A charged hadronic string model within the R.P. Feynman proper time paradigm and vacuum field theory approach

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Abstract. We study a novel charged hadronic string model within the least action principle and the vacuum field theory approach based on the classical R.P. Feynman's proper time paradigm. It is stated that the hadronic string model allows the conformal local coordinates, with respect to which the resulting string dynamics is described by means of the linear second order elliptic equation under the Dirac type constraint, suitable for the canonical quantization and demonstrating a related gauge type invariance of the model. The related Lagrangian and Hamiltonian aspects of the string model, interacting with ambient both electrical and electromagnetic potential fields, are analyzed in detail.

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1 Introduction: the classical least action principle

Bosonic string theory, which is the most basic form of string theory, describes the propagation of one-dimensional relativistic extended objects, the fundamental strings, and their interactions by joining and splitting. Quantum field theories of point particles are naturally obtained by starting with a classical action, and quantizing the fluctuations around a given classical solution of the equations of motion. An analogous string field theory exists, but is still poorly understood. In such a theory one should have operators creating a loop in space, which is certainly more difficult to describe mathematically. Rather the practical way to handle string theory is to follow the propagation in time of a single string in a fixed reference space-time. The classical least action principle is considered in modern physics [67] as a fundamental tool for deriving true and physically sound equations governing the dynamics of the corresponding physical objects. By means of this principle there has been described many physical models [54, 37, 38, 49] including those of classical mechanics, electrodynamics and Einsteinian gravity theory. As it was mentioned in many classical manuals [10, 11, 31, 49, 53, 54], a suitable and physically motivated method of choosing the corresponding Lagrangian functions proves nowadays to be open for studying. Application of the least action principle is strongly complicated by inconsistencies often accompanying the derived physical statements which are considered to be well understood and checked by means of other physical theories. In particular, in modern electrodynamics of a charged point particle moving under influence of an external electromagnetic field, there is well known misreading [54, 38, 49] related to the charged point particle energy expression. Namely, the latter being obtained by means of the classical least action principle, gives rise to the charged particle "*dynamical*" mass expression not depending on the external potential energy. This fact was discussed also in the physics literature, for instance in [24], where there are also described other physically reasonable examples. Taking this into account and being motivated by R.P. Feynman's considerations of the problem in [38, 34, 35, 50] as well as a recently devised vacuum field theory approach [12, 13, 15, 21, 16, 17, 18, 20] to a physically reasonable formulation of the corresponding least action principle for describing the charged point particle electrodynamics, we have revisited in [14] the approach, based on the Feynman proper time paradigm [38, 34, 35, 50] and applied it to describing the dynamics of a charged point particle, having stated its complete physical adequacy. Based on this experience, the devised vacuum field theory approach is applied to describing space-time dynamics of a charged hadronic string model under influence of an external vacuum field potential. We analyze also in detail the related Lagrangian and Hamiltonian string model description, in particular, we state that with respect to some conformal local coordinates the resulting space-time dynamics is described by means of a linear second order elliptic equation under the corresponding Dirac type constraints, well fitting [29, 30] for its quantization.A novel charged hadronic string model: the least action principle analysis within the R.P. Feynman proper time paradigm

A classical relativistic hadronic string model was first proposed in [6, 59, 42] and studied in [5], making use of the least action principle and related Lagrangian and Hamiltonian formalisms. We will not discuss here this classical string model approach and will not comment the physical problems accompanying it, especially those related to its diverse quantization versions, but proceed to formulating a novel relativistic hadronic string model within the vacuum field theory approach, devised in [16, 17, 13]. For a uniformly charged string, interacting with an ambient electric potential field, the classical least action principle is, following [5, 17, 20], formulated as

$$\delta S^{(\tau)} = 0, \quad S^{(\tau)} := -\int_{s(\tau_1)}^{s(\tau_2)} \int_{\sigma_1(s)}^{\sigma_2(s)} q\bar{W}(x(\xi))d\Sigma^{(2)}(\xi). \tag{1}$$

Here $\bar{W} : M^4 \to \mathbb{R}$ is a vacuum field potential function, $q \in \mathbb{R}\backslash\{0\}$ is elementary charge parameter, characterizing the interaction of the vacuum medium with our charged hadronic string object, and the differential 2-form $d\Sigma^{(2)} := (|\dot{\xi}|^2|\xi'|^2 - \langle\dot{\xi}|\xi'\rangle^2_{\mathbb{E}^4})^{1/2}d\sigma \wedge ds = \sqrt[2]{g(\xi)}d\sigma \wedge ds$, $g(\xi) := \det(g_{ij}(\xi)|_{i,j=\overline{1,2}})$, $|\dot{\xi}|^2 := \langle\dot{\xi}|\dot{\xi}\rangle_{\mathbb{E}^4}$, $|\xi'|^2 := \langle\xi'|\xi'\rangle_{\mathbb{E}^4}$, where $\dot{\xi} := \partial\xi/\partial s$, $\xi' := \partial\xi/\partial\sigma$ denote the corresponding partial derivatives, being related with the induced positive definite Riemannian infinitesimal metrics $dz^2 := \langle d\xi|d\xi\rangle_{\mathbb{E}^4} = g_{11}(\xi)d\sigma^2 + g_{12}(\xi)d\sigma ds + g_{21}(\xi)dsd\sigma + g_{22}(\xi)ds^2$ on the string, meaning [1, 5, 32, 69] the infinitesimal two-dimensional surface element, parameterized by local variables $(s,\sigma) \in \mathbb{R}^2_\xi$ and embedded into the four-dimensional Euclidean space-time $\mathbb{R} \times \mathbb{E}^3$ with coordinates $\xi := (\tau(s,\sigma), r(s,\sigma)) \in \mathbb{R} \times \mathbb{E}^3$ subject to the proper time reference frame $\mathcal{K}_r$. The related boundary conditions are chosen as

$$\delta\xi(s, \sigma(s)) = 0 \tag{2}$$

at string parameter $\sigma(s) \in \mathbb{R}$ for all $s \in \mathbb{R}$. The action functional expression (1) is strongly motivated by the extended string action functional

$$S^{(\tau)} := -q\int_{\sigma_1}^{\sigma_2} dl(\sigma)\int_{t(\sigma,\tau_1)}^{t(\sigma,\tau_2)} \bar{W}dt(\tau,\sigma), \tag{3}$$

where the laboratory reference time parameter $t(\tau,\sigma) \in \mathbb{R}$ is related to the proper time string reference frame parameter $\tau \in \mathbb{R}$ by means of the standard Euclidean infinitesimal relationship

$$dt(\tau,\sigma) := (1 + |\dot{r}_\perp|^2(\tau,\sigma))^{1/2}d\tau, \quad |\dot{r}_\perp|^2 := \langle\dot{r}_\perp|\dot{r}_\perp\rangle_{\mathbb{E}^3}, \tag{4}$$

with $\sigma \in [\sigma_1, \sigma_2] \subset \mathbb{R}$, being a spatial variable, parameterizing the string length measure $dl(\sigma)$ on the real axis $\mathbb{R}$, $\dot{r}_\perp(\tau,\sigma) := \hat{N}\,\dot{r}(\tau,\sigma) \in \mathbb{E}^3$, being orthogonal to the string velocity component, where, by definition,

$$\hat{N} := (1 - |r'|^{-2} r' \otimes r'), \quad |r'|^{-2} := \langle r'|r'\rangle_{\mathbb{E}^3}^{-1}, \tag{5}$$

being the corresponding orthogonal projector operator on $\mathbb{E}^3$ to the string direction, expressed, for brevity, by means of the standard tensor product " $\otimes$ " on the Euclidean space $\mathbb{E}^3$. The result of recalculating the expression (3) gives rise to the following functional expression

$$S^{(\tau)} = -\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} q\bar{W}(|r'|^2(1+|\dot{r}|^2) - \langle \dot{r}|r'\rangle_{\mathbb{E}^3}^2)^{1/2} d\ \sigma, \tag{6}$$

where we made use of the infinitesimal measure representation $dl(\sigma) = \langle r'|r'\rangle_{\mathbb{E}^3}^{1/2} d\sigma$, $\sigma \in [\sigma_1, \sigma_2]$. If we introduce on the string world surface local coordinates $(s(\tau,\sigma), \sigma) \in \mathbb{E}^2$ and the related Euclidean string position vector $\xi := (\tau, r(s,\sigma)) \in \mathbb{R} \times \mathbb{E}^3$, the string action functional (6) reduces equivalently to that of (1).

2 The Lagrangian analysis within the R.P. Feynman proper time paradigm

First obtain the Euler equations corresponding to (6) with respect to the special [5, 32] internal conformal variables $(s,\sigma) \in \mathbb{R}^2$ on the world space-time string surface $\Sigma^{(2)}$, with respect to which the metrics on it becomes equal to $dz^2 = |\xi'|^2 d\sigma^2 + |\dot{\xi}|^2 ds^2$, where $\langle \xi'|\dot{\xi}\rangle_{\mathbb{E}^4} = 0 = |\xi'|^2 - |\dot{\xi}|^2$ is the imposed space-time constraint, and the corresponding infinitesimal world surface measure $d\Sigma^{(2)}$ becomes $d\Sigma^{(2)} = |\xi'||\dot{\xi}| d\sigma \wedge ds$. As a result of simple calculations one obtains the linear second order partial differential equation

$$\partial(\bar{W}\dot{\xi})/\partial s + \partial(\bar{W}\xi')/\partial\sigma = \partial(|\xi'|\ |\dot{\xi}|\bar{W})/\partial\sigma \tag{7}$$

under the suitably chosen boundary conditions

$$\xi' - \dot{\xi}\ \dot{\sigma} = 0 \tag{8}$$

for all $s \in \mathbb{R}$. It is interesting to mention that equation (7) is of *elliptic type*, contrary to the case considered before in [5]. This is, evidently, owing to the fact that the resulting metrics on the string world surface proves to be Euclidean, as we took into account that the real space-time string motion is, in reality, realized with respect to its proper time reference frame $\mathcal{K}_r$, not dependent on the string motion observation data, measured with respect to any external laboratory reference frame $\mathcal{K}$. The latter can be used for physically motivated evidence of the dynamical stability of the relativistic charged string object, modeling a charged hadronic particle [4, 43, 59, 74] with non-trivial internal structure.

The differential equation (7) strongly depends on the vacuum field potential function $\bar{W} : M^4 \to \mathbb{R}$, which, as a function of the Minkowski four-vector variable $x := (t(s,\sigma), r) \in M^4$ of the laboratory reference frame $\mathcal{K}$, should be expressed as that of the variables $(s,\sigma) \in \mathbb{E}^2$, making use of the infinitesimal relationship (4) in the following form:

$$dt = \langle \hat{N}\partial\xi/\partial\tau|\hat{N}\partial\xi/\partial\tau\rangle_{\mathbb{E}^3}^{1/2}(\frac{\partial\tau}{\partial s}ds + \frac{\partial\tau}{\partial\sigma}d\sigma), \tag{9}$$

defined on the string world surface $\Sigma^{(2)}$. The function $\bar{W} : M^4 \to \mathbb{R}$ itself should be simultaneously found, following ideas of [24, 71] and recent results of [12, 13, 15, 16, 17, 18], by means of a suitable solution to the Maxwell equation $\partial^2 W/\partial t^2 - \Delta W = \rho$, where $\rho \in \mathbb{R}$ is an ambient charge density and, by definition, $\bar{W}(r(t)) := \lim_{r \to r(t)} W(r,t)|$, with $r(t) \in \mathbb{E}^3$ being the position of the string element with a proper time reference frame $\mathcal{K}_r$ coordinates $(\tau,\sigma) \in \mathbb{E}^2$ at the time moment $t = t(\tau,\sigma) \in \mathbb{R}$.

3 The Hamiltonian analysis within the R.P. Feynman proper time paradigm

We proceed now to constructing the Hamiltonian equations for our string model, making use of the general action functional (6) in the following form:

$$S^{(\tau)} = -\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} q\bar{W}|r'|(1+|\dot{r}|^2 - |r'|^{-2}\langle r'|\dot{r}\rangle_{\mathbb{E}^3}^2)^{1/2} d\ \sigma. \tag{10}$$

It is easy to calculate that the generalized momentum density

$$p \quad : \quad = \partial\mathcal{L}^{(\tau)}/\partial\dot{r} = \frac{-q\bar{W}|r'|(\dot{r} - r'|r'|^{-2}\langle r'|\dot{r}\rangle_{\mathbb{E}^3})}{(|\dot{r}|^2 + 1 - |r'|^{-2}\langle r'|\dot{r}\rangle^2_{\mathbb{E}^3})^{1/2}} =$$
$$= \quad \frac{-q\bar{W}|r'|\hat{N}dr/d\tau}{(|\dot{r}|^2 + 1 - |r'|^{-2}\langle r'|\dot{r}\rangle^2_{\mathbb{E}^3})^{1/2}} = -q|r'|\bar{W}\hat{N}dr/dt \; = |r'|\hat{N}\;\left(-q\bar{W}u\right) \tag{11}$$

satisfies the dynamical equation

$$\begin{aligned} dp/d\tau \quad : \quad & = \delta\mathcal{L}^{(\tau)}/\delta r = -q(|r'|^2(|\dot{r}|^2+1) - \langle r'|\dot{r}\rangle^2_{\mathbb{E}^3})^{1/2}\;\nabla\bar{W} + \\ & + \frac{\partial}{\partial\sigma}\left\{\frac{q\bar{W}(1+|\dot{r}|^2\hat{T})r'}{(1+|r'|^{-2}|\dot{r}|^2\langle\hat{T}r'|r'\rangle_{\mathbb{E}^3})^{1/2}}\right\}, \end{aligned} \tag{12}$$

where we denoted by

$$\mathcal{L}^{(\tau)} := -q\bar{W}(|r'|^2(1+|\dot{r}|^2) - \langle\dot{r}|r'\rangle^2_{\mathbb{E}^3})^{1/2} = -q\bar{W}(|r'|^2 + |\dot{r}|^2\langle r'|\hat{T}_{\dot{r}}r'\rangle_{\mathbb{E}^3})^{1/2} \tag{13}$$

the corresponding Lagrangian function, and for any vector $w \in \mathbb{E}^3$

$$\hat{T}_w := 1 - |w|^{-2}\; w\otimes w, \qquad |w|^2 := \langle w|w\rangle^2_{\mathbb{E}^3}, \tag{14}$$

the usual projector operator on $\mathbb{E}^3$. As a result of (12) one finds that

$$\begin{aligned} dp/dt = -|r'|\;\nabla\left(q\bar{W}\right) + (1-|u|^2)\langle u|r'\rangle^2_{\mathbb{E}^3})^{-1/2}\times \\ \times\frac{\partial}{\partial\sigma}\left\{\frac{q\bar{W}(1-|u|^2+|r'|^{-2}\langle u|r'\rangle^2_{\mathbb{E}^3}+|u|^2\hat{T}_u)r'}{(1-|u|^2+|r'|^{-2}\langle u|r'\rangle^2_{\mathbb{E}^3})^{1/2}}\right\}, \end{aligned} \tag{15}$$

where we took into account that owing to (4)

$$\dot{r} = dr/d\tau = dr/dt\;\cdot dt/d\tau \; = u(1-|u|^2 + |r'|^{-2}\langle u|r'\rangle^2_{\mathbb{E}^3})^{-1/2}. \tag{16}$$

The Lagrangian function is degenerate [1, 3, 5, 20, 32], satisfying the constraint

$$\langle p|r'\rangle_{\mathbb{E}^3} = 0 \tag{17}$$

for all $\tau \in \mathbb{R}$, meaning that the physical momentum vector $p \in \mathbb{E}^3$ is locally orthogonal to the string $\Sigma^{(2)}$ spatial surface. Concerning the Hamiltonian formulation of the dynamics (12), we construct the corresponding Hamiltonian functional density as

$$\begin{aligned} \mathcal{H} := \textstyle\int_{\sigma_1}^{\sigma_2}(\langle p|\dot{r}\rangle_{\mathbb{E}^3} - \mathcal{L}^{(\tau)})d\sigma = \\ = \textstyle\int_{\sigma_1}^{\sigma_2}\left(\frac{-q\bar{W}|r'|(|\dot{r}|^2 - \;|r'|^{-2}\langle r'|\dot{r}\rangle^2_{\mathbb{E}^3})}{(|\dot{r}|^2+1-|r'|^{-2}\langle r'|\dot{r}\rangle^2_{\mathbb{E}^3})^{1/2}} + \frac{q\bar{W}|r'|(|\dot{r}|^2+1-|r'|^{-2}\langle r'|\dot{r}\rangle^2_{\mathbb{E}^3})}{(|\dot{r}|^2+1-|r'|^{-2}\langle r'|\dot{r}\rangle^2_{\mathbb{E}^3})^{1/2}}\right)d\sigma = \\ = \textstyle\int_{\sigma_1}^{\sigma_2}\left(\frac{q\bar{W}|r'|}{(|\dot{r}|^2+1-|r'|^{-2}\langle r'|\dot{r}\rangle^2_{\mathbb{E}^3})^{1/2}}\right)d\sigma = \;\int_{\sigma_1}^{\sigma_2}(-q^2\bar{W}^2|r'|^2 + |p|^2)^{1/2}d\sigma, \end{aligned} \tag{18}$$

satisfying the canonical Hamiltonian equations

$$dr/d\tau \; = \delta\mathcal{H}/\delta p, \quad dp/d\tau \; = -\delta\mathcal{H}/\delta r, \tag{19}$$

where

$$d\mathcal{H}/d\tau = 0, \tag{20}$$

holding only with respect to the proper time reference frame $\mathcal{K}_r$ time parameter $\tau \in$ R.P. Now making use of identity (17) the Hamiltonian functional (18) can be equivalently represented [5] in the symbolic form as

$$\mathcal{H} = \int_{\sigma_1}^{\sigma_2}|q\bar{W}r' \pm ip|_{\mathbb{C}^3}d\sigma, \tag{21}$$

where $i := \sqrt{-1}$ and $|\cdot|_{\mathbb{C}^3}$ denotes the norm on the complex space $\mathbb{C}^3$. It is worthwhile to mention here that a pair of dynamic variables $(r,p) \in T^*(\Sigma^{(2)})$ is canonical, that is, their Poisson brackets [1, 3, 20] satisfy the following relationships:

$$\{r(\tau,\sigma), p(\tau,\sigma'\} = \delta(\sigma-\sigma'), \tag{22}$$

$$\{r(\tau,\sigma), r(\tau,\sigma'\} = 0 = \{p(\tau,\sigma), p(\tau,\sigma'\}$$

for all string coordinates (τ,σ) and $(\tau,\sigma') \in \Sigma^{(2)}$. Moreover, concerning the result obtained above, we need to mention here that one can not construct the suitable Hamiltonian function expression and relationship of type (20) with respect to the laboratory reference frame $\mathcal{K}$, since expression (21) is not defined on the whole for a separate laboratory time parameter $t \in \mathbb{R}$ locally dependent both on the spatial parameter $\sigma \in \mathbb{R}$ and the proper time reference frame time parameter $\tau \in \mathbb{R}$. Thereby, one can formulate the following proposition.

Proposition 1 *The hadronic string model (1) allows on the related world space-time surface the conformal local coordinates, with respect to which the resulting dynamics is described by means of the linear second order elliptic equation (7). Subject to the proper time Euclidean reference frame $\mathcal{K}_r$ coordinates the corresponding dynamics is equivalent to the Hamiltonian equations*

$$dr/d\tau = \{\mathcal{H}, r\}, \quad dp/d\tau = \{\mathcal{H}, p\}, \tag{23}$$

with respect to the canonical Poisson structure (22) and the Hamiltonian functional (18) .

It is worth remarking here, that any quantization scheme of the Hamiltonian expression (21) under the derived above constraint (17) should be performed within the classical Dirac type reduction [29] scheme.

4 A charged hadronic string interaction with a moving external point charge

We proceed now to construct the action functional expression for a charged string model under an external electromagnetic field, generated by a point charged particle q_f, moving with some velocity $u_f := dr_f/dt \in \mathbb{E}^3$ subject to a laboratory reference frame $\mathcal{K}$. To solve this problem we make use of the trick, passing to the string, considered with respect to the proper time reference frame $\mathcal{K}_r$ moving under the external vacuum field potential $\tilde{W}(\tilde{t}, r)$, measured in the reference frame $\tilde{\mathcal{K}}_f$, specified by its own Euclidean coordinates $(\tilde{t}, r) \in \mathbb{E}^4$, which simultaneously moves with velocity $u_f = dr_f/dt \in \mathbb{E}^3$, measured in the laboratory reference frame $\mathcal{K}$. As a result of this reasoning, we can write down the action functional:

$$S^{(\tau)} = -\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} q\tilde{W}(|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f|r'\rangle_{\mathbb{E}^3}^2)^{1/2} d\,\sigma, \tag{24}$$

giving rise to the following dynamical equation

$$\begin{aligned} dP/d\tau &= \delta\mathcal{L}^{(\tau)}/\delta r = -q(|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f|r'\rangle_{\mathbb{E}^3}^2)^{1/2}\ \nabla\tilde{W} + \\ &+ \frac{\partial}{\partial\sigma}\left\{\frac{q\tilde{W}(1+|\dot{r}-\dot{r}_f|^2\hat{T}_{\dot{r}-\dot{r}_f})r'}{(|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f|r'\rangle_{\mathbb{E}^3}^2)^{1/2}}\right\} \end{aligned} \tag{25}$$

with the generalized momentum density

$$P := \frac{-q\tilde{W}|r'|^2\ \hat{N}(\dot{r}-\dot{r}_f)}{(|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f|r'\rangle_{\mathbb{E}^3\mathbb{E}^3}^2)^{1/2}}. \tag{26}$$

Owing to the vacuum field potential Lorentz transform $\tilde{W}(1-|u_f|^2)^{1/2} = \bar{W}$, one can define the string momentum density

$$\begin{aligned} p \quad : &= \frac{-q\tilde{W}|r'|^2\hat{N}\dot{r}}{(|r'|^2(1+|\dot{r}-\dot{r}_f|^2) - \langle \dot{r}-\dot{r}_f|r'\ \rangle_{\mathbb{E}^3}^2)^{1/2}} = \\ &= \frac{-q\tilde{W}|r'|\hat{N}dr/d\tau}{(1+|\dot{r}-\dot{r}_f|^2 - |r'|^{-2}\langle \dot{r}-\dot{r}_f|r'\rangle_{\mathbb{E}^3}^2)^{1/2}} = -q\tilde{W}|r'|\hat{N}\tilde{u} = -q\bar{W}|r'|\hat{N}u, \end{aligned} \tag{27}$$

as the local string momentum density, and

$$q|r'|A \quad : \quad = \frac{q\tilde{W}|r'|^2\hat{N}\dot{r}_f}{(|r'|^2(1+|\dot{r}-\dot{r}_f|^2)-\ \langle\dot{r}-\dot{r}_f|r'\rangle^2_{\mathbb{E}^3})^{1/2}} = \tag{28}$$

$$= \quad \frac{q_f\tilde{W}|r'|\hat{N}dr_f/d\tau}{(1+|\dot{r}-\dot{r}_f|^2-|r'|^{-2}\langle\dot{r}-\dot{r}_f|r'\ \rangle^2_{\mathbb{E}^3})^{1/2}} = q\tilde{W}|r'|\hat{N}\tilde{u}_f = q\bar{W}|r'|\hat{N}u_f,$$

as the external vector magnetic potential density, where $q\in\mathbb{R}$ is a uniform charge density, distributed along the string length. Thus, equation (25) reduces to

$$\frac{d}{d\tilde{t}}(p+q|r'|A) = -q|r'|\ \nabla\tilde{W}+ \tag{29}$$

$$+\frac{\partial}{\partial\sigma}\left\{\frac{q\tilde{W}(1-|u'-u'_f|^2+|r'|^{-2}\langle u'-u'_f|r'\rangle^2_{\mathbb{E}^3}+|u'-u'_f|^2\hat{T}_{u'-u'_f})r'\ )}{(1-|u'-u'_f|^2+|r'|^{-2}\langle u'-u'_f|r'\rangle^2_{\mathbb{E}^3})^{1/2}}\right\}$$

with respect to the moving reference frame $\tilde{\mathcal{K}}_f$, or equivalently, reduces to

$$\begin{aligned}&\tfrac{d}{dt}(p+q|r'|A) = -q|r'|\ \nabla\bar{W}(1-|u_f|^2)+\\ &+(1-|u_f|^2)(1-|u_f|^2-|u\ -u_f|^2+|r'|^{-2}\langle u-u_f|r'\rangle^2_{\mathbb{E}^3})^{-1/2}\times\\ &\times\tfrac{\partial}{\partial\sigma}\left\{\tfrac{q|r'|\bar{W}(1-|u_f|^2-|u-u_f|^2+|r'|^{-2}\langle u-u_f|r'\rangle^2_{\mathbb{E}^3}+|u-u_f|^2\hat{T}_{u-u_f})r'}{(1-|u_f|^2-|u-u_f|^2+|r'|^{-2}\langle u-u_f|r'\rangle^2_{\mathbb{E}^3})^{1/2}}\right\}\end{aligned} \tag{30}$$

with respect to the moving laboratory frame $\mathcal{K}$. The latter can be easily rewritten also as the generalized Lorentz type force expression

$$\begin{aligned}&dp/dt = q|r'|E+q|r'|u\times B-q|r'|\ \nabla\langle u-u_f|A\rangle_{\mathbb{E}^3}+\\ &+(1-|u_f|^2)(1-|u_f|^2-|u\ -u_f|^2+|r'|^{-2}\langle u-u_f|r'\rangle^2_{\mathbb{E}^3})^{-1/2}\times\\ &\times\tfrac{\partial}{\partial\sigma}\left\{\tfrac{q|r'|\bar{W}(1-|u_f|^2-|u-u_f|^2+|r'|^{-2}\langle u-u_f|r'\rangle^2_{\mathbb{E}^3}+|u-u_f|^2\hat{T}_{u-u_f})r'}{(1-|u_f|^2-|u-u_f|^2+|r'|^{-2}\langle u-u_f|r'\rangle^2_{\mathbb{E}^3})^{1/2}}\right\},\end{aligned} \tag{31}$$

where $B=\ \nabla\times A$ means, as usual, the external magnetic field and $E\ =\ -\partial A\ /\partial t-\nabla\bar{W}$ means the corresponding electric field, acting on the string $\Sigma^{(2)}$. Making use of the standard scheme, one can derive, as above, the Hamiltonian interpretation of dynamical equations (25), but which will not be here discussed.

5 A charged hadronic string interaction with ambient electromagnetic field

Let us now consider the classical least action principle for a charged string interacting with an ambient electromagnetic field:

$$\delta S^{(\tau)}=0,\quad S^{(\tau)}:=-q\int_{s(\tau_1)}^{s(\tau_2)}\int_{\sigma_1(s)}^{\sigma_2(s)}[\bar{W}(x(\xi))+\langle dx(\xi)/d\tau|A(x(\xi))\rangle]d\Sigma^{(2)}(\xi). \tag{32}$$

Here $\bar{W}:M^4\to\mathbb{R}$ is an electric field potential function, $A:M^4\to\mathbb{E}^3$ is a magnetic field potential function, $q\in\mathbb{R}\backslash\{0\}$ is elementary charge parameter, characterizing the interaction of the vacuum medium with our charged hadronic string object, and as before, the differential 2-form $d\Sigma^{(2)}:=(|\dot{\xi}|^2|\xi'|^2-\langle\dot{\xi}|\xi'\rangle^2_{\mathbb{E}^4})^{1/2}d\sigma\wedge ds=\sqrt[2]{g(\xi)}d\sigma\wedge ds$, $g(\xi):=\det(\,g_{ij}(\xi)|_{i,j=\overline{1,2}})$, $|\dot{\xi}|^2:=\langle\dot{\xi}|\dot{\xi}\rangle_{\mathbb{E}^4}$, $|\xi'|^2:=\ \langle\xi'|\xi'\rangle_{\mathbb{E}^4}$, being related to the induced positive definite Riemannian infinitesimal metrics $dz^2\ :=\ \langle d\xi|d\xi\rangle_{\mathbb{E}^4}\ =\ g_{11}(\xi)d\sigma^2+g_{12}(\xi)d\sigma ds+g_{21}(\xi)dsd\sigma\ +g_{22}(\xi)ds^2$ on the string, meaning the infinitesimal two-dimensional surface element, parameterized by local parameters $(s,\sigma)\in\mathbb{R}^2_\xi$, embedded into the four-dimensional Euclidean space-time $\mathbb{R}\times\mathbb{E}^3$ with space-time coordinate $\xi\ :=\ (\tau(s,\sigma),r(s,\sigma))\ \in\ \mathbb{R}\times\mathbb{E}^3\ \simeq\ \mathbb{E}^4$ subject to the proper time reference frame $\mathcal{K}_r$. The related boundary conditions are usually chosen as

$$\delta\xi(s,\sigma(s))=0 \tag{33}$$

at string parameter $\sigma(s) \in \mathbb{R}$ for all $s \in \mathbb{R}$. The action functional expression (32) is naturally motivated by the extended string action functional

$$S^{(\tau)} := -q \int_{\sigma_1}^{\sigma_2} dl(\sigma) \int_{t(\sigma,\tau_1)}^{t(\sigma,\tau_2)} [\bar{W}(t,r) \ + \langle dr/dt|A(t,r)\rangle] dt(\tau,\sigma), \tag{34}$$

where the laboratory reference time parameters $(t(\tau,\sigma), r(\tau,\sigma)) \in \mathbb{R} \times \mathbb{E}^3$ is related to the proper time string reference frame parameter $\tau \in \mathbb{R}$ by means of the standard Euclidean infinitesimal relationship

$$dt(\tau,\sigma) := (1 + |\dot{r}_\perp|^2(\tau,\sigma))^{1/2} d\tau, \quad |\dot{r}_\perp|^2 := \langle \dot{r}_\perp|\dot{r}_\perp\rangle_{\mathbb{E}^3}, \tag{35}$$

with $\sigma \in [\sigma_1, \sigma_2] \subset \mathbb{R}$, as before, parameterizing the string length measure $dl(\sigma)$ on the real axis $\mathbb{R}$, $\dot{r}_\perp(\tau,\sigma) := \hat{N}\ \dot{r}(\tau,\sigma) \in \mathbb{E}^3$, being the orthogonal to the string velocity component, where, by definition,

$$\hat{N} := (1 - |r'|^{-2} r' \otimes r'), \quad |r'|^{-2} := \langle r'|r'\rangle_{\mathbb{E}^3}^{-1}, \tag{36}$$

being the corresponding orthogonal projector operator on $\mathbb{E}^3$ to the string direction, expressed, for brevity, by means of the standard tensor product " $\otimes$ " on the Euclidean space $\mathbb{E}^3$. The result of recalculating the expression (3) gives rise to the following functional expression

$$\begin{aligned} S^{(\tau)} &= -q \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} (\bar{W} + \langle dr/dt|A\rangle)|r'|(1 + |\dot{r}|^2 - |r'|^{-2}\langle \dot{r}|r'\rangle_{\mathbb{E}^3}^2)^{1/2} d\ \sigma = \\ &= -q \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \bar{W}|r'|(1 + |\dot{r}|^2 - |r'|^{-2}\langle \dot{r}|r'\rangle_{\mathbb{E}^3}^2)^{1/2} d\ \sigma - \\ &\quad -q \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \langle \dot{r}|A\rangle)|r'|\ d\ \sigma, \end{aligned} \tag{37}$$

where we made use of the infinitesimal measure representation $dl(\sigma) = \langle r'|r'\rangle_{\mathbb{E}^3}^{1/2} d\sigma$, $\sigma \in [\sigma_1, \sigma_2]$. If we introduce on the string world surface local coordinates $(s(\tau,\sigma), \sigma) \in \mathbb{E}^2$ and the related Euclidean string position vector $\xi := (\tau, r(s(\tau,\sigma),\sigma)) \in \mathbb{E}^4$, the string action functional (37) reduces equivalently to that of (32). Based on this expression, one can write down the related string momentum density

$$\begin{aligned} P := \partial\mathcal{L}^{(\tau)}/\partial\dot{r} = -\tfrac{q\bar{W}|r'|(\dot{r} - r'|r'|^{-2}\langle r'|\dot{r}\rangle_{\mathbb{E}^3}}{(|\dot{r}|^2 + 1 - |r'|^{-2}\langle r'|\dot{r}\rangle_{\mathbb{E}^3}^2)^{1/2}} - qA|r'| = \\ = -q|r'|(\hat{N}\bar{W}u + A), \end{aligned} \tag{38}$$

satisfying the dynamical equation

$$\begin{aligned} dP/d\tau := \delta\mathcal{L}^{(\tau)}/\delta r = -q|r'|(\ |\dot{r}|^2 + 1\ - |r'|^{-2}\langle r'|\dot{r}\rangle_{\mathbb{E}^3}^2)^{1/2}\ \nabla\bar{W}(\tau,r)) - \\ -q|r'|\nabla A - q\langle \dot{r}|A\rangle r' + \tfrac{\partial}{\partial\sigma}\left\{\tfrac{q\bar{W}(1+|\dot{r}|^2\hat{T})r'}{(1+|r'|^{-2}|\dot{r}|^2\langle \hat{T}r'|r'\rangle_{\mathbb{E}^3})^{1/2}}\right\}, \end{aligned} \tag{39}$$

where we denoted by

$$\mathcal{L}^{(\tau)} := -q\bar{W}|r'|(1 + |\dot{r}|^2 - |r'|^{-2}\langle \dot{r}|r'\rangle_{\mathbb{E}^3}^2)^{1/2} - q\langle \dot{r}|A\rangle)|r'| \tag{40}$$

the corresponding Lagrangian density for the action functional (37). Based on the expressions (38) and (40) one easily obtains the string Hamiltonian function

$$\mathcal{H} = \int_{\sigma_1}^{\sigma_2} d\sigma[(P + q|r'|A)^2 - q^2|r'|^2\bar{W}^2]^{1/2}, \tag{41}$$

generalizing the result (18) and satisfying the conservation condition $dH/dt = 0$ for all temporal values $t \in \mathbb{R}$. What is interesting, the dynamical equation (39) is equivalent to that of (25) on taking into account the vector field potential definition (30). Moreover, one can observe that the Hamiltonian expression (41) can be equivalently rewritten as

$$\mathcal{H} = \int_{\sigma_1}^{\sigma_2} |q\bar{W}r' \pm i(P + q|r'|A)|_{\mathbb{C}^3} d\sigma, \tag{42}$$

where, as before, $i := \sqrt{-1}$ and $|\cdot|_{\mathbb{C}^3}$ denotes the norm on the complex space $\mathbb{C}$, and a pair of dynamic variables $(r, P) \in T^*(\Sigma^{(2)})$ is canonical, that is their Poisson brackets satisfy the following relationships:

$$\begin{aligned} \{r(\tau,\sigma), P(\tau,\sigma'\} &= \delta(\sigma - \sigma'), \\ \{r(\tau,\sigma), r(\tau,\sigma'\} &= 0 = \{P(\tau,\sigma), P(\tau,\sigma'\} \end{aligned} \tag{43}$$

for all string coordinates (τ, σ) and $(\tau, \sigma') \in \Sigma^{(2)}$. The above Hamiltonian function (42) generalizes the previously obtained expression (21) and takes into account the interaction of our string with ambient electromagnetic field explicitly. Summarizing the statements above, one can formulate the following proposition.

Proposition 2 *The hadronic string model (32) allows on the related world space-time surface the conformal local coordinates, whose dynamics is described subject to the proper time Euclidean reference frame $\mathcal{K}_r$ by means of the Hamiltonian equations*

$$dr/d\tau = \{\mathcal{H}, r\}, \quad dP/d\tau = \{\mathcal{H}, P\}, \tag{44}$$

with respect to the canonical Poisson structure (43) and the Hamiltonian functional (42).

Here we need to underline that with respect to the proper time reference frame, the charged string model, supplemented with the Maxwell electrodynamics, allows both the Lagrangian and Hamiltonian physically reasonable formulations regarding their canonical Poisson structure, suitable for the canonical quantization procedure.

6 Conclusion

Based on the vacuum field theory approach, devised recently in [16, 17, 13], we revisited the alternative charged hadronic string model, having succeeded in treating their Lagrangian and Hamiltonian dynamic properties. The obtained results, as compared with classical ones, make it possible to argue a physically motivated choice of a charged string model. Another important aspect of the developed vacuum field theory approach to studying extended relativistic string models consists in singling out the decisive role of the R.P. Feynman proper time paradigm, related to rest reference frame $\mathcal{K}_r$, subject to which the relativistic object motion, in reality, is realized. Namely, with respect to the proper time reference frame evolution parameter, the charged string model allows both the Lagrangian and Hamiltonian physically reasonable formulations regarding their canonical Poisson structure, well suitable for the canonical quantization procedure. The deeper physical nature of this fact remains, as of today, is not enough understood and needs additional investigations. We would like only to recall here very interesting reasonings of R.P. Feynman, who argued in [38, 39] that the relativistic expressions have physical sense only with respect to the proper time reference frames, associated with respectively observed physical objects. In a sequel of our work we plan to analyze our relativistic charged string model subject to its canonical quantization devised in [64] and make a step toward the related vacuum quantum field theory of infinite many particle systems.

7 Supplement: The Maxwell electrodynamics within the vacuum field theory approach

We start from the following field theoretical model [12, 13, 15] of the microscopic vacuum medium structure, considered as some physical reality imbedded into the standard three-dimensional Euclidean space reference frame marked by three spatial coordinates $r \in \mathbb{E}^3$, endowed, as before, with the standard scalar product $\langle\cdot|\cdot\rangle_{\mathbb{E}^3}$, and parameterized by means of the scalar temporal parameter $t \in \mathbb{R}$. First we will describe the physical vacuum medium endowing it with an everywhere smooth enough four-vector potential function $(W, A) : M^4 \to T^*(M^4)$, defined in the Minkowski space M^4 and naturally related to light propagation properties. The material objects, imbedded into the vacuum medium, we will model (classically here) by means of the scalar charge density function $\rho : M^4 \to \mathbb{R}$ and the vector current density $J : M^4 \to \mathbb{E}^3$, being also everywhere smooth enough functions.

1. The *first* vacuum field theory principle regarding the vacuum we accept is formulated as follows: the four-vector function $(W, A) : M^4 \to T^*(M^4)$ satisfies the standard Lorentz type conserved continuity relationship
$$\frac{1}{c}\frac{\partial W}{\partial t} + \langle \nabla | A \rangle_{\mathbb{E}^3} = 0, \tag{45}$$
where, by definition, $\nabla := \partial/\partial r$ is the usual gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$ and $c > 0$ denotes the light velocity in the vacuum.

2. The *second* vacuum field theory principle we accept is an evolution wave relationship on the scalar potential $W : M^4 \to \mathbb{R}$:
$$\frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} - \nabla^2 W = \rho, \tag{46}$$
assuming the linear law of the small vacuum uniform and isotropic perturbation propagations in the space-time, understood here, evidently, as a first (linear) approximation in the case of weak enough fields.

3. The *third* vacuum principle is similar to the first one and means simply the conserved continuity condition for the charge and current density functions:
$$\partial \rho/\partial t + \langle \nabla | J \rangle_{\mathbb{E}^3} = 0. \tag{47}$$

We need to note here that the vacuum field perturbations velocity parameter $c > 0$, used above, coincides with the vacuum light velocity, as we are trying to derive successfully from these first principles the well-known Maxwell electromagnetism field equations, to analyze the related Lorentz forces and special relativity relationships. To do this, we first combine equations (45) and (46):
$$\frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = -\langle \nabla | \frac{1}{c}\frac{\partial A}{\partial t} \rangle_{\mathbb{E}^3} = \langle \nabla | \nabla W \rangle_{\mathbb{E}^3} + \rho,$$
whence
$$\langle \nabla | -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla W \rangle_{\mathbb{E}^3} = \rho. \tag{48}$$
Having put, by definition,
$$E := -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla W, \tag{49}$$
we obtain the first material Maxwell equation
$$\langle \nabla | E \rangle_{\mathbb{E}^3} = \rho \tag{50}$$
for the electric field $E : M^4 \to \mathbb{E}^3$. Having now applied the rotor-operation $\nabla \times$ to expression (49) we obtain the first Maxwell field equation
$$\frac{1}{c}\frac{\partial B}{\partial t} - \nabla \times E = 0 \tag{51}$$
on the magnetic field vector function $B : M^4 \to \mathbb{E}^3$, defined as
$$B := \nabla \times A. \tag{52}$$

Remark 1 *It is useful to remark that* the second field theory principle *is exactly equivalent to the experimentally stated physical relationships (49) and (50) for the electric field $E : M^4 \to \mathbb{E}^3$. Really, having applied the operator $\nabla \times$ to the left-hand side of (49), one obtains the wave relationship (46).*

To derive the second Maxwell field equation we will make use of (52), (45) and (49):

$$\begin{aligned}\nabla \times B &= \nabla \times (\nabla \times A) = \nabla \langle \nabla | A \rangle_{\mathbb{E}^3} - \nabla^2 A = \\ &= \nabla(-\frac{1}{c}\frac{\partial W}{\partial t}) - \nabla^2 A = \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla W - \frac{1}{c}\frac{\partial A}{\partial t} + \frac{1}{c}\frac{\partial A}{\partial t}\right) - \nabla^2 A = \\ &= \frac{1}{c}\frac{\partial E}{\partial t} + (\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A). \end{aligned} \tag{53}$$

We have from (49), (50) and (47) that

$$\langle \nabla | \frac{1}{c}\frac{\partial E}{\partial t}\rangle_{\mathbb{E}^3} = \frac{1}{c}\frac{\partial \rho}{\partial t} = -\frac{1}{c}\langle \nabla | J \rangle_{\mathbb{E}^3},$$

or

$$\langle \nabla | -\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla(\frac{1}{c}\frac{\partial W}{\partial t}) + \frac{1}{c}J\rangle_{\mathbb{E}^3} = 0. \tag{54}$$

Now making use of (45), from (54) we obtain that

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{1}{c}(J + \nabla \times S) \tag{55}$$

for some smooth vector function $S : M^4 \to \mathbb{E}^3$. Here we need to note that continuity equation (47) is defined, concerning the current density vector $J : M^4 \to \mathbb{R}^3$, up to a vorticity expression, that is $J \simeq J + \nabla \times S$ and equation (55) can finally be rewritten as the next wave equation

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \frac{1}{c}J \tag{56}$$

on the magnetic potential function $A : M^4 \to \mathbb{E}^3$. Having substituted (56) into (53) we obtain the second Maxwell field equation

$$\nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{1}{c}J. \tag{57}$$

In addition, from (52) one also finds the magnetic no-charge relationship

$$\langle \nabla | B \rangle_{\mathbb{E}^3} = 0. \tag{58}$$

Thus, we have derived all the Maxwell electromagnetic field equations from our three main principles (45), (46) and (47). The success of our undertaking will be more impressive if we adapt our results to those following from the well-known relativity theory in the case of point charges or masses. Below we will try to demonstrate the corresponding derivations based on some completely new physical conceptions of the vacuum medium first discussed in [66, 13].

It is interesting to observe a partial case of the first field theory vacuum principle (45), equivalent to the following local conservation law for the scalar potential field function $W : M^4 \to \mathbb{R}$:

$$\frac{d}{dt}\int_{\Omega_t} W(t, r\ )d^3r = 0, \tag{59}$$

where $\Omega_t \subset \mathbb{E}^3$ is an arbitrary open domain in space $\mathbb{E}^3$ with the smooth enough boundary $\partial\Omega_t$ for all $t \in \mathbb{R}$ and d^3r denotes the standard volume measure in $\mathbb{R}^3$ in a vicinity of the point $r(t) \in \Omega_t$. Having calculated expression (59) we obtain the following continuity equation

$$\frac{1}{c}\frac{\partial W}{\partial t} + \langle \nabla | \frac{v}{c}W\rangle_{\mathbb{E}^3} = 0, \tag{60}$$

where $\nabla := \partial/\partial r$ is, as above, the gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$, $v := dr(t)/dt$ is the velocity vector of the corresponding vacuum medium change influenced by an external charge particle, carrying the potential field energy W. Comparing now equations (45), (60) and using equation (47) we can make the suitable very important identification:

$$A = \frac{v}{c}W, \tag{61}$$

well known from the classical electrodynamics [54] and superconductivity theory [38, 51]. Thus, we are faced with a new physical interpretation of the conservative electromagnetic field theory when the vector potential $A : M^4 \to \mathbb{E}^3$ is completely determined via expression (61) by the scalar field potential function $W : M^4 \to \mathbb{R}$. It is also evident that all the Maxwell electromagnetism with filed equations (57) and (58) derived above, hold as well in the case (61), as it was first demonstrated in [?]

Consider now the conservation equation (59) jointly with the related integral "*vacuum momentum*" conservation condition

$$\frac{d}{dt}\int_{\Omega_t}\left(c^{-2}Wv\right)d^3r = 0, \quad \Omega_t|_{t=0} = \Omega_0, \tag{62}$$

where, as above, $\Omega_t \subset \mathbb{E}^3$ is for any time $t \in \mathbb{R}$ an open domain with the smooth boundary $\partial\Omega_t$, whose evolution is governed both by the transport equation

$$dr/dt = v(t,r) \tag{63}$$

for all $r \in \Omega_t, t \in \mathbb{R}$, and by the initial state of the boundary $\partial\Omega_0$. As a result of relation (62) one obtains the new continuity equation

$$\frac{d(vW)}{dt} + vW\langle\nabla|v\rangle_{\mathbb{E}^3} = 0. \tag{64}$$

Now making use of (60) in the equivalent form

$$\frac{dW}{dt} + W\langle\nabla|v\rangle_{\mathbb{E}^3} = 0,$$

we finally obtain a very interesting local conservation relationship

$$dv/dt = 0 \tag{65}$$

on the vacuum matter perturbations velocity $v = dr(t)/dt$, which holds for all values of the time parameter $t \in \mathbb{R}$. As it is easy to observe, the obtained relationship completely coincides with the well-known hydrodynamic type equation [56] of ideal compressible liquid without any external exertion, that is, any external forces and field "pressure" are equally identical to zero. We received a natural enough result, confirming that the propagation velocity of the vacuum field matter is strait linear, constant and equals exactly $v = c$, that is the light velocity c in the vacuum, if to take into account the starting wave equation (46) owing to which the small vacuum field matter perturbations propagate in the space with the light velocity.

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