{2.106}$$

where we used $(\gamma^2)^T = \gamma^2$, $(a_p^s)^* = a_p^{s\dagger}$, and $(b_p^s)^* = b_p^{s\dagger}$ because $a_p^s$ and $(b_p^s)$ are not matrices. For $\overline{\psi}$, we find

$$\mathcal{C}\overline{\psi}(x)\mathcal{C}^\dagger = \mathcal{C}\psi^\dagger(x)\mathcal{C}^\dagger\gamma^0 = ie^{-i\beta_c}(-\gamma^2\psi)^T\gamma^0 = -ie^{-i\beta_c}(\gamma^0\gamma^2\psi)^T \tag{2.107}$$

Notice that

$$\mathcal{C}^2\psi(x)\mathcal{C}^{\dagger 2} = -e^{i\beta_c}\gamma^2\mathcal{C}\psi^*(x)\mathcal{C}^\dagger = e^{i\beta_c}e^{-i\beta_c}\gamma^2(-\gamma^2)\psi(x) = -(\gamma^2)^2\psi(x) = \psi(x). \tag{2.108}$$

Under charge conjugation, the transformation properties of the fermion bilinears are

$$\begin{aligned}
\mathcal{C}\overline{\psi}_1(x)\psi_2(x)\mathcal{C}^\dagger &= +\overline{\psi}_2(x)\psi_1(x) && (2.109)\\
\mathcal{C}\overline{\psi}_1(x)\psi_2(x)\mathcal{C}^\dagger &= +\overline{\psi}_2(x)\gamma_5\psi_1(x) && (2.110)\\
\mathcal{C}\overline{\psi}_1(x)\gamma^\mu(x)\psi_2(x)\mathcal{C}^\dagger &= -\overline{\psi}_2(x)\gamma^\mu\psi_1(x) && (2.111)\\
\mathcal{C}\overline{\psi}_1(x)\gamma^\mu\gamma_5(x)\psi_2(x)\mathcal{C}^\dagger &= +\overline{\psi}_2(x)\gamma^\mu\gamma_5\psi_1(x) && (2.112)\\
\mathcal{C}\overline{\psi}_1(x)\sigma^{\mu\nu}\psi_2(x)\mathcal{C}^\dagger &= -\overline{\psi}_2(x)\sigma^{\mu\nu}\psi_1(x), && (2.113)\\
\mathcal{C}\overline{\psi}_1(x)\sigma^{\mu\nu}\gamma_5\psi_2(x)\mathcal{C}^\dagger &= -\overline{\psi}_2(x)\sigma^{\mu\nu}\gamma_5\psi_1(x), && (2.114)
\end{aligned}$$

where we have used

$$\gamma^0\gamma^2(\gamma^\mu)^T = -\gamma^\mu\gamma^0\gamma^2. \tag{2.115}$$

Giving the derivation of $\overline{\psi}_1(x)\gamma^\mu\psi_2(x)$ as an example,

$$\begin{aligned}
\mathcal{C}\overline{\psi}_1\gamma^\mu\psi_2\mathcal{C}^\dagger &= \mathcal{C}\overline{\psi}_1\mathcal{C}^\dagger\mathcal{C}\gamma^\mu\mathcal{C}^\dagger\mathcal{C}\psi_2\mathcal{C}^\dagger\\
&= -e^{-i\beta_c}(\gamma^0\gamma^2\psi_1)^T\gamma^\mu e^{i\beta_c}(\overline{\psi}_2\gamma^0\gamma^2)^T\\
&= \left[\overline{\psi}_2\gamma^0\gamma^2(\gamma^\mu)^T\gamma^0\gamma^2\psi_1\right]^T\\
&= -\overline{\psi}_2\gamma^\mu\gamma^0\gamma^2\gamma^0\gamma^2\psi_1\\
&= \overline{\psi}_2\gamma^\mu\gamma^0\gamma^0\ \gamma^2\gamma^2\psi_1.\\
&= -\overline{\psi}_2\gamma^\mu\psi_1, && (2.116)
\end{aligned}$$

where a minus sign appears on the third step due to fermion anticommutation and applying Eq. (2.115) results in a minus sign on the fourth step.

### 2.3.6 Gluon field

The Lagrangian of quantum chromodynamics (QCD) based on an internal $SU(3)_C$ symmetry with $C$ denoting color is

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \sum_f \bar{q}_f(i\partial_\mu - m_f + g_sG^a_\mu t^a)\gamma^\mu q_f, \tag{2.117}$$

where $g_s$ is the strong interaction coupling constant, $q_f$ is a quark field with flavor index $f = u,\ d,\ c,\ s,\ t,\ b$ , and $q_f$ has an implicit color index $i = 1,\ 2,\ 3$, $m_f$ is the mass of $q_f$, $G_\mu^a$ with $a = 1, 2, ..., 8$ are the gluon fields, $t^a = \frac{\lambda^a}{2}$ where $\lambda^a$ are the Gell-Mann matrices [85]

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{2.118}$$

and

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g_s f^{abc} G_\mu^b G_\nu^c, \tag{2.119}$$

where $f^{abc}$ are the structure constants of $SU(3)$ defined as

$$[t_a, t_b] = i f^{abc} t_c, \tag{2.120}$$

given by [86]

$$\begin{aligned} f^{123} &= 1 \\ f^{147} &= -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} \\ f^{458} &= f^{678} = \frac{\sqrt{3}}{2}. \end{aligned} \tag{2.121}$$

The quantum numbers of the gluon are the same as that of the photon, so that $G_\mu = G_\mu^a t^a$ should transform in the same way as the photon does under P, C, and T as shown in Eq. (2.43) - Eq. (2.45). We write down the transformation properties of $G_\mu = \lambda^a G_\mu^a$ for convenience:

$$\mathcal{P} G_\mu(t, \boldsymbol{x}) \mathcal{P}^\dagger = (-1)^\mu G_\mu(t, -\boldsymbol{x}), \tag{2.122}$$
$$\mathcal{T} G_\mu(t, \boldsymbol{x}) \mathcal{T}^\dagger = (-1)^\mu G_\mu(-t, \boldsymbol{x}), \tag{2.123}$$
$$\mathcal{C} G_\mu(t, \boldsymbol{x}) \mathcal{C}^\dagger = -G_\mu(t, \boldsymbol{x}), \tag{2.124}$$

Here we also need to figure out the transformation properties of $G_\mu^a$ and $G_{\mu\nu}^a$. Since P transformation has no effect on $t^a$, for $G_\mu^a$, we have

$$\mathcal{P} G_\mu^a(t, \boldsymbol{x}) \mathcal{P}^\dagger = (-1)^\mu G_\mu^a(t, -\boldsymbol{x}). \tag{2.125}$$

The Gell-Mann matrices are Hermitian, and they are either symmetric and real, or antisymmetric and pure imaginary, so we have $(\lambda^a)^T = (\lambda^a)^* = c_a \lambda^a$, with $c_a = +1$ for $a = 1, 3, 4, 6, 8$ and $c_a = -1$ for $a = 2, 5, 7$. When applying T or C transformation to the Lagrangian, $\lambda^a$ will turn $(\lambda^a)^*$ under T and $(\lambda^a)^T$ under C. Accounting for the transformation properties of $G_\mu = G_\mu^a t^a$ under T and C as Eq. (2.123) and Eq. (2.124) respectively, we should have

$$\mathcal{T} G_\mu^a(t, \boldsymbol{x}) \mathcal{T}^\dagger = c_a (-1)^\mu G_\mu^a(-t, \boldsymbol{x}), \tag{2.126}$$

$$\mathcal{C} G_\mu^a(t, \boldsymbol{x}) \mathcal{C}^\dagger = -c_a G_\mu^a(t, \boldsymbol{x}). \tag{2.127}$$

Finally for $G_{\mu\nu}^a$ defined in Eq. (2.119), using the transformation properties of $\partial_\mu$ and $G_\mu^a$, we have

$$\mathcal{P} G_{\mu\nu}^a(t, \boldsymbol{x}) \mathcal{P}^\dagger = (-1)^\mu (-1)^\nu G_{\mu\nu}^a(t, -\boldsymbol{x}), \tag{2.128}$$

$$\mathcal{T} G_{\mu\nu}^a(t, \boldsymbol{x}) \mathcal{T}^\dagger = -c_a (-1)^\mu (-1)^\nu G_{\mu\nu}^a(-t, \boldsymbol{x}), \tag{2.129}$$

$$\mathcal{C} G_{\mu\nu}^a(t, \boldsymbol{x}) \mathcal{C}^\dagger = -c_a G_{\mu\nu}^a(t, \boldsymbol{x}), \tag{2.130}$$

where $c_a = +1$ for $a = 1, 3, 4, 6, 8$ and $c_a = -1$ for $a = 2, 5, 7$.

Given these transformation properties, it is obvious that the Lagrangian remains invariant under P, C, and T.

### 2.3.7 CPT theorem

Any of the discrete symmetries P, C, or T, or their combinations, may be violated in nature. However, the combined operation of P, C, and T is an exact symmetry of any interaction and under the three transformations all physical laws must be invariant, which is known as the "CPT theorem" [5, 6, 8, 7]. The CPT theorem can be stated that any theory, of which the Lagrangian is Hermitian, Lorentz invariant, local, and with commutation or anti-commutation relations obeying the spin-statistics theorem under a general set of conditions, is asserted to be CPT invariant.

Basic consequences of CPT symmetry include the equality of the masses as well as the lifetimes or decay widths of a particle and its antiparticle. The CPT theorem also guarantees that for a system with CPT symmetry, the violation (conservation) of one discrete symmetry implies the violation (conservation) of the complementary ones. Thus when dealing with the CP transformation acting on the operators of a Lagrangian with the basic conditions to guarantee CPT symmetry, we can examine their T transformation property, rather than analyzing their behaviors under the combined P and C transformations, which can be very convenient.

### 2.3.8 Summary of C, P, and T in QFT

The transformation properties of various fields discussed here are to be used in Ch. 7 where specific CP violating operators are investigated.

Table 2.2: Transformation properties for fermion bilinears, $\partial_\mu$, $A_\mu$, and $G_\mu$.

| | $\overline{\psi}\psi$ | $\overline{\psi}\gamma_5\psi$ | $\overline{\psi}\gamma^\mu\psi$ | $\overline{\psi}\gamma^\mu\gamma_5\psi$ | $\overline{\psi}\sigma^{\mu\nu}\psi$ | $\overline{\psi}\sigma^{\mu\nu}\gamma_5\psi$ | $\partial_\mu$ | $A_\mu/G_\mu$ |
|---|---|---|---|---|---|---|---|---|
| $P$ | $+1$ | $-1$ | $(-1)^\mu$ | $-(-1)^\mu$ | $(-1)^\mu(-1)^\nu$ | $-(-1)^\mu(-1)^\nu$ | $(-1)^\mu$ | $(-1)^\mu$ |
| $T$ | $+1$ | $+1$ | $(-1)^\mu$ | $(-1)^\mu$ | $-(-1)^\mu(-1)^\nu$ | $-(-1)^\mu(-1)^\nu$ | $-(-1)^\mu$ | $(-1)^\mu$ |
| $C$ | $+1$ | $+1$ | $-1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ |
| $CPT$ | $+1$ | $-1$ | $-1$ | $-1$ | $+1$ | $-1$ | $-1$ | $-1$ |

We list the transformation properties of fermion bilinears, $\partial_\mu$, photon field $A_\mu$, and gluon field $G_\mu = G_\mu^a \lambda^a$ in Table 2.2. Notice that for quark bilinears $\bar{\psi}_1(x)\Gamma^{\mu(\nu)}\psi_2(x)$, under C transformation, it turns in to $\bar{\psi}_2(x)\Gamma^{\mu(\nu)}\psi_1(x)$ with the specific C transformation factor listed in Table 2.2.

CHAPTER 3

# CP VIOLATION IN AND BEYOND THE STANDARD MODEL

The CP transformation combines the transformations of charge conjugation C and parity P. In this chapter, we will first discuss different mechanisms of CP violation in meson decays and consider the observables that can reveal each of them. In this thesis we focus on the possibility of studying CP violation through a charge asymmetry in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot — a probe of direct CP violation, as we discuss in the next section. Finally we discuss sources of CP violation both within and beyond the SM.

## 3.1 CP-violating observables

CP violation can be experimentally tested in many processes, such as meson decays, electric dipole moments (EDM), neutrino oscillations, and so forth. Note that meson decays and neutrino oscillations can occur without breaking CP, but there are experimental tests that can be made by combining meson decay and neutrino oscillation measurements. CP violation has been observed in meson decays, but not in neutrino oscillations thus far. In the case of permanent EDM searches, experimental searches have all yielded null results thus far. The limit on the magnitude of an EDM that comes out from such a search is an indication of that experiment's sensitivity. We know that CP is broken is the SM, but the permanent EDM that comes from SM effects is extremely small, much smaller than any experiment can yet detect. Thus if an experiment finds a nonzero EDM at current levels of sensitivity, then we have found new physics. We will focus on CP violation in meson decays, because this is central to my research project. This part is largely based on Branco, Lavoura, and Silva [80] and the review "CP violation in the quark sector" of PDG [41].

Generally, there are three types of CP violation in meson decays: direct CP violation (which reflects CP violation in the decay amplitudes), indirect CP violation (which reflects CP violation in meson mixing), and CP violation that appears through the interference of meson mixing and decay. Note that all three types of CP violation only appear for $|\Delta F| = 1$ decays, where $F$ is a flavor quantum number, say $S$, $C$, or $B$, which represents strange number, charm number or beauty number, respectively. The topic of my thesis, the charge asymmetry of $\eta \to \pi^+\pi^-\pi^0$ is a direct CP violating observable, and its test is like EDM searches in that if an experiment accurately finds a nonzero result for it we have found physics beyond the SM.

### 3.1.1 Direct CP violation

For a meson $M$ and its antiparticle $\overline{M}$, considering the transitions of the initial state meson $M$ and its CP conjugate $\overline{M}$ to a final state $f$ and its CP conjugate $\bar{f}$, respectively, we can define the decay amplitudes

$$\begin{aligned} A_f &= \langle f|\mathcal{H}|M\rangle, \quad \overline{A}_{\bar{f}} = \langle \bar{f}|\mathcal{H}|\overline{M}\rangle, \\ A_{\bar{f}} &= \langle \bar{f}|\mathcal{H}|M\rangle, \quad \overline{A}_f = \langle f|\mathcal{H}|\overline{M}\rangle, \end{aligned} \tag{3.1}$$

where $\mathcal{H}$ is the Hamiltonian involving the transitions of initial states to final states. Under a CP transformation,

$$CP|M\rangle = e^{i\xi_M}|\overline{M}\rangle, \quad CP|f\rangle = e^{i\xi_f}|\overline{f}\rangle, \tag{3.2}$$

where $\xi_M$ and $\xi_f$ are real numbers. If CP is conserved by the dynamics, which means

$$(CP)\mathcal{H}(CP)^\dagger = \mathcal{H} \quad \text{or} \quad [CP, \mathcal{H}] = 0, \tag{3.3}$$

we have

$$\overline{A}_{\bar{f}} = e^{i(\xi_f - \xi_M)} A_f, \tag{3.4}$$

which means $A_f$ and $\overline{A}_{\bar{f}}$ have the same magnitude but can differ in an overall phase [80]. Since the observables are related to the magnitudes of $A_f$ and $\overline{A}_{\bar{f}}$, the phase is unobservable. CP violation appears if

$$|\overline{A}_{\bar{f}}/A_f| \neq 1; \tag{3.5}$$

this is often called direct CP violation, which is the only possible CP violating effect in charged meson decays. The existence of this effect can be determined through a rate asymmetry A, which can be defined as

$$\mathcal{A} = \frac{\Gamma(\overline{M}^0 \to \bar{f}) - \Gamma(M^0 \to f)}{\Gamma(\overline{M}^0 \to \bar{f}) + \Gamma(M^0 \to f)} = \frac{|\overline{A}_{\bar{f}}/A_f|^2 - 1}{|\overline{A}_{\bar{f}}/A_f|^2 + 1}. \tag{3.6}$$

There are three kinds of phases that could appear in $A_f$ and $\overline{A}_{\bar{f}}$. The phase that appears as a complex conjugate in the CP-conjugate amplitude is usually called a weak, or CP-odd phase, whereas the phases that are the same in $A_f$ and $\overline{A}_{\bar{f}}$ are called strong, or CP-even phases. In addition to the weak and strong phases, as can be seen from Eq. (3.4), there is another global relative phase between $A_f$ and $\overline{A}_{\bar{f}}$ that is called a spurious phase, which appears because of an arbitrary choice of convention. A spurious phase does not originate from any dynamics, and it does not induce any CP violation, so that we can conveniently set it to zero.

Suppose the decay amplitudes are expressed as a sum of two interfering amplitudes

$$\begin{aligned} A_f &= |a_1|e^{i(\delta_1+\phi_1)} + |a_2|e^{i(\delta_2+\phi_2)}, \\ \overline{A}_{\bar{f}} &= |a_1|e^{i(\delta_1-\phi_1+\theta)} + |a_2|e^{i(\delta_2-\phi_2+\theta)} \end{aligned} \tag{3.7}$$

where $\delta_1$ and $\delta_2$ are strong phases, $\phi_1$ and $\phi_2$ are weak phases, and $\theta$ is a spurious phase and we can set $\theta = 0$ without loss of generality. We have [41]

$$\mathcal{A} = \frac{-2|a_1 a_2| \sin(\delta_2 - \delta_1) \sin(\phi_2 - \phi_1)}{|a_1|^2 + |a_2|^2 + 2|a_1 a_2| \cos(\delta_2 - \delta_1) \cos(\phi_2 - \phi_1)}. \tag{3.8}$$

Thus a non-zero rate asymmetry, and thus CP violation, will appear when the interfering amplitudes have different strong phases ($\delta_1 \neq \delta_2$) and different weak phases ($\phi_1 \neq \phi_2$). The absolute value of the asymmetry reaches its maximum value 1 in the limiting case $|\delta_1 - \delta_2| = |\phi_1 - \phi_2| = \pi/2$ and $|a_1| = |a_2|$.

There are other probes of direct CP violation: $\epsilon'$ in the B system [87, 88]; effects in the angular distribution in $B \to V_1 V_2$ [89]; a population asymmetry about the mirror line in $|A(B \to f_{CP})|^2 + |A(\overline{B} \to f_{CP})|^2$ decay [90], and so forth.

The topic of this thesis is CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay, which is an observable of direct CP violation that can be probed through a population asymmetry [90]. We will introduce the method of studying a population asymmetry in the distribution of the Dalitz plot, which is first proposed by Gardner [90] and has been used to study untagged B decay [53] and untagged D meson decay [91], and which we have extended to $\eta \to \pi^+\pi^-\pi^0$ decay [92].

**Studying CP violation through a Dalitz plot**

Considering a spinless meson, with mass $M$ and four momentum $P^\mu = (M, 0)$, decaying to three pseudo-scalar particles with masses $m_i$ and four momenta $p_i^\mu = (E_i,\ \boldsymbol{p})$, where $i = 1,\ 2,\ 3$, we can define the Lorentz invariant masses

$$m_{ij}^2 = (p_i + p_j)^2 = (P - p_k)^2 = M^2 + m_k^2 - 2ME_k, \tag{3.9}$$

with $i \neq j \neq k$. With the relation

$$m_{12}^2 + m_{23}^2 + m_{13}^2 = M^2 + m_1^2 + m_2^2 + m_3^2, \tag{3.10}$$

there are two independent invariant values of $m_{ij}^2$. If we choose to use $m_{12}^2$ and $m_{23}^2$, starting with $m_{12}^2$, the minimum value of $m_{12}^2$ is $(m_1 + m_2)^2$ and its maximum is $(M - m_3)^2$. Given a value of $m_{12}^2$, the minimum and maximum values of $m_{23}^2$ can be obtained when $\boldsymbol{p}_2$ is parallel or antiparallel to $\boldsymbol{p}_3$ [41], respectively:

$$\begin{aligned} (m_{23}^2)_{max} &= (E_2^* + E_3^*)^2 - (\sqrt{E_2^* - m_2^2} - \sqrt{E_3^* - m_3^2})^2, \\ (m_{23}^2)_{min} &= (E_2^* + E_3^*)^2 - (\sqrt{E_2^* - m_2^2} + \sqrt{E_3^* - m_3^2})^2, \end{aligned} \tag{3.11}$$

where $E_2^* = (m_{12}^2 - m_1^2 + m_2^2)/(2m_{12})$ and $E_3^* = (M^2 - m_{12}^2 - m_3^2)/(2m_{12})$ are the energies of $m_2$ and $m_3$ in the rest frame of $m_{12}$ [41].

The decay width can be expressed as

$$\Gamma = \int \frac{1}{256\pi^3 M^3} |\mathcal{M}|^2 dm_{12}^2 dm_{23}^2, \tag{3.12}$$

where $|\mathcal{M}|^2$, with invariant amplitude $\mathcal{M}$, reflects the dynamics. R. H. Dalitz introduced the convenient technique [93] to analyze three-body decays through the scatter plot in $m_{12}^2$ versus $m_{23}^2$, which is called the Dalitz plot. If $|\mathcal{M}|^2$ is a constant, or equivalently if the decay is through non-resonant processes, the events distribution will be uniform across the kinematically allowed region in the Dalitz plot. However if a decay is dominated by resonant processes, there will be a varying population distribution in the Dalitz plot, with peaks around the masses of the final-state resonances. Given the Dalitz plot of $D^0 \to K^-\pi^+\eta$ shown in Fit. 3.1.1 from Belle [94] as an illustration, it includes six resonances, which are $a_0(980)^+$, $a_2(1320)^+$, $\overline{K}^*(892)^0$, $\overline{K}^*(1410)^0$, $K^*(1680)$, and $K_2^*(1980)$ [94].

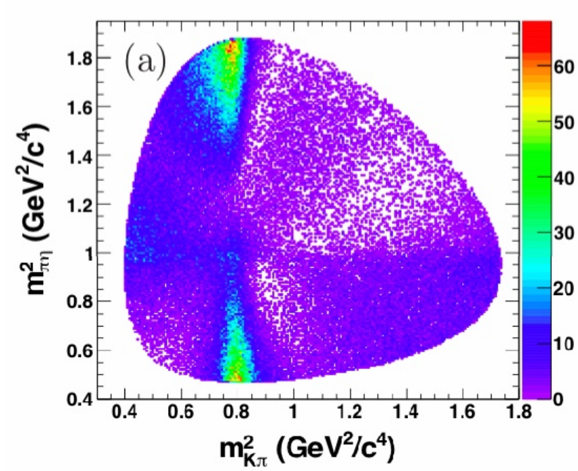


Figure 3.1: Dalitz plot of $D^0 \to K^-\pi^+\eta$ from Belle [94].

The Dalitz plot technique is also widely used in studying CP violation.

For the untagged meson decay, where the amplitudes of decaying particle and its antiparticle are combined, where we give $B^0(\overline{B}^0) \to \pi^+\pi^-\pi^0$ decay as an example. Defining the Mandelstam variables as

$$s = (p_{\pi^+} + p_{\pi^-})^2, \quad ,t = (p_{\pi^-} + p_{\pi^0})^2, \quad ,u = (p_{\pi^+} + p_{\pi^0})^2, \tag{3.13}$$

the untagged decay width can be expressed as

$$\Gamma_{3\pi} = \int \frac{dtdu}{256\pi^3 m_{B^0}^3}(|\mathcal{M}_{3\pi}|^2 + |\overline{\mathcal{M}}_{3\pi}|^2), \tag{3.14}$$

where $\mathcal{M}_{3\pi}$ stands for the amplitude of $B^0 \to \pi^+\pi^-\pi^0$ and $\overline{\mathcal{M}}_{3\pi}$ represents the amplitude of $\overline{B}^0 \to \pi^+\pi^-\pi^0$. Under CP, we have [53]

$$\begin{aligned} & |\mathcal{M}_{3\pi}(p_{\pi^+}, p_{\pi^-}, p_{\pi^0})|^2 + |\overline{\mathcal{M}}_{3\pi}(p_{\pi^+}, p_{\pi^-}, p_{\pi^0})|^2 \\ \overset{CP}{\longleftrightarrow} \quad & |\mathcal{M}_{3\pi}(p_{\pi^-}, p_{\pi^+}, p_{\pi^0})|^2 + |\overline{\mathcal{M}}_{3\pi}(p_{\pi^-}, p_{\pi^+}, p_{\pi^0})|^2. \end{aligned} \tag{3.15}$$

So the untagged decay width, that integrating $|\mathcal{M}_{3\pi}|^2 + |\overline{\mathcal{M}}_{3\pi}|^2$ over the physical phase space, as in Eq. (3.14), is a CP even quantity because the integration region is $p_{\pi^+} \leftrightarrow p_{\pi^-}$ symmetric [53]. However, as first proposed by Gardner [90], the CP transformation corresponds to the mirror transformation of the Dalitz plot with $t$ versus $u$, equivalent with $p_{\pi^+} \leftrightarrow p_{\pi^-}$, thus the direct CP violation can be observed through the population asymmetry across the mirror line ($u = t$) of the Dalitz plot [90, 53] as

$$\mathcal{A}_{3\pi} \equiv \frac{\Gamma_{3\pi}[u > t] - \Gamma_{3\pi}[u < t]}{\Gamma_{3\pi}[u > t] + \Gamma_{3\pi}[u < t]}, \tag{3.16}$$

or equivalently,

$$\mathcal{A}_{LR} = \frac{N_+ - N_-}{N_+ + N_-}, \tag{3.17}$$

where $N_+$ ($N_-$) stands for the number of events that $\pi^+$ has greater(smaller) energy than $\pi^-$. We see that the test of the CP violating observable $A_{3\pi}$ or $A_{LR}$ needs not separate the initial particles $B^0$ and $\overline{B}^0$. However the theoretical dynamics [53] is embedded in $\mathcal{M}_{3\pi}$ and $\overline{\mathcal{M}}_{3\pi}$ in Eq. (3.14), of which some theoretical parameters can be determined by the study of $B^0(\overline{B}^0) \to \pi^+\pi^-\pi^0$ decay [90, 53].

The topic of this thesis, $\eta \to \pi^+\pi^-\pi^0$ is analogous to the untagged $B$ meson decaying to $\pi^+\pi^-\pi^0$, though there is no $\overline{\mathcal{M}}_{3\pi}$ because $\eta$ is its own antiparticle. The Dalitz plot for $\eta \to \pi^+\pi^-\pi^0$ with $t$ versus $u$ is shown in Fig. 3.2. For $\eta \to \pi^+\pi^-\pi^0$

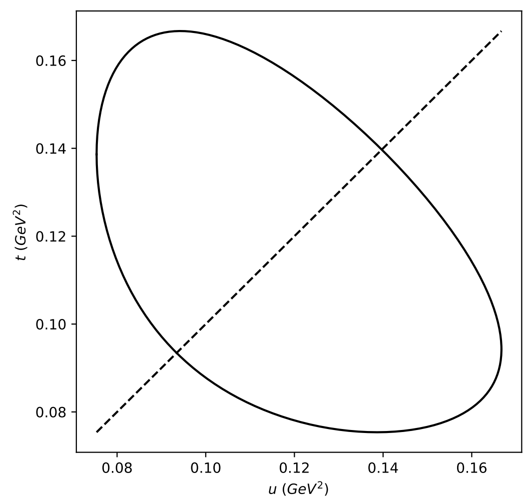


Figure 3.2: Dalitz plot with $t$ versus $u$ for $\eta \to \pi^+\pi^-\pi^0$. The dashed line indicates the $t = u$ mirror line.

decay, it is convenient to introduce the normalized variables commonly defined as [95]

$$\begin{aligned} X &\equiv \rho \sin\phi \equiv \sqrt{3}\frac{T_{\pi^+} - T_{\pi^-}}{Q_\eta} = \frac{\sqrt{3}}{2m_\eta Q_\eta}(u - t), \\ Y &\equiv \rho \cos\phi \equiv \frac{3T_{\pi^0}}{Q_\eta} - 1 = \frac{3}{2m_\eta Q_\eta}[(m_\eta - m_{\pi^0})^2 - s] - 1, \end{aligned} \tag{3.18}$$

where ($\rho$, $\phi$) is the polar coordinate of the Dalitz-Fabri plot [93, 96], $T_i$ are the kinematic energies of the pions in the rest frame of $\eta$, and $Q_\eta = T_{\pi^+} + T_{\pi^-} + T_{\pi^0} =$

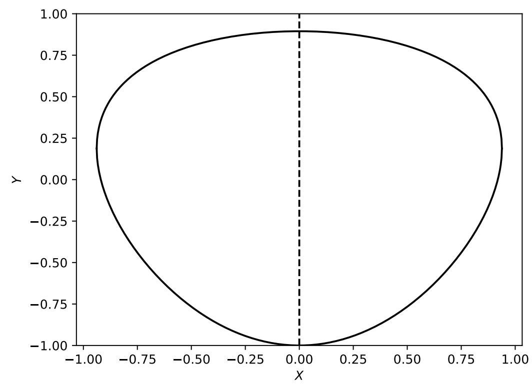


Figure 3.3: Dalitz plot with $X$ versus $Y$ for $\eta \to \pi^+\pi^-\pi^0$. The dashed line indicates the $X = 0$ mirror line.

$m_\eta - m_{\pi^+} - m_{\pi^-} - m_{\pi^0}$. The Dalitz plot with $Y$ versus $X$ is depicted in Fig. 3.1.1. Under CP, $t \leftrightarrow u$ implies

$$X \xrightarrow{CP} -X, \quad Y \xrightarrow{CP} Y. \tag{3.19}$$

Accordingly, the left-right asymmetry in Eq. (3.16) or (3.17) can be observed from the asymmetry across the mirror line $X = 0$. The absolute squared amplitude can be expressed in the polynomial expansion around $(X, Y) = (0, 0)$ as [72]

$$|A(X,Y)|^2 \simeq N(1+aY+bY^2+cX+dX^2+eXY+fY^3+gX^2Y+hXY^2+lX^3+\dots), \tag{3.20}$$

where $N$ is the normalization factor, and $a$, $b$, $\dots$ are usually called Dalitz plot parameters, which can be set by fitting the Dalitz plot distribution using the above formula. The non-zero coefficients multiplying odd powers of $X$ ($c$, $e$, $h$, or $l$) would show proof of the existence of CP violation in this decay.

In Ch. 6, we are going to introduce the method of studying isospin-dependent patterns of CP violating amplitudes through the distribution asymmetry of the Dalitz plot [92].

### 3.1.2 Indirect CP violation

In this part, we will discuss the mixing and decays of neutral mesons $M$ and $\overline{M}$, of which $M$ may refer to either $K^0$, $D^0$, $B_d^0$, or $B_s^0$. Under a CP transformation, assuming $(CP)^2 = 1$, we can define

$$CP|M\rangle = e^{i\xi}|\overline{M}\rangle, \qquad CP|\overline{M}\rangle = e^{-i\xi}|M\rangle. \tag{3.21}$$

Because of the weak interaction, $M$ and $\overline{M}$ both mix and decay to other states. A state that is initially a superposition of $M$ and $\overline{M}$ evolves in time with the form

$$a(t)|M\rangle + b(t)|\overline{M}\rangle + c_1(t)|f_1\rangle + c_2(t)|f_2\rangle + ... \tag{3.22}$$

where $f_1$, $f_2$... are all possible final states to which $M$ and $\overline{M}$ can decay. When we only want to figure out $a(t)$ and $b(t)$, rather than the values of $c_i(t)$, and if the time scale we consider is much larger than the typical strong interaction scale, a simplified formalism [97, 98] can be used, which is determined by a $2 \times 2$ Hamiltonian $\boldsymbol{H}$ that is not Hermitian. It can be written in terms of Hermitian matrices $\boldsymbol{T}$ and $\boldsymbol{\Gamma}$ as

$$\boldsymbol{H} = \boldsymbol{T} + i\boldsymbol{\Gamma}. \tag{3.23}$$

$\boldsymbol{T}$ and $\boldsymbol{\Gamma}$ are associated with $(M, \overline{M}) \leftrightarrow (M, \overline{M})$ transitions. Assuming the system is invariant under CPT, the eigenvectors of $\boldsymbol{H}$ can be written as

$$|M_H\rangle = p|M\rangle + q|\overline{M}\rangle, \quad |M_L\rangle = p|M\rangle - q|\overline{M}\rangle \tag{3.24}$$

where $p$ and $q$ are complex parameters with the normalization condition $|p|^2+|q^2| = 1$. $M_H$ and $M_L$ have well defined mass and decay widths, with the label representing heavy or light (notice that in neutral $K$ meson decays, $K_L$ and $K_S$ are used instead, with $K_L$ for the long-lived state and $K_S$ for the short-lived state). We define

$$m \equiv \frac{m_H + m_L}{2}, \qquad \Gamma \equiv \frac{\Gamma_H + \Gamma_L}{2}, \tag{3.25}$$

$$\Delta m \equiv m_H - m_L, \qquad \Delta\Gamma = \Gamma_H - \Gamma_L. \tag{3.26}$$

Diagonalizing $\boldsymbol{H}$ yields

$$\left(\frac{q}{p}\right)^2 = \frac{T_{12}^* - (i/2)\Gamma_{12}^*}{T_{12} - (i/2)\Gamma_{12}}. \tag{3.27}$$

The CP invariance condition reads [80]

$$T_{12}^* = e^{2i\xi}T_{12}, \quad \Gamma_{12}^* = e^{2i\xi}\Gamma_{12}, \tag{3.28}$$

which yields

$$\left|\frac{q}{p}\right| = |e^{i\xi}| = 1. \tag{3.29}$$

Note that this result does not require CPT symmetry. If Eq. 3.29 is not satisfied, there exists CP violation in the mixing between $M$ and $\overline{M}$, which is usually called indirect CP violation.

### 3.1.3 Interference CP violation

CP violation can also exist from the interference between a decay without mixing, that is a direct $M \to f$ decay, and a decay with mixing, $M \to \overline{M} \to f$. The complex parameter that manifests this kind of CP violation is defined as

$$\lambda_f \equiv \frac{q}{p}\frac{\overline{A}_f}{A_f}, \tag{3.30}$$

and the condition for the presence of CP violation is

$$\mathrm{Im}(\lambda_f) \neq 0. \tag{3.31}$$

This form of CP violation can be observed from an asymmetry in $M$ and $\overline{M}$ decaying to CP eigenstates [99, 100, 80].

### 3.2 CP violation in the SM

#### 3.2.1 CKM matrix

The Standard Model Lagrangian is determined by its gauge symmetry with the gauge group $SU_C(3) \times SU_L(2) \times U_Y(1)$, renormalizability, and its particle content, that is, quark fields, lepton fields, Higgs field, and gauge fields. Quark and lepton fields carry a generation index 1, 2, 3 running over the three generations of up-type quarks $\mathbf{u} = (u,\ c,\ t)$, down-type quarks $\mathbf{d} = (d,\ s,\ b)$, charged leptons $\mathbf{e} = (e,\ \mu,\ \tau)$, and neutrinos $\nu = (\nu_e,\ \nu_\mu,\ \nu_\tau)$. The left-handed fermions are doublets of $SU_L(2)$,

$$q_{Lp} = \begin{pmatrix} u^i_{Lp} \\ d_{Lp} \end{pmatrix}, \quad l_{Lp} = \begin{pmatrix} \nu_{Lp} \\ e_{Lp} \end{pmatrix}, \tag{3.32}$$

while the right-handed fields $u_{Rp}$, $d_{Rp}$, and $e_{Rp}$ are $SU(2)_L$ singlets, where the superscript $p$ stands for the generation index which we will leave as implicit for convenience in the following discussion. The Higgs field $\varphi$ denotes an $SU_L(2)$ doublet of scalar fields that can be expressed as

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}, \tag{3.33}$$

and

$$\tilde{\varphi} = i\sigma^2(\varphi^\dagger)^T = \begin{pmatrix} \varphi^0 \\ -\varphi^- \end{pmatrix}, \tag{3.34}$$

where $\sigma^i$ are Pauli matrixes. The left and right handed quarks are in the fundamental representation of $SU_C(3)$. The gauge bosons associated with the gauge groups $SU_C(3)$, $SU_L(2)$, and $U_Y(1)$ are $G^a_\mu$, $W^i_\mu$, and $B_\mu$, respectively, with $a = 1,\ 2, ...8$ and $i = 1,\ 2,\ 3$.

The standard mass dimension-four Lagrangian $\mathcal{L}^{(4)}_{SM}$ is

$$\begin{aligned} \mathcal{L}^{(4)}_{SM} &= -\frac{1}{4}(G^A_{\mu\nu}G^{A\mu\nu} + W^I_{\mu\nu}W^{I\mu\nu} + B_{\mu\nu}B^{\mu\nu}) \\ &+\bar{q}_L i\not{D} q_L + \bar{u}_R i\not{D} u_R + \bar{d}_R \not{D} d_R + \bar{l}_L i\not{D} l_L + \bar{e}_R i\not{D} e_R + D_\mu\varphi^\dagger D^\mu\varphi \\ &+\mu^2\varphi^\dagger\varphi - \frac{\lambda}{2}(\varphi^\dagger\varphi)^2 - \bar{q}_L Y_d \varphi d_R - \bar{q}_L Y_u \tilde{\varphi} u_R - \bar{l}_L Y_e \varphi e_R + \text{H.c.} \\ &-\frac{\epsilon^{\mu\nu\alpha\beta}}{64\pi^2}(g_s^2\theta G^a_{\mu\nu}G^a_{\alpha\beta} + g^2\theta_w W^i_{\mu\nu}W^i_{\alpha\beta}), \end{aligned} \tag{3.35}$$

where

$$D_\mu = \partial_\mu - ig_s G^a_\mu T^a - igW^i_\mu S^i - ig'YB_\mu, \tag{3.36}$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu \tag{3.37}$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ijk} W^j_\mu W^k_\nu \tag{3.38}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{3.39}$$

Here $T^a = \frac{1}{2}\lambda^a$ and $S^i = \frac{1}{2}\sigma^i$ are SU(3) and SU(2) generators, where $\lambda^a$ and $\sigma^a$ are Gell-Mann and Pauli matrices, respectively. We relate these gauge fields to those of the physical basis via

$$W^1_\mu = \frac{1}{\sqrt{2}}(W^+_\mu + W^-_\mu), \tag{3.40}$$

$$W^2_\mu = \frac{i}{\sqrt{2}}(W^+_\mu - W^-_\mu), \tag{3.41}$$

$$W^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu, \tag{3.42}$$

$$B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu, \tag{3.43}$$

$$c_w \equiv \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \quad , \quad s_w \equiv \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \tag{3.44}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the four-dimensional Levi-Civita symbol with $\epsilon^{0123} = +1$. Note that the terms with $\theta$ and $\theta_W$ in the last line of Eq. (3.35) break P and CP, and including them in the SM Lagrangian is not standard, because the CP-violating effects they would generate are immeasurably small. The mass-dimension 4 SM Lagrangian is renormalizable, whereas renormalizability is not guaranteed in the higher mass-dimensional Lagrangians of SM effective field theory. Using the relations above, we have

$$-igW^i_\mu S^i - ig'YB_\mu = -ig(W^+_\mu S^+ + W^-_\mu S^-) - i\frac{g}{c_w}Z_\mu(S^3 - Qs_w^2) - ieA_\mu Q, \tag{3.45}$$

where

$$S^\pm \equiv \frac{S^1 \pm S^2}{\sqrt{2}}, \quad Q = S^3 + Y, \quad g = \frac{e}{s_w}. \tag{3.46}$$

The $U(1)_Y$ hypercharge of $q_L$, $u_R$, $d_R$, $l_L$, $e_R$, $\nu_R$, and $\varphi$ is shown in Table 3.1 [101]. The $SU(2) \times U(1)$ Yukawa couplings involving the quarks are

Table 3.1: Hypercharge of particle content of the SM.

| particle | $l^i_{Lp}$ | $e_{Rp}$ | $q^i_{Lp}$ | $u_{Rp}$ | $d_{Rp}$ | $\varphi^i$ |
|---|---|---|---|---|---|---|
| $Y$ | $-\frac{1}{2}$ | $-1$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $\frac{1}{2}$ |

$$\mathcal{L}_Y = -\bar{q}_L Y_d \varphi d_R - \bar{q}_L Y_u \tilde{\varphi} u_R + h.c. \tag{3.47}$$

where $Y^d$ and $Y^u$ are $3\times3$, complex matrices in the quark generation space, and they need not be either Hermitian or symmetric. Under a CP transformation the operators in Eq. (3.47) interchange with their Hermitian conjugates, without changing the couplings. So the CP operation is equivalent to the substitution[81]

$$Y^{ij}_d \to (Y^{ij}_d)^*, \quad Y^{ij}_u \to (Y^{ij}_u)^*. \tag{3.48}$$

Since there is no reason to require $Y_d^{ij}$ and $Y_u^{ij}$ to be real-valued, it seems Eq. (3.47) breaks CP symmetry. However, we can try to eliminate this by applying chiral transformations that are independent for left- and right-handed fields. Defining unitary matrices $U^u$, $W^u$, $U^d$, and $W^d$ by [81]

$$Y_u Y_u^\dagger = U_u (D_u)^2 U_u^\dagger, \quad Y_u^\dagger Y_u = W_u (D_u)^2 W_u^\dagger, \tag{3.49}$$
$$Y_d Y_d^\dagger = U_d (D_d)^2 U_d^\dagger, \quad Y_d^\dagger Y_d = W_d (D_d)^2 W_d^\dagger, \tag{3.50}$$

where $D_u^2$ and $D_d^2$ are diagonal matrices with positive eigenvalues. Then we have

$$\begin{aligned} Y_u &= U_u D_u W_u^\dagger, \\ Y_d &= U_d D_d W_d^\dagger, \end{aligned} \tag{3.51}$$

where $D_u$ and $D_d$ are also diagonal matrices and their diagonal elements are the square root of eigenvalues of Eq. (3.49) and Eq. (3.50), respectively [81]. Now making the transformations

$$\begin{aligned} u_R^i &\to W_u^{ij} u_R^j, \qquad d_R^i \to W_d^{ij} d_R^j, \\ u_L^i &\to U_u^{ij} u_L^j, \qquad d_L^i \to U_d^{ij} d_L^j. \end{aligned} \tag{3.52}$$

We find $W_u$, $W_d$, $U_u$, and $U_d$ are eliminated from Eq. (3.47), as well as other couplings involving quark currents [81] in the SM Lagrangian, Eq. (3.35), except for those with $W_\mu^\pm$. For example, for the quark bilinear coupled with $W_\mu^+$ from $\bar{q}_L i\not{D} q_L$ in Eq. (3.35), since $W_u$, $W_d$, $U_u$, and $U_d$ are matrices in the quark generation space, they commute with the covariant derivative. Using Eq. (3.52), we find

$$\frac{1}{\sqrt{2}} \bar{u}_L^i \gamma^\mu d_L^i \to \frac{1}{\sqrt{2}} \bar{u}_L^i \gamma^\mu (U_u^\dagger U_d)^{ij} d_L^j, \tag{3.53}$$

which means the charged weak interaction links the three $u_L^i$ quarks with a unitary rotation of the three $d_L^i$ quarks by a unitary matrix

$$V = U_u^\dagger U_d \equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{3.54}$$

The matrix $V$ is called the Cabibbo-Kobayashi-Maskawa(CKM) matrix. We can see that the weak charged interaction mixes quark flavors. Defining [102, 103, 104]

$$\begin{aligned} \lambda &= \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}, \qquad A\lambda^2 = \lambda \frac{V_{cb}}{V_{us}}, \\ V_{ub}^* &= A\lambda^3(\rho + i\eta) = \frac{A\lambda^3(\bar{\rho} + i\bar{\eta})\sqrt{1 - A^2\lambda^4}}{\sqrt{1-\lambda^2}[1 - A^2\lambda^4(\bar{\rho} + i\bar{\eta})]}, \end{aligned} \tag{3.55}$$

where $\bar{\rho}$ and $\bar{\eta}$ can be expressed as $\bar{\rho} = \rho(1 - \lambda^2/2 + \dots)$, $\bar{\eta} = \eta(1 - \lambda^2/2 + \dots)$, the CKM matrix can be conveniently expressed in Wolfenstein parametrization [102] as

$$V = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4), \tag{3.56}$$

where $\lambda \approx 0.23$ [41] plays the role of an expansion parameter and $\eta$ represents the CP violating phase. If we replace $\rho$ and $\eta$ in the CKM matrix with $r\bar{h}o$ and $e\bar{t}a$, it becomes unitary to all orders in lambda. [41].

For a general unitary $n \times n$ complex matrix contains $2n^2$ real parameters, it contains $n(n-1)/2$ independent rotation angles and $(n-1)(n-2)/2$ independent phases [79]. Thus when there are 2 generations with $u$, $d$, $c$, and $s$, $V$ is a $2 \times 2$ real matrix with one rotation angle [81] called the Cabibbo angle. When there are 3 generations, the CKM matrix contains 3 rotation angles, of which one is the Cabibbo angle, and one phase, which is the CP violating source of the SM and called KM phase.

The CKM matrix which mixes the 3-generation quarks is the dominant source of the CP violation in meson decays; moreover, there is no compellingly significant, direct evidence for more CP violating sources beyond the SM in $K$, $B$, and $D$ meson decays. However, the CP violation within the SM cannot explain the observed baryon asymmetry in universe[31, 32, 105, 15]. The search for new sources and the precision of tests of CPV in the SM suggests that we can have subdominant CPV from new physics.

### 3.2.2 The strong CP problem

In the SM, there is an intriguing problem of the strong and electroweak interactions: the "strong CP problem". Standard Model gauge symmetry and renormalizabilty should suffice to yield a term that breaks CP, that is,

$$\mathcal{L}_\theta = \theta_{QCD} \frac{g_s^2}{32\pi} G^{a\mu\nu} \tilde{G}^a_{\mu\nu}, \tag{3.57}$$

where $\tilde{G}^a_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} G^{a\alpha\beta}$. According to the CP transformation properties in Ch. 2, under parity P, charge conjugation C, and time reversal T, we have

$$G^{a\mu\nu} \tilde{G}^a_{\mu\nu} \xrightarrow{P} (-1)^\mu (-1)^\nu (-1)^\alpha (-1)^\beta G^{a\mu\nu} \tilde{G}^a_{\mu\nu}, \tag{3.58}$$

$$G^{a\mu\nu} \tilde{G}^a_{\mu\nu} \xrightarrow{C} (-1)(-1) G^{a\mu\nu} \tilde{G}^a_{\mu\nu} = +G^{a\mu\nu} \tilde{G}^a_{\mu\nu}, \tag{3.59}$$

$$G^{a\mu\nu} \tilde{G}^a_{\mu\nu} \xrightarrow{T} (-1)^\mu (-1)^\nu (-1)^\alpha (-1)^\beta G^{a\mu\nu} \tilde{G}^a_{\mu\nu}. \tag{3.60}$$

Since $\epsilon_{\mu\nu\alpha\beta}(-1)^\mu(-1)^\nu(-1)^\alpha(-1)^\beta = -1$, the $\theta$ term $\mathcal{L}_\theta$ is P and T violating, whereas it is C conserving. This is incomprehensible, since all of our experimental results are in good agreement with the assumption that QCD automatically conserves C, P, and T. This problem is called the "strong CP problem".

The $\theta$ term in Eq. (3.57) is closely related to the $U(1)_A$ problem [106, 107] of QCD, which is posed as that of a missing light state [106], or of why the $\eta'$ is so heavy [108].As shown by 't Hooft [21, 20, 109], including instantons can give rise to the $U(1)_A$ symmetry breaking without producing a Goldstone boson, thus solving the $U(1)_A$ problem. This solution to the $U(1)_A$ problem forces the QCD Lagrangian to include the $\theta$ term. Note that there is another solution with large $N_f$ (the number of quark flavors) limit in QCD that does not need the inclusion of instantons raised by

Witten [110] and Veneziano [111]. This approach relates the $\eta'$ mass to the Witten-Veneziano relation $\chi_t = F_\pi^2/12\left(m_\eta^2 + m_{\eta'}^2 - 2m_K^2\right)$ [110, 111, 112], where $\chi_t$ is the topological susceptibility which is defined as [110, 111]

$$\chi_t \equiv \frac{1}{4}\langle\left(\epsilon^{\mu\nu\alpha\beta}G^a_{\mu\nu}G_{\alpha\beta}\right)\left(\epsilon^{\rho\sigma\kappa\lambda}G^a_{\rho\sigma}G_{\kappa\lambda}\right)\rangle. \tag{3.61}$$

If $\epsilon^{\mu\nu\alpha\beta}G^a_{\mu\nu}G^a_{\alpha\beta}$ has an effect, then $\chi_t \neq 0$ and the mass of $\eta'$ can be large. Solving $\chi_t$ from the Witten-Veneziano relation gives $\chi_t = (174\ \text{MeV})^4$ [110, 111, 112].

We see that the $\theta$ term is a total derivative:

$$G_{\mu\nu}\tilde{G}^{\mu\nu} = \partial_\mu J^\mu, \tag{3.62}$$

with

$$J^\mu \equiv \epsilon^{\mu\nu\rho\sigma}(G^a_\nu G^a_{\rho\sigma} - \frac{2}{3}f^{abc}A^a_\nu A^b_\rho A^c_\sigma). \tag{3.63}$$

One might think that we can adopt $J_\mu = 0$ for the gluon field at spatial infinity, so that the inclusion of $\mathcal{L}_\theta$ would have no effect on physics in QCD. This is not true due to the complex topological structure of the QCD vacuum state, and the gauge fields do not fall off to zero at infinity. This is consistent with our conclusion that the $\theta$ term should appear in the QCD Lagrangian so that it can explain the $\eta'$ puzzle.

When including the weak interactions, the strong CP problem becomes more entangled. Considering the Yukawa coupling after electroweak symmetry breaking, when the Higgs field $\varphi^0$ obtains a vacuum expectation value (VEV) with the form $\langle\varphi\rangle = (0, v)^T$, then quarks obtain their masses from Eq. (3.47)

$$-\frac{v}{\sqrt{2}}\bar{d}^i_L Y^{ij}_d d^j_R - \frac{v}{\sqrt{2}}\bar{u}^i_L Y^{ij}_u u^j_R + h.c. \tag{3.64}$$

After diagonalizing $Y^u$ and $Y^d$ using Eq. (3.51) and making the transformation of $u_L$, $u_R$, $d_L$ and $d_R$ as in Eq. (3.52), we have

$$-\frac{v}{\sqrt{2}}\bar{d}^i_L D^{ii}_d d^i_R - \frac{v}{\sqrt{2}}\bar{u}^i_L D^{ii}_u u^i_R + h.c. \tag{3.65}$$

with

$$m^i_d = \frac{v}{\sqrt{2}}D^{ii}_d, m^i_u = \frac{v}{\sqrt{2}}D^{ii}_u. \tag{3.66}$$

Writing the up quark mass term as an example,

$$\mathcal{L}^u_m = -\frac{v}{\sqrt{2}}\bar{u}_L D_u u_R - \frac{v}{\sqrt{2}}\bar{u}_R D^\dagger_u u_L = -\frac{v}{2\sqrt{2}}[\bar{u}(D_u + D^\dagger_u)u + \bar{u}(D_u - D^\dagger_u)\gamma_5 u] \tag{3.67}$$

The terms proportional to $\bar{u}\gamma_5 u$ can be removed by the chiral rotation [79]

$$u^i \to e^{-i\frac{1}{2}\alpha_i\gamma_5}u^i, \tag{3.68}$$

when denoting the matrix elements of $\frac{v}{\sqrt{2}}D^{ii}_u$ as $m^i_u e^{i\alpha^i}$. However, the current connected to this rotation is not conserved

$$\partial^\mu j^{5i}_\mu = \partial^\mu\bar{u}^i\gamma\gamma_{\mu 5}u^i = 2m^i_u i\bar{u}\gamma_5 u + \frac{g_s^2}{16\pi^2}G\cdot\tilde{G} \neq 0 \tag{3.69}$$

This induces a contribution to the $\theta$ term with [79]

$$\bar{\theta} = \theta_{QCD} - \theta_M, \tag{3.70}$$

with

$$\theta_M = \arg\det(M_u M_d), \tag{3.71}$$

where $M_u$ and $M_d$ are diagonal matrices with $m_u^i$ and $m_d^i$ as their diagonal elements, respectively.

The most pertinent restriction on the $\theta$ term comes from the neutron electric dipole momentum (nEDM). The relation between the nEDM and the $\theta$ term is [113]

$$d_n^{QCD} \approx -\bar{\theta}(0.9 - 1.2) \times 10^{-16} \text{ e} \cdot \text{cm}. \tag{3.72}$$

The most recent result for the value of nEDM is [42]

$$d_n = (0.0 \pm 1.1_{\text{stat}} \pm 0.2_{\text{sys}}) \times 10^{-26} \text{ e} \cdot \text{cm}, \tag{3.73}$$

then we have

$$|\bar{\theta}| < 1.5 \times 10^{-10}. \tag{3.74}$$

This is strange that a dimensionless parameter should be so tiny, and it is considered to be unnatural because it is so small for no apparent reason, which is the other puzzle of the "strong CP problem".

The most widely accepted solution to the "strong CP problem" is the axion, which is generated by the spontaneously breaking of another $U(1)_A$ symmetry added to the SM proposed by Peccei and Quinn [78, 114]. The Peccei-Quinn mechanism can be equivalently comprehended as promoting $\bar{\theta}$ from a constant coefficient to a dynamical field – the axion, which was first proposed by Weinberg [115] and Wilczek [116]. A general review can be found in Brubaker [117].

## 3.3 CP violation beyond the SM

Accounting for the failure to give adequate explanation to the baryon asymmetry in universe, there should exist new CP violating mechanisms beyond the SM. In this part, we briefly introduce some popular beyond the SM models like the two Higgs doublet model [118, 119, 120, 121], the left-right symmetric model [122, 123, 124, 125], and weak scale supersymmetry[126, 127, 128, 129, 130, 131].

### 3.3.1 two Higgs-doublet model

The SM consists of only one doublet of the scalar sector with weak hypercharge $Y = 1/2$. Adding a second Higgs doublet is one of the simplest extensions of the SM. Our description of the two-Higgs-doublet model (2HDM) is taken from Branco, Lavoura, and Silva [80].

The 2HDM has gauge group $SU(2) \times U(1)$ and the usual fermion content. The scalar sector of the model consists two doublets, $\phi_a$, $a = 1, 2$ with hypercharge $Y = 1/2$ as

$$\phi_a = \begin{pmatrix} \varphi_a^+ \\ \varphi_a^0 \end{pmatrix}, \tag{3.75}$$

and $\tilde{\phi}_a$ with $Y = -1/2$ as

$$\tilde{\phi}_a \equiv i\sigma^2(\phi_a^\dagger)^T = \begin{pmatrix} \varphi_a^0 \\ -\varphi_a^- \end{pmatrix}. \tag{3.76}$$

With the assumption that the vacuum preserves a gauge symmetry related to electromagnetism, we have

$$\langle 0|\phi_a|0\rangle = \begin{pmatrix} 0 \\ v_a e^{i\theta_a} \end{pmatrix}, \tag{3.77}$$

where $v_a$ are real and non-negative and

$$v^2 \equiv v_1^2 + v_2^2 = \frac{4m_W^2}{g^2} \tag{3.78}$$

We can make the VEV of $\varphi_1^0$ real and positive using a $U(1)$ gauge transformation without loss of generality. Thus $\theta_1$=0 and setting $\theta_2 = \theta$, we have

$$\langle 0|\phi_1|0\rangle = \begin{pmatrix} 0 \\ v_a \end{pmatrix}, \qquad \langle 0|\phi_2|0\rangle = \begin{pmatrix} 0 \\ v_2 e^{i\theta} \end{pmatrix}. \tag{3.79}$$

The most general scalar potential in the 2HDM is

$$\begin{aligned} V &= m_1\phi_1^\dagger\phi_1 + m_2\phi_2^\dagger\phi_2 + (m_3\phi_1^\dagger\phi_2 + h.c.) \\ &\quad +\lambda_1(\phi_1^\dagger\phi_1)^2 + \lambda_2(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) \\ &\quad +\{\lambda_5(\phi_1^\dagger\phi_2)^2 + [\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)](\phi_1^\dagger\phi_2) + h.c.\}, \end{aligned} \tag{3.80}$$

where the coupling constants $m_1$, $m_2$, $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$ are real, $m_3 = |m_3|e^{i\delta_3}$, $\lambda_5 = |\lambda_5|e^{i\delta_5}$, $\lambda_6 = |\lambda_6|e^{i\delta_6}$, and $\lambda_7 = |\lambda_7|e^{i\delta_7}$ are complex, and we take $m_3$, $\lambda_5$, $\lambda_6$ and $\lambda_7$ to be nonnegative without loss of generality. The vacuum expectation value of the potential is

$$\begin{aligned} V_0 \equiv \langle 0|V|0\rangle &= m_1 v_1^2 + m_2 v_2^2 + 2|m_3|v_1 v_2\cos(\delta_3+\theta) \\ &\quad + \lambda_1 v_1^4 + \lambda_2 v_2^4 + (\lambda_3+\lambda_4)v_1^2 v_{2^2} + 2|\lambda_5|v_1^2 v_2^2\cos(\delta_5+2\theta) \\ &\quad + 2|\lambda_6|v_1^3 v_2\cos(\delta_6+\theta) + 2|\lambda_7|v_1 v_2^3\cos(\delta_7+\theta). \end{aligned} \tag{3.81}$$

The stability of the vacuum demands that

$$\begin{aligned} 0 &= \frac{-1}{2v_1v_2}\frac{\partial V_0}{\partial\theta} \\ &= |m_3|\sin(\delta_3+\theta) + 2|\lambda_5|v_1v_2\sin(\delta_5+2\theta) + |\lambda_6|v_1^2 sin(\delta_6+\theta) + |\lambda_7|v_2^2\sin(\delta_7+\theta). \end{aligned} \tag{3.82}$$

When defining the CP transformation in this model, the gauge-kinetic terms of the Lagrangian being CP invariant is required. Then we have

$$(\mathcal{CP})\phi_a(\mathcal{CP})^\dagger = U_{ab}^{CP}(\phi_b^\dagger)^T(t, -\boldsymbol{r}), \tag{3.83}$$

where the $2 \times 2$ unitary matrix $U^{CP}$ is determined by $v_a e^{i\theta_a} = U^{CP}_{ab} v_b e^{-i\theta_b}$ as $U^{CP} = diag(1, e^{2i\theta})$.

Consider a discrete symmetry under which $\phi_2 \to -\phi_2$, the scalar potential then has $m_3 = \lambda_6 = \lambda_7 = 0$. Then there is only one $\theta$-dependent term in Eq. (3.81) and the minimum is obtained when

$$\cos(\delta_5 + 2\theta) = -1. \tag{3.84}$$

Using the CP transformation property as Eq. (3.83) and the expression of $U^{CP}$, we have

$$(\mathcal{CP})(\phi_1^\dagger\phi_2)^2(\mathcal{CP})^\dagger = e^{4i\theta}(\phi_2^\dagger\phi_1). \tag{3.85}$$

Then the $\lambda_5$-term in the scalar potential as Eq. (3.80) is CP invariant if the following relation is satisfied

$$e^{i(\delta_5+4\theta)} = e^{-i\delta_5}. \tag{3.86}$$

Actually Eq. (3.84) and Eq. (3.86) are equivalent. So there is no CP violation under this model when not considering the Yukawa couplings. However when allowing the discrete symmetry $\phi_2 \to -\phi_2$ to be softly broken, which means it can be added and not impact the renormalizability of the theory, the situation changes. Under this circumstance, there will be another term present in the potential, the $m_3$-term. Now CP invariance requires

$$e^{2i(\delta_5+2\theta)} = 1, \tag{3.87}$$

$$e^{2i(\delta_3+\theta)} = 1. \tag{3.88}$$

Meanwhile, $\theta$ is determined by the stability condition in Eq. (3.82), with $a_6 = a_7 = 0$, which generally shows $\theta$ satisfies neither Eq. (3.87) nor Eq. (3.88). Therefore, CP is violated.

As a conclusion, when allowing the CP symmetry to be softly broken in the scalar potential, generally there would be CP violation in the 2HDM, when only analyzing the scalar potential. It should be noticed that when considering the Yukawa coupling or the scalars coupling with weak gauge bosons, there should be more CP violation sources, as discussed in Ref. [79, 80].

### 3.3.2 Left-right symmetric model

In the SM, the left-handed fermions are in doublets while the right-handed fermions are singlets, which makes the Lagrangian parity not symmetric, since parity transformation will interchange left- and right-handed fermions . The main motivation of the left-right symmetric model (LRSM) [132, 133, 134, 135] is to make an extension of the SM in which parity can be spontaneously broken. Thus the gauge group of the LRSM is $SU(2)_L \times SU(2)_R \times U(1)$.

The lepton fields and quark fields are in doublets of $SU(2)_L$ and $SU(2)_R$, respectively

$$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, L_R = \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}, \tag{3.89}$$

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, Q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}. \tag{3.90}$$

The gauge bosons $W_{Lk}$ and $W_{Rk}$ with $k = 1, 2, 3$ are associated with the gauge groups $SU(2)_L$ and $SU(2)_R$, respectively. Under parity $P$, we have

$$L_L \leftrightarrow L_R, \qquad Q_L \leftrightarrow Q_R, \qquad W_{Lk} \leftrightarrow W_{Rk}. \tag{3.91}$$

Then the covariant derivative can be expressed as

$$\partial_\mu - ig\sum_{k=1}^{3}(W_{Lk}T_{Lk} + W_{Rk}T_{Rk}) - ig'BY, \tag{3.92}$$

where $T_{Lk}$ and $T_{Rk}$ are the generators of $SU(1)_L$ and $SU(2)_R$. The hypercharge $Y$ takes different values in the LRSM from those of the SM. When making the extension $Q = T_{L3} + T_{R3} + Y$ for the electric charge, the hypercharge acquires a simple physical meaning with the general formula $Y = (B - L)/2$, where $B$ and $L$ represents the baryon number and the lepton number, respectively.

Under the restrictions that the Higgs sector should lead to an appropriate symmetry breaking of $SU(2)_L \times SU(2)_R \times U(1)$, and it should give quarks and charged leptons a mass, while giving either zero or naturally small masses to the neutrinos, the Higgs sector should contain a scalar multiplet $\phi$,

$$\phi = \begin{pmatrix} \varphi_1^{0\dagger} & \varphi_2^+ \\ -\varphi_1^- & \varphi_2^0 \end{pmatrix}, \tag{3.93}$$

with

$$\tilde{\phi} \equiv \sigma^2(\phi^\dagger)^T\sigma^2 = \begin{pmatrix} \varphi_2^{0\dagger} & \varphi_1^+ \\ -\varphi_2^- & \varphi_1^0 \end{pmatrix}, \tag{3.94}$$

with hypercharge $Y = 0$, as well as a triplet of $SU(2)_L$ which is a singlet of $SU(2)_R$, and a triplet of $SU(2)_R$ which is a singlet of $SU(2)_L$, termed as $\Delta_L$ and $\Delta_R$, separately:

$$\Delta_L = \begin{pmatrix} \Delta_L^+/\sqrt{2} & -\Delta_L^{++} \\ \Delta_L^0 & -\Delta_L^+/\sqrt{2} \end{pmatrix}, \tag{3.95}$$

$$\Delta_R = \begin{pmatrix} \Delta_R^+/\sqrt{2} & -\Delta_R^{++} \\ \Delta_R^0 & -\Delta_R^+/\sqrt{2} \end{pmatrix}, \tag{3.96}$$

with hypercharge $Y = (B - L)/2 = 1$. The $SU(2)_L$ doublets are

$$\begin{pmatrix} \varphi_1^+ \\ \varphi_1^0 \end{pmatrix} \text{ and } \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix}, \tag{3.97}$$

and the $SU(2)_R$ doublets are

$$\begin{pmatrix} \varphi_1^+ \\ -\varphi_2^{0\dagger} \end{pmatrix} \text{ and } \begin{pmatrix} \varphi_2^+ \\ -\varphi_1^{0\dagger} \end{pmatrix}. \tag{3.98}$$

The Yukawa coupling of $\phi$ and $\tilde{\phi}$ will generate Dirac masses for all fermions, including neutrinos. While the Yukawa couplings of the triplets can generate $|\Delta L| = 2$ Majorana masses, leading to naturally small neutrino masses through the seesaw mechanism [136, 137]. The VEV of $\varphi_1^0$ and $\varphi_2^0$ are $k_1$ and $k_2$, respectively, both of which are generally complex. The VEV of $\Delta_L^0$ and $\Delta_R^0$ are $F_L$ and $F_R$, respectively. All other Higgs fields are assumed to have vanishing VEV.

The Yukawa couplings of quarks are expressed as

$$\mathcal{L}_Y^{(q)} = \overline{Q}_L(\phi\Gamma_1 + \tilde{\phi}\Gamma_2)Q_R + \overline{Q}_R(\phi^\dagger\Gamma_1^\dagger + \tilde{\phi}^\dagger\Gamma_2^\dagger)Q_L, \tag{3.99}$$

where $\Gamma_1$ and $\Gamma_2$ are $n_g \times n_g$ matrices in generation space, with $n_g$ the number of generations of quarks. Assuming under parity $\phi \xrightarrow{P} \phi^\dagger$,the parity invariance restricts the Yukawa-coupling matrices to be Hermitian, i.e. $\Gamma_1 = \Gamma_1^\dagger$, $\Gamma_2 = \Gamma_2^\dagger$. The mass matrices for the up-type and down-type quarks, defined from $\bar{u}_L M_u u_R$ and $\bar{d}_L M_d d_R$, are

$$M_u = k_1^*\Gamma_1 + k_2^*\Gamma_2, M_d = k_2\Gamma_1 + k_1\Gamma_2. \tag{3.100}$$

Defining

$$T_{L\pm} \equiv \frac{T_{L1} \pm iT_{L2}}{\sqrt{2}}, T_{R\pm} = \frac{T_{R1} \pm iT_{R2}}{\sqrt{2}}, \tag{3.101}$$

$$W_{L\pm} \equiv \frac{W_{L1} \pm iW_{L2}}{\sqrt{2}}, W_{R\pm} = \frac{W_{R1} \pm iW_{R2}}{\sqrt{2}}, \tag{3.102}$$

the charged-current Lagrangian written in the quark mass eigenstates is

$$\frac{g}{\sqrt{2}}\bar{u}\gamma^\mu(W_{L\mu}^+ V_L \frac{1-\gamma_5}{2} + W_{R\mu}^+ V_R \frac{1+\gamma_5}{2})d + H.c., \tag{3.103}$$

where the charged-current mixing matrices

$$V_L \equiv U_L^{u\dagger} U_L^d, \qquad V_R \equiv U_R^{u\dagger} U_R^d, \tag{3.104}$$

are $n_g \times n_g$ unitary matrices.

As in the neutral-current Lagrangian does not change its form when expressed in the quark mass eigenstates, due to the fact that all quark fields of a given charge and helicity have the same $T_{L3}$, $T_{R3}$, and $Y$. The total number of meaningful phases in the mixing matrices $V_L$ and $V_R$ are [80]

$$N_{phase}^{LR} = n_g^2 - n_g + 1. \tag{3.105}$$

Therefore, even when there is only one generation, there will be a phase that gives rise to CP violation. This is very different from the SM, because in that case there exists only one phase that leads to CP violation for $n_g = 3$.

Above all, there are three types of CP violation sources in the LRSM [79]: the VEVs of $\varphi_1^0$ and $\varphi_2^0$, which can turn out to be complex; the left-handed quark mixing matrices $V_L$ corresponding to the usual CKM matrix in the SM; and its right-handed analogue $V_R$ existing due to the presence of the right-handed currents.

### 3.3.3 Weak scale supersymmetry

In this part, we will give a very brief introduction to weak scale supersymmetry.

Supersymmetry(SUSY) is a generalization of the space-time symmetries in quantum field theory. In SUSY, fermions can transfer to bosons and vice versa [138]. It also forms a bridge to gravity, providing a unification of particle physics and gravity [139] at the Plank energy scale, $M_P \sim 10^{19}$ TeV. So SUSY is the ultimate symmetry. SUSY is also an important possible "loophole" of a no-go theorem in theoretical physics, the Coleman-Mandula theorem [140, 141], stating that combining space-time and internal symmetries in any but a trivial way is impossible, which also means that the symmetry group of a consistent 4-dimensional quantum field theory is a direct product of the internal symmetry group and Poincaré group [140]. SUSY can be considered as a possible way to evade the Coleman-Mandula theorem because it contains additional generators – supercharges, providing a nontrivial extension of the Poincaré algebra, namely the supersymmetry algebra [142, 143].

In SUSY, each state has a superpartner, a state whose spin differs by half a unit. Since no superpartners have ever been observed, supersymmetry must be a broken symmetry. This can be achieved by setting the superpartners with sufficiently heavy masses so that they could escape detection by experiments, say well in excess of 100 GeV [79]. The SUSY Lagrangian should contain soft-supersymmetry-breaking terms, whose total mass dimension is less than 4, which still maintains the stability of the gauge hierarchy [41].

There are many additional gateways for CP violation to enter in the SUSY theory. For example, CP odd operators can arise both in chirality conserving and changing couplings, flavor-diagonal couplings can also give rise to CP violation and so on. The CP violating sources can enter through the superpotential, which introduces the non-gauge interactions, and/or the soft breaking terms.

The minimal supersymmetric extension of the SM is usually called the minimal supersymmetric Standard Model (MSSM) [144, 139], which considers the minimum number of new particle states and new interactions consistent with phenomenology [145]. In MSSM, $R$ parity is introduced as a multiplicative quantum number, assigning it to be $+1$ for ordinary fields and $-1$ for their superpartners. The ordinary fields are quarks, leptons, gluons, gauge bosons, Higgs, and gravitons, and their superpartners are named squarks, sleptons, gluinos, gauginos, higgsinos, and gravitinos, respectively. Each of the field in the SM is extended into a superfield, which combine fields differing by half a unit in their spin, i.e. the fields and the relevant superpartners. The MSSM consists of the two-Higgs-doublet extension of the SM and their superpartners. In MSSM, the flavor-changing neutral currents (FCNC) even happen in the strong sector, but they arise radiatively and are reduced significantly in strength [79]. In general FCNC do not conserve CP. Moreover, in the MSSM, CP violation occurs even in flavor diagonal transitions [79]. Explicitly, there are two observable phases that induce CP violation in the MSSM, one is the usual KM phase with its origin in the misalignment of the mass matrices for up and down type quarks, which also migrates into the squark mass matrices and controls the CP properties of the quark-squark-gluino couplings; the other one reflects soft SUSY breaking, which is strongly constrained by the experimental bound on the neutron EDM [79]. Comparing with the SM, the existence of CP violation in flavor-diagonal transitions from KM ansatz in MSSM allows EDM to arise on an observable level. MSSM contributions to $\Delta b \neq 0 \neq \Delta s$ (where $b$ and $s$ stand for beauty number and strange number, respectively) rare decays can be sizable and remarkable, but they can only modify the SM predictions by a factor of two difference in the rate at most. However, non-minimal implementation of SUSY can give rise to quite dramatic modifications [79]. When we once enter the non-minimal SUSY models, even by a slightly modification of the MSSM, there will be considerable extra sources of CP violation. Those extra CP-violating sources can have observable effects likely or quite possibly for the EDMs of neutrons and electrons, as well as very distinct CP violating scenarios compared with the MSSM, that due to $|\Delta B| = 2$ dynamics being modified by SUSY, dramatic deviations from the KM expectations occur [79].

### 3.4 Summary

In this chapter, we introduced some model independent CP violating observables, CP violating mechanisms within the SM, as well as some popular models beyond the SM that can have new CP violating sources. The topic of my thesis, CP violation in $\eta \to \pi^+\pi^-\pi^0$ is a kind of direct CP violating process that can be observed through a production asymmetry of its Dalitz plot [90]. The KM phase as a crucial CP violating source of the SM does not support $\eta \to \pi^+\pi^-\pi^0$, which is a flavor-diagonal process, and the $\theta$ term in the SM, which is P and T violating, does not contribute directly to $\eta \to \pi^+\pi^-\pi^0$ whose CP violation property is C and CP violating. Of all the CP violation mechanisms in the beyond the SM scenarios introduced in this chapter, the only possible new source that can contribute to CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay is the CP violation in flavor-diagonal transitions from KM ansatz in MSSM [79], but it contributes mainly to EDM, so its contribution to $\eta \to \pi^+\pi^-\pi^0$ decay would be very small. In the next chapter, we will discuss another very important beyond the SM theory, the SM effective field theory, from which we will investigate out CP violating sources that can contribute to $\eta \to \pi^+\pi^-\pi^0$ decay shown in Ch. 7.

CHAPTER 4

# BEYOND THE STANDARD MODEL PHYSICS AS AN EFFECTIVE FIELD THEORY

## 4.1 Effective field theory

The Standard Model (SM) [146, 147, 148, 149] of particle physics has been very successful in describing the strong and electroweak interactions. In the year 2012, the ATLAS [150] and CMS [151] collaborations at CERN the Large Hadron Collider (LHC) discovered a scalar particle with a mass around 126 GeV and with properties that were roughly consistent with the SM Higgs boson. With that discovery, all the particles included in the SM theory were verified; as of this writing no particles beyond the SM have been found as yet. Nevertheless the SM is not perfect regarding the many features of Nature that it does not explain, such as the strong CP problem, neutrino masses, dark matter and dark energy, the baryon asymmetry in universe, and the precise values of the fermions' masses (e.g. the top quark is so much heavier than the other quarks). New physics beyond the SM should exist without doubt, however at a sufficiently high energy scale (because no new physics has been detected yet), say at around 500 GeV [152] or at the TeV scale [153], and/or interact with the SM particles with very weak couplings. A theory of new physics above a certain scale, for which the SM is no longer applicable, should

- contain the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry of the SM,
- incorporate all the SM degrees of freedom either as fundamental or composite fields,
- reduce to the SM at low energies.

All the above properties can be satisfied by an effective field theory (EFT) [154, 155, 156, 157], that the low-energy physics can be described using an effective Lagrangian without the necessity of considering degrees of freedom of new physics appearing at high energies [154]; and it is convenient to use it as a phenomenological tool to study new physics. The main advantage of an EFT is that it is general and model-independent. An EFT usually can be constructed in one of two ways: the top-down approach, where the ultraviolet physics is known and the EFT can be formulated by integrating out the heavy degrees of freedom, or the bottom-up procedure, where the underlying high energy physics is unknown. The bottom-up EFT Lagrangian can be constructed by enumerating the most general possible operators consistent with the present degrees of freedom, say, e.g., forming the operators with the SM fields, and imposing the symmetries required.

Integrating out heavy fields means the process of removing particular fields present in the Lagrangian. Suppose $\Phi$ [1] is a heavy, real scalar boson with mass $M$ that will be integrated out and $S[\Phi,\ \phi]$ is the action containing $\Phi$ and its interactions with the SM fields $\phi$. If we work at energies at which the $\Phi$ can no longer be produced as a physical (i.e., on its mass shell) state, then the resulting effective action after integrating out $\Phi$ is given by

$$e^{iS_{\text{eff}}[\phi]} = \int \mathcal{D}\Phi e^{iS[\Phi,\ \phi]}. \tag{4.1}$$

We can compute the effective action to one-loop order by expanding the action around the minimum of $\Phi$ as

$$S[\Phi_c + \eta,\ \phi] = S[\Phi_c,\ \phi] + \frac{1}{2}\eta^2 \frac{\delta^2 S}{\delta \Phi^2} + \mathcal{O}(\eta^3), \tag{4.2}$$

where the minimum $\Phi_c$ is determined by

$$\left.\frac{\delta S[\Phi,\ \phi]}{\delta \Phi}\right|_{\Phi=\Phi_c} = 0. \tag{4.3}$$

Then the integral can be computed as [158]

$$e^{iS_{\text{eff}}[\phi]} = \int \mathcal{D}\eta e^{iS[\Phi_c+\eta,\ \phi]} \approx e^{iS[\Phi_c,\ \phi]} \left[\det\left(-\frac{\delta^2 S}{\delta \Phi^2}\ |_{\Phi_c}\right)\right]^{-\frac{1}{2}}. \tag{4.4}$$

And the effective action is given by [158]

$$S_{\text{eff}}[\phi] \approx S[\Phi, \phi] + \frac{i}{2}\text{Tr}\ \ln\left(-\frac{\delta^2 S}{\delta \Phi^2}\ |_{\Phi_c}\right). \tag{4.5}$$

The first term is the tree-level piece, which can be obtained by solving the heavy field's equation of motion and plugging it back into the action, while the second term is the one-loop piece. This result comes from the path-integral formalism. It can also be computed through the perturbation theory where the effect of the path-integral is represented by a sum of the Feynman diagrams [159]. Given the scattering process $\bar{\nu} + d \to e^- + u$ through the exchange of a virtual charged weak gauge boson $W$ with the Feynman diagram depicted in Fig. 4.1 as an example, the interactions from the SM Lagrangian responsible for this process come from

$$\frac{g}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L + \bar{u}_L\gamma^\mu d_L)W^+_\mu + h.c. \tag{4.6}$$

Then the amplitude is expressed as

$$\frac{g^2}{2}(\bar{u}_L\gamma^\mu d_L)\frac{-i}{q^2 - m_W^2}(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2})(\bar{e}_L\gamma^\nu \nu_L), \tag{4.7}$$

[1]The procedure and notations are chosen to coincide with Henning, Lu, and Murayama [158].

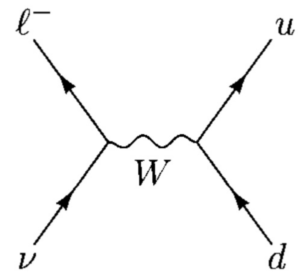


Figure 4.1: Feynman diagram for $\bar{\nu} + d \to e^- + u$ through $W$ exchange.

where $q$ is the 4-momentum of the $W$ and $m_W$ denotes the mass of the $W$. At energies much smaller than the mass of the $W$ gauge boson, we can neglect the $q^2$ dependence as well as the $q^\mu q^\nu / m_W^2$ term and describe the diagram by an effective Lagrangian

$$\Delta\mathcal{L}_W = i\frac{g^2}{2m_W^2}(\bar{u}_L\gamma^\mu d_L)(\bar{e}_L\gamma^\mu \nu_L). \tag{4.8}$$

The coefficient is usually written in terms of the Fermi constant

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}. \tag{4.9}$$

We have introduced some general features of EFT. Now we are going to discuss a very important kind of effective field theory explicitly, the Standard Model effective field theory (SMEFT), which is closely related to Chapter 7 of this thesis.

## 4.2 Standard Model effective field theory

Accounting for the success of the SM at low energy, assuming new physics appears only at very high energies, we can construct an EFT Lagrangian extended from the SM in an expansion in $1/\Lambda$, where $\Lambda$ represents the energy scale that new physics occurs, by imposing the SM gauge symmetry $SU(3)_C \times SU(2) \times U(1)_Y$ and using the SM fields for the new terms. We have

$$\begin{aligned}\mathcal{L}_{eff} &= \mathcal{L}_{SM}^{(4)} + \frac{1}{\Lambda}\mathcal{L}^{(5)} + \frac{1}{\Lambda^2}\mathcal{L}^{(6)} + \cdots \\ &= \mathcal{L}_{SM} + \frac{1}{\Lambda}\sum_k C_k^{(5)}\mathcal{O}_k^{(5)} + \frac{1}{\Lambda^2}\sum_k C_k^{(6)}\mathcal{O}_k^{(6)} + \ldots, \end{aligned} \tag{4.10}$$

where $\mathcal{L}_{SM}^{(4)}$ is the renormalizable Lagrangian of the SM which contains all possible dimension-four operators, $\mathcal{O}_k^{(n)}$ denotes dimension-n operators, and $C_k^{(n)}$ represents the corresponding dimensionless effective coupling constants (Wilson coefficients [160]). This scenario is usually called SMEFT. At low energies, the new physics interactions are suppressed by inverse powers of $\Lambda$, such that the theory will be well-described by the renormalizable SM Lagrangian at an energy scale $\mu \ll \Lambda$.

The SM fields are left-handed doublets of $SU(2)_L$,

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \tag{4.11}$$

right-handed singlets $u_{Rp}$, $d_{Rp}$, and $e_{Rp}$, where $p = 1,\ 2,\ 3$ are generation indices, $SU(3)_C$ gauge field $G^A_\mu$, electroweak gauge fields $W^I_\mu$ and $B_\mu$, and the Higgs field $\varphi$, which are introduced in Sec. 3.2.1. A field's mass-dimension refers to its dimension in terms of the dimension of mass $[M]$. Giving the SM standard Lagrangian to be mass-dimension 4 as Eq. (3.35), the mass dimensions of the Higgs field $\varphi$, fermion fields $\psi$ (quark and lepton fields), gauge fields $G^A_\mu$, $W^I_\mu$, and $B_\mu$ are

$$[\varphi] = 1, \quad [\psi] = \frac{3}{2}, \quad [G^a_\mu] = 1, \quad ,[W^I_\mu] = 1, \quad \text{and} \quad [B_\mu] = 1. \tag{4.12}$$

In the following operators, $p,\ r,\ s,$ and $t$ represents generation indices, and the chirality of fermions is left implicit, so that $l_p$ and $q_p$ indicate left-handed doublets $l_{Lp}$ and $q_{Lp}$, and $u_p$, $d_p$ and $e_p$ mean right-handed singlets $u_{Rp}$, $d_{Rp}$, and $e_{Rp}$, respectively. Taking the potential of the Higgs field from the mass-dimension 4 SM Lagrangian in Eq. (3.35),

$$V(\varphi) = \mu^2 \varphi^\dagger \varphi - \frac{\lambda}{2}(\varphi^\dagger \varphi)^2, \tag{4.13}$$

with $\mu^2 > 0$, the Higgs field can obtain a vacuum expectation value (VEV) $v = \sqrt{\mu^2/\lambda}$ by taking the minimum of this potential, while the electroweak symmetry of the SM is spontaneously broken.

There is only one mass-dimension-5 operator [161, 162, 163, 164, 101]

$$\mathcal{O}^{(5)} = \epsilon_{jk}\epsilon_{mn}\varphi^j\varphi^m(l^k_p)^T C l^n_r \equiv (\tilde{\varphi}^\dagger l_p)^T C(\tilde{\varphi}^\dagger l_r), \tag{4.14}$$

where $C$ denotes the charge conjugation matrix with $C = i\gamma_2\gamma_0$ in the Dirac representation [101], $\varphi$ is the Higgs doublet, $\tilde{\varphi} = i\sigma^2\varphi$, $l$ is the left-handed lepton doublet, and $p,\ r$ are the generation indices. The charge conjugation commutes with the generator of Lorentz boots, which is essential to make the operator in Eq. (4.14) Lorentz invariant [165]. The lepton number of $(l_p)^T$ and $l_r$ are both 1, so that $\mathcal{O}_5$ violates lepton number by 2 units. Eq. (4.14) generates Majorana masses for the neutrinos after the electroweak symmetry is spontaneously broken once the Higgs field obtains its VEV.

Complete sets of the possible dimension-6 operators are given by Buchmuller and Wyler [164] and Grzadkowski et al. [101]. Ref. [101] identified and removed some redundant operators by applying equations of motion(EOM), integration by parts, and Fierz identities in comparison to Ref. [164]. The classes contributing to the complete set of dimension-6 operators are $X^3$, $\varphi^6$, $\varphi^4 D^2$, $\psi^2\varphi^3$, $X^2\varphi^2$, $\psi^2 X\varphi$, $\psi^2\varphi^2 D$ and $\psi^4$, where $X$ denotes the gauge field strength tensor $G^A_{\mu\nu}$, $W^I_{\mu\nu}$, and $B_{\mu\nu}$, $D$ stands for the covariant derivative $D_\mu = \partial_\mu - ig_s G^a_\mu T^a - igW^i_\mu S^i - ig'YB_\mu$ and $\psi$ represents the fermion fields, i.e., the quark and lepton fields. From Ref. [101], when we take the

Table 4.1: Mass-dimension 6 SMEFT operators from Ref. [101] other than the four-fermion ones, which are listed in Table 4.2.

| $X^3$ | | $\varphi^6$ and $\varphi^4 D^2$ | | $\psi^2\varphi^3$ | |
|---|---|---|---|---|---|
| $Q_G$ | $f^{ABC}G_\mu^{A\nu}G_\nu^{B\rho}G_\rho^{C\mu}$ | $Q_\varphi$ | $(\varphi^\dagger\varphi)^3$ | $Q_{e\varphi}$ | $(\varphi^\dagger\varphi)(\bar{l}_p e_r \varphi)$ |
| $Q_{\widetilde{G}}$ | $f^{ABC}\widetilde{G}_\mu^{A\nu}G_\nu^{B\rho}G_\rho^{C\mu}$ | $Q_{\varphi\Box}$ | $(\varphi^\dagger\varphi)(\varphi^\dagger\varphi)$ | $Q_{u\varphi}$ | $(\varphi^\dagger\varphi)(\bar{q}_p u_r \widetilde{\varphi})$ |
| $Q_W$ | $\epsilon^{IJK}W_\mu^{I\nu}W_\nu^{J\rho}W_\rho^{K\mu}$ | $Q_{\varphi D}$ | $\left(\varphi^\dagger D_\mu\varphi\right)^* \left(\varphi^\dagger D_\mu\varphi\right)$ | $Q_{d\varphi}$ | $(H^\dagger H)(\bar{q}_p d_r \varphi)$ |
| $Q_{\widetilde{W}}$ | $\epsilon^{IJK}\widetilde{W}_\mu^{I\nu}W_\nu^{J\rho}W_\rho^{K\mu}$ | | | | |

| $X^2\varphi^2$ | | $\psi^2 X\varphi$ | | $\psi^2\varphi^2 D$ | |
|---|---|---|---|---|---|
| $Q_{\varphi G}$ | $\varphi^\dagger\varphi\, G_{\mu\nu}^A G^{A\mu\nu}$ | $Q_{eW}$ | $(\bar{l}_p\sigma^{\mu\nu}e_r)\tau^I\varphi W_{\mu\nu}^\varphi$ | $Q_{\varphi l}^{(1)}$ | $(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{l}_p\gamma^\mu l_r)$ |
| $Q_{\varphi\widetilde{G}}$ | $\varphi^\dagger\varphi\, \widetilde{G}_{\mu\nu}^A G^{A\mu\nu}$ | $Q_{eB}$ | $(\bar{l}_p\sigma^{\mu\nu}e_r)\varphi B_{\mu\nu}$ | $Q_{\varphi l}^{(3)}$ | $(\varphi^\dagger i\overleftrightarrow{D}_\mu^I\varphi)(\bar{l}_p\tau^I\gamma^\mu l_r)$ |
| $Q_{\varphi W}$ | $\varphi^\dagger\varphi\, W_{\mu\nu}^I W^{I\mu\nu}$ | $Q_{uG}$ | $(\bar{q}_p\sigma^{\mu\nu}T^A u_r)\widetilde{\varphi}\, G_{\mu\nu}^A$ | $Q_{\varphi e}$ | $(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{e}_p\gamma^\mu e_r)$ |
| $Q_{\varphi\widetilde{W}}$ | $\varphi^\dagger\varphi\, \widetilde{W}_{\mu\nu}^I W^{I\mu\nu}$ | $Q_{uW}$ | $(\bar{q}_p\sigma^{\mu\nu}u_r)\tau^I\widetilde{\varphi}\, W_{\mu\nu}^I$ | $Q_{\varphi q}^{(1)}$ | $(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{q}_p\gamma^\mu q_r)$ |
| $Q_{\varphi B}$ | $\varphi^\dagger\varphi\, B_{\mu\nu}B^{\mu\nu}$ | $Q_{uB}$ | $(\bar{q}_p\sigma^{\mu\nu}u_r)\widetilde{\varphi}\, B_{\mu\nu}$ | $Q_{\varphi q}^{(3)}$ | $(\varphi^\dagger i\overleftrightarrow{D}_\mu^I\varphi)(\bar{q}_p\tau^I\gamma^\mu q_r)$ |
| $Q_{\varphi\widetilde{B}}$ | $\varphi^\dagger\varphi\, \widetilde{B}_{\mu\nu}B^{\mu\nu}$ | $Q_{dG}$ | $(\bar{q}_p\sigma^{\mu\nu}T^A d_r)\varphi\, G_{\mu\nu}^A$ | $Q_{\varphi u}$ | $(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{u}_p\gamma^\mu u_r)$ |
| $Q_{\varphi WB}$ | $\varphi^\dagger\tau^I\varphi\, W_{\mu\nu}^I B^{\mu\nu}$ | $Q_{dW}$ | $(\bar{q}_p\sigma^{\mu\nu}d_r)\tau^I\varphi\, W_{\mu\nu}^I$ | $Q_{\varphi d}$ | $(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{d}_p\gamma^\mu d_r)$ |
| $Q_{\varphi\widetilde{W}B}$ | $\varphi^\dagger\tau^I\varphi\, \widetilde{W}_{\mu\nu}^I B^{\mu\nu}$ | $Q_{dB}$ | $(\bar{q}_p\sigma^{\mu\nu}d_r)\varphi\, B_{\mu\nu}$ | $Q_{\varphi ud}$ + h.c. | $i(\widetilde{\varphi}^\dagger D_\mu\varphi)(\bar{u}_p\gamma^\mu d_r)$ |

generation number to be 1, there are 59 Hermitian operators that conserves baryon number and 5 operators that violate baryon number. We list the operators from Ref. [101] in Table 4.1 and Table 4.2.

We will study CP violation in the mass-dimension 6 SMEFT operators without baryon number and lepton number violation in Chapter 7.

## 4.3 Spontaneously broken electroweak effective field theory

In the SM, the Higgs is a fundamental particle whose dynamics is weakly coupled, and electroweak symmetry is spontaneously broken to $U(1)_{em}$ when the Higgs field develops a vacuum expectation value. Although the LHC experiments [150, 151] observed the Higgs boson of the SM to some precision and the Higgs mechanism is certainly at work; the precise mechanism of electroweak symmetry breaking remains mysterious. Whether the underlying dynamics of Higgs field is strongly or weakly coupled is not completed determined yet [166, 167, 168]. SMEFT has the hidden assumption that the electroweak symmetry is broken in the same way as the SM. Meanwhile, data on the Higgs parameters [169] allow deviations from the SM of the order of 10% [170].

The simplest option for a weak-coupled scenario is to introduce a scalar doublet for the Higgs field. In addition to the required 3 Goldstone bosons, another massive

Table 4.2: Mass-dimension 6 SMEFT operators composed of four fermions from Ref. [101].

| $(\bar{L}L)(\bar{L}L)$ | |
|---|---|
| $Q_{ll}$ | $(\bar{l}_p\gamma^\mu l_r)(\bar{l}_s\gamma_\mu l_t)$ |
| $Q_{qq}^{(1)}$ | $(\bar{q}_p\gamma^\mu q_r)(\bar{q}_s\gamma_\mu q_t)$ |
| $Q_{qq}^{(3)}$ | $(\bar{q}_p\gamma^\mu \tau^I q_r)(\bar{q}_s\gamma_\mu \tau^I q_t)$ |
| $Q_{lq}^{(1)}$ | $(\bar{l}_p\gamma^\mu l_r)(\bar{q}_s\gamma_\mu q_t)$ |
| $Q_{lq}^{(3)}$ | $(\bar{l}_p\gamma^\mu \tau^I l_r)(\bar{q}_s\gamma_\mu \tau^I q_t)$ |

| $(\bar{R}R)(\bar{R}R)$ | |
|---|---|
| $Q_{ee}$ | $(\bar{e}_p\gamma^\mu e_r)(\bar{e}_s\gamma_\mu e_t)$ |
| $Q_{uu}$ | $(\bar{u}_p\gamma^\mu u_r)(\bar{u}_s\gamma_\mu u_t)$ |
| $Q_{dd}$ | $(\bar{d}_p\gamma^\mu d_r)(\bar{d}_s\gamma_\mu d_t)$ |
| $Q_{eu}$ | $(\bar{e}_p\gamma^\mu e_r)(\bar{u}_s\gamma_\mu u_t)$ |
| $Q_{ed}$ | $(\bar{e}_p\gamma^\mu e_r)(\bar{d}_s\gamma_\mu d_t)$ |
| $Q_{ud}^{(1)}$ | $(\bar{u}_p\gamma^\mu u_r)(\bar{d}_s\gamma_\mu d_t)$ |
| $Q_{ud}^{(8)}$ | $(\bar{u}_p\gamma^\mu T^A u_r)(\bar{d}_s\gamma_\mu T^A d_t)$ |

| $(\bar{L}L)(\bar{R}R)$ | |
|---|---|
| $Q_{le}$ | $(\bar{l}_p\gamma^\mu l_r)(\bar{e}_s\gamma_\mu e_t)$ |
| $Q_{lu}$ | $(\bar{l}_p\gamma^\mu l_r)(\bar{u}_s\gamma_\mu u_t)$ |
| $Q_{ld}$ | $(\bar{l}_p\gamma^\mu l_r)(\bar{d}_s\gamma_\mu d_t)$ |
| $Q_{qe}$ | $(\bar{q}_p\gamma^\mu q_r)(\bar{e}_s\gamma_\mu e_t)$ |
| $Q_{qu}^{(1)}$ | $(\bar{q}_p\gamma^\mu q_r)(\bar{u}_s\gamma_\mu u_t)$ |
| $Q_{qu}^{(8)}$ | $(\bar{q}_p\gamma^\mu T^A q_r)(\bar{u}_s\gamma_\mu T^A u_t)$ |
| $Q_{qd}^{(1)}$ | $(\bar{q}_p\gamma^\mu q_r)(\bar{d}_s\gamma_\mu d_t)$ |
| $Q_{qd}^{(8)}$ | $(\bar{q}_p\gamma^\mu T^A q_r)(\bar{d}_s\gamma_\mu T^A d_t)$ |

| $(\bar{L}R)(\bar{R}L)$ and $(\bar{L}R)(\bar{L}R)$ | |
|---|---|
| $Q_{ledq}$ | $(\bar{l}_p^j e_r)(\bar{d}_s q_{tj})$ |
| $Q_{quqd}^{(1)}$ | $(\bar{q}_p^j u_r)\epsilon_{jk}(\bar{q}_s^k d_t)$ |
| $Q_{quqd}^{(8)}$ | $(\bar{q}_p^j T^A u_r)\epsilon_{jk}(\bar{q}_s^k T^A d_t)$ |
| $Q_{lequ}^{(1)}$ | $(\bar{l}_p^j e_r)\epsilon_{jk}(\bar{q}_s^k u_t)$ |
| $Q_{lequ}^{(3)}$ | $(\bar{l}_p^j \sigma_{\mu\nu} e_r)\epsilon_{jk}(\bar{q}_s^k \sigma^{\mu\nu} u_t)$ |

| $B$−violating | |
|---|---|
| $Q_{duq}$ | $\epsilon^{\alpha\beta\gamma}\epsilon_{jk}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(q_s^{\gamma j})^T C l_t^k\right]$ |
| $Q_{qqu}$ | $\epsilon^{\alpha\beta\gamma}\epsilon_{jk}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(u_s^\gamma)^T C e_t\right]$ |
| $Q_{qqq}^{(1)}$ | $\epsilon^{\alpha\beta\gamma}\epsilon_{jk}\epsilon_{mn}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$ |
| $Q_{qqq}^{(3)}$ | $\epsilon^{\alpha\beta\gamma}(\tau^I\epsilon)_{jk}(\tau^I\epsilon)_{mn}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$ |
| $Q_{duu}$ | $\epsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]$ |

scalar field, the Higgs field, is obtained, with its mass taken as a free parameter. The SM electroweak symmetry breaking mechanism is one weak-coupling scenario. Introducing a new scalar particle – the Higgs boson, and treating it as a fundamental particle has a problem of naturalness, and additional stabilization mechanism is needed so as to make the theory meaningful [166]. Supersymmetry may be the most concrete framework to understand the lightness of of a weakly coupled Higgs [166].

The alternatives are strong-coupled scenarios with dynamical symmetry breaking, in which case electroweak symmetry is spontaneously broken by a trigger of condensate that is generated by new interactions strongly coupled at the electroweak scale [166]. Analogous to how chiral symmetry is broken in QCD, the strong-coupled scenarios has a feature of compositeness and large number of bound states. Generally, strong-coupled scenarios allow for a light scalar with mass naturally at the order of electroweak scale [171] when it is interpreted as a pseudo-Goldstone boson of a spontaneously broken symmetry [169, 172, 173, 174, 175, 176]. Such scenario can be called strongly-interacting light Higgs (SILH) [173, 177] and can be studied through an effective field theory involving electroweak chiral Lagrangian [178, 167].

CHAPTER 5

# MESON DECAYS AT LOW ENERGY

Quantum chromodynamics (QCD) is the quantum field theory considered as the fundamental theory of the strong interactions. Due to the running of the strong coupling constant, the coupling decreases logarithmically as the energy increases, leading to asymptotic freedom [179]; and it increases as the energy decreases, which means the coupling becomes large at low energy, and leads quarks to be confined. Thus, with quarks and gluons as its fundamental degrees of freedom, QCD is not practical to use at low energy because of quark confinement. Meanwhile the perturbative methods for calculating QCD processes, which are successful at high energies, cannot be applied at low energies because the strong coupling constants are no longer small quantities that can be treated perturbatively. However, we can use effective field theory (EFT). Chiral perturbation theories (ChPT) is one of the most prominent effective theories to study QCD at low energy.

In this chapter I will first introduce some basics of ChPT that can be used to study meson decays, and then I will introduce the theoretical analysis of $\eta \to 3\pi$ decay. The ChPT part is mainly based on Scherer and Schindler [180], Bernard and Meißner [181], and Ecker [182].

## 5.1 Chiral symmetry of low energy QCD

The QCD Lagrangian satisfies $SU(3)_C$ gauge invariance, and it is given by [180]

$$\mathcal{L}_{QCD} = \sum_{f=u,d,s,c,b,t} \bar{q}_f (i \not{D} - m_f) q_f - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \tag{5.1}$$

where $q_f$ represents the color triplet of each quark flavor $f$ as

$$q_f = \begin{pmatrix} q_{f,1} \\ q_{f,2} \\ q_{f,3} \end{pmatrix} \tag{5.2}$$

and

$$\begin{aligned} D_\mu &= \partial_\mu - i g_s G^a_\mu t_a, && (5.3) \\ G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f_{abc} G^a_\mu G^c_\nu, && (5.4) \end{aligned}$$

where $t_a = \lambda_a/2$ and $\lambda_a$ are the Gell-Mann matrices with $a = 1,\ 2, \ldots 8$ with the following commutation and anticommutation relations

$$\begin{aligned} [\frac{\lambda_a}{2}, \frac{\lambda_b}{2}] &= i f_{abc} \frac{\lambda_c}{2}, \\ \{\lambda_a, \lambda_b\} &= \frac{4}{3}\delta_{ab} + 2 d_{abc} \lambda_c, && (5.5) \end{aligned}$$

where $f_{abc}$ are totally antisymmetric real constants and $d_{abc}$ are totally symmetric real constants. The QCD Lagrangian is invariant under the local $SU(3)$ transformation

$$\begin{aligned} q_f(x) &\mapsto q'_f(x) = \exp(-i\Theta_a(x)t_a)q_f(x) \equiv U(x)q_f(x), \\ G_\mu(x) &\equiv G^a_\mu(x)t^a \mapsto G'_\mu = U(x)G_\mu(x)U^\dagger(x) + \frac{i}{g_s}\partial_\mu U(x)U^\dagger(x), \end{aligned} \tag{5.6}$$

where $\Theta_a(x)$ are smooth, real functions in Minkowski space [180]. The 6-flavor quarks are usually divided into light quarks $u,\ d,\ s$ and heavy quarks $c,\ b,\ t$ according to their masses [180], in comparison to the typical hadronic scale of $\sim 1$ GeV, where [41]

$$\begin{aligned} &m_u = (2.16^{+0.49}_{-0.26})\ \text{MeV},\ m_d = (4.67^{+0.48}_{-0.17})\ \ \text{MeV},\ m_s = (93^{+11}_{-5})\ \text{MeV}, \\ &m_c = 1.27 \pm 0.02\ \text{GeV},\ m_b = 4.18^{+0.03}_{-0.02}\ \text{GeV},\ m_t = (172.76 \pm 0.30)\ \text{GeV}. \end{aligned} \tag{5.7}$$

It could be a good starting point to discuss the low energy limit of QCD from $\mathcal{L}^0_{QCD}$, which contains only the light-flavored quarks and works in the chiral limit, namely that $m_u,\ m_d,\ m_s \to 0$, so that

$$\mathcal{L}^0_{QCD} = \sum_{f=u,d,s} \bar{q}_f i \not{D} q_f - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}. \tag{5.8}$$

Introducing the chiral projection operators

$$P_L = \frac{1-\gamma_5}{2}, \quad P_R = \frac{1+\gamma_5}{2}, \tag{5.9}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ with $\{\gamma_5, \gamma_\mu\} = 0$, we have the relations

$$P_L + P_R = 1, \quad (P_L)^2 = P_L, \quad (P_R)^2 = P_R, \quad P_L P_R = P_R P_L = 0. \tag{5.10}$$

The projection operators $P_L$ and $P_R$ project the Dirac field variable $q$ to its chiral components $q_L$ and $q_R$

$$q_L = P_L q, \quad q_R = P_R q. \tag{5.11}$$

For the quark current $\bar{q}\Gamma q$ where $\Gamma = 1,\ \gamma_5,\ \gamma_\mu,\ \gamma_\mu\gamma_5,$ or $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, we have

$$\bar{q}\Gamma q = \begin{cases} \bar{q}_L\Gamma q_L + \bar{q}_R\Gamma q_R & \text{for } \Gamma \in \gamma_\mu,\ \gamma_\mu\gamma_5 \\ \bar{q}_R\Gamma q_L + \bar{q}_L\Gamma q_R & \text{for } \ \Gamma \in 1,\ \gamma_5,\ \sigma_{\mu\nu} \end{cases} \tag{5.12}$$

The low energy QCD Lagrangian in the chiral limit can be written as

$$\mathcal{L}^0_{QCD} = \sum_{f=u,d,s} (\bar{q}_{R,f} i \not{D} q_{R,f} + \bar{q}_{L,f} i \not{D} q_{L,f}) - \frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}, \tag{5.13}$$

where the left-handed and right-handed quarks are decoupled. Thus $\mathcal{L}^0_{QCD}$ is invariant under the transformations which act differently on left-handed and right-handed fields: [180]

$$\begin{aligned} q_L = \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} &= \exp\left(-i\sum_{a=1}^{8}\Theta_{La}t_a\right)e^{-i\Theta_L}\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \\ q_R = \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} &= \exp\left(-i\sum_{a=1}^{8}\Theta_{Ra}t_a\right)e^{-i\Theta_R}\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \end{aligned} \tag{5.14}$$

where $t_a = \lambda_a/2$ act in flavor space, $U_L$ and $U_R$ are unitary independent $3 \times 3$ matrices indicating the transformations of $U(3)_L$ and $U(3)_R$, and we have decomposed the $U(3)_L \times U(3)_R$ transformation into $SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$ transformation. QCD at chiral limit has a global $SU(3)_L \times SU(3)_R \times U(1)_V \times U(1)_A$ symmetry. The $U(1)_V$ is a symmetry of QCD even when quark masses are included [112], and it generates baryon number conservation. The invariance of $\mathcal{L}^0_{QCD}$ under $SU(3)_L \times SU(3)_R$ is often referred to as a chiral symmetry. According to Noether's theorem [183, 184], one might expect 9 left-handed and 9 right-handed conserved currents

$$\begin{aligned} L^\mu_a &= \bar{q}_L\gamma^\mu t_a q_L, \\ R^\mu_a &= \bar{q}_R\gamma^\mu t_a q_R, \\ L^\mu &= \bar{q}_L\gamma^\mu q_L, \\ R^\mu &= \bar{q}_R\gamma^\mu q_R. \end{aligned} \tag{5.15}$$

The $U(1)_A$ under which $q_{L(R)} \to \exp(i\Theta_{L(R)}\gamma_5)q_{L(R)}$ is not an exact symmetry of QCD with massless quarks at the quantum level due to the Abelian anomaly [21]. We can form axial-vector currents from the linear combinations of the above chiral currents

$$\begin{aligned} V^\mu_a &= R^\mu_a + L^\mu_a = \bar{q}\gamma^\mu t_a q, \\ A^\mu_a &= R^\mu_a - L^\mu_a = \bar{q}\gamma^\mu\gamma_5 t_a q, \end{aligned}$$

The chiral symmetry is a symmetry of the Lagrangian of QCD with massless quarks, but it is not a symmetry of the ground state, resulting in a spontaneously symmetry breaking, which can be easily seen from the fact that hadrons with exactly the same quantum numbers and the same mass but opposite parity do not exist [181]. This leads to a quark condensate in the vacuum, $\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{q}_L q_R + \bar{q}_R q_L|0\rangle$, which connects the left-handed with the right-handed quark fields. The chiral symmetry $SU(3)_L \times SU(3)_R$ is spontaneously broken to $SU(3)_V$. According to the Goldstone theorem [185, 186], there are 8 massless pseudoscalar Goldstone bosons arising from such spontaneous symmetry breaking. We can collect the Goldstone fields in a unitary matrix $U(\phi)$ with the chiral transformation behavior

$$U(\phi) \mapsto U_R U(\phi) U_L^{-1}, \tag{5.16}$$

and $U(\phi)$ can be conveniently parametrized in the exponential form

$$U(\phi) = \exp\left(i\phi/F_0\right), \tag{5.17}$$

$$\phi = \lambda_a\phi^a \equiv \begin{pmatrix} \pi_3 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi_3 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta_8 \end{pmatrix}, \tag{5.18}$$

where $F_0$ is the meson decay constant in the chiral limit. To the lowest order of $\pi^0 - \eta$ mixing angle, which breaks isospin symmetry, $\pi_3$ and $\eta_8$ can be expressed in terms of the physical $\pi^0$ and $\eta$ mesons as [64]

$$\begin{aligned} \pi_3 &= \pi^0 \cos(\epsilon) - \eta \sin(\epsilon), \\ \eta_8 &= \pi^0 \sin(\epsilon) + \eta \cos(\epsilon), \end{aligned} \tag{5.19}$$

where the corresponding lowest order mixing angle is given by

$$\tan(2\epsilon) = \frac{\sqrt{3}}{2} \frac{m_d - m_u}{m_s - \hat{m}} \tag{5.20}$$

with $\hat{m} = (m_u + m_d)/2$.

Including the finite masses of $u$, $d$ and $s$ quarks explicitly breaks chiral symmetry since the mass terms mix the left- and right-handed fields

$$\mathcal{L}_M = -\bar{q}Mq = -(\bar{q}_L M q_R + \bar{q}_R M q_L) \tag{5.21}$$

with

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{5.22}$$

The vector and axial vector currents are no longer conserved except for the baryon number current $V^\mu$ [180]

$$\partial_\mu V_a^\mu = i\bar{q}[M,\ t_a]q, \tag{5.23}$$

$$\begin{aligned} \partial_\mu A_a^\mu &= i\bar{q}\{M,\ t_a\}q, \\ \partial_\mu A^\mu &= 2i\bar{q}\gamma_5 M q + \frac{3g_s^2}{2\pi^2}\epsilon_{\mu\nu\rho\sigma} G_a^{\mu\nu} G_a^{\rho\sigma}, \end{aligned} \tag{5.24}$$

where we include the axial anomaly of $U(1)_A$ [180].

When considering only $u$ and $d$ quarks where $L_{QCD}^0$ is said to have $SU(2)_L \times SU(2)_R$ symmetry, rewriting the mass term makes the strong isospin violation explicit

$$\mathcal{L}_M^{ud} = m_u \bar{u}u + m_d \bar{d}d = \frac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d) + \frac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d), \tag{5.25}$$

where the first term is an isoscalar and the second one is an isovector.

By applying the Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism [187], the Green functions, which are vacuum (in QCD) matrix elements of time-ordered products of color-singlet, Hermitian products of light-quark fields, can be connected to physical processes concerning mesons and their interactions with the electroweak gauge fields of the SM [180], when the scalar and pseudoscalar densities are also included:

$$\begin{aligned} S_a^\mu &= \bar{q}t_a q \\ P_a^\mu &= i\bar{q}\gamma_5 t_a q \\ S^\mu &= \bar{q}q \\ P^\mu &= i\bar{q}\gamma^\mu\gamma_5 q. \end{aligned} \tag{5.26}$$

For example, the Green functions $\langle 0|T[P_a(x)J^\mu(y)P_b(z)]|0\rangle$ and $\langle 0|T[P_a(w)P_b(x)P_c(y)P_d(z)]|0\rangle$ are related to the pion electromagnetic form factor ($J^\mu$ represents the electromagnetic current) and pion-pion scattering, respectively.

Following Gasser and Leutwyler [188, 189], we can extend the chirally invariant low energy QCD Lagrangian by introducing the coupling of quark currents to external hermitian matrix fields $v_\mu$, $a_\mu$, $s$ and $p$ which are vector field, axial-vector field, scalar field and pseudoscalar field, respectively:

$$\begin{aligned}\mathcal{L}_{QCD} &= \mathcal{L}^0_{QCD} + \mathcal{L}_{ext} \\ &= \mathcal{L}^0_{QCD} + \bar{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \bar{q}(s - i\gamma_5 p)q \\ &= \mathcal{L}^0_{QCD} + \bar{q}_L\gamma^\mu(v_\mu - a_\mu)q_L + \bar{q}_R\gamma^\mu(v_\mu + a_\mu)q_R - \bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R,\end{aligned} \tag{5.27}$$

with the following chiral transformation properties

$$\begin{aligned} r_\mu \equiv v_\mu + a_\mu &\mapsto U_R r_\mu U_R^\dagger + iU_R\partial_\mu U_R^\dagger \\ l_\mu \equiv v_\mu - a_\mu &\mapsto U_L l_\mu U_L^\dagger + iU_L\partial_\mu U_L^\dagger \\ s + ip &\mapsto U_R(s+ip)U_L^\dagger. \end{aligned} \tag{5.28}$$

The global chiral symmetry is promoted to a local one after including the external fields [188, 190, 182, 191]. The general functional relates to the vacuum-to-vacuum transition amplitude when including the external fields [180]

$$\exp\left(iZ[v,a,s,p]\right) = \langle 0,\mathrm{out}|0,\mathrm{in}\rangle_{v,a,s,p}, \tag{5.29}$$

and the Green functions can be involved in the general functional

$$\exp\left(iZ[v,a,s,p]\right) = \langle 0|T\exp\left[i\int d^4x\mathcal{L}_{ext}(x)\right]|0\rangle_0, \tag{5.30}$$

where the subscript 0 means that the quark field operators in $\mathcal{L}_{ext}$ and the ground state $|0\rangle$ refer to the chiral limit. The external fields should be distinguished from dynamical fields in that the former do not propagate and are introduced to generate Green functions of quarks. The external photons and weak gauge bosons can be incorporated in the gauge fields $v_\mu$, $a_\mu$ [182]

$$\begin{aligned} r_\mu &= v_\mu + a_\mu = -eQA_\mu^{\mathrm{ext}} + \cdots \\ l_\mu &= v_\mu - a_\mu = -eQA_\mu^{\mathrm{ext}} - \frac{e}{\sqrt{2}\sin\theta_W}(W_\mu^{\mathrm{ext},+}T^+ + h.c.) + \cdots, \\ \text{with} \quad & Q = \frac{1}{3}\mathrm{diag}(2,-1,-1),\ T^+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{5.31}$$

where $Q$ is the quark charge currents and $V_{ij}$ are the CKM matrix elements. In addition to generating Green functions of scalar quark currents, $s$ also provides a convenient way to incorporate explicit chiral symmetry breaking through quark masses.

The relevant physically Green functions are functional derivatives of the general functional $Z[v, a, s, p]$ at $v = a = p = 0$ and $s = \mathrm{diag}(m_u, m_d, m_s)$.

Though there is no physical realization of the tensor field coupled to the tensor quark bilinear $\bar{q}\sigma^{\mu\nu}q$ in QCD, the tensor source is important when studying new physics beyond the SM. The QCD Lagrangian with an external tensor field and the relevant ChPT terms are introduced in Cata and Mateu [191]. We will discuss this part explicitly in Ch. 8.

## 5.2 Chiral perturbation theory with mesons

The general functional admits a Taylor expansion in powers of the external momenta and quark masses [192, 190, 193]. Such an expansion is most conveniently made in an effective field theory using the pseudo-Goldstone bosons, which are outcomes of the spontaneous chiral symmetry breaking of low-energy QCD, as the dynamical degrees of freedom. When choosing the effective Lagrangian properly, the resulting Green functions are identical to those from low-energy QCD [192, 190]. Counting the quark masses as quadratic powers of the momenta, the expansion of the effective Lagrangian starts at $O(p^2)$ and is in the form

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots \tag{5.32}$$

which generates the expansion of the generating functional as

$$Z = Z_2 + Z_4 + \cdots \tag{5.33}$$

Such a procedure to analyze the low-energy structure of the SM is called chiral perturbation theory (ChPT) [192, 188, 189]. The subscript $2k$ denotes the order of momenta $\mathcal{O}(p^{2k})$. For example, the subscript 2, represents each operator in the Lagrangian contains either two derivatives or one quark mass term. According to the Feynman rules, derivatives generate four-momenta. On account of Eq. (5.39) discussed later with the assumption that the chiral condensate in Eq. (5.38) does not vanish in the chiral limit, together with the on-shell condition $p^2 = M^2$, quark mass $m_q \sim M^2 \sim q^2$ with $M$ as meson masses. Note that there is an alternate formulation called "generalized chiral perturbation" theory [194, 195, 196], in which chiral condensate could be small in the chiral limit, making the power counting of the possible contributions somewhat different. A definitive test of the two scenarios can be made through a precise measurement of the pionium atom ($\pi^+\pi^-$) lifetime, and that experimental effort is still ongoing [197]. We will proceed with the usual approach. The chiral orders in Eq. (5.32) are all even because Lorentz indices of derivatives should be contracted and the quark-mass terms are counted as $\mathcal{O}(p^2)$ [180].

The effective chiral Lagrangian is constructed from the field $U(\phi)$ as defined in Eq. (5.18) and the external fields $v$, $a$, $s$ and $p$, which are subjected to the general chiral counting rules [182]

$$\begin{aligned} U &\sim O(p^0), \\ v_\mu,\ a_\mu &\sim O(p), \\ s,\ p &\sim O(p^2). \end{aligned} \tag{5.34}$$

These can be understood by the definitions of $U$ and the covariant derivative $D_\mu U$ as Eq. (5.36), and $s \sim m_q$. The leading-order chiral effective Lagrangian takes the form

$$\mathcal{L}_2 = \frac{F_0^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle, \tag{5.35}$$

where $\chi = 2B_0(s+ip)$, $\langle A\rangle$ indicates the trace of matrix $A$, and the covariant derivative is defined as

$$D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu, \tag{5.36}$$

with the chiral transformation property

$$D_\mu U \mapsto U_R(D_\mu U)U_L^\dagger. \tag{5.37}$$

There are only two low-energy constants in the lowest-order chiral Lagrangian, i.e. $F_0$ and $B_0$, which relate to the pion decay constant and to the quark condensate in the chiral limit:

$$\begin{aligned} F_\pi &= F_0[1+O(m_q)] \\ \langle 0|\bar{u}u|0\rangle &= -F_0^2 B_0[1+O(m_q)], \end{aligned} \tag{5.38}$$

where $F_\pi$ is the physical pion decay constant, with value 92.4 MeV [181] determined from leptonic pion decay [198]. Upon setting the external fields as $v_\mu = a_\mu = p = 0$ and $s = \text{diag}(m_u,\ m_d\ , m_s)$, when neglecting $\pi^0 - \eta$ mixing, i.e., the mixing angle $\epsilon = 0$, $\pi_3 = \pi^0$, and $\eta_8 = \eta$ from Eq. (5.19), and expanding $U(\phi)$ to $\phi^2$, we have [182]

$$\begin{aligned} \text{Tr}(B_0 s\phi^2) &= 2(m_u+m_d)\pi^+\pi^- + 2(m_u+m_s)K^+K^- + 2(m_d+m_s)\overline{K}^0K^0 \\ &+ (m_u+m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u-m_d)\pi^0\eta + \frac{1}{3}(m_u+m_d+m_s)\eta^2. \end{aligned} \tag{5.39}$$

Then we obtain the masses of the Goldstone bosons to the leading order in quark masses [182]:

$$\begin{aligned} M_{\pi^+}^2 &= 2\hat{m}B_0 \\ M_{\pi^0}^2 &= 2\hat{m}B_0 + O[\frac{(m_u-m_d)^2}{m_s-\hat{m}}] \\ M_{K^+}^2 &= (m_u+m_s)B_0 \\ M_{K^0}^2 &= (m_d+m_s)B_0 \\ M_\eta^2 &= \frac{2}{3}(\hat{m}+2m_s) + O[\frac{(m_u-m_d)^2}{m_s-\hat{m}}] \end{aligned} \tag{5.40}$$

with $\hat{m} \equiv \frac{1}{2}(m_u+m_d)$. These are called the Gell-Mann–Oakes–Renner relations [199]. If we take the isospin-symmetric limit $m_u = m_d = \hat{m}$, the $\pi^0\eta$ term vanishes and there is no $\pi^0 - \eta$ mixing. The squared masses satisfy the Gell-Mann–Okubo relation [200, 201]

$$4M_K^2 = 4(\hat{m}+m_s)B_0 = 3M_\eta^2 + M_\pi^2. \tag{5.41}$$

The most general Lagrangian of the next-to-leading order, which respects the local chiral gauge invariance, Lorentz invariance, parity(P) and charge conjugation(C), can be expressed as [188, 182, 193, 180]

$$\begin{aligned}
\mathcal{L}_4 &= L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\
&+ L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D^\mu U^\dagger D_\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+ L_5\langle D^\mu U^\dagger D_\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+ L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U\chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&- iL_9\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle + L_{10}\langle U^\dagger F^R_{\mu\nu} U F^{L\mu\nu}\rangle \\
&+ H_1\langle F_{R\mu\nu}F^{\mu\nu}_R + F_{L\mu\nu}F^{\mu\nu}_L\rangle + H_2\langle \chi^\dagger\chi\rangle. \qquad (5.42)
\end{aligned}$$

where the field strength tensors are of the form

$$\begin{aligned}
F^{\mu\nu}_R &= \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \\
F^{\mu\nu}_L &= \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]. \qquad (5.43)
\end{aligned}$$

The numerical values of the low-energy constants cannot be determined by chiral symmetry. In principle, they contain information on the underlying dynamics and should be calculable from QCD. In practice, they are either extracted from experiments or theoretically estimated using lattice QCD or with additional model dependent assumptions. [182, 180].

Power counting is very important when calculating processes to a certain order $\mathcal{O}(p^{2n})$. Considering a connected Feynman diagram with $N_L$ loops and $N_{2k}$ vertices from $\mathcal{L}_{2k}$, the chiral dimension of this diagram is given by [202, 180]

$$D = 2N_L + 2 + \sum_{k=1}^{\infty}(2k-2)N_{2k}. \qquad (5.44)$$

Therefore, up to order $\mathcal{O}(p^4)$ or next-to-leader order (NLO), we have [193]

$$\begin{aligned}
D = 2: &\qquad N_L = 0,\ k = 1; \qquad (5.45)\\
D = 4: &\qquad N_L = 0,\ k = 2, \qquad (5.46)\\
&\qquad N_L = 1,\ k = 1. \qquad (5.47)
\end{aligned}$$

### 5.3 Theoretical analysis of $\eta \to 3\pi$

Let us first take a glance at $\eta \to \pi^0\pi^0\pi^0$ where all three pions are identical.

The quantum numbers of $\eta$ is $I^G(J^{PC}) = 0^+(0^{-+})$. In $\eta \to \pi^0\pi^0\pi^0$, two pions of the final state would seem to be able to form states of total isospin $I_{2\pi} = 0,\ 1,\ 2$. Coupling with the remaining pion to make the total isospin of the final $3\pi^0$ system to be $I_{3\pi} = 0$ only exists when $I_{2\pi} = 1$. However $(\pi^0\pi^0)_{I=1}$ does not exist according to Bose-Einstein statistics (or Clebsch-Gordan coefficients) because the two-pion state must be symmetric under particle exchange. As a result $\eta \to \pi^0\pi^0\pi^0$ must violate isospin conservation. Based on angular momentum conservation and that the pions are spinless, the orbital angular momentum of the two coupled pions and the one between the two coupled pions and the third or "bachelor" pion should be the same. Then we have that the parity of the $J = 0$ 3-pion final state is [53],

$$P|[\pi_1(\boldsymbol{p})\pi_2(-\boldsymbol{p})]_l\pi_3(\boldsymbol{p}')_l\rangle = -|[\pi_1(\boldsymbol{p})\pi_2(-\boldsymbol{p})]_l\pi_3(\boldsymbol{p}')_l\rangle, \tag{5.48}$$

so that $\eta \to \pi^0\pi^0\pi^0$ conserves parity. Since the final pions are all identical and not distinguishable, there is no charge conjugation violation in this process, and there is no CP violation either.

As for $\eta \to \pi^+\pi^-\pi^0$, the final-state pions can form total isospin 0, 1, 2, and 3. For the $I = 0$ final state which conserves isospin, we can write

$$\begin{aligned}(3\pi)_{I=0} &= \frac{1}{\sqrt{3}}\left\{[(\pi^+\pi^0)_{I=1}|\pi^-\rangle + |\pi^-\rangle(\pi^+\pi^0)_{I=1}] - (\pi^+\pi^-)_{I=1}|\pi^0\rangle\right. \\ &\quad \left.+[(\pi^-\pi^0)_{I=1}|\pi^+\rangle + |\pi^+\rangle(\pi^-\pi^0)_{I=1}]\right\},\end{aligned} \tag{5.49}$$

with

$$\begin{aligned}(\pi^+\pi^0)_{I=1} &= \frac{1}{\sqrt{2}}(|\pi^+\rangle|\pi^0\rangle - |\pi^0\rangle|\pi^+\rangle) \\ (\pi^+\pi^-)_{I=1} &= \frac{1}{\sqrt{2}}(|\pi^+\rangle|\pi^-\rangle - |\pi^-\rangle|\pi^+\rangle) \\ (\pi^-\pi^0)_{I=1} &= \frac{1}{\sqrt{2}}(-|\pi^-\rangle|\pi^0\rangle + |\pi^0\rangle|\pi^-\rangle).\end{aligned} \tag{5.50}$$

Then the full wave functions for the $(\pi^+\pi^-\pi^0)_{I=0}$ system is

$$\begin{aligned}(\pi^+\pi^-\pi^0)_{I=0} &= \frac{1}{\sqrt{6}}\left[\left(|\pi^+\rangle|\pi^0\rangle - |\pi^0\rangle|\pi^+\rangle\right)|\pi^-\rangle + |\pi^-\rangle\left(|\pi^+\rangle|\pi^0\rangle - |\pi^0\rangle|\pi^+\rangle\right)\right. \\ &+ \left(-|\pi^-\rangle|\pi^0\rangle + |\pi^0\rangle|\pi^-\rangle\right)|\pi^+\rangle + |\pi^+\rangle\left(-|\pi^-\rangle|\pi^0\rangle + |\pi^0\rangle|\pi^-\rangle\right) \\ &- \left.\left(|\pi^+\rangle|\pi^-\rangle - |\pi^-\rangle|\pi^+\rangle\right)|\pi^0\rangle\right]\end{aligned} \tag{5.51}$$

The wave function is antisymmetric by exchanging $\pi^+ \leftrightarrow \pi^-$, which indicates that under charge conjugation we have

$$\mathcal{C}[(\pi^+\pi^-\pi^0)_{I=0}] = -(\pi^+\pi^-\pi^0)_{I=0}. \tag{5.52}$$

Since $C(\eta) = +1$, the process $\eta \to \pi^+\pi^-\pi^0$ should violate either charge conjugation symmetry or isospin conservation or both. Moreover, the relation between the charge conjugation eigenvalue and the total isospin of the final $\pi^+\pi^-\pi^0$ state can be expressed as [52]

$$C = -(-1)^I \tag{5.53}$$

which indicates that for the $I = 0$ final state, $I$ is conserved while $C$ should be broken, whereas for $I = 1$ and $I = 3$ final states, $C$ is conserved while $I$ should be broken, and for the $I = 2$ final state, both $C$ and $I$ should be broken. According to the parity of the 3-pion final states given by Eq. (5.48), parity is conserved in the decay

$\eta \to \pi^+\pi^-\pi^0$. Thus if there exists $CP$ violation in this decay, it would be $C$ and $CP$ violation.

In 1965, Lee [52], Bernstein et al. [203] and Nauenberg [56] supposed that $\eta \to \pi^+\pi^-\pi^0$ could happen via second order EM interaction to a $C = +1$ and $I = 1$ final state, and C breaking interaction to a $C = -1$ and $I = 0$ or $I = 2$ final states. The interference between the $C = +1$ and $C = -1$ amplitudes would result in asymmetry in the energy distribution of $\pi^+$ and $\pi^-$. Refs. [52, 56, 57] assumed simple phenomenological forms of the C-conserving and C-breaking amplitudes and calculated the size of the $\pi^+$ and $\pi^-$ energy distribution asymmetry across the mirror line of the Dalitz plot, as discussed in Sec. 3.1.1.

There has been no more consideration of CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay after 1966. However, at that early time the quark model was not well established and well accepted. Now we know that in $\eta \to \pi^+\pi^-\pi^0$ decay, the amplitude with $C = +1$ and $I = 1$ is dominant, and within the SM there are two $\Delta I = 1$ sources, one is from strong interaction (QCD) as

$$\mathcal{H}_{\mathrm{QCD}}(x) = \frac{m_d - m_u}{2}(\bar{d}d - \bar{u}u)(x), \tag{5.54}$$

the other is from the electromagnetic interaction as

$$\mathcal{H}_{\mathrm{QED}}(x) = -\frac{e^2}{2}\int dy D^{\mu\nu}(x-y)T(j_\mu(x)j_\nu(y)), \tag{5.55}$$

where $D^{\mu\nu}(x-y)$ is the photon propagator and $j_\mu(x)$ is the current density containing charged fields. The tree level electromagnetic contribution was studied by Sutherland and Bell [204, 205] and found to be much too small compared to the decay rate. Baur et al. [206] studied the one-loop level electromagnetic contribution neglecting terms proportional to $e^2(m_d - m_u)$ and found it to be very small, at the percent level with respect to the decay width. Ditsche [207] re-evaluated the one-loop level contribution by including terms proportional to $e^2(m_d - m_u)$ and found those terms to be of the same order as those considered in Ref. [206]. The dominant contribution is from the QCD $\Delta I = 1$ source. The decay will vanish if $m_u = m_d$ and the amplitude of $\eta \to 3\pi$ should be proportional to $(m_u - m_d)$. The amplitude can be expressed as

$$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi}M(s,t,u), \tag{5.56}$$

with

$$\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}. \tag{5.57}$$

Using ChPT, the leading-order (LO) amplitude was given by Osborn and Wallace [208]. Gasser and Leutwyler [61] and Bijnens [64] gave the next-to-leading-order (NLO) amplitude considering $\pi\pi$ scattering unitarity, and Bijnens and Ghorbani [64] gave the next-to-next-leading-order (NNLO) description of this process. In addition to ChPT, the SM contribution is also studied in frameworks tailored to address various final-state-interaction effects [209, 210, 211, 62, 212, 213, 214, 215, 216, 217, 218, 219].

The $\eta \to \pi^+\pi^-\pi^0$ decay has aroused much modern interest because we can extract the $u$ quark and $d$ quark mass difference from it. However it has not been well considered that this channel can also appear through the $C$ and $CP$ violating interactions to $I = 0$ and $I = 2$ final states. Previous theoretical literatures studying this decay channel treated those $C$ and $CP$ violating amplitudes to be negligible by default. Since the SM CP violating contribution to this channel should be vanishingly small, and there is no direct connection to the popular and highly studied permanent EDM searches, searching for and studying C and CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay is very important to understand new CP violating mechanisms beyond the SM.

In this chapter, we discussed the chiral symmetry of QCD and the effective field theory that is prominently used to describe QCD at low energy – ChPT, as well as a general theoretical analysis of $\eta \to 3\pi$. In the next chapter, we are going to introduce a new method to study the patterns of CP violating amplitudes of $\eta \to \pi^+\pi^-\pi^0$ decay through the mirror symmetry breaking of its Dalitz plot.

CHAPTER 6

# PATTERNS OF $CP$ VIOLATION FROM MIRROR SYMMETRY BREAKING IN THE $\eta \to \pi^+\pi^-\pi^0$ DALITZ PLOT

A violation of mirror symmetry in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot has long been recognized as a signal of $C$ and $CP$ violation. Here we show how the isospin of the underlying $C$- and $CP$-violating structures can be reconstructed from their kinematic representation in the Dalitz plot. Our analysis of the most recent experimental data reveals, for the first time, that the $C$- and $CP$-violating amplitude with total isospin $I = 2$ is much more severely suppressed than that with total isospin $I = 0$.

This chapter has been published as a Regular Article in Physical Review D. [92] in collaboration with Dr. Susan Gardner.

## 6.1 Introduction

The decay $\eta \to 3\pi$ first came to prominence after the observation of $K_L \to \pi^+\pi^-$ decay and the discovery of $CP$ violation in 1964 [4], because it could be used to test whether $K_L \to \pi^+\pi^-$ decay was generated by $CP$ violation in the weak interactions [54, 55]. Rather, $CP$ violation could arise from the interference of the $CP$-conserving weak interaction with a new, "strong" interaction that breaks $C$ and $CP$; this new interaction could be identified through the appearance of a charge asymmetry in the momentum distribution of $\pi^+$ and $\pi^-$ in $\eta \to \pi^+\pi^-\pi^0$ decay [54, 52, 56]. Since $\eta \to \pi^+\pi^-\pi^0$ breaks G parity, isospin $I$ and/or charge-conjugation $C$ must be broken in order for the process to occur. Thus a charge asymmetry could arise from the interference of a $C$-conserving, but isospin-breaking amplitude with a isospin-conserving, but $C$-violating one [52]. Numerical estimates were made by assuming that the isospin-violating contributions were driven by electromagnetism [52, 56, 57]. Since that early work, our understanding of these decays within the Standard Model (SM) has changed completely: the weak interaction does indeed break $CP$ symmetry, through flavor-changing transitions characterized by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Moreover, isospin breaking in the strong interaction, mediated by the up-down quark mass difference [58, 59, 60], is now known to provide the driving effect in mediating $\eta \to 3\pi$ decay [61, 62, 63, 64], with isospin-breaking, electromagnetic effects playing a much smaller role [204, 205, 206, 207].

Modern theoretical studies of $\eta \to 3\pi$ decay focus on a complete description of the final-state interactions within the SM, in order to extract the isospin-breaking, light-quark mass ratio $Q \equiv \sqrt{(m_s^2 - \hat{m}^2)/(m_d^2 - m_u^2)}$, with $\hat{m} = (m_d + m_u)/2$, precisely [62, 63, 64, 213, 215, 216, 218, 217, 219]. There has been no further theoretical study of $CP$ violation in $\eta \to \pi^+\pi^-\pi^0$ decay since 1966. Since the $\eta$ meson carries neither spin nor flavor, searches for new physics in this system possess special features. For example, $\eta \to \pi^+\pi^-\pi^0$ decay must be parity $P$ conserving if the $\pi$ and $\eta$ mesons have the same intrinsic parity, so that $C$ violation in this process implies that $CP$ is violated as well. There has been, moreover, much effort invested in the possibility of flavor-diagonal $CP$ violation via a non-zero permanent electric dipole moment (EDM), which is $P$ and time-reversal T violating, or $P$ and $CP$ violating if CPT symmetry is assumed. Studies of flavor-diagonal, $C$ and $CP$ violating processes are largely lacking. We believe that the study of the Dalitz plot distribution in $\eta \to \pi^+(p_{\pi^+})\pi^-(p_{\pi^-})\pi^0(p_{\pi^0})$ decay is an ideal arena in which to search for $C$ and $CP$ violation beyond the SM. Were we to plot the Dalitz distribution in terms of the Mandelstam variables $t \equiv (p_{\pi^-} + p_{\pi^0})^2$ and $u \equiv (p_{\pi^+} + p_{\pi^0})^2$, the charge asymmetry we have noted corresponds to a failure of mirror symmetry, i.e., of $t \leftrightarrow u$ exchange, in the Dalitz plot. In contrast to that $C$ and $CP$ violating observable, a nucleon EDM could be mediated by a minimal $P$- and T-violating interaction, the mass-dimension-four $\bar{\theta}$ term of the SM, and not new weak-scale physics. The $\bar{\theta}$ term can also generate $\eta \to \pi\pi$ and $\eta/\eta' \to 4\pi^0$ decay, breaking $P$ and $CP$ explicitly, so that limits on the decay rate constrains the square of a $CP$-violating parameter [46, 220, 49, 51]. Since the $\bar{\theta}$ term is $C$ even, it cannot contribute to the charge asymmetry, at least at tree level. Moreover, SM weak interactions do not support flavor-diagonal $C$ and $CP$ violation. Note that the charge asymmetry is linear in $CP$-violating parameters.

The appearance of a charge asymmetry and thus of $C$ (and CP) violation in $\eta \to \pi^+(p_{\pi^+})\pi^-(p_{\pi^-})\pi^0(p_{\pi^0})$ decay can be probed experimentally through the measurement of a left-right asymmetry, $A_{LR}$ [69]:

$$A_{LR} \equiv \frac{N_+ - N_-}{N_+ + N_-} \equiv \frac{1}{N_{\rm tot}}(N_+ - N_-)\,, \tag{6.1}$$

where $N_\pm$ is the number of events with $u \gtrless t$, so that the $\pi^+$ has more (less) energy than the $\pi^-$ if $u > (<)t$ in the $\eta$ rest system. A number of experiments have been conducted over the years to test for a charge asymmetry in $\eta \to \pi^+\pi^-\pi^0$ decay, with early experiments finding evidence for a nonzero asymmetry [65, 66, 67], but with possible systematic problems becoming apparent only later, as, e.g., in Ref. [70]. Other experiments find no evidence for a charge asymmetry and $C$ violation [71, 69, 72, 73, 74, 70], and we note that new, high-statistics experiments are planned [75, 76, 77]. It is also possible to form asymmetries that probe the isospin of the $C$-violating final state: a sextant asymmetry $A_S$, sensitive to the $I = 0$ state [56, 52], and a quadrant asymmetry $A_Q$, sensitive to the $I = 2$ final state [52, 69]. These asymmetries are more challenging to measure and are only poorly known [69]. In this paper we develop a method to discriminate between the possible $I = 0$ and $I = 2$ final states by considering the pattern of mirror-symmetry-breaking events they engender in the Dalitz plot. Mirror-symmetry breaking as a probe of $CP$ violation has also been studied in untagged, heavy-flavor decays [221, 90, 53, 91], with Ref. [53] analyzing how different $CP$-violating mechanisms populate the Dalitz plot. We also note Refs. [39, 40] for Dalitz studies of $CP$ violation in heavy-flavor decays.

## 6.2 Theoretical Framework

The $\eta \to 3\pi$ decay amplitude in the SM can be expressed as [61, 62]

$$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u), \tag{6.2}$$

where we employ the Mandelstam variables $u$, $t$, and $s = (p_{\pi^+} + p_{\pi^-})^2$ and work to leading order in strong-interaction isospin breaking. Since $C = -(-1)^I$ in $\eta \to 3\pi$ decay [52], the $C$- and $CP$-even transition amplitude with a $\Delta I = 1$ isospin-breaking prefactor must have $I = 1$. The amplitude $M(s,t,u)$ thus corresponds to the total isospin $I = 1$ component of the $\pi^+\pi^-\pi^0$ state and can be expressed as [62, 222]

$$\begin{aligned} M_1^C(s,t,u) &= M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) \\ &+ M_2(t) + M_2(u) - \frac{2}{3}M_2(s)\,, \end{aligned} \tag{6.3}$$

where $M_I$(z) is an amplitude with $\pi - \pi$ rescattering in the $z$-channel with isospin $I$. This decomposition can be recovered under isospin symmetry in chiral perturbation theory (ChPT) up to next-to-next-to-leading order (NNLO), $\mathcal{O}(p^6)$, because the only absorptive parts that can appear are in the $\pi - \pi$ $S-$ and $P$-wave amplitudes [64]. An analogous relationship exists in $\eta \to 3\pi^0$ decay [62], though there is no Dalitz plot asymmetry and hence no effect linear in $CP$ violation in that case because the final-state particles are all identical.

Since we are considering $C$ and $CP$ violation, additional amplitudes can appear — namely, total $I = 0$ and $I = 2$ amplitudes. The complete amplitude is thus

$$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M_1^C(s,t,u) + \alpha M_0^{\not C}(s,t,u) + \beta M_2^{\not C}(s,t,u)\,, \tag{6.4}$$

where $\alpha$ and $\beta$ are unknown, low-energy constants — complex numbers to be determined by fits to the experimental event populations in the Dalitz plot. If they are determined to be non-zero, they signal the appearance of $C$- and $CP$-violation. To construct $A_{LR}$ in Eq. (6.1), we compute

$$N_\pm = \frac{1}{256\pi^3 M_\eta^3}\int_{u \gtrless t} dtdu\,|A(s,t,u)|^2\,, \tag{6.5}$$

using Eq. (6.4) and working to leading order in $CP$ violation. Since the phase space is symmetric and the $CP$-violating terms are antisymmetric under $u \leftrightarrow t$ exchange, we see that the $CP$-violating terms leave the total decay rate unchanged in $\mathcal{O}(\alpha)$, $\mathcal{O}(\beta)$.

We now turn to the amplitudes $M_{0,2}^{\not C}(s,t,u)$. Here, too, we introduce functions $M_I(z)$ for amplitudes that contain $\pi - \pi$ scattering in the $z$ channel with isospin $I$. After using angular-momentum conservation and the Clebsch-Gordon coefficients for the addition of the possible isospin states, as shown in the appendix, we have

$$M_0^{\not C}(s,t,u) = (s-t)M_1'(u) + (u-s)M_1'(t) - (u-t)M_1'(s) \tag{6.6}$$

and

$$M_2^{\not C}(s,t,u) = (s-t)M_1''(u) + (u-s)M_1''(t) + 2(u-t)M_1''(s) + \sqrt{5}[M_2''(u) - M_2''(t)]\,, \quad (6.7)$$

where the superscripts distinguish the functions that appear in each state of total isospin. In what follows we do not compute $M_I'(z)$ and $M_I''(z)$ explicitly, but, rather, estimate them. With this, we can use the experimental studies we consider in this paper to set limits on the possibilities, by constraining $\alpha$ and $\beta$. For context we note that the particular new-physics operators that would give $C$- and $CP$-violation are not well-established, though examples have been discussed in the literature [223, 224, 225, 226, 227]. From the viewpoint of SM effective field theory (SMEFT) [164, 101], we also know that there are many more examples, even in leading-mass dimension, than have been discussed thus far [228]. Nevertheless, we can draw conclusions about $M_I'(z)$ and $M_I''(z)$ irrespective of the choice of new-physics operator. In particular, since the operators that mediate $I = 0$ or $I = 2$ amplitudes break $C$, they cannot mediate a $\eta \to \pi^0$ transition, as we suppose that the neutral meson states remain of definite $C$-parity in the presence of $C$-violation. Thus if we were to evaluate the decay diagrams in NLO ChPT in these exotic cases, they would have the same decay topology as the diagrams that appear in that order in the SM amplitude for $\eta \to \pi^+\pi^-\pi^0$ decay. Thus there is a one-to-one map of the two-body rescattering terms in the SM to the $C$- and $CP$-violating amplitudes. To proceed, we assume that the phases of the functions $M_I(z)$, $M_I'(z)$ and $M_I''(z)$ arise from the strong-interaction dynamics of final-state, $\pi - \pi$ scattering of isospin $I$ in channel z, making the phase of each function common to all three isospin amplitudes. Such treatments are familiar from the search for non-SM $CP$ violation, such as in the study of $B \to \pi(\rho \to \pi\pi)$ decays [229, 230, 231, 79]. Moreover, at the low energies we consider here, the scattering of the two-pions in the final state is predominantly elastic, as mixing with other final-states can only occur through G-parity breaking. Regardless of the total isospin of the final state pions, the effective Hamiltonian that mediates the decay separates into a $C$- and/or $I$-breaking piece and a $C$- and $I$-conserving piece. Working to leading order in $C$- and/or $I$-breaking, and assuming that the final-state interactions are two-body only, Watson's theorem [232], familiar from $K \to \pi\pi$ decays [79], also applies to this case and makes the phase of the function $M_I(z)$ common to the three cases. However, the functions $M_I(z)$, $M_I'(z)$ and $M_I''(z)$ could differ by polynomial prefactors that depend on $z$. Nevertheless, we believe these effects are relatively negligible, because the energy release in $\eta \to \pi^+\pi^-\pi^0$ decay is small. We illustrate this explicitly later in this section.

We wish to study the possible patterns of $C$- and $CP$-violation across the Dalitz plot, so that we now turn to the explicit evaluation of Eq. (6.4) and its associated Dalitz distribution. Much effort has been devoted to the evaluation of the SM contribution, with work in ChPT [208, 61, 64], as well as in frameworks tailored to address various final-state-interaction effects [209, 210, 211, 62, 212, 213, 214, 215, 216, 218, 217, 219]. In what follows we employ a next-to-leading-order (NLO) ChPT analysis [61, 64] because it is the simplest choice in which the $C$- and $CP$-violating coefficients $\alpha$ and $\beta$ can have both real and imaginary parts. A comparison of the NLO and NNLO analyses of Bijnens et al. [64], noting Table I of Ref. [70], shows that this is an acceptable choice. We thus think it is rich enough to give a basic view as to how our idea works. To compute the $C$-violating amplitudes, we decompose the $I = 1$ amplitude into the isospin basis $M_I(z)$. As well known [233, 64, 217, 218], the isospin decompositions involving the $\pi - \pi$ rescattering functions $J^r_{PQ}(s)$ are unique, whereas the polynomial parts of the amplitude are not, due to the relation $s + t + u = 3s_0$, where $s_0 = (M^2_\eta + 2M^2_{\pi^+} + M^2_{\pi^0})/3$. Thus there are $M_I(z)$ redefinitions that leave the $I = 1$ amplitude invariant, as discussed in Ref. [233]. However, since we assume that strong rescattering effects dominate $M_I(x)$, we can demand that the $I = 0, 2$ amplitudes remain invariant also. As a result, only the redefinition $M_0(s) - \frac{4}{3}\delta_1$ and $M_2(z)+\delta_1$, with $\delta_1$ an arbitrary constant, survives. In what follows we adopt the NLO analyses of Refs. [61, 64], and our isospin decomposition of Ref. [61] is consistent with that in Bijnens and Ghorbani [64] — its detailed form can be found in the information in the appendix. Small differences in the numerical predictions exist, however, due to small differences in the inputs used [61, 64], and we study their impact explicitly. Returning to the would-be NLO ChPT computation of the total $I = 0, 2$ amplitudes, we note that $C$- and $CP$-violating four-quark operators are generated by operators in mass dimension 8 in SMEFT [228], so that these amplitudes start beyond $\mathcal{O}(p^4)$, though this is not at odds with pulling out a strong rescattering function. The $p^2$-dependence of a $C$- and $CP$-violating operator from physics beyond the SM would in part be realized as dimensionless ratios involving the new physics scale and would appear in the pre-factors $\alpha$ or $\beta$ as appropriate.

Irrespective of the particular new-physics operator, we note, by analyticity, that the $M_I(x)$ for the total $I = 0, 2$ amplitudes could differ from the SM form, which is driven by the strong $\pi - \pi$ phase shifts, by a multiplicative polynomial factor, nominally of form $1 + C^I_1 x/M^2_\pi + C^I_2 x^2/M^4_\pi + \ldots$, where $C^I_1$ and $C^I_2$ are constants. (We note polynomials of similar origin appear in the time-like pion form factor [234].) We emphasize that in assuming that strong-interaction phases dominate we suppose these corrections to be unimportant. We believe this to be an excellent approximation, which we illustrate through a plot of the functions $M_I(s)$, as shown in Fig. 6.1. The physics of $\pi-\pi$ scattering make the functions $M_I(s)$ vary substantially with $s$, whereas $s$ itself only changes by about a factor of 2 in $\eta \to 3\pi$ decay. As a result we expect that the ignored polynomial factors are numerically unimportant, so that their neglect does not impact the conclusions of this paper.

### 6.3 Results

The Dalitz distribution in $\eta \to \pi^+\pi^-\pi^0$ is usually described in terms of variables $X$ and $Y$ [95]:

$$X \equiv \sqrt{3}\frac{T_{\pi^+} - T_{\pi^-}}{Q_\eta} = \frac{\sqrt{3}}{2M_\eta Q_\eta}(u - t), \tag{6.8}$$

$$Y \equiv \frac{3T_{\pi^0}}{Q_\eta} - 1 = \frac{3}{2M_\eta Q_\eta}[(M_\eta - M_{\pi^0})^2 - s] - 1\,, \tag{6.9}$$

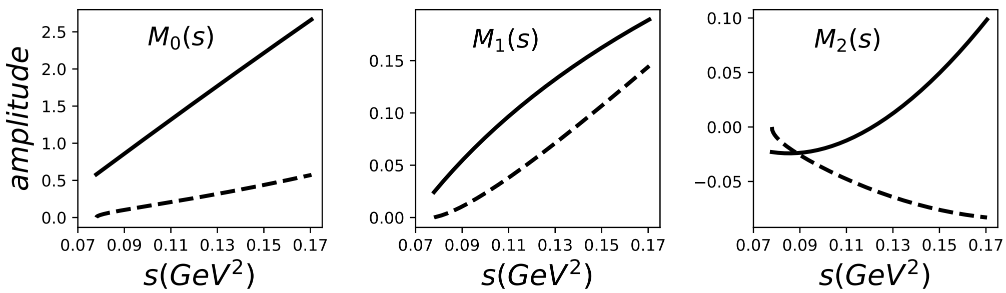


Figure 6.1: Amplitudes of $M_I(s)$ from Eqs. (A.9), (A.10) and (A.11). The solid lines represent the real part of $M_I(s)$ and the dashed lines denote the imaginary part. Their $s$-dependence is driven by that of the $\pi$-$\pi$ phase shift [61]. Note that the $M_0(s)$ and $M_2(s)$ amplitudes are dimensionless, whereas $M_1(s)$ has units of $\mathrm{Ge}V^{-2}$. Since the form $(u-t)M_1(s)$ appears in the final $C$- and $CP$-violating amplitudes, we note that $M_2(s)$ is typically a factor of a few larger across the Dalitz plot.

where $Q_\eta = T_{\pi^+} + T_{\pi^-} + T_{\pi^0} = M_\eta - 2M_{\pi^+} - M_{\pi^0}$, and $T_{\pi^i}$ is the $\pi^i$ kinetic energy in the $\eta$ rest frame. The decay probability can be parametrized in a polynomial expansion around the center point $(X,Y) = (0,0)$ [70]:

$$\begin{aligned} |A(s,t,u)|^2 &= N(1 + aY + bY^2 + cX + dX^2 + eXY \\ &+ fY^3 + gX^2Y + hXY^2 + lX^3 + \ldots)\,. \end{aligned} \tag{6.10}$$

Since the $C$ transformation on the decay amplitude is equivalent to $t \leftrightarrow u$ exchange [53], we see that the appearance of terms that are odd in $X$ would indicate both $C$ and $CP$ violation. The KLOE-2 collaboration [70] has provided a more precise estimate of the $C$-even parameters in Eq. (6.10) and bounded the $C$-odd ones. Returning to Eq. (6.4), we see that the $C$- and $CP$-violating contributions to the decay probability are

$$\begin{aligned} \frac{1}{\xi}|A(s,t,u)|^2_{\not{C}} &= M_1^C[\alpha M_0^{\not{C}} + \beta M_2^{\not{C}}]^* + H.c. \\ &= 2\mathrm{Re}(\alpha)[\mathrm{Re}(M_1^C)\mathrm{Re}(M_0^{\not{C}}) + \mathrm{Im}(M_1^C)\mathrm{Im}(M_0^{\not{C}})] \\ &- 2\mathrm{Im}(\alpha)[\mathrm{Re}(M_1^C)\mathrm{Im}(M_0^{\not{C}}) - \mathrm{Im}(M_1^C)\mathrm{Re}(M_0^{\not{C}})] \\ &+ 2\mathrm{Re}(\beta)[\mathrm{Re}(M_1^C)\mathrm{Re}(M_2^{\not{C}}) + \mathrm{Im}(M_1^C)\mathrm{Im}(M_2^{\not{C}})] \\ &- 2\mathrm{Im}(\beta)[\mathrm{Re}(M_1^C)\mathrm{Im}(M_2^{\not{C}}) - \mathrm{Im}(M_1^C)\mathrm{Re}(M_2^{\not{C}})]\,, \end{aligned} \tag{6.11}$$

where $\xi \equiv -(M_K^2/M_\pi^2)(M_K^2 - M_\pi^2)/(3\sqrt{3}F_\pi^2 Q^2)$, and the existing experimental assessments of $|A(s,t,u)|^2_{\not{C}}$ correspond to the set of odd $X$ polynomials in $|A(s,t,u)|^2$. The parameter $N$ drops out in the evaluation of the asymmetries, and the parameters $c, e, h,$ and $l$ are taken from the first line of Table 4.6 in the Ph.D. thesis of Caldeira Balkeståhl [68],

$$\begin{aligned} c &= (-4.34 \pm 3.39) \times 10^{-3}, \quad e = (2.52 \pm 3.20) \times 10^{-3}, \\ h &= (1.07 \pm 0.90) \times 10^{-2}, \quad l = (1.08 \pm 6.54) \times 10^{-3}\,, \end{aligned} \tag{6.12}$$

which fleshes out Ref. [70] — the results emerge from a global fit to the Dalitz distribution. There is a typographical error in the sign of $c$ in Ref. [70]. We now turn to the extraction of Re($\alpha$), Im($\alpha$), Re($\beta$), and Im($\beta$) using the experimental data and Eqs. (6.3,6.6,6.7) using the $M_I(z)$ amplitudes from $\mathcal{O}(p^4)$ ChPT [61, 64]. We evaluate the denominators of the possible charge asymmetries by computing $\xi^2|M_1^C(z,t,u)|^2$ only.

Herewith we collect the parameters needed for our analysis. We compute the phase space with physical masses, so that $s+t+u=3s_0$, but the decay amplitudes [61, 64] on which we rely, namely, $M(s,t,u)$ in Eq. (6.2), should be in the isospin limit, implying some adjustment of the input parameters may be needed. We adopt the hadron masses and $\sqrt{2}F_\pi = (130.2 \pm 1.7) \times 10^{-3}$ GeV from Ref. [41] for both amplitudes. For the Gasser and Leutwyler (GL) amplitude [61] we use $M_\pi \equiv \sqrt{(2M_{\pi^\pm}^2 + M_{\pi^0}^2)/3}$, $M_K \equiv \sqrt{(M_{K^+}^2 + M_{K^0}^2)/2}$, where we discuss our treatment of the two-particle thresholds in the supplement, with $F_0 = F_\pi$, $F_K/F_\pi = 1.1928 \pm 0.0026$ [41], and $L_3 = (-3.82 \pm 0.30) \times 10^{-3}$ from the NLO fit with the scale $\mu = 0.77$ GeV [235]. We use these parameters in the prefactor in Eq. (6.2) also, as well as $Q = 22.0$ [217], to find $\xi = -0.137$. For the Bijnens and Ghorbani (BG) amplitude through $\mathcal{O}(p^4)$ [64], we use $M(s,t,u) = M^{(2)}(s) + M^{(4)}(s,t,u)$ and multiply the prefactor in Eq. (6.2) by $-(3F_\pi^2)/(M_\eta^2 - M_\pi^2)$ to yield that in Ref. [64]. In the $\mathcal{O}(p^2)$ term, which contributes to $M_0(s)$, $M^{(2)}(s) = (4M_\pi^2 - s)/F_\pi^2$, and we use $M_\pi$ and $F_\pi$ as defined for the GL amplitude [61]. In the $\mathcal{O}(p^4)$ term, we use $M_{\pi^0}$ and $M_{K^0}$ as indicated, as well as $\Delta = M_\eta^2 - M_{\pi^0}^2$ and $L_3$, $L_5$, $L_7$, $L_8$ from fit 10 of Ref. [236].

We solve for $\alpha$ and $\beta$ in two different ways for each of the decay amplitudes [61, 64]. We begin with the GL amplitude [61], first making a Taylor expansion of Eq. (6.11) to cubic power in $X$ and $Y$ about $(X,\ Y) = (0,\ 0)$. We then equate coefficients associated with the $X$, $XY$, $XY^2$, and $X^3$ terms to $c$, $e$, $h$ and $l$, respectively, and then solve the four equations to obtain Re($\alpha$), Im($\alpha$), Re($\beta$), and Im($\beta$). The resulting values of $\alpha$ and $\beta$ are

$$\begin{aligned}
\mathrm{Re}(\alpha) &= 16 \pm 24\,, \\
\mathrm{Re}(\beta) &= (-1.5 \pm 2.7) \times 10^{-3}\,, \\
\mathrm{Im}(\alpha) &= -20 \pm 29\,, \\
\mathrm{Im}(\beta) &= (-1.3 \pm 4.7) \times 10^{-3}\,.
\end{aligned} \tag{6.13}$$

In the first row of Fig. 6.2 we compare the resulting assessment of Eq. (6.11) with the KLOE-2 results. Large discrepancies exist, particularly at large values of $X$ and/or $Y$, where the empirical Dalitz plot [70] shows considerable stength. Thus we turn to a second procedure, in which we make a global fit of $\alpha$ and $\beta$ in Eq. (6.11) to the KLOE-2 results. That is, we assess the Dalitz distribution $N(X,Y)$ and its error by using the Dalitz plot parameters in Eq. (6.12), discretized onto a $(X,Y)$ mesh with 682 points. To determine $N(X,Y)$ and its error we use the odd-$X$ terms in Eq. (6.10) with the normalization factor $N = 0.0474$ as per the GL amplitude [61] and compute the covariance matrix using Eq. (6.12) and the correlation matrix given in Table 4.3 of Ref. [68]. We then fit $|A(s,t,u)|^2_{\not{C}}$ using the GL amplitude [61] to $N(X,Y)$ using

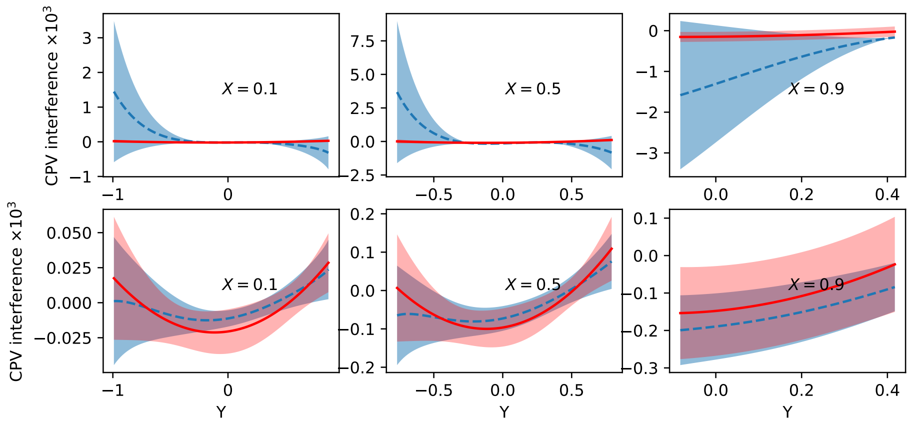


Figure 6.2: Results for the $C$- and $CP$-violating (CPV) interference term, $|A(s,t,u)|^2_{\not{C}}$ in Eq. (6.11), using the GL amplitude [61] and two methods for the determination of $\alpha$ and $\beta$: (i) a Taylor expansion (top row) and (ii) a global fit (bottom row) as described in text using the GL decay amplitude [61]. The blue dashed lines with a one-$\sigma$ error band (dark) are our results, and the red solid lines with a one-$\sigma$ error band (light) are the KLOE-2 results, as per Eq. (6.12) [68].

a $\chi^2$ optimization to find

$$\begin{aligned} \mathrm{Re}(\alpha) &= -0.65 \pm 0.80\,, \\ \mathrm{Im}(\alpha) &= 0.44 \pm 0.74\,, \\ \mathrm{Re}(\beta) &= (-6.3 \pm 14.7) \times 10^{-4}\,, \\ \mathrm{Im}(\beta) &= (2.2 \pm 2.0) \times 10^{-3}\,, \end{aligned} \tag{6.14}$$

and we show the results of this method in the second row of Fig. 6.2. Enlarging the $(X, Y)$ mesh to 1218 points incurs changes within $\pm 1$ of the last significant figure. The comparison with experiment shows that the fitting procedure is the right choice. We draw the same conclusion from the use of the BG amplitude [64], noting that the global fit in that case (with $N = 0.0508$) gives

$$\begin{aligned} \mathrm{Re}(\alpha) &= -0.79 \pm 0.91\,, \\ \mathrm{Im}(\alpha) &= 0.61 \pm 0.93\,, \\ \mathrm{Re}(\beta) &= (-1.4 \pm 2.3) \times 10^{-3}\,, \\ \mathrm{Im}(\beta) &= (2.3 \pm 1.4) \times 10^{-3}\,, \end{aligned} \tag{6.15}$$

so that the results are compatible within errors. Using these solutions, we obtain $A_{LR} = (-7.18 \pm 4.51) \times 10^{-4}$ using Ref. [61] and $A_{LR} = (-7.20 \pm 4.52) \times 10^{-4}$ using Ref. [64]. These compare favorably with $A_{LR} = (-7.29 \pm 4.81) \times 10^{-4}$ that we determine using the complete set of Dalitz plot parameters and the covariance matrix we construct given the information in Ref. [68]. We note that our $A_{LR}$ as evaluated from the Dalitz plot parameters, which are fitted from the binned data, is a little different from the reported value using the unbinned data, i.e., $(-5.0 \pm 4.5\ ^{+5.0}_{-11}) \times 10^{-4}$, reported by KLOE-2 [70]. The discrepancy is not significant, and we suppose its origin could arise from the slight mismatch between the theoretically accessible phase space and the experimentally probed one, or other experimental issues. Although NLO ChPT does not describe the $CP$-conserving Dalitz distribution well [64], we find it can confront the existing $CP$-violating observables successfully.

We have shown that the empirical Dalitz plot distribution can be used to determine $\alpha$ and $\beta$. These, in turn, limit the strength of $C$-odd and $CP$-odd operators that can arise from physics beyond the SM [223, 224, 225, 226, 227, 228]. That $\beta$ is so much smaller than $\alpha$ can be, in part, understood from the differing behavior of the $M_I(z)$, as illustrated in Fig. 6.1, which follows because the L=0, I=2 $\pi - \pi$ phase shift is larger than the L=1, I=1 one for the kinematics of interest [237, 61, 238, 239], making it easier to veto the $I = 2$ operators. Crudely, the ratio of $\beta$ to $\alpha$ we have found is that of the SM electromagnetic interactions that would permit an $I = 2$ amplitude to appear in addition to an $I = 0$ one. The utility of our Dalitz analysis is underscored by our results for the quadrant asymmetry $A_Q$ and sextant asymmetry $A_S$ defined in Fig. 6.3. Using Ref. [61] and Eq. (6.14), e.g., we find $A_Q = (2.85 \pm 3.72) \times 10^{-4}$, and $A_S = (3.87 \pm 4.04) \times 10^{-4}$; the asymmetries by themselves hide the nature of the underlying strong amplitudes. For reference we note the KLOE-2 results using unbinned data [70]: $A_Q = (1.8 \pm 4.5\ ^{+4.8}_{-2.3}) \times 10^{-4}$ and $A_S = (-0.4 \pm 4.5\ ^{+3.1}_{-3.5}) \times 10^{-4}$, with which our results are compatible within errors.

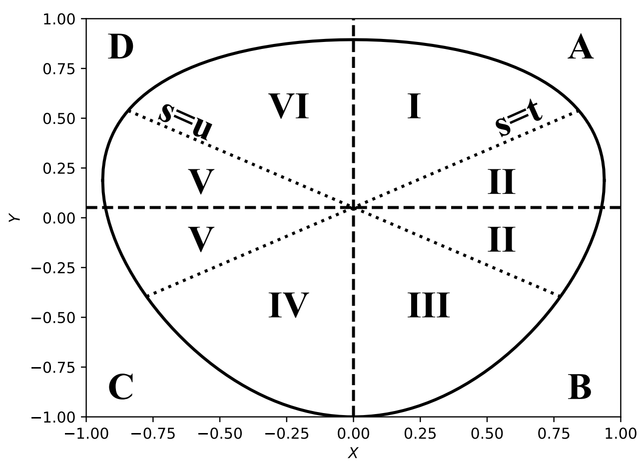


Figure 6.3: The Dalitz plot geometry in $\eta \to \pi^+\pi^-\pi^0$ decay, broken into regions for probes of its symmetries. The solid line is the boundary of the physically accessible region. The asymmetry $A_{LR}$, Eq. (6.1), compares the population $N_+$ ($X > 0$) with $N_-$ ($X < 0$). The quadrant asymmetry $A_Q$ probes $I = 2$ contributions, $N_{tot}A_Q \equiv N(\text{A}) + N(\text{C}) - N(\text{B}) - N(\text{D})$ [52], and the sextant asymmetry $A_S$ probes $I = 0$ contributions, $N_{\text{tot}}A_S \equiv N(\text{I}) + N(\text{I}II) + N(\text{V}) - N(\text{I}I) - N(\text{IV}) - N(\text{V}I)$ [56, 52]. All asymmetries probe $C$ and $CP$ violation.

## 6.4 Summary

We propose an innovative way of probing $C$- and $CP$-violation in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot. Working to leading order in charge conjugation $C$ and isospin $I$ breaking, we have shown that the strong amplitudes associated with the appearance of $C$- and $CP$-violation can be estimated from the SM amplitude for $\eta \to \pi^+\pi^-\pi^0$ if the decomposition of Eq. (6.3) holds [62]. We have illustrated this in NLO ChPT, though the use of more sophisticated theoretical analyses would also be possible. New-physics contributions that differ in their isospin can thus be probed through the kinematic pattern they imprint in the Dalitz plot. Our method opens a new window on the study of $C$- and $CP$-violation in $\eta \to \pi^+\pi^-\pi^0$ decay, and it holds promise for the high-statistics experiments of the future [75, 76, 77].

## 6.5 Appendix

We begin by showing how Eqs. (6.6) and (6.7) emerge from elementary considerations. Working in the isospin limit, a $|\pi^+\pi^-\pi^0\rangle$ state with $J = 0$ must obey Bose symmetry, so that it is proportional to

$$|\pi^+\pi^-\rangle_{\rm s}|\pi^0\rangle + |\pi^+\pi^0\rangle_{\rm s}|\pi^-\rangle + |\pi^-\pi^0\rangle_{\rm s}|\pi^+\rangle\,, \tag{A.1}$$

where "s" denotes a symmetrized combination of distinct pion states. In what follows, as in the $CP$-conserving case, Eq. (6.3) [62], we consider $S$- and $P$-wave $\pi - \pi$ amplitudes only. The symmetrized two-pion states can be written as a $S$-wave $I = 0$ or $I = 2$ state or as a $P$-wave $I = 1$ state. We choose $|\pi^i\rangle \equiv |I = 1, I_3 = i\rangle$. For S-waves, we write

$$|\pi^i\pi^j\rangle_{\rm s} \equiv \frac{1}{\sqrt{2}}\{|\pi^i\pi^j\rangle + |\pi^j\pi^i\rangle\}\,, \tag{A.2}$$

whereas for $P$-waves we note

$$\underbrace{|\pi^i\pi^j\rangle_{I=1,L=1}|\pi^k\rangle}_{L=1} \tag{A.3}$$

with $i + j + k = 0$ contributes to the $J = 0$ state. Defining

$$|\pi^i\pi^j\rangle_{\rm a} \equiv \frac{1}{\sqrt{2}}\{|\pi^i\pi^j\rangle - |\pi^j\pi^i\rangle\}\,, \tag{A.4}$$

we see, e.g.,

$$|(\pi^+\pi^-)_{I=1}\rangle_{\rm s}|\pi^0\rangle = |\pi^+\pi^-\rangle_a(p_{\pi^+} - p_{\pi^-})\cdot p_{\pi^0}|\pi^0\rangle\,. \tag{A.5}$$

We can also label particular $\eta \to \pi^+\pi^-\pi^0$ decay amplitudes by the isospin of the two-pion state, as used in Eq. (6.3). Enumerating the possibilities, we find

$$|(\pi^+\pi^-)_{I=0}\rangle|\pi^0\rangle \to M_0(s)\,, \tag{A.6}$$

which contributes to the total $I=1$ amplitude, $M_1^C$, only, as well as

$$\begin{aligned}
|(\pi^+\pi^-)_{I=1}\rangle|\pi^0\rangle(p_{\pi^+}-p_{\pi^-})\cdot p_{\pi^0} &\to M_1(s)\frac{u-t}{2}\,,\\
|(\pi^+\pi^0)_{I=1}\rangle|\pi^-\rangle(p_{\pi^+}-p_{\pi^0})\cdot p_{\pi^-} &\to M_1(u)\frac{s-t}{2}\,,\\
|(\pi^-\pi^0)_{I=1}\rangle|\pi^+\rangle(p_{\pi^0}-p_{\pi^-})\cdot p_{\pi^+} &\to M_1(t)\frac{u-s}{2}\,,
\end{aligned} \tag{A.7}$$

which contribute to the amplitudes with total $I=0,1$, and 2, and

$$\begin{aligned}
|(\pi^+\pi^-)_{I=2}\rangle|\pi^0\rangle &\to M_2(s)\,,\\
|(\pi^+\pi^0)_{I=2}\rangle|\pi^-\rangle &\to M_2(u)\,,\\
|(\pi^-\pi^0)_{I=2}\rangle|\pi^+\rangle &\to M_2(t)\,,
\end{aligned} \tag{A.8}$$

which contribute to the total $I=1$ and 2 amplitudes. Using the Clebsch-Gordan coefficients tabulated in Ref. [41], we find, after redefining $M_1/2\sqrt{2}\to M_1$ and $M_2\sqrt{3/10}\to M_2$, that $M_1^C(s,t,u)$, $M_0^{\not C}(s,t,u)$, and $M_2^{\not C}(s,t,u)$ are precisely as given in Eqs. (6.3) [62], (6.6), and (6.7). Note that only the $C$-odd amplitudes are odd under $t\leftrightarrow u$ as needed. Adding the possible total $I$ amplitudes in leading order in $C$, $CP$, and $I$ violation, with their associated coefficients, yields Eq. (6.4).

We now turn to our isospin decomposition of the $\eta\to\pi^+\pi^-\pi^0$ amplitude of Gasser and Leutwyler through $\mathcal{O}(p^4)$ [61]:

$$\begin{aligned}
M_0(s) &= \left[\frac{2(s-s_0)}{\Delta}+\frac{5}{3}\right]\frac{1}{2F_\pi^2}(2s-M_\pi^2)J^r_{\pi\pi}(s)\\
&+ \frac{1}{6F_\pi^2\Delta}(4M_K^2-3M_\eta^2-M_\pi^2)(s-2M_\pi^2)J^r_{\pi\pi}(s)\\
&+ \frac{1}{4F_\pi^2\Delta}\Big[-6s^2+s(5M_\pi^2+4M_K^2+3M_\eta^2)-4M_K^2(M_\eta^2+\frac{1}{3}M_\pi^2)\Big]J^r_{KK}(s)\\
&+ \frac{M_\pi^2}{3F_\pi^2\Delta}\left(2s-\frac{11}{3}M_\pi^2+M_\eta^2\right)J^r_{\pi\eta}(s)-\frac{M_\pi^2}{2F_\pi^2}J^r_{\eta\eta}(s)\\
&- \frac{3s}{8F_\pi^2}\frac{(3s-4M_K^2)}{(s-4M_K^2)}\left(J^r_{KK}(s)-J^r_{KK}(0)-\frac{1}{8\pi^2}\right)\\
&+ \Bigg[1+a_1+3a_2\Delta+a_3(9M_\eta^2-M_\pi^2)+\frac{2}{3}d_1\\
&\qquad +\frac{8M_\pi^2}{3\Delta}d_2\Bigg]\left(1+3\frac{s-s_0}{\Delta}\right)+a_4-\frac{8}{3}\frac{M_\pi^2}{\Delta}d_1\\
&- \frac{3}{\Delta}(2\mu_\pi+\mu_K)(s-s_0)+\left(\frac{4L_3}{F_0^2\Delta}-\frac{1}{64\pi^2F_\pi^2\Delta}\right)\left(\frac{4}{3}s^2-9s_0s+9s_0^2\right)\\
&- \frac{1}{64\pi^2F_\pi^2\Delta}3(s-s_0)(4M_\pi^2+2M_K^2)\,,
\end{aligned} \tag{A.9}$$

$$M_1(z) = \frac{1}{4\Delta F_\pi^2}\Big[(z-4M_\pi^2)J^r_{\pi\pi}(z)+\left(\frac{1}{2}z-2M_K^2\right)J^r_{KK}(z)\Big]\,, \tag{A.10}$$

and

$$M_2(z) = \left(1 - \frac{3}{2}\frac{z - s_0}{\Delta}\right)\left[-\frac{1}{2F_\pi^2}(z - 2M_\pi^2)J^r_{\pi\pi}(z) + \frac{1}{4F_\pi^2}(3z - 4M_K^2)J^r_{KK}(z) + \frac{M_\pi^2}{3F_\pi^2}J^r_{\pi\eta}(z)\right] + \left(\frac{1}{64\pi^2 F_\pi^2\Delta} - \frac{4L_3}{F_0^2\Delta}\right)z^2\,, \tag{A.11}$$

where $\Delta = M_\eta^2 - M_\pi^2$, $M_\pi^2 = (2M_{\pi^+}^2 + M_{\pi^0}^2)/3$, and $M_K^2 = (M_{K^+}^2 + M_{\bar{K}^0}^2)/2$. We refer to Ref. [189] for $J^r_{PQ}(z)$, noting Eqs. (8.8-8.10) and (A.11), where $P$ and $Q$ denote the mesons $\pi$, $K$, or $\eta$, and to Ref. [61] for $a_i$ and $d_i$. We note that the $J^r_{PQ}(z)$ carry renormalization-scale $\mu$ dependence, though cancelling that dependence is beyond the scope of our current approach — we note a similar issue arises in the use of the pion form factor in the analysis of $B \to \pi(\rho \to \pi\pi)$ decay [231]. For this choice of $M_\pi$ and the use of physical phase space we need to evaluate the possible two-particle thresholds with care. The rescattering function $J^r_{\pi\pi}(z)$ contains $\sigma(s) = \sqrt{1 - 4m_\pi^2/z}$. If we use $m_\pi^2 = M_\pi^2$, then for $M_I(z)$ with $z = t$ or $u$ evaluated at its minimum value the argument of the square root is less than zero. To avoid this problem, we use $\sigma(z) = \sqrt{1 - (M_{\pi^\pm} + M_{\pi^0})^2/z}$ for $z = t$ or $u$. For $M_I(s)$, though, $s_{\min} = 4M_{\pi^+}^2$ and this problem does not occur. However, for consistency we use $\sigma(s) = \sqrt{1 - 4M_{\pi^+}^2/s}$ for $M_I(s)$. Moreover, we note $J^r_{\pi\eta}(s)$ contains $\nu(s) = \sqrt{(s - (M_\eta - m_\pi)^2)(s - (M_\eta + m_\pi)^2)}$. If we use $m_\pi = M_\pi$, then for $M_I(s)$ at the maximum of $s$, we once again find the argument of the square root to be less than zero. To avoid this, we use $\nu(s) = \sqrt{(s - (M_\eta - M_{\pi^0})^2)(s - (M_\eta + M_{\pi^0})^2)}$ for $M_I(s)$. To be consistent, we use $\nu(z) = \sqrt{(z - (M_\eta - M_{\pi^+})^2)(z - (M_\eta + M_{\pi^+})^2)}$ for $M_I(z)$ with $z = t$ or $u$. As a check of our assessments we have extracted the $C$- and $CP$-conserving Dalitz plot parameters from this amplitude. Describing the $CP$-conserving piece of $|A(s,t,u)|^2$ by $N(1 + aY + bY^2 + dX^2 + fY^3 + gX^2Y)$, recalling Eq. (6.10), we find using a global fit that $a = -1.326$, $b = 0.426$, $d = 0.086$, $f = 0.017$, and $g = -0.072$. These results compare favorably to the global fit results of Ref. [218]; namely, $a = -1.328$, $b = 0.429$, $d = 0.090$, $f = 0.017$, and $g = -0.081$. That work also uses the decay amplitude of Ref. [61] through $\mathcal{O}(p^4)$ and the same value of $L_3$ but includes electromagnetic corrections through $\mathcal{O}(e^2p^2)$ as well.

In evaluating the BG amplitude [64] we note that an overall 2 should not appear on the second right-hand side of Eq.(3.23); this is needed for the result to agree with that of Ref. [61].

Values of the strong functions associated with the $CP$-violating parameters Re($\alpha$), Im($\alpha$), Re($\beta$), Im($\beta$) in Eq. (6.11) on our analysis grids in $(X, Y)$ are available upon request.

CHAPTER 7

# LEADING-DIMENSION, EFFECTIVE OPERATORS WITH CP AND C OR P VIOLATION IN STANDARD MODEL EFFECTIVE FIELD THEORY

In this chapter, we will show our enumeration of the leading-dimension, CP-violating effective operators that share the gauge symmetry and particle content of the Standard Model (SM), carefully separating the operators that are P-odd from those that are C-odd just below the electroweak scale. The P-odd and CP-odd effective operators that generate permanent electric dipole moments have been the subject of much investigation; we now consider C-odd and CP-odd operators systematically as well. We will discuss the operators contributing to flavor-changing and flavor-conserving interactions separately.

This chapter forms the basis for an forthcoming paper with Dr. Susan Gardner, which we would place on the arXiv and submit for publication as a regular journal article.

## 7.1 Introduction

We believe that new CP violating mechanisms exist beyond the Standard Model (SM), while it has not been found yet. Generally new physics are presumed to contain new particles with much higher mass or weakly coupled with well-known fundamental particles. Assuming new physics enter at high energies well in excess of the electroweak scale, new physics can affect physics at energies much lower than the new physics scale through an effective field theory (EFT) [240]. Considering the great success of the SM in describing and predicting experimental observations so far, the Standard Model Effective Field Theory (SMEFT) [164, 101], an effective field theory which satisfies SM gauge symmetry and contains SM fields only, is a pertinent method to study low energy physics with underlying new physics effects. The SMEFT Lagrangian is composed of the usual renormalizable dimension-4 SM Lagrangian supplemented with higher dimensional operators. The dimension of SM operators with different baryon violation $\Delta B$ and lepton violation $\Delta L$ is investigated by Kobach [241] that in the SMEFT when the electroweak symmetry is unbroken, the operators with $(\Delta B - \Delta L)/2$ even (odd) are of even (odd) dimension.

The SMEFT has been used in processes like B physics[242] which is flavor-changing and neutron electric dipole moment (NEDM)[243, 244, 45, 245] which is flavor-conserving and P and CP violating. Starting from SMEFT operators, Jenkins et al.[246, 247] classified operators up to dimension six in the low-energy effective field theory (LEFT) and showed the matching from SMEFT onto these operators. The LEFT works at energies much smaller than the electroweak scale and its gauge group is $SU_C(3) \times U_Q(1)$. Jenkins et al. [246] gave the numbers of CP-even and CP-odd operators up to dimension 6, but did not give explicit forms of CP-odd operators, and they did not separate P-odd and C-odd sources either.

In search of new CP violating mechanisms from SMEFT, P-odd and CP-odd operators have been studied extensively [244, 243, 245, 45], basically due to their connections to the permanent EDM of the neutron. However there is no systematic study of C-odd and CP-odd operators so far.

In this work, we show our studies about the CP-odd operators starting from the compendium of Grzadkowski et al. [101], where an updated complete set of mass-dimension 6 operators from SMEFT is given. Considering one symmetry breaking at a time, which is CP symmetry breaking in our case, we will not include baryon number violation, lepton number violation operators [248], or neutrino-mixing here. In our analysis, we first write down the mass-dimension-6 CP-odd operators in all fundamental fields from the SM at energies above the weak scale, then we integrate out the weak gauge bosons after electroweak symmetry is broken, which is essential to get operators of specific P or C. We then give a complete list of the lowest-mass-dimension C-odd and CP-odd operators, as well as the P-odd and CP-odd operators for flavor-changing and flavor-conserving interactions respectively. We find that the flavor-changing and flavor-conserving CP violating operators are really different regarding either C violating or P violating cases. Our analysis is more general and complete that includes CP-odd operators which is explicitly P-odd or C-odd for both flavor-changing and flavor-conserving interactions. Our list of C-odd and CP-odd operators for flavor-conserving interactions is without precedent, and it gives direct insight into studies of low energy C-odd and CP-odd flavor-conserving processes such as $\eta \to \pi^+\pi^-\pi^0$ [92].

### 7.2 Preamble

The Standard Model is the fundamental theory which explains how the elementary particles interact through three of the four fundamental forces in universe–strong, electromagnetic, and weak interactions. The SM Lagrangian is invariant under gauge group $SU_C(3) \times SU_L(2) \times U_Y(1)$ and it is constructed of quark fields, lepton fields, gauge boson fields and a Higgs field. Quark and lepton fields carry a generation index $p = 1,\ 2,\ 3$, running over the three generations of up-type quarks $u_p = (u,\ c,\ t)$, down-type quarks $d_p = (d,\ s,\ b)$, charged leptons $e_p = (e,\ \mu,\ \tau)$, and neutrinos $\nu_p = (\nu_e,\ \nu_\mu,\ \nu_\tau)$. The left-handed fermions are doublets of $SU_L(2)$,

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \tag{1}$$

where as the right-handed fields $u_R$, $d_R$, and $e_R$ are singlets. The left-handed and right-handed quarks are in the fundamental representation of $SU_C(3)$ and also carry color indices $\alpha = 1,\ 2,\ 3$, which should be contracted and will be left implicit. The Higgs field $\varphi$ denotes an $SU_L(2)$ doublet of scalar fields. The gauge bosons associated with the gauge groups $SU_C(3)$, $SU_L(2)$, and $U_Y(1)$ are $G^A_\mu$, $W^I_\mu$ and $B_\mu$, respectively, with $A = 1,\ 2, ...8$ and $I = 1,\ 2,\ 3$.

We choose the Higgs field as the $SU(2)$ doublet expressed as

$$\varphi = \frac{v}{\sqrt{2}} U(x) \begin{pmatrix} 0 \\ 1 + \frac{h(x)}{v} \end{pmatrix}, \tag{2}$$

where $h(x)$ is real and represents fluctuations around the vacuum where the Higgs field acquires the vacuum expectation value (VEV) $v = \sqrt{\mu^2/\lambda}$, and U(x) is an SU(2) matrix which encodes the Goldstone bosons. Since the Goldstone bosons are not physical degrees of freedom, we choose unitary gauge to set $U(x)$ to one. Although the Higgs mechanism is surely at work, the way how electroweak symmetry is broken remains a mystery. We take the popular assumption that the electroweak symmetry is spontaneously breaking when Higgs field acquires its VEV. As pointed out by Jenkins, Manohar, and Stoffer [247], if the electroweak symmetry breaking (EWSB) is not perfectly Higgs-like, the Wilson coefficients for all the new LEFT operators that are only $SU(3)_C \times U(1)_Q$ at the matching scale (electroweak scale) can be nonzero, and the non-SM EWSB can be incorporated as boundary conditions on the running of their effective field theory [247]. Independent of this, as discussed by Buchalla et al. [166, 178, 167, 170, 168], the non-SM effects in the Higgs sector are only about 10 % in size.

The mass-dimension of fermion fields, gauge boson fields and Higgs field is 3/2, 1 and 1, respectively. The hypercharge of $l^j_{Lp}$, $e_{Rp}$, $q^j_{Lp}$, $u_{Rp}$, $d_{Rp}$, and $\varphi^j$ are $-\frac{1}{2}$, $-1$, $\frac{1}{6}$, $\frac{2}{3}$, $-\frac{1}{3}$ and $\frac{1}{2}$, respectively, where $j = 1,\ 2$ is the isospin indices.

The standard dimension-four Lagrangian $\mathcal{L}^{(4)}_{SM}$ [101] is

$$\begin{aligned} \mathcal{L}^{(4)}_{SM} &= -\frac{1}{4}(G^A_{\mu\nu}G^{A\mu\nu} + W^I_{\mu\nu}W^{I\mu\nu} + B_{\mu\nu}B^{\mu\nu}) \\ &\quad +\bar{q}_{Lp} i\not{D} q_{Lp} + \bar{u}_{Rp} i\not{D} u_{Rr} + \bar{d}_{Rp} i\not{D} d_{Rp} + \bar{l}_{Lp} i\not{D} l_{Lp} + \bar{e}_{Rp} i\not{D} e_{Rp} + D_\mu\varphi^\dagger D^\mu\varphi \\ &\quad +\mu^2\varphi^\dagger\varphi - \frac{\lambda}{2}(\varphi^\dagger\varphi)^2 - \bar{q}_{Lp}Y^u\tilde{\varphi}u_{Rp} - \bar{l}_{Lp}Y^e\varphi e_{Rp} + \text{H.c.} \\ &\quad -\frac{g_s^2\theta}{32\pi^2}G^A_{\mu\nu}\tilde{G}^A_{\mu\nu}, \end{aligned} \tag{3}$$

where $\tilde{\varphi} = i\tau^2\varphi^*$ and $\tilde{X}_{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}X_{\alpha\beta}$. Moreover,

$$G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu + g_s f^{ABC} G^B_\mu G^C_\nu, \tag{4}$$

$$W^I_{\mu\nu} = \partial_\mu W^I_\nu - \partial_\nu W^i_\mu + g\epsilon^{IJK} W^J_\mu W^K_\nu, \tag{5}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{6}$$

$$D_\mu = \partial_\mu - ig_s G^A_\mu T^A - igW^I_\mu S^I - ig'YB_\mu, \tag{7}$$

where $T^A = \frac{1}{2}\lambda^A$ and $S^I = \frac{1}{2}\tau^I$ are $SU(3)$ and $SU(2)$ generators, with $\lambda^A$ and $\tau^I$ the

Gell-Mann color and Pauli isospin matrices, respectively; and

$$c_w \equiv \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} = \frac{e}{g'}, \quad s_w \equiv \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} = \frac{e}{g}, \tag{8}$$

$$W_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-), \tag{9}$$

$$W_\mu^2 = \frac{i}{\sqrt{2}}(W_\mu^+ - W_\mu^-), \tag{10}$$

$$W_\mu^3 = c_w Z_\mu + s_w A_\mu, \tag{11}$$

$$B_\mu = c_w A_\mu - s_w Z_\mu, \tag{12}$$

where $\theta_W$ is the weak coupling angle and $e = -|e|$ is the electron electric charge. Our sign convention of $D_\mu$ and $X_{\mu\nu}$ is consistent with Peskin and Schroeder [81] and Buchmuller and Wyler [164]. Note that Ref. [101, 244, 41] used different conventions [1].

The CP violating mechanisms within the SM come from the non-zero phase of Cabibbo-Kobayashi-Maskawa (CKM) matrix elements and $\theta$ term. The former mainly contributes to specific flavor-changing processes and the latter has the so-called "strong CP problem". We believe new CP violating sources should exist beyond the Standard Model. And we will study higher dimensional operators from SMEFT.

Assuming new physics enters above the energy scale $\Lambda$, according to SMEFT, at energy $E \ll \Lambda$, the SM Lagrangian can be extended as

$$\mathcal{L}_{SM} = \mathcal{L}_{SM}^{(4)} + \sum_i \frac{C_i}{\Lambda^{D-4}}\mathcal{O}^D, \tag{13}$$

where the new operators $\mathcal{O}^D$ have mass dimension $D > 4$ and $C_i$ are the corresponding dimensionless coefficients (Wilson coefficients). It is natural to treat the energy scales for lepton-number violation and baryon-number violation to be much larger than the energy scale that conserve both, so operators violate lepton-number or baryon-number can be much suppressed comparing to lowest mass dimension operators beyond SM that conserve lepton and baryon numbers [246].

The only dimension 5 operator obeying the SM gauge symmetry constraints is [161]

$$\mathcal{O}^5 = \epsilon_{jk}\epsilon_{mn}\varphi^j\varphi^m(l_p^k)^T C(l_r^n) \equiv (\tilde{\varphi}^\dagger l_p)^T C(\tilde{\varphi}^\dagger l_r), \tag{14}$$

where $C$ is the charge conjugation matrix. This operator breaks lepton number by 2. The operator picks out neutrino and Higgs interactions when we choose the unitary gauge in Eq. (2). If the Higgs acquires its VEV which induces the electroweak symmetry to be spontaneously broken, this operator generates neutrino masses and mixing.

[1] In Ref. [101, 41], $D_\mu = \partial_\mu + ig_s G_\mu^A T^A + igW_\mu^I S^I + ig'YB_\mu$, $G_{\mu\nu}^A = \partial_\mu G_\nu^A - \partial_\nu G_\mu^A - g_s f^{ABC} G_\mu^B G_\nu^C$, and $W_{\mu\nu}^I = \partial_\mu W_\nu^I - \partial_\nu W_\mu^i - g\epsilon^{IJK} W_\mu^J W_\nu^K$. In Ref. [244], $D_\mu = \partial_\mu - ig_s G_\mu^A T^A - igW_\mu^I S^I - ig'YB_\mu$, $G_{\mu\nu}^A = \partial_\mu G_\nu^A - \partial_\nu G_\mu^A - g_s f^{ABC} G_\mu^B G_\nu^C$, and $W_{\mu\nu}^I = \partial_\mu W_\nu^I - \partial_\nu W_\mu^i - g\epsilon^{IJK} W_\mu^J W_\nu^K$.

Thus the first operators with mass-dimension bigger than four, conserving baryon numbers ($B$) and lepton numbers ($L$), meanwhile respect the SM gauge symmetries, will come from dimension-six operators. At energy scales beyond the SM, Buchmuller and Wyler [164] and Grzadkowski et al. [101] gave the complete set of dimension-six operators. With the constraint of baryon number conservation, Buchmuller and Wyler [164] gave 80 operators (excluding flavor structure and Hermitian conjugates) in SMEFT, while Grzadkowski et al. [101] found 59 such operators (the number would be 76 when counting Hermitian conjugates). In contrast with Buchmuller and Wyler [164], Grzadkowski et al. [101] added one missing operator [249, 250, 101] and removed some redundant operators considering equations of motion (EOM), total derivatives, Fierz identities and so forth. We will study the discrete transformation (C, P and T) properties of the $B$ conserving and $L$ conserving dimension-six operators from Grzadkowski et al. [101].

The paper is organized as follows. In Section 7.3, we show general analysis of the CP odd operators from mass dimension 6 SMEFT [101] at energy higher than the electroweak scale. In Section 7.4, we list the CP odd operators after electroweak symmetry breaking where the Higgs field obtains its VEV and after rotating $SU(2)_L \times U(1)_Y$ gauge fields to physical ones. When going down to energy just below masses of weak gauge bosons, in Section 7.5, we respectively show the lowest mass-dimension C-odd and CP-odd operators for flavor-changing and flavor-conserving interactions after integrating out weak gauge bosons and give more discussions therein. Since P odd and CP-odd operators are more prevalently studied before, as a completeness of lowest mass-dimension CP odd operators beyond SM, we list our results of the lowest mass-dimension P-odd and CP-odd operators for flavor-changing and flavor-conserving interactions in Appendix 7.8.2 and 7.8.3 respectively and discuss our analysis comparing with previous literatures.

### 7.3 Dimension-6 CP Violating Operators from SMEFT

According to the CPT theorem that any Lorentz covariant local quantum field theories must respect CPT symmetry [5, 6, 7, 8, 251], operators that break T also break CP. So first we will pick out the operators that break T from Ref. [101] on the basis of the transformation properties of SM fields listed in Appendix 7.8.1. Since the operators listed in Ref. [101] have no coefficients involved, we write down the effective Lagrangians with specific low energy coefficients. Following Ref. [101], the subscripts of Lagrangians mean which classes the operators belong, with $X$ stands for $X_{\mu\nu} \equiv \{G^A_{\mu\nu},\ W^I_{\mu\nu},\ B_{\mu\nu}\}$, $\psi$ fermion fields, $\varphi$ the Higgs field, and $D$ the covariant derivative $D_\mu$, respectively. The mass-dimensions of $X_{\mu\nu}$, $\psi$, $\varphi$ and $D$ are 2, 3/2, 1 and 1, respectively.

Operators that are explicitly T-odd from Ref. [101] are

$$\mathcal{L}_{X^3} = C_{G^3} f^{ABC}\epsilon^{\mu\nu\alpha\beta}G^A_{\alpha\beta}G^B_{\mu\rho}G^{C\rho}_{\nu} + C_{W^3}\epsilon^{IJK}\epsilon^{\mu\nu\alpha\beta}W^I_{\alpha\beta}W^J_{\mu\rho}W^{K\rho}_{\nu} \tag{15}$$

$$\begin{aligned}\mathcal{L}_{X^2\varphi^2} &= C_{G^2\varphi^2}\epsilon^{\mu\nu\alpha\beta}G^A_{\alpha\beta}G^A_{\mu\nu}(\varphi^\dagger\varphi) + C_{W^2\varphi^2}\epsilon^{\mu\nu\alpha\beta}W^I_{\alpha\beta}W^I_{\mu\nu}(\varphi^\dagger\varphi)\\ &+ C_{B^2\varphi^2}\epsilon^{\mu\nu\alpha\beta}B_{\alpha\beta}B_{\mu\nu}(\varphi^\dagger\varphi) + C_{WB\varphi^2}\epsilon^{\mu\nu\alpha\beta}W^I_{\alpha\beta}B_{\mu\nu}(\varphi^\dagger\tau^I\varphi),\end{aligned} \tag{16}$$

$$\begin{aligned}\mathcal{L}_{\psi^2\varphi^3} &= (\varphi^\dagger\varphi)\bar{l}_{Lp}i\mathrm{Im}(C^{pr}_{le\varphi^3})\varphi e_{Rr} + (\varphi^\dagger\varphi)(\bar{q}_{Lp}i\mathrm{Im}(C^{pr}_{qu\varphi^3})\tilde{\varphi}u_{Rr}\\ &+ \bar{q}_{Lp}i\mathrm{Im}(C^{pr}_{qd\varphi^3})\varphi d_{Rr}) + H.c.\end{aligned} \tag{17}$$

$$\begin{aligned}\mathcal{L}_{\psi^2X\varphi} &= \bar{l}_{Lp}\sigma^{\mu\nu}i\mathrm{Im}(C^{pr}_{leW\varphi})\tau^I W^I_{\mu\nu}\varphi e_{Rr} + \bar{l}_{Lp}\sigma^{\mu\nu}i\mathrm{Im}(C^{pr}_{leB\varphi})B_{\mu\nu}\varphi e_{Rr}\\ &+ \bar{q}_{Lp}\sigma^{\mu\nu}\left[i\mathrm{Im}(C^{pr}_{quG\varphi})T^AG^A_{\mu\nu} + i\mathrm{Im}(C^{pr}_{quB\varphi})B_{\mu\nu} + i\mathrm{Im}(C^{pr}_{quW\varphi})\tau^IW^I_{\mu\nu}\right]\tilde{\varphi}u_{Rr}\\ &+ \bar{q}_{Lp}\sigma^{\mu\nu}\left[i\mathrm{Im}(C^{pr}_{qdG\varphi})T^AG^A_{\mu\nu} + i\mathrm{Im}(C^{pr}_{qdB\varphi})B_{\mu\nu} + i\mathrm{Im}(C^{pr}_{quW\varphi})\tau^IW^I_{\mu\nu}\right]\varphi d_{Rr}\\ &+ H.c.\end{aligned} \tag{18}$$

$$\begin{aligned}\mathcal{L}_{\psi^2\varphi^2D} &= (\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{l}_{Lp}i\mathrm{Im}(C^{(1)pr}_{l^2\varphi^2D})\gamma^\mu l_{Lr}) + (\varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi)(\bar{l}_{Lp}\tau^Ii\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2D})\gamma^\mu l_{Lr})\\ &+ (\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{e}_{Rp}i\mathrm{Im}(C^{pr}_{e^2\varphi^2D})\gamma^\mu e_{Rr})\\ &+ (\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{q}_{Lp}i\mathrm{Im}(C^{(1)pr}_{q^2\varphi^2D})\gamma^\mu q_{Lr}) + (\varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi)(\bar{q}_{Lp}\tau^Ii\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2D})\gamma^\mu q_{Lr})\\ &+ (\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{u}_{Rp}i\mathrm{Im}(C^{pr}_{u^2\varphi^2D})\gamma^\mu u_{Rr}) + (\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{d}_{Rp}i\mathrm{Im}(C^{pr}_{d^2\varphi^2D})\gamma^\mu d_{Rr})\\ &+ (i\tilde{\varphi}^\dagger D_\mu\varphi)(\bar{u}_{Rp}i\mathrm{Im}(C^{pr}_{ud\varphi^2D})\gamma^\mu d_{Rr}) + H.c.\end{aligned} \tag{19}$$

$$\begin{aligned}\mathcal{L}_{\psi^4} &= i\mathrm{Im}(C^{prst}_{l^4})(\bar{l}_{Lp}\gamma_\mu l_{Lr})(\bar{l}_{Ls}\gamma^\mu l_{Lt}) + i\mathrm{Im}(C^{(1)prst}_{q^4})(\bar{q}_{Lp}\gamma_\mu q_{Lr})(\bar{q}_{Ls}\gamma^\mu q_{Lt})\\ &+ i\mathrm{Im}(C^{(3)prst}_{q^4})(\bar{q}_{Lp}\gamma_\mu\tau^I q_{Lr})(\bar{q}_{Ls}\gamma^\mu\tau^I q_{Lt})\\ &+ i\mathrm{Im}(C^{(1)prst}_{l^2q^2})(\bar{l}_{Lp}\gamma_\mu l_{Lr})(\bar{q}_{Ls}\gamma^\mu q_{Lt}) + i\mathrm{Im}(C^{(3)prst}_{l^2q^2})(\bar{l}_{Lp}\gamma_\mu\tau^I l_{Lr})(\bar{q}_{Ls}\gamma^\mu\tau^I q_{Lt})\\ &+ i\mathrm{Im}(C^{prst}_{e^4})(\bar{e}_{Rp}\gamma_\mu e_{Rr})(\bar{e}_{Rs}\gamma^\mu e_{Rt})\\ &+ i\mathrm{Im}(C^{prst}_{u^4})(\bar{u}_{Rp}\gamma_\mu u_{Rr})(\bar{u}_{Rs}\gamma^\mu u_{Rt}) + i\mathrm{Im}(C^{prst}_{d^4})(\bar{d}_{Rp}\gamma_\mu d_{Rr})(\bar{d}_{Rs}\gamma^\mu d_{Rt})\\ &+ i\mathrm{Im}(C^{prst}_{e^2u^2})(\bar{e}_{Rp}\gamma_\mu e_{Rr})(\bar{u}_{Rs}\gamma^\mu u_{Rt}) + i\mathrm{Im}(C^{prst}_{e^2d^2})(\bar{e}_{Rp}\gamma_\mu e_{Rr})(\bar{d}_{Rs}\gamma^\mu d_{Rt})\\ &+ i\mathrm{Im}(C^{(1)prst}_{u^2d^2})(\bar{u}_{Rp}\gamma_\mu u_{Rr})(\bar{d}_{Rs}\gamma^\mu d_{Rt}) + i\mathrm{Im}(C^{(8)prst}_{u^2d^2})(\bar{u}_{Rp}\gamma_\mu T^A u_{Rr})(\bar{d}_{Rs}\gamma^\mu T^A d_{Rt})\\ &+ i\mathrm{Im}((C^{prst}_{l^2e^2}))(\bar{l}_{Lp}\gamma_\mu l_{Lr})(\bar{e}_{Rs}\gamma^\mu e_{Rt}) + i\mathrm{Im}(C^{prst}_{l^2u^2})(\bar{l}_{Lp}\gamma_\mu l_{Lr})(\bar{u}_{Rs}\gamma^\mu u_{Rt})\\ &+ i\mathrm{Im}(C^{prst}_{l^2d^2})(\bar{l}_{Lp}\gamma_\mu l_{Lr})(\bar{d}_{Rs}\gamma^\mu d_{Rt})\\ &+ i\mathrm{Im}(C^{prst}_{q^2e^2})(\bar{q}_{Lp}\gamma_\mu q_{Lr})(\bar{e}_{Rs}\gamma^\mu e_{Rt}) + i\mathrm{Im}(C^{(1)prst}_{q^2u^2})(\bar{q}_{Lp}\gamma_\mu q_{Lr})(\bar{u}_{Rs}\gamma^\mu u_{Rt})\\ &+ i\mathrm{Im}(C^{(8)prst}_{q^2u^2})(\bar{q}_{Lp}\gamma_\mu T^A q_{Lr})(\bar{u}_{Rs}\gamma^\mu T^A u_{Rt})\\ &+ i\mathrm{Im}(C^{(1)prst}_{q^2d^2})(\bar{q}_{Lp}\gamma_\mu q_{Lr})(\bar{d}_{Rs}\gamma^\mu d_{Rt}) + i\mathrm{Im}(C^{(8)prst}_{q^2d^2})(\bar{q}_{Lp}\gamma_\mu T^A q_{Lr})(\bar{d}_{Rs}\gamma^\mu T^A d_{Rt})\\ &+ i\mathrm{Im}(C^{prst}_{ledq})(\bar{l}^j_{Lp}e_{Rr})(\bar{d}_{Rs}q^j_{Lt}) + i\mathrm{Im}(C^{(1)prst}_{lequ})(\bar{l}^j_{Lp}e_{Rr})\epsilon_{jk}(\bar{q}^k_{Ls}u_{Rt})\\ &+ i\mathrm{Im}(C^{(3)prst}_{lequ})(\bar{l}^j_{Lp}\sigma_{\mu\nu}e_{Rr})\epsilon_{jk}(\bar{q}^k_{Ls}\sigma^{\mu\nu}u_{Rt})\\ &+ i\mathrm{Im}(C^{(1)prst}_{quqd})(\bar{q}^j_{Lp}u_{Rr})\epsilon_{jk}(\bar{q}^k_{Ls}d_{Rt}) + i\mathrm{Im}(C^{(8)prst}_{quqd})(\bar{q}^j_{Lp}T^A u_{Rr})\epsilon_{jk}(\bar{q}^k_{Ls}T^A d_{Rt})\\ &+ H.c.\end{aligned} \tag{20}$$

Definitions of $\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi$ and $\varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi$ can be found in Ref. [101] and in the footnote [2]. The factor $1/\Lambda^2$ are left implicit in the coefficient of each operator. The subscripts $p$, $r$, $s$, and $t$ are generation indices. The operators listed in Ref. [101] contain no coefficients. We add relevant dimensionless coefficients to the operators with

[2] $\varphi^\dagger\overleftarrow{D}_\mu\varphi \equiv (D_\mu\varphi)^\dagger\varphi$, $\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi \equiv i\varphi^\dagger\left(D_\mu - \overleftarrow{D}_\mu\right)\varphi$, and $\varphi^\dagger\overleftrightarrow{D}^I_\mu\varphi \equiv i\varphi^\dagger\left(\tau^ID_\mu - \overleftarrow{D}_\mu\tau^I\right)\varphi$.

subscripts according to their constituents and superscripts as matrix or tensor indices if the coefficient is not a number. For example, the coefficient $C_{qu\varphi^3}^{pr}$ added to operator $(\varphi^\dagger\varphi)(\bar{q}_{Lp}\tilde{\varphi}u_{Rr})$ indicates the operator is composed of $q_L$, $u_R$ and three Higgs fields, the matrix indices $pr$ tells the generation of the first and the second fermion to be $p$ and $r$, respectively. Following the convention of Ref. [101], the superscripts (1), (3) and (8) of some coefficients are used to distinguish operators with the same constituents but different fermion-bilinear structures, where operators with notation (3) contains $\tau^I$ or $\sigma^{\mu\nu}$ and operators with (8) contains $T^A$.

### 7.4 Effective CPV Operators after Electroweak Symmetry Breaking

When the Higgs field acquires its VEV, the electroweak symmetry is spontaneously broken. Using Eq. (9) – (12) we can rotate $W_\mu^i$ and $B_\mu$ to the physical fields $W_\mu^\pm$, $Z_\mu$ and $A_\mu$ basis, thus $\mathcal{L}_{X^3}^T$, $\mathcal{L}_{X^2\varphi^2}^T$, $\mathcal{L}_{\psi^2\varphi^3}^T$, $\mathcal{L}_{\psi^2X\varphi}^T$, $\mathcal{L}_{\psi^2\varphi^2D}^T$ and $\mathcal{L}_{\psi^4}$ turn to

$$\begin{aligned}
\mathcal{L}^T_{X^3} &= C_{G^3} f^{ABC} \epsilon^{\mu\nu\alpha\beta} G^A_{\alpha\beta} G^B_{\mu\rho} G^{C\rho}_{\nu} \\
&+ 2iC_{W^3}\epsilon^{\mu\nu\alpha\beta}\Big\{[\partial_\alpha W^-_\beta - \partial_\beta W^-_\alpha + ig(c_w Z_\alpha + s_w A_\alpha)W^-_\beta - igW^-_\alpha(c_w Z_\beta + s_w A_\beta)] \\
&\quad [\partial_\mu W^+_\rho - \partial_\rho W^+_\mu + igW^+_\mu(c_w Z_\rho + s_w A_\rho) - ig(c_w Z_\mu + s_w A_\mu)W^+_\rho] \\
&\quad [c_w Z^\rho_\nu + s_w F^\rho_\nu + ig(W^-_\nu W^{+\rho} - W^+_\nu W^{-\rho})] \\
&- [\partial_\alpha W^+_\beta - \partial_\beta W^+_\alpha + igW^+_\alpha(c_w Z_\beta + s_w A_\beta) - ig(c_w Z_\alpha + s_w A_\alpha)W^+_\beta] \\
&\quad [\partial_\mu W^-_\rho - \partial_\rho W^-_\mu + ig(c_w Z_\mu + s_w A_\mu)W^-_\rho - igW^-_\mu(c_w Z_\rho + s_w A_\rho)] \\
&\quad [c_w Z^\rho_\nu + s_w F^\rho_\nu + ig(W^-_\nu W^{+\rho} - W^+_\nu W^{-\rho})] \\
&+ [c_w Z_{\alpha\beta} + s_w F_{\alpha\beta} + ig(W^-_\alpha W^+_\beta - W^+_\alpha W^-_\beta)] \\
&\quad [\partial_\mu W^-_\rho - \partial_\rho W^-_\mu + ig(c_w Z_\mu + s_w A_\mu)W^-_\rho - igW^-_\mu(c_w Z_\rho + s_w A_\rho)] \\
&\quad [\partial_\nu W^{+\rho} - \partial^\rho W^+_\nu + igW^+_\nu(c_w Z^\rho + s_w A^\rho) - ig(c_w Z_\nu + s_w A_\nu)W^{+\rho}]\Big\} \qquad (21)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^T_{X^2\varphi^2} &= \frac{v^2}{2}\epsilon^{\mu\nu\alpha\beta}\Big\{C_{G^2\varphi^2} G^A_{\mu\nu} G^A_{\alpha\beta} + 8C_{W^2\varphi^2}[\partial_\alpha W^+_\beta + igW^+_\alpha(c_w Z_\beta + s_w A_\beta)]\times \\
&\quad [\partial_\mu W^-_\nu - igW^-_\mu(c_w Z_\nu + s_w A_\nu)] \\
&+ (C_{W^2\varphi^2} {c_w}^2 + C_{B^2\varphi^2} {s_w}^2 + C_{WB\varphi^2} c_w s_w) Z_{\alpha\beta} Z_{\mu\nu} \\
&+ (C_{W^2\varphi^2} {s_w}^2 + C_{B^2\varphi^2} {c_w}^2 - C_{WB\varphi^2} c_w s_w) F_{\alpha\beta} F_{\mu\nu} \\
&+ [2c_w s_w(C_{W^2\varphi^2} - C_{B^2\varphi^2}) - C_{WB\varphi^2}({c_w}^2 - {s_w}^2)] Z_{\alpha\beta} F_{\mu\nu} \\
&+ (4iC_{W^2\varphi^2} g\ c_w + 2iC_{WB\varphi^2} g s_w) W^-_\alpha W^+_\beta Z_{\mu\nu} \\
&+ (4iC_{W^2\varphi^2} g\ s_w - 2iC_{WB\varphi^2} g c_w) W^-_\alpha W^+_\beta F_{\mu\nu} - 4C_{W^2\varphi^2} g^2 W^-_\alpha W^+_\beta W^-_\mu W^+_\nu\Big\} \qquad (22)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^T_{\psi^2\varphi^3} &= \frac{v^3}{2\sqrt{2}}\Big[i\mathrm{Im}(C^{pr}_{le\varphi^3})(\bar{e}_{Lp} e_{Rr} - \bar{e}_{Rr} e_{Lp}) + i\mathrm{Im}(C^{pr}_{qu\varphi^3})(\bar{u}_{Lp} u_{Rr} - \bar{u}_{Rr} u_{Lp}) \\
&+ i\mathrm{Im}(C^{pr}_{qd\varphi^3})(\bar{d}_{Lp} d_{Rr} - \bar{d}_{Rr} d_{Lp})\Big] \qquad (23)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^T_{\psi^2 X\varphi} &= 2vi\mathrm{Im}(C^{pr}_{leW\varphi})\left\{\bar{\nu}_{Lp}\sigma^{\mu\nu} e_{Rr}[\partial_\mu W^+_\nu + igW^+_\mu(c_w Z_\nu + s_w A_\nu)]\right. \\
&- \left.\bar{e}_{Rr}\sigma^{\mu\nu}\nu_{Lp}[\partial_\mu W^-_\nu - igW^-_\mu(c_w Z_\nu + s_w A_\nu)]\right\} \\
&+ \frac{1}{\sqrt{2}} vi(\bar{e}_{Lp}\sigma^{\mu\nu} e_{Rr} - \bar{e}_{Rr}\sigma^{\mu\nu} e_{Lp})\left[C^{pr}_{leZ\varphi} Z_{\mu\nu} + C^{pr}_{leF\varphi} F_{\mu\nu} - 2ig\mathrm{Im}(C^{pr}_{leW\varphi}) W^-_\mu W^+_\nu\right] \\
&+ \frac{1}{\sqrt{2}} v\Big[i\mathrm{Im}(C^{pr}_{quG\varphi})(\bar{u}_{Lp}\sigma^{\mu\nu} T^A G^A_{\mu\nu} u_{Rr} - \bar{u}_{Rr}\sigma^{\mu\nu} T^A G^A_{\mu\nu} u_{Lp}) \\
&\quad + i\mathrm{Im}(C^{pr}_{qdG\varphi})(\bar{d}_{Lp}\sigma^{\mu\nu} T^A G^A_{\mu\nu} d_{Rr} - \bar{d}_{Rr}\sigma^{\mu\nu} T^A G^A_{\mu\nu} d_{Lp})\Big] \\
&+ \frac{1}{\sqrt{2}} vi(\bar{u}_{Lp}\sigma^{\mu\nu} u_{Rr} - \bar{u}_{Rr}\sigma^{\mu\nu} u_{Lp})[C^{pr}_{quZ\varphi} Z_{\mu\nu} + C^{pr}_{quF\varphi} F_{\mu\nu} + 2\mathrm{Im}(C^{pr}_{quW\varphi}) igW^-_\mu W^+_\nu] \\
&+ \frac{1}{\sqrt{2}} vi(\bar{d}_{Lp}\sigma^{\mu\nu} d_{Rr} - \bar{d}_{Rr}\sigma^{\mu\nu} d_{Lp})[C^{pr}_{qdZ\varphi} Z_{\mu\nu} + C^{pr}_{qdF\varphi} F_{\mu\nu} - 2\mathrm{Im}(C^{pr}_{qdW\varphi}) igW^-_\mu W^+_\nu] \\
&+ 2vi\mathrm{Im}(C^{pr}_{quW\varphi})\left\{\bar{d}_{Lp}\sigma^{\mu\nu} u_{Rr}[\partial_\mu W^-_\nu - igW^-_\mu(c_w Z_\nu + s_w A_\nu)]\right. \\
&- \left.\bar{u}_{Rr}\sigma^{\mu\nu} d_{Lp}[\partial_\mu W^+_\nu + igW^+_\mu(c_w Z_\nu + s_w A_\nu)]\right\} \\
&+ 2vi\mathrm{Im}(C^{pr}_{qdW\varphi})\left\{\bar{u}_{Lp}\sigma^{\mu\nu} d_{Rr}[\partial_\mu W^+_\nu + igW^+_\mu(c_w Z_\nu + s_w A_\nu)]\right. \\
&- \left.\bar{d}_{Rr}\sigma^{\mu\nu} u_{Lp}[\partial_\mu W^-_\nu - igW^-_\mu(c_w Z_\nu + s_w A_\nu)]\right\} \qquad (24)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}^T_{\psi^2\varphi^2 D} &= -\frac{v^2}{2} i\mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) g_Z Z_\mu (\bar{\nu}_{Lp}\gamma^\mu \nu_{Lr} - \bar{\nu}_{Lr}\gamma^\mu \nu_{Lp} + \bar{e}_{Lp}\gamma^\mu e_{Lr} - \bar{e}_{Lr}\gamma^\mu e_{Lp}) \\
&+ \frac{v^2}{2} i\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D}) \{ -g_Z Z_\mu (\bar{\nu}_{Lp}\gamma^\mu \nu_{Lr} - \bar{\nu}_{Lr}\gamma^\mu \nu_{Lp} - \bar{e}_{Lp}\gamma^\mu e_{Lr} + \bar{e}_{Lr}\gamma^\mu e_{Lp}) \\
&\quad + \sqrt{2} g [(\overline{e}_{Lp}\gamma^\mu \nu_{Lr} - \overline{e}_{Lr}\gamma^\mu \nu_{Lp}) W^-_\mu + (\overline{\nu}_{Lp}\gamma^\mu e_{Lr} - \overline{\nu}_{Lr}\gamma^\mu e_{Lp}) W^+_\mu ] \Big\} \\
&- \frac{v^2}{2} i\mathrm{Im}(C^{pr}_{e^2\varphi^2 D}) g_Z Z_\mu (\bar{e}_{Rp}\gamma^\mu e_{Rr} - \bar{e}_{Rr}\gamma^\mu e_{Rp}) \\
&- \frac{v^2}{2} i\mathrm{Im}(C^{(1)pr}_{q^2\varphi^2 D}) g_Z Z_\mu (\bar{u}_{Lp}\gamma^\mu u_{Lr} - \bar{u}_{Lr}\gamma^\mu u_{Lp} + \bar{d}_{Lp}\gamma^\mu d_{Lr} - \bar{d}_{Lr}\gamma^\mu d_{Lp}) \\
&+ \frac{v^2}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) \{ -g_Z Z_\mu (\bar{u}_{Lp}\gamma^\mu u_{Lr} - \bar{u}_{Lr}\gamma^\mu u_{Lp} - \bar{d}_{Lp}\gamma^\mu d_{Lr} + \bar{d}_{Lr}\gamma^\mu d_{Lp}) \\
&\quad + \sqrt{2} g [(\overline{d}_{Lp}\gamma^\mu u_{Lr} - \overline{d}_{Lr}\gamma^\mu u_{Lp}) W^-_\mu + (\overline{u}_{Lp}\gamma^\mu d_{Lr} - \overline{u}_{Lr}\gamma^\mu d_{Lp}) W^+_\mu ] \Big\} \\
&- \frac{v^2}{2} i\mathrm{Im}(C^{pr}_{u^2\varphi^2 D}) g_Z Z_\mu (\bar{u}_{Rp}\gamma^\mu u_{Rr} - \bar{u}_{Rr}\gamma^\mu u_{Rp}) \\
&- \frac{v^2}{2} i\mathrm{Im}(C^{pr}_{d^2\varphi^2 D}) g_Z Z_\mu (\bar{d}_{Rp}\gamma^\mu d_{Rr} - \bar{d}_{Rr}\gamma^\mu d_{Rp}) \\
&+ \frac{v^2}{2\sqrt{2}} i\mathrm{Im}(C^{pr}_{ud\varphi^2 D}) g \{ W^+_\mu [\bar{u}_{Rp}\gamma^\mu d_{Rr}] - W^-_\mu [\bar{d}_{Rr}\gamma^\mu u_{Rp}] \}
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
\mathcal{L}_{\psi^4} &= i\mathrm{Im}(C_{l^4}^{prst})\,[(\bar{\nu}_{Lp}\gamma_\mu\nu_{Lr})(\bar{\nu}_{Ls}\gamma^\mu\nu_{Lt}) + (\bar{e}_{Lp}\gamma_\mu e_{Lr})(\bar{e}_{Ls}\gamma^\mu e_{Lt})\\
&+ (\bar{\nu}_{Lp}\gamma_\mu\nu_{Lr})(\bar{e}_{Ls}\gamma^\mu e_{Lt}) + (\bar{e}_{Lp}\gamma_\mu e_{Lr})(\bar{\nu}_{Ls}\gamma^\mu\nu_{Lt})]\\
&+ i\mathrm{Im}(C_{q^4}^{(1)prst} + C_{q^4}^{(3)prst})[(\bar{u}_{Lp}\gamma_\mu u_{Lr})(\bar{u}_{Ls}\gamma^\mu u_{Lt}) + (\bar{d}_{Lp}\gamma_\mu d_{Lr})(\bar{d}_{Ls}\gamma^\mu d_{Lt})]\\
&+ i\mathrm{Im}(C_{q^4}^{(1)prst} - C_{q^4}^{(3)prst})[(\bar{u}_{Lp}\gamma_\mu u_{Lr})(\bar{d}_{Ls}\gamma^\mu d_{Lt}) + (\bar{d}_{Lp}\gamma_\mu d_{Lr})(\bar{u}_{Ls}\gamma^\mu u_{Lt})]\\
&+ 2i\mathrm{Im}(C_{q^4}^{(3)prst})[(\bar{u}_{Lp}\gamma_\mu d_{Lr})(\bar{d}_{Ls}\gamma^\mu u_{Lt}) + (\bar{d}_{Lp}\gamma_\mu u_{Lr})(\bar{u}_{Ls}\gamma^\mu d_{Lt})]\\
&+ i\mathrm{Im}(C_{l^2q^2}^{(1)prst} + C_{l^2q^2}^{(3)prst})[(\bar{\nu}_{Lp}\gamma_\mu\nu_{Lr})(\bar{u}_{Ls}\gamma^\mu u_{Lt}) + (\bar{e}_{Lp}\gamma_\mu e_{Lr})(\bar{d}_{Ls}\gamma^\mu d_{Lt})]\\
&+ i\mathrm{Im}(C_{l^2q^2}^{(1)prst} - C_{l^2q^2}^{(3)prst})[(\bar{\nu}_{Lp}\gamma_\mu\nu_{Lr})(\bar{d}_{Ls}\gamma^\mu d_{Lt}) + (\bar{e}_{Lp}\gamma_\mu e_{Lr})(\bar{u}_{Ls}\gamma^\mu u_{Lt})]\\
&+ 2i\mathrm{Im}(C_{l^2q^2}^{(3)prst})[(\bar{\nu}_{Lp}\gamma_\mu e_{Lr})(\bar{d}_{Ls}\gamma^\mu u_{Lt}) + (\bar{e}_{Lp}\gamma_\mu \nu_{Lr})(\bar{u}_{Ls}\gamma^\mu d_{Lt})]\\
&+ i\mathrm{Im}(C_{e^4}^{prst})(\bar{e}_{Rp}\gamma_\mu e_{Rr})(\bar{e}_{Rs}\gamma^\mu e_{Rt})\\
&+ i\mathrm{Im}(C_{u^4}^{prst})(\bar{u}_{Rp}\gamma_\mu u_{Rr})(\bar{u}_{Rs}\gamma^\mu u_{Rt}) + i\mathrm{Im}(C_{d^4}^{prst})(\bar{d}_{Rp}\gamma_\mu d_{Rr})(\bar{d}_{Rs}\gamma^\mu d_{Rt})\\
&+ i\mathrm{Im}(C_{e^2u^2}^{prst})(\bar{e}_{Rp}\gamma_\mu e_{Rr})(\bar{u}_{Rs}\gamma^\mu u_{Rt}) + i\mathrm{Im}(C_{e^2d^2}^{prst})(\bar{e}_{Rp}\gamma_\mu e_{Rr})(\bar{d}_{Rs}\gamma^\mu d_{Rt})\\
&+ i\mathrm{Im}(C_{u^2d^2}^{(1)prst})(\bar{u}_{Rp}\gamma_\mu u_{Rr})(\bar{d}_{Rs}\gamma^\mu d_{Rt}) + i\mathrm{Im}(C_{u^2d^2}^{(8)prst})(\bar{u}_{Rp}\gamma_\mu T^A u_{Rr})(\bar{d}_{Rs}\gamma^\mu T^A d_{Rt})\\
&+ i\mathrm{Im}(C_{l^2e^2}^{prst})[(\bar{\nu}_{Lp}\gamma_\mu\nu_{Lr})(\bar{e}_{Rs}\gamma^\mu e_{Rt}) + (\bar{e}_{Lp}\gamma_\mu e_{Lr})(\bar{e}_{Rs}\gamma^\mu e_{Rt})]\\
&+ i\mathrm{Im}(C_{l^2u^2}^{prst})[(\bar{\nu}_{Lp}\gamma_\mu\nu_{Lr})(\bar{u}_{Rs}\gamma^\mu u_{Rt}) + (\bar{e}_{Lp}\gamma_\mu e_{Lr})(\bar{u}_{Rs}\gamma^\mu u_{Rt})]\\
&+ i\mathrm{Im}(C_{l^2d^2}^{prst})[(\bar{\nu}_{Lp}\gamma_\mu\nu_{Lr})(\bar{d}_{Rs}\gamma^\mu d_{Rt}) + (\bar{e}_{Lp}\gamma_\mu e_{Lr})(\bar{d}_{Rs}\gamma^\mu d_{Rt})]\\
&+ i\mathrm{Im}(C_{q^2e^2}^{prst})[(\bar{u}_{Lp}\gamma_\mu u_{Lr})(\bar{e}_{Rs}\gamma^\mu e_{Rt}) + (\bar{d}_{Lp}\gamma_\mu d_{Lr})(\bar{e}_{Rs}\gamma^\mu e_{Rt})]\\
&+ i\mathrm{Im}(C_{q^2u^2}^{(1)prst})[(\bar{u}_{Lp}\gamma_\mu u_{Lr})(\bar{u}_{Rs}\gamma^\mu u_{Rt}) + (\bar{d}_{Lp}\gamma_\mu d_{Lr})(\bar{u}_{Rs}\gamma^\mu u_{Rt})]\\
&+ i\mathrm{Im}(C_{q^2u^2}^{(8)prst})[(\bar{u}_{Lp}\gamma_\mu T^A u_{Lr})(\bar{u}_{Rs}\gamma^\mu T^A u_{Rt}) + (\bar{d}_{Lp}\gamma_\mu T^A d_{Lr})(\bar{u}_{Rs}\gamma^\mu T^A u_{Rt})]\\
&+ i\mathrm{Im}(C_{q^2d^2}^{(1)prst})[(\bar{u}_{Lp}\gamma_\mu u_{Lr})(\bar{d}_{Rs}\gamma^\mu d_{Rt}) + (\bar{d}_{Lp}\gamma_\mu d_{Lr})(\bar{d}_{Rs}\gamma^\mu d_{Rt})]\\
&+ i\mathrm{Im}(C_{q^2d^2}^{(8)prst})[(\bar{u}_{Lp}\gamma_\mu T^A u_{Lr})(\bar{d}_{Rs}\gamma^\mu T^A d_{Rt}) + (\bar{d}_{Lp}\gamma_\mu T^A d_{Lr})(\bar{d}_{Rs}\gamma^\mu T^A d_{Rt})]\\
&+ i\mathrm{Im}(C_{ledq}^{prst})[(\bar{\nu}_{Lp} e_{Rr})(\bar{d}_{Rs} u_{Lt}) + (\bar{e}_{Lp} e_{Rr})(\bar{d}_{Rs} d_{Lt})]\\
&+ i\mathrm{Im}(C_{lequ}^{(1)prst})[(\bar{\nu}_{Lp} e_{Rr})(\bar{d}_{Ls} u_{Rt}) - (\bar{e}_{Lp} e_{Rr})(\bar{u}_{Ls} u_{Rt})]\\
&+ i\mathrm{Im}(C_{lequ}^{(3)prst})[(\bar{\nu}_{Lp}\sigma^{\mu\nu} e_{Rr})(\bar{d}_{Ls}\sigma^{\mu\nu} u_{Rt}) - (\bar{e}_{Lp}\sigma^{\mu\nu} e_{Rr})(\bar{u}_{Ls}\sigma^{\mu\nu} u_{Rt})]\\
&+ i\mathrm{Im}(C_{quqd}^{(1)prst})[(\bar{u}_{Lp} u_{Rr})(\bar{d}_{Ls} d_{Rt}) - (\bar{d}_{Lp} u_{Rr})(\bar{u}_{Ls} d_{Rt})]\\
&+ i\mathrm{Im}(C_{quqd}^{(8)prst})[(\bar{u}_{Lp} T^A u_{Rr})(\bar{d}_{Ls} T^A d_{Rt}) - (\bar{d}_{Lp} T^A u_{Rr})(\bar{u}_{Ls} T^A d_{Rt})]\\
&+ H.c. \qquad (26)
\end{aligned}
$$

where we used $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ in $\mathcal{L}^T_{\psi^2 X\varphi}$, $g_Z = g/c_w = \sqrt{g^2+g'^2}$, $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$, $F_{\mu\nu} = \partial_\mu F_\nu - \partial_\nu F_\mu$, and

$$
\begin{aligned}
C_{leZ\varphi}^{pr} &= -[c_w\ \mathrm{Im}(C_{leW\varphi}^{pr}) + s_w\ \mathrm{Im}(C_{leB\varphi}^{pr})],\\
C_{leF\varphi}^{pr} &= [-s_w\ \mathrm{Im}(C_{leW\varphi}^{pr}) + c_w\ \mathrm{Im}(C_{leB\varphi}^{pr})],\\
C_{quZ\varphi}^{pr} &= [c_w \mathrm{Im}(C_{quW\varphi}^{pr}) - s_w \mathrm{Im}(C_{quB\varphi}^{pr})],\\
C_{quF\varphi}^{pr} &= [s_w \mathrm{Im}(C_{quW\varphi}^{pr}) + c_w \mathrm{Im}(C_{quB\varphi}^{pr})],\\
C_{qdZ\varphi}^{pr} &= -[c_w\ \mathrm{Im}(C_{qdW\varphi}^{pr}) + s_w\ \mathrm{Im}((C_{qdB\varphi}^{pr})\Gamma_B^d)],\\
C_{qdF\varphi}^{pr} &= [-s_w\ \mathrm{Im}(C_{qdW\varphi}^{pr}) + c_w\ \mathrm{Im}(C_{qdB\varphi}^{pr})]. \qquad (27)
\end{aligned}
$$

### 7.5 Lowest Mass-dimensional C-odd and CP-odd Operators after Integrating Out Weak Gauge Bosons

From the mass-dimension-4 SM Lagrangian in Eq. (3), the charged and neutral current weak interactions are [81]

$$\mathcal{H} = \frac{g}{\sqrt{2}}(J^-_\mu W^{+\mu} + J^+_\mu W^{-\mu}) + g_Z J^0_\mu Z^\mu \tag{28}$$

where $g_Z = g/\cos\theta_W$ and

$$\begin{aligned}
J^-_\mu &= \bar{\nu}_{Lp}\gamma_\mu e_{Lp} + \bar{u}_{Lp}\gamma_\mu V_{px} d_{Lx} &(29)\\
J^+_\mu &= \bar{e}_{Lp}\gamma_\mu \nu_{Lp} + \bar{d}_{Lx}\gamma_\mu V^*_{px} u_{Lp} &(30)\\
J^0_\mu &= \bar{\nu}_{Lp}\gamma_\mu(\frac{1}{2})\nu_{Lp} + \bar{e}_{Lp}\gamma_\mu(-\frac{1}{2} + s^2_w)e_{Lp} + \bar{e}_{RL}\gamma_\mu(s^2_w)e_{RL}\\
&\quad +\bar{u}_{Lp}\gamma_\mu(\frac{1}{2} - \frac{2}{3}s^2_w)u_{Lp} + \bar{d}_{Lp}\gamma_\mu(-\frac{1}{2} + \frac{1}{3}s^2_w)d_{Lp}\\
&\quad +\bar{u}_{Rp}\gamma_\mu(-\frac{2}{3}s^2_w)u_{Rp} + \bar{d}_{Rp}\gamma_\mu(\frac{1}{3}s^2_w)d_{Rp} &(31)
\end{aligned}$$

where $V_{px}$ are the Cabibbo-Kobayashi-Maskawa(CKM) matrix elements mixing down-type quarks, with $V_{px}d_{Lx} = V_{pd}d_L + V_{ps}s_L + V_{pb}b_L$ [246]. The neutrinos of different generations can also mix through the Pontecorvo-Maki-Nakagawa-Sakata matrix. Since we focus on quark interactions in this work, we neglect the neutrino transitions here.

The weak gauge fields couple both to vector and axial-vector current, so operators containing weak gauge fields do not have definite C or P transformation properties. At energy $E < M_W$, we can integrate out the weak gauge bosons and replace them with relevant fermion bilinear as follows

$$\begin{aligned}
W^+_\mu &\to \frac{4G_F}{\sqrt{2}}J^+_\mu, &(32)\\
W^-_\mu &\to \frac{4G_F}{\sqrt{2}}J^-_\mu, &(33)\\
Z_\mu &\to \frac{8G_F}{\sqrt{2}}J^0_\mu, &(34)
\end{aligned}$$

where $G_F$ is the Fermi weak interaction constant and we used $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M^2_W}$.

In our following analysis, we will keep the VEV of the Higgs field in our operators and count its dimension to the operator's total mass-dimension based on the fact that the operators are inherited from SMEFT, and thus they retain the dimension counting theorem of Kobach [241] for SMEFT operators, that operators with $\Delta B - \Delta L = 0$ are of even dimension, where $B$ and $L$ are baryon and lepton numbers, respectively.

Jenkins et al. [246] gave effective operators up to dimension six in the LEFT which is $SU_L(3) \times U_Q(1)$ invariant and showed the matching from SMEFT onto these operators. Comparing with their $\Delta B = 0$ and $\Delta L = 0$ LEFT operators listed in Table 7 of Ref. [246], our operators without weak gauge bosons in Eq. (21) $\sim$

Eq. (26) are consistent with theirs at energies just below the weak scale, which is the energy scale we study, though with a little difference in convention. Explicitly, in Eq. (24), our operators contains Higgs VEV and we treat them as mass-dimension-6 operators, while Jenkins et al. [246] did not show the VEV explicitly and claimed them as dimension-five operators. After integrating out $W_\mu^\pm$ in the last line of Eq. (25), we have $\sqrt{2}v^2 G_F(\bar{\nu}_{Lp}\gamma_\mu e_{Lr})(\bar{d}_{Rs}\gamma^\mu u_{Lt})+h.c.$ and $\sqrt{2}v^2 G_F(\bar{u}_{Lp}\gamma_\mu d_{Lr})(\bar{d}_{Rs}\gamma^\mu u_{Lt})+h.c.$, where the later would induce another operator with extra color structure [244] $(\bar{u}_{Lp}\gamma_\mu T^A d_{Lr})(\bar{d}_{Rs}\gamma^\mu T^A u_{Lt})$ that is absent at electroweak scale due to its gauge-symmetry-breaking properties. All these three operators without $v^2 G_F$ were included in Table 7 of Ref. [246]. Ref. [246] contains $(\bar{u}_{Lp}\gamma^\mu T^A u_{Lr})(\bar{d}_{Ls}\gamma^\mu T^A d_{Lt})$ which is not shown in our list, but we have $(\bar{u}_{Lp}\gamma_\mu d_{Lr})(\bar{d}_{Ls}\gamma^\mu u_{Lt})$ which is not included in Ref. [246]. However, using $T^A_{\alpha\beta}T^A_{\kappa\lambda} = \frac{1}{2}\delta_{\alpha\lambda}\delta_{\kappa\beta} - \frac{1}{6}\delta_{\alpha\beta}\delta_{\kappa\lambda}$ with $\alpha$, $\beta$, $\kappa$ and $\lambda$ as color indices, together with left-handed Fierz identity $(\bar{\psi}_{1L}\gamma_\mu\psi_{2L})(\bar{\psi}_{3L}\gamma^\mu\psi_{4L}) = (\bar{\psi}_{1L}\gamma_\mu\psi_{4L})(\bar{\psi}_{3L}\gamma^\mu\psi_{2L})$ [252, 101], these two operators can be related as

$$\begin{aligned}(\bar{u}^\alpha_{Lp}\gamma^\mu T^A_{\alpha\beta}u^\beta_{Lr})(\bar{d}^\kappa_{Ls}\gamma^\mu T^A_{\kappa\lambda}d^\lambda_{Lt}) &= \frac{1}{2}(\bar{u}^\alpha_{Lp}\gamma^\mu u^\beta_{Lr})(\bar{d}^\beta_{Ls}\gamma^\mu d^\alpha_{Lt}) - \frac{1}{6}(\bar{u}^\alpha_{Lp}\gamma^\mu u^\alpha_{Lr})(\bar{d}^\beta_{Ls}\gamma^\mu d^\beta_{Lt}) \\ &= \frac{1}{2}(\bar{u}_{Lp}\gamma^\mu d_{Lt})(\bar{d}_{Ls}\gamma^\mu u_{Lr}) - \frac{1}{6}(\bar{u}_{Lp}\gamma^\mu u_{Lr})(\bar{d}_{Ls}\gamma^\mu d_{Lt}),\end{aligned} \tag{35}$$

so these two operators are actually alternating.

When going down to much lower energy scales, Ref. [246] had extra operators that are generated by QCD and QED

$$\begin{aligned}&(\bar{e}_{Lp}e_{Rr})(\bar{e}_{Ls}e_{Rt}),\\ &(\bar{e}_{Lp}e_{Rr})(\bar{d}_{Ls}d_{Rt}),\\ &(\bar{e}_{Lp}\sigma^{\mu\nu}e_{Rr})(\bar{d}_{Ls}\sigma_{\mu\nu}d_{Rt}),\\ &(\bar{u}_{Lp}u_{Rr})(\bar{u}_{Ls}u_{Rt}),\\ &(\bar{u}_{Lp}T^A u_{Rr})(\bar{u}_{Ls}T^A u_{Rt}),\\ &(\bar{d}_{Lp}d_{Rr})(\bar{d}_{Ls}d_{Rt}),\\ &(\bar{d}_{Lp}T^A d_{Rr})(\bar{d}_{Ls}T^A d_{Rt}),\\ &(\bar{e}_{Lp}e_{Rr})(\bar{u}_{Rs}u_{Lt}),\end{aligned} \tag{36}$$

together with their Hermitian conjugates. We do not have these operators because they have non-zero hypercharge and no matching from SMEFT at tree level. These mass-dimension 6 operators can contribute to both C- and CP-odd, and P- and CP-odd flavor-changing interactions and P-odd and CP odd flavor-conserving interactions. Notice that there could be extra LEFT operators contributing to flavor conserving C- and CP-odd interactions but their mass-dimension have to be higher than 6. We will add these missing operators with some annotation in our following discussion about CP odd operators.
We will not consider the LEFT operators that have no matching to SMEFT for now. (IF choose this, later parts need modifications)

After replacing the $W_\mu^\pm$ and $Z_\mu$ fields with relevant fermion bilinears, we can separate each operator with definite P and C transformation properties. We will show C-odd CP-odd, versus P-odd CP odd operators for flavor-changing and flavor-conserving interactions respectively. We find that the C-odd CP-odd and P-odd CP-odd operators in flavor-changing case are analogous, however they are very different in flavor-conserving cases. Since P odd and CP odd operators have been extensively studied by other work, we focus on the lowest-mass dimensional C-odd and CP-odd operators, meanwhile we list the P odd and CP odd operators in Appendices to construct a complete list of CP odd operators from SMEFT.

### 7.5.1 Lowest Mass-dimensional C-odd and CP-odd Operators Contributing to Flavor-Changing Interactions

Some constraints of the generation indices are required to guarantee the operators to be flavor-changing and non-vanishing. For example $p \neq r$ for the operators in classes $\psi^2\varphi^3$ and $\psi^2 X\varphi$. As for the operators in class $\psi^4$, given $Q_{uu} = (\bar{u}_{Rp}\gamma_\mu u_{Rr})(\bar{u}_{Rs}\gamma^\mu u_{Rt})$ as an example, the relations $p \neq r$ and $s \neq t$, or, $p \neq t$ and $r \neq s$ should be applied.

The classes $X^3$ and $X^2\varphi^2$ contain no fermions. The C-odd and CP-odd flavor-changing operators come from classes $\psi^2\varphi^3$, $\psi^2 X\varphi$, $\psi^4$, and $\psi^2\varphi^2 D$.

The C-odd and CP-odd operators with the lowest mass-dimension are

$$
\begin{aligned}
\mathcal{L}^{\not{C}}_{\psi^2\varphi^3} &= \frac{v^3}{4\sqrt{2}}\Big[i\mathrm{Im}(C^{pr}_{le\varphi^3})(\bar{e}_p e_r - \bar{e}_r e_p) + i\mathrm{Im}(C^{pr}_{qu\varphi^3})(\bar{u}_p u_r - \bar{u}_r u_p) \\
&+ i\mathrm{Im}(C^{pr}_{qd\varphi^3})(\bar{d}_p d_r - \bar{d}_r d_p)\Big], \qquad (37) \\
\mathcal{L}^{\not{C}}_{\psi^2 X\varphi} &= \frac{1}{2\sqrt{2}}v[iC^{pr}_{leF\varphi}(\bar{e}_p\sigma^{\mu\nu}e_r - \bar{e}_r\sigma^{\mu\nu}e_p)F_{\mu\nu} \\
&+ \frac{1}{2\sqrt{2}}v\left[i\mathrm{Im}(C^{pr}_{quG\varphi})(\bar{u}_p\sigma^{\mu\nu}T^A u_r - \bar{u}_r\sigma^{\mu\nu}T^A u_p)G^A_{\mu\nu}\right. \\
&+ \left. i\mathrm{Im}(C^{pr}_{qdG\varphi})(\bar{d}_p\sigma^{\mu\nu}T^A d_r - \bar{d}_r\sigma^{\mu\nu}T^A d_p)G^A_{\mu\nu}\right] \\
&+ \frac{1}{2\sqrt{2}}viC^{pr}_{quF\varphi}(\bar{u}_p\sigma^{\mu\nu}u_r - \bar{u}_r\sigma^{\mu\nu}u_p)F_{\mu\nu} \\
&+ \frac{1}{2\sqrt{2}}viC^{pr}_{qdF\varphi}(\bar{d}_p\sigma^{\mu\nu}d_r - \bar{d}_r\sigma^{\mu\nu}d_p)F_{\mu\nu}, \qquad (38)
\end{aligned}
$$

where $C^{pr}_{leF\varphi}$ and $C^{pr}_{quF\varphi}$ are defined in Eq. (27), together with the operators from class $\psi^4$ listed in Table 7.1 and Table 7.2. Their mass-dimension is 6.

For operators in $\mathcal{L}_{\psi^2\varphi^2 D}$, note that the original mass-dimension 6 operators have an implicit factor $1/\Lambda^2$, after integrating out the weak gauge bosons, each operator contains $G_F/\Lambda^2 \sim 1/\Lambda^4$, so the operators become mass-dimension 8. In the SM, $v^2 G_F = \frac{\sqrt{2}}{4} \sim \mathcal{O}(1)$, so the mass-dimension 8 operators in Eq. (39) below can give comparative numerical effects as the mass-dimension 6 operators listed before

Table 7.1: Flavor-changing C-odd and CP-odd operators from class $\psi^4$.

| 1: lepton-only interaction | |
|---|---|
| operator | coefficient |
| $(\bar\nu_p\gamma_\mu\nu_r)(\bar\nu_s\gamma^\mu\nu_t)-(\bar\nu_t\gamma^\mu\nu_s)(\bar\nu_r\gamma_\mu\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^4}^{prst})$ |
| $(\bar\nu_p\gamma_\mu\gamma_5\nu_r)(\bar\nu_s\gamma^\mu\gamma_5\nu_t)-(\bar\nu_t\gamma^\mu\gamma_5\nu_s)(\bar\nu_r\gamma_\mu\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^4}^{prst})$, |
| $(\bar e_p\gamma_\mu e_r)(\bar e_s\gamma^\mu e_t)-(\bar e_t\gamma^\mu e_s)(\bar e_r\gamma_\mu e_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^4}^{prst}+C_{e^4}^{prst}+C_{l^2e^2}^{prst})$ |
| $(\bar e_p\gamma_\mu\gamma_5 e_r)(\bar e_s\gamma^\mu\gamma_5 e_t)-(\bar e_t\gamma^\mu\gamma_5 e_s)(\bar e_r\gamma_\mu\gamma_5 e_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^4}^{prst}+C_{e^4}^{prst}-C_{l^2e^2}^{prst})$ |
| $(\bar\nu_p\gamma_\mu\nu_r)(\bar e_s\gamma^\mu e_t)-(\bar e_t\gamma^\mu e_s)(\bar\nu_r\gamma_\mu\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^4}^{prst}-C_{l^4}^{tsrp}+C_{e^4}^{prst})$ |
| $(\bar\nu_p\gamma_\mu\gamma_5\nu_r)(\bar e_s\gamma^\mu\gamma_5 e_t)-(\bar e_t\gamma^\mu\gamma_5 e_s)(\bar\nu_r\gamma_\mu\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^4}^{prst}-C_{l^4}^{tsrp}-C_{e^4}^{prst})$ |

| 2: lepton-quark interaction | |
|---|---|
| $(\bar\nu_p\gamma_\mu\nu_r)(\bar u_s\gamma^\mu u_t)-(\bar u_t\gamma^\mu u_s)(\bar\nu_r\gamma_\mu\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^2q^2}^{(1)prst}+C_{l^2q^2}^{(3)prst}+C_{l^2u^2}^{prst})$ |
| $(\bar\nu_p\gamma_\mu\gamma_5\nu_r)(\bar u_s\gamma^\mu\gamma_5 u_t)-(\bar u_t\gamma^\mu\gamma_5 u_s)(\bar\nu_r\gamma_\mu\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^2q^2}^{prst}+C_{l^2q^2}^{(3)prst}-C_{l^2u^2}^{prst})$ |
| $(\bar\nu_p\gamma_\mu\nu_r)(\bar d_s\gamma^\mu d_t)-(\bar d_t\gamma^\mu d_s)(\bar\nu_r\gamma_\mu\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^2q^2}^{(1)prst}-C_{l^2q^2}^{(3)prst}+C_{l^2d^2}^{prst})$ |
| $(\bar\nu_p\gamma_\mu\gamma_5\nu_r)(\bar d_s\gamma^\mu\gamma_5 d_t)-(\bar d_t\gamma^\mu\gamma_5 d_s)(\bar\nu_r\gamma_\mu\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^2q^2}^{(1)prst}-C_{l^2q^2}^{(3)prst}-C_{l^2d^2}^{prst})$ |
| $(\bar e_p\gamma_\mu e_r)(\bar u_s\gamma^\mu u_t)-(\bar u_t\gamma^\mu u_s)(\bar e_r\gamma_\mu e_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^2q^2}^{(1)prst}-C_{l^2q^2}^{(3)prst}$ $+C_{e^2u^2}^{prst}+C_{l^2u^2}^{prst}-C_{q^2e^2}^{tsrp})$ |
| $(\bar e_p\gamma_\mu\gamma_5 e_r)(\bar u_s\gamma^\mu\gamma_5 u_t)-(\bar u_t\gamma^\mu\gamma_5 u_s)(\bar e_r\gamma_\mu\gamma_5 e_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^2q^2}^{(1)prst}-C_{l^2q^2}^{(3)prst}$ $+C_{e^2u^2}^{prst}-C_{l^2u^2}^{prst}+C_{q^2e^2}^{tsrp})$ |
| $(\bar e_p\gamma_\mu e_r)(\bar d_s\gamma^\mu d_t)-(\bar d_t\gamma^\mu d_s)(\bar e_r\gamma_\mu e_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^2q^2}^{(1)prst}+C_{l^2q^2}^{(3)prst}$ $+C_{e^2d^2}^{prst}+C_{l^2d^2}^{prst}-C_{q^2e^2}^{tsrp})$ |
| $(\bar e_p\gamma_\mu\gamma_5 e_r)(\bar d_s\gamma^\mu\gamma_5 d_t)-(\bar d_t\gamma^\mu\gamma_5 d_s)(\bar e_r\gamma_\mu\gamma_5 e_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{l^2q^2}^{(1)prst}+C_{l^2q^2}^{(3)prst}$ $+C_{e^2d^2}^{prst}-C_{l^2d^2}^{prst}+C_{q^2e^2}^{tsrp})$ |
| $(\bar\nu_p\gamma_\mu e_r)(\bar d_s\gamma^\mu u_t)-(\bar u_t\gamma^\mu d_s)(\bar e_r\gamma_\mu\nu_p)$ | $\frac{1}{2}i\mathrm{Im}(C_{l^2q^2}^{(3)prst})$ |
| $(\bar\nu_p\gamma_\mu\gamma_5 e_r)(\bar d_s\gamma^\mu\gamma_5 u_t)-(\bar u_t\gamma^\mu\gamma_5 d_s)(\bar e_r\gamma_\mu\gamma_5\nu_p)$ | $\frac{1}{2}i\mathrm{Im}(C_{l^2q^2}^{(3)prst})$ |
| $(\bar e_p\gamma_\mu\nu_r)(\bar u_s\gamma^\mu d_t)-(\bar d_t\gamma^\mu u_s)(\bar\nu_r\gamma_\mu e_p)$ | $\frac{1}{2}i\mathrm{Im}(C_{l^2q^2}^{(3)prst})$ |
| $(\bar e_p\gamma_\mu\gamma_5\nu_r)(\bar u_s\gamma^\mu\gamma_5 d_t)-(\bar d_t\gamma^\mu\gamma_5 u_s)(\bar\nu_r\gamma_\mu\gamma_5 e_p)$ | $\frac{1}{2}i\mathrm{Im}(C_{l^2q^2}^{(3)prst})$ |
| $(\bar\nu_p e_r)(\bar d_s u_t)-(\bar u_t d_s)(\bar e_r\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{ledq}^{prst}+C_{lequ}^{(1)prst})$ |
| $(\bar\nu_p\gamma_5 e_r)(\bar d_s\gamma_5 u_t)-(\bar u_t\gamma_5 d_s)(\bar e_r\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(-C_{ledq}^{prst}+C_{lequ}^{(1)prst})$ |
| $(\bar e_p e_r)(\bar d_s d_t)-(\bar d_t d_s)(\bar e_r e_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{ledq}^{prst})$ |
| $(\bar e_p\gamma_5 e_r)(\bar d_s\gamma_5 d_t)-(\bar d_t\gamma_5 d_s)(\bar e_r\gamma_5 e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{ledq}^{prst})$ |
| $(\bar e_p e_r)(\bar u_s u_t)-(\bar u_t u_s)(\bar e_r e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{lequ}^{prst})$ |
| $(\bar e_p\gamma_5 e_r)(\bar u_s\gamma_5 u_t)-(\bar u_t\gamma_5 u_s)(\bar e_r\gamma_5 e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{lequ}^{(1)prst})$ |
| $(\bar\nu_p\sigma_{\mu\nu}e_r)(\bar d_s\sigma^{\mu\nu}u_t)-(\bar u_t\sigma^{\mu\nu}d_s)(\bar e_r\sigma_{\mu\nu}\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{lequ}^{(3)prst})$ |
| $(\bar\nu_p\sigma_{\mu\nu}\gamma_5 e_r)(\bar d_s\sigma^{\mu\nu}\gamma_5 u_t)-(\bar u_t\sigma^{\mu\nu}\gamma_5 d_s)(\bar e_r\sigma_{\mu\nu}\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{lequ}^{(3)prst})$ |
| $(\bar e_p\sigma_{\mu\nu}e_r)(\bar u_s\sigma_{\mu\nu}u_t)-(\bar u_t\sigma_{\mu\nu}u_s)(\bar e_r\sigma_{\mu\nu}e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{lequ}^{(3)prst})$ |
| $(\bar e_p\sigma_{\mu\nu}\gamma_5 e_r)(\bar u_s\sigma_{\mu\nu}\gamma_5 u_t)-(\bar u_t\sigma_{\mu\nu}\gamma_5 u_s)(\bar e_r\sigma_{\mu\nu}\gamma_5 e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{lequ}^{(3)prst})$ |

Table 7.2: Flavor-changing C-odd and CP-odd operators from class $\psi^4$ (Continued).

| 3: quark-only interaction | |
|---|---|
| $(\bar{u}_p\gamma_\mu u_r)(\bar{u}_s\gamma^\mu u_t) - (\bar{u}_t\gamma^\mu u_s)(\bar{u}_r\gamma_\mu u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{q^4}^{(1)prst} + C_{q^4}^{(3)prst} + C_{u^4}^{prst} + C_{q^2u^2}^{(1)prst})$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 u_r)(\bar{u}_s\gamma^\mu\gamma_5 u_t) - (\bar{u}_t\gamma^\mu\gamma_5 u_s)(\bar{u}_r\gamma_\mu\gamma_5 u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{q^4}^{(1)prst} + C_{q^4}^{(3)prst} + C_{u^4}^{prst} - C_{q^2u^2}^{(1)prst})$ |
| $(\bar{d}_p\gamma_\mu d_r)(\bar{d}_s\gamma^\mu d_t) - (\bar{d}_t\gamma^\mu d_s)(\bar{d}_r\gamma_\mu d_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{q^4}^{(1)prst} + C_{q^4}^{(3)prst} + C_{d^4}^{prst} + C_{q^2d^2}^{(1)prst})$ |
| $(\bar{d}_p\gamma_\mu\gamma_5 d_r)(\bar{d}_s\gamma^\mu\gamma_5 d_t) - (\bar{d}_t\gamma^\mu\gamma_5 d_s)(\bar{d}_r\gamma_\mu\gamma_5 d_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{q^4}^{(1)prst} + C_{q^4}^{(3)prst} + C_{d^4}^{prst} - C_{q^2d^2}^{(1)prst})$ |
| $(\bar{u}_p\gamma_\mu u_r)(\bar{d}_s\gamma^\mu\ d_t) - (\bar{d}_t\gamma^\mu d_s)(\bar{u}_r\gamma_\mu u_p)$ | $\frac{1}{4}i\mathrm{Im}\left(C_{q^4}^{(1)prst} - C_{q^2u^2}^{(3)prst} - C_{q^4}^{(1)tsrp} + C_{q^4}^{(3)tsrp} + C_{u^2d^2}^{(1)prst} - C_{q^2u^2}^{(1)tsrp} + C_{q^2d^2}^{(1)prst}\right)$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 u_r)(\bar{d}_s\gamma^\mu\gamma_5 d_t) - (\bar{d}_t\gamma^\mu\gamma_5 d_s)(\bar{u}_r\gamma_\mu\gamma_5 u_p)$ | $\frac{1}{4}i\mathrm{Im}\ (C_{q^4}^{(1)prst} - C_{q^4}^{(3)prst} - C_{q^4}^{(1)tsrp} + C_{q^4}^{(3)tsrp} + C_{u^2d^2}^{(1)prst} + C_{q^2u^2}^{(1)tsrp} - C_{q^2d^2}^{(1)prst})$ |
| $(\bar{u}_p\gamma_\mu T^A u_r)(\bar{u}_s\gamma^\mu T^A u_t) - (\bar{u}_t\gamma^\mu T^A u_s)(\bar{u}_r\gamma_\mu T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{q^2u^2}^{(8)prst})$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 T^A u_r)(\bar{u}_s\gamma^\mu\gamma_5 T^A u_t) - (\bar{u}_t\gamma^\mu\gamma_5 T^A u_s)(\bar{u}_r\gamma_\mu\gamma_5 T^A u_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{q^2u^2}^{(8)prst})$ |
| $(\bar{d}_p\gamma_\mu T^A d_r)(\bar{d}_s\gamma^\mu T^A d_t) - (\bar{d}_t\gamma^\mu T^A d_s)(\bar{d}_r\gamma_\mu T^A d_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{q^2d^2}^{(8)prst})$ |
| $(\bar{d}_p\gamma_\mu\gamma_5 T^A d_r)(\bar{d}_s\gamma^\mu\gamma_5 T^A d_t) - (\bar{d}_t\gamma^\mu\gamma_5 T^A d_s)(\bar{d}_r\gamma_\mu\gamma_5 T^A d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{q^2d^2}^{(8)prst})$ |
| $(\bar{u}_p\gamma_\mu T^A u_r)(\bar{d}_s\gamma^\mu T^A d_t) - (\bar{d}_t\gamma^\mu T^A d_s)(\bar{u}_r\gamma_\mu T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{u^2d^2}^{(8)prst} - C_{q^2u^2}^{(8)tsrp} + C_{q^2d^2}^{(8)prst})$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 T^A u_r)(\bar{d}_s\gamma^\mu\gamma_5 T^A d_t) - (\bar{d}_t\gamma^\mu\gamma_5 T^A d_s)(\bar{u}_r\gamma_\mu\gamma_5 T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{u^2d^2}^{(8)prst} + C_{q^2u^2}^{(8)tsrp} - C_{q^2d^2}^{(8)prst})$ |
| $(\bar{u}_p\gamma_\mu d_r)(\bar{d}_s\gamma^\mu u_t) - (\bar{u}_t\gamma^\mu d_s)(\bar{d}_r\gamma_\mu u_p)$ | $\frac{1}{2}i\mathrm{Im}(C_{q^4}^{(3)prst})$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 d_r)(\bar{d}_s\gamma^\mu\gamma_5 u_t) - (\bar{u}_t\gamma^\mu\gamma_5 d_s)(\bar{d}_r\gamma_\mu\gamma_5 u_p)$ | $\frac{1}{2}i\mathrm{Im}(C_{q^4}^{(3)prst})$ |
| $(\bar{d}_p\gamma_\mu u_r)(\bar{u}_s\gamma^\mu d_t) - (\bar{d}_t\gamma^\mu u_s)(\bar{u}_r\gamma_\mu d_p)$ | $\frac{1}{2}i\mathrm{Im}(C_{q^4}^{(3)prst})$ |
| $(\bar{d}_p\gamma_\mu\gamma_5 u_r)(\bar{u}_s\gamma^\mu\gamma_5 d_t) - (\bar{d}_t\gamma^\mu\gamma_5 u_s)(\bar{u}_r\gamma_\mu\gamma_5 d_p)$ | $\frac{1}{2}i\mathrm{Im}(C_{q^4}^{(3)prst})$ |
| $(\bar{u}_p u_r)(\bar{d}_s d_t) - (\bar{d}_t d_s)(\bar{u}_r u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(1)prst})$ |
| $(\bar{u}_p\gamma_5 u_r)(\bar{d}_s\gamma_5 d_t) - (\bar{d}_t\gamma_5 d_s)(\bar{u}_r\gamma_5 u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(1)prst})$ |
| $(\bar{d}_p u_r)(\bar{u}_s d_t) - (\bar{d}_t u_s)(\bar{u}_r d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(1)prst})$ |
| $(\bar{d}_p\gamma_5 u_r)(\bar{u}_s\gamma_5 d_t) - (\bar{d}_t\gamma_5 u_s)(\bar{u}_r\gamma_5 d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(1)prst})$ |
| $(\bar{u}_p T^A u_r)(\bar{d}_s T^A d_t) - (\bar{d}_t T^A d_s)(\bar{u}_r T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(8)prst}$ |
| $(\bar{u}_p\gamma_5 T^A u_r)(\bar{d}_s\gamma_5 T^A d_t) - (\bar{d}_t\gamma_5 T^A d_s)(\bar{u}_r\gamma_5 T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(8)prst})$ |
| $(\bar{d}_p T^A u_r)(\bar{u}_s T^A d_t) - (\bar{d}_t T^A u_s)(\bar{u}_r T^A d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(8)prst})$ |
| $(\bar{d}_p\gamma_5 T^A u_r)(\bar{u}_s T^A\gamma_5 d_t) - (\bar{d}_t\gamma_5 T^A u_s)(\bar{u}_r\gamma_5 T^A d_p)$ | $\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(8)prst})$ |

$$
\begin{aligned}
\mathcal{L}^{\not{C}}_{\psi^2\varphi^2 D} \;\; = \;\; & -\sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) + \mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D})] \,\{(\bar{\nu}_p\gamma^\mu\nu_r - \bar{\nu}_r\gamma^\mu\nu_p)\times \\
& g_Z[\frac{1}{4}\bar{\nu}_s\gamma_\mu\nu_s + (-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\mu e_s + (\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\mu u_s + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\mu d_s] \\
& +\frac{1}{4}(\bar{\nu}_p\gamma^\mu\gamma_5\nu_r - \bar{\nu}_r\gamma^\mu\gamma_5\nu_p)g_Z(\bar{\nu}_s\gamma_\mu\gamma_5\nu_s + \bar{e}_s\gamma_\mu\gamma_5 e_s + \bar{u}_s\gamma_\mu\gamma_5 u_s - \bar{d}_s\gamma_\mu\gamma_5 d_s)\Big\} \\
+ \;\; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{pr}_{e^2\varphi^2 D})](\bar{e}_p\gamma^\mu e_r - \bar{e}_r\gamma^\mu e_p)\times \\
& g_Z\left[\frac{1}{4}\bar{\nu}_s\gamma_\mu\nu_s + (-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\mu e_s + (\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\mu u_s + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\mu d_s\right] \\
+ \;\; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) + \mathrm{Im}(C^{pr}_{e^2\varphi^2 D})]\frac{1}{4}(\bar{e}_p\gamma^\mu\gamma_5 e_r - \bar{e}_r\gamma^\mu\gamma_5 e_p)\times \\
& g_Z\left(\bar{\nu}_s\gamma_\mu\gamma_5\nu_s + \bar{e}_s\gamma_\mu\gamma_5 e_s + \bar{u}_s\gamma_\mu\gamma_5 u_s - \bar{d}_s\gamma_\mu\gamma_5 d_s\right) \\
+ \;\; & \frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D})g\,[(\bar{e}_p\gamma^\mu\nu_r - \bar{e}_r\gamma^\mu\nu_p)(\bar{\nu}_s\gamma_\mu e_s + \bar{u}_s\gamma_\mu V_{sx} d_x) \\
& +(\bar{\nu}_p\gamma^\mu e_r - \bar{\nu}_r\gamma^\mu e_p)(\bar{e}_s\gamma_\mu\nu_s + \bar{d}_x\gamma_\mu V^*_{sx} u_s) \\
& +(\bar{e}_p\gamma^\mu\gamma_5\nu_r - \bar{e}_r\gamma^\mu\gamma_5\nu_p)(\bar{\nu}_s\gamma_\nu\gamma_5 e_s + \bar{u}_s\gamma_\nu\gamma_5 V_{sx} d_x) \\
& +(\bar{\nu}_p\gamma^\mu\gamma_5 e_r - \bar{\nu}_r\gamma^\mu\gamma_5 e_p)(\bar{e}_s\gamma_\mu\gamma_5\nu_s + \bar{d}_x\gamma_\mu\gamma_5 V^*_{sx} u_s)] \\
- \;\; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(1)pr}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{pr}_{u^2\varphi^2 D})](\bar{u}_p\gamma^\mu u_r - \bar{u}_r\gamma^\mu u_p)\times \\
& g_Z\left[\frac{1}{4}\bar{\nu}_s\gamma_\mu\nu_s + (-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\mu e_s + (\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\mu u_s + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\mu d_s\right] \\
- \;\; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(1)pr}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{pr}_{u^2\varphi^2 D})](\bar{u}_p\gamma^\mu\gamma_5 u_r - \bar{u}_r\gamma^\mu\gamma_5 u_p)\times \\
& g_Z\frac{1}{4}\left(\bar{\nu}_s\gamma_\mu\gamma_5\nu_s + \bar{e}_s\gamma_\mu\gamma_5 e_s + \bar{u}_s\gamma_\mu\gamma_5 u_s - \bar{d}_s\gamma_\mu\gamma_5 d_s\right) \\
+ \;\; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{pr}_{d^2\varphi^2 D})](\bar{d}_p\gamma^\mu d_r - \bar{d}_r\gamma^\mu d_p)\times \\
& g_Z\left[\frac{1}{4}\bar{\nu}_s\gamma_\mu\nu_s + (-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\mu e_s + (\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\mu u_s + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\mu d_s\right] \\
+ \;\; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{(1)pr}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{pr}_{d^2\varphi^2 D})](\bar{d}_p\gamma^\mu\gamma_5 d_r - \bar{d}_r\gamma^\mu\gamma_5 d_p)\times \\
& g_Z\frac{1}{4}\left(\bar{\nu}_s\gamma_\mu\gamma_5\nu_s + \bar{e}_s\gamma_\mu\gamma_5 e_s + \bar{u}_s\gamma_\mu\gamma_5 u_s - \bar{d}_s\gamma_\mu\gamma_5 d_s\right) \\
+ \;\; & \frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D})g\left[(\bar{d}_p\gamma^\mu u_r - \bar{d}_r\gamma^\mu u_p)(\bar{\nu}_s\gamma_\mu e_s + \bar{u}_s\gamma_\mu V_{sx} d_x)\right. \\
& \left.+(\bar{u}_p\gamma^\mu d_r - \bar{u}_r\gamma^\mu d_p)(\bar{e}_s\gamma_\mu\nu_s + \bar{d}_x\gamma_\mu V^*_{sx} u_s)]\right] \\
+ \;\; & \frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D})g\left[(\bar{d}_p\gamma^\mu\gamma_5 u_r - \bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{\nu}_s\gamma_\mu\gamma_5 e_s + \bar{u}_s\gamma_\mu\gamma_5 V_{sx} d_x)\right. \\
& \left.+(\bar{u}_p\gamma^\mu\gamma_5 d_r - \bar{u}_r\gamma^\mu\gamma_5 d_p)(\bar{e}_s\gamma_\mu\gamma_5\nu_s + \bar{d}_x\gamma_\mu\gamma_5 V^*_{sx} u_s)\right] \\
+ \;\; & \frac{v^2 G_F}{4} i\mathrm{Im}(C^{pr}_{ud\varphi^2 D})g\left[(\bar{u}_p\gamma^\mu d_r)(\bar{e}_s\gamma_\mu\nu_s + \bar{d}_x\gamma_\mu V^*_{sx} u_s)\right. \\
& -(\bar{d}_r\gamma^\mu u_p)(\bar{\nu}_s\gamma_\mu e_s + \bar{u}_s\gamma_\mu V_{sx} d_x) - (\bar{u}_p\gamma^\mu\gamma_5 d_r)(\bar{e}_s\gamma_\mu\gamma_5\nu_s + \bar{d}_x\gamma_\mu\gamma_5 V^*_{sx} u_s) \\
& \left.+(\bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{\nu}_s\gamma_\mu\gamma_5 e_s + \bar{u}_s\gamma_\mu\gamma_5 V_{sx} d_x)\right] \qquad (39)
\end{aligned}
$$

The mass-dimension 8 operators in Eq. (40) containing $vG_F$ from class $\psi^2 X\varphi$ can also give important contributions:

$$
\begin{aligned}
\mathcal{L}^{\not{C}}_{\psi^2X\varphi} \;=\;& \sqrt{2}vG_F i\mathrm{Im}(C^{pr}_{leW\varphi})\left\{(\bar\nu_p\sigma^{\mu\nu}e_r)[\partial_\mu(\bar e_s\gamma_\nu\nu_s+\bar d_{Lx}\gamma_\nu V^*_{sx}u_s)\right.\\
&+ig(\bar e_s\gamma_\mu\nu_s+\bar d_{Lx}\gamma_\mu V^*_{sx}u_{Ls})s_wA_\nu]\\
&-(\bar e_r\sigma^{\mu\nu}\nu_p)[\partial_\mu(\bar\nu_s\gamma_\nu e_s+\bar u_s\gamma_\nu V_{sx}d_{Lx})-ig(\bar\nu_s\gamma_\mu e_s+\bar u_s\gamma_\mu V_{sx}d_{Lx})s_wA_\nu]\\
&-(\bar\nu_p\sigma^{\mu\nu}\gamma_5e_r)[\partial_\mu(\bar e_s\gamma_\nu\gamma_5\nu_s+\bar d_{Lx}\gamma_\nu\gamma_5V^*_{sx}u_s)+ig(\bar e_s\gamma_\mu\gamma_5\nu_s+\bar d_{Lx}\gamma_\mu\gamma_5V^*_{sx}u_{Ls})s_wA_\nu]\\
&\left.+(\bar e_r\sigma^{\mu\nu}\gamma_5\nu_p)[\partial_\mu(\bar\nu_s\gamma_\nu\gamma_5e_s+\bar u_s\gamma_\nu\gamma_5V_{sx}d_{Lx})-ig(\bar\nu_s\gamma_\mu\gamma_5e_s+\bar u_s\gamma_\mu\gamma_5V_{sx}d_{Lx})s_wA_\nu]\right\}\\
+\;& 2vG_FiC^{pr}_{leZ\varphi}\left\{(\bar e_p\sigma^{\mu\nu}e_r-\bar e_r\sigma^{\mu\nu}e_p)\times\right.\\
&\partial_\mu\left[\frac14\bar\nu_s\gamma_\nu\nu_s+(-\frac14+s_w^2)\bar e_s\gamma_\nu e_s+(\frac14-\frac23s_w^2)\bar u_s\gamma_\nu u_s+(-\frac14+\frac13s_w^2)\bar d_s\gamma_\nu d_s\right]\\
&\left.+\frac14(\bar e_p\sigma^{\mu\nu}\gamma_5e_r+\bar e_r\sigma^{\mu\nu}\gamma_5e_p)\partial_\mu(\bar\nu_s\gamma_\mu\gamma_5\nu_s+\bar e_s\gamma_\mu\gamma_5e_s+\bar u_s\gamma_\mu\gamma_5u_s-\bar d_s\gamma_\mu\gamma_5d_s)\right\}\\
+\;& 2vG_FiC^{pr}_{quZ\varphi}\left\{(\bar u_p\sigma^{\mu\nu}u_r-\bar u_r\sigma^{\mu\nu}u_p)\times\right.\\
&\partial_\mu\left[\frac14\bar\nu_s\gamma_\nu\nu_s+(-\frac14+s_w^2)\bar e_s\gamma_\nu e_s+(\frac14-\frac23s_w^2)\bar u_s\gamma_\nu u_s+(-\frac14+\frac13s_w^2)\bar d_s\gamma_\nu d_s\right]\\
&\left.+\frac14(\bar u_p\sigma^{\mu\nu}\gamma_5u_r+\bar u_r\sigma^{\mu\nu}\gamma_5u_p)\partial_\mu(\bar\nu_s\gamma_\mu\gamma_5\nu_s+\bar e_s\gamma_\mu\gamma_5e_s+\bar u_s\gamma_\mu\gamma_5u_s-\bar d_s\gamma_\mu\gamma_5d_s)\right\}\\
+\;& 2vG_FiC^{pr}_{qdZ\varphi}\left\{(\bar d_p\sigma^{\mu\nu}d_r-\bar d_r\sigma^{\mu\nu}d_p)\times\right.\\
&\partial_\mu\left[\frac14\bar\nu_s\gamma_\nu\nu_s+(-\frac14+s_w^2)\bar e_s\gamma_\nu e_s+(\frac14-\frac23s_w^2)\bar u_s\gamma_\nu u_s+(-\frac14+\frac13s_w^2)\bar d_s\gamma_\nu d_s\right]\\
&\left.+\frac14(\bar d_p\sigma^{\mu\nu}\gamma_5d_r+\bar d_r\sigma^{\mu\nu}\gamma_5d_p)\partial_\mu(\bar\nu_s\gamma_\mu\gamma_5\nu_s+\bar e_s\gamma_\mu\gamma_5e_s+\bar u_s\gamma_\mu\gamma_5u_s-\bar d_s\gamma_\mu\gamma_5d_s)\right\}\\
+\;& \sqrt{2}vG_Fi[\mathrm{Im}(C^{pr}_{quW\varphi})-\mathrm{Im}(C^{rp}_{qdW\varphi})]\left\{(\bar d_p\sigma^{\mu\nu}u_r)\left[\partial_\mu(\bar\nu_s\gamma_\nu e_s+\bar u_s\gamma_\nu V_{sx}d_x)\right.\right.\\
&-ig(\bar\nu_s\gamma_\mu e_s+\bar u_s\gamma_\mu V_{sx}d_x)s_wA_\nu]-(\bar u_r\sigma^{\mu\nu}d_p)\left[\partial(\bar e_s\gamma_\nu\nu_s+\bar d_x\gamma_\nu V^*_{sx}u_s)\right.\\
&\left.\left.+ig(\bar e_s\gamma_\mu\nu_s+\bar d_x\gamma_\mu V^*_{sx}u_s)s_wA_\nu\right]\right\}\\
-\;& \sqrt{2}vG_Fi[\mathrm{Im}(C^{pr}_{quW\varphi})+\mathrm{Im}(C^{rp}_{qdW\varphi})]\left\{(\bar d_p\sigma^{\mu\nu}\gamma_5u_r)\left[\partial(\bar\nu_s\gamma_\nu\gamma_5e_s+\bar u_s\gamma_\nu\gamma_5V_{sx}d_x)\right.\right.\\
&-ig(\bar\nu_s\gamma_\mu\gamma_5e_s+\bar u_s\gamma_\mu\gamma_5V_{sx}d_x)s_wA_\nu]+(\bar u_r\sigma^{\mu\nu}\gamma_5d_p)\left[\partial_\mu(\bar e_s\gamma_\nu\gamma_5\nu_s+\bar d_x\gamma_\nu\gamma_5V^*_{sx}u_s)\right.\\
&\left.\left.+ig(\bar e_s\gamma_\mu\gamma_5\nu_s+\bar d_x\gamma_\mu\gamma_5V^*_{sx}u_s)s_wA_\nu\right]\right\} \qquad (40)
\end{aligned}
$$

The flavor-changing P-odd and CP-odd operators with the lowest mass-dimension which are listed in Appendix 7.8.2 also come from classes $\psi^2\varphi^3$, $\psi^2X\varphi$, $\psi^4$, and $\psi^2\varphi^2D$. We find that the flavor-changing P-odd and CP-odd operators and C-odd and CP-odd operators are related either with the same factors or with factors composed from different linear combinations of the same low-energy coefficients. Explicitly, the relevant P-odd and CP-odd operators and C-odd and CP-odd operators from classes $\psi^2\varphi^3$, $\psi^2X\varphi$, $\psi^2\varphi^2D$ and many from $\psi^4$ share the same factors, the rest from $\psi^4$ have factors with different linear combinations of the same low-energy coefficients, for example one of the C-odd and CP-odd operator $[(\bar u_p\gamma_\mu u_r)(\bar u_s\gamma^\mu u_t)-(\bar u_t\gamma^\mu u_s)(\bar u_r\gamma_\mu u_p)]$ has factor $\frac14 i\mathrm{Im}(C^{(1)prst}_{q^4}+C^{(3)prst}_{q^4}+C^{prst}_{u^4}+C^{(1)prst}_{q^2u^2})$, and one of its relevant P-odd and CP-odd operators $[(\bar u_p\gamma_\mu\gamma_5u_r)(\bar u_s\gamma^\mu u_t)-(\bar u_t\gamma^\mu u_s)(\bar u_r\gamma_\mu\gamma_5u_p)]$ has factor $\frac14 i\mathrm{Im}(-C^{(1)prst}_{q^4}-C^{(3)prst}_{q^4}+C^{prst}_{u^4}-C^{(1)prst}_{q^2u^2})$, as shown in Table 7.2 and Table 7.6. The situation will be very different for flavor-conserving interactions which will be discussed later.

The operators we listed also contain possible charged-lepton-flavor-violation (CLFV) interactions. De Gouvea and Jenkins [253] investigated some operators that can contribute to CLFV processes like $\mu \to e\gamma$ and $\mu \to e$-conversion which are included in our Eq. (38), lepton-only interaction ands lepton-quark interactions in our four-fermion C-odd and CP-odd operators listed above, but they did not discuss the CP properties of these operators, and we have more operators contributing to CLFV interactions.

### 7.5.2 Lowest Mass-dimensional C-odd and CP-odd Operators Contributing to Flavor-conserving Interactions

When only considering flavor-conserving interactions, many CP-odd operators existing in flavor-changing case obviously vanish, e.g. in Eq. (25), $-\frac{v^2}{2}i\mathrm{Im}(C^{pr}_{d^2\varphi^2 D})g_Z Z_\mu$ $(\bar{d}_{Rp}\gamma^\mu d_{Rr} - \bar{d}_{Rr}\gamma^\mu d_{Rp}) = 0$ when $p = r$. For operators involving two fermion bilinears, operators like $(\bar{u}_{Lp}\gamma_\mu u_{Lp})(\bar{d}_{Lr}\gamma^\mu d_r) - (\bar{d}_r\gamma^\mu d_r)(\bar{u}_p\gamma_\mu u_p)$ do not vanish naively when considering non-factorizable effects [254]. However, it is reasonable to presume extra interactions of non-factorization are from electromagnetic or strong interactions that are C invariant, which makes the CP odd operators like $(\bar{u}_{Lp}\gamma_\mu u_{Lp})(\bar{d}_{Lr}\gamma^\mu d_r) -$ $(\bar{d}_r\gamma^\mu d_r)(\bar{u}_p\gamma_\mu u_p)$ still vanish.

Before integrating out the weak gauge fields $W^\pm_\mu$ and $Z_\mu$, the CP-odd operators in $\mathcal{L}^T_{X^3}$, $\mathcal{L}^T_{X^2\varphi^2}$, $\mathcal{L}^T_{\psi^2\varphi^3}$, and $\mathcal{L}^T_{\psi^2 X\varphi}$ remain the same as Eq.(21), Eq. (22), Eq. (23) with $p = r$ and Eq. (24). As for $\mathcal{L}^T_{\psi^2\varphi^2 D}$ and $\mathcal{L}^T_{\psi^4}$, we have

$$
\begin{aligned}
\mathcal{L}^T_{\psi^2\varphi^2 D} &= \frac{v^2}{2}i\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D})\sqrt{2}g[(\bar{e}_{Lp}\gamma^\mu\nu_{Lr} - \bar{e}_{Lr}\gamma^\mu\nu_{Lp})W^-_\mu + (\bar{\nu}_{Lp}\gamma^\mu e_{Lr} - \bar{\nu}_{Lr}\gamma^\mu e_{Lp})W^+_\mu] \\
&+ \frac{v^2}{2}i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D})\sqrt{2}g[(\bar{d}_{Lp}\gamma^\mu u_{Lr} - \bar{d}_{Lr}\gamma^\mu u_{Lp})W^-_\mu + (\bar{u}_{Lp}\gamma^\mu d_{Lr} - \bar{u}_{Lr}\gamma^\mu d_{Lp})W^+_\mu] \\
&+ \frac{v^2}{2\sqrt{2}}i\mathrm{Im}(C^{pr}_{ud\varphi^2 D})g\{W^+_\mu\bar{u}_{Rp}\gamma^\mu d_{Rr} - W^-_\mu\bar{d}_{Rr}\gamma^\mu u_{Rp}\} \qquad (41) \\
\mathcal{L}^T_{\psi^4} &= i\mathrm{Im}(C^{(1)pprr}_{quqd})\left[(\bar{u}_{Lp}u_{Rp})(\bar{d}_{Lr}d_{Rr}) - (\bar{d}_{Rr}d_{Lr})(\bar{u}_{Rp}u_{Lp})\right. \\
&\quad \left. -(\bar{d}_{Lp}u_{Rr})(\bar{u}_{Lr}d_{Rp}) + (\bar{d}_{Rp}u_{Lr})(\bar{u}_{Rr}d_{Lp})\right] \\
&\quad +i\mathrm{Im}(C^{(8)pprr}_{quqd})\left[(\bar{u}_{Lp}T^A u_{Rp})(\bar{d}_{Lr}T^A d_{Rr}) - (\bar{d}_{Rr}T^A d_{Lr})(\bar{u}_{Rp}T^A u_{Lp})\right. \\
&\qquad \left. -(\bar{d}_{Lp}T^A u_{Rr})(\bar{u}_{Lr}T^A d_{Rp}) + (\bar{d}_{Rp}T^A u_{Lr})(\bar{u}_{Rr}T^A d_{Lp})\right] \qquad (42)
\end{aligned}
$$

Next we will investigate the C-odd and CP-odd operators after integrating out the weak gauge bosons. We note that for the flavor-changing C-odd and CP-odd operators in Eq. (39), the terms that do not obviously vanish for flavor-conserving interactions are

$$\begin{aligned}
\mathcal{L}_{\psi^2\varphi D} \sim\ & \frac{v^2G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) g\left[(\bar{d}_p\gamma^\mu u_r)(\bar{u}_r\gamma_\mu V_{rp}d_p) - (\bar{d}_r\gamma^\mu u_p)(\bar{u}_p\gamma_\mu V_{pr}d_r)\right.\\
& \left.+(\bar{d}_r\gamma_\mu V^*_{pr}u_p)(\bar{u}_p\gamma^\mu d_r) - (\bar{d}_p\gamma_\mu V^*_{rp}u_r)(\bar{u}_r\gamma^\mu d_p))\right]\\
+\ & \frac{v^2G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) g\left[(\bar{d}_p\gamma^\mu\gamma_5 u_r)(\bar{u}_r\gamma_\mu\gamma_5 V_{rp}d_p) - (\bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu\gamma_5 V_{pr}d_r)\right.\\
& \left.+(\bar{d}_r\gamma_\mu\gamma_5 V^*_{pr}u_p)(\bar{u}_p\gamma^\mu\gamma_5 d_r) - (\bar{d}_p\gamma_\mu\gamma_5 V^*_{rp}u_r)(\bar{u}_r\gamma^\mu\gamma_5 d_p)\right]\\
+\ & \frac{v^2G_F}{4} i\mathrm{Im}(C^{pr}_{ud\varphi^2 D}) g\left[(\bar{u}_p\gamma^\mu d_r)(\bar{d}_r\gamma_\mu V^*_{pr}u_p) - (\bar{d}_r\gamma^\mu u_p)(\bar{u}_p\gamma_\mu V_{pr}d_r)\right.\\
& \left.-(\bar{d}_r\gamma_\mu\gamma_5 V^*_{pr}u_p)(\bar{u}_p\gamma^\mu\gamma_5 d_r) + (\bar{u}_p\gamma_\mu\gamma_5 V_{pr}d_r)(\bar{d}_r\gamma^\mu\gamma_5 u_p)\right]\\
=\ & \frac{v^2G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) g\left\{\left(V_{rp} - V^*_{rp}\right)\left[(\bar{d}_p\gamma^\mu u_r)(\bar{u}_r\gamma_\mu d_p) + (\bar{d}_p\gamma^\mu\gamma_5 u_r)(\bar{u}_r\gamma_\mu\gamma_5 d_p)\right]\right.\\
& -\left(V_{pr} - V^*_{pr}\right)\left[(\bar{d}_r\gamma^\mu u_p)(\bar{u}_p\gamma_\mu d_r) + (\bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu\gamma_5 d_r)\right]\\
+\ & \frac{v^2G_F}{4} i\mathrm{Im}(C^{pr}_{ud\varphi^2 D}) g\left(V^*_{pr} - V_{pr}\right)\left[(\bar{u}_p\gamma^\mu d_r)(\bar{d}_r\gamma_\mu u_p) + (\bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu\gamma_5 d_r)\right]
\end{aligned} \tag{43}$$

which are actually P-even and C-even so CP-even. So there is no contribution to flavor-conserving C-odd and CP-odd operators from $\mathcal{L}_{\psi^2\varphi^2 D}$.

The C-odd and CP-odd terms with the lowest mass-dimension come from $\mathcal{L}^{\not C}_{X^2\varphi^2}$ and $\mathcal{L}^{\not C}_{\psi^2 X\varphi}$:

$$\begin{aligned}
\mathcal{L}^{\not C}_{X^2\varphi^2} =\ & \frac{2v^2G_F}{\sqrt{2}}\epsilon^{\mu\nu\alpha\beta}[2c_w s_w(C_{W^2\varphi^2} - C_{B^2\varphi^2}) - C_{WB\varphi^2}({c_w}^2 - {s_w}^2)] \times\\
& \partial_\alpha[-\frac{1}{2}(\bar{u}_p\gamma_\beta\gamma_5 u_p) + \frac{1}{2}(\bar{d}_p\gamma_\beta\gamma_5 d_p)]F_{\mu\nu}
\end{aligned} \tag{44}$$

$$
\begin{aligned}
\mathcal{L}^{\not{C}}_{\psi^2 X\varphi} \;=\;& \frac{G_F}{\sqrt{2}} vi\mathrm{Im}(C^{pp}_{leZ\varphi})(\bar{e}_p\sigma^{\mu\nu}\gamma_5 e_p)\times \\
& \partial_\mu(-\bar{\nu}_r\gamma_\nu\gamma_5\nu_r + \bar{e}_r\gamma_\nu\gamma_5 e_r - \bar{u}_r\gamma_\nu\gamma_5 u_r + \bar{d}_r\gamma_\nu\gamma_5 d_r) \\
+\;& \frac{2G_F}{\sqrt{2}} vi\mathrm{Im}(C^{pp}_{leW\varphi})\,\{(\bar{\nu}_p\sigma^{\mu\nu}e_p)[\partial_\mu(\bar{e}_p\gamma_\nu\nu_p) + ig(\bar{e}_p\gamma_\mu\nu_p)s_w A_\nu] \\
& -(\bar{e}_p\sigma^{\mu\nu}\nu_p)[\partial_\mu(\bar{e}_p\gamma_\nu e_p) - ig(\bar{\nu}_p\gamma_\mu e_p)s_w A_\nu] \\
& -(\bar{\nu}_p\sigma^{\mu\nu}\gamma_5 e_p)[\partial_\mu(\bar{e}_p\gamma_\nu\gamma_5\nu_p) + ig(\bar{e}_p\gamma_\mu\gamma_5\nu_p)s_w A_\nu] \\
& -(\bar{e}_p\sigma^{\mu\nu}\gamma_5\nu_p)[\partial_\mu(\bar{\nu}_p\gamma_\nu\gamma_5 e_p) - ig(\bar{\nu}_p\gamma_\mu\gamma_5 e_p)s_w A_\nu]\} \\
+\;& \frac{G_F}{\sqrt{2}} vi[(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)C^{pp}_{quZ\varphi} + (\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)C^{pp}_{qdZ\varphi}]\times \\
& \partial_\mu(-\bar{u}_r\gamma_\nu\gamma_5 u_r + \bar{d}_r\gamma_\nu\gamma_5 d_r - \bar{\nu}_r\gamma_\nu\gamma_5\nu_r + \bar{e}_r\gamma_\nu\gamma_5 e_r) \\
+\;& \frac{2G_F}{\sqrt{2}} vi[\mathrm{Im}(C^{pr}_{quW\varphi}) - \mathrm{Im}(C^{rp}_{qdW\varphi})]\,\Big\{(\bar{d}_p\sigma^{\mu\nu}u_r)\times \\
& [\partial_\mu(\bar{u}_r\gamma_\nu V_{u_r d_p} d_p) - ig(\bar{u}_r\gamma_\mu V_{u_r d_p} d_p)s_w A_\nu] \\
& -(\bar{u}_r\sigma^{\mu\nu}d_p)[\partial_\mu(\bar{d}_p V^*_{u_r d_p}\gamma_\nu u_r) + ig(\bar{d}_p V^*_{u_r d_p}\gamma_\mu u_r)s_w A_\nu]\Big\} \\
-\;& \frac{2G_F}{\sqrt{2}} vi[\mathrm{Im}(C^{pr}_{quW\varphi}) + \mathrm{Im}(C^{rp}_{qdW\varphi})]\,\Big\{(\bar{d}_p\sigma^{\mu\nu}\gamma_5 u_r)\times \\
& [\partial_\mu(\bar{u}_r\gamma_\nu\gamma_5 V_{u_r d_p} d_p) - ig(\bar{u}_r\gamma_\mu\gamma_5 V_{u_r d_p} d_p)s_w A_\nu] \\
& +(\bar{u}_r\sigma^{\mu\nu}\gamma_5 d_p)[\partial_\mu(\bar{d}_p V^*_{u_r d_p}\gamma_\nu\gamma_5 u_r) + ig(\bar{d}_p V^*_{u_r d_p}\gamma_\mu\gamma_5 u_r)s_w A_\nu]\Big\} \qquad (45)
\end{aligned}
$$

The dimension of the lowest mass dimension C-odd and CP-odd operators for flavor-conserving interactions is 8. The mass dimension of our operators which conserve B and L are consistent with the SMEFT dimension theorem of Kobach [241].

We list the lowest mass-dimensional C-odd and CP-odd operators for quark-flavor-conserving interactions analyzed from mass-dimension 6 SMEFT operators in Table 7.3.

The flavor-conserving P-odd and CP-odd operators have been studied by other literatures, e.g. de Vries et al. [244]. We list our results in Appendix 7.8.3 as part of the complete list of lowest mass-dimensional flavor-conserving CP-odd operators and compare our list with those previous works therein. The dominant P-odd CP-odd operators are from classes $X^3$, $X^2\varphi^2$, $\psi^2\varphi^3$, $\psi^2 X\varphi$, $\psi^2\varphi D$ and $\psi^4$ and their lowest mass-dimension is 6, whereas the dominant C-odd CP-odd operators come from classes $X^2\varphi^2$ and $\psi^2 X\varphi$ and they are mass-dimension 8. There is only one correspondence from the P-odd and CP operators to the C-odd and CP-odd operators that share the same factor, that is Eq. (44) and Eq. (C.10). So most flavor-conserving P-odd and CP-odd operators and C-odd and CP-odd operators will be probed through very different experimental effects.

Our lowest-dimensional quark-flavor-conserving C-odd and CP-odd operators arise from mass-dimension 6 SMEFT operators and their mass-dimension is 8 after integrating out the weak gauge bosons. One of them contains photon field strength tensor with $v^2 G_F \sim \mathcal{O}(1)$ in SM and some constants $vG_F\partial_\mu \sim vG_F p \sim p/v$, where $p$ represents the momentum of fermion current. Considering electroweak symmetry

Table 7.3: Lowest mass-dimensional C-odd and CP-odd operators contributing to flavor-conserving interactions

| | | |
|---|---|---|
| $1_a$ | $\frac{v^2}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha(\bar{u}_p\gamma_\beta\gamma_5 u_p)F_{\mu\nu}$ | $-\frac{4G_F}{\sqrt{2}}[2c_w s_w(C_{W^2\varphi^2}-C_{B^2\varphi^2})-C_{WB\varphi^2}({c_w}^2-{s_w}^2)]$ |
| $1_b$ | $\frac{v^2}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha(\bar{d}_p\gamma_\beta\gamma_5 d_p)F_{\mu\nu}$ | $\frac{4G_F}{\sqrt{2}}[2c_w s_w(C_{W^2\varphi^2}-C_{B^2\varphi^2})-C_{WB\varphi^2}({c_w}^2-{s_w}^2)]$ |
| $2_a$ | $\frac{v}{\sqrt{2}}(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)\partial_\mu(\bar{u}_r\gamma_\nu\gamma_5 u_r)$ | $-G_F iC^{pr}_{quZ\varphi}$ |
| $2_b$ | $\frac{v}{\sqrt{2}}(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)\partial_\mu(\bar{d}_r\gamma_\nu\gamma_5 d_r)$ | $G_F iC^{pr}_{qdZ\varphi}$ |
| $2_c$ | $\frac{v}{\sqrt{2}}(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)\partial_\mu(\bar{u}_r\gamma_\nu\gamma_5 u_r)$ | $-G_F iC^{pr}_{quZ\varphi}$ |
| $2_d$ | $\frac{v}{\sqrt{2}}(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)\partial_\mu(\bar{d}_r\gamma_\nu\gamma_5 d_r)$ | $G_F iC^{pr}_{qdZ\varphi}$ |
| $3_a$ | $\frac{v}{\sqrt{2}}\Big[V_{u_r d_p}(\bar{d}_p\sigma^{\mu\nu}u_r)\partial_\mu(\bar{u}_r\gamma_\nu d_p)$ $-V^*_{u_r d_p}(\bar{u}_r\sigma^{\mu\nu}d_p)\partial_\mu(\bar{d}_p\gamma_\nu u_r)\Big]$ | $2G_F i[\mathrm{Im}(C^{pr}_{quW\varphi})-\mathrm{Im}(C^{rp}_{qdW\varphi})]$ |
| $3_b$ | $\frac{v}{\sqrt{2}}\Big[V_{u_r d_p}(\bar{d}_p\sigma^{\mu\nu}\gamma_5 u_r)\partial_\mu(\bar{u}_r\gamma_\nu\gamma_5 d_p)$ $+V^*_{u_r d_p}(\bar{u}_r\sigma^{\mu\nu}\gamma_5 d_p)\partial_\mu(\bar{d}_p\gamma_\nu\gamma_5 u_r)\Big]$ | $-2G_F i[\mathrm{Im}(C^{pr}_{quW\varphi})+\mathrm{Im}(C^{rp}_{qdW\varphi})]$ |
| $4_a$ | $\frac{v}{\sqrt{2}}\Big[V_{u_r d_p}(\bar{d}_p\sigma^{\mu\nu}u_r)(\bar{u}_r\gamma_\mu d_p)A_\nu$ $+V^*_{u_r d_p}(\bar{u}_r\sigma^{\mu\nu}d_p)(\bar{d}_p\gamma_\mu u_r)A_\nu\Big]$ | $2G_F g s_w[\mathrm{Im}(C^{pr}_{quW\varphi})-\mathrm{Im}(C^{rp}_{qdW\varphi})]$ |
| $4_b$ | $\frac{v}{\sqrt{2}}\Big[V_{u_r d_p}(\bar{d}_p\sigma^{\mu\nu}\gamma_5 u_r)(\bar{u}_r\gamma_\mu\gamma_5 d_p)A_\nu$ $-V^*_{u_r d_p}(\bar{u}_r\sigma^{\mu\nu}\gamma_5 d_p)(\bar{d}_p\gamma_\mu\gamma_5 u_r)A_\nu\Big]$ | $-2G_F g s_w[\mathrm{Im}(C^{pr}_{quW\varphi})+\mathrm{Im}(C^{rp}_{qdW\varphi})]$ |

breaking effects induced by new physics [166, 178, 170, 168] may be different from the SM, we believe there are more mass-dimension 8 C-odd and CP-odd flavor-conserving operators from the SMEFT that can have important contributions. Therefore we will investigate the mass-dimension 8 SMEFT operators and list the C-odd and CP-odd flavor-conserving operators as follows.

### Other Possible C-odd and CP-odd Mass-Dimension 8 Operators from SMEFT

Lehman and Martin [255] and Henning et al. [256] gave the operators with possible class types and listed their constituents of fundamental fields using SMEFT, but did not show the explicit structures of these operators. Lehman and Martin [255] had 931 operators for $N_f = 1$ when including hermitian conjugates. Henning et al. [256] found 895 B-conserving operators for $N_f = 1$ and 36971 for $N_f = 3$, with $N_f$ the generation number. We will try to pick the possible mass-dimension 8 flavor-conserving operators that are C-odd and CP-odd.

Here for simplicity we will not consider lepton interactions. Since we finally focus on effective operators just below the weak scale, where the weak gauge bosons $W^\pm$, $Z$ are integrated out, we just need to investigate the operators with quarks, gluons and photons. Applying the fundamental fields' transformation properties listed in Appendix 7.8.1, we found quark flavor-conserving C-odd and CP-odd operators from $\psi^4 X$, $\psi^2 X^2\varphi$, $\psi^2 X^2 D$, $\psi^2 X\varphi D^2$, $\psi^4 D^2$, $\psi^4\varphi D$,$\psi^4\varphi^2$, $\psi^2\varphi^2 D^3$, $\psi^2\varphi^3 D^2$ and $\psi^2\varphi^4 D$. We show the complete list with some useful notes in the Appendix 7.8.5. Each low energy operator can come from some different SM gauge invariant mass-dimension 8 operators. For simplicity, we just show their compositions of fundamental fields and omit their coefficients there. Operators without the imaginary unit $i$ indicates the operator is C-odd and CP-odd with the real part of coefficients, for example hermitian operators, while operators with $i$ are C-odd and CP-odd only when adding $i$ times the imaginary parts of the coefficients to them.

The flavor-conserving mass-dimension 8 SMEFT operators should be more suppressed comparing with the mass-dimension 8 operators we found originated from mass-dimension 6 SMEFT operators. The operators in classes $\psi^2 X\varphi^2 D$ and $\psi^4\varphi D$ should be dominant among all flavor-conserving mass-dimension 8 SMEFT operators and have the same structure as Eq. (44) and Eq. (45), which are originated from mass-dimension 6 SMEFT operators in class $X^2\varphi^2$ and $\psi^2 X\varphi$, respectively. We show their forms explicitly in this section.

For operators in class $\psi^4\varphi D$, the fermion chiralities can only be $LLLR$ and $RRRL$ and the covariant derivative should act on either the Higgs field or fermion with the dominant chirality [255]. We show flavor-conserving C-odd and CP-odd operators we found from mass-dimension 8 SMEFT which are new or already found in our analysis of the mass-dimension 6 SMEFT operators.

i) Operators that are new:

$$\psi^2 X\varphi^2 D: \qquad v^2\epsilon^{\mu\nu\alpha\beta}\partial_\alpha(\bar{u}_p\gamma_\beta\gamma_5 T^A u_p)G^A_{\mu\nu}, \tag{46}$$

$$v^2\epsilon^{\mu\nu\alpha\beta}\partial_\alpha(\bar{d}_p\gamma_\beta\gamma_5 T^A d_p)G^A_{\mu\nu}. \tag{47}$$

$$\psi^4\varphi D: \qquad iv[(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_r)\partial_\mu(\bar{d}_r\gamma_\nu\gamma_5 d_p)+(\bar{d}_r\sigma^{\mu\nu}\gamma_5 d_p)\partial_\mu(\bar{d}_p\gamma_\nu\gamma_5 d_r)], \tag{48}$$

$$iv[(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_r)\partial_\mu(\bar{u}_r\gamma_\nu\gamma_5 u_p)+(\bar{u}_r\sigma^{\mu\nu}\gamma_5 u_p)\partial_\mu(\bar{u}_p\gamma_\nu\gamma_5 u_r)], \tag{49}$$

$$iv[(\bar{u}_p\sigma^{\mu\nu} u_r)\partial_\mu(\bar{u}_r\gamma_\nu u_p)-(\bar{u}_r\sigma^{\mu\nu} u_p)\partial_\mu(\bar{u}_p\gamma_\nu u_r)], \tag{50}$$

$$iv[(\bar{d}_p\sigma^{\mu\nu} d_r)\partial_\mu(\bar{d}_r\gamma_\nu d_p)-(\bar{d}_r\sigma^{\mu\nu} d_p)\partial_\mu(\bar{d}_p\gamma_\nu d_r)], \tag{51}$$

The above operators are similar to those found in previous part of this work, but with different quark-flavors.

$$iv[(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_r)(\bar{d}_r\overleftarrow{D}_\mu\gamma_\nu\gamma_5 d_p) + (\bar{d}_r\sigma^{\mu\nu}\gamma_5 d_p)(\bar{d}_p\gamma_\nu\gamma_5 D_\mu d_r)], \tag{52}$$

$$iv[(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_r)(\bar{u}_r\overleftarrow{D}_\mu\gamma_\nu\gamma_5 u_p) + (\bar{u}_r\sigma^{\mu\nu}\gamma_5 u_p)(\bar{u}_p\gamma_\nu\gamma_5 D_\mu u_r)], \tag{53}$$

$$iv[(\bar{u}_p\sigma^{\mu\nu} u_r)(\bar{u}_r\overleftarrow{D}_\mu\gamma_\nu u_p) - (\bar{u}_r\sigma^{\mu\nu} u_p)(\bar{u}_p\gamma_\nu D_\mu u_r)], \tag{54}$$

$$iv[(\bar{d}_p\sigma^{\mu\nu} d_r)(\bar{d}_r\overleftarrow{D}_\mu\gamma_\nu d_p) - (\bar{d}_r\sigma^{\mu\nu} d_p)(\bar{d}_p\gamma_\nu D_\mu d_r)], \tag{55}$$

$$iv(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)(\bar{u}_r\overleftarrow{D}_\mu\gamma_\nu\gamma_5 u_r) + iv(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)(\bar{u}_r\gamma_\nu\gamma_5 D_\mu u_r), \tag{56}$$

$$iv(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)(\bar{d}_r\overleftarrow{D}_\mu\gamma_\nu\gamma_5 d_r) + iv(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)(\bar{d}_r\gamma_\nu\gamma_5 D_\mu d_r), \tag{57}$$

$$iv(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)(\bar{u}_r\overleftarrow{D}_\mu\gamma_\nu\gamma_5 u_r) + iv(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)(\bar{u}_r\gamma_\nu\gamma_5 D_\mu u_r), \tag{58}$$

$$iv(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)(\bar{d}_r\overleftarrow{D}_\mu\gamma_\nu\gamma_5 d_r) + iv(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)(\bar{d}_r\gamma_\nu\gamma_5 D_\mu d_r), \tag{59}$$

$$iv[(\bar{u}_p\sigma^{\mu\nu}\gamma_5 d_r)(\bar{d}_r\overleftarrow{D}_\mu\gamma_\nu\gamma_5 u_p) + (\bar{d}_r\sigma^{\mu\nu}\gamma_5 u_p)\partial_\mu(\bar{u}_p\gamma_\nu\gamma_5 D_\mu d_r)], \tag{60}$$

$$iv[(\bar{u}_p\sigma^{\mu\nu} d_r)(\bar{d}_r\overleftarrow{D}_\mu\gamma_\nu u_p) - (\bar{d}_r\sigma^{\mu\nu} u_p)\partial_\mu(\bar{u}_p\gamma_\nu D_\mu d_r)], \tag{61}$$

$$v\partial_\mu(\bar{u}_p u_p)(\bar{u}_r\gamma_\mu u_r), \tag{62}$$

$$iv\partial_\mu(\bar{u}_p\gamma_5 u_p)(\bar{u}_r\gamma_\mu\gamma_5 u_r), \tag{63}$$

$$v[\partial_\mu(\bar{u}_p u_r)(\bar{u}_r\gamma_\mu u_p) + (\bar{u}_p\gamma_\mu u_r)\partial_\mu(\bar{u}_r u_p)], \tag{64}$$

$$iv[\partial_\mu(\bar{u}_p\gamma_5 u_r)(\bar{u}_r\gamma_\mu\gamma_5 u_p) + (\bar{u}_p\gamma_\mu\gamma_5 u_r)\partial_\mu(\bar{u}_r\gamma_5 u_p)], \tag{65}$$

$$v\partial_\mu(\bar{u}_p u_p)(\bar{d}_r\gamma_\mu d_r), \tag{66}$$

$$iv\partial_\mu(\bar{u}_p\gamma_5 u_p)(\bar{d}_r\gamma_\mu\gamma_5 d_r), \tag{67}$$

$$v\partial_\mu(\bar{d}_p d_p)(\bar{d}_r\gamma_\mu d_r), \tag{68}$$

$$iv\partial_\mu(\bar{d}_p\gamma_5 d_p)(\bar{d}_r\gamma_\mu\gamma_5 d_r), \tag{69}$$

$$v[\partial_\mu(\bar{d}_p d_r)(\bar{d}_r\gamma_\mu d_p) + (\bar{d}_p\gamma_\mu d_r)\partial_\mu(\bar{d}_r d_p)], \tag{70}$$

$$iv[\partial_\mu(\bar{d}_p\gamma_5 d_r)(\bar{d}_r\gamma_\mu\gamma_5 d_p) + (\bar{d}_p\gamma_\mu\gamma_5 d_r)\partial_\mu(\bar{d}_r\gamma_5 d_p)], \tag{71}$$

$$v\partial_\mu(\bar{d}_p d_p)(\bar{u}_r\gamma_\mu u_r), \tag{72}$$

$$iv\partial_\mu(\bar{d}_p\gamma_5 d_p)(\bar{u}_r\gamma_\mu\gamma_5 u_r), \tag{73}$$

$$v\left[\partial_\mu(\bar{u}_p d_r)(\bar{d}_r\gamma_\mu u_p) + (\bar{u}_p\gamma_\mu d_r)\partial_\mu(\bar{d}_r u_p)\right], \tag{74}$$

$$iv\left[\partial_\mu(\bar{u}_p\gamma_5 d_r)(\bar{d}_r\gamma_\mu\gamma_5 u_p) + (\bar{u}_p\gamma_\mu\gamma_5 d_r)\partial_\mu(\bar{d}_r\gamma_5 u_p)\right], \tag{75}$$

$$v[\bar{u}_p(D_\mu + \overleftarrow{D}_\mu)u_p](\bar{u}_r\gamma_\mu u_r), \tag{76}$$

$$v[\bar{u}_p(D_\mu + \overleftarrow{D}_\mu)u_p](\bar{d}_r\gamma_\mu d_r), \tag{77}$$

$$v[\bar{d}_p(D_\mu + \overleftarrow{D}_\mu)d_p](\bar{d}_r\gamma_\mu d_r), \tag{78}$$

$$v[\bar{d}_p(D_\mu + \overleftarrow{D}_\mu)d_p](\bar{u}_r\gamma_\mu u_r), \tag{79}$$

$$v[(\bar{u}_p D_\mu u_r)(\bar{u}_r \gamma_\mu u_p) + (\bar{u}_r \overleftarrow{D}_\mu u_p)(\bar{u}_p \gamma_\mu u_r)], \tag{80}$$
$$v[(\bar{d}_p D_\mu d_r)(\bar{d}_r \gamma_\mu d_p) + (\bar{d}_r \overleftarrow{D}_\mu d_p)(\bar{d}_p \gamma_\mu d_r)], \tag{81}$$
$$v\left[(\bar{u}_p D_\mu d_r)(\bar{d}_r \gamma_\mu u_p) + (\bar{u}_p \gamma_\mu d_r)(\bar{d}_r \overleftarrow{D}_\mu u_p)\right], \tag{82}$$
$$v(\bar{u}_p (D_\mu - \overleftarrow{D}_\mu)\gamma_5 u_p)(\bar{u}_r \gamma_\mu \gamma_5 u_r), \tag{83}$$
$$v(\bar{u}_p (D_\mu - \overleftarrow{D}_\mu)\gamma_5 u_p)(\bar{d}_r \gamma_\mu \gamma_5 d_r), \tag{84}$$
$$v(\bar{d}_p (D_\mu - \overleftarrow{D}_\mu)\gamma_5 d_p)(\bar{d}_r \gamma_\mu \gamma_5 d_r), \tag{85}$$
$$v(\bar{d}_p (D_\mu - \overleftarrow{D}_\mu)\gamma_5 d_p)(\bar{u}_r \gamma_\mu \gamma_5 u_r), \tag{86}$$
$$v[(\bar{u}_p D_\mu \gamma_5 u_r)(\bar{u}_r \gamma_\mu \gamma_5 u_p) - (\bar{u}_r \overleftarrow{D}_\mu \gamma_5 u_p)(\bar{u}_p \gamma_\mu \gamma_5 u_r)], \tag{87}$$
$$v[(\bar{d}_p D_\mu \gamma_5 d_r)(\bar{d}_r \gamma_\mu \gamma_5 d_p) - (\bar{d}_r \overleftarrow{D}_\mu \gamma_5 d_p)(\bar{d}_p \gamma_\mu \gamma_5 d_r)], \tag{88}$$
$$v[(\bar{u}_p D_\mu \gamma_5 d_r)(\bar{d}_r \gamma_\mu \gamma_5 u_p) - (\bar{u}_p \gamma_\mu \gamma_5 d_r)(\bar{d}_r \overleftarrow{D}_\mu \gamma_5 u_p)]. \tag{89}$$

ii) Operators that are already found from analyzing mass-dimension 6 SMEFT in this work:

$$\psi^2 X \varphi^2 D: \qquad v^2 \epsilon^{\mu\nu\alpha\beta} \partial_\alpha (\bar{u}_p \gamma_\beta \gamma_5 u_p) F_{\mu\nu}, \tag{90}$$
$$v^2 \epsilon^{\mu\nu\alpha\beta} \partial_\alpha (\bar{d}_p \gamma_\beta \gamma_5 d_p) F_{\mu\nu}. \tag{91}$$

$$\psi^4 \varphi D: \qquad iv(\bar{u}_p \sigma^{\mu\nu} \gamma_5 u_p) \partial_\mu (\bar{u}_r \gamma_\nu \gamma_5 u_r), \tag{92}$$
$$iv(\bar{u}_p \sigma^{\mu\nu} \gamma_5 u_p) \partial_\mu (\bar{d}_r \gamma_\nu \gamma_5 d_r), \tag{93}$$
$$iv(\bar{d}_p \sigma^{\mu\nu} \gamma_5 d_p) \partial_\mu (\bar{u}_r \gamma_\nu \gamma_5 u_r), \tag{94}$$
$$iv(\bar{d}_p \sigma^{\mu\nu} \gamma_5 d_p) \partial_\mu (\bar{d}_r \gamma_\nu \gamma_5 d_r), \tag{95}$$
$$iv[(\bar{u}_p \sigma^{\mu\nu} \gamma_5 d_r) \partial_\mu (\bar{d}_r \gamma_\nu \gamma_5 u_p) + (\bar{d}_r \sigma^{\mu\nu} \gamma_5 u_p) \partial_\mu (\bar{u}_p \gamma_\nu \gamma_5 d_r)], \tag{96}$$
$$iv[(\bar{u}_p \sigma^{\mu\nu} d_r) \partial_\mu (\bar{d}_r \gamma_\nu u_p) - (\bar{d}_r \sigma^{\mu\nu} u_p) \partial_\mu (\bar{u}_p \gamma_\nu d_r)]. \tag{97}$$

**Previous Literatures about C-odd and CP-odd Operators**

In this section, we will compare our C-odd and CP-odd analysis with those from other papers with lowest mass-dimension.

Khriplovich [223] started with the $T$ odd and $P$ odd interaction

$$V_d = \frac{1}{2} d \bar{\psi} \gamma_5 \sigma_{\mu\nu} \psi F_{\mu\nu}, \tag{98}$$

with $d$ the electric dipole moment and $F_{\mu\nu}$ the electromagnetic field strength tensor, then substituted the vector potential with an axial-vector current to get the following mass-dimension 7 T-odd, P-even (C-odd) operators

$$\frac{G_F}{\sqrt{2}} \frac{q_1}{2m_p} \bar{\psi}_1 i\gamma_5 \sigma^{\mu\nu} (p_1' - p_1)_\nu \psi_1 \bar{\psi}_2 \gamma_\mu \gamma_5 \psi_2, \tag{99}$$
$$\frac{G_F}{\sqrt{2}} \frac{q_2}{2m_p} \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_1 \bar{\psi}_2 i\gamma_5 \sigma^{\mu\nu} (p_2' - p_2)_\nu \psi_2, \tag{100}$$

where $m_p$ is the proton mass and $q_{1,2}$ are dimensionless. Actually Eq. (99) and Eq. (100) are equivalent when considering non-factorization effects [254] do not contribute.

Conti and Khriplovich [224] used identical operator with Eq. (99), though with different factors

$$\frac{4\pi\beta}{\mu^2-k^2}\frac{1}{2m}i\bar{\psi}_1\gamma_5\sigma^{\mu\rho}k_\rho\psi_1\bar{\psi}_2\gamma_\mu\gamma_5\psi_2, \tag{101}$$

where $\beta$ a the dimensionless coupling constant, $k=p_1'-p_1$, $\mu$ is the mass of an axial boson which mediates the interaction, $m$ equals to the proton mass $m_p$.

Engel et al. [225] gave two mass-dimension 7 T-odd and P-even operators which are $SU(3)_C\times U(1)_Q$-invariant

$$C_7\left(\frac{1}{\Lambda^3}\right)\bar{q}_1\gamma_5 D^\mu q_2\bar{q}_3\gamma_5\gamma_\mu q_4+H.c., \tag{102}$$

$$C_7'\left(\frac{1}{\Lambda^3}\right)\bar{q}\sigma^{\mu\nu}\lambda^A qG^{A\mu\rho}F_\rho^\nu, \tag{103}$$

where $C_7$ and $C_7'$ are dimensionless constants, $q_1=q_2$, $q_3=q_4\neq q_1$ or $q_1=q_4$, $q_2=q_3\neq q_1$. The momenta of Eq. (99) are results of acting derivatives to the quark fields. Replacing the derivative of Eq. (99) with covariant derivative, then using Gordon decomposition which relates $D_\mu-\overleftarrow{D_\mu}$ to $\sigma_{\mu\nu}(D^\nu+\overleftarrow{D}^\nu)$ will result in Eq. (102).

Ramsey-Musolf [226] listed three TVPC operators with dimension 7,

$$\begin{aligned}
\mathcal{O}_7^{ff'} &= C_7^{ff'}\bar{\psi}_f\overleftrightarrow{D}^\mu\gamma_5\psi_f\bar{\psi}_{f'}\gamma_\mu\gamma_5\psi_{f'} && (104)\\
\mathcal{O}_7^{\gamma g'} &= C_7^{\gamma g'}\bar{\psi}\sigma_{\mu\nu}\lambda^A\psi F^{\mu\alpha}G_\alpha^{A\nu} && (105)\\
\mathcal{O}_7^{\gamma Z'} &= C_7^{\gamma Z'}\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\alpha}Z_\alpha^\nu && (106)
\end{aligned}$$

where $f$ and $f'$ are distinct fermions. The first two operators cited Ref. [224, 225]. Eq. (104) [226] equals to Eq. (108) [225] when $q_1=q_2\neq q_3=q_4$,

Over all, the lowest mass-dimension T-odd and P-odd (C-even) operators from these literatures are

$$\frac{G_F}{\sqrt{2}}\frac{q_1}{2m_p}\bar{\psi}_1 i\gamma_5\sigma^{\mu\nu}(p_1'-p_1)_\nu\psi_1\bar{\psi}_2\gamma_\mu\gamma_5\psi_2\ , \tag{107}$$

$$C_7\left(\frac{1}{\Lambda^3}\right)\bar{q}_1\gamma_5 D^\mu q_2\bar{q}_3\gamma_5\gamma_\mu q_4+H.c., \tag{108}$$

$$C_7'\left(\frac{1}{\Lambda^3}\right)\bar{q}\sigma_{\mu\nu}\lambda^A qG^{A\mu\rho}F_\rho^\nu, \tag{109}$$

$$C_7^{\gamma Z'}\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\alpha}Z_\alpha^\nu. \tag{110}$$

and they are mass-dimension 7.

Ref. [257] gave a complete set of dimension-7 operators from the Standard Model effective field theory, but they eliminated $\psi^2X^2$ operators due to hypercharge constraints [257]. Eq. (109) [225] and Eq. (110) [226] are not $SU(3)_C\times SU(2)_L\times U(1)_Y$

invariant. Meanwhile, the mass dimension of the above operators is inconsistent with the dimension of Kobach [241].

Comparing with our analysis, operators in Eq. (107) - Eq. (108) are equivalent to some of our operators in Eq. (45) (see Appendix 7.8.4) and Eq. (83)-Eq. (89). The operator in Eq. (109) is consistent with the operator we found from mass-dimension 8 SMEFT operators in class $\psi^2 X^2 \varphi$ as Eq. (E.63) and Eq. (E.64) listed in Appendix 7.8.5, which contain the Higgs field. The operator in Eq. (110) contains $Z$ boson, which is not under our consideration at low energy, but can also come from class $\psi^2 X^2 \varphi$.

Our results are from a systematic analysis from the SMEFT with mass-dimension consistent with the dimension theorem of SMEFT by Kobach [241], and we have more operators as in Eq. (44), Eq. (45), Eq. (46) - Eq. (82,) and operators listed in Appendix 7.8.5 that have not been considered before.

### 7.6 Constraints on Low-Energy Coefficients

There are some phenomenological studies regarding constraints of C odd and CP operators from contributing to EDM at loop level [224, 225, 226, 227] and flavor-changing processes like $b \to s$ transition [242]. Here we focus on the experimental probes of C-odd and CP-odd interactions. We pick $B \to \pi^+\pi^-\pi^0$ as an example of flavor-changing interactions and $\eta \to \pi^+\pi^-\pi^0$ as a typical illustration of flavor-conserving C-odd and CP-odd process.

#### 7.6.1 Untagged Decay $B \to \pi^+\pi^-\pi^0$

Direct CP violation can exist in untagged, neutral $B$-meson decays to certain self-conjugate final states containing more than two hadrons [90]. We will discuss about the operators acting in $B^0(\bar{B}^0) \to \pi^+\pi^-\pi^0$ that induce a CP asymmetry. The decay $B^0(\bar{B}^0) \to \pi^+\pi^-\pi^0$ can happen through the dominating intermediate states $\rho\pi$ and $\sigma\pi$. We can evaluate the resonance contributions to the $B^0(\bar{B}^0) \to \pi^+\pi^-\pi^0$ decay by using a product ansatz [258] as

$$A_R(B \to \pi^+\pi^-\pi^0) = \frac{1}{2}|\epsilon_{ijk}|\langle (R \to \pi^j\pi^k)\pi^i|\mathcal{H}_{eff}|B\rangle = \frac{1}{2}|\epsilon_{ijk}|\langle R\pi^i|\mathcal{H}_{eff}|B\rangle \Gamma^R_{\pi^j\pi^k}, \tag{111}$$

where $\pi^i = \pi^+,\ \pi^-$ or $\pi^0$ for $i = 1, 2, 3$, the matrix elements $\langle R\pi^i|\mathcal{H}_{eff}|B\rangle$ can be calculated in factorization assumption [259, 260, 261, 262, 263] , and $\Gamma^R_{\pi^j\pi^k}$ stands for the vertex function of $R \to \pi^j\pi^k$ which can be found in Ref. [231] for $\sigma\pi$ and $\rho\pi$ modes. The vertex function $\Gamma^R_{\pi^j\pi^k}$ conserves $CP$. So the matrix elements $\langle R\pi^i|\mathcal{H}_{eff}|B\rangle$ will contribute to the CP asymmetry in the untagged neutral B meson decay. The details of calculating the matrix elements in $B^0(\bar{B}^0) \to \pi^+\pi^-\pi^0$ can be found in Ref. [261, 53]. Gardner and Tandean [53] showed exactly how the matrix elements behave under $CP$ and discuss how the population asymmetry emerges due to the amplitude interferences of distinct intermediate states whose angular momenta are either with the same parity or with different parity. Here we show that the CP-odd mechanism in $B^0(\bar{B}^0) \to \pi^+\pi^-\pi^0$are actually C-odd which was not declared before and we will give the list of CP-odd combinations from the $\Delta b = -1$ and $\Delta b = +1$ operators, with $b$ the bottom quark number.

Considering the match between the flavor quantum numbers of the quark currents and those of $B^0$, $\overline{B}^0$, $\rho^\pm$, $\rho^0$, $\sigma$, $\pi^\pm$ and $\pi^0$,the key matrix elements contributing to $\overline{B}^0 \to \rho^-\pi^+$ and $B^0 \to \rho^+\pi^-$ are

$$\langle\rho^-(p,\epsilon)|\bar{d}\gamma_\mu u|0\rangle\langle\pi^+(p_\pi)|\bar{u}\gamma^\mu b|\overline{B}^0(k)\rangle, \tag{112}$$
$$\langle\rho^+(p,\epsilon)|\bar{u}\gamma_\mu d|0\rangle\langle\pi^-(p_\pi)|\bar{b}\gamma^\mu u|B^0(k)\rangle. \tag{113}$$

For $\overline{B}^0 \to \rho^+\pi^-$ and $B^0 \to \rho^-\pi^+$, the relevant matrix elements that contribute are

$$\langle\pi^-(p_\pi)|\bar{d}\gamma_\mu\gamma_5 u|0\rangle\langle\rho^+(p,\epsilon)|\bar{u}\gamma^\mu\gamma_5 b|\overline{B}^0(k)\rangle, \tag{114}$$
$$\langle\pi^+(p_\pi)|\bar{u}\gamma_\mu\gamma_5 d|0\rangle\langle\rho^-(p,\epsilon)|\bar{b}\gamma^\mu\gamma_5 u|B^0(k)\rangle. \tag{115}$$

For $\overline{B}^0 \to \rho^0\pi^0$ and $B^0 \to \rho^0\pi^0$, the relevant matrix elements that contribute are

$$\langle\rho^0(p,\epsilon)|\bar{u}\gamma_\mu u|0\rangle\langle\pi^0(p_\pi)|\bar{d}\gamma^\mu b|\overline{B}^0(k)\rangle, \tag{116}$$
$$\langle\rho^0(p,\epsilon)|\bar{u}\gamma_\mu u|0\rangle\langle\pi^0(p_\pi)|\bar{b}\gamma^\mu d|B^0(k)\rangle, \tag{117}$$
$$\langle\pi^0(p_\pi)|\bar{u}\gamma_\mu\gamma_5 u|0\rangle\langle\rho^0(p,\epsilon)|\bar{d}\gamma^\mu\gamma_5 b|\overline{B}^0(k)\rangle, \tag{118}$$
$$\langle\pi^0(p_\pi)|\bar{u}\gamma_\mu\gamma_5 u|0\rangle\langle\rho^0(p,\epsilon)|\bar{b}\gamma^\mu\gamma_5 d|B^0(k)\rangle. \tag{119}$$

As for $\overline{B}^0 \to \sigma\pi^0$ and $B^0 \to \sigma\pi^0$, the relevant matrix elements that contribute are

$$\langle\pi^0(p_\pi)|\bar{d}\gamma_\mu\gamma_5 d|0\rangle\langle\sigma(p)|\bar{d}\gamma^\mu\gamma_5 b|\overline{B}^0(k)\rangle, \tag{120}$$
$$\langle\pi^0(p_\pi)|\bar{d}\gamma_\mu\gamma_5 d|0\rangle\langle\sigma(p)|\bar{b}\gamma^\mu\gamma_5 d|B^0(k)\rangle, \tag{121}$$
$$\langle\sigma(p)|\bar{d}\gamma_\mu d|0\rangle\langle\pi^0(p_\pi)|\bar{d}\gamma^\mu b|\overline{B}^0(k)\rangle, \tag{122}$$
$$\langle\sigma(p)|\bar{d}\gamma_\mu d|0\rangle\langle\pi^0(p_\pi)|\bar{b}\gamma^\mu d|B^0(k)\rangle. \tag{123}$$

Since

$$(\bar{d}\gamma_\mu u)(\bar{u}\gamma^\mu b) - (\bar{b}\gamma^\mu u)(\bar{u}\gamma_\mu d) \tag{124}$$
$$(\bar{d}\gamma_\mu\gamma_5 u)(\bar{u}\gamma^\mu\gamma_5 b) - (\bar{b}\gamma^\mu\gamma_5 u)(\bar{u}\gamma_\mu\gamma_5 d) \tag{125}$$
$$(\bar{u}\gamma_\mu u)(\bar{d}\gamma^\mu b) - (\bar{b}\gamma^\mu d)(\bar{u}\gamma_\mu u) \tag{126}$$
$$(\bar{u}\gamma_\mu\gamma_5 u)(\bar{d}\gamma^\mu\gamma_5 b) - (\bar{b}\gamma^\mu\gamma_5 d)(\bar{u}\gamma_\mu\gamma_5 u) \tag{127}$$
$$(\bar{d}\gamma_\mu d)(\bar{d}\gamma^\mu b) - (\bar{d}\gamma^\mu b)(\bar{d}\gamma_\mu d) \tag{128}$$
$$(\bar{d}\gamma_\mu\gamma_5 d)(\bar{d}\gamma^\mu\gamma_5 b) - (\bar{d}\gamma^\mu\gamma_5 b)(\bar{d}\gamma_\mu\gamma_5 d) \tag{129}$$

that contribute to the tree level matrix elements in the untagged process $B^0(\overline{B}^0) \to (\rho^\pm\pi^\mp, \rho^0\pi^0, \sigma\pi^0) \to \pi^+\pi^-\pi^0$ are all C-odd and CP-odd, the CP-odd mechanism in $B^0(\bar{B}^0) \to \pi^+\pi^-\pi^0$are actually C-odd.

In the SM and SMEFT, the effective, weak interactive Hamiltonian for $b \to qq'\bar{q}'$, where $q = d,\ s,\ q' = u,\ d,\ s,\ c$, and its CP conjugate is [264, 261]

$$\begin{aligned}\mathcal{H}_{\rm eff} &= \frac{G_F}{\sqrt{2}}\left[V_{ub}V^*_{uq}(C_1O^u_1 + C_2O^u_2) + V_{cb}V^*_{cq}\,(C_1O^c_1 + C_2O^c_2)\right.\\ &\qquad \left. -V_{tb}V^*_{tq}\left(\sum_{i=3}^{10} C_iO_i + C_gO_g\right)\right] + H.c. + H_{new}, \end{aligned} \tag{130}$$

where $G_F$ is the Fermi coupling constant, the factor $V_{ij}$ are CKM matrix elements, and $C_i$ are the Wilson coefficients evaluated at the renormalization scale $\mu$, $O_{1,\cdots,10}$ are four-quark operators, and $O_g$ is the chromomagnetic penguin operator. Given $b \to d$ transition as an example, $O_i$ and $O_g$ are

$$\begin{aligned}
O^u_1 &= \left(\bar{d}_{L\alpha}\gamma^\mu u_{L\alpha}\right)\left(\bar{u}_{L\beta}\gamma^\mu b_{L\beta}\right),\\
O^u_2 &= \left(\bar{d}_{L\alpha}\gamma^\mu u_{L\beta}\right)\left(\bar{u}_{L\beta}\gamma^\mu b_{L\alpha}\right),\\
O^c_1 &= \left(\bar{d}_{L\alpha}\gamma^\mu c_{L\alpha}\right)\left(\bar{c}_{L\beta}\gamma^\mu b_{L\beta}\right),\\
O^c_2 &= \left(\bar{d}_{L\alpha}\gamma^\mu c_{L\beta}\right)\left(\bar{c}_{L\beta}\gamma^\mu b_{L\alpha}\right),\\
O_3 &= \left(\bar{d}_{L\alpha}\gamma^\mu b_{L\alpha}\right)\sum_{q'}\left(\bar{q}'_{L\beta}\gamma^\mu q'_{L\beta}\right),\\
O_4 &= \left(\bar{d}_{L\alpha}\gamma^\mu b_{L\beta}\right)\sum_{q'}\left(\bar{q}'_{L\beta}\gamma^\mu q'_{L\alpha}\right),\\
O_5 &= \left(\bar{d}_{L\alpha}\gamma^\mu b_{L\alpha}\right)\sum_{q'}\left(\bar{q}'_{R\beta}\gamma^\mu q'_{R\beta}\right),\\
O_6 &= \left(\bar{d}_{L\alpha}\gamma^\mu b_{L\beta}\right)\sum_{q'}\left(\bar{q}'_{R\beta}\gamma^\mu q'_{R\alpha}\right),\\
O_7 &= \frac{3}{2}\left(\bar{d}_{L\alpha}\gamma^\mu b_{L\alpha}\right)\sum_{q'}\left(e_{q'}\bar{q}'_{R\beta}\gamma^\mu q'_{R\beta}\right),\\
O_8 &= \frac{3}{2}\left(\bar{d}_{L\alpha}\gamma^\mu b_{L\beta}\right)\sum_{q'}\left(e_{q'}\bar{q}'_{R\beta}\gamma^\mu q'_{R\alpha}\right),\\
O_9 &= \frac{3}{2}\left(\bar{d}_{L\alpha}\gamma^\mu b_{L\alpha}\right)\sum_{q'}\left(e_{q'}\bar{q}'_{L\beta}\gamma^\mu q'_{L\beta}\right),\\
O_{10} &= \frac{3}{2}\left(\bar{d}_{L\alpha}\gamma^\mu b_{L\beta}\right)\sum_{q'}\left(e_{q'}\bar{q}'_{L\beta}\gamma^\mu q'_{L\alpha}\right),\\
O_g &= (g_s/8\pi^2)m_b\bar{d}_{L\alpha}\sigma^{\mu\nu}T^A_{\alpha\beta}b_{R\beta}G^A_{\mu\nu}.
\end{aligned}$$

The usual tree-level W-exchange contribution in the effective theory corresponds to $O_1$, $O_2$ emerges due to the QCD corrections, $O_3, \ldots, O_6$ arise from the QCD-penguin diagrams and operator mixing due to QCD corrections, and $O_7, \ldots, O_{10}$ arise from the electroweak-penguin diagrams [261]. In spite of QCD corrections, the dimension 6 four-quark operators from our analysis in Eq. (26) and Eq. (25) after integrating out weak gauge bosons contains $O^{u,c}_1,\ O_3,\ O_5$, Eq. (24) contains $\frac{1}{\sqrt{2}}viC^{13}_{qdF\varphi}(\bar{d}_L\sigma^{\mu\nu}T^Ab_R -$

$\bar{b}_R\sigma^{\mu\nu}T^A d_L)G^A_{\mu\nu}$ which is the same form as $O_g$ that can generate quark chromo-electric dipole moment. $H_new$ contains more mass-dimension-6 operators from our analysis which are

$$\frac{v^3}{2\sqrt{2}} i\mathrm{Im}(C^{13}_{qd\varphi^3})(\bar{q}_{Lq}b_R - \bar{b}_R q_{Lq}), \tag{131}$$

$$\frac{1}{\sqrt{2}} v i\mathrm{Im}C^{13}_{qdF\varphi}(\bar{q}_{Lq}\sigma^{\mu\nu}b_R - \bar{b}_R\sigma^{\mu\nu}q_{Lq})F_{\mu\nu}, \tag{132}$$

$$i\mathrm{Im}C^{q3pp}_{d^4}\left[(\bar{q}_{Rq}\gamma_\mu b_R)(\bar{d}_{Rp}\gamma^\mu d_{Rp}) - (\bar{b}_R\gamma_\mu d_{Rq})(\bar{d}_{Rp}\gamma^\mu d_{Rp})\right], \tag{133}$$

$$i\mathrm{Im}C^{(1)ppq3}_{u^2d^2}\left[(\bar{u}_{Rp}\gamma^\mu u_{Rp})(\bar{d}_{Rq}\gamma_\mu b_R) - (\bar{u}_{Rp}\gamma^\mu u_{Rp})(\bar{b}_R\gamma_\mu d_{Rq})\right], \tag{134}$$

$$i\mathrm{Im}C^{(8)ppq3}_{u^2d^2}\left[(\bar{u}_{Rp}\gamma^\mu T^A u_{Rr})(\bar{d}_{Rq}\gamma_\mu T^A b_R) - (\bar{u}_{Rr}\gamma^\mu T^A u_{Rp})(\bar{b}_R\gamma_\mu T^A d_{Rq})\right], \tag{135}$$

$$i\mathrm{Im}C^{(8)q3pp}_{q^2u^2}\left[(\bar{d}_{Lq}\gamma_\mu T^A b_{Lq})(\bar{u}_{Rp}\gamma^\mu T^A u_{Rp}) - (\bar{b}_L\gamma_\mu T^A d_{Lq})(\bar{u}_{Rp}\gamma^\mu T^A u_{Rp})\right], \tag{136}$$

$$i\mathrm{Im}C^{(1)ppq3}_{q^2d^2}\left[(\bar{u}_{Lp}\gamma^\mu u_{Lr})(\bar{d}_{Rq}\gamma_\mu b_R) - (\bar{u}_{Lr}\gamma^\mu u_{Lp})(\bar{b}_R\gamma^\mu T^A d_{Rq})\right], \tag{137}$$

$$i\mathrm{Im}C^{(8)ppq3}_{q^2d^2}\left[(\bar{u}_{Lp}\gamma_\mu T^A u_{Lr})(\bar{d}_{Rq}\gamma^\mu T^A b_R) - (\bar{u}_{Lr}\gamma_\mu T^A u_{Lp})(\bar{b}_R\gamma^\mu T^A d_{Rq})\right] \tag{138}$$

$$i\mathrm{Im}C^{(8)q3pp}_{q^2d^2}\left[(\bar{d}_{Lq}\gamma_\mu T^A b_L)(\bar{d}_{Rp}\gamma^\mu T^A b_{Rp}) - (\bar{b}_L\gamma_\mu T^A d_{Lq})(\bar{d}_{Rp}\gamma^\mu T^A b_{Rp})\right], \tag{139}$$

$$i\mathrm{Im}C^{(1)q3pp}_{quqd}\left[(\bar{u}_{Lp}u_{Lp})(\bar{d}_{Lq}b_R) - (\bar{u}_{Lp}u_{Lp})(\bar{b}_R d_{Lq})\right], \tag{140}$$

$$i\mathrm{Im}C^{(8)q3pp}_{quqd}\left[(\bar{u}_{Lp}T^A u_{Lp})(\bar{d}_{Lq}T^A b_R) - (\bar{u}_{Lp}T^A u_{Lp})(\bar{b}_R T^A d_{Lq})\right], \tag{141}$$

$$i\mathrm{Im}C^{(1)qpp3}_{quqd}\left[(\bar{d}_{Lq}u_{Rr})(\bar{u}_{Ls}b_R) - (\bar{u}_{Rr}d_{Lq})(\bar{b}_R u_{Ls})\right], \tag{142}$$

$$i\mathrm{Im}C^{(8)qpp3}_{quqd}\left[(\bar{d}_{Lq}T^A u_{Rp})(\bar{u}_{Lp}T^A b_R) - (\bar{u}_{Rp}T^A d_{Lq})(\bar{b}_R T^A u_{Lp})\right], \tag{143}$$

$$\begin{aligned}&-\frac{4v^2G_F}{\sqrt{2}} i\mathrm{Im}C^{q3}_{d^2\varphi^2D}g_Z\left[(\bar{d}_{Rq}\gamma_\mu b_R) - (\bar{b}_R\gamma_\mu d_{Rq})\right]\times\\ &\left[\bar{u}_{Lp}\gamma_\mu(\frac{1}{2}-\frac{2}{3}s_w^2)u_{Lp} + \bar{d}_{Lp}\gamma_\mu(-\frac{1}{2}+\frac{1}{3}s_w^2)d_{Lp}\right.\\ &\left.+\bar{u}_{Rp}\gamma_\mu(-\frac{2}{3}s_w^2)u_{Rp} + \bar{d}_{Rp}\gamma_\mu(\frac{1}{3}s_w^2)d_{Rp}\right],\end{aligned} \tag{144}$$

$$\begin{aligned}&\frac{2v^2G_F}{\sqrt{2}} i\mathrm{Im}C^{p3}_{ud\varphi^2D}g\left[(\bar{u}_{Rp}\gamma^\mu b_R)\left(\bar{d}_{Lq}\gamma_\mu V^*_{rd}u_{Lr}\right)\right.\\ &\left.-\left(\bar{b}_R\gamma^\mu u_{Rp}\right)(\bar{u}_{Lr}\gamma_\mu V_{rd}d_{Lq})\right],\end{aligned} \tag{145}$$

$$\begin{aligned}&\frac{2v^2G_F}{\sqrt{2}} i\mathrm{Im}C^{pq}_{ud\varphi^2D}g\left[(\bar{u}_{Rp}\gamma^\mu d_{Rq})\left(\bar{b}_L\gamma_\mu V^*_{rb}u_{Lr}\right)\right.\\ &\left.-\left(\bar{d}_{Rq}\gamma^\mu u_{Rp}\right)(\bar{u}_{Lr}\gamma_\mu V_{rb}b_L)\right].\end{aligned} \tag{146}$$

In the following we will show operators from SMEFT contributing to $b \to d$ P- and CP-violating processes like $B^0(\overline{B}^0) \to \pi^+\pi^-$ as well as C- and CP-violating processes like $B^0(\bar{B}^0) \to \pi^+\pi^-\pi^0$ through $\rho\pi$ or $\sigma\pi$, where $d_1 \equiv d$ and $d_3 \equiv b$. The P-odd and CP-odd operators that can contribute to $b \to d$ transition are

$$\mathcal{L}_{\psi^2\varphi^3}: \qquad \frac{v^3}{4\sqrt{2}} i\mathrm{Im}(C^{pr}_{qd\varphi^3})(\bar{d}_1\gamma_5 d_3 + \bar{d}_3\gamma_5 d_1), \tag{147}$$

$$\mathcal{L}_{\psi^2 X\varphi}: \qquad \frac{1}{2\sqrt{2}} v i\mathrm{Im}(C^{pr}_{qdG\varphi})(\bar{d}_1\sigma^{\mu\nu}\gamma_5 T^A d_3 + \bar{d}_3\sigma^{\mu\nu}\gamma_5 T^A d_1) G^A_{\mu\nu}, \tag{148}$$

$$\frac{1}{2\sqrt{2}} v i C^{pr}_{qdF\varphi}(\bar{d}_1\sigma^{\mu\nu}\gamma_5 d_3 + \bar{d}_3\sigma^{\mu\nu}\gamma_5 d_1) F_{\mu\nu}, \tag{149}$$

$\mathcal{L}_{\psi^4}$ :

$$\frac{1}{4} i\mathrm{Im}(-C^{(1)13pp}_{q^4} - C^{(3)13pp}_{q^4} + C^{13pp}_{d^4} - C^{(1)13pp}_{q^2d^2}) \left[(\bar{d}_1\gamma_\mu\gamma_5 d_3)(\bar{d}_p\gamma^\mu d_p) - (\bar{d}_p\gamma^\mu d_p)(\bar{d}_3\gamma_\mu\gamma_5 d_1)\right], \tag{150}$$

$$\frac{1}{4} i\mathrm{Im}(-C^{(1)13pp}_{q^4} - C^{(3)13pp}_{q^4} + C^{13pp}_{d^4} + C^{(1)13pp}_{q^2d^2}) \left[(\bar{d}_1\gamma_\mu d_3)(\bar{d}_p\gamma^\mu\gamma_5 d_p) - (\bar{d}_p\gamma^\mu\gamma_5 d_p)(\bar{d}_3\gamma_\mu d_1)\right], \tag{151}$$

$$\frac{1}{4} i\mathrm{Im}(-C^{(1)1pp3}_{q^4} - C^{(3)1pp3}_{q^4} + C^{1pp3}_{d^4} - C^{(1)1pp3}_{q^2d^2}) \left[(\bar{d}_1\gamma_\mu\gamma_5 d_p)(\bar{d}_p\gamma^\mu d_3) - (\bar{d}_3\gamma^\mu d_p)(\bar{d}_p\gamma_\mu\gamma_5 d_1)\right], \tag{152}$$

$$\frac{1}{4} i\mathrm{Im}(-C^{(1)1pp3}_{q^4} - C^{(3)1pp3}_{q^4} + C^{1pp3}_{d^4} + C^{(1)1pp3}_{q^2d^2}) \left[(\bar{d}_1\gamma_\mu d_p)(\bar{d}_p\gamma^\mu\gamma_5 d_3) - (\bar{d}_3\gamma^\mu\gamma_5 d_p)(\bar{d}_p\gamma_\mu d_1)\right], \tag{153}$$

$$\frac{1}{4} i\mathrm{Im}\left(-C^{(1)pp13}_{q^4} + C^{(3)pp13}_{q^4} + C^{(1)31pp}_{q^4} - C^{(3)31pp}_{q^4} + C^{(1)pp13}_{u^2d^2} + C^{(1)31pp}_{q^2u^2} - C^{(1)pp13}_{q^2d^2}\right) \times \left[(\bar{u}_p\gamma_\mu\gamma_5 u_p)(\bar{d}_1\gamma^\mu d_3) - (\bar{d}_3\gamma^\mu d_1)(\bar{u}_p\gamma_\mu\gamma_5 u_p)\right], \tag{154}$$

$$\frac{1}{4} i\mathrm{Im}\left(-C^{(1)pp13}_{q^4} + C^{(3)pp13}_{q^4} + C^{(1)31pp}_{q^4} - C^{(3)31pp}_{q^4} + C^{(1)pp13}_{u^2d^2} - C^{(1)31pp}_{q^2u^2} + C^{(1)pp13}_{q^2d^2}\right) \times \left[(\bar{u}_p\gamma_\mu u_p)(\bar{d}_1\gamma^\mu\gamma_5 d_3) - (\bar{d}_1\gamma^\mu\gamma_5 d_s)(\bar{u}_r\gamma_\mu u_p)\right], \tag{155}$$

$$-\frac{1}{4} i\mathrm{Im}(C^{(8)13pp}_{q^2d^2} \left[(\bar{d}_1\gamma_\mu\gamma_5 T^A d_3)(\bar{d}_p\gamma^\mu T^A d_p) - (\bar{d}_p\gamma^\mu T^A d_p)(\bar{d}_3\gamma_\mu\gamma_5 T^A d_1)\right], \tag{156}$$

$$\frac{1}{4} i\mathrm{Im}(C^{(8)13pp}_{q^2d^2} \left[(\bar{d}_1\gamma_\mu T^A d_3)(\bar{d}_p\gamma^\mu\gamma_5 T^A d_p) - (\bar{d}_p\gamma^\mu\gamma_5 T^A d_p)(\bar{d}_3\gamma_\mu T^A d_1)\right], \tag{157}$$

$$-\frac{1}{4} i\mathrm{Im}(C^{(8)1pp3}_{q^2d^2} \left[(\bar{d}_1\gamma_\mu\gamma_5 T^A d_p)(\bar{d}_p\gamma^\mu T^A d_3) - (\bar{d}_3\gamma^\mu T^A d_p)(\bar{d}_p\gamma_\mu\gamma_5 T^A d_1)\right], \tag{158}$$

$$\frac{1}{4} i\mathrm{Im}(C^{(8)1pp3}_{q^2d^2} \left[(\bar{d}_1\gamma_\mu T^A d_p)(\bar{d}_p\gamma^\mu\gamma_5 T^A d_3) - (\bar{d}_3\gamma^\mu\gamma_5 T^A d_p)(\bar{d}_p\gamma_\mu T^A d_1)\right], \tag{159}$$

$$\frac{1}{4} i\mathrm{Im}(C^{(8)pp13}_{u^2d^2} + C^{(8)31pp}_{q^2u^2} - C^{(8)pp13}_{q^2d^2}) \left[(\bar{u}_p\gamma_\mu\gamma_5 T^A u_p)(\bar{d}_1\gamma^\mu T^A d_3) - (\bar{d}_3\gamma^\mu T^A d_1)(\bar{u}_p\gamma_\mu\gamma_5 T^A u_p)\right], \tag{160}$$

$$\frac{1}{4} i\mathrm{Im}(C^{(8)pp13}_{u^2d^2} - C^{(8)31pp}_{q^2u^2} + C^{(8)pp13}_{q^2d^2}) \left[(\bar{u}_p\gamma_\mu T^A u_p)(\bar{d}_1\gamma^\mu\gamma_5 T^A d_3) - (\bar{d}_3\gamma^\mu\gamma_5 T^A d_1)(\bar{u}_p\gamma_\mu T^A u_p)\right], \tag{161}$$

$$-\frac{1}{2} i\mathrm{Im}(C^{(3)p31p}_{q^4}) \left[(\bar{u}_p\gamma_\mu\gamma_5 d_3)(\bar{d}_1\gamma^\mu u_p) - (\bar{u}_p\gamma^\mu d_1)(\bar{d}_3\gamma_\mu\gamma_5 u_p)\right], \tag{162}$$

$$-\frac{1}{2} i\mathrm{Im}(C^{(3)p31p}_{q^4}) \left[(\bar{u}_p\gamma_\mu d_3)(\bar{d}_1\gamma^\mu\gamma_5 u_p) - (\bar{u}_p\gamma^\mu\gamma_5 d_1)(\bar{d}_3\gamma_\mu u_p)\right], \tag{163}$$

$$-\frac{1}{2}i\mathrm{Im}(C_{q^4}^{(3)1pp3})\left[(\bar{d}_1\gamma_\mu\gamma_5 u_p)(\bar{u}_p\gamma^\mu d_3) - (\bar{d}_3\gamma^\mu u_p)(\bar{u}_p\gamma_\mu\gamma_5 d_1)\right], \tag{164}$$

$$-\frac{1}{2}i\mathrm{Im}(C_{q^4}^{(3)1pp3})\left[(\bar{d}_1\gamma_\mu u_p)(\bar{u}_p\gamma^\mu\gamma_5 d_3) - (\bar{d}_3\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu d_1)\right], \tag{165}$$

$$\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(1)pp13})\left[(\bar{u}_p\gamma_5 u_p)(\bar{d}_1 d_3) + (\bar{d}_3 d_1)(\bar{u}_p\gamma_5 u_p)\right], \tag{166}$$

$$\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(1)pp13})\left[(\bar{u}_p u_r)(\bar{d}_1\gamma_5 d_3) + (\bar{d}_3\gamma_5 d_1)(\bar{u}_p u_p)\right], \tag{167}$$

$$-\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(1)1pp3})\left[(\bar{d}_1\gamma_5 u_p)(\bar{u}_s d_3) + (\bar{d}_3 u_p)(\bar{u}_p\gamma_5 d_1)\right], \tag{168}$$

$$-\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(1)prst})\left[(\bar{d}_1 u_p)(\bar{u}_p\gamma_5 d_3) + (\bar{d}_3\gamma_5 u_p)(\bar{u}_p d_1)\right], \tag{169}$$

$$\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(8)pp13})\left[(\bar{u}_p\gamma_5 T^A u_p)(\bar{d}_1 T^A d_3) + (\bar{d}_3 T^A d_1)(\bar{u}_p\gamma_5 T^A u_p)\right], \tag{170}$$

$$\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(8)pp13})\left[(\bar{u}_p T^A u_p)(\bar{d}_1\gamma_5 T^A d_3) + (\bar{d}_3\gamma_5 T^A d_1)(\bar{u}_p T^A u_p)\right], \tag{171}$$

$$-\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(8)1pp3})\left[(\bar{d}_1\gamma_5 T^A u_p)(\bar{u}_p T^A d_3) + (\bar{d}_3 T^A u_p)(\bar{u}_p\gamma_5 T^A d_1)\right], \tag{172}$$

$$-\frac{1}{4}i\mathrm{Im}(C_{quqd}^{(8)1pp3})\left[(\bar{d}_1 T^A u_p)(\bar{u}_p T^A\gamma_5 d_3) + (\bar{d}_3\gamma_5 T^A u_p)(\bar{u}_p T^A d_1)\right]. \tag{173}$$

$\mathcal{L}_{\psi^2\varphi^2 D}$ :

$$-\sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)13}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{(1)13}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{13}_{d^2\varphi^2 D})]g_Z \left\{(\bar{d}_1\gamma^\mu\gamma_5 d_3)\times \left[(\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_p\gamma_\mu u_p + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_r\gamma_\mu d_r\right] - \left[(\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_p\gamma_\mu u_p + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_r\gamma_\mu d_r\right](\bar{d}_3\gamma^\mu\gamma_5 d_1)\right\}, \tag{174}$$

$$-\sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)13}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{(1)13}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{pr}_{d^2\varphi^2 D})]g_Z \times \left\{(\bar{d}_1\gamma^\mu d_3)\frac{1}{4}\left(\bar{u}_p\gamma_\mu\gamma_5 u_p - \bar{d}_r\gamma_\mu\gamma_5 d_r\right) - \frac{1}{4}\left(\bar{u}_p\gamma_\mu\gamma_5 u_p - \bar{d}_r\gamma_\mu\gamma_5 d_r\right)(\bar{d}_3\gamma^\mu d_1)\right\}, \tag{175}$$

$$-\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)1r}_{q^2\varphi^2 D})g\left[(\bar{d}_1\gamma^\mu\gamma_5 u_r)(\bar{u}_r\gamma_\mu V_{r3}d_3) - (\bar{d}_3\gamma_\mu V^*_{r3}u_r)(\bar{u}_r\gamma^\mu\gamma_5 d_1)\right], \tag{176}$$

$$-\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)3r}_{q^2\varphi^2 D})g\left[(\bar{d}_3\gamma^\mu\gamma_5 u_r)(\bar{u}_r\gamma_\mu V_{r1}d_1) - (\bar{d}_1\gamma_\mu V^*_{r1}u_r)(\bar{u}_r\gamma^\mu\gamma_5 d_3)\right], \tag{177}$$

$$-\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)p3}_{q^2\varphi^2 D})g\left[(\bar{u}_p\gamma^\mu\gamma_5 d_3)(\bar{d}_1\gamma_\mu V^*_{p1}u_p) - (\bar{u}_p\gamma_\mu V_{p1}d_1)(\bar{d}_3\gamma^\mu\gamma_5 u_p)\right], \tag{178}$$

$$-\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)p1}_{q^2\varphi^2 D})g\left[(\bar{u}_p\gamma^\mu\gamma_5 d_1)(\bar{d}_3\gamma_\mu V^*_{p3}u_p) - (\bar{u}_p\gamma_\mu V_{p3}d_3)(\bar{d}_1\gamma^\mu\gamma_5 u_p)\right], \tag{179}$$

$$-\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)1r}_{q^2\varphi^2 D})g\left[(\bar{d}_1\gamma^\mu u_r)(\bar{u}_r\gamma_\mu\gamma_5 V_{r3}d_3) - (\bar{d}_3\gamma_\mu V^*_{r3}u_r)(\bar{u}_r\gamma^\mu\gamma_5 d_1)\right], \tag{180}$$

$$-\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)3r}_{q^2\varphi^2 D})g\left[(\bar{d}_3\gamma^\mu u_r)(\bar{u}_r\gamma_\mu\gamma_5 V_{r1}d_1) - (\bar{d}_1\gamma_\mu V^*_{r1}u_r)(\bar{u}_r\gamma^\mu\gamma_5 d_3)\right], \tag{181}$$

$$-\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)p3}_{q^2\varphi^2 D})g\left[(\bar{u}_p\gamma^\mu\gamma_5 d_3)(\bar{d}_1\gamma_\mu V^*_{p1}u_p) - (\bar{u}_p\gamma_\mu V_{p1}d_1)(\bar{d}_3\gamma^\mu\gamma_5 u_p)\right], \tag{182}$$

$$-\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)p1}_{q^2\varphi^2 D})g\left[(\bar{u}_p\gamma^\mu\gamma_5 d_1)(\bar{d}_3\gamma_\mu V^*_{p3}u_p) - (\bar{u}_p\gamma_\mu V_{p3}d_3)(\bar{d}_1\gamma^\mu\gamma_5 u_p)\right], \tag{183}$$

$$\frac{v^2 G_F}{4} i\mathrm{Im}(C^{p3}_{ud\varphi^2 D})g\left[(\bar{u}_p\gamma^\mu\gamma_5 d_3)(\bar{d}_1\gamma_\mu V^*_{p1}u_p) - (\bar{u}_p\gamma_\mu V_{p1}d_1)(\bar{d}_3\gamma^\mu\gamma_5 u_p) - (\bar{u}_p\gamma^\mu d_3)(\bar{d}_1\gamma_\mu\gamma_5 V^*_{p1}u_p) + (\bar{u}_p\gamma_\mu\gamma_5 V_{p1}d_1)(\bar{d}_3\gamma^\mu u_p)\right], \tag{184}$$

$$\frac{v^2 G_F}{4} i\mathrm{Im}(C^{p1}_{ud\varphi^2 D})g\left[(\bar{u}_p\gamma^\mu\gamma_5 d_1)(\bar{d}_3\gamma_\mu V^*_{p3}u_p) - (\bar{u}_p\gamma_\mu V_{p3}d_3)(\bar{d}_1\gamma^\mu\gamma_5 u_p) - (\bar{u}_p\gamma^\mu d_1)(\bar{d}_3\gamma_\mu\gamma_5 V^*_{p3}u_p) + (\bar{u}_p\gamma_\mu\gamma_5 V_{p3}d_3)(\bar{d}_1\gamma^\mu u_p)\right]. \tag{185}$$

The C-odd and CP-odd operators that can contribute to $b \to d$ transition are

$$\mathcal{L}_{\psi^2\varphi^3}: \quad \frac{v^3}{4\sqrt{2}} i\mathrm{Im}(C^{13}_{qd\varphi^3})(\bar{d}_1 d_3 - \bar{d}_3 d_1), \tag{186}$$

$$\mathcal{L}_{\psi^2 X\varphi}: \quad \frac{1}{2\sqrt{2}} vi\mathrm{Im}(C^{13}_{qdG\varphi})(\bar{d}_1\sigma^{\mu\nu}T^A d_3 - \bar{d}_3\sigma^{\mu\nu}T^A d_1)G^A_{\mu\nu} \tag{187}$$

$$\frac{1}{2\sqrt{2}} vi C^{13}_{qdF\varphi}(\bar{d}_1\sigma^{\mu\nu}d_3 - \bar{d}_3\sigma^{\mu\nu}d_1)F_{\mu\nu} \tag{188}$$

$$\mathcal{L}_{\psi^4}: \quad \frac{1}{4} i\mathrm{Im}(C_{q^4}^{(1)13pp} + C_{q^4}^{(3)13pp} + C_{d^4}^{13pp} + C_{q^2d^2}^{(1)13pp}) \left[(\bar{d}_1\gamma_\mu d_3)(\bar{d}_p\gamma^\mu d_p) - (\bar{d}_p\gamma^\mu d_p)(\bar{d}_3\gamma_\mu d_1)\right], \tag{189}$$

$$\frac{1}{4} i\mathrm{Im}(C_{q^4}^{(1)13pp} + C_{q^4}^{(3)13pp} + C_{d^4}^{13pp} - C_{q^2d^2}^{(1)13pp}) \left[(\bar{d}_1\gamma_\mu\gamma_5 d_3)(\bar{d}_p\gamma^\mu\gamma_5 d_p) - (\bar{d}_p\gamma^\mu\gamma_5 d_p)(\bar{d}_3\gamma_\mu\gamma_5 d_1)\right], \tag{190}$$

$$\frac{1}{4} i\mathrm{Im}(C_{q^4}^{(1)1pp3} + C_{q^4}^{(3)1pp3} + C_{d^4}^{1pp3} + C_{q^2d^2}^{(1)1pp3}) \left[(\bar{d}_1\gamma_\mu d_p)(\bar{d}_p\gamma^\mu d_3) - (\bar{d}_3\gamma^\mu d_p)(\bar{d}_p\gamma_\mu d_1)\right], \tag{191}$$

$$\frac{1}{4} i\mathrm{Im}(C_{q^4}^{(1)1pp3} + C_{q^4}^{(3)1pp3} + C_{d^4}^{1pp3} - C_{q^2d^2}^{(1)1pp3}) \left[(\bar{d}_1\gamma_\mu\gamma_5 d_p)(\bar{d}_p\gamma^\mu\gamma_5 d_3) - (\bar{d}_3\gamma^\mu\gamma_5 d_p)(\bar{d}_p\gamma_\mu\gamma_5 d_1)\right], \tag{192}$$

$$\frac{1}{4} i\mathrm{Im}\left(C_{q^4}^{(1)pp13} - C_{q^2u^2}^{(3)pp13} - C_{q^4}^{(1)31pp} + C_{q^4}^{(3)31pp} + C_{u^2d^2}^{(1)pp13} - C_{q^2u^2}^{(1)31pp} + C_{q^2d^2}^{(1)pp13}\right) \left[(\bar{u}_p\gamma_\mu u_p)(\bar{d}_1\gamma^\mu\ d_3) - (\bar{d}_3\gamma^\mu d_1)(\bar{u}_p\gamma_\mu u_p)\right], \tag{193}$$

$$\frac{1}{4} i\mathrm{Im}\left(C_{q^4}^{(1)pp13} - C_{q^4}^{(3)pp13} - C_{q^4}^{(1)31pp} + C_{q^4}^{(3)31pp} + C_{u^2d^2}^{(1)pp13} + C_{q^2u^2}^{(1)31pp} - C_{q^2d^2}^{(1)pp13}\right) \left[(\bar{u}_p\gamma_\mu\gamma_5 u_p)(\bar{d}_1\gamma^\mu\gamma_5 d_3) - (\bar{d}_3\gamma^\mu\gamma_5 d_1)(\bar{u}_p\gamma_\mu\gamma_5 u_p)\right], \tag{194}$$

$$\frac{1}{4} i\mathrm{Im}(C_{q^2d^2}^{(8)13pp}) \left[(\bar{d}_1\gamma_\mu T^A d_3)(\bar{d}_p\gamma^\mu T^A d_p) - (\bar{d}_p\gamma^\mu T^A d_p)(\bar{d}_3\gamma_\mu T^A d_1)\right], \tag{195}$$

$$-\frac{1}{4} i\mathrm{Im}(C_{q^2d^2}^{(8)13pp}) \left[(\bar{d}_1\gamma_\mu\gamma_5 T^A d_3)(\bar{d}_p\gamma^\mu\gamma_5 T^A d_p) - (\bar{d}_p\gamma^\mu\gamma_5 T^A d_p)(\bar{d}_3\gamma_\mu\gamma_5 T^A d_1)\right], \tag{196}$$

$$\frac{1}{4} i\mathrm{Im}(C_{q^2d^2}^{(8)1pp3}) \left[(\bar{d}_1\gamma_\mu T^A d_p)(\bar{d}_p\gamma^\mu T^A d_3) - (\bar{d}_3\gamma^\mu T^A d_p)(\bar{d}_p\gamma_\mu T^A d_1)\right], \tag{197}$$

$$-\frac{1}{4} i\mathrm{Im}(C_{q^2d^2}^{(8)1pp3}) \left[(\bar{d}_1\gamma_\mu\gamma_5 T^A d_p)(\bar{d}_p\gamma^\mu\gamma_5 T^A d_3) - (\bar{d}_3\gamma^\mu\gamma_5 T^A d_p)(\bar{d}_p\gamma_\mu\gamma_5 T^A d_1)\right], \tag{198}$$

$$\frac{1}{4} i\mathrm{Im}(C_{u^2d^2}^{(8)pp13} - C_{q^2u^2}^{(8)31pp} + C_{q^2d^2}^{(8)pp13}) \left[(\bar{u}_p\gamma_\mu T^A u_p)(\bar{d}_1\gamma^\mu T^A d_3) - (\bar{d}_3\gamma^\mu T^A d_1)(\bar{u}_p\gamma_\mu T^A u_p)\right], \tag{199}$$

$$\frac{1}{4} i\mathrm{Im}(C_{u^2d^2}^{(8)pp13} + C_{q^2u^2}^{(8)31pp} - C_{q^2d^2}^{(8)pp13}) \left[(\bar{u}_p\gamma_\mu\gamma_5 T^A u_p)(\bar{d}_1\gamma^\mu\gamma_5 T^A d_3) - (\bar{d}_3\gamma^\mu\gamma_5 T^A d_1)(\bar{u}_p\gamma_\mu\gamma_5 T^A u_p)\right], \tag{200}$$

$$\frac{1}{2} i\mathrm{Im}(C_{q^4}^{(3)p31p}) \left[(\bar{u}_p\gamma_\mu d_3)(\bar{d}_1\gamma^\mu u_p) - (\bar{u}_p\gamma^\mu d_1)(\bar{d}_3\gamma_\mu u_p)\right], \tag{201}$$

$$\frac{1}{2} i\mathrm{Im}(C_{q^4}^{(3)p31p}) \left[(\bar{u}_p\gamma_\mu\gamma_5 d_3)(\bar{d}_1\gamma^\mu\gamma_5 u_p) - (\bar{u}_p\gamma^\mu\gamma_5 d_1)(\bar{d}_3\gamma_\mu\gamma_5 u_p)\right], \tag{202}$$

$$\frac{1}{2} i\mathrm{Im}(C_{q^4}^{(3)1pp3}) \left[(\bar{d}_1\gamma_\mu u_p)(\bar{u}_p\gamma^\mu d_3) - (\bar{d}_3\gamma^\mu u_p)(\bar{u}_p\gamma_\mu d_1)\right], \tag{203}$$

$$\frac{1}{2} i\mathrm{Im}(C_{q^4}^{(3)1pp3}) \left[(\bar{d}_1\gamma_\mu\gamma_5 u_p)(\bar{u}_p\gamma^\mu\gamma_5 d_3) - (\bar{d}_3\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu\gamma_5 d_1)\right], \tag{204}$$

$$\frac{1}{4}i\mathrm{Im}(C^{(1)pp13}_{quqd})\left[(\bar{u}_p u_p)(\bar{d}_1 d_3)-(\bar{d}_3 d_1)(\bar{u}_p u_p)\right], \tag{205}$$

$$\frac{1}{4}i\mathrm{Im}(C^{(1)pp13}_{quqd})\left[(\bar{u}_p \gamma_5 u_p)(\bar{d}_1 \gamma_5 d_3)-(\bar{d}_3 \gamma_5 d_1)(\bar{u}_p \gamma_5 u_p)\right], \tag{206}$$

$$-\frac{1}{4}i\mathrm{Im}(C^{(1)1pp3}_{quqd})\left[(\bar{d}_1 u_p)(\bar{u}_p d_3)-(\bar{d}_3 u_p)(\bar{u}_p d_1)\right], \tag{207}$$

$$-\frac{1}{4}i\mathrm{Im}(C^{(1)1pp3}_{quqd})\left[(\bar{d}_1 \gamma_5 u_p)(\bar{u}_p \gamma_5 d_3)-(\bar{d}_3 \gamma_5 u_p)(\bar{u}_p \gamma_5 d_1)\right], \tag{208}$$

$$\frac{1}{4}i\mathrm{Im}(C^{(8)pp13}_{quqd})\left[(\bar{u}_p T^A u_p)(\bar{d}_1 T^A d_3)-(\bar{d}_3 T^A d_1)(\bar{u}_p T^A u_p)\right], \tag{209}$$

$$\frac{1}{4}i\mathrm{Im}(C^{(8)pp13}_{quqd})\left[(\bar{u}_p \gamma_5 T^A u_p)(\bar{d}_1 \gamma_5 T^A d_3)-(\bar{d}_3 \gamma_5 T^A d_1)(\bar{u}_p \gamma_5 T^A u_p)\right], \tag{210}$$

$$-\frac{1}{4}i\mathrm{Im}(C^{(8)1pp3}_{quqd})\left[(\bar{d}_1 T^A u_p)(\bar{u}_p T^A d_3)-(\bar{d}_3 T^A u_p)(\bar{u}_p T^A d_1)\right], \tag{211}$$

$$\frac{1}{4}i\mathrm{Im}(C^{(8)1pp3}_{quqd})\left[(\bar{d}_1 \gamma_5 T^A u_p)(\bar{u}_p T^A \gamma_5 d_3)-(\bar{d}_3 \gamma_5 T^A u_p)(\bar{u}_p \gamma_5 T^A d_1)\right]. \tag{212}$$

$$\mathcal{L}_{\psi^2\varphi^2 D}:$$

$$\sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)13}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{(1)13}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{13}_{d^2\varphi^2 D})] g_Z \left\{ (\bar{d}_1\gamma^\mu d_3)\times \right.$$
$$\left[ (\frac{1}{4} - \frac{2}{3}s_w^2)\bar{u}_p\gamma_\mu u_p + (-\frac{1}{4} + \frac{1}{3}s_w^2)\bar{d}_r\gamma_\mu d_r \right]$$
$$\left. - \left[ (\frac{1}{4} - \frac{2}{3}s_w^2)\bar{u}_p\gamma_\mu u_p + (-\frac{1}{4} + \frac{1}{3}s_w^2)\bar{d}_r\gamma_\mu d_r \right] (\bar{d}_3\gamma^\mu d_1) \right\}, \tag{213}$$

$$\sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)13}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{(1)13}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{pr}_{d^2\varphi^2 D})] g_Z \left\{ (\bar{d}_1\gamma^\mu\gamma_5 d_3)\times \right.$$
$$\left. \frac{1}{4}\left(\bar{u}_p\gamma_\mu\gamma_5 u_p - \bar{d}_r\gamma_\mu\gamma_5 d_r\right) - \frac{1}{4}\left(\bar{u}_p\gamma_\mu\gamma_5 u_p - \bar{d}_r\gamma_\mu\gamma_5 d_r\right)(\bar{d}_3\gamma^\mu\gamma_5 d_1) \right\}, \tag{214}$$

$$\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)1p}_{q^2\varphi^2 D}) g \left[ (\bar{d}_1\gamma^\mu u_p)(\bar{u}_p\gamma_\mu V_{p3} d_3) - (\bar{d}_3\gamma_\mu V^*_{p3} u_p)(\bar{u}_p\gamma^\mu d_1) \right], \tag{215}$$

$$\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)3p}_{q^2\varphi^2 D}) g \left[ (\bar{d}_3\gamma^\mu u_p)(\bar{u}_p\gamma_\mu V_{p1} d_1) - (\bar{d}_1\gamma_\mu V^*_{p1} u_p)(\bar{u}_p\gamma^\mu d_3) \right], \tag{216}$$

$$\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)p3}_{q^2\varphi^2 D}) g \left[ (\bar{u}_p\gamma^\mu d_3)(\bar{d}_1\gamma_\mu V^*_{p1} u_p) - (\bar{u}_p\gamma_\mu V_{p1} d_1)(\bar{d}_3\gamma^\mu u_p) \right], \tag{217}$$

$$\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)p1}_{q^2\varphi^2 D}) g \left[ (\bar{u}_p\gamma^\mu d_1)(\bar{d}_3\gamma_\mu V^*_{p3} u_p) - (\bar{u}_p\gamma_\mu V_{p3} d_3)(\bar{d}_1\gamma^\mu u_p) \right], \tag{218}$$

$$\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)1p}_{q^2\varphi^2 D}) g \left[ (\bar{d}_1\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu\gamma_5 V_{p3} d_3) \right.$$
$$\left. -(\bar{d}_3\gamma_\mu\gamma_5 V^*_{p3} u_p)(\bar{u}_p\gamma^\mu\gamma_5 d_1) \right], \tag{219}$$

$$\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)3p}_{q^2\varphi^2 D}) g \left[ (\bar{d}_3\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu\gamma_5 V_{p1} d_1) \right.$$
$$\left. -(\bar{d}_1\gamma_\mu\gamma_5 V^*_{p1} u_p)(\bar{u}_p\gamma^\mu\gamma_5 d_3) \right], \tag{220}$$

$$\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)p3}_{q^2\varphi^2 D}) g \left[ (\bar{u}_p\gamma^\mu\gamma_5 d_3)(\bar{d}_1\gamma_\mu\gamma_5 V^*_{p1} u_p) \right.$$
$$\left. -(\bar{u}_p\gamma_\mu\gamma_5 V_{sx} d_1)(\bar{d}_3\gamma^\mu\gamma_5 u_p) \right], \tag{221}$$

$$\frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)p1}_{q^2\varphi^2 D}) g \left[ (\bar{u}_p\gamma^\mu\gamma_5 d_1)(\bar{d}_3\gamma_\mu\gamma_5 V^*_{p3} u_p) \right.$$
$$\left. -(\bar{u}_p\gamma_\mu\gamma_5 V_{p3} d_3)(\bar{d}_1\gamma^\mu\gamma_5 u_p) \right], \tag{222}$$

$$\frac{v^2 G_F}{4} i\mathrm{Im}(C^{p3}_{ud\varphi^2 D}) g \left[ (\bar{u}_p\gamma^\mu d_3)(\bar{d}_1\gamma_\mu V^*_{p1} u_p) - (\bar{u}_p\gamma_\mu V_{p1} d_1)(\bar{d}_3\gamma^\mu u_p) \right.$$
$$\left. -(\bar{u}_p\gamma^\mu\gamma_5 d_3)(\bar{d}_1\gamma_\mu\gamma_5 V^*_{p1} u_p) + (\bar{u}_p\gamma_\mu\gamma_5 V_{p1} d_1)(\bar{d}_3\gamma^\mu\gamma_5 u_p) \right], \tag{223}$$

$$\frac{v^2 G_F}{4} i\mathrm{Im}(C^{p1}_{ud\varphi^2 D}) g \left[ (\bar{u}_p\gamma^\mu d_1)(\bar{d}_3\gamma_\mu V^*_{p3} u_p) - (\bar{u}_p\gamma_\mu V_{p3} d_3)(\bar{d}_1\gamma^\mu u_p) \right.$$
$$\left. -(\bar{u}_p\gamma^\mu\gamma_5 d_1)(\bar{d}_3\gamma_\mu\gamma_5 V^*_{p3} u_p) + (\bar{u}_p\gamma_\mu\gamma_5 V_{p3} d_3)(\bar{d}_1\gamma^\mu\gamma_5 u_p) \right]. \tag{224}$$

We see that the operators contributing to P- and CP-violating $b \to d$ decay like $B^0(\overline{B}^0) \to \pi^+\pi^0$ and C-and CP-violating process like $B^0(\bar{B}^0) \to \pi^+\pi^-\pi^0$, the operators are related either with the same factors or with factors composed of different linear combinations of low-energy coefficients.

### 7.6.2 $\eta \to \pi^+\pi^-\pi^0$ Decays

$\eta \to \pi^+\pi^-\pi^0$ decay can only occur via isospin breaking or $C$ and $CP$ violating processes [52]. It is well known that this decay is dominantly contributed through isospin breaking by 1 ($\Delta I = 1$) strong interaction. While the theoretical $C$ and $CP$ violating patterns have not been further investigated since 1966 [54, 56, 57]. The SM model $CP$ violating mechanism, $CKM$ matrix, does not have substantial contribution to this decay channel since $\eta \to \pi^+\pi^-\pi^0$ decay is flavor-conserving. In our recent work [92], we studied about the C and CP violating amplitudes with isospin $I = 0$ and $I = 2$ and we found that the $I = 2$ amplitude is much suppressed. In this section, we will look at the isospin structures of some of the C-odd and CP-odd operators we found that can contribute to $\eta \to \pi^+\pi^-\pi^0$.

The lowest dimensional $C$ and $CP$ odd operators from Eq. (45) would contribute to $\eta \to \pi^+\pi^-\pi^0$. According to $\eta \to \pi^+\pi^-\pi^0$ the theorem $C = -(-1)^I$ by Lee [52], the $C$ violating process contains $I = 0$ and $I = 2$ (which meanwhile breaks isospin by $\Delta I = 2$) for the final $\pi^+\pi^-\pi^0$ states. Thus we need to figure out the isospin structure of our lowest-mass-dimension flavor-conserving $C$ and $CP$ odd operators .

The quarks that act in $\eta \to \pi^+\pi^-\pi^0$ are $u$, $d$, and $s$. Strange quark $s$ has isospin 0. Up and down quarks can be treated as isospin doublet with $I = \frac{1}{2}$, and $I_3 = +\frac{1}{2}$ and $-\frac{1}{2}$, separately

$$q = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad \bar{q} = \begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}, \tag{225}$$

A system of quark-antiquark pair composed of $u$ and $d$ has isospin states $\theta_{(I,I_3)}$ [265]

$$\begin{cases} \theta_{(1,1)} = -u\bar{d} \\ \theta_{(1,0)} = \sqrt{\frac{1}{2}}(u\bar{u} - d\bar{d}) \\ \theta_{(1,-1)} = d\bar{u} \end{cases}$$
$$\theta_{(0,0)} = \sqrt{\tfrac{1}{2}}(u\bar{u} + d\bar{d}). \tag{226}$$

From the above equations, we also have

$$\begin{aligned} u\bar{u} &= \frac{1}{\sqrt{2}}(\theta_{(1,0)} + \theta_{(0,0)}), \\ d\bar{d} &= \frac{1}{\sqrt{2}}(\theta_{(0,0)} - \theta_{(1,0)}). \end{aligned} \tag{227}$$

We pick operators that can contribute to $\eta \to \pi^+\pi^-\pi^0$ from Eq. (44) and Eq. (45) showing below as examples to illustrate our idea:

$$
\begin{aligned}
& \frac{2v^2 G_F}{\sqrt{2}}\epsilon^{\mu\nu\alpha\beta}[2c_w s_w(C_{W^2\varphi^2}-C_{B^2\varphi^2})-C_{WB\varphi^2}({c_w}^2-{s_w}^2)]\partial_\alpha[-\frac{1}{2}(\bar{u}\gamma_\beta\gamma_5 u)+\frac{1}{2}(\bar{d}\gamma_\beta\gamma_5 d)]F_{\mu\nu}, \\
& \frac{2v^2 G_F}{\sqrt{2}}\epsilon^{\mu\nu\alpha\beta}[2c_w s_w(C_{W^2\varphi^2}-C_{B^2\varphi^2})-C_{WB\varphi^2}({c_w}^2-{s_w}^2)]\frac{1}{2}\partial_\alpha(\bar{s}\gamma_\beta\gamma_5 s)F_{\mu\nu}, \\
& \frac{G_F}{\sqrt{2}}vi[C^{11}_{quZ\varphi}(\bar{u}\sigma^{\mu\nu}\gamma_5 u)+C^{11}_{qdZ\varphi}(\bar{d}\sigma^{\mu\nu}\gamma_5 d)+C^{22}_{qdZ\varphi}(\bar{s}\sigma^{\mu\nu}\gamma_5 s)]\partial_\mu(-\bar{u}\gamma_\nu\gamma_5 u+\bar{d}\gamma_\nu\gamma_5 d+\bar{s}\gamma_\nu\gamma_5 s), \\
& \frac{2G_F}{\sqrt{2}}vi[\mathrm{Im}(C^{11}_{quW\varphi})-\mathrm{Im}(C^{11}_{qdW\varphi})]\left[(\bar{d}\sigma^{\mu\nu}u)\partial_\mu(\bar{u}\gamma_\nu V_{ud}d)-(\bar{u}\sigma^{\mu\nu}d)\partial_\mu(\bar{d}V^*_{ud}\gamma_\nu u)\right], \\
& \frac{2G_F}{\sqrt{2}}vi[\mathrm{Im}(C^{21}_{quW\varphi})-\mathrm{Im}(C^{12}_{qdW\varphi})]\left[(\bar{s}\sigma^{\mu\nu}u)\partial_\mu(\bar{u}\gamma_\nu V_{us}s)-(\bar{u}\sigma^{\mu\nu}s)\partial_\mu(\bar{s}V^*_{us}\gamma_\nu u)\right], \\
- & \frac{2G_F}{\sqrt{2}}vi[\mathrm{Im}(C^{11}_{quW\varphi})+\mathrm{Im}(C^{11}_{qdW\varphi})]\left[(\bar{d}\sigma^{\mu\nu}\gamma_5 u)\partial_\mu(\bar{u}\gamma_\nu\gamma_5 V_{ud}d)+(\bar{u}\sigma^{\mu\nu}\gamma_5 d)\partial_\mu(\bar{d}V^*_{ud}\gamma_\nu\gamma_5 u)\right], \\
- & \frac{2G_F}{\sqrt{2}}vi[\mathrm{Im}(C^{21}_{quW\varphi})+\mathrm{Im}(C^{12}_{qdW\varphi})]\left[(\bar{s}\sigma^{\mu\nu}\gamma_5 u)\partial_\mu(\bar{u}\gamma_\nu\gamma_5 V_{us}s)+(\bar{u}\sigma^{\mu\nu}\gamma_5 s)\partial_\mu(\bar{s}V^*_{us}\gamma_\nu\gamma_5 u)\right],
\end{aligned}
$$

Using Eq. (226) and Eq. (227), the isospin structures noted as $\vartheta_{\Delta|I|}$ from combinations of operators in Eq. (45) that may contribute to $\eta \to \pi^+\pi^-\pi^0$ are

$$\begin{aligned}
\vartheta_{|\Delta I|=0} &= \frac{2v^2G_F}{\sqrt{2}}\epsilon^{\mu\nu\alpha\beta}[2c_ws_w(C_{W^2\varphi^2}-C_{B^2\varphi^2})-C_{WB\varphi^2}({c_w}^2-{s_w}^2)]\times \\
&\quad \frac{1}{2}\partial_\alpha(\bar{s}\gamma_\beta\gamma_5 s)F_{\mu\nu}, \qquad (228)\\
\vartheta_{|\Delta I|=1} &= \frac{2v^2G_F}{\sqrt{2}}\epsilon^{\mu\nu\alpha\beta}[2c_ws_w(C_{W^2\varphi^2}-C_{B^2\varphi^2})-C_{WB\varphi^2}({c_w}^2-{s_w}^2)]\times \\
&\quad \partial_\alpha[-\frac{1}{2}(\bar{u}\gamma_\beta\gamma_5 u)+\frac{1}{2}(\bar{d}\gamma_\beta\gamma_5 d)]F_{\mu\nu}, \qquad (229)\\
\vartheta_{|\Delta I|=1} &= \frac{G_F}{\sqrt{2}}viC^{22}_{qdZ\varphi}(\bar{s}\sigma^{\mu\nu}\gamma_5 s)\partial_\mu(-\bar{u}\gamma_\nu\gamma_5 u+\bar{d}\gamma_\nu\gamma_5 d), \qquad (230)\\
\vartheta_{|\Delta I|=0} &= C^a_0\frac{G_F}{\sqrt{2}}vi[(\bar{u}\sigma^{\mu\nu}\gamma_5 u)+(\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(\bar{s}\gamma_\nu\gamma_5 s), \qquad (231)\\
\vartheta_{|\Delta I|=1} &= C^a_1\frac{G_F}{\sqrt{2}}vi[(\bar{u}\sigma^{\mu\nu}\gamma_5 u)-(\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(\bar{s}\gamma_\nu\gamma_5 s), \qquad (232)\\
\vartheta_{|\Delta I|=1} &= \frac{2G_F}{\sqrt{2}}vi[\mathrm{Im}(C^{11}_{quW\varphi})-\mathrm{Im}(C^{11}_{qdW\varphi})]\times \\
&\quad \left[(\bar{d}\sigma^{\mu\nu}u)\partial_\mu(\bar{u}\gamma_\nu V_{ud}d)-(\bar{u}\sigma^{\mu\nu}d)\partial_\mu(\bar{d}V^*_{ud}\gamma_\nu u)\right], \qquad (233)\\
\vartheta_{|\Delta I|=0} &= \frac{2G_F}{\sqrt{2}}vi[\mathrm{Im}(C^{21}_{quW\varphi})-\mathrm{Im}(C^{12}_{qdW\varphi})]\times \\
&\quad [(\bar{s}\sigma^{\mu\nu}u)\partial_\mu(\bar{u}\gamma_\nu V_{us}s)-(\bar{u}\sigma^{\mu\nu}s)\partial_\mu(\bar{s}V^*_{us}\gamma_\nu u)], \qquad (234)\\
\vartheta_{|\Delta I|=1} &= -\frac{2G_F}{\sqrt{2}}vi[\mathrm{Im}(C^{21}_{quW\varphi})+\mathrm{Im}(C^{rp}_{qdW\varphi})]\times \\
&\quad [(\bar{s}\sigma^{\mu\nu}\gamma_5 u)\partial_\mu(\bar{u}\gamma_\nu\gamma_5 V_{us}s)+(\bar{u}\sigma^{\mu\nu}\gamma_5 s)\partial_\mu(\bar{s}V^*_{us}\gamma_\nu\gamma_5 u)], \qquad (235)\\
\vartheta_{|\Delta I|=1} &= C^b_1\frac{G_F}{\sqrt{2}}vi[(\bar{u}\sigma^{\mu\nu}\gamma_5 u)+(\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(-\bar{u}\gamma_\nu\gamma_5 u+\bar{d}\gamma_\nu\gamma_5 d), \qquad (236)\\
\vartheta_{|\Delta I|=0} &= C^b_0\frac{G_F}{\sqrt{2}}vi\left\{[(\bar{u}\sigma^{\mu\nu}\gamma_5 u)-(\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(-\bar{u}\gamma_\nu\gamma_5 u+\bar{d}\gamma_\nu\gamma_5 d)\right. \\
&\quad \left.+\left[(\bar{d}\sigma^{\mu\nu}\gamma_5 u)\partial_\mu(\bar{u}\gamma_\nu\gamma_5 d)+(\bar{u}\sigma^{\mu\nu}\gamma_5 d)\partial_\mu(\bar{d}\gamma_\nu\gamma_5 u)\right]\right\}, \qquad (237)\\
\vartheta_{|\Delta I|=2} &= C^b_2\frac{G_F}{\sqrt{2}}vi\left\{[(\bar{u}\sigma^{\mu\nu}\gamma_5 u)-(\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(-\bar{u}\gamma_\nu\gamma_5 u+\bar{d}\gamma_\nu\gamma_5 d)\right. \\
&\quad \left.-2\left[(\bar{d}\sigma^{\mu\nu}\gamma_5 u)\partial_\mu(\bar{u}\gamma_\nu\gamma_5 d)+(\bar{u}\sigma^{\mu\nu}\gamma_5 d)\partial_\mu(\bar{d}\gamma_\nu\gamma_5 u)\right]\right\}, \qquad (238)
\end{aligned}$$

where

$$C_0^a = \frac{1}{2}\left(C^{11}_{quZ\varphi} + C^{11}_{qdZ\varphi}\right), \tag{239}$$

$$C_1^a = \frac{1}{2}\left(C^{11}_{quZ\varphi} - C^{11}_{qdZ\varphi}\right), \tag{240}$$

$$C_1^b = \frac{1}{2}\left(C^{11}_{quZ\varphi} + C^{11}_{qdZ\varphi}\right), \tag{241}$$

$$C_0^b = \frac{1}{3}\left\{C^{11}_{quZ\varphi} - C^{11}_{qdZ\varphi} - 2V_{ud}[\mathrm{Im}(C^{21}_{quW\varphi}) + \mathrm{Im}(C^{12}_{qdW\varphi})]\right\}, \tag{242}$$

$$C_2^b = \frac{1}{3}\left\{2[\mathrm{Im}(C^{21}_{quW\varphi}) + \mathrm{Im}(C^{12}_{qdW\varphi})] + \frac{1}{2}(C^{11}_{quZ\varphi} - \frac{3}{2}C^{11}_{qdZ\varphi})\right\}. \tag{243}$$

The operators with $|\Delta I| = 0$ and $|\Delta I| = 2$ can contribute directly to the C- and CP-violating amplitudes [92] with final states' total isospin to be $I = 0$ and $I = 2$, respectively. The operators with $\Delta I = 1$ can also contribute to the final state with total isospin $I = 0$ and $I = 2$ by adding an extra $|\Delta I| = 1$ factor, which could come from the SM that is proportional to $(m_u - m_d)$, or from electromagnetic source that is of order $\mathcal{O}(e)$.

### 7.7 Summary

With the motivation that C and CP violating mechanisms are not sufficiently studied and could give hint to new physics beyond the SM, we investigated the lowest mass-dimension C-odd and CP-odd operators from the SMEFT. We started from the energy scale high above the weak gauge boson masses, showed CP-odd operators of mass-dimension 6 SMEFT after electroweak symmetry is breaking and rotating the $SU_L(2)$ gauge fields to physical weak gauge boson fields. After integrating out the weak gauge bosons, we separated the P-odd and CP-odd, meanwhile C-odd and CP-odd operators and listed them respectively for flavor-changing and flavor-conserving interactions. We found that the lowest mass-dimension C-odd and CP-odd operators for flavor-changing processes is 6, while for flavor-conserving processes it is 8, some of which can be mass-dimension 6 in numerical effect. We found that the P-odd CP-odd operators and C-odd CP-odd ones in flavor-changing case are strong related, either with the same factors or with factors composed with different linear combinations from the same low-energy coefficients However they are very different and have little connection in flavor-conserving case.

We showed some applications of our results to some experimental probes like flavor-changing $B$ meson decay and $\eta \to \pi^+\pi^-\pi^0$ decay which is flavor-conserving. Our investigation of C and CP violating sources of flavor-conserving interactions are precedent and will open a new window to studies of C and CP violating flavor-conserving interactions.

## 7.8 Appendix

### 7.8.1 Discrete Symmetries — Definitions and Other Essentials

Here we give a compact list of the transformation properties of scalar fields, dirac fields and gauge fields etc. A more detailed discussion can be found in 2.3.

For scalar fields $\phi(t, \mathbf{x})$, we have the following transformation properties [80]

$$\mathbf{P}\phi(t,\mathbf{x})\mathbf{P}^{-1} = \exp(i\alpha_p)\phi(t,-\mathbf{x}), \tag{A.1}$$

$$\mathbf{C}\phi(t,\mathbf{x})\mathbf{C}^{-1} = \exp(i\alpha_c)\phi^\dagger(t,\mathbf{x}), \tag{A.2}$$

$$\mathbf{T}\phi(t,\mathbf{x})\mathbf{T}^{-1} = \exp(i\alpha_t)\phi(-t,\mathbf{x}), \tag{A.3}$$

where $\alpha_p$, $\alpha_c$ and $\alpha_t$ are arbitrary phases.

The discrete-symmetry transformations of a four-component fermion field $\psi(x)$ are given by [80, 84]

$$\mathbf{C}\psi(x)\mathbf{C}^{-1} = \eta_c C\gamma^0\psi^*(x) \equiv \eta_c i\gamma^2\psi^*(x) = \eta_c i(\bar{\psi}(x)\gamma^0\gamma^2)^T, \tag{A.4}$$

$$\mathbf{C}\bar{\psi}(x)\mathbf{C}^{-1} = \eta_c^* i\psi^T(x)\gamma^2\gamma^0 = \eta_c^* i(\gamma^0\gamma^2\psi(x))^T \tag{A.5}$$

$$\mathbf{P}\psi(t,\mathbf{x})\mathbf{P}^{-1} = \eta_p\gamma^0\psi(t,-\mathbf{x}), \tag{A.6}$$

$$\mathbf{P}\bar{\psi}(t,\mathbf{x})\mathbf{P}^{-1} = \eta_p^*\bar{\psi}(t,-\mathbf{x})\gamma^0, \tag{A.7}$$

$$\mathbf{T}\psi(t,\mathbf{x})\mathbf{T}^{-1} = \eta_t\gamma^1\gamma^3\psi(-t,\mathbf{x}), \tag{A.8}$$

$$\mathbf{T}\bar{\psi}(t,\mathbf{x})\mathbf{T}^{-1} = -\eta_t^*\bar{\psi}(-t,\mathbf{x})\gamma^1\gamma^3, \tag{A.9}$$

where $\eta_c$, $\eta_p$, and $\eta_t$ denote unimodular phase factors of the charge-conjugation $C$, parity $P$, and time-reversal $T$ transformations, respectively, and we have chosen the Dirac-Pauli representation for the gamma matrices. The transformation properties are consistent with those of Ref. [81] when setting $\eta_c = -1$, $\eta_p = +1$, $\eta_t = +1$, and with those of Ref. [80], though we have chosen a specific representation of the gamma matrices. More details can be found in Ref. [81, 80, 84].

For fermion bilinears, $\partial_\mu$, photon field $A_\mu$, and gluon field $G_\mu = G_\mu^A T^a$, their transformation properties are listed in Table 7.4, where $(-1)^\mu = +1$ for $\mu = 0$ and

Table 7.4: Transformation properties for fermion bilinears, $\partial_\mu$, $A_\mu$, and $G_\mu$

| | $\bar{\psi}\psi$ | $\bar{\psi}\gamma_5\psi$ | $\bar{\psi}\gamma^\mu\psi$ | $\bar{\psi}\gamma^\mu\gamma_5\psi$ | $\bar{\psi}\sigma^{\mu\nu}\psi$ | $\bar{\psi}\sigma^{\mu\nu}\gamma_5\psi$ | $\partial_\mu$ | $A_\mu/G_\mu$ |
|---|---|---|---|---|---|---|---|---|
| **P** | $+1$ | $-1$ | $(-1)^\mu$ | $-(-1)^\mu$ | $(-1)^\mu(-1)^\nu$ | $-(-1)^\mu(-1)^\nu$ | $(-1)^\mu$ | $(-1)^\mu$ |
| **T** | $+1$ | $+1$ | $(-1)^\mu$ | $(-1)^\mu$ | $-(-1)^\mu(-1)^\nu$ | $-(-1)^\mu(-1)^\nu$ | $-(-1)^\mu$ | $(-1)^\mu$ |
| **C** | $+1$ | $+1$ | $-1$ | $+1$ | $-1$ | $-1$ | $+1$ | $-1$ |
| **CPT** | $+1$ | $-1$ | $-1$ | $-1$ | $+1$ | $-1$ | $-1$ | $-1$ |

$(-1)^\mu = -1$ for $\mu = 1,\ 2,\ 3$, and $G_\mu = T^A G_\mu^A$. Note that for quark bilinears $\bar{\psi}_1(x)\Gamma^{\mu(\nu)}\psi_2(x)$, under $C$ transformation, it turns to $\bar{\psi}_2(x)\Gamma^{\mu(\nu)}\psi_1(x)$ with specific $C$ transformation factor listed in Table 7.4. For a complex number $c$, $\mathbf{T}c\mathbf{T}^{-1} = c^*$. Since for $A = 2,\ 5,\ 7$, the SU(3) generators $T^A$ are pure imaginary matrices, so

$\mathbf{T}T^A\mathbf{T}^{-1} = T^{A*} = -T^A$, while for $A = 1,\ 3,\ 4,\ 6,\ 8$ $T^A$ stays the same under $\mathbf{T}$, we have

$$\begin{aligned}
\mathbf{T}G^A_\mu\mathbf{T}^{-1} &= -(-1)^\mu \qquad \text{when } A = 2,\ 5,\ 7, \\
\mathbf{T}G^A_\mu\mathbf{T}^{-1} &= (-1)^\mu \qquad \text{when } A = 1,\ 3,\ 4,\ 6,\ 8.
\end{aligned} \tag{A.10}$$

We also have

$$\begin{aligned}
\mathbf{T}G^A_{\mu\nu}\mathbf{T}^{-1} &= (-1)^\mu(-1)^\nu \qquad \text{when } A = 2,\ 5,\ 7, \\
\mathbf{T}G^A_{\mu\nu}\mathbf{T}^{-1} &= -(-1)^\mu(-1)^\nu \qquad \text{when } A = 1,\ 3,\ 4,\ 6,\ 8,
\end{aligned} \tag{A.11}$$

where $G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - gsf^{ABC}G^B_\mu G^C_\nu$. Notice that $f^{ABC}$ are nonzero only when containing odd number of the indices $2,\ 5,\ 7$.

Similarly, since among the $SU(2)$ generators $\tau^2$ is pure imaginary, we have

$$\begin{aligned}
\mathbf{T}W^I_\mu\mathbf{T}^{-1} &= -(-1)^\mu \ \text{ for } I = 2, \\
\mathbf{T}W^I_\mu\mathbf{T}^{-1} &= (-1)^\mu \text{ for } I = 1,\ 3, \\
\mathbf{T}W^I_{\mu\nu}\mathbf{T}^{-1} &= (-1)^\mu(-1)^\nu \ \text{ for } I = 2, \\
\mathbf{T}W^I_{\mu\nu}\mathbf{T}^{-1} &= -(-1)^\mu(-1)^\nu \text{ for } I = 1,\ 3.
\end{aligned} \tag{A.12}$$

### 7.8.2 P-odd and CP-odd Operators with Lowest Mass-dimension for Flavor-changing Process

The P-odd CP-odd flavor-changing operators come from class $\psi^2\varphi^3$, $\psi^2X\varphi$, $\psi^4$, $\psi^2\varphi^2D$ and $\psi^2X\varphi$.

$$\begin{aligned}
\mathcal{L}^{\not{P}}_{\psi^2\varphi^3} &= \frac{v^3}{4\sqrt{2}}\Big[i\mathrm{Im}(C^{pr}_{le\varphi^3})(\bar{e}_p\gamma_5 e_r + \bar{e}_r\gamma_5 e_p) + i\mathrm{Im}(C^{pr}_{qu\varphi^3})(\bar{u}_p\gamma_5 u_r + \bar{u}_r\gamma_5 u_p) \\
&\quad + i\mathrm{Im}(C^{pr}_{qd\varphi^3})(\bar{d}_p\gamma_5 d_r + \bar{d}_r\gamma_5 d_p)\Big], \qquad \text{(B.1)} \\
\mathcal{L}^{\not{P}}_{\psi^2X\varphi} &= \frac{1}{2\sqrt{2}}viC^{pr}_{leF\varphi}(\bar{e}_p\sigma^{\mu\nu}\gamma_5 e_r + \bar{e}_r\sigma^{\mu\nu}\gamma_5 e_p)F_{\mu\nu} \\
&+ \frac{1}{2\sqrt{2}}v\left[i\mathrm{Im}(C^{pr}_{quG\varphi})(\bar{u}_p\sigma^{\mu\nu}\gamma_5\lambda^A u_r + \bar{u}_r\sigma^{\mu\nu}\gamma_5\lambda^A u_p)G^A_{\mu\nu}\right. \\
&\quad \left. + i\mathrm{Im}(C^{pr}_{qdG\varphi})(\bar{d}_p\sigma^{\mu\nu}\gamma_5\lambda^A d_r + \bar{d}_r\sigma^{\mu\nu}\gamma_5\lambda^A d_p)G^A_{\mu\nu}\right] \\
&+ \frac{1}{2\sqrt{2}}viC^{pr}_{quF\varphi}(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_r + \bar{u}_r\sigma^{\mu\nu}\gamma_5 u_p)F_{\mu\nu} \\
&+ \frac{1}{2\sqrt{2}}viC^{pr}_{qdF\varphi}(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_r + \bar{d}_r\sigma^{\mu\nu}\gamma_5 d_p)F_{\mu\nu}, \qquad \text{(B.2)}
\end{aligned}$$

together with the operators from class $\psi^4$ listed in Table 7.5 and Table 7.6.

Table 7.5: Flavor-changing P-odd and CP-odd operators from class $\psi^4$.

| 1: lepton-only interaction | |
|---|---|
| operator | coefficient |
| $(\bar\nu_p\gamma_\mu\gamma_5\nu_r)(\bar\nu_s\gamma^\mu\nu_t)-(\bar\nu_t\gamma^\mu\nu_s)(\bar\nu_r\gamma_\mu\gamma_5\nu_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{prst}_{l^4})$ |
| $(\bar\nu_p\gamma_\mu\nu_r)(\bar\nu_s\gamma^\mu\gamma_5\nu_t)-(\bar\nu_t\gamma^\mu\gamma_5\nu_s)(\bar\nu_r\gamma_\mu\nu_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{prst}_{l^4})$ |
| $(\bar e_p\gamma_\mu\gamma_5 e_r)(\bar e_s\gamma^\mu e_t)-(\bar e_t\gamma^\mu e_s)(\bar e_r\gamma_\mu\gamma_5 e_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{prst}_{l^4}+C^{prst}_{e^4}-C^{prst}_{l^2e^2})$ |
| $(\bar e_p\gamma_\mu e_r)(\bar e_s\gamma^\mu\gamma_5 e_t)-(\bar e_t\gamma^\mu\gamma_5 e_s)(\bar e_r\gamma_\mu e_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{prst}_{l^4}+C^{prst}_{e^4}+C^{prst}_{l^2e^2})$ |
| $(\bar\nu_p\gamma_\mu\gamma_5\nu_r)(\bar e_s\gamma^\mu e_t)-(\bar e_t\gamma^\mu e_s)(\bar\nu_r\gamma_\mu\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{prst}_{l^4}+C^{tsrp}_{l^4}-C^{prst}_{l^2e^2})$ |
| $(\bar\nu_p\gamma_\mu\nu_r)(\bar e_s\gamma^\mu\gamma_5 e_t)-(\bar e_t\gamma^\mu\gamma_5 e_s)(\bar\nu_r\gamma_\mu\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{prst}_{l^4}+C^{tsrp}_{l^4}+C^{prst}_{l^2e^2})$ |

| 2: lepton-quark interaction | |
|---|---|
| $(\bar\nu_p\gamma_\mu\gamma_5\nu_r)(\bar u_s\gamma^\mu u_t)-(\bar u_t\gamma^\mu u_s)(\bar\nu_r\gamma_\mu\gamma_5\nu_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{prst}_{l^2q^2}+C^{(3)prst}_{l^2q^2}+C^{prst}_{l^2u^2})$ |
| $(\bar\nu_p\gamma_\mu\nu_r)(\bar u_s\gamma^\mu\gamma_5 u_t)-(\bar u_t\gamma^\mu\gamma_5 u_s)(\bar\nu_r\gamma_\mu\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{prst}_{l^2q^2}-C^{(3)prst}_{l^2q^2}+C^{prst}_{l^2u^2})$ |
| $(\bar\nu_p\gamma_\mu\gamma_5\nu_r)(\bar d_s\gamma^\mu d_t)-(\bar d_t\gamma^\mu d_s)(\bar\nu_r\gamma_\mu\gamma_5\nu_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{prst}_{l^2q^2}-C^{(3)prst}_{l^2q^2}+C^{prst}_{l^2d^2})$ |
| $(\bar\nu_p\gamma_\mu\nu_r)(\bar d_s\gamma^\mu\gamma_5 d_t)-(\bar d_t\gamma^\mu\gamma_5 d_s)(\bar\nu_r\gamma_\mu\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{prst}_{l^2q^2}+C^{(3)prst}_{l^2q^2}+C^{prst}_{l^2d^2})$ |
| $(\bar e_p\gamma_\mu\gamma_5 e_r)(\bar u_s\gamma^\mu u_t)-(\bar u_t\gamma^\mu u_s)(\bar e_r\gamma_\mu\gamma_5 e_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{(1)prst}_{l^2q^2}+C^{(3)prst}_{l^2q^2}$ <br> $+C^{prst}_{e^2u^2}-C^{prst}_{l^2u^2}-C^{tsrp}_{q^2e^2})$ |
| $(\bar e_p\gamma_\mu e_r)(\bar u_s\gamma^\mu\gamma_5 u_t)-(\bar u_t\gamma^\mu\gamma_5 u_s)(\bar e_r\gamma_\mu e_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{(1)prst}_{l^2q^2}+C^{(3)prst}_{l^2q^2}$ <br> $+C^{prst}_{e^2u^2}+C^{prst}_{l^2u^2}+C^{tsrp}_{q^2e^2})$ |
| $(\bar e_p\gamma_\mu\gamma_5 e_r)(\bar d_s\gamma^\mu d_t)-(\bar d_t\gamma^\mu d_s)(\bar e_r\gamma_\mu\gamma_5 e_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{(1)prst}_{l^2q^2}-C^{(3)prst}_{l^2q^2}$ <br> $+C^{prst}_{e^2d^2}-C^{prst}_{l^2d^2}-C^{tsrp}_{q^2e^2})$ |
| $(\bar e_p\gamma_\mu e_r)(\bar d_s\gamma^\mu\gamma_5 d_t)-(\bar d_t\gamma^\mu\gamma_5 d_s)(\bar e_r\gamma_\mu e_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{(1)prst}_{l^2q^2}-C^{(3)prst}_{l^2q^2}$ <br> $+C^{prst}_{e^2d^2}+C^{prst}_{l^2d^2}+C^{tsrp}_{q^2e^2})$ |
| $(\bar\nu_p\gamma_\mu\gamma_5 e_r)(\bar d_s\gamma^\mu u_t)-(\bar u_t\gamma^\mu d_s)(\bar e_r\gamma_\mu\gamma_5\nu_p)$ | $-\frac{1}{2}i\mathrm{Im}(C^{(3)prst}_{l^2q^2})$ |
| $(\bar\nu_p\gamma_\mu e_r)(\bar d_s\gamma^\mu\gamma_5 u_t)-(\bar u_t\gamma^\mu\gamma_5 d_s)(\bar e_r\gamma_\mu\nu_p)$ | $-\frac{1}{2}i\mathrm{Im}(C^{(3)prst}_{l^2q^2})$ |
| $(\bar e_p\gamma_\mu\gamma_5\nu_r)(\bar u_s\gamma^\mu d_t)-(\bar d_t\gamma^\mu u_s)(\bar\nu_r\gamma_\mu\gamma_5 e_p)$ | $-\frac{1}{2}i\mathrm{Im}(C^{(3)prst}_{l^2q^2})$ |
| $(\bar e_p\gamma_\mu\nu_r)(\bar u_s\gamma^\mu\gamma_5 d_t)-(\bar d_t\gamma^\mu\gamma_5 u_s)(\bar\nu_r\gamma_\mu e_p)$ | $-\frac{1}{2}i\mathrm{Im}(C^{(3)prst}_{l^2q^2})$ |
| $(\bar\nu_p\gamma_5 e_r)(\bar d_s u_t)+(\bar u_t d_s)(\bar e_r\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{prst}_{ledq}+C^{(1)prst}_{lequ})$ |
| $(\bar\nu_p e_r)(\bar d_s\gamma_5 u_t)+(\bar u_t\gamma_5 d_s)(\bar e_r\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{prst}_{ledq}+C^{(1)prst}_{lequ})$ |
| $(\bar e_p\gamma_5 e_r)(\bar d_s d_t)+(\bar d_t d_s)(\bar e_r\gamma_5 e_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{prst}_{ledq})$ |
| $(\bar e_p e_r)(\bar d_s\gamma_5 d_t)+(\bar d_t\gamma_5 d_s)(\bar e_r e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{prst}_{ledq})$ |
| $(\bar e_p\gamma_5 e_r)(\bar u_s u_t)+(\bar u_t u_s)(\bar e_r\gamma_5 e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(1)prst}_{lequ})$ |
| $(\bar e_p e_r)(\bar u_s\gamma_5 u_t)+(\bar u_t\gamma_5 u_s)(\bar e_r e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(1)prst}_{lequ})$ |
| $(\bar\nu_p\sigma_{\mu\nu}\gamma_5 e_r)(\bar d_s\sigma^{\mu\nu}u_t)+(\bar u_t\sigma^{\mu\nu}d_s)(\bar e_r\sigma_{\mu\nu}\gamma_5\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(3)prst}_{lequ})$ |
| $(\bar\nu_p\sigma_{\mu\nu}e_r)(\bar d_s\sigma^{\mu\nu}\gamma_5 u_t)+(\bar u_t\sigma^{\mu\nu}\gamma_5 d_s)(\bar e_r\sigma_{\mu\nu}\nu_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(3)prst}_{lequ})$ |
| $(\bar e_p\sigma_{\mu\nu}\gamma_5 e_r)(\bar u_s\sigma_{\mu\nu}u_t)-(\bar u_t\sigma_{\mu\nu}u_s)(\bar e_r\sigma_{\mu\nu}\gamma_5 e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(3)prst}_{lequ})$ |
| $(\bar e_p\sigma_{\mu\nu}e_r)(\bar u_s\sigma_{\mu\nu}\gamma_5 u_t)-(\bar u_t\sigma_{\mu\nu}\gamma_5 u_s)(\bar e_r\sigma_{\mu\nu}e_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(3)prst}_{lequ})$ |

The mass-dimension 8 operators containing $v^2 G_F$ below can give comparative contributions as the mass-dimension 6 operators listed previously.

$$
\begin{aligned}
\mathcal{L}^{P}_{\psi^2\varphi^2 D} \;=\; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) + \mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D})]\,\{(\bar\nu_p\gamma^\mu\gamma_5\nu_r - \bar\nu_r\gamma^\mu\gamma_5\nu_p)\times \\
& g_Z\left[\frac{1}{4}\bar\nu_s\gamma_\mu\nu_s + (-\frac{1}{4}+s_w^2)\bar e_s\gamma_\mu e_s + (\frac{1}{4}-\frac{2}{3}s_w^2)\bar u_s\gamma_\mu u_s + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar d_s\gamma_\mu d_s\right] \\
& +\frac{1}{4}(\bar\nu_p\gamma^\mu\nu_r - \bar\nu_r\gamma^\mu\nu_p)g_Z(\bar\nu_s\gamma_\mu\gamma_5\nu_s + \bar e_s\gamma_\mu\gamma_5 e_s + \bar u_s\gamma_\mu\gamma_5 u_s - \bar d_s\gamma_\mu\gamma_5 d_s)\Big\} \\
- \; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) + \mathrm{Im}(C^{pr}_{e^2\varphi^2 D})](\bar e_p\gamma^\mu\gamma_5 e_r - \bar e_r\gamma^\mu\gamma_5 e_p)\times \\
& g_Z\left[\frac{1}{4}\bar\nu_s\gamma_\mu\nu_s + (-\frac{1}{4}+s_w^2)\bar e_s\gamma_\mu e_s + (\frac{1}{4}-\frac{2}{3}s_w^2)\bar u_s\gamma_\mu u_s + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar d_s\gamma_\mu d_s\right] \\
+ \; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{pr}_{e^2\varphi^2 D})]\frac{1}{4}(\bar e_p\gamma^\mu e_r - \bar e_r\gamma^\mu e_p) \\
& g_Z\left(\bar\nu_s\gamma_\mu\gamma_5\nu_s + \bar e_s\gamma_\mu\gamma_5 e_s + \bar u_s\gamma_\mu\gamma_5 u_s - \bar d_s\gamma_\mu\gamma_5 d_s\right) \\
- \; & \frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)pr}_{l^2\varphi^2 D})g\,[(\bar e_p\gamma^\mu\gamma_5\nu_r - \bar e_r\gamma^\mu\gamma_5\nu_p)\partial_\mu(\bar\nu_s\gamma_\mu e_s + \bar u_s\gamma_\mu V_{sx} d_x) \\
& +(\bar\nu_p\gamma^\mu\gamma_5 e_r - \bar\nu_r\gamma^\mu\gamma_5 e_p)(\bar e_s\gamma_\mu\nu_s + \bar d_x\gamma_\mu V^*_{sx} u_s) \\
& +(\bar e_p\gamma^\mu\nu_r - \bar e_r\gamma^\mu\nu_p)\partial_\mu(\bar\nu_s\gamma_\nu\gamma_5 e_s + \bar u_s\gamma_\nu\gamma_5 V_{sx} d_x) \\
& +(\bar\nu_p\gamma^\mu e_r - \bar\nu_r\gamma^\mu e_p)(\bar e_s\gamma_\mu\gamma_5\nu_s + \bar d_x\gamma_\mu\gamma_5 V^*_{sx} u_s)] \\
+ \; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(1)pr}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{pr}_{u^2\varphi^2 D})](\bar u_p\gamma^\mu\gamma_5 u_r - \bar u_r\gamma^\mu\gamma_5 u_p)\times \\
& g_Z\left[\frac{1}{4}\bar\nu_s\gamma_\mu\nu_s + (-\frac{1}{4}+s_w^2)\bar e_s\gamma_\mu e_s + (\frac{1}{4}-\frac{2}{3}s_w^2)\bar u_s\gamma_\mu u_s + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar d_s\gamma_\mu d_s\right] \\
+ \; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(1)pr}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) + \mathrm{Im}(C^{pr}_{u^2\varphi^2 D})](\bar u_p\gamma^\mu u_r - \bar u_r\gamma^\mu u_p)\times \\
& g_Z\frac{1}{4}\left(\bar\nu_s\gamma_\mu\gamma_5\nu_s + \bar e_s\gamma_\mu\gamma_5 e_s + \bar u_s\gamma_\mu\gamma_5 u_s - \bar d_s\gamma_\mu\gamma_5 d_s\right) \\
- \; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) + \mathrm{Im}(C^{pr}_{d^2\varphi^2 D})](\bar d_p\gamma^\mu\gamma_5 d_r - \bar d_r\gamma^\mu\gamma_5 d_p)\times \\
& g_Z\left[\frac{1}{4}\bar\nu_s\gamma_\mu\nu_s + (-\frac{1}{4}+s_w^2)\bar e_s\gamma_\mu e_s + (\frac{1}{4}-\frac{2}{3}s_w^2)\bar u_s\gamma_\mu u_s + (-\frac{1}{4}+\frac{1}{3}s_w^2)\bar d_s\gamma_\mu d_s\right] \\
- \; & \sqrt{2}v^2 G_F i[\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) - \mathrm{Im}(C^{(1)pr}_{l^2\varphi^2 D}) - \mathrm{Im}(C^{pr}_{d^2\varphi^2 D})](\bar d_p\gamma^\mu d_r - \bar d_r\gamma^\mu d_p)\times \\
& g_Z\frac{1}{4}\left(\bar\nu_s\gamma_\mu\gamma_5\nu_s + \bar e_s\gamma_\mu\gamma_5 e_s + \bar u_s\gamma_\mu\gamma_5 u_s - \bar d_s\gamma_\mu\gamma_5 d_s\right) \\
- \; & \frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D})g\left[(\bar d_p\gamma^\mu\gamma_5 u_r - \bar d_r\gamma^\mu\gamma_5 u_p)(\bar\nu_s\gamma_\mu e_s + \bar u_s\gamma_\mu V_{sx} d_x)\right. \\
& +(\bar u_p\gamma^\mu\gamma_5 d_r - \bar u_r\gamma^\mu\gamma_5 d_p)(\bar e_s\gamma_\mu\nu_s + \bar d_x\gamma_\mu V^*_{sx} u_s)]] \\
- \; & \frac{v^2 G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D})g\left[(\bar d_p\gamma^\mu u_r - \bar d_r\gamma^\mu u_p)(\bar\nu_s\gamma_\mu\gamma_5 e_s + \bar u_s\gamma_\mu\gamma_5 V_{sx} d_x)\right. \\
& +(\bar u_p\gamma^\mu d_r - \bar u_r\gamma^\mu d_p)(\bar e_s\gamma_\mu\gamma_5\nu_s + \bar d_x\gamma_\mu\gamma_5 V^*_{sx} u_s)] \\
+ \; & \frac{v^2 G_F}{4} i\mathrm{Im}(C^{pr}_{ud\varphi^2 D})g\left[(\bar u_p\gamma^\mu\gamma_5 d_r)(\bar e_s\gamma_\mu\nu_s + \bar d_x\gamma_\mu V^*_{sx} u_s)\right. \\
& -(\bar d_r\gamma^\mu\gamma_5 u_p)(\bar\nu_s\gamma_\mu e_s + \bar u_s\gamma_\mu V_{sx} d_x) - (\bar u_p\gamma^\mu d_r)(\bar e_s\gamma_\mu\gamma_5\nu_s + \bar d_x\gamma_\mu\gamma_5 V^*_{sx} u_s) \\
& +(\bar d_r\gamma^\mu u_p)(\bar\nu_s\gamma_\mu\gamma_5 e_s + \bar u_s\gamma_\mu\gamma_5 V_{sx} d_x)]
\end{aligned} \tag{B.3}
$$

The mass-dimension 8 operators below containing $vG_F$ can also give important contributions:

Table 7.6: Flavor-changing P-odd and CP-odd operators from class $\psi^4$ (Continued).

| 3: quark-only interaction | |
|---|---|
| $(\bar{u}_p\gamma_\mu\gamma_5 u_r)(\bar{u}_s\gamma^\mu u_t)-(\bar{u}_t\gamma^\mu u_s)(\bar{u}_r\gamma_\mu\gamma_5 u_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{(1)prst}_{q^4}-C^{(3)prst}_{q^4}+C^{prst}_{u^4}-C^{(1)prst}_{q^2u^2})$ |
| $(\bar{u}_p\gamma_\mu u_r)(\bar{u}_s\gamma^\mu\gamma_5 u_t)-(\bar{u}_t\gamma^\mu\gamma_5 u_s)(\bar{u}_r\gamma_\mu u_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{(1)prst}_{q^4}-C^{(3)prst}_{q^4}+C^{prst}_{u^4}+C^{(1)prst}_{q^2u^2})$ |
| $(\bar{d}_p\gamma_\mu\gamma_5 d_r)(\bar{d}_s\gamma^\mu d_t)-(\bar{d}_t\gamma^\mu d_s)(\bar{d}_r\gamma_\mu\gamma_5 d_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{(1)prst}_{q^4}-C^{(3)prst}_{q^4}+C^{prst}_{d^4}-C^{(1)prst}_{q^2d^2})$ |
| $(\bar{d}_p\gamma_\mu d_r)(\bar{d}_s\gamma^\mu\gamma_5 d_t)-(\bar{d}_t\gamma^\mu\gamma_5 d_s)(\bar{d}_r\gamma_\mu d_p)$ | $\frac{1}{4}i\mathrm{Im}(-C^{(1)prst}_{q^4}-C^{(3)prst}_{q^4}+C^{prst}_{d^4}+C^{(1)prst}_{q^2d^2})$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 u_r)(\bar{d}_s\gamma^\mu d_t)-(\bar{d}_t\gamma^\mu d_s)(\bar{u}_r\gamma_\mu\gamma_5 u_p)$ | $\frac{1}{4}i\mathrm{Im}\left(-C^{(1)prst}_{q^4}+C^{(3)prst}_{q^4}+C^{(1)tsrp}_{q^4}-C^{(3)tsrp}_{q^4}+C^{(1)prst}_{u^2d^2}+C^{(1)tsrp}_{q^2u^2}-C^{(1)prst}_{q^2d^2}\right)$ |
| $(\bar{u}_p\gamma_\mu u_r)(\bar{d}_s\gamma^\mu\gamma_5 d_t)-(\bar{d}_t\gamma^\mu\gamma_5 d_s)(\bar{u}_r\gamma_\mu u_p)$ | $\frac{1}{4}i\mathrm{Im}\left(-C^{(1)prst}_{q^4}+C^{(3)prst}_{q^4}+C^{(1)tsrp}_{q^4}-C^{(3)tsrp}_{q^4}+C^{(1)prst}_{u^2d^2}-C^{(1)tsrp}_{q^2u^2}+C^{(1)prst}_{q^2d^2}\right)$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 T^A u_r)(\bar{u}_s\gamma^\mu T^A u_t)-(\bar{u}_t\gamma^\mu T^A u_s)(\bar{u}_r\gamma_\mu\gamma_5 T^A u_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(8)tsrp}_{q^2u^2})$ |
| $(\bar{u}_p\gamma_\mu T^A u_r)(\bar{u}_s\gamma^\mu\gamma_5 T^A u_t)-(\bar{u}_t\gamma^\mu\gamma_5 T^A u_s)(\bar{u}_r\gamma_\mu T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(8)tsrp}_{q^2u^2})$ |
| $(\bar{d}_p\gamma_\mu\gamma_5 T^A d_r)(\bar{d}_s\gamma^\mu T^A d_t)-(\bar{d}_t\gamma^\mu T^A d_s)(\bar{d}_r\gamma_\mu\gamma_5 T^A d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(8)prst}_{q^2d^2}$ |
| $(\bar{d}_p\gamma_\mu T^A d_r)(\bar{d}_s\gamma^\mu\gamma_5 T^A d_t)-(\bar{d}_t\gamma^\mu\gamma_5 T^A d_s)(\bar{d}_r\gamma_\mu T^A d_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(8)prst}_{q^2d^2}$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 T^A u_r)(\bar{d}_s\gamma^\mu T^A d_t)-(\bar{d}_t\gamma^\mu T^A d_s)(\bar{u}_r\gamma_\mu\gamma_5 T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(8)prst}_{u^2d^2}+C^{(8)tsrp}_{q^2u^2}-C^{(8)prst}_{q^2d^2})$ |
| $(\bar{u}_p\gamma_\mu T^A u_r)(\bar{d}_s\gamma^\mu\gamma_5 T^A d_t)-(\bar{d}_t\gamma^\mu\gamma_5 T^A d_s)(\bar{u}_r\gamma_\mu T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(8)prst}_{u^2d^2}-C^{(8)tsrp}_{q^2u^2}+C^{(8)prst}_{q^2d^2})$ |
| $(\bar{u}_p\gamma_\mu\gamma_5 d_r)(\bar{d}_s\gamma^\mu u_t)-(\bar{u}_t\gamma^\mu d_s)(\bar{d}_r\gamma_\mu\gamma_5 u_p)$ | $-\frac{1}{2}i\mathrm{Im}(C^{(3)prst}_{q^4})$ |
| $(\bar{u}_p\gamma_\mu d_r)(\bar{d}_s\gamma^\mu\gamma_5 u_t)-(\bar{u}_t\gamma^\mu\gamma_5 d_s)(\bar{d}_r\gamma_\mu u_p)$ | $-\frac{1}{2}i\mathrm{Im}(C^{(3)prst}_{q^4})$ |
| $(\bar{d}_p\gamma_\mu\gamma_5 u_r)(\bar{u}_s\gamma^\mu d_t)-(\bar{d}_t\gamma^\mu u_s)(\bar{u}_r\gamma_\mu\gamma_5 d_p)$ | $-\frac{1}{2}i\mathrm{Im}(C^{(3)prst}_{q^4})$ |
| $(\bar{d}_p\gamma_\mu u_r)(\bar{u}_s\gamma^\mu\gamma_5 d_t)-(\bar{d}_t\gamma^\mu\gamma_5 u_s)(\bar{u}_r\gamma_\mu d_p)$ | $-\frac{1}{2}i\mathrm{Im}(C^{(3)prst}_{q^4})$ |
| $(\bar{u}_p\gamma_5 u_r)(\bar{d}_s d_t)+(\bar{d}_t d_s)(\bar{u}_r\gamma_5 u_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(1)prst}_{quqd})$ |
| $(\bar{u}_p u_r)(\bar{d}_s\gamma_5 d_t)+(\bar{d}_t\gamma_5 d_s)(\bar{u}_r u_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(1)prst}_{quqd})$ |
| $(\bar{d}_p\gamma_5 u_r)(\bar{u}_s d_t)+(\bar{d}_t u_s)(\bar{u}_r\gamma_5 d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(1)prst}_{quqd})$ |
| $(\bar{d}_p u_r)(\bar{u}_s\gamma_5 d_t)+(\bar{d}_t\gamma_5 u_s)(\bar{u}_r d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(1)prst}_{quqd})$ |
| $(\bar{u}_p\gamma_5 T^A u_r)(\bar{d}_s T^A d_t)+(\bar{d}_t T^A d_s)(\bar{u}_r\gamma_5 T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(8)prst}_{quqd})$ |
| $(\bar{u}_p T^A u_r)(\bar{d}_s\gamma_5 T^A d_t)+(\bar{d}_t\gamma_5 T^A d_s)(\bar{u}_r T^A u_p)$ | $\frac{1}{4}i\mathrm{Im}(C^{(8)prst}_{quqd})$ |
| $(\bar{d}_p\gamma_5 T^A u_r)(\bar{u}_s T^A d_t)+(\bar{d}_t T^A u_s)(\bar{u}_r\gamma_5 T^A d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(8)prst}_{quqd})$ |
| $(\bar{d}_p T^A u_r)(\bar{u}_s T^A\gamma_5 d_t)+(\bar{d}_t\gamma_5 T^A u_s)(\bar{u}_r T^A d_p)$ | $-\frac{1}{4}i\mathrm{Im}(C^{(8)prst}_{quqd})$ |

$$
\begin{aligned}
\mathcal{L}^{\mathcal{P}}_{\psi^2 X\varphi} &= \sqrt{2}vG_F i\mathrm{Im}(C^{pr}_{leW\varphi})\,\{(\bar{\nu}_p\sigma^{\mu\nu}\gamma_5 e_r)\times \\
& [\partial_\mu(\bar{e}_s\gamma_\nu\nu_s+\bar{d}_{Lx}\gamma_\nu V^*_{sx}u_s)+ig(\bar{e}_s\gamma_\mu\nu_s+\bar{d}_{Lx}\gamma_\mu V^*_{sx}u_{Ls})s_wA_\nu] \\
& +(\bar{e}_r\sigma^{\mu\nu}\gamma_5\nu_p)[\partial_\mu(\bar{\nu}_s\gamma_\nu e_s+\bar{u}_s\gamma_\nu V_{sx}d_{Lx})-ig(\bar{\nu}_s\gamma_\mu e_s+\bar{u}_s\gamma_\mu V_{sx}d_{Lx})s_wA_\nu] \\
& -(\bar{\nu}_p\sigma^{\mu\nu}e_r)[\partial_\mu(\bar{e}_s\gamma_\nu\gamma_5\nu_s+\bar{d}_{Lx}\gamma_\nu\gamma_5V^*_{sx}u_s)+ig(\bar{e}_s\gamma_\mu\gamma_5\nu_s+\bar{d}_{Lx}\gamma_\mu\gamma_5V^*_{sx}u_{Ls})s_wA_\nu] \\
& +(\bar{e}_r\sigma^{\mu\nu}\nu_p)[\partial_\mu(\bar{\nu}_s\gamma_\nu\gamma_5e_s+\bar{u}_s\gamma_\nu\gamma_5V_{sx}d_{Lx})-ig(\bar{\nu}_s\gamma_\mu\gamma_5e_s+\bar{u}_s\gamma_\mu\gamma_5V_{sx}d_{Lx})s_wA_\nu]\} \\
&+ 2vG_F iC^{pr}_{leZ\varphi}\,\{(\bar{e}_p\sigma^{\mu\nu}\gamma_5e_r+\bar{e}_r\sigma^{\mu\nu}\gamma_5e_p)\times \\
& \partial_\mu\left[\frac{1}{4}\bar{\nu}_s\gamma_\nu\nu_s+(-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\nu e_s+(\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\nu u_s+(-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\nu d_s\right] \\
& \left.+\frac{1}{4}(\bar{e}_p\sigma^{\mu\nu}e_r-\bar{e}_r\sigma^{\mu\nu}e_p)\partial_\mu(\bar{\nu}_s\gamma_\mu\gamma_5\nu_s+\bar{e}_s\gamma_\mu\gamma_5e_s+\bar{u}_s\gamma_\mu\gamma_5u_s-\bar{d}_s\gamma_\mu\gamma_5d_s)\right\} \\
&+ 2vG_F iC^{pr}_{quZ\varphi}\,\{(\bar{u}_p\sigma^{\mu\nu}\gamma_5u_r+\bar{u}_r\sigma^{\mu\nu}\gamma_5u_p)\times \\
& \partial_\mu\left[\frac{1}{4}\bar{\nu}_s\gamma_\nu\nu_s+(-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\nu e_s+(\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\nu u_s+(-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\nu d_s\right] \\
& \left.+\frac{1}{4}(\bar{u}_p\sigma^{\mu\nu}u_r-\bar{u}_r\sigma^{\mu\nu}u_p)\partial_\mu(\bar{\nu}_s\gamma_\mu\gamma_5\nu_s+\bar{e}_s\gamma_\mu\gamma_5e_s+\bar{u}_s\gamma_\mu\gamma_5u_s-\bar{d}_s\gamma_\mu\gamma_5d_s)\right\} \\
&+ 2vG_F iC^{pr}_{qdZ\varphi}\,\left\{(\bar{d}_p\sigma^{\mu\nu}\gamma_5d_r+\bar{d}_r\sigma^{\mu\nu}\gamma_5d_p)\times\right. \\
& \partial_\mu\left[\frac{1}{4}\bar{\nu}_s\gamma_\nu\nu_s+(-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\nu e_s+(\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\nu u_s+(-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\nu d_s\right] \\
& \left.+\frac{1}{4}(\bar{d}_p\sigma^{\mu\nu}d_r-\bar{d}_r\sigma^{\mu\nu}d_p)\partial_\mu(\bar{\nu}_s\gamma_\mu\gamma_5\nu_s+\bar{e}_s\gamma_\mu\gamma_5e_s+\bar{u}_s\gamma_\mu\gamma_5u_s-\bar{d}_s\gamma_\mu\gamma_5d_s)\right\} \\
&+ \sqrt{2}vG_F i[\mathrm{Im}(C^{pr}_{quW\varphi})+\mathrm{Im}(C^{rp}_{qdW\varphi})]\,\left\{(\bar{d}_p\sigma^{\mu\nu}\gamma_5u_r)\,[\partial_\mu(\bar{\nu}_s\gamma_\nu e_s+\bar{u}_s\gamma_\nu V_{sx}d_x)\right. \\
& -ig(\bar{\nu}_s\gamma_\mu e_s+\bar{u}_s\gamma_\mu V_{sx}d_x)s_wA_\nu]+(\bar{u}_r\sigma^{\mu\nu}\gamma_5d_p)\,\left[\partial(\bar{e}_s\gamma_\nu\nu_s+\bar{d}_x\gamma_\nu V^*_{sx}u_s)\right. \\
& \left.\left.+ig(\bar{e}_s\gamma_\mu\nu_s+\bar{d}_x\gamma_\mu V^*_{sx}u_s)s_wA_\nu\right]\right\} \\
&- \sqrt{2}vG_F i[\mathrm{Im}(C^{pr}_{quW\varphi})-\mathrm{Im}(C^{rp}_{qdW\varphi})]\,\left\{(\bar{d}_p\sigma^{\mu\nu}u_r)\,[\partial(\bar{\nu}_s\gamma_\nu\gamma_5e_s+\bar{u}_s\gamma_\nu\gamma_5V_{sx}d_x)\right. \\
& -ig(\bar{\nu}_s\gamma_\mu\gamma_5e_s+\bar{u}_s\gamma_\mu\gamma_5V_{sx}d_x)s_wA_\nu]-(\bar{u}_r\sigma^{\mu\nu}d_p)\,\left[\partial_\mu(\bar{e}_s\gamma_\nu\gamma_5\nu_s+\bar{d}_x\gamma_\nu\gamma_5V^*_{sx}u_s)\right. \\
& \left.\left.+ig(\bar{e}_s\gamma_\mu\gamma_5\nu_s+\bar{d}_x\gamma_\mu\gamma_5V^*_{sx}u_s)s_wA_\nu\right]\right\} \qquad \text{(B.4)}
\end{aligned}
$$

### 7.8.3 P-odd and CP-odd Operators with Lowest Mass-dimension for Flavor-conserving Processes

The P-odd and CP-odd operators for flavor-conserving processes with the lowest dimension come from classes $X^3$, $X^2\varphi^2$, $\psi^2\varphi^3$, $\psi^2 X\varphi$, $\psi^4$ and $\psi^2\varphi^2 D$. They are

$$
\begin{aligned}
\mathcal{L}^{\not{P}}_{X^3} &= C_{G^3} f^{ABC}\epsilon^{\mu\nu\alpha\beta}G^A_{\alpha\beta}G^B_{\mu\rho}G^{C\rho}_{\nu}, && \text{(C.5)}\\
\mathcal{L}^{\not{P}}_{X^2\varphi^2} &= \frac{v^2}{2}\epsilon^{\mu\nu\alpha\beta}\left\{C_{G^2\varphi^2}G^A_{\mu\nu}G^A_{\alpha\beta}\right.\\
&\quad \left.+(C_{W^2\varphi^2}{s_w}^2+C_{B^2\varphi^2}{c_w}^2+C_{WB\varphi^2}c_w s_w)F_{\alpha\beta}F_{\mu\nu}\right\}, && \text{(C.6)}\\
\mathcal{L}^{\not{P}}_{\psi^2\varphi^3} &= \frac{v^3}{2\sqrt{2}}\Big[i\mathrm{Im}(C^{pp}_{le\varphi^3})(\bar{e}_p\gamma_5 e_p)+i\mathrm{Im}(C^{pp}_{qu\varphi^3})(\bar{u}_p\gamma_5 u_p)\\
&\quad +i\mathrm{Im}(C^{pp}_{qd\varphi^3})(\bar{d}_p\gamma_5 d_p)\Big], && \text{(C.7)}\\
\mathcal{L}^{\not{P}}_{\psi^2X\varphi} &= \frac{1}{\sqrt{2}}v[i\mathrm{Im}(C^{pp}_{quG\varphi})(\bar{u}_p\sigma^{\mu\nu}\gamma_5\lambda^A G^A_{\mu\nu}u_p)+i\mathrm{Im}(C^{pp}_{qdG\varphi})/v(\bar{d}_p\sigma^{\mu\nu}\gamma_5\lambda^A G^A_{\mu\nu}d_p)]\\
&+ \frac{1}{\sqrt{2}}viC^{pp}_{quF\varphi}(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)F_{\mu\nu}-\frac{1}{2}viC^{pp}_{qdF\varphi}(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)F_{\mu\nu}, && \text{(C.8)}\\
\mathcal{L}^{\not{P}}_{\psi^4} &= \frac{1}{2}i\mathrm{Im}(C^{(1)pprr}_{quqd})\left[(\bar{u}_p\gamma_5 u_p)(\bar{d}_r d_r)+(\bar{u}_p u_p)(\bar{d}_r\gamma_5 d_r)-(\bar{d}_p\gamma_5 u_r)(\bar{u}_r d_p)\right.\\
&\quad \left.-(\bar{d}_p u_r)(\bar{u}_r\gamma_5 d_p)\right]\\
&\quad +\frac{1}{2}i\mathrm{Im}(C^{(8)pprr}_{quqd})\left[(\bar{u}_p\gamma_5 T^A u_p)(\bar{d}_r T^A d_r)+(\bar{u}_p T^A u_p)(\bar{d}_r\gamma_5 T^A d_r)\right.\\
&\quad \left.-(\bar{d}_p\gamma_5 T^A u_r)(\bar{u}_r T^A d_p)-(\bar{d}_p T^A u_r)(\bar{u}_r\gamma_5 T^A d_p)\right]. && \text{(C.9)}
\end{aligned}
$$

The mass-dimension of the above P-odd and CP-odd operators for flavor-conserving interactions is 6.

The mass-dimension 8 operators involving $v^2 G_F$ can effectively give numerical comparative contributions. Such operators can be from class $X^2\varphi^2$ and $\psi^2\varphi D$.

$$
\begin{aligned}
\mathcal{L}^{P}_{X^2\varphi^2} &= \frac{2v^2G_F}{\sqrt{2}}\epsilon^{\mu\nu\alpha\beta}[2c_w s_w(C_{W^2\varphi^2}-C_{B^2\varphi^2})-C_{WB\varphi^2}({c_w}^2-{s_w}^2)]\times\\
&\quad \partial_\alpha[-\frac{1}{2}(\bar{u}_p\gamma_\beta u_p)+\frac{1}{2}(\bar{d}_p\gamma_\beta d_p)]F_{\mu\nu} && \text{(C.10)}
\end{aligned}
$$

The flavor-changing P-odd and CP-odd operators of class $\psi^2\varphi D$ in Eq. (B.3) that do not vanish for flavor-conserving interactions are

$$
\begin{aligned}
\mathcal{L}_{\psi^2\varphi D} \quad \sim \quad & -\frac{v^2G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) g \left[(\bar{d}_p\gamma^\mu\gamma_5 u_r)(\bar{u}_r\gamma_\mu V_{rp}d_p) - (\bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu V_{pr}d_r)\right. \\
& +(\bar{u}_p\gamma^\mu\gamma_5 d_r)(\bar{d}_r\gamma_\mu V^*_{pr}u_p) - (\bar{u}_r\gamma^\mu\gamma_5 d_p)(\bar{d}_p\gamma_\mu V^*_{rp}u_r)]] \\
& -\frac{v^2G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) g \left[(\bar{d}_p\gamma^\mu u_r)(\bar{u}_r\gamma_\mu\gamma_5 V_{rp}d_p) - (\bar{d}_r\gamma^\mu u_p)(\bar{u}_p\gamma_\mu\gamma_5 V_{pr}d_r)\right. \\
& \left.+(\bar{u}_p\gamma^\mu d_r)(\bar{d}_r\gamma_\mu\gamma_5 V^*_{pr}u_p) - (\bar{u}_r\gamma^\mu d_p)(\bar{d}_p\gamma_\mu\gamma_5 V^*_{rp}u_r)\right] \\
& +\frac{v^2G_F}{4} i\mathrm{Im}(C^{pr}_{ud\varphi^2 D}) g \left[(\bar{u}_p\gamma^\mu\gamma_5 d_r)(\bar{d}_r\gamma_\mu V^*_{pr}u_p) - (\bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu V_{pr}d_r)\right. \\
& \left.-(\bar{u}_p\gamma^\mu d_r)(\bar{d}_r\gamma_\mu\gamma_5 V^*_{rp}u_p) + (\bar{d}_r\gamma^\mu u_p)(\bar{u}_p\gamma_\mu\gamma_5 V_{pr}d_r)\right] \\
= \quad & -\frac{v^2G_F}{2} i\mathrm{Im}(C^{(3)pr}_{q^2\varphi^2 D}) g \left\{\left(V_{rp} - V^*_{rp}\right)\left[(\bar{d}_p\gamma^\mu\gamma_5 u_r)(\bar{u}_r\gamma_\mu d_p) + (\bar{u}_r\gamma^\mu\gamma_5 d_p)(\bar{d}_p\gamma_\mu u_r)\right]\right. \\
& \left.-\left(V_{pr} - V^*_{pr}\right)\left[(\bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma_\mu d_r) + (\bar{u}_p\gamma^\mu\gamma_5 d_r)(\bar{d}_r\gamma_\mu u_p)\right]\right\} \\
& +\frac{v^2G_F}{4} i\mathrm{Im}(C^{pr}_{ud\varphi^2 D}) g \left(V^*_{pr} + V_{pr}\right)\left[(\bar{u}_p\gamma^\mu\gamma_5 d_r)(\bar{d}_r\gamma^\mu u_p) - (\bar{d}_r\gamma^\mu\gamma_5 u_p)(\bar{u}_p\gamma^\mu d_r)\right].
\end{aligned}
\tag{C.11}
$$

The above operators containing factor $C^{(3)pr}_{q^2\varphi^2 D}$ are P-odd and C-odd, and thus CP-even. So the flavor-conserving P-odd and CP-odd operators from $\mathcal{L}_{\psi^2\varphi^2 D}$ are

$$
\begin{aligned}
\mathcal{L}^{\not P}_{\psi^2\varphi^2 D} \quad = \quad & \frac{v^2G_F}{4} g i\mathrm{Im}(C^{pr}_{ud\varphi^2 D})(V^*_{u_pd_r} + V_{u_pd_r}) \times \\
& \left[(\bar{d}_r\gamma_\mu u_p)(\bar{u}_p\gamma^\mu\gamma_5 d_r) - (\bar{d}_r\gamma_\mu\gamma_5 u_p)(\bar{u}_p\gamma^\mu d_r)\right] \\
+ \quad & \frac{v^2G_F}{4} g i\mathrm{Im}(C^{(8)pr}_{ud\varphi^2 D})(V^*_{u_pd_r} + V_{u_pd_r}) \times \\
& \left[(\bar{d}_r\gamma_\mu T^A u_p)(\bar{u}_p\gamma^\mu\gamma_5 T^A d_r) - (\bar{d}'_r\gamma_\mu\gamma_5 T^A u_p)(\bar{u}_p\gamma^\mu T^A d_r)\right],
\end{aligned}
\tag{C.12}
$$

where we add extra operators with the coefficient $C^{(8)pr}_{ud\varphi^2 D}$ with similar field structure but with additional color structure comparing to the original one, which appears from QCD when going down to very low energy, or LEFT. These operators do not show up at electroweak scale due to its gauge-symmetry-breaking properties [244]. The operators in Eq. (C.12) are P-odd and CP-odd, and they are mass-dimension 8, but since $v^2G_F \sim \mathcal{O}(1)$, they can be mass-dimension 6 numerically.

The operators in class $\psi^2 X\varphi$ containing $vG_F$ are also mass-dimension 8 that can give important contributions:

$$
\begin{aligned}
\mathcal{L}^{P}_{\psi^2 X\varphi} \;=\;& \sqrt{2}vG_F i\mathrm{Im}(C^{pp}_{leW\varphi})\,\{(\bar{\nu}_p\sigma^{\mu\nu}\gamma_5 e_p)[\partial_\mu(\bar{e}_p\gamma_\nu\nu_p)+ig(\bar{e}_p\gamma_\mu\nu_p)s_wA_\nu] \\
& +(\bar{e}_p\sigma^{\mu\nu}\gamma_5\nu_p)[\partial_\mu(\bar{\nu}_p\gamma_\nu e_p)-ig(\bar{\nu}_p\gamma_\mu e_p)s_wA_\nu] \\
& -(\bar{\nu}_p\sigma^{\mu\nu}e_p)[\partial_\mu(\bar{e}_p\gamma_\nu\gamma_5\nu_p)+ig(\bar{e}_p\gamma_\mu\gamma_5\nu_p)s_wA_\nu] \\
& +(\bar{e}_p\sigma^{\mu\nu}\nu_p)[\partial_\mu(\bar{\nu}_p\gamma_\nu\gamma_5 e_p)-ig(\bar{\nu}_p\gamma_\mu\gamma_5 e_p)s_wA_\nu]\} \\
+\;& 4vG_F iC^{pp}_{leZ\varphi}\,\{(\bar{e}_p\sigma^{\mu\nu}\gamma_5 e_p) \\
& \partial_\mu\left[\frac{1}{4}\bar{\nu}_s\gamma_\nu\nu_s+(-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\nu e_s+(\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\nu u_s+(-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\nu d_s\right]\Big\} \\
+\;& 4vG_F iC^{pp}_{quZ\varphi}\,\{(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_p)\times \\
& \partial_\mu\left[\frac{1}{4}\bar{\nu}_s\gamma_\nu\nu_s+(-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\nu e_s+(\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\nu u_s+(-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\nu d_s\right]\Big\} \\
+\;& 4vG_F iC^{pr}_{qdZ\varphi}\,\{(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_p)\times \\
& \partial_\mu\left[\frac{1}{4}\bar{\nu}_s\gamma_\nu\nu_s+(-\frac{1}{4}+s_w^2)\bar{e}_s\gamma_\nu e_s+(\frac{1}{4}-\frac{2}{3}s_w^2)\bar{u}_s\gamma_\nu u_s+(-\frac{1}{4}+\frac{1}{3}s_w^2)\bar{d}_s\gamma_\nu d_s\right]\Big\} \\
+\;& \sqrt{2}vG_F i[\mathrm{Im}(C^{pr}_{quW\varphi})+\mathrm{Im}(C^{rp}_{qdW\varphi})]\times \\
& \big\{(\bar{d}_p\sigma^{\mu\nu}\gamma_5 u_r)\,[\partial_\mu(\bar{u}_r\gamma_\nu V_{rp}d_p)-ig(\bar{u}_r\gamma_\mu V_{rp}d_p)s_wA_\nu] \\
& +(\bar{u}_r\sigma^{\mu\nu}\gamma_5 d_p)\,\big[\partial_\mu(\bar{d}_p\gamma_\nu V^*_{rp}u_r)+ig(\bar{d}_p\gamma_\mu V^*_{rp}u_r)s_wA_\nu\big]\big\} \\
-\;& \sqrt{2}vG_F i[\mathrm{Im}(C^{pr}_{quW\varphi})-\mathrm{Im}(C^{rp}_{qdW\varphi})]\times \\
& \big\{(\bar{d}_p\sigma^{\mu\nu}u_r)\,[\partial_\mu(\bar{u}_r\gamma_\nu\gamma_5 V_{rp}d_p)-ig(\bar{u}_r\gamma_\mu\gamma_5 V_{rp}d_p)s_wA_\nu] \\
& -(\bar{u}_r\sigma^{\mu\nu}d_p)\,\big[\partial_\mu(\bar{d}_p\gamma_\nu\gamma_5 V^*_{rp}u_r)+ig(\bar{d}_p\gamma_\mu\gamma_5 V^*_{rp}u_r)s_wA_\nu\big]\big\}\,.
\end{aligned}
\tag{C.13}
$$

Comparing with de Vries [244] where they analyzed the flavor-conserving P-odd and CP-odd operators with light quarks, our operators in Eq.(C.9) is identical with their effective dimension-6 P-odd and CP-odd operators when the generation indices of our operators is $p = 1$ and $r = 1$. For our operators in Eq. (C.6) and Eq. (C.7), Ref. [244] claimed that they can be absorbed to the $\theta$ and Yukawa terms separately with redefinitions of the coefficients in the mass-dimension-4 SM Lagrangian. For the operator in Eq. (C.8), Ref. [244] added coefficients with $\frac{1}{v}$ so the VEV of Higgs in their operators with the same structure does not show up.

As for the operator in Eq. (C.12), the operator in Ref. [244] with the same structure do not contain $v^2$ explicitly as discussed above.

### 7.8.4 Discussion of the Flavor-conserving Lowest-mass-dimension C-odd and CP-odd Operators

Here we show the derivation that the operator in Eq. (107) is equivalent with the operators with derivatives in Eq. (45) of our work, and how they can evolve to Eq. (108).

For operators in our Eq. (45) with the form $(\bar{q}_1\sigma^{\mu\nu}\gamma_5 q_1)\partial_\mu(\bar{q}_2\gamma_\nu\gamma_5 q_2)$ (the subscripts 1, 2 here do not indicate quark generation, they are used to distinguish different quark field), when using derivative by parts and consider the total derivative does not contribute, we have

$$
\begin{aligned}
(\bar{q}_1\sigma^{\mu\nu}\gamma_5 q_1)\partial_\mu(\bar{q}_2\gamma_\nu\gamma_5 q_2) &= -\partial_\mu(\bar{q}_1\sigma^{\mu\nu}\gamma_5 q_1)(\bar{q}_2\gamma_\nu\gamma_5 q_2) \\
&= -\left[\bar{q}_1\sigma^{\mu\nu}\gamma_5 i(p'_{1\mu}-p_{1\mu})q_1\right](\bar{q}_2\gamma_\nu\gamma_5 q_2) \\
&= \left[\bar{q}_1\sigma^{\mu\nu}\gamma_5 i(p'_{1\nu}-p_{1\nu})q_1\right](\bar{q}_2\gamma_\nu\gamma_5 q_2) \qquad \text{(D.14)}
\end{aligned}
$$

where $p'_1$ and $p_1$ are momenta of $\bar{q}_1$ and $q_1$, respectively. Thus $(\bar{q}_1\sigma^{\mu\nu}\gamma_5 q_1)\partial_\mu(\bar{q}_2\gamma_\nu\gamma_5 q_2)$ in our Eq. (45) is equivalent with Eq. (107).

In Eq. (45), we have $i(\bar{q}_1\sigma^{\mu\nu}\gamma_5 q_1)\partial_\mu(\bar{q}_2\gamma_\nu\gamma_5 q_2)$, $i(\bar{q}_1\sigma^{\mu\nu}\gamma_5 q_2)\partial_\mu(\bar{q}_2\gamma_\nu\gamma_5 q_1)+h.c.$ and $i(\bar{q}_1\sigma^{\mu\nu}q_2)\partial_\mu(\bar{q}_2\gamma_\nu q_1)+h.c.$, from Eq. (D.14), they can equivalently expressed as

$$
\begin{aligned}
i(\bar{q}_1\sigma^{\mu\nu}\gamma_5 q_1)\partial_\mu(\bar{q}_2\gamma_\nu\gamma_5 q_2) &= i[\bar{q}_1\sigma^{\mu\nu}\gamma_5(\partial_\nu+\overleftarrow{\partial}_\nu)q_1](\bar{q}_2\gamma_\mu\gamma_5 q_2), \qquad \text{(D.15)} \\
i(\bar{q}_1\sigma^{\mu\nu}\gamma_5 q_2)\partial_\mu(\bar{q}_2\gamma_\nu\gamma_5 q_1)+h.c. &= i[\bar{q}_1\sigma^{\mu\nu}\gamma_5(\partial_\nu+\overleftarrow{\partial}_\nu)q_2](\bar{q}_2\gamma_\mu\gamma_5 q_1) \\
&\quad +h.c., \qquad \text{(D.16)} \\
i(\bar{q}_1\sigma^{\mu\nu}q_2)\partial_\mu(\bar{q}_2\gamma_\nu q_1)+h.c. &= i[\bar{q}_1\sigma^{\mu\nu}(\partial_\nu+\overleftarrow{\partial}_\nu)q_2](\bar{\psi}_2\gamma_\mu q_1)+h.c. \qquad \text{(D.17)}
\end{aligned}
$$

Replacing the derivatives $\partial_\mu$ with covariant derivatives $D_\mu$, then using Gordon decompositions

$$
\begin{aligned}
\bar{q}_1\sigma^{\mu\nu}(D_\nu+\overleftarrow{D}_\nu)q_2 &= \bar{q}_1(m_1+m_2)\gamma^\mu q_2-\bar{q}_1(iD^\mu-i\overleftarrow{D}^\mu)q_2, \qquad \text{(D.18)} \\
\bar{q}_1\sigma^{\mu\nu}\gamma_5(D_\nu+\overleftarrow{D}_\nu)q_2 &= \bar{q}_1(m_1-m_2)\gamma^\mu\gamma_5 q_2-\bar{q}_1(iD^\mu-i\overleftarrow{D}^\mu)\gamma_5 q_2, \qquad \text{(D.19)}
\end{aligned}
$$

which are derived from equation of motion $(i\not{D}-m)\psi=0$, we have

$$
i[\bar{q}_1\sigma^{\mu\nu}\gamma_5(D_\nu+\overleftarrow{D}_\nu)q_1](\bar{q}_2\gamma_\mu\gamma_5 q_2)=[\bar{q}_1(D^\mu-\overleftarrow{D^\mu})q_1](\bar{q}_2\gamma_\mu\gamma_5 q_2) \qquad \text{(D.20)}
$$

$$
\begin{aligned}
& i[\bar{q}_1\sigma^{\mu\nu}(D_\nu+\overleftarrow{D}_\nu)q_2](\bar{q}_2\gamma_\mu q_1)-i[\bar{q}_2\sigma^{\mu\nu}(D_\nu+\overleftarrow{D}_\nu)q_1](\bar{q}_1\gamma_\nu q_2) \\
=\ & [\bar{q}_1(D^\mu-\overleftarrow{D}^\mu)q_2](\bar{q}_2\gamma_\mu q_1)-[\bar{q}_2(D^\mu-\overleftarrow{D}^\mu)q_1](\bar{q}_1\gamma_\mu q_2), \qquad \text{(D.21)}
\end{aligned}
$$

and

$$
\begin{aligned}
& i[\bar{q}_1\sigma^{\mu\nu}(D_\nu+\overleftarrow{D}_\nu)\gamma_5 q_2](\bar{q}_2\gamma_\mu\gamma_5 q_1)+[\bar{q}_2\sigma^{\mu\nu}\gamma_5(D_\nu+\overleftarrow{D}_\nu)q_1](\bar{q}_1\gamma_\mu\gamma_5 q_2) \\
=\ & [\bar{q}_1(D^\mu-\overleftarrow{D^\mu})\gamma_5 q_2](\bar{q}_2\gamma_\mu\gamma_5 q_1)+[\bar{q}_2(D^\mu-\overleftarrow{D^\mu})\gamma_5 q_p](\bar{q}_p\gamma_\mu\gamma_5 q_r), \qquad \text{(D.22)}
\end{aligned}
$$

where Eq. (D.20) is equivalent with Eq. (108) when $q_1=q_2\neq q_3=q_4$ and Eq. (D.22) is a linear combination of Eq. (108) when $q_1=q_3\neq q_2=q_4$, and we have one more which is Eq. (D.21) which can not obtained from Eq. (108).

In part 7.5.2, there could also be new flavor-conserving C-odd and CP-odd operators in $\psi^4\varphi D$ in the form $iv(\bar{q}_1\sigma^{\mu\nu}D_\nu q_2)(\bar{q}_2\gamma_\mu q_1)+h.c.$, $iv(\bar{q}_1\sigma^{\mu\nu}\gamma_5 D_\nu q_2)(\bar{q}_2\gamma_\mu\gamma_5 q_1)+h.c.$, but actually they are equivalent with operators $v(\bar{q}_1 D_\mu q_2)(\bar{q}_2\gamma_\mu q_1)+h.c.$ and

$v(\bar{q}_1 D_\mu \gamma_5 q_2)(\bar{q}_2\gamma_\mu\gamma_5 q_1) + h.c.$ which are already included there. We show the derivation here. Using the equation of motion $(i\not{D} - m)\psi = 0$, we have

$$
\begin{aligned}
& iv(\bar{q}_1\sigma^{\mu\nu}D_\nu q_2)(\bar{q}_2\gamma_\mu q_1) + h.c.) \\
= \; & iv\left[\bar{q}_1\frac{i}{2}(2\gamma^\mu\gamma^\nu - 2g^{\mu\nu})D_\nu q_2\right](\bar{q}_2\gamma_\mu q_1) - iv\left[\bar{q}_2\overleftarrow{D}_\nu\frac{i}{2}(2g^{\mu\nu} - 2\gamma^\nu\gamma^\mu)q_1\right](\bar{q}_1\gamma_\mu q_2) \\
= \; & iv\left[\bar{q}_1(m_2\gamma^\mu - iD^\mu)q_2\right](\bar{q}_2\gamma_\mu q_1) - iv\left[\bar{q}_2(i\overleftarrow{D}^\mu + m_2\gamma^\mu)q_1\right](\bar{q}_1\gamma_\mu q_2), \\
= \; & iv\left[(\bar{q}_1 D^\mu q_2)(\bar{q}_2\gamma_\mu q_1) + (q_2 D^\mu q_1)(\bar{q}_1\gamma_\mu q_2)\right], \qquad \text{(D.23)} \\
& iv(\bar{q}_1\sigma^{\mu\nu}\gamma_5 D_\nu q_2)(\bar{q}_2\gamma_\mu\gamma_5 q_1) + h.c.) \\
= \; & iv\left[\bar{q}_1\frac{i}{2}(2\gamma^\mu\gamma^\nu - 2g^{\mu\nu})\gamma_5 D_\nu q_2\right](\bar{q}_2\gamma_\mu\gamma_5 q_1) \\
+ \; & iv\left[\bar{q}_2\overleftarrow{D}_\nu\gamma_5\frac{i}{2}(2g^{\mu\nu} - 2\gamma^\nu)\gamma^\mu q_1\right](\bar{q}_1\gamma_\mu\gamma_5 q_2) \\
= \; & iv\left[\bar{q}_1(m_2\gamma^\mu\gamma_5 - iD^\mu\gamma_5)q_2\right](\bar{q}_2\gamma_\mu\gamma_5 q_1) + iv\left[\bar{q}_2(i\overleftarrow{D}^\mu - m_2\gamma^\mu)\gamma_5 q_1\right](\bar{q}_1\gamma_\mu\gamma_5 q_2), \\
= \; & iv\left[(\bar{q}_1 D^\mu\gamma_5 q_2)(\bar{q}_2\gamma_\mu\gamma_5 q_1) - (q_2\overleftarrow{D}^\mu\gamma_5 q_1)(\bar{q}_1\gamma_\mu\gamma_5 q_2)\right], \qquad \text{(D.24)}
\end{aligned}
$$

### 7.8.5 Notes for Mass-dimension 8 SMEFT Operators

We show some of our notes when deriving the C-odd and CP-odd operators from the mass-dimension 8 SMEFT and the list the flavor-conserving C-odd and CP-odd operators we found.

According to the fundamental fields' transformation properties listed in Appendix 7.8.1, operators from classes $X^4$, $X^3\varphi^2$, $X^2\varphi^4$, $X^2\varphi^2 D^2$, $\varphi^8$, $\varphi^6 D^2$, $\varphi^4 D^4$ and $\psi^2\varphi^5$ obviously do not contain quark-flavor-conserving C-odd and CP-odd operators we are searching for. The C-odd and CP-odd operators could probably come from classes $X\varphi^4 D^2$, $\psi^4\varphi^2$, $\psi^4 X$, $\psi^2 X^2\varphi$, $\psi^2 X\varphi^3$, $\psi^4\varphi D$, $\psi^2\varphi^4 D$, $\psi^2\varphi^3 D^2$, $\psi^2\varphi^2 D^3$, $\psi^2 X^2 D$, $\psi^2 X\varphi^2 D$, $\psi^2 X\varphi D^2$, and $\psi^4 D^2$.

For operators composed of odd number of $D_\mu$, Higgs and only one quark-bilinear, acting $D_\mu$ on any quark field would produce Equations of Motion (EOMs) [101] and move $D\psi$ to $\varphi\psi$, i.e. to other classes with lower derivatives. Thus for such classes it is appropriate to consider the bachelor covariant-derivative on scalar fields only. After electroweak symmetry is spontaneously broken, since

$$
\begin{aligned}
D_\mu\varphi &= (\partial_\mu - ig\frac{\tau^I}{2}W^I_\mu - ig'YB_\mu)\varphi \\
&\sim \frac{1}{\sqrt{2}}v(-\partial\mu + \frac{1}{2}igW^3_\mu - \frac{1}{2}ig'B_\mu) \\
&= \frac{1}{\sqrt{2}}v(-\partial\mu + \frac{1}{2}i\sqrt{g^2+g'^2}Z_\mu), \qquad \text{(E.25)}
\end{aligned}
$$

where $Z_\mu$ will be integrated out at energies lower than the weak gauge bosons and result in higher dimensional operators, and the minus sign accompanying the derivative appears when we use derivative by parts, given total derivatives do not contribute, and it results with a derivative to the quark-current. Also, we have

$$\begin{aligned}
D_\mu D_\nu \varphi &= \Big[\partial_\mu\partial_\nu - \partial_\mu(\frac{i}{2}g\tau^3 W^3_\nu + \frac{i}{2}g' B_\nu) - \partial_\nu(\frac{i}{2}g\tau^3 W^3_\mu + \frac{i}{2}g' B_\mu) \\
&- \frac{1}{4}g^2\tau^I\tau^J W^I_\mu W^J_\nu - \frac{1}{4}g'^2 B_\mu B_\nu - \frac{1}{4}gg'\tau^3 W^3_\mu B_\nu - \frac{1}{4}gg'\tau^3 W^3_\nu B_\mu\Big]\varphi \\
&= \frac{1}{\sqrt{2}}v\Big[\partial_\mu\partial_\nu - \frac{1}{2}i\sqrt{g^2+g'^2}(Z_\mu\partial\nu + Z_\nu\partial_\mu) + \frac{1}{4}g^2(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu) \\
&\quad -\frac{1}{4}g'^2 B_\mu B_\nu + \frac{1}{4}gg'(W^3_\mu B_\nu + B_\mu W^3_\nu)\Big] \\
&= \frac{1}{\sqrt{2}}v\Big\{\partial_\mu\partial_\nu - \frac{1}{2}i\sqrt{g^2+g'^2}(Z_\mu\partial\nu + Z_\nu\partial_\mu) + \frac{1}{4}g^2(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu) \\
&- \frac{1}{4}g'^2[c_w^2 A_\mu A_\nu + s_w^2 Z_\mu Z_\nu - s_w c_w(Z_\mu A_\nu + A_\mu Z_\nu)] \\
&+ \frac{1}{4}gg'[(c_w^2 - s_w^2)(Z_\mu A_\nu + A_\mu Z_\nu) + 2s_w c_w(A_\mu A_\nu - Z_\mu Z_\nu)]\Big\} \qquad \text{(E.26)}
\end{aligned}$$

Thus after integrating out weak gauge bosons, the remaining pieces of $D_\mu D_\nu\varphi$ contributing to dimension-8 C-odd and CP-odd operators are $\frac{1}{\sqrt{2}}v[\partial_\mu\partial_\nu - \frac{1}{4}g'^2 c_w^2 A_\mu A_\nu + \frac{1}{2}gg' s_w c_w A_\mu A_\nu] = \frac{1}{\sqrt{2}}v[\partial_\mu\partial_\nu + \frac{1}{4}\frac{g^2 g'^2}{g^2+g'^2}A_\mu A_\nu]$.

The following are some important notes for classes $X\varphi^4 D^2$, $\psi^4\varphi^2$, $\psi^2 X^2\varphi$, $\psi^2 X^2\varphi$, $\psi^2 X\varphi^3$, $\psi^2 X^2 D$ and $\psi^4 D^2$, where "none" means there is no flavor-conserving C-odd and CP-odd operators in that class.

- $X\varphi^4 D^2$: none (For example $[(\varphi^\dagger D_\mu\varphi)^*(\varphi^\dagger D_\nu\varphi) - (\varphi^\dagger D_\nu\varphi)^*(\varphi^\dagger D_\mu\varphi)]F^{\mu\nu}$ does not contribute after electroweak symmetry is spontaneously broken, since $D_\mu$ acting on $X_{\mu\nu}$ would result in EOMs.)

- $\psi^4\varphi^2$: none

- $\psi^2 X^2\varphi$: Note that $v\bar{u}_p\sigma_{\mu\nu}u_p F^{\mu\alpha}F_\alpha{}^\nu$=0, since $F^{\mu\alpha}F_\alpha{}^\nu = \partial^\mu F^\alpha\partial_\alpha F^\nu + \partial^\alpha F^\mu\partial^\nu F_\alpha - \partial^\alpha F^\mu\partial_\alpha F^\nu - \partial^\mu F^\alpha\partial^\nu F_\alpha$ is symmetric while $\sigma_{\mu\nu}$ is antisymmetric, analogous for $v\epsilon^{\mu\nu\alpha\beta}\bar{u}_p\sigma_{\mu\nu}\gamma_5 u_p F^{\alpha\rho}F^\beta_\rho$

- $\psi^2 X\varphi^3$: none (For example $v^3\bar{u}_p\sigma^{\mu\nu}u_p F_{\mu\nu}$ is T even, $v^3\epsilon^{\mu\nu\alpha\beta}\bar{u}_p\sigma_{\mu\nu}u_p F_{\alpha\beta}$ is T odd and P odd,
  $iv^3\epsilon^{\mu\nu\alpha\beta}\bar{u}_p\sigma_{\mu\nu}\gamma_5 u_p F_{\alpha\beta}$ is T even)

- $\psi^2 X^2 D$: Note that acting $D_\mu$ to either $\bar{\psi}\gamma^\mu\psi$ or $X_{\mu\nu}$ would result in EOMs [101] and move operators to other classes.

- $\psi^4 D^2$: Note that $(\bar{u}_p\sigma^{\mu\nu}u_r)(\bar{u}_r\overleftarrow{D}_\mu D_\nu u_p)$ would result in total derivative plus EOMs and antisymmetric quark bilinear times symmetric quark bilinear, i.e. $(\bar{u}_p\sigma^{\mu\nu}u_r)(\bar{u}_r D_\mu D_\nu u_p) = 0$). Also, note that $(\bar{u}_p\gamma^\mu D_\nu u_r)(\bar{u}_r\gamma^\nu D_\mu u_p)$ is equivalent with the addition of total derivative and EOMs.
  $(\bar{u}_p\gamma^\mu\overleftrightarrow{D}_\nu u_r)(\bar{u}_r\gamma^\nu\overleftrightarrow{D}_\mu u_p)$ is hermitian and T even. $(\bar{u}_p\gamma^\mu\gamma_5\overleftrightarrow{D}_\nu u_p)(\bar{u}_r\gamma^\nu\overleftrightarrow{D}_\mu u_r)$ is hermitian, T even, P odd and C odd.

We show the flavor-conserving C-odd and CP-odd operators that we found as follows. Operators without the imaginary unit $i$ indicates the operator is C-odd and CP-odd with the real part of coefficients, for example hermitian operators, while operators with $i$ are C-odd and CP-odd only when adding $i$ times the imaginary parts of the coefficients to them.

i) Operators that are new:

$$\psi^4 X: \quad (\bar{u}_p\gamma_\mu u_p)(\bar{u}_r\gamma_\nu u_r)F^{\mu\nu}\ (p\neq r), \tag{E.27}$$

$$(\bar{u}_p\gamma_\mu\gamma_5 u_p)(\bar{u}_r\gamma_\nu\gamma_5 u_r)F^{\mu\nu}\ (p\neq r), \tag{E.28}$$

$$(\bar{d}_p\gamma_\mu d_p)(\bar{d}_r\gamma_\nu d_r)F^{\mu\nu}\ (p\neq r), \tag{E.29}$$

$$(\bar{d}_p\gamma_\mu\gamma_5 d_p)(\bar{d}_r\gamma_\nu\gamma_5 d_r)F^{\mu\nu}\ (p\neq r), \tag{E.30}$$

$$i[(\bar{u}_p\gamma_\mu d_r)(\bar{d}_r\gamma_\nu u_p)-(\bar{d}_r\gamma_\mu u_p)(\bar{u}_p\gamma_\nu d_r)]F^{\mu\nu}, \tag{E.31}$$

$$i[(\bar{u}_p\gamma_\mu d_r)(\bar{d}_r\gamma_\nu u_p)-(\bar{d}_r\gamma_\mu u_p)(\bar{u}_p\gamma_\nu d_r)]F^{\mu\nu}, \tag{E.32}$$

$$i[(\bar{u}_p\gamma_\mu u_r)(\bar{u}_r\gamma_\nu u_p)-(\bar{u}_r\gamma_\mu u_p)(\bar{u}_p\gamma_\nu u_r)]F^{\mu\nu}\ (p\neq r), \tag{E.33}$$

$$i[(\bar{u}_p\gamma_\mu\gamma_5 u_r)(\bar{u}_r\gamma_\nu\gamma_5 u_p)-(\bar{u}_r\gamma_\mu\gamma_5 u_p)(\bar{u}_p\gamma_\nu\gamma_5 u_r)]F^{\mu\nu}\ (p\neq r), \tag{E.34}$$

$$i[(\bar{d}_p\gamma_\mu d_r)(\bar{d}_r\gamma_\nu d_p)-(\bar{d}_r\gamma_\mu d_p)(\bar{d}_p\gamma_\nu d_r)]F^{\mu\nu}\ (p\neq r), \tag{E.35}$$

$$i[(\bar{d}_p\gamma_\mu\gamma_5 d_r)(\bar{d}_r\gamma_\nu\gamma_5 d_p)-(\bar{d}_r\gamma_\mu\gamma_5 d_p)(\bar{d}_p\gamma_\nu\gamma_5 d_r)]F^{\mu\nu}\ (p\neq r), \tag{E.36}$$

$$(\bar{u}_p\sigma_{\mu\nu}u_p)(\bar{u}_r u_r)F^{\mu\nu}\ (p\neq r), \tag{E.37}$$

$$(\bar{d}_p\sigma_{\mu\nu}d_p)(\bar{d}_r d_r)F^{\mu\nu}\ (p\neq r), \tag{E.38}$$

$$i[(\bar{u}_p\sigma_{\mu\nu}u_r)(\bar{u}_r u_p)-(\bar{u}_r\sigma_{\mu\nu}u_p)(\bar{u}_p u_r)]F^{\mu\nu}\ (p\neq r), \tag{E.39}$$

$$i[(\bar{u}_p\sigma_{\mu\nu}\gamma_5 u_r)(\bar{u}_r\gamma_5 u_p)-(\bar{u}_r\sigma_{\mu\nu}\gamma_5 u_p)(\bar{u}_p\gamma_5 u_r)]F^{\mu\nu}\ (p\neq r), \tag{E.40}$$

$$i[(\bar{d}_p\sigma_{\mu\nu}d_r)(\bar{d}_r d_p)-(\bar{d}_r\sigma_{\mu\nu}d_p)(\bar{d}_p d_r)]F^{\mu\nu}\ (p\neq r), \tag{E.41}$$

$$i[(\bar{d}_p\sigma_{\mu\nu}\gamma_5 d_r)(\bar{d}_r\gamma_5 d_p)-(\bar{d}_r\sigma_{\mu\nu}\gamma_5 d_p)(\bar{d}_p\gamma_5 d_r)]F^{\mu\nu}\ (p\neq r), \tag{E.42}$$

$$i[(\bar{u}_p\sigma_{\mu\nu}d_r)(\bar{d}_r u_p)-(\bar{d}_r\sigma_{\mu\nu}u_p)(\bar{u}_p d_r)]F^{\mu\nu}, \tag{E.43}$$

$$i[(\bar{u}_p\sigma_{\mu\nu}\gamma_5 d_r)(\bar{d}_r\gamma_5 u_p)-(\bar{d}_r\sigma_{\mu\nu}\gamma_5 u_p)(\bar{u}_p\gamma_5 d_r)]F^{\mu\nu}, \tag{E.44}$$

$$(\bar{u}_p\gamma^\mu u_p)(\bar{u}_r\gamma^\nu T^A u_r)G^A_{\mu\nu}\ (p\neq r), \tag{E.45}$$

$$(\bar{u}_p\gamma^\mu\gamma_5 u_p)(\bar{u}_r\gamma^\nu\gamma_5 T^A u_r)G^A_{\mu\nu}\ (p\neq r), \tag{E.46}$$

$$(\bar{d}_p\gamma^\mu d_p)(\bar{d}_r\gamma^\nu T^A d_r)G^A_{\mu\nu}\ (p\neq r), \tag{E.47}$$

$$(\bar{d}_p\gamma_\mu\gamma_5 d_p)(\bar{d}_r\gamma_\nu\gamma_5 T^A d_r)G^A_{\mu\nu}\ (p\neq r), \tag{E.48}$$

$$i[(\bar{u}_p\gamma_\mu d_r)(\bar{d}_r\gamma_\nu T^A u_p)-(\bar{d}_r\gamma_\mu u_p)(\bar{u}_p\gamma_\nu T^A d_r)]G^A_{\mu\nu} \tag{E.49}$$

$$i[(\bar{u}_p\gamma_\mu d_r)(\bar{d}_r\gamma_\nu T^A u_p)-(\bar{d}_r\gamma_\mu u_p)(\bar{u}_p\gamma_\nu T^A d_r)]G^A_{\mu\nu} \tag{E.50}$$

$$i[(\bar{u}_p\gamma_\mu u_r)(\bar{u}_r\gamma_\nu T^A u_p)-(\bar{u}_r\gamma_\mu u_p)(\bar{u}_p\gamma_\nu T^A u_r)]G^A_{\mu\nu}\ (p\neq r), \tag{E.51}$$

$$i[(\bar{u}_p\gamma_\mu\gamma_5 u_r)(\bar{u}_r\gamma_\nu\gamma_5 T^A u_p)-(\bar{u}_r\gamma_\mu\gamma_5 u_p)(\bar{u}_p\gamma_\nu\gamma_5 T^A u_r)]G^A_{\mu\nu}\ (p\neq r), \tag{E.52}$$

$$i[(\bar{d}_p\gamma_\mu d_r)(\bar{d}_r\gamma_\nu d_p)-(\bar{d}_r\gamma_\mu d_p)(\bar{d}_p\gamma_\nu d_r)]G^A_{\mu\nu}\ (p\neq r), \tag{E.53}$$

$$i[(\bar{d}_p\gamma_\mu\gamma_5 d_r)(\bar{d}_r\gamma_\nu\gamma_5 d_p)-(\bar{d}_r\gamma_\mu\gamma_5 d_p)(\bar{d}_p\gamma_\nu\gamma_5 d_r)]G^A_{\mu\nu}\ (p\neq r), \tag{E.54}$$

$$(\bar{u}_p\sigma_{\mu\nu}u_p)(\bar{u}_r T^A u_r)G^A_{\mu\nu}\ (p\neq r), \tag{E.55}$$

$$(\bar{d}_p\sigma_{\mu\nu}d_p)(\bar{d}_r T^A d_r)G^A_{\mu\nu}\ (p\neq r), \tag{E.56}$$

$$i[(\bar{u}_p\sigma^{\mu\nu}u_r)(\bar{u}_r T^A u_p)-(\bar{u}_r\sigma^{\mu\nu}u_p)(\bar{u}_p T^A u_r)]G^A_{\mu\nu}, \tag{E.57}$$

$$i[(\bar{u}_p\sigma^{\mu\nu}\gamma_5 u_r)(\bar{u}_r\gamma_5 T^A u_p)-(\bar{u}_r\sigma^{\mu\nu}\gamma_5 u_p)(\bar{u}_p\gamma_5 T^A u_r)]G^A_{\mu\nu}, \tag{E.58}$$

$$i[(\bar{d}_p\sigma^{\mu\nu}d_r)(\bar{d}_r T^A d_p)-(\bar{d}_r\sigma^{\mu\nu}d_p)(\bar{d}_p T^A d_r)]G^A_{\mu\nu}, \tag{E.59}$$

$$i[(\bar{d}_p\sigma^{\mu\nu}\gamma_5 d_r)(\bar{d}_r\gamma_5 T^A d_p)-(\bar{d}_r\sigma^{\mu\nu}\gamma_5 T^A d_p)(\bar{d}_p\gamma_5 d_r)]G^A_{\mu\nu}, \tag{E.60}$$

$$i[(\bar{u}_p\sigma^{\mu\nu}d_r)(\bar{d}_r T^A u_p) - (\bar{d}_r\sigma^{\mu\nu}u_p)(\bar{u}_p T^A d_r)]G^A_{\mu\nu}, \tag{E.61}$$
$$i[(\bar{u}_p\sigma^{\mu\nu}\gamma_5 d_r)(\bar{d}_r\gamma_5 T^A u_p) - (\bar{d}_r\sigma^{\mu\nu}\gamma_5 u_p)(\bar{u}_p\gamma_5 T^A d_r)]G^A_{\mu\nu}. \tag{E.62}$$

$$\psi^2 X^2\varphi: \qquad v\bar{u}_p\sigma_{\mu\nu}T^A u_p F^{\mu\alpha}G_\alpha^{\ \nu}, \tag{E.63}$$
$$\bar{d}_p\sigma_{\mu\nu}T^A d_p F^{\mu\alpha}G_\alpha^{\ \nu}. \tag{E.64}$$

$$\psi^2 X^2 D: \qquad (\bar{u}_p\gamma_\mu D_\nu u_p)G^{A\mu\rho}G^{A\nu}_\rho, \tag{E.65}$$
$$(\bar{d}_p\gamma_\mu D_\nu d_p)G^{A\mu\rho}G^{A\nu}_\rho, \tag{E.66}$$
$$(\bar{u}_p\gamma_\mu D_\nu u_p)F^{\mu\rho}F_\rho^{\ \nu}, \tag{E.67}$$
$$(\bar{d}_p\gamma_\mu D_\nu d_p)F^{\mu\rho}F_\rho^{\ \nu}, \tag{E.68}$$
$$(\bar{u}_p\gamma_\mu T^A D_\nu u_p)F^{\mu\rho}G^{A\nu}_\rho, \tag{E.69}$$
$$(\bar{d}_p\gamma_\mu T^A D_\nu d_p)F^{\mu\rho}G^{A\nu}_\rho. \tag{E.70}$$

$$\psi^2 X\varphi D^2: \qquad v\partial_\nu(\bar{d}_p D_\mu d_p)F^{\mu\nu} + v\partial_\nu(\bar{d}_p\overleftarrow{D}_\mu d_p)F^{\mu\nu}, \tag{E.71}$$
$$v\partial_\nu(\bar{u}_p D_\mu u_p)F^{\mu\nu} + v\partial_\nu(\bar{u}_p\overleftarrow{D}_\mu u_p)F^{\mu\nu}, \tag{E.72}$$
$$v\partial_\nu(\bar{u}_p T^A D_\mu u_p)G^{A\mu\nu} + v\partial_\nu(\bar{u}_p\overleftarrow{D}_\mu T^A u_p)G^{A\mu\nu}, \tag{E.73}$$
$$v\partial_\nu(\bar{d}_p T^A D_\mu d_p)G^{A\mu\nu} + v\partial_\nu(\bar{d}_p T^A\overleftarrow{D}_\mu T^A d_p)G^{A\mu\nu}, \tag{E.74}$$
$$i\epsilon^{\mu\nu\alpha\beta}v\partial_\nu(\bar{d}_p\gamma_5 D_\mu d_p)F_{\alpha\beta} - i\epsilon^{\mu\nu\alpha\beta}v\partial_\nu(\bar{d}_p\overleftarrow{D}_\mu\gamma_5 d_p)F_{\alpha\beta}, \tag{E.75}$$
$$i\epsilon^{\mu\nu\alpha\beta}v\partial_\nu(\bar{u}_p\gamma_5 D_\mu u_p)F_{\alpha\beta} - i\epsilon^{\mu\nu\alpha\beta}v\partial_\nu(\bar{u}_p\overleftarrow{D}_\mu\gamma_5 u_p)F_{\alpha\beta}, \tag{E.76}$$
$$i\epsilon^{\mu\nu\alpha\beta}v\partial_\nu(\bar{u}_p\gamma_5 T^A D_\mu u_p)G_{A\alpha\beta} - i\epsilon^{\mu\nu\alpha\beta}v\partial_\nu(\bar{u}_p\overleftarrow{D}_\mu\gamma_5 T^A u_p)G_{A\alpha\beta}, \tag{E.77}$$
$$i\epsilon^{\mu\nu\alpha\beta}v\partial_\nu(\bar{d}_p\gamma_5 T^A D_\mu d_p)G_{A\alpha\beta} - i\epsilon^{\mu\nu\alpha\beta}v\partial_\nu(\bar{d}_p\overleftarrow{D}_\mu\gamma_5 T^A d_p)G_{A\alpha\beta}. \tag{E.78}$$

$$\psi^4 D^2: \quad \epsilon^{\mu\nu\alpha\beta}(\bar{u}_p\gamma^\mu\gamma_5 \overleftrightarrow{D}_\nu u_p)(\bar{u}_r\gamma^\alpha \overleftrightarrow{D}_\beta u_r), \tag{E.79}$$

$$\epsilon^{\mu\nu\alpha\beta}(\bar{d}_p\gamma^\mu\gamma_5 \overleftrightarrow{D}_\nu d_p)(\bar{d}_r\gamma^\alpha \overleftrightarrow{D}_\beta d_r), \tag{E.80}$$

$$\epsilon^{\mu\nu\alpha\beta}(\bar{u}_p\gamma^\mu\gamma_5 \overleftrightarrow{D}_\nu u_r)(\bar{u}_r\gamma^\alpha \overleftrightarrow{D}_\beta u_p) + \epsilon^{\mu\nu\alpha\beta}(\bar{u}_r\gamma^\mu\gamma_5 \overleftrightarrow{D}_\nu u_p)(\bar{u}_p\gamma^\alpha \overleftrightarrow{D}_\beta u_r), \tag{E.81}$$

$$\epsilon^{\mu\nu\alpha\beta}(\bar{d}_p\gamma^\mu\gamma_5 \overleftrightarrow{D}_\nu d_r)(\bar{d}_r\gamma^\alpha \overleftrightarrow{D}_\beta d_p) + \epsilon^{\mu\nu\alpha\beta}(\bar{d}_r\gamma^\mu\gamma_5 \overleftrightarrow{D}_\nu d_p)(\bar{d}_p\gamma^\alpha \overleftrightarrow{D}_\beta d_r), \tag{E.82}$$

$$\epsilon^{\mu\nu\alpha\beta}(\bar{u}_p\gamma^\mu\gamma_5 \overleftrightarrow{D}_\nu d_r)(\bar{d}_r\gamma^\alpha \overleftrightarrow{D}_\beta u_p) + \epsilon^{\mu\nu\alpha\beta}(\bar{d}_r\gamma^\mu\gamma_5 \overleftrightarrow{D}_\nu u_p)(\bar{u}_p\gamma^\alpha \overleftrightarrow{D}_\beta d_r), \tag{E.83}$$

$$\epsilon^{\mu\nu\alpha\beta}(\bar{u}_p\sigma_{\mu\nu}u_r)(\bar{u}_r \overleftarrow{D}_\alpha D_\beta u_p) + \epsilon^{\mu\nu\alpha\beta}(\bar{u}_r\sigma_{\mu\nu}u_p)(\bar{u}_p \overleftarrow{D}_\beta D_\alpha u_r), \tag{E.84}$$

$$\epsilon^{\mu\nu\alpha\beta}(\bar{d}_p\sigma_{\mu\nu}d_r)(\bar{d}_r \overleftarrow{D}_\alpha D_\beta d_p) + \epsilon^{\mu\nu\alpha\beta}(\bar{d}_r\sigma_{\mu\nu}d_p)(\bar{d}_p \overleftarrow{D}_\beta D_\alpha d_r), \tag{E.85}$$

$$i\epsilon^{\mu\nu\alpha\beta}(\bar{u}_p\sigma_{\mu\nu}\gamma_5 u_r)(\bar{u}_r \overleftarrow{D}_\alpha D_\beta u_p) + i\epsilon^{\mu\nu\alpha\beta}(\bar{u}_r\sigma_{\mu\nu}\gamma_5 u_p)(\bar{u}_p \overleftarrow{D}_\beta D_\alpha u_r), \tag{E.86}$$

$$i\epsilon^{\mu\nu\alpha\beta}(\bar{d}_p\sigma_{\mu\nu}\gamma_5 d_r)(\bar{d}_r \overleftarrow{D}_\alpha D_\beta d_p) + i\epsilon^{\mu\nu\alpha\beta}(\bar{d}_r\sigma_{\mu\nu}\gamma_5 d_p)(\bar{d}_p \overleftarrow{D}_\beta D_\alpha d_r), \tag{E.87}$$

$$\epsilon^{\mu\nu\alpha\beta}(\bar{u}_p\sigma_{\mu\nu}d_r)(\bar{d}_r \overleftarrow{D}_\alpha D_\beta u_p) \quad + \epsilon^{\mu\nu\alpha\beta}(\bar{d}_r\sigma_{\mu\nu}u_p)(\bar{u}_p \overleftarrow{D}_\beta D_\alpha d_r, ), \tag{E.88}$$

$$i\epsilon^{\mu\nu\alpha\beta}(\bar{u}_p\sigma_{\mu\nu}\gamma_5 d_r)(\bar{d}_r \overleftarrow{D}_\alpha D_\beta u_p) + i\epsilon^{\mu\nu\alpha\beta}(\bar{d}_r\sigma_{\mu\nu}\gamma_5 u_p)(\bar{u}_p \overleftarrow{D}_\beta D_\alpha d_r). \tag{E.89}$$

$\psi^2 X\varphi^2 D$ : Eq. (46) and Eq. (47).
$\psi^4\varphi D$ : Eq. (48) - Eq. (89).

$$\psi^2\varphi^2 D^3: \quad \frac{1}{4\sqrt{2}}v\frac{g^2g'^2}{g^2+g'^2}(\bar{u}_p\gamma^\mu D^\nu u_p)A_\mu A_\nu, \tag{E.90}$$

$$\frac{1}{4\sqrt{2}}v\frac{g^2g'^2}{g^2+g'^2}(\bar{d}_p\gamma^\mu D^\nu d_p)A_\mu A_\nu. \tag{E.91}$$

$$\psi^2\varphi^3 D^2: \quad v^3\frac{1}{8\sqrt{2}}\frac{g^2g'^2}{g^2+g'^2}(\bar{u}_p\sigma^{\mu\nu}u_p)A_\mu A_\nu, \tag{E.92}$$

$$v^3\frac{1}{8\sqrt{2}}\frac{g^2g'^2}{g^2+g'^2}(\bar{d}_p\sigma^{\mu\nu}d_p)A_\mu A_\nu. \tag{E.93}$$

ii) Operators that are already found from analyzing mass-dimension 6 SMEFT in this work:
$\psi^2 X\varphi^2 D$ : Eq. (90) and Eq. (91).
$\psi^4\varphi D$ : Eq. (92) - Eq. (97).

iii) Operators that are total derivatives and may not really contribute:

$$\psi^2\varphi^4 D: \quad v^4\partial_\mu(\bar{u}_p\gamma^\mu u_p), \tag{E.94}$$

$$v^4\partial_\mu(\bar{d}_p\gamma^\mu u_p). \tag{E.95}$$

$$\psi^2\varphi^3D^2: \qquad \frac{1}{2\sqrt{2}}v^3\partial_\mu\partial_\nu(\bar{u}_p\sigma^{\mu\nu}u_p), \tag{E.96}$$

$$\frac{1}{2\sqrt{2}}v^3\partial_\mu\partial_\nu(\bar{d}_p\sigma^{\mu\nu}d_p). \tag{E.97}$$

$$\psi^2\varphi^2D^3: \qquad \frac{1}{\sqrt{2}}v\partial_\mu\partial_\nu(\bar{u}_p\gamma^\mu D^\nu u_p), \tag{E.98}$$

$$\frac{1}{\sqrt{2}}v\partial_\mu\partial_\nu(\bar{d}_p\gamma^\mu D^\nu d_p). \tag{E.99}$$

CHAPTER 8

# MATCHING QUARK-LEVEL OPERATORS ONTO OPERATORS IN THE CHIRAL PERTURBATION THEORY

We have derived the C and CP violating operators that can contribute to a charge asymmetry in $\eta \to \pi^+\pi^-\pi^0$ decay in Ch. 7. The operators we found are written in quark degrees of freedom. It is more practical to re-express them in terms of meson degrees of freedom, in order to facilitate the computation of their contribution to meson decay amplitudes in ChPT.

In this chapter, we introduce the method of matching operators at the quark-level onto ChPT in which the operators are on the meson-level. Such a method can be used to calculate C and CP violating amplitudes for $\eta \to \pi^+\pi^-\pi^0$.

## 8.1 Chiral Lagrangian with tensor sources

The CP odd operators we obtained in Ch. 7 contain quark bilinears with tensor Lorentz structures. Before determining how operators can be realized in meson degrees of freedom, the chiral Lagrangian with tensor sources need to be introduced first.

The extension of ChPT to include tensor sources was developed by Catà and Mateu [191]. Here we briefly discuss the derivation and result. As an extension of the external field analysis of Gasser and Leutwyler [188] we discussed in Ch. 5, including an external tensor field to $\mathcal{L}_{\rm ext}$ in Eq. (5.27), we have

$$\begin{aligned}\mathcal{L}_{\rm ext} &= \bar{q}\gamma_\mu(v^\mu + \gamma_5 a^\mu)q - \bar{q}(s - i\gamma_5 p)q + \bar{q}\sigma_{\mu\nu}\bar{t}^{\mu\nu}q \\ &= \bar{q}_R\gamma_\mu(v^\mu + a^\mu)q_R + \bar{q}_L\gamma_\mu(v^\mu - a^\mu)q_L - \bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R \\ &\quad +\bar{q}_R\sigma^{\mu\nu}t_{\mu\nu}q_L + \bar{q}_L\sigma^{\mu\nu}t^\dagger_{\mu\nu}q_R, \end{aligned} \tag{1}$$

where the tensor part has the relation

$$\begin{aligned}\bar{t}^{\mu\nu} &= P_L^{\mu\nu\lambda\rho}t_{\lambda\rho} + P_R^{\mu\nu\lambda\rho}t^\dagger_{\lambda\rho}, \\ t^{\mu\nu} &= P_L^{\mu\nu\lambda\rho}\bar{t}^{\lambda\rho}, \end{aligned} \tag{2}$$

with the projection operators defined as

$$\begin{aligned}P_R^{\mu\nu\lambda\rho} &= \frac{1}{4}(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho} + i\varepsilon^{\mu\nu\lambda\rho}), \\ P_L^{\mu\nu\lambda\rho} &= (P_R^{\mu\nu\lambda\rho})^\dagger, \end{aligned} \tag{3}$$

which has the general projector properties

$$\begin{aligned}P_R^{\mu\nu\lambda\rho}P_R^{\lambda\rho\alpha\beta} &= P_R^{\mu\nu\alpha\beta}, \\ P_L^{\mu\nu\lambda\rho}P_L^{\lambda\rho\alpha\beta} &= P_R^{\mu\nu\alpha\beta}, \\ P_L^{\mu\nu\lambda\rho}P_R^{\lambda\rho\alpha\beta} &= 0. \end{aligned} \tag{4}$$

$P_R^{\mu\nu\lambda\rho}$ and $P_L^{\mu\nu\lambda\rho}$ are analogs of $P_{L,R}$ on account of the following discussion. Since $\gamma_5\sigma^{\mu\nu} = \sigma^{\mu\nu}\gamma_5 = \frac{i}{2}\varepsilon^{\mu\nu\lambda\rho}\sigma_{\lambda\rho}$, where $\varepsilon^{\mu\nu\lambda\rho}$ is an antisymmetric symbol with $\varepsilon^{0123} = 1$, we have

$$\begin{aligned}
\sigma_{\mu\nu}\bar{t}^{\mu\nu} &= \sigma_{\mu\nu}\left[\frac{1}{4}(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho} - i\varepsilon^{\mu\nu\lambda\rho})t_{\lambda\rho} + \frac{1}{4}(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho} + i\varepsilon^{\mu\nu\lambda\rho})t^\dagger_{\lambda\rho}\right] \\
&= \frac{1}{4}(\sigma^{\lambda\rho} - \sigma^{\rho\lambda} - 2\sigma^{\lambda\rho}\gamma_5)t_{\lambda\rho} + \frac{1}{4}(\sigma^{\lambda\rho} - \sigma^{\rho\lambda} + 2\sigma^{\lambda\rho}\gamma_5)t^\dagger_{\lambda\rho} \\
&= \frac{1}{2}\sigma^{\lambda\rho}t_{\lambda\rho}(1-\gamma_5) + \frac{1}{2}\sigma^{\lambda\rho}t^\dagger_{\lambda\rho}(1+\gamma_5),
\end{aligned}$$

such that

$$\begin{aligned}
\sigma_{\mu\nu}\bar{t}^{\mu\nu}P_L &= \sigma^{\mu\nu}t_{\mu\nu}\frac{1}{2}(1-\gamma_5) = \sigma^{\mu\nu}t_{\mu\nu}P_L \\
\sigma_{\mu\nu}\bar{t}^{\mu\nu}P_R &= \sigma^{\mu\nu}t^\dagger_{\mu\nu}\frac{1}{2}(1+\gamma_5) = \sigma^{\mu\nu}t^\dagger_{\mu\nu}P_R. \qquad (5)
\end{aligned}$$

Eq. (2) and Eq. (5) indicate that $t_{\mu\nu}$ and $t^\dagger_{\mu\nu}$ are the left- and right-handed projections of the tensor field.

As discussed in Ch. 5, introducing external fields result in promoting the chiral symmetry to a local one [188, 190, 182, 191], and the transformation properties of the external fields are

$$\begin{aligned}
r_\mu &\mapsto U_R r_\mu U_R^\dagger + iU_R\partial_\mu U_R^\dagger \\
l_\mu &\mapsto U_L l_\mu U_L^\dagger + iU_L\partial_\mu U_L^\dagger \\
\chi &\mapsto U_R\chi U_L^\dagger, \\
t_{\mu\nu} &\mapsto U_R t_{\mu\nu} U_L^\dagger, \qquad (6)
\end{aligned}$$

where $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$ and $\chi = 2B_0(s+ip)$, together with the ones for covariant derivatives on the Goldstone fields as

$$\begin{aligned}
D_\mu U &= \partial_\mu U - ir_\mu U + iUl_\mu, \qquad & D_\mu U \to U_R(D_\mu U)U_L^\dagger, \\
D_\mu U^\dagger &= \partial_\mu U^\dagger + iU^\dagger r_\mu - il_\mu U^\dagger, \qquad & D_\mu U^\dagger \to U_L(D_\mu U^\dagger)U_R^\dagger. \qquad (7)
\end{aligned}$$

For the right- and left-handed fields, we can naturally get the field strength tensors as [191] naturally

$$[D_\mu, D_\nu]X = iXF_L^{\mu\nu} - iF_R^{\mu\nu}X, \qquad (8)$$

with

$$F_L^{\mu\nu} = \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu], \qquad F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]. \qquad (9)$$

The set $(U, F_{L,R}^{\mu\nu}, \chi, t_{\mu\nu})$, along with their adjoints and covariant derivatives are the building blocks to construct a theory with chiral symmetry. These building blocks transform differently under chiral group. Constructing the following set of Hermitian and anti-Hermitian terms can make the building blocks transform in the same manner

$$\begin{aligned}
u_\mu &= i\{u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\} \equiv iu^\dagger D_\mu U u^\dagger, \\
h_{\mu\nu} &= \nabla_\mu u_\nu + \nabla_\nu u_\mu, \\
f_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u, \\
t_\pm^{\mu\nu} &= u^\dagger t^{\mu\nu}u^\dagger \pm ut^{\mu\nu\dagger}u, \\
\chi_\pm &= u^\dagger\chi u^\dagger \pm u\chi^\dagger u, \qquad (10)
\end{aligned}$$

with $u(\phi)$ defined as

$$u(\phi) = \exp\left(\frac{i}{2F_0}\lambda_a\phi^a\right), \qquad (11)$$

and its transformation behavior is [191]

$$u(\phi) \mapsto U_R u(\phi)h^\dagger = hu(\phi)U_L^\dagger, \qquad (12)$$

where $U_R \in SU(3)_R$, $h \in SU(3)_V$, and we have $U(\phi) = u^2(\phi)$. Their identical transformation manner under the chiral group is [266, 191]

$$hXh^\dagger, \quad \text{with} \quad X = u_\mu, f_\pm^{\mu\nu}, t_\pm^{\mu\nu}, \chi_\pm \qquad (13)$$

The C, P transformation properties and hermitian conjugates of ($u_\mu$, $h_{\mu\nu}$, $f_\pm^{\mu\nu}$, $t_\pm^{\mu\nu}$ and $\chi_\pm$) are listed in table 8.1 [191], where we used the P and C transforming conventions of Ref. [267] which are consistent with ours discussed in Sec. 2.3.

Table 8.1: Transformation properties of operators in Eq. (10) under P and C and their hermitian conjugates.

| | P | C | h.c. |
|---|---|---|---|
| $u_\mu$ | $-(-1)^\mu u_\mu$ | $(u_\mu)^T$ | $u_\mu$ |
| $h_{\mu\nu}$ | $-(-1)^\mu(-1)^\nu h_{\mu\nu}$ | $(h_{\mu\nu})^T$ | $h_{\mu\nu}$ |
| $f_\pm^{\mu\nu}$ | $\pm(-1)^\mu(-1)^\nu f_\pm^{\mu\nu}$ | $\mp(f_\pm^{\mu\nu})^T$ | $f_\pm^{\mu\nu}$ |
| $\chi_\pm$ | $\pm\chi_\pm$ | $(\chi_\pm)^T$ | $\pm\chi_\pm$ |
| $t_\pm^{\mu\nu}$ | $\pm t_\pm^{\mu\nu}$ | $-(t_\pm^{\mu\nu})^T$ | $\pm t_\pm^{\mu\nu}$ |

The power counting of tensor field can be conveniently chosen to be [191]

$$t_{\mu\nu} \sim \mathcal{O}(p^2). \qquad (14)$$

Operators with the tensor fields first appear at $\mathcal{O}(p^4)$, constituting the lowest order chiral Lagrangian with external tensor fields to be [191]

$$\mathcal{L}_4^{\text{tensor}} = \Lambda_1\langle t_+^{\mu\nu}f_{\mu\nu}^+\rangle - i\Lambda_2\langle t_+^{\mu\nu}u_\mu u_\nu\rangle + \Lambda_3\langle t_+^{\mu\nu}t_{\mu\nu}^+\rangle + \Lambda_4\langle t_+^{\mu\nu}\rangle^2 \qquad (15)$$

where $\langle\cdots\rangle$ stands for the trace in $n_f$ space. Note that we have the traceless properties $\langle r_\mu\rangle = 0 = F_R^{\mu\nu}$, $\langle l_\mu\rangle = 0 = F_L^{\mu\nu}$ and $\langle u_\mu\rangle = 0 = f_\pm^{\mu\nu}$.

### 8.2 Matching quark-level operators onto operators in ChPT

Following de Rafael [268] and Georgi [269], the chiral realization can be made by taking appropriate variations of the effective actions with respect to the external fields.

For example, from Eq.(1) and $\mathcal{L}^{(2)}$ from Eq. (5.35), the matching of vector quark bilinear to lowest order can be obtained as [268]:

$$J_L^\mu \equiv \bar{q}_L\gamma^\mu q_L \doteq \frac{\delta\mathcal{L}^{(2)}}{\delta l_\mu} + \mathcal{O}(p^4) = \frac{i}{2}F_0^2\langle(D_\mu U^\dagger)U\rangle + \mathcal{O}(p^4) \tag{16}$$

$$J_R^\mu \equiv \bar{q}_R\gamma^\mu q_R \doteq \frac{\delta\mathcal{L}^{(2)}}{\delta r_\mu} + \mathcal{O}(p^4) = \frac{i}{2}F_0^2\langle(D_\mu U)U^\dagger\rangle + \mathcal{O}(p^4). \tag{17}$$

Then expanding $U$ in powers of $\phi$-matrix fields, we have:

$$J_L^\mu = \langle f_\pi \frac{1}{\sqrt{2}}D^\mu\phi - \frac{i}{2}\left[\phi(D^\mu\phi) - (D^\mu\phi)\phi\right] + \mathcal{O}(\phi^3)\rangle \tag{18}$$

$$J_R^\mu = \langle -f_\pi \frac{1}{\sqrt{2}}D^\mu\phi - \frac{i}{2}\left[\phi(D^\mu\phi) - (D^\mu\phi)\phi\right] + \mathcal{O}(\phi^3)\rangle. \tag{19}$$

Note that this "bosonization" procedure in ChPT always involves treating each quark bilinear as independent object to be bosonized, which is equivalent to a factorization ansatz [270].

Dekens et al. [271] investigated an extended matching for tensor quark bilinears based on the ChPT operators with tensor external fields from Catà and Mateu [191]. Accounting for contributions beyond the SM, the external fields $S \equiv s + ip$ [1], $r_\mu$, $l_\mu$ and $t_{\mu\nu}$ can be split into two parts as [271]

$$S \to S + \tilde{S}, \quad r_\mu \to r_\mu + \tilde{r}_\mu, \quad l_\mu \to l_\mu + \tilde{l}_\mu, \quad t_{\mu\nu} \to t_{\mu\nu} + \tilde{t}_{\mu\nu}. \tag{20}$$

where $S$, $r_\mu$, $l_\mu$, and $t_{\mu\nu}$ describe the quark mass matrix and coupling to gauge fields, which can be fixed to their physical values. For example in QED with $D_\mu = \partial_\mu - ieQA_\mu$ where we use the convention consistent with Peskin and Schroeder [81], we have

$$S \mapsto M_q, \qquad r_\mu \mapsto eQA_\mu, \qquad l_\mu \mapsto eQA_\mu, \qquad t_{\mu\nu} \mapsto 0. \tag{21}$$

$\tilde{S}$, $\tilde{r}_\mu$, $\tilde{l}_\mu$ and $\tilde{t}_{\mu\nu}$ are spurions that contain the contributions from higher dimensional operators [271], and they have the same power counting as $S$, $r_\mu$, $l_\mu$ and $t_{\mu\nu}$, respectively.

Dekens et al. [271] give the results for $\bar{q}_L\tilde{S}q_R$, $\bar{q}_L\sigma^{\mu\nu}\tilde{t}^\dagger_{\mu\nu}q_R$ and $\bar{q}_R\gamma^\mu\tilde{r}_\mu q_R$ to order $\mathcal{O}(p^4)$ explicitly and note that for $\bar{q}_L(\tilde{S})^\dagger q_R$ and $\bar{q}_R\sigma^{\mu\nu}\tilde{t}_{\mu\nu}q_L$, the results are the Hermitian conjugates of the one of $\bar{q}_L\tilde{S}q_R$ and $\bar{q}_L\sigma^{\mu\nu}\tilde{t}^\dagger_{\mu\nu}q_R$ respectively, and the result for $\bar{q}_L\gamma^\mu\tilde{l}_\mu q_L$ would be making the exchange $r_\mu \leftrightarrow l_\mu$, $U \leftrightarrow U^\dagger$ and $\chi \leftrightarrow \chi^\dagger$. We show the

[1] Dekens et al. [271] used a different notation with $S^\dagger = -(s+ip)$ and for the usual ChPT scalar source they have $\chi = -2B_0S^\dagger$.

resulting replacements together as follows [2].
Scalar:

$$\bar{q}_R\tilde{S}q_L \to -2B_0\Big[\frac{1}{4}F_0^2\langle\tilde{S}U^\dagger\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle\tilde{S}U^\dagger\rangle + L_5\langle\tilde{S}U^\dagger D_\mu U^\dagger D^\mu U\rangle + 2L_6\langle U^\dagger\chi+\chi^\dagger U\rangle\langle\tilde{S}U^\dagger\rangle + 2L_7\langle U^\dagger\chi-\chi^\dagger U\rangle\langle\tilde{S}U^\dagger\rangle + 2L_8\langle\tilde{S}U^\dagger\chi U^\dagger\rangle + H_2\langle\chi^\dagger\tilde{S}\rangle\Big] + \mathcal{O}(p^6)\,. \tag{22}$$

$$\bar{q}_L\tilde{S}^\dagger q_R \to -2B_0\Big[\frac{1}{4}F_0^2\langle\tilde{S}^\dagger U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle\tilde{S}^\dagger U\rangle + L_5\langle\tilde{S}^\dagger U D_\mu U^\dagger D^\mu U\rangle + 2L_6\langle U^\dagger\chi+\chi^\dagger U\rangle\langle\tilde{S}^\dagger U\rangle - 2L_7\langle U^\dagger\chi-\chi^\dagger U\rangle\langle\tilde{S}^\dagger U\rangle + 2L_8\langle\tilde{S}^\dagger U\chi^\dagger U\rangle + H_2\langle\tilde{S}^\dagger\chi\rangle\Big] + \mathcal{O}(p^6)\,. \tag{23}$$

Vector:

$$\begin{aligned}\bar{q}_R\gamma^\mu\tilde{r}_\mu q_R \to\ & \frac{i}{2}F_0^2\langle\tilde{r}^\mu D_\mu U U^\dagger\rangle + 4iL_1\langle D_\nu U^\dagger D^\nu U\rangle\langle\tilde{r}^\mu D_\mu U U^\dagger\rangle \\ & + 4iL_2\langle D^\mu U^\dagger D^\nu U\rangle\langle\tilde{r}_\mu D_\nu U U^\dagger\rangle + 2iL_3\Big\langle\Big(U^\dagger\tilde{r}_\mu D^\mu U - D^\mu U^\dagger\tilde{r}_\mu U\Big)D_\nu U^\dagger D^\nu U\Big\rangle \\ & + 2iL_4\langle\tilde{r}^\mu D_\mu U U^\dagger\rangle\langle U^\dagger\chi+\chi^\dagger U\rangle + iL_5\Big\langle\Big(U^\dagger\tilde{r}_\mu D^\mu U - D^\mu U^\dagger\tilde{r}_\mu U\Big)(U^\dagger\chi+\chi^\dagger U)\Big\rangle \\ & + L_9\Big[-\langle\tilde{r}_\mu F_R^{\mu\nu}D_\nu U U^\dagger\rangle - \langle\tilde{r}_\mu U D_\nu U^\dagger F_R^{\mu\nu}\rangle + \langle\tilde{r}_\mu D_\nu U F_L^{\mu\nu}U^\dagger\rangle + \langle\tilde{r}_\mu U F_L^{\mu\nu}D_\nu U^\dagger\rangle\Big] \\ & - iL_9\langle\tilde{r}^\mu D^\nu(D_\mu U D_\nu U^\dagger - D_\nu U D_\mu U^\dagger\rangle + 2L_{10}\langle\tilde{r}_\mu D_\nu(U F_L^{\mu\nu}U^\dagger)\rangle \\ & + 4H_1\langle\tilde{r}_\mu D_\nu F_R^{\mu\nu}\rangle + \epsilon\ \text{terms} + \mathcal{O}(p^6)\,,\end{aligned} \tag{24}$$

$$\begin{aligned}\bar{q}_L\gamma^\mu\tilde{l}_\mu q_L \to\ & \frac{i}{2}F_0^2\langle D_\mu U^\dagger U\tilde{l}^\mu\rangle + 4iL_1\langle D_\nu U^\dagger D^\nu U\rangle\langle D_\mu U^\dagger U\tilde{l}^\mu\rangle \\ & + 4iL_2\langle D^\mu U D^\nu U^\dagger\rangle\langle D_\nu U^\dagger U\tilde{l}^\mu\rangle + 2iL_3\Big\langle\Big(U\tilde{l}_\mu D^\mu U^\dagger - D^\mu U\tilde{l}^\mu U^\dagger\Big)D_\nu U^\dagger D^\nu U\Big\rangle \\ & + 2iL_4\langle D_\mu U^\dagger U\tilde{l}^\mu\rangle\langle U^\dagger\chi+\chi^\dagger U\rangle + iL_5\Big\langle\Big(U\tilde{l}_\mu D^\mu U^\dagger - D^\mu U\tilde{l}^\mu U^\dagger\Big)(U^\dagger\chi+\chi^\dagger U)\Big\rangle \\ & + L_9\Big[-\langle\tilde{l}_\mu F_L^{\mu\nu}D_\nu U^\dagger U\rangle - \langle\tilde{l}_\mu U^\dagger D_\nu U F_L^{\mu\nu}\rangle + \langle\tilde{l}_\mu D_\nu U^\dagger F_R^{\mu\nu}U\rangle + \langle\tilde{l}_\mu U^\dagger F_R^{\mu\nu}D_\nu U\rangle\Big] \\ & - iL_9\langle\tilde{l}^\mu D^\nu(D_\mu U^\dagger D_\nu U - D_\nu U^\dagger D_\mu U\rangle + 2L_{10}\langle\tilde{l}_\mu D_\nu(U^\dagger F_R^{\mu\nu}U)\rangle \\ & + 4H_1\langle\tilde{l}_\mu D_\nu F_L^{\mu\nu}\rangle + \epsilon\ \text{terms} + \mathcal{O}(p^6)\,,\end{aligned} \tag{25}$$

where $\mathcal{O}(p^4)$ terms involving $\epsilon_{\alpha\beta\lambda\sigma}$ is ommitted as it belongs to the odd intrinsic parity sector and is related to the $U(1)_A$ anomaly [272, 273, 274, 275] as noted by Ref. [271].
Tensor:

$$\bar{q}_R\sigma^{\mu\nu}\tilde{t}_{\mu\nu}q_L \to \Lambda_1\langle\tilde{t}_{\mu\nu}(F_L^{\mu\nu}U^\dagger + U^\dagger F_R^{\mu\nu})\rangle - i\Lambda_2\langle\tilde{t}^{\mu\nu}D_\nu U^\dagger U D_\mu U^\dagger\rangle + \mathcal{O}(p^6)\,. \tag{26}$$

$$\bar{q}_L\sigma^{\mu\nu}\tilde{t}^\dagger_{\mu\nu}q_R \to \Lambda_1\langle\tilde{t}^\dagger_{\mu\nu}(U F_L^{\mu\nu} + F_R^{\mu\nu}U)\rangle + i\Lambda_2\langle\tilde{t}^{\dagger\mu\nu}D_\mu U U^\dagger D_\nu U\rangle + \mathcal{O}(p^6)\,. \tag{27}$$

[2] Dekens. et al. [271] used $t_{\mu\nu} \to t^\dagger_{\mu\nu}$ for the external tensor field compared with the notations of Catà and Mateu [191]. We follow the notation of $t_{\mu\nu}$ and $t^\dagger_{\mu\nu}$ of Catà and Mateu [191] and we have modified the results of Dekens. et al. [271] accordingly.

### 8.3 Evaluations of operators contributing to $\eta \to \pi^+\pi^-\pi^0$

Now we transition to the evaluations of the operators we have in Ch. 7 that can contribute to $\eta \to \pi^+\pi^-\pi^0$ using the matching procedure discussed in the previous section.

As an illustration, we pick an operator with $|\Delta I| = 0$ from Ch. 7

$$\vartheta = C_0^a \frac{G_F}{\sqrt{2}} vi[(\bar{u}\sigma^{\mu\nu}\gamma_5 u) + (\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(\bar{s}\gamma_\nu\gamma_5 s), \tag{28}$$

where $C_0^a = \frac{1}{2}\left(C_{quZ\varphi}^{11} + C_{qdZ\varphi}^{11}\right)$. We first write the operator in chiral basis

$$\begin{aligned}(\bar{q}_t\sigma^{\mu\nu}\gamma_5 q_w) &= (\bar{q}_{Lt}\sigma^{\mu\nu}q_{Rw}) - (\bar{q}_{Rt}\sigma^{\mu\nu}q_{Lw}),\\ (\bar{q}_t\gamma_\nu\gamma_5 q_w) &= (\bar{q}_{Rt}\gamma_\nu q_{Rw}) - (\bar{q}_{Lt}\gamma_\nu q_{Lw}).\end{aligned} \tag{29}$$

Then despite the coefficient $C_0^a \frac{G_F}{\sqrt{2}} vi$, we have

$$\begin{aligned}&\left[(\bar{u}\sigma^{\mu\nu}\gamma_5 u + (\bar{d}\sigma^{\mu\nu}\gamma_5 d))\right]\partial_\mu(\bar{s}\gamma_\nu\gamma_5 s)\\ = &\left[(\bar{u}_L\sigma^{\mu\nu}u_R) - (\bar{u}_R\sigma^{\mu\nu}u_L) + (\bar{d}_L\sigma^{\mu\nu}d_R) - (\bar{d}_R\sigma^{\mu\nu}d_L)\right]\partial_\mu\left[(\bar{s}_R\gamma_\nu s_R) - (\bar{s}_L\gamma_\nu s_L)\right]\end{aligned} \tag{30}$$

where $(\bar{q}_{Lt}\sigma^{\mu\nu}q_{Rw})$, $(\bar{q}_{Rt}\sigma^{\mu\nu}q_{Lw})$, $(\bar{q}_{Rt}\gamma_\nu q_{Rw})$ and $(\bar{q}_{Lt}\gamma_\nu q_{Lw})$ can be replaced using Eq. (27), Eq. (26), Eq. (24) and Eq. (25), respectively. Since we want to compute $\eta \to \pi^+\pi^-\pi^0$ decay, there are no photons in the final state. Neglecting contributions from photons which could be more suppressed, the matching operator from ChPT to lowest order would be

$$\begin{aligned}&C_0^a \frac{G_F}{\sqrt{2}} vi \cdot i\Lambda_2\left[\left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{11} + \left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{22}\right]\cdot\\ &\frac{i}{2F_0^2}\partial_\mu\left(\partial_\nu U U^\dagger - \partial_\mu U^\dagger U\right)_{33},\end{aligned} \tag{31}$$

which is $\mathcal{O}(p^4)$.

Before continuing the analysis, we want to mention that since the C and CP violating flavor-conserving operators containing four quarks all have a tensor structure analogous to Eq. (28) that can be matched to operators similar to Eq. (31) and it can always be expressed as $(D_\mu U U^\dagger D_\nu U)_{tw}$ or $(D_\nu U^\dagger U D_\mu U^\dagger)_{tw}$ times $\partial_\mu(D_\nu U U^\dagger)_{wt}$ or $\partial_\mu(D_\nu U^\dagger U)_{wt}$ with $t$, $w$ as flavor indices, thus the derived interactions in the freedom of mesons should contain at least three meson fields because there are three derivatives of $U^{(\dagger)}$. This is consistent with the fact that there is no C and CP violating mixing of $\eta$ and $\pi^0$ because they have the same C, which means C and CP violating $\eta - \pi^0$ transition does not exist. We also checked that for $\eta \to \pi^+\pi^-$, Eq. (31) gives a zero result, which is predictable since a C and CP violating operator can not contribute to a P and CP violating process.

Then we need to expand $U$ as Eq. (5.17) - Eq. (5.19) and pick the terms containing just $\eta$, $\pi^+$, $\pi^-$ and $\pi^0$ to leading order. Neglecting interactions with virtual photons which is suppressed by $\mathcal{O}(e)$ at least, we take $r_\mu = 0$ and $l_\mu = 0$ and expand $U$ to

$\phi^4$, then pick the terms containing $\eta$, $\pi^+$, $\pi^-$ and $\pi^0$ only. The resulted matching operator of Eq. (28) is

$$\begin{aligned}
\vartheta^{0,a}_{\text{ChPT}} &= C_0^a \frac{G_F v}{\sqrt{2}} \frac{i\Lambda_2}{2F_0^2} \frac{16\sqrt{3}}{F_0^4} \left[\cos^2(\epsilon) \left(\pi^+\partial_\mu\pi^-\partial_\nu\pi^0 - \pi^+\partial_\nu\pi^-\partial_\mu\pi^0 - \pi^-\partial_\mu\pi^+\partial_\nu\pi^0\right.\right. \\
&+ \left. \pi^-\partial_\nu\pi^+\partial_\mu\pi^0 + \pi^0\partial_\mu\pi^+\partial_\nu\pi^- - \pi^0\partial_\nu\pi^+\partial_\mu\pi^-\right)\partial_\nu\partial_\mu\eta \\
&+ \sin^2(\epsilon)\left(-\eta\partial_\mu\pi^+\partial_\nu\pi^- + \eta\partial_\nu\pi^+\partial_\mu\pi^- + \pi^+\partial_\mu\eta\partial_\nu\pi^- - \pi^+\partial_\nu\eta\partial_\mu\pi^-\right. \\
&- \left.\left. \pi^-\partial_\mu\eta\partial_\nu\pi^+ + \pi^-\partial_\nu\eta\partial_\mu\pi^+\right)\partial_\nu\partial_\mu\pi^0\right] \qquad (32)
\end{aligned}$$

where $\vartheta^{I',a}_{\text{ChPT}}$ indicates the isospin of the operator is $|\Delta I| = I'$ and $a, b, \ldots$ is used as a sequence index of operators with the same $|\Delta I|$. Calculating the matrix element $\langle\pi^+\pi^-\pi^0|\vartheta^{0,a}_{\text{ChPT}}|\eta\rangle$, we have

$$\begin{aligned}
\langle\pi^+\pi^-\pi^0|\vartheta^{0,a}_{\text{ChPT}}|\eta\rangle &= C_0^a \frac{G_F v}{\sqrt{2}} \frac{i\Lambda_2}{2F_0^2} \frac{16\sqrt{3}}{F_0^4} \left\{\cos^2(\epsilon)\left[(p_{\pi^+}\cdot p_\eta)(p_{\pi^-}\cdot p_\eta)\right.\right. \\
&- (p_{\pi^-}\cdot p_\eta)(p_{\pi^+}\cdot p_\eta) - (p_{\pi^+}\cdot p_\eta)(p_{\pi^+}\cdot p_\eta) + (p_{\pi^-}\cdot p_\eta)(p_{\pi^+}\cdot p_\eta) \\
&+ \left.(p_{\pi^+}\cdot p_\eta)(p_{\pi^-}\cdot p_\eta) - (p_{\pi^+}\cdot p_\eta)(p_{\pi^-}\cdot p_\eta)\right] \\
&+ \sin^2(\epsilon)\left[-(p_{\pi^+}\cdot p_{\pi^0})(p_{\pi^-}\cdot p_{\pi^0}) + -(p_{\pi^+}\cdot p_{\pi^0})(p_{\pi^-}\cdot p_{\pi^0})\right. \\
&- -(p_{\pi^+}\cdot p_{\pi^0})(p_\eta\cdot p_{\pi^0}) + (p_{\pi^+}\cdot p_{\pi^0})(p_\eta\cdot p_{\pi^0}) \\
&+ \left.\left. -(p_{\pi^-}\cdot p_{\pi^0})(p_\eta\cdot p_{\pi^0}) + (p_{\pi^-}\cdot p_{\pi^0})(p_\eta\cdot p_{\pi^0})\right]\right\} \\
&= C_0^a \frac{G_F v}{\sqrt{2}} \frac{i\Lambda_2}{2F_0^2} \frac{16\sqrt{3}}{F_0^4} \left[\cos^2(\epsilon)\cdot 0 + \sin^2(\epsilon)\cdot 0\right] \\
&= 0 \qquad (33)
\end{aligned}$$

Thus this $\mathcal{O}(p^4)$ operator matched from Eq. (28) vanishes, and we need to consider it in higher order for its contribution.

The $\mathcal{O}(p^4)$ results of the rest operators with definite isospin that can contribute to $\eta \to \pi^+\pi^-\pi^0$ listed in Ch. 7 are shown as follows.

$$\vartheta^a_{|\Delta I|=1} = C_1^a \frac{G_F}{\sqrt{2}} v i[(\bar{u}\sigma^{\mu\nu}\gamma_5 u) - (\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(\bar{s}\gamma_\nu\gamma_5 s) \qquad (34)$$

$$\begin{aligned}
&\Rightarrow -C_1^a \frac{G_F v}{\sqrt{2}} \frac{i\Lambda_2}{2F_0^2}\left[\left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{11}\right. \\
&\quad - \left.\left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{22}\right]\cdot\partial_\mu\left(\partial_\nu U U^\dagger - \partial_\mu U^\dagger U\right)_{33} \\
&\sim C_0^a \frac{G_F v}{\sqrt{2}} \frac{i\Lambda_2}{2F_0^2} \frac{16}{3F_0^4}\cos(\epsilon)\sin(\epsilon)\left(\eta\partial_\mu\pi^+\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^0\right. \\
&\quad \left.-\eta\partial_\nu\pi^+\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^0 + \pi^0\partial_\mu\pi^+\partial_\nu\pi^-\partial_\nu\partial_\mu\eta - \pi^0\partial_\nu\pi^+\partial_\mu\pi^-\partial_\nu\partial_\mu\eta\right) \qquad (35)
\end{aligned}$$

$$\langle\pi^+\pi^-\pi^0|\vartheta^{1,a}_{\text{ChPT}}|\eta\rangle = 0 \qquad (36)$$

$$
\begin{aligned}
\vartheta^b_{|\Delta I|=1} &= \frac{G_F}{\sqrt{2}} v i C^{22}_{qdZ\varphi} (\bar{s}\sigma^{\mu\nu}\gamma_5 s)\partial_\mu(-\bar{u}\gamma_\nu\gamma_5 u + \bar{d}\gamma_\nu\gamma_5 d) \qquad (37)\\
&\Rightarrow -C^{22}_{qdZ\varphi} \frac{G_F v}{\sqrt{2}} \frac{i\Lambda_2}{2F_0^2} \left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{11} \\
&\quad \partial_\mu \left[-\left(\partial_\nu U U^\dagger - \partial_\mu U^\dagger U\right)_{11} + \left(\partial_\nu U U^\dagger - \partial_\mu U^\dagger U\right)_{22}\right] \\
&\sim 0 \qquad (38)
\end{aligned}
$$

$$
\langle \pi^+\pi^-\pi^0 | \vartheta^{1,b}_{\text{ChPT}} | \eta \rangle = 0. \qquad (39)
$$

$$
\begin{aligned}
\vartheta^c_{|\Delta I|=1} &= \frac{2G_F}{\sqrt{2}} v i [\text{Im}(C^{11}_{quW\varphi}) - \text{Im}(C^{11}_{qdW\varphi})] \left[(\bar{d}\sigma^{\mu\nu} u)\partial_\mu(\bar{u}\gamma_\nu V_{ud} d)\right.\\
&\quad \left. -(\bar{u}\sigma^{\mu\nu} d)\partial_\mu(\bar{d}V^*_{ud}\gamma_\nu u)\right], \qquad (40)\\
&\Rightarrow -\frac{2G_F}{\sqrt{2}} v [\text{Im}(C^{11}_{quW\varphi}) - \text{Im}(C^{11}_{qdW\varphi})] V_{ud} \frac{i\Lambda_2}{2F_0^2} \\
&\quad \left[\left(\partial_\mu U U^\dagger \partial_\nu U - \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{21} \cdot \partial_\mu \left(\partial_\nu U U^\dagger + \partial_\mu U^\dagger U\right)_{12}\right.\\
&\quad \left. - \left(\partial_\mu U U^\dagger \partial_\nu U - \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{12} \cdot \partial_\mu \left(\partial_\nu U U^\dagger + \partial_\mu U^\dagger U\right)_{21}\right]\\
&\sim \frac{2G_F}{\sqrt{2}} v [\text{Im}(C^{11}_{quW\varphi}) - \text{Im}(C^{11}_{qdW\varphi})] V_{ud} \frac{i\Lambda_2}{2F_0^2} \frac{8}{F_0^4} \sin(\epsilon)\cos(\epsilon)\\
&\quad \left(-\eta\partial_\mu\pi^+\partial_\nu\pi^0\partial_\nu\partial_\mu\pi^- + \eta\partial_\nu\pi^+\partial_\mu\pi^0\partial_\nu\partial_\mu\pi^-\right.\\
&\quad +\eta\partial_\mu\pi^-\partial_\nu\pi^0\partial_\nu\partial_\mu\pi^+ - \eta\partial_\nu\pi^-\partial_\mu\pi^0\partial_\nu\partial_\mu\pi^+\\
&\quad +\pi^+\partial_\mu\eta\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^0 - \pi^+\partial_\nu\eta\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^0\\
&\quad -\pi^+\partial_\mu\pi^-\partial_\nu\pi^0\partial_\nu\partial_\mu\eta + \pi^+\partial_\nu\pi^-\partial_\mu\pi^0\partial_\nu\partial_\mu\eta\\
&\quad -\pi^-\partial_\mu\eta\partial_\nu\pi^+\partial_\nu\partial_\mu\pi^0 + \pi^-\partial_\nu\eta\partial_\mu\pi^+\partial_\nu\partial_\mu\pi^0\\
&\quad +\pi^-\partial_\mu\pi^+\partial_\nu\pi^0\partial_\nu\partial_\mu\eta - \pi^-\partial_\nu\pi^+\partial_\mu\pi^0\partial_\nu\partial_\mu\eta\\
&\quad +\pi^0\partial_\mu\eta\partial_\nu\pi^+\partial_\nu\partial_\mu\pi^- - \pi^0\partial_\mu\eta\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^+\\
&\quad \left.-\pi^0\partial_\nu\eta\partial_\mu\pi^+\partial_\nu\partial_\mu\pi^- + \pi^0\partial_\nu\eta\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^+\right)\\
&\qquad (41)
\end{aligned}
$$

$$
\langle \pi^+\pi^-\pi^0 | \vartheta^{1,c}_{\text{ChPT}} | \eta \rangle = 0 \qquad (42)
$$

$$
\begin{aligned}
\vartheta^{b}_{|\Delta I|=0} &= \frac{2G_F}{\sqrt{2}} vi[\mathrm{Im}(C^{21}_{quW\varphi}) - \mathrm{Im}(C^{12}_{qdW\varphi})]\,[(\bar{s}\sigma^{\mu\nu}u)\partial_\mu(\bar{u}\gamma_\nu V_{us}s) \\
&\quad -(\bar{u}\sigma^{\mu\nu}s)\partial_\mu(\bar{s}V^{*}_{us}\gamma_\nu u)] \qquad (43)\\
&\Rightarrow -\frac{2G_F}{\sqrt{2}} v[\mathrm{Im}(C^{21}_{quW\varphi}) - \mathrm{Im}(C^{12}_{qdW\varphi})]V_{us}\frac{i\Lambda_2}{2F_0^2} \\
&\quad \left[\left(\partial_\mu UU^\dagger\partial_\nu U - \partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{31}\cdot\partial_\mu\left(\partial_\nu UU^\dagger + \partial_\mu U^\dagger U\right)_{13}\right. \\
&\quad \left. -\left(\partial_\mu UU^\dagger\partial_\nu U - \partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{13}\cdot\partial_\mu\left(\partial_\nu UU^\dagger + \partial_\mu U^\dagger U\right)_{31}\right] \\
&\sim 0 \qquad (44)
\end{aligned}
$$

$$
\langle\pi^+\pi^-\pi^0|\vartheta^{0,b}_{\mathrm{ChPT}}|\eta\rangle = 0. \qquad (45)
$$

$$
\begin{aligned}
\vartheta^{d}_{|\Delta I|=1} &= -\frac{2G_F}{\sqrt{2}} vi[\mathrm{Im}(C^{21}_{quW\varphi}) + \mathrm{Im}(C^{12}_{qdW\varphi})]\,[(\bar{s}\sigma^{\mu\nu}\gamma_5 u)\partial_\mu(\bar{u}\gamma_\nu\gamma_5 V_{us}s) \\
&\quad +(\bar{u}\sigma^{\mu\nu}\gamma_5 s)\partial_\mu(\bar{s}V^{*}_{us}\gamma_\nu\gamma_5 u)] \qquad (46)\\
&\Rightarrow \frac{2G_F}{\sqrt{2}} v[\mathrm{Im}(C^{21}_{quW\varphi}) + \mathrm{Im}(C^{12}_{qdW\varphi})]V_{us}\frac{i\Lambda_2}{2F_0^2} \\
&\quad \left[\left(\partial_\mu UU^\dagger\partial_\nu U + \partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{31}\cdot\partial_\mu\left(\partial_\nu UU^\dagger - \partial_\mu U^\dagger U\right)_{13}\right. \\
&\quad \left. +\left(\partial_\mu UU^\dagger\partial_\nu U + \partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{13}\cdot\partial_\mu\left(\partial_\nu UU^\dagger - \partial_\mu U^\dagger U\right)_{31}\right] \\
&\sim 0 \qquad (47)
\end{aligned}
$$

$$
\langle\pi^+\pi^-\pi^0|\vartheta^{1,d}_{\mathrm{ChPT}}|\eta\rangle = 0. \qquad (48)
$$

$$
\begin{aligned}
\vartheta^{e}_{|\Delta I|=1} &= C^b_1\frac{G_F}{\sqrt{2}} vi[(\bar{u}\sigma^{\mu\nu}\gamma_5 u) + (\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(-\bar{u}\gamma_\nu\gamma_5 u + \bar{d}\gamma_\nu\gamma_5 d), \qquad (49)\\
&\Rightarrow -C^b_1\frac{G_F v}{\sqrt{2}}\frac{i\Lambda_2}{2F_0^2}\left[\left(\partial_\mu UU^\dagger\partial_\nu U + \partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{11}\right. \\
&\quad \left. -\left(\partial_\mu UU^\dagger\partial_\nu U + \partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{22}\right]\cdot \\
&\quad \partial_\mu\left[-\left(\partial_\nu UU^\dagger - \partial_\mu U^\dagger U\right)_{11} + \left(\partial_\nu UU^\dagger - \partial_\nu U^\dagger U\right)_{22}\right] \\
&\sim C^b_1\frac{G_F v}{\sqrt{2}}\frac{i\Lambda_2}{2F_0^2}\frac{16}{F_0^4}\sin(\epsilon)\cos(\epsilon)\left(-\eta\partial_\mu\pi^+\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^0 + \eta\partial_\nu\pi^+\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^0\right. \\
&\quad +\pi^+\partial_\mu\eta\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^0 - \pi^+\partial_\nu\eta\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^0 \\
&\quad -\pi^+\partial_\mu\pi^-\partial_\nu\pi^0\partial_\nu\partial_\mu\eta + \pi^+\partial_\nu\pi^-\partial_\mu\pi^0\partial_\nu\partial_\mu\eta \\
&\quad -\pi^-\partial_\mu\eta\partial_\nu\pi^+\partial_\nu\partial_\mu\pi^0 + \pi^-\partial_\nu\eta\partial_\mu\pi^+\partial_\nu\partial_\mu\pi^0 \\
&\quad +\pi^-\partial_\mu\pi^+\partial_\nu\pi^0\partial_\nu\partial_\mu\eta - \pi^-\partial_\nu\pi^+\partial_\mu\pi^0\partial_\nu\partial_\mu\eta \\
&\quad \left. -\pi^0\partial_\mu\pi^+\partial_\nu\pi^-\partial_\nu\partial_\mu\eta + \pi^0\partial_\nu\pi^+\partial_\mu\pi^-\partial_\nu\partial_\mu\eta\right) \qquad (50)
\end{aligned}
$$

$$\langle \pi^+\pi^-\pi^0|\vartheta^{1,e}_{\text{ChPT}}|\eta\rangle = 0. \tag{51}$$

$$\begin{aligned}
\vartheta^{c}_{|\Delta I|=0} &= C_0^b\frac{G_F}{\sqrt{2}}vi\left\{[(\bar{u}\sigma^{\mu\nu}\gamma_5 u)-(\bar{d}\sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(-\bar{u}\gamma_\nu\gamma_5 u+\bar{d}\gamma_\nu\gamma_5 d)\right. \\
&\quad \left.+\left[(\bar{d}\sigma^{\mu\nu}\gamma_5 u)\partial_\mu(\bar{u}\gamma_\nu\gamma_5 d)+(\bar{u}\sigma^{\mu\nu}\gamma_5 d)\partial_\mu(\bar{d}\gamma_\nu\gamma_5 u)\right]\right\} \qquad (52) \\
&\Rightarrow -C_0^b\frac{G_F v}{\sqrt{2}}\frac{i\Lambda_2}{2F_0^2}\left\{\left[\left(\partial_\mu UU^\dagger\partial_\nu U+\partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{11}\right.\right. \\
&\quad \left.-\left(\partial_\mu UU^\dagger\partial_\nu U+\partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{22}\right]\cdot \\
&\quad \partial_\mu\left[-\left(\partial_\nu UU^\dagger-\partial_\mu U^\dagger U\right)_{11}+\left(\partial_\nu UU^\dagger-\partial_\nu U^\dagger U\right)_{22}\right] \\
&\quad +*\left[\left(\partial_\mu UU^\dagger\partial_\nu U+\partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{21}\cdot\partial_\mu\left(\partial_\nu UU^\dagger-\partial_\mu U^\dagger U\right)_{12}\right. \\
&\quad \left.\left.+\left(\partial_\mu UU^\dagger\partial_\nu U+\partial_\nu U^\dagger U\partial_\mu U^\dagger\right)_{12}\cdot\partial_\mu\left(\partial_\nu UU^\dagger-\partial_\mu U^\dagger U\right)_{21}\right]\right\} \\
&\sim C_0^b\frac{G_F v}{\sqrt{2}}\frac{i\Lambda_2}{2F_0^2}\frac{8}{\sqrt{3}F_0^4}\left\{\cos^2(\epsilon)\cdot\left(2\partial_\mu\pi^+\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^0+\partial_\mu\pi^+\partial_\nu\pi^0\partial_\nu\partial_\mu\pi^-\right.\right. \\
&\quad -2\partial_\nu\pi^+\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^0-\partial_\nu\pi^+\partial_\mu\pi^0\partial_\nu\partial_\mu\pi^- \\
&\quad \left.-\partial_\mu\pi^-\partial_\nu\pi^0\partial_\nu\partial_\mu\pi^++\partial_\nu\pi^-\partial_\mu\pi^0\partial_\nu\partial_\mu\pi^+\right)\eta \\
&\quad \sin^2(\epsilon)\cdot\left(\partial_\mu\eta\partial_\nu\pi^+\partial_\nu\partial_\mu\pi^--\partial_\mu\eta\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^+\right. \\
&\quad -\partial_\nu\eta\partial_\mu\pi^+\partial_\nu\partial_\mu\pi^-+\partial_\nu\eta\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^+ \\
&\quad \left.\left.-2\partial_\mu\pi^+\partial_\nu\pi^-\partial_\nu\partial_\mu\eta+2\partial_\nu\pi^+\partial_\mu\pi^-\partial_\nu\partial_\mu\eta\right)\pi^0\right\} \qquad (53)
\end{aligned}$$

$$\langle \pi^+\pi^-\pi^0|\vartheta^{0,c}_{\text{ChPT}}|\eta\rangle = 0. \tag{54}$$

$$
\begin{aligned}
\vartheta^a_{|\Delta I|=2} &= C_2^b \frac{G_F}{\sqrt{2}} v i \left\{[(\bar u \sigma^{\mu\nu}\gamma_5 u) - (\bar d \sigma^{\mu\nu}\gamma_5 d)]\partial_\mu(-\bar u\gamma_\nu\gamma_5 u + \bar d\gamma_\nu\gamma_5 d) \right.\\
&\left. -2\left[(\bar d\sigma^{\mu\nu}\gamma_5 u)\partial_\mu(\bar u\gamma_\nu\gamma_5 d) + (\bar u\sigma^{\mu\nu}\gamma_5 d)\partial_\mu(\bar d\gamma_\nu\gamma_5 u)\right]\right\} \qquad (55)\\
&\Rightarrow -C_2^b \frac{G_F v}{\sqrt{2}}\frac{i\Lambda_2}{2F_0^2}\left\{\left[\left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{11}\right.\right.\\
&- \left.\left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{22}\right]\cdot\\
&\partial_\mu\left[-\left(\partial_\nu U U^\dagger - \partial_\mu U^\dagger U\right)_{11} + \left(\partial_\nu U U^\dagger - \partial_\nu U^\dagger U\right)_{22}\right]\\
&-2 * \left[\left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{21}\cdot\partial_\mu\left(\partial_\nu U U^\dagger - \partial_\mu U^\dagger U\right)_{12}\right.\\
&\left.\left. + \left(\partial_\mu U U^\dagger \partial_\nu U + \partial_\nu U^\dagger U \partial_\mu U^\dagger\right)_{12}\cdot\partial_\mu\left(\partial_\nu U U^\dagger - \partial_\mu U^\dagger U\right)_{21}\right]\right\}\\
&\sim C_2^b \frac{G_F v}{\sqrt{2}}\frac{i\Lambda_2}{2F_0^2}\frac{16\sqrt{3}}{F_0^4}\left[\cos^2(\epsilon)\left(\partial_\mu\pi^+\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^0 - \partial_\mu\pi^+\partial_\nu\pi^0\partial_\nu\partial_\mu\pi^-\right.\right.\\
&-\partial_\nu\pi^+\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^0 + \partial_\nu\pi^+\partial_\mu\pi^0\partial_\nu\partial_\mu\pi^-\\
&\left.+\partial_\mu\pi^-\partial_\nu\pi^0\partial_\nu\partial_\mu\pi^+ - \partial_\nu\pi^-\partial_\mu\pi^0\partial_\nu\partial_\mu\pi^+\right)\eta\\
&+\sin^2(\epsilon)\left(-\partial_\mu\eta\partial_\nu\pi^+\partial_\nu\partial_\mu\pi^- + \partial_\mu\eta\partial_\nu\pi^-\partial_\nu\partial_\mu\pi^+\right.\\
&+\partial_\nu\eta\partial_\mu\pi^+\partial_\nu\partial_\mu\pi^- - \partial_\nu\eta\partial_\mu\pi^-\partial_\nu\partial_\mu\pi^+\\
&\left.\left.-\partial_\mu\pi^+\partial_\nu\pi^-\partial_\nu\partial_\mu\eta + \partial_\nu\pi^+\partial_\mu\pi^-\partial_\nu\partial_\mu\eta\right)\pi^0\right] \qquad (56)
\end{aligned}
$$

$$
\langle\pi^+\pi^-\pi^0|\vartheta^{2,a}_{\rm ChPT}|\eta\rangle = 0. \qquad (57)
$$

where

$$
\begin{aligned}
C_0^a &= \frac{1}{2}\left(C^{11}_{quZ\varphi} + C^{11}_{qdZ\varphi}\right), \qquad (58)\\
C_1^a &= \frac{1}{2}\left(C^{11}_{quZ\varphi} - C^{11}_{qdZ\varphi}\right), \qquad (59)\\
C_1^b &= \frac{1}{2}\left(C^{11}_{quZ\varphi} + C^{11}_{qdZ\varphi}\right), \qquad (60)\\
C_0^b &= \frac{1}{3}\left\{C^{11}_{quZ\varphi} - C^{11}_{qdZ\varphi} - 2V_{ud}[\mathrm{Im}(C^{21}_{quW\varphi}) + \mathrm{Im}(C^{12}_{qdW\varphi})]\right\}, \qquad (61)\\
C_2^b &= \frac{1}{3}\left\{2[\mathrm{Im}(C^{21}_{quW\varphi}) + \mathrm{Im}(C^{12}_{qdW\varphi})] + \frac{1}{2}(C^{11}_{quZ\varphi} - \frac{3}{2}C^{11}_{qdZ\varphi})\right\}. \qquad (62)
\end{aligned}
$$

Thus all the quark-only operators with definite isospin that can contribute to $\eta \to \pi^+\pi^-\pi^0$ listed in Ch. 7 vanish at $\mathcal{O}(p^4)$. This conclusion is important to indicate that the real and imaginary parts of CP violating amplitudes can appear at the same order in ChPT, which is structurally analogous to that of the CP conserving amplitude calculation from the SM. This supports the validity of our claim in studying the patterns of CP violation in $\eta \to \pi^+\pi^-\pi^0$ decays in Ch. 6 or in Gardner and Shi [92], that the dominant strong phases of the ChPT amplitudes for the SM CP-conserving $\eta \to \pi^+\pi^-\pi^0$ can be used to determine the strong phases associated with the new physics amplitudes. For the evaluation of $\eta \to \pi^+\pi^-\pi^0$ decay using the new operators we found in Ch. 7, in future work, we can get the $\pi - \pi$ scattering amplitudes $M_I(z)$ with $I = 1,\ 2$ from any of our new operators in $\mathcal{O}(p^4)$ in a one–loop graph taken to $\mathcal{O}(p^6)$.

CHAPTER 9

# SUMMARY AND OUTLOOK

We have found $\eta \to \pi^+\pi^-\pi^0$ decay is an ideal process for studying new CP violating mechanisms beyond the SM, since the CPV contribution from the SM to this channel is very small as discussed in Ch. 1 and 3, and as we have established in Ch. 7, and there is no obvious direct relation to limits from permanent EDM searches.

We first studied patterns of CP violating amplitudes from analyzing the possibility of an asymmetric energy distribution in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot. We found that the possibility of a CPV $I = 2$ amplitude is much more suppressed than that of a CPV $I = 0$ amplitude. We have shown that our method is a discriminating way of separating $I = 0$ and $I = 2$ final states by only using the distribution of mirror symmetry breaking in the Dalitz plot.

To discover the CP violating sources contributing to $\eta \to \pi^+\pi^-\pi^0$, we have investigated the C, P and CP properties of mass-dimension 6, baryon number conserving SMEFT operators. Working at energies just below the masses of weak gauge bosons, we carefully separate the P-odd CP-odd and C-odd CP-odd operators for flavor-changing and flavor-conserving interactions respectively. We have found that in flavor-changing processes the P-odd CP-odd operators and C-odd CP-odd operators from SMEFT are strongly related, either with the same low energy coefficients or different linear combinations of the same low energy coefficients, and their lowest mass-dimensions are both 6. However, for flavor-conserving processes, the P-odd CP-odd operators and C-odd CP-odd operators are very different, with only one operator which is mass-dimension 8 from each set having the same low energy coefficients. Moreover, we note that the lowest mass-dimension for P-odd CP-odd operators from SMEFT is 6, while the one for C-odd and CP-odd operators is 8. The only common operator for P-odd CP-odd and C-odd CP-odd flavor-conserving interactions will show some connection from nEDM to $\eta \to \pi^+\pi^-\pi^0$. We have also determined the C-odd and CP-odd operators with definite isospin that can contribute to $\eta \to \pi^+\pi^-\pi^0$ decay.

Finally, we show how the new C and CP violating operators with quarks degrees of freedom can be matched onto ChPT operators with mesons as the degrees of freedom. In this way we can evaluate the ChPT CP-violating amplitudes of $\eta \to \pi^+\pi^-\pi^0$. We found the operators that can contribute to $\eta \to \pi^+\pi^-\pi^0$ decay all vanish at lowest order, i.e. $\mathcal{O}(p^4)$ of ChPT. These results are consistent with our procedure in Gardner and Shi [92] that the strong phases in the ChPT SM $\eta \to \pi^+\pi^-\pi^0$ amplitudes can be used to determine the strong phases in the CP-violating new physics amplitudes. Note for the new operators we found in Ch. 7, we can also determine the imaginary parts of the $\pi - \pi$ rescattering amplitudes $M_1(z)$ and $M_2(z)$ from any of the new physics operators in $\mathcal{O}(p^4)$ in a one–loop graph taken to $\mathcal{O}(p^6)$ as an additional check of our procedures. With the low energy coefficients constrained by experiments, we are hopeful that we can figure out which of the new physics operators can give the biggest effect.

In the future, we plan to write a separate paper on our extension of Watson's theorem [232, 276, 79] to study two body rescattering within three body final states, with the supporting result that our $\mathcal{O}(p^4)$ study of the new operators is consistent with that result and with the SM computation of the strong phases [92]. We could also use empirical information on the two-pion phase shifts in $L$, $I$ (where $L$ and $I$ stand for orbital angular momentum and isospin, respectively) to make a new and possibly better prediction of the final-state phases in $\eta \to \pi^+\pi^-\pi^0$ decay.

# BIBLIOGRAPHY


[1] T. D. Lee and Chen-Ning Yang. Question of Parity Conservation in Weak Interactions. *Phys. Rev.*, 104:254–258, 1956.

[2] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson. Experimental Test of Parity Conservation in $\beta$ Decay. *Phys. Rev.*, 105:1413–1414, 1957.

[3] R. L. Garwin, L. M. Lederman, and Marcel Weinrich. Observations of the Failure of Conservation of Parity and Charge Conjugation in Meson Decays: The Magnetic Moment of the Free Muon. *Phys. Rev.*, 105:1415–1417, 1957.

[4] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay. Evidence for the $2\pi$ Decay of the $K_2^0$ Meson. *Phys. Rev. Lett.*, 13:138–140, 1964.

[5] Julian S. Schwinger. The Theory of quantized fields. I. *Phys. Rev.*, 82:914–927, 1951.

[6] Gerhart Luders. On the Equivalence of Invariance under Time Reversal and under Particle-Antiparticle Conjugation for Relativistic Field Theories. *Kong. Dan. Vid. Sel. Mat. Fys. Med.*, 28N5(5):1–17, 1954.

[7] Wolfgang Pauli. *Niels Bohr and the Development of Physics*. McGraw-Hill Co., New York, 1955.

[8] J. S. Bell. Time reversal in field theory. *Proceedings of the Royal Society*, 1955. [Proc. Roy. Soc. Lond.A231,479(1955)].

[9] Raymond Frederick Streater and Arthur S Wightman. *PCT, Spin and Statistics, and All That*, volume 52. Princeton University Press, 2000.

[10] S. Singh. *Big bang: The most important scientific discovery of all time and why you need to know about it*. Fourth Estate, 2004.

[11] Simon Singh. *Big Bang: the Origin of the Universe*. New York: Fourth Estate, 2004.

[12] Abhay Ashtekar, Tomasz Pawlowski, and Parampreet Singh. Quantum nature of the big bang. *Phys. Rev. Lett.*, 96:141301, 2006.

[13] Abhay Ashtekar, Tomasz Pawlowski, and Parampreet Singh. Quantum Nature of the Big Bang: Improved dynamics. *Phys. Rev.*, D74:084003, 2006.

[14] Laurent Canetti, Marco Drewes, and Mikhail Shaposhnikov. Matter and Antimatter in the Universe. *New J. Phys.*, 14:095012, 2012.

[15] A. D. Sakharov. Violation of CP Invariance, C Asymmetry, and Baryon Asymmetry of the Universe. *Pisma Zh. Eksp. Teor. Fiz.*, 5:32–35, 1967. [Usp. Fiz. Nauk161,61(1991)].

[16] Gary Steigman. Neutrinos And Big Bang Nucleosynthesis. *Adv. High Energy Phys.*, 2012:268321, 2012.

[17] N. Aghanim et al. Planck 2018 results. VI. Cosmological parameters. 2018.

[18] Gary Steigman. The cosmological evolution of the average mass per baryon. *Journal of Cosmology and Astroparticle Physics*, 2006(10):016, 2006.

[19] Glennys R. Farrar and M. E. Shaposhnikov. Baryon asymmetry of the universe in the minimal standard model. *Physical Review Letters*, 70(19):2833–2836, May 1993.

[20] Gerard 't Hooft. Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle. *Phys. Rev.*, D14:3432–3450, 1976. [Erratum: Phys. Rev.D18,2199(1978)].

[21] Gerard 't Hooft. Symmetry Breaking Through Bell-Jackiw Anomalies. *Phys. Rev. Lett.*, 37:8–11, 1976.

[22] N. S. Manton. Topology in the Weinberg-Salam Theory. *Phys. Rev.*, D28:2019, 1983.

[23] Frans R. Klinkhamer and N. S. Manton. A Saddle Point Solution in the Weinberg-Salam Theory. *Phys. Rev.*, D30:2212, 1984.

[24] V. A. Kuzmin, V. A. Rubakov, and M. E. Shaposhnikov. On the Anomalous Electroweak Baryon Number Nonconservation in the Early Universe. *Phys. Lett.*, 155B:36, 1985.

[25] V. A. Rubakov and M. E. Shaposhnikov. Electroweak baryon number nonconservation in the early universe and in high-energy collisions. *Usp. Fiz. Nauk*, 166:493–537, 1996. [Phys. Usp.39,461(1996)].

[26] Makoto Kobayashi and Toshihide Maskawa. CP Violation in the Renormalizable Theory of Weak Interaction. *Prog. Theor. Phys.*, 49:652–657, 1973.

[27] Andrei D. Linde. On the Vacuum Instability and the Higgs Meson Mass. *Phys. Lett.*, 70B:306–308, 1977.

[28] D. A. Kirzhnits and Andrei D. Linde. Symmetry Behavior in Gauge Theories. *Annals Phys.*, 101:195–238, 1976.

[29] Sidney R. Coleman and Erick J. Weinberg. Radiative Corrections as the Origin of Spontaneous Symmetry Breaking. *Phys. Rev.*, D7:1888–1910, 1973.

[30] Andrew G. Cohen, D. B. Kaplan, and A. E. Nelson. Progress in electroweak baryogenesis. *Ann. Rev. Nucl. Part. Sci.*, 43:27–70, 1993.

[31] M. B. Gavela, P. Hernandez, J. Orloff, and O. Pene. Standard model CP violation and baryon asymmetry. *Mod. Phys. Lett.*, A9:795–810, 1994.

[32] M. B. Gavela, P. Hernandez, J. Orloff, O. Pene, and C. Quimbay. Standard model CP violation and baryon asymmetry. Part 2: Finite temperature. *Nucl. Phys.*, B430:382–426, 1994.

[33] Patrick Huet and Eric Sather. Electroweak baryogenesis and standard model CP violation. *Phys. Rev.*, D51:379–394, 1995.

[34] Glennys R. Farrar and M. E. Shaposhnikov. Baryon asymmetry of the universe in the standard model. *Physical Review D*, 50(2):774–818, Jul 1994.

[35] M. Gurtler, Ernst-Michael Ilgenfritz, and A. Schiller. Where the electroweak phase transition ends. *Phys. Rev.*, D56:3888–3895, 1997.

[36] F. Csikor, Z. Fodor, and J. Heitger. Endpoint of the hot electroweak phase transition. *Phys. Rev. Lett.*, 82:21–24, 1999.

[37] Y. Aoki, F. Csikor, Z. Fodor, and A. Ukawa. The Endpoint of the first order phase transition of the SU(2) gauge Higgs model on a four-dimensional isotropic lattice. *Phys. Rev.*, D60:013001, 1999.

[38] M. Laine and K. Rummukainen. What's new with the electroweak phase transition? *Nucl. Phys. Proc. Suppl.*, 73:180–185, 1999. [,180(1998)].

[39] I. Bediaga, I. I. Bigi, A. Gomes, G. Guerrer, J. Miranda, and A. C. dos Reis. On a CP anisotropy measurement in the Dalitz plot. *Phys. Rev.*, D80:096006, 2009.

[40] I. Bediaga, J. Miranda, A. C. dos Reis, I. I. Bigi, A. Gomes, J. M. Otalora Goicochea, and A. Veiga. Second Generation of 'Miranda Procedure' for CP Violation in Dalitz Studies of $B$ (and $D$ and $\tau$) Decays. *Phys. Rev.*, D86:036005, 2012.

[41] M. Tanabashi, K. Hagiwara, K. Hikasa, K. Nakamura, Y. Sumino, F. Takahashi, J. Tanaka, K. Agashe, G. Aielli, C. Amsler, and et al. Review of particle physics. *Physical Review D*, 98(3), Aug 2018.

[42] C. Abel, S. Afach, N.J. Ayres, C.A. Baker, G. Ban, G. Bison, K. Bodek, V. Bondar, M. Burghoff, E. Chanel, and et al. Measurement of the permanent electric dipole moment of the neutron. *Physical Review Letters*, 124(8), Feb 2020.

[43] J. de Vries, R. G. E. Timmermans, E. Mereghetti, and U. van Kolck. The Nucleon Electric Dipole Form Factor From Dimension-Six Time-Reversal Violation. *Phys. Lett.*, B695:268–274, 2011.

[44] J. de Vries, R. Higa, C. P. Liu, E. Mereghetti, I. Stetcu, R. G. E. Timmermans, and U. van Kolck. Electric Dipole Moments of Light Nuclei From Chiral Effective Field Theory. *Phys. Rev.*, C84:065501, 2011.

[45] W. Dekens, J. de Vries, J. Bsaisou, W. Bernreuther, C. Hanhart, Ulf-G. Meißner, A. Nogga, and A. Wirzba. Unraveling models of CP violation through electric dipole moments of light nuclei. *JHEP*, 07:069, 2014.

[46] Antonio Pich and Eduardo de Rafael. Strong CP violation in an effective chiral Lagrangian approach. *Nucl. Phys.*, B367:313–333, 1991.

[47] F. Ambrosino, A. Antonelli, M. Antonelli, C. Bacci, M. Barva, P. Beltrame, G. Bencivenni, S. Bertolucci, C. Bini, C. Bloise, and et al. Upper limit on the $\eta \to \pi^+\pi^-$ branching ratio with the kloe detector. *Physics Letters B*, 606(3-4):276–280, Jan 2005.

[48] A. M. Blik, A. M. Gorin, S. V. Donskov, S. Inaba, V. N. Kolosov, M. E. Ladygin, A. A. Lednev, V. A. Lishin, I. V. Manuilov, Yu. V. Mikhailov, and et al. Searches for rare and forbidden neutral decays of  mesons at the gams-4 facility. *Physics of Atomic Nuclei*, 70(4):693–701, Apr 2007.

[49] Alexey S. Zhevlakov, Mikhail Gorchtein, Astrid N. Hiller Blin, Thomas Gutsche, and Valery E. Lyubovitskij. Bounds on rare decays of $\eta$ and $\eta'$ mesons from the neutron EDM. *Phys. Rev.*, D99(3):031703, 2019.

[50] S. Prakhov, W. B. Tippens, C. Allgower, V. Bekrenev, E. Berger, W. J. Briscoe, M. Clajus, J. R. Comfort, K. Craig, D. Grosnick, and et al. Search for the cp forbidden decay $\eta \to 4\pi^0$. *Physical Review Letters*, 84(21):4802–4805, May 2000.

[51] Feng-Kun Guo, Bastian Kubis, and Andreas Wirzba. Anomalous decays of eta' and eta into four pions. *Phys. Rev.*, D85:014014, 2012.

[52] T. D. Lee. Possible C-Noninvariant Effects in the $3\pi$ Decay Modes of $\eta^0$ and $\omega^0$. *Phys. Rev.*, 139:B1415–B1420, 1965.

[53] Susan Gardner and Jusak Tandean. Observing direct CP violation in untagged B meson decays. *Phys. Rev.*, D69:034011, 2004.

[54] T. D. Lee and L. Wolfenstein. Analysis of CP Noninvariant Interactions and the $K_1^0$, $K_2^0$ System. *Phys. Rev.*, 138:B1490–B1496, 1965.

[55] J. Prentki and M. J. G. Veltman. Possibility of CP violation in semistrong interactions. *Phys. Lett.*, 15:88–90, 1965.

[56] M. Nauenberg. The $\eta \to \pi^+\pi^-\pi^0$ Decay with C-Violation. *Phys. Lett.*, 17:329, 1965.

[57] Barbara Barrett, Maurice Jacob, Michael Nauenberg, and Tran N. Truong. Consequences of C - Violating Interactions in $\eta^0$ and $X^0$ Decays. *Phys. Rev.*, 141:1342–1349, 1966.

[58] David J. Gross, S. B. Treiman, and Frank Wilczek. Light Quark Masses and Isospin Violation. *Phys. Rev.*, D19:2188, 1979.

[59] Paul Langacker and Heinz Pagels. Light Quark Mass Spectrum in Quantum Chromodynamics. *Phys. Rev.*, D19:2070, 1979.

[60] H. Leutwyler. The Ratios of the light quark masses. *Phys. Lett.*, B378:313–318, 1996.

[61] J. Gasser and H. Leutwyler. $\eta \to 3\pi$ to One Loop. *Nucl. Phys.*, B250:539–560, 1985.

[62] A. V. Anisovich and H. Leutwyler. Dispersive analysis of the decay $\eta \to 3\pi$. *Phys. Lett.*, B375:335–342, 1996.

[63] Johan Bijnens and Jurg Gasser. Eta decays at and beyond $p^4$ in chiral perturbation theory. *Phys. Scripta*, T99:34–44, 2002.

[64] Johan Bijnens and Karim Ghorbani. $\eta \to 3\pi$ at Two Loops In Chiral Perturbation Theory. *JHEP*, 11:030, 2007.

[65] C. Baltay. Search for C Violation in $\eta \to \pi^+\pi^-\pi^0$ (CLPWY: Columbia-Berkeley-Purdue-Wisconsin-Yale Collaboration). *Phys. Rev.*, 149:1044, 1966.

[66] M. Gormley, E. Hyman, Won-Yong Lee, T. Nash, J. Peoples, C. Schultz, and S. Stein. Experimental determination of the Dalitz-plot distribution of the decays $\eta \to \pi^+\pi^-\pi^0$ and $\eta \to \pi^+\pi^-\gamma$, and the branching ratio $\eta \to \pi^+\pi^-\gamma/\eta \to \pi^+$. *Phys. Rev.*, D2:501–505, 1970.

[67] M. Gormley, E. Hyman, W. Lee, T. Nash, J. Peoples, C. Schultz, and S. Stein. Experimental Test of C Invariance in $\eta \to \pi^+\pi^-\pi^0$. *Phys. Rev. Lett.*, 21:402–406, 1968.

[68] Li Caldeira Balkeståhl. *Measurement of the Dalitz Plot Distribution for $\eta \to \pi^+\pi^-\pi^0$ with KLOE.* PhD thesis, Uppsala U., 2016.

[69] J. G. Layter, J. A. Appel, A. Kotlewski, Won-Yong Lee, S. Stein, and J. J. Thaler. Measurement of the charge asymmetry in the decay $\eta \to \pi^+\pi^-\pi^0$. *Phys. Rev. Lett.*, 29:316–319, 1972.

[70] A. Anastasi et al. Precision measurement of the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot distribution with the KLOE detector. *JHEP*, 05:019, 2016.

[71] A. Larribe et al. Test of C Invariance in the $3\pi$ Decay Mode of the $\eta$ Meson. *Phys. Lett.*, 23:600–604, 1966.

[72] J. G. Layter, J. A. Appel, A. Kotlewski, Won-Yong Lee, S. Stein, and J. J. Thaler. Study of dalitz-plot distributions of the decays $\eta \to \pi^+\pi^-\pi^0$ and $\eta \to \pi^+\pi^-\gamma$. *Phys. Rev.*, D7:2565–2568, 1973.

[73] M. R. Jane et al. A Measurement of the Charge Asymmetry in the Decay $\eta \to \pi^+\pi^-\pi^0$. *Phys. Lett.*, 48B:260–264, 1974.

[74] F. Ambrosino et al. Determination of $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot slopes and asymmetries with the KLOE detector. *JHEP*, 05:006, 2008.

[75] Liping Gan. Test Fundamental Symmetries via Precision Measurements of $\pi^0, \eta$, and $\eta'$ Decays. *JPS Conf. Proc.*, 13:020063, 2017.

[76] Corrado Gatto, Brenda Fabela Enriquez, and Maria Isabel Pedraza Morales. The REDTOP project: Rare Eta Decays with a TPC for Optical Photons. *PoS*, ICHEP2016:812, 2016.

[77] J. Beacham et al. Physics Beyond Colliders at CERN: Beyond the Standard Model Working Group Report. *J. Phys.*, G47(1):010501, 2020.

[78] R. D. Peccei and Helen R. Quinn. CP Conservation in the Presence of Instantons. *Phys. Rev. Lett.*, 38:1440–1443, 1977.

[79] Ikaros I. Bigi and A. I. Sanda. *CP violation*. Cambridge University Press, 2009. p.224 [Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.9,1(2009)].

[80] Gustavo C. Branco, Luis Lavoura, and Joao P. Silva. *CP Violation*. Oxford University Press, 1999.

[81] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995.

[82] T. D. Lee and G. C. Wick. Space Inversion, Time Reversal, and Other Discrete Symmetries in Local Field Theories. *Phys. Rev.*, 148:1385–1404, 1966. [,445(1966)].

[83] P. A. M. Dirac. *The Principle of Quantum Mechanics*. Oxford University Press, 1958.

[84] Susan Gardner and Xinshuai Yan. CPT, CP, and C transformations of fermions, and their consequences, in theories with B-L violation. *Phys. Rev.*, D93(9):096008, 2016.

[85] Murray Gell-Mann. Symmetries of baryons and mesons. *Phys. Rev.*, 125:1067–1084, 1962.

[86] Steven Weinberg. *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 2005.

[87] L. Wolfenstein. Cp violation in b0b0 mixing. *Nuclear Physics B*, 246(1):45 – 51, 1984.

[88] I.I. Bigi. Looking from the east at an elephant trotting west:11with due apologies to sir charles laughton. direct cp violation in b0 decays. *Physics Letters B*, 535(1):155 – 158, 2002.

[89] Nita Sinha and Rahul Sinha. Determination of the cp violation angle $\gamma$ using b d* v modes. *Physical review letters*, 80(17):3706, 1998.

[90] Susan Gardner. Direct CP violation in untagged $B$ meson decays. *Phys. Lett.*, B553:261–266, 2003.

[91] Alexey A. Petrov. Hunting for CP violation with untagged charm decays. *Phys. Rev.*, D69:111901, 2004.

[92] S. Gardner and J. Shi. Patterns of $CP$ violation from mirror symmetry breaking in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot. *Physical Review D*, 101(11):115038, 2020.

[93] R. H. Dalitz. On the analysis of tau-meson data and the nature of the tau-meson. *Phil. Mag. Ser.7*, 44:1068–1080, 1953.

[94] Y.Q. Chen et al. Dalitz analysis of $D^0 \to K^-\pi^+\eta$ decays at Belle. *Phys. Rev. D*, 102(1):012002, 2020.

[95] Steven Weinberg. New Test for $\Delta I = 1/2$ in $K^+$ Decay. *Phys. Rev. Lett.*, 4:87–89, 1960. [Erratum: Phys. Rev. Lett.4,585(1960)].

[96] E. Fabri. A study of tau-meson decay. *Nuovo Cim.*, 11:479–491, 1954.

[97] V. Weisskopf and Eugene P. Wigner. Calculation of the natural brightness of spectral lines on the basis of Dirac's theory. *Z. Phys.*, 63:54–73, 1930.

[98] V. Weisskopf and E. Wigner. Over the natural line width in the radiation of the harmonius oscillator. *Z. Phys.*, 65:18–29, 1930.

[99] Ashton B. Carter and A.I. Sanda. CP Violation in B Meson Decays. *Phys. Rev. D*, 23:1567, 1981.

[100] Ikaros I. Y. Bigi and A. I. Sanda. Notes on the Observability of CP Violations in B Decays. *Nucl. Phys.*, B193:85–108, 1981.

[101] B. Grzadkowski, M. Iskrzynski, M. Misiak, and J. Rosiek. Dimension-Six Terms in the Standard Model Lagrangian. *JHEP*, 10:085, 2010.

[102] Lincoln Wolfenstein. Parametrization of the Kobayashi-Maskawa Matrix. *Phys. Rev. Lett.*, 51:1945, 1983.

[103] Andrzej J. Buras, Markus E. Lautenbacher, and Gaby Ostermaier. Waiting for the top quark mass, $K^+ \to \pi^+\nu\bar{\nu}$, $B_s^0 - \overline{B}_s^0$ mixing and CP asymmetries in B decays. *Phys. Rev.*, D50:3433–3446, 1994.

[104] J. Charles, A. Höcker, H. Lacker, S. Laplace, F. R. Le Diberder, J. Malclés, J. Ocariz, M. Pivk, and L. Roos. Cp violation and the ckm matrix: assessing the impact of the asymmetric b factories. *The European Physical Journal C*, 41(1):1–131, May 2005.

[105] Antonio Riotto. Theories of baryogenesis. In *Proceedings, Summer School in High-energy physics and cosmology: Trieste, Italy, June 29-July 17, 1998*, pages 326–436, 1998.

[106] Steven Weinberg. The U(1) Problem. *Phys. Rev.*, D11:3583–3593, 1975.

[107] R. J. Crewther. Status of the U(1) Problem. *Riv. Nuovo Cim.*, 2N8:63–117, 1979.

[108] Ta-Pei Cheng and Ling-Fong Li. *Gauge theory of elementary particle physics*. Clarendon press Oxford, 1984.

[109] Gerard 't Hooft. How Instantons Solve the U(1) Problem. *Phys. Rept.*, 142:357–387, 1986.

[110] Edward Witten. Current algebra theorems for the u (1)"goldstone boson". *Nuclear Physics B*, 156(2):269–283, 1979.

[111] Gabriele Veneziano. U (1) without instantons. *Nuclear Physics B*, 159(1-2):213–224, 1979.

[112] Matthew D Schwartz. *Quantum field theory and the standard model*. Cambridge University Press, 2014.

[113] T.E. Chupp. Electric dipole moments of atoms, molecules, nuclei, and particles. *Reviews of Modern Physics*, 91(1), 2019.

[114] R. D. Peccei and Helen R. Quinn. Constraints imposed by CP conservation in the presence of instantons. *Physical Review D*, 16(6):1791–1797, 1977.

[115] Steven Weinberg. A new light boson? *Physical Review Letters*, 40(4):223–226, 1978.

[116] Frank Wilczek. Problem of Strong p and t Invariance in the Presence of Instantons. *Physical Review Letters*, 40(5):279–282, 1978.

[117] Benjamin M Brubaker. First results from the haystac axion search. *arXiv preprint arXiv:1801.00835*, 2018.

[118] TD Lee. A theory of spontaneous t violation. *Physical Review D*, 8(4):1226, 1973.

[119] Tsung-Dao Lee. Cp nonconservation and spontaneous symmetry breaking. *Physics Reports*, 9(2):143–177, 1974.

[120] Pierre Fayet. A gauge theory of weak and electromagnetic interactions with spontaneous parity breaking. *Nuclear Physics B*, 78(1):14–28, 1974.

[121] Ricardo A Flores and Marc Sher. Higgs masses in the standard, multi-higgs and supersymmetric models. *Annals of Physics*, 148(1):95–134, 1983.

[122] Jogesh C Pati and Abdus Salam. Lepton number as the fourth "color". *Physical Review D*, 10(1):275, 1974.

[123] Rabindra N Mohapatra and Jogesh C Pati. Left-right gauge symmetry and an "isoconjugate" model of CP violation. *Physical Review D*, 11(3):566, 1975.

[124] Rabindra N Mohapatra and Goran Senjanović. Neutrino mass and spontaneous parity nonconservation. *Physical Review Letters*, 44(14):912, 1980.

[125] Rabindra N Mohapatra and Goran Senjanović. Neutrino masses and mixings in gauge models with spontaneous parity violation. *Physical Review D*, 23(1):165, 1981.

[126] Edward Witten. Dynamical breaking of supersymmetry. *Nuclear Physics B*, 188(3):513–554, 1981.

[127] Savas Dimopoulos and Howard Georgi. Softly broken supersymmetry and su (5). *Nuclear Physics B*, 193(1):150–162, 1981.

[128] N Sakai. Naturalnes in supersymmetric guts. *Zeitschrift für Physik C Particles and Fields*, 11(2):153–157, 1981.

[129] Romesh K Kaul. Gauge hierarchy in a supersymmetric model. *Physics Letters B*, 109(1-2):19–24, 1982.

[130] Romesh K Kaul and Parthasarathi Majumdar. Cancellation of quadratically divergent mass corrections in globally supersymmetric spontaneously broken gauge theories. *Nuclear Physics B*, 199(1):36–58, 1982.

[131] Leonard Susskind. The gauge hierarchy problem, technicolor, supersymmetry, and all that. *Physics Reports*, 104(2-4):181–193, 1984.

[132] R. N. Mohapatra and Jogesh C. Pati. A Natural Left-Right Symmetry. *Phys. Rev.*, D11:2558, 1975.

[133] Rabindra N. Mohapatra and Jogesh C. Pati. Left-Right Gauge Symmetry and an Isoconjugate Model of CP Violation. *Phys. Rev.*, D11:566–571, 1975.

[134] Jogesh C. Pati and Abdus Salam. Lepton Number as the Fourth Color. *Phys. Rev.*, D10:275–289, 1974. [Erratum: Phys. Rev.D11,703(1975)].

[135] G. Senjanovic and Rabindra N. Mohapatra. Exact Left-Right Symmetry and Spontaneous Violation of Parity. *Phys. Rev.*, D12:1502, 1975.

[136] M. Gell-Mann, Slansky R., and Ramond P. *Supergravity*. North Holland, Amsterdam.

[137] Tsutomu Yanagida. Horizontal Symmetry and Masses of neutrinos. *Conf. Proc.*, C7902131:95–99, 1979.

[138] Gordon L. Kane and M. Shifman, editors. *The supersymmetric world: The beginning of the theory*. 2000.

[139] Hans Peter Nilles. Supersymmetry, Supergravity and Particle Physics. *Phys. Rept.*, 110:1–162, 1984.

[140] Sidney Coleman and Jeffrey Mandula. All possible symmetries of the s matrix. *Physical Review*, 159(5):1251, 1967.

[141] Oskar Pelc and LP Horwitz. Generalization of the coleman–mandula theorem to higher dimension. *Journal of Mathematical Physics*, 38(1):139–172, 1997.

[142] Yu.A. Golfand and E.P. Likhtman. Extension of the Algebra of Poincare Group Generators and Violation of p Invariance. *JETP Lett.*, 13:323–326, 1971.

[143] Rudolf Haag, Jan T. Łopuszański, and Martin Sohnius. All possible generators of supersymmetries of the s-matrix. *Nuclear Physics B*, 88(2):257 – 274, 1975.

[144] Stephen P. Martin. A supersymmetry primer.

[145] Howard Baer and Xerxes Tata. *Weak scale supersymmetry: From superfields to scattering events*. Cambridge University Press, 2006.

[146] S. L. Glashow. Partial Symmetries of Weak Interactions. *Nucl. Phys.*, 22:579–588, 1961.

[147] Abdus Salam. Weak and Electromagnetic Interactions. *Conf. Proc.*, C680519:367–377, 1968.

[148] Steven Weinberg. A Model of Leptons. *Phys. Rev. Lett.*, 19:1264–1266, 1967.

[149] Gerard 't Hooft and M. J. G. Veltman. Regularization and Renormalization of Gauge Fields. *Nucl. Phys.*, B44:189–213, 1972.

[150] Georges Aad et al. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. *Phys. Lett.*, B716:1–29, 2012.

[151] Serguei Chatrchyan et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. *Phys. Lett.*, B716:30–61, 2012.

[152] Sudip Jana and S. Nandi. New physics scale from higgs observables with effective dimension-6 operators. *Physics Letters B*, 783:51 – 58, 2018.

[153] T.L. Barklow, S. Dawson, Haber H.E., and Siegrist J.L. *Electroweak Symmetry Breaking and New Physics at the TeV Scale*, volume 16. World Scientific, 1997.

[154] Aneesh V Manohar. Effective field theories. In *Perturbative and nonperturbative aspects of quantum field theory*, pages 311–362. Springer, 1977.

[155] Cliff P Burgess. An introduction to effective field theory. *Annu. Rev. Nucl. Part. Sci.*, 57:329–362, 2007.

[156] H. Georgi. Effective field theory. *Ann. Rev. Nucl. Part. Sci.*, 43:209–252, 1993.

[157] Howard M Georgi. *Weak interactions and modern particle theory.* Benjamin-Cummings, 1984.

[158] Brian Henning, Xiaochuan Lu, and Hitoshi Murayama. How to use the Standard Model effective field theory. *JHEP*, 01:023, 2016.

[159] J. F. Donoghue, E. Golowich, and Barry R. Holstein. *Dynamics of the standard model.* Cambridge University Press, 2nd edition, 2014. Subjects: Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology.

[160] Kenneth G. Wilson. Non-lagrangian models of current algebra. *Physical Review*, 179(5):1499–1512, 1969.

[161] Steven Weinberg. Baryon and Lepton Nonconserving Processes. *Phys. Rev. Lett.*, 43:1566–1570, 1979.

[162] Frank Wilczek. Operator analysis of nucleon decay. *Physical Review Letters*, 43(21):1571–1573, 1979.

[163] HA Weldon and A Zee. Operator analysis of new physics. *Nuclear Physics B*, 173(2):269–290, 1980.

[164] W. Buchmuller and D. Wyler. Effective Lagrangian Analysis of New Interactions and Flavor Conservation. *Nucl. Phys.*, B268:621–653, 1986.

[165] Anthony Zee. *Quantum field theory in a nutshell*, volume 7. Princeton university press, 2010.

[166] Gerhard Buchalla and Oscar Cata. Effective Theory of a Dynamically Broken Electroweak Standard Model at NLO. *JHEP*, 07:101, 2012.

[167] Gerhard Buchalla, Oscar Cata, and Claudius Krause. A Systematic Approach to the SILH Lagrangian. *Nucl. Phys.*, B894:602–620, 2015.

[168] G. Buchalla, O. Cata, A. Celis, and C. Krause. Standard Model Extended by a Heavy Singlet: Linear vs. Nonlinear EFT. *Nucl. Phys.*, B917:209–233, 2017.

[169] V. Khachatryan, A. M. Sirunyan, A. Tumasyan, W. Adam, T. Bergauer, M. Dragicevic, J. Erö, M. Friedl, R. Frühwirth, and et al. Precise determination of the mass of the Higgs boson and tests of compatibility of its couplings with the standard model predictions using proton collisions at 7 and 8 TeV. *The European Physical Journal C*, 75(5), May 2015.

[170] G. Buchalla, O. Cata, A. Celis, and C. Krause. Note on Anomalous Higgs-Boson Couplings in Effective Field Theory. *Phys. Lett.*, B750:298–301, 2015.

[171] Howard Georgi, David B. Kaplan, and Peter Galison. Calculation of the composite higgs mass. *Physics Letters B*, 143(1-3):152–154, Aug 1984.

[172] David B. Kaplan, Howard Georgi, and Savas Dimopoulos. Composite higgs scalars. *Physics Letters B*, 136(3):187–190, Mar 1984.

[173] Gian Francesco Giudice, Christophe Grojean, Alex Pomarol, and Riccardo Rattazzi. The strongly-interacting light higgs. *Journal of High Energy Physics*, 2007(06):045–045, Jun 2007.

[174] Roberto Contino, Christophe Grojean, Mauro Moretti, Fulvio Piccinini, and Riccardo Rattazzi. Strong double higgs production at the lhc. *Journal of High Energy Physics*, 2010(5), May 2010.

[175] Roberto Contino. The Higgs as a Composite Nambu-Goldstone Boson. *Physics of the Large and the Small*, Mar 2011.

[176] Roberto Contino. New Physics at the LHC: Strong vs. weak symmetry breaking. *Nuovo Cim.*, 32:11–18, 2009.

[177] Roberto Contino, Margherita Ghezzi, Christophe Grojean, Margarete Mühlleitner, and Michael Spira. Effective lagrangian for a light higgs-like scalar. *Journal of High Energy Physics*, 2013(7), Jul 2013.

[178] Gerhard Buchalla, Oscar Catà, and Claudius Krause. Complete Electroweak Chiral Lagrangian with a Light Higgs at NLO. *Nucl. Phys.*, B880:552–573, 2014. [Erratum: Nucl. Phys.B913,475(2016)].

[179] David J. Gross. Ultraviolet behavior of non-abelian gauge theories. *Physical Review Letters*, 30(26):1343–1346, 1973.

[180] Stefan Scherer and Matthias R. Schindler. A Primer for Chiral Perturbation Theory. *Lect. Notes Phys.*, 830:pp.1–338, 2012.

[181] Veronique Bernard and Ulf-G. Meissner. Chiral perturbation theory. *Ann. Rev. Nucl. Part. Sci.*, 57:33–60, 2007.

[182] G. Ecker. Chiral perturbation theory. *Prog. Part. Nucl. Phys.*, 35:1–80, 1995.

[183] Emmy Noether. Invariante variationsprobleme. *Königlich Gesellschaft der Wissenschaften Göttingen Nachrichten Mathematik-physik Klasse*, 2:235–267, 1918.

[184] Emmy Noether. Invariant variation problems. *Transport Theory and Statistical Physics*, 1(3):186–207, 1971.

[185] J. Goldstone. Field Theories with Superconductor Solutions. *Nuovo Cim.*, 19:154–164, 1961.

[186] Jeffrey Goldstone, Abdus Salam, and Steven Weinberg. Broken Symmetries. *Phys. Rev.*, 127:965–970, 1962.

[187] H. Lehmann, K. Symanzik, and W. Zimmermann. On the formulation of quantized field theories. *Nuovo Cim.*, 1:205–225, 1955.

[188] J. Gasser and H. Leutwyler. Chiral Perturbation Theory to One Loop. *Annals Phys.*, 158:142, 1984.

[189] J. Gasser and H. Leutwyler. Chiral Perturbation Theory: Expansions in the Mass of the Strange Quark. *Nucl. Phys.*, B250:465–516, 1985.

[190] H. Leutwyler. On the foundations of chiral perturbation theory. *Annals Phys.*, 235:165–203, 1994.

[191] O. Cata and V. Mateu. Chiral perturbation theory with tensor sources. *JHEP*, 09:078, 2007.

[192] Steven Weinberg. Phenomenological Lagrangians. *Physica*, A96:327–340, 1979.

[193] J. Bijnens, G. Ecker, and J. Gasser. Chiral perturbation theory. In *2nd DAPHNE Physics Handbook:125-144*, pages 125–144, 1994.

[194] M. Knecht and J. Stern. Generalized chiral perturbation theory. In *2nd DAPHNE Physics Handbook:169-190*, pages 169–190, 1994.

[195] NH Fuchs, H Sazdjian, and J Stern. How to probe the scale of $\langle \bar{q}q \rangle$ in chiral perturbation theory. *Physics Letters B*, 269(1-2):183–188, 1991.

[196] J Stern, H Sazdjian, and NH Fuchs. What $\pi$- $\pi$ scattering tells us about chiral perturbation theory. *Physical Review D*, 47(9):3814, 1993.

[197] B Adeva, L Afanasyev, A Anania, S Aogaki, A Benelli, V Brekhovskikh, T Cechak, M Chiba, PV Chliapnikov, P Doskarova, et al. First measurement of a long-lived $\pi$+ $\pi$- atom lifetime. *Physical review letters*, 122(8):082003, 2019.

[198] J. Gasser and G. R. S. Zarnauskas. On the pion decay constant. *Phys. Lett.*, B693:122–128, 2010.

[199] Murray Gell-Mann, Robert J Oakes, and Brian Renner. Behavior of current divergences under su 3$\times$ su 3. *Physical Review*, 175(5):2195, 1968.

[200] Murray Gell-Mann. Model of the Strong Couplings. *Phys. Rev.*, 106:1296–1300, 1957.

[201] Susumu Okubo. Note on unitary symmetry in strong interactions. *Prog. Theor. Phys.*, 27:949–966, 1962.

[202] Steven Weinberg. Phenomenological lagrangians. *Physica A: Statistical Mechanics and its Applications*, 96(1-2):327–340, Apr 1979.

[203] J. Bernstein, G. Feinberg, and T. D. Lee. Possible C, $T$ Noninvariance in the Electromagnetic Interaction. *Phys. Rev.*, 139:B1650–B1659, 1965. [,381(1965)].

[204] D. G. Sutherland. Current algebra and the decay $\eta \to 3\pi$. *Phys. Lett.*, 23:384–385, 1966.

[205] J. S. Bell and D. G. Sutherland. Current algebra and $\eta \to 3\pi$. *Nucl. Phys.*, B4:315–325, 1968.

[206] R. Baur, J. Kambor, and D. Wyler. Electromagnetic corrections to the decays $\eta \to 3\pi$. *Nucl. Phys.*, B460:127–142, 1996.

[207] Christoph Ditsche, Bastian Kubis, and Ulf-G. Meissner. Electromagnetic corrections in $\eta \to 3\pi$ decays. *Eur. Phys. J.*, C60:83–105, 2009.

[208] H. Osborn and D. J. Wallace. *eta* - $X$ mixing, $\eta \to 3\pi$ and chiral lagrangians. *Nucl. Phys.*, B20:23–44, 1970.

[209] A. Neveu and J. Scherk. Final-state interaction and current algebra in $K_{3\pi}$ and $\eta_{3\pi}$ decays. *Annals Phys.*, 57:39–64, 1970.

[210] C. Roiesnel and Tran N. Truong. Resolution of the $\eta \to 3\pi$ Problem. *Nucl. Phys.*, B187:293–300, 1981.

[211] J. Kambor, C. Wiesendanger, and D. Wyler. Final state interactions and Khuri-Treiman equations in $\eta \to 3\pi$ decays. *Nucl. Phys.*, B465:215–266, 1996.

[212] B. Borasoy and R. Nissler. Hadronic eta and eta-prime decays. *Eur. Phys. J.*, A26:383–398, 2005.

[213] Gilberto Colangelo, Stefan Lanz, and Emilie Passemar. A New Dispersive Analysis of $\eta \to 3\pi$. *PoS*, CD09:047, 2009.

[214] Sebastian P. Schneider, Bastian Kubis, and Christoph Ditsche. Rescattering effects in $\eta \to 3\pi$ decays. *JHEP*, 02:028, 2011.

[215] Karol Kampf, Marc Knecht, Jiri Novotny, and Martin Zdrahal. Analytical dispersive construction of $\eta \to 3\pi$ amplitude: first order in isospin breaking. *Phys. Rev.*, D84:114015, 2011.

[216] Peng Guo, Igor V. Danilkin, Diane Schott, C. Fernández-Ramírez, V. Mathieu, and Adam P. Szczepaniak. Three-body final state interaction in $\eta \to 3\pi$. *Phys. Rev.*, D92(5):054016, 2015.

[217] Gilberto Colangelo, Stefan Lanz, Heinrich Leutwyler, and Emilie Passemar. $\eta \to 3\pi$: Study of the Dalitz plot and extraction of the quark mass ratio $Q$. *Phys. Rev. Lett.*, 118(2):022001, 2017.

[218] M. Albaladejo and B. Moussallam. Extended chiral Khuri-Treiman formalism for $\eta \to 3\pi$ and the role of the $a_0(980)$, $f_0(980)$ resonances. *Eur. Phys. J.*, C77(8):508, 2017.

[219] Gilberto Colangelo, Stefan Lanz, Heinrich Leutwyler, and Emilie Passemar. Dispersive analysis of $\eta \to 3\pi$. *Eur. Phys. J.*, C78(11):947, 2018.

[220] Thomas Gutsche, Astrid N. Hiller Blin, Sergey Kovalenko, Serguei Kuleshov, Valery E. Lyubovitskij, Manuel J. Vicente Vacas, and Alexey Zhevlakov. CP-violating decays of the pseudoscalars eta and eta' and their connection to the electric dipole moment of the neutron. *Phys. Rev.*, D95(3):036022, 2017.

[221] Gustavo Burdman and John F. Donoghue. B meson CP violation without flavor identification. *Phys. Rev.*, D45:187–192, 1992.

[222] Stefan Lanz. *Determination of the quark mass ratio Q from* $\eta \to 3\pi$. PhD thesis, Bern U., 2011.

[223] I. B. Khriplovich. What do we know about T odd but P even interaction? *Nucl. Phys.*, B352:385–401, 1991.

[224] R. S. Conti and I. B. Khriplovich. New limits on T odd, P even interactions. *Phys. Rev. Lett.*, 68:3262–3265, 1992.

[225] Jonathan Engel, Paul H. Frampton, and Roxanne P. Springer. Effective Lagrangians and parity conserving time reversal violation at low-energies. *Phys. Rev.*, D53:5112–5114, 1996.

[226] M. J. Ramsey-Musolf. Electric dipole moments and the mass scale of new T violating, P conserving interactions. *Phys. Rev. Lett.*, 83:3997–4000, 1999. [Erratum: Phys. Rev. Lett.84,5681(2000)].

[227] A. Kurylov, G. C. McLaughlin, and M. J. Ramsey-Musolf. Constraints on T odd, P even interactions from electric dipole moments, revisited. *Phys. Rev.*, D63:076007, 2001.

[228] Susan Gardner and Jun Shi. Leading-dimension, effective operators with cp and c or p violation in standard model effective field theory. *in preparation*, 2020.

[229] Harry J. Lipkin, Yosef Nir, Helen R. Quinn, and A. Snyder. Penguin trapping with isospin analysis and CP asymmetries in B decays. *Phys. Rev.*, D44:1454–1460, 1991.

[230] Arthur E. Snyder and Helen R. Quinn. Measuring CP asymmetry in $B \to \rho\pi$ decays without ambiguities. *Phys. Rev.*, D48:2139–2144, 1993.

[231] Susan Gardner and Ulf-G. Meissner. Rescattering and chiral dynamics in $B \to \rho\pi$ decay. *Phys. Rev.*, D65:094004, 2002.

[232] Kenneth M. Watson. Some general relations between the photoproduction and scattering ofmesons. *Physical Review*, 95(1):228–236, Jul 1954.

[233] Johan Bijnens, Pierre Dhonte, and Fredrik Persson. $K \to 3\pi$ decays in chiral perturbation theory. *Nucl. Phys.*, B648:317–344, 2003.

[234] Susan Gardner and Heath Bland O'Connell. $\rho - \omega$ mixing and the pion form-factor in the timelike region. *Phys. Rev.*, D57:2716–2726, 1998. [Erratum: Phys. Rev.D62,019903(2000)].

[235] Johan Bijnens and Gerhard Ecker. Mesonic low-energy constants. *Ann. Rev. Nucl. Part. Sci.*, 64:149–174, 2014.

[236] G. Amoros, J. Bijnens, and P. Talavera. QCD isospin breaking in meson masses, decay constants and quark mass ratios. *Nucl. Phys.*, B602:87–108, 2001.

[237] B. Hyams et al. $\pi\pi$ Phase Shift Analysis from 600-MeV to 1900-MeV. *Nucl. Phys.*, B64:134–162, 1973.

[238] G. Colangelo, J. Gasser, and H. Leutwyler. $\pi\pi$ scattering. *Nucl. Phys.*, B603:125–179, 2001.

[239] B. Ananthanarayan, G. Colangelo, J. Gasser, and H. Leutwyler. Roy equation analysis of pi pi scattering. *Phys. Rept.*, 353:207–279, 2001.

[240] Thomas Appelquist and J. Carazzone. Infrared Singularities and Massive Fields. *Phys. Rev.*, D11:2856, 1975.

[241] Andrew Kobach. Baryon Number, Lepton Number, and Operator Dimension in the Standard Model. *Phys. Lett.*, B758:455–457, 2016.

[242] Jason Aebischer, Wolfgang Altmannshofer, Diego Guadagnoli, Mril Reboud, Peter Stangl, and David M. Straub. B-decay discrepancies after Moriond 2019. 2019.

[243] Haipeng An, Xiangdong Ji, and Fanrong Xu. P-odd and CP-odd Four-Quark Contributions to Neutron EDM. *JHEP*, 02:043, 2010.

[244] J. de Vries, E. Mereghetti, R. G. E. Timmermans, and U. van Kolck. The Effective Chiral Lagrangian From Dimension-Six Parity and Time-Reversal Violation. *Annals Phys.*, 338:50–96, 2013.

[245] Y. T. Chien, V. Cirigliano, W. Dekens, J. de Vries, and E. Mereghetti. Direct and indirect constraints on CP-violating Higgs-quark and Higgs-gluon interactions. 2015.

[246] Elizabeth E. Jenkins, Aneesh V. Manohar, and Peter Stoffer. Low-Energy Effective Field Theory below the Electroweak Scale: Operators and Matching. *JHEP*, 03:016, 2018.

[247] Elizabeth E. Jenkins, Aneesh V. Manohar, and Peter Stoffer. Low-Energy Effective Field Theory below the Electroweak Scale: Anomalous Dimensions. *JHEP*, 01:084, 2018.

[248] Andre de Gouvea and James Jenkins. A Survey of Lepton Number Violation Via Effective Operators. *Phys. Rev.*, D77:013008, 2008.

[249] W. Buchmüller, B. Lampe, and N. Vlachos. Contact interactions and the callan-gross relation. *Physics Letters B*, 197(3):379–382, Oct 1987.

[250] C. Arzt, M.B. Einhorn, and J. Wudka. Patterns of deviation from the standard model. *Nuclear Physics B*, 433(1):41–66, Jan 1995.

[251] V. Alan Kostelecky. The Status of CPT. In *Physics beyond the standard model. Proceedings, 5th International WEIN Symposium, Santa Fe, USA, June 14-19, 1998*, pages 588–600, 1998.

[252] C. C. Nishi. Simple derivation of general Fierz-like identities. *Am. J. Phys.*, 73:1160–1163, 2005.

[253] A. de Gouvea. (Charged) lepton flavor violation. *Nucl. Phys. Proc. Suppl.*, 188:303–308, 2009.

[254] M. Diehl and G. Hiller. New ways to explore factorization in b decays. *JHEP*, 06:067, 2001.

[255] Landon Lehman and Adam Martin. Low-derivative operators of the Standard Model effective field theory via Hilbert series methods. *JHEP*, 02:081, 2016.

[256] Brian Henning, Xiaochuan Lu, Tom Melia, and Hitoshi Murayama. 2, 84, 30, 993, 560, 15456, 11962, 261485, ...: Higher dimension operators in the SM EFT. *JHEP*, 08:016, 2017.

[257] Landon Lehman. Extending the Standard Model Effective Field Theory with the Complete Set of Dimension-7 Operators. *Phys. Rev.*, D90(12):125023, 2014.

[258] Ulf-G. Meissner and Susan Gardner. $B \to 3\pi$ decay and chiral dynamics. *Eur. Phys. J.*, A18:543–545, 2003.

[259] Ahmed Ali and C. Greub. An analysis of two-body nonleptonic B decays involving light mesons in the standard model. *Phys. Rev.*, D57:2996–3016, 1998.

[260] Yaw-Hwang Chen, Hai-Yang Cheng, B. Tseng, and Kwei-Chou Yang. Charmless hadronic two-body decays of B(u) and B(d) mesons. *Phys. Rev.*, D60:094014, 1999.

[261] Ahmed Ali, G. Kramer, and Cai-Dian Lu. Experimental tests of factorization in charmless nonleptonic two-body B decays. *Phys. Rev.*, D58:094009, 1998.

[262] Hai-Yang Cheng and Kwei-Chou Yang. Implications of recent measurements of hadronic charmless B decays. *Phys. Rev.*, D62:054029, 2000.

[263] A. Deandrea, Raoul Gatto, M. Ladisa, G. Nardulli, and Pietro Santorelli. Measuring $B \to \rho\pi$ decays and the unitarity angle alpha. *Phys. Rev.*, D62:036001, 2000.

[264] Gerhard Buchalla, Andrzej J. Buras, and Markus E. Lautenbacher. Weak decays beyond leading logarithms. *Rev. Mod. Phys.*, 68:1125–1144, 1996.

[265] F. Halzen and Alan D. Martin. *Quarks and leptons: an introductory course in modern particle physics*. 1984.

[266] Johan Bijnens, Gilberto Colangelo, and Gerhard Ecker. The Mesonic chiral Lagrangian of order $p^6$. *JHEP*, 02:020, 1999.

[267] G. Ecker, J. Gasser, A. Pich, and E. [De Rafael]. The role of resonances in chiral perturbation theory. *Nuclear Physics B*, 321(2):311 – 342, 1989.

[268] Eduardo de Rafael. Chiral Lagrangians and kaon CP violation. In *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 94): CP Violation and the limits of the Standard Model Boulder, Colorado, May 29-June 24, 1994*, 1995.

[269] H. Georgi. *Weak Interactions and Modern Particle Theory*. 1984.

[270] Manfred Bauer, B Stech, and M Wirbel. Exclusive non-leptonic decays ofd-, d s-andb-mesons. *Zeitschrift für Physik C Particles and Fields*, 34(1):103–115, 1987.

[271] Wouter Dekens, Elizabeth E. Jenkins, Aneesh V. Manohar, and Peter Stoffer. Non-perturbative effects in $\mu \to e\gamma$. *JHEP*, 01:088, 2019.

[272] J. Wess and B. Zumino. Consequences of anomalous ward identities. *Physics Letters B*, 37(1):95–97, Nov 1971.

[273] Edward Witten. Global aspects of current algebra. *Nuclear Physics B*, 223(2):422–432, Aug 1983.

[274] Aneesh Manohar and Gregory Moore. Anomalous inequivalence of phenomenological theories. *Nuclear Physics B*, 243(1):55–64, Aug 1984.

[275] R. Kaiser and H. Leutwyler. Large $N_c$ in chiral perturbation theory. *The European Physical Journal C*, 17(4):623–649, Dec 2000.

[276] S. Gardner, Ulf-G. Meißner, and G. Valencia. Watson's theorem and electromagnetism in k decay. *Physics Letters B*, 508(1-2):44–50, May 2001.

# VITA

I was born in Shandong province of China. The following is my educational experience

- **Aug. 2015 – present:**
  University of Kentucky, Lexington, KY, United States of America
  Ph.D. candidate in High Energy Physics
  Advisor: Professor Susan Gardner

- **Sept. 2010 – July 2015:**
  Institute of High Energy Physics, CAS, Beijing, China
  Ph.D. in Theoretical Physics (2015)
  Thesis: Research on Properties of Hyperon Resonances
  Advisor: Professor Bing-Song Zou

- **Sept. 2006 – July 2010:**
  Shandong University, Weihai, Shandong, China
  B.S. in Physics (2010)
  Thesis: Study of proton-antiproton annihilation producing $\phi$ meson
  Advisor: Professor Bing-Song Zou, Associate Professor Shao-Xia Chen