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Construction and Analysis of Higher Order Superintegrable Systems

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Abstract

In this thesis, we develop novel techniques to generate superintegrable systems and compute their symmetry algebras. We develop an efficient method to calculate the compatibility equations of the potential when separation in Cartesian coordinates is assumed. This is an algorithmic approach which determines the coefficients of the integral by iterated univariate integrations. We are able to achieve this derivation without making any additional assumptions on the form of the integral such as assuming the linear compatibility equation is zero trivially. We produce an analysis of rational solutions to the compatibility equations which allows us to complete the classification of models which possess an additional fourth-order integral. We show that the so-called standard potentials arise from the existence of multiple ladder operator pairs and that the full symmetry algebra of quantum superintegrable models, depending on the parameters in the potential, generally requires addition integrals beyond the classical limit for closure. We demonstrate this in detail for the singular oscillator model with specially chosen coefficients on the reciprocal quadratic terms. For systems that separate in polar coordinates, ladder operators can still be used but are in general algebraic not polynomial. We show how this can be applied systematically to the TTW and PVZ models with some subtleties regarding the reduction of order of the integral which do not occur in the Cartesian-separable case. Finally, we introduce an algebraic framework to derive superintegrable extensions of exactly solvable models. Our method constructs from scratch, i.e. without the use of coupling-constant metamorphosis, the Darboux surface metrics and some associated superintegrable systems. We derive known potentials involving the third and sixth Painlevé transcendents via one-dimensional operator and find a new pseudo-Euclidean model with a potential that depends on the fifth Painlevé transcendent. Our results show that many of the laborious derivations of third- and fourth-order superintegrable models can be greatly simplified, and thus opens the way for analysis of superintegrability with higher transcendental potentials.

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Publications Included in this Thesis

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List of Abbreviations and Symbols

Operations

$\llbracket *, * \rrbracket$	Poisson bracket
$[*, *]$	Commutator
$\{*, *\}$	Symmetrised product
\cdot	Dot product

Mathematical Constants

π	Archimedes	$\frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \dots$
i	Cardan	$\angle \frac{1}{2}\pi$
e	Bernoulli	$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$

Sequences and Functions

\mathcal{A}_n	Noumi-Yamada transcendents
\mathcal{B}_n	Bernoulli numbers
\mathcal{H}_n	Hermite polynomials
\mathcal{L}_n	Laguerre polynomials
\mathcal{P}_n	Jacobi polynomials
\mathcal{Q}_n	Okamoto polynomials
\mathcal{U}_n	Umemura polynomials

Chapter 1

Introduction

The Kepler and isotropic oscillator models have for a long time been regarded as exceptional. In classical mechanics, they can be completely understood within a geometric framework. Indeed, they are the only systems which give rise to conic section orbits under the influence of a centripetal force [3], a result going back to Newton, the former with the attraction being at a focus and the latter at the centre. Bertrand, albeit with gaps, upgraded this result by showing that they can be characterised by having stable, periodic orbits [4]. In quantum mechanics, the energy levels of the hydrogen atom and the harmonic oscillator can be specified by a single quantum number, giving rise to degeneracy well in excess of their spherical symmetry. It was not commonly recognised at the time that this was due to parabolic and elliptical symmetry, respectively, [5, 6] and hence the degeneracy was often called accidental [7]. These systems were studied in a group-theoretic framework [8, 9]. However, the group-theoretic approach does not single out the Kepler and oscillator as unique in the classical case [10] nor is its relevance clear in the quantum case when the commutation relations are non-linear.

The seminal papers of Smorodinsky, Winternitz and collaborators [11–13] systematically organised these observations and initiated the classification project of superintegrable systems, notwithstanding some earlier noteworthy contributions [14]. These are systems which possess overabundant, not necessarily Lie, symmetry. These differ from integrable systems in that the solution to the equations of motion can be reduced from quadratures to a system of algebraic or transcendental equations. In two dimensions, with second-order constants of motion, this corresponds to separability in two or more orthogonal coordinate systems. The Smorodinsky-Winternitz systems comprise four distinct models, of which their orbits in the classical case lie on quartic curves and in the quantum case, their spectrum can be calculated exactly with the corresponding Schrödinger equation being reducible to the confluent hypergeometric equation. Kalnins, Pogosyan and Miller Jr generalised these results to complex Euclidean, spherical and hyperbolic space [15–17]. The multiseparable systems in real Euclidean 3-space were classified by Evans [18] and extended to the complexes and the sphere, substantially but not completely, by Kalnins, Kress and Miller and collaborators [19, 20]. The Smorodinsky-Winternitz systems contain the Kepler and oscillator models as important special cases, but in the general case preserve the same interesting geometric and algebraic properties [21, 22].

The commutation or Poisson bracket relations of superintegrable systems describe a polynomial algebra. The finite-dimensional representation theory of quadratic algebras can be used to describe the degeneracy levels of the system [23, 24] and the recurrence relations for the eigenfunctions of the separation equations [25–27] which are usually exactly solvable or quasi-exactly solvable.

The direct method for finding superintegrable systems involves solving a system of compatibility equations for the potential function. At the first- and second-order these are linear partial differential equations. For higher-order, there are additional non-linear equations. Thus, the solutions are divided into the standard cases, those which can be derived from the linear equations, and the exotic cases, those which are true solutions to the non-linear equations. The exotic potentials have been worked out in many cases, namely third [28–30], fourth [31] and some of fifth order [32, 33] for systems that are separable in Cartesian coordinates and third [34] and fourth order [35, 36] in polar coordinates. The non-linear equations are found to satisfy the Painlevé property [37], indeed, known solutions can be expressed in all six of the Painlevé transcendents. As we proceed to higher orders, the computations naturally become more involved. Some work has been made in the attempt to ascertain general patterns at arbitrary order [38, 39]. In classical mechanics, action-angle variable formalism allows for the additional constants of motion to be derived via quadratures [40, 41]. Systematic methods for constructing quantum superintegrable models of arbitrary order were few, limited at first to the singular anisotropic oscillator [42] and other systems with separation in Cartesian coordinates [43]. The discovery of the Tremblay-Turbiner-Winternitz model [44] presented a novel step in this direction. Several techniques were developed to prove its superintegrability [45–49]. The search for superintegrable systems has also been extended to other geometries [50–52], with vector potentials [53], with many-body interactions [54] and with reflection operators [55, 56].

The aim of this thesis is to develop techniques to derive higher order superintegrable systems and calculate their symmetry algebras. In Chapter 2, we give the relevant mathematical background which we shall make use of throughout this thesis. In Chapter 3, we explore in detail the TTW model on constant-curvature space together with a sister model. We construct their integrals via formal ladder operators and prove that the ones found are indeed those of minimal order and generate all others. In Chapter 4, we develop an original technique to calculate the compatibility equations of systems separable in Cartesian coordinates. We apply this method to finding rational solutions and thereby complete the classification of systems with an independent fourth-order integral. In Chapter 5, we use ladder operators and supersymmetry to extend these results to fifth- and sixth-order. In Chapter 6, we study a particularly simple model, the singular harmonic oscillator in one dimension. We show that its dynamic algebra is parameter-dependent in the quantum case, giving rise to an ensemble of distinct ladder operator pairs. We give an example of a superintegrable system in two dimensions, which is a special case of the Smorodinsky-Winternitz system, that was first observed by Gravel [29] to have a third-order integral independent of the generic model, whose complete symmetry algebra we compute. In Chapter 7, we study the most general three-generator polynomial algebra in order to consolidate all of the component potentials which have so far been discovered in third- and fourth-order superintegrability into a single equation. We show how to construct from the polynomial algebra

superintegrable extensions which generalise previous results in the literature and unite the Cartesian and polar coordinate techniques of the other chapters. Moreover, we add a new model to the catalogue, which may be written in terms of Painlevé III or V depending on the values of the parameters, which resides in pseudo-Euclidean space.

Chapter 2

Superintegrability in Classical and Quantum Mechanics

We give an overview of classical non-relativistic mechanics and separation of variables. We then discuss a wide class of superintegrable systems in d -dimensions and show that their formation can be reduced to elementary one-dimensional components.

2.1 Rational Mechanics

The mechanics of punctual systems is an analytic theory that models the motion of inanimate macroscopic bodies whose relative distances are much greater than the size of the individual objects of consideration. The principle developers of the theory were Newton, Euler and Lagrange. We consider only non-dissipative forces and non-relativistic motion. While the Eulerian calculus is best suited for problems particular to the field, the Hamiltonian formulation was given renewed import with quantum mechanics and we shall therefore follow it.

For an ensemble of point-masses, a configuration is an assignment of a unique position to each entity. Each point-mass may be assumed to roam free or confined by (holonomic) constraints to a surface or curve. Let d be the total number of degrees of freedom. The space of all possible configurations is an analytic manifold with local coordinates $\mathbf{q} = (q_1, q_2, \dots, q_d)$. The present state of motion of a particular configuration is indicated by the momenta $\mathbf{p} = (p_1, p_2, \dots, p_d)$. Together, they form a mobile (\mathbf{q}, \mathbf{p}) which is a point on phase space. By dynamical quantity, we mean an analytic function on phase space and of time. We shall not consider any quantities that have explicit time-dependence. The first dynamical quantity of interest is the *kinetic energy*

$$T = \frac{1}{2} \sum_{i,j=1}^d g^{ij}(\mathbf{q}) p_i p_j \quad (2.1)$$

where g^{ij} is a contravariant metric tensor on the configuration space. The second is the *potential*

energy $V = V(\mathbf{q})$ which is assumed to depend only on position. The *Hamiltonian*

$$H = T + V \quad (2.2)$$

synthesises both the geometric and dynamical information proper to the system. A mobile follows a *trajectory* which is the integral curve of the *equations of motion*

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}. \quad (2.3)$$

Two Hamiltonians give rise to equivalent dynamics if they differ by a constant shift. We define the Poisson bracket

$$[[A, B]] := \frac{\partial A}{\partial \mathbf{q}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{q}} \quad (2.4)$$

A dynamical quantity has the rate of change

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + [[A, H]]. \quad (2.5)$$

If the right-hand side is zero, then A is an *integral* of (2.3). This is the case for the Hamiltonian when it does not depend on time explicitly (which is the case throughout this thesis), and its constant value E is called the *energy*. In the components of the mobile, we have

$$[[q_i, q_j]] = 0, \quad [[p_i, p_j]] = 0, \quad [[q_i, p_j]] = \delta_{ij} \quad (2.6)$$

which are called the *canonical relations*. A *canonical transformation* is a transformation $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P})$ that preserves the canonical relations. One of the simplest forms of canonical transformations are the *coordinate transformations*

$$\mathbf{Q} = \mathbf{f}(\mathbf{q}), \quad \mathbf{P} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{q}} \right)^{-1} \mathbf{p}.$$

The kinetic energy is expressed in the new mobile as

$$T = \frac{1}{2} \sum_{i,j=1}^d G^{ij} P_i P_j, \quad G^{ij} = \sum_{k,\ell=1}^d \frac{\partial f_i}{\partial q_k} \frac{\partial f_j}{\partial q_\ell} g^{k\ell}.$$

Another important transformation is *coupling-constant metamorphosis* [57]. If a Hamiltonian is of the form $H = H_0 + \alpha U(\mathbf{q})$ and I is a time-independent integral which depends analytically on α then $H' = H_0/U$ is a new Hamiltonian with integral $I' = I(\alpha \mapsto -H')$. Time-dependent integrals are lost however.

Example 2.1. Consider the Smorodinsky-Winternitz Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + (\alpha + \alpha')(q_1^2 + q_2^2) + \frac{\beta}{q_1^2} + \frac{\gamma}{q_2^2} + \delta$$

The Hamiltonian depends on three parameters $\alpha + \alpha'$, β , γ in addition to the trivial additive constant δ . Ordinarily we would suppress this constant, but it must be carried through so that the transformation

is invertible. We take here $U = q_1^2 + q_2^2$. To simplify the resulting expression we perform a coordinate transformation $Q_1 = \frac{1}{2}(q_1^2 - q_2^2)$, $Q_2 = q_1 q_2$ so that the new Hamiltonian becomes

$$H' = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2\sqrt{Q_1^2 + Q_2^2}} \left(\frac{\beta}{\sqrt{Q_1^2 + Q_2^2} + Q_1} + \frac{\gamma}{\sqrt{Q_1^2 + Q_2^2} - Q_1} + \delta \right) + \alpha'.$$

It is seen that the (extended) harmonic oscillator is dual to the (extended) Kepler model. \square

Let A, B, C be dynamical quantities and α, β constants. The following properties are straightforward to verify:

- (i) $\llbracket A, B \rrbracket = -\llbracket B, A \rrbracket$ (Anti-symmetry);
- (ii) $\llbracket A, \alpha B + \beta C \rrbracket = \alpha \llbracket A, B \rrbracket + \beta \llbracket A, C \rrbracket$ (Bilinearity);
- (iii) $\llbracket A, \llbracket B, C \rrbracket \rrbracket + \llbracket B, \llbracket C, A \rrbracket \rrbracket + \llbracket C, \llbracket A, B \rrbracket \rrbracket = 0$ (Jacobi's identity);
- (iv) $\llbracket A, BC \rrbracket = \llbracket A, B \rrbracket C + B \llbracket A, C \rrbracket$ (Product rule);
- (v) If $\frac{dA}{dt} = \frac{dB}{dt} = 0$ then $\frac{d}{dt} \llbracket A, B \rrbracket = 0$ (Poisson's theorem).

We say that two dynamical quantities A, B are *in involution* if $\llbracket A, B \rrbracket = 0$.

Definition 2.1. A Hamiltonian is *integrable* if there exists d functionally independent integrals that are in pairwise involution.

Theorem 2.1 (Jacobi-Liouville-Donkin). *Let H be integrable with integrals I_1, I_2, \dots, I_d . If the assignment of the integrals to constant values*

$$\alpha_i = I_i(\mathbf{q}, \mathbf{p}, t), \quad i = 1, 2, \dots, d$$

be solved for the momenta

$$\mathbf{p} = \mathbf{f}(\mathbf{q}, \alpha_1, \alpha_2, \dots, \alpha_d, t)$$

then the differential for

$$\mathbf{p} \cdot d\mathbf{q} - H dt \tag{2.7}$$

will be exact.

Proof. See, for example, Whittaker [58]. \square

If S is a primitive of (2.7) then

$$\frac{\partial S}{\partial q_i} = p_i, \quad \frac{\partial S}{\partial t} = -H. \tag{2.8}$$

We can combine these conditions into a single partial differential equation

$$\frac{\partial S}{\partial t} + H \left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}} \right) = 0 \tag{2.9}$$

which is called the *Hamilton-Jacobi equation*.

Theorem 2.2 (Jacobi). *Let $S = S(\mathbf{q}, \alpha_1, \alpha_2, \dots, \alpha_d, t)$ be a solution of the Hamilton-Jacobi equation depending on d parameters $\alpha_1, \alpha_2, \dots, \alpha_d$, then the mobile is determined as a function of time by the $2d$ equations*

$$\begin{cases} p_i = \frac{\partial S}{\partial q_i}, \\ \beta_i = \frac{\partial S}{\partial \alpha_i}, \end{cases} \quad i = 1, 2, \dots, d$$

where $\beta_1, \beta_2, \dots, \beta_d$ are additional constants.

Proof. Gelfand and Fomin give two proofs of this classic result [59]. □

Thus integrability facilitates the resolution of the dynamical problem via only half as many quadratures as typically necessary for the order of the system of differential equations. The additional constants $\beta_1, \beta_2, \dots, \beta_d$ will not necessarily correspond to integrals since they will in general be *multivalued* functions of the mobile.

Definition 2.2. The *symmetry (dynamic) algebra* is the set of all time-independent (time-dependent) integrals of H together with the Poisson bracket $[[\cdot, \cdot]]$.

The advantage in restricting oneself to time-independent integrals is that they encapsulate the geometric information of the system, often in a very simple algebraic form. The time-dependent integrals typically involve transcendental functions. From the theory of ordinary differential equations [60], there can be at most $2d - 1$ generators of the symmetry algebra and $2d$ generators of the dynamic algebra. In practice, the Hamilton-Jacobi equation is solved by additive separation of variables:

$$S = \sum_{i=1}^d S_i(q_i) - Et$$

This places restrictions on the form of the metric g and the potential V in the coordinates. These conditions were found by Levi-Civita and its explicit form is given by Havas [61]. We shall be interested in the special case where the metric is orthogonal, that is, when the kinetic energy takes the form

$$T = \frac{1}{2} \sum_{i=1}^d g_i(\mathbf{q}) p_i^2$$

Theorem 2.3 (Stäckel). *The Hamilton-Jacobi equation is separable if and only if there exists an invertible matrix $\Sigma = [\sigma_{ij}(q_i)]_{i,j=1}^d$ and functions $V_i(q_i)$ such that*

$$\sum_{i=1}^d g_i \sigma_{ij} = \delta_{j1}, \quad \sum_{i=1}^d g_i V_i = V.$$

Proof. See [5]. □

In this case, the Hamilton-Jacobi equation reduces to the system of ordinary differential equations

$$\frac{1}{2} \left(\frac{dS_i}{dq_i} \right)^2 + V_i(q_i) = \sum_{j=1}^d \alpha_j \sigma_{ij}(q_i), \quad i = 1, 2, \dots, d \quad (2.10)$$

where $\alpha_1, \alpha_2, \dots, \alpha_d$ are separation constants. We may invert these linear equations to find $\alpha_i = I_i(\mathbf{q}, \mathbf{p}, t)$ as quadratic integrals in pairwise involution [62].

2.2 Superintegrability

Here we introduce superintegrability following the usual exposition which may be found in standard texts [5, 63]. The integral

$$\oint \mathbf{p} \cdot d\mathbf{q} - H dt$$

taken along a closed path in phase space is time-independent [64]. We suppose that the Hamilton-Jacobi equation additively separates and that the orbit consists of libration motions in each coordinate, i.e. the range is confined to closed and bounded intervals. Then we may construct the action variables

$$I_i = \frac{1}{2\pi} \oint p_i dq_i \quad (2.11)$$

where the path chosen is along a period of the orbit in the q_i coordinate and $p_i = \frac{dS_i}{dq_i}$ is given by (2.10). The angle variables θ_i are defined to be the canonical conjugates of I_i , i.e. $[[\theta_i, I_j]] = \delta_{ij}$. They are found by taking

$$\theta_i = \left(\frac{\partial S_i}{\partial I_i} \right)_{q_i}. \quad (2.12)$$

The angle variables are multivalued functions of the coordinates and momenta, uniquely determined up to an integer multiple of 2π .

Since the action variables are constants of the motion, Hamilton's equations expressed in these coordinates gives

$$\frac{\partial H}{\partial \theta_i} = 0, \quad i = 1, 2, \dots, d$$

so $H = H(I_1, I_2, \dots, I_d)$ can be expressed purely in terms of the action variables. A special case of a Hamiltonian expressed in action-angle variables occurs when it takes on the additive form

$$H = F(\omega_1 I_1 + \omega_2 I_2 + \dots + \omega_d I_d) \quad (2.13)$$

where F is some analytic function and $\omega_1, \omega_2, \dots, \omega_d$ are constants. In the general case, such motion is called *quasi-periodic*. As

$$\frac{d\theta_1}{d\theta_2} = \frac{\omega_1}{\omega_2},$$

it is seen that $\omega_2 \theta_1 - \omega_1 \theta_2$ has no time-dependence. While we already have similar conserved quantities β_i defined by Jacobi's theorem, the action-angle variables show what modifications are required in

order for the trajectory to be *degenerate* or partially periodic. If we have integers n_1, n_2, \dots, n_j such that

$$n_1\omega_1 + n_2\omega_2 + \dots + n_d\omega_d = 0$$

then

$$\sin(n_1\theta_1 + n_2\theta_2 + \dots + n_d\theta_d)$$

will be a new integral, functionally independent of the action variables. This motivates the following definition:

Definition 2.3. An integrable Hamiltonian that possesses additional functionally independent integrals is *superintegrable*.

Superintegrability is the site of exceptionally well-behaved motion. This being motion, in the case of confining potentials, which is closed, bounded and stable.

2.3 Quantum Mechanics

The principles of modern quantum mechanics were developed in large part by the Göttingen school [65, 66]. It encodes a mathematical description of the actions of measurement and observation on a microscopic configuration. We will make some departures from the usual prescriptions and consider the theory from the point of view of formal eigenvalue problems which shall better accommodate special singular oscillators which we treat of in chapter 5.

In quantum mechanics we do not speak of mobiles and dynamical quantities but rather of the *state* of the system and *observables* respectively. A state vector is denoted $|\psi\rangle$ and resides in a linear space \mathfrak{S} over the complex numbers called the *state space*. The state space is endowed with a correlation

$$\dagger : \mathfrak{S} \rightarrow \mathfrak{S}^\dagger, \quad |\psi\rangle \mapsto \langle\psi|$$

i.e. a semilinear map. The correlation induces a sesquilinear inner product $\langle\psi|\varphi\rangle := \langle\psi|(|\varphi\rangle)$. The Hermitian conjugate of a linear operator A on the state space is denoted A^\dagger . An observable is a Hermitian operator. Let $[A, B] := AB - BA$ and $\{A, B\} := \frac{1}{2}(AB + BA)$ denote the commutator and symmetriser of two operators A, B . The time evolution of an operator A is governed by the *Heisenberg equation*

$$i\hbar \frac{dA}{dt} = [A, H] \tag{2.14}$$

where H is the Hamiltonian of the system. The commutator satisfies the same properties (i)–(v) that Poisson brackets satisfies but note that the order of multiplication in (iv) is essential. The commutation relations between the position and the momenta are supposed to be

$$[q_j, q_k] = 0, \quad [q_j, p_k] = 0, \quad [q_j, p_k] = i\hbar\delta_{jk}, \quad i = 1, 2, \dots, d.$$

The commutation relations may be naturally generalised to any observable which is analytic in the coordinates and polynomial in the momenta [67] by taking

$$[f(\mathbf{q}), p_j] = i\hbar \frac{\partial f}{\partial q_j}.$$

Integrals, integrability and superintegrability are carried over as before, replacing Poisson brackets with the commutator. There is however no intrinsic characterisation of functional independence. In the literature, it is often supposed that some form of algebraic independence may be a suitable substitute, however this is not sufficient [68]. Independence of observables can only be determined by their action on state vectors. The matrix elements have time-dependent factors which alter the phase of the state, however these can always be suppressed for time-independent operators. This may be achieved more formally by the gauge transformation

$$A \mapsto \exp\left(-\frac{itH}{\hbar}\right) A \exp\left(\frac{itH}{\hbar}\right).$$

The Schrödinger realisation

$$p_i \mapsto g_i(\mathbf{q}) - i\hbar \frac{\partial}{\partial q_i}, \quad |\psi\rangle \mapsto \psi(\mathbf{q})$$

where g_i are select functions which preserve the canonical relations, is of great utility in deriving the states. Typically, in this realisation, one has

$$\langle \psi | \varphi \rangle = \int \bar{\psi}(\mathbf{q}) \varphi(\mathbf{q}) v(\mathbf{q}) d^d \mathbf{q}$$

where the integration runs over a suitable domain. However, we will need to consider states that are not square-integrable. More precisely, our state spaces shall correspond to the lowest weight modules of the Hamiltonian. The eigenvalue condition $H\psi = E\psi$ transforms into the *Schrödinger equation*

$$-\frac{1}{2}\hbar^2 \Delta \psi + V(\mathbf{q})\psi = E\psi. \quad (2.15)$$

A particular solution is usually looked for in the separable form

$$\psi = \psi_1(q_1)\psi_2(q_2) \cdots \psi_d(q_d)$$

from which the general solution may be obtained by linear superposition.

2.4 Factorisation Methods

Ladder operator techniques have been firmly established as a versatile tool in quantum mechanics. Schrödinger [69, 70] translated this approach into a factorisation method for solving Sturm-Liouville eigenvalue problems, whose range of applicability was comprehensively classified by Infeld and Hull [71]. An alternative route to the problem, using the concept of shape-invariance and supersymmetry [72], initially lead to the same outcome [73] but has many more applications, such as soliton theory [74] or models with reflection operators [75]. In particular, it serves as an operator formalism for the classic work of Darboux [76], Crum and others [77] on the generation of new exactly solvable Sturm-Liouville equations. Let

$$H = \frac{1}{2}p^2 + V(q) \quad (2.16)$$

be a one-dimensional Hamiltonian. Then H admits the factorisation

$$H = E + D^\dagger D$$

for a first-order operator

$$D = \frac{1}{\sqrt{2}} (p + i\hbar\varpi)$$

if and only if ϖ satisfies the Riccati equation

$$\frac{\hbar^2}{2} \left(\frac{d\varpi}{dq} + \varpi^2 \right) + E = V(q) \quad (2.17)$$

or equivalently, by taking the logarithmic substitution $\varpi = \frac{1}{\psi} \frac{d\psi}{dq}$, ψ is a solution to the Sturm-Liouville or time-independent Schrödinger equation

$$-\frac{\hbar^2}{2} \frac{d^2\psi}{dq^2} + V(q)\psi = E\psi. \quad (2.18)$$

We can construct a new Hamiltonian

$$H' = E + DD^\dagger$$

which differs from the old by

$$H' = H - \hbar^2 \frac{d\varpi}{dq}.$$

The operators D, D^\dagger intertwine the old and new Hamiltonians

$$DH = H'D, \quad HD^\dagger = D^\dagger H'$$

So that if $|\varphi\rangle$ is an eigenstate of H with energy \tilde{E} then $D|\varphi\rangle$ is an eigenstate of H' with the same energy. However, the state with energy E will have been deleted since $D|\psi\rangle = 0$.

The Darboux transformation can be iterated any number of times, generating a sequence of Hamiltonians

$$D^{(j)} H^{(j)} = H^{(j+1)} D^{(j)}.$$

The Crum formula [78] gives the transformed Hamiltonian at each iteration

$$H^{(k)} = H - \hbar^2 \frac{d^2}{dq^2} \log \text{Wr}(\psi, \psi', \psi'', \dots, \psi^{(k-1)}) \quad (2.19)$$

where $\psi, \psi', \psi'', \dots$ are solutions to Schrödinger's equation with distinct energies E, E', E'', \dots and Wr is the Wronskian determinant. If any of the energies, say $E = E' = E'' = \dots = E^{(l)}$ for $l < k$, were to coincide, we must instead choose generalised eigenfunctions

$$-\frac{\hbar^2}{2} \frac{d^2\psi^{(j)}}{dq^2} + V\psi^{(j)} = E\psi^{(j)} + \psi^{(j-1)}, \quad j = 1, 2, \dots, l$$

as our seed functions [79]. In this case, the transformation is called confluent. Thus from one Hamiltonian we generate a whole family of Hamiltonians with similar spectral properties.

The following publication has been incorporated as Chapter 3.

[1] I. Marquette, A. Parr, ‘Superintegrability and deformed oscillator realizations of quantum TTW Hamiltonians on constant-curvature manifolds and with reflections in a plane’, *J. Phys. A: Math. Theor.* 57 (13) (2024). doi : 10.1088/1751-8121/ad2e3f.

Contributor	Statement of contribution	%
Anthony Parr	writing of text	80
	proof-reading	70
	theoretical derivations	100
	numerical calculations	100
	preparation of figures	N/A
	initial concept	70
Ian Marquette	writing of text	20
	proof-reading	30
	supervision, guidance	100
	theoretical derivations	0
	numerical calculations	0
	preparation of figures	N/A
	initial concept	30

Chapter 3

Constant-Curvature TTW and PVZ Models

Throughout this chapter, the classical analogues are not being considered and we have used the Schrödinger realisation $p \mapsto -i\partial$ with $\hbar = 1$.

3.1 Introduction

Several construction procedures have been developed for finding higher order integrals with a potential specified up to a few rational or integral parameters (see, e.g., [36, 38, 42, 46, 47, 49, 53, 80–84] and references therein). For separable systems, the symmetry algebra corresponds to the dynamic algebra of a Hamiltonian of lower dimension. The algebra usually takes the form of a generalised Heisenberg algebra [82, 83] but obtaining explicit formulas is non-trivial. In the quantum case, these algebras are built up from ladder operators and the recurrence relations [47] of orthogonal polynomials and special functions from the Askey-Wilson scheme.

The expansion of integrals as a finite power series of the Hamiltonian and other known integrals so as to represent the whole as a differential operator of reduced order has proved effective [46, 49, 81]. This technique we shall designate the expansion method. The expansion method is powerful in that no knowledge of the eigenfunctions are required. Its application to the resolution of higher order integrals [46] was met with difficulties in producing explicit results, because the equations involved were of fourth degree. However, this can be simplified if separability is taken into account.

It is our purpose here to apply the expansion and ladder operator methods to the construction of the symmetry algebra of certain separable Hamiltonians. We shall, throughout, work over a two-dimensional space of constant curvature κ . Positive curvature corresponds to elliptical space, negative curvature to hyperbolic space and no curvature to flat Euclidean space. The Hamiltonian families of interest here are of radial harmonic oscillator type

$$H = \frac{1}{2} \left[-\partial_r^2 - \frac{c}{s} \partial_r + \frac{M^2}{s^2} + \left(\omega^2 - \frac{1}{4} \kappa^2 \right) \frac{s^2}{c^2} \right] \quad (3.1)$$

where: $\omega > 0$; M^2 is a second-order differential operator which depends on the angular variable θ

and a rational parameter $k \in \mathbb{Q}_{>0}$;

$$s = \sum_{i=0}^{\infty} \frac{(-1)^i \kappa^i r^{2i+1}}{(2i+1)!} = \begin{cases} \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}}, & \kappa > 0, \\ r, & \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa} r)}{\sqrt{-\kappa}}, & \kappa < 0, \end{cases}$$

and

$$c = \sum_{i=0}^{\infty} \frac{(-1)^i \kappa^i r^{2i}}{(2i)!} = \begin{cases} \cos(\sqrt{\kappa} r), & \kappa > 0, \\ 1, & \kappa = 0, \\ \cosh(\sqrt{-\kappa} r), & \kappa < 0, \end{cases}$$

are the trigonometric, flat or hyperbolic sine and cosine respectively. The expansion method, as applied to this Hamiltonian, consists in writing

$$\partial_r^n = \sum_{\ell, m} [F_{0\ell mn}(s, c) + F_{1\ell mn}(s, c)\partial_r] (M^2)^\ell H^m.$$

We may thus consolidate any differential operator in the variables r, θ into the form

$$I = I_0(r, \theta, \partial_\theta, H) + I_1(r, \theta, \partial_\theta, \rho(H))\partial_r \quad (3.2)$$

where I_0 and I_1 are meromorphic in the first two arguments and polynomial in the last two, arranged so that the H -terms are ordered to the right of the variables and operators. Here, ρ denotes multiplication from the right, i.e. $\rho(H)\partial_r = \partial_r H$. This procedure can be extended to a large class of superintegrable systems.

The chapter is organised in the following way. In section 3.2, we introduce the two classes of models which we are of focus for this chapter, the Tremblay-Turbiner-Winternitz (TTW) and Post-Vinet-Zhedanov (PVZ) models, and discuss the different forms that their symmetries will take. The main result of our research is that we determine the full symmetry algebra of these models where the ladder operator method does not automatically yield the lowest-order integrals (see, e.g., Theorem 3.3). The approach we shall take in finding their integrals I is to expand as a power series in s . In section 3.3, we find that an infinite expansion corresponds to an infinite-order differential operator. The finite expansion enforces upon the operator the constraints that it is a linear expansion of formal operators L_i satisfying

$$[M^2, L_i] = 4piL_i(pi + M) \quad (3.3)$$

where $k = p/q$ in reduced form. The ladder operators L_i decompose into a product of an angular operator J and radial operator K , which we determine by similar expansion techniques in section 3.4. In section 3.5, the generators of the symmetry algebra are presented, their commutation and closure relations are given and finite-dimensional irreducible representations are derived. The latter will give us the spectrum of the Hamiltonian, obtained through purely algebraic means.

3.2 Models, Eigenfunctions and Spectra

In this section, we discuss the TTW and PVZ models which are of focus for this chapter and give their spectral decomposition, which will be used for comparison with the representation theory of their symmetry algebras to be developed in section 3.5. We shall discuss the differences between how their symmetry algebras will be constructed.

The celebrated TTW model [44, 85] has the Pöschl-Teller operator as angular component:

$$M_{\text{TTW}}^2 = -\partial_\theta^2 + \frac{k^2}{4} \left[\frac{4\alpha^2 - 1}{\sin^2(k\theta)} + \frac{4\beta^2 - 1}{\cos^2(k\theta)} \right], \quad (3.4)$$

where $\alpha, \beta > -1$. The Hamiltonian (3.1) is separable in polar coordinates. Therefore the eigenfunctions can be determined analytically by simultaneously diagonalizing H and M^2 . The eigenfunctions of (3.4) are

$$\Theta_\ell(\theta) = [\sin(k\theta)]^{\alpha+\frac{1}{2}} [\cos(k\theta)]^{\beta+\frac{1}{2}} \mathcal{P}_\ell^{\alpha,\beta}(\cos(2k\theta)), \quad \ell = 0, 1, 2, \dots$$

where \mathcal{P} is a Jacobi polynomial. The corresponding eigenvalues are $k^2(2\ell + \alpha + \beta + 1)^2$.

This model has attracted intensive interest [45, 86–93] as a higher-order superintegrable system distinct from the anisotropic oscillator. The TTW model includes, for $k = 1, 2, 3$, the Smorodinsky-Winternitz model [11], the BC_2 system of Type V in the classification of Olshanetsky and Perelomov [94], and the three-body model introduced by Wolfes [95] respectively. The TTW systems in their full generality were introduced in 2009 by Tremblay, Turbiner and Winternitz where they verified the model was superintegrable for $k = 4$ and conjectured superintegrability for all rational k . Kalnins, Kress and Miller Jr. [47] and Quesne [45] proposed various approaches to demonstrating superintegrability. Studies of this model have generated new problem-solving techniques [46, 47] which have been applied to other models that are structurally very similar to the TTW model [96]. The ladder operator method of Kalnins *et al.* involved decomposing the integral into formal components $L_{\pm 1}$ of the form $\frac{1}{2}I_I \pm \frac{1}{2}I_{II}M$, I_I, I_{II} being differential operators. Then $I_I = L_1 + L_{-1}$, $I_{II} = (L_1 - L_{-1})/M$ are integrals of H . Even for $k = 1$, this does not yield the lowest degree operators. Indeed, there is a differential operator I_{III} and a polynomial P such that

$$I_I - pI_{II} = I_{III}(M^2 - p^2) + P(H). \quad (3.5)$$

The existence of the integral I_{III} was inferred by Kalnins *et al.* from algebraic considerations and computed an expression for $P(H)$. That M^2, H, I_{III}, H generate by multiplication the full symmetry algebra for all k is not evident from their analysis. That this is the case we demonstrate in section 3.5.3. Moreover, we derive equation (3.5) from the explicit differential operator form. We also provide an algebraic derivation of the spectrum and a generalisation of the results for curved spaces. The classical analogue [85] is well-understood and has been shown to be superintegrable on spaces with constant [91, 97] and non-constant curvature [52]. Anharmonic generalisations [98] have been postulated, but it is not known at the outset which varieties are superintegrable.

We shall also consider a model discovered by Post, Vinet and Zhedanov [55] which has angular momentum component

$$M_{\text{PVZ}}^2 = -\partial_\theta^2 + k^2 \left[\frac{\alpha^2 - 2\alpha \cos(2k\theta)R}{\sin^2(2k\theta)} + \frac{\beta^2 - 2\beta \sin 2(k\theta)}{\cos^2(2k\theta)} \right], \quad (3.6)$$

where $\alpha, \beta > -1$. Here, R is a reflection operator on functions of θ , i.e., $(Rf)(\theta) = f(-\theta)$. The PVZ model is similar to the TTW model but exhibits its own peculiar properties. In particular, M^2 is a perfect square:

$$M_{\text{PVZ}} = \left[\partial_\theta - \frac{k\beta}{\cos(2k\theta)} \right] R - \frac{k\alpha}{\sin(2k\theta)}. \quad (3.7)$$

The eigenfunctions of (3.6) are

$$\Theta_\ell(\theta) = [\sin(2k\theta)]^{\frac{\alpha}{2}} [\cos(2k\theta)]^{\frac{\beta}{2}} [1 + \sin(2k\theta)]^{\frac{1}{2}} \mathcal{P}_\ell(\sin(2k\theta); \alpha, \beta | -1), \quad \ell = 0, 1, 2, \dots$$

where \mathcal{P} is a Little Jacobi polynomial [99]. The eigenvalues of (3.7) are alternating in sign and are given by

$$(-1)^{\ell+1} k(2\ell + \alpha + \beta + 1).$$

A symmetry algebra had been obtained for this system utilizing the ladder operator method. The ladder operators satisfy generalised Heisenberg algebra relations:

$$\begin{aligned} ML_1 &= (-1)^q L_1(M + 2p), \\ ML_{-1} &= (-1)^q L_{-1}(M - 2p), \\ [L_1, L_{-1}] &= P(M, H) \end{aligned}$$

for some polynomial P . The algebra stated by Post *et al.* was generated by M, H, L_1, L_{-1} for q even and M, H, L_2, L_{-2} for q odd. However, for q odd, we have the property that

$$L_1 = \tilde{I}_I(M + p) + \xi_+, \quad L_{-1} = \tilde{I}_{II}(M - p) + \xi_-$$

where $\tilde{I}_I, \tilde{I}_{II}$ are differential-difference operators and ξ_\pm are constants. Then the full symmetry algebra is given by $M, H, \tilde{I}_I, \tilde{I}_{II}$, which we show in detail in section 3.5.2. The model for even q does not possess this property, and forms only a simple polynomial algebra satisfying the generalised Heisenberg relations.

The reduction properties associated with the TTW model and the PVZ model for q even, do not appear to necessarily be specific to these models alone but a consequence of the commutation identities associated with the ladder operators, which are common to a variety of systems.

The eigenfunctions for both systems can be expressed as $\Psi_{m,\ell}(r, \theta) = S_{m,\ell}(r)\Theta_\ell(\theta)$ where the radial part is found to be

$$S_{m,\ell}(r) = \begin{cases} s^{k(2\ell+\alpha+\beta+1)} c^{\frac{1}{2} + \frac{\omega}{|\kappa|}} \mathcal{P}_m^{k(2\ell+\alpha+\beta+1), \frac{\omega}{|\kappa|}}(c^2 - \kappa s^2), & \kappa \neq 0 \\ r^{k(2\ell+\alpha+\beta+1)} \mathcal{L}_m^{k(2\ell+\alpha+\beta+1)}(\omega r^2), & \kappa = 0 \end{cases} \quad m = 0, 1, 2, \dots$$

and \mathcal{L} is a Laguerre polynomial. The spectrum is

$$E_{m,\ell} = \omega \epsilon_{m,\ell} + \frac{1}{2} \kappa \epsilon_{m,\ell}^2, \quad \epsilon_{m,\ell} = 2m + 1 + k(2\ell + \alpha + \beta + 1). \quad (3.8)$$

The spectrum of H therefore splits into finite degeneracy levels $\{E_{m,\ell}\}_{qm+p\ell=n}$. That is, the eigenvalues are invariant under the transformation $(m, \ell) \mapsto (m \pm p, \ell \mp q)$ which is the implicit evidence for integral operators being built up out of ladder operators, that raise or lower the quantum numbers of the eigenfunctions.

3.3 General Form of the Symmetry

In this section, we demonstrate from the expansion method that every integral of the TTW and PVZ systems must be a linear combination of a product of ladder operators $J^u K^{ku}$ where $J^u = J^u(\theta, \partial_\theta, R, M)$ and $K^{ku} = K^{ku}(r, \partial_r, M, H)$ which satisfy the commutation relations:

$$[M^2, J^u] = 4kuJ^u(ku + M), \quad [K^u, H] = \frac{2u(u + M)}{s^2} K^u. \quad (3.9)$$

In the case of the TTW model, each equation is really two, with M being a formal operator.

For an operator I consolidated in the manner of (3.2), we write I_r, I_θ, \dots as its partial derivatives with respect to just the variables r, θ ignoring any of the operator terms. Then $[I, H] = 0$ generates the following system:

$$\begin{aligned} \frac{[M^2, I_0]}{s^2} &= I_{0,r,r} + \frac{c}{s} I_{0,r} + \frac{2}{s^2} \rho M^2 I_{1,r} - 4I_{1,r} H \\ &\quad + \frac{2s}{c^3} \left(\omega^2 - \frac{1}{4} \kappa^2 \right) (sc I_{1,r} + I_1) - \frac{2c}{s^3} I_1 M^2, \end{aligned} \quad (3.10a)$$

$$\frac{[M^2, I_1]}{s^2} = I_{1,r,r} - \frac{c}{s} I_{1,r} + \frac{1}{s^2} I_1 + 2I_{0,r}, \quad (3.10b)$$

For $\kappa = 0$, taking a power series expansion in r to solve the system (3.10a)–(3.10b) naturally suggests itself. For general κ , we find that the appropriate *Ansatz* is:

$$I_0 = \sum_i s^{-i} I_{0,i}(\theta, \partial_\theta, R, H), \quad I_1 = \sum_i c s^{-i+1} I_{1,i}(\theta, \partial_\theta, R, H).$$

We obtain the recurrence equations:

$$\begin{aligned} 0 &= \kappa(i+1)(i+2)I_{0,i+2} - [i^2 - M^2 + \rho(M^2)]I_{0,i} \\ &\quad + 2(i+2) \left(\omega^2 - \frac{1}{4} \kappa^2 + 2\kappa H \right) I_{1,i+4} \\ &\quad - 2(i+1)[2H + \kappa\rho(M^2)]I_{1,i+2} + 2iI_{1,i}M^2, \end{aligned} \quad (3.11a)$$

$$0 = \kappa i(i+1)I_{1,i+2} - [i^2 - M^2 + \rho(M^2)]I_{1,i} + 2iI_{0,i}. \quad (3.11b)$$

Combining (3.11a) with (3.11b) in order to eliminate the $(I_{0,i})$ -sequence, we get

$$\begin{aligned} 0 &= i(i+2)[4\omega^2 + 8\kappa H - \kappa^2(i+2)^2]I_{1,i+4} \\ &\quad + 2i(i+1)\{\kappa[(i+1)^2 + 1 - M^2 - \rho(M^2)] - 4H\}I_{1,i+2} \\ &\quad + \{4i^2\rho(M^2) - [i^2 - M^2 + \rho(M^2)]^2\}I_{1,i} \end{aligned} \quad (3.12)$$

It is not possible for this sequence to admit infinitely many non-zero values without I having infinite degree in H, M^2 . Therefore, we must truncate the series.

If $I_{1,m} \neq 0$ but $I_{1,i} = 0$ for $i < m$, then equation (3.12) says

$$[4\omega^2 + 8\kappa H - \kappa^2(m-2)^2](m-2)(m-4)I_{1,m} = 0,$$

so m is 2 or 4. Furthermore, $I_{0,i} = I_{1,i} = 0$ for odd i .

In order to cut off the upper bound on the coefficients, we require that for some sufficiently large integer μ

$$\{[\mu^2 - M^2 + \rho(M^2)]^2 - 4\mu^2\rho(M^2)\}I_{1,\mu} = 0.$$

Equation (3.12) gives

$$\prod_{i=\frac{1}{2}j}^{\frac{1}{2}\mu} \{[4i^2 - M^2 + \rho(M^2)]^2 - 16i^2\rho(M^2)\}I_{1,j} = 0$$

and, from (3.11b), we have

$$\prod_{i=1}^{\frac{1}{2}\mu} \{[4i^2 - M^2 + \rho(M^2)]^2 - 16i^2\rho(M^2)\}[M^2, I] = 0$$

with μ even. From here we wish to show I is a linear combination of operators L_i satisfying (3.3) where i is any integer with magnitude less than or equal to $\frac{1}{2}\mu$. Let Q_{i^2} satisfy

$$[M^2, [M^2, Q_{i^2}]] - 16i^2 \{M^2, Q_{i^2}\} + 16i^4 Q_{i^2} = 0. \quad (3.13)$$

for a non-zero integer i . Then

$$L_i := \frac{1}{4i}[M^2, Q_{i^2}] - iQ_{i^2} + Q_{i^2}M$$

satisfies (3.3). Conversely, any L_i satisfying this commutation relation also satisfies (3.13) in place of Q_{i^2} . Now, if

$$[M^2, [M^2, L]] - 16i^2 \{M^2, L\} + 16i^2 L = L_j$$

where $i \neq j$, then L resolves into homogeneous solutions $L_{\pm i}$ plus the particular solution

$$L_j [16(i^2 - j^2)(i^2 - j^2 - 2jM - M^2)]^{-1}.$$

This achieves the desired decomposition. It is evident that the $\{L_i\}_i$ are linearly independent as operators with coefficients formally rational in M, H . It follows that each L_i must commute with H also. Furthermore, there is a finite basis of operators L_i satisfying (3.3) and we may choose a basis $J^{i/p}(\theta, \partial_\theta, R, M)$ consisting of operators with no dependence upon r, ∂_r . Then $L_i = J^{i/p}K^i$ where $K^i = K^i(r, \partial_r, M, H)$. These must satisfy (3.9). The basis $J^{i/p}$ can be further decomposed into operators satisfying either one of the equations

$$[M, J^u] = 2kuJ^u \quad \text{or} \quad \{M, J^u\} = -4kuJ^u. \quad (3.14)$$

However, for the TTW model, this is merely an ambiguity on the sign of M acting on the eigenfunctions of M . We take its action to always be of the same sign, i.e., we formally set $MJ^u = J^u(M + 2ku)$. The operators J, K modify respectively the quantum numbers ℓ, m of the eigenfunctions of H .

3.4 Ladder Operators

In the previous section we established that the integral must be built up from linear combinations of ladder operators. We shall in this section derive the ladder operators for the values of the exponents in which they exist. This is the integers. We consider the angular ladder operators J for the PVZ and TTW model and finally, the radial ladder operator K which is the same for both. The result of this section is that the form of the integral must be

$$I = \sum_i J^{iq} K^{ip} \chi_i(M, H)$$

where we permit χ_i to be rational functions insofar as I is a differential or differential-difference operator. Having calculated the ladder operators, we determine what forms of coefficients are acceptable, that being when their fractional residue in M, H (treated as formal operators) on the whole vanishes. The TTW model has the additional constraint that I must be even in M .

We use the expansion method to simplify the angular partial derivatives. For the TTW model, we write

$$\partial_\theta^m = \sum_\ell [F_{2\ell m}(\sin(k\theta), \cos(k\theta)) + F_{3\ell m}(\sin(k\theta), \cos(k\theta))\partial_\theta](M_{\text{TTW}}^2)^\ell$$

in order to realise J^u as a first-order operator in the form

$$J^u(\theta, \partial_\theta, M) = J_0^u(\theta)(M) + J_1^u(\theta)(\rho_M)\partial_\theta. \quad (3.15)$$

For the PVZ model, we do the same by writing

$$\partial_\theta^m = \sum_\ell [F_{4\ell m}(\sin(2k\theta), \cos(2k\theta)) + F_{5\ell m}(\sin(2k\theta), \cos(2k\theta))R]M_{\text{PVZ}}^\ell$$

Hence, J^u can be written as

$$J^u(\theta, \partial_\theta, R) = J_0^u(\theta)(M) + J_1^u(\theta)(\rho_M)R. \quad (3.16)$$

We can employ an analogous reduction for K^u :

$$K^u(r, \partial_r, M) = K_0^u(r, M, H) + K_1^u(r, M, H)\partial_r. \quad (3.17)$$

In [47], the reducibility of the integrals was induced from simple cases (i.e., $k = 1$), but there was no way to verify that the results obtained were in fact of minimal degree. This is then an important problem: to develop systematically a means to obtain the lowest-order integrals. Here we will develop the method for the TTW and PVZ models on constant-curvature space based on the explicit results of sections 3.4.1–3.

For polynomials in M, H , we consider their divisibility where M, H are taken to be formal operators. The greatest common divisors of a family of polynomials $P = \{p_i(M, H)\}_i$ are the polynomials Q which divide P such that all the divisors of P are divisors of Q . For a given representative $q \in Q$, we shall write $\text{gcd } P \propto q$. The units are the non-zero real numbers. A formal differential operator of the form $\sum_i \sum_{0 \leq a, b \leq 1} f_{abi}(r, \theta) \partial_r^a \partial_\theta^b g_{abi}(M, H)$ where g_{abi} are polynomials. For the PVZ model, we substitute R for ∂_θ .

3.4.1 PVZ Angular Operators

We determine the coefficients in (3.16) by finding solutions to (3.14) which are polynomials in M . The resultant system is

$$\begin{aligned} J_0^{u'}(-\theta) &= \frac{k\alpha}{\sin(2k\theta)} [J_1^u(-\theta) - J_1^u(\theta)] \\ &\quad - J_1^u(-\theta)M - vJ_1^u(\theta)(M + 2ku); \end{aligned} \quad (3.18a)$$

$$\begin{aligned} J_1^{u'}(-\theta) &= \frac{k\alpha}{\sin(2k\theta)} [J_0^u(-\theta) - J_0^u(\theta)] - \frac{2k\beta}{\cos(2k\theta)} J_1^u(-\theta) \\ &\quad + J_0^u(-\theta)M - vJ_0^u(\theta)(M + 2ku), \end{aligned} \quad (3.18b)$$

where $v = \pm 1$ is to be determined. To solve this differential-difference system, we develop the coefficients into a sinusoidal power series with a cosine gauge on J_1^u :

$$J_0^u(\theta)(M) = \sum_{i=V}^U [\sin(2k\theta)]^i J_{0,i}^u(M), \quad J_1^u(\theta)(M) = \cos(2k\theta) \sum_{i=\tilde{V}}^{\tilde{U}} [\sin(2k\theta)]^i J_{1,i}^u(M).$$

Substituting these series into (3.18a)–(3.18b), we get the recurrence relations

$$\begin{aligned} 0 &= \{[(-1)^i - v]M/k - 2uv\} J_{0,i}^u - [(-1)^i + 1]\alpha J_{0,i+1}^u - 2(-1)^i(i+1)J_{1,i+1}^u \\ &\quad - 2(-1)^i\beta J_{1,i}^u + 2(-1)^i i J_{1,i-1}^u; \end{aligned} \quad (3.19a)$$

$$0 = 2(-1)^i(i+1)J_{0,i+1}^u + \{[(-1)^i + v]M/k + 2uv\} J_{1,i}^u + [(-1)^i + 1]\alpha J_{1,i+1}^u. \quad (3.19b)$$

For finite-order in M , the system (3.19a)–(3.19b) requires $J_{0,i}^u = J_{1,i}^u = 0$ for all but finite i . The appropriate limits are $U = |u|$, $\tilde{U} = |u| - 1$, $V = \tilde{V} = 0$. In particular,

$$\begin{aligned} 0 &= \{[(-1)^u - v]M/k - 2uv\} J_{0,|u|}^u + 2(-1)^u |u| J_{1,|u|-1}^u, \\ 0 &= 2(-1)^u |u| J_{0,|u|}^u + \{[(-1)^u - v]M/k - 2uv\} J_{1,|u|-1}^u, \end{aligned}$$

which, for $[J_{0,|u|}^u(M)]^2 + [J_{1,|u|-1}^u(M)]^2 \neq 0$, implies

$$\{[(-1)^u - v]M/k - 2uv\}^2 = 4u^2$$

so $v = (-1)^u$. Thus, $\{M, J^u\} = -4kuJ^u$ when u is odd and $[M, J^u] = 2kuJ^u$ when u is even. Up to proportionality the solution is unique, so we designate our principle solutions to be:

$$J^{\pm 1} = [\sin(2k\theta) \pm \cos(2k\theta)R](M \pm k) + k(\alpha \pm \beta) \quad (3.20)$$

and define the rest by $J^{\pm 2j} = (J^{\mp 1} J^{\pm 1})^j$ and $J^{\pm 2j \pm 1} = J^{\pm 1} J^{\pm 2j} = J^{\mp 2j} J^{\pm 1}$. In particular,

$$\begin{aligned} J^{\pm 2} &= [\cos(4k\theta) \mp \sin(4k\theta)R](M \pm k)(M \pm 3k) \\ &\quad \mp k \sin(2k\theta)[k\alpha + \beta(M \pm 2k)] \pm k \cos(2k\theta)R[k\beta + \alpha(M \pm 2k)] \\ &\quad + k^2(\alpha - \beta)(\alpha + \beta); \end{aligned} \quad (3.21)$$

Note that in (3.20) but not in (3.21), all the coefficients except the constant term are divisible by a linear factor in M . This, in fact, occurs for every odd u .

Lemma 3.1. For all odd integers u , $J^u(\theta)(-ku) = J_{0,0}^u(-ku)$.

Proof. This is clearly true for $|u| = 1$ and we assume the lemma holds for $|u| = 2j - 1$, where j is a positive integer. Using $J^{\pm 2j \pm 1} = J^{\pm 1} J^{\mp 2j \pm 1} J^{\pm 1}$, we have

$$J^{\pm 2j \pm 1}(\theta)(\mp(2j + 1)k) = J_{0,0}^{\mp 2j \pm 1}(\pm(2j - 1)k)\Phi^{\pm 1}(\mp(2j + 1)k) \quad (3.22)$$

which is a constant. \square

The combinations $(J^u)^2$ and $J^{-u}J^u$ for u odd and even respectively, commute with M . We designate them both by $\Phi^u(M)$, and is appropriately called the structure function. By induction, we compute:

$$\Phi^{\pm j}(M) = \prod_{i=0}^{\lfloor j/2 \rfloor - 1} [k^2(\alpha \pm \beta)^2 - (M \pm k \pm 4ki)^2] \prod_{i=0}^{\lfloor j/2 \rfloor - 1} [k^2(\alpha \mp \beta)^2 - (M \pm 3k \pm 4ki)^2]. \quad (3.23)$$

For u even, we have $\Phi^{-u}(M) = \Phi^u(M - 2ku)$ and $\Phi^u(M) = \Phi^j(-M - 2ku)$ for u odd. With knowledge of the structure function, we may now use equation (3.22) to compute $J_{0,0}^{\pm(2j+1)}(\mp k(2j+1))$. The result of the computation, consolidated into a single formula, is:

$$J_{0,0}^u(-ku) = k^{|u|} \left[\alpha + (-1)^{\frac{u-1}{2}} \beta \right] \prod_{i=1}^{\frac{|u|-1}{2}} \left\{ \left[\alpha + (-1)^{\frac{u-1}{2}-i} \beta \right]^2 - 4i^2 \right\}$$

where u is any odd integer. By the lemma, it follows that

$$[J^u - J_{0,0}^u(-ku)](M + ku)^{-1} \quad (3.24)$$

is polynomial in M . We now need to prove that there are no redundant factors in J^u generally. First, we compute some explicit terms in J^u so as to illustrate the phenomenon which we shall exploit. For a concise expression, we use

$$\exp(\theta R) = \cos(\theta) + \sin(\theta)R.$$

Then for $j > 1$:

$$\begin{aligned} J^{\pm 2j \mp 1} &= \pm \text{Re} e^{\mp 2(2j-1)k\theta R} \prod_{i=1}^{2j-1} [M \pm (2i-1)k] \\ &\quad + k(\alpha \pm \beta) e^{\pm 4(j-1)k\theta R} \prod_{i=2}^{2j-1} [M \pm (2i-1)k] \\ &\quad + k(\alpha \pm \beta) e^{\mp 4(j-1)k\theta R} \prod_{i=1}^{2(j-1)} [M \pm (2i-1)k] \\ &\quad \pm k^2(\alpha \pm \beta)^2 \text{Re} e^{\pm 2(2j-3)k\theta R} \prod_{i=2}^{2(j-1)} [M \pm (2i-1)k] \\ &\quad \pm 2(j-1)k^2(\alpha^2 - \beta^2) \text{Re} e^{\mp 2(2j-3)k\theta R} \prod_{i=2}^{2(j-1)} [M \pm (2i-1)k] + \dots \end{aligned} \quad (3.25a)$$

$$\begin{aligned}
J^{\pm 2j} &= e^{\mp 4jk\theta R} \prod_{i=1}^{2j} [M \pm (2i-1)k] \\
&\pm k(\alpha \pm \beta) R e^{\pm 2(2j-1)k\theta R} \prod_{i=2}^{2j} [M \pm (2i-1)k] \\
&\pm k(\alpha \mp \beta) R e^{\mp 2(2j-1)k\theta R} \prod_{i=1}^{2j-1} [M \pm (2i-1)k] \\
&+ k^2(\alpha^2 - \beta^2) e^{\pm 4(j-1)k\theta R} \prod_{i=2}^{2j-1} [M \pm (2i-1)k] \\
&+ k^2(2j-1)(\alpha^2 - \beta^2) e^{\mp 4(j-1)k\theta R} \prod_{i=2}^{2j-1} [M \pm (2i-1)k] + \dots
\end{aligned} \tag{3.25b}$$

where the ellipses indicates trigonometric functions with lower frequency. We observe in the above that the first few terms in (3.25a) and (3.25b) consist of products of $M + (2i-1)k$ where the number of factors is decreasing along with the frequency of the $e^{k\ell\theta R}$ terms which they multiply. In particular, the common divisor of the second and third terms is the same as that of the fourth and fifth terms. This is why J^u for u even cannot be reduced after removing the constant term, but J^u for u odd can be.

Theorem 3.1. *Suppose $\alpha + \beta, \alpha - \beta$ are not even integers. Then for all odd integer u ,*

$$\gcd\{J_{0,i}^u, J_{1,i}^u\}_{i \geq 0} \propto 1$$

and for all non-zero even integer u ,

$$\gcd\{J_{0,i+1}^u, J_{1,i}^u\}_{i \geq 0} \propto 1$$

Proof. From (3.25a) and (3.25b),

$$\gcd\{J_{0,|u|}^u(M), J_{0,|u|-1}^u(M)\} \propto \prod_{i=1}^{|u|} [M + \operatorname{sgn}(u)(2i-1)k]$$

so we need to show that there are non-zero terms in J^u when M is evaluated at each of the roots. By writing

$$J^u = J^{-u+\operatorname{sgn}(u)(2j+1)} J^{\operatorname{sgn}(u)(2j+1)}, \quad 0 \leq j < |u|,$$

we have

$$J_{0,|u|-2j-1}^u(-\operatorname{sgn}(u)(2j+1)k) \neq 0$$

since $J_{0,0}^{\operatorname{sgn}(u)(2j+1)}(-\operatorname{sgn}(u)(2j+1)k) \neq 0$ by the hypothesis. Similarly, by writing

$$J^{u-\operatorname{sgn}(u)(2j+1)} J^{2u-\operatorname{sgn}(u)(2j+1)}, \quad 0 \leq j < |u|$$

we get

$$J_{0,|u|-2j-1}^u(-2uk + \operatorname{sgn}(u)(2j+1)k) \neq 0.$$

The result follows. \square

Theorem 3.1 will be combined with similar results for the K operators to determine that only the factor $M + ku$ will divide the non-constant terms of $J^u K^{ku}$ for u odd but this is indivisible for u even.

3.4.2 TTW Angular Operators

Here we will solve equation (3.9) for M^2 given by equation (3.4). Substituting (3.15) into (3.9) produces the system

$$\begin{aligned} 0 &= J_0^{u''} + u(u + \rho_M)J_0^u - 2J_1^{u'}M^2 + \frac{k^2(4\alpha^2 - 1)}{2\sin^2(k\theta)}[J_1^{u'} - k\cot(k\theta)J_1^u] \\ &\quad + \frac{k^2(4\beta^2 - 1)}{2\cos^2(k\theta)}[J_1^{u'} + k\tan(k\theta)J_1^u], \\ 0 &= J_1^{u''} + u(u + \rho_M)J_1^u + 2J_0^{u'}. \end{aligned}$$

We develop the coefficients as a cosinusoidal power series

$$J_0^u(\theta)(M) = \sum_{i=V}^U [\cos(2k\theta)]^i J_{0,i}^u(M), \quad J_1^u = \sin(2k\theta) \sum_{i=\tilde{V}}^{\tilde{U}} [\cos(2k\theta)]^i J_{1,i}^u(M) \quad (3.26)$$

to obtain the recurrence relations:

$$\begin{aligned} 0 &= k(i+1)(i+2)J_{0,i+2}^u + [u(ku + M) - ki^2]J_{0,i}^u \\ &\quad + (i+1)[M^2 - k^2(2\alpha^2 + 2\beta^2 - 1)]J_{1,i+1}^u \\ &\quad + k^2(\beta^2 - \alpha^2)(2i+1)J_{1,i}^u - M^2iJ_{1,i-1}^u \end{aligned} \quad (3.27a)$$

$$\begin{aligned} 0 &= k(i+1)(i+2)J_{1,i+2}^u + [u(ku + M) - k(i+1)^2]J_{1,i}^u \\ &\quad - (i+1)J_{0,i+1}^u. \end{aligned} \quad (3.27b)$$

Again, we require $U, \tilde{U}, V, \tilde{V}$ to be finite. The necessary limits are $U = |u|, \tilde{U} = |u| - 1, V = \tilde{V} = 0$.

We take as our principles

$$J^{\pm 1} = \sin(2k\theta)\partial_\theta(k \pm M) \pm \cos(2k\theta)M(k \pm M) + k^2(\alpha^2 - \beta^2)$$

and set $J^{\pm j} = (J^{\pm 1})^j$. From $J^{-1} = J[1; -M]$ we get $J^{-u} = J[u; -M]$. In particular, we obtain a differential operator with no formal part if we consider symmetric combinations:

$$J^u f(M) + J^{-u} f(-M) \quad (3.28)$$

where f is polynomial. We will need to determine, as with the PVZ model, to what extent f may be rational. Indeed, setting $f(M) = M^{-1}$ in (3.28) leaves no residue.

For $j > 1$, we have:

$$\begin{aligned} J^j &= [\sin(2jk\theta)\partial_\theta + \cos(2jk\theta)M] \prod_{i=1}^{2j-1} (M + ki) \\ &\quad + 2k^2(\alpha^2 - \beta^2)[(j-1)\sin[2(j-1)k\theta]\partial_\theta \\ &\quad + \cos[2(j-1)k\theta](jM + 2jk - k)] \prod_{i=2}^{2j-2} (M + ki) + \dots \end{aligned} \quad (3.29)$$

where the ellipses denote trigonometric terms of lower frequency. The structure function $\Phi^u(M)$ is defined as $J^{-u}J^u$. For positive values, we determine:

$$\Phi^j(M) = \prod_{i=0}^{j-1} [(M + 2ki + k)^2 - k^2(\alpha + \beta)^2][(M + 2ki + k)^2 - k^2(\alpha - \beta)^2].$$

Observe $\Phi^{-j}(M) = \Phi^j(-M) = \Phi^j(M - 2jk)$.

The divisibility property which held for odd integers in section 3.4.1 this time holds for all integers.

Lemma 3.2. *For all non-zero integer u , $J^u(\theta)(-ku) = J_{0,0}^u(-ku)$.*

Proof. We know that $J_{0,|u|}^u(-ku) = J_{1,|u|-1}^u(-ku) = 0$. Now, suppose that $J_{0,j}^u(-ku) = J_{1,j-1}^u(-ku) = 0$ for all $j > i$ where $0 < i < |u|$. Equations (3.27b) and (3.27a) simplify to

$$\begin{aligned} 0 &= -ki^2 J_{0,i}^u(-ku) - k^2 u^2 i J_{0,i-1}^u(-ku); \\ 0 &= -ki^2 J_{1,i-1}^u(-ku) - i J_{0,i}^u(-ku). \end{aligned}$$

This system is non-singular so $J_{0,i}^u(-ku) = J_{1,i-1}^u(-ku) = 0$ and therefore the result holds by induction. \square

Consequently,

$$[J^{\pm u} - J_{0,0}^{\pm u}(-ku)](M \pm ku)^{-1}$$

is of the form $\pm D_0 + D_1 M$ where D_0, D_1 are differential operators. We wish to find an explicit formula for $J_{0,0}^u(-ku)$. To this end, we express the structure function in terms of the coefficients:

$$\begin{aligned} \Phi^j(M) &= J_{0,0}^j(-M - 2jk)J_{0,0}^j(M) - 2kJ_{1,0}^j(-M - 2jk)J_{0,1}^j(M) \\ &\quad + [k^2(2\alpha^2 + 2\beta^2 - 1) - M^2]J_{1,0}^j(-M - 2jk)J_{1,0}^j(M). \end{aligned}$$

By Lemma 3.2

$$\Phi^j(-jk) = [J_{0,0}^j(-jk)]^2 \tag{3.30}$$

This gives $J_{0,0}^j(-jk)$ up to a sign. By $J^j = J^1 J^{j-1}$, we have:

$$\begin{aligned} J_{0,0}^j(M) &= J_{0,0}^1(M + 2k(j-1))J_{0,0}^{j-1}(M) - 2kJ_{1,0}^1(M + 2k(j-1))J_{0,1}^{j-1}(M) \\ &\quad + [k^2(2\alpha^2 + 2\beta^2 - 1) - M^2]J_{1,0}^1(M + 2k(j-1))J_{1,0}^{j-1}(M), \\ J_{0,1}^j(M) &= J_{0,1}^1(M + 2k(j-1))J_{0,0}^{j-1}(M) + J_{0,0}^1(M + 2k(j-1))J_{0,1}^{j-1}(M) \\ &\quad + 2k^2(\alpha^2 - \beta^2)J_{1,0}^1(M + 2k(j-1))J_{1,0}^{j-1}(M) \\ &\quad + [k^2(2\alpha^2 + 2\beta^2 - 1) - M^2]J_{1,0}^1(M + 2k(j-1))J_{1,1}^{j-1}(M), \\ J_{1,0}^j(M) &= J_{1,0}^1(M + 2k(j-1))J_{0,0}^{j-1}(M) + J_{0,0}^1(M + 2k(j-1))J_{1,0}^{j-1}(M) \\ &\quad - 2kJ_{1,0}^1(M + 2k(j-1))J_{1,1}^{j-1}(M), \\ J_{1,1}^j(M) &= J_{0,0}^1(M + 2k(j-1))J_{1,1}^{j-1}(M) + 2kJ_{1,0}^1(M + 2k(j-1))J_{1,0}^{j-1}(M) \\ &\quad + J_{0,1}^1(M + 2k(j-1))J_{1,0}^{j-1}(M) + J_{1,0}^1(M + 2k(j-1))J_{0,1}^{j-1}(M). \end{aligned}$$

This indicates that in powers of α :

$$\begin{aligned} J_{0,0}^j &= k^{2j}\alpha^{2j} + \mathcal{O}(\alpha^{2j-2}); & J_{0,1}^j &= \mathcal{O}(\alpha^{2j-2}); \\ J_{1,0}^j &= \mathcal{O}(\alpha^{2j-2}); & J_{1,1}^j &= \mathcal{O}(\alpha^{2j-2}); \end{aligned}$$

for all $j \geq 1$. This enables us to choose the correct square-root in (3.30). At last, we calculate

$$J_{0,0}^j(-jk) = k^{2j}(\alpha^2 - \beta^2)^{\frac{1-(-1)^j}{2}} \prod_{i=0}^{\lfloor j/2 \rfloor - 1} \{[(2i-j+1)^2 - \alpha^2 - \beta^2]^2 - 4\alpha^2\beta^2\}. \quad (3.31)$$

The result below shows that the coefficients, except in extraordinary cases, have no common factor.

Theorem 3.2. *Suppose that $\alpha + \beta, \alpha - \beta$ are not integers. Then $\gcd\{J_{0,i}^u, J_{1,i}^u\}_{i \geq 0} \propto 1$.*

Proof. We shall assume u is positive without loss of generality. From (3.29), we have

$$\gcd\{J_{0,2u}^u, J_{1,2u-1}^u\} \propto \prod_{i=1}^{2u-1} (M + ki).$$

By $J^u = J^{u-j}J^j$ we get

$$J_{0,2u-2j}^u(-jk) \neq 0$$

for $1 \leq j \leq u$. Now, $J^u(-k(j+u)) = J^{u-j}(k(j-u))J^j(-k(j+u))$ so

$$J_{0,2j}^u(-k(j+u)) \neq 0.$$

This shows that the coefficients cannot all be equal to zero simultaneously. □

Corollary 3.1. *Let u, v be integers with $|u| \geq |v|$. Then the operator*

$$[J^u - J_{0,0}^{|u|-|v|}(k(|v| - |u|))J^v](M + u + v)^{-1}$$

is polynomial in M .

3.4.3 Radial Operators

For the determination of the radial operators K^u , we resume the procedure of section 3.3, namely by expanding as a power series in s :

$$K_0^u(r, M, H) = \sum_{i=0}^{|u|} s^{-2i} K_{0,i}^u(M, H), \quad K_1^u(r, M, H) = \sum_{i=0}^{|u|-1} cs^{-1-2i} K_{1,i}^u(M, H).$$

Equation (3.9) translates to the system

$$\begin{aligned} 0 &= \kappa(i + \frac{1}{2})(i + 1)K_{0,i+1}^u - [i^2 - u(u + M)]K_{0,i}^u \\ &\quad + (i + 1)(\omega^2 - \frac{1}{4}\kappa^2 + 2\kappa H)K_{1,i+1}^u \\ &\quad - (i + \frac{1}{2})(\kappa M^2 + 2H)K_{1,i}^u + iM^2K_{1,i-1}^u, \end{aligned} \quad (3.32a)$$

$$0 = \kappa i(i + \frac{1}{2})K_{1,i}^u - [i^2 - u(u + M)]K_{1,i-1}^u + iK_{0,i}^u. \quad (3.32b)$$

We take as our principles

$$K^{\pm 1} = \frac{(1 \pm M)c}{s} \partial_r + H + \frac{1}{2} \kappa M(M \pm 1) - \frac{M(M \pm 1)}{s^2}.$$

Note that $K^{-1}(r, \partial_r, -M) = K^1(r, \partial_r, -M)$. The effect of curvature is manifested in the $\frac{1}{2} \kappa M(M \pm 1)$ term, but otherwise the form of the operators is the same as that on flat space [46]. Unlike in the computation of the eigenfunctions in section 3.2, there is no need to divide into curved and non-curved space. This will serve only to increase the order of the symmetry algebra, but not its overall structure. Indeed, the calculations to compute the integrals remains the same since the K -operators only depend explicitly on integer powers of κ, c, s, M, H , whereas the Jacobi polynomials degenerate to the Laguerre polynomials only by a limit process. We set

$$K^{\pm j}(r, \partial_r, M) = K^{\pm 1}(r, \partial_r, M \pm 2(j-1))K^{\pm(j-1)}(r, \partial_r, M)$$

which gives the identity $K^{-u} = K^u(r, \partial_r, -M)$. Generally, we have

$$\begin{aligned} K^j &= \frac{(-2)^{j-1}}{s^{2j-1}} \left(\frac{c}{M} \partial_r - \frac{1}{s} \right) \prod_{i=0}^{2j-1} (M+i) \\ &+ (2j-1) \frac{(-2)^{j-2}}{s^{2j-3}} \left(\frac{c}{M} \partial_r - \frac{1}{s} \right) [H + \frac{1}{2} \kappa M(M+2j-1)] \prod_{i=1}^{2j-2} (M+i) \\ &- \frac{(-2)^{j-2}}{s^{2j-3}} \left(\frac{c}{M} \partial_r + \frac{1}{s} \right) [H + \frac{1}{2} \kappa M(M+1)] \prod_{i=2}^{2j-1} (M+i) + \mathcal{O} \left(\frac{1}{s^{2j-4}} \right) \end{aligned} \quad (3.33)$$

for positive integer j . The operators $K^{-u}(r, \partial_r, M+2u)K^u(r, \partial_r, M)$ commute with M, H and are thus polynomials in those operators. We designate this operator by the structure function $\Psi^u(M, H)$. We have the formula

$$\Psi^j(M, H) = \prod_{i=0}^{j-1} \left\{ [H - \frac{1}{2} \kappa(M+2i+1)]^2 - \omega^2(M+2i+1)^2 \right\}. \quad (3.34)$$

Observe that $\Psi^{-j}(M, H) = \Psi^j(-M, H) = \Psi^j(M-2j, H)$. The base terms have the recurrence

$$\begin{aligned} K_{0,0}^j(M, H) &= K_{0,0}^1(M+2j-2, H)K_{0,0}^{j-1}(M, H) \\ &+ (\omega^2 - \frac{1}{4} \kappa^2 + 2\kappa H)K_{1,0}^1(M+2j-2, H)K_{1,0}^{j-1}(M, H); \\ K_{1,0}^j(M, H) &= K_{1,0}^1(M+2j-2, H)K_{0,0}^{j-1}(M, H) + [\kappa K_{1,0}^1(M+2j-2, H) \\ &+ K_{0,0}^1(M+2j-2, H)]K_{1,0}^{j-1}(M, H) \end{aligned}$$

and a simple closed-form expression is readily obtainable:

$$\begin{aligned} K_{1,0}^j(M, H) &= \frac{1}{2\sqrt{\omega^2 + 2\kappa H}} \\ &\times \left\{ \prod_{i=0}^{j-1} [H + \frac{1}{2} \kappa(M+2i+1)^2 + \sqrt{\omega^2 + 2\kappa H} (M+2i+1)] \right. \\ &\left. - \prod_{i=0}^{j-1} [H + \frac{1}{2} \kappa(M+2i+1)^2 - \sqrt{\omega^2 + 2\kappa H} (M+2i+1)] \right\}; \end{aligned} \quad (3.35)$$

$$\begin{aligned}
K_{0,0}^j(M, H) &= -\frac{1}{2}\kappa K_{1,0}^j(M, H) \\
&+ \frac{1}{2} \prod_{i=0}^{j-1} [H + \frac{1}{2}\kappa(M + 2i + 1)^2 + \sqrt{\omega^2 + 2\kappa H} (M + 2i + 1)] \\
&+ \frac{1}{2} \prod_{i=0}^{j-1} [H + \frac{1}{2}\kappa(M + 2i + 1)^2 - \sqrt{\omega^2 + 2\kappa H} (M + 2i + 1)].
\end{aligned} \tag{3.36}$$

We see that $K_{1,0}^j(-j, H) = 0$ and the latter becomes

$$K_{0,0}^j(-j, H) = H^{\frac{1-(-1)^j}{2}} \prod_{i=0}^{\lfloor j/2 \rfloor - 1} \left\{ \left[H - \frac{1}{2}\kappa(2i - j + 1)^2 \right]^2 - \omega^2(2i - j + 1)^2 \right\}. \tag{3.37}$$

Proposition 3.1. *For all non-zero integers u , $K^u(-u, H) = K_{0,0}^u(-u, H)$.*

Proof. By equation (3.33),

$$K_{0,|u|}^u(-u, H) = K_{1,|u|-1}^u(-u, H) = 0.$$

Now, suppose that for some $0 < i < |u|$, $K_{0,j}^u(-u, H) = K_{1,j-1}^u(-u, H) = 0$ for all $j > i$. Then (3.32a) and (3.32b) read

$$\begin{aligned}
0 &= iK_{0,i}^u(-u, H) - i^2K_{1,i-1}^u(-u, H); \\
&= iu^2K_{1,i-1}^u(-u, H) - i^2K_{0,i}^u(-u, H).
\end{aligned}$$

This equation is non-singular, so $K_{0,i}^u(-u, H) = K_{1,i-1}^u(-u, H) = 0$. This proves the result. \square

Lemma 3.3. *Let u be a non-zero integer, then*

$$L_j := \gcd\{K_{0,i}^u(M, H), K_{1,i-1}^u(M, H)\}_{i \geq |u|-j+1} \propto \prod_{i=j}^{2|u|-j} [M + \operatorname{sgn}(u)i], \quad j = 1, \dots, |u|.$$

Proof. Proposition 3.1 tells us that $M + u$ divides $L_{|u|}$. We suppose, without loss of generality, that u is positive and $j < u$. We write

$$K^u(r, \partial_r, M) = K^{u-i}(r, \partial_r, M + 2i)K^i(r, \partial_r, M)$$

to obtain

$$K^u(r, \partial_r, -j) = K^{u-j}(r, \partial_r, j)K_{0,0}^j(-j, H). \tag{3.38}$$

It follows that $M + j \mid L_j$ and $M + j \nmid L_{j+1}$. Similarly, by writing

$$K^u = K^j(r, \partial_r, M + 2u - 2j)K^{u-j}$$

it is apparent that

$$K^u(r, \partial_r, j - 2u) = K_{0,0}^j(-j, H)K^{u-j}(r, \partial_r, j - 2u) \tag{3.39}$$

so $M + 2u - j \mid L_j$ and $M + 2u - j \nmid L_{j+1}$. We know that $M + u \mid L_u$. Thus,

$$\prod_{i=j}^{2u-j} (M + i)$$

divides L_j for all $j = 1, \dots, |u|$. By (3.33), we know that equality holds for $j = 1$, and so it must hold for all j . \square

Corollary 3.2. *Let u, v be integers with $|u| \geq |v|$. Then the operator*

$$(K^u - K_{0,0}^{|u|-|v|}(|v| - |u|, H)K^v)(M + u + v)^{-1}$$

is polynomial in M .

We shall use these calculations to begin deriving the linear combinations which are admissible for an integral of H . Since the coefficients of the J -operators have no common factor, it is only necessary to first consider K -operators by themselves and then extend to their products.

Theorem 3.3. *Let u be a positive integer greater than one and let $a, b_{\pm}, c_{\pm}, \dots, z_{\pm}$ be a series of $2j + 1$ rational functions of M, H such that*

$$Q = a + K^u b_+ + K^{-u} b_- + K^{2u} c_+ + K^{-2u} c_- + \dots + K^{ju} z_+ + K^{-ju} z_-$$

is an operator polynomial in M, ∂_r . Then $z_{\pm} \prod_{i=0}^{2j-1} (M \pm iu)$ is a polynomial function of M, H .

Proof. By denominator, we refer to the minimal polynomial ζ_{\pm} in M such that $z_{\pm} \zeta_{\pm}$ is also polynomial. Up to leading order, we have

$$Q = (-2)^{ju-1} \left[\left(\frac{M^{-1}c}{s^{2ju-1}} \partial_r - \frac{1}{s^{2ju}} \right) \prod_{i=0}^{2ju-1} (M + i) z_+ - \left(\frac{M^{-1}c}{s^{2ju-1}} \partial_r + \frac{1}{s^{2ju}} \right) \prod_{i=0}^{2ju-1} (M - i) z_- \right] + \mathcal{O} \left(\frac{1}{s^{2ju-2}} \right)$$

so

$$z_+ \prod_{i=1}^{2ju-1} (M + i) - z_- \prod_{i=1}^{2ju-1} (M - i)$$

and

$$z_+ \prod_{i=0}^{2ju-1} (M + i) + z_- \prod_{i=0}^{2ju-1} (M - i)$$

must be polynomials in M . The denominators of z_{\pm} therefore must divide $\prod_{i=0}^{2ju-1} (M \pm i)$ respectively. If $j = 1$ then we are done, otherwise, we shall proceed inductively on $j > 1$. The common factor here between the two denominators is M , so we deduce the existence of polynomial operators

$$Q' = a' + K^u b'_+ + K^{-u} b'_- + \dots + K^{(j-1)u} y'_+ + K^{-(j-1)u} y'_- + K^{ju} z'_+$$

and

$$Q'' = a'' + K^u b_+'' + K^{-u} b_-'' + \dots + K^{(j-1)u} y_+'' + K^{-(j-1)u} y_-'' + K^{-ju} z_-''$$

such that $Q = (Q' + Q'')M^{-1}$. That the leading order terms of Q' may be polynomial in M and ∂_r , the denominator of z_+' must divide $\prod_{i=u}^{(2j-1)u} (M + i)$, by Lemma 3.3.

Let $Q''' = Q'(M + u)[M + (2j - 1)u]$ and denote by $(\cdot)'''$ all the coefficients therein. This ensures

$$K_{0,2(j-1)u}^{ju} z_+'''$$

and

$$K_{1,2(j-1)u-1}^{ju} z_+'''$$

are polynomials. Then the denominators of y_{\pm}''' must divide $\prod_{i=0}^{2(j-1)u-1} (M \pm i)$. In particular, the denominators of y_+''' and z_+''' share no common factor. We may thus resolve Q''' into a sum of two polynomial operators

$$Q^{iv} = a^{iv} + K^u b_+^{iv} + K^{-u} b_-^{iv} + \dots + K^{(j-1)u} y_+^{iv} + K^{ju} z_+^{iv}$$

and

$$Q^v = a^v + K^u b_+^v + K^{-u} b_-^v + \dots + K^{(j-1)u} y_+^v + K^{-(j-1)u} y_-^v.$$

Let Q^{vi} be the result of substituting $M - 2u$ for M in Q^{iv} and denote by $(\cdot)^{vi}$ all the coefficients therein. Then

$$\begin{aligned} Q^{vi} K^{-u} &= b_+^{vi} \Psi^{-u}(M, H) + K^u c_+^{vi} \Psi^{-u}(M, H) + K^{-u} a^{vi} \\ &\quad + \dots + K^{(j-1)u} z_+^{vi} \Psi^{-u}(M, H) + K^{-(j-1)u} x_-^{vi} \end{aligned}$$

is a polynomial operator. The denominator of $z_+^{vi} \Psi^{-u}(M, H)$ thus divides

$$\prod_{i=0}^{2j-3} (M + iu).$$

There are no common factors between this product and $\Phi^{-u}(M, H)$. So the denominator of z_+^{iv} divides

$$\prod_{i=2}^{2j-1} (M + iu).$$

But the denominator of $z_+''' = z_+^{iv}$, by definition of Q''' , cannot contain the factor $M + (2j - 1)u$. As $z_+ = z_+''' M^{-1} (M + u)^{-1} [M + (2j - 1)u]^{-1}$, the result is proven. \square

3.5 Symmetry Algebra

The construction of the symmetry algebra for superintegrable systems allows us to describe the degeneracies of the Hamiltonian. Each degenerate family consolidates into the whole eigenspace of the Hamiltonian, and so gives us the whole spectrum.

For $J^u K^{ku}$ to be well-defined, then u and ku must both be integers, so u is an integral multiple of q . Therefore the general form of an integral of H is

$$I = \sum_{i \in \mathbb{Z}} J^{iq} K^{ip} \chi_i \quad (3.40)$$

where χ_i are formal rational functions, with only finitely many that are non-zero, of M, H such that I is a polynomial in those operators, and, for the TTW model, even in M . Let $\chi_N^2 + \chi_{-N}^2 \neq 0$ and $\chi_i = 0$ for $|i| > N$. Then by Theorem 3.3, we require

$$\frac{\chi_{\pm N} \prod_{i=0}^{2N-1} (M \pm ip)}{\gcd\{J_{0,j}^{\pm Nq}(M), J_{1,j}^{\pm Nq}(M)\}_{j \geq 0}} \quad (3.41)$$

to be a polynomial in M . By Theorems 3.1 and 3.2, the denominator in (3.41) is a unit. So, by linear independence of $\{J^{iq} K^{ip}\}_i$, the integral must be of order $\geq 2N(p+q-1)$ for the TTW model and $\geq N(2p+q-2)$ for the PVZ model. Whether or not we can eliminate any of the factors in the product must depend, by Corollary 3.2, on

So far the representation theory for the TTW and the PVZ model has not been developed. In this section we will also provide a description of the representations from the symmetry algebra in the context of deformed oscillator realisations. This is also interesting in its own right as the construction of representations has been mainly limited to quadratic algebras such as Racah or some cubic algebras as in the Heun-Lie algebras.

The deformed oscillator realisation [100] was introduced within the context of quadratically super-integrable systems. Here we will rely on the analogous construction for polynomial algebras of three generators of arbitrarily large order. This realisation consists in the system generated by the three operators $\mathfrak{N}, \mathfrak{b}, \mathfrak{b}^\dagger$ satisfying the relations

$$[\mathfrak{b}, \mathfrak{N}] = \mathfrak{b}, \quad [\mathfrak{N}, \mathfrak{b}^\dagger] = \mathfrak{b}^\dagger, \quad \mathfrak{b}^\dagger \mathfrak{b} = \Xi(\mathfrak{N}), \quad \mathfrak{b} \mathfrak{b}^\dagger = \Xi(\mathfrak{N} + 1)$$

where Ξ is, for our purposes, an analytic function. The finite-dimensional irreducible representations of this algebra are, up to a translation of \mathfrak{N} , given by

$$\mathfrak{N}|i\rangle = i|i\rangle, \quad \mathfrak{b}|i\rangle = \sqrt{\Xi(i)} |i-1\rangle, \quad \mathfrak{b}^\dagger|i\rangle = \sqrt{\Xi(i+1)} |i+1\rangle,$$

where $\{|i\rangle\}_{i=0}^n$ is the basis of the module. The constraints on finite dimension imply that $\Xi(0) = \Xi(n) = 0$ with no intermediate roots of Ξ . The central elements, namely $\mathbb{R}[H]$ act as constants on the representation, and we denote the eigenvalue of H as E . From one of the two roots of Ξ , we may determine the eigenvalue E of H . The energy spectrum will include as a subset the physical spectrum (3.8) found by analysis. However, even the non-physical states can still be interesting in its links with finding new special functions.

3.5.1 PVZ q even

For q even, $2p + q \geq N(2p + q - 2)$ for all $N > 1$. Thus we need only consider the form

$$I = \phi_0(M, H)M^{-1}(M^2 - p^2)^{-1} + J^q K^p \phi_1(M, H)M^{-1}(M + p)^{-1} \\ + J^{-q} K^{-p} \phi_{-1}(M, H)M^{-1}(M - p)^{-1}$$

to find the smallest integral with ϕ_0 a polynomial and $\phi_{\pm 1}$ at most quadratic. Let us show that an integral of order $2p + q - 2$ or $2p + q - 1$ is impossible. We note that

$$\operatorname{Res}_{M=0} I = k^q (2q - 1)!! [e^{-2p\theta R} \phi_1(0, H) - e^{2p\theta R} \phi_{-1}(0, H)] K^p(r, \partial_r, 0)/p + \dots$$

where the ellipses indicate lower frequency terms, cannot be zero unless M divides $\phi_{\pm 1}$. Similarly,

$$\operatorname{Res}_{M=\pm p} I = \pm k^q e^{\pm 2p\theta R} \phi_{\mp 1}(\pm p, H) K_{0,0}^p(-p, H) \prod_{i=1}^q (2i - 1 - q)/p + \dots$$

requires that $\Phi_{\mp 1}(\pm p, H) = 0$. Therefore, H is a superintegrable Hamiltonian of order $2p + q$ with generators $M, J^q K^p, J^{-q} K^{-p}$. We set

$$A = \frac{M}{2p}, \quad B = J^q K^p, \quad C = J^{-q} K^{-p},$$

and $\Lambda_{\pm} = \Phi^{\pm q}(2pA)\Psi^{\pm p}(2pA, H)$. The symmetry algebra relations are

$$[A, B] = B, \quad (3.42a)$$

$$[A, C] = -C, \quad (3.42b)$$

$$[B, C] = \Lambda_- - \Lambda_+. \quad (3.42c)$$

The algebra of order $4p + 2q - 1$ if $\kappa \neq 0$ and of order $2p + 2q - 1$ if $\kappa = 0$. The enveloping algebra \mathfrak{U} of A, B, C, H , regarded as purely abstract operators, joined by the relations (3.42a), (3.42b), (3.42c) with H central, satisfies the PBW property, i.e., $\mathfrak{U} = \mathbb{R}[A]\mathbb{R}[B]\mathbb{R}[C]\mathbb{R}[H]$. Furthermore, the operator

$$D = 2\{B, C\} - \Lambda_+ - \Lambda_-$$

commutes with A, B, C , hence $\mathfrak{U} = \mathbb{R}[A]\mathbb{R}[B](\mathbb{R} + C\mathbb{R})\mathbb{R}[H, D]$. The center is then seen to be $\mathbb{R}[H, D]$ however the differential operator realisation annihilates D .

The realisation as a deformed oscillator algebra is straightforward:

$$A = \mathfrak{N} + u,$$

$$B = \mathfrak{b}^\dagger + \mathfrak{b},$$

$$C = \mathfrak{b}^\dagger - \mathfrak{b},$$

$$\Xi(\mathfrak{N}) = \Phi^{-q}(2p[\mathfrak{N} + u])\Psi^p(-2p[\mathfrak{N} + u], E),$$

$$\Xi(\mathfrak{N} + 1) = \Phi^q(2p[\mathfrak{N} + u])\Psi^p(2p[\mathfrak{N} + u], E).$$

where u is an arbitrary constant.

3.5.2 PVZ q odd

For q odd, $2p + q - 1 \geq N(2p + q - 2)$ for all $N > 1$. So as before, we consider

$$I = \phi_0(M, H)M^{-1}(M^2 - p^2)^{-1} + J^q K^p \phi_1(M, H)M^{-1}(M + p)^{-1} \\ + J^{-q} K^{-p} \phi_{-1}(M, H)M^{-1}(M - p)^{-1}.$$

By a similar argument to section 3.5.1, $\phi_0, \phi_{\pm 1}$ must divide M . Lemma 3.1 guarantees that we may take $\phi_{\pm 1}$ to be any real number proportion of M . Therefore, H is a superintegrable Hamiltonian of order $2p + q - 1$. If we write

$$X = [J^q K^p - J_{0,0}^q(-p)K_{0,0}^p(-p, H)](M + p)^{-1}, \\ Y = [J^{-q} K^{-p} - J_{0,0}^{-q}(p)K_{0,0}^p(-p, H)](M - p)^{-1},$$

then X, Y, M, H together generate all the integrals of H . The Hamiltonian is therefore superintegrable of order $2p + q - 1$. We take

$$A = \frac{M}{2p}; \quad B = p(Y + X); \quad C = p(Y - X)$$

and

$$\Lambda_{\pm} = \frac{\Phi^{\pm q}(2pA)\Psi^p(\pm 2pA, H) - [J_{0,0}^{\pm q}(\mp p)K_{0,0}^p(-p, H)]^2}{(2A \pm 1)^2}.$$

The algebra relations are

$$\{A, B\} = \frac{1}{2}C - \frac{1}{2}[J_{0,0}^q(-p) + J_{0,0}^{-q}(p)]K_{0,0}^p(-p, H), \\ \{A, C\} = \frac{1}{2}B + \frac{1}{2}[J_{0,0}^q(-p) - J_{0,0}^{-q}(p)]K_{0,0}^p(-p, H), \\ \{B, C\} = \Lambda_+ - \Lambda_-.$$

This is a symmetry algebra of order $4p + 2q - 3$ if $\kappa \neq 0$ and of order $2p + 2q - 3$ if $\kappa = 0$. We have the Casimir operator of the enveloping algebra

$$D = B^2 + C^2 - 2\Lambda_+ - 2\Lambda_-.$$

The element D evaluates to zero in the differential-difference operator realisation.

When $k = 1$, we have

$$\{B, C\} = -\alpha\beta(4\omega^2 - \kappa^2 + 4\kappa H) + 2A[4\omega^2 + 4\kappa H + \kappa^2(\alpha^2 + \beta^2 - 2)] \\ + 4\kappa^2\alpha\beta A^2 - 16\kappa^2 A^3.$$

For $\kappa = 0$, $\Lambda_+ - \Lambda_-$ is of first-order, and the system generated by A, B, C is a Bannai-Ito algebra over the ring $\mathbb{R}[H]$:

$$\{A, B\} = \frac{1}{2}C - \alpha H; \\ \{A, C\} = \frac{1}{2}B + \beta H; \\ \{B, C\} = (4\omega^2 - \kappa^2)(2A - \alpha\beta); \\ 0 = (4\omega^2 - \kappa^2)(4A^2 + 1 - \alpha^2 - \beta^2) + B^2 + C^2 - 4H^2.$$

In order to produce a deformed oscillator realisation, we require a ‘fermionic’ number operator in contrast to the ‘bosonic’ operator \mathfrak{N} that alternates the sign of $\mathfrak{b}, \mathfrak{b}^\dagger$ for left- and right-multiplication. This is achieved if we take $(-1)^{\mathfrak{N}} = \cos(\pi\mathfrak{N})$, giving:

$$\begin{aligned}
A &= -(\mathfrak{N} + u) \cos(\pi\mathfrak{N}), \\
B &= \mathfrak{b}[2(\mathfrak{N} + u) - 1]^{-1} + [2(\mathfrak{N} + u) - 1]^{-1} \mathfrak{b}^\dagger \\
&\quad + K_{0,0}^p(-p, E) \left[\frac{J_{0,0}^q(-p)}{2(\mathfrak{N} + u) \cos(\pi\mathfrak{N}) - 1} - \frac{J_{0,0}^{-q}(p)}{2(\mathfrak{N} + u) \cos(\pi\mathfrak{N}) + 1} \right] \\
C &= \cos(\pi\mathfrak{N}) \{ \mathfrak{b}[2(\mathfrak{N} + u) - 1]^{-1} - [2(\mathfrak{N} + u) - 1]^{-1} \mathfrak{b}^\dagger \} \\
&\quad - K_{0,0}^p(-p, E) \left[\frac{J_{0,0}^q(-p)}{2(\mathfrak{N} + u) \cos(\pi\mathfrak{N}) - 1} + \frac{J_{0,0}^{-q}(p)}{2(\mathfrak{N} + u) \cos(\pi\mathfrak{N}) + 1} \right], \\
\Xi(\mathfrak{N}) &= \Phi^{-q \cos(\pi\mathfrak{N})}(-2p[\mathfrak{N} + u] \cos[\pi\mathfrak{N}]) \Psi^p(-2p[\mathfrak{N} + u], E), \\
\Xi(\mathfrak{N} + 1) &= \Phi^{q \cos(\pi\mathfrak{N})}(-2p[\mathfrak{N} + u] \cos[\pi\mathfrak{N}]) \Psi^p(2p[\mathfrak{N} + u], E).
\end{aligned}$$

3.5.3 TTW on constant-curvature space

For the TTW model, the operators

$$\begin{aligned}
A &= \frac{M^2}{4p^2} - \frac{1}{4}; \\
B &= p^2 J^q K^p M^{-1} (M + p)^{-1} + p^2 J^{-q} K^{-p} M^{-1} (M - p)^{-1} \\
&\quad - 2p^2 J_{0,0}^q(-p) K_{0,0}^p(-p, H) (M^2 - p^2)^{-1} \\
C &= p J^q K^p M^{-1} - p J^{-q} K^{-p} M^{-1};
\end{aligned}$$

together with H , generate all the integrals of this model and thus the full symmetry algebra. So H is superintegrable of order $2p + 2q - 2$ (this being the order of B). The symmetry algebra relations are

$$[A, B] = C; \quad (3.43a)$$

$$[A, C] = 4\{A, B\} + 2J_{0,0}^q(-p) K_{0,0}^p(-p, H); \quad (3.43b)$$

$$[B, C] = -2B^2 + \Lambda \quad (3.43c)$$

where

$$\Lambda = \frac{2p^3[(M-p)^2 \Phi^q(M) \Psi^p(M, H) - (M+p)^2 \Phi^q(-M) \Psi^p(-M, H) + 4pM \Phi^q(-p) \Psi^p(-p, H)]}{M(M^2 - p^2)^2}. \quad (3.44)$$

is a polynomial in A, H . This algebra is of order: $2p + 2q - 2$ if $\kappa \neq 0$; $p + 2q - 2$ if $\kappa = 0, k \neq 1$; or 2 if $\kappa = 0, k = 1$. There is a Casimir

$$D = C^2 + 4B^2 - 4\{A, B^2\} - 4J_{0,0}^q(-p) K_{0,0}^p(-p, H) B + \Omega \quad (3.45)$$

where

$$\Omega = \frac{2p^2[(M-p)^3 \Phi^q(M) \Psi^p(M, H) + (M+p)^3 \Phi^q(-M) \Psi^p(-M, H) - 2\Phi^q(-p, H) \Psi^p(-p, H) M(M^2 + 3p^2)]}{M(M^2 - p^2)^2} \quad (3.46)$$

is a polynomial in A, H . In the differential operator realisation, $D \equiv 0$.

For $k = 1$, these polynomials evaluate to

$$\begin{aligned}\Lambda &= 8H^2 + 16\sigma(\alpha^2 + \beta^2 - 2 - 2A) + 16\kappa H(\alpha^2 + \beta^2 - 2 - 2A) \\ &\quad + 2\kappa^2[(\alpha^2 - \beta^2)^2 + 12A^2 - 2(\alpha^2 + \beta^2)(4A + 3) + 28A + 12]; \\ \Omega &= 8(\alpha^2 + \beta^2)(4\sigma A - H^2) - 4(\alpha^2 - \beta^2)^2\sigma - 64\sigma A^2 \\ &\quad + 4\kappa H[8(\alpha^2 + \beta^2 - 2)A - (\alpha^2 - \beta^2)^2 - 16A^2] \\ &\quad + 4\kappa^2 A[16A^2 + (\alpha^2 - \beta^2)^2(A - 1) - 44A + 48 - 2(\alpha^2 + \beta^2)(4A + 3)].\end{aligned}$$

If, in addition, $\kappa = 0$, then Λ is of first-order in A and we obtain a Racah algebra over the ring $\mathbb{R}[H]$:

$$\begin{aligned}[A, B] &= C, \\ [A, C] &= 4\{A, B\} + 2H(\alpha^2 - \beta^2), \\ [B, C] &= -2B^2 + 8H^2 + 16\sigma(\alpha^2 + \beta^2 - 2 - 2A), \\ 0 &= C^2 + 4\beta^2 - 4\{A, B^2\} - 4H(\alpha^2 - \beta^2)B \\ &\quad + 8(\alpha^2 + \beta^2)(4\sigma A - H^2) - 4(\alpha^2 - \beta^2)^2\sigma - 64\sigma A^2.\end{aligned}$$

A deformed oscillator realisation is readily given by

$$\begin{aligned}A &= (\mathfrak{N} + u)^2 - \frac{1}{4}; \\ B &= \mathfrak{b}^\dagger(\mathfrak{N} + u)^{-1}[2(\mathfrak{N} + u) + 1]^{-1} + [2(\mathfrak{N} + u) + 1]^{-1}(\mathfrak{N} + u)^{-1}\mathfrak{b} \\ &\quad - 2J_{0,0}^q(-p)K_{0,0}^p(-p, E)[4(\mathfrak{N} + u)^2 - 1]^{-1}, \\ C &= \mathfrak{b}^\dagger(\mathfrak{N} + u)^{-1} - (\mathfrak{N} + u)^{-1}\mathfrak{b}, \\ \Xi(\mathfrak{N}) &= 4\Phi^q(-2p[\mathfrak{N} + u])\Psi^p(-2p[\mathfrak{N} + u], E), \\ \Xi(\mathfrak{N} + 1) &= 4\Phi^q(2p[\mathfrak{N} + u])\Psi^p(2p[\mathfrak{N} + u], E).\end{aligned}$$

3.5.4 Spectrum

The deformed oscillator realisations of both models relate the roots of Ξ to those of $\Phi^{\pm q}, \Psi^p$. The roots of these structure functions supply constraints on n, u and allow us to determine the value of E corresponding to a $n + 1$ -dimensional representations.

Let us consider the case where $\Phi^{-q}(2pu) = \Phi^q(2p[n + u]) = 0$ for the TTW model and the PVZ model with q even and $\Phi^{-q}(2pu) = \Phi^{(-1)^n q}(2p[-1]^n[n + u]) = 0$ otherwise. Then E is free to take any value but at least one of $\alpha, \beta, \alpha + \beta, \alpha - \beta$ must be an integer (an even integer, if PVZ). Second, we have $\Psi^p(-2pu, E) = \Psi^p(2p[n + u], E) = 0$. This yields the two equations:

$$0 = E^2 - (\omega^2 + \kappa E)(2pu - 2i - 1)^2 + \frac{1}{4}\kappa^2(2pu - 2i - 1)^4, \quad (3.47)$$

$$0 = E^2 - (\omega^2 + \kappa E)(2pn + 2pu + 2i' + 1)^2 + \frac{1}{4}\kappa^2(2pn + 2pu + 2i' + 1)^4, \quad (3.48)$$

for some $0 \leq i, i' < p$. Eliminating E results in an equation of fourth degree in u . There is first, the double root:

$$u_0 = \frac{i - i' - pn}{2p}.$$

Setting $u = u_0$ makes (3.47) identical to (3.48). There are two corresponding values for E , given by

$$E = \frac{1}{2}\kappa(pn + i + i' + 1)^2 \pm \omega(pn + i + i' + 1).$$

There is another value eigenvalue:

$$E = \frac{1}{2}\kappa(pn + i + i' + 1)^2 - \frac{\omega^2}{2\kappa}$$

which only exists for non-zero curvature. This occurs when u satisfies the two other roots of the resultant:

$$u_{\pm} = u_0 \pm \frac{\omega}{2p}.$$

Finally, the case where $\Phi^{-q}(2pu) = \Psi^q(2p[n + u]) = 0$ or $\Psi^p(-2pu, E) = 0$ and $\Phi^q(2p[n + u]) = 0$ for the TTW model and the PVZ model with q even and $\Phi^{(-1)^n q}(2p[-1]^n[n + u]) = 0$ otherwise. This yields eight different sequences of eigenvalues:

$$E_{m,\ell} = \pm\omega\epsilon_{m,\ell} + \frac{1}{2}\kappa\epsilon_{m,\ell}^2, \quad \epsilon_{m,\ell} = 2m + 1 + k(2\ell + \gamma + 1)$$

where $\gamma \in \{\alpha + \beta, \alpha - \beta, -\alpha + \beta, -\alpha - \beta\}$. The physical energies correspond to a choice of positive sign for α, β, ω .

3.6 Conclusion

We have applied the expansion method to the TTW and PVZ models. This has proved successful on account of their separability. We obtained ODE in the radial variable with operator coefficients. The solution lead to identification of an additional constraint between the angular operator and the integral, so that the integral decomposes into ladder operators. We analyzed the properties of these operators in order to compute the symmetry algebras. This shows how the expansion method can lead to the resolution of a range of superintegrable systems. The approach here has also proved more effective in analyzing the ladder operators as we have not needed to consider the action on a basis of eigenfunctions. Moreover, we have shown that the ladder operators do not give the generators directly but have given a method to reduce them down to basic operators.

We have also found the degenerate energy spectrum of the two Hamiltonians via the algebraic method. That is, the three generator polynomial algebra formed by the integrals and the deformed oscillator realisation. It is not known in general if the spectrum can always be recovered in this way for superintegrable systems, even on two-dimensional spaces. This chapter has given examples where this is the case for a symmetry algebra which may be of arbitrarily large order.

The following publication has been incorporated as Chapter 4.

[2] I. Marquette, A. Parr, ‘Polynomially superintegrable Hamiltonians separating in Cartesian coordinates’, *J. Phys. A: Math. Theor.* 58 (26) (2025). doi:10.1088/1751-8121/ade57e.

Contributor	Statement of contribution	%
Anthony Parr	writing of text	100
	proof-reading	90
	theoretical derivations	100
	numerical calculations	100
	preparation of figures	N/A
	initial concept	90
Ian Marquette	writing of text	0
	proof-reading	10
	supervision, guidance	100
	theoretical derivations	0
	numerical calculations	0
	preparation of figures	N/A
	initial concept	10

Chapter 4

Superintegrable Systems Separating in Cartesian Coordinates

4.1 Introduction

Scalar potentials whose corresponding Hamiltonians admit more integrals than degrees of freedom have been partitioned into two classes: those which satisfy linear differential equations, called ‘standard’, and those which do not, called ‘exotic’. The harmonic oscillator and the Kepler model are well-known members of the first class, and in more recent times, the Smorodinsky-Winternitz, Tremblay-Turbiner-Winternitz (TTW) and Post-Winternitz models have been added to this list [11, 38, 44]. It is known that systems with third-order integrals lead to standard potentials with wave functions involving exceptional orthogonal polynomials [101, 102]. The exotic class includes potentials that are algebraic in the classical case or satisfy the Painlevé property in the quantum case.

It is difficult, however, to study even separable Hamiltonians with higher order symmetries because of the increasing non-linearity of the resulting PDE system. Thus, research has been pressured towards cases where many of the coefficients can be set to zero [33]. It is the aim of this chapter to present a general method valid for Cartesian-separable higher-order superintegrable Hamiltonians by which these additional assumptions may be dismissed and we can obtain a system of ODE leading to the complete determination of the potential energy.

We consider a natural Hamiltonian, either classical or quantum, on a flat configuration space with d degrees of freedom that is separable in Cartesian coordinates, i.e. of the form

$$H = \sum_{i=1}^d \left[\frac{1}{2} p_i^2 + V_i(q_i) \right] \quad (4.1)$$

where $p_1, p_2, \dots, p_d, q_1, q_2, \dots, q_d$ are the canonical momenta and coordinates respectively and $V(\mathbf{q}) = \sum_{i=1}^d V_i(q_i)$ is the potential energy, V_i we call a potential summand. This Hamiltonian possesses d second-order integrals

$$H_i = \frac{1}{2} p_i^2 + V_i, \quad i = 1, 2, \dots, d. \quad (4.2)$$

For two dimensions, such a Hamiltonian is one of four types that is second-order integrable, the others being separable in polar, parabolic or elliptical coordinates. For $d > 2$, separability is a special case of second-order integrability [13, 18, 103]. We wish to address the case where the Hamiltonian is (minimally and polynomially) superintegrable, i.e. it possesses at least one integral X which is of n th order in the momenta that is not a polynomial in the known integrals H_1, H_2, \dots, H_d .

Problem 1. Given a Hamiltonian of the form (4.1), to find which assignments of the potential summands V_1, V_2, \dots, V_d such that H is superintegrable.

The simplicity of Cartesian separability greatly facilitates the search for superintegrable systems. Indeed, for classical systems, the use of action-angle variables has been very fruitful in making general statements and deriving diverse families of superintegrable models [104–106]. Such methods however fail for quantum systems because the integrals are usually algebraic in the momenta.

The direct method for systematically finding superintegrable models is to expand the integral into its coefficients and solve the partial differential equations that result from requiring it to be in involution (in the classical case) or commute (in the quantum case) with the Hamiltonian. These PDE form an overdetermined system. To ensure compatibility, there is an additional series of equations that the potential and the coefficients of the integral must satisfy. Since the leading-order terms of any integral must be a polynomial combination of the linear and angular momenta, the only unknowns of the first compatibility equation (which is linear in the potential) are constant parameters. However, the other compatibility equations involve functional unknowns which must be determined by integrating the potential and the previous set of coefficients of the integral. A variation of this approach is to expand in terms of powers of the known integrals [49, 107].

The essence of our approach is to treat the homogeneous components rather than the individual coefficients as the fundamental units of the problem. The system of PDE relates one order to all the higher orders and this can be consolidated into a very simple form. We have developed an algorithm that can solve each equation in succession. This gives us a formula for the integral as a rational function of the form

$$X = \text{polynomial in the momenta} + \text{reciprocal powers in the momenta.}$$

The coefficients of the reciprocal powers need to be equal to zero in order for it to be a polynomial integral. Out of these constraints arise a series of equations for the potential summands which are equivalent to the compatibility equations obtained from the direct method. This technique requires we treat the momenta as commuting variables rather than as differential operators. We therefore recast quantum mechanics as a deformation of classical mechanics by assigning Planck's reduced constant \hbar as the deformation parameter. This allows for a unified treatment of classical and quantum regimes via McCoy's formulas [108].

In section 4.2, we derive the PDE for the integral components as the interpretation of Problem 1. In section 4.3, we establish the framework which will enable us to solve these equations. The things to be determined are a sequence of operators which can be solved algorithmically. In section 4.4, we

demonstrate this algorithm for a few cases (sufficient to calculate any integral and the determining equations for the potential up to tenth order). Section 4.5 will deal with the general construction of the determining equations. Section 4.6 recapitulates the process. In section 4.7, we restrict ourselves to two dimensions and give the linear, quadratic and cubic compatibility equations explicitly, valid for all orders. As an application we give all the fourth-order standard potentials, with a new family discovered.

4.2 Description of the Problem

Identifying \hbar with the deformation parameter means that at $\hbar = 0$, we should obtain the equations for classical motion.

4.2.1 Discussion of the Classical Case

Let us for the moment restrict ourselves to classical models where the system of equations related to this problem can easily be written down. To be in involution with the Hamiltonian (4.1) is equivalent to the following condition:

$$0 = \sum_{i=1}^d \left(p_i \frac{\partial X}{\partial q_i} - V'_i \frac{\partial X}{\partial p_i} \right). \quad (4.3)$$

When $d = 1$, the only solution to (4.3) is an arbitrary function of H . Let us then take $d > 1$. By considering polynomial integrals, we eliminate the momenta as variables. Let us write

$$X := \sum_{i_1+i_2+\dots+i_d \leq n} \gamma_{\mathbf{i}}(\mathbf{q}) \prod_{j=1}^d p_j^{i_j}, \quad (4.4)$$

where $\mathbf{i} = (i_1, i_2, \dots, i_d)$ is a tuple of non-negative integers. Then the single equation (4.3) becomes a system of equations

$$0 = \sum_{j=1}^d \left[\frac{\partial \gamma_{\mathbf{i}-\mathbf{e}_j}}{\partial q_j} - (i_j + 1) V'_j \gamma_{\mathbf{i}+\mathbf{e}_j} \right]. \quad (4.5)$$

where \mathbf{e}_j is an elementary unit vector. It is clear we may assume X is even or odd under time reversal, i.e. $\gamma_{\mathbf{i}} = 0$ if $n - i_1 - i_2 - \dots - i_d$ is odd. For odd n , there is an additional condition on the linear terms

$$0 = \sum_{j=1}^d V'_j \gamma_{\mathbf{e}_j}.$$

Equation (4.5) overdetermines the coefficients on the left. This results in additional compatibility equations. In two dimensions, we may express these explicitly:

$$0 = \sum_{i=0}^{n-k+1} (-1)^i \frac{\partial^{n-k+1}}{\partial q_1^{n-k+1-i} \partial q_2^i} \left[(i+1) V'_1 \gamma_{i+1, n-k+1} + (n-k+2) V'_2 \gamma_{i, n-k+2} \right]. \quad (4.6)$$

Such compatibility equations are the conditions which limit the form of the potential energy. The assumption of separability of the potential reduces the compatibility equations in the most general

case from a PDE to a system of ODE in the potential summands provided the coefficients are also determined. If we are to determine the potentials which lead to superintegrability we must have a constructive method for the integral. Equation (4.5) allows us to determine one order in terms of another. We therefore write our integral as $X = X_0 + X_1 + X_2 + \dots$ where X_i is a homogeneous polynomial in the momenta of order $n - i$. Equation (4.3) becomes

$$0 = \sum_{j=1}^d \left(p_j \frac{\partial X_k}{\partial q_j} - V_j' \frac{\partial X_{k-2}}{\partial p_j} \right). \quad (4.7)$$

A similar system of equations was derived by Holt [109] for his truncation program. Our restriction to Cartesian separability is enough that equation (4.7) can be solved.

4.2.2 Extension to the Quantum Case

Before we proceed to our analysis, we shall generalise (4.7) to the quantum case. We shall do this by defining a total symbol, also denoted X . This allows us to speak of division and differentiation of the momenta. The usual procedure for this purpose is to define a Moyal bracket [110], but this necessitates the Weyl ordering. To preserve the standard ordering (momenta on the right), we make a different choice of operations. Let f, g be two functions on phase space, polynomial in the momenta and smooth in the coordinates. We define

$$(f, g) := \sum_{i_1+i_2+\dots+i_d \geq 1} \frac{1}{i_1! i_2! \dots i_d!} \left(\frac{\hbar}{i} \right)^{i_1+i_2+\dots+i_d-1} \\ \times \left(\frac{\partial^{i_1+i_2+\dots+i_d} f}{\partial q_1^{i_1} \partial q_2^{i_2} \dots \partial q_d^{i_d}} \frac{\partial^{i_1+i_2+\dots+i_d} g}{\partial p_1^{i_1} \partial p_2^{i_2} \dots \partial p_d^{i_d}} \right. \\ \left. - \frac{\partial^{i_1+i_2+\dots+i_d} f}{\partial p_1^{i_1} \partial p_2^{i_2} \dots \partial p_d^{i_d}} \frac{\partial^{i_1+i_2+\dots+i_d} g}{\partial q_1^{i_1} \partial q_2^{i_2} \dots \partial q_d^{i_d}} \right), \quad (4.8)$$

$$f^* := \sum_{i_1, i_2, \dots, i_d \geq 0} \frac{1}{i_1! i_2! \dots i_d!} \left(\frac{\hbar}{i} \right)^{i_1+i_2+\dots+i_d} \frac{\partial^{2i_1+2i_2+\dots+2i_d} \bar{f}}{\partial q_1^{i_1} \partial q_2^{i_2} \dots \partial q_d^{i_d} \partial p_1^{i_1} \partial p_2^{i_2} \dots \partial p_d^{i_d}} \quad (4.9)$$

where \bar{f} is the complex conjugate of f . We can simplify (4.9) by introducing the operator

$$a_0 := \frac{\hbar}{2i} \sum_{i=1}^d \frac{\partial^2}{\partial q_i \partial p_i}. \quad (4.10)$$

Then by the multinomial theorem, (4.9) can be written as

$$f^* = \sum_{i \geq 0} \frac{2^i}{i!} a_0^i \bar{f} = \exp(2a_0) \bar{f} \quad (4.11)$$

These expansions are well-defined provided f, g are polynomial in the momenta. We have

$$(\cdot, \cdot) = \llbracket \cdot, \cdot \rrbracket + \mathcal{O}(\hbar). \quad (4.12)$$

Moreover, it follows from McCoy's formulas [108] that

$$f \circ g - g \circ f = i\hbar(f, g) \quad (4.13)$$

where \circ represents non-commutative multiplication of observables. By a similar induction, f^* gives precisely the Hermitean conjugate. Throughout we shall use (\cdot, \cdot) doubly as the Poisson bracket and normalised quantum bracket. The coordinates and their conjugate momenta are always understood to commute under $[\cdot, \cdot]$.

Theorem 4.1. *Problem 1 is equivalent to finding $n + 1$ functions X_0, X_1, \dots, X_n on $2d$ -dimensional phase space such that:*

(i) X_k is a homogeneous polynomial of order $n - k$ in the momenta for each k .

(ii) They constitute a non-trivial solution to the system of PDE

$$\sum_{i=1}^d \left(p_i \frac{\partial X_k}{\partial q_i} + \frac{\hbar}{2i} \frac{\partial^2 X_{k-1}}{\partial q_i^2} \right) = \sum_{i=1}^d \sum_{j=1}^{k-1} \frac{1}{j!} \left(\frac{\hbar}{i} \right)^{j-1} V_i^{(j)} \frac{\partial^j X_{k-1-j}}{\partial p_i^j}. \quad (4.14)$$

Proof. We need to find a function such that $(X, H) = 0$. Using the definition (4.8) of the deformed Poisson bracket applied to (4.1), we get

$$\sum_{i=1}^d \left[\frac{\partial X}{\partial q_i} p_i + \frac{\hbar}{2i} \frac{\partial^2 X}{\partial q_i^2} - \sum_{j=1}^n \frac{1}{j!} \left(\frac{\hbar}{i} \right)^{j-1} V_i^{(j)} \frac{\partial^j X}{\partial p_i^j} \right] = 0. \quad (4.15)$$

Expanding into homogeneous components, each term that appears is homogeneous. Therefore, all the terms of a particular order are independently zero. We see that $p_i \frac{\partial X_k}{\partial q_i}$ is of order $n - k + 1$ while $\frac{\partial^j X_k}{\partial p_i^j}$ is of order $n - k - j$. Equation (4.14) gives all the terms that are of order $n - k + 1$. \square

Theorem 2 in [38] states there is no loss of generality in taking our integral to be the particular Hermitean form $X = \frac{1}{2}(Y + Y^*)$ where Y is real and even or odd in the quantum momenta according to the parity of n . Writing $Y = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} Y_i$ where Y_i is the homogeneous component in the momenta of order $n - 2i$, we get

$$X_{2i} = Y_i + \sum_{j=0}^{i-1} \frac{2^{2i-2j-1}}{(2i-2j)!} a_0^{2i-2j} Y_j, \quad (4.16)$$

$$X_{2i+1} = \sum_{j=0}^i \frac{2^{2i-2j}}{(2i-2j+1)!} a_0^{2i-2j+1} Y_j. \quad (4.17)$$

There are now $\lfloor \frac{1}{2}n \rfloor + 1$ independent equations to solve. In the case $\hbar = 0$, a_0 vanishes and $X_{2i} = Y_i$ and $X_{2i+1} = 0$. This aligns with the classical case after assuming the integral was even or odd under time reversal.

4.3 Iterative Integration of the Equations

Let us define the partial differential operator

$$L := \sum_{i=1}^d p_i \frac{\partial}{\partial q_i}. \quad (4.18)$$

A solution to equation (4.14) requires the iterative inversion of this operator.

4.3.1 The Leading Order Term

When $k = 0$, equation (4.14), for a Hermitean integral, has

$$LY_0 = 0$$

The kernel of L coincides with the integrals of a free particle Hamiltonian, or in geometric language the Killing tensors of the Euclidean metric. An integral of arbitrary order can be expressed as a polynomial in the first-order integrals. These are the linear momenta p_1, p_2, \dots, p_d and the angular momenta $m_{ij} := q_i p_j - q_j p_i$. The expression is not in general unique thanks to the dependence relations

$$p_i m_{jk} + p_j m_{ki} + p_k m_{ij} = 0. \quad (4.19)$$

4.3.2 Algorithm for Integration

We shall endeavour to reduce Y to a series of ‘integration constants’ Z_0, Z_1, \dots that are functions of the linear and angular momenta. Our first assignment is $Y_0 = Z_0$ so Z_0 must be a non-zero polynomial in p_i, m_{ij} . For $k = 2$ in equation (4.14), we obtain another equation

$$LY_1 = \sum_{i=1}^d \left(V_i' \frac{\partial}{\partial p_i} - \frac{\hbar}{2i} a_0 \frac{\partial^2}{\partial q_i^2} \right) Z_0 \quad (4.20)$$

The homogeneous solution is designated Z_1 . We now need to find the particular solution. Differential operators that are a linear combination of

$$F = \frac{f(p_i, q_i)}{k!} \frac{\partial^{k+\ell}}{\partial p_i^k \partial q_i^\ell}. \quad (4.21)$$

we will call *additively separable*. The set of additively separable operators forms a Lie algebra under the commutator.

Theorem 4.2. *For any additively separable operator F , there exists another additively separable operator G such that $[L, G] = F$.*

Proof. Given the linearity of solutions, we may without loss of generality assume F is in the simple form (4.21). We observe that

$$\left[L, \frac{\partial^{k+\ell}}{\partial p_i^k \partial q_i^\ell} \right] = -k \frac{\partial^{k+\ell}}{\partial p_i^{k-1} \partial q_i^{\ell+1}}$$

Repeating the commutators, we have the nilpotency property

$$\text{ad}_L^{k+1} \left(\frac{\partial^{k+\ell}}{\partial p_i^k \partial q_i^\ell} \right) = 0. \quad (4.22)$$

We require an operator G such that $[L, G] = F$. Here, we use integration by parts with the nilpotency property (4.22) ensuring that the process will always terminate after a finite number of iterations. The solution is plainly

$$G = \frac{\int f dq_i}{k! p_i} \frac{\partial^{k+\ell}}{\partial p_i^k \partial q_i^\ell} + \frac{\iint f d^2 q_i}{(k-1)! p_i^2} \frac{\partial^{k+\ell}}{\partial p_i^{k-1} \partial q_i^{\ell+1}} + \dots + \frac{\int \dots \int f d^{k+1} q_i}{p_i^{k+1}} \frac{\partial^{k+\ell}}{\partial q_i^{\ell+1}} \quad (4.23)$$

□

As an application of (4.23), we look for operators a_j such that

$$[L, a_j] = \frac{1}{j!} \left(\frac{\hbar}{i} \right)^{j-1} \sum_{i=1}^d V_i^{(j)} \frac{\partial^j}{\partial p_i^j}, \quad j \geq 1. \quad (4.24)$$

We integrate to find

$$a_j := \left(\frac{\hbar}{i} \right)^{j-1} \sum_{i=1}^d \left(\frac{\int V_i}{p_i^{j+1}} \frac{\partial^j}{\partial q_i^j} + \sum_{k=1}^j \frac{1}{k!} \frac{V_i^{(k-1)}}{p_i^{j-k+1}} \frac{\partial^j}{\partial p_i^k \partial q_i^{j-k}} \right), \quad j \geq 1. \quad (4.25)$$

Equation (4.14) may then be consolidated into the algebraic form

$$LX_k = \sum_{i=0}^{k-1} [L, a_i] X_{k-1-i}. \quad (4.26)$$

using

$$[L, a_0] = -\frac{\hbar}{2i} \sum_{i=1}^d \frac{\partial^2}{\partial q_i^2}.$$

where a_0 is given by (4.10).

4.3.3 The Next-to-Leading Order Term

With the new notation, equation (4.20) simplifies to the following

$$LY_1 = ([L, a_1] + [L, a_0]a_0 - La_0^2)Z_0.$$

We are now ready to make our next integration

$$Y_1 = Z_1 + (a_1 - \frac{1}{2}a_0^2)Z_0 \quad (4.27)$$

Let us consider the the Smorodinsky-Winternitz model

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1(q_1^2 + q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\beta_3}{q_2^2}. \quad (4.28)$$

Now that the potential is given, we can calculate (4.25)

$$a_1 = \frac{\beta_1 q_1^4 - 3\beta_2}{3q_1 p_1^2} \frac{\partial}{\partial q_1} + \frac{\beta_1 q_1^4 + \beta_2}{q_1^2 p_1} \frac{\partial}{\partial p_1} + \frac{\beta_1 q_2^4 - 3\beta_3}{3q_2 p_2^2} \frac{\partial}{\partial q_2} + \frac{\beta_1 q_2^4 + \beta_3}{q_2^2 p_2} \frac{\partial}{\partial p_2}$$

The Hamiltonian (4.28) is second-order superintegrable. In order to determine the integral *a priori* we need to find Z_0, Z_1 . This is the task of the compatibility equations, which we derive in Section 4.6. For the present moment, let us set $Z_0 = m_{12}^2$ which coincides with the leading-order term of the integral. Equation (4.27) gives us

$$Y_1 = Z_1 - \frac{1}{2}\hbar^2 + \frac{2\beta_1 m_{12}^4}{3p_1^2 p_2^2} + 2(q_1^2 p_2^2 - q_2^2 p_1^2) \left(\frac{\beta_3}{q_2^2 p_2^2} - \frac{\beta_2}{q_1^2 p_1^2} \right)$$

We must choose Z_1 so that the fractional terms in Y_1 are annihilated. This is done if we take

$$Z_1 = \frac{2\beta_2 p_2^2}{p_1^2} + \frac{2\beta_1 p_1^2}{p_2^2} - \frac{2\beta_1 m_{12}^4}{3p_1^2 p_2^2}.$$

Then our integral is

$$X = q_2^2 p_1^2 + q_1^2 p_2^2 - 2q_1 q_2 p_1 p_2 + i\hbar(q_1 p_1 + q_2 p_2) + \frac{1}{2}\hbar^2 + \frac{2\beta_2 q_2^2}{q_1^2} + \frac{2\beta_3 q_1^2}{q_2^2}$$

which we calculated by using $X = \frac{1}{2}(Y + Y^*)$ and (4.11). We see then that (4.27) splits the polynomial components of the integral into rational parts. The integration constants Z_k are found by eliminating all the residues. For any other choice of Z_0 in our example, besides p_1^2, p_2^2 , we would not have been able to find a polynomial form for Y_1 .

4.3.4 General Solution as a Recurrence Relation

To systematically derive each term successively, we consider the following recurrence relation

$$[L, b_{i,k}] = \sum_{j=0}^i b_{i,k-1} [L, a_{i-j}], \quad b_{i,0} = \delta_{i,0}. \quad (4.29)$$

In Section 4.4, we shall solve (4.29). Supposing for now though that (4.29) has a solution, the convolutive sum

$$Z_k := \sum_{i+j \leq 2k} (-1)^j b_{i,j} X_{2k-i-j} \quad (4.30)$$

satisfies $LZ_k = 0$. We then look for a recursive inverse. If we substitute (4.16) and (4.17), we obtain

$$Y_k = \sum_{i=0}^k W_i Z_{k-i} \quad (4.31)$$

where $W_0 = 1$ and

$$W_k = \sum_{j=0}^{k-1} \left[\sum_{i=1}^{2k-2j} (-1)^{i+1} b_{2k-i-2j,i} + \sum_{i=0}^{2k-2j-1} \sum_{\ell=2j+1}^{2k-i} \frac{(-1)^{i+1} 2^{\ell-2j-1}}{(\ell-2j)!} b_{2k-i-\ell,i} a_0^{\ell-2j} \right] W_j \quad (4.32)$$

for $k \geq 1$. It follows by this inversion that Z_0, Z_1, Z_2, \dots form a sequence of independent functions of the linear and angular momenta which generate the integral X . As it stands, these functions develop into an infinite sequence, and we must impose extra constraints so that Y_0, Y_1, Y_2, \dots truncates.

4.4 Determination of the Constituent Operators

In this section, we present an algorithmic solution to (4.29). This is taken up to the point that is necessary to determine W_k for $k \leq 5$. This is sufficient to determine a formula for an integral in terms of the potential function of order less than eleven. It also allows us to calculate the first five compatibility equations which define the potentials for integrals up to tenth order.

Let s be an indeterminate and write $A := \sum_{i \geq 0} a_i s^i$. Since A is additively separable, we take for granted the existence of a formal differential operator A_i that satisfies $[L, A_i] = \frac{i}{(i+1)!} (\text{ad } A)^i [L, A]$ with $A_0 = 0$. Let t be another indeterminate and write $A' := \sum_{j \geq 1} A_j t^j$. Repeating this process, we set A'_i to be a separable operator such that $[L, A'_i] = \frac{i}{(i+1)!} (\text{ad } A')^i [L, A']$ with $A'_0 = 0$. The individual coefficients

$$a_{i,j} := \frac{1}{i!} \frac{d^i A_j}{ds^i} \Big|_{s=0}$$

$$a_{i,j,k} := \frac{1}{i!j!} \frac{\partial^{i+j} A'_k}{\partial s^i \partial t^j} \Big|_{s,t=0}$$

may, in principle, be calculated explicitly however a general formula has evaded us.

Let $B := \sum_{i,j \geq 0} b_{i,j} s^i t^j$. Then equation (4.29) becomes

$$[L, B] = tB[L, A], \quad B(t=0) = 1. \quad (4.33)$$

We shall show how (4.33) can be reduced to operators of the form $a_i, a_{i,j}, a_{i,j,k}, \dots$ which can be calculated *ex post facto*.

Lemma 4.1. *For $k \geq 1$, the following identity holds*

$$kA^{k-1}[L, A] = [L, A^k] + \sum_{i=0}^{k-1} \frac{k!}{(k-1-i)!} [L, A_i] A^{k-1-i}. \quad (4.34)$$

Proof. For $j \geq -1$, define

$$c_{j,k,\ell} := \sum_{m=0}^{\ell} (-1)^m (\ell - m - 1) \binom{k}{\ell - m} \binom{j+m}{m}$$

This sequence satisfies the recursion

$$\sum_{j=i+1}^{k-\ell-1} c_{j,k,\ell} = c_{i,k,\ell+1}$$

In particular,

$$c_{-1,k,\ell} = (\ell - 1) \binom{k}{\ell}.$$

We have:

$$\begin{aligned}
 kA^{k-1}[L, A] - \sum_{i=0}^{k-1} A^{k-1-i}[L, A]A^i &= \sum_{i=0}^{k-1} A^{k-1-i} [A^i, [L, A]] \\
 &= \sum_{i=0}^{k-1} \sum_{j=0}^{i-1} A^{k-2-j}[A, [L, A]]A^j = \sum_{j=0}^{k-2} (k-1-j)A^{k-2-j}[A, [L, A]]A^j \\
 &= \sum_{j=0}^{k-2} (j+1)A^j[A, [L, A]]A^{k-2-j} = \sum_{j=0}^{k-2} c_{j,k,1}A^j[A, [L, A]]A^{k-2-j} \\
 &= \sum_{j=0}^{k-2} c_{j,k,1}[A, [L, A]]A^{k-2} + \sum_{j=1}^{k-2} \sum_{i=0}^{j-1} c_{j,k,1}A^i[A, [A, [L, A]]]A^{k-3-i} \\
 &= c_{-1,k,2}[A, [L, A]]A^{k-2} + \sum_{i=0}^{k-3} c_{i,k,2}A^i[A, [A, [L, A]]]A^{k-3-i} \\
 &= \sum_{i=1}^{k-1} c_{-1,k,i+1} \operatorname{ad}_A^i([L, A])A^{k-1-i}
 \end{aligned}$$

Substituting $c_{-1,k,i}$ yields the lemma. □

For C, D, E be formal power series such that

$$C = D + s^k t^\ell E.$$

We denote this relation as $C = D + \mathcal{O}(s^k t^\ell)$.

Theorem 4.3. *Let $B'_i, i \geq 0$ be a sequence of operators satisfying $[L, B'_i] = B'_{i-1}[L, A']$ with $B'_0 = 1$. Then a solution to (4.33) satisfies*

$$B = (B'_0 + tB'_1 + t^2B'_2 + \cdots + t^{k-1}B'_{k-1})e^{tA} + \mathcal{O}(s^k t^{2k}).$$

Proof. By definition,

$$\left[L, \sum_{i=0}^{k-1} t^i B'_i \right] = t \sum_{i=0}^{k-2} t^i B'_i [L, A']$$

From the lemma, we have

$$[L, e^{tA}] = te^{tA}[L, A] - t[L, A']e^{tA}$$

Combining these two identities, we get

$$\left[L, \sum_{i=0}^{k-1} t^i B'_i e^{tA} \right] = t \sum_{i=0}^{k-1} t^i B'_i e^{tA} [L, A] - t^k B'_{k-1} [L, A'] e^{tA}.$$

As $[a_0, [L, a_0]] = 0$, we have $A_i|_{s=0} = 0$. So $A' = \mathcal{O}(st)$, and it follows $B'_k = \mathcal{O}(s^k t^k)$. Then the last term is $\mathcal{O}(s^k t^{2k})$ so this expansion is a valid solution of (4.33) up to this degree. □

Theorem 4.3 gives us a way to compute $b_{i,j}$ rapidly. If we take

$$B_i = \frac{1}{i!} \frac{\partial^i B}{\partial t^i} \Big|_{t=0}$$

then $[L, B_i] = B_{i-1}[L, A]$ with $B_0 = 1$. This is the same recurrence relation satisfied by B'_i , with A' swapped for A . Using $B'_0 = 1$, we have

$$B = e^{tA} + \mathcal{O}(st^2) \quad (4.35)$$

which means that

$$b_{0,0} = 1, \quad b_{0,1} = a_0, \quad b_{0,2} = \frac{1}{2}a_0^2, \quad b_{1,0} = 0, \quad b_{1,1} = a_1.$$

Then W_1 is readily found by (4.32) to be

$$W_1 = a_1 - \frac{1}{2}a_0^2$$

which matches (4.27). To go further, we note that $B_1 = A$ so $B'_1 = A'$ by analogy. Using Theorem 4.3 again,

$$B = (1 + tA')e^{tA} + \mathcal{O}(s^2t^4) \quad (4.36)$$

from which we calculate

$$W_2 = \frac{1}{2}a_1^2 + \frac{1}{2}[a_2, a_0] - \frac{1}{3}a_0^2a_1 - \frac{1}{3}a_0a_1a_0 + \frac{1}{6}a_1a_0^2 + \frac{5}{24}a_0^4 + a_3 - a_{2,1} + a_{1,2} \quad (4.37)$$

The last two operators are found to be

$$\begin{aligned} a_{2,1} = \sum_{i=1}^d \left[\frac{3V_i \int V_i - 3 \int V_i^2}{2p_i^4} \frac{\partial}{\partial q_i} + \frac{V_i' \int V_i}{2p_i^3} \frac{\partial}{\partial p_i} - \hbar^2 \left(\frac{9 \int V_i}{4p_i^4} \frac{\partial^3}{\partial q_i^3} + \frac{3V_i}{4p_i^4} \frac{\partial^2}{\partial q_i^2} \right. \right. \\ \left. \left. + \frac{7V_i}{4p_i^3} \frac{\partial^3}{\partial p_i \partial q_i^2} + \frac{V_i'}{2p_i^3} \frac{\partial^2}{\partial p_i \partial q_i} + \frac{5V_i'}{8p_i^2} \frac{\partial^3}{\partial p_i^2 \partial q_i} + \frac{V_i''}{8p_i^2} \frac{\partial^2}{\partial p_i^2} + \frac{V_i''}{8p_i} \frac{\partial^3}{\partial p_i^3} \right) \right] \\ a_{1,2} = \hbar^2 \sum_{i=1}^d \left(-\frac{\int V_i}{2p_i^4} \frac{\partial^3}{\partial q_i^3} + \frac{V_i}{2p_i^4} \frac{\partial^2}{\partial q_i^2} - \frac{5V_i}{6p_i^3} \frac{\partial^3}{\partial p_i \partial q_i^2} + \frac{V_i'}{4p_i^4} \frac{\partial}{\partial q_i} - \frac{5V_i'}{12p_i^2} \frac{\partial^3}{\partial q_i \partial p_i^2} \right. \\ \left. + \frac{V_i''}{12p_i^3} \frac{\partial}{\partial p_i} - \frac{V_i''}{12p_i^2} \frac{\partial^2}{\partial p_i^2} - \frac{V_i''}{12p_i} \frac{\partial^3}{\partial p_i^3} \right) \end{aligned}$$

Then Y_2 is determined from (4.31). Thus far we have determined

$$B = 1 + tA + t^2\left(\frac{1}{2}A^2 + A_1\right) + t^3\left(\frac{1}{6}A^3 + A_1A + A_2\right) + \mathcal{O}(t^4)$$

up to third order in t . Reading off the quadratic and cubic terms, and using the analogy,

$$B'_2 = \frac{1}{2}(A')^2 + A'_1, \quad B'_3 = \frac{1}{6}(A')^3 + A'_1A' + A'_2$$

we then generate more terms in the expansion

$$B = \left\{ 1 + tA' + t^2 \left[\frac{1}{2}(A')^2 + A'_1 \right] + t^3 \left[\frac{1}{6}(A')^3 + A'_1A' + A'_2 \right] \right\} e^{tA} + \mathcal{O}(s^4t^8)$$

This is sufficient to determine W_3, W_4, W_5 in terms of a -operators. They require the explicit calculation of $a_{i,j}, a_{i,j,k}$ for $i + j + k < 10, i + j + k \neq 8$. It is no challenge in this algebraic exercise to continue working out the components of B to an arbitrary degree, besides the limitations set upon us by our computers.

4.5 Assignment of the Correction Functions

We have shown how to realise W_0, W_1, W_2, \dots as differential operators whose only momenta dependence in the coefficients is by reciprocal powers. Since Y_k is a homogeneous polynomial of order $n - 2k$, $W_k Z_0$ must be homogeneous to the same degree, and in particular

$$\left(\prod_{i=1}^d p_i^{2k} \right) W_k$$

has polynomial coefficients in the momenta. It is sufficient to take

$$Z_k = \sum_{i_1+i_2+\dots+i_d+j_{12}+j_{13}+\dots+j_{d-1,d}=n+2(d-1)k} \alpha_{\mathbf{i};\mathbf{j}} \prod_{a=1}^d p_a^{i_a-2k} \prod_{b<c} m_{bc}^{j_{bc}}. \quad (4.38)$$

where $\alpha_{\mathbf{i};\mathbf{j}}$ are (not necessarily unique) constants and $\mathbf{j} = (j_{k\ell})_{1 \leq k < \ell \leq d}$. We set

$$R_{kij} := \text{Res}_{p_j=0} \frac{Y_k}{p_j^{i+1}} \prod_{\ell=1}^d p_\ell^{2k}. \quad (4.39)$$

It is necessary and sufficient for X to be a polynomial integral that

$$R_{kij} = 0, \quad 0 \leq i \leq 2k - 1, 1 \leq j \leq d, 1 \leq k \leq \lceil \frac{1}{2}n \rceil \quad (4.40)$$

These are the compatibility equations. They govern the assignment of the parameters $\alpha_{\mathbf{i};\mathbf{j}}$ and the potential function V . There will be a total of

$$N(n, d) := d \sum_{k=1}^{\lceil \frac{1}{2}n \rceil} \sum_{i=1}^{2k} \binom{n + 2(d-1)k - i + d - 1}{d-2}$$

integro-differential equations involving the d summands of the potential. In particular,

$$N(n, 2) = \begin{cases} \frac{1}{2}n(n+2), & n \text{ even,} \\ \frac{1}{2}(n+1)(n+3), & n \text{ odd;} \end{cases}$$

$$N(n, 3) = \begin{cases} \frac{3}{8}n(n+2)(4n+5), & n \text{ even,} \\ \frac{3}{8}(n+1)(n+3)(4n+7), & n \text{ odd;} \end{cases}$$

$$N(n, 4) = \begin{cases} \frac{1}{12}n(n+2)(45n^2 + 122n + 62), & n \text{ even,} \\ \frac{1}{4}(n+1)(n+3)(15n^2 + 60n + 53), & n \text{ odd.} \end{cases}$$

Let $\Pi(n, d)$ be the number of $\alpha_{i,j}$ to determine. Eliminating redundant parameters via (4.19), we calculate

$$\begin{aligned}\Pi(n, 2) &= \sum_{k=0}^{\lceil \frac{1}{2}n \rceil} \binom{n+2k+2}{2} \\ \Pi(n, 3) &= \sum_{k=0}^{\lceil \frac{1}{2}n \rceil} \left[2 \binom{n+4k+4}{4} - \binom{n+4k+3}{3} \right] \\ \Pi(n, 4) &= \sum_{k=0}^{\lceil \frac{1}{2}n \rceil} \left[2 \binom{n+6k+7}{7} - \binom{n+6k+5}{5} \right]\end{aligned}$$

4.6 Calculation of the Potential and the Integral

We give here an outline about how to proceed in finding the potential and the integral. First it is necessary to find the appropriate algebraic expressions for W_1, W_2, \dots in terms of $a_i, a_{i,j}, a_{i,j,k}, \dots$ as given in Section 4.4. Once these are found, the particular a -operators which appear in these expressions should then be computed. The inversion formula (4.23) is sufficient. No variables are mixed, so this amounts to computing expressions of the form

$$\iint \dots \int V_i^{(k)} V_i^{(\ell)} \dots V_i^{(m)}$$

and such like. Using integration by parts, we can reduce to a few canonical forms which in the following sections are denoted $Q_{i,j}$. Once this is achieved, we compute the compatibility equations (4.39). Then we solve for V for all the possible assignments of the α -parameters such that Z_0 is not identically zero. Once V and $\alpha_{i,j}$ are determined, we can compute Y using equation (28) and take the self-adjoint part to find X .

4.7 Equations for the Potential in Two Dimensions

For two dimensions, it is possible for us to express each of the compatibility equations as the integral develops in a concise form that is valid for arbitrary order. Indeed, let us take

$$\xi_{ki1}(q_1) := \sum_{j=0}^{n+2k-i} \alpha_{i,j,n+2k-i-j} q_1^{n+2k-i-j} \quad (4.41a)$$

$$\xi_{kj2}(q_2) := \sum_{i=0}^{n+2k-j} \alpha_{i,j,n+2k-i-j} (-1)^i q_2^{n+2k-i-j} \quad (4.41b)$$

Then (4.38) can be alternately written as

$$Z_k = \sum_{i=0}^{n+2k} \sum_{j=0}^{n+2k-i} \frac{(-q_2)^j}{j!} \xi_{ki1}^{(j)}(q_1) p_1^{i+j-2k} p_2^{n-i-j} \quad (4.42a)$$

$$= (-1)^n \sum_{j=0}^{n+2k} (-1)^j \sum_{i=0}^{n+2k-j} \frac{(-q_1)^i}{i!} \xi_{kj2}^{(i)}(q_2) p_1^{n-i-j} p_2^{i+j-2k} \quad (4.42b)$$

The component in R_{ki1} which comes from Z_k is

$$\xi_{ki1}(q_1) + \sum_{\ell=1}^i \frac{(-q_2)^\ell}{\ell!} \xi_{k,i-\ell,1}^{(\ell)}(q_1).$$

A similar expression can be obtained for R_{ki2} . Each equation $R_{kij} = 0$ can then be repurposed as a definition for $\xi_{kij}(q_j)$ in order to eliminate this term from subsequent equations. We designate S_{kij} as the equation $R_{kij} = 0$ once this elimination has been performed.

4.7.1 Linear Equations

We calculate

$$S_{10i} : \xi'_{00i}(q_i) Q_{i,1} + \xi_{10i}(q_i) = 0 \quad (4.43)$$

$$S_{11i} : [\xi_{01i}(q_i) Q_{i,1}]' + \xi_{11i}(q_i) = 0 \quad (4.44)$$

where $Q_{i,1} := \int V_i dq_i$. These are required to hold for all integrals regardless of order. When the coefficients are not all equal to zero, the solution is a rational function with no simple poles.

Theorem 4.4. *The potential summand V_i is a solution to the linear compatibility equations for non-trivial values of the parameters if $\frac{\partial Z_0}{\partial q_i} \Big|_{p_i=0} \neq 0$ or $\frac{\partial Z_0}{\partial p_i} \Big|_{p_i=0} \neq 0$.*

Proof. We require that $\xi'_{00i}(q_i) \neq 0$ or $\xi_{01i}(q_i) \neq 0$. From (4.42a), we have

$$\begin{aligned} \frac{\partial Z_0}{\partial q_1} \Big|_{p_1=0} &= \xi'_{001}(q_1) p_2^n & \frac{\partial Z_0}{\partial q_2} &= (-1)^n \xi'_{002}(q_2) p_1^n \\ \frac{\partial Z_0}{\partial p_1} \Big|_{p_1=0} &= [\xi_{011}(q_1) - q_2 \xi'_{001}(q_1)] p_2^{n-1} & \frac{\partial Z_0}{\partial p_2} &= (-1)^{n+1} [\xi_{012}(q_2) + q_1 \xi'_{002}(q_1)] p_1^{n-1} \end{aligned}$$

The theorem follows naturally. \square

The above result allows us to identify whether the linear compatibility equations hold just from inspection of the leading-order term.

Working out the compatibility equation from (4.6) for $k = 2$, after some simplification, gives

$$\begin{aligned} 0 &= \frac{d^{n+1}}{dq_1^{n+1}} [\xi_{011}(q_1) Q_{1,1}] - q_2 \frac{d^{n+1}}{dq_1^{n+1}} [\xi'_{001}(q_1) Q_{1,1}] \\ &\quad + \frac{d^{n+1}}{dq_2^{n+1}} [\xi_{012}(q_2) Q_{2,1}] + q_1 \frac{d^{n+1}}{dq_2^{n+1}} [\xi'_{002}(q_2) Q_{2,1}]. \end{aligned} \quad (4.45)$$

Equation (4.45) is manifestly the same as the four simultaneous equations $S_{101}, S_{111}, S_{102}, S_{112} = 0$.

4.7.2 Quadratic Equations

We compute

$$S_{20i} : \xi'_{00i}(q_i) \left(\frac{3}{2} Q_{i,2} - V_i Q_{i,1} \right) - \frac{1}{2} \xi''_{00i}(q_i) Q_{i,1}^2 - \frac{1}{4} \hbar^2 \xi'''_{00i}(q_i) Q_{i,1} + \xi_{20i}(q_i) = 0 \quad (4.46)$$

$$S_{21i} : \left[\xi_{01i}(q_i) \left(\frac{3}{2} Q_{i,2} - V_i Q_{i,1} \right) - \frac{1}{2} \xi'_{01i}(q_i) Q_{i,1}^2 - \frac{1}{4} \hbar^2 \xi''_{01i}(q_i) Q_{i,1} \right]' + \xi_{21i}(q_i) = 0 \quad (4.47)$$

$$S_{22i} : \xi'_{02i}(q_i) \left(\frac{3}{2} Q_{i,2} - \frac{1}{4} \hbar^2 V_i' \right) + \frac{1}{2} \xi''_{02i}(q_i) \left(Q_{i,1}^2 - \hbar^2 V_i \right) + [\xi'_{12i}(q_i) - \frac{1}{4} \hbar^2 \xi'''_{02i}(q_i)] Q_{i,1} + \xi_{22i}(q_i) = 0 \quad (4.48)$$

$$S_{23i} : \left\{ \xi_{03i}(q_i) \left(\frac{3}{2} Q_{i,2} - \frac{1}{4} \hbar^2 V_i' \right) + \frac{1}{2} \xi'_{03i}(q_i) \left(Q_{i,1}^2 - \hbar^2 V_i \right) + [\xi_{13i}(q_i) - \frac{1}{4} \hbar^2 \xi''_{03i}(q_i)] Q_{i,1} \right\}' + \xi_{23i}(q_i) = 0 \quad (4.49)$$

where $Q_{i,2} := \int V_i^2 dq_i$. These equations hold for all n but are superfluous for $n < 3$.

As a special case, let us take $Z_0 = \frac{1}{n-4} p_1^2 p_2^2 m^{n-4}$. Then $\xi_{02i}(q_i) = \frac{1}{n-4} q_i^{n-4}$ and differentiating (4.48), we obtain

$$\begin{aligned} 0 = & \hbar^2 q_i^{n-4} Q_{i,1}''' + 2\hbar^2(n-5)q_i^{n-5} Q_{i,1}'' - 6q_i^{n-4} \left(Q_{i,1}' \right)^2 - 4(n-5)q_i^{n-5} Q_{i,1} Q_{i,1}' \\ & + 2(n-5)q_i^{n-6} Q_{i,1}^2 + Q_{i,1}' [(n-5)(n-8)\hbar^2 q_i^{n-6} - 4q_i \xi'_{12i}(q_i)] \\ & - 2Q_{i,1} [(n-5)(n-6)\hbar^2 q_i^{n-7} - 2(n-5)\xi'_{12i}(q_i) + 2q_i \xi''_{12i}(q_i)] \\ & + 4(n-5)\xi_{22i}(q_i) - 4q_i \xi'_{22i}(q_i). \end{aligned} \quad (4.50)$$

Since $\xi_{00i} = \xi_{01i} = 0$, (4.46) and (4.47) give $\xi_{20i} = \xi_{21i} = 0$ and in particular $\alpha_{02,n+2} = \alpha_{12,n+1} = \alpha_{20,n+2} = \alpha_{21,n+1} = 0$. So ξ_{22i} is of degree n or less. This proves Conjecture 2 of Escobar-Ruiz, Linares and Winternitz [39]. For $n = 3, 4$, the solutions give the exotic potentials of Gravel [29] and Marquette, Sajedi and Winternitz [31].

4.7.3 Cubic Equations

The next set of equations is computed likewise. Taking the auxiliary function

$$Q_{i,3} = \int \left[V_i^3 + \frac{1}{4} \hbar^2 (V_i')^2 \right] dq_i$$

we obtain

$$\begin{aligned} S_{30i} : & \xi'_{00i}(q_i) \left[\frac{5}{2} Q_{i,3} - \frac{3}{2} V_i Q_{i,2} - \frac{1}{2} V_i^2 Q_{i,1} + \frac{1}{2} V_i' Q_{i,1}^2 + \hbar^2 \left(\frac{1}{4} V_i'' Q_{i,1} - \frac{1}{2} V_i V_i' \right) \right] \\ & + \xi''_{00i}(q_i) \left[V_i Q_{i,1}^2 - \frac{3}{2} Q_{i,1} Q_{i,2} + \hbar^2 \left(\frac{3}{4} V_i' Q_{i,1} - \frac{1}{2} V_i^2 \right) \right] \\ & + \xi'''_{00i}(q_i) \left[\frac{1}{6} Q_{i,1}^3 + \hbar^2 \left(V_i Q_{i,1} - \frac{5}{4} Q_{i,2} \right) \right] + \frac{1}{4} \hbar^2 \xi''''_{00i}(q_i) Q_{i,1}^2 \\ & + \frac{1}{16} \hbar^4 \xi''''''_{00i}(q_i) Q_{i,1} + \xi_{30i}(q_i) = 0 \end{aligned}$$

$$\begin{aligned}
S_{31i} : & \left\{ \xi_{01i}(q_i) \left[\frac{5}{2} Q_{i,3} - \frac{3}{2} V_i Q_{i,2} - \frac{1}{2} V_i^2 Q_{i,1} + \frac{1}{2} V_i' Q_{i,1}^2 + \hbar^2 \left(\frac{1}{4} V_i'' Q_{i,1} - \frac{1}{2} V_i V_i' \right) \right] \right. \\
& + \xi'_{01i}(q_i) \left[V_i Q_{i,1}^2 - \frac{3}{2} Q_{i,1} Q_{i,2} + \hbar^2 \left(\frac{3}{4} V_i' Q_{i,1} - \frac{1}{2} V_i^2 \right) \right] \\
& + \xi''_{01i}(q_i) \left[\frac{1}{6} Q_{i,1}^3 + \hbar^2 \left(V_i Q_{i,1} - \frac{5}{4} Q_{i,2} \right) \right] + \frac{1}{4} \hbar^2 \xi'''_{01i}(q_i) Q_{i,1}^2 \\
& \left. + \frac{1}{16} \hbar^4 \xi''''_{01i}(q_i) Q_{i,1} \right\}' + \xi_{31i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{32i} : & \xi'_{02i}(q_i) \left[\frac{5}{2} Q_{i,3} - \frac{3}{2} V_i^2 Q_{i,1} + \hbar^2 \left(\frac{1}{4} V_i'' Q_{i,1} - \frac{1}{2} V_i V_i' \right) \right] - \xi''_{02i}(q_i) \left[V_i Q_{i,1}^2 \right. \\
& + \hbar^2 \left(\frac{1}{2} V_i^2 - \frac{3}{4} V_i' Q_{i,1} \right) \left. \right] - \xi'''_{02i}(q_i) \left[\frac{1}{3} Q_{i,1}^3 + \hbar^2 \left(\frac{5}{4} Q_{i,2} - \frac{3}{4} V_i Q_{i,1} \right) - \frac{1}{16} \hbar^4 V_i' \right] \\
& + \frac{1}{8} \hbar^4 \xi''''_{02i}(q_i) V_i + \frac{1}{16} \hbar^4 \xi''''_{02i}(q_i) Q_{i,1} + \xi'_{12i}(q_i) \left(\frac{3}{2} Q_{i,2} - V_i Q_{i,1} \right) \\
& - \frac{1}{2} \xi''_{12i}(q_i) Q_{i,1}^2 - \frac{1}{4} \hbar^2 \xi'''_{12i}(q_i) Q_{i,1} + \xi_{32i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{33i} : & \left\{ \xi_{03i}(q_i) \left[\frac{5}{2} Q_{i,3} - \frac{3}{2} V_i^2 Q_{i,1} + \hbar^2 \left(\frac{1}{4} V_i'' Q_{i,1} - \frac{1}{2} V_i V_i' \right) \right] - \xi'_{03i}(q_i) \left[V_i Q_{i,1}^2 \right. \right. \\
& + \hbar^2 \left(\frac{1}{2} V_i^2 - \frac{3}{4} V_i' Q_{i,1} \right) \left. \right] - \xi''_{03i}(q_i) \left[\frac{1}{3} Q_{i,1}^3 + \hbar^2 \left(\frac{5}{4} Q_{i,2} - \frac{3}{4} V_i Q_{i,1} \right) - \frac{1}{16} \hbar^4 V_i' \right] \\
& + \frac{1}{8} \hbar^4 \xi''''_{03i}(q_i) V_i + \frac{1}{16} \hbar^4 \xi''''_{03i}(q_i) Q_{i,1} + \xi_{13i}(q_i) \left(\frac{3}{2} Q_{i,2} - V_i Q_{i,1} \right) \\
& \left. - \frac{1}{2} \xi'_{13i}(q_i) Q_{i,1}^2 - \frac{1}{4} \hbar^2 \xi''_{13i}(q_i) Q_{i,1} \right\}' + \xi_{33i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{34i} : & \xi'_{04i}(q_i) \left(\frac{5}{2} Q_{i,3} - \frac{5}{4} \hbar^2 V_i V_i' + \frac{1}{16} \hbar^4 V_i'''' \right) + \xi''_{04i}(q_i) \left[\frac{3}{2} Q_{i,1} Q_{i,2} \right. \\
& - \hbar^2 \left(\frac{3}{2} V_i^2 + \frac{1}{4} V_i' Q_{i,1} \right) + \frac{1}{4} \hbar^4 V_i'' \left. \right] + \xi'''_{04i}(q_i) \left[\frac{1}{6} Q_{i,1}^3 - \hbar^2 \left(\frac{5}{4} Q_{i,2} + \frac{1}{2} V_i Q_{i,1} \right) \right. \\
& + \frac{3}{8} \hbar^4 V_i' \left. \right] - \frac{1}{4} \hbar^2 \xi''''_{04i}(q_i) \left(Q_{i,1}^2 - \hbar^2 V_i \right) + \xi'_{14i}(q_i) \left(\frac{3}{2} Q_{i,2} - \frac{1}{4} \hbar^2 V_i' \right) \\
& + \xi''_{14i}(q_i) \left(\frac{1}{2} Q_{i,1}^2 - \frac{1}{2} \hbar^2 V_i \right) + \left[\frac{1}{16} \hbar^4 \xi''''_{04i}(q_i) - \frac{1}{4} \hbar^2 \xi'''_{14i}(q_i) + \xi'_{24i}(q_i) \right] Q_{i,1} \\
& + \xi_{34i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{35i} : & \left\{ \xi_{05i}(q_i) \left(\frac{5}{2} Q_{i,3} - \frac{5}{4} \hbar^2 V_i V_i' + \frac{1}{16} \hbar^4 V_i'''' \right) + \xi'_{05i}(q_i) \left[\frac{3}{2} Q_{i,1} Q_{i,2} \right. \right. \\
& - \hbar^2 \left(\frac{3}{2} V_i^2 + \frac{1}{4} V_i' Q_{i,1} \right) + \frac{1}{4} \hbar^4 V_i'' \left. \right] + \xi''_{05i}(q_i) \left[\frac{1}{6} Q_{i,1}^3 - \hbar^2 \left(\frac{5}{4} Q_{i,2} + \frac{1}{2} V_i Q_{i,1} \right) \right. \\
& + \frac{3}{8} \hbar^4 V_i' \left. \right] - \frac{1}{4} \hbar^2 \xi'''_{05i}(q_i) \left(Q_{i,1}^2 - \hbar^2 V_i \right) + \xi_{15i}(q_i) \left(\frac{3}{2} Q_{i,2} - \frac{1}{4} \hbar^2 V_i' \right) \\
& + \xi'_{15i}(q_i) \left(\frac{1}{2} Q_{i,1}^2 - \frac{1}{2} \hbar^2 V_i \right) + \left[\frac{1}{16} \hbar^4 \xi''''_{05i}(q_i) - \frac{1}{4} \hbar^2 \xi''_{15i}(q_i) + \xi_{25i}(q_i) \right] Q_{i,1} \left. \right\}' \\
& + \xi_{35i}(q_i) = 0
\end{aligned}$$

The quartic and quintic equations have been calculated and placed in Appendix A. We give here their auxiliary functions

$$Q_{i,4} = \int \left[V_i^4 + \hbar^2 V_i (V_i')^2 + \frac{1}{20} \hbar^4 (V_i'')^2 \right] dq_i$$

$$Q_{i,5} = \int \left[V_i^5 + \frac{5}{2} \hbar^2 V_i^2 (V_i')^2 + \frac{1}{4} \hbar^4 V_i (V_i'')^2 + \frac{1}{112} \hbar^6 (V_i''')^2 \right] dq_i.$$

Given that if H is superintegrable so must $H + \alpha$, we would expect that the auxiliary functions possess a similar symmetry under translation. Indeed, we note that for $V_i \mapsto V_i + \alpha$, $Q_{i,j} \mapsto Q_{i,j} + j\alpha Q_{i,j-1} + \dots$.

We expect the general expansion

$$\begin{aligned} Q_{i,j} = \int & \left[V_i^j + \frac{1}{4} \binom{j}{3} \hbar^2 V_i^{j-3} (V_i')^2 + \frac{1}{20} \binom{j}{4} \hbar^4 V_i^{j-4} (V_i'')^2 \right. \\ & \left. + \frac{1}{112} \binom{j}{5} \hbar^6 V_i^{j-5} (V_i''')^2 + \dots \right] dq_i \end{aligned} \quad (4.51)$$

though we have as yet no way of determining the law which produces the next coefficient.

4.8 Rational Potentials

The classification of superintegrable systems with higher-order integrals has focused mainly on exotic potentials, i.e., potential for which the linear equations S_{10i} , S_{11i} vanish identically. We shall provide some details on obtaining potentials in the case where S_{10i} , S_{11i} do not vanish. We perform a complete classification of fourth-order standard models as an application of these compatibility equations. The generalisation to higher order follows naturally, the only difficulty being algebraic complexity.

Lemma 4.2. *If V_i is a solution to the linear compatibility equations for $n = 3, 4$ then $Q_{i,1}$ and $Q_{i,2}$ must be rational. If $n = 5, 6$ then $Q_{i,3}$ must be rational also.*

Proof. If S_{10i} or S_{11i} possess non-trivial parameters then so do S_{j0i} or S_{j1i} for $j = 2, 3$. We then use S_{j0i} or S_{j1i} to solve for $Q_{i,j}$ in terms of V_i , its derivatives, $Q_{i,k}$ for $k < j$, ratios of ξ_{jki} and its derivatives. The lemma follows by induction. \square

Lemma 4.3. *If V_i is a solution to the linear compatibility equations for $n \leq 4$, then it must be of the form*

$$V_i = \beta_1 q_i^2 + \beta_2 q_i + \sum_{j=1}^{n-1} \frac{\gamma_j}{(q_i - \delta_j)^2}$$

Proof. As S_{j1i} is of the same form as S_{j0i} but differentiated, we may assume without loss of generality that S_{j0i} holds non-trivially, and prove that V_i cannot be otherwise. Suppose that up to leading-order, $V_i = \beta q_i^k + \dots$ where $k > 2$ and $\xi_{00i} = \gamma q_i^{\ell+1}/(\ell+1) + \dots$. Then from S_{20i} the leading-order term from the potential summand is

$$-\frac{\beta^2 \gamma (2k + 2k^2 + \ell + 2k\ell) q_i^{2k+\ell+1}}{2(k+1)^2(2k+1)} + \dots$$

which must be annihilated by $\xi_{20i} \sim \mathcal{O}(q_i^{n+2})$ where $n \leq 4$. However, $2k + \ell + 1 \geq 7$ so this is not possible.

From Lemma 4.2, we know that $Q_{i,1}$ is rational so V_i cannot have simple poles. Suppose $V_i \sim \delta q_i^{-k}$ for $k > 2$ as $q_i \rightarrow 0$. Then the lowest-order term in S_{20i} is

$$-\frac{\delta^2 \gamma q_i^{1-2k+\ell} (2k - 2k^2 - \ell + 2k\ell)}{2(k-1)^2(2k-1)}$$

The coefficient is not zero for any integer value of $k > 2$ when $\ell = 0, 1, 2, 3$. Since $k \geq 3$, the exponent $1 - 2k + \ell \leq -2$ so cannot be annihilated by ξ_{20i} . Then V_i has only poles of second-order. These are exactly the simple poles of $Q_{i,1}$. From S_{10i} , the number of these is equal to $\ell \leq n - 1$. \square

Theorem 4.5. *If V_i for $i = 1, 2$ is a solution to the linear compatibility equations and $H = \frac{1}{2}(p_1^2 + p_2^2) + V_1 + V_2$ is superintegrable of at least fourth-order then the potential summands must be of the form*

$$\begin{aligned} \text{(I)} \quad & \beta q_i, \\ \text{(II)} \quad & \beta_1 q_i^2 + \frac{\beta_2}{q_i^2}, \\ \text{(III)} \quad & \beta_1 \left(q_i^2 + \frac{\beta_2^2}{q_i^2} \right) + \hbar^2 \left[\frac{2(q_i^2 + \beta_2)}{(q_i - \beta_2)^2} - \frac{1}{8q_i^2} \right], \\ \text{(IV)} \quad & \frac{3\hbar^2 q_i(q_i^3 + 2\beta)}{(q_i^3 - \beta)^2}. \end{aligned}$$

Proof. For $n = 1, 2$, we can solve the linear compatibility equations directly. The answers match with the known solutions, being Types I and II.

For $n = 3, 4$, subject to the constraints of Lemma 4.3, we look for systems of the following form

$$V_i = \beta_1 q_i^2 + \beta_2 q_i + \frac{\beta_3}{q_i^2} + \frac{\beta_4}{(q_i - \beta_6)^2} + \frac{\beta_5}{(q_i - \beta_7)^2} \quad (4.52)$$

We then eliminate all the simple poles of $Q'_{i,j}$ as per Lemma 4.2. This amounts to a series of algebraic equations involving $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7$. The four families listed above present all the admissible solutions. \square

The Type II models have been well studied [21]. Type III is a Darboux transformation of Type II (after a reparametrisation) [111]. Type III only appears in a special case in Gravel's classification, when it is a Darboux transformation of the simple harmonic oscillator [112]. As for Type IV, this can be constructed via a conformal Darboux transformation of the free particle [113].

To generalise to higher order, one needs to consider rational functions with more poles as per Lemma 4.3. However, the algebraic conditions become more complicated in this case and a general solution for any order higher than four is yet to be had. Some cases will be given in Chapter 5.

4.8.1 List of Potentials

Once the forms I–IV are assigned, the compatibility equations reduce to a system in terms of rational functions which has at most four parameters occurring in a non-linear manner. This can readily be solved. The non-zero α_{ijk} allow us to determine Z_0, Z_1, Z_2 . One can calculate the integral via

$$Y = W_0 Z_0$$

for $n = 1$,

$$Y = W_0(Z_0 + Z_1) + W_1 Z_0$$

for $n = 2, 3$ and

$$Y = W_0(Z_0 + Z_1 + Z_2) + W_1(Z_0 + Z_1) + W_2 Z_0$$

for $n = 4$. Then one takes $X = \frac{1}{2}(Y + Y^*)$. We list below the potentials found for non-zero α_{ijk} . Models are to be distinguished either by their form or by the number of integrals they possess (that is, if there are additional integrals for special values of their parameters). At first-order, we obtain Hooke's law

$$(1) \beta(q_1^2 + q_2^2)$$

as well as an oblique linear potential (cf. 3a). At second-order, we have the Smorodinsky-Winternitz systems and their special cases:

$$(2) \beta_1(q_1^2 + q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\beta_3}{q_2^2}$$

$$(3) \beta_1(q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2} + \beta_3 q_2$$

$$(2a) \frac{\beta}{q_1^2}$$

$$(3a) \beta q_2$$

At third-order are the Gravel models:

$$(2ai) \frac{\hbar^2}{q_1^2}$$

$$(4a) \hbar^2 \left[\frac{q_1^2 + q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} \right]$$

$$(2b) \frac{\beta}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

$$(4b) \hbar^2 \left[\frac{q_1^2 + q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} + \frac{1}{q_2^2} \right]$$

$$(2bi) \frac{\hbar^2}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

$$(5a) \hbar^2 \left[\frac{q_1^2 + q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} + \frac{2(q_2^2 + \beta)}{(q_2^2 - \beta)^2} \right]$$

$$(2c) \beta(q_1^2 + q_2^2) + \frac{\hbar^2}{q_1^2}$$

$$(5b) \hbar^2 \left[\frac{q_1^2 + q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} + \frac{2(q_2^2 - \beta)}{(q_2^2 + \beta)^2} \right]$$

$$(2d) \beta(q_1^2 + q_2^2) + \frac{\hbar^2}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

At fourth-order, we obtain the potentials:

$$(4) \beta_1 \left(q_1^2 + q_2^2 + \frac{\beta_2^2}{q_1^2} \right) + \frac{\beta_3}{q_2^2} + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} \right]$$

$$(4c) \hbar^2 \left[\frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} - \frac{1}{8q_1^2} \right]$$

$$(4d) \hbar^2 \left[\frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} - \frac{1}{8q_1^2} + \frac{1}{q_2^2} \right]$$

$$(5) \beta_1 \left(q_1^2 + q_2^2 + \frac{\beta_2^2}{q_1^2} + \frac{\beta_3^2}{q_2^2} \right) + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} + \frac{2(q_2^2 + \beta_3)}{(q_2^2 - \beta_3)^2} - \frac{1}{8q_2^2} \right]$$

$$(6) \frac{3\hbar^2 q_1(q_1^3 + 2\beta_1)}{(q_1^3 - \beta_1)^2} + \frac{\beta_2}{q_2^2}$$

$$(7) \hbar^2 \left[\frac{3q_1(q_1^3 + 2\beta_1)}{(q_1^3 - \beta_1)^2} + \frac{2(q_2^2 + \beta_2)}{(q_2^2 - \beta_2)^2} - \frac{1}{8q_2^2} \right]$$

$$(6a) \frac{3\hbar^2 q_1(q_1^3 + 2\beta)}{(q_1^3 - \beta)^2}$$

$$(8) \hbar^2 \left[\frac{3q_1(q_1^3 + 2\beta_1)}{(q_1^3 - \beta_1)^2} + \frac{3q_2(q_2^3 + 2\beta_2)}{(q_2^3 - \beta_2)^2} \right]$$

$$(6b) \frac{3\hbar^2 q_1(q_1^3 + 2\beta)}{(q_1^3 - \beta)^2} + \frac{\hbar^2}{q_2^2}$$

It is of interest to note the following rational solutions, which have summands that are of Types I–IV, but their linear compatibility equations happen to be zero identically. Consequently, they are properly speaking exotic models. At third-order these are

$$(3b) \frac{\hbar^2}{q_1^2} + \beta q_2$$

$$(10) \beta(q_1^2 + 9q_2^2) + \frac{\hbar^2}{q_1^2}$$

$$(4e) \hbar^2 \left[\frac{9(q_1^2 + q_2^2)}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} + \frac{1}{q_1^2} \right]$$

$$(11) \hbar^2 \left[\frac{q_1^2 + 9q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} \right]$$

$$(9) \beta(q_1^2 + 9q_2^2)$$

and at fourth-order

$$(12) \hbar^2 \left[\frac{2(q_2^2 + \beta_1)}{(q_2^2 - \beta_1)^2} - \frac{1}{8q_2^2} \right] + \beta_2 q_2$$

$$(14a) \beta_1(q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

$$(13) \frac{3\hbar^2 q_2(q_2^3 + 2\beta_1)}{(q_2^3 - \beta_1)^2} + \beta_2 q_2$$

$$(15) \beta_1(q_1^2 + 16q_2^2) + \frac{\beta_2}{q_1^2}$$

$$(16) \beta_1(9q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2}$$

$$(14) \beta_1(q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\beta_3}{q_2^2}$$

$$(17) \beta_1(9q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

$$(18) \beta_1 \left(q_1^2 + 4q_2^2 + \frac{\beta_2^2}{q_1^2} \right) + \frac{\beta_3}{q_2^2} + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} \right]$$

$$(18a) \beta_1 \left(q_1^2 + 4q_2^2 + \frac{\beta_2^2}{q_1^2} \right) + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} \right]$$

$$(19) \beta_1 \left(q_1^2 + 16q_2^2 + \frac{\beta_2^2}{q_1^2} \right) + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} \right]$$

$$(20) \hbar^2 \left[\frac{4q_1^2 + q_2^2}{32\beta_1^2} + \frac{2(q_1^2 + \beta_1)}{(q_1^2 - \beta_1)^2} \right] + \frac{\beta_2}{q_2^2}$$

$$(21) \quad \hbar^2 \left[\frac{9(4q_1^2 + q_2^2)}{32\beta_1^2} + \frac{2(q_1^2 + \beta_1)}{(q_1^2 - \beta_1)^2} + \frac{1}{q_1^2} \right] + \frac{\beta_2}{q_2^2}$$

$$(22) \quad \hbar^2 \left[\frac{4q_1^2 + 9q_2^2}{32\beta_1^2} + \frac{2(q_1^2 + \beta_1)}{(q_1^2 - \beta_1)^2} \right] + \frac{\beta_2}{q_2^2}$$

as well as two degenerate cases

$$(\alpha) \quad V_2(q_2) \qquad (\beta) \quad \frac{\hbar^2}{q_1^2} + V_2(q_2)$$

which have a first- and third-order integral [28, 114] respectively for V_2 arbitrary. The former is not in general superintegrable.

These models were found through a direct calculation from the compatibility equations. In Appendix B, we have listed all of the integrals we have obtained up to fourth order, independent of the trivial ones (H_1 , H_2 and their products).

4.9 Conclusion

We have expounded a new method for finding Cartesian-separable superintegrable models and constructing their integrals. Instead of dealing with individual coefficients, we have reduced the number of equations and outlined a systematic algorithm for deriving each homogeneous component. The residue becomes the compatibility equations, recast into integral rather than differential equations. These compatibility equations have not been able to be derived using the ordinary methods in their full generality. While we can in theory calculate each compatibility equation successively, the computations become very involved. More work is needed in order to obtain general results about the compatibility equations of arbitrarily large order and in particular the determination of their auxiliary functions, which will yield more information about their rational potentials.

We have constrained ourselves to the case of Cartesian separability in order that the integration by parts can be achieved in finite steps. Whether it can be extended to other cases is a question for further investigation.

Chapter 5

Rational Extensions of the Harmonic Oscillator

5.1 Introduction

In the classification of orthogonal polynomials satisfying second-order ODE by Bochner [115], there are four kinds, namely Hermite, Laguerre, Bessel and Jacobi polynomials. Each of these families corresponds, up to gauge and variable transformation, to the eigenspace of an exactly solvable Hamiltonian—simple harmonic oscillator, singular oscillator, Morse potential and Pöschl-Teller respectively. Relaxing some of the initial hypotheses of Bochner, one obtains the exceptional orthogonal polynomials, which were introduced by Gómez-Ullate, Kamran and Milson [116]. These differ from the classic families in that they do not possess polynomials at all degrees but are a missing a finite set.

Much interest has been excited by these newly discovered families of exactly solvable models and their exceptional eigenfunctions especially as to how they can be generated by methods of supersymmetry [117–119]. The resulting potentials are connected to rational solutions of Painlevé IV, Painlevé V, Painlevé VI [120] and their higher order analogues [121].

Our aim here is to provide a systematic method of deriving non-degenerate standard superintegrable systems that separate in Cartesian coordinates. By non-degenerate, we mean one possessing a non-zero leading order quadratic term in the potential. The algebraic technique offered in the chapter preceding is too cumbersome to be extendable to higher orders. The approach adopted here is that of realisations of polynomial Heisenberg algebras or ladder operator pairs. These have been studied by several authors [122–126], particularly in their connection to Painlevé transcendents. The degenerate case requires a different kind of polynomial algebra [126, 127].

In section 5.2 we shall discuss the basic properties of ladder operators and how to derive their realisations. In section 5.3, we shall show how one can generate a superintegrable model from multiple one-dimensional Hamiltonians that have ladder operators with commensurate step-sizes. In sections 5.4–6 we shall adapt this intuition to the cases of higher order superintegrable models and show how we may obtain rational potentials with third, fourth, fifth and sixth order ladder operators, and select

for those potentials which are standard in the superintegrable sense.

5.2 Ladder Operators

Let us take $H = \frac{1}{2}p^2 + V(q)$. We look for an operator L such that

$$[L, H] = \hbar\lambda L.$$

We say that L is a ladder operator with step-size λ . It is a lowering operator if $\lambda > 0$ and a raising operator for $\lambda < 0$. More explicitly, L is a solution to the PDE

$$\frac{\partial L}{\partial q} p + \frac{\hbar}{2i} \frac{\partial^2 L}{\partial q^2} - \sum_{i \geq 1} \frac{1}{i!} \left(\frac{\hbar}{i} \right)^{i-1} \frac{d^i V}{dq^i} \frac{\partial^i L}{\partial p^i} + i\lambda L = 0. \quad (5.1)$$

For the component-wise expression, we write

$$L = \sum_{j=0}^n L_j p^{n-j} (-i)^j \quad (5.2)$$

Then

$$\frac{dL_i}{dq} + \frac{\hbar}{2} \frac{d^2 L_{i-1}}{dq^2} + \sum_{j=1}^{i-1} \binom{n-i+j+1}{j} \hbar^{j-1} \frac{d^j V}{dq^j} L_{i-j-1} = \lambda L_{i-1} \quad (5.3)$$

where $i = 0, 1, \dots, n+1$ with $L_i = 0$ for $i < 0$ or $i > n$. Let $L^{(n)}$ be a particular solution to (5.1) which is $O(p^n)$. We repeatedly integrate (5.3) to find

$$L_0^{(n)} = 1$$

$$L_1^{(n)} = \lambda q$$

$$L_2^{(n)} = \frac{1}{2} \lambda^2 q^2 - nV$$

$$L_3^{(n)} = \frac{1}{6} \lambda^3 q^3 - \frac{1}{2} \hbar \lambda^2 q - \lambda Q_1 - \lambda(n-1)qV - \frac{1}{2} \hbar n(n-2) \frac{dV}{dq}$$

$$L_4^{(n)} = \frac{1}{24} \lambda^4 q^4 - \frac{1}{2} \hbar \lambda^3 q^2 + \frac{1}{2} n(n-2)V^2 - \lambda^2 q Q_1 - \frac{1}{2} \lambda [\lambda(n-2)q^2 - 2\hbar] V \\ - \frac{1}{2} \hbar \lambda q(n-1)(n-3) \frac{dV}{dq} - \frac{1}{12} \hbar^2 n(n-2)(2n-5) \frac{d^2 V}{dq^2}$$

$$L_5^{(n)} = \frac{1}{120} \lambda^5 q^5 - \frac{1}{4} \hbar \lambda^4 q^3 + \frac{1}{2} \hbar^2 \lambda^3 q + \frac{3}{2} \lambda Q_2 + \lambda(n-3)VQ_1 + \frac{1}{2} \lambda q(n-1)(n-3)V^2 \\ + \frac{1}{2} \hbar n(n-2)(n-4)V \frac{dV}{dq} - \frac{1}{2} \lambda^2 (\lambda q^2 - 3\hbar) Q_1 - \frac{1}{6} \lambda^2 q [\lambda(n-3)q^2 - 3\hbar(n-1)] V \\ - \frac{1}{4} \hbar \lambda [\lambda(n-2)(n-4)q^2 + 3\hbar] \frac{dV}{dq} - \frac{1}{12} \hbar^2 \lambda(n-1)(n-3)(2n-7)q \frac{d^2 V}{dq^2} \\ - \frac{1}{24} \hbar^3 n(n-2)^2(n-4) \frac{d^3 V}{dq^3}$$

and so on where Q_1, Q_2, \dots are auxiliary functions which satisfy

$$\frac{dQ_i}{dq} = V^i + \frac{\hbar^2}{4} \binom{i}{3} V^{i-3} \left(\frac{dV}{dq} \right)^2 + \frac{\hbar^4}{20} \binom{i}{4} V^{i-4} \left(\frac{d^2 V}{dq^2} \right)^2 + \dots$$

The general form of an n th order ladder operator is then given by

$$\alpha_0 L^{(n)} + i\alpha_1 L^{(n-1)} - \alpha_2 L^{(n-2)} + i\alpha_3 L^{(n-3)} + \cdots + \alpha_n L^{(0)}$$

for some constants $\alpha_0 \neq 0, \alpha_1, \alpha_2, \alpha_3, \dots$. There is one compatibility equation on the potential, namely

$$\alpha_0 L_{n+1}^{(n)} + \alpha_1 L_n^{(n-1)} + \alpha_2 L_{n-1}^{(n-2)} + \alpha_3 L_{n-2}^{(n-3)} + \cdots + \alpha_n L_1^{(0)} + \alpha_{n+1} = 0. \quad (5.4)$$

Now (5.3) is satisfied, by definition, for $i = 0, 1, \dots, n$. If we define L_{n+1} by (5.3) for $i = n + 1$ then the condition for L to be n th order is simply

$$\frac{dL_{n+1}}{dq} = 0$$

which is clearly equivalent to (5.4). Translating q, V , we may assign α_1, α_2 any value we please. For $n = 1$, we have

$$V(q) = \frac{1}{2}\lambda^2 q^2$$

and for $n = 2$,

$$V = \frac{1}{8}\lambda^2 q^2 + \frac{\beta}{q^2}$$

We denote by $V := \mathcal{A}_n(q)$ the general solution of (5.4) with $\alpha_1 = \alpha_2 = 0, \lambda = 1$. Each function \mathcal{A}_n must necessarily contain within its parameters as special cases: \mathcal{A}_m for $m < n$ and m, n have the same parity; and $k^{-1}\mathcal{A}_{n/k}\left(\frac{q}{\sqrt{k}}\right)$ whenever $k \mid n$. We list some of the basic facts about ladder operators.

Theorem 5.1. *Let $H = \frac{1}{2}p^2 + V(q)$ be a one-dimensional Hamiltonian and L_1, L_2 satisfy $[L_i, H] = \hbar\lambda_i L_i$ such that $L_1 L_2 = \Phi(H)L_3$ for some L_3 . The following properties hold*

- (i) $[H, L_3] = (\lambda_1 + \lambda_2)L_3$.
- (ii) If $L_2 = L_1^\dagger$, then $\lambda_2 = -\lambda_1$.
- (iii) If $[L_1, L_3] = 0$ then $L_2 L_1 = \Phi(H - \hbar\lambda_1)L_3$.
- (iv) If $[L_1, L_3] = 0$ then $L_1^k L_2^k = \Psi(H)L_3^k$ where $\Psi(H) = \prod_{j=0}^{k-1} \Phi(H + \hbar(k-1)\lambda_1 + \hbar j\lambda_2)$.
- (v) If $H = D^\dagger D + \alpha, H' = DD^\dagger + \beta, L'_i = DL_i D^\dagger, i = 1, 2, 3$ for some first-order operator D then L'_i is a ladder operator of H' with the same step-size and direction as L_i and $L'_1 L'_2 = (H' - \hbar\lambda_1 - \beta)\Phi(H' + \alpha - \beta)L'_3$.

Proof. The first part follows from the derivation property. The second from the fact that H is Hermitian. For the third part, we write $L_2 L_1 = \Psi(H)L_3$ and observe that

$$\Phi(H)L_3 L_1 = L_1 L_2 L_1 = L_1 \Psi(H)L_3 = \Psi(H + \hbar\lambda_1)L_1 L_3.$$

The last two parts are by direct calculation. □

Theorem 5.2. *Any Hermitian operator invariant under time-reversal that commutes with a one-dimensional Hamiltonian H is a polynomial in H .*

Proof. Let X be such an operator and suppose that its order is $2m$ and any operator satisfying the above hypotheses with order $2k < 2m$ is a polynomial in H . Then we may write

$$X = \sum_i \{f_i(q), p^{2m-2i}\}$$

where the f_i are real. Since X commutes with H , f_0 must be constant. Let $Y := X - 2^m f_0 H^m$. Now, either $Y = 0$ or Y is of order less than X . But Y is invariant under time-reversal and Hermitian. The result must follow. \square

Corollary 5.1. *$L^\dagger L$ is a polynomial in H .*

Corollary 5.2. *L is a product of first-order Darboux transformations.*

Proof. Since $L^\dagger L$ is a polynomial in H , the kernel of L must be a span of eigenfunctions and generalised eigenfunctions of H . Thus, it may be so factorised by state-deleting transformations. \square

Consider a ladder operator thus factorised $L = D_n D_{n-1} \cdots D_1$ and a sequence of Hamiltonians $D_i H_i = H_{i+1} D_i$ with $H_{n+1} = H_1 + \hbar\lambda$. There exist functions ϖ_i with energies E_i satisfying

$$V_i = \frac{\hbar^2}{2} \left(\varpi_i^2 + \frac{d\varpi_i}{dq} \right) + E_i \quad (5.5)$$

and

$$\hbar^2 \frac{d\varpi_i}{dq} = V_i - V_{i+1} \quad (5.6)$$

where $V_{n+1} = V_1 + \hbar\lambda$. Equations (5.5) and (5.6) form what is called a cyclic dressing chain by Veselov and Shabat [128]. Let us examine the case for $n = 3$. Setting $v_1 = \varpi_1 + \varpi_2, v_2 = \varpi_2 + \varpi_3, v_3 = \varpi_3 + \varpi_1$, we have

$$\frac{dv_1}{dq} + v_1(v_2 - v_3) = \varepsilon_1 \quad (5.7a)$$

$$\frac{dv_2}{dq} + v_2(v_3 - v_1) = \varepsilon_2 \quad (5.7b)$$

$$\frac{dv_3}{dq} + v_3(v_1 - v_2) = \varepsilon_3 \quad (5.7c)$$

for constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$ related by

$$\frac{1}{2}\hbar^2\varepsilon_1 = E_1 - E_2$$

$$\frac{1}{2}\hbar^2\varepsilon_2 = E_2 - E_3$$

$$\frac{1}{2}\hbar^2\varepsilon_3 = E_3 - \hbar\lambda - E_1$$

Since E_1 is arbitrary, we may set it to zero. Summing (5.7a)–(5.7c), we obtain

$$\frac{dv_1}{dq} + \frac{dv_2}{dq} + \frac{dv_3}{dq} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = -\frac{2\lambda}{\hbar},$$

so, up to translation in q ,

$$v_1 + v_2 + v_3 = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)q \quad (5.8)$$

In order to determine the original potential, we write

$$\varpi_1 = \frac{1}{2}(v_1 - v_2 + v_3) = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)q - v_2 \quad (5.9)$$

We now find an equation with v_2 on its own. Using (5.7b) and (5.8), we may eliminate v_1, v_3 :

$$v_1 = -\frac{1}{2v_2} \frac{dv_2}{dq} + \frac{\varepsilon_2}{2v_2} + \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)q - \frac{1}{2}v_2 \quad (5.10a)$$

$$v_3 = \frac{1}{2v_2} \frac{dv_2}{dq} - \frac{\varepsilon_2}{2v_2} + \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)q - \frac{1}{2}v_2 \quad (5.10b)$$

and from (5.7a) and (5.7c) we may eliminate their first derivatives. Differentiating (5.7b), we obtain

$$\begin{aligned} \frac{d^2v_2}{dq^2} &= \frac{1}{2v_2} \frac{dv_2}{dq} - \frac{\varepsilon_2^2}{2v_2} + \frac{3}{2}v_2^3 - 2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)qv_2^2 \\ &\quad + \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2q^2v_2 + (\varepsilon_1 - \varepsilon_3)v_2 \end{aligned} \quad (5.11)$$

After the substitution $q = \beta x, y = \beta v_2$ with $\beta^2 = \frac{\hbar}{\lambda}$, $y = y(x)$ satisfies the Painlevé IV equation. Evidently, $D_1 D_3 D_2$ and $D_2 D_1 D_3$ are also third-order ladder operators so v_1, v_3 must also be Painlevé IV transcendents (albeit with different parameters). Equations (5.10a) and (5.10b) define Bäcklund transformations, i.e. symmetries of (5.11) and may be applied iteratively to discover new solutions given a simple seed solution [129]. If we look for a linear solution, we find

$$v_2 = -\frac{2\lambda q}{\hbar} \implies V = \frac{1}{2}\lambda^2 q^2 + \frac{1}{2}\hbar\lambda$$

or

$$v_2 = -\frac{2\lambda q}{3\hbar} \implies V = \frac{1}{18}\lambda^2 q^2 - \frac{1}{6}\hbar\lambda$$

Theorem 5.3. *Suppose $V(q) \sim \frac{1}{2}\omega^2 q^2$ as $q \rightarrow \infty$ and that H possesses a ladder operator of order k with step-size $j\omega$, $j > 0$. Then j is an integer of the same parity as k and less than or equal to k .*

Proof. Let L be of k th order such that $[H, L] = j\hbar\omega L$ and monic in p . From (5.3), we see that the leading-order term of L_i is $a_i(\omega q)^i$ where a_i satisfies the recurrence relation

$$(i+1)a_{i+1} - ja_i + (k-i+1)a_{i-1} = 0, \quad a_0 = 1, a_1 = j.$$

The roots of a_{k+1} give us the possible step-sizes up to sign. This is a polynomial in j of degree $k+1$. The roots must be the same as that for the simple harmonic oscillator. If K is the first-order ladder operator with step-size $\hbar\omega$, then $H^u L^v$ will be of order $2u+v$ with step-size $v\omega$. So the roots are $j = \pm 1, \pm 3, \pm 5, \dots, \pm k$ for k odd and $j = 0, \pm 2, \pm 4, \pm 6, \dots, \pm k$ for k even. \square

We saw that Darboux transformations may transform a Hamiltonian into a similar one with a ladder operator of the same order. However, in the general case, the construction from Theorem 5.1 (v) is needed. We give here a reduction theorem to show when this construction becomes redundant.

Theorem 5.4. *Let D be a first-order Darboux transformation of H that deletes the state ψ with energy E and suppose L also annihilates ψ . Then the partner Hamiltonian H' has a ladder operator of the same order and step-size as L and annihilates $DL^\dagger\psi$.*

Proof. Since $L\psi = 0$, we may factorise $L = KD$. Then $KH' = (H + \hbar\lambda)K$. We take $L' = DK$. Hence

$$L'(DL^\dagger\psi) = DLL^\dagger\psi = D\Phi(H)\psi = \Phi(H')D\psi = 0.$$

□

5.3 Construction of Integrals

Suppose $H = H_1 + H_2$ is a two-dimensional Cartesian-separable Hamiltonian and H_1, H_2 both have ladder operators L_1, L_2 of orders n_1, n_2 both with step-size λ . We assume suitable translations of q_i, V_i and take the ladder operators to be monic so that:

$$L_i = p_i^{n_i} - i\lambda q_i p_i^{n_i-1} - \left(\frac{1}{2}\lambda^2 q_i^2 - n_i V_i\right) p_i^{n_i-2} + O\left(p_i^{n_i-3}\right), \quad i = 1, 2.$$

and similarly for L' . We have the following integrals of H :

$$\begin{aligned} A &= H_1 - H_2 = \frac{1}{2}(p_1^2 - p_2^2) + V_1 - V_2 \\ B &= i(L_1 L_2^\dagger - L_1^\dagger L_2) = 2\lambda(q_1 p_2 - q_2 p_1) p_1^{n_1-1} p_2^{n_2-1} + \dots \\ C &= -2\lambda(L_1 L_2^\dagger + L_1^\dagger L_2) \\ &= -4\lambda p_1^{n_1} p_2^{n_2} + 4\lambda \left[\frac{1}{2}\lambda^2 (q_1 p_2 - q_2 p_1)^2 - n_1 V_1 p_2^2 - n_2 V_2 p_1^2 \right] p_1^{n_1-2} p_2^{n_2-2} + \dots \end{aligned}$$

where the ellipses signifies terms of lower order in the momenta. So A is second-order, B is of order $n_1 + n_2 - 1$ and C is of order $n_1 + n_2$. If n_1, n_2 are both even, we can decrease the order of C by taking

$$X := C + 2^{\frac{n_1+n_2}{2}+1} \lambda H_1^{\frac{n_1}{2}} H_2^{\frac{n_2}{2}} = \lambda^3 (q_1 p_2 - q_2 p_1)^2 p_1^{n_1-2} p_2^{n_2-2} + \dots$$

which is of order $n_1 + n_2 - 2$. Thus, we can readily determine the following superintegrable systems. For third-order we have only one case

$$\begin{aligned} 1. & \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2 q_1^2 + \omega \mathcal{A}_3 \left(q_2 \sqrt{\omega} \right) \\ & \bullet \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2 \left(q_1^2 + \frac{1}{9}q_2^2 \right) \end{aligned}$$

which was found by Gravel [29]. The 1:3 oscillator is clearly not a standard potential but just a rational subcase of an exotic potential. For fourth-order there are

$$\begin{aligned} 1. & \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2 q_1^2 + \omega \mathcal{A}_4 \left(q_2 \sqrt{\omega} \right) \\ & \bullet \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2 \left(q_1^2 + \frac{1}{16}q_2^2 \right) + \frac{\alpha}{q_2^2} \end{aligned}$$

$$2. \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{8} \omega^2 q_1^2 + \frac{\alpha}{q_1^2} + \omega \mathcal{A}_3 (q_2 \sqrt{\omega})$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 \left(\frac{1}{4} q_1^2 + \frac{1}{9} q_2^2 \right) + \frac{\alpha}{q_1^2}$$

$$3. \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{8} \omega^2 q_1^2 + \frac{\alpha}{q_1^2} + \omega \mathcal{A}_4 (q_2 \sqrt{\omega})$$

all of which were determined by Marquette *et al.* [31]. The only exotic systems that we have missed are those of degenerate type. At fifth-order

$$1. \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 q_1^2 + \omega \mathcal{A}_5 (q_2 \sqrt{\omega})$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 \left(q_1^2 + \frac{1}{25} q_2^2 \right)$$

$$2. \frac{1}{2} (p_1^2 + p_2^2) + \omega \mathcal{A}_3 (q_1 \sqrt{\omega}) \pm \omega \mathcal{A}_3 (\pm q_2 \sqrt{\omega})$$

The latter was calculated directly by Abouamal *et al.* [33]. For sixth order, we have

$$1. \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{8} \omega^2 q_1^2 + \frac{\alpha}{q_1^2} + \omega \mathcal{A}_5 (q_2 \sqrt{\omega})$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 \left(\frac{1}{4} q_1^2 + \frac{1}{25} q_2^2 \right) + \frac{\alpha}{q_1^2}$$

$$2. \frac{1}{2} (p_1^2 + p_2^2) + \omega \mathcal{A}_4 (q_1 \sqrt{\omega}) \pm \omega \mathcal{A}_3 (\pm q_2 \sqrt{\omega})$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 \left(\frac{1}{16} q_1^2 + \frac{1}{9} q_2^2 \right) + \frac{\alpha}{q_1^2}$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{32} \omega^2 q_1^2 + \frac{\alpha}{q_1^2} + \omega \mathcal{A}_3 (q_2 \sqrt{\omega})$$

$$3. \frac{1}{2} (p_1^2 + p_2^2) + \omega \mathcal{A}_4 (q_1 \sqrt{\omega}) \pm \omega \mathcal{A}_4 (\pm q_2 \sqrt{\omega})$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{32} \omega^2 q_1^2 + \frac{\alpha}{q_1^2} + \omega \mathcal{A}_4 (q_2 \sqrt{\omega})$$

$$4. \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 q_1^2 + \omega \mathcal{A}_6 (q_2 \sqrt{\omega})$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 \left(q_1^2 + \frac{1}{36} q_2^2 \right) + \frac{\alpha}{q_2^2}$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{2} \omega^2 q_1^2 + \frac{1}{2} \omega \mathcal{A}_3 \left(q_2 \sqrt{\frac{\omega}{2}} \right)$$

$$5. \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{8} \omega^2 q_1^2 + \frac{\alpha}{q_1^2} + \omega \mathcal{A}_6 (q_2 \sqrt{\omega})$$

$$\bullet \frac{1}{2} (p_1^2 + p_2^2) + \frac{1}{8} \omega^2 q_1^2 + \frac{\alpha}{q_1^2} + \frac{1}{2} \omega \mathcal{A}_3 \left(q_2 \sqrt{\frac{\omega}{2}} \right)$$

This procedure however does not explain the presence of superintegrable systems with leading order terms that are of higher order in the angular momenta. In the third- and fourth-order case, these were the standard potentials.

Our resolution to this problem is thus. Suppose that we had another pair of ladder operators K_1, K_2 of the same order as L_1, L_2 respectively, but of a step-size $\kappa \neq \pm\lambda$. We further suppose that

$$K_i = p_i^{n_i} - i\mu q_i p_i^{n_i-1} + \dots$$

Then

$$\begin{aligned} Y &:= \kappa B - i\lambda[K_1 K_2^\dagger - K_1^\dagger K_2] \\ &= \frac{1}{3}\kappa\lambda(\kappa^2 - \lambda^2)(q_1 p_2 - q_2 p_1)^3 p_1^{n_1-3} p_2^{n_2-3} + \alpha_1(q_1 p_2 - q_2 p_1) p_1^{n_1-1} p_2^{n_2-3} \\ &\quad + \alpha_2(q_1 p_2 - q_2 p_1) p_1^{n_1-3} p_2^{n_2-1} + \alpha_3 p_1^{n_1} p_2^{n_2-3} + \alpha_4 p_1^{n_1-3} p_2^{n_2} + \dots \end{aligned}$$

for some constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, will be of order $n_1 + n_2 - 3$. If n_1, n_2 are again even, by a similar but slightly more involved elimination, we can construct an integral of order $n_1 + n_2 - 4$, of the form

$$\begin{aligned} Z &:= \frac{1}{12}\kappa^3\lambda^3(\kappa^2 - \lambda^2)(q_1 p_2 - q_2 p_1)^4 p_1^{n_1-4} p_2^{m-4} + \frac{1}{2}(q_1 p_2 - q_2 p_1)^2(\alpha_1 p_1^{n_1-2} p_2^{m-4} + \alpha_2 p_1^{n_1-4} p_2^{m-2}) \\ &\quad + (q_1 p_2 - q_2 p_1)(\alpha_3 p_1^{n_1-1} p_2^{m-4} + \alpha_4 p_1^{n_1-4} p_2^{m-1}) + \dots \end{aligned}$$

Clearly, we cannot do this indefinitely, but are restricted in the number of possible reductions by $\min\{n_1, n_2\}$. We develop the following schema

Table 5.1. Possible orders of an integral constructible from ladder operators

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2		2	4	4	6	6	8	8	10	10
3			5/3	6	7/5	8/6	9/7	10/8	11/9	12/10
4				6/4	8	8/6	10	10/8	12	12/10
5					9/7/5	10/8	11/9/7	12/10	13/11/9	14/12/10
6						10/8/6	12/10	12/10/8	14/12	14/12/10
7							13/11/9/7	14/12	15/13/11/9	16/14/12
8								14/12/10/8	14/12	16/14/12/10
9									17/15/13/11/9	18/16/14
10										18/16/14/12/10

To construct this table, we use Theorem 5.3. This was only proven for systems asymptotically quadratic. We can only conjecture that it holds in more exotic cases. For example, a system with sixth-order ladder operators may have step-sizes one, two or three times a particular value, whereas a system with third-order ladder operators may only have one or three times. Taking $n_1 = 6, n_2 = 3$ we therefore have 6 or 8 as the possible orders.

5.4 Rational Models with Third-Order Ladder Operators

The simple harmonic oscillator

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$$

has, as already mentioned, a first-order ladder operator. This can be used to generate its eigenfunctions

$$\psi_n = \mathcal{H}_n \left(q \sqrt{\frac{\omega}{\hbar}} \right) \exp \left(-\frac{\omega q^2}{2\hbar} \right)$$

and spectrum

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right)$$

as discussed in any introductory quantum mechanics textbook. If we apply a Darboux transformations to this model we obtain odd-order ladder operators. In special cases, we may reduce these odd-order ladder operators to third-order.

The rational solutions of Painlevé IV are generated by two polynomial sequences [130, 131] called the generalised Hermite polynomials

$$\mathcal{H}_{m,n}(x) := \text{Wr}[\mathcal{H}_m(x), \mathcal{H}_{m+1}(x), \dots, \mathcal{H}_{m+n-1}(x)], \quad \mathcal{H}_{m,0}(x) := 1$$

and the generalised Okamoto polynomials

$$\mathcal{Q}_{m,n}(x) := \text{Wr}[\mathcal{H}_1(x), \mathcal{H}_4(x), \dots, \mathcal{H}_{3m-2}(x), \mathcal{H}_2(x), \mathcal{H}_5(x), \dots, \mathcal{H}_{3n-1}(x)], \quad \mathcal{Q}_{0,0}(x) := 1$$

Our definitions here differ from the usual ones given in the literature by a scaling factor of the polynomial and the variable.

Theorem 5.5. *The Hamiltonian*

$$H_{m,n;1} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - \hbar^2 \frac{d^2}{dq^2} \log \mathcal{H}_{m,n} \left(q \sqrt{\frac{\omega}{\hbar}} \right) + n\hbar\omega$$

has third-order ladder operators of step-size $\pm\omega$.

The proof of this theorem takes its inspiration from Bermudez [123]. Bermudez's argument has a gap however which we have salvaged by means of Theorem 5.4.

Proof. Denote the eigenfunctions by $\psi_k = \mathcal{H}_k \left(q \sqrt{\frac{\omega}{\hbar}} \right) \exp \left(-\frac{\omega q^2}{2\hbar} \right)$. We have

$$\text{Wr}[\psi_m, \psi_{m+1}, \dots, \psi_{m+n-1}] = \exp \left(-\frac{n\omega q^2}{2\hbar} \right) \mathcal{H}_{m,n} \left(q \sqrt{\frac{\omega}{\hbar}} \right)$$

so by the Crum formula $H_{m,n;1}$ is the Hamiltonian obtained from the simple harmonic oscillator by deleting the states $\psi_m, \psi_{m+1}, \dots, \psi_{m+n-1}$. Let

$$D_{m,n} := p - i\omega q + i\hbar \frac{d}{dq} \log \frac{\mathcal{H}_{m,n} \left(q \sqrt{\frac{\omega}{\hbar}} \right)}{\mathcal{H}_{m,n-1} \left(q \sqrt{\frac{\omega}{\hbar}} \right)} \quad (5.12)$$

Then

$$D_{m,n}H_{m,n-1;1} = H_{m,n;1}D_{m,n}$$

We define

$$\Lambda_{m,0} = (p - i\omega q) \left[H_{m,0;1} - \left(m + \frac{1}{2} \right) \hbar\omega \right]$$

Then $\Lambda_{m,0}$ is a third-order ladder operator of $H_{m,0}$, of step-size ω , and annihilates the state ψ_m . Now, suppose that $\Lambda_{m,n}$ for general n is of third-order and annihilates

$$D_{m,n}D_{m,n-1} \cdots D_{m,1}\psi_{m+n} \neq 0$$

By Theorem 5.4, there exists a ladder operator $\Lambda_{m,n+1}$ such that

$$D_{m,n+1}\Lambda_{m,n}D_{m,n+1}^\dagger = \Lambda_{m,n+1} \left[H_{m,n+1;1} - \left(m + n + \frac{1}{2} \right) \hbar\omega \right]$$

which completes the induction. □

We read off from (5.12)

$$\varpi = -\frac{\omega q}{\hbar} + \frac{d}{dq} \log \frac{\mathcal{H}_{m,n+1} \left(q\sqrt{\frac{\omega}{\hbar}} \right)}{\mathcal{H}_{m,n} \left(q\sqrt{\frac{\omega}{\hbar}} \right)}$$

and using (5.9), we have

$$y = -\frac{d}{dx} \log \frac{\mathcal{H}_{m,n+1}(x)}{\mathcal{H}_{m,n}(x)}$$

is a solution to the Painlevé IV equation. This forms part of the $-1/x$ and $-2x$ hierarchies so-called [132]. The third hierarchy makes use of the Okamoto polynomials.

Theorem 5.6. *The Hamiltonian*

$$H_{m,n,3} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2 - \hbar^2 \frac{d^2}{dq^2} \log \mathcal{Q}_{m,n} \left(q\sqrt{\frac{\omega}{\hbar}} \right) + (m+n)\hbar\omega$$

has third-order ladder operators of step-size $\pm 3\omega$.

Proof. Clearly, $H_{m,n,3}$ is the Hamiltonian attained by deleting the eigenfunctions

$$\psi_1, \psi_4, \dots, \psi_{3m-2}, \psi_2, \psi_5, \dots, \psi_{3n-1}.$$

Now,

$$\Lambda_{0,0} := (p - i\omega q)^3$$

is third-order, is a ladder operator of $H_{0,0,3}$ with step-size 3ω , and annihilates ψ_1, ψ_2 . Set

$$D_{m,n,1} := p - i\omega q + i\hbar \frac{d}{dq} \log \frac{\mathcal{Q}_{m,n} \left(q\sqrt{\frac{\omega}{\hbar}} \right)}{\mathcal{Q}_{m-1,n} \left(q\sqrt{\frac{\omega}{\hbar}} \right)}$$

$$D_{m,n,2} := p - i\omega q + i\hbar \frac{d}{dq} \log \frac{\mathcal{Q}_{m,n} \left(q\sqrt{\frac{\omega}{\hbar}} \right)}{\mathcal{Q}_{m,n-1} \left(q\sqrt{\frac{\omega}{\hbar}} \right)}$$

Then $D_{m,n,1}H_{m-1,n;3} = H_{m,n;3}D_{m,n,1}$, $D_{m,n,2}H_{m,n-1;3} = H_{m,n;3}D_{m,n,2}$. The eigenfunctions of $H_{m,n,3}$ are

$$\psi_{m,n,k} = \frac{\text{Wr} \left[\left\{ \mathcal{H}_{3k-2} \left(q\sqrt{\frac{\omega}{\hbar}} \right), \mathcal{H}_{3\ell-1} \left(q\sqrt{\frac{\omega}{\hbar}} \right) \right\}_{k,\ell=1}^{m,n}, \mathcal{H}_k \left(q\sqrt{\frac{\omega}{\hbar}} \right) \right]}{\mathcal{Q}_{m,n} \left(q\sqrt{\frac{\omega}{\hbar}} \right)} \exp \left(-\frac{\omega q^2}{\hbar} \right)$$

Now suppose that $\Lambda_{m,n}$ is a third-order ladder operator of $H_{m,n,3}$ with step-size 3ω and annihilates the states $\psi_{m,n,3m+1}$ and $\psi_{m,n,3n+2}$. By Theorem 5.4, there exists ladder operators $\Lambda_{m+1,n}$, $\Lambda_{m,n+1}$ such that

$$D_{m+1,n,1}\Lambda_{m,n}D_{m+1,n,1}^\dagger = \Lambda_{m+1,n} \left[H_{m+1,n,3} - 3 \left(m + \frac{1}{2} \right) \hbar\omega \right]$$

and

$$D_{m,n+1,2}\Lambda_{m,n}D_{m,n+1,2}^\dagger = \Lambda_{m,n+1} \left[H_{m,n+1,3} - \left(3n + \frac{5}{2} \right) \hbar\omega \right].$$

Now, $\Lambda_{m+1,n}$ annihilate the states

$$\begin{aligned} D_{m+1,n,1}\psi_{m,n,3m+4} &\propto \psi_{m+1,n,3(m+1)+1}, \\ D_{m+1,n,1}\psi_{m,n,3n+2} &\propto \psi_{m+1,n,3n+2} \end{aligned}$$

while $\Lambda_{m,n+1}$ annihilates the states

$$\begin{aligned} D_{m,n+1,2}\psi_{m,n,3m+1} &\propto \psi_{m,n+1,3m+1} \\ D_{m,n+1,2}\psi_{m,n,3n+5} &\propto \psi_{m,n+1,3(n+1)+2} \end{aligned}$$

This completes the induction. □

As a result of the theorems above, the Hamiltonians

$$\begin{aligned} H_{m,0,1} &= H_{0,0,3} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2 \\ H_{1,1,1} &= H_{1,0,3} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2 + \frac{\hbar^2}{q^2} + \hbar\omega \\ H_{1,2,1} &= H_{1,1,3} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2 + \frac{4\hbar^2\omega(2\omega q^2 - \hbar)}{(2\omega q^2 + \hbar)^2} + 2\hbar\omega \\ H_{2,1,1} &= H_{0,1,3} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2 + \frac{4\hbar^2\omega(2\omega q^2 + \hbar)}{(2\omega q^2 - \hbar)^2} + \hbar\omega \end{aligned}$$

have third-order ladder operators of step-sizes $\pm\omega$ and $\pm 3\omega$. To construct all of the standard non-degenerate superintegrable models at third-order in two-dimensions is simply to take combinations of any of the above three one-dimensional models. We therefore have six third-order superintegrable models, viz.

$$\begin{array}{lll} H_{1,0,1;1} + H_{1,1,1;2}, & H_{1,0,1;1} + H_{1,2,1;2}, & H_{1,1,1;1} + H_{1,1,1;2}, \\ H_{1,1,1;1} + H_{1,2,1;2}, & H_{1,2,1;1} + H_{1,2,1;2}, & H_{1,2,1;1} + H_{2,1,1;2} \end{array}$$

where, for example,

$$\begin{aligned} H_{1,2,1;1} + H_{1,2,1;2} &= \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2) + \frac{4\hbar^2\omega(2\omega q_1^2 - \hbar)}{(2\omega q_1^2 + \hbar)} \\ &\quad + \frac{4\hbar^2\omega(2\omega q_2^2 - \hbar)}{(2\omega q_2^2 + \hbar)} + 4\hbar\omega. \end{aligned}$$

This agrees with Gravel's results [29] and our own.

5.5 Rational Models with Fourth-Order Ladder Operators

It was shown by Adler [133] that the fourth-order dressing chain can be transformed, by a non-linear substitution, into the Painlevé V equation. The rational solutions can be written in terms of Laguerre polynomials, on account of

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 - \hbar^2}{8q^2}$$

which has second-order lowering and raising operators and eigenfunctions given by

$$\psi_{n,\pm}(q) = \mathcal{L}_n^{\left(\pm\frac{\mu}{\hbar}\right)}\left(\frac{\omega q^2}{\hbar}\right) \exp\left(-\frac{\omega q^2}{2\hbar}\right).$$

The two polynomial sequences that are of interest are the generalised Laguerre polynomials [134]

$$\mathcal{L}_{m,n}^{(\alpha)}(x) := \text{Wr} \left[\mathcal{L}_m^{(\alpha)}(x), \mathcal{L}_{m+1}^{(\alpha)}(x), \dots, \mathcal{L}_{m+n-1}^{(\alpha)}(x) \right], \quad \mathcal{L}_{m,0}^{(\alpha)}(x) = 1$$

and the generalised Umemura polynomials [135]

$$\mathcal{U}_{m,n}^{(\alpha)}(x) := x^{\alpha n} \text{Wr} \left\{ \mathcal{L}_{2k-1}^{(\alpha)}(x), x^{-\alpha} \mathcal{L}_{2\ell-1}^{(-\alpha)}(x) \right\}_{k,\ell=1}^{m,n}, \quad \mathcal{U}_{0,0}^{(\alpha)}(x) := 1$$

The latter are, in general, Laurent polynomials.

Theorem 5.7. *The Hamiltonian*

$$H_{m,n,2} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4(\mu + n\hbar)^2 - \hbar^2(2n-1)^2}{8q^2} - \hbar^2 \frac{d^2}{dq^2} \log \mathcal{L}_{m,n}^{\left(\frac{\mu}{\hbar}\right)}\left(\frac{\omega q^2}{\hbar}\right) + n\hbar\omega$$

has fourth-order ladder operators of step-size $\pm 2\omega$.

Proof. The proof is analogous to that of Theorem 5.5. □

Theorem 5.8. *The Hamiltonian*

$$\begin{aligned} H_{m,n,4} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4[\mu + (m-n)\hbar]^2 - \hbar^2}{8q^2} - \frac{\hbar^2}{q^2} \left[\binom{m}{2} + \binom{n}{2} - mn \right] \\ &\quad - \hbar^2 \frac{d^2}{dq^2} \log \mathcal{U}_{m,n}^{\left(\frac{\mu}{\hbar}\right)}\left(\frac{\omega q^2}{\hbar}\right) + (m+n)\hbar\omega \end{aligned}$$

has fourth-order ladder operators of step-size $\pm 4\omega$.

Proof. The proof is analogous to that of Theorem 5.6, starting from the fact that the square of the second-order ladder operator of $H_{0,0,4}$ annihilates $\mathcal{L}_1^{\left(\pm\frac{\mu}{\hbar}\right)}\left(\frac{\omega q^2}{\hbar}\right)$. \square

We now have

$$H_{1,0,2} = H_{0,0,4} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 - \hbar^2}{8q^2}$$

$$H_{1,1,2} = H_{1,0,4} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4(\mu + \hbar)^2 - \hbar^2}{8q^2} + \frac{2\hbar^2\omega(q^2 + \mu + \hbar)}{(\omega q^2 - \mu - \hbar)^2} + \hbar\omega$$

have fourth-order ladder operators of step-size $\pm 2\omega$ and $\pm 4\omega$. We therefore have two non-degenerate standard fourth-order superintegrable systems

$$H_{1,0,2;1} + H_{1,1,2;2}, \quad H_{1,1,2;1} + H_{1,1,2;2}$$

where, for example,

$$H_{1,0,2;1} + H_{1,1,2;2} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2) + \frac{4\mu_1^2 - \hbar^2}{8q_1^2}$$

$$+ \frac{4(\mu_2 + \hbar)^2 - \hbar^2}{8q_2^2} + \frac{2\hbar^2\omega(q_2^2 + \mu_2 + \hbar)}{(\omega q_2^2 - \mu_2 - \hbar)^2} + \hbar\omega$$

This agrees with our previous calculations.

5.6 Rational Models with Fifth- or Sixth-Order Ladder Operators

The extension of Theorem 5.6 and Theorem 5.8 to fifth- and sixth-order ladder operators with step-sizes $\pm 5\omega$ and $\pm 6\omega$ is straightforward. For example, it is easily seen that the models

$$H_{k,\ell,m,n,5} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 - \hbar^2 \frac{d^2}{dq^2} \log \mathcal{Q}_{k,\ell,m,n} \left(q \sqrt{\frac{\omega}{\hbar}} \right) + (k + \ell + m + n)\hbar\omega$$

and

$$H_{k,\ell,m,n,6} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4[\mu + (k + \ell - m - n)\hbar]^2 - \hbar^2}{8q^2}$$

$$+ \frac{\hbar^2}{2q^2} [k + \ell + m + n - (k + \ell - m - n)^2]$$

$$- \hbar^2 \frac{d^2}{dq^2} \log \mathcal{U}_{k,\ell,m,n}^{\left(\frac{\mu}{\hbar}\right)} \left(\frac{\omega q^2}{\hbar} \right) + (k + \ell + m + n)\hbar\omega$$

have ladder operators of fifth and sixth order with step-sizes $\pm 5\omega$ and $\pm 6\omega$ respectively, where $\mathcal{Q}_{k,\ell,m,n}$ and $\mathcal{U}_{k,\ell,m,n}^{\left(\frac{\mu}{\hbar}\right)}$ are the 5-Okamoto and 3-Umemura polynomials

$$\mathcal{Q}_{k,\ell,m,n}(x) := \text{Wr} \left\{ \mathcal{H}_{5a-4}(x), \mathcal{H}_{5b-3}(x), \mathcal{H}_{5c-2}(x), \mathcal{H}_{5d-1}(x) \right\}_{a,b,c,d=1}^{k,\ell,m,n}$$

$$\mathcal{U}_{k,\ell,m,n}^{(\alpha)}(x) := x^{\alpha(m+n)} \text{Wr} \left\{ \mathcal{L}_{3a-2}^{(\alpha)}(x), \mathcal{L}_{3b-1}^{(\alpha)}(x), x^{-\alpha} \mathcal{L}_{3c-2}^{(\alpha)}(x), x^{-\alpha} \mathcal{L}_{3d-1}^{(\alpha)}(x) \right\}_{a,b,c,d=1}^{k,\ell,m,n}$$

To shorten the case-checking that is required to ascertain which models are standard, we shall borrow from the algebraic approach employed in Chapter 4. We take as our *Ansatz*

$$\begin{aligned} V(q) &= \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 - \hbar^2}{8q^2} - \hbar^2 \frac{d^2}{dq^2} \log(q^2 - \alpha)(q^2 - \beta) \\ &= \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 - \hbar^2}{8q^2} + 2\hbar^2 \left[\frac{q^2 + \alpha}{(q^2 - \alpha)^2} + \frac{q^2 + \beta}{(q^2 - \beta)^2} \right] \end{aligned}$$

Then we determine α, β by eliminating the simple poles of V^2 . We have

$$\begin{aligned} \operatorname{Res}_{q=\pm\sqrt{\alpha}} V(q)^2 &= \frac{\pm 2\hbar^2}{\alpha^{3/2}(\alpha - \beta)} (\alpha^5 \omega^2 - 3\alpha^4 \beta \omega^2 + 3\alpha^3 \beta^2 \omega^2 - \alpha^3 \mu^2 - 4\alpha^3 \hbar^2 \\ &\quad - \alpha^2 \beta^3 \omega^2 + 3\alpha^2 \beta \mu^2 - 12\alpha^2 \beta \hbar^2 - 3\alpha \beta^2 \mu^2 + \beta^3 \mu^2) \\ \operatorname{Res}_{q=\pm\sqrt{\beta}} V(q)^2 &= \frac{\mp 2\hbar^2}{\beta^{3/2}(\alpha - \beta)} (\beta^5 \omega^2 - 3\alpha \beta^4 \omega^2 + 3\alpha^2 \beta^3 \omega^2 - \beta^3 \mu^2 - 4\beta^3 \hbar^2 \\ &\quad - \alpha^3 \beta^2 \omega^2 + 3\alpha \beta^2 \mu^2 - 12\alpha \beta^2 \hbar^2 - 3\alpha^2 \beta \mu^2 + \alpha^3 \mu^2) \end{aligned}$$

After eliminating β , we find that the resultant admits the factorisation

$$\begin{aligned} &32768\hbar^6 \alpha^9 (\alpha^2 \omega^2 + 2\alpha \mu \omega + \mu^2 - \mu \hbar - 2\alpha \omega \hbar) (\alpha^2 \omega^2 - 2\alpha \mu \omega + \mu^2 + \mu \hbar - 2\alpha \omega \hbar) \\ &\quad \times (\alpha^2 \omega^2 - 2\alpha \mu \omega + \mu^2 - \mu \hbar + 2\alpha \omega \hbar) (\alpha^2 \omega^2 + 2\alpha \mu \omega + \mu^2 + \mu \hbar + 2\alpha \omega \hbar) (\alpha^2 \omega^2 - \mu^2 + \hbar^2) \end{aligned}$$

So there are two solutions, up to permutation of α, β and of alternation of sign in ω, μ :

- (i) $\omega\alpha = \sqrt{\mu^2 - \hbar^2}, \omega\beta = -\sqrt{\mu^2 - \hbar^2}$;
- (ii) $\omega\alpha = \mu + \hbar + \sqrt{\hbar(\mu + \hbar)}, \omega\beta = \mu + \hbar - \sqrt{\hbar(\mu + \hbar)}$.

Thus,

$$\begin{aligned} H &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 - \hbar^2}{8q^2} + \frac{4\hbar^2 \omega^2 q^2 (\omega^2 q^4 + 3\mu^2 - 3\hbar^2)}{(\omega^2 q^4 - \mu^2 + \hbar^2)^2} \\ &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 + 7\hbar^2}{8q^2} - \hbar^2 \frac{d^2}{dq^2} \log \mathcal{U}_{1,1}^{\left(\frac{\mu}{\hbar}\right)} \left(\frac{\omega q^2}{\hbar} \right) \end{aligned} \quad (5.13)$$

or

$$\begin{aligned} H &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 - \hbar^2}{8q^2} \\ &\quad + \frac{4\hbar^2 \omega [\omega^3 q^6 - \omega^2 (\mu + \hbar) q^4 - \omega (\mu + \hbar) (\mu - 2\hbar) q^2 + \mu (\mu + \hbar)^2]}{[\omega^2 q^4 - 2\omega (\mu + \hbar) q^2 + \mu (\mu + \hbar)]^2} \\ &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 - \hbar^2}{8q^2} - \hbar^2 \frac{d^2}{dq^2} \log \mathcal{L}_2^{\left(\frac{\mu}{\hbar}-1\right)} \left(\frac{\omega q^2}{\hbar} \right) \end{aligned} \quad (5.14)$$

are our only possibilities. On the other hand, the ladder operators of $H_{1,1,4}$ of step-sizes $\pm 2\omega, \pm 4\omega, \pm 6\omega$ are of orders six, four and six, while those of $H_{2,1,2}$ are of orders four, six and six. So these are indeed the desired solutions. Similarly, the ladder operators of $H_{1,0,2}, H_{1,1,2}$ which are of step-size $\pm 6\omega$ are

both of sixth-order. Thus, there are seven standard non-degenerate sixth-order models:

$$\begin{array}{lll} H_{1,0,2;1} + H_{1,1,4;2}, & H_{1,0,2;1} + H_{2,1,2;2}, & H_{1,1,2;1} + H_{1,1,4;2}, \\ H_{1,1,2;1} + H_{2,1,2;2}, & H_{1,1,4;1} + H_{1,1,4;2}, & H_{1,1,4;1} + H_{2,1,2;2}, \\ H_{2,1,2;1} + H_{2,1,2;2}. & & \end{array}$$

To find the ones which admit odd-order ladder operators, it is sufficient to check when the coefficient of the q^{-2} -term is 0, \hbar^2 , $3\hbar^2$ (see Theorem 6.2 below). The possibilities are, in addition to $H_{1,0,1}$, $H_{1,1,1}$, $H_{1,2,1}$, $H_{2,1,1}$,

$$\begin{aligned} H_{1,0,1,0,5} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{3\hbar^2}{q^2} + 2\hbar\omega \\ H_{1,3,1} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{\hbar^2}{q^2} + \frac{4\hbar^2\omega(2\omega q^2 - 3\hbar)}{(2\omega q^2 + 3\hbar)^2} + 3\hbar\omega \\ H_{3,1,1} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{\hbar^2}{q^2} + \frac{4\hbar^2\omega(2\omega q^2 + 3\hbar)}{(2\omega q^2 - 3\hbar)^2} + \hbar\omega \\ H_{2,2,1} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{16\hbar^2\omega^2 q^2 (4\omega^2 q^4 - 9\hbar^2)}{(4\omega^2 q^4 + 3\hbar^2)^2} + 2\hbar\omega \\ H_{1,4,1} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{8\hbar^2\omega(8\omega^3 q^6 + 12\hbar\omega^2 q^4 + 18\hbar^2\omega q^2 - 9\hbar^3)}{(4\omega^2 q^4 - 12\hbar\omega q^2 + 3\hbar^2)^2} + 4\hbar\omega \\ H_{2,0,3} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{8\hbar^2\omega(8\omega^3 q^6 - 4\hbar\omega^2 q^4 + 10\hbar^2\omega q^2 - \hbar^3)}{(4\omega^2 q^4 - 4\hbar\omega q^2 - \hbar^2)^2} + 2\hbar\omega \\ H_{1,0,1,1,5} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{\hbar^2}{q^2} + \frac{8\hbar^2\omega(8\omega^3 q^6 + 4\hbar\omega^2 q^4 - 14\hbar^2\omega q^2 - 3\hbar^3)}{(4\omega^2 q^4 + 4\hbar\omega q^2 + 3\hbar^2)^2} + 3\hbar\omega \\ H_{0,1,0,1,5} &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{\hbar^2}{q^2} + \frac{8\hbar^2\omega(8\omega^3 q^6 - 4\hbar\omega^2 q^4 - 14\hbar^2\omega q^2 + 3\hbar^3)}{(4\omega^2 q^4 - 4\hbar\omega q^2 + 3\hbar^2)^2} + 2\hbar\omega \end{aligned}$$

There are correspondingly fifty-three standard non-degenerate fifth-order Cartesian-separable super-integrable Hamiltonians in two dimensions:

$$\begin{array}{lll} H_{1,0,1;1} + H_{1,0,1,0,5;2}, & H_{1,0,1;1} + H_{1,3,1;2}, & H_{1,0,1;1} + H_{2,2,1;2}, \\ H_{1,0,1;1} + H_{1,4,1;2}, & H_{1,0,1;1} + H_{2,0,3;2}, & H_{1,0,1;1} + H_{1,0,1,1,5;2}, \\ H_{1,1,1;1} + H_{1,0,1,0,5;2}, & H_{1,1,1;1} + H_{1,3,1;2}, & H_{1,1,1;1} + H_{2,2,1;2}, \\ H_{1,1,1;1} + H_{1,4,1;2}, & H_{1,1,1;1} + H_{2,0,3;2}, & H_{1,1,1;1} + H_{1,0,1,1,5;2}, \\ H_{1,0,1,0,5;1} + H_{1,0,1,0,5;2}, & H_{1,0,1,0,5;1} + H_{1,3,1;2}, & H_{1,0,1,0,5;1} + H_{2,2,1;2}, \\ H_{1,0,1,0,5;1} + H_{1,4,1;2}, & H_{1,0,1,0,5;1} + H_{2,0,3;2}, & H_{1,0,1,0,5;1} + H_{1,0,1,1,5;2}, \\ H_{1,2,1;1} + H_{1,0,1,0,5;2}, & H_{1,2,1;1} + H_{1,3,1;2}, & H_{1,2,1;1} + H_{2,2,1;2}, \\ H_{1,2,1;1} + H_{1,4,1;2}, & H_{1,2,1;1} + H_{2,0,3;2}, & H_{1,2,1;1} + H_{1,0,1,1,5;2}, \end{array}$$

$$\begin{array}{lll}
H_{2,1,1;1} + H_{1,3,1;2}, & H_{2,1,1;1} + H_{1,4,1;2}, & H_{2,1,1;1} + H_{2,0,3;2}, \\
H_{2,1,1;1} + H_{1,0,1,1,5;2}, & H_{1,3,1;1} + H_{1,3,1;2}, & H_{1,3,1;1} + H_{3,1,1;2}, \\
H_{1,3,1;1} + H_{2,2,1;2}, & H_{1,3,1;1} + H_{1,4,1;2}, & H_{1,3,1;1} + H_{2,0,3;2}, \\
H_{1,3,1;1} + H_{1,0,1,1,5;2}, & H_{3,1,1;1} + H_{1,4,1;2}, & H_{3,1,1;1} + H_{2,0,3;2}, \\
H_{3,1,1;1} + H_{1,0,1,1,5;2}, & H_{2,2,1;1} + H_{2,2,1;2}, & H_{2,2,1;1} + H_{1,4,1;2}, \\
H_{2,2,1;1} + H_{2,0,3;2}, & H_{2,2,1;1} + H_{1,0,1,1,5;2}, & H_{1,4,1;1} + H_{1,4,1;2}, \\
H_{1,4,1;1} + H_{1,4,1;2}^-, & H_{1,4,1;1} + H_{2,0,3;2}, & H_{1,4,1;1} + H_{2,0,3;2}^-, \\
H_{1,4,1;1} + H_{1,0,1,1,5;2}, & H_{1,4,1;1} + H_{0,1,0,1,5;2}, & H_{2,0,3;1} + H_{2,0,3;2}, \\
H_{2,0,3;1} + H_{2,0,3;2}^-, & H_{2,0,3;1} + H_{1,0,1,1,5;2}, & H_{2,0,3;1} + H_{0,1,0,1,5;2}, \\
H_{1,0,1,1,5;1} + H_{1,0,1,1,5;2}, & H_{1,0,1,1,5;1} + H_{0,1,0,1,5;2} &
\end{array}$$

where H^- indicates $\omega \mapsto -\omega$.

5.7 Conclusion

We have shown that the problem of Cartesian-separable superintegrability in the non-degenerate case can be understood within a one-dimensional operator algebra framework. We have been able to obtain the previous results of Chapter 4 as well as extend them to higher orders. However, unlike Chapter 4, we are unable to provide proofs of completeness. Thus, this approach, while promising, still remains at the conjectural stage.

Chapter 6

Fine Structure of the Singular Oscillator

6.1 Introduction

In this chapter, we examine in detail how the existence of multiple ladder operators gives rise to new symmetry algebras. Multiple ladder operator pairs has been noticed by several authors [136, 137] however there has not been a consideration as to how they form a complete algebraic system. Rather they have been taken as repeated instantiations of separate polynomial Heisenberg algebras. As can be seen in the appendix, systems whose potential summands are Darboux transforms of the free particle or oscillator possess integrals which exceed the classical limit on the number of generators (i.e. twice the dimension less one). Therefore the previous attempts at describing the symmetry algebras of these systems are incomplete [138, 139]. We will not address every such system but confine our attention to just the singular oscillator Hamiltonian

$$H[\mu] := \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{4\mu^2 - \hbar^2}{8q^2} + \omega\mu$$

The parametrisation chosen here is that which is most convenient for expressing the eigenstates. It will be assumed that $\omega > 0$ and $\mu \geq 0$. While generally the ladder operator is of second-order, we know that for $\mu = \frac{1}{2}\hbar$, H is the simple harmonic oscillator which has a first-order ladder operator. In this case, the first-order ladder operator generates the second-order one, however there are additional values of μ (so that the model is purely quantum) where there are additional operators which cannot be totally reduced in this way.

This chapter is organised in the following way. In section 6.2, we discuss the basic spectral facts about the singular oscillator. In section 6.3 we introduce shifting operators that translate the parameter μ which we use to build new ladder operators for particular values of μ . This procedure is well understood. We extend the known facts by showing the effects of these ladder operators on the eigenstates of the Hamiltonian. Furthermore we give a proof that the set of ladder operators is complete and of minimal order. In section 6.4, we apply these observations to the two-dimensional isotropic oscillator.

6.2 Eigenstates, Spectrum and Ladder Operators

The second-order lowering operator can be written down as

$$\Delta = H - i\omega\{q, p\} - \omega^2 q^2 - \omega\mu \quad (6.1)$$

The operator has step-size 2ω . The characteristic polynomial is

$$\Delta^\dagger \Delta = (H - \hbar\omega)[H - \omega(\hbar + 2\mu)]. \quad (6.2)$$

So there are two ground states. Noting that $H[-\mu] = H[\mu] - 2\omega\mu$, we may without loss of generality, confine our attention to the state with energy $\omega(\hbar + 2\mu)$ for now. The eigenfunctions take the form

$$\psi_{n,\mu}(q) = q^{\frac{1}{2} + \frac{\mu}{\hbar}} \mathcal{L}_n^{\left(\frac{\mu}{\hbar}\right)} \left(\frac{\omega q^2}{\hbar} \right) \exp\left(-\frac{\omega q^2}{2\hbar}\right)$$

The matrix elements of the eigenfunctions are calculated to be

$$\begin{aligned} H\psi_{n,\mu} &= \omega[\hbar(2n+1) + 2\mu]\psi_{n,\mu}, \\ \Delta\psi_{n,\mu} &= 2\omega(\mu + n\hbar)\psi_{n-1,\mu} \\ \Delta^\dagger\psi_{n,\mu} &= 2\hbar\omega(n+1)\psi_{n+1,\mu} \end{aligned}$$

There is the orthogonality relation

$$2 \left(\frac{\omega}{\hbar} \right)^{\frac{\mu}{\hbar} + 1} \int_0^\infty \psi_{m,\mu}^*(q) \psi_{n,\mu}(q) q^{\frac{\mu}{\hbar} + 1} \exp\left(-\frac{\omega q^2}{\hbar}\right) dq = \Gamma\left(1 + \frac{\mu}{\hbar}\right) \binom{n + \frac{\mu}{\hbar}}{n} \delta_{m,n} \quad (6.3)$$

for $\omega > 0, \mu > -\hbar$. For $\mu < -\hbar$, μ/\hbar not an integer, the orthogonality relation can be extended by analytic continuation. However, the corresponding inner product will not be positive definite. In fact,

$$\text{sgn}\langle m|n\rangle = \text{sgn}\Gamma\left(1 + \frac{\mu}{\hbar} + n\right) \delta_{m,n}.$$

The generalised integration of Dereziński, Gaß and Vättö [140] achieves the same effect as our inner product in this case. When $\mu = -k\hbar$, k a positive integer, the dynamic algebra generated by $H, \Delta, \Delta^\dagger$ admits a finite-dimensional irreducible representation. However, this cannot be the Schrödinger realisation, since on the one hand

$$\Delta\Delta^\dagger\psi_{k-1,\mu} = 0$$

but on the other

$$\Delta^\dagger\psi_{k-1,\mu} \neq 0.$$

In this case, we have $\langle n|n\rangle = 0$ for $n \geq k$.

6.3 Additional Ladder Operators

Using the ground state eigenfunction, we construct the Darboux transformation

$$D[\mu] = \frac{1}{\sqrt{2}} \left[p - i\omega q + \frac{i(\hbar + 2\mu)}{2q} \right] \quad (6.4)$$

This transformation is a shifting operator on μ

$$D[\mu]H[\mu] = H[\mu + \hbar]D[\mu] \quad (6.5)$$

We deduce the identities

$$\begin{aligned} D^\dagger[\mu]D[\mu] &= H[\mu] - \omega(\hbar + 2\mu), \\ D[\mu]D^\dagger[\mu] &= H[\mu + \hbar] - \omega(\hbar + 2\mu), \\ D[-\mu]H[\mu] &= \{H[\mu - \hbar] + 2\hbar\omega\}D[-\mu] \\ D^\dagger[-\mu]D[-\mu] &= H[\mu] - \hbar\omega, \\ D[-\mu]D^\dagger[-\mu] &= H[\mu - \hbar] + \hbar\omega, \\ D[-\mu - \hbar]D[\mu] &= \Delta[\mu^2], \\ D[\mu]\Delta[\mu^2] &= \Delta[(\mu + \hbar)^2]D[\mu] \\ D^\dagger[-\mu - 2\hbar]\Delta[\mu] &= D[\mu + \hbar]D^\dagger[-\mu - \hbar] \end{aligned}$$

The shifting operator can be used to construct new ladder operators

$$\Lambda^{(i)} = \prod_{k=1-j}^{i-1} D\left[-\left(k + \frac{1}{2}\right)\hbar\right] \prod_{k=i}^{j-1} D^\dagger\left[\left(k - \frac{1}{2}\right)\hbar\right] \quad (6.6)$$

for $i = 1, 2, \dots, j$ which act on the special Hamiltonian

$$H\left[\hbar\left(j - \frac{1}{2}\right)\right] = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2 + \frac{\hbar^2}{q^2}\binom{j}{2} + \hbar\omega\left(j - \frac{1}{2}\right).$$

We have

$$\begin{aligned} \Lambda^{(i)}H\left[\left(j - \frac{1}{2}\right)\hbar\right] &= \prod_{k=1-j}^{i-1} D\left[-\left(k + \frac{1}{2}\right)\hbar\right] H\left[\left(i - \frac{1}{2}\right)\hbar\right] \prod_{k=i}^{j-1} D^\dagger\left[\left(k - \frac{1}{2}\right)\hbar\right] \\ &= \prod_{k=1-j}^{i-1} D\left[-\left(k + \frac{1}{2}\right)\hbar\right] \left\{ H\left[-\left(i - \frac{1}{2}\right)\hbar\right] + (2i - 1)\hbar\omega \right\} \\ &\quad \times \prod_{k=i}^{j-1} D^\dagger\left[\left(k - \frac{1}{2}\right)\hbar\right] \\ &= \left\{ H\left[\left(j - \frac{1}{2}\right)\hbar\right] + (2i - 1)\hbar\omega \right\} \Lambda^{(i)} \end{aligned}$$

so $\Lambda^{(i)}$ is of step-size $(2i - 1)\omega$ but of order $2j - 1$. In a similar way, we determine

$$\Lambda^{(i_1)}\Lambda^{(i_2)} = \Delta^{i_1+i_2-1} \prod_{k=i_1}^{j-1} [H - \hbar\omega(2i_2 + 2k - 1)] \prod_{k=i_2}^{j-1} (H - 2k\hbar\omega)$$

For $i_1 > i_2$,

$$\begin{aligned} \Lambda^{(i_1)}[\Lambda^{(i_2)}]^\dagger &= \Delta^{i_1-i_2} \prod_{k=1}^{j-1} [H - (2k - 2i_2)\hbar\omega] \prod_{k=i_1}^{j-1} [H - (2k - 2i_2 + 1)\hbar\omega] \\ [\Lambda^{(i_2)}]^\dagger \Lambda^{(i_1)} &= \Delta^{i_1-i_2} \prod_{k=i_1}^{j-1} (H - 2k\hbar\omega) \prod_{k=1}^{i_2+j-1} [H - (2k + 2i_1 - 2i_2 - 1)\hbar\omega] \end{aligned}$$

The characteristic polynomials are

$$[\Lambda^{(i)}]^\dagger \Lambda^{(i)} = \prod_{k=i}^{j-1} (H - 2k\hbar\omega) \prod_{k=1}^{i+j-1} [H - (2k-1)\hbar\omega] \quad (6.7)$$

For $i < j$

$$\Lambda^{(i)} \Delta = \Lambda^{(i+1)} (H - 2j\hbar\omega)$$

$$\Delta \Lambda^{(i)} = \Lambda^{(i+1)} (H - 2i\hbar\omega)$$

and $\Lambda^{(j)} \Delta = \Delta \Lambda^{(j)}$. Finally, we have

$$\Lambda^{(i)} \Delta^\dagger = \begin{cases} [\Lambda^{(1)}]^\dagger (H + \hbar\omega), & i = 1; \\ \Lambda^{(i-1)} (H + \hbar\omega), & i > 1 \end{cases}$$

$$\Delta^\dagger \Lambda^{(i)} = \begin{cases} [\Lambda^{(1)}]^\dagger [H - \hbar\omega(2j+1)], & i = 1; \\ \Lambda^{(i-1)} [H - \hbar\omega(2j+2i-1)], & i > 1 \end{cases}$$

Let $\psi_{n,\mu}$ denote the eigenfunction of $H[\mu]$ with μ -dependence explicit. We calculate

$$D[\mu]\psi_{n,\mu} = \begin{cases} i\omega\psi_{n-1,\mu+\hbar}\sqrt{2}, & n > 0; \\ 0, & n = 0. \end{cases}$$

$$D^\dagger[\mu]\psi_{n,\mu+\hbar} = -i\hbar(n+1)\psi_{n+1,\mu}\sqrt{2}$$

$$D[-\mu]\psi_{n,\mu} = -i(\mu+n\hbar)\psi_{n,\mu-\hbar}\sqrt{2}$$

$$D^\dagger[-\mu]\psi_{n,\mu-\hbar} = i\omega\psi_{n,\mu}\sqrt{2}$$

So then for $\mu = (j - \frac{1}{2})\hbar$, there is an additional chain of eigenfunctions $\psi_{n,-\mu}$ that are interlaced with the usual states $\psi_{n,\mu}$. The additional matrix elements are

$$H\psi_{n,-\mu} = \hbar\omega(2n+1)\psi_{n,-\mu}$$

$$\Lambda^{(i)}\psi_{n,\mu} = (-i\hbar\sqrt{2})^{2j-1} \psi_{n+j-i,-\mu} \prod_{k=1}^{j-i} (n+k) \prod_{k=1}^{i+j-1} \left(n+k-i+\frac{1}{2}\right)$$

$$[\Lambda^{(i)}]^\dagger \psi_{n,\mu} = (-i\hbar\sqrt{2})^{2j-1} \psi_{n+i+j-1,-\mu} \prod_{k=1}^{i+j-1} (n+k) \prod_{k=1}^{j-i} \left(n+i+k-\frac{1}{2}\right)$$

$$\Lambda^{(i)}\psi_{n,-\mu} = \begin{cases} (i\omega\sqrt{2})^{2j-1} \psi_{n-i-j+1,\mu}, & n \geq 2i-1; \\ 0, & \text{otherwise} \end{cases}$$

$$[\Lambda^{(j)}]^\dagger \psi_{n,-\mu} = \begin{cases} (i\omega\sqrt{2})^{2j-1} \psi_{n-j+i,\mu}, & n \geq j-i; \\ 0, & \text{otherwise.} \end{cases}$$

For $j = 1$, the Hamiltonian is just a translation of the simple harmonic oscillator. The interlacing here simply corresponds to the quadratic relation that exists between Laguerre and Hermite polynomials

$$\begin{aligned}\psi_{n,-\mu} &= \frac{(-1)^n}{4^n n!} \mathcal{H}_{2n} \left(q \sqrt{\frac{\omega}{\hbar}} \right) \exp \left(-\frac{\omega q^2}{2\hbar} \right) \\ \psi_{n,\mu} &= \frac{(-1)^n}{2^{2n+1} n!} \mathcal{H}_{2n+1} \left(q \sqrt{\frac{\omega}{\hbar}} \right) \exp \left(-\frac{\omega q^2}{2\hbar} \right)\end{aligned}$$

For $j > 1$, the eigenfunctions $\psi_{n,-\mu}$ are not square-integrable on account of the singularity at the origin.

Definition 6.1. A ladder operator L is *reducible* if there exists a polynomial Φ and another ladder operator L' such that $L = \Phi[H]L'$.

Theorem 6.1. *If a ladder operator L is reducible then its characteristic polynomial has repeated factors.*

Proof. Write $L = L'\Phi[H]$ for some L', Φ . Then $L^\dagger L = \Phi[H]^2(L')^\dagger L'$. \square

It follows from (6.7) that the new ladder operators are irreducible.

Definition 6.2. A family of ladder operators L, L', L'', \dots are called *generators* if every ladder operator can be expressed as monomials in terms of them and the Hamiltonian.

We now show that there are no other values of μ where there are additional ladder operators, and in particular that these ladder operators are of minimal order. Expressed in a different way, this sets a lower bound on the order of such ladder operators.

Theorem 6.2. *Let $H = \frac{1}{2}p^2 + \frac{4\mu^2 - \hbar^2}{8q^2} + c_{-1}q^{-1} + c_1q + c_2q^2 + c_3q^3 + \dots$ be the Laurent series expansion of a Hamiltonian about $q = 0$. Suppose that H admits ladder operators of odd order $2j - 1$. Then $2\mu\hbar$ is an odd integer less than or equal to $2j - 1$.*

Proof. Let L be such a ladder operator and assume it is monic. From (5.3), we again see that the lowest-order term of L_i is $a_i(\hbar/q)^i$ where

$$ia_i + a_{i-1} \binom{i}{2} + \frac{4\mu^2 - \hbar^2}{8\hbar^2} \sum_{k=1}^{i-1} (k+1)! \binom{2j-i+k}{k} a_{i-k-1} = 0, \quad a_0 = 1, a_1 = 0.$$

For $i > 1$, a_i is of degree $2i$ in μ . The roots of a_{2j} are the admissible values of μ . We already know that $\mu = \pm(2i - 1)\hbar/2$, $i = 1, 2, \dots, j$ are solutions and it follows there can be no others. \square

Corollary 6.1. *If $2\mu\hbar = 2j - 1$ is an odd integer greater than unity, the ladder operators of H are generated by $\Delta, \Lambda^{(1)}, \Lambda^{(2)}, \dots, \Lambda^{(j)}$ and their conjugates. If $\mu = \frac{1}{2}\hbar$, then the only generators are $\Lambda^{(1)}, [\Lambda^{(1)}]^\dagger$. Otherwise, the generators are Δ, Δ^\dagger .*

Proof. Every ladder operator of even step-size is either reducible to Δ^i or $(\Delta^\dagger)^i$ whereas every ladder operator of odd step-size is reducible to $\Lambda^{(i)}, [\Lambda^{(i)}]^\dagger$, $i \leq j$ or $\Lambda^{(j)}\Delta^i, [\Lambda^{(j)}\Delta^i]^\dagger$. \square

6.4 Two-Dimensional Singular Oscillator

As is clear from the considerations outlined in section 5.3, the integrals of minimal-order of

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + k^2 q_2^2) + \frac{4\mu_1^2 - \hbar^2}{2q_1^2} + \frac{4\mu_2^2 - \hbar^2}{2q_2^2}, \quad k \in \mathbb{Q} \quad (6.8)$$

must be constructed by a superposition of ladder operator products.

Example 6.1. Let us now consider the Hamiltonian

$$K = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2) + \frac{\hbar^2}{q_1^2} + \frac{\hbar^2}{q_2^2}.$$

This is a special case of a Smorodinsky-Winternitz system. As such, it has the three first integrals generated from the even step ladder operators

$$\begin{aligned} A &= H_1 - H_2 && \text{(second-order)} \\ B &= 2\Delta_1\Delta_2^\dagger + 2\Delta_1^\dagger\Delta_2 - 4(H_1 - \frac{3}{2}\hbar\omega)(H_2 - \frac{3}{2}\hbar\omega) && \text{(second-order)} \\ C &= 8i\omega(\Delta_1\Delta_2^\dagger - \Delta_1^\dagger\Delta_2) && \text{(third-order)} \end{aligned}$$

where H_1, H_2 are one-dimensional singular oscillators with $\mu = \frac{3}{2}\hbar$ for both. Gravel observed that this special case admits an additional third-order integral. This can be realised in terms of odd step ladder operators

$$D = i(3\Lambda_1^{(1)}[\Lambda_2^{(1)}]^\dagger - 3[\Lambda_1^{(1)}]^\dagger\Lambda_2^{(1)} - \Lambda_1^{(2)}[\Lambda_2^{(2)}]^\dagger + [\Lambda_1^{(2)}]^\dagger\Lambda_2^{(2)}) \quad \text{(third-order)}$$

To complete the symmetry algebra, we need three more generators

$$\begin{aligned} E &= -6\omega(\Lambda_1^{(1)}[\Lambda_2^{(1)}]^\dagger + [\Lambda_1^{(1)}]^\dagger\Lambda_2^{(1)} - \Lambda_1^{(2)}[\Lambda_2^{(2)}]^\dagger - [\Lambda_1^{(2)}]^\dagger\Lambda_2^{(2)}) && \text{(fourth-order)} \\ F &= -12\omega^2 i(\Lambda_1^{(1)}[\Lambda_2^{(1)}]^\dagger - [\Lambda_1^{(1)}]^\dagger\Lambda_2^{(1)} - 3\Lambda_1^{(2)}[\Lambda_2^{(2)}]^\dagger + 3[\Lambda_1^{(2)}]^\dagger\Lambda_2^{(2)}) && \text{(fifth-order)} \\ G &= 24\omega^3(\Lambda_1^{(1)}[\Lambda_2^{(1)}]^\dagger + [\Lambda_1^{(1)}]^\dagger\Lambda_2^{(1)} - 9\Lambda_1^{(2)}[\Lambda_2^{(2)}]^\dagger - 9[\Lambda_1^{(2)}]^\dagger\Lambda_2^{(2)}) && \text{(sixth-order)} \end{aligned}$$

We find the following commutation relations. Even times even

$$[A, B] = i\hbar C, \quad [A, C] = -16\omega^2 i\hbar(B - K^2 + A^2), \quad [B, C] = 32\omega^2 i\hbar(\{A, B\} + 5\hbar^2\omega^2 A).$$

Even times odd

$$\begin{aligned} [A, D] &= i\hbar E, & [A, E] &= i\hbar F, \\ [A, F] &= i\hbar G, & [A, G] &= -18\omega^2 i\hbar(8\omega^2 D + 5F), \\ [B, D] &= 0, & [B, E] &= -48\omega^2 i\hbar\{A, D\}, \\ [B, F] &= -64\omega^2 i\hbar\{A, E\}, & [B, G] &= 48\omega^2 i\hbar\{A, 16\omega^2 D - F\}, \\ [C, D] &= 48\omega^2 i\hbar\{A, D\}, & [C, E] &= 16\omega^2 i\hbar\{A, E\}, \\ [C, F] &= -16\omega^2 i\hbar\{A, 48\omega^2 D + F\}, & [C, G] &= -16\omega^2 i\hbar\{A, 112\omega^2 E + 3G\} \end{aligned}$$

Odd times odd

$$\begin{aligned}
[D, E] &= 9\omega^2 i \hbar \{A, B^2 + 2\hbar^2 \omega^2 B - 31\hbar^4 \omega^4\} \\
[D, F] &= 18\omega^2 i \hbar \{A, B\} - 7\hbar^2 \omega^2 A, C\} \\
[D, G] &= -36\omega^4 i \hbar [12\{AK^2 - A^3, B\} + 13\{A, B^2\} \\
&\quad - 92\hbar^2 \omega^2 A^3 + 156\hbar^2 \omega^2 K^2 A + 110\hbar^2 \omega^2 \{A, B\} - 335\hbar^4 \omega^4 A] \\
[E, F] &= -108\omega^4 i \hbar [4\{AK^2 - A^3, B\} + \{A, B^2\} - 52\hbar^2 \omega^2 A^3 \\
&\quad + 84\hbar^2 \omega^2 AK^2 + 30\hbar^2 \omega^2 \{A, B\} + 45\hbar^4 \omega^4 A] \\
[E, G] &= 216\omega^4 i \hbar \{2A^3 - 2K^2 A - \{A, B\} - 7\hbar^2 \omega^2 A, C\} \\
[F, G] &= 432\omega^6 i \hbar \{A, 16A^4 - 32A^2 K^2 + 16K^4 + 160\hbar^2 \omega^2 A^2 \\
&\quad - 320\hbar^2 \omega^2 K^2 + 3B^2 + 6\hbar^2 \omega^2 B - 557\hbar^4 \omega^4\}
\end{aligned}$$

There are three closure relations

$$\begin{aligned}
0 &= C^2 + \omega^2(16B^2 + 32\{K^2 - A^2, B\} + 416\hbar^2 \omega^2 K^2 - 96\hbar^2 \omega^2 A^2 - 400\hbar^4 \omega^4) \\
0 &= G^2 + 40\omega^2 F^2 + 48\omega^4 E^2 - 3456\omega^6 D^2 - 20736\hbar^2 \omega^8 \{B, K^2 - 2A^2\} \\
&\quad - 2304\omega^6 (K^2 - A^2)^3 + 11520\hbar^2 \omega^8 (13K^4 - 36K^2 A^2 + 23A^4) \\
&\quad - 2304\hbar^4 \omega^{10} (712K^2 + 269A^2) - 145152\hbar^4 \omega^{10} B + 1327104\hbar^6 \omega^{12} \\
0 &= \{G, E\} + 66\omega^2 \{F, D\} + 1080\omega^4 D^2 + \{B, 144\omega^4 (K^2 - A^2)^2 \\
&\quad + 1008\hbar^2 \omega^6 (3K^2 - 8A^2) + 27936\hbar^4 \omega^8\} - 33696\hbar^6 \omega^{10} \\
&\quad + 144\hbar^2 \omega^6 (5K^4 - 14K^2 A^2 + 9A^4) + 144\hbar^4 \omega^8 (445K^2 - 76A^2)
\end{aligned}$$

□

In the isotropic case, we may obtain the lowest-order integrals systematically by using radial symmetry.

Theorem 6.3. Let $H_{i,j} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega^2(q_1^2 + q_2^2) + \frac{\hbar^2}{q_1^2} \binom{i}{2} + \frac{\hbar^2}{q_2^2} \binom{j}{2}$ for positive integers $j \geq i$. In addition to the second- and third-order integrals from being a special case of a Smorodinsky-Winternitz system, the Hamiltonian $H_{i,j}$ possesses $j - i + 1$ independent integrals of order $2j - 1$ when $j > 1$.

Proof. We start from the fact the simple harmonic oscillator $H_{1,1}$ has a first-order integral $X_{1,1,1} := m_{12}$. The Smorodinsky-Winternitz system is separable in polar coordinates so it admits the additional Darboux transformation

$$\tilde{D}_{i,j} := m_{12} - i\hbar i q_2 q_1^{-1} + i\hbar j q_1 q_2^{-1}$$

This has the intertwining relations

$$\tilde{D}_{k,j} H_{i,j} = \begin{cases} H_{i-1,j+1} \tilde{D}_{k,j}, & k = 1 - i; \\ H_{i+1,j+1} \tilde{D}_{k,j}, & k = i. \end{cases}$$

Recall, we have $D_2 H_{i,j} = H_{i,j+1} D_2$ (suppressing the arguments for brevity). We define

$$\begin{aligned} X_{1,j+1,1} &:= \tilde{D}_{0,j} X_{1,j,1} \tilde{D}_{0,j}^\dagger \\ X_{i+1,j+1,k} &:= \tilde{D}_{i,j} X_{i,j,k} \tilde{D}_{i,j}^\dagger \\ X_{i,j+1,k+1} &:= D_2 X_{i,j,k} D_2^\dagger \end{aligned}$$

At the leading order, we have $X_{i,j,k} = \left\{ m_{12}^{2j+1-2k}, p_2^{2k-2} \right\} + \dots$ for $k = 1, 2, \dots, j - i + 1$ so they are all independent. \square

6.5 Conclusion

In this chapter, we have for the first time stated a symmetry algebra of a two-dimensional superintegrable system which requires not three but seven generators for closure. This suggests that quantum superintegrability is very poorly understood compared with classical superintegrability [68]. Furthermore, the notion of a polynomial Heisenberg algebra needs to be expanded in order to account for additional ladder operators.

We may generalise this to angular models. In Theorem 3.2, we noted that there were cases in the coefficients of the TTW model where the independence of the ladder operators breaks down, and we can find operators of lesser order or more varied step-sizes.

Chapter 7

Polynomial Algebras and Superintegrable Extensions

7.1 Introduction

In the previous chapters it was established that the approach of considering the one-dimensional components of a superintegrable system in isolation is far more effective in deriving and describing the model. The various approaches offered for classical systems, in general though not necessarily, involve canonical transformations which lead to non-polynomial integrals [40]. Adapting it to quantum models is therefore not at all straightforward [141]. The recurrence relation approach of Kalnins, Kress and Miller Jr [47], which we improved upon and developed systematically in Chapter 3, is not sufficiently comprehensive to deliver for us a unified treatment of superintegrability. The extension method of Chanu, Degiovanni and Rastelli [142, 143] was promising, however the range of applicability of the original implementation was limited and modifications were required in order to address many of the well-known superintegrable systems [90].

The remarkable effectiveness and generality of action-angle variables has suggested to us that the quantum analogue must itself be very simple. We here shall recommend the method of constructing superintegrable models by polynomial deformations of the Racah-Wilson algebra

$$\begin{aligned} [A, B] &= i\hbar C \\ [A, C] &= i\hbar [P(A) + \alpha B + 2\beta\{A, B\}] \\ [B, C] &= i\hbar [Q(A) - \{R(A), B\} - \beta B^2] \end{aligned} \tag{7.1}$$

where P, Q, R are polynomial functions and A, B, C are all self-adjoint. If A is a Hamiltonian and belongs to an algebra of this form then for select χ, U the *extended Hamiltonian*

$$H = \left\{ \left(\frac{d\chi}{dx} \right)^2, \frac{1}{2} p_x^2 + A + U(x) \right\} \tag{7.2}$$

will admit an integral independent of H, A . In particular, when A is integrable, H is superintegrable. The polynomial algebra (7.1) is well understood to be the crux which unites superintegrability with

exact solvability [144, 145], as the representations of (7.1) correspond to the recurrence relations of orthogonal polynomials in the Askey-Wilson scheme. The quadratic form of (7.1) was discussed by Granovskii, Lutzenko and Zhedanov [146] in order to understand an observation by Manning [147] regarding the connection between quantum exact solvability and the classical correspondence where the trajectories can be described in terms of elementary functions. This can be seen as follows. If we take the classical analogue of (7.1), namely

$$\begin{aligned} \llbracket A, B \rrbracket &= C \\ \llbracket A, C \rrbracket &= P(A) + \alpha B + 2\beta AB \\ \llbracket B, C \rrbracket &= Q(A) - R(A)B - \beta B^2 \end{aligned} \tag{7.3}$$

then

$$\frac{d^2 B}{dt^2} = P(A) + (\alpha + 2\beta A)B$$

which is a linear equation since A is a constant of the motion. If $\alpha = \beta = 0$, then B is a linear function of t whereas if $\alpha \neq 0$ or $\beta \neq 0$ then B is exponential or trigonometric. Conversely, only when $\alpha < 0, \beta = 0$ or $\beta \neq 0$ do we have a discrete spectrum for A upon computing the representations of (7.1).

Not all of the coefficients of (7.1) are independent. In section 7.2, we shall derive the dependency relations in terms of sequences defined by recurrence relations. This is to improve upon the calculations of Isaac and Marquette [84] who gave only indirect formulas for these sequences. In section 7.3 we shall derive the irreducible A -diagonal representations of (7.1) explicitly. In section 7.4 we derive one-dimensional realisations of (7.1) and show how they lead to the known exactly solvable models as well as some new ones. That is, we show that they all lead to the Painlevé potentials which have been derived in the course of classifying third- and fourth-order superintegrable systems. This extends the work of Marquette, Sajedi and Winternitz [126] where they derived all of the Cartesian separable exotic potentials except for the Painlevé III potential [37]. Our formulation includes in addition the Painlevé VI potential derived by Escobar-Ruiz *et al.* [35] and a new Morse-like potential which is reducible to either Painlevé III or Painlevé V. In section 7.5, we show how to apply this to superintegrable extensions.

7.2 Determination of Relations

The Jacobi identity

$$[A, [B, C]] + [B, [A, C]] + [C, [A, B]] = 0 \tag{7.4}$$

implies the relation:

$$[P(A), B] = i\hbar\{R(A), C\} \tag{7.5}$$

We will show how we may determine R from this equation given P . We proceed inductively. Using the derivation property, we may write $[A^{i+1}, B]$ in two different ways

$$[A^{i+1}, B] = A[A^i, B] + [A, B]A^i = [A^i, B]A + A^i[A, B].$$

So

$$[A^{i+1}, B] = i\hbar\{A^i, C\} + \{A, [A^i, B]\}. \quad (7.6)$$

We note that

$$\begin{aligned} \{A, \{A^j, C\}\} &= \frac{1}{4}(A^{j+1}C + ACA^j + A^jCA + CA^{j+1}) \\ &= \frac{1}{2}\{A^{j+1}, C\} + \frac{1}{4}(ACA^j + A^jCA) \\ &= \{A^{j+1}, C\} + \frac{1}{4}[[A, C], A^j] \\ &= \{A^{j+1}, C\} + \frac{1}{4}i\hbar[P(A) + \alpha B + 2\beta\{A, B\}, A^j] \\ &= \{A^{j+1}, C\} - \frac{1}{4}i\hbar\alpha[A^j, B] - \frac{1}{2}i\hbar\beta\{A, [A^j, B]\} \\ &= \{A^{j+1}, C\} - \frac{1}{2}\hbar^2\beta\{A^j, C\} - \frac{1}{4}i\hbar\alpha[A^j, B] - \frac{1}{2}i\hbar\beta[A^{j+1}, B] \end{aligned} \quad (7.7)$$

where in the last line we used (7.6). Now, we set

$$[A^i, B] = i\hbar \sum_{j=0}^{i-1} a_{i,j} \{A^j, C\}$$

Combining (7.6) and (7.7), we obtain the recurrence relation

$$a_{i+1,j} = \delta_{i,j} + a_{i,j-1} - \frac{1}{2}\hbar^2\beta a_{i,j} + \frac{1}{4}\hbar^2\alpha \sum_{k=j+1}^{i-1} a_{i,k} a_{k,j} + \frac{1}{2}\hbar^2\beta \sum_{k=j}^{i-1} a_{i,k} a_{k+1,j} \quad (7.8)$$

In particular $a_{i+1,i} = 1 + a_{i,i-1}$ and using $a_{1,0} = 1$, we get $a_{i,i-1} = i$. Continuing, we have

$$\begin{aligned} a_{i+1,i-1} &= a_{i,i-2} - \frac{1}{2}\hbar^2\beta a_{i,i-1} + \frac{1}{2}\hbar^2\beta a_{i,i-1}^2 \\ &= a_{i,i-2} + \hbar^2\beta \binom{i}{2} \end{aligned}$$

Thus,

$$a_{i,i-2} = \hbar^2\beta \binom{i}{3}$$

Similarly,

$$\begin{aligned} a_{i,i-3} &= \frac{1}{2}\hbar^2\alpha \binom{i}{3} + 2\hbar^4\beta^2 \left[\binom{i}{4} + 2 \binom{i}{5} \right] \\ a_{i,i-4} &= \hbar^4\alpha\beta \left[\binom{i}{4} + 4 \binom{i}{5} \right] + \hbar^6\beta^3 \left[7 \binom{i}{5} + 34 \binom{i}{6} + 34 \binom{i}{7} \right] \\ a_{i,i-5} &= \hbar^4\alpha^2 \binom{i}{5} + \frac{1}{2}\hbar^6\alpha\beta^2 \left[7 \binom{i}{5} + 68 \binom{i}{6} + 102 \binom{i}{7} \right] \\ &\quad + \hbar^8\beta^4 \left[38 \binom{i}{6} + 324 \binom{i}{7} + 744 \binom{i}{8} + 496 \binom{i}{9} \right] \end{aligned}$$

and so on to as many terms as necessary. This allows us to compute R in terms of the coefficients of P . For $\beta = 0$, we can determine this sequence directly. We take as our *Ansatz*

$$a_{i,j} = g_{i-j} \alpha^{(i-j-1)/2} \hbar^{i-j-1} \binom{i}{i-j}$$

Then

$$\begin{aligned} a_{i+\ell+2,i+1} - a_{i+\ell+1,i} &= \frac{1}{4} \hbar^2 \alpha \sum_{k=1}^{\ell-1} a_{i+\ell+1,i+k+1} a_{i+k+1,i+1} \\ &= \frac{1}{4} \hbar^{\ell+1} \alpha^{(\ell+1)/2} \sum_{k=1}^{\ell} g_{\ell-k} g_k \binom{i+\ell+1}{\ell-k} \binom{i+k+1}{k} \\ &= \frac{1}{4} \hbar^{\ell+1} \alpha^{(\ell+1)/2} \sum_{k=1}^{\ell} g_{\ell-k} g_k \binom{i+\ell+1}{\ell} \binom{\ell}{k} \end{aligned}$$

So we have the convolution identity

$$g_{\ell+1} = \frac{1}{4} \sum_{k=1}^{\ell} g_{\ell-k} g_k \binom{\ell}{k}$$

where $g_0 = 0$. Defining the generating function

$$G = \sum_{\ell=0}^{\infty} \frac{g_{\ell}}{\ell!} s^{\ell}$$

and using $g_1 = 1$, we obtain the differential equation

$$\frac{dG}{ds} = \frac{1}{4} G^2 + 1, \quad G(0) = 0$$

That is,

$$G = 2 \tan \left(\frac{s}{2} \right) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1} 4(4^{\ell} - 1) \mathcal{B}_{2\ell}}{(2\ell)!} s^{2\ell-1}$$

where \mathcal{B}_i are Bernoulli numbers. Thus,

$$g_{\ell} = \frac{(-1)^{\frac{\ell+1}{2}} 4(2^{\ell+1} - 1) \mathcal{B}_{\ell+1}}{\ell + 1}.$$

For the inverse problem, we write

$$i\hbar \{A^i, C\} = \sum_{j=1}^{i+1} b_{i,j} [A^j, B].$$

The coefficients are determined via

$$\sum_{k=j+1}^{i+1} b_{i,k} a_{k,j} = \delta_{i,j}.$$

We calculate

$$b_{i,i+1} = \frac{1}{i+1}$$

$$b_{i,i} = -\frac{1}{6}\hbar^2\beta(i-1)$$

$$b_{i,i-1} = -\frac{1}{6}\hbar^2\alpha i - \frac{1}{180}\hbar^4\beta^2i(i^2-4)$$

and so on. We can also find a recurrence relation directly. Consider

$$\begin{aligned} \{A, [A^j, B]\} &= \frac{1}{2}(A^{j+1}B - ABA^j + A^jBA - BA^{j+1}) \\ &= \frac{1}{2}[A^{j+1}, B] + \frac{1}{2}(A^jBA - ABA^j) \\ &= [A^{j+1}, B] + \frac{1}{2}(A^j[B, A] - [A, B]A^j) \\ &= [A^{j+1}, B] - i\hbar\{A^j, C\} \end{aligned} \quad (7.9)$$

Rearranging (7.7) for $\{A^{j+1}, C\}$ and using (7.9), we obtain

$$b_{i+1,j} = b_{i,j-1} - \sum_{k=j-1}^{i+1} b_{i,k}b_{k,j} + \frac{1}{2}\hbar^2\beta b_{i,j} - \frac{1}{2}\hbar^2\alpha\delta_{i,j} - \frac{1}{2}\hbar^2\beta\delta_{i+1,j} \quad (7.10)$$

The presentation of (7.1) can be simplified if we know α or β is non-zero. If $\beta \neq 0$, then we may assume P is a constant, otherwise we make the substitutionn

$$\tilde{A} := A, \quad \tilde{B} := B + \frac{P(A) - P\left(-\frac{\alpha}{2\beta}\right)}{2\beta A + \alpha}, \quad \tilde{C} := C.$$

Similarly, if $\beta = 0, \alpha \neq 0$, we can assume $P = 0$ or else take

$$\tilde{A} := A, \quad \tilde{B} := B + \alpha^{-1}P(A), \quad \tilde{C} := C.$$

If the algebra is realised by dynamical quantities or quantum observables, then this transformation does not in general preserve the order of the objects of consideration. To show that $[\tilde{B}, \tilde{C}]$ maintains the same general form, we need to, for a given polynomial X , find Y, Z such that

$$[X(A), C] = i\hbar[Y(A) + \{Z(A), B\}]. \quad (7.11)$$

Given such a relationship, we will be able to determine \tilde{Q} such that

$$\begin{aligned} [\tilde{A}, \tilde{B}] &= i\hbar\tilde{C} \\ [\tilde{A}, \tilde{C}] &= i\hbar\left[\alpha\tilde{B} + 2\beta\{\tilde{A}, \tilde{B}\} + \gamma\right] \\ [\tilde{B}, \tilde{C}] &= i\hbar\left[\tilde{Q}(\tilde{A}) - \beta\tilde{B}^2\right] \end{aligned}$$

It is clear that \tilde{R} must be zero by the Jacobi identity. Component-wise, we wish to find $c_{i,j}, d_{i,j}$ such that

$$[A^i, C] = i\hbar\left[P(A)\sum_{j=0}^{i-1}c_{i,j}A^j + \sum_{j=0}^i d_{i,j}\{A^j, B\}\right]$$

We have

$$\begin{aligned}
[A^{i+1}, C] &= \{A, [A^i, C]\} + \{A^i, [A, C]\} \\
&= i\hbar \left\{ A, P(A) \sum_{j=0}^{i-1} c_{i,j} A^j + \sum_{j=0}^i d_{i,j} \{A^j, B\} \right\} + i\hbar \{A^i, P(A) + \alpha B + 2\beta \{A, B\}\} \\
&= i\hbar \left[A^i P(A) + P(A) \sum_{j=0}^{i-1} c_{i,j} A^{j+1} + \sum_{j=0}^i d_{i,j} \{A, \{A^j, B\}\} \right. \\
&\quad \left. + \alpha \{A^i, B\} + 2\beta \{A, \{A^i, B\}\} \right].
\end{aligned}$$

Now,

$$\begin{aligned}
\{A, \{A^j, B\}\} &= \frac{1}{2} \{A^{j+1}, B\} + \frac{1}{4} A B A^j + \frac{1}{4} A^j B A \\
&= \{A^{j+1}, B\} - [A^j, [A, B]] \\
&= \{A^{j+1}, B\} - i\hbar [A^j, C]
\end{aligned}$$

which gives us the recurrence relations

$$c_{i+1,j} = c_{i,j-1} + \delta_{i,j} + 2\hbar^2 \beta c_{i,j} + \hbar^2 \sum_{k=j+1}^i d_{i,k} c_{k,j} \quad (7.12)$$

$$d_{i+1,j} = d_{i,j-1} + \alpha \delta_{i,j} + 2\beta \delta_{i+1,j} + 2\hbar^2 \beta d_{i,j} + \hbar^2 \sum_{k=j}^i d_{i,k} c_{k,j} \quad (7.13)$$

We may then calculate the solution to (7.11) for any given order of X .

7.3 Representations

In this section, we consider representations as quantum observables, finite- or infinite-dimensional, induced from the eigenstates of A . The method of calculating the representations is that of Granovskii *et al.* [146]. Let $|\phi\rangle$ be an eigenstate of A with eigenvalue μ . The triple $(|\phi\rangle, B|\phi\rangle, C|\phi\rangle)$ is A -invariant, with matrix representation (taking the elements to be formally independent)

$$A \mapsto \begin{bmatrix} \mu & 0 & i\hbar P(\mu) \\ 0 & \mu & i\hbar(\alpha + 2\beta\mu) \\ 0 & i\hbar & \mu - \hbar^2\beta \end{bmatrix}$$

The characteristic equation reads

$$0 = (\rho - \mu)[\rho^2 - (2\mu - \hbar^2\beta)\rho + \mu(\mu - \hbar^2\beta) + \hbar^2(\alpha + 2\beta\mu)]$$

If $\beta \neq 0$, then the three solutions can be written as

$$\begin{aligned}
\rho_n &= -\frac{1}{2}\hbar^2\beta \left[\left(n + \delta - \frac{(-\alpha)^{\frac{1}{2}}}{\hbar\beta} \right)^2 - \frac{1}{4} \right] - \frac{\alpha}{2\beta}, \\
\rho_{n\pm 1} &= -\frac{1}{2}\hbar^2\beta \left[\left(n + \delta \pm 1 - \frac{(-\alpha)^{\frac{1}{2}}}{\hbar\beta} \right)^2 - \frac{1}{4} \right] - \frac{\alpha}{2\beta}
\end{aligned} \quad (7.14)$$

where $\rho_n = \mu$, and n, δ are auxiliary parameters. Similarly, if $\beta = 0$ and $\alpha \neq 0$, we have

$$\rho_n = \hbar(n + \delta)(-\alpha)^{\frac{1}{2}}, \quad \rho_{n\pm 1} = \hbar(n + \delta \pm 1)(-\alpha)^{\frac{1}{2}}$$

which corresponds to the limiting case in (7.14). In this way, we generate a chain of eigenstates with n serving as the quantum number up to translation. Indeed, we can label the eigenstates just generated as $|n-1\rangle, |n\rangle := |\phi\rangle, |n+1\rangle$. For $\alpha = \beta = 0$, we do not obtain new eigenstates. There is no discrete parametrisation in this case and the spectrum of A will be continuous. Otherwise, we have that A is diagonal with B, C tridiagonal

$$A|n\rangle = \rho_n|n\rangle$$

$$B|n\rangle = \pm\sigma_{n+1}^*|n+1\rangle + \tau_n|n\rangle + \sigma_n|n-1\rangle$$

$$C|n\rangle = \pm i \left[\hbar\beta \left(n + \delta + \frac{1}{2} \right) - (-\alpha)^{\frac{1}{2}} \right] \sigma_{n+1}^*|n+1\rangle - i \left[\hbar\beta \left(n + \delta - \frac{1}{2} \right) - (-\alpha)^{\frac{1}{2}} \right] \sigma_n|n-1\rangle$$

where we assume, without loss of generality, $\alpha < 0$. From $[A, C]$, we determine

$$\tau_n = -P(\rho_n)(\alpha + 2\beta\rho_n)^{-1}.$$

From $[B, C]$, we find

$$\begin{aligned} & \pm 2 \left[\beta \left(n + \delta + \frac{1}{2} \right) - \frac{(-\alpha)^{\frac{1}{2}}}{\hbar} \right] |\sigma_{n+1}|^2 \mp 2 \left[\beta \left(n + \delta - \frac{1}{2} \right) - \frac{(-\alpha)^{\frac{1}{2}}}{\hbar} \right] |\sigma_n|^2 \\ & = Q(\rho_n) - R(\rho_n)\tau_n - \beta\tau_n^2. \end{aligned}$$

To determine σ_n up to a phase we need an extra condition. Here, we make use of a well-known fact.

Theorem 7.1. *The Racah-Wilson algebra has a central element*

$$\Omega = C^2 - \{2P(A) + \hbar^2\beta R(A), B\} - (\alpha + \hbar^2\beta^2)B^2 - 2\beta\{A, B^2\} - \frac{1}{4}\hbar^2 R(A)^2 + 2S(A) \quad (7.15)$$

where $S(A)$ is a polynomial which satisfies

$$[S(A), B] = i\hbar\{Q(A), C\}. \quad (7.16)$$

Proof. Since $C = -i[A, B]/\hbar$, we only need to check that K commutes with A, B . First, we note that

$$\begin{aligned} 4\{\{A, B\}, C\} - 4\{A, \{B, C\}\} &= [B, [A, C]] \\ &= i\hbar[B, P(A) + \alpha B + 2\beta\{A, B\}] \\ &= \hbar^2\{R(A), C\} + 2\hbar^2\beta\{B, C\} \end{aligned}$$

Then it follows

$$\begin{aligned} [A, M] &= 2i\hbar\{P(A) + \alpha B + 2\beta\{A, B\}, C\} - i\hbar\{2P(A) + \beta\hbar^2 R(A), C\} \\ &\quad - 2i\hbar(\alpha + \hbar^2\beta^2)\{B, C\} - 4i\hbar\beta\{A, \{B, C\}\} \\ &= 4i\hbar(\{\{A, B\}, C\} - \{A, \{B, C\}\}) - i\hbar^3(\{R(A), C\} + 2\beta^2\{B, C\}) = 0. \end{aligned}$$

Similarly, we calculate

$$\begin{aligned}
[B, M] &= 2i\hbar\{Q(A) - \{R(A), B\} - \beta B^2, C\} + 2i\hbar\{\{R(A), C\}, B\} + \frac{1}{2}\hbar^2\beta[R(A), B^2] \\
&\quad + 2\beta i\hbar\{B^2, C\} + \frac{1}{4}\hbar^2[R(A)^2, B] - 2i\hbar\{Q(A), C\} \\
&= 2i\hbar(\{\{R(A), C\}, B\} - \{\{R(A), B\}, C\}) + \frac{1}{2}\hbar^2\beta[R(A), B^2] + \frac{1}{4}\hbar^2[R(A)^2, B] \\
&= \frac{1}{2}i\hbar[R(A), [C, B]] + \frac{1}{2}\hbar^2\beta[R(A), B^2] + \frac{1}{4}\hbar^2[R(A)^2, B] \\
&= \frac{1}{2}\hbar^2[R(A), Q(A) - \{R(A), B\} - \beta B^2] + \frac{1}{2}\hbar^2\beta[R(A), B^2] + \frac{1}{4}\hbar^2[R(A)^2, B] = 0.
\end{aligned}$$

□

In an irreducible representation, the central element Ω will take a specific value, typically zero for convenience, and is identical to the closure relation of the algebra. We obtain

$$\begin{aligned}
0 &= \pm 2 \left[\hbar\beta \left(n + \delta + \frac{1}{4} \right) - (-\alpha)^{\frac{1}{2}} \right]^2 |\sigma_{n+1}|^2 \pm 2 \left[\hbar\beta \left(n + \delta - \frac{1}{4} \right) - (-\alpha)^{\frac{1}{2}} \right]^2 |\sigma_n|^2 \\
&\quad - [2P(\rho_n) + \hbar^2\beta R(\rho_n)]\tau_n - (\alpha + \hbar^2\beta^2 + 2\beta\rho_n)\tau_n^2 - \frac{1}{4}\hbar^2 R(\rho_n)^2 + 2S(\rho_n).
\end{aligned}$$

So $|\sigma_n|^2$ is rational in n with denominator

$$\left[\hbar\beta(n + \delta) - (-\alpha)^{\frac{1}{2}} \right] \left[\hbar\beta \left(n + \delta - \frac{1}{2} \right) - (-\alpha)^{\frac{1}{2}} \right]^2 \left[\hbar\beta(n + \delta - 1) - (-\alpha)^{\frac{1}{2}} \right].$$

A more convenient notation to express this representation is via a deformed oscillator algebra. Let \mathfrak{N} be the number operator $\mathfrak{N}|n\rangle = n|n\rangle$ and $\mathfrak{a}, \mathfrak{b}$ are ladder operators which satisfy

$$[\mathfrak{a}, \mathfrak{N}] = \mathfrak{a}, \quad [\mathfrak{N}, \mathfrak{b}] = \mathfrak{b}, \quad \mathfrak{a}\mathfrak{b} = \Phi(\mathfrak{N} + 1), \quad \mathfrak{b}\mathfrak{a} = \Phi(\mathfrak{N}) \quad (7.17)$$

where Φ is the *characteristic polynomial*. If we write

$$\begin{aligned}
A &= \rho_{\mathfrak{N}} \\
B &= \left\{ \mathfrak{a} \left[\hbar\beta \left(\mathfrak{N} + \delta - \frac{1}{2} \right) - (-\alpha)^{\frac{1}{2}} \right]^{-1} \right. \\
&\quad \left. + \left[\hbar\beta \left(\mathfrak{N} + \delta - \frac{1}{2} \right) - (-\alpha)^{\frac{1}{2}} \right]^{-1} \mathfrak{b} \right\} \left[\hbar\beta(\mathfrak{N} + \delta) - (-\alpha)^{\frac{1}{2}} \right]^{-1} \\
&\quad - P(\rho_{\mathfrak{N}})(\alpha + 2\beta\rho_{\mathfrak{N}})^{-1} \\
C &= (-i\mathfrak{a} \pm i\mathfrak{b}) \left[\hbar\beta(\mathfrak{N} + \delta) - (-\alpha)^{\frac{1}{2}} \right]^{-1}
\end{aligned} \quad (7.18)$$

then $\Phi(n)$ is clearly proportional to the numerator of $|\sigma_n|^2$. Note the similarity of (7.18) with the form of the integrals obtained in Chapter 3. It is this connection which leads us to conclude that the polynomial algebra which offers a far more convenient setting to study superintegrability and allows us to avoid passing through formal operators.

7.4 One-dimensional Realisations

Here we look for a realisation of (7.1) where $A = \frac{1}{2}p_y^2 + V(y)$ and B is invariant under time-reversal. In order to fulfil all of the relations it is sufficient to define C by $[A, B]/(i\hbar)$ and solve for B by computing

$$[A, [A, B]] = -\hbar^2 [P(A) + \alpha B + 2\beta\{A, B\}] \quad (7.19)$$

We apply the Jacobi identity (7.4) and equation (7.5) to obtain

$$[A, [B, [A, B]]] = [B, [A, [A, B]]] = \hbar^2 [A, \{R(A), B\} + \beta B^2]$$

So

$$[B, [A, B]] = \hbar^2 (\{R(A), B\} + \beta B^2 - Q)$$

where $[A, Q] = 0$. By hypothesis, A is a one-dimensional Hamiltonian so Theorem 5.2 implies Q is a polynomial in A . Thus, the third equation in (7.1) will hold by merely imposing the condition (7.19), though the coefficients of Q must be determined *ex post facto* by explicit computation. We consider two cases, namely when B does not depend on the momentum and when B is quadratic in the momentum.

7.4.1 Elementary Realisations

Here, we take

$$B = a(y), \quad P(A) = \gamma_0 A + \gamma_1$$

from which it follows $R(A) = \gamma_0$. Granovskii *et al.* [146] already considered this case but we shall go over it for completeness. They also considered the case of coupling-constant metamorphoses to the extent that there were no quantum corrections in the potential (i.e. the classical potential matched the quantum potential exactly). One readily finds $C = -\{a', p_y\}$. The second condition in (7.1) gives

$$\gamma_0 = 2 \frac{d^2 a}{dy^2} - 2\beta a \quad (7.20a)$$

$$\gamma_1 = -\alpha a - \frac{da}{dy} \frac{dV}{dy} - 2V \frac{d^2 a}{dy^2} + \frac{1}{4} \hbar^2 \frac{d^4 a}{dy^4} \quad (7.20b)$$

Multiplying (7.20a) and (7.20b) by $\frac{da}{dy}$ and then integrating, we obtain

$$\left(\frac{da}{dy}\right)^2 = \beta a^2 + \gamma_0 a + \delta_0 \quad (7.21a)$$

$$0 = \frac{1}{2} \alpha a^2 + \gamma_1 a + \delta_1 + \left(\frac{da}{dy}\right)^2 V + \frac{1}{8} \hbar^2 \left[\left(\frac{d^2 a}{dy^2}\right)^2 - 2 \frac{da}{dy} \frac{d^3 a}{dy^3} \right] \quad (7.21b)$$

From (7.21a) we determine $Q(A) = -\delta_0$. From (7.21b) we see that V is a rational function of a . For perspicacity, we divide the solutions according to when the right-hand side of (7.21a) is (a) constant; (b) linear; (c) quadratic with repeated roots; or (d) quadratic with distinct roots.

(a) $a = y$

We obtain

$$A = \frac{1}{2}p_y^2 - \frac{1}{2}\alpha y^2 - \gamma_1 y + \delta, \quad Q = -1$$

with $\beta = 0, \gamma_0 = 0$.

(b) $a = y^2$

We obtain

$$A = \frac{1}{2}p_y^2 - \frac{1}{8}\alpha y^2 + \frac{\delta}{y^2} - \frac{1}{4}\gamma_1, \quad Q = 0$$

where $\beta = 0, \gamma_0 = 4$.

(c) $a = e^{y\sqrt{\beta}}$

We obtain

$$A = \frac{1}{2}p_y^2 - \frac{\gamma_1}{\beta}e^{-y\sqrt{\beta}} + \delta e^{-2y\sqrt{\beta}} - \frac{\alpha}{2\beta} + \frac{1}{8}\hbar^2\beta, \quad Q = 0$$

with $\gamma_0 = 0$.

(d) $a = \cosh\left(y\sqrt{\beta}\right)$

We obtain

$$A = \frac{1}{2}p_y^2 + \frac{\delta - \gamma_1\beta^{-1}\cosh\left(y\sqrt{\beta}\right)}{\sinh^2\left(y\sqrt{\beta}\right)} - \frac{\alpha}{2\beta} + \frac{1}{8}\hbar^2\beta, \quad Q = \beta$$

with $\gamma_0 = 0$.

7.4.2 Second-Order Realisations

Here we take

$$B = \{a(y), A\} + b(y), \quad P(A) = \gamma_0 A^2 + \gamma_1 A + \gamma_2$$

so $R(A) = 2\gamma_0 A + \gamma_1$. The following system of equations is needed to be satisfied in order to fulfil (7.19):

$$\gamma_0 = 2\frac{d^2 a}{dy^2} - 2\beta a \tag{7.22a}$$

$$\gamma_1 = 2\frac{d^2 b}{dy^2} - 2\beta b - \alpha a - \frac{da}{dy}\frac{dV}{dy} - 2V\frac{d^2 a}{dy^2} + \hbar^2\left(\frac{5}{4}\frac{d^4 a}{dy^4} - \beta\frac{d^2 a}{dy^2}\right) \tag{7.22b}$$

$$\begin{aligned} \gamma_2 = & -\alpha b - \frac{db}{dy}\frac{dV}{dy} - 2V\frac{d^2 b}{dy^2} + \hbar^2\left(\frac{1}{2}\beta\frac{da}{dy}\frac{dV}{dy} + \beta V\frac{d^2 a}{dy^2} - V\frac{d^4 a}{dy^4}\right. \\ & \left. - \frac{1}{2}\frac{dV}{dy}\frac{d^3 a}{dy^3} + \frac{1}{4}\frac{d^4 b}{dy^4}\right) + \frac{1}{8}\hbar^4\left(\frac{d^6 a}{dy^6} - \beta\frac{d^4 a}{dy^4}\right) \end{aligned} \tag{7.22c}$$

Equations (7.22a) and (7.22b) admit first integrals

$$\left(\frac{da}{dy}\right)^2 = \beta a^2 + \gamma_0 a + \delta_0 \quad (7.23a)$$

$$\begin{aligned} \frac{da}{dy} \frac{db}{dy} &= \beta ab + \frac{1}{4}\alpha a^2 + \frac{1}{2}\gamma_1 a + \frac{1}{2}\gamma_0 b + \frac{1}{2}V \left(\frac{da}{dy}\right)^2 \\ &+ \hbar^2 \left[\frac{1}{4}\beta \left(\frac{da}{dy}\right)^2 + \frac{5}{16} \left(\frac{d^2a}{dy^2}\right)^2 - \frac{5}{8} \frac{da}{dy} \frac{d^3a}{dy^3} \right] + \delta_1 \end{aligned} \quad (7.23b)$$

Eliminating V from (7.22c) and taking a as our new variable, we find

$$\begin{aligned} 0 &= \frac{1}{4}\hbar^2(\beta a^2 + \gamma_0 a + \delta_0)^2 \frac{d^4b}{da^4} + \frac{3}{4}\hbar^2(2\beta a + \gamma_0)(\beta a^2 + \gamma_0 a + \delta_0) \frac{d^3b}{da^3} \\ &- 6(\beta a^2 + \gamma_0 a + \delta_0) \frac{db}{da} \frac{d^2b}{da^2} + (2\beta a + \gamma_0) \left[2b \frac{d^2b}{da^2} - \left(\frac{db}{da}\right)^2 \right] \\ &+ 2\beta b \frac{db}{da} + [(\alpha + \frac{3}{2}\hbar^2\beta^2)a^2 + (2\gamma_1 + \frac{3}{2}\hbar^2\beta\gamma_0)a + 8\delta_1 - \hbar^2\beta\delta_0 + \hbar^2\gamma_0^2] \frac{d^2b}{da^2} \\ &+ (\alpha a + \gamma_1) \frac{db}{da} - \alpha b - \gamma_2 \end{aligned}$$

There is a first integral

$$\begin{aligned} 0 &= \frac{1}{4}\hbar^2(\beta a^2 + \gamma_0 a + \delta_0)^2 \frac{d^3b}{da^3} + \frac{1}{4}\hbar^2(2\beta a + \gamma_0)(\beta a^2 + \gamma_0 a + \delta_0) \frac{d^2b}{da^2} \\ &- \beta \left(3a \frac{db}{da} - b \right) \left(a \frac{db}{da} - b \right) - \gamma_0 \frac{db}{da} \left(3a \frac{db}{da} - 2b \right) - 3\delta_0 \left(\frac{db}{da} \right)^2 \\ &+ \alpha a \left(a \frac{db}{da} - b \right) + \gamma_1 \left(2a \frac{db}{da} - b \right) + (4\delta_1 + \frac{1}{4}\hbar^2\gamma_0^2 - \beta\hbar^2\delta_0) \frac{db}{da} \\ &- \gamma_2 a + \delta_2 \end{aligned} \quad (7.24)$$

from which we determine

$$\begin{aligned} Q(A) &= -3\delta_0 A^2 + \left[-4\delta_1 + \frac{3}{4}\hbar^2(4\beta\delta_0 - \gamma_0^2) \right] A \\ &+ \delta_2 + \frac{1}{4}\hbar^2(\alpha\delta_0 + 4\beta\delta_1 - \gamma_0\gamma_1) + \frac{1}{8}\hbar^4\beta(\gamma_0^2 - 4\beta\delta_0) \end{aligned}$$

Equation (7.24) is a Chazy equation (Class I in Cosgrove's list [148]). There is a second integral

$$\begin{aligned} 0 &= \frac{1}{8}\hbar^2(\beta a^2 + \gamma_0 a + \delta_0)^2 \left(\frac{d^2b}{da^2}\right)^2 - \beta \frac{db}{da} \left(a \frac{db}{da} - b\right)^2 - \gamma_0 \left(\frac{db}{da}\right)^2 \left(a \frac{db}{da} - b\right) \\ &- \delta_0 \left(\frac{db}{da}\right)^3 + \frac{1}{2}\alpha \left(a \frac{db}{da} - b\right)^2 + \gamma_1 \frac{db}{da} \left(a \frac{db}{da} - b\right) \\ &+ \left(2\delta_1 + \frac{1}{8}\hbar^2\gamma_0^2 - \frac{1}{2}\hbar^2\beta\delta_0\right) \left(\frac{db}{da}\right)^2 - \gamma_2 \left(a \frac{db}{da} - b\right) + \delta_2 \frac{db}{da} + \delta_3 \end{aligned} \quad (7.25)$$

which is called SD-I by Cosgrove and Scoufis [149]. The equation cannot be reduced in order but can be reduced in degree (i.e. to an equation that is linear in the second-derivative). Depending on the values of the parameters, we have a correspondence between the solutions of (7.25) and elliptic functions or one of the six Painlevé transcendents, which are defined as the general solution to the equations

$$\frac{d^2P_1}{d\tau^2} = 6P_1^2 + \tau$$

$$\begin{aligned}
\frac{d^2 P_{II}}{d\tau^2} &= 2P_{II}^3 + \tau P_{II} + \kappa \\
\frac{d^2 P_{III}}{d\tau^2} &= \frac{1}{P_{III}} \left(\frac{dP_{III}}{d\tau} \right)^2 - \frac{1}{\tau} \frac{dP_{III}}{d\tau} + \kappa_1 P_{III}^3 + \frac{1}{\tau} (\kappa_2 P_{III}^2 + \kappa_3) + \frac{\kappa_4}{P_{III}} \\
\frac{d^2 P_{IV}}{d\tau^2} &= \frac{1}{2P_{IV}} \left(\frac{dP_{IV}}{d\tau} \right)^2 + \frac{3}{2} P_{IV}^3 + 4\tau P_{IV}^2 + 2(\tau^2 - \kappa_1) P_{IV} + \frac{\kappa_2}{P_{IV}} \\
\frac{d^2 P_V}{d\tau^2} &= \left(\frac{1}{2P_V} + \frac{1}{P_V - 1} \right) \left(\frac{dP_V}{d\tau} \right)^2 - \frac{1}{\tau} \frac{dP_V}{d\tau} + \frac{(P_V - 1)^2}{\tau^2} \left(\kappa_1 P_V + \frac{\kappa_2}{P_V} \right) \\
&\quad + \frac{\kappa_3 P_V}{\tau} + \frac{\kappa_4 P_V (P_V + 1)}{P_V - 1} \\
\frac{d^2 P_{VI}}{d\tau^2} &= \frac{1}{2} \left(\frac{1}{P_{VI}} + \frac{1}{P_{VI} - 1} + \frac{1}{P_{VI} - \tau} \right) \left(\frac{dP_{VI}}{d\tau} \right)^2 - \left(\frac{1}{\tau} + \frac{1}{\tau - 1} + \frac{1}{P_{VI} - \tau} \right) \frac{dP_{VI}}{d\tau} \\
&\quad + \frac{P_{VI}(P_{VI} - 1)(P_{VI} - \tau)}{\tau^2(\tau - 1)^2} \left[\kappa_1 + \frac{\kappa_2 \tau}{P_{VI}^2} + \frac{\kappa_3(\tau - 1)}{(P_{VI} - 1)^2} + \frac{\kappa_4 P_{VI}(P_{VI} - 1)}{(P_{VI} - \tau)^2} \right]
\end{aligned}$$

Since the Painlevé transcendents are considered canonical special functions in their own right, it is appropriate to consider such a presentation as the *solution* of (7.25). Our formulas, given below, are taken wholly from Cosgrove and Scoufis [149] by modifying our parameters to fit with theirs. The presentation is related to the Hamiltonians of Okamoto [150]. By introducing two auxiliary functions u, v we can write $b = b(u, v, a)$ in such a way that the equations of motion

$$f(a) \frac{du}{da} = \frac{\partial b}{\partial v}, \quad f(a) \frac{dv}{da} = -\frac{\partial b}{\partial u}, \quad \frac{db}{da} = \frac{\partial b}{\partial a}$$

hold. Then u will satisfy a Painlevé equation up to a very simple transformation and v can also be shown to belong to one of the fifty Painlevé-type equations [60].

For a given solution b to (7.25), we obtain the potential via

$$\begin{aligned}
(\beta a^2 + \gamma_0 a + \delta_0) V &= 2(\beta a^2 + \gamma_0 a + \delta_0) \frac{db}{da} - (2\beta a + \gamma_0) b + \left(-\frac{1}{2} \alpha + \frac{1}{8} \hbar^2 \beta^2 \right) a^2 \\
&\quad + \left(-\gamma_1 + \frac{1}{8} \hbar^2 \beta \gamma_0 \right) a - 2\delta_1 + \frac{1}{32} \hbar^2 (24\beta \delta_0 - 5\gamma_0^2).
\end{aligned} \tag{7.26}$$

In the classical limit, (7.24) and (7.25) are two polynomial equations in $b, \frac{db}{da}$ or equivalently V, b . Eliminating the latter, we obtain V as an algebraic function of a . In general the resultant is a quintic equation but can be reduced in order if some of the parameters are known to be zero.

(e) $a = y$

Here we have $\beta = \gamma_0 = 0, \delta_0 = 1$.

For $\alpha = \gamma_1 = \gamma_2 = 0$, b vanishes from (7.25) and can be written as a first-order ODE in V . We take

$$b = -\frac{1}{2} \hbar^2 \zeta(y - \varepsilon) + \frac{2}{3} \delta_1 y, \quad V = \hbar^2 \wp(y - \varepsilon) - \frac{2}{3} \delta_1$$

where ζ, \wp are Weierstrass elliptic functions with the invariants

$$g_2 = \frac{16(4\delta_1^2 + 3\delta_2)}{3\hbar^4}, \quad g_3 = \frac{32(16\delta_1^3 + 18\delta_1\delta_2 + 27\delta_3)}{27\hbar^6}.$$

For $\alpha = \gamma_1 = 0, \gamma_2 \neq 0$, we take

$$\begin{aligned}\gamma_2 &= \frac{\hbar^4}{8\varepsilon^5}, & u &= P_I \left(\frac{y}{\varepsilon} - \frac{8(4\delta_1^2 + 3\delta_2)\varepsilon^4}{3\hbar^4} \right), \\ b &= -\frac{\hbar^2}{4\varepsilon}(u')^2 + \frac{\hbar^2}{\varepsilon}u^3 + \left(\frac{\hbar^2 y}{2\varepsilon^2} - \frac{16\varepsilon^3\delta_1}{3\hbar^2} - \frac{4\varepsilon^3\delta_2}{\hbar^2} \right)u + \frac{2\delta_1 y}{3} - \frac{128\varepsilon^5\delta_1^3}{27\hbar^4} - \frac{8\delta_3\varepsilon^5}{\hbar^4}, \\ V &= \frac{\hbar^2 u}{\varepsilon^2} - \frac{2\delta_1}{3}.\end{aligned}$$

For $\alpha = \gamma_2 = 0, \gamma_1 \neq 0$, we take

$$\begin{aligned}\gamma_1 &= \frac{\hbar^2}{4\varepsilon^3}, & u &= P_{II} \left(\frac{y}{\varepsilon} + \frac{8\delta_1\varepsilon^2}{\hbar^2}; \kappa \right), & \delta_3 &= -\frac{\hbar^6}{512\varepsilon^6}(2\kappa \pm 1)^2, \\ b &= -\frac{\hbar^2}{4\varepsilon}(u')^2 + \frac{\hbar^2 u^4}{4\varepsilon} + \left(\frac{\hbar^2 y}{4\varepsilon^2} + 2\delta_1\varepsilon \right)u^2 + \left(\frac{\hbar^2 \kappa}{2\varepsilon} \pm \frac{\hbar^2}{4\varepsilon} \right)u \\ &\quad + \frac{\hbar^2 y^2}{16\varepsilon^3} + \delta_1 y + \frac{4\delta_1^2\varepsilon^3}{\hbar^2} + \frac{4\delta_2\varepsilon^3}{\hbar^2}, \\ V &= \pm \frac{\hbar^2}{2\varepsilon^2}u' + \frac{\hbar^2}{2\varepsilon^2}u^2.\end{aligned}$$

For $\gamma_1 = \gamma_2 = 0, \alpha \neq 0$, we take

$$\begin{aligned}\alpha &= -\frac{\hbar^2}{\varepsilon^4}, & u &= P_{IV} \left(\frac{y}{\varepsilon}; \kappa_1, \kappa_2 \right), & \delta_2 &= \frac{1}{24\varepsilon^4} [2\hbar^4(\kappa_1 \mp 1)^2 - 32\delta_1^2\varepsilon^4 - 3\hbar^4\kappa_2], \\ \delta_3 &= \frac{1}{216\varepsilon^6} (\hbar^2\kappa_1 + 2\delta_1\varepsilon^2 \pm \hbar^2)(2\hbar^4 \pm 16\hbar^2\delta_1\varepsilon^2 + 32\delta_1^2\varepsilon^4 \mp 4\hbar^4\kappa_1 \\ &\quad - 16\hbar^2\varepsilon^2\delta_1\kappa_1 + 2\hbar^4\kappa_1^2 + 9\hbar^4\kappa_2), \\ b &= -\frac{\hbar^2}{16\varepsilon u}(u')^2 + \frac{\hbar^2 u^3}{16\varepsilon} + \frac{\hbar^2 y}{4\varepsilon^2}u^2 + \frac{\hbar^2}{4\varepsilon^3}(y^2 \pm \varepsilon^2 - \kappa_1\varepsilon^2)u \\ &\quad + \frac{y}{6\varepsilon^2}(4\delta_1\varepsilon^2 \pm \hbar^2 - \kappa_1\hbar^2) - \frac{\hbar^2\kappa_2}{8\varepsilon u} \\ V &= \pm \frac{\hbar^2}{2\varepsilon^2}u' + \frac{\hbar^2}{2\varepsilon^2}u^2 + \frac{\hbar^2 y}{\varepsilon^3}u + \frac{\hbar^2 y^2}{2\varepsilon^4} - \frac{2\delta_1}{3} - \frac{\hbar^2\kappa_1}{3\varepsilon^2} \pm \frac{\hbar^2}{3\varepsilon^2}.\end{aligned}$$

which is the same potential we derived in section 5.2 for a third-order ladder operator.

(f) $a = y^2$

Here we have $\beta = 0, \gamma_0 = 4$.

For $\alpha = \gamma_1 = 0$, we take

$$\begin{aligned}
u &= P_{\text{III}}(y; \kappa_1, \kappa_2, \kappa_3, \kappa_4), & \delta_2 &= \frac{\hbar^2}{4}(2\kappa_3\sqrt{\kappa_1} + \kappa_2\kappa_3) \\
\delta_3 &= -\frac{\hbar^4}{32}(\hbar^2\kappa_1\kappa_3^2 - 8\hbar^2\kappa_1\kappa_4 - 4\delta_1\kappa_1\kappa_4 - 4\hbar^2\kappa_2\kappa_4\sqrt{\kappa_1} - \hbar^2\kappa_2^2\kappa_4) \\
b &= -\frac{\hbar^2 y^2}{8u^2}(u')^2 + \frac{\hbar^2 y}{4u}u' + \frac{1}{8}\hbar^2\kappa_1 y^2 u^2 + \frac{\hbar^2}{4}(\kappa_2 y + 2\sqrt{\kappa_1})u \\
&\quad - \frac{\delta_1}{2} - \frac{5\hbar^2}{8} - \frac{\hbar^2\kappa_3 y}{4u} - \frac{\hbar^2\kappa_4 y^2}{8u^2} \\
V &= \frac{\hbar^2}{8u^2}(u')^2 + \frac{\hbar^2}{4}\left(2\sqrt{\kappa_1} - \frac{1}{yu}\right)u' + \frac{3\hbar^2\kappa_1}{8}u^2 + \frac{\hbar^2\kappa_2}{4y}u + \frac{\hbar^2\kappa_3}{4yu} + \frac{\hbar^2\kappa_4}{8u^2}
\end{aligned}$$

For $\alpha \neq 0, \gamma_1 = \gamma_2 = 0$, we take

$$\begin{aligned}
u &= P_{\text{V}}(y^2; \kappa_1, \kappa_2, \kappa_3, \kappa_4), & \alpha &= 8\hbar^2\kappa_4, \\
\delta_1 &= \frac{\hbar^2}{4\kappa_4}(\kappa_3^2 - 8\kappa_4 + 8\kappa_4\sqrt{2\kappa_1} - 8\kappa_1\kappa_4 + 8\kappa_2\kappa_4), \\
\delta_2 &= 2\hbar^4\kappa_3(1 + 2\kappa_1 + 2\kappa_2 - 2\sqrt{2\kappa_1}), \\
\delta_3 &= \frac{\hbar^6}{16\kappa_4}(\kappa_3^2 - 16\kappa_2\kappa_4)(16\kappa_4\sqrt{2\kappa_1} - \kappa_3^2 - 8\kappa_4 - 16\kappa_1\kappa_4), \\
b &= -\frac{\hbar^2}{2u}\left(\frac{y^2 u'}{u-1} - u\right)^2 + \frac{\hbar^2}{2}(u-1)\left(\sqrt{2\kappa_1} - 1\right)^2 + \frac{\hbar^2\kappa_2(u-1)}{u} \\
&\quad - \frac{\hbar^2\kappa_3 y^2(u+1)}{2(u-1)} - \frac{\hbar^2\kappa_4 y^4 u}{(u-1)^2} - \frac{\hbar^2}{8\kappa_4}(\kappa_3^2 - 4\kappa_4 + 8\kappa_4\sqrt{2\kappa_1} - 8\kappa_1\kappa_4 + 8\kappa_2\kappa_4), \\
V &= \frac{\hbar^2 y^2(2u-1)}{2u(u-1)^2}(u')^2 - \hbar^2\left(\frac{1}{u-1} + 2\sqrt{2\kappa_1}\right)u' + \frac{2\hbar^2\kappa_1}{y^2}u^2 + \frac{\hbar^2\left(\sqrt{2\kappa_1} - 3\kappa_1\right)}{y^2}u \\
&\quad - \frac{\hbar^2}{2y^2}\left(\sqrt{2\kappa_1} - 1\right)^2 + \frac{\hbar^2\kappa_2(u-1)}{y^2 u} + \frac{\hbar^2\kappa_3(3u-1)}{2(u-1)} + \frac{\hbar^2\kappa_4 y^2 u(2u-1)}{(u-1)^2} \\
&\quad - \hbar^2\kappa_4 y^2 - \frac{\hbar^2}{8y^2}
\end{aligned}$$

$$(g) \quad a = e^{y\sqrt{\beta}}$$

Here we have $\beta \neq 0, \gamma_0 = \delta_0 = 0$. We set $\gamma_1 = 0$.

For $\delta_1 = 0$, we take

$$\begin{aligned} u &= P_{\text{III}} \left(e^{-\frac{1}{2}y\sqrt{\beta}}; \kappa_1, \kappa_2, \kappa_3, \kappa_4 \right), \quad \gamma_2 = -\frac{1}{256} \hbar^4 \beta^3 \kappa_3 (\kappa_2 + 2\sqrt{\kappa_1}), \\ \delta_2 &= -\frac{1}{256} \hbar^4 \beta^3 \kappa_1 \kappa_4, \\ \delta_3 &= -\frac{\hbar^4 \beta^2}{8192} (\hbar^2 \beta^2 \kappa_1 \kappa_3^2 - 16\kappa_1 \kappa_4 \alpha - 4\hbar^2 \beta^2 \kappa_1 \kappa_4 - 4\hbar^2 \beta^2 \kappa_2 \kappa_4 \sqrt{\kappa_1} - \hbar^2 \beta^2 \kappa_2^2 \kappa_4), \\ b &= \frac{\hbar^2 \beta}{32u^2} (u')^2 - \frac{\hbar^2 \beta}{16u} e^{\frac{1}{2}y\sqrt{\beta}} u' - \frac{\hbar^2 \beta \kappa_1}{32} u^2 - \frac{\hbar^2 \beta}{16} (2\sqrt{\kappa_1} + \kappa_2) u e^{\frac{1}{2}y\sqrt{\beta}} \\ &\quad + \frac{16\alpha + \hbar^2 \beta^2}{32\beta} e^{y\sqrt{\beta}} + \frac{\hbar^2 \beta \kappa_3}{16u} e^{\frac{1}{2}y\sqrt{\beta}} + \frac{\hbar^2 \beta \kappa_4}{32u^2} \\ V &= \frac{1}{8} \hbar^2 \beta u' \sqrt{\kappa_1} e^{-y\sqrt{\beta}} + \frac{1}{8} \hbar^2 \beta \kappa_1 u^2 e^{-y\sqrt{\beta}} \\ &\quad + \frac{\hbar^2 \beta}{8} (\sqrt{\kappa_1} + \kappa_2) u e^{-\frac{1}{2}y\sqrt{\beta}} - \frac{\alpha}{2\beta} + \frac{\hbar^2 \beta}{8} \end{aligned}$$

For $\delta_1 \neq 0$ (this solution is also valid when $\delta_1 = 0, \delta_2 \neq 0$), we take

$$\begin{aligned} u &= P_{\text{V}} \left(e^{-y\sqrt{\beta}}; \kappa_1, \kappa_2, \kappa_3, \kappa_4 \right), \quad \gamma_2 = -\frac{\hbar^4 \beta^3 \kappa_3}{32} \left[\left(\sqrt{2\kappa_1} - 1 \right)^2 + 2\kappa_2 \right], \\ \delta_1 &= \frac{1}{8} \hbar^2 \beta^2 \kappa_4, \\ \delta_2 &= \frac{\hbar^2 \beta}{64} \left(\hbar^2 \beta^2 \kappa_3^2 - 16\alpha \kappa_4 - 4\hbar^2 \beta^2 \kappa_4 + 8\hbar^2 \beta^2 \kappa_4 \sqrt{2\kappa_1} - 8\hbar^2 \beta^2 \kappa_1 \kappa_4 + 8\hbar^2 \beta^2 \kappa_2 \kappa_4 \right), \\ \delta_3 &= \frac{\hbar^2}{256} \left(16\alpha^2 \kappa_4 - 2\alpha \beta^2 \kappa_3^2 \hbar^2 + 8\alpha \beta^2 \kappa_4 \hbar^2 + 16\alpha \beta^2 \kappa_1 \kappa_4 \hbar^2 - 16\alpha \beta^2 \kappa_2 \kappa_4 \hbar^2 \right. \\ &\quad - 16\alpha \beta^2 \sqrt{2\kappa_1} \kappa_4 \hbar^2 - \beta^4 \kappa_3^2 \hbar^4 - 2\beta^4 \kappa_1 \kappa_3^2 \hbar^4 + 2\beta^4 \kappa_2 \kappa_3^2 \hbar^4 + 2\beta^4 \sqrt{2\kappa_1} \kappa_3^2 \hbar^4 \\ &\quad + \beta^4 \kappa_4 \hbar^4 + 4\beta^4 \kappa_1^2 \kappa_4 \hbar^4 + 4\beta^4 \kappa_2^2 \kappa_4 \hbar^4 - 8\sqrt{2} \beta^4 \kappa_1^{3/2} \kappa_4 \hbar^4 + 12\beta^4 \kappa_1 \kappa_4 \hbar^4 \\ &\quad \left. + 4\beta^4 \kappa_2 \kappa_4 \hbar^4 + 8\beta^4 \kappa_1 \kappa_2 \kappa_4 \hbar^4 - 8\beta^4 \sqrt{2\kappa_1} \kappa_2 \kappa_4 \hbar^4 - 4\beta^4 \sqrt{2\kappa_1} \kappa_4 \hbar^4 \right), \\ b &= \frac{\hbar^2 \beta e^{-y\sqrt{\beta}}}{8u(u-1)^2} (u')^2 - \frac{\hbar^2 \beta}{4(u-1)} u' + \frac{\hbar^2 \beta}{4} \left(\sqrt{2\kappa_1} - \kappa_1 \right) (u-1) e^{y\sqrt{\beta}} \\ &\quad - \frac{\hbar^2 \beta (u-1)}{4u} \kappa_2 e^{y\sqrt{\beta}} + \frac{\hbar^2 \beta \kappa_3 (u+1)}{8(u-1)} + \frac{\hbar^2 \beta \kappa_4 u e^{-y\sqrt{\beta}}}{4(u-1)^2} \\ &\quad + \left(\frac{\alpha}{2\beta} + \frac{\hbar^2 \beta}{8} \right) e^{y\sqrt{\beta}}, \\ V &= \frac{\hbar^2 \beta e^{-2y\sqrt{\beta}}}{4u(u-1)} (u')^2 - \hbar^2 \beta u' e^{-y\sqrt{\beta}} \sqrt{\frac{\kappa_1}{2}} + \frac{\hbar^2 \beta \kappa_1}{2} u(u-1) + \frac{\hbar^2 \beta \kappa_2 (u-1)}{2u} \\ &\quad + \frac{\hbar^2 \beta \kappa_3}{4} e^{-y\sqrt{\beta}} + \frac{\hbar^2 \beta \kappa_4 (u+1)}{4(u-1)} e^{-2y\sqrt{\beta}} - \frac{\alpha}{2\beta} + \frac{\hbar^2 \beta}{8} \end{aligned}$$

$$(h) \ a = \cosh\left(y\sqrt{\beta}\right)$$

We have $\gamma_0 = 0, \delta_0 = -\beta$. We set $\gamma_1 = 0$ and take

$$\begin{aligned} u &= P_{VI}\left(\sin^2\left(\frac{1}{2}y\sqrt{\beta}\right); \kappa_1, \kappa_2, \kappa_3, \kappa_4\right), \quad \gamma_2 = -\frac{1}{8}\hbar^4\beta^3(\kappa_2 + \kappa_3)\left(\sqrt{2\kappa_1} - \kappa_1 - \kappa_4\right), \\ \delta_1 &= \frac{1}{4}\left(-3\alpha - \beta^2\kappa_1\hbar^2 + \beta^2\kappa_2\hbar^2 - \beta^2\kappa_3\hbar^2 + \beta^2\kappa_4\hbar^2 + \beta^2\sqrt{2\kappa_1}\hbar^2 - 2\beta^2\hbar^2\right), \\ \delta_2 &= \frac{1}{16\beta}\left(12\alpha^2 + 8\alpha\beta^2\kappa_1\hbar^2 - 8\alpha\beta^2\kappa_2\hbar^2 + 8\alpha\beta^2\kappa_3\hbar^2 - 8\alpha\beta^2\kappa_4\hbar^2 - 8\alpha\beta^2\sqrt{2\kappa_1}\hbar^2 \right. \\ &\quad + 8\alpha\beta^2\hbar^2 + \beta^4\kappa_1^2\hbar^4 + \beta^4\kappa_2^2\hbar^4 + \beta^4\kappa_3^2\hbar^4 + \beta^4\kappa_4^2\hbar^4 - 2\sqrt{2}\beta^4\kappa_1^{3/2}\hbar^4 + 2\beta^4\kappa_1\hbar^4 \\ &\quad - 4\beta^4\kappa_2\hbar^4 - 4\beta^4\kappa_1\kappa_2\hbar^4 + 4\beta^4\sqrt{2\kappa_1}\kappa_2\hbar^4 + 4\beta^4\kappa_3\hbar^4 + 4\beta^4\kappa_1\kappa_3\hbar^4 + 2\beta^4\kappa_2\kappa_3\hbar^4 \\ &\quad \left. - 4\beta^4\sqrt{2\kappa_1}\kappa_3\hbar^4 + 2\beta^4\kappa_1\kappa_4\hbar^4 + 4\beta^4\kappa_2\kappa_4\hbar^4 - 4\beta^4\kappa_3\kappa_4\hbar^4 - 2\beta^4\sqrt{2\kappa_1}\kappa_4\hbar^4\right), \\ \delta_3 &= \frac{1}{32\beta^2}\left(-\beta^6\kappa_2^2\hbar^6 - \beta^6\kappa_1\kappa_2^2\hbar^6 + \beta^6\sqrt{2\kappa_1}\kappa_2^2\hbar^6 - \beta^6\kappa_3^2\hbar^6 - \beta^6\kappa_1\kappa_3^2\hbar^6 - 2\beta^6\kappa_1\kappa_2\kappa_3\hbar^6 \right. \\ &\quad + \beta^6\sqrt{2\kappa_1}\kappa_3^2\hbar^6 + \beta^6\kappa_2\kappa_4^2\hbar^6 - \beta^6\kappa_3\kappa_4^2\hbar^6 + \beta^6\kappa_1^2\kappa_2\hbar^6 - 2\sqrt{2}\beta^6\kappa_1^{3/2}\kappa_2\hbar^6 \\ &\quad + 2\beta^6\kappa_1\kappa_2\hbar^6 - \beta^6\kappa_1^2\kappa_3\hbar^6 + 2\sqrt{2}\beta^6\kappa_1^{3/2}\kappa_3\hbar^6 - 2\beta^6\kappa_1\kappa_3\hbar^6 - 2\beta^6\kappa_2\kappa_3\hbar^6 \\ &\quad + 2\beta^6\sqrt{2\kappa_1}\kappa_2\kappa_3\hbar^6 + \beta^6\kappa_2^2\kappa_4\hbar^6 + \beta^6\kappa_3^2\kappa_4\hbar^6 + 2\beta^6\kappa_1\kappa_2\kappa_4\hbar^6 - 2\beta^6\sqrt{2\kappa_1}\kappa_2\kappa_4\hbar^6 \\ &\quad - 2\beta^6\kappa_1\kappa_3\kappa_4\hbar^6 + 2\beta^6\kappa_2\kappa_3\kappa_4\hbar^6 + 2\beta^6\sqrt{2\kappa_1}\kappa_3\kappa_4\hbar^6 - \alpha\beta^4\kappa_1^2\hbar^4 - \alpha\beta^4\kappa_2^2\hbar^4 \\ &\quad - \alpha\beta^4\kappa_3^2\hbar^4 - \alpha\beta^4\kappa_4^2\hbar^4 + 2\sqrt{2}\alpha\beta^4\kappa_1^{3/2}\hbar^4 - 2\alpha\beta^4\kappa_1\hbar^4 + 4\alpha\beta^4\kappa_2\hbar^4 + 4\alpha\beta^4\kappa_1\kappa_2\hbar^4 \\ &\quad - 4\alpha\beta^4\sqrt{2\kappa_1}\kappa_2\hbar^4 - 4\alpha\beta^4\kappa_3\hbar^4 - 4\alpha\beta^4\kappa_1\kappa_3\hbar^4 - 2\alpha\beta^4\kappa_2\kappa_3\hbar^4 + 4\alpha\beta^4\sqrt{2\kappa_1}\kappa_3\hbar^4 \\ &\quad - 2\alpha\beta^4\kappa_1\kappa_4\hbar^4 - 4\alpha\beta^4\kappa_2\kappa_4\hbar^4 + 4\alpha\beta^4\kappa_3\kappa_4\hbar^4 + 2\alpha\beta^4\sqrt{2\kappa_1}\kappa_4\hbar^4 - 4\alpha^2\beta^2\hbar^2 \\ &\quad \left. - 4\alpha^2\beta^2\kappa_3\hbar^2 + 4\alpha^2\beta^2\kappa_4\hbar^2 + 4\alpha^2\beta^2\sqrt{2\kappa_1}\hbar^2 - 4\alpha^3 + 4\alpha^2\beta^2\kappa_2\hbar^2 - 4\alpha^2\beta^2\kappa_1\hbar^2\right), \end{aligned}$$

$$\begin{aligned} b &= \frac{\hbar^2\beta\sin^4\left(y\sqrt{\beta}\right)}{32u(u-1)\left[2u-1+\cos\left(y\sqrt{\beta}\right)\right]}(u')^2 + \frac{\hbar^2\beta\sin^2\left(y\sqrt{\beta}\right)}{8u-4+4\cos\left(y\sqrt{\beta}\right)}u' \\ &\quad + \frac{\hbar^2\beta}{4}\left(\sqrt{2\kappa_1}-\kappa_1\right)(2u-1) - \frac{\hbar^2\beta\kappa_2}{4u}\left[u-1+\cos\left(y\sqrt{\beta}\right)\right] \\ &\quad - \frac{\hbar^2\beta\kappa_3}{4(u-1)}\left[u+\cos\left(y\sqrt{\beta}\right)\right] - \frac{\hbar^2\beta\kappa_4\left[1+(2u-1)\cos\left(y\sqrt{\beta}\right)\right]}{4\left[2u-1+\cos\left(y\sqrt{\beta}\right)\right]} \\ &\quad + \frac{\alpha}{2\beta}\cos\left(y\sqrt{\beta}\right), \end{aligned}$$

$$\begin{aligned}
V = & -\frac{\hbar^2 \beta (2u-1) \sin^2(y\sqrt{\beta})}{16u(u-1) \left[2u-1 + \cos(y\sqrt{\beta})\right]} (u')^2 - \frac{\alpha}{2\beta} + \frac{\hbar^2 \beta}{8} - \frac{3\hbar^2 \beta}{8 \sin^2(y\sqrt{\beta})} \\
& + \frac{\hbar^2 \beta u'}{4 \left[u - \sin^2\left(\frac{1}{2}y\sqrt{\beta}\right)\right]} \left[\cos(y\sqrt{\beta}) - 2u\sqrt{2\kappa_1} + 2\sqrt{2\kappa_1} \sin^2\left(\frac{1}{2}y\sqrt{\beta}\right)\right] \\
& + \frac{1}{2} \hbar^2 \beta (2u-1) \left(\sqrt{2\kappa_1} - \kappa_1\right) \cot(y\sqrt{\beta}) \csc(y\sqrt{\beta}) \\
& + \frac{1}{2} \hbar^2 \beta \left(\sqrt{2\kappa_1} - \kappa_1 + 4u\kappa_1 - 4u^2\kappa_1\right) \csc^2(y\sqrt{\beta}) - \frac{\hbar^2 \beta \kappa_2 (u-1)}{4u \cos^2\left(\frac{1}{2}y\sqrt{\beta}\right)} \\
& + \frac{\hbar^2 \beta \kappa_3 u}{4(u-1) \sin^2\left(\frac{1}{2}y\sqrt{\beta}\right)} + \frac{\hbar^2 \beta \kappa_4 (2u-1)}{2 \left[2u-1 + \cos(y\sqrt{\beta})\right]}
\end{aligned}$$

7.5 Extended Hamiltonians

Suppose

$$H = \left\{ (\chi')^2, \frac{1}{2} p_x^2 + A + U(x) \right\}$$

admits operators J, K , which depend only on x, p_x, A , that satisfy

$$\begin{aligned}
[H, J] &= i\hbar K (\chi')^2, \\
[H, K] &= i\hbar [(\alpha + 2\beta A)J + F(H, A) + i\hbar \beta K] (\chi')^2.
\end{aligned} \tag{7.27}$$

Then the operators

$$\begin{aligned}
X &= A, \\
Y &= KC - JP(A) - [(\alpha + 2\beta A)J + F(H, A)]B, \\
Z &= K[P(A) + \alpha B + 2\beta\{A, B\}] - [(\alpha + 2\beta A)J + F]C
\end{aligned}$$

will commute with H making it superintegrable. Furthermore, we will have the commutation relations

$$\begin{aligned}
[X, Y] &= i\hbar Z \\
[X, Z] &= i\hbar [\alpha Y + 2\beta\{X, Y\} - F(H, X)P(X)]
\end{aligned}$$

7.5.1 Elementary Realisations

We can construct a realisation of (7.27) as follows. Take

$$J = a(x), \quad F = \mu_0 A + \mu_1 H + \mu_2$$

Then $K = -\{a', p_x\} - i\hbar a' \chi'' / \chi'$. We write

$$u_0 A + \{u_1, H\} + i\hbar \{u_2, p_x\} + u_3 := \frac{[H, K]}{i\hbar} - [\gamma_0 H + \gamma_1 A + \gamma_2 + i\hbar \beta K + (\alpha + 2\beta A)J]$$

Setting $u_i \equiv 0$ gives us four equations. There is one independence relation, namely

$$2(\chi')^2 u_1' + 2u_2 + (\chi')^2 \left[\frac{u_0}{(\chi')^2} \right]' = 0$$

so we only have three independent conditions. Rearranging the terms in u_0, u_2, u_3 , we have:

$$\mu_0 = -2\beta a - 2a'' \quad (7.28a)$$

$$\mu_1 = \frac{2a''}{(\chi')^2} - \frac{2a'\chi''}{(\chi')^3} \quad (7.28b)$$

$$\mu_2 = -\alpha a - a'U' - 2a''U + \hbar^2 \left[\frac{a'\chi''\chi'''}{(\chi')^2} - \frac{a'(\chi'')^3}{(\chi')^3} + \frac{a''(\chi'')^2}{(\chi')^2} + \frac{1}{4}a'''' \right] \quad (7.28c)$$

Equation (7.28b) has the general solution

$$a = \frac{1}{4}\mu_1\chi^2 + \delta_0\chi + \delta_1$$

so

$$(\chi')^2 = \frac{(a')^2}{\mu_1(a - \delta_1) + \delta_1^2}$$

while (7.28c) is essentially the same as (7.20b) with an additional quantum correction. After relabeling the parameters, we find:

(i) $a = x$

$$H = \left\{ \frac{1}{\mu_1 x + \nu_1}, \frac{1}{2}p_x^2 + A - \frac{1}{2}\alpha x^2 - \mu_2 x + \nu_2 \right\} + \frac{\hbar^2 \mu_1^2}{8(\mu_1 x + \nu_1)^3}$$

Depending on μ_1, μ_2 we have, up to equivalence, two metrics if A is a one-dimensional Hamiltonian:

$$dx^2 + dy^2, \quad x(dx^2 + dy^2)$$

That is, the Euclidean plane and Darboux surface I.

(j) $a = x^2$

$$H = \left\{ \frac{4x^2}{\mu_1 x^2 + \nu_1}, \frac{1}{2}p_x^2 + A - \frac{1}{8}\alpha x^2 + \frac{\nu_2}{x^2} - \frac{1}{4}\mu_2 \right\} + \frac{2\hbar^2 \nu_1^2}{(\mu_1 x^2 + \nu_1)^3}$$

We have three metrics

$$dx^2 + dy^2, \quad x^{-2}(dx^2 + dy^2), \quad (1 + x^{-2})(dx^2 + dy^2)$$

which correspond to the Euclidean plane, the Poincaré half-plane and Darboux surface II respectively.

(k) $a = e^{x\sqrt{-\beta}}$

$$H = \left\{ \frac{-\beta e^{2x\sqrt{-\beta}}}{\mu_1 e^{x\sqrt{-\beta}} + \nu_1}, \frac{1}{2}p_x^2 + A + \frac{\mu_2}{\beta} e^{-x\sqrt{-\beta}} + \nu_2 e^{-2x\sqrt{-\beta}} \right. \\ \left. + \frac{\alpha}{2\beta} - \frac{1}{8}\hbar^2\beta - \frac{\hbar^2\beta (\mu_1 e^{x\sqrt{-\beta}} + 2\nu_1)^2}{8(\mu_1 e^{x\sqrt{-\beta}} + \nu_1)^2} \right\}$$

If we set $\mu_1 = 0, v_1 = \beta^2, \xi = e^{-x\sqrt{-\beta}}, p_x = -\sqrt{-\beta} \{\xi, p_\xi\}$ then

$$H = \frac{1}{2}p_\xi^2 - \frac{\mu_2}{\beta^2\xi} - \frac{v_2}{\beta} - \frac{1}{\beta\xi^2} \left(A + \frac{\alpha}{2\beta} \right) \quad (7.29)$$

which is a Kepler-Coulomb extension. Alternatively, if we take $\mu_1 = \frac{1}{4}\beta^2, v_1 = 0, \xi = e^{-\frac{1}{2}x\sqrt{-\beta}}, p_x = -\frac{1}{2}\sqrt{-\beta} \{\xi, p_\xi\}$, then

$$H = \frac{1}{2}p_\xi^2 + \left(\frac{\hbar^2}{2} - \frac{4v_2}{\beta} \right) \xi^2 - \frac{4\mu_2}{\beta^2} - \frac{4}{\beta\xi^2} \left(A + \frac{\alpha}{2\beta} - \frac{3\hbar^2\beta}{32} \right) \quad (7.30)$$

which is a singular oscillator extension. Thus, we have only two metrics in this case

$$d\xi^2 + \xi^{-2}dy^2, \quad (e^x + e^{2x})(dx^2 + dy^2)$$

the former being the Euclidean plane in polar coordinates and the latter is Darboux surface III.

$$(1) a = \cos(x\sqrt{\beta})$$

$$H = \left\{ \frac{\beta \sin^2(x\sqrt{\beta})}{\mu_1 \cos(x\sqrt{\beta}) + v_1}, \frac{1}{2}p_x^2 + \frac{v_2 - \mu_2\beta^{-1} \cos(x\sqrt{\beta})}{\sin^2(x\sqrt{\beta})} + \frac{\alpha}{2\beta} - \frac{1}{8}\hbar^2\beta \right. \\ \left. + \frac{\hbar^2\beta \left[\mu_1^2 \sin^2(x\sqrt{\beta}) - 4\mu_1 v_1 \cos(x\sqrt{\beta}) - 4v_1^2 \right]}{8 \left[\mu_1 \cos(x\sqrt{\beta}) + v_1 \right]^2} \right\}$$

Let us take $\mu_1 = 0, v_1 = \beta, \sin(x\sqrt{\beta}) \cosh(\xi\sqrt{\beta}) = 1, p_x = \left\{ \cosh(\xi\sqrt{\beta}), p_\xi \right\}$. We get

$$H = \frac{1}{2}p_\xi^2 + \left(A + \frac{\alpha}{2\beta} \right) \operatorname{sech}^2(\xi\sqrt{\beta}) + v_2 - \frac{3\hbar^2\beta}{8} + \frac{\mu_2}{\beta} \tanh(\xi\sqrt{\beta}) \quad (7.31)$$

This is an extension by a Rosen-Morse potential. Let us take $\mu_1 = v_1 = 2\beta, \sin\left(\frac{1}{2}x\sqrt{\beta}\right) \cosh\left(\frac{1}{2}\xi\sqrt{\beta}\right) = 1, p_x = \left\{ \cosh\left(\frac{1}{2}\xi\sqrt{\beta}\right), p_\xi \right\}$. We find

$$H = \frac{1}{2}p_\xi^2 - \frac{3\hbar^2\beta}{32} - \frac{\mu_2}{4\beta} + \frac{v_2}{4} + \left(A + \frac{\alpha}{2\beta} - \frac{3\hbar^2\beta}{32} \right) \operatorname{sech}^2\left(\frac{1}{2}\xi\sqrt{\beta}\right) \\ + \left(\frac{\mu_2}{4\beta} + \frac{v_2}{4} - \frac{\hbar^2\beta}{8} \right) \operatorname{csch}^2\left(\frac{1}{2}\xi\sqrt{\beta}\right) \quad (7.32)$$

which is an extension by the Pöschl-Teller model. We therefore have, up to translation and scaling, two metrics

$$d\xi^2 + \frac{dy^2}{\sin^2\xi}, \quad \csc x(\cot x + v \csc x)(dx^2 + dy^2)$$

which corresponds to the sphere and Darboux surface IV.

7.5.2 Superintegrable Models

We denote a substitution of A into any of the forms of H by $(*) + (*)$.

Example 7.1. The superposition (i)+(a) corresponds to the system

$$H = \left\{ \frac{1}{\mu_1 x + \nu_1}, \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 - \frac{1}{2} \alpha (x^2 + y^2) - \mu_2 x - \gamma_1 y + \nu_2 \right\} + \frac{\hbar^2 \mu_1^2}{8(\mu_1 x + \nu_1)^3}$$

When $\mu_1 = 0, \nu_1 = 1$, H is a simple isotropic harmonic oscillator for $\alpha \neq 0$ and a Galilean potential when $\alpha = 0$. For $\mu_1 \neq 0$, H is a coupling-constant metamorphosis of the same onto a Darboux surface. \square

We enumerate the possible combinations that lead to quadratic superintegrability in the table below. These are all well-known systems, and we follow the designations as they are given in [51, 151].

Table 7.1. Quadratically superintegrable systems that are one-dimensional extensions

Model	Name	Model	Name
(i)+(a)	$\begin{cases} \text{E3}', & \mu_1 = 0, \nu_1 \neq 0 \\ \text{D1C}, & \mu_1 \neq 0 \end{cases}$	(i)+(b)	$\begin{cases} \text{E2}, & \mu_1 = 0, \nu_1 \neq 0 \\ \text{D1B}, & \mu_1 \neq 0 \end{cases}$
(j)+(a)	$\begin{cases} \text{E2}, & \mu_1 \neq 0, \nu_1 = 0 \\ \text{S1}, & \mu_1 = 0, \nu_1 \neq 0 \\ \text{D2A}, & \mu_1, \nu_1 \neq 0 \end{cases}$	(j)+(b)	$\begin{cases} \text{E1}, & \mu_1 \neq 0, \nu_1 = 0 \\ \text{S2}, & \mu_1 = 0, \nu_1 \neq 0 \\ \text{D2B}, & \mu_1, \nu_1 \neq 0 \end{cases}$
(k)+(c)	$\begin{cases} \text{E8}, & \mu_1 \neq 0, \nu_1 = 0 \\ \text{E17}, & \mu_1 = 0, \nu_1 \neq 0 \\ \text{D3D}, & \mu_1, \nu_1 \neq 0 \end{cases}$	(k)+(d)	$\begin{cases} \text{E1}, & \mu_1 \neq 0, \nu_1 = 0 \\ \text{E16}, & \mu_1 = 0, \nu_1 \neq 0 \\ \text{D3C}, & \mu_1, \nu_1 \neq 0 \end{cases}$
(l)+(c)	$\begin{cases} \text{S2}, & \mu_1 \neq 0, \nu_1 = \pm \mu_1 \\ \text{S4}, & \mu_1 = 0, \nu_1 \neq 0 \\ \text{D4C}, & \mu_1 \neq 0, \nu_1 \neq \pm \mu_1 \end{cases}$	(l)+(d)	$\begin{cases} \text{S9}, & \mu_1 \neq 0, \nu_1 = \pm \mu_1 \\ \text{S7}, & \mu_1 = 0, \nu_1 \neq 0 \\ \text{D4A}, & \mu_1 \neq 0, \nu_1 \neq \pm \mu_1 \end{cases}$

Table 7.1 encompasses all of the models which on a Darboux surface or a surface of constant-curvature which have separation in subgroup-separable coordinates. The other admissible superpositions are (e)+(e), (e)+(f), (f)+(f), (i)+(e), (i)+(f), (j)+(e), (j)+(f), (k)+(g), (k)+(h), (l)+(g) and (l)+(h). The known third- and fourth-order superintegrable systems on the Euclidean plane are included within this catalogue.

Example 7.2. The exotic third-order superintegrable models [29] with separation in Cartesian coordinates are (i)+(e) when $\mu_1 = 0$ and (e)+(e) when $\alpha = \gamma_1 = 0$ for each component. \square

Example 7.3. The exotic fourth-order superintegrable models [31] with separation in Cartesian coordinates are (i)+(f) for $\mu_1 = 0$ and (j)+(e), (j)+(f) when $\nu_1 = 0$. \square

Example 7.4. The exotic fourth-order superintegrable models with separation in polar coordinates are (k)+(g) and (k)+(h) when either $\mu_1 = 0$ or $\nu_1 = 0$. They are third-order superintegrable when $\gamma_2 = 0$. The model (k)+(h) appears in [34, 35] however we believe that (k)+(g) is new. It is only real when $\beta > 0$ and the metric is indefinite. \square

7.5.3 Addition Laws

In this section we show how we may dilate the parameters to produce new superintegrable models. Suppose B_i, C_i are given for $i = 1, 2$ such that

$$[A, B_i] = i\hbar C_i, \quad [A, C_i] = i\hbar[P_i(A) + \alpha_i B_i + 2\beta_i \{A, B_i\}] \quad (7.33)$$

then we may construct a third pair B_3, C_3 , also satisfying (7.33). It is enough to give the explicit form for B_3 . For $\beta_1 = \beta_2 = 0$, we take

$$B_3 = \{C_1, C_2\} + \sqrt{\alpha_1 \alpha_2} \{B_1, B_2\} + \sqrt{\frac{\alpha_1}{\alpha_2}} \{B_1, P_2(A)\} + \sqrt{\frac{\alpha_2}{\alpha_1}} \{P_1(A), B_2\} \quad (7.34)$$

and

$$\alpha_3 = (\sqrt{\alpha_1} + \sqrt{\alpha_2})^2, \quad \beta_3 = 0, \quad P_3(A) = \frac{(\sqrt{\alpha_1} + \sqrt{\alpha_2})^2}{\sqrt{\alpha_1 \alpha_2}} P_1(A) P_2(A)$$

while for $\beta_1, \beta_2 \neq 0$ and assuming $\alpha_i = \frac{1}{4}\hbar^2 \beta_i^2$, $i = 1, 2$ we take

$$\begin{aligned} B_3 = & \sqrt{\beta_1 \beta_2} \{ \{A, C_1\}, C_2 \} + \sqrt{\beta_1 \beta_2} \{ C_1, \{A, C_2\} \} + \frac{1}{4} \sqrt{\beta_1 \beta_2} \left(\beta_1 - \sqrt{\beta_1 \beta_2} + \beta_2 \right) \hbar^2 \{ C_1, C_2 \} \\ & + 4\beta_1 \beta_2 \{ \{A, B_1\}, \{A, B_2\} \} + \frac{1}{2} \beta_1 \beta_2^2 \hbar^2 \{ \{A, B_1\}, B_2 \} + \frac{1}{2} \beta_1^2 \beta_2 \hbar^2 \{ B_1, \{A, B_2\} \} \\ & + \frac{1}{16} \beta_1^2 \beta_2^2 \hbar^4 \{ B_1, B_2 \} + 2\beta_2 \{ P_1(A), \{A, B_2\} \} + 2\beta_1 \{ \{A, B_1\}, P_2(A) \} \\ & + \frac{1}{4} \beta_2^2 \hbar^2 \{ P_1(A), B_2 \} + \frac{1}{4} \beta_1^2 \hbar^2 \{ B_1, P_2(A) \} + P_1(A) P_2(A) \end{aligned} \quad (7.35)$$

and

$$\alpha_3 = \frac{1}{4} \hbar^2 \left(\sqrt{\beta_1} + \sqrt{\beta_2} \right)^4, \quad \beta_3 = \left(\sqrt{\beta_1} + \sqrt{\beta_2} \right)^2, \quad P_3(A) = 0$$

We may not have constructed the least-order operator in this way. Nevertheless we have shown that the model $(x)^m$ where $x = a, b, c, d, e, f, g, h$ which is attained by dividing α (if $\beta = 0$) or β if non-zero by m^2 in A , admits operators B, C so that the polynomial algebra relations (7.1) are fulfilled. In $\beta \neq 0$, α must be shifted appropriately.

Let us now concentrate on a realisation of (7.27). For all intents and purposes, we may treat A as an abstract symbol. For $\alpha \neq 0, \beta = 0$, we have

$$\left[H, K \pm i(-\alpha)^{\frac{1}{2}} J \mp i(-\alpha)^{-\frac{1}{2}} F \right] = \mp \hbar (-\alpha)^{\frac{1}{2}} \left[K \pm i(-\alpha)^{\frac{1}{2}} J \mp i(-\alpha)^{-\frac{1}{2}} F \right] (\chi')^2 \quad (7.36)$$

Setting $L^\pm = K \pm i(-\alpha)^{\frac{1}{2}} J \mp i(-\alpha)^{-\frac{1}{2}} F$ and writing $H = H[A]$ to make the dependence on A explicit, (7.36) becomes

$$H[A]L^\pm = L^\pm H \left[A \mp \hbar(-\alpha)^{\frac{1}{2}} \right] \quad (7.37)$$

Similarly, for $\beta \neq 0$, we have

$$[H, L^\pm] = \left(-\frac{1}{2}\hbar^2\beta \mp \hbar M \right) L^\pm (\chi')^2 \quad (7.38)$$

where

$$L^\pm = (\pm M + \frac{1}{2}\hbar\beta)K + i(\pm M - \frac{1}{2}\hbar\beta)(\pm M + \frac{1}{2}\hbar\beta)J - iF, \quad M^2 = -\alpha - 2\beta A + \frac{1}{4}\hbar^2\beta^2$$

In this case we write $H = H[M^2]$. Then (7.38) expresses the relation

$$H[M^2]L^\pm = L^\pm H[(M \pm \hbar\beta)^2] \quad (7.39)$$

So (7.27) algebraically summarises two species of shape-invariance: those of linear type (as in (7.37)) and of quadratic type (as in (7.39)). Concatenating the shifting operators L^\pm , we can construct higher order realisations $(y)^n$ of (7.27), which consist essentially of the Hamiltonians (i), (j), (k) and (l) but with α or β divided by n^2 for integer n . Furthermore, we can combine this with the addition laws of (7.34) and (7.35) to construct higher order superintegrable systems.

Example 7.5. We take (i)^m+(a)ⁿ and $\mu_1 = 0$. This gives the anisotropic oscillator

$$\frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{2}\alpha \left(\frac{x^2}{m^2} + \frac{y^2}{n^2} \right) + \lambda_1 x + \lambda_2 y$$

□

Example 7.6. We take (k)^m+(c)ⁿ in the oscillator case. We have the system

$$\frac{1}{2}p_\xi^2 + \lambda_1 \xi^2 - \frac{4m^2}{\beta \xi^2} \left[\frac{1}{2}p_y^2 + \frac{\hbar^2 \beta}{32m^2} + \sigma_1 \exp\left(-\frac{y\sqrt{\beta}}{n}\right) + \sigma_2 \exp\left(-\frac{2y\sqrt{\beta}}{n}\right) \right]$$

Let us set $\eta = y\sqrt{\beta}/(2m)$ so that the system is in a reduced form

$$\frac{1}{2}p_\xi^2 + \lambda_1 \xi^2 - \frac{1}{\xi^2} \left[\frac{1}{2}p_\eta^2 + \lambda_2 e^{-\frac{2m\eta}{n}} + \lambda_3 e^{-\frac{4m\eta}{n}} \right]$$

This model has already been shown to be superintegrable in the $\lambda_1 = 0$ case by Campoamor-Stursburg [152].

□

Example 7.7. We take (l)^m+(d)ⁿ in the Pöschl-Teller case. We obtain the superintegrable system

$$\frac{1}{2}p_\xi^2 + \frac{1}{\sin^2\left(\frac{\rho\sqrt{\beta}}{2m}\right)} \left[\frac{1}{2}p_y^2 - \frac{\hbar^2 \beta}{32m^2} + \frac{\sigma_1}{\sin^2\left(\frac{y\sqrt{\beta}}{2n}\right)} + \frac{\sigma_2}{\cos^2\left(\frac{y\sqrt{\beta}}{2n}\right)} \right] + \frac{\lambda_3}{\cos^2\left(\frac{\xi\sqrt{\beta}}{2m}\right)}$$

where we have made a complex-linear transformation to convert the ξ -component into trigonometric form. Setting $r = \xi\sqrt{\beta} l(2m)$, $\theta = y\sqrt{\beta} l(2n)$ and scaling, we obtain

$$\frac{1}{2}p_r^2 + \frac{1}{\sin^2 r} \left[\frac{1}{2}p_\theta^2 + \frac{\lambda_1}{\sin^2\left(\frac{m\theta}{n}\right)} + \frac{\lambda_2}{\cos^2\left(\frac{m\theta}{n}\right)} - \frac{\hbar^2}{8} \right] + \frac{\lambda_3}{\cos^2 r}$$

which is the TTW model on the sphere that we considered in detail in Chapter 3. □

Indeed, the analysis of Chapter 3 may be generalised to a wide variety of systems.

7.6 Conclusion

In summary, we have shown that the three-generator polynomial algebra lies at the centre of one-dimensional exact solvability and two-dimensional superintegrability. In particular, we have shown that the Chazy or Painlevé potentials that we derived belong to a higher echelon of exact solvability beyond oscillator, Morse and Pöschl-Teller. Indeed, as P, Q are second-order, the algebra corresponds exactly to the quadratic Racah-Wilson type [145, 153] and therefore the coefficients of the representation correspond to recurrence relations of Askey-Wilson polynomials [144].

Chapter 8

Conclusion

In this thesis, we have discovered many new aspects of higher-order superintegrability.

In Chapter 3, we gave an operator-theoretic approach to deriving the integrals of the TTW model. We proved that the reduction approach of Kalnins, Kress and Miller Jr [47] was indeed valid and not capable of further divisions. Further, we generalised it to the PVZ model which produced a two-dimensional realisation of a deformed Bannai-Ito algebra.

In Chapters 4 and 5, we classified standard superintegrable systems with separation in Cartesian coordinates up to sixth-order. Alas, we have not been able to establish definitive evidence that our lists in Chapter 5 are indeed complete. Notwithstanding, the ladder operator approach proved to be much more powerful and efficient than the technique presented in Chapter 4. We aim in future works to extend our technique to the determination of new integrable systems under other constraints.

We have shown that these two-dimensional systems give rise to intricate dynamic and symmetry algebras. We gave one example in Chapter 6 but it would be interesting to investigate more complicated cases such as rational extensions of the harmonic oscillator or special cases of the TTW model.

In Chapter 7 we have accomplished a complete unification of the ladder operator methods of Chapters 5 and 6 with the formal approach of Chapter 3. We presented a very simple derivation of the third- and fourth-order exotic models. The polynomial algebra approach could prove to be a systematic method of obtaining higher order Painlevé transcendents [121, 154]. So far, the method of polynomial algebra realisations has only been used as a generating tool and its deeper properties have not been examined. We have reported on new addition laws but there are also other aspects of interest such as the effect of Darboux transformations and the consequences that brings to enlarging the complete algebra. We derived the four Darboux surfaces which, together with the surfaces of constant-curvature, comprise every Riemannian surface which admits a Killing vector and a functionally independent second-order Killing tensor. The generalisation to third-order Killing tensors was achieved by [155] and it is likely that study of higher order shift operators may lead to the determination of new geometries with analogous properties.

We have confined our attention to subgroup separability. However, the case of non-subgroup separability is a fertile ground for interesting results [53]. In classical mechanics, quadratic super-

integrability for systems which separate in non-subgroup coordinates is a consequence of addition theorems of elliptic and hyper-elliptic functions [156, 157]. In quantum mechanics, it is associated with quasi-exact solvability [25, 158, 159] but the connection has not been definitively established. As far as higher-order superintegrability is concerned [160, 161], this remains an open problem for further investigation.

Bibliography

- [1] I. Marquette, A. Parr, ‘Superintegrability and deformed oscillator realizations of quantum TTW Hamiltonians on constant-curvature manifolds and with reflections in a plane’, *J. Phys. A: Math. Theor.* 57 (13) (2024). doi:10.1088/1751-8121/ad2e3f.
- [2] I. Marquette, A. Parr, ‘Polynomially superintegrable Hamiltonians separating in Cartesian coordinates’, *J. Phys. A: Math. Theor.* 58 (26) (2025). doi:10.1088/1751-8121/ade57e.
- [3] T. Simpson, *The Doctrine and Application of Fluxions*, Vol. 1, Nourse, J., 1750.
- [4] L. De Leenheer, J. Musgrove, T. Schimleck, ‘A Comprehensive Proof of Bertrand’s Theorem’, *SIAM Review* 65 (2) (2023) 563–588. doi:10.1137/21M1436658.
- [5] L. A. Pars, *A Treatise on Analytical Dynamics*, Ox Bow Press, 1965.
- [6] J. M. Jauch, E. H. Hill, ‘On the Problem of Degeneracy in Quantum Mechanics’, *Phys. Rev.* 57 (7) (1940) 641–645. doi:10.1103/PhysRev.57.641.
- [7] H. V. McIntosh, ‘On Accidental Degeneracy in Classical and Quantum Mechanics’, *Am. J. Phys.* 27 (1959) 620–625. doi:10.1119/1.1934944.
- [8] V. Bargmann, M. Moshinsky, ‘Group Theory of Harmonic Oscillators: (I). The Collective Modes’, *Nucl. Phys.* 18 (1960) 697–712. doi:10.1016/0029-5582(60)90438-7.
- [9] M. Bander, M. Itzykson, ‘Group Theory and the Hydrogen Atom (I)’, *Rev. Mod. Phys.* 38 (2) (1960) 330–345. doi:10.1103/RevModPhys.38.330.
- [10] D. M. Fradkin, ‘Existence of the Dynamic Symmetries O_4 and SU_3 for All Classical Central Potential Problems’, *Prog. Theor. Phys.* 37 (5) (1967) 798–812. doi:10.1143/PTP.37.798.
- [11] J. Friš, V. Mandrosov, Y. A. Smorodinsky, M. Uhlír, P. Winternitz, ‘On Higher Symmetries in Quantum Mechanics’, *Phys. Lett.* 16 (3) (1965) 354–356. doi:10.1016/0031-9163(65)90885-1.
- [12] J. Friš, V. Mandrosov, Y. A. Smorodinsky, M. Uhlír, P. Winternitz, ‘Symmetry groups in classical and quantum mechanics’, *Sov. J. Nucl. Phys.* 4 (1967) 444–450.

- [13] A. A. Makarov, J. A. Smorodinsky, K. Valiev, P. Winternitz, 'A systematic search for non-relativistic systems with dynamical symmetries', *Nuovo Cim. A LII A (4)* (1967) 1061–1084. doi:10.1007/BF02755212.
- [14] A. V. Tsiganov, 'The Drach superintegrable systems', *J. Phys. A: Math. Gen.* 33 (41) (2000) 7407–7422. doi:10.1088/0305-4470/33/41/310.
- [15] E. G. Kalnins, W. Miller, Jr, G. S. Pogosyan, 'Completeness of multiseparable superintegrability on the complex 2-sphere', *J. Phys. A: Math. Gen.* 33 (38) (2000) 6791–6806. doi:10.1088/0305-4470/33/38/310.
- [16] E. G. Kalnins, G. S. Pogosyan, W. Miller, Jr, 'Completeness of Multiseparable Superintegrability in Two Dimensions', *Phys. Atom. Nucl.* 65 (6) (2002) 1033–1035. doi:10.1134/1.1490105.
- [17] E. G. Kalnins, W. Miller, Jr, G. S. Pogosyan, 'Superintegrability on the two-dimensional hyperboloid', *J. Math. Phys.* 38 (10) (1997) 5416–5433. doi:10.1063/1.531951.
- [18] N. W. Evans, 'Superintegrability in classical mechanics', *Phys. Rev. A* 41 (10) (1990) 5666–5676. doi:10.1103/PhysRevA.41.5666.
- [19] E. G. Kalnins, J. M. Kress, W. Miller, Jr, 'Second order superintegrable systems in conformally flat spaces. IV. The classical 3D Stäckel transform and 3D classification theory', *J. Math. Phys.* 47 (4) (2006). doi:10.1063/1.2191789.
- [20] M. A. Escobar-Ruiz, W. Miller, Jr, 'Toward a classification of semidegenerate 3D superintegrable systems', *J. Phys. A: Math. Theor.* 50 (9) (2017). doi:10.1088/1751-8121/aa5843.
- [21] N. W. Evans, 'Super-integrability of the Winternitz system', *Phys. Lett. A* 147 (8,9) (1990) 483–486. doi:10.1016/0375-9601(90)90611-Q.
- [22] P. E. Verrier, N. W. Evans, 'A new superintegrable Hamiltonian', *J. Math. Phys.* 49 (2) (2008). doi:10.1063/1.2840465.
- [23] C. Daskaloyannis, 'Generalized deformed oscillator and nonlinear algebras', *J. Phys. A: Math. Gen.* 24 (15) (1991) L789–L794. doi:10.1088/0305-4470/24/15/001.
- [24] D. Bonatsos, C. Daskaloyannis, K. Kokkotas, 'Deformed oscillator algebras for two-dimensional quantum superintegrable systems', *Phys. Rev. A* 50 (5) (1994) 3700–3709. doi:10.1103/PhysRevA.50.3700.
- [25] P. Létourneau, L. Vinet, 'Superintegrable Systems: Polynomial Algebras and Quasi-Exactly Solvable Hamiltonians', *Ann. Phys.* 243 (1) (1994) 144–168. doi:10.1006/aphy.1995.1094.

- [26] E. G. Kalnins, W. Miller, Jr, G. S. Pogosyan, ‘Superintegrability and associated polynomial solutions: Euclidean space and the sphere in two dimensions’, *J. Math. Phys.* 37 (12) (1996) 6439–6467. doi:10.1063/1.531786.
- [27] M. A. Escobar-Ruiz, E. G. Kalnins, W. Miller, Jr, ‘Separation equations for 2D superintegrable systems on constant curvature spaces’, *J. Phys. A: Math. Theor.* 50 (38) (2017). doi:10.1088/1751-8121/aa8489.
- [28] S. Gravel, P. Winternitz, ‘Superintegrable systems with third-order integrals in classical and quantum mechanics’, *J. Math. Phys.* 43 (12) (2002) 5902–5912. doi:10.1063/1.1514385.
- [29] S. Gravel, ‘Hamiltonians separable in Cartesian coordinates and third-order integrals of motion’, *J. Math. Phys.* 45 (3) (2004) 1003–1019. doi:10.1063/1.1633352.
- [30] I. Marquette, P. Winternitz, ‘Superintegrable systems with third-order integrals of motion’, *J. Phys. A: Math. Theor.* 41 (30) (2008). doi:10.1088/1751-8113/41/30/304031.
- [31] I. Marquette, M. Sajedi, P. Winternitz, ‘Fourth order superintegrable systems separating in Cartesian coordinates I. Exotic quantum potentials’, *J. Phys. A: Math. Theor.* 50 (31) (2017). doi:10.1088/1751-8121/aa7a67.
- [32] I. Yurdeşen, A. M. Escobar-Ruiz, I. Palma Y Meza Montoya, ‘Doubly Exotic N th-Order Superintegrable Classical Systems Separating in Cartesian Coordinates’, *SIGMA* 18 (039) (2022). doi:10.3842/SIGMA.2022.039.
- [33] I. Abouamal, P. Winternitz, ‘Fifth-order superintegrable quantum systems separating in Cartesian coordinates: Doubly exotic potentials’, *J. Math. Phys.* 59 (2) (2018). doi:10.1063/1.5007252.
- [34] F. Tremblay, P. Winternitz, ‘Third-order superintegrable systems separating in polar coordinates’, *J. Phys. A: Math. Theor.* 43 (17) (2010). doi:10.1088/1751-8113/43/17/175206.
- [35] A. M. Escobar-Ruiz, J. C. López Vieyra, P. Winternitz, ‘Fourth order superintegrable systems separating in polar coordinates. I. Exotic potentials’, *J. Phys. A: Math. Theor.* 50 (49) (2017). doi:10.1088/1751-8121/aa9203.
- [36] A. M. Escobar-Ruiz, J. C. López Vieyra, P. Winternitz, I. Yurdeşen, ‘Fourth-order superintegrable systems separating in polar coordinates. II. Standard potentials’, *J. Phys. A: Math. Theor.* 51 (45) (2018). doi:10.1088/1751-8121/aae291.
- [37] I. Marquette, P. Winternitz, *Integrability, Supersymmetry and Coherent States*, CRM Series in Mathematical Physics, Springer, 2019, Ch. ‘Higher Order Quantum Superintegrability: a new “Painlevé conjecture”’, pp. 103–131. doi:10.1007/978-3-030-20087-9.
- [38] S. Post, P. Winternitz, ‘General N th order integrals of motion in the Euclidean plane’, *J. Phys. A: Math. Theor.* 48 (40) (2015). doi:10.1088/1751-8113/48/40/405201.

- [39] A. M. Escobar-Ruiz, R. Linares, P. Winternitz, ‘New infinite families of N th-order superintegrable systems separating in Cartesian coordinates’, *J. Phys. A: Math. Theor.* 53 (44) (2020). doi:10.1088/1751-8121/abb341.
- [40] E. G. Kalnins, J. M. Kress, W. Miller, Jr, ‘Families of classical subgroup separable superintegrable systems’, *J. Phys. A: Math. Theor.* 43 (9) (2010). doi:10.1088/1751-8113/43/9/092001.
- [41] A. V. Tsiganov, ‘Superintegrable systems with algebraic and rational integrals of motion’, *Theor. Math. Phys.* 199 (2) (2019) 659–674. doi:10.1134/S0040577919050040.
- [42] N. W. Evans, P. E. Verrier, ‘Superintegrability of the caged anisotropic oscillator’, *J. Math. Phys.* 49 (9) (2008). doi:10.1063/1.2988133.
- [43] I. Marquette, ‘Higher order superintegrability, Painlevé transcendents and representations of polynomial algebras’, *J. Phys.: Conf. Ser.* 1194 (2019). doi:10.1088/1742-6596/1194/1/012074.
- [44] F. Tremblay, A. V. Turbiner, P. Winternitz, ‘An infinite family of solvable and integrable quantum systems on a plane’, *J. Phys. A: Math. Theor.* 42 (24) (2009). doi:10.1088/1751-8113/42/24/242001.
- [45] C. Quesne, ‘Superintegrability of the Tremblay-Turbiner-Winternitz quantum Hamiltonians on a plane for odd k ’, *J. Phys. A: Math. Theor.* 43 (8) (2010). doi:10.1088/1751-8113/43/8/082001.
- [46] E. G. Kalnins, J. M. Kress, W. Miller, Jr, ‘Superintegrability and higher order integrals for quantum systems’, *J. Phys. A: Math. Theor.* 43 (26) (2010). doi:10.1088/1751-8113/43/26/265205.
- [47] E. G. Kalnins, J. M. Kress, W. Miller, Jr, ‘A Recurrence Relation Approach to Higher Order Quantum Superintegrability’, *SIGMA* 7 (031) (2011). doi:https://doi.org/10.3842/SIGMA.2011.031.
- [48] T. Hakobyan, O. Lechtenfeld, A. Nersessian, A. Saghatelian, V. Yeghikyan, ‘Action-angle variables and novel superintegrable systems’, *Phys. Part. Nucl.* 43 (5) (2012) 577–582. doi:10.1134/S1063779612050152.
- [49] B. K. Berntson, I. Marquette, W. Miller, Jr, ‘A new way to classify 2D higher order quantum superintegrable systems’, *J. Phys. A: Math. Theor.* 53 (49) (2020). doi:10.1088/1751-8121/abc04a.
- [50] E. G. Kalnins, J. M. Kress, P. Winternitz, ‘Superintegrability in a two-dimensional space of nonconstant curvature’, *J. Math. Phys.* 43 (2) (2002) 970–983. doi:10.1063/1.1429322.

- [51] E. G. Kalnins, J. M. Kress, W. Miller, Jr, P. Winternitz, ‘Superintegrable Systems in Darboux Spaces’, *J. Math. Phys.* 44 (12) (2003) 5811–5848. doi:10.1063/1.1619580.
- [52] A. Ballesteros, A. Enciso, F. J. Herranz, O. Ragnisco, ‘A maximally superintegrable system on an n -dimensional space of nonconstant curvature’, *Phys. D: Nonl. Phen.* 237 (4) (2008) 505–509. doi:10.1016/j.physd.2007.09.021.
- [53] S. Bertrand, O. Kubů, L. Šnobl, ‘On superintegrability of 3D axially-symmetric non-subgroup-type systems with magnetic fields’, *J. Phys. A: Math. Theor.* 54 (1) (2021). doi:10.1088/1751-8121/abc4b8.
- [54] M. A. Agrotis, P. A. Damianou, C. Sophocleous, ‘The Toda lattice is super-integrable’, *Phys. A: Stat. Mech. App.* 365 (1) (2006) 235–243. doi:10.1016/j.physa.2006.01.001.
- [55] S. Post, L. Vinet, A. Zhedanov, ‘An infinite family of superintegrable Hamiltonians with reflection in the plane’, *J. Phys. A: Math. Theor.* 44 (50) (2011). doi:10.1088/1751-8113/44/50/505201.
- [56] S. Ghazouani, S. Insaf, ‘Superintegrability of the Dunkl-Coulomb problem in three-dimensions’, *J. Phys. A: Math. Theor.* 53 (3) (2019). doi:10.1088/1751-8121/ab4a2d.
- [57] J. Hietarinta, B. Grammaticos, B. Dorizzi, A. Ramani, ‘Coupling-Constant Metamorphosis and Duality between Integrable Hamiltonian Systems’, *Phys. Rev. Lett.* 53 (18) (1984) 1707–1710. doi:10.1103/PhysRevLett.53.1707.
- [58] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 2nd Edition, Cambridge: at the University Press, 1917.
- [59] I. M. Gelfand, S. V. Fomin, *Calculus of Variations*, Dover Publications, Inc., 2000.
- [60] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, Inc., 1956.
- [61] P. Havas, ‘Separation of variables in the Hamilton–Jacobi, Schrödinger, and related equations. I. Complete separation’, *J. Math. Phys.* 16 (1975) 1461–1468.
- [62] E. G. Kalnins, W. Miller, Jr, ‘Conformal Killing Tensors and Variable Separation for Hamilton–Jacobi Equations’, *SIAM J. Math. Anal.* 14 (1) (1983).
- [63] L. D. Landau, E. M. Lifshitz, *Mechanics*, Third Edition, Course of Theoretical Physics, Butterworth-Heinemann, 1976.
- [64] E. Cartan, *Lessons on Integral Invariants*, Hermann, 1922.
- [65] W. Heisenberg, *Sources of Quantum Mechanics*, Dover Publications, Inc., 1968, Ch. ‘Quantum-Theoretical Re-Interpretation of Kinematic and Mechanical Relations’, pp. 261–276.

- [66] M. Born, J. P., *Sources of Quantum Mechanics*, Dover Publications, Inc., 1968, Ch. ‘On Quantum Mechanics’, pp. 277–306.
- [67] P. A. M. Dirac, *The Principles of Quantum Mechanics*, Fourth Edition, Oxford at the Clarendon Press, 1967.
- [68] S. Gravel, *Superintegrability in Classical and Quantum Systems*, Vol. 37 of CRM Proceedings and Lecture Notes, AMS, 2004, Ch. ‘Cubic Integrals of Motion and Quantum Superintegrability’, pp. 53–64. doi:10.1090/crmp/037.
- [69] E. Schrödinger, ‘A Method of Determining Quantum-Mechanical Eigenvalues and Eigenfunctions’, *Proc. Roy. Ir. Acad. Sec. A: Math. Phys. Sc.* 40 (1940) 9–16.
- [70] E. Schrödinger, ‘Further Studies on Solving Eigenvalue Problems by Factorization’, *Proc. Roy. Ir. Acad. Sec. A: Math. Phys. Sc.* 46 (1941) 183–206.
- [71] L. Infeld, T. E. Hull, ‘The Factorization Method’, *Rev. Mod. Phys.* 23 (1) (1951) 21–68. doi:10.1103/RevModPhys.23.21.
- [72] L. E. Gendenshtein, ‘Derivation of exact spectra of the Schrödinger equation by means of supersymmetry’, *JETP Lett.* 38 (6) (1983) 356–359.
- [73] A. Stahlhofen, ‘Remarks on the equivalence between the shape-invariance condition and the factorisation condition’, *J. Phys. A: Math. Gen.* 22 (8) (1989) 1053–1058. doi:10.1088/0305-4470/22/8/019.
- [74] V. B. Matveev, M. A. Salle, *Darboux Transformations and Solitons*, Springer Series in Nonlinear Dynamics, Springer-Verlag, 1991.
- [75] S. Post, L. Vinet, A. Zhedanov, ‘Supersymmetric quantum mechanics with reflections’, *J. Phys. A: Math. Theor.* 44 (43) (2011). doi:10.1088/1751-8113/44/43/435301.
- [76] G. Darboux, *Leçons sur la Théorie Générale des Surfaces*, Vol. 2, Gauthier-Villars, 1915.
- [77] V. E. Adler, ‘A modification of Crum’s method’, *Theor. Math. Phys.* 101 (1994) 1381–1386. doi:10.1007/BF01035458.
- [78] A. Shabat, ‘The infinite-dimensional dressing dynamical system’, *Inv. Prob.* 8 (2) (1991) 303–308. doi:10.1088/0266-5611/8/2/009.
- [79] A. Schulze-Halberg, ‘Wronskian representation for confluent supersymmetric transformation chains of arbitrary order’, *Eur. Phys. J. Plus* 128 (68) (2013). doi:10.1140/epjp/i2013-13068-2.
- [80] A. M. Escobar-Ruiz, P. Winternitz, I. Yurdeşen, ‘General N th-order superintegrable systems separating in polar coordinates’, *J. Phys. A: Math. Theor.* 51 (40) (2018). doi:10.1088/1751-8121/aadc23.

- [81] B. K. Berntson, I. Marquette, W. Miller, Jr, *Quantum Theory and Symmetries*, CRM Series in Mathematical Physics, Springer, 2020, Ch. ‘A New Approach to Analysis of 2D Higher Order Quantum Superintegrable Systems’, pp. 111–120. doi:10.1007/978-3-030-55777-5.
- [82] J. A. Calzada, Ş. Kuru, J. Negro, ‘Superintegrable Lissajous systems on the sphere’, *Eur. Phys. J. Plus* 129 (164) (2014). doi:10.1140/epjp/i2014-14164-5.
- [83] I. Marquette, C. Quesne, ‘Deformed oscillator algebra approach of some quantum superintegrable Lissajous systems on the sphere and of their rational extensions’, *J. Math. Phys.* 56 (6) (2015). doi:10.1063/1.4922020.
- [84] P. S. Isaac, I. Marquette, ‘On realizations of polynomial algebras with three generators via deformed oscillator algebras’, *J. Phys. A: Math. Theor.* 47 (20) (2014). doi:10.1088/1751-8113/47/20/205203.
- [85] F. Tremblay, A. Turbiner, P. Winternitz, ‘Periodic orbits for an infinite family of classical superintegrable systems’, *J. Phys. A: Math. Theor.* 43 (1) (2010). doi:10.1088/1751-8113/43/1/015202.
- [86] C. Quesne, ‘ $\mathcal{N} = 2$ supersymmetric extension of the Tremblay-Turbiner-Winternitz Hamiltonians on a plane’, *J. Phys. A: Math. Theor.* 43 (30) (2010). doi:10.1088/1751-8113/43/30/305202.
- [87] J. A. Calzada, E. Celeghini, M. A. del Olmo, M. A. Velasco, ‘Algebraic Aspects of Tremblay-Turbiner-Winternitz Hamiltonian Systems’, *J. Phys.: Conf. Ser.* 343 (2012). doi:10.1088/1742-6596/343/1/012029.
- [88] C. Gonera, ‘On the superintegrability of the TTW model’, *Phys. Lett. A* 376 (35) (2012) 2341–2343. doi:10.1016/j.physleta.2012.06.006.
- [89] E. Celeghini, Ş. Kuru, J. Negro, M. A. del Olmo, ‘A unified approach to quantum and classical TTW systems based on factorizations’, *Ann. Phys.* 332 (2012) 27–37. doi:10.1016/j.aop.2013.01.008.
- [90] C. M. Chanu, L. Degiovanni, G. Rastelli, ‘The Tremblay-Turbiner-Winternitz system as extended Hamiltonian’, *J. Math. Phys.* 55 (12) (2014). doi:10.1063/1.4903508.
- [91] M. F. Rañada, ‘The Tremblay–Turbiner–Winternitz system on spherical and hyperbolic spaces: superintegrability, curvature-dependent formalism and complex factorization’, *J. Phys. A: Math. Theor.* 47 (16) (2014). doi:10.1088/1751-8113/47/16/165203.
- [92] T. Hakobyan, O. Lechtenfeld, A. Nersessian, ‘Superintegrability of generalized Calogero models with oscillator or Coulomb potential’, *Phys. Rev. D* 90 (101701(R)) (2014). doi:10.1103/PhysRevD.90.101701.

- [93] M. F. Rañada, ‘Quasi-bi-Hamiltonian structures, complex functions and superintegrability: the Tremblay-Turbiner-Winternitz (TTW) and the Post-Winternitz (PW) systems’, *J. Phys. A: Math. Theor.* 50 (31) (2017). doi:10.1088/1751-8121/aa7951.
- [94] M. A. Olshanetsky, A. M. Perelomov, ‘Quantum integrable systems related to Lie algebras’, *Phys. Rep.* 94 (6) (1982) 313–404. doi:10.1016/0370-1573(83)90018-2.
- [95] J. Wolfes, ‘On the three-body linear problem with three-body interaction’, *J. Math. Phys.* 15 (9) (1972) 1420–1424. doi:10.1063/1.1666826.
- [96] S. Post, S. Tsujimoto, L. Vinet, ‘Families of superintegrable Hamiltonians constructed from exceptional polynomials’, *J. Phys. A: Math. Theor.* 45 (40) (2012). doi:10.1088/1751-8113/45/40/405202.
- [97] J. F. Cariñena, M. F. Rañada, M. Santander, ‘The Harmonic Oscillator on Three-Dimensional Spherical and Hyperbolic Spaces: Curvature Dependent Formalism and Quantization’, *Int. J. Theor. Phys.* 50 (2011) 2170–2178. doi:10.1007/s10773-011-0688-z.
- [98] M. A. Rodríguez, P. Tempesta, ‘On higher-dimensional superintegrable systems: a new family of classical and quantum Hamiltonian models’, *J. Phys. A: Math. Theor.* 55 (50) (2022). doi:10.1088/1751-8121/acaada.
- [99] R. Koekoek, P. A. Lesky, R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q -analogues*, Springer Monographs in Mathematics, Springer, 2010. doi:10.1007/978-3-642-05014-5.
- [100] D. Bonatsos, C. Daskaloyannis, P. Kolokotronis, ‘Generalized Deformed Oscillators and Algebras’ (1995). doi:10.48550/arXiv.hep-th/9512083.
- [101] I. Marquette, C. Quesne, ‘New ladder operators for a rational extension of the harmonic oscillator and superintegrability of some two dimensional systems’, *J. Math. Phys.* 54 (10) (2013). doi:10.1063/1.4823771.
- [102] V. Hussin, I. Marquette, K. Zelaya, ‘Third-order ladder operators, generalized Okamoto and exceptional orthogonal polynomials’, *J. Phys. A: Math. Theor.* 55 (4) (2022). doi:10.1088/1751-8121/ac43cc.
- [103] L. Šnobl, ‘On the incompleteness of the classification of quadratically integrable Hamiltonian systems in the three-dimensional Euclidean space’, *J. Phys. A: Math. Theor.* 58 (11) (2025). doi:10.1088/1751-8121/adbe19.
- [104] E. Onofri, M. Pauri, ‘Search for periodic Hamiltonian flows: A generalized Bertrand’s theorem’, *J. Math. Phys.* 19 (9) (1978) 1850–58. doi:10.1063/1.523926.
- [105] Y. A. Grigoriev, A. V. Tsiganov, ‘On superintegrable systems separable in Cartesian coordinates’, *Phys. Lett. A* 382 (32) (2018) 2092–2096. doi:10.1016/j.physleta.2018.05.039.

- [106] W. Miller, Jr, ‘Complete Sets of Invariants for Classical Systems’, in: Tempesta, P. and Winternitz, P. and Harnad, J. and Miller, Jr, W. and Pogosyan, G. and Rodriguez, M. (Ed.), *Superintegrability in Classical and Quantum Systems*, Vol. 37 of CRM Proceedings & Lecture Notes, Centre de Recherches Mathématiques Université de Montréal, American Mathematical Society, 2004, pp. 125–135. doi:10.1090/crmp/037.
- [107] E. G. Kalnins, J. M. Kress, W. Miller, Jr, ‘Superintegrability and higher order constants for quantum systems’, *J. Phys. A: Math. Theor.* 43 (26) (2010). doi:10.1088/1751-8113/43/26/265205.
- [108] N. H. McCoy, ‘On Commutation Formulas in the Algebra of Quantum Mechanics’, *Trans. Am. Math. Soc.* 31 (4) (1929) 793–806. doi:10.1090/S0002-9947-1929-1501512-1.
- [109] C. R. Holt, ‘Construction of new integrable Hamiltonians in two degrees of freedom’, *J. Math. Phys.* 23 (6) (1982). doi:10.1063/1.525492.
- [110] J. Hietarinta, ‘Classical versus quantum integrability’, *J. Math. Phys.* 25 (6) (1983) 1833–1840. doi:10.1063/1.526373.
- [111] I. Marquette, ‘Cubic algebras, induced representations and general solution of the exceptional Laguerre equation X_1 ’, *Phys. D: Nonl. Phen.* 473 (134547) (2025). doi:10.1016/j.physd.2025.134547.
- [112] S. Y. Dubov, V. M. Eleonskii, N. E. Kulagin, ‘Equidistant spectra of anharmonic oscillators’, *Chaos* 4 (1) (1994) 47–53. doi:10.1063/1.166056.
- [113] J. W. E. Harrow, A. N. W. Hone, ‘Confluent Darboux Transformations and Wronskians for Algebraic Solutions of the Painlevé III (D_7) Equation’, *Math.* 13 (14) (2025). doi:10.3390/math13142236.
- [114] J. Hietarinta, ‘Solvability in quantum mechanics and classically superfluous invariants’, *J. Phys. A: Math. Gen.* 22 (5) (1989) L143–L147. doi:10.1088/0305-4470/22/5/002.
- [115] S. Bochner, ‘Über Sturm-Liouvillesche Polynomsysteme’, *Math. Z.* 29 (1929) 730–736. doi:10.1007/BF01180560}.
- [116] D. Gómez-Ullate, N. Kamran, R. Milson, ‘An extended class of orthogonal polynomials defined by a Sturm–Liouville problem’, *J. Math. Anal. Appl.* 359 (1) (2009) 352–367. doi:10.1016/j.jmaa.2009.05.052.
- [117] C. Quesne, ‘Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry’, *J. Phys. A: Math. Theor.* 41 (39) (2008). doi:10.1088/1751-8113/41/39/392001.
- [118] Y. Grandati, ‘Solvable rational extensions of the isotonic oscillator’, *Ann. Phys.* 326 (8) (2011) 2074–2090. doi:10.1016/j.aop.2011.03.001.

- [119] D. Fernández, N. Fernández-García, ‘Higher-order supersymmetric quantum mechanics’, *AIP Conf. Proc.* 744 (1) (2004) 236–273. doi:10.1063/1.1853203.
- [120] I. Marquette, S. Post, L. Ritter, ‘A family of fourth-order superintegrable systems with rational potentials related to Painlevé VI’, *J. Phys. A: Math. Theor.* 55 (15) (2022). doi:10.1088/1751-8121/ac550a.
- [121] P. A. Clarkson, D. Gómez-Ullate, Y. Grandati, R. Milson, ‘Cyclic Maya diagrams and rational solutions of higher order Painlevé systems’, *Stud. App. Math.* 144 (3) (2020). doi:10.1111/sapm.12300.
- [122] J. M. Carballo, D. J. Fernández, J. Negro, L. M. Nieto, ‘Polynomial Heisenberg Algebras’, *J. Phys. A: Math. Gen.* 37 (43) (2004) 10349–10362. doi:10.1088/0305-4470/37/43/022.
- [123] D. Bermúdez, *Polynomial Heisenberg algebras and Painlevé equations*, Ph.D. thesis, Department of Physics, Cinvestav (2013).
- [124] I. Marquette, ‘Representations of quadratic Heisenberg-Weyl algebras and polynomials in the fourth Painlevé transcendent’, *AIMS Mathematics* 9 (10) (2024) 26836–26853. doi:{10.3934/math.20241306}.
- [125] I. Marquette, ‘An infinite family of superintegrable systems from higher order ladder operators and supersymmetry’, *J. Phys.: Conf. Ser.* 284 (2011). doi:10.1088/1742-6596/284/1/012047.
- [126] I. Marquette, M. Sajedi, P. Winternitz, ‘Two-dimensional superintegrable systems from operator algebras in one dimension’, *J. Phys. A: Math. Theor.* 52 (11) (2019). doi:10.1088/1751-8121/ab01a2.
- [127] F. Güngör, Ş. Kuru, J. Negro, L. M. Nieto, ‘Heisenberg-type higher order symmetries of superintegrable systems separable in Cartesian coordinates’, *Nonl.* 30 (5) (2017). doi:10.1088/1361-6544/aa6445.
- [128] A. P. Veselov, A. B. Shabat, ‘Dressing chains and the spectral theory of the Schrödinger operator’, *Func. Anal. Its App.* 27 (2) (1993) 81–96. doi:10.1007/BF01085979.
- [129] A. P. Bassom, P. A. Clarkson, A. C. Hicks, ‘Bäcklund Transformations and Solution Hierarchies for the Fourth Painlevé Equation’, *Stud. App. Math.* 95 (1) (1995) 1–126. doi:10.1002/sapm19959511.
- [130] M. Noumi, Y. Yamada, ‘Symmetries in the Fourth Painlevé Equation and Okamoto Polynomials’, *Nagoya Math. J.* 153 (1999) 53–86. doi:10.1017/S0027763000006899.
- [131] K. Kajiwara, Y. Ohta, ‘Determinant structure of the rational solutions for Painlevé IV equation’, *J. Phys. A: Math. Gen.* 31 (10) (1998). doi:10.1088/0305-4470/31/10/017.

- [132] P. A. Clarkson, ‘The fourth Painlevé equation and associated special polynomials’, *J. Math. Phys.* 44 (11) (2003) 5350–5374. doi:10.1063/1.1603958.
- [133] V. E. Adler, ‘Recuttings of polygons’, *Func. Anal. Its App.* 27 (1993) 141–143. doi:10.1007/BF01085984.
- [134] P. A. Clarkson, C. Dunning, ‘Rational solutions of the fifth Painlevé equation. Generalized Laguerre polynomials’, *Stud. App. Math.* 152 (1) (2024) 453–507. doi:10.1111/sapm.12649.
- [135] M. Noumi, Y. Yamada, ‘Umemura polynomials for the Painlevé V equation’, *Phys. Lett. A* 247 (1–2) (1998) 65–69. doi:10.1016/S0375-9601(98)00625-2.
- [136] I. Marquette, C. Quesne, ‘New ladder operators for a rational extension of the harmonic oscillator and superintegrability of some two-dimensional systems’, *J. Math. Phys.* 54 (10) (2013). doi:10.1063/1.4823771.
- [137] J. F. Cariñena, M. S. Plyushchay, ‘ABC of ladder operators for rationally extended quantum harmonic oscillator systems’, *J. Phys. A: Math. Theor.* 50 (27) (2017). doi:10.1088/1751-8121/aa739b.
- [138] I. Marquette, ‘Superintegrability and higher order polynomial algebras’, *J. Phys. A: Math. Theor.* 43 (13) (2010). doi:10.1088/1751-8113/43/13/135203.
- [139] D. Latini, I. Marquette, Y. Zhang, ‘Polynomial algebras of superintegrable systems separating in Cartesian coordinates from higher order ladder operators’, *Phys. D: Nonl. Phen.* 440 (133464) (2022). doi:10.1016/j.physd.2022.133464.
- [140] J. Dereziński, C. Gaß, J. M. Vättö, ‘Confluent functions, Laguerre polynomials and their (generalized) bilinear integrals’ (2025). doi:10.48550/arXiv.2404.16539.
- [141] E. G. Kalnins, J. M. Kress, W. Miller, Jr, ‘Tools for Verifying Classical and Quantum Superintegrability’, *SIGMA* 6 (022) (2010). doi:10.3842/SIGMA.2010.066.
- [142] C. M. Chanu, L. Degiovanni, G. Rastelli, ‘First Integrals of Extended Hamiltonians in $n + 1$ Dimensions Generated by Powers of an Operator’, *SIGMA* 7 (038) (2011). doi:10.3842/SIGMA.2011.038.
- [143] C. M. Chanu, L. Degiovanin, G. Rastelli, ‘Superintegrable Extensions of Superintegrable Systems’, *SIGMA* 8 (070) (2012). doi:10.3842/SIGMA.2012.070.
- [144] E. G. Kalnins, W. Miller, Jr, S. Post, ‘Wilson polynomials and the generic superintegrable system on the 2-sphere’, *J. Phys. A: Math. Theor.* 40 (38) (2007) 11525–11538. doi:10.1088/1751-8113/40/38/005.

- [145] V. X. Genest, L. Vinet, A. Zhedanov, ‘Superintegrability in two dimensions and the Racah-Wilson algebra’, *Lett. Math. Phys.* 104 (8) (2014) 931–952. doi:10.1007/s11005-014-0697-y.
- [146] Y. I. Granovskii, I. M. Lutzenko, A. S. Zhedanov, ‘Mutual Integrability, Quadratic Algebras, and Dynamical Symmetry’, *Ann. Phys.* 217 (1992) 1–20. doi:10.1016/0003-4916(92)90336-K.
- [147] M. F. Manning, ‘Exact Solutions of the Schrödinger Equation’, *Phys. Rev.* 48 (1935). doi:10.1103/PhysRev.48.161.
- [148] C. M. Cosgrove, ‘Chazy Classes IX–XI Of Third-Order Differential Equations’, *Stud. Appl. Math.* 104 (3) (2000) 171–228. doi:10.1111/1467-9590.00134.
- [149] C. M. Cosgrove, G. Scoufis, ‘Painlevé Classification of a Class of Differential Equations of the Second Order and Second Degree’, *Stud. App. Math.* 88 (1) (1993) 1–87. doi:10.1002/sapm199388125.
- [150] K. Okamoto, *The Painlevé Property: One Century Later*, First Edition, CRM Series in Mathematical Physics, Springer, 1999, Ch. ‘The Hamiltonians Associated to the Painlevé Equations’, pp. 735–787. doi:10.1007/978-1-4612-1532-5.
- [151] E. G. Kalnins, J. M. Kress, G. S. Pogosyan, W. Miller, Jr, ‘Completeness of superintegrability in two-dimensional constant-curvature spaces’, *J. Phys. A: Math. Gen.* 34 (22) (2001) 4705–4720. doi:10.1088/0305-4470/34/22/311.
- [152] R. Campoamor-Stursberg, ‘Superposition of super-integrable pseudo-Euclidean potentials in $N = 2$ with a fundamental constant of motion of arbitrary order in the momenta’, *J. Math. Phys.* 55 (4) (2014). doi:10.1063/1.4871546.
- [153] V. X. Genest, L. Vinet, A. Zhedanov, ‘The Racah algebra and superintegrable models’, *J. Phys.: Conf. Ser.* 512 (2012). doi:10.1088/1742-6596/512/1/012011.
- [154] N. A. Kudryashov, ‘The first and second Painlevé equations of higher order and some relations between them’, *Phys. Lett. A* 224 (6) (1997) 353–360. doi:10.1016/S0375-9601(96)00795-5.
- [155] V. S. Matveev, V. V. Shevchishin, ‘Two-dimensional superintegrable metrics with one linear and one cubic integral’, *J. Geom. Phys.* 61 (8) (2011) 1353–1377. doi:10.1016/j.geomphys.2011.02.012.
- [156] A. V. Tsiganov, ‘On the superintegrable Richelot systems’, *J. Phys. A: Math. Theor.* 43 (5) (2010). doi:10.1088/1751-8113/43/5/055201.
- [157] A. V. Tsiganov, ‘Elliptic curve arithmetic and superintegrable systems’, *Phys. Scr.* 94 (8) (2019). doi:10.1088/1402-4896/ab0297.

- [158] A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics*, Institute of Physics Publishing, 1994. doi:10.1201/9780203741450.
- [159] E. G. Kalnins, W. Miller, Jr, G. S. Pogosyan, *Symmetries and Overdetermined Systems of Partial Differential Equations*, Vol. 144 of The IMA Volumes in Mathematics and its Applications, Springer New York, NY, 2008, Ch. ‘Exact and Quasi-Exact Solvability of Second-Order Superintegrable Quantum Systems’, pp. 431–444. doi:10.1007/978-0-387-73831-4.
- [160] I. Popper, S. Post, P. Winternitz, ‘Third-order superintegrable systems separable in parabolic coordinates’, *J. Math. Phys.* 53 (6) (2012). doi:10.1063/1.4729248.
- [161] A. Marchesiello, S. Post, L. Šnobl, ‘Third-order superintegrable systems with potentials satisfying only nonlinear equations’, *J. Math. Phys.* 56 (10) (2015). doi:10.1063/1.4933218.

Appendix A

List of Equations

A.1 Quartic Equations

$$\begin{aligned} S_{40i} : & \xi'_{00i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} - \frac{1}{4} \hbar^2 V''' Q_{i,1}^2 + \frac{5}{4} \hbar^2 Q_{i,2} V'' - \frac{1}{6} Q_{i,1}^3 V'' - \frac{1}{8} \hbar^2 Q_{i,1} V'^2 \right. \\ & + \frac{3}{2} Q_{i,1} Q_{i,2} V' - \frac{1}{2} V^3 Q_{i,1} - \frac{3}{4} V^2 Q_{i,2} - \frac{5}{2} V Q_{i,3} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V''' - \frac{3}{2} \hbar^2 V^2 V' \\ & \left. - \frac{3}{16} \hbar^4 V' V'' \right) + \xi''_{00i}(q_i) \left(-\frac{5}{16} \hbar^4 V''' Q_{i,1} - \hbar^2 Q_{i,1}^2 V'' + \frac{15}{4} \hbar^2 Q_{i,2} V' - \frac{1}{2} Q_{i,1}^3 V' \right. \\ & + 3V Q_{i,1} Q_{i,2} - \frac{9}{8} Q_{i,2}^2 - \frac{5}{2} Q_{i,1} Q_{i,3} + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 - \hbar^2 V^3 \left. \right) \\ & + \xi'''_{00i}(q_i) \left(-\frac{11}{16} \hbar^4 Q_{i,1} V'' - \frac{13}{8} \hbar^2 Q_{i,1}^2 V' + \frac{3}{8} \hbar^2 V^2 Q_{i,1} + \frac{33}{8} \hbar^2 V Q_{i,2} - \frac{1}{2} V Q_{i,1}^3 \right. \\ & \left. - \frac{35}{8} \hbar^2 Q_{i,3} + \frac{3}{4} Q_{i,1}^2 Q_{i,2} + \hbar^4 V V' \right) + \xi''''_{00i}(q_i) \left(-\frac{13}{16} \hbar^4 Q_{i,1} V' - \frac{5}{4} \hbar^2 V Q_{i,1}^2 \right. \\ & + \frac{13}{8} \hbar^2 Q_{i,1} Q_{i,2} - \frac{1}{24} Q_{i,1}^4 + \frac{1}{2} \hbar^4 V^2 \left. \right) + \xi''''''_{00i}(q_i) \left(-\frac{9}{16} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} \right. \\ & \left. - \frac{1}{8} \hbar^2 Q_{i,1}^3 \right) - \frac{3}{32} \hbar^4 \xi''''''_{00i}(q_i) Q_{i,1}^2 - \frac{1}{64} \hbar^6 \xi''''''_{00i}(q_i) Q_{i,1} + \xi_{40i}(q_i) = 0 \end{aligned}$$

$$\begin{aligned} S_{41i} : & \left[\xi_{01i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} - \frac{1}{4} \hbar^2 V''' Q_{i,1}^2 + \frac{5}{4} \hbar^2 Q_{i,2} V'' - \frac{1}{6} Q_{i,1}^3 V'' - \frac{1}{8} \hbar^2 Q_{i,1} V'^2 \right. \right. \\ & + \frac{3}{2} Q_{i,1} Q_{i,2} V' - \frac{1}{2} V^3 Q_{i,1} - \frac{3}{4} V^2 Q_{i,2} - \frac{5}{2} V Q_{i,3} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V''' - \frac{3}{2} \hbar^2 V^2 V' \\ & \left. - \frac{3}{16} \hbar^4 V' V'' \right) + \xi'_{01i}(q_i) \left(-\frac{5}{16} \hbar^4 V''' Q_{i,1} - \hbar^2 Q_{i,1}^2 V'' + \frac{15}{4} \hbar^2 Q_{i,2} V' - \frac{1}{2} Q_{i,1}^3 V' \right. \\ & + 3V Q_{i,1} Q_{i,2} - \frac{9}{8} Q_{i,2}^2 - \frac{5}{2} Q_{i,1} Q_{i,3} + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 - \hbar^2 V^3 \left. \right) \\ & + \xi''_{01i}(q_i) \left(-\frac{11}{16} \hbar^4 Q_{i,1} V'' - \frac{13}{8} \hbar^2 Q_{i,1}^2 V' + \frac{3}{8} \hbar^2 V^2 Q_{i,1} + \frac{33}{8} \hbar^2 V Q_{i,2} - \frac{1}{2} V Q_{i,1}^3 \right. \\ & \left. - \frac{35}{8} \hbar^2 Q_{i,3} + \frac{3}{4} Q_{i,1}^2 Q_{i,2} + \hbar^4 V V' \right) + \xi'''_{01i}(q_i) \left(-\frac{13}{16} \hbar^4 Q_{i,1} V' - \frac{5}{4} \hbar^2 V Q_{i,1}^2 \right. \\ & + \frac{13}{8} \hbar^2 Q_{i,1} Q_{i,2} - \frac{1}{24} Q_{i,1}^4 + \frac{1}{2} \hbar^4 V^2 \left. \right) + \xi''''_{01i}(q_i) \left(-\frac{9}{16} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} \right. \\ & \left. - \frac{1}{8} \hbar^2 Q_{i,1}^3 \right) - \frac{3}{32} \hbar^4 \xi''''_{01i}(q_i) Q_{i,1}^2 - \frac{1}{64} \hbar^6 \xi''''_{01i}(q_i) Q_{i,1} \left. \right]' + \xi_{41i}(q_i) = 0 \end{aligned}$$

$$\begin{aligned}
S_{42i} : & \xi'_{02i}(q_i) \left(-\frac{1}{16} \hbar^4 V''' Q_{i,1} - \frac{1}{8} \hbar^2 V''' Q_{i,1}^2 + \hbar^2 V Q_{i,1} V'' + \frac{3}{8} \hbar^2 Q_{i,2} V'' + \frac{5}{8} \hbar^2 Q_{i,1} V'^2 \right. \\
& \left. + \frac{3}{2} V Q_{i,1}^2 V' - V^3 Q_{i,1} - \frac{9}{4} V^2 Q_{i,2} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V''' - \frac{5}{2} \hbar^2 V^2 V' - \frac{3}{16} \hbar^4 V' V'' \right) \\
& + \xi''_{02i}(q_i) \left(-\frac{5}{16} \hbar^4 V''' Q_{i,1} - \frac{1}{4} \hbar^2 Q_{i,1}^2 V'' + \frac{13}{4} \hbar^2 V Q_{i,1} V' + \frac{9}{8} \hbar^2 Q_{i,2} V' + \frac{1}{2} Q_{i,1}^3 V' \right. \\
& \left. + \frac{3}{2} V^2 Q_{i,1}^2 - \frac{3}{2} V Q_{i,1} Q_{i,2} - \frac{9}{8} Q_{i,2}^2 + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 - 2 \hbar^2 V^3 \right) \\
& + \xi'''_{02i}(q_i) \left(-\frac{11}{16} \hbar^4 Q_{i,1} V'' + \frac{19}{8} \hbar^2 V^2 Q_{i,1} + \frac{9}{8} \hbar^2 V Q_{i,2} + V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} \right. \\
& \left. - \frac{3}{4} Q_{i,1}^2 Q_{i,2} + \hbar^4 V V' \right) + \xi''''_{02i}(q_i) \left(-\frac{13}{16} \hbar^4 Q_{i,1} V' + \frac{1}{2} \hbar^2 V Q_{i,1}^2 + \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} \right. \\
& \left. + \frac{1}{8} Q_{i,1}^4 + \frac{1}{2} \hbar^4 V^2 \right) + \xi''''''_{02i}(q_i) \left(-\frac{1}{2} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} + \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{1}{64} \hbar^6 V' \right) \\
& + \xi''''''''_{02i}(q_i) \left(-\frac{1}{32} \hbar^4 Q_{i,1}^2 - \frac{1}{32} \hbar^6 V \right) - \frac{1}{64} \hbar^6 \xi''''''''_{02i}(q_i) Q_{i,1} + \xi'_{12i}(q_i) \left(\frac{1}{4} \hbar^2 Q_{i,1} V'' \right. \\
& \left. + \frac{1}{2} Q_{i,1}^2 V' - \frac{1}{2} V^2 Q_{i,1} - \frac{3}{2} V Q_{i,2} + \frac{5}{2} Q_{i,3} - \frac{1}{2} \hbar^2 V V' \right) + \xi''_{12i}(q_i) \left(\frac{3}{4} \hbar^2 Q_{i,1} V' \right. \\
& \left. + V Q_{i,1}^2 - \frac{3}{2} Q_{i,1} Q_{i,2} - \frac{1}{2} \hbar^2 V^2 \right) + \xi'''_{12i}(q_i) \left(\hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} + \frac{1}{6} Q_{i,1}^3 \right) \\
& + \frac{1}{4} \hbar^2 \xi'''_{12i}(q_i) Q_{i,1}^2 + \frac{1}{16} \hbar^4 \xi''''_{12i}(q_i) Q_{i,1} + \xi_{42i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{43i} : & \left[\xi_{03i}(q_i) \left(-\frac{1}{16} \hbar^4 V''' Q_{i,1} - \frac{1}{8} \hbar^2 V''' Q_{i,1}^2 + \hbar^2 V Q_{i,1} V'' + \frac{3}{8} \hbar^2 Q_{i,2} V'' + \frac{5}{8} \hbar^2 Q_{i,1} V'^2 \right. \right. \\
& \left. \left. + \frac{3}{2} V Q_{i,1}^2 V' - V^3 Q_{i,1} - \frac{9}{4} V^2 Q_{i,2} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V''' - \frac{5}{2} \hbar^2 V^2 V' - \frac{3}{16} \hbar^4 V' V'' \right) \right. \\
& \left. + \xi'_{03i}(q_i) \left(-\frac{5}{16} \hbar^4 V''' Q_{i,1} - \frac{1}{4} \hbar^2 Q_{i,1}^2 V'' + \frac{13}{4} \hbar^2 V Q_{i,1} V' + \frac{9}{8} \hbar^2 Q_{i,2} V' + \frac{1}{2} Q_{i,1}^3 V' \right. \right. \\
& \left. \left. + \frac{3}{2} V^2 Q_{i,1}^2 - \frac{3}{2} V Q_{i,1} Q_{i,2} - \frac{9}{8} Q_{i,2}^2 + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 - 2 \hbar^2 V^3 \right) \right. \\
& \left. + \xi''_{03i}(q_i) \left(-\frac{11}{16} \hbar^4 Q_{i,1} V'' + \frac{19}{8} \hbar^2 V^2 Q_{i,1} + \frac{9}{8} \hbar^2 V Q_{i,2} + V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} \right. \right. \\
& \left. \left. - \frac{3}{4} Q_{i,1}^2 Q_{i,2} + \hbar^4 V V' \right) + \xi'''_{03i}(q_i) \left(-\frac{13}{16} \hbar^4 Q_{i,1} V' + \frac{1}{2} \hbar^2 V Q_{i,1}^2 + \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} \right. \right. \\
& \left. \left. + \frac{1}{8} Q_{i,1}^4 + \frac{1}{2} \hbar^4 V^2 \right) + \xi''''_{03i}(q_i) \left(-\frac{1}{2} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} + \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{1}{64} \hbar^6 V' \right) \right. \\
& \left. + \xi''''''_{03i}(q_i) \left(-\frac{1}{32} \hbar^4 Q_{i,1}^2 - \frac{1}{32} \hbar^6 V \right) - \frac{1}{64} \hbar^6 \xi''''''_{03i}(q_i) Q_{i,1} + \xi_{13i}(q_i) \left(\frac{1}{4} \hbar^2 Q_{i,1} V'' \right. \right. \\
& \left. \left. + \frac{1}{2} Q_{i,1}^2 V' - \frac{1}{2} V^2 Q_{i,1} - \frac{3}{2} V Q_{i,2} + \frac{5}{2} Q_{i,3} - \frac{1}{2} \hbar^2 V V' \right) + \xi'_{13i}(q_i) \left(\frac{3}{4} \hbar^2 Q_{i,1} V' \right. \right. \\
& \left. \left. + V Q_{i,1}^2 - \frac{3}{2} Q_{i,1} Q_{i,2} - \frac{1}{2} \hbar^2 V^2 \right) + \xi''_{13i}(q_i) \left(\hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} + \frac{1}{6} Q_{i,1}^3 \right) \right. \\
& \left. + \frac{1}{4} \hbar^2 \xi''_{13i}(q_i) Q_{i,1}^2 + \frac{1}{16} \hbar^4 \xi'''_{13i}(q_i) Q_{i,1} \right]' + \xi_{43i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{44i} : & \xi'_{04i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} + \frac{5}{4} \hbar^2 V Q_{i,1} V'' + \frac{5}{8} \hbar^2 Q_{i,1} V'^2 - \frac{5}{2} V^3 Q_{i,1} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V'''' \right. \\
& \left. - \frac{5}{2} \hbar^2 V^2 V' - \frac{3}{16} \hbar^4 V' V'' \right) + \xi''_{04i}(q_i) \left(-\frac{5}{16} \hbar^4 V'''' Q_{i,1} + \frac{1}{4} \hbar^2 Q_{i,1}^2 V'' + 4 \hbar^2 V Q_{i,1} V' \right. \\
& \left. - \frac{3}{2} V^2 Q_{i,1}^2 - \frac{3}{2} V Q_{i,1} Q_{i,2} + \frac{9}{8} Q_{i,2}^2 + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 - 2 \hbar^2 V^3 \right) \\
& + \xi'''_{04i}(q_i) \left(-\frac{5}{8} \hbar^4 Q_{i,1} V'' + \frac{3}{4} \hbar^2 Q_{i,1}^2 V' + \frac{11}{4} \hbar^2 V^2 Q_{i,1} - \frac{1}{2} V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} \right. \\
& \left. - \frac{3}{4} Q_{i,1}^2 Q_{i,2} - \frac{1}{64} \hbar^6 V'''' + \frac{19}{16} \hbar^4 V V' \right) + \xi''''_{04i}(q_i) \left(-\frac{9}{16} \hbar^4 Q_{i,1} V' + \hbar^2 V Q_{i,1}^2 \right. \\
& \left. - \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} - \frac{1}{8} Q_{i,1}^4 - \frac{1}{16} \hbar^6 V'' + \frac{3}{4} \hbar^4 V^2 \right) + \xi''''''_{04i}(q_i) \left(-\frac{3}{16} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} \right. \\
& \left. + \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{3}{32} \hbar^6 V' \right) + \xi''''''''_{04i}(q_i) \left(\frac{1}{32} \hbar^4 Q_{i,1}^2 - \frac{1}{16} \hbar^6 V \right) - \frac{1}{64} \hbar^6 \xi''''''''_{04i}(q_i) Q_{i,1} \\
& + \xi'_{14i}(q_i) \left(\frac{1}{4} \hbar^2 Q_{i,1} V'' - \frac{3}{2} V^2 Q_{i,1} + \frac{5}{2} Q_{i,3} - \frac{1}{2} \hbar^2 V V' \right) + \xi''_{14i}(q_i) \left(\frac{3}{4} \hbar^2 Q_{i,1} V' \right. \\
& \left. - V Q_{i,1}^2 - \frac{1}{2} \hbar^2 V^2 \right) + \xi'''_{14i}(q_i) \left(\frac{3}{4} \hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} - \frac{1}{3} Q_{i,1}^3 + \frac{1}{16} \hbar^4 V' \right) \\
& + \frac{1}{8} \hbar^4 \xi''''_{14i}(q_i) V + \frac{1}{16} \hbar^4 \xi''''''_{14i}(q_i) Q_{i,1} + \xi'_{24i}(q_i) \left(\frac{3}{2} Q_{i,2} - V Q_{i,1} \right) - \frac{1}{2} \xi''_{24i}(q_i) Q_{i,1}^2 \\
& - \frac{1}{4} \xi'''_{24i}(q_i) \hbar^2 Q_{i,1} + \xi_{44i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{45i} : & \left[\xi_{05i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} + \frac{5}{4} \hbar^2 V Q_{i,1} V'' + \frac{5}{8} \hbar^2 Q_{i,1} V'^2 - \frac{5}{2} V^3 Q_{i,1} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V'''' \right. \right. \\
& \left. - \frac{5}{2} \hbar^2 V^2 V' - \frac{3}{16} \hbar^4 V' V'' \right) + \xi'_{05i}(q_i) \left(-\frac{5}{16} \hbar^4 V'''' Q_{i,1} + \frac{1}{4} \hbar^2 Q_{i,1}^2 V'' + 4 \hbar^2 V Q_{i,1} V' \right. \\
& \left. - \frac{3}{2} V^2 Q_{i,1}^2 - \frac{3}{2} V Q_{i,1} Q_{i,2} + \frac{9}{8} Q_{i,2}^2 + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 - 2 \hbar^2 V^3 \right) \\
& + \xi''_{05i}(q_i) \left(-\frac{5}{8} \hbar^4 Q_{i,1} V'' + \frac{3}{4} \hbar^2 Q_{i,1}^2 V' + \frac{11}{4} \hbar^2 V^2 Q_{i,1} - \frac{1}{2} V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} \right. \\
& \left. - \frac{3}{4} Q_{i,1}^2 Q_{i,2} - \frac{1}{64} \hbar^6 V'''' + \frac{19}{16} \hbar^4 V V' \right) + \xi'''_{05i}(q_i) \left(-\frac{9}{16} \hbar^4 Q_{i,1} V' + \hbar^2 V Q_{i,1}^2 \right. \\
& \left. - \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} - \frac{1}{8} Q_{i,1}^4 - \frac{1}{16} \hbar^6 V'' + \frac{3}{4} \hbar^4 V^2 \right) + \xi''''_{05i}(q_i) \left(-\frac{3}{16} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} \right. \\
& \left. + \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{3}{32} \hbar^6 V' \right) + \xi''''''_{05i}(q_i) \left(\frac{1}{32} \hbar^4 Q_{i,1}^2 - \frac{1}{16} \hbar^6 V \right) - \frac{1}{64} \hbar^6 \xi''''''_{05i}(q_i) Q_{i,1} \\
& + \xi_{15i}(q_i) \left(\frac{1}{4} \hbar^2 Q_{i,1} V'' - \frac{3}{2} V^2 Q_{i,1} + \frac{5}{2} Q_{i,3} - \frac{1}{2} \hbar^2 V V' \right) + \xi'_{15i}(q_i) \left(\frac{3}{4} \hbar^2 Q_{i,1} V' \right. \\
& \left. - V Q_{i,1}^2 - \frac{1}{2} \hbar^2 V^2 \right) + \xi''_{15i}(q_i) \left(\frac{3}{4} \hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} - \frac{1}{3} Q_{i,1}^3 + \frac{1}{16} \hbar^4 V' \right) \\
& + \frac{1}{8} \hbar^4 \xi'''_{15i}(q_i) V + \frac{1}{16} \hbar^4 \xi''''_{15i}(q_i) Q_{i,1} + \xi_{25i}(q_i) \left(\frac{3}{2} Q_{i,2} - V Q_{i,1} \right) - \frac{1}{2} \xi'_{25i}(q_i) Q_{i,1}^2 \\
& \left. - \frac{1}{4} \xi''_{25i}(q_i) \hbar^2 Q_{i,1} \right]' + \xi_{45i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{46i} : & \xi'_{06i}(q_i) \left(\frac{35}{8} Q_{i,4} - \frac{1}{64} \hbar^6 V'''''' + \frac{7}{16} \hbar^4 V V''' - \frac{35}{8} \hbar^2 V^2 V' + \frac{7}{16} \hbar^4 V' V'' \right) \\
& + \xi''_{06i}(q_i) \left(\frac{1}{16} \hbar^4 V''' Q_{i,1} - \frac{5}{4} \hbar^2 V Q_{i,1} V' - \frac{3}{8} \hbar^2 Q_{i,2} V' + \frac{9}{8} Q_{i,2}^2 + \frac{5}{2} Q_{i,1} Q_{i,3} \right. \\
& \left. - \frac{3}{32} \hbar^6 V'''' + 2 \hbar^4 V V'' + \frac{27}{32} \hbar^4 V'^2 - 4 \hbar^2 V^3 \right) + \xi'''_{06i}(q_i) \left(\frac{1}{4} \hbar^4 Q_{i,1} V'' - \frac{1}{8} \hbar^2 Q_{i,1}^2 V' \right. \\
& \left. - \frac{3}{2} \hbar^2 V^2 Q_{i,1} - \frac{3}{4} \hbar^2 V Q_{i,2} - \frac{35}{8} \hbar^2 Q_{i,3} + \frac{3}{4} Q_{i,1}^2 Q_{i,2} - \frac{15}{64} \hbar^6 V'''' + \frac{61}{16} \hbar^4 V V' \right) \\
& + \xi''''_{06i}(q_i) \left(\frac{7}{16} \hbar^4 Q_{i,1} V' - \frac{1}{4} \hbar^2 V Q_{i,1}^2 - \frac{13}{8} \hbar^2 Q_{i,1} Q_{i,2} + \frac{1}{24} Q_{i,1}^4 - \frac{5}{16} \hbar^6 V'' + \frac{7}{4} \hbar^4 V^2 \right) \\
& + \xi''''''_{06i}(q_i) \left(\frac{3}{8} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} - \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{15}{64} \hbar^6 V' \right) + \xi''''''''_{06i}(q_i) \left(\frac{3}{32} \hbar^4 Q_{i,1}^2 \right. \\
& \left. - \frac{3}{32} \hbar^6 V \right) + \xi''''''''''_{06i}(q_i) \left(-\frac{1}{64} \hbar^6 Q_{i,1} \right) + \xi'_{16i}(q_i) \left(\frac{5}{2} Q_{i,3} + \frac{1}{16} \hbar^4 V'''' - \frac{5}{4} \hbar^2 V V' \right) \\
& + \xi''_{16i}(q_i) \left(-\frac{1}{4} \hbar^2 Q_{i,1} V' + \frac{3}{2} Q_{i,1} Q_{i,2} + \frac{1}{4} \hbar^4 V'' - \frac{3}{2} \hbar^2 V^2 \right) + \xi'''_{16i}(q_i) \left(-\frac{1}{2} \hbar^2 V Q_{i,1} \right. \\
& \left. - \frac{5}{4} \hbar^2 Q_{i,2} + \frac{1}{6} Q_{i,1}^3 + \frac{3}{8} \hbar^4 V' \right) + \xi''''_{16i}(q_i) \left(\frac{1}{4} \hbar^4 V - \frac{1}{4} \hbar^2 Q_{i,1}^2 \right) + \frac{1}{16} \hbar^4 \xi''''''_{16i}(q_i) Q_{i,1} \\
& + \xi'_{26i}(q_i) \left(\frac{3}{2} Q_{i,2} - \frac{1}{4} \hbar^2 V' \right) + \xi''_{26i}(q_i) \left(\frac{1}{2} Q_{i,1}^2 - \frac{1}{2} \hbar^2 V \right) - \frac{1}{4} \hbar^2 \xi''''_{26i}(q_i) Q_{i,1} \\
& + \xi'_{36i}(q_i) Q_{i,1} + \xi_{46i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{47i} : & \left[\xi_{07i}(q_i) \left(\frac{35}{8} Q_{i,4} - \frac{1}{64} \hbar^6 V'''''' + \frac{7}{16} \hbar^4 V V''' - \frac{35}{8} \hbar^2 V^2 V' + \frac{7}{16} \hbar^4 V' V'' \right) \right. \\
& + \xi'_{07i}(q_i) \left(\frac{1}{16} \hbar^4 V''' Q_{i,1} - \frac{5}{4} \hbar^2 V Q_{i,1} V' - \frac{3}{8} \hbar^2 Q_{i,2} V' + \frac{9}{8} Q_{i,2}^2 + \frac{5}{2} Q_{i,1} Q_{i,3} \right. \\
& \left. - \frac{3}{32} \hbar^6 V'''' + 2 \hbar^4 V V'' + \frac{27}{32} \hbar^4 V'^2 - 4 \hbar^2 V^3 \right) + \xi''_{07i}(q_i) \left(\frac{1}{4} \hbar^4 Q_{i,1} V'' - \frac{1}{8} \hbar^2 Q_{i,1}^2 V' \right. \\
& \left. - \frac{3}{2} \hbar^2 V^2 Q_{i,1} - \frac{3}{4} \hbar^2 V Q_{i,2} - \frac{35}{8} \hbar^2 Q_{i,3} + \frac{3}{4} Q_{i,1}^2 Q_{i,2} - \frac{15}{64} \hbar^6 V'''' + \frac{61}{16} \hbar^4 V V' \right) \\
& + \xi'''_{07i}(q_i) \left(\frac{7}{16} \hbar^4 Q_{i,1} V' - \frac{1}{4} \hbar^2 V Q_{i,1}^2 - \frac{13}{8} \hbar^2 Q_{i,1} Q_{i,2} + \frac{1}{24} Q_{i,1}^4 - \frac{5}{16} \hbar^6 V'' + \frac{7}{4} \hbar^4 V^2 \right) \\
& + \xi''''_{07i}(q_i) \left(\frac{3}{8} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} - \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{15}{64} \hbar^6 V' \right) + \xi''''''_{07i}(q_i) \left(\frac{3}{32} \hbar^4 Q_{i,1}^2 \right. \\
& \left. - \frac{3}{32} \hbar^6 V \right) + \xi''''''''_{07i}(q_i) \left(-\frac{1}{64} \hbar^6 Q_{i,1} \right) + \xi_{17i}(q_i) \left(\frac{5}{2} Q_{i,3} + \frac{1}{16} \hbar^4 V'''' - \frac{5}{4} \hbar^2 V V' \right) \\
& + \xi'_{17i}(q_i) \left(-\frac{1}{4} \hbar^2 Q_{i,1} V' + \frac{3}{2} Q_{i,1} Q_{i,2} + \frac{1}{4} \hbar^4 V'' - \frac{3}{2} \hbar^2 V^2 \right) + \xi''_{17i}(q_i) \left(-\frac{1}{2} \hbar^2 V Q_{i,1} \right. \\
& \left. - \frac{5}{4} \hbar^2 Q_{i,2} + \frac{1}{6} Q_{i,1}^3 + \frac{3}{8} \hbar^4 V' \right) + \xi'''_{17i}(q_i) \left(\frac{1}{4} \hbar^4 V - \frac{1}{4} \hbar^2 Q_{i,1}^2 \right) + \frac{1}{16} \hbar^4 \xi''''_{17i}(q_i) Q_{i,1} \\
& + \xi'_{27i}(q_i) \left(\frac{3}{2} Q_{i,2} - \frac{1}{4} \hbar^2 V' \right) + \xi''_{27i}(q_i) \left(\frac{1}{2} Q_{i,1}^2 - \frac{1}{2} \hbar^2 V \right) - \frac{1}{4} \hbar^2 \xi''''_{27i}(q_i) Q_{i,1} \\
& \left. + \xi_{37i}(q_i) Q_{i,1} \right]' + \xi_{47i}(q_i) = 0
\end{aligned}$$

A.2 Quintic Equations

$$\begin{aligned}
S_{50i} : & \xi'_{00i}(q_i) \left(\frac{1}{64} \hbar^6 V'''''' Q_{i,1} + \frac{3}{32} \hbar^4 V'''''' Q_{i,1}^2 + \frac{3}{16} \hbar^4 V V'''''' Q_{i,1} - \frac{21}{32} \hbar^4 V'''''' Q_{i,2} \right. \\
& + \frac{1}{8} \hbar^2 V'''''' Q_{i,1}^3 + \frac{1}{2} \hbar^2 V V'''''' Q_{i,1}^2 - \frac{13}{8} \hbar^2 V'''''' Q_{i,1} Q_{i,2} + \frac{1}{24} V'''''' Q_{i,1}^4 \\
& + \frac{1}{32} \hbar^4 Q_{i,1} V''^2 + \frac{1}{8} \hbar^2 V^2 Q_{i,1} V'' - \frac{7}{8} \hbar^2 V Q_{i,2} V'' + \frac{35}{8} \hbar^2 Q_{i,3} V'' + \frac{1}{6} V Q_{i,1}^3 V'' \\
& - \frac{3}{4} Q_{i,1}^2 Q_{i,2} V'' - \frac{5}{8} \hbar^2 V Q_{i,1} V'^2 - \frac{3}{16} \hbar^2 Q_{i,2} V'^2 + \frac{9}{8} Q_{i,2}^2 V' + \frac{5}{2} Q_{i,1} Q_{i,3} V' \\
& + \frac{1}{4} \hbar^4 V'''' Q_{i,1} V' + \frac{1}{4} \hbar^2 Q_{i,1}^2 V' V'' - \frac{5}{8} V^4 Q_{i,1} - \frac{3}{4} V^3 Q_{i,2} - \frac{5}{4} V^2 Q_{i,3} - \frac{35}{8} V Q_{i,4} \\
& + \frac{63}{8} Q_{i,5} - \frac{1}{32} \hbar^6 V V'''''' + \frac{1}{4} \hbar^4 V^2 V'''' + \frac{9}{32} \hbar^4 V'^3 - 4 \hbar^2 V^3 V' + \frac{3}{64} \hbar^6 V'''''' V' \\
& \left. - \frac{1}{16} \hbar^6 V'''''' V'' - \frac{7}{4} \hbar^4 V V' V'' \right) + \xi''_{00i}(q_i) \left(\frac{7}{64} \hbar^6 V'''''' Q_{i,1} + \frac{9}{16} \hbar^4 V'''''' Q_{i,1}^2 \right. \\
& + \frac{15}{16} \hbar^4 V V'''''' Q_{i,1} - \frac{105}{32} \hbar^4 V'''''' Q_{i,2} + \frac{5}{8} \hbar^2 V'''''' Q_{i,1}^3 + 2 \hbar^2 V Q_{i,1}^2 V'' - \frac{13}{2} \hbar^2 Q_{i,1} Q_{i,2} V'' \\
& + \frac{1}{6} Q_{i,1}^4 V'' + \frac{3}{4} \hbar^2 Q_{i,1}^2 V'^2 + \frac{3}{8} \hbar^2 V^2 Q_{i,1} V' - \frac{21}{8} \hbar^2 V Q_{i,2} V' + \frac{105}{8} \hbar^2 Q_{i,3} V' \\
& + \frac{1}{2} V Q_{i,1}^3 V' - \frac{9}{4} Q_{i,1}^2 Q_{i,2} V' + \frac{19}{16} \hbar^4 Q_{i,1} V' V'' + \frac{9}{4} V Q_{i,2}^2 + 5 V Q_{i,1} Q_{i,3} - \frac{15}{4} Q_{i,2} Q_{i,3} \\
& - \frac{35}{8} Q_{i,1} Q_{i,4} - \frac{3}{16} \hbar^6 V V'''''' - \frac{1}{8} \hbar^6 V''^2 + \hbar^4 V^2 V'' - 3 \hbar^4 V V'^2 + \frac{15}{64} \hbar^6 V'''''' V' - 2 \hbar^2 V^4 \left. \right) \\
& + \xi'''_{00i}(q_i) \left(\frac{11}{32} \hbar^6 V'''''' Q_{i,1} + \frac{47}{32} \hbar^4 V'''''' Q_{i,1}^2 + \frac{13}{8} \hbar^4 V Q_{i,1} V'' - \frac{55}{8} \hbar^4 Q_{i,2} V'' \right. \\
& + \frac{31}{24} \hbar^2 Q_{i,1}^3 V'' + \frac{43}{32} \hbar^4 Q_{i,1} V'^2 + \frac{11}{4} \hbar^2 V Q_{i,1}^2 V' - \frac{81}{8} \hbar^2 Q_{i,1} Q_{i,2} V' + \frac{1}{4} Q_{i,1}^4 V' \\
& + \frac{3}{4} \hbar^2 V^3 Q_{i,1} - \frac{3}{4} \hbar^2 V^2 Q_{i,2} + \frac{55}{4} \hbar^2 V Q_{i,3} + \frac{1}{4} V^2 Q_{i,1}^3 - \frac{9}{4} V Q_{i,1}^2 Q_{i,2} - \frac{105}{8} \hbar^2 Q_{i,4} \\
& + \frac{9}{8} Q_{i,1} Q_{i,2}^2 + \frac{5}{4} Q_{i,1}^2 Q_{i,3} - \frac{17}{32} \hbar^6 V V'''''' + \frac{23}{8} \hbar^4 V^2 V' + \frac{5}{8} \hbar^6 V' V'' \left. \right) \\
& + \xi''''_{00i}(q_i) \left(\frac{5}{8} \hbar^6 V'''''' Q_{i,1} + \frac{17}{8} \hbar^4 Q_{i,1}^2 V'' + \frac{9}{8} \hbar^4 V Q_{i,1} V' - \frac{15}{2} \hbar^4 Q_{i,2} V' + \frac{11}{8} \hbar^2 Q_{i,1}^3 V' \right. \\
& + \frac{3}{4} \hbar^2 V^2 Q_{i,1}^2 - \frac{29}{4} \hbar^2 V Q_{i,1} Q_{i,2} + \frac{1}{6} V Q_{i,1}^4 + \frac{15}{8} \hbar^2 Q_{i,2}^2 + 5 \hbar^2 Q_{i,1} Q_{i,3} - \frac{1}{4} Q_{i,1}^3 Q_{i,2} \\
& - \frac{7}{8} \hbar^6 V V'' + \frac{15}{32} \hbar^6 V'^2 + \frac{5}{4} \hbar^4 V^3 \left. \right) + \xi''''''_{00i}(q_i) \left(\frac{23}{32} \hbar^6 Q_{i,1} V'' + \frac{29}{16} \hbar^4 Q_{i,1}^2 V' \right. \\
& - \frac{1}{16} \hbar^4 V^2 Q_{i,1} - \frac{69}{16} \hbar^4 V Q_{i,2} + \frac{3}{4} \hbar^2 V Q_{i,1}^3 + \frac{63}{16} \hbar^4 Q_{i,3} - \hbar^2 Q_{i,1}^2 Q_{i,2} + \frac{1}{120} Q_{i,1}^5 \\
& \left. - \frac{15}{16} \hbar^6 V V' \right) + \xi''''''''_{00i}(q_i) \left(\frac{17}{32} \hbar^6 Q_{i,1} V' + \frac{7}{8} \hbar^4 V Q_{i,1}^2 - \frac{17}{16} \hbar^4 Q_{i,1} Q_{i,2} + \frac{1}{24} \hbar^2 Q_{i,1}^4 \right. \\
& \left. - \frac{5}{16} \hbar^6 V^2 \right) + \xi''''''''''_{00i}(q_i) \left(\frac{1}{4} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} + \frac{1}{16} \hbar^4 Q_{i,1}^3 \right) + \frac{1}{32} \hbar^6 \xi''''''''''''_{00i}(q_i) Q_{i,1}^2 \\
& + \frac{1}{256} \hbar^8 \xi''''''''''''''_{00i}(q_i) Q_{i,1} + \xi_{50i}(q_i)
\end{aligned}$$

$$\begin{aligned}
S_{51i} : & \left[\xi_{01i}(q_i) \left(\frac{1}{64} \hbar^6 V'''''' Q_{i,1} + \frac{3}{32} \hbar^4 V'''''' Q_{i,1}^2 + \frac{3}{16} \hbar^4 V V'''' Q_{i,1} - \frac{21}{32} \hbar^4 V'''' Q_{i,2} \right. \right. \\
& + \frac{1}{8} \hbar^2 V'''' Q_{i,1}^3 + \frac{1}{2} \hbar^2 V V'''' Q_{i,1}^2 - \frac{13}{8} \hbar^2 V'''' Q_{i,1} Q_{i,2} + \frac{1}{24} V'''' Q_{i,1}^4 \\
& + \frac{1}{32} \hbar^4 Q_{i,1} V''^2 + \frac{1}{8} \hbar^2 V^2 Q_{i,1} V'' - \frac{7}{8} \hbar^2 V Q_{i,2} V'' + \frac{35}{8} \hbar^2 Q_{i,3} V'' + \frac{1}{6} V Q_{i,1}^3 V'' \\
& - \frac{3}{4} Q_{i,1}^2 Q_{i,2} V'' - \frac{5}{8} \hbar^2 V Q_{i,1} V'^2 - \frac{3}{16} \hbar^2 Q_{i,2} V'^2 + \frac{9}{8} Q_{i,2}^2 V' + \frac{5}{2} Q_{i,1} Q_{i,3} V' \\
& + \frac{1}{4} \hbar^4 V'''' Q_{i,1} V' + \frac{1}{4} \hbar^2 Q_{i,1}^2 V' V'' - \frac{5}{8} V^4 Q_{i,1} - \frac{3}{4} V^3 Q_{i,2} - \frac{5}{4} V^2 Q_{i,3} - \frac{35}{8} V Q_{i,4} \\
& + \frac{63}{8} Q_{i,5} - \frac{1}{32} \hbar^6 V V'''''' + \frac{1}{4} \hbar^4 V^2 V'''' + \frac{9}{32} \hbar^4 V'^3 - 4 \hbar^2 V^3 V' + \frac{3}{64} \hbar^6 V'''''' V' \\
& \left. - \frac{1}{16} \hbar^6 V'''' V'' - \frac{7}{4} \hbar^4 V V' V'' \right) + \xi'_{01i}(q_i) \left(\frac{7}{64} \hbar^6 V'''''' Q_{i,1} + \frac{9}{16} \hbar^4 V'''' Q_{i,1}^2 \right. \\
& + \frac{15}{16} \hbar^4 V V'''' Q_{i,1} - \frac{105}{32} \hbar^4 V'''' Q_{i,2} + \frac{5}{8} \hbar^2 V'''' Q_{i,1}^3 + 2 \hbar^2 V Q_{i,1}^2 V'' - \frac{13}{2} \hbar^2 Q_{i,1} Q_{i,2} V'' \\
& + \frac{1}{6} Q_{i,1}^4 V'' + \frac{3}{4} \hbar^2 Q_{i,1}^2 V'^2 + \frac{3}{8} \hbar^2 V^2 Q_{i,1} V' - \frac{21}{8} \hbar^2 V Q_{i,2} V' + \frac{105}{8} \hbar^2 Q_{i,3} V' \\
& + \frac{1}{2} V Q_{i,1}^3 V' - \frac{9}{4} Q_{i,1}^2 Q_{i,2} V' + \frac{19}{16} \hbar^4 Q_{i,1} V' V'' + \frac{9}{4} V Q_{i,2}^2 + 5 V Q_{i,1} Q_{i,3} - \frac{15}{4} Q_{i,2} Q_{i,3} \\
& \left. - \frac{35}{8} Q_{i,1} Q_{i,4} - \frac{3}{16} \hbar^6 V V'''''' - \frac{1}{8} \hbar^6 V''^2 + \hbar^4 V^2 V'' - 3 \hbar^4 V V'^2 + \frac{15}{64} \hbar^6 V'''''' V' - 2 \hbar^2 V^4 \right) \\
& + \xi''_{01i}(q_i) \left(\frac{11}{32} \hbar^6 V'''''' Q_{i,1} + \frac{47}{32} \hbar^4 V'''' Q_{i,1}^2 + \frac{13}{8} \hbar^4 V Q_{i,1} V'' - \frac{55}{8} \hbar^4 Q_{i,2} V'' \right. \\
& + \frac{31}{24} \hbar^2 Q_{i,1}^3 V'' + \frac{43}{32} \hbar^4 Q_{i,1} V'^2 + \frac{11}{4} \hbar^2 V Q_{i,1}^2 V' - \frac{81}{8} \hbar^2 Q_{i,1} Q_{i,2} V' + \frac{1}{4} Q_{i,1}^4 V' \\
& + \frac{3}{4} \hbar^2 V^3 Q_{i,1} - \frac{3}{4} \hbar^2 V^2 Q_{i,2} + \frac{55}{4} \hbar^2 V Q_{i,3} + \frac{1}{4} V^2 Q_{i,1}^3 - \frac{9}{4} V Q_{i,1}^2 Q_{i,2} - \frac{105}{8} \hbar^2 Q_{i,4} \\
& \left. + \frac{9}{8} Q_{i,1} Q_{i,2}^2 + \frac{5}{4} Q_{i,1}^2 Q_{i,3} - \frac{17}{32} \hbar^6 V V'''''' + \frac{23}{8} \hbar^4 V^2 V' + \frac{5}{8} \hbar^6 V' V'' \right) \\
& + \xi'''_{01i}(q_i) \left(\frac{5}{8} \hbar^6 V'''' Q_{i,1} + \frac{17}{8} \hbar^4 Q_{i,1}^2 V'' + \frac{9}{8} \hbar^4 V Q_{i,1} V' - \frac{15}{2} \hbar^4 Q_{i,2} V' + \frac{11}{8} \hbar^2 Q_{i,1}^3 V' \right. \\
& + \frac{3}{4} \hbar^2 V^2 Q_{i,1}^2 - \frac{29}{4} \hbar^2 V Q_{i,1} Q_{i,2} + \frac{1}{6} V Q_{i,1}^4 + \frac{15}{8} \hbar^2 Q_{i,2}^2 + 5 \hbar^2 Q_{i,1} Q_{i,3} - \frac{1}{4} Q_{i,1}^3 Q_{i,2} \\
& \left. - \frac{7}{8} \hbar^6 V V'' + \frac{15}{32} \hbar^6 V'^2 + \frac{5}{4} \hbar^4 V^3 \right) + \xi''''_{01i}(q_i) \left(\frac{23}{32} \hbar^6 Q_{i,1} V'' + \frac{29}{16} \hbar^4 Q_{i,1}^2 V' \right. \\
& - \frac{1}{16} \hbar^4 V^2 Q_{i,1} - \frac{69}{16} \hbar^4 V Q_{i,2} + \frac{3}{4} \hbar^2 V Q_{i,1}^3 + \frac{63}{16} \hbar^4 Q_{i,3} - \hbar^2 Q_{i,1}^2 Q_{i,2} + \frac{1}{120} Q_{i,1}^5 \\
& \left. - \frac{15}{16} \hbar^6 V V' \right) + \xi''''''_{01i}(q_i) \left(\frac{17}{32} \hbar^6 Q_{i,1} V' + \frac{7}{8} \hbar^4 V Q_{i,1}^2 - \frac{17}{16} \hbar^4 Q_{i,1} Q_{i,2} + \frac{1}{24} \hbar^2 Q_{i,1}^4 \right. \\
& \left. - \frac{5}{16} \hbar^6 V^2 \right) + \xi''''''''_{01i}(q_i) \left(\frac{1}{4} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} + \frac{1}{16} \hbar^4 Q_{i,1}^3 \right) + \frac{1}{32} \hbar^6 \xi''''''''''_{01i}(q_i) Q_{i,1}^2 \\
& \left. + \frac{1}{256} \hbar^8 \xi''''''''''''_{01i}(q_i) Q_{i,1} \right]' + \xi_{51i}(q_i)
\end{aligned}$$

$$\begin{aligned}
S_{52i} : \xi'_{02i}(q_i) & \left(\frac{1}{64} \hbar^6 V'''''' Q_{i,1} + \frac{1}{16} \hbar^4 V'''''' Q_{i,1}^2 - \frac{3}{16} \hbar^4 V V'''' Q_{i,1} - \frac{5}{16} \hbar^4 V'''' Q_{i,2} \right. \\
& + \frac{1}{24} \hbar^2 V'''' Q_{i,1}^3 - \frac{3}{4} \hbar^2 V V'''' Q_{i,1}^2 - \frac{3}{8} \hbar^2 V'''' Q_{i,1} Q_{i,2} - \frac{19}{32} \hbar^4 Q_{i,1} V''^2 + \frac{3}{8} \hbar^2 V^2 Q_{i,1} V'' \\
& + \frac{33}{8} \hbar^2 V Q_{i,2} V'' + \frac{5}{8} \hbar^2 Q_{i,3} V'' - \frac{1}{2} V Q_{i,1}^3 V'' - \frac{3}{4} \hbar^2 V Q_{i,1} V'^2 + \frac{57}{16} \hbar^2 Q_{i,2} V'^2 \\
& - \frac{1}{2} Q_{i,1}^3 V'^2 + \frac{9}{2} V Q_{i,1} Q_{i,2} V' - \frac{11}{16} \hbar^4 V'''' Q_{i,1} V' - \frac{9}{4} \hbar^2 Q_{i,1}^2 V' V'' - \frac{9}{8} V^4 Q_{i,1} \\
& - \frac{3}{2} V^3 Q_{i,2} - \frac{15}{4} V^2 Q_{i,3} + \frac{63}{8} Q_{i,5} - \frac{1}{32} \hbar^6 V V'''''' + \frac{3}{4} \hbar^4 V^2 V'''' - \frac{13}{2} \hbar^2 V^3 V' \\
& + \frac{3}{64} \hbar^6 V'''' V' - \frac{1}{16} \hbar^6 V'''' V'' - \frac{9}{16} \hbar^4 V V' V'' \left. \right) + \xi''_{02i}(q_i) \left(\frac{7}{64} \hbar^6 V'''''' Q_{i,1} \right. \\
& + \frac{5}{16} \hbar^4 V'''''' Q_{i,1}^2 - \frac{9}{8} \hbar^4 V V'''' Q_{i,1} - \frac{25}{16} \hbar^4 V'''' Q_{i,2} - \frac{1}{24} \hbar^2 V'''' Q_{i,1}^3 - 4 \hbar^2 V Q_{i,1}^2 V'' \\
& - \frac{1}{4} \hbar^2 Q_{i,1} Q_{i,2} V'' - \frac{1}{6} Q_{i,1}^4 V'' - \frac{31}{8} \hbar^2 Q_{i,1}^2 V'^2 - \frac{3}{8} \hbar^2 V^2 Q_{i,1} V' + \frac{129}{8} \hbar^2 V Q_{i,2} V' \\
& + \frac{15}{8} \hbar^2 Q_{i,3} V' - 2 V Q_{i,1}^3 V' + \frac{3}{2} Q_{i,1}^2 Q_{i,2} V' - \frac{27}{8} \hbar^4 Q_{i,1} V' V'' - \frac{1}{2} V^3 Q_{i,1}^2 \\
& + \frac{21}{4} V^2 Q_{i,1} Q_{i,2} - \frac{5}{2} V Q_{i,1} Q_{i,3} - \frac{15}{4} Q_{i,2} Q_{i,3} - \frac{3}{16} \hbar^6 V V'''' - \frac{1}{8} \hbar^6 V''^2 + \frac{7}{2} \hbar^4 V^2 V'' \\
& - \frac{15}{16} \hbar^4 V V'^2 + \frac{15}{64} \hbar^6 V'''' V' - 4 \hbar^2 V^4 \left. \right) + \xi'''_{02i}(q_i) \left(\frac{11}{32} \hbar^6 V'''''' Q_{i,1} + \frac{21}{32} \hbar^4 V'''' Q_{i,1}^2 \right. \\
& - \frac{25}{8} \hbar^4 V Q_{i,1} V'' - \frac{103}{32} \hbar^4 Q_{i,2} V'' - \frac{7}{12} \hbar^2 Q_{i,1}^3 V'' - \frac{23}{8} \hbar^4 Q_{i,1} V'^2 - \frac{65}{8} \hbar^2 V Q_{i,1}^2 V' \\
& + \frac{3}{2} \hbar^2 Q_{i,1} Q_{i,2} V' - \frac{1}{2} Q_{i,1}^4 V' + \frac{5}{8} \hbar^2 V^3 Q_{i,1} + \frac{87}{8} \hbar^2 V^2 Q_{i,2} + \frac{15}{8} \hbar^2 V Q_{i,3} - \frac{5}{4} V^2 Q_{i,1}^3 \\
& + 3 V Q_{i,1}^2 Q_{i,2} - \frac{105}{8} \hbar^2 Q_{i,4} - \frac{5}{4} Q_{i,1}^2 Q_{i,3} - \frac{17}{32} \hbar^6 V V'''' + 8 \hbar^4 V^2 V' + \frac{5}{8} \hbar^6 V' V'' \left. \right) \\
& + \xi''''_{02i}(q_i) \left(\frac{5}{8} \hbar^6 V'''''' Q_{i,1} + \frac{11}{16} \hbar^4 Q_{i,1}^2 V'' - \frac{71}{16} \hbar^4 V Q_{i,1} V' - \frac{109}{32} \hbar^4 Q_{i,2} V' - \frac{29}{24} \hbar^2 Q_{i,1}^3 V' \right. \\
& - \frac{27}{8} \hbar^2 V^2 Q_{i,1}^2 + \frac{21}{8} \hbar^2 V Q_{i,1} Q_{i,2} - \frac{1}{2} V Q_{i,1}^4 + \frac{15}{8} \hbar^2 Q_{i,2}^2 + \frac{5}{8} \hbar^2 Q_{i,1} Q_{i,3} + \frac{1}{2} Q_{i,1}^3 Q_{i,2} \\
& - \frac{7}{8} \hbar^6 V V'' + \frac{15}{32} \hbar^6 V'^2 + 3 \hbar^4 V^3 \left. \right) + \xi''''''_{02i}(q_i) \left(\frac{23}{32} \hbar^6 Q_{i,1} V'' + \frac{5}{16} \hbar^4 Q_{i,1}^2 V' - \frac{7}{4} \hbar^4 V^2 Q_{i,1} \right. \\
& - \frac{59}{32} \hbar^4 V Q_{i,2} - \frac{25}{24} \hbar^2 V Q_{i,1}^3 + \frac{63}{16} \hbar^4 Q_{i,3} + \frac{5}{8} \hbar^2 Q_{i,1}^2 Q_{i,2} - \frac{1}{30} Q_{i,1}^5 - \frac{15}{16} \hbar^6 V V' \left. \right) \\
& + \xi''''''''_{02i}(q_i) \left(\frac{17}{32} \hbar^6 Q_{i,1} V' - \frac{1}{16} \hbar^4 V Q_{i,1}^2 - \frac{13}{32} \hbar^4 Q_{i,1} Q_{i,2} - \frac{1}{12} \hbar^2 Q_{i,1}^4 - \frac{5}{16} \hbar^6 V^2 \right) \\
& + \xi''''''''''_{02i}(q_i) \left(\frac{15}{64} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} - \frac{1}{32} \hbar^4 Q_{i,1}^3 + \frac{1}{256} \hbar^8 V' \right) + \xi''''''''''''_{02i}(q_i) \left(\frac{1}{64} \hbar^6 Q_{i,1}^2 \right. \\
& + \frac{1}{128} \hbar^8 V \left. \right) + \frac{1}{256} \hbar^8 \xi''''''''''''_{02i}(q_i) Q_{i,1} + \xi'_{12i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} - \frac{1}{4} \hbar^2 V'''' Q_{i,1}^2 \right. \\
& + \frac{5}{4} \hbar^2 Q_{i,2} V'' - \frac{1}{6} Q_{i,1}^3 V'' - \frac{1}{8} \hbar^2 Q_{i,1} V'^2 + \frac{3}{2} Q_{i,1} Q_{i,2} V' - \frac{1}{2} V^3 Q_{i,1} - \frac{3}{4} V^2 Q_{i,2} \\
& - \frac{5}{2} V Q_{i,3} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V'''' - \frac{3}{2} \hbar^2 V^2 V' - \frac{3}{16} \hbar^4 V' V'' \left. \right) + \xi''_{12i}(q_i) \left(-\frac{5}{16} \hbar^4 V'''' Q_{i,1} \right. \\
& - \hbar^2 Q_{i,1}^2 V'' + \frac{15}{4} \hbar^2 Q_{i,2} V' - \frac{1}{2} Q_{i,1}^3 V' + 3 V Q_{i,1} Q_{i,2} - \frac{9}{8} Q_{i,2}^2 - \frac{5}{2} Q_{i,1} Q_{i,3} + \frac{1}{2} \hbar^4 V V'' \\
& - \frac{9}{32} \hbar^4 V'^2 - \hbar^2 V^3 \left. \right) + \xi'''_{12i}(q_i) \left(-\frac{11}{16} \hbar^4 Q_{i,1} V'' - \frac{13}{8} \hbar^2 Q_{i,1}^2 V' + \frac{3}{8} \hbar^2 V^2 Q_{i,1} \right. \\
& + \frac{33}{8} \hbar^2 V Q_{i,2} - \frac{1}{2} V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} + \frac{3}{4} Q_{i,1}^2 Q_{i,2} + \hbar^4 V V' \left. \right) + \xi''''_{12i}(q_i) \left(-\frac{13}{16} \hbar^4 Q_{i,1} V' \right. \\
& - \frac{5}{4} \hbar^2 V Q_{i,1}^2 + \frac{13}{8} \hbar^2 Q_{i,1} Q_{i,2} - \frac{1}{24} Q_{i,1}^4 + \frac{1}{2} \hbar^4 V^2 \left. \right) + \xi''''''_{12i}(q_i) \left(-\frac{9}{16} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} \right. \\
& - \frac{1}{8} \hbar^2 Q_{i,1}^3 \left. \right) - \frac{3}{32} \hbar^4 \xi''''''''_{12i}(q_i) Q_{i,1}^2 - \frac{1}{64} \hbar^6 \xi''''''''_{12i}(q_i) Q_{i,1} + \xi_{52i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{53i} : & \left[\xi_{03i}(q_i) \left(\frac{1}{64} \hbar^6 V'''''' Q_{i,1} + \frac{1}{16} \hbar^4 V'''''' Q_{i,1}^2 - \frac{3}{16} \hbar^4 V V'''' Q_{i,1} - \frac{5}{16} \hbar^4 V'''' Q_{i,2} \right. \right. \\
& + \frac{1}{24} \hbar^2 V'''' Q_{i,1}^3 - \frac{3}{4} \hbar^2 V V'''' Q_{i,1}^2 - \frac{3}{8} \hbar^2 V'''' Q_{i,1} Q_{i,2} - \frac{19}{32} \hbar^4 Q_{i,1} V''^2 + \frac{3}{8} \hbar^2 V^2 Q_{i,1} V'' \\
& + \frac{33}{8} \hbar^2 V Q_{i,2} V'' + \frac{5}{8} \hbar^2 Q_{i,3} V'' - \frac{1}{2} V Q_{i,1}^3 V'' - \frac{3}{4} \hbar^2 V Q_{i,1} V'^2 + \frac{57}{16} \hbar^2 Q_{i,2} V'^2 \\
& - \frac{1}{2} Q_{i,1}^3 V'^2 + \frac{9}{2} V Q_{i,1} Q_{i,2} V' - \frac{11}{16} \hbar^4 V'''' Q_{i,1} V' - \frac{9}{4} \hbar^2 Q_{i,1}^2 V' V'' - \frac{9}{8} V^4 Q_{i,1} \\
& - \frac{3}{2} V^3 Q_{i,2} - \frac{15}{4} V^2 Q_{i,3} + \frac{63}{8} Q_{i,5} - \frac{1}{32} \hbar^6 V V'''''' + \frac{3}{4} \hbar^4 V^2 V'''' - \frac{13}{2} \hbar^2 V^3 V' \\
& + \frac{3}{64} \hbar^6 V'''' V' - \frac{1}{16} \hbar^6 V'''' V'' - \frac{9}{16} \hbar^4 V V' V'' \left. \right) + \xi'_{03i}(q_i) \left(\frac{7}{64} \hbar^6 V'''''' Q_{i,1} \right. \\
& + \frac{5}{16} \hbar^4 V'''''' Q_{i,1}^2 - \frac{9}{8} \hbar^4 V V'''' Q_{i,1} - \frac{25}{16} \hbar^4 V'''' Q_{i,2} - \frac{1}{24} \hbar^2 V'''' Q_{i,1}^3 - 4 \hbar^2 V Q_{i,1}^2 V'' \\
& - \frac{1}{4} \hbar^2 Q_{i,1} Q_{i,2} V'' - \frac{1}{6} Q_{i,1}^4 V'' - \frac{31}{8} \hbar^2 Q_{i,1}^2 V'^2 - \frac{3}{8} \hbar^2 V^2 Q_{i,1} V' + \frac{129}{8} \hbar^2 V Q_{i,2} V' \\
& + \frac{15}{8} \hbar^2 Q_{i,3} V' - 2 V Q_{i,1}^3 V' + \frac{3}{2} Q_{i,1}^2 Q_{i,2} V' - \frac{27}{8} \hbar^4 Q_{i,1} V' V'' - \frac{1}{2} V^3 Q_{i,1}^2 \\
& + \frac{21}{4} V^2 Q_{i,1} Q_{i,2} - \frac{5}{2} V Q_{i,1} Q_{i,3} - \frac{15}{4} Q_{i,2} Q_{i,3} - \frac{3}{16} \hbar^6 V V'''' - \frac{1}{8} \hbar^6 V''^2 + \frac{7}{2} \hbar^4 V^2 V'' \\
& - \frac{15}{16} \hbar^4 V V'^2 + \frac{15}{64} \hbar^6 V'''' V' - 4 \hbar^2 V^4 \left. \right) + \xi''_{03i}(q_i) \left(\frac{11}{32} \hbar^6 V'''''' Q_{i,1} + \frac{21}{32} \hbar^4 V'''''' Q_{i,1}^2 \right. \\
& - \frac{25}{8} \hbar^4 V Q_{i,1} V'' - \frac{103}{32} \hbar^4 Q_{i,2} V'' - \frac{7}{12} \hbar^2 Q_{i,1}^3 V'' - \frac{23}{8} \hbar^4 Q_{i,1} V'^2 - \frac{65}{8} \hbar^2 V Q_{i,1}^2 V' \\
& + \frac{3}{2} \hbar^2 Q_{i,1} Q_{i,2} V' - \frac{1}{2} Q_{i,1}^4 V' + \frac{5}{8} \hbar^2 V^3 Q_{i,1} + \frac{87}{8} \hbar^2 V^2 Q_{i,2} + \frac{15}{8} \hbar^2 V Q_{i,3} - \frac{5}{4} V^2 Q_{i,1}^3 \\
& + 3 V Q_{i,1}^2 Q_{i,2} - \frac{105}{8} \hbar^2 Q_{i,4} - \frac{5}{4} Q_{i,1}^2 Q_{i,3} - \frac{17}{32} \hbar^6 V V'''' + 8 \hbar^4 V^2 V' + \frac{5}{8} \hbar^6 V' V'' \left. \right) \\
& + \xi'''_{03i}(q_i) \left(\frac{5}{8} \hbar^6 V'''''' Q_{i,1} + \frac{11}{16} \hbar^4 Q_{i,1}^2 V'' - \frac{71}{16} \hbar^4 V Q_{i,1} V' - \frac{109}{32} \hbar^4 Q_{i,2} V' - \frac{29}{24} \hbar^2 Q_{i,1}^3 V' \right. \\
& - \frac{27}{8} \hbar^2 V^2 Q_{i,1}^2 + \frac{21}{8} \hbar^2 V Q_{i,1} Q_{i,2} - \frac{1}{2} V Q_{i,1}^4 + \frac{15}{8} \hbar^2 Q_{i,2}^2 + \frac{5}{8} \hbar^2 Q_{i,1} Q_{i,3} + \frac{1}{2} Q_{i,1}^3 Q_{i,2} \\
& - \frac{7}{8} \hbar^6 V V'' + \frac{15}{32} \hbar^6 V'^2 + 3 \hbar^4 V^3 \left. \right) + \xi''''_{03i}(q_i) \left(\frac{23}{32} \hbar^6 Q_{i,1} V'' + \frac{5}{16} \hbar^4 Q_{i,1}^2 V' - \frac{7}{4} \hbar^4 V^2 Q_{i,1} \right. \\
& - \frac{59}{32} \hbar^4 V Q_{i,2} - \frac{25}{24} \hbar^2 V Q_{i,1}^3 + \frac{63}{16} \hbar^4 Q_{i,3} + \frac{5}{8} \hbar^2 Q_{i,1}^2 Q_{i,2} - \frac{1}{30} Q_{i,1}^5 - \frac{15}{16} \hbar^6 V V' \left. \right) \\
& + \xi''''''_{03i}(q_i) \left(\frac{17}{32} \hbar^6 Q_{i,1} V' - \frac{1}{16} \hbar^4 V Q_{i,1}^2 - \frac{13}{32} \hbar^4 Q_{i,1} Q_{i,2} - \frac{1}{12} \hbar^2 Q_{i,1}^4 - \frac{5}{16} \hbar^6 V^2 \right) \\
& + \xi''''''''_{03i}(q_i) \left(\frac{15}{64} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} - \frac{1}{32} \hbar^4 Q_{i,1}^3 + \frac{1}{256} \hbar^8 V' \right) + \xi''''''''''_{03i}(q_i) \left(\frac{1}{64} \hbar^6 Q_{i,1}^2 \right. \\
& + \frac{1}{128} \hbar^8 V \left. \right) + \frac{1}{256} \hbar^8 \xi''''''''''_{03i}(q_i) Q_{i,1} + \xi_{13i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} - \frac{1}{4} \hbar^2 V'''' Q_{i,1}^2 \right. \\
& + \frac{5}{4} \hbar^2 Q_{i,2} V'' - \frac{1}{6} Q_{i,1}^3 V'' - \frac{1}{8} \hbar^2 Q_{i,1} V'^2 + \frac{3}{2} Q_{i,1} Q_{i,2} V' - \frac{1}{2} V^3 Q_{i,1} - \frac{3}{4} V^2 Q_{i,2} \\
& - \frac{5}{2} V Q_{i,3} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V'''' - \frac{3}{2} \hbar^2 V^2 V' - \frac{3}{16} \hbar^4 V' V'' \left. \right) + \xi'_{13i}(q_i) \left(-\frac{5}{16} \hbar^4 V'''' Q_{i,1} \right. \\
& - \hbar^2 Q_{i,1}^2 V'' + \frac{15}{4} \hbar^2 Q_{i,2} V' - \frac{1}{2} Q_{i,1}^3 V' + 3 V Q_{i,1} Q_{i,2} - \frac{9}{8} Q_{i,2}^2 - \frac{5}{2} Q_{i,1} Q_{i,3} + \frac{1}{2} \hbar^4 V V'' \\
& - \frac{9}{32} \hbar^4 V'^2 - \hbar^2 V^3 \left. \right) + \xi''_{13i}(q_i) \left(-\frac{11}{16} \hbar^4 Q_{i,1} V'' - \frac{13}{8} \hbar^2 Q_{i,1}^2 V' + \frac{3}{8} \hbar^2 V^2 Q_{i,1} \right. \\
& + \frac{33}{8} \hbar^2 V Q_{i,2} - \frac{1}{2} V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} + \frac{3}{4} Q_{i,1}^2 Q_{i,2} + \hbar^4 V V' \left. \right) + \xi'''_{13i}(q_i) \left(-\frac{13}{16} \hbar^4 Q_{i,1} V' \right. \\
& - \frac{5}{4} \hbar^2 V Q_{i,1}^2 + \frac{13}{8} \hbar^2 Q_{i,1} Q_{i,2} - \frac{1}{24} Q_{i,1}^4 + \frac{1}{2} \hbar^4 V^2 \left. \right) + \xi''''_{13i}(q_i) \left(-\frac{9}{16} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} \right. \\
& \left. - \frac{1}{8} \hbar^2 Q_{i,1}^3 \right) - \frac{3}{32} \hbar^4 \xi''''''_{13i}(q_i) Q_{i,1}^2 - \frac{1}{64} \hbar^6 \xi''''''''_{13i}(q_i) Q_{i,1} \left. \right]' + \xi_{53i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{54i} : \xi'_{04i}(q_i) & \left(\frac{1}{64} \hbar^6 V'''''' Q_{i,1} + \frac{1}{32} \hbar^4 V'''''' Q_{i,1}^2 - \frac{3}{8} \hbar^4 V V'''' Q_{i,1} - \frac{3}{32} \hbar^4 V'''' Q_{i,2} \right. \\
& - \frac{5}{8} \hbar^2 V V'''' Q_{i,1}^2 - \frac{21}{32} \hbar^4 Q_{i,1} V''^2 + \frac{25}{8} \hbar^2 V^2 Q_{i,1} V'' + \frac{15}{8} \hbar^2 V Q_{i,2} V'' + \frac{15}{4} \hbar^2 V Q_{i,1} V'^2 \\
& + \frac{15}{16} \hbar^2 Q_{i,2} V'^2 + \frac{15}{4} V^2 Q_{i,1}^2 V' - \frac{7}{8} \hbar^4 V'''' Q_{i,1} V' - \frac{5}{4} \hbar^2 Q_{i,1}^2 V' V'' - \frac{15}{8} V^4 Q_{i,1} \\
& - \frac{15}{4} V^3 Q_{i,2} + \frac{63}{8} Q_{i,5} - \frac{1}{32} \hbar^6 V V'''''' + \frac{7}{8} \hbar^4 V^2 V'''' - \frac{35}{4} \hbar^2 V^3 V'' + \frac{3}{64} \hbar^6 V'''' V' \\
& - \frac{1}{16} \hbar^6 V'''' V'' - \frac{7}{16} \hbar^4 V V' V'' \left. \right) + \xi''_{04i}(q_i) \left(\frac{7}{64} \hbar^6 V'''''' Q_{i,1} + \frac{1}{8} \hbar^4 V'''' Q_{i,1}^2 \right. \\
& - \frac{33}{16} \hbar^4 V V'''' Q_{i,1} - \frac{15}{32} \hbar^4 V'''' Q_{i,2} - \frac{1}{8} \hbar^2 V'''' Q_{i,1}^3 - 2 \hbar^2 V Q_{i,1}^2 V'' + \frac{3}{4} \hbar^2 Q_{i,1} Q_{i,2} V'' \\
& - \frac{13}{8} \hbar^2 Q_{i,1}^2 V'^2 + 11 \hbar^2 V^2 Q_{i,1} V' + 6 \hbar^2 V Q_{i,2} V' + \frac{3}{2} V Q_{i,1}^3 V' + \frac{3}{4} Q_{i,1}^2 Q_{i,2} V' \\
& - \frac{69}{16} \hbar^4 Q_{i,1} V' V'' + 3 V^3 Q_{i,1}^2 - \frac{9}{4} V Q_{i,2}^2 - 3 V^2 Q_{i,1} Q_{i,2} - \frac{3}{16} \hbar^6 V V'''' - \frac{1}{8} \hbar^6 V''^2 \\
& + 4 \hbar^4 V^2 V'' - \frac{9}{16} \hbar^4 V V'^2 + \frac{15}{64} \hbar^6 V'''' V' - 6 \hbar^2 V^4 \left. \right) + \xi'''_{04i}(q_i) \left(\frac{11}{32} \hbar^6 V'''' Q_{i,1} \right. \\
& + \frac{5}{32} \hbar^4 V'''' Q_{i,1}^2 - \frac{21}{4} \hbar^4 V Q_{i,1} V'' - \frac{15}{16} \hbar^4 Q_{i,2} V'' - \frac{3}{8} \hbar^2 Q_{i,1}^3 V'' - 4 \hbar^4 Q_{i,1} V'^2 \\
& - \frac{19}{8} \hbar^2 V Q_{i,1}^2 V' + \frac{9}{4} \hbar^2 Q_{i,1} Q_{i,2} V' + \frac{1}{4} Q_{i,1}^4 V' + 6 \hbar^2 V^3 Q_{i,1} + \frac{33}{8} \hbar^2 V^2 Q_{i,2} \\
& + \frac{7}{4} V^2 Q_{i,1}^3 + \frac{3}{4} V Q_{i,1}^2 Q_{i,2} - \frac{105}{8} \hbar^2 Q_{i,4} - \frac{9}{4} Q_{i,1} Q_{i,2}^2 - \frac{17}{32} \hbar^6 V V'''' + \frac{73}{8} \hbar^4 V^2 V' \\
& + \frac{5}{8} \hbar^6 V' V'' \left. \right) + \xi''''_{04i}(q_i) \left(\frac{5}{8} \hbar^6 V'''' Q_{i,1} - \frac{1}{8} \hbar^4 Q_{i,1}^2 V'' - \frac{57}{8} \hbar^4 V Q_{i,1} V' - \frac{15}{16} \hbar^4 Q_{i,2} V' \right. \\
& - \frac{1}{2} \hbar^2 Q_{i,1}^3 V' + 3 \hbar^2 V Q_{i,1} Q_{i,2} + \frac{1}{2} V Q_{i,1}^4 - \frac{7}{8} \hbar^6 V V'' + \frac{15}{32} \hbar^6 V'^2 + \frac{7}{2} \hbar^4 V^3 \left. \right) \\
& + \xi''''''_{04i}(q_i) \left(\frac{45}{64} \hbar^6 Q_{i,1} V'' - \frac{17}{32} \hbar^4 Q_{i,1}^2 V' - \frac{85}{32} \hbar^4 V^2 Q_{i,1} - \frac{15}{32} \hbar^4 V Q_{i,2} - \frac{1}{4} \hbar^2 V Q_{i,1}^3 \right. \\
& + \frac{63}{16} \hbar^4 Q_{i,3} + \frac{3}{4} \hbar^2 Q_{i,1}^2 Q_{i,2} + \frac{1}{20} Q_{i,1}^5 + \frac{1}{256} \hbar^8 V'''' - \frac{63}{64} \hbar^6 V V' \left. \right) \\
& + \xi''''''_{04i}(q_i) \left(\frac{15}{32} \hbar^6 Q_{i,1} V' - \frac{9}{16} \hbar^4 V Q_{i,1}^2 + \frac{1}{64} \hbar^8 V'' - \frac{3}{8} \hbar^6 V^2 \right) \\
& + \xi''''''''_{04i}(q_i) \left(\frac{5}{32} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} - \frac{1}{16} \hbar^4 Q_{i,1}^3 + \frac{3}{128} \hbar^8 V' \right) + \frac{1}{64} \hbar^8 \xi''''''''_{04i}(q_i) V \\
& + \frac{1}{256} \hbar^8 \xi''''''''_{04i}(q_i) Q_{i,1} + \xi'_{14i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''''' Q_{i,1} - \frac{1}{8} \hbar^2 V'''' Q_{i,1}^2 + \hbar^2 V Q_{i,1} V'' \right. \\
& + \frac{3}{8} \hbar^2 Q_{i,2} V'' + \frac{5}{8} \hbar^2 Q_{i,1} V'^2 + \frac{3}{2} V Q_{i,1}^2 V' - V^3 Q_{i,1} - \frac{9}{4} V^2 Q_{i,2} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V'''' \\
& - \frac{5}{2} \hbar^2 V^2 V' - \frac{3}{16} \hbar^4 V' V'' \left. \right) + \xi''_{14i}(q_i) \left(-\frac{5}{16} \hbar^4 V'''' Q_{i,1} - \frac{1}{4} \hbar^2 Q_{i,1}^2 V'' + \frac{13}{4} \hbar^2 V Q_{i,1} V' \right. \\
& + \frac{9}{8} \hbar^2 Q_{i,2} V' + \frac{1}{2} Q_{i,1}^3 V' + \frac{3}{2} V^2 Q_{i,1}^2 - \frac{3}{2} V Q_{i,1} Q_{i,2} - \frac{9}{8} Q_{i,2}^2 + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 \\
& - 2 \hbar^2 V^3 \left. \right) + \xi'''_{14i}(q_i) \left(-\frac{11}{16} \hbar^4 Q_{i,1} V'' + \frac{19}{8} \hbar^2 V^2 Q_{i,1} + \frac{9}{8} \hbar^2 V Q_{i,2} + V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} \right. \\
& - \frac{3}{4} Q_{i,1}^2 Q_{i,2} + \hbar^4 V V' \left. \right) + \xi''''_{14i}(q_i) \left(-\frac{13}{16} \hbar^4 Q_{i,1} V' + \frac{1}{2} \hbar^2 V Q_{i,1}^2 + \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} + \frac{1}{8} Q_{i,1}^4 \right. \\
& + \frac{1}{2} \hbar^4 V^2 \left. \right) + \xi''''''_{14i}(q_i) \left(-\frac{1}{2} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} + \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{1}{64} \hbar^6 V' \right) \\
& + \xi''''''''_{14i}(q_i) \left(-\frac{1}{32} \hbar^4 Q_{i,1}^2 - \frac{1}{32} \hbar^6 V \right) + \xi''''''''_{14i}(q_i) \left(-\frac{1}{64} \hbar^6 Q_{i,1} \right) + \xi'_{24i}(q_i) \left(\frac{1}{4} \hbar^2 Q_{i,1} V'' \right. \\
& + \frac{1}{2} Q_{i,1}^2 V' - \frac{1}{2} V^2 Q_{i,1} - \frac{3}{2} V Q_{i,2} + \frac{5}{2} Q_{i,3} - \frac{1}{2} \hbar^2 V V' \left. \right) + \xi''_{24i}(q_i) \left(\frac{3}{4} \hbar^2 Q_{i,1} V' + V Q_{i,1}^2 \right. \\
& - \frac{3}{2} Q_{i,1} Q_{i,2} - \frac{1}{2} \hbar^2 V^2 \left. \right) + \xi'''_{24i}(q_i) \left(\hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} + \frac{1}{6} Q_{i,1}^3 \right) + \frac{1}{4} \hbar^2 \xi''''_{24i}(q_i) Q_{i,1}^2 \\
& + \frac{1}{16} \hbar^4 \xi''''''_{24i}(q_i) Q_{i,1} + \xi_{54i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{55i} : & \left[\xi_{05i}(q_i) \left(\frac{1}{64} \hbar^6 V'''''' Q_{i,1} + \frac{1}{32} \hbar^4 V'''''' Q_{i,1}^2 - \frac{3}{8} \hbar^4 V V'''' Q_{i,1} - \frac{3}{32} \hbar^4 V'''' Q_{i,2} \right. \right. \\
& - \frac{5}{8} \hbar^2 V V'''' Q_{i,1}^2 - \frac{21}{32} \hbar^4 Q_{i,1} V''^2 + \frac{25}{8} \hbar^2 V^2 Q_{i,1} V'' + \frac{15}{8} \hbar^2 V Q_{i,2} V'' + \frac{15}{4} \hbar^2 V Q_{i,1} V'^2 \\
& + \frac{15}{16} \hbar^2 Q_{i,2} V'^2 + \frac{15}{4} V^2 Q_{i,1}^2 V' - \frac{7}{8} \hbar^4 V'''' Q_{i,1} V' - \frac{5}{4} \hbar^2 Q_{i,1}^2 V' V'' - \frac{15}{8} V^4 Q_{i,1} \\
& - \frac{15}{4} V^3 Q_{i,2} + \frac{63}{8} Q_{i,5} - \frac{1}{32} \hbar^6 V V'''''' + \frac{7}{8} \hbar^4 V^2 V'''' - \frac{35}{4} \hbar^2 V^3 V' + \frac{3}{64} \hbar^6 V'''' V' \\
& - \frac{1}{16} \hbar^6 V'''' V'' - \frac{7}{16} \hbar^4 V V' V'' \left. \right) + \xi'_{05i}(q_i) \left(\frac{7}{64} \hbar^6 V'''''' Q_{i,1} + \frac{1}{8} \hbar^4 V'''' Q_{i,1}^2 \right. \\
& - \frac{33}{16} \hbar^4 V V'''' Q_{i,1} - \frac{15}{32} \hbar^4 V'''' Q_{i,2} - \frac{1}{8} \hbar^2 V'''' Q_{i,1}^3 - 2 \hbar^2 V Q_{i,1}^2 V'' + \frac{3}{4} \hbar^2 Q_{i,1} Q_{i,2} V'' \\
& - \frac{13}{8} \hbar^2 Q_{i,1}^2 V'^2 + 11 \hbar^2 V^2 Q_{i,1} V' + 6 \hbar^2 V Q_{i,2} V' + \frac{3}{2} V Q_{i,1}^3 V' + \frac{3}{4} Q_{i,1}^2 Q_{i,2} V' \\
& - \frac{69}{16} \hbar^4 Q_{i,1} V' V'' + 3 V^3 Q_{i,1}^2 - \frac{9}{4} V Q_{i,2}^2 - 3 V^2 Q_{i,1} Q_{i,2} - \frac{3}{16} \hbar^6 V V'''' - \frac{1}{8} \hbar^6 V''^2 \\
& + 4 \hbar^4 V^2 V'' - \frac{9}{16} \hbar^4 V V'^2 + \frac{15}{64} \hbar^6 V'''' V' - 6 \hbar^2 V^4 \left. \right) + \xi''_{05i}(q_i) \left(\frac{11}{32} \hbar^6 V'''' Q_{i,1} \right. \\
& + \frac{5}{32} \hbar^4 V'''' Q_{i,1}^2 - \frac{21}{4} \hbar^4 V Q_{i,1} V'' - \frac{15}{16} \hbar^4 Q_{i,2} V'' - \frac{3}{8} \hbar^2 Q_{i,1}^3 V'' - 4 \hbar^4 Q_{i,1} V'^2 \\
& - \frac{19}{8} \hbar^2 V Q_{i,1}^2 V' + \frac{9}{4} \hbar^2 Q_{i,1} Q_{i,2} V' + \frac{1}{4} Q_{i,1}^4 V' + 6 \hbar^2 V^3 Q_{i,1} + \frac{33}{8} \hbar^2 V^2 Q_{i,2} \\
& + \frac{7}{4} V^2 Q_{i,1}^3 + \frac{3}{4} V Q_{i,1}^2 Q_{i,2} - \frac{105}{8} \hbar^2 Q_{i,4} - \frac{9}{4} Q_{i,1} Q_{i,2}^2 - \frac{17}{32} \hbar^6 V V'''' + \frac{73}{8} \hbar^4 V^2 V' \\
& + \frac{5}{8} \hbar^6 V' V'' \left. \right) + \xi'''_{05i}(q_i) \left(\frac{5}{8} \hbar^6 V'''' Q_{i,1} - \frac{1}{8} \hbar^4 Q_{i,1}^2 V'' - \frac{57}{8} \hbar^4 V Q_{i,1} V' - \frac{15}{16} \hbar^4 Q_{i,2} V' \right. \\
& - \frac{1}{2} \hbar^2 Q_{i,1}^3 V' + 3 \hbar^2 V Q_{i,1} Q_{i,2} + \frac{1}{2} V Q_{i,1}^4 - \frac{7}{8} \hbar^6 V V'' + \frac{15}{32} \hbar^6 V'^2 + \frac{7}{2} \hbar^4 V^3 \left. \right) \\
& + \xi''''_{05i}(q_i) \left(\frac{45}{64} \hbar^6 Q_{i,1} V'' - \frac{17}{32} \hbar^4 Q_{i,1}^2 V' - \frac{85}{32} \hbar^4 V^2 Q_{i,1} - \frac{15}{32} \hbar^4 V Q_{i,2} - \frac{1}{4} \hbar^2 V Q_{i,1}^3 \right. \\
& + \frac{63}{16} \hbar^4 Q_{i,3} + \frac{3}{4} \hbar^2 Q_{i,1}^2 Q_{i,2} + \frac{1}{20} Q_{i,1}^5 + \frac{1}{256} \hbar^8 V'''' - \frac{63}{64} \hbar^6 V V' \left. \right) \\
& + \xi''''''_{05i}(q_i) \left(\frac{15}{32} \hbar^6 Q_{i,1} V' - \frac{9}{16} \hbar^4 V Q_{i,1}^2 + \frac{1}{64} \hbar^8 V'' - \frac{3}{8} \hbar^6 V^2 \right) \\
& + \xi''''''''_{05i}(q_i) \left(\frac{5}{32} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} - \frac{1}{16} \hbar^4 Q_{i,1}^3 + \frac{3}{128} \hbar^8 V' \right) + \frac{1}{64} \hbar^8 \xi''''''''''_{05i}(q_i) V \\
& + \frac{1}{256} \hbar^8 \xi''''''''''_{05i}(q_i) Q_{i,1} + \xi_{15i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} - \frac{1}{8} \hbar^2 V'''' Q_{i,1}^2 + \hbar^2 V Q_{i,1} V'' \right. \\
& + \frac{3}{8} \hbar^2 Q_{i,2} V'' + \frac{5}{8} \hbar^2 Q_{i,1} V'^2 + \frac{3}{2} V Q_{i,1}^2 V' - V^3 Q_{i,1} - \frac{9}{4} V^2 Q_{i,2} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V'''' \\
& - \frac{5}{2} \hbar^2 V^2 V' - \frac{3}{16} \hbar^4 V' V'' \left. \right) + \xi'_{15i}(q_i) \left(-\frac{5}{16} \hbar^4 V'''' Q_{i,1} - \frac{1}{4} \hbar^2 Q_{i,1}^2 V'' + \frac{13}{4} \hbar^2 V Q_{i,1} V' \right. \\
& + \frac{9}{8} \hbar^2 Q_{i,2} V' + \frac{1}{2} Q_{i,1}^3 V' + \frac{3}{2} V^2 Q_{i,1}^2 - \frac{3}{2} V Q_{i,1} Q_{i,2} - \frac{9}{8} Q_{i,2}^2 + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 \\
& - 2 \hbar^2 V^3 \left. \right) + \xi''_{15i}(q_i) \left(-\frac{11}{16} \hbar^4 Q_{i,1} V'' + \frac{19}{8} \hbar^2 V^2 Q_{i,1} + \frac{9}{8} \hbar^2 V Q_{i,2} + V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} \right. \\
& - \frac{3}{4} Q_{i,1}^2 Q_{i,2} + \hbar^4 V V' \left. \right) + \xi'''_{15i}(q_i) \left(-\frac{13}{16} \hbar^4 Q_{i,1} V' + \frac{1}{2} \hbar^2 V Q_{i,1}^2 + \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} + \frac{1}{8} Q_{i,1}^4 \right. \\
& + \frac{1}{2} \hbar^4 V^2 \left. \right) + \xi''''_{15i}(q_i) \left(-\frac{1}{2} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} + \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{1}{64} \hbar^6 V' \right) \\
& + \xi''''''_{15i}(q_i) \left(-\frac{1}{32} \hbar^4 Q_{i,1}^2 - \frac{1}{32} \hbar^6 V \right) + \xi''''''''_{15i}(q_i) \left(-\frac{1}{64} \hbar^6 Q_{i,1} \right) + \xi_{25i}(q_i) \left(\frac{1}{4} \hbar^2 Q_{i,1} V'' \right. \\
& + \frac{1}{2} Q_{i,1}^2 V' - \frac{1}{2} V^2 Q_{i,1} - \frac{3}{2} V Q_{i,2} + \frac{5}{2} Q_{i,3} - \frac{1}{2} \hbar^2 V V' \left. \right) + \xi'_{25i}(q_i) \left(\frac{3}{4} \hbar^2 Q_{i,1} V' + V Q_{i,1}^2 \right. \\
& - \frac{3}{2} Q_{i,1} Q_{i,2} - \frac{1}{2} \hbar^2 V^2 \left. \right) + \xi''_{25i}(q_i) \left(\hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} + \frac{1}{6} Q_{i,1}^3 \right) + \frac{1}{4} \hbar^2 \xi''''_{25i}(q_i) Q_{i,1}^2 \\
& + \frac{1}{16} \hbar^4 \xi''''''_{25i}(q_i) Q_{i,1} \left. \right]' + \xi_{55i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{56i} : \xi'_{06i}(q_i) & \left(\frac{1}{64} \hbar^6 V'''''' Q_{i,1} - \frac{7}{16} \hbar^4 V V'''' Q_{i,1} - \frac{21}{32} \hbar^4 Q_{i,1} V''^2 + \frac{35}{8} \hbar^2 V^2 Q_{i,1} V'' \right. \\
& + \frac{35}{8} \hbar^2 V Q_{i,1} V'^2 - \frac{7}{8} \hbar^4 V'''' Q_{i,1} V' - \frac{35}{8} V^4 Q_{i,1} + \frac{63}{8} Q_{i,5} - \frac{1}{32} \hbar^6 V V'''''' + \frac{7}{8} \hbar^4 V^2 V'''' \\
& - \frac{35}{4} \hbar^2 V^3 V' + \frac{3}{64} \hbar^6 V'''' V' - \frac{1}{16} \hbar^6 V'''' V'' - \frac{7}{16} \hbar^4 V V' V'' \left. \right) + \xi''_{06i}(q_i) \left(\frac{7}{64} \hbar^6 V'''''' Q_{i,1} \right. \\
& - \frac{1}{16} \hbar^4 V'''''' Q_{i,1}^2 - \frac{19}{8} \hbar^4 V V'''' Q_{i,1} + \frac{5}{4} \hbar^2 V Q_{i,1}^2 V'' + \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} V'' + \frac{5}{8} \hbar^2 Q_{i,1}^2 V'^2 \\
& + \frac{31}{2} \hbar^2 V^2 Q_{i,1} V' - \frac{3}{4} \hbar^2 V Q_{i,2} V' - \frac{69}{16} \hbar^4 Q_{i,1} V' V'' - \frac{5}{2} V^3 Q_{i,1}^2 - \frac{9}{4} V^2 Q_{i,1} Q_{i,2} \\
& - \frac{5}{2} V Q_{i,1} Q_{i,3} + \frac{15}{4} Q_{i,2} Q_{i,3} - \frac{3}{16} \hbar^6 V V'''' - \frac{1}{8} \hbar^6 V''^2 + 4 \hbar^4 V^2 V'' - \frac{9}{16} \hbar^4 V V'^2 \\
& + \frac{15}{64} \hbar^6 V'''' V' - 6 \hbar^2 V^4 \left. \right) + \xi'''_{06i}(q_i) \left(\frac{21}{64} \hbar^6 V'''' Q_{i,1} - \frac{5}{16} \hbar^4 V'''' Q_{i,1}^2 - \frac{89}{16} \hbar^4 V Q_{i,1} V'' \right. \\
& + \frac{1}{8} \hbar^2 Q_{i,1}^3 V'' - \frac{123}{32} \hbar^4 Q_{i,1} V'^2 + \frac{17}{4} \hbar^2 V Q_{i,1}^2 V' + \frac{9}{8} \hbar^2 Q_{i,1} Q_{i,2} V' + \frac{69}{8} \hbar^2 V^3 Q_{i,1} \\
& - \frac{3}{4} \hbar^2 V^2 Q_{i,2} - \frac{3}{4} V^2 Q_{i,1}^3 - \frac{3}{2} V Q_{i,1}^2 Q_{i,2} - \frac{105}{8} \hbar^2 Q_{i,4} - \frac{5}{4} Q_{i,1}^2 Q_{i,3} + \frac{1}{256} \hbar^8 V'''''' \\
& - \frac{39}{64} \hbar^6 V V'''' + \frac{307}{32} \hbar^4 V^2 V' + \frac{15}{32} \hbar^6 V' V'' \left. \right) + \xi''''_{06i}(q_i) \left(\frac{17}{32} \hbar^6 V'''' Q_{i,1} - \frac{11}{16} \hbar^4 Q_{i,1}^2 V'' \right. \\
& - \frac{103}{16} \hbar^4 V Q_{i,1} V' + \frac{3}{32} \hbar^4 Q_{i,2} V' + \frac{3}{8} \hbar^2 Q_{i,1}^3 V' + \frac{27}{8} \hbar^2 V^2 Q_{i,1}^2 + \frac{19}{8} \hbar^2 V Q_{i,1} Q_{i,2} \\
& - \frac{1}{6} V Q_{i,1}^4 - \frac{15}{8} \hbar^2 Q_{i,2}^2 - \frac{5}{8} \hbar^2 Q_{i,1} Q_{i,3} - \frac{1}{2} Q_{i,1}^3 Q_{i,2} + \frac{3}{128} \hbar^8 V'''' - \frac{5}{4} \hbar^6 V V'' \\
& + \frac{3}{16} \hbar^6 V'^2 + 4 \hbar^4 V^3 \left. \right) + \xi''''''_{06i}(q_i) \left(\frac{31}{64} \hbar^6 Q_{i,1} V'' - \frac{25}{32} \hbar^4 Q_{i,1}^2 V' - \frac{61}{32} \hbar^4 V^2 Q_{i,1} \right. \\
& + \frac{3}{16} \hbar^4 V Q_{i,2} + \frac{5}{8} \hbar^2 V Q_{i,1}^3 + \frac{63}{16} \hbar^4 Q_{i,3} + \frac{5}{8} \hbar^2 Q_{i,1}^2 Q_{i,2} - \frac{1}{30} Q_{i,1}^5 + \frac{15}{256} \hbar^8 V'''' \\
& - \frac{105}{64} \hbar^6 V V' \left. \right) + \xi''''''''_{06i}(q_i) \left(\frac{7}{32} \hbar^6 Q_{i,1} V' - \frac{1}{2} \hbar^4 V Q_{i,1}^2 + \frac{13}{32} \hbar^4 Q_{i,1} Q_{i,2} + \frac{1}{12} \hbar^2 Q_{i,1}^4 \right. \\
& + \frac{5}{64} \hbar^8 V'' - \frac{5}{8} \hbar^6 V^2 \left. \right) + \xi''''''''''_{06i}(q_i) \left(\frac{1}{64} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} - \frac{1}{32} \hbar^4 Q_{i,1}^3 + \frac{15}{256} \hbar^8 V' \right) \\
& + \xi''''''''''''_{06i}(q_i) \left(\frac{3}{128} \hbar^8 V - \frac{1}{64} \hbar^6 Q_{i,1}^2 \right) + \frac{1}{256} \hbar^8 \xi''''''''''''_{06i}(q_i) Q_{i,1} + \xi'_{16i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} \right. \\
& + \frac{5}{4} \hbar^2 V Q_{i,1} V'' + \frac{5}{8} \hbar^2 Q_{i,1} V'^2 - \frac{5}{2} V^3 Q_{i,1} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V'''' - \frac{5}{2} \hbar^2 V^2 V' \\
& - \frac{3}{16} \hbar^4 V' V'' \left. \right) + \xi''_{16i}(q_i) \left(-\frac{5}{16} \hbar^4 V'''' Q_{i,1} + \frac{1}{4} \hbar^2 Q_{i,1}^2 V'' + 4 \hbar^2 V Q_{i,1} V' - \frac{3}{2} V^2 Q_{i,1}^2 \right. \\
& - \frac{3}{2} V Q_{i,1} Q_{i,2} + \frac{9}{8} Q_{i,2}^2 + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 - 2 \hbar^2 V^3 \left. \right) + \xi'''_{16i}(q_i) \left(-\frac{5}{8} \hbar^4 Q_{i,1} V'' \right. \\
& + \frac{3}{4} \hbar^2 Q_{i,1}^2 V' + \frac{11}{4} \hbar^2 V^2 Q_{i,1} - \frac{1}{2} V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} - \frac{3}{4} Q_{i,1}^2 Q_{i,2} - \frac{1}{64} \hbar^6 V'''' \\
& + \frac{19}{16} \hbar^4 V V' \left. \right) + \xi''''_{16i}(q_i) \left(-\frac{9}{16} \hbar^4 Q_{i,1} V' + \hbar^2 V Q_{i,1}^2 - \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} - \frac{1}{8} Q_{i,1}^4 - \frac{1}{16} \hbar^6 V'' \right. \\
& + \frac{3}{4} \hbar^4 V^2 \left. \right) + \xi''''''_{16i}(q_i) \left(-\frac{3}{16} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} + \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{3}{32} \hbar^6 V' \right) \\
& + \xi''''''''_{16i}(q_i) \left(\frac{1}{32} \hbar^4 Q_{i,1}^2 - \frac{1}{16} \hbar^6 V \right) - \frac{1}{64} \hbar^6 \xi''''''''''_{16i}(q_i) Q_{i,1} + \xi'_{26i}(q_i) \left(\frac{1}{4} \hbar^2 Q_{i,1} V'' \right. \\
& - \frac{3}{2} V^2 Q_{i,1} + \frac{5}{2} Q_{i,3} - \frac{1}{2} \hbar^2 V V' \left. \right) + \xi''_{26i}(q_i) \left(\frac{3}{4} \hbar^2 Q_{i,1} V' - V Q_{i,1}^2 - \frac{1}{2} \hbar^2 V^2 \right) \\
& + \xi'''_{26i}(q_i) \left(\frac{3}{4} \hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} - \frac{1}{3} Q_{i,1}^3 + \frac{1}{16} \hbar^4 V' \right) + \frac{1}{8} \hbar^4 \xi''''_{26i}(q_i) V \\
& + \frac{1}{16} \hbar^4 \xi''''''_{26i}(q_i) Q_{i,1} + \xi'_{36i}(q_i) \left(\frac{3}{2} Q_{i,2} - V Q_{i,1} \right) - \frac{1}{2} \xi''_{36i}(q_i) Q_{i,1}^2 - \frac{1}{4} \hbar^2 \xi'''_{36i}(q_i) Q_{i,1} \\
& + \xi_{56i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{57i} : & \left[\xi_{07i}(q_i) \left(\frac{1}{64} \hbar^6 V'''''' Q_{i,1} - \frac{7}{16} \hbar^4 V V'''' Q_{i,1} - \frac{21}{32} \hbar^4 Q_{i,1} V''^2 + \frac{35}{8} \hbar^2 V^2 Q_{i,1} V'' \right. \right. \\
& + \frac{35}{8} \hbar^2 V Q_{i,1} V'^2 - \frac{7}{8} \hbar^4 V'''' Q_{i,1} V' - \frac{35}{8} V^4 Q_{i,1} + \frac{63}{8} Q_{i,5} - \frac{1}{32} \hbar^6 V V'''''' + \frac{7}{8} \hbar^4 V^2 V'''' \\
& - \frac{35}{4} \hbar^2 V^3 V' + \frac{3}{64} \hbar^6 V'''' V' - \frac{1}{16} \hbar^6 V'''' V'' - \frac{7}{16} \hbar^4 V V' V'' \left. \right) + \xi'_{07i}(q_i) \left(\frac{7}{64} \hbar^6 V'''''' Q_{i,1} \right. \\
& - \frac{1}{16} \hbar^4 V'''' Q_{i,1}^2 - \frac{19}{8} \hbar^4 V V'''' Q_{i,1} + \frac{5}{4} \hbar^2 V Q_{i,1}^2 V'' + \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} V'' + \frac{5}{8} \hbar^2 Q_{i,1}^2 V'^2 \\
& + \frac{31}{2} \hbar^2 V^2 Q_{i,1} V' - \frac{3}{4} \hbar^2 V Q_{i,2} V' - \frac{69}{16} \hbar^4 Q_{i,1} V' V'' - \frac{5}{2} V^3 Q_{i,1}^2 - \frac{9}{4} V^2 Q_{i,1} Q_{i,2} \\
& - \frac{5}{2} V Q_{i,1} Q_{i,3} + \frac{15}{4} Q_{i,2} Q_{i,3} - \frac{3}{16} \hbar^6 V V'''' - \frac{1}{8} \hbar^6 V''^2 + 4 \hbar^4 V^2 V'' - \frac{9}{16} \hbar^4 V V'^2 \\
& + \frac{15}{64} \hbar^6 V'''' V' - 6 \hbar^2 V^4 \left. \right) + \xi''_{07i}(q_i) \left(\frac{21}{64} \hbar^6 V'''' Q_{i,1} - \frac{5}{16} \hbar^4 V'''' Q_{i,1}^2 - \frac{89}{16} \hbar^4 V Q_{i,1} V'' \right. \\
& + \frac{1}{8} \hbar^2 Q_{i,1}^3 V'' - \frac{123}{32} \hbar^4 Q_{i,1} V'^2 + \frac{17}{4} \hbar^2 V Q_{i,1}^2 V' + \frac{9}{8} \hbar^2 Q_{i,1} Q_{i,2} V' + \frac{69}{8} \hbar^2 V^3 Q_{i,1} \\
& - \frac{3}{4} \hbar^2 V^2 Q_{i,2} - \frac{3}{4} V^2 Q_{i,1}^3 - \frac{3}{2} V Q_{i,1}^2 Q_{i,2} - \frac{105}{8} \hbar^2 Q_{i,4} - \frac{5}{4} Q_{i,1}^2 Q_{i,3} + \frac{1}{256} \hbar^8 V'''''' \\
& - \frac{39}{64} \hbar^6 V V'''' + \frac{307}{32} \hbar^4 V^2 V' + \frac{15}{32} \hbar^6 V' V'' \left. \right) + \xi'''_{07i}(q_i) \left(\frac{17}{32} \hbar^6 V'''' Q_{i,1} - \frac{11}{16} \hbar^4 Q_{i,1}^2 V'' \right. \\
& - \frac{103}{16} \hbar^4 V Q_{i,1} V' + \frac{3}{32} \hbar^4 Q_{i,2} V' + \frac{3}{8} \hbar^2 Q_{i,1}^3 V' + \frac{27}{8} \hbar^2 V^2 Q_{i,1}^2 + \frac{19}{8} \hbar^2 V Q_{i,1} Q_{i,2} \\
& - \frac{1}{6} V Q_{i,1}^4 - \frac{15}{8} \hbar^2 Q_{i,2}^2 - \frac{5}{8} \hbar^2 Q_{i,1} Q_{i,3} - \frac{1}{2} Q_{i,1}^3 Q_{i,2} + \frac{3}{128} \hbar^8 V'''' - \frac{5}{4} \hbar^6 V V'' \\
& + \frac{3}{16} \hbar^6 V'^2 + 4 \hbar^4 V^3 \left. \right) + \xi''''_{07i}(q_i) \left(\frac{31}{64} \hbar^6 Q_{i,1} V'' - \frac{25}{32} \hbar^4 Q_{i,1}^2 V' - \frac{61}{32} \hbar^4 V^2 Q_{i,1} \right. \\
& + \frac{3}{16} \hbar^4 V Q_{i,2} + \frac{5}{8} \hbar^2 V Q_{i,1}^3 + \frac{63}{16} \hbar^4 Q_{i,3} + \frac{5}{8} \hbar^2 Q_{i,1}^2 Q_{i,2} - \frac{1}{30} Q_{i,1}^5 + \frac{15}{256} \hbar^8 V'''' \\
& - \frac{105}{64} \hbar^6 V V' \left. \right) + \xi''''''_{07i}(q_i) \left(\frac{7}{32} \hbar^6 Q_{i,1} V' - \frac{1}{2} \hbar^4 V Q_{i,1}^2 + \frac{13}{32} \hbar^4 Q_{i,1} Q_{i,2} + \frac{1}{12} \hbar^2 Q_{i,1}^4 \right. \\
& + \frac{5}{64} \hbar^8 V'' - \frac{5}{8} \hbar^6 V^2 \left. \right) + \xi''''''''_{07i}(q_i) \left(\frac{1}{64} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} - \frac{1}{32} \hbar^4 Q_{i,1}^3 + \frac{15}{256} \hbar^8 V' \right) \\
& + \xi''''''''''_{07i}(q_i) \left(\frac{3}{128} \hbar^8 V - \frac{1}{64} \hbar^6 Q_{i,1}^2 \right) + \frac{1}{256} \hbar^8 \xi''''''''''_{07i}(q_i) Q_{i,1} + \xi_{17i}(q_i) \left(-\frac{1}{16} \hbar^4 V'''' Q_{i,1} \right. \\
& + \frac{5}{4} \hbar^2 V Q_{i,1} V'' + \frac{5}{8} \hbar^2 Q_{i,1} V'^2 - \frac{5}{2} V^3 Q_{i,1} + \frac{35}{8} Q_{i,4} + \frac{1}{8} \hbar^4 V V'''' - \frac{5}{2} \hbar^2 V^2 V' \\
& - \frac{3}{16} \hbar^4 V' V'' \left. \right) + \xi'_{17i}(q_i) \left(-\frac{5}{16} \hbar^4 V'''' Q_{i,1} + \frac{1}{4} \hbar^2 Q_{i,1}^2 V'' + 4 \hbar^2 V Q_{i,1} V' - \frac{3}{2} V^2 Q_{i,1}^2 \right. \\
& - \frac{3}{2} V Q_{i,1} Q_{i,2} + \frac{9}{8} Q_{i,2}^2 + \frac{1}{2} \hbar^4 V V'' - \frac{9}{32} \hbar^4 V'^2 - 2 \hbar^2 V^3 \left. \right) + \xi''_{17i}(q_i) \left(-\frac{5}{8} \hbar^4 Q_{i,1} V'' \right. \\
& + \frac{3}{4} \hbar^2 Q_{i,1}^2 V' + \frac{11}{4} \hbar^2 V^2 Q_{i,1} - \frac{1}{2} V Q_{i,1}^3 - \frac{35}{8} \hbar^2 Q_{i,3} - \frac{3}{4} Q_{i,1}^2 Q_{i,2} - \frac{1}{64} \hbar^6 V'''' \\
& + \frac{19}{16} \hbar^4 V V' \left. \right) + \xi'''_{17i}(q_i) \left(-\frac{9}{16} \hbar^4 Q_{i,1} V' + \hbar^2 V Q_{i,1}^2 - \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} - \frac{1}{8} Q_{i,1}^4 - \frac{1}{16} \hbar^6 V'' \right. \\
& + \frac{3}{4} \hbar^4 V^2 \left. \right) + \xi''''_{17i}(q_i) \left(-\frac{3}{16} \hbar^4 V Q_{i,1} + \frac{21}{32} \hbar^4 Q_{i,2} + \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{3}{32} \hbar^6 V' \right) \\
& + \xi''''''_{17i}(q_i) \left(\frac{1}{32} \hbar^4 Q_{i,1}^2 - \frac{1}{16} \hbar^6 V \right) - \frac{1}{64} \hbar^6 \xi''''''_{17i}(q_i) Q_{i,1} + \xi_{27i}(q_i) \left(\frac{1}{4} \hbar^2 Q_{i,1} V'' \right. \\
& - \frac{3}{2} V^2 Q_{i,1} + \frac{5}{2} Q_{i,3} - \frac{1}{2} \hbar^2 V V' \left. \right) + \xi'_{27i}(q_i) \left(\frac{3}{4} \hbar^2 Q_{i,1} V' - V Q_{i,1}^2 - \frac{1}{2} \hbar^2 V^2 \right) \\
& + \xi''_{27i}(q_i) \left(\frac{3}{4} \hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} - \frac{1}{3} Q_{i,1}^3 + \frac{1}{16} \hbar^4 V' \right) + \frac{1}{8} \hbar^4 \xi'''_{27i}(q_i) V \\
& + \frac{1}{16} \hbar^4 \xi''''_{27i}(q_i) Q_{i,1} + \xi_{37i}(q_i) \left(\frac{3}{2} Q_{i,2} - V Q_{i,1} \right) - \frac{1}{2} \xi'_{37i}(q_i) Q_{i,1}^2 - \frac{1}{4} \hbar^2 \xi''_{37i}(q_i) Q_{i,1} \left. \right]' \\
& + \xi_{57i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{58i} : \xi'_{08i}(q_i) & \left(\frac{63}{8} Q_{i,5} + \frac{1}{256} \hbar^8 V'''''' - \frac{9}{64} \hbar^6 V V'''' + \frac{63}{32} \hbar^4 V^2 V'' + \frac{35}{32} \hbar^4 V'^3 - \frac{105}{8} \hbar^2 V^3 V' \right. \\
& - \frac{9}{32} \hbar^6 V'''' V' - \frac{39}{64} \hbar^6 V'''' V'' + \frac{63}{16} \hbar^4 V V' V'' \left. \right) + \xi''_{08i}(q_i) \left(-\frac{1}{64} \hbar^6 V'''' Q_{i,1} \right. \\
& + \frac{7}{16} \hbar^4 V V'''' Q_{i,1} + \frac{3}{32} \hbar^4 V'''' Q_{i,2} - \frac{35}{8} \hbar^2 V^2 Q_{i,1} V' - \frac{15}{8} \hbar^2 V Q_{i,2} V' - \frac{5}{8} \hbar^2 Q_{i,3} V' \\
& + \frac{7}{16} \hbar^4 Q_{i,1} V' V'' + \frac{15}{4} Q_{i,2} Q_{i,3} + \frac{35}{8} Q_{i,1} Q_{i,4} + \frac{1}{32} \hbar^8 V'''''' - \frac{15}{16} \hbar^6 V V'''' - \frac{11}{8} \hbar^6 V''^2 \\
& + 10 \hbar^4 V^2 V'' + \frac{135}{16} \hbar^4 V V'^2 - \frac{105}{64} \hbar^6 V'''' V' - 10 \hbar^2 V^4 \left. \right) + \xi'''_{08i}(q_i) \left(-\frac{3}{32} \hbar^6 V'''' Q_{i,1} \right. \\
& + \frac{1}{32} \hbar^4 V'''' Q_{i,1}^2 + 2 \hbar^4 V Q_{i,1} V'' + \frac{3}{8} \hbar^4 Q_{i,2} V'' + \frac{27}{32} \hbar^4 Q_{i,1} V'^2 - \frac{5}{8} \hbar^2 V Q_{i,1}^2 V' \\
& - \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} V' - 4 \hbar^2 V^3 Q_{i,1} - \frac{9}{4} \hbar^2 V^2 Q_{i,2} - \frac{5}{4} \hbar^2 V Q_{i,3} - \frac{105}{8} \hbar^2 Q_{i,4} + \frac{9}{8} Q_{i,1} Q_{i,2}^2 \\
& + \frac{5}{4} Q_{i,1}^2 Q_{i,3} + \frac{7}{64} \hbar^8 V'''''' - \frac{87}{32} \hbar^6 V V'''' + \frac{179}{8} \hbar^4 V^2 V' - \frac{15}{4} \hbar^6 V' V'' \left. \right) \\
& + \xi''''_{08i}(q_i) \left(-\frac{1}{4} \hbar^6 V'''' Q_{i,1} + \frac{1}{8} \hbar^4 Q_{i,1}^2 V'' + \frac{33}{8} \hbar^4 V Q_{i,1} V' + \frac{7}{8} \hbar^4 Q_{i,2} V' - \frac{1}{24} \hbar^2 Q_{i,1}^3 V' \right. \\
& - \frac{3}{4} \hbar^2 V^2 Q_{i,1}^2 - \frac{3}{4} \hbar^2 V Q_{i,1} Q_{i,2} - \frac{15}{8} \hbar^2 Q_{i,2}^2 - 5 \hbar^2 Q_{i,1} Q_{i,3} + \frac{1}{4} Q_{i,1}^3 Q_{i,2} + \frac{7}{32} \hbar^8 V'''' \\
& - \frac{35}{8} \hbar^6 V V'' - \frac{69}{32} \hbar^6 V'^2 + \frac{33}{4} \hbar^4 V^3 \left. \right) + \xi''''''_{08i}(q_i) \left(-\frac{3}{8} \hbar^6 Q_{i,1} V'' + \frac{1}{4} \hbar^4 Q_{i,1}^2 V' \right. \\
& + \frac{17}{8} \hbar^4 V^2 Q_{i,1} + \hbar^4 V Q_{i,2} - \frac{1}{12} \hbar^2 V Q_{i,1}^3 + \frac{63}{16} \hbar^4 Q_{i,3} - \hbar^2 Q_{i,1}^2 Q_{i,2} + \frac{1}{120} Q_{i,1}^5 \\
& + \frac{35}{128} \hbar^8 V'''' - \frac{135}{32} \hbar^6 V V' \left. \right) + \xi''''''''_{08i}(q_i) \left(-\frac{11}{32} \hbar^6 Q_{i,1} V' + \frac{1}{4} \hbar^4 V Q_{i,1}^2 + \frac{17}{16} \hbar^4 Q_{i,1} Q_{i,2} \right. \\
& - \frac{1}{24} \hbar^2 Q_{i,1}^4 + \frac{7}{32} \hbar^8 V'' - \frac{19}{16} \hbar^6 V^2 \left. \right) + \xi''''''''''_{08i}(q_i) \left(-\frac{3}{16} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} + \frac{1}{16} \hbar^4 Q_{i,1}^3 \right. \\
& + \frac{7}{64} \hbar^8 V' \left. \right) + \xi''''''''''''_{08i}(q_i) \left(\frac{1}{32} \hbar^8 V - \frac{1}{32} \hbar^6 Q_{i,1}^2 \right) + \frac{1}{256} \hbar^8 \xi''''''''''''_{08i}(q_i) Q_{i,1} \\
& + \xi'_{18i}(q_i) \left(\frac{35}{8} Q_{i,4} - \frac{1}{64} \hbar^6 V'''''' + \frac{7}{16} \hbar^4 V V'''' - \frac{35}{8} \hbar^2 V^2 V' + \frac{7}{16} \hbar^4 V' V'' \right) \\
& + \xi''_{18i}(q_i) \left(\frac{1}{16} \hbar^4 V'''' Q_{i,1} - \frac{5}{4} \hbar^2 V Q_{i,1} V' - \frac{3}{8} \hbar^2 Q_{i,2} V' + \frac{9}{8} Q_{i,2}^2 + \frac{5}{2} Q_{i,1} Q_{i,3} - \frac{3}{32} \hbar^6 V'''' \right. \\
& + 2 \hbar^4 V V'' + \frac{27}{32} \hbar^4 V'^2 - 4 \hbar^2 V^3 \left. \right) + \xi'''_{18i}(q_i) \left(\frac{1}{4} \hbar^4 Q_{i,1} V'' - \frac{1}{8} \hbar^2 Q_{i,1}^2 V' - \frac{3}{2} \hbar^2 V^2 Q_{i,1} \right. \\
& - \frac{3}{4} \hbar^2 V Q_{i,2} - \frac{35}{8} \hbar^2 Q_{i,3} + \frac{3}{4} Q_{i,1}^2 Q_{i,2} - \frac{15}{64} \hbar^6 V'''' + \frac{61}{16} \hbar^4 V V' \left. \right) + \xi''''_{18i}(q_i) \left(\frac{7}{16} \hbar^4 Q_{i,1} V' \right. \\
& - \frac{1}{4} \hbar^2 V Q_{i,1}^2 - \frac{13}{8} \hbar^2 Q_{i,1} Q_{i,2} + \frac{1}{24} Q_{i,1}^4 - \frac{5}{16} \hbar^6 V'' + \frac{7}{4} \hbar^4 V^2 \left. \right) + \xi''''''_{18i}(q_i) \left(\frac{3}{8} \hbar^4 V Q_{i,1} \right. \\
& + \frac{21}{32} \hbar^4 Q_{i,2} - \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{15}{64} \hbar^6 V' \left. \right) + \xi''''''''_{18i}(q_i) \left(\frac{3}{32} \hbar^4 Q_{i,1}^2 - \frac{3}{32} \hbar^6 V \right) \\
& - \frac{1}{64} \hbar^6 \xi''''''''''_{18i}(q_i) Q_{i,1} + \xi'_{28i}(q_i) \left(\frac{5}{2} Q_{i,3} + \frac{1}{16} \hbar^4 V'''' - \frac{5}{4} \hbar^2 V V' \right) + \xi''_{28i}(q_i) \left(-\frac{1}{4} \hbar^2 Q_{i,1} V' \right. \\
& + \frac{3}{2} Q_{i,1} Q_{i,2} + \frac{1}{4} \hbar^4 V'' - \frac{3}{2} \hbar^2 V^2 \left. \right) + \xi'''_{28i}(q_i) \left(-\frac{1}{2} \hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} + \frac{1}{6} Q_{i,1}^3 + \frac{3}{8} \hbar^4 V' \right) \\
& + \xi''''_{28i}(q_i) \left(\frac{1}{4} \hbar^4 V - \frac{1}{4} \hbar^2 Q_{i,1}^2 \right) + \frac{1}{16} \hbar^4 \xi''''''_{28i}(q_i) Q_{i,1} + \xi'_{38i}(q_i) \left(\frac{3}{2} Q_{i,2} - \frac{1}{4} \hbar^2 V' \right) \\
& + \xi''_{38i}(q_i) \left(\frac{1}{2} Q_{i,1}^2 - \frac{1}{2} \hbar^2 V \right) - \frac{1}{4} \hbar^2 \xi'''_{38i}(q_i) Q_{i,1} + \xi'_{48i}(q_i) Q_{i,1} + \xi_{58i}(q_i) = 0
\end{aligned}$$

$$\begin{aligned}
S_{59i} : & \left[\xi_{09i}(q_i) \left(\frac{63}{8} Q_{i,5} + \frac{1}{256} \hbar^8 V'''''' - \frac{9}{64} \hbar^6 V V'''''' + \frac{63}{32} \hbar^4 V^2 V'''' + \frac{35}{32} \hbar^4 V'^3 - \frac{105}{8} \hbar^2 V^3 V' \right. \right. \\
& - \frac{9}{32} \hbar^6 V'''' V' - \frac{39}{64} \hbar^6 V'''' V'' + \frac{63}{16} \hbar^4 V V' V'' \left. \right) + \xi'_{09i}(q_i) \left(-\frac{1}{64} \hbar^6 V'''''' Q_{i,1} \right. \\
& + \frac{7}{16} \hbar^4 V V'''' Q_{i,1} + \frac{3}{32} \hbar^4 V'''' Q_{i,2} - \frac{35}{8} \hbar^2 V^2 Q_{i,1} V' - \frac{15}{8} \hbar^2 V Q_{i,2} V' - \frac{5}{8} \hbar^2 Q_{i,3} V' \\
& + \frac{7}{16} \hbar^4 Q_{i,1} V' V'' + \frac{15}{4} Q_{i,2} Q_{i,3} + \frac{35}{8} Q_{i,1} Q_{i,4} + \frac{1}{32} \hbar^8 V'''''' - \frac{15}{16} \hbar^6 V V'''' - \frac{11}{8} \hbar^6 V''^2 \\
& + 10 \hbar^4 V^2 V'' + \frac{135}{16} \hbar^4 V V'^2 - \frac{105}{64} \hbar^6 V'''' V' - 10 \hbar^2 V^4 \left. \right) + \xi''_{09i}(q_i) \left(-\frac{3}{32} \hbar^6 V'''''' Q_{i,1} \right. \\
& + \frac{1}{32} \hbar^4 V'''' Q_{i,1}^2 + 2 \hbar^4 V Q_{i,1} V'' + \frac{3}{8} \hbar^4 Q_{i,2} V'' + \frac{27}{32} \hbar^4 Q_{i,1} V'^2 - \frac{5}{8} \hbar^2 V Q_{i,1}^2 V' \\
& - \frac{3}{8} \hbar^2 Q_{i,1} Q_{i,2} V' - 4 \hbar^2 V^3 Q_{i,1} - \frac{9}{4} \hbar^2 V^2 Q_{i,2} - \frac{5}{4} \hbar^2 V Q_{i,3} - \frac{105}{8} \hbar^2 Q_{i,4} + \frac{9}{8} Q_{i,1} Q_{i,2}^2 \\
& + \frac{5}{4} Q_{i,1}^2 Q_{i,3} + \frac{7}{64} \hbar^8 V'''''' - \frac{87}{32} \hbar^6 V V'''' + \frac{179}{8} \hbar^4 V^2 V' - \frac{15}{4} \hbar^6 V' V'' \left. \right) \\
& + \xi'''_{09i}(q_i) \left(-\frac{1}{4} \hbar^6 V'''''' Q_{i,1} + \frac{1}{8} \hbar^4 Q_{i,1}^2 V'' + \frac{33}{8} \hbar^4 V Q_{i,1} V' + \frac{7}{8} \hbar^4 Q_{i,2} V' - \frac{1}{24} \hbar^2 Q_{i,1}^3 V' \right. \\
& - \frac{3}{4} \hbar^2 V^2 Q_{i,1}^2 - \frac{3}{4} \hbar^2 V Q_{i,1} Q_{i,2} - \frac{15}{8} \hbar^2 Q_{i,2}^2 - 5 \hbar^2 Q_{i,1} Q_{i,3} + \frac{1}{4} Q_{i,1}^3 Q_{i,2} + \frac{7}{32} \hbar^8 V'''''' \\
& - \frac{35}{8} \hbar^6 V V'' - \frac{69}{32} \hbar^6 V'^2 + \frac{33}{4} \hbar^4 V^3 \left. \right) + \xi''''_{09i}(q_i) \left(-\frac{3}{8} \hbar^6 Q_{i,1} V'' + \frac{1}{4} \hbar^4 Q_{i,1}^2 V' \right. \\
& + \frac{17}{8} \hbar^4 V^2 Q_{i,1} + \hbar^4 V Q_{i,2} - \frac{1}{12} \hbar^2 V Q_{i,1}^3 + \frac{63}{16} \hbar^4 Q_{i,3} - \hbar^2 Q_{i,1}^2 Q_{i,2} + \frac{1}{120} Q_{i,1}^5 \\
& + \frac{35}{128} \hbar^8 V'''''' - \frac{135}{32} \hbar^6 V V' \left. \right) + \xi''''''_{09i}(q_i) \left(-\frac{11}{32} \hbar^6 Q_{i,1} V' + \frac{1}{4} \hbar^4 V Q_{i,1}^2 + \frac{17}{16} \hbar^4 Q_{i,1} Q_{i,2} \right. \\
& - \frac{1}{24} \hbar^2 Q_{i,1}^4 + \frac{7}{32} \hbar^8 V'''' - \frac{19}{16} \hbar^6 V^2 \left. \right) + \xi''''''''_{09i}(q_i) \left(-\frac{3}{16} \hbar^6 V Q_{i,1} - \frac{9}{32} \hbar^6 Q_{i,2} + \frac{1}{16} \hbar^4 Q_{i,1}^3 \right. \\
& + \frac{7}{64} \hbar^8 V' \left. \right) + \xi''''''''''_{09i}(q_i) \left(\frac{1}{32} \hbar^8 V - \frac{1}{32} \hbar^6 Q_{i,1}^2 \right) + \frac{1}{256} \hbar^8 \xi''''''''''_{09i}(q_i) Q_{i,1} \\
& + \xi_{19i}(q_i) \left(\frac{35}{8} Q_{i,4} - \frac{1}{64} \hbar^6 V'''''' + \frac{7}{16} \hbar^4 V V'''' - \frac{35}{8} \hbar^2 V^2 V' + \frac{7}{16} \hbar^4 V' V'' \right) \\
& + \xi'_{19i}(q_i) \left(\frac{1}{16} \hbar^4 V'''' Q_{i,1} - \frac{5}{4} \hbar^2 V Q_{i,1} V' - \frac{3}{8} \hbar^2 Q_{i,2} V' + \frac{9}{8} Q_{i,2}^2 + \frac{5}{2} Q_{i,1} Q_{i,3} - \frac{3}{32} \hbar^6 V'''''' \right. \\
& + 2 \hbar^4 V V'' + \frac{27}{32} \hbar^4 V'^2 - 4 \hbar^2 V^3 \left. \right) + \xi''_{19i}(q_i) \left(\frac{1}{4} \hbar^4 Q_{i,1} V'' - \frac{1}{8} \hbar^2 Q_{i,1}^2 V' - \frac{3}{2} \hbar^2 V^2 Q_{i,1} \right. \\
& - \frac{3}{4} \hbar^2 V Q_{i,2} - \frac{35}{8} \hbar^2 Q_{i,3} + \frac{3}{4} Q_{i,1}^2 Q_{i,2} - \frac{15}{64} \hbar^6 V'''' + \frac{61}{16} \hbar^4 V V' \left. \right) + \xi'''_{19i}(q_i) \left(\frac{7}{16} \hbar^4 Q_{i,1} V' \right. \\
& - \frac{1}{4} \hbar^2 V Q_{i,1}^2 - \frac{13}{8} \hbar^2 Q_{i,1} Q_{i,2} + \frac{1}{24} Q_{i,1}^4 - \frac{5}{16} \hbar^6 V'' + \frac{7}{4} \hbar^4 V^2 \left. \right) + \xi''''_{19i}(q_i) \left(\frac{3}{8} \hbar^4 V Q_{i,1} \right. \\
& + \frac{21}{32} \hbar^4 Q_{i,2} - \frac{1}{8} \hbar^2 Q_{i,1}^3 - \frac{15}{64} \hbar^6 V' \left. \right) + \xi''''''_{19i}(q_i) \left(\frac{3}{32} \hbar^4 Q_{i,1}^2 - \frac{3}{32} \hbar^6 V \right) \\
& - \frac{1}{64} \hbar^6 \xi''''''''_{19i}(q_i) Q_{i,1} + \xi_{29i}(q_i) \left(\frac{5}{2} Q_{i,3} + \frac{1}{16} \hbar^4 V'''' - \frac{5}{4} \hbar^2 V V' \right) + \xi'_{29i}(q_i) \left(-\frac{1}{4} \hbar^2 Q_{i,1} V' \right. \\
& + \frac{3}{2} Q_{i,1} Q_{i,2} + \frac{1}{4} \hbar^4 V'' - \frac{3}{2} \hbar^2 V^2 \left. \right) + \xi''_{29i}(q_i) \left(-\frac{1}{2} \hbar^2 V Q_{i,1} - \frac{5}{4} \hbar^2 Q_{i,2} + \frac{1}{6} Q_{i,1}^3 + \frac{3}{8} \hbar^4 V' \right) \\
& + \xi'''_{29i}(q_i) \left(\frac{1}{4} \hbar^4 V - \frac{1}{4} \hbar^2 Q_{i,1}^2 \right) + \frac{1}{16} \hbar^4 \xi''''_{29i}(q_i) Q_{i,1} + \xi_{39i}(q_i) \left(\frac{3}{2} Q_{i,2} - \frac{1}{4} \hbar^2 V' \right) \\
& + \xi'_{39i}(q_i) \left(\frac{1}{2} Q_{i,1}^2 - \frac{1}{2} \hbar^2 V \right) - \frac{1}{4} \hbar^2 \xi''_{39i}(q_i) Q_{i,1} + \xi_{49i}(q_i) Q_{i,1} \left. \right]' + \xi_{58i}(q_i) = 0
\end{aligned}$$

Appendix B

List of Integrals

We give here all of the integrals of order less than or equal to four up to linear independence for the Hamiltonians listed in Section 4.8.1. Throughout, we use $\{a, b\} = \frac{1}{2}(a \circ b + b \circ a)$ for the symmetrised product. To avoid repetition, the trivial integrals $H_1, H_2, \{H_1, H_1\}, \{H_1, H_2\}, \{H_2, H_2\}$ are omitted.

$$(1) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta(q_1^2 + q_2^2).$$

There is a single first-order integral m_{12} . The second-order integrals are $\{m_{12}, m_{12}\}, (H_1 - H_2, m_{12})$. The third-order integrals are

$$\{m_{12}, \{m_{12}, m_{12}\}\}, \{(H_1 - H_2, m_{12}), m_{12}\}, \{H_1, m_{12}\}, \{H_2, m_{12}\}.$$

The fourth-order integrals are

$$\{\{m_{12}, m_{12}\}, \{m_{12}, m_{12}\}\}, \{H_1, \{m_{12}, m_{12}\}\}, \{H_2, \{m_{12}, m_{12}\}\}, \\ \{H_1, (H_1 - H_2, m_{12})\}, \{H_2, (H_1 - H_2, m_{12})\}, \{(H_1 - H_2, m_{12}), \{m_{12}, m_{12}\}\}.$$

$$(2) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1(q_1^2 + q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\beta_3}{q_2^2}$$

There is one second-order integral

$$X = \{m_{12}, m_{12}\} + \frac{1}{2}\hbar^2 + \frac{2\beta_2 q_2^2}{q_1^2} + \frac{2\beta_3 q_1^2}{q_2^2}$$

There is one third-order integral $(H_1 - H_2, X)$ and three fourth-order integrals

$$\{X, X\}, \{H_1, X\}, \{H_2, X\}.$$

$$(2a) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\beta}{q_1^2}$$

This is a special case of (2) with $\beta_1 = 0, \beta_2 = \beta, \beta_3 = 0$. It has a first-order integral p_2 . There is an additional second-order integral $(p_2, X^{(1)})$. There are three more third-order integrals

$$p_2^3, \{X^{(1)}, p_2\}, \{H_1, p_2\}$$

There are three more fourth-order integrals

$$\{X, (p_2, X)\}, \{(p_2, X), H_1\}, \{(p_2, X), H_2\}$$

$$(2ai) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\hbar^2}{q_1^2}$$

This is a special case of (2a) with $\beta = \hbar^2$. It has four additional third-order integrals

$$X^{(1)} = \{m_{12}, \{m_{12}, m_{12}\}\} - \hbar^2 q_2 \left\{ p_1, 2 + \frac{3q_2^2}{q_1^2} \right\} + \frac{\hbar^2}{q_1} \{p_2, 2q_1^2 + 3q_2^2\}$$

and $(p_2, X^{(1)})$, $(p_2, (p_2, X^{(1)}))$, $(p_2, (p_2, (p_2, X^{(1)})))$. There are four more fourth-order integrals

$$\{p_2, X^{(1)}\}, \{p_2, (p_2, X^{(1)})\}, \{p_2, (p_2, (p_2, X^{(1)}))\}, \{p_2, (p_2, (p_2, (p_2, X^{(1)})))\}$$

$$(2b) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\beta}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

This is a special case of (2) with $\beta_1 = 0$, $\beta_2 = \beta$, $\beta_3 = \hbar^2$. It has two additional third-order integrals

$$X^{(1)} = \{p_2, \{m_{12}, m_{12}\}\} - \frac{2\hbar^2}{q_2} \{p_1, q_1\} + \left\{ p_2, \frac{\hbar^2}{2} + \frac{3\hbar^2 q_1^2}{q_2^2} + \frac{2\beta q_2^2}{q_1^2} \right\}$$

$$X^{(2)} = p_2^3 + 3\hbar^2 \left\{ p_2, \frac{1}{q_2^2} \right\}$$

There are two more fourth-order integrals $(X, X^{(1)})$, $(X, X^{(2)})$.

$$(2bi) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\hbar^2}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

This is a special case of (2b) with $\beta = \hbar^2$. There are three more third-order integrals

$$X''' = \{m_{12}, \{m_{12}, m_{12}\}\} - \frac{\hbar^2}{q_2} \left\{ p_1, 3q_1^2 + 2q_2^2 + \frac{3q_2^4}{q_1^2} \right\} + \frac{\hbar^2}{q_1} \left\{ p_2, 3q_2^2 + 2q_1^2 + \frac{3q_1^4}{q_2^2} \right\}$$

$$X'''' = \{p_1, \{m_{12}, m_{12}\}\} + \hbar^2 \left\{ p_1, \frac{1}{2} + \frac{3\hbar^2 q_2^2}{q_1^2} + \frac{2q_1^2}{q_2^2} \right\} - \frac{2\hbar^2}{q_1} \{p_2, q_2\}$$

$$X''''' = p_1^3 + 3\hbar^2 \left\{ p_1, \frac{1}{q_1^2} \right\}$$

There are consequently three more fourth-order integrals (X, X''''') , (X, X''''''') , $(H_1 - H_2, X''''')$.

$$(2c) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta(q_1^2 + q_2^2) + \frac{\hbar^2}{q_1^2}$$

This is a special case of (2) with $\beta_1 = \beta$, $\beta_2 = \hbar^2$, $\beta_3 = 0$. There are two more third-order integrals

$$X^{(1)} = \{m_{12}, \{m_{12}, m_{12}\}\} - \hbar^2 q_2 \left\{ p_1, 2 + \frac{3q_2^2}{q_1^2} \right\} + \frac{\hbar^2}{q_1} \{p_2, 2q_1^2 + 3q_2^2\}$$

$$X^{(2)} = \{p_1^2, m_{12}\} - q_2 \left\{ p_1, 2\beta q_1^2 + \frac{3\hbar^2}{q_1^2} \right\} + \left(2\beta q_1^3 + \frac{\hbar^2}{q_1} \right) p_2$$

and two more fourth-order integrals $(H_1 - H_2, X^{(1)})$, $(H_1 - H_2, X^{(2)})$.

$$(2d) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta(q_1^2 + q_2^2) + \frac{\hbar^2}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

This is a special case of (2) with $\beta_1 = \beta$, $\beta_2 = \hbar^2$, $\beta_3 = \hbar^2$. There is one more third-order integral

$$X^{(1)} = \{m_{12}, \{m_{12}, m_{12}\}\} - \frac{\hbar^2}{q_2} \left\{ p_1, 3q_1^2 + 2q_2^2 + \frac{3q_2^4}{q_1^2} \right\} + \frac{\hbar^2}{q_1} \left\{ p_2, 2q_1^2 + 3q_2^2 + \frac{3q_1^4}{q_2^2} \right\}$$

and one more fourth-order integral $(H_1 - H_2, X^{(1)})$.

$$(3) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1(q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2} + \beta_3 q_2$$

There is one second-order integral

$$X = \{p_1, m_{12}\} - \frac{2\beta_2 q_2}{q_1^2} + \frac{1}{2} q_1^2 (4q_2 \beta_1 + \beta_3)$$

There is one third-order integral $(H_1 - H_2, X)$ and three fourth-order integrals

$$\{X, X\}, \{H_1, X\}, \{H_2, X\}.$$

$$(3a) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta q_2$$

This is a special case of (3) with $\beta_1 = \beta_2 = 0$, $\beta_3 = \beta$. It has a first-order integral p_1 . It has another second-order integral (p_1, X) . There are three more third-order integrals are fourfold

$$p_1^3, \{X, p_1\}, \{(p_1, X), p_1\}.$$

There are three more fourth-order integrals

$$\{(p_1, X), X\}, \{(p_1, X), H_1\}, \{(p_1, X), H_2\}$$

$$(3b) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{\hbar^2}{q_1^2} + \beta q_2$$

This is a special case of (3) with $\beta_1 = 0$, $\beta_2 = \hbar^2$, $\beta_3 = \beta$. There are two more third-order integrals

$$X^{(1)} = \{p_1^2, m_{12}\} + \left\{ p_1, \frac{1}{2}\beta q_1^2 - \frac{3\hbar^2 q_2}{q_1^2} \right\} + \frac{\hbar^2 p_2}{q_1}$$

$$X^{(2)} = p_1^3 + 3\hbar^2 \left\{ p_1, \frac{1}{q_1^2} \right\}$$

and two more fourth-order integrals $(H_1 - H_2, X^{(1)})$, $(X, X^{(1)})$.

$$(4) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1 \left(q_1^2 + q_2^2 + \frac{\beta_2^2}{q_1^2} \right) + \frac{\beta_3}{q_2^2} + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} \right]$$

There are two fourth-order integrals

$$\begin{aligned}
X^{(1)} = & \{ \{m_{12}, m_{12}\}, \{m_{12}, m_{12}\} - 2\beta_2 p_2^2 \} + \left\{ p_1^2, -\beta_2 \hbar^2 + \frac{16\beta_2 q_2^4 \hbar^2}{(q_1^2 - \beta_2)^2} + \frac{8q_2^4 \hbar^2}{q_1^2 - \beta_2} \right. \\
& + \left. \frac{8\beta_1 \beta_2^2 q_2^4 - q_2^4 \hbar^2}{2q_1^2} - 4\beta_1 \beta_2 q_2^4 + 4\beta_3 q_1^2 + 5q_2^2 \hbar^2 \right\} + \left\{ p_1 p_2, -\frac{48\beta_2 q_2^3 q_1 \hbar^2}{(q_1^2 - \beta_2)^2} \right. \\
& + \left. \frac{2q_1 (4\beta_2 \beta_3 + 4\beta_1 \beta_2 q_2^4 - 5q_2^2 \hbar^2)}{q_2} - \frac{16q_2^3 q_1 \hbar^2}{q_1^2 - \beta_2} + \frac{q_2^3 \hbar^2 - 8\beta_1 \beta_2^2 q_2^3}{q_1} - \frac{8\beta_3 q_1^3}{q_2} \right\} \\
& + \left\{ p_2^2, -\frac{q_1^2 (8\beta_2 \beta_3 + 4\beta_1 \beta_2 q_2^4 - 5q_2^2 \hbar^2)}{q_2^2} + \frac{32\beta_2^2 q_2^2 \hbar^2}{(q_1^2 - \beta_2)^2} + \frac{40\beta_2 q_2^2 \hbar^2}{q_1^2 - \beta_2} \right. \\
& + \left. \frac{1}{2} (-2\beta_2 \hbar^2 + 8\beta_1 \beta_2^2 q_2^2 + 15q_2^2 \hbar^2) + \frac{\beta_2 q_2^2 \hbar^2 - 8\beta_1 \beta_2^3 q_2^2}{2q_1^2} + \frac{4\beta_3 q_1^4}{q_2^2} \right\} \\
& + \frac{64\beta_2^2 q_2^4 \hbar^4}{(q_1^2 - \beta_2)^4} + \frac{16q_2^2 (9\beta_2^2 \hbar^4 + 4\beta_2 q_2^2 \hbar^4)}{(q_1^2 - \beta_2)^3} - \frac{2q_1^2 (4\beta_2 \beta_3^2 + \beta_1 \beta_2 q_2^4 \hbar^2 - 3\beta_3 q_2^2 \hbar^2)}{q_2^4} \\
& + \frac{\hbar^2 (-8\beta_2 \beta_3 + 24\beta_1 \beta_2 q_2^4 + 8\beta_1 \beta_2^2 q_2^2 + 8\beta_3 q_2^2 + q_2^2 \hbar^2)}{4q_2^2} \\
& + \frac{4 (4\beta_2^2 \hbar^4 + 8\beta_2^2 \beta_3 \hbar^2 + 33\beta_2 q_2^2 \hbar^4 + 3q_2^4 \hbar^4)}{(q_1^2 - \beta_2)^2} \\
& + \frac{2 (10\beta_2^2 \hbar^4 + 16\beta_2^2 \beta_3 \hbar^2 + 6\beta_2 q_2^2 \hbar^4 - 8\beta_1 \beta_2^2 q_2^4 \hbar^2 + q_2^4 \hbar^4)}{\beta_2 (q_1^2 - \beta_2)} \\
& + \frac{\beta_2^2 \hbar^4 - 8\beta_1 \beta_2^4 \hbar^2 - 3\beta_2 q_2^2 \hbar^4 + 24\beta_1 \beta_2^3 q_2^2 \hbar^2 + 68\beta_1 \beta_2^2 q_2^4 \hbar^2 - 32\beta_1^2 \beta_2^4 q_2^4 - 8q_2^4 \hbar^4}{4\beta_2 q_1^2} \\
& + \frac{-16\beta_1 \beta_2^2 q_2^4 \hbar^2 + 64\beta_1^2 \beta_2^4 q_2^4 + q_2^4 \hbar^4}{16q_1^4} + \frac{4\beta_3^2 q_1^4}{q_2^4}
\end{aligned}$$

$$\begin{aligned}
X^{(2)} = & \{p_1^2, \{m_{12}, m_{12}\}\} + \left\{ p_1^2, \frac{16\beta_2 q_2^2 \hbar^2}{(q_1^2 - \beta_2)^2} + \frac{1}{2} (\hbar^2 - 8\beta_1 \beta_2 q_2^2) + \frac{8q_2^2 \hbar^2}{q_1^2 - \beta_2} \right. \\
& + \left. \frac{8\beta_1 \beta_2^2 q_2^2 - q_2^2 \hbar^2}{2q_1^2} + \frac{2q_1^2 (\beta_3 + \beta_1 q_2^4)}{q_2^2} \right\} + \left\{ p_1 p_2, -\frac{24\beta_2 q_2 q_1 \hbar^2}{(q_1^2 - \beta_2)^2} \right. \\
& - \left. \frac{8q_2 q_1 \hbar^2}{q_1^2 - \beta_2} + \frac{q_2 \hbar^2 - 8\beta_1 \beta_2^2 q_2}{2q_1} - 4\beta_1 q_2 q_1^3 + 8\beta_1 \beta_2 q_2 q_1 \right\} \\
& + \left\{ p_2^2, \frac{8\beta_2^2 \hbar^2}{(q_1^2 - \beta_2)^2} + \frac{8\beta_2 \hbar^2}{q_1^2 - \beta_2} + 2\beta_1 q_1^4 - 4\beta_1 \beta_2 q_1^2 + \frac{\hbar^2}{2} \right\} \\
& + \frac{64\beta_2^2 q_2^2 \hbar^4}{(q_1^2 - \beta_2)^4} + \frac{8(3\beta_2^2 \hbar^4 + 8\beta_2 q_2^2 \hbar^4)}{(q_1^2 - \beta_2)^3} + \frac{\hbar^2 (\beta_3 + \beta_1 q_2^4 + 2\beta_1 \beta_2 q_2^2)}{q_2^2} \\
& - \frac{\beta_1 q_1^2 (8\beta_2 \beta_3 + 3q_2^2 \hbar^2)}{q_2^2} + \frac{-\beta_2 \hbar^4 + 8\beta_1 \beta_2^3 \hbar^2 + 136\beta_1 \beta_2^2 q_2^2 \hbar^2 - 64\beta_1^2 \beta_2^4 q_2^2 - 16q_2^2 \hbar^4}{8\beta_2 q_1^2} \\
& + \frac{-16\beta_1 \beta_2^2 q_2^2 \hbar^2 + 64\beta_1^2 \beta_2^4 q_2^2 + q_2^2 \hbar^4}{16q_1^4} + \frac{2(8\beta_2^2 \beta_3 \hbar^2 + \beta_2 q_2^2 \hbar^4 + q_2^4 \hbar^4)}{\beta_2 q_2^2 (q_1^2 - \beta_2)} \\
& + \frac{2(8\beta_2^2 \beta_3 \hbar^2 + 11\beta_2 q_2^2 \hbar^4 + 8\beta_1 \beta_2^2 q_2^4 \hbar^2 + 6q_2^4 \hbar^4)}{q_2^2 (q_1^2 - \beta_2)^2} + \frac{4\beta_1 \beta_3 q_1^4}{q_2^2}
\end{aligned}$$

$$(4a) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{q_1^2 + q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} \right]$$

This is a special case of (4) where $\beta_1 = \frac{\hbar^2}{8\beta^2}$, $\beta_2 = \beta$, $\beta_3 = 0$. There are two more third-order integrals

$$\begin{aligned}
X''' = & \{m_{12}, \{m_{12}, m_{12}\} - 3\beta p_2^2\} + \hbar^2 \left\{ p_1, -\frac{6q_2^3}{q_1^2 - \beta} - \frac{12\beta q_2^3}{(q_1^2 - \beta)^2} - \frac{q_2 (8\beta - 3q_2^2)}{4\beta} \right\} \\
& + \hbar^2 \left\{ p_2, \frac{6q_1 q_2^2}{q_1^2 - \beta} + \frac{24\beta q_1 q_2^2}{(q_1^2 - \beta)^2} + \frac{q_1 (8\beta - 3q_2^2)}{4\beta} \right\}
\end{aligned}$$

$$\begin{aligned}
X'''' = & \{p_1^2, m_{12}\} + \hbar^2 \left\{ p_1, -\frac{q_2 q_1^2}{4\beta^2} + \frac{q_2}{\beta} - \frac{6q_2}{q_1^2 - \beta} - \frac{12\beta q_2}{(q_1^2 - \beta)^2} \right\} \\
& + \hbar^2 \left\{ p_2, \frac{q_1^3}{4\beta^2} - \frac{q_1}{\beta} + \frac{2q_1}{q_1^2 - \beta} + \frac{8\beta q_1}{(q_1^2 - \beta)^2} \right\}
\end{aligned}$$

and two more fourth-order integrals $(H_1 - H_2, X''')$, $(H_1 - H_2, X'''')$.

$$(4b) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{q_1^2 + q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} + \frac{1}{q_2^2} \right]$$

This is a special case of (4) where $\beta_1 = \frac{\hbar^2}{8\beta^2}$, $\beta_2 = \beta$, $\beta_3 = \hbar^2$. There is a third-order integral

$$X''' = \{m_{12}, \{m_{12}, m_{12}\} - 3\beta p_2^2\} + \hbar^2 \left\{ p_1, \frac{12\beta^2 - 8\beta q_2^2 + 3q_2^4}{4\beta q_2} - \frac{6q_2^3}{q_1^2 - \beta} - \frac{12\beta q_2^3}{(q_1^2 - \beta)^2} - \frac{3q_1^2}{q_2} \right\} \\ + \hbar^2 \left\{ p_2, -\frac{q_1(36\beta^2 - 8\beta q_2^2 + 3q_2^4)}{4\beta q_2^2} + \frac{6q_2^2 q_1}{q_1^2 - \beta} + \frac{24\beta q_2^2 q_1}{(q_1^2 - \beta)^2} + \frac{3q_1^3}{q_2^2} \right\}$$

and one more fourth-order integral ($H_1 - H_2, X'''$)

$$(4c) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} - \frac{1}{8q_1^2} \right]$$

This is a special case of (4) where $\beta_1 = 0$, $\beta_2 = \beta$, $\beta_3 = 0$. There is a first-order integral p_2 , a third-order integral p_1^3 and two additional fourth-order integrals ($p_2, X^{(1)}, (p_2, (p_2, (p_2, X^{(1)})))$).

$$(4d) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} - \frac{1}{8q_1^2} + \frac{1}{q_2^2} \right]$$

This is a special case of (4) where $\beta_1 = 0$, $\beta_2 = \beta$, $\beta_3 = \hbar^2$. There is a third-order integral

$$X''' = p_2^3 + 3\hbar^2 \left\{ p_2, \frac{1}{q_2^2} \right\}$$

and one more fourth-order integral

$$X'''' = \{p_1, \{m_{12}, \{m_{12}, m_{12}\} - \beta p_2^2\}\} + \hbar^2 \left\{ p_1^2, -\frac{8q_2^3}{q_1^2 - \beta} - \frac{16\beta q_2^3}{(q_1^2 - \beta)^2} \right. \\ \left. - \frac{2q_2^2 - \beta}{q_2} + \frac{q_2^3}{2q_1^2} - \frac{3q_1^2}{q_2} \right\} + \hbar^2 \left\{ p_1 p_2, \frac{12q_2^2 q_1}{q_1^2 - \beta} + \frac{36\beta q_2^2 q_1}{(q_1^2 - \beta)^2} \right. \\ \left. - \frac{q_1(3\beta - 2q_2^2)}{q_2^2} + \frac{3q_1^3}{q_2^2} - \frac{3q_2^2}{4q_1} \right\} + \hbar^2 \left\{ p_2^2, -\frac{16\beta^2 q_2}{(q_1^2 - \beta)^2} \right. \\ \left. - \frac{20\beta q_2}{q_1^2 - \beta} - \frac{\beta q_2}{4q_1^2} - \frac{15q_2}{4} \right\} + \hbar^4 \left[-\frac{64\beta^2 q_2^3}{(q_1^2 - \beta)^4} - \frac{8q_2(9\beta^2 + 8\beta q_2^2)}{(q_1^2 - \beta)^3} \right. \\ \left. - \frac{2(10\beta^2 + 3\beta q_2^2 + q_2^4)}{\beta q_2(q_1^2 - \beta)} - \frac{2(8\beta^2 + 33\beta q_2^2 + 6q_2^4)}{q_2(q_1^2 - \beta)^2} + \frac{-2\beta^2 + 3\beta q_2^2 + 16q_2^4}{8\beta q_1^2 q_2} \right. \\ \left. + \frac{3(2\beta - 7q_2^2)}{4q_2^3} - \frac{q_2^3}{16q_1^4} - \frac{9q_1^2}{2q_2^3} \right]$$

$$(4e) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{9(q_1^2 + q_2^2)}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} + \frac{1}{q_1^2} \right]$$

This is a special case of (4) where $\beta_1 = \frac{9\hbar^2}{8\beta^2}$, $\beta_2 = \beta$, $\beta_3 = 0$. There is one third-order integral

$$X''' = \{p_1^2, m_{12}\} + \hbar^2 q_2 \left\{ p_1, \frac{6}{\beta} - \frac{3}{q_1^2} - \frac{9q_1^2}{4\beta^2} - \frac{6(q_1^2 + \beta)}{(q_1^2 - \beta)^2} \right\} \\ + \hbar^2 p_2 \left[\frac{1}{q_1} - \frac{6q_1}{\beta} + \frac{9q_1^3}{4\beta^2} + \frac{2q_1(q_1^2 + 3\beta)}{(q_1^2 - \beta)^2} \right]$$

and one more fourth-order integral ($H_1 - H_2, X'''$).

$$(5) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1 \left(q_1^2 + q_2^2 + \frac{\beta_2^2}{q_1^2} + \frac{\beta_3^2}{q_2^2} \right) + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} + \frac{2(q_2^2 + \beta_3)}{(q_2^2 - \beta_3)^2} - \frac{1}{8q_2^2} \right]$$

There is one fourth-order integral

$$\begin{aligned}
X = & \{ \{m_{12}, m_{12}\}, \{m_{12}, m_{12}\} - 2\beta_2 p_2^2 - 2\beta_3 p_1^2 \} + \left\{ p_1^2, \frac{8q_2^2 (q_2^2 \hbar^2 - 2\beta_3 \hbar^2)}{q_1^2 - \beta_2} + \frac{16q_2^2 (\beta_2 q_2^2 \hbar^2 - 2\beta_2 \beta_3 \hbar^2)}{(q_1^2 - \beta_2)^2} \right. \\
& + q_2^2 (8\beta_1 \beta_2 \beta_3 - 4\beta_1 \beta_3 q_1^2 + 5\hbar^2) + \frac{8(5\beta_3 q_1^2 \hbar^2 - 2\beta_2 \beta_3 \hbar^2)}{q_2^2 - \beta_3} + \frac{2\beta_3 q_2^2 \hbar^2 + 8\beta_1 \beta_2^2 q_2^4 - 16\beta_1 \beta_2 \beta_3 q_2^2 + q_2^4 (-\hbar^2)}{2q_1^2} \\
& + \left. \frac{1}{2} (-2\beta_2 \hbar^2 - 2\beta_3 \hbar^2 + 8\beta_1 \beta_3^2 q_1^2 + 15q_1^2 \hbar^2) + \frac{16(2\beta_3^2 q_1^2 \hbar^2 - \beta_2 \beta_3^2 \hbar^2)}{(q_2^2 - \beta_3)^2} + \frac{\beta_3 q_1^2 \hbar^2 - 8\beta_1 \beta_3^3 q_1^2}{2q_2^2} - 4\beta_1 \beta_2 q_2^4 \right\} \\
& + \left\{ p_1 p_2, -2q_1 q_2 (8\beta_1 \beta_2 \beta_3 - 4\beta_1 \beta_3 q_1^2 + 5\hbar^2) - \frac{16(q_1 q_2^3 \hbar^2 - \beta_3 q_1 q_2 \hbar^2)}{q_1^2 - \beta_2} - \frac{48(\beta_2 q_1 q_2^3 \hbar^2 - \beta_2 \beta_3 q_1 q_2 \hbar^2)}{(q_1^2 - \beta_2)^2} \right. \\
& - \frac{48(\beta_3 q_1^3 q_2 \hbar^2 - \beta_2 \beta_3 q_1 q_2 \hbar^2)}{(q_2^2 - \beta_3)^2} + \frac{-\beta_3 q_2 \hbar^2 - 8\beta_1 \beta_2^2 q_2^3 + 8\beta_1 \beta_2^2 \beta_3 q_2 + q_2^3 \hbar^2}{q_1} - \frac{16(q_1^3 q_2 \hbar^2 - \beta_2 q_1 q_2 \hbar^2)}{q_2^2 - \beta_3} \\
& + \left. \frac{-\beta_2 q_1 \hbar^2 - 8\beta_1 \beta_3^2 q_1^3 + 8\beta_1 \beta_2 \beta_3^2 q_1 + q_1^3 \hbar^2}{q_2} + 8\beta_1 \beta_2 q_1 q_2^3 \right\} + \{ p_2^2, -\beta_2 \hbar^2 - \beta_3 \hbar^2 \\
& + \frac{1}{2} q_2^2 (8\beta_1 \beta_2^2 - 8\beta_1 \beta_2 q_1^2 + 15\hbar^2) + \frac{8(5\beta_2 q_2^2 \hbar^2 - 2\beta_2 \beta_3 \hbar^2)}{q_1^2 - \beta_2} + \frac{16(\beta_3 q_1^4 \hbar^2 - 2\beta_2 \beta_3 q_1^2 \hbar^2)}{(q_2^2 - \beta_3)^2} \\
& + \frac{16(2\beta_2^2 q_2^2 \hbar^2 - \beta_2^2 \beta_3 \hbar^2)}{(q_1^2 - \beta_2)^2} + \frac{2\beta_2 q_1^2 \hbar^2 + 8\beta_1 \beta_3^2 q_1^4 - 16\beta_1 \beta_2 \beta_3^2 q_1^2 + q_1^4 (-\hbar^2)}{2q_2^2} + \frac{8(q_1^4 \hbar^2 - 2\beta_2 q_1^2 \hbar^2)}{q_2^2 - \beta_3} \\
& + \left. \frac{\beta_2 q_2^2 \hbar^2 - 8\beta_1 \beta_2^2 q_2^2}{2q_1^2} - 4\beta_1 \beta_3 q_1^4 + 8\beta_1 \beta_2 \beta_3 q_1^2 + 5q_1^2 \hbar^2 \right\} + \frac{2475\beta_2 \beta_3^2 \hbar^4}{4(q_1^2 - \beta_2)^3} + q_2^4 \left(\frac{4\beta_1^2 \beta_2^6}{(q_1^2 - \beta_2)^4} \right. \\
& - \frac{8\beta_1^2 \beta_2^5}{(q_1^2 - \beta_2)^3} + \frac{4\beta_1^2 \beta_2^4}{q_1^4} + \frac{12\beta_1^2 \beta_2^4}{(q_1^2 - \beta_2)^2} - \frac{\hbar^2 \beta_1 \beta_2^4}{(q_1^2 - \beta_2)^4} - \frac{16\beta_1^2 \beta_2^3}{q_1^2 - \beta_2} + \frac{2\hbar^2 \beta_1 \beta_2^3}{(q_1^2 - \beta_2)^3} - \frac{\hbar^2 \beta_1 \beta_2^2}{q_1^4} \\
& - \frac{3\hbar^2 \beta_1 \beta_2^2}{(q_1^2 - \beta_2)^2} + \frac{64\hbar^4 \beta_2^2}{(q_1^2 - \beta_2)^4} + \frac{4\hbar^2 \beta_1 \beta_2}{q_1^2 - \beta_2} + \frac{64\hbar^4 \beta_2}{(q_1^2 - \beta_2)^3} + \frac{17\hbar^2 \beta_1 \beta_2 - 8\beta_1^2 \beta_2^3}{q_1^2} + \frac{3(8\beta_1^2 \beta_2^3 - 7\hbar^2 \beta_1 \beta_2)}{q_1^2 - \beta_2} \\
& - \frac{8\beta_1^2 \beta_2^3 - \hbar^2 \beta_1 \beta_2}{q_1^2 - \beta_2} + \frac{6(2\beta_1^2 \beta_2^4 - 3\hbar^2 \beta_1 \beta_2^2)}{(q_1^2 - \beta_2)^2} - \frac{3(8\beta_1^2 \beta_2^4 - 7\hbar^2 \beta_1 \beta_2^2)}{(q_1^2 - \beta_2)^2} + \frac{3(8\beta_1^2 \beta_2^5 - 7\hbar^2 \beta_1 \beta_2^3)}{(q_1^2 - \beta_2)^3} \\
& - \frac{16\beta_1^2 \beta_2^5 - 19\hbar^2 \beta_1 \beta_2^3}{(q_1^2 - \beta_2)^3} - \frac{3(8\beta_1^2 \beta_2^6 - 7\hbar^2 \beta_1 \beta_2^4)}{(q_1^2 - \beta_2)^4} - \frac{2(18\beta_1^2 \beta_2^6 - 19\hbar^2 \beta_1 \beta_2^4)}{(q_1^2 - \beta_2)^4} + \frac{2(28\beta_1^2 \beta_2^6 - 29\hbar^2 \beta_1 \beta_2^4)}{(q_1^2 - \beta_2)^4} \\
& + \frac{12\hbar^4}{(q_1^2 - \beta_2)^2} + \frac{\hbar^4}{16q_1^4} + \frac{2\hbar^4}{(q_1^2 - \beta_2)\beta_2} - \frac{2\hbar^4}{q_1^2 \beta_2} \Big) + \frac{3(107\hbar^4 \beta_2 - 5\hbar^4 \beta_3)}{4(q_1^2 - \beta_2)} + \frac{-\beta_2 \hbar^4 - \beta_3 \hbar^4}{4(q_1^2 - \beta_2)} \\
& + \frac{\beta_2 \hbar^4 + \beta_3 \hbar^4}{4q_1^2} + \frac{3(101\hbar^4 \beta_2^2 - 59\hbar^4 \beta_2 \beta_3)}{4(q_1^2 - \beta_2)^2} + \frac{2(\hbar^2 \beta_1 \beta_2^3 + \hbar^2 \beta_1 \beta_3 \beta_2^2)}{q_1^2 - \beta_2} - \frac{2(\hbar^2 \beta_1 \beta_2^3 + \hbar^2 \beta_1 \beta_3 \beta_2^2)}{q_1^2} \\
& + \frac{3\hbar^4 \beta_2^4 - 29\hbar^4 \beta_2^3 \beta_3}{4(q_1^2 - \beta_2)^4} - \frac{2(\hbar^2 \beta_1 \beta_2^5 + \hbar^2 \beta_1 \beta_3 \beta_2^4)}{(q_1^2 - \beta_2)^3} + \frac{\beta_2^2 \hbar^4 - 72\beta_3^2 \hbar^4 + \beta_2 \beta_3 \hbar^4}{4(q_1^2 - \beta_2)^2} \\
& - \frac{2(-9\beta_3^2 \hbar^4 + \beta_1 \beta_2^4 \hbar^2 + \beta_1 \beta_2^3 \beta_3 \hbar^2)}{(q_1^2 - \beta_2)^2} + 2(2\hbar^4 + \beta_1 \beta_2^2 \hbar^2 + \beta_1 \beta_3^2 \hbar^2 - q_1^2 \beta_1 \beta_2 \hbar^2 + 3q_1^2 \beta_1 \beta_3 \hbar^2 - 2\beta_1 \beta_2 \beta_3 \hbar^2) \\
& + \frac{\beta_2^3 \hbar^4 - 2475\beta_2 \beta_3^2 \hbar^4 - 191\beta_2^2 \beta_3 \hbar^4}{4(q_1^2 - \beta_2)^3} + \frac{2(\hbar^2 \beta_1 \beta_2^5 + \hbar^2 \beta_1 \beta_3 \beta_2^4 - 9\hbar^4 \beta_3^2 \beta_2)}{(q_1^2 - \beta_2)^3} + \frac{-\beta_2^3 \hbar^4 + 72\beta_2 \beta_3^2 \hbar^4 - \beta_2^2 \beta_3 \hbar^4}{4(q_1^2 - \beta_2)^3} \\
& + \frac{16(9q_1^2 \beta_3^2 \hbar^4 - 3\beta_2 \beta_3^2 \hbar^4 + 4q_1^4 \beta_3 \hbar^4 - 8q_1^2 \beta_2 \beta_3 \hbar^4)}{(q_2^2 - \beta_3)^3} - \frac{2(\hbar^2 \beta_1 \beta_2^3 + \hbar^2 \beta_1 \beta_3 \beta_2^2 - 16\hbar^2 \beta_1 \beta_3^2 \beta_2)}{q_1^2 - \beta_2} \\
& + \frac{\beta_2^4 \hbar^4 - 72\beta_2^2 \beta_3^2 \hbar^4 + \beta_2^2 \beta_3 \hbar^4}{4(q_1^2 - \beta_2)^4} - \frac{2(\hbar^2 \beta_1 \beta_2^6 + \hbar^2 \beta_1 \beta_3 \beta_2^5 - 9\hbar^4 \beta_3^2 \beta_2^2)}{(q_1^2 - \beta_2)^4} + \frac{2(\hbar^2 \beta_1 \beta_2^4 + \hbar^2 \beta_1 \beta_3 \beta_2^3 + 16\hbar^2 \beta_1 \beta_3^2 \beta_2^2)}{(q_1^2 - \beta_2)^2}
\end{aligned}$$

$$\begin{aligned}
& - \frac{2(4\hbar^2\beta_1\beta_2^6 + \hbar^2\beta_1\beta_3\beta_2^5 + \hbar^2\beta_1\beta_3^2\beta_2^4)}{(q_1^2 - \beta_2)^4} + \frac{10\hbar^2\beta_1\beta_2^6 + 4\hbar^2\beta_1\beta_3\beta_2^5 - \hbar^4\beta_2^4 + 2\hbar^2\beta_1\beta_3^2\beta_2^4 + 7\hbar^4\beta_3\beta_2^3}{(q_1^2 - \beta_2)^4} \\
& + \frac{q_1^4\hbar^4 - 2q_1^2\beta_2\hbar^4 - 16q_1^4\beta_1\beta_3^2\hbar^2 + 32q_1^2\beta_1\beta_2\beta_3^2\hbar^2 + 64q_1^4\beta_1^2\beta_3^4 - 128q_1^2\beta_1^2\beta_2\beta_3^4}{16q_2^4} \\
& + \frac{-8\hbar^4q_1^8 - 32\beta_1^2\beta_3^4q_1^8 + 68\hbar^2\beta_1\beta_3^2q_1^8 + 128\beta_1^2\beta_2\beta_3^4q_1^6 + 24\hbar^2\beta_1\beta_3^3q_1^6 - 272\hbar^2\beta_1\beta_2\beta_3^2q_1^6 + 32\hbar^4\beta_2q_1^6 - 3\hbar^4\beta_3q_1^6}{4q_2^2(q_1^2 - \beta_2)^2\beta_3} \\
& + \frac{-160\beta_1^2\beta_2^2\beta_3^4q_1^4 - 8\hbar^2\beta_1\beta_3^4q_1^4 - 56\hbar^2\beta_1\beta_2\beta_3^3q_1^4 - 40\hbar^4\beta_2^2q_1^4 + \hbar^4\beta_3^2q_1^4 + 340\hbar^2\beta_1\beta_2^2\beta_3^2q_1^4 + 7\hbar^4\beta_2\beta_3q_1^4 + 64\beta_1^2\beta_2^2\beta_3^4q_1^2}{4q_2^2(q_1^2 - \beta_2)^2\beta_3} \\
& + \frac{-112\hbar^2\beta_1\beta_2\beta_3^4q_1^2 + 16\hbar^4\beta_2^2q_1^2 + 40\hbar^2\beta_1\beta_2^2\beta_3^3q_1^2 - 136\hbar^2\beta_1\beta_2^2\beta_3^2q_1^2 + 14\hbar^4\beta_2\beta_3^2q_1^2 - 5\hbar^4\beta_2^2\beta_3q_1^2 - 8\hbar^2\beta_1\beta_2^2\beta_3^4}{4q_2^2(q_1^2 - \beta_2)^2\beta_3} \\
& + \frac{-8\hbar^2\beta_1\beta_2^3\beta_3^3 + \hbar^4\beta_2^2\beta_3^2 + \hbar^4\beta_2^3\beta_3}{4q_2^2(q_1^2 - \beta_2)^2\beta_3} + q_2^2 \left(-\frac{8\beta_1^2\beta_3\beta_2^6}{(q_1^2 - \beta_2)^4} + \frac{16\beta_1^2\beta_3\beta_2^5}{(q_1^2 - \beta_2)^3} - \frac{8\beta_1^2\beta_3\beta_2^4}{q_1^4} - \frac{24\beta_1^2\beta_3\beta_2^4}{(q_1^2 - \beta_2)^2} + \frac{2\hbar^2\beta_1\beta_3\beta_2^4}{(q_1^2 - \beta_2)^4} \right. \\
& + \frac{32\beta_1^2\beta_3\beta_2^3}{q_1^2 - \beta_2} - \frac{4\hbar^2\beta_1\beta_3\beta_2^3}{(q_1^2 - \beta_2)^3} + \frac{87\hbar^4\beta_2^3}{4(q_1^2 - \beta_2)^4} + \frac{2\hbar^2\beta_1\beta_3\beta_2^2}{q_1^4} + \frac{6\hbar^2\beta_1\beta_3\beta_2^2}{(q_1^2 - \beta_2)^2} - \frac{265\hbar^4\beta_3\beta_2^2}{2(q_1^2 - \beta_2)^4} - \frac{8\hbar^2\beta_1\beta_3\beta_2}{q_1^2 - \beta_2} + \frac{991\hbar^4\beta_3\beta_2}{4(q_1^2 - \beta_2)^3} \\
& + \frac{531\hbar^4\beta_2}{4(q_1^2 - \beta_2)^2} - \frac{57\hbar^4\beta_3}{2(q_1^2 - \beta_2)^2} - \frac{\hbar^4\beta_3}{8q_1^4} - \frac{3\hbar^4(\beta_2 + 6\beta_3)}{4(q_1^2 - \beta_2)^2} + 2(3\hbar^2\beta_1\beta_2 - \hbar^2\beta_1\beta_3) + \frac{3\hbar^4(\beta_2^2 + 6\beta_3\beta_2)}{4(q_1^2 - \beta_2)^3} \\
& + \frac{3(191\hbar^4\beta_2^2 - 495\hbar^4\beta_2\beta_3)}{4(q_1^2 - \beta_2)^3} - \frac{3\hbar^4(\beta_2^2 + 6\beta_3\beta_2)}{4(q_1^2 - \beta_2)^4} + \frac{2(8\beta_1^2\beta_3\beta_2^3 + 3\hbar^2\beta_1\beta_2^2 - 17\hbar^2\beta_1\beta_3\beta_2)}{q_1^2 - \beta_2} \\
& + \frac{2(8\beta_1^2\beta_3\beta_2^3 + 3\hbar^2\beta_1\beta_2^2 - 17\hbar^2\beta_1\beta_3\beta_2)}{q_1^2} - \frac{6(8\beta_1^2\beta_3\beta_2^3 + \hbar^2\beta_1\beta_2^2 - 7\hbar^2\beta_1\beta_3\beta_2)}{q_1^2 - \beta_2} - \frac{2(12\beta_1^2\beta_3\beta_2^4 + 3\hbar^2\beta_1\beta_2^3 - 2\hbar^2\beta_1\beta_3\beta_2^2)}{(q_1^2 - \beta_2)^2} \\
& + \frac{3(3\beta_3\hbar^4 + 2\beta_1\beta_2^3\hbar^2 - 14\beta_1\beta_2^2\beta_3\hbar^2 + 16\beta_1^2\beta_2^4\beta_3)}{(q_1^2 - \beta_2)^2} + \frac{2(16\beta_1^2\beta_3\beta_2^5 + 3\hbar^2\beta_1\beta_2^4 - 19\hbar^2\beta_1\beta_3\beta_2^3)}{(q_1^2 - \beta_2)^3} \\
& - \frac{3(16\beta_1^2\beta_3\beta_2^5 + 2\hbar^2\beta_1\beta_2^4 - 14\hbar^2\beta_1\beta_3\beta_2^3 + 3\hbar^4\beta_3\beta_2)}{(q_1^2 - \beta_2)^3} + \frac{3(16\beta_1^2\beta_3\beta_2^5 + 2\hbar^2\beta_1\beta_2^5 - 14\hbar^2\beta_1\beta_3\beta_2^4 + 3\hbar^4\beta_3\beta_2^2)}{(q_1^2 - \beta_2)^4} \\
& + \frac{2(36\beta_1^2\beta_3\beta_2^6 + 6\hbar^2\beta_1\beta_2^5 - 37\hbar^2\beta_1\beta_3\beta_2^4)}{(q_1^2 - \beta_2)^4} - \frac{112\beta_1^2\beta_3\beta_2^6 + 18\hbar^2\beta_1\beta_2^5 - 114\hbar^2\beta_1\beta_3\beta_2^4 + 21\hbar^4\beta_2^3}{(q_1^2 - \beta_2)^4} + \frac{12\hbar^4}{q_1^2 - \beta_2} - \frac{3\hbar^4}{4q_1^2} \\
& - \frac{4\hbar^4\beta_3}{(q_1^2 - \beta_2)\beta_2} + \frac{4\hbar^4\beta_3}{q_1^2\beta_2} \Big) + \frac{1}{q_2^2 - \beta_3} \left(\frac{765\beta_2^2\beta_3^3\hbar^4}{(q_1^2 - \beta_2)^4} + \frac{2475\beta_2\beta_3^3\hbar^4}{2(q_1^2 - \beta_2)^3} + \frac{225\beta_3^3\hbar^4}{(q_1^2 - \beta_2)^2} + \frac{319\beta_2^2\beta_3^2\hbar^4}{2(q_1^2 - \beta_2)^4} + \frac{2101\beta_2^2\beta_3^2\hbar^4}{2(q_1^2 - \beta_2)^3} \right. \\
& + \frac{1947\beta_2\beta_3^2\hbar^4}{2(q_1^2 - \beta_2)^2} + \frac{165\beta_3^2\hbar^4}{2(q_1^2 - \beta_2)} - \frac{4\beta_2\beta_3\hbar^4}{q_1^2 - \beta_2} + \frac{4\beta_2\beta_3\hbar^4}{q_1^2} + 16\beta_3\hbar^4 + \frac{3(214\beta_2\beta_3 - 55\beta_3^2)\hbar^4}{2(q_1^2 - \beta_2)} + \frac{\beta_2^2\beta_3(7\beta_2^2 + 9\beta_3^2)\hbar^4}{2(q_1^2 - \beta_2)^4} \\
& - \frac{\beta_2\beta_3(7\beta_2^2 + 9\beta_3^2)\hbar^4}{2(q_1^2 - \beta_2)^3} + \frac{\beta_3(7\beta_2^2 + 9\beta_3^2)\hbar^4}{2(q_1^2 - \beta_2)^2} + \frac{3(202\beta_2^2\beta_3 - 649\beta_2\beta_3^2)\hbar^4}{2(q_1^2 - \beta_2)^2} - \frac{q_1^4\hbar^4}{4\beta_3} - \frac{4q_1^2\beta_2\hbar^4}{\beta_3} - \frac{32\beta_1\beta_2^3\beta_3\hbar^2}{q_1^2} \\
& + \frac{32\beta_1\beta_2^3\beta_3\hbar^2}{q_1^2 - \beta_2} + 32\beta_1\beta_2^2\beta_3\hbar^2 - 20q_1^4\beta_1\beta_3\hbar^2 + 8q_1^2\beta_1\beta_2\beta_3\hbar^2 + 16q_1^4\beta_1^2\beta_3^3 - 32q_1^2\beta_1^2\beta_2\beta_3^3 + \frac{-\hbar^4\beta_2^5 - \hbar^4\beta_3\beta_2^4}{4(q_1^2 - \beta_2)^4} \\
& + \frac{1}{4}(45q_1^2\hbar^4 - 15\beta_2\hbar^4 + 257\beta_3\hbar^4) + \frac{\hbar^4\beta_2^5 + \hbar^4\beta_3\beta_2^4}{4(q_1^2 - \beta_2)^4} + \frac{13\hbar^4\beta_2^4\beta_3 - 15\hbar^4\beta_2^5}{4(q_1^2 - \beta_2)^4} + \frac{9q_1^4\hbar^4 - \beta_3^2\hbar^4 + 3q_1^2\beta_3\hbar^4 - \beta_2\beta_3\hbar^4}{4\beta_3} \\
& + \frac{15\hbar^4\beta_2^5 - 17\hbar^4\beta_3\beta_2^4 - 638\hbar^4\beta_3^2\beta_2^3}{4(q_1^2 - \beta_2)^4} + \frac{\hbar^4\beta_2^2\beta_3 - 63\hbar^4\beta_3^3}{2(q_1^2 - \beta_2)^2} + \frac{27\hbar^4\beta_3^3 - 32\hbar^2\beta_1\beta_2^4\beta_3}{(q_1^2 - \beta_2)^2} + \frac{-32\hbar^2\beta_1\beta_3\beta_2^5 + 3\hbar^4\beta_3\beta_2^3 - 990\hbar^4\beta_3^3\beta_2}{(q_1^2 - \beta_2)^3} \\
& + \frac{-495\beta_2\beta_3^3\hbar^4 - 2101\beta_2^2\beta_3^2\hbar^4 + 2\beta_2^3\beta_3\hbar^4}{2(q_1^2 - \beta_2)^3} + \frac{32\hbar^2\beta_1\beta_2^5\beta_3 - 27\hbar^4\beta_2\beta_3^3}{(q_1^2 - \beta_2)^3} + \frac{63\hbar^4\beta_2\beta_3^3 - \hbar^4\beta_2^3\beta_3}{2(q_1^2 - \beta_2)^3} \\
& + \frac{-57\beta_2\beta_3\hbar^4 - 32\beta_1\beta_2\beta_3^3\hbar^2 - 32\beta_1\beta_2^2\beta_3\hbar^2}{q_1^2 - \beta_2} - \frac{4(\hbar^4\beta_2\beta_3 - 8\hbar^2\beta_1\beta_2\beta_3^3)}{q_1^2 - \beta_2} + \frac{\hbar^4\beta_2^4\beta_3 - 63\hbar^4\beta_2^2\beta_3^3}{2(q_1^2 - \beta_2)^4} \\
& + 4(-4\beta_1^2\beta_3^3q_1^4 + \hbar^2\beta_1\beta_3q_1^4 + 8\beta_1^2\beta_2\beta_3^3q_1^2 - 2\hbar^2\beta_1\beta_2\beta_3q_1^2) + \frac{27\hbar^4\beta_2^2\beta_3^3 - 32\hbar^2\beta_1\beta_2^6\beta_3}{(q_1^2 - \beta_2)^4}
\end{aligned}$$

$$\begin{aligned}
& + \frac{-225\beta_3^3\hbar^4 - 111\beta_2^2\beta_3\hbar^4 - 32\beta_1\beta_2^2\beta_3^3\hbar^2 + 32\beta_1\beta_2^4\beta_3\hbar^2}{(q_1^2 - \beta_2)^2} - \frac{4(\hbar^4\beta_2^2\beta_3 - 8\hbar^2\beta_1\beta_2^2\beta_3^3)}{(q_1^2 - \beta_2)^2} \\
& + \frac{160\hbar^2\beta_1\beta_3\beta_2^6 - 2\hbar^2\beta_1\beta_3^2\beta_2^5 - 2\hbar^2\beta_1\beta_3^3\beta_2^4 - 3\hbar^4\beta_3\beta_2^4}{(q_1^2 - \beta_2)^4} + \frac{-128\hbar^2\beta_1\beta_3\beta_2^6 + 2\hbar^2\beta_1\beta_3^2\beta_2^5 + 2\hbar^2\beta_1\beta_3^3\beta_2^4 - 765\hbar^4\beta_3^3\beta_2^2}{(q_1^2 - \beta_2)^4} \\
& + \frac{1}{(q_2^2 - \beta_3)^2} \left(\frac{57}{4}q_1^4\hbar^4 + \frac{9033\beta_2^2\beta_3^4\hbar^4}{4(q_1^2 - \beta_2)^4} + \frac{1485\beta_2\beta_3^4\hbar^4}{(q_1^2 - \beta_2)^3} + \frac{675\beta_3^4\hbar^4}{(q_1^2 - \beta_2)^2} - \frac{4\beta_2\beta_3^2\hbar^4}{q_1^2 - \beta_2} + \frac{4\beta_2\beta_3^3\hbar^4}{q_1^2} - 117\beta_3^2\hbar^4 - 24q_1^2\beta_2\hbar^4 \right. \\
& + 99q_1^2\beta_3\hbar^4 - 33\beta_2\beta_3\hbar^4 + 2\beta_1\beta_3^4\hbar^2 - 6q_1^2\beta_1\beta_3^3\hbar^2 + 2\beta_1\beta_2\beta_3^3\hbar^2 - \frac{32\beta_1\beta_2^3\beta_3^2\hbar^2}{q_1^2} + \frac{32\beta_1\beta_2^3\beta_3^2\hbar^2}{q_1^2 - \beta_2} + 32\beta_1\beta_2^2\beta_3^2\hbar^2 \\
& - 18q_1^4\beta_1\beta_3^2\hbar^2 + 4q_1^2\beta_1\beta_2\beta_3^2\hbar^2 + 12q_1^4\beta_1^2\beta_3^4 - 24q_1^2\beta_1^2\beta_2\beta_3^4 + \frac{1}{4}(-9q_1^4\hbar^4 + \beta_3^2\hbar^4 - 3q_1^2\beta_3\hbar^4 + \beta_2\beta_3\hbar^4) \\
& \left. + \frac{3}{4}(257\beta_3^2\hbar^4 + 45q_1^2\beta_3\hbar^4 - 15\beta_2\beta_3\hbar^4) - \frac{4\beta_3^2(8\hbar^2\beta_1\beta_2\beta_3^2 - \hbar^4\beta_2)}{q_1^2 - \beta_2} - \frac{4\beta_3^2(8\hbar^2\beta_1\beta_2\beta_3^2 - \hbar^4\beta_2^2)}{(q_1^2 - \beta_2)^2} \right. \\
& - \frac{3(79\hbar^4\beta_2\beta_3^2 - 120\hbar^4\beta_3^3)}{2(q_1^2 - \beta_2)} + \frac{9(107\hbar^4\beta_2\beta_3^2 - 40\hbar^4\beta_3^3)}{2(q_1^2 - \beta_2)} - \frac{3(\hbar^4\beta_2^2\beta_3^2 - 1416\hbar^4\beta_2\beta_3^3)}{2(q_1^2 - \beta_2)^2} + \frac{9(101\hbar^4\beta_2^2\beta_3^2 - 472\hbar^4\beta_2\beta_3^3)}{2(q_1^2 - \beta_2)^2} \\
& - \beta_3(24\beta_1^2\beta_3^3q_1^4 - 21\hbar^2\beta_1\beta_3q_1^4 - 48\beta_1^2\beta_2\beta_3^3q_1^2 - 6\hbar^2\beta_1\beta_3^2q_1^2 + 42\hbar^2\beta_1\beta_2\beta_3q_1^2 + 2\hbar^2\beta_1\beta_3^3 + 2\hbar^2\beta_1\beta_2\beta_3^2) \\
& + \frac{4584\hbar^4\beta_2^2\beta_3^3 - 13\hbar^4\beta_2^2\beta_3^2}{2(q_1^2 - \beta_2)^3} + \frac{3(15\hbar^4\beta_3\beta_2^5 - 23\hbar^4\beta_2^3\beta_3^4 - 464\hbar^4\beta_3^3\beta_2^3)}{4(q_1^2 - \beta_2)^4} + \frac{696\beta_2^3\beta_3^3\hbar^4 + 35\beta_2^4\beta_3^2\hbar^4}{2(q_1^2 - \beta_2)^4} + \frac{\hbar^4\beta_2^2\beta_3^2 - 27\hbar^4\beta_3^4}{2(q_1^2 - \beta_2)^2} \\
& + \frac{9\beta_3^4\hbar^4 + 7\beta_2^2\beta_3^3\hbar^4}{2(q_1^2 - \beta_2)^2} + \frac{9\hbar^4\beta_3^4 - 32\hbar^2\beta_1\beta_2^4\beta_3^2}{(q_1^2 - \beta_2)^2} + \frac{-32\hbar^2\beta_1\beta_2^3\beta_3^2 + 9\hbar^4\beta_3^2\beta_3^2 - 2970\hbar^4\beta_3^4\beta_2}{(q_1^2 - \beta_2)^3} + \frac{-9\beta_2\beta_3^4\hbar^4 - 7\beta_2^3\beta_3^3\hbar^4}{2(q_1^2 - \beta_2)^3} \\
& + \frac{32\hbar^2\beta_1\beta_2^5\beta_3^2 - 9\hbar^4\beta_2\beta_3^4}{(q_1^2 - \beta_2)^3} + \frac{27\hbar^4\beta_2\beta_3^4 - \hbar^4\beta_2^3\beta_3^2}{2(q_1^2 - \beta_2)^3} + \frac{3(990\beta_2\beta_3^4\hbar^4 - 1528\beta_2^2\beta_3^3\hbar^4 + \beta_2^3\beta_3^2\hbar^4)}{2(q_1^2 - \beta_2)^3} \\
& + \frac{-171\beta_2\beta_3^2\hbar^4 + 32\beta_1\beta_2\beta_3^4\hbar^2 - 32\beta_1\beta_2^3\beta_3^2\hbar^2}{q_1^2 - \beta_2} - 3(-4\beta_1^2\beta_3^3q_1^4 + \hbar^2\beta_1\beta_2^3q_1^4 + 8\beta_1^2\beta_2\beta_3^4q_1^2 - 2\hbar^2\beta_1\beta_2\beta_3^2q_1^2) \\
& + \frac{\hbar^4\beta_2^4\beta_3^2 - 27\hbar^4\beta_2^2\beta_3^3}{2(q_1^2 - \beta_2)^4} + \frac{7\hbar^4\beta_3^2\beta_2^4 + 9\hbar^4\beta_3^4\beta_2^2}{2(q_1^2 - \beta_2)^4} + \frac{9\hbar^4\beta_2^2\beta_3^4 - 32\hbar^2\beta_1\beta_2^6\beta_3^2}{(q_1^2 - \beta_2)^4} + \frac{-177\hbar^4\beta_3\beta_2^5 - 5\hbar^4\beta_3^2\beta_2^4 + 147\hbar^4\beta_3^4\beta_2^2}{4(q_1^2 - \beta_2)^4} \\
& + \frac{-675\beta_3^4\hbar^4 - 333\beta_2^2\beta_3^2\hbar^4 + 32\beta_1\beta_2^2\beta_3^4\hbar^2 + 32\beta_1\beta_2^4\beta_3^2\hbar^2}{(q_1^2 - \beta_2)^2} + \frac{160\hbar^2\beta_1\beta_2^3\beta_3^6 + 2\hbar^2\beta_1\beta_3^3\beta_2^5 + 2\hbar^2\beta_1\beta_3^4\beta_2^4 - 3\hbar^4\beta_2^3\beta_3^4}{(q_1^2 - \beta_2)^4} \\
& + \frac{-128\hbar^2\beta_1\beta_2^3\beta_3^6 - 2\hbar^2\beta_1\beta_3^3\beta_2^5 + 33\hbar^4\beta_3\beta_2^5 - 2\hbar^2\beta_1\beta_3^4\beta_2^4 - 2295\hbar^4\beta_3^4\beta_2^2}{(q_1^2 - \beta_2)^4} \left. + \frac{1}{(q_2^2 - \beta_3)^4} \left(\frac{765\hbar^4\beta_2^2\beta_3^6}{(q_1^2 - \beta_2)^4} + \frac{2475\hbar^4\beta_2\beta_3^6}{4(q_1^2 - \beta_2)^3} \right) \right. \\
& + \frac{225\hbar^4\beta_3^6}{(q_1^2 - \beta_2)^2} + \frac{2(\hbar^2\beta_1\beta_2^5 + \hbar^2\beta_1\beta_3\beta_2^4)\beta_3^5}{(q_1^2 - \beta_2)^4} - \frac{4(8\hbar^2\beta_1\beta_2\beta_3^2 - \hbar^4\beta_2)\beta_3^4}{q_1^2 - \beta_2} - \frac{4(8\hbar^2\beta_1\beta_2^2\beta_3^2 - \hbar^4\beta_2^2)\beta_3^4}{(q_1^2 - \beta_2)^2} \\
& - \frac{(\hbar^4\beta_2^5 + \hbar^4\beta_3\beta_2^4)\beta_3^3}{4(q_1^2 - \beta_2)^4} - (24\beta_1^2\beta_3^3q_1^4 - 21\hbar^2\beta_1\beta_3q_1^4 - 48\beta_1^2\beta_2\beta_3^3q_1^2 - 6\hbar^2\beta_1\beta_3^2q_1^2 + 42\hbar^2\beta_1\beta_2\beta_3q_1^2 + 2\hbar^2\beta_1\beta_3^3 \\
& + 2\hbar^2\beta_1\beta_2\beta_3^2)\beta_3^3 + \frac{113}{8}\hbar^4q_1^4\beta_3^2 + \frac{35}{8}\hbar^4q_1^2\beta_2\beta_3^2 + \frac{1}{4}(-9q_1^4\hbar^4 + \beta_3^2\hbar^4 - 3q_1^2\beta_3\hbar^4 + \beta_2\beta_3\hbar^4)\beta_3^2 + (4\beta_1^2\beta_3^4q_1^4 - \hbar^2\beta_1\beta_3^2q_1^4 \\
& - 8\beta_1^2\beta_2\beta_3^4q_1^2 + 2\hbar^2\beta_1\beta_2\beta_3^2q_1^2)\beta_3^2 + \frac{3}{8}(49\hbar^4q_1^4\beta_3^2 - 75\hbar^4q_1^2\beta_2\beta_3^2) + \frac{1}{2}(147\beta_3^4\hbar^4 - 177q_1^2\beta_3^3\hbar^4 + 59\beta_2\beta_3^3\hbar^4 - 135q_1^2\beta_2\beta_3^2\hbar^4) \\
& + \frac{1}{4}(257\beta_3^4\hbar^4 + 45q_1^2\beta_3^3\hbar^4 - 15\beta_2\beta_3^3\hbar^4 + 135q_1^4\beta_2^2\hbar^4 - 147q_1^2\beta_2\beta_3^2\hbar^4) + \frac{\hbar^4(161\beta_2\beta_3^4 - 30\beta_3^5)}{2(q_1^2 - \beta_2)} - \frac{3\hbar^4(79\beta_2\beta_3^4 - 10\beta_3^5)}{2(q_1^2 - \beta_2)} \\
& + \frac{\hbar^4(143\beta_2^2\beta_3^4 - 354\beta_2\beta_3^5)}{2(q_1^2 - \beta_2)^2} - \frac{3\hbar^4(\beta_2^2\beta_3^4 - 118\beta_2\beta_3^5)}{2(q_1^2 - \beta_2)^2} - \frac{\hbar^4(13\beta_2^3\beta_3^4 - 382\beta_2^2\beta_3^5)}{2(q_1^2 - \beta_2)^3} + \frac{15\hbar^4\beta_3^3\beta_2^5 - 3\hbar^4\beta_3^4\beta_2^4 - 116\hbar^4\beta_3^5\beta_2^3}{4(q_1^2 - \beta_2)^4} \\
& + \frac{-59\hbar^4\beta_3^3\beta_2^5 + 2\hbar^4\beta_3^4\beta_2^4 + 58\hbar^4\beta_3^5\beta_2^3}{2(q_1^2 - \beta_2)^4} + \frac{\hbar^4(1485\beta_2\beta_3^6 - 764\beta_2^2\beta_3^5 + 2\beta_2^3\beta_3^4)}{4(q_1^2 - \beta_2)^3} + \frac{6(\hbar^4\beta_2^3\beta_3^4 - 165\hbar^4\beta_2\beta_3^6)}{(q_1^2 - \beta_2)^3} \\
& + \frac{2(16\hbar^2\beta_1\beta_2\beta_3^6 + 17\hbar^4\beta_2\beta_3^4)}{q_1^2 - \beta_2} + 2(10q_1^4\beta_1^2\beta_3^6 + \hbar^2\beta_1\beta_3^6 - 20q_1^2\beta_1^2\beta_2\beta_3^6 - 3\hbar^2q_1^2\beta_1\beta_3^5 + \hbar^2\beta_1\beta_2\beta_3^5 - 69\hbar^4\beta_3^4 - 10\hbar^2q_1^4\beta_1\beta_3^4)
\end{aligned}$$

$$+20\hbar^2 q_1^2 \beta_1 \beta_2 \beta_3^4 + 39\hbar^4 q_1^2 \beta_3^3 - 13\hbar^4 \beta_2 \beta_3^3) + \frac{-225\hbar^4 \beta_3^6 + 32\hbar^2 \beta_1 \beta_2^2 \beta_3^6 - 74\hbar^4 \beta_2^2 \beta_3^4}{(q_1^2 - \beta_2)^2}$$

$$+ \frac{-2\hbar^2 \beta_1 \beta_2^4 \beta_3^6 - 765\hbar^4 \beta_2^2 \beta_3^6 - 2\hbar^2 \beta_1 \beta_2^5 \beta_3^5 + 26\hbar^4 \beta_2^5 \beta_3^3}{(q_1^2 - \beta_2)^4}$$

$$(5a) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{q_1^2 + q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} + \frac{2(q_2^2 + \beta)}{(q_2^2 - \beta)^2} \right]$$

This is a special case of (5) where $\beta_1 = \frac{\hbar^2}{8\beta^2}$, $\beta_2 = \beta_3 = \beta$. There is a third-order integral

$$X^{(1)} = \{m_{12}, \{m_{12}, m_{12}\} - 3\beta p_1^2 - 3\beta p_2^2\} + \hbar^2 \left\{ p_1, -\frac{3q_2 q_1^2 (23\beta^2 + 10\beta q_2^2 - q_2^4)}{4\beta (\beta - q_2^2)^2} \right.$$

$$+ \frac{12(3\beta^2 q_2 - \beta q_2^3)}{(q_1^2 - \beta)^2} + \frac{q_2 (52\beta^3 + 67\beta^2 q_2^2 - 26\beta q_2^4 + 3q_2^6)}{4\beta (\beta - q_2^2)^2} + \left. \frac{6(3\beta q_2 - q_2^3)}{q_1^2 - \beta} \right\}$$

$$+ \hbar^2 \left\{ p_2, \frac{3q_2^2 q_1 (23\beta^2 + 10\beta q_1^2 - q_1^4)}{4\beta (\beta - q_1^2)^2} + \frac{138\beta^2 q_1 - 127\beta q_1^3}{4(\beta - q_1^2)^2} - \frac{5(19\beta^2 q_1 - 6\beta q_1^3)}{2(\beta - q_1^2)^2} \right.$$

$$\left. - \frac{12(3\beta^2 q_1 - \beta q_1^3)}{(q_2^2 - \beta)^2} + \frac{8q_1^5}{(q_1^2 - \beta)^2} - \frac{3(2\beta q_1^5 + q_1^7)}{4\beta (\beta - q_1^2)^2} - \frac{6(3\beta q_1 - q_1^3)}{q_2^2 - \beta} \right\}$$

and one more fourth-order integral ($H_1 - H_2, X^{(1)}$).

$$(5b) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{q_1^2 + q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} + \frac{2(q_2^2 - \beta)}{(q_2^2 + \beta)^2} \right]$$

This is a special case of (5) where $\beta_1 = \frac{\hbar^2}{8\beta^2}$, $\beta_2 = \beta$, $\beta_3 = -\beta$. There is a third-order integral

$$X^{(1)} = \{m_{12}, \{m_{12}, m_{12}\} + 3\beta p_1^2 - 3\beta p_2^2\} + \hbar^2 \left\{ p_1, \frac{3q_2 q_1^2 (23\beta^2 - 10\beta q_2^2 - q_2^4)}{4\beta (\beta + q_2^2)^2} - \frac{12(3\beta^2 q_2 + \beta q_2^3)}{(q_1^2 - \beta)^2} \right.$$

$$\left. - \frac{q_2 (68\beta^3 - 35\beta^2 q_2^2 - 10\beta q_2^4 - 3q_2^6)}{4\beta (\beta + q_2^2)^2} - \frac{6(3\beta q_2 + q_2^3)}{q_1^2 - \beta} \right\} + \hbar^2 \left\{ p_2, -\frac{3q_1^3 (7\beta^2 - 10\beta q_2^2 - q_2^4)}{4\beta (\beta + q_2^2)^2} \right.$$

$$\left. + \frac{q_1 (68\beta^3 - 83\beta^2 q_2^2 - 10\beta q_2^4 - 3q_2^6)}{4\beta (\beta + q_2^2)^2} + \frac{6q_1 (\beta + q_2^2)}{q_1^2 - \beta} + \frac{24\beta q_1 (\beta + q_2^2)}{(q_1^2 - \beta)^2} \right\}$$

and one more fourth-order integral ($H_1 - H_2, X^{(1)}$).

$$(6) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{3\hbar^2 q_1 (q_1^3 + 2\beta_1)}{(q_1^3 - \beta_1)^2} + \frac{\beta_2}{q_2^2}$$

There are three fourth-order integrals

$$\begin{aligned}
X^{(1)} = & \{ \{m_{12}, m_{12}\}, \{m_{12}, m_{12}\} \} - 4\beta_1 \{p_2^3, m_{12}\} \\
& + \left\{ p_1^2, \frac{36\beta_1 q_1 q_2^4 \hbar^2}{(q_1^3 - \beta_1)^2} + \frac{12q_1 q_2^4 \hbar^2}{q_1^3 - \beta_1} + 4\beta_2 q_1^2 + 5q_2^2 \hbar^2 \right\} \\
& + \left\{ p_1 p_2, -\frac{108\beta_1 q_2^3 q_1^2 \hbar^2}{(q_1^3 - \beta_1)^2} - \frac{24q_2^3 q_1^2 \hbar^2}{q_1^3 - \beta_1} - \frac{8\beta_2 q_1^3}{q_2} + \frac{8\beta_1 \beta_2}{q_2} - 10q_2 q_1 \hbar^2 \right\} \\
& + \left\{ p_2^2, \frac{108\beta_1^2 q_2^2 \hbar^2}{(q_1^3 - \beta_1)^2} + \frac{108\beta_1 q_2^2 \hbar^2}{q_1^3 - \beta_1} + \frac{4\beta_2 q_1^4}{q_2^2} - \frac{16\beta_1 \beta_2 q_1}{q_2^2} + 5q_1^2 \hbar^2 + 12q_2^2 \hbar^2 \right\} \\
& + \frac{1}{2} \hbar^2 (4\beta_2 + 5\hbar^2) + \frac{324\beta_1^2 q_2^4 q_1^2 \hbar^4}{(q_1^3 - \beta_1)^4} + \frac{54(9\beta_1^2 q_1 q_2^2 \hbar^4 + 4\beta_1 q_1^2 q_2^4 \hbar^4)}{(q_1^3 - \beta_1)^3} + \frac{6\beta_2 q_1^2 \hbar^2}{q_2^2} \\
& + \frac{18(3\beta_1 \hbar^4 + 4\beta_1 \beta_2 \hbar^2 + q_1 q_2^2 \hbar^4)}{q_1^3 - \beta_1} + \frac{18(3\beta_1^2 \hbar^4 + 4\beta_1^2 \beta_2 \hbar^2 + 21\beta_1 q_1 q_2^2 \hbar^4 + 2q_1^2 q_2^4 \hbar^4)}{(q_1^3 - \beta_1)^2} \\
& + \frac{4\beta_2^2 q_1^4}{q_2^4} - \frac{16\beta_1 \beta_2^2 q_1}{q_2^4}
\end{aligned}$$

$$\begin{aligned}
X^{(2)} = & \{p_1^2, \{m_{12}, m_{12}\}\} + \left\{ p_1^2, \frac{36\beta_1 q_1 q_2^2 \hbar^2}{(q_1^3 - \beta_1)^2} + \frac{12q_1 q_2^2 \hbar^2}{q_1^3 - \beta_1} + \frac{2\beta_2 q_1^2}{q_2^2} + \frac{\hbar^2}{2} \right\} \\
& + \left\{ p_1 p_2, -\frac{54\beta_1 q_2 q_1^2 \hbar^2}{(q_1^3 - \beta_1)^2} - \frac{12q_2 q_1^2 \hbar^2}{q_1^3 - \beta_1} \right\} + \left\{ p_2^2, \frac{18\beta_1^2 \hbar^2}{(q_1^3 - \beta_1)^2} + \frac{18\beta_1 \hbar^2}{q_1^3 - \beta_1} + \frac{\hbar^2}{2} \right\} \\
& + \frac{324\beta_1^2 q_2^2 q_1^2 \hbar^4}{(q_1^3 - \beta_1)^4} + \frac{27(3\beta_1^2 q_1 \hbar^4 + 8\beta_1 q_1^2 q_2^2 \hbar^4)}{(q_1^3 - \beta_1)^3} + \frac{\beta_2 \hbar^2}{q_2^2} + \frac{3(12\beta_1 \beta_2 \hbar^2 + q_1 q_2^2 \hbar^4)}{q_2^2 (q_1^3 - \beta_1)} \\
& + \frac{9(4\beta_1^2 \beta_2 \hbar^2 + 7\beta_1 q_1 q_2^2 \hbar^4 + 4q_1^2 q_2^4 \hbar^4)}{q_2^2 (q_1^3 - \beta_1)^2}
\end{aligned}$$

$$\begin{aligned}
X^{(3)} = & \{p_1^2 p_2, m_{12}\} + \left\{ p_1^2, \frac{2\beta_2 q_1}{q_2^2} - \frac{q_2^2 \hbar^2}{2\beta_1} \right\} + \left\{ p_1 p_2, -\frac{27\beta_1 q_1 q_2 \hbar^2}{(q_1^3 - \beta_1)^2} - \frac{9q_1 q_2 \hbar^2}{q_1^3 - \beta_1} + \frac{q_1 q_2 \hbar^2}{\beta_1} \right\} \\
& + \left\{ p_2^2, \frac{18\beta_1 q_1^2 \hbar^2}{(q_1^3 - \beta_1)^2} + \frac{3q_1^2 \hbar^2}{q_1^3 - \beta_1} - \frac{q_1^2 \hbar^2}{2\beta_1} \right\} + \frac{\hbar^4}{4\beta_1} + \frac{81\beta_1^2 \hbar^4}{2(q_1^3 - \beta_1)^3} - \frac{\beta_2 q_1^2 \hbar^2}{\beta_1 q_2^2} \\
& - \frac{9(-9\beta_1 q_2^2 \hbar^4 - 8\beta_1 \beta_2 q_1^2 \hbar^2 + 2q_1 q_2^4 \hbar^4)}{2q_2^2 (q_1^3 - \beta_1)^2} - \frac{3(-3\beta_1 q_2^2 \hbar^4 - 4\beta_1 \beta_2 q_1^2 \hbar^2 + 2q_1 q_2^4 \hbar^4)}{2\beta_1 q_2^2 (q_1^3 - \beta_1)}
\end{aligned}$$

$$(6a) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{3\hbar^2 q_1 (q_1^3 + 2\beta)}{(q_1^3 - \beta)^2}$$

This is a special case of (6) where $\beta_1 = \beta, \beta_2 = 0$. There is a first-order integral p_2 , a third-order integral p_2^3 and three more fourth-order integrals

$$(p_2, X^{(1)}), (p_2, (p_2, (p_2, X^{(1)}))), (p_2, X^{(3)})$$

$$(6b) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{3\hbar^2 q_1 (q_1^3 + 2\beta)}{(q_1^3 - \beta)^2} + \frac{\hbar^2}{q_2^2}$$

This is a special case of (6) where $\beta_1 = \beta, \beta_2 = \hbar^2$. There is a third-order integral

$$X''' = p_2^3 + 3\hbar^2 \left\{ p_2, \frac{1}{q_2} \right\}$$

and one more fourth-order integral

$$\begin{aligned} X'''' = & \{p_1, \{m_{12}, \{m_{12}, m_{12}\}\} - \beta p_2^3\} + \hbar^2 \left\{ p_1^2, -\frac{12q_1q_2^3}{q_1^3 - \beta} - \frac{36\beta q_1q_2^3}{(q_1^3 - \beta)^2} - 2q_2 - \frac{3q_1^2}{q_2} \right\} \\ & + \hbar^2 \left\{ p_1p_2, \frac{18q_2^2q_1^2}{q_1^3 - \beta} + \frac{81\beta q_2^2q_1^2}{(q_1^3 - \beta)^2} - \frac{3\beta}{q_2^2} + \frac{3q_1^3}{q_2^2} + 2q_1 \right\} \\ & + \hbar^2 \left\{ p_2^2, -\frac{54\beta^2q_2}{(q_1^3 - \beta)^2} - \frac{54\beta q_2}{q_1^3 - \beta} - 6q_2 \right\} + \hbar^4 \left(-\frac{324\beta^2q_1^2q_2^3}{(q_1^3 - \beta)^4} - \frac{27(9\beta^2q_1q_2 + 8\beta q_1^2q_2^3)}{(q_1^3 - \beta)^3} \right. \\ & \left. - \frac{9(6\beta^2 + 21\beta q_1q_2^2 + 4q_1^2q_2^4)}{q_2(q_1^3 - \beta)^2} - \frac{9(6\beta + q_1q_2^2)}{q_2(q_1^3 - \beta)} - \frac{15}{2q_2} - \frac{9q_1^2}{2q_2^3} \right) \end{aligned}$$

$$(7) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{3q_1(q_1^3 + 2\beta_1)}{(q_1^3 - \beta_1)^2} + \frac{2(q_2^2 + \beta_2)}{(q_2^2 - \beta_2)^2} - \frac{1}{8q_2^2} \right]$$

There is one fourth-order integral

$$\begin{aligned}
X = & \{ \{m_{12}, m_{12}\}, \{m_{12}, m_{12}\} - 2\beta_2 p_1^2 \} - 4\beta_1 \{p_2^3, m_{12}\} \\
& + \hbar^2 \left\{ p_1^2, -\beta_2 + q_1^2 \left(\frac{\beta_2^5}{2q_2^2 (q_2^2 - \beta_2)^4} - \frac{45\beta_2^2 q_2^4 - 29\beta_2^3 q_2^2}{(q_2^2 - \beta_2)^4} + \frac{-5\beta_2^4 + 21\beta_2 q_2^6 + 15q_2^8}{2(q_2^2 - \beta_2)^4} \right) \right. \\
& + \left. \frac{12q_1 (q_2^4 - 2\beta_2 q_2^2)}{q_1^3 - \beta_1} + \frac{36q_1 (\beta_1 q_2^4 - 2\beta_1 \beta_2 q_2^2)}{(q_1^3 - \beta_1)^2} + 5q_2^2 \right\} \\
& + \hbar^2 \left\{ p_1 p_2, -\frac{q_1^3 (-\beta_2^2 + 34\beta_2 q_2^2 + 15q_2^4)}{q_2 (q_2^2 - \beta_2)^2} - \frac{24q_2 q_1^2 (q_2^2 - \beta_2)}{q_1^3 - \beta_1} - \frac{108q_1^2 (\beta_1 q_2^3 - \beta_1 \beta_2 q_2)}{(q_1^3 - \beta_1)^2} \right. \\
& + \left. \frac{\beta_1 (-\beta_2^2 + 34\beta_2 q_2^2 + 15q_2^4)}{q_2 (q_2^2 - \beta_2)^2} - 10q_2 q_1 \right\} + \hbar^2 \left\{ p_2^2, -\beta_2 + q_1^4 \left(\frac{15q_2^6}{2(q_2^2 - \beta_2)^4} \right. \right. \\
& - \left. \left. \frac{-10\beta_2^3 + 6\beta_2 q_2^4 + 11\beta_2^2 q_2^2}{(q_2^2 - \beta_2)^4} - \frac{\beta_2^4}{2q_2^2 (q_2^2 - \beta_2)^4} \right) + q_1 \left(\frac{30\beta_1 q_2^6}{(q_2^2 - \beta_2)^4} \right. \right. \\
& - \left. \left. \frac{4(10\beta_1 \beta_2^3 + 15\beta_1 q_2^6 - 6\beta_1 \beta_2 q_2^4 - 11\beta_1 \beta_2^2 q_2^2)}{(q_2^2 - \beta_2)^4} + \frac{2\beta_1 \beta_2^4}{q_2^2 (q_2^2 - \beta_2)^4} \right) + \frac{36(3\beta_1 q_2^2 - \beta_1 \beta_2)}{q_1^3 - \beta_1} \right. \\
& + \left. \frac{36(3\beta_1^2 q_2^2 - \beta_1^2 \beta_2)}{(q_1^3 - \beta_1)^2} + 5q_1^2 + 12q_2^2 \right\} + \hbar^4 \left(\frac{\beta_2^5}{4q_2^2 (q_2^2 - \beta_2)^4} \right. \\
& + \left. \frac{54(4q_1^2 \beta_1 q_2^4 + 9q_1 \beta_1^2 q_2^2 - 8q_1^2 \beta_1 \beta_2 q_2^2 - 3q_1 \beta_1^2 \beta_2)}{(q_1^3 - \beta_1)^3} + \frac{324(q_1^2 q_2^4 \beta_1^2 - 2q_1^2 q_2^2 \beta_1^2 \beta_2)}{(q_1^3 - \beta_1)^4} \right. \\
& - \left. \frac{3(133q_2^4 \beta_2^2 - 67q_2^2 \beta_2^3)}{2(q_2^2 - \beta_2)^4} + \frac{3(6q_1 q_2^8 + 63\beta_1 q_2^6 - 14q_1 \beta_2 q_2^6 + 10q_1 \beta_2^2 q_2^4 + 117\beta_1 \beta_2 q_2^4 - 2q_1 \beta_2^3 q_2^2 + 9\beta_1 \beta_2^2 q_2^2 + 3\beta_1 \beta_2^3)}{q_2^2 (q_1^3 - \beta_1) (q_2^2 - \beta_2)^2} \right. \\
& + \left. \frac{9(4q_1^2 q_2^{10} + 42q_1 \beta_1 q_2^8 - 16q_1^2 \beta_2 q_2^8 + 21\beta_1^2 q_2^6 + 20q_1^2 \beta_2^2 q_2^6 - 98q_1 \beta_1 \beta_2 q_2^6 - 8q_1^2 \beta_2^3 q_2^4 + 70q_1 \beta_1 \beta_2^2 q_2^4)}{q_2^2 (q_1^3 - \beta_1)^2 (q_2^2 - \beta_2)^2} \right. \\
& + \left. \frac{9(39\beta_1^2 \beta_2 q_2^4 - 14q_1 \beta_1 \beta_2^3 q_2^2 + 3\beta_1^2 \beta_2^2 q_2^2 + \beta_1^2 \beta_2^3)}{q_2^2 (q_1^3 - \beta_1)^2 (q_2^2 - \beta_2)^2} + \frac{5(5q_2^8 + 73\beta_2 q_2^6 + \beta_2^4)}{4(q_2^2 - \beta_2)^4} \right. \\
& + q_1^4 \left(\frac{225q_2^4}{16(q_2^2 - \beta_2)^4} + \frac{135\beta_2 q_2^2}{4(q_2^2 - \beta_2)^4} + \frac{147\beta_2^2}{8(q_2^2 - \beta_2)^4} - \frac{9\beta_2^3}{4(q_2^2 - \beta_2)^4 q_2^2} + \frac{\beta_2^4}{16(q_2^2 - \beta_2)^4 q_2^4} \right. \\
& + q_1 \left(-\frac{225\beta_1 q_2^4}{4(q_2^2 - \beta_2)^4} + \frac{135\beta_1 \beta_2 q_2^2}{(q_2^2 - \beta_2)^4} + \frac{147\beta_1 \beta_2^2}{2(q_2^2 - \beta_2)^4} - \frac{3(90\beta_1 \beta_2 q_2^2 + 49\beta_1 \beta_2^2)}{(q_2^2 - \beta_2)^4} \right. \\
& + \left. \frac{9\beta_1 \beta_2^3}{(q_2^2 - \beta_2)^4 q_2^2} - \frac{\beta_1 \beta_2^4}{4(q_2^2 - \beta_2)^4 q_2^4} \right) + q_1^2 \left(\frac{45q_2^6}{4(q_2^2 - \beta_2)^4} - \frac{177\beta_2^2 q_2^2}{2(q_2^2 - \beta_2)^4} \right. \\
& \left. + \frac{3(33q_2^4 \beta_2 - 7\beta_2^3)}{(q_2^2 - \beta_2)^4} - \frac{3\beta_2^4}{4(q_2^2 - \beta_2)^4 q_2^2} \right) \Big)
\end{aligned}$$

$$(8) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{3q_1(q_1^3 + 2\beta_1)}{(q_1^3 - \beta_1)^2} + \frac{3q_2(q_2^3 + 2\beta_2)}{(q_2^3 - \beta_2)^2} \right]$$

There is one fourth-order integral

$$\begin{aligned}
X = & \{ \{m_{12}, m_{12}\}, \{m_{12}, m_{12}\} + 5\hbar^2 \} + 4\{\beta_2 p_1^3 - \beta_1 p_2^3, m_{12}\} + \hbar^2 \left\{ p_1^2, \frac{12q_1 q_2^4 (2\beta_1 + q_1^3)}{(q_1^3 - \beta_1)^2} \right. \\
& - \frac{48\beta_2 q_1 q_2 (2\beta_1 + q_1^3)}{(q_1^3 - \beta_1)^2} + \frac{108\beta_2^2 q_1^2}{(q_2^3 - \beta_2)^2} + \frac{108\beta_2 q_1^2}{q_2^3 - \beta_2} + 12q_1^2 \left. \right\} + \hbar^2 \left\{ p_1 p_2, -\frac{12q_2^2 q_1^3 (7\beta_2 + 2q_2^3)}{(q_2^3 - \beta_2)^2} \right. \\
& - \frac{24q_1^2 (q_2^3 - \beta_2)}{q_1^3 - \beta_1} - \frac{108q_1^2 (\beta_1 q_2^3 - \beta_1 \beta_2)}{(q_1^3 - \beta_1)^2} + \frac{12\beta_1 q_2^2 (7\beta_2 + 2q_2^3)}{(q_2^3 - \beta_2)^2} \left. \right\} + \hbar^2 \left\{ p_2^2, \frac{12q_2 q_1^4 (2\beta_2 + q_2^3)}{(q_2^3 - \beta_2)^2} \right. \\
& - \frac{48\beta_1 q_2 q_1 (2\beta_2 + q_2^3)}{(q_2^3 - \beta_2)^2} + \frac{108\beta_1^2 q_2^2}{(q_1^3 - \beta_1)^2} + \frac{108\beta_1 q_2^2}{q_1^3 - \beta_1} + 12q_2^2 \left. \right\} + \hbar^4 \left(\frac{36q_2^2 q_1^4 (2\beta_2 + q_2^3)^2}{(q_2^3 - \beta_2)^4} \right. \\
& + \frac{324q_1^2 (\beta_1^2 q_2^4 - 4\beta_1^2 \beta_2 q_2)}{(q_1^3 - \beta_1)^4} + \frac{18q_2 q_1^2 (7\beta_2^2 + 19\beta_2 q_2^3 + q_2^6)}{(q_2^3 - \beta_2)^3} - \frac{144\beta_1 q_2^2 q_1 (2\beta_2 + q_2^3)^2}{(q_2^3 - \beta_2)^4} \\
& + \frac{54 (4\beta_1 q_1^2 q_2^4 + 9\beta_1^2 q_1 q_2^2 - 16\beta_1 \beta_2 q_2^2 q_2)}{(q_1^3 - \beta_1)^3} + \frac{17\beta_2^2 + 506\beta_2 q_2^3 + 17q_2^6}{2 (q_2^3 - \beta_2)^2} \\
& + \frac{18 (15\beta_1 \beta_2^2 + 15\beta_1 q_2^6 - 2\beta_2 q_1 q_2^5 + 78\beta_1 \beta_2 q_2^3 + \beta_2^2 q_1 q_2^2 + q_1 q_2^8)}{(q_1^3 - \beta_1) (q_2^3 - \beta_2)^2} \\
& \left. + \frac{18 (15\beta_1^2 \beta_2^2 + 21\beta_1 q_1 q_2^8 - 12\beta_2 q_1^2 q_2^7 + 15\beta_1^2 q_2^6 - 42\beta_1 \beta_2 q_1 q_2^5 + 18\beta_2^2 q_1^2 q_2^4 + 78\beta_1^2 \beta_2 q_2^3 + 21\beta_1 \beta_2^2 q_1 q_2^2 - 8\beta_2^3 q_1^2 q_2 + 2q_1^2 q_2^{10})}{(q_1^3 - \beta_1)^2 (q_2^3 - \beta_2)^2} \right)
\end{aligned}$$

$$(9) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta(q_1^2 + 9q_2^2)$$

There is one third-order integral

$$X = \{p_1^2, m_{12}\} + 6\beta q_2 \{p_1, q_1^2\} - \frac{2}{3}\beta q_1^3 p_2$$

and one fourth-order integral $(H_1 - H_2, X)$.

$$(10) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta(q_1^2 + 9q_2^2) + \frac{\hbar^2}{q_1^2}$$

There is one third-order integral

$$X = \{p_1^2, m_{12}\} + 3q_2 \left\{ p_1, 2\beta q_1^2 - \frac{\hbar^2}{q_1^2} \right\} + \left(\frac{\hbar^2}{q_1} - \frac{2\beta q_1^3}{3} \right) p_2$$

and one fourth-order integral $(H_1 - H_2, X)$.

$$(11) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{q_1^2 + 9q_2^2}{8\beta^2} + \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} \right]$$

There is one third-order integral

$$X = \{p_1^2, m_{12}\} + 3\hbar^2 q_2 \left\{ p_1, \frac{q_1^2}{4\beta^2} - \frac{2(q_1^2 + \beta)}{(q_1^2 - \beta)^2} \right\} + \hbar^2 q_1 p_2 \left[\frac{2(q_1^2 + 3\beta)}{(q_1^2 - \beta)^2} - \frac{q_1^2}{12\beta^2} \right]$$

and one fourth-order integral $(H_1 - H_2, X)$.

$$(12) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{2(q_1^2 + \beta_1)}{(q_1^2 - \beta_1)^2} - \frac{1}{8q_1^2} \right] + \beta_2 q_2$$

There are two fourth-order integrals

$$\begin{aligned} X^{(1)} = & \{p_1^2, \{m_{12}, m_{12}\}\} + \left\{ p_1^2, \frac{16\beta_1 q_2^2 \hbar^2}{(q_1^2 - \beta_1)^2} + \frac{1}{2} (2\beta_1 \beta_2 q_2 + \hbar^2) + \frac{8q_2^2 \hbar^2}{q_1^2 - \beta_1} \right. \\ & \left. + \beta_2 (-q_2) q_1^2 - \frac{q_2^2 \hbar^2}{2q_1^2} \right\} + \left\{ p_1 p_2, -\frac{24\beta_1 q_2 q_1 \hbar^2}{(q_1^2 - \beta_1)^2} - \frac{8q_2 q_1 \hbar^2}{q_1^2 - \beta_1} + \beta_2 q_1^3 \right. \\ & \left. - \beta_1 \beta_2 q_1 + \frac{q_2 \hbar^2}{2q_1} \right\} + \frac{\hbar^2 p_2^2 (\beta_1^2 + 14\beta_1 q_1^2 + q_1^4)}{2(q_1^2 - \beta_1)^2} + \frac{64\beta_1^2 q_2^2 \hbar^4}{(q_1^2 - \beta_1)^4} \\ & + \frac{2(11\beta_1 \hbar^4 + 6q_2^2 \hbar^4)}{(q_1^2 - \beta_1)^2} + \frac{8(3\beta_1^2 \hbar^4 + 8\beta_1 q_2^2 \hbar^4)}{(q_1^2 - \beta_1)^3} - \frac{17}{4} \beta_2 q_2 \hbar^2 \\ & + \frac{-\beta_1 \hbar^4 - 2\beta_1^2 \beta_2 q_2 \hbar^2 - 16q_2^2 \hbar^4}{8\beta_1 q_1^2} - \frac{2(-\beta_1 \hbar^4 + 2\beta_1^2 \beta_2 q_2 \hbar^2 - q_2^2 \hbar^4)}{\beta_1 (q_1^2 - \beta_1)} \\ & + \frac{1}{4} \beta_2^2 q_1^4 - \frac{1}{2} \beta_1 \beta_2^2 q_1^2 + \frac{q_2^2 \hbar^4}{16q_1^4} \end{aligned}$$

$$\begin{aligned} X^{(2)} = & \{p_1^3, m_{12}\} + \left\{ p_1^2, -\frac{16\beta_1 q_2 \hbar^2}{(q_1^2 - \beta_1)^2} - \frac{8q_2 \hbar^2}{q_1^2 - \beta_1} + \frac{1}{2} \beta_2 q_1^2 + \frac{q_2 \hbar^2}{2q_1^2} \right\} \\ & + \hbar^2 \left\{ p_1 p_2, \frac{-\beta_1^2 + 34\beta_1 q_1^2 + 15q_1^4}{4q_1 (q_1^2 - \beta_1)^2} \right\} + \frac{\beta_2 \hbar^2}{4} - \frac{64\beta_1^2 q_2 \hbar^4}{(q_1^2 - \beta_1)^4} \\ & - \frac{64\beta_1 q_2 \hbar^4}{(q_1^2 - \beta_1)^3} + \frac{2q_2 \hbar^4}{\beta_1 q_1^2} - \frac{2(q_2 \hbar^4 - 2\beta_1^2 \beta_2 \hbar^2)}{\beta_1 (q_1^2 - \beta_1)} - \frac{4(3q_2 \hbar^4 - \beta_1^2 \beta_2 \hbar^2)}{(q_1^2 - \beta_1)^2} - \frac{q_2 \hbar^4}{16q_1^4} \end{aligned}$$

$$(13) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{3\hbar^2 q_1 (q_1^3 + 2\beta_1)}{(q_1^3 - \beta_1)^2} + \beta_2 q_2$$

There are three fourth-order integrals

$$\begin{aligned} X^{(1)} = & \{p_1^2, \{m_{12}, m_{12}\}\} + \left\{ p_1^2, \frac{36\beta_1 q_1 q_2^2 \hbar^2}{(q_1^3 - \beta_1)^2} + \frac{12q_1 q_2^2 \hbar^2}{q_1^3 - \beta_1} - \beta_2 q_1^2 q_2 + \frac{\hbar^2}{2} \right\} \\ & + \left\{ p_1 p_2, -\beta_1 \beta_2 - \frac{54\beta_1 q_2 q_1^2 \hbar^2}{(q_1^3 - \beta_1)^2} - \frac{12q_2 q_1^2 \hbar^2}{q_1^3 - \beta_1} + \beta_2 q_1^3 \right\} \\ & + \frac{p_2^2 \hbar^2 (\beta_1^2 + 34\beta_1 q_1^3 + q_1^6)}{2(q_1^3 - \beta_1)^2} + \frac{324\beta_1^2 q_1^2 q_2^2 \hbar^4}{(q_1^3 - \beta_1)^4} + \frac{27(3\beta_1^2 q_1 \hbar^4 + 8\beta_1 q_1^2 q_2^2 \hbar^4)}{(q_1^3 - \beta_1)^3} \\ & - \frac{13}{2} \beta_2 q_2 \hbar^2 + \frac{3(q_1 \hbar^4 - 6\beta_1 \beta_2 q_2 \hbar^2)}{q_1^3 - \beta_1} + \frac{9(7\beta_1 q_1 \hbar^4 - 2\beta_1^2 \beta_2 q_2 \hbar^2 + 4q_1^2 q_2^2 \hbar^4)}{(q_1^3 - \beta_1)^2} \\ & + \frac{1}{4} \beta_2^2 q_1^4 - \beta_1 \beta_2^2 q_1 \end{aligned}$$

$$X^{(2)} = \{p_1^3, m_{12}\} + \left\{ p_1^2, -\frac{36\beta_1 q_1 q_2 \hbar^2}{(q_1^3 - \beta_1)^2} - \frac{12q_1 q_2 \hbar^2}{q_1^3 - \beta_1} + \frac{1}{2}\beta_2 q_1^2 \right\} + \hbar^2 \left\{ p_1 p_2, \frac{3q_1^2 (7\beta_1 + 2q_1^3)}{(q_1^3 - \beta_1)^2} \right\} \\ + \frac{\beta_2 \hbar^2}{4} - \frac{324\beta_1^2 q_1^2 q_2 \hbar^4}{(q_1^3 - \beta_1)^4} - \frac{216\beta_1 q_1^2 q_2 \hbar^4}{(q_1^3 - \beta_1)^3} + \frac{9\beta_1 \beta_2 \hbar^2}{q_1^3 - \beta_1} - \frac{9(4q_1^2 q_2 \hbar^4 - \beta_1^2 \beta_2 \hbar^2)}{(q_1^3 - \beta_1)^2}$$

$$X''' = p_1^3 p_2 + \left\{ p_1^2, \frac{q_2 \hbar^2}{\beta_1} + \beta_2 q_1 \right\} + \hbar^2 \left\{ p_1 p_2, \frac{q_1 (17\beta_1^2 + 11\beta_1 q_1^3 - q_1^6)}{\beta_1 (q_1^3 - \beta_1)^2} \right\} - \frac{\beta_2 q_1^2 \hbar^2}{2\beta_1} \\ + \frac{18(\beta_1 \beta_2 q_1^2 \hbar^2 + q_1 q_2 \hbar^4)}{(q_1^3 - \beta_1)^2} + \frac{3(\beta_1 \beta_2 q_1^2 \hbar^2 + 2q_1 q_2 \hbar^4)}{\beta_1 (q_1^3 - \beta_1)}$$

$$(14) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1(q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\beta_3}{q_2^2}$$

There is one fourth-order integral

$$X = \{p_1^2, m_{12}\} + \left\{ p_1^2, \frac{4\beta_2 q_2^2}{q_1^2} + q_1^2 \left(\frac{2\beta_3}{q_2^2} - 4\beta_1 q_2^2 \right) + \frac{\hbar^2}{2} \right\} + \left\{ p_1 p_2, \frac{4q_2 (\beta_1 q_1^4 - \beta_2)}{q_1} \right\} \\ + \frac{\hbar^2 p_2^2}{2} + 3\beta_1 q_1^2 \hbar^2 - 2\beta_1 q_2^2 (4\beta_2 + \hbar^2) + \frac{\beta_3 \hbar^2}{q_2^2} + \frac{\beta_2 \hbar^2}{q_1^2} + 4\beta_1^2 q_2^2 q_1^4 + \frac{4\beta_2^2 q_2^2}{q_1^4}$$

$$(14a) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1(q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

This is a special case of (14) where $\beta_3 = \hbar^2$. There is another fourth-order integral

$$X^{(1)} = \{p_1 p_2^2, m_{12}\} - \left(8\beta_1 q_2^3 + \frac{\hbar^2}{q_2} \right) p_1^2 + \left\{ p_1 p_2, q_2 \left(8\beta_1 q_2^2 + \frac{3\hbar^2}{q_2^2} \right) \right\} + 2 \left(\beta_1 q_1^2 - \frac{\beta_2}{q_1^2} \right) \{p_2^2, q_2\} \\ + 4\beta_1 q_2 \hbar^2 + \frac{2\beta_1 q_1^2 (8\beta_1 q_2^4 + \hbar^2)}{q_2} - \frac{2\beta_2 (8\beta_1 q_2^4 + \hbar^2)}{q_1^2 q_2} - \frac{3\hbar^4}{2q_2^3}$$

$$(15) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1(q_1^2 + 16q_2^2) + \frac{\beta_2}{q_1^2}$$

There is one fourth-order integral

$$X = \{p_1^3, m_{12}\} + 4q_2 \left\{ p_1^2, 3\beta_1 q_1^2 - \frac{\beta_2}{q_1^2} \right\} + 2p_2 \left\{ p_1, \frac{\beta_2}{q_1} - \beta_1 q_1^3 \right\} \\ + 2\beta_1 q_2 (4\beta_2 + 3\hbar^2) - 4\beta_1^2 q_2 q_1^4 - \frac{4\beta_2^2 q_2}{q_1^4}$$

$$(16) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1(9q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2}$$

There is one fourth-order integral

$$X = \{p_1 p_2^2, m_{12}\} + \frac{8}{3}\beta_1 q_2^3 p_1^2 - 24\beta_1 \{p_1 p_2, q_1 q_2^2\} + 2 \left(9\beta_1 q_1^2 - \frac{\beta_2}{q_1^2} \right) \{p_2^2, q_2\} \\ - 12\beta_1 q_2 \hbar^2 - 48\beta_1^2 q_1^2 q_2^3 + \frac{16\beta_1 \beta_2 q_2^3}{3q_1^2}$$

$$(17) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1(9q_1^2 + 4q_2^2) + \frac{\beta_2}{q_1^2} + \frac{\hbar^2}{q_2^2}$$

There is one fourth-order integral

$$\begin{aligned} X = & \{p_1 p_2^2, m_{12}\} + \left(\frac{8}{3} \beta_1 q_2^3 - \frac{\hbar^2}{q_2} \right) p_1^2 + \left\{ p_1 p_2, \frac{3\hbar^2 q_1}{q_2^2} - 24\beta_1 q_1 q_2^2 \right\} \\ & + 2 \left(9\beta_1 q_1^2 - \frac{\beta_2}{q_1^2} \right) \{p_2^2, q_2\} - \frac{3\hbar^2 (8\beta_1 q_2^4 + \hbar^2)}{2q_2^3} - \frac{6\beta_1 q_1^2 (8\beta_1 q_2^4 - 3\hbar^2)}{q_2} \\ & + \frac{2\beta_2 (8\beta_1 q_2^4 - 3\hbar^2)}{3q_1^2 q_2} \end{aligned}$$

$$(18) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1 \left(q_1^2 + 4q_2^2 + \frac{\beta_2^2}{q_1^2} \right) + \frac{\beta_3}{q_2^2} + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} \right] \quad (1)$$

There is one fourth-order integral

$$\begin{aligned} X = & \{p_1^2, \{m_{12}, m_{12}\}\} + \left\{ p_1^2, \frac{16\beta_2 q_2^2 \hbar^2}{(q_1^2 - \beta_2)^2} + \frac{8q_2^2 \hbar^2}{q_1^2 - \beta_2} + \frac{8\beta_1 \beta_2^2 q_2^2 - q_2^2 \hbar^2}{2q_1^2} \right. \\ & \left. - \frac{2q_1^2 (2\beta_1 q_2^4 - \beta_3)}{q_2^2} + \frac{\hbar^2}{2} \right\} + \left\{ p_1 p_2, -\frac{24\beta_2 q_2 q_1 \hbar^2}{(q_1^2 - \beta_2)^2} - \frac{8q_2 q_1 \hbar^2}{q_1^2 - \beta_2} \right. \\ & \left. + \frac{q_2 \hbar^2 - 8\beta_1 \beta_2^2 q_2}{2q_1} + 4\beta_1 q_2 q_1^3 \right\} + \frac{p_2^2 \hbar^2 (\beta_2^2 + 14\beta_2 q_1^2 + q_1^4)}{2(q_1^2 - \beta_2)^2} + \frac{64\beta_2^2 q_2^2 \hbar^4}{(q_1^2 - \beta_2)^4} \\ & + \frac{8(3\beta_2^2 \hbar^4 + 8\beta_2 q_2^2 \hbar^4)}{(q_1^2 - \beta_2)^3} + 3\beta_1 q_1^2 \hbar^2 - \frac{-\beta_3 \hbar^2 + 17\beta_1 q_2^4 \hbar^2 + 8\beta_1^2 \beta_2^2 q_2^4}{q_2^2} \\ & + \frac{-\beta_2 \hbar^4 + 8\beta_1 \beta_2^3 \hbar^2 + 128\beta_1 \beta_2^2 q_2^2 \hbar^2 - 16q_2^2 \hbar^4}{8\beta_2 q_1^2} + \frac{-16\beta_1 \beta_2^2 q_2^2 \hbar^2 + 64\beta_1^2 \beta_2^4 q_2^2 + q_2^2 \hbar^4}{16q_1^4} \\ & + \frac{2(8\beta_2^2 \beta_3 \hbar^2 + 11\beta_2 q_2^2 \hbar^4 + 6q_2^4 \hbar^4)}{q_2^2 (q_1^2 - \beta_2)^2} + \frac{2(8\beta_2^2 \beta_3 \hbar^2 + \beta_2 q_2^2 \hbar^4 - 16\beta_1 \beta_2^2 q_2^4 \hbar^2 + q_2^4 \hbar^4)}{\beta_2 q_2^2 (q_1^2 - \beta_2)} \\ & + 4\beta_1^2 q_1^4 q_2^2 \end{aligned}$$

$$(18a) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1 \left(q_1^2 + 4q_2^2 + \frac{\beta_2^2}{q_1^2} \right) + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} \right]$$

This is a special case of (18) where $\beta_3 = 0$. There is one more fourth-order integral

$$\begin{aligned}
X = & \{p_1^3, m_{12}\} + q_2 \left\{ p_1^2, 4\beta_1\beta_2 - \frac{16\beta_2\hbar^2}{(q_1^2 - \beta_2)^2} - \frac{8\hbar^2}{q_1^2 - \beta_2} + \frac{\hbar^2 - 8\beta_1\beta_2^2}{2q_1^2} \right\} \\
& + \left\{ p_1p_2, \frac{12\beta_2q_1\hbar^2}{(q_1^2 - \beta_2)^2} + \frac{4q_1\hbar^2}{q_1^2 - \beta_2} + \frac{8\beta_1\beta_2^2 - \hbar^2}{4q_1} + 2\beta_1q_1^3 - 4\beta_1\beta_2q_1 \right\} \\
& - \frac{64\beta_2^2q_2\hbar^4}{(q_1^2 - \beta_2)^4} - \frac{64\beta_2q_2\hbar^4}{(q_1^2 - \beta_2)^3} - \frac{12q_2\hbar^4}{(q_1^2 - \beta_2)^2} - \frac{2(q_2\hbar^4 - 8\beta_1\beta_2^2q_2\hbar^2)}{\beta_2(q_1^2 - \beta_2)} \\
& + \frac{16\beta_1\beta_2^2q_2\hbar^2 - 64\beta_1^2\beta_2^4q_2 - q_2\hbar^4}{16q_1^4} + \frac{-17\beta_1\beta_2^2q_2\hbar^2 + 8\beta_1^2\beta_2^4q_2 + 2q_2\hbar^4}{\beta_2q_1^2} \\
& + 4\beta_1^2q_1^4q_2 - 8\beta_1^2\beta_2q_1^2q_2
\end{aligned}$$

$$(19) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1 \left(q_1^2 + 16q_2^2 + \frac{\beta_2^2}{q_1^2} \right) + \hbar^2 \left[\frac{2(q_1^2 + \beta_2)}{(q_1^2 - \beta_2)^2} - \frac{1}{8q_1^2} \right]$$

There is one fourth-order integral

$$\begin{aligned}
X = & \{p_1^3, m_{12}\} + q_2 \left\{ p_1^2, -\frac{16\beta_2\hbar^2}{(q_1^2 - \beta_2)^2} - \frac{8\hbar^2}{q_1^2 - \beta_2} + \frac{\hbar^2 - 8\beta_1\beta_2^2}{2q_1^2} + 12\beta_1q_1^2 \right\} \\
& + \left\{ p_1p_2, \frac{12\beta_2q_1\hbar^2}{(q_1^2 - \beta_2)^2} + \frac{4q_1\hbar^2}{q_1^2 - \beta_2} + \frac{8\beta_1\beta_2^2 - \hbar^2}{4q_1} - 2\beta_1q_1^3 \right\} - \frac{64\beta_2^2q_2\hbar^4}{(q_1^2 - \beta_2)^4} \\
& - \frac{64\beta_2q_2\hbar^4}{(q_1^2 - \beta_2)^3} + \beta_1q_2(8\beta_1\beta_2^2 + 21\hbar^2) - \frac{2(q_2\hbar^4 - 48\beta_1\beta_2^2q_2\hbar^2)}{\beta_2(q_1^2 - \beta_2)} \\
& - \frac{4(3q_2\hbar^4 - 16\beta_1\beta_2^2q_2\hbar^2)}{(q_1^2 - \beta_2)^2} - \frac{2(8\beta_1\beta_2^2q_2\hbar^2 - q_2\hbar^4)}{\beta_2q_1^2} \\
& + \frac{16\beta_1\beta_2^2q_2\hbar^2 - 64\beta_1^2\beta_2^4q_2 - q_2\hbar^4}{16q_1^4} - 4\beta_1^2q_1^4q_2
\end{aligned}$$

$$(20) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{4q_1^2 + q_2^2}{32\beta_1^2} + \frac{2(q_1^2 + \beta_1)}{(q_1^2 - \beta_1)^2} \right] + \frac{\beta_2}{q_2^2}$$

There is one fourth-order integral

$$\begin{aligned}
X = & \{p_1^2p_2, m_{12}\} + \left(\frac{2\beta_2}{q_2^2} - \frac{q_2^2\hbar^2}{16\beta_1^2} \right) \{p_1^2, q_1\} - \hbar^2 \left\{ p_1p_2, \frac{q_2(20\beta_1^3 - 6\beta_1q_1^4 + 33\beta_1^2q_1^2 + q_1^6)}{4\beta_1^2(q_1^2 - \beta_1)^2} \right\} \\
& + \frac{p_2^2q_1\hbar^2(20\beta_1^3 - 6\beta_1q_1^4 + 17\beta_1^2q_1^2 + q_1^6)}{4\beta_1^2(q_1^2 - \beta_1)^2} + \frac{12\beta_1q_1\hbar^4}{(q_1^2 - \beta_1)^3} - \frac{q_1^3\hbar^2(q_2^4\hbar^2 - 32\beta_1^2\beta_2)}{64\beta_1^4q_2^2} \\
& + \frac{q_1\hbar^2(-32\beta_1^2\beta_2 - 2\beta_1q_2^2\hbar^2 + q_2^4\hbar^2)}{16\beta_1^3q_2^2} + \frac{32\beta_1^2\beta_2q_1\hbar^2 - q_1q_2^4\hbar^4}{8\beta_1^2q_2^2(q_1^2 - \beta_1)} \\
& + \frac{6\beta_1q_1q_2^2\hbar^4 + 32\beta_1^2\beta_2q_1\hbar^2 - q_1q_2^4\hbar^4}{2\beta_1q_2^2(q_1^2 - \beta_1)^2}
\end{aligned}$$

$$(21) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{9(4q_1^2 + q_2^2)}{32\beta_1^2} + \frac{2(q_1^2 + \beta_1)}{(q_1^2 - \beta_1)^2} + \frac{1}{q_1^2} \right] + \frac{\beta_2}{q_2^2}$$

There is one fourth-order integral

$$\begin{aligned} X = & \{p_1^2 p_2, m_{12}\} + \left(\frac{2\beta_2}{q_2^2} - \frac{9\hbar^2 q_2^2}{16\beta_1^2} \right) \{p_1^2, q_1\} \\ & - \hbar^2 \left\{ p_1 p_2, \frac{3q_2(4\beta_1^4 - 14\beta_1 q_1^6 + 31\beta_1^2 q_1^4 - 8\beta_1^3 q_1^2 + 3q_1^8)}{4\beta_1^2 q_1^2 (q_1^2 - \beta_1)^2} \right\} \\ & + \frac{p_2^2 \hbar^2 (4\beta_1^4 - 42\beta_1 q_1^6 + 69\beta_1^2 q_1^4 - 8\beta_1^3 q_1^2 + 9q_1^8)}{4\beta_1^2 q_1 (q_1^2 - \beta_1)^2} + \frac{3\hbar^4}{2q_1^3} \\ & + \frac{12\beta_1 q_1 \hbar^4}{(q_1^2 - \beta_1)^3} - \frac{9q_1^3 \hbar^2 (9q_2^4 \hbar^2 - 32\beta_1^2 \beta_2)}{64\beta_1^4 q_2^2} + \frac{3q_1 \hbar^2 (-32\beta_1^2 \beta_2 - 3\beta_1 q_2^2 \hbar^2 + 9q_2^4 \hbar^2)}{8\beta_1^3 q_2^2} \\ & + \frac{32\beta_1^2 \beta_2 \hbar^2 - 9q_2^4 \hbar^4}{16\beta_1^2 q_1 q_2^2} + \frac{32\beta_1^2 \beta_2 q_1 \hbar^2 - 9q_1 q_2^4 \hbar^4}{8\beta_1^2 q_2^2 (q_1^2 - \beta_1)} + \frac{6\beta_1 q_1 q_2^2 \hbar^4 + 32\beta_1^2 \beta_2 q_1 \hbar^2 - 9q_1 q_2^4 \hbar^4}{2\beta_1 q_2^2 (q_1^2 - \beta_1)^2} \end{aligned}$$

$$(22) \quad H = \frac{1}{2}(p_1^2 + p_2^2) + \hbar^2 \left[\frac{4q_1^2 + 9q_2^2}{32\beta_1^2} + \frac{2(q_1^2 + \beta_1)}{(q_1^2 - \beta_1)^2} \right] + \frac{\beta_2}{q_2^2}$$

There is one fourth-order integral

$$\begin{aligned} X = & \{p_1^2 p_2, m_{12}\} + \left(\frac{2\beta_2}{q_2^2} - \frac{9\hbar^2 q_2^2}{16\beta_1^2} \right) \{p_1^2, q_1\} + \hbar^2 \left\{ p_1 p_2, \frac{3q_2(-8\beta_1^3 - 2\beta_1 q_1^4 - 7\beta_1^2 q_1^2 + q_1^6)}{4\beta_1^2 (q_1^2 - \beta_1)^2} \right\} \\ & + \frac{p_2^2 q_1 \hbar^2 (72\beta_1^3 + 2\beta_1 q_1^4 + 23\beta_1^2 q_1^2 - q_1^6)}{12\beta_1^2 (q_1^2 - \beta_1)^2} + \frac{12\beta_1 q_1 \hbar^4}{(q_1^2 - \beta_1)^3} + \frac{3q_1 \hbar^4}{8\beta_1^2} + \frac{q_1^3 \hbar^2 (9q_2^4 \hbar^2 - 32\beta_1^2 \beta_2)}{192\beta_1^4 q_2^2} \\ & - \frac{q_1 (-32\beta_1^2 \beta_2 \hbar^2 - 6\beta_1 q_2^2 \hbar^4 + 9q_2^4 \hbar^4)}{2\beta_1 q_2^2 (q_1^2 - \beta_1)^2} + \frac{32\beta_1^2 \beta_2 q_1 \hbar^2 - 9q_1 q_2^4 \hbar^4}{8\beta_1^2 q_2^2 (q_1^2 - \beta_1)} \end{aligned}$$

I wouldn't say when you've seen one Western you've seen the lot; but when you've seen the lot you get the feeling you've seen one.

Katharine Whitehorn,
Sunday Best (1976) 'Decoding the West'