SOME ASPECTS OF MODIFIED GAUSS-BONNET GRAVITY

By

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CERTIFICATE

I certify that the research work presented in this thesis is the original work of **Hafiza Ismat Fatima** D/O **Mian Muhammad Siddique** and is carried out under my supervision. I endorse its evaluation for the award of **Ph.D.** degree through the official procedure of **University of the Punjab**.

Prof. Dr. Muhammad Sharif
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DECLARATION

I, **Hafiza Ismat Fatima** D/O **Mian Muhammad Siddique**, hereby declare that the matter printed in this thesis is my original work. This thesis does not contain any material that has been submitted for the award of any other degree in any university and to the best of my knowledge, neither does this thesis contain any material published or written previously by any other person, except due reference is made in the text of this thesis. Most of the contents have been appeared as my research papers.

Hafiza Ismat Fatima

DEDICATED

To

My Loving Parents

Respected Parents-in-Law

and Beloved Husband

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All praises and gratitude to **Allah Almighty**, the most **Affectionate**, the **Merciful**. Lord of all the worlds. Owner of the day of judgment. I courteously bow before Him for granting and bestowing me the light of knowledge. He has enabled me to complete this work through His countless mercy. All praises and respect to the beloved Holy Prophet **Hazrat Muhammad (PBUH)**, Who is forever a torch of knowledge and guidance for humanity.

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Abstract

This thesis deals with some cosmic aspects in the context of modified Gauss-Bonnet gravity. Firstly, we explore static spherically symmetric wormhole solutions in galactic halo region as well as using conformal Killing vectors technique. The effective energy-momentum tensor leads to the violation of energy conditions while normal matter satisfies these conditions. We use Navarro-Frenk-White energy density profile to examine possible existence of traversable wormholes in galactic halo region. We find physically acceptable wormhole solutions threaded by normal matter for all values of r. We also investigate stability of the resulting wormhole solutions. For conformally symmetric traversable wormholes, it is found that all shape functions satisfy flaring-out condition except phantom case with non-static conformal symmetry.

Secondly, we study the dynamics of self-gravitating objects for spherical and axial systems. We construct structure scalars through orthogonal splitting of the Riemann tensor and deduce a complete set of equations governing the evolution of dissipative anisotropic fluid in terms of these scalars. In spherically symmetric system, we investigate some particular fluid models according to various dynamical conditions and find that our results are consistent with general relativity for constant $f(G)$ model. Any other choice of the model leads to irregular distribution of dark energy and deviates from general relativity. We also explore different causes of density inhomogeneity which turns out to be a necessary condition in the presence of dark sources.

In axially symmetric system with shear, it is found that dark sources affect thermodynamics of the system, evolution of kinematical quantities as well as density inhomogeneity. For the shear-free case, we study both non-geodesic as well as geodesic fluids with and without dissipation. The non-geodesic (non-dissipative) fluid gives inhomogeneous expansion while geodesic fluid leads the system either to vorticity-free or expansion-free. The vorticity-free non-dissipative geodesic fluid reduces the axial system to FRW model with homogeneous distribution of dark sources while expansion-free geodesic fluid does not exist even in the presence of dark sources.

Abbreviations

In this thesis, the metric signatures will be $(-,+,+,+)$ and Greek indices will vary from 0 to 3, if different it will be mentioned. Also, the unit system $\kappa^2 = \frac{8\pi G}{c} = 1$ will be used, if not then it will be mentioned. We shall use the following abbreviations.

CKVs:	Conformal Killing Vectors
DE:	Dark Energy
GB:	Gauss-Bonnet
GR:	General Relativity
ΛCDM:	Λ Cold Dark Matter
NEC:	Null Energy Condition
WEC:	Weak Energy Condition
EoS:	Equation of State

Introduction

Cosmology deals with the origin, composition and evolution of the universe right from the big-bang to today and onwards to the future. The fundamental theory which generates a relation between geometry and matter contents is the theory of general relativity. Cosmic models derived on the basis of this theory represent a complete picture of the universe. Recent cosmological observations from Supernovae type Ia [1] and its cross-comparison with foreground stellar galactic distributions indicate that our universe is expanding at an accelerating rate with time. Other observational evidences from cosmic microwave background, galaxy redshift surveys and large scale structure [2] also favor this phenomenon. These cosmological surveys proposed an obscure type of energy which is considered to be a pivotal constituent for cosmic accelerated expansion. This is termed as DE possessing repulsive force pushing various cosmic objects far away from each other against their gravitational force.

The ΛCDM model is a straightforward explanation of DE in GR but it suffers problems like cosmic coincidence and fine-tuning. This leads to two well-known approaches to modify GR. The first approach extends the matter part while the second modifies the geometric part. Modified GB gravity or $f(G)$ gravity is one of the modified versions of GR which is obtained by adding an arbitrary function $f(G)$ of GB quadratic invariant G

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\rho\nu\sigma}R^{\mu\rho\nu\sigma}, \tag{0.0.1}$$

in the action [3]. This theory efficiently elucidates cosmic accelerated expansion and transition phases of the universe from deceleration to acceleration [4]. The solar system experiments are essential constrains on modified gravity that measure the deviation of these theories from GR. A viable gravity model must satisfy these constraints and this theory passes all solar system tests [5]. This also avoids all possible four types of finite time future singularities [6], explains black hole thermodynamics [7] and studies DE as well as inflationary era [8].

Traversable wormholes are regarded as hypothetical paths (tunnels or bridges) connecting two different regions of the same spacetime (intra-universe wormhole) or two different spacetimes (inter-universe wormhole). These paths provide free passage for observers from which they may traverse easily. The history of wormhole can be traced back to 1916 when Flamm analyzed a newly described Schwarzschild solution [9] and found that this solution represents wormhole. Later in 1935, Einstein and Rosen presented wormhole type solutions known as Einstein-Rosen bridge [10]. In the modern age, Morris and Thorne [11] brought back the idea of wormhole geometry in 1988 which rekindled the researchers to study traversable wormholes. They established wormhole solutions from the Einstein field equations as static spherically symmetric metric and proposed that wormholes might be the objects of nature like stars and black holes.

In GR, the violation of NEC is the necessary condition for wormhole solutions. Its violation leads to exotic matter (having negative energy density) which allows two way travel. The search for a realistic model supporting the energy conditions or to minimize the utilization of exotic matter has a significant role in the wormhole history. For this purpose, various strategies are adopted such as brane wormholes [12], generalized Chaplygin gas [13] as well as modified theories of gravity. These approaches may cure the violation of energy conditions and lead to the realistic wormholes. In fact, the effective energy-momentum tensor in these theories is responsible for energy conditions violation (extra curvature terms or modified terms take part in the violation) while normal matter satisfies these conditions. Böhmer et al. [14] discussed traversable wormholes using conformal symmetry. Rahaman et al. [15] described the interior of a relativistic star using CKVs under dark effects of generalized teleparallel gravity. Sharif and Ikram [16] investigated wormholes by considering traceless, barotropic as well as isotropic fluids in $f(G)$ gravity and found physically acceptable wormhole solutions. However, no remarkable work has been done in finding the wormhole solutions using CKVs technique in modified theories of gravity.

Self-gravitation is one of the principal features of stellar structures in the universe which keeps these structures together under the influence of their own gravity. In the absence of self-gravitation, all celestial bodies like stars, galaxies and cluster of galaxies will expand and vanish. Gravitational collapse is the source of energy behind structure formation in the universe where over time, once vastly distributed matter collapses to high density pockets ultimately leading to the hierarchy of all cosmological structures like galaxies, black holes and all types of stars. Physical quantities like pressure anisotropy, the Weyl tensor, energy density inhomogeneity and dissipation characterize self-gravitating fluids in various evolving stellar bodies. The pressure anisotropy is considered as an important parameter in the stellar evolution which has significant effects in controlling hydrostatic equilibrium. Herrera and Santos [17] have studied significant role of local pressure anisotropy in the evolution of self-gravitating fluids. The physical relevances of density inhomogeneity, the Weyl tensor and heat dissipation have been illustrated in [18].

Ellis [19] developed a set of equations comprising dynamical quantities (physical as well as kinematical quantities) associated with evolving configuration. These equations govern the evolution and structure of astronomical objects. Herrera et al. [20] formulated a set of equations governing the evolution of different spherically symmetric self-gravitating systems. Di Prisco et al. [21] investigated the effects of charge on the dissipative gravitational collapse by using dynamical equations. Sharif and Manzoor [22] explored self-gravitating fluid models in Brans-Dicke theory using spherical as well as cylindrical symmetries. They deduced a set of governing equations to study the dynamics of anisotropic dissipative fluids.

Scalar expressions with various combinations of physical quantities are known as structure scalars which have individual physical meanings. These scalars can control as well as simplify the complexities arising during evolution of the system. Herrera et al. [23] studied spherically symmetric self-gravitating fluids using structure scalars generated through orthogonal splitting of the Reimann tensor. Herrera et al. [24] investigated the effects of electric charge and cosmological constant on the scalars corresponding to a spherical configuration. Sharif and his collaborators [25] explored the role of scalars in charged plane symmetry as well as $f(R)$ gravity.

Generally, it is assumed that astrophysical objects are endowed with angular momentum, e.g., stellar compact objects (like white dwarfs, neutron stars) are in rotational motion and can deviate from spherical symmetry. This deviation is likely to be incidental rather than basic features of these systems which induces the idea of axially symmetry for celestial objects. Some authors [26] studied the evolution of axially symmetric self-gravitating fluids with perfect matter configurations. The assumption of perfect fluid seems to be a stringent restriction for axially symmetric sources even in static case [27, 28]. Herrera et al. [28] studied static interior solutions of axially symmetric model through structure scalars by considering anisotropic fluid. Sharif and Nasir [29] extended this work in $f(R)$ gravity. Sharif et al. [30] explored axial and reflection symmetric systems with anisotropic matter configurations in GR as well as in self-interacting Brans-Dicke theory.

Stellar objects undergo various phases during their evolution due to different kinematical factors. Shear tensor ia one of these factors which measures distortion in configuration preserving its volume. The role of shear tensor during the evolution of stellar objects and the consequences emerging from its vanishing have attracted many researchers. Glass [31] observed that shear-free condition leaves perfect fluid irrotational if and only if magnetic part of the Weyl tensor vanishes. Collins and Wainwright [32] hypothesized that spherical system with perfect fluid transformed to FRW universe during the collapse process under this condition. Tomimura and Nunes [33] investigated a radiating collapse with heat flow of shear-free geodesic fluid. Herrera et al. [34] studied stability of spherically symmetric anisotropic matter distribution using this condition. Herrera and his collaborators [35] provided a comprehensive analysis of shear-free rotating fluid in the presence of pressure anisotropy and heat dissipation.

In this thesis, we investigate static spherically symmetric traversable wormholes in galactic halo region as well as wormholes admitting conformal symmetry in the background of $f(G)$ gravity. We also explore some dynamical aspects of relativistic self-gravitating models. The arrangement of the thesis is as follows.

- Chapter **One** presents an overview of basic concepts and definitions related to this thesis.

- Chapter **Two** explores static spherically symmetric wormhole solutions in galactic halo region as well as wormholes admitting conformal symmetry.

- Chapter **Three** studies the evolution of spherically symmetric self-gravitating system in the presence of dark sources through $f(G)$ gravity. We also develop structure scalars and discuss density inhomogeneity as well as all static inhomogeneous solutions for anisotropic spherical models.

- Chapter **Four** investigates dynamics of axial and reflection symmetric system in $f(G)$ gravity with and without shear stress.

- Chapter **Five** provides summary of the results and gives some ideas for future research.

Chapter 1

Preliminaries

This chapter provides some basic material about this thesis.

1.1 Modified Gauss-Bonnet Gravity

Modified theories of gravity have been the subject of great interest in modern cosmology that provide a convincing way for settling the issue of late cosmic accelerated expansion. Such theories involve higher order derivatives that allow the field equations to be higher than second order. The inclusion of a particular linear combination of the Ricci scalar (R), Ricci ($R_{\mu\nu}$) and Reimann ($R_{\mu\rho\nu\sigma}$) tensors (Eq.(0.0.1)) in the action gives rise to GB gravity. The corresponding action for higher dimensions ($D \geq 5$) is given as

$$S = \frac{1}{2}\int d^D x[R + G]\sqrt{-g} + S_{_M}, \tag{1.1.1}$$

where $S_{_M}$ is the matter action. The term G in Eq.(0.0.1) is known as GB quadratic invariant which is a topological term. This does not contribute to the field equations in four-dimensional spacetime. In order to obtain non-trivial results in four-dimensional spacetime, the scalar G in Eq.(1.1.1) is replaced by an arbitrary function $f(G)$ and the resulting action is named as modified GB gravity. The action for such theory is given by [3]

$$S=\frac{1}{2}\int d^4x[R+f(G)]\sqrt{-g}+S_{_M}, \qquad (1.1.2)$$

Varying this action with respect to the metric tensor $g_{\mu\nu}$, we obtain the modified field equations as

$$\begin{aligned} &R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+8[R_{\mu\rho\nu\sigma}+R_{\rho\nu}g_{\sigma\mu}-R_{\rho\sigma}g_{\nu\mu}-R_{\mu\nu}g_{\sigma\rho}+R_{\mu\sigma}g_{\nu\rho}\\ &+\frac{1}{2}R(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho})]\nabla^\rho\nabla^\sigma f_{_G}+(Gf_{_G}-f)g_{\mu\nu}=T_{\mu\nu}, \end{aligned} \qquad (1.1.3)$$

where $f_{_G}$ denotes derivative of f with respect to G and $T_{\mu\nu}$ is the energy-momentum tensor. In terms of Einstein tensor, Eq.(1.1.3) can be written as

$$\mathcal{G}_{\mu\nu}=\overset{(\mathrm{eff})}{T}_{\mu\nu}=\overset{(\mathrm{m})}{T}_{\mu\nu}+\overset{(\mathrm{GB})}{T}_{\mu\nu}, \qquad (1.1.4)$$

where the notation (eff) (shorten for effective) denotes combined effects of matter and dark sources (GB terms), $\mathcal{G}_{\mu\nu}$ is the Einstein tensor, $\overset{(\mathrm{m})}{T}_{\mu\nu}$ is the energy-momentum tensors for matter and

$$\begin{aligned} \overset{(\mathrm{GB})}{T}_{\mu\nu} &= 8[R_{\mu\rho\nu\sigma}+R_{\rho\nu}g_{\sigma\mu}-R_{\rho\sigma}g_{\nu\mu}-R_{\mu\nu}g_{\sigma\rho}+R_{\mu\sigma}g_{\nu\rho}\\ &+ \frac{1}{2}R(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho})]\nabla^\rho\nabla^\sigma f_{_G}+(Gf_{_G}-f)g_{\mu\nu}, \end{aligned} \qquad (1.1.5)$$

is the energy-momentum tensor contributing the gravitational effects due to $f(G)$ extra dark source terms.

In order to present $f(G)$ as acceptable theory, one should consider viable $f(G)$ models. A viable model not only helps to shed light over current cosmic acceleration but also obeys the requirements imposed by the solar system experiments with relativistic background. A modified gravity model should avoid ghosts (instabilities) to preserve precise cosmological dynamics. A ghost mode is a propagating degree of freedom with a kinetic term in the action having opposite sign, i.e., there exist particles which propagate with negative energy. Ghost often appears, while dealing with modified gravity theories that indicates DE as a source behind current cosmic acceleration. A viable $f(G)$ model should satisfy the following conditions [5]

- $f(G)$ and all of its derivatives $(f_{_G},\ f_{_{GG}},\ f_{_{GGG}}...)$ are regular.
- $f_{_{GG}} > 0,\ \forall\ G$ and $f_{_{GG}} \to 0$ as $|G| \to +\infty$.
- The condition $\dot{f}_{_G} > 0$ is required to avoid ghost.

We shall use two $f(G)$ models in this thesis to discuss some cosmological aspects. One of the $f(G)$ models is [6]

$$f(G) = \alpha G^l(1 + \beta G^m), \tag{1.1.6}$$

where $\alpha,\ \beta,\ m$ and l are arbitrary constants with $l > 0$. This model satisfies the above conditions and cures four types of finite-time future singularities. The other model is [4]

$$f(G) = \alpha G^n. \tag{1.1.7}$$

This model satisfies all the required conditions to be viable. The model parameter n has some significant effects on $R + f(G)$ cosmology. For $n < 0$, this model describes transition from non-phantom to phantom phases while $0 < n < \frac{1}{2}$ gives transition from decelerated to accelerated universe.

1.2 Wormholes

A wormhole looks like a tunnel, bridge or tube with two open ends which may link long (a billion light years or more) or short distances (a few meters) in time. Conventionally, all known wormhole solutions belong to inter-universe wormholes and the simplest example of such wormholes is the Schwarzschild wormhole. However, this is not traversable as it possesses an anti-horizon (surface from which objects can come out but cannot go in) which is unstable against small disturbances. Even when light passes through such anti-horizon, it changes to a horizon thereby closing and eventually destroying the wormhole. In such situation, an object fails to link with other end of wormhole.

Morris and Thorne [11] presented the idea of traversable wormhole in a way that a human may traverse from one side of the wormhole to the other. They generalized Schwarzschild wormhole to eliminate the event horizon and proposed static spherically symmetric spacetime representing wormhole geometry given by

$$ds^2 = -e^{2\mathcal{R}(r)}dt^2 + \left(\frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{1.2.1}$$

where $\mathcal{R}(r)$ and $b(r)$ are known as redshift and shape functions, respectively. The redshift (or potential) function determines gravitational redshift of a light particle (photon) and the shape function defines shape of the wormhole. The radial coordinate r bears non-monotonical behavior as it goes down from infinity to the lowest value of r which defines the wormhole throat r_0 (say) and then returns from throat to infinity. Here we list some essential properties which need to be fulfilled by a spherical symmetric traversable wormhole [11].

- A horizon prevents two-way travel through a wormhole. To avoid the horizon of wormhole, the magnitude of redshift function must be finite everywhere. This gives rise to no-horizon condition for a wormhole which is imposed on redshift function.

- The shape function must satisfy the flaring-out condition on throat, i.e., to have proper shape for a wormhole, the ratio of radial coordinate to the shape function at that coordinate must be 1. This yields $b(r_0) = r_0$ and $b'(r_0) < 1$.

- The radial distance $d(r) = \pm \int_{r_0}^{r} \frac{dr}{\sqrt{1-\frac{b}{r}}}$, for $r \geq r_0$ should be finite throughout the space. The $+$ sign represents upper part of spacetime while $-$ indicates the lower one which are joined through the wormhole throat.

- At large distances, the asymptotic flatness should be accomplished by the spacetime, i.e., $\frac{b(r)}{r} \rightarrow 0$ as $r \rightarrow \infty$.

Schwarzschild solution represents a particular type of wormhole which depends only on the mass of wormhole while Morris-Thorne solution (1.2.1) represents a class of wormhole solutions for arbitrarily large number of redshift as well as shape function satisfying the above conditions.

In order to keep wormhole throat open, the energy-momentum tensor must be negative which enables one to traverse through it. In this case, NEC (sum of energy density and pressure of matter) violates and the matter is termed as exotic (normal matter satisfies this condition). In GR, the violation of this condition is the necessary tool for the existence of wormholes. In order to minimize the usage of exotic matter at throat region and thereby obtaining realistic wormhole, modified theories provide the effective energy-momentum tensor which violates WEC.

Energy conditions (arising from the Raychaudhuri equation) are used to work out various important results in different physical scenarios, in particular, to describe physically realistic matter configuration. In order to discuss NEC and WEC, we assume the energy-momentum tensor $T_{\mu\nu} = diag(\rho,\ p_1,\ p_2,\ p_3)$, where ρ is the energy density and $p_1,\ p_2,\ p_3$ denote pressures. The relationship between Raychaudhuri equation and attractive nature of gravity with timelike vector U^μ yields the inequality $T_{\mu\nu}U^\mu U^\nu \geq 0$ which defines WEC. In component form of the energy-momentum tensor, we have $\rho \geq 0,\ \rho + p_i \geq 0,\ i = 1, 2, 3$. In modified theories, the usual energy-momentum tensor becomes effective $\overset{(\text{eff})}{T}_{\mu\nu}$ and the corresponding matter components as $\overset{(\text{eff})}{\rho}$, $\overset{(\text{eff})}{p}_1$, $\overset{(\text{eff})}{p}_2$ and $\overset{(\text{eff})}{p}_3$. The WEC implies NEC $T_{\mu\nu}k^\mu k^\nu \geq 0$ for any null vector k^μ, which gives $\rho + p_i \geq 0$ and hence $\overset{(\text{eff})}{\rho} + \overset{(\text{eff})}{p}_i \geq 0$.

1.3 Galactic Halo Region

Galactic halo region is a large spheroidal region aggregating scattered stars, dust and gas surrounding a galaxy. This is a much larger region (several times greater than the mass of the rest of the galaxy) containing large amount of non-luminous (dark) matter - also known as dark halo or extended halo. The plot of the velocity (orbital speed) of stars in a galaxy versus radius (different distances from the center) gives the galaxy rotation curves (also called velocity curves). It is usually provided graphically as a plot, and data observed from each side of a spiral galaxy is generally asymmetric so that data from each side is averaged to create the curve. A flat rotation curve is one in which the velocity is constant over some range of radii which are observed in galaxies with a central bulge in their disk (i.e. stars are observed to revolve around the center of these galaxies at a constant speed over a large range of distances from the center of galaxy). A flat rotation curve implies that mass continues to increase linearly with radius. The non-luminous matter (dark matter which does not interact with normal matter and is detected only through its gravitational effects on visible matter) is considered responsible for these flat rotation curves.

The N-body simulation is the simulation for a dynamical system comprising N-particles, usually under physical forces (such as gravity). Navarro et al. [36] used N-body simulations to confront the structure of dark halos in the standard ΛCDM cosmology which led to the density profile

$$\rho = \frac{k}{\frac{r}{r_c}(1+\frac{r}{r_c})^2}, \tag{1.3.1}$$

where k and r_c represent characteristic density and scale radius, respectively. This density expression is known as Navarro-Frenk-White energy density. Rahaman et al. [37] used this density profile to check the possible existence of wormholes with these strange halos. They also discussed some characteristics of these galactic halos which support wormhole geometry and checked the equilibrium condition for this geometry.

1.4 Conformal Symmetry

It is always difficult to find exact solutions of the Einstein field equations unless some certain symmetry restrictions are imposed on spacetime geometry. These restrictions are expressed in terms of isometries (Killing vectors) possessed by a spacetime metric. Various symmetries arising from geometrical viewpoint are known as collineations defined by

$$\mathcal{L}_\xi \Phi = \Theta, \tag{1.4.1}$$

where $\mathcal{L}$ is the Lie derivative, ξ is collineation (symmetry) vector, Φ is tensor field which can be $g_{\mu\nu},\ R_{\mu\nu}, R^{\eta}_{\mu\nu\sigma},\ \Gamma^{\sigma}_{\mu\nu}$ and Θ is the tensor with same index symmetries as

Φ. One can deduce the known collineations by substituting particular forms of Φ and Θ. Amongst them, CKVs are the best for deeper insight into spacetime geometry which are obtained by replacing $\Phi = g_{\mu\nu}$ and $\Theta = \varphi g_{\mu\nu}$ (φ is an arbitrary function (conformal factor)) in Eq.(1.4.1). This provides inheritance symmetry which helps to find exact solutions from highly non-linear field equations. Suppose that the vector field ξ generates conformal symmetry so that metric tensor $g_{\mu\nu}$ is conformally mapped onto itself along ξ, then from Eq.(1.4.1), we obtain

$$\mathcal{L}_\xi g_{\mu\nu} = \varphi g_{\mu\nu}, \tag{1.4.2}$$

For $\varphi = 0$, this equation yields Killing vectors, $\varphi = constant$ (real) gives homotheties (homothetic vector field) and the general choice $\varphi = \varphi(t, X)$ produces CKVs. Equation (1.4.2) can also be written as

$$g_{\mu\nu,\alpha}\xi^\alpha + g_{\alpha\nu}\xi^\alpha_{,\mu} + g_{\mu\alpha}\xi^\alpha_\nu = \varphi g_{\mu\nu}. \tag{1.4.3}$$

1.5 Dynamical Quantities

During stellar evolution, a system bears various phases like it can expand or contract, its shape can be distorted or it can take a rotation. Dynamical quantities play a vital role in the description of the system in such situations. These quantities include physical and kinematical variables given as follows.

1.5.1 Kinematical Quantities

Kinematical quantities describe the features of fluid motion comprising four acceleration, expansion parameter, shear as well as vorticity tensors [19].

(i) Four Acceleration

The combined effects of gravitational as well as inertial forces on the fluid can be described by four acceleration which is the rate at which the four velocity (u_μ) of matter changes with respect to time. It is given as

$$a_\mu = u_{\mu;\nu}u^\nu, \quad a^2 = a^\mu a_\mu, \tag{1.5.1}$$

where the symbol ; denotes covariant derivative. It vanishes for fluid particles moving with uniform velocity and the corresponding fluid is called geodesic fluid which plays a significant role in the evolution of the system.

(ii) Expansion Parameter

The volume expansion and contraction of fluid can be measured by expansion parameter ϑ. It evaluates the rate of change of distance of neighboring fluid particles with respect to time. It is a scalar quantity defined by

$$\vartheta = u^\mu_{;\mu}. \tag{1.5.2}$$

The positive values of this parameter ($\vartheta > 0$) define the expanding behavior of matter (distance of neighboring fluid particles is increasing with time) and negative values ($\vartheta < 0$) define the contracting behavior of matter (distance of neighboring fluid particles is decreasing with time).

(iii) Shear Tensor

The shear tensor is a symmetric tensor which is used to evaluate distortion appearing in the fluid due to motion defined as

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{3}\vartheta h_{\mu\nu}, \tag{1.5.3}$$

where $u_{(\mu;\nu)} = \frac{1}{2}(u_{\mu;\nu}+u_{\nu;\mu})$ is the symmetric part of four velocity and $h_{\mu\nu} = g_{\mu\nu}+u_\mu u_\nu$ is the projection tensor. The trace of shear tensor is called shear scalar defined as $\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$. The condition $\sigma_{\mu\nu} = 0 \Leftrightarrow \sigma = 0$ defines a shear-free system which provides interesting consequences for stellar evolution.

(iv) Vorticity Vector

The local spinning of the system is defined by the vorticity vector

$$w_\mu = \frac{1}{2}\eta_{\mu\nu\alpha\beta}u^{\nu;\alpha}u^\beta = \frac{1}{2}\eta_{\mu\nu\alpha\beta}\Omega^{\nu\alpha}u^\beta, \tag{1.5.4}$$

where $\eta_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor and

$$\Omega_{\mu\nu} = u_{[\mu;\nu]} + a_{[\mu}u_{\nu]} \tag{1.5.5}$$

defines vorticity tensor. Here $u_{[\mu;\nu]} = \frac{1}{2}(u_{\mu;\nu} - u_{\nu;\mu})$ is the anti-symmetric part of the four velocity. The trace of vorticity tensor is vorticity scalar defined as $\Omega^2 = \Omega_{\mu\nu}\Omega^{\mu\nu}$. If the system exhibits non-spinning behavior, then

$$\Omega = 0 \quad \Leftrightarrow \quad \Omega_{\mu\nu} = 0 \quad \Leftrightarrow \quad w_\mu = 0. \tag{1.5.6}$$

1.5.2 The Weyl Tensor

The Weyl tensor or the Weyl curvature tensor is the four indexed tensor described as a combination of the Riemann tensor, Ricci tensor and Ricci scalar. It explains how an object is distorted due to the effects of tidal force and is given as

$$C^\mu_{\rho\sigma\nu} = R^\mu_{\rho\sigma\nu} - \frac{1}{2}R^\mu_\sigma g_{\rho\nu} + \frac{1}{2}R_{\rho\sigma}\delta^\mu_\nu - \frac{1}{2}R_{\rho\nu}\delta^\mu_\sigma + \frac{1}{2}R^\mu_\nu g_{\rho\sigma} + \frac{1}{6}R(\delta^\mu_\sigma g_{\rho\nu} - \delta^\mu_\nu g_{\rho\sigma}). \tag{1.5.7}$$

Its vanishing leads to conformally flatness condition of the fluid distribution. This tensor can be decomposed into two parts, magnetic $M_{\mu\nu}$ and electric $\mathbb{E}_{\mu\nu}$ parts

$$\overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\nu} = C_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta}, \quad \overset{(\mathrm{eff})}{M}_{\mu\nu} = \frac{1}{2}\eta_{\mu\alpha\delta\gamma}C^{\delta\gamma}_{\nu\lambda}u^{\alpha}u^{\lambda}. \tag{1.5.8}$$

Just analogous to electric and magnetic components of electromagnetic force, the decomposition of the Weyl tensor into electric and magnetic parts is an attempt to represent constituents of gravitational force in different directions.

1.5.3 Heat Dissipation

Heat dissipation (due to emission of photons or neutrinos which are massless particles) during collapse cannot be overemphasized and is, indeed, a characteristic process during stellar evolution. It is irreversible process in which energy of a system is transformed from initial to final state so that the initial one has more capability to do any mechanical work. For example, heat dissipation as it is a transformation of energy from a hotter to cooler body. It plays an important role in the evolution and formation of various astrophysical objects, e.g., dissipation due to neutrino emission of gravitational binding energy leads to the formation of neutron stars or black holes [38]. Gravitational collapse is a highly dissipative process which leads to structure formation of the universe [39].

1.5.4 Density Inhomogeneity

The energy density inhomogeneity is indeed unavoidable during stellar evolution. The homogeneous or non-homogeneous behavior of energy density depends upon derivative of density variable. If the derivative vanishes ($\overset{(\mathrm{eff})}{\rho}_{,\nu} = 0$), there will be homogenous distribution of density, otherwise, there will be density inhomogeneity. The collection of dynamical variables which causes density inhomogeneity is known as density inhomogeneity factors. The vanishing of such combination of dynamical variables is necessary and sufficient for homogeneity of energy density [23].

1.5.5 Transport Equation

The heat transport equation elucidates dissipation process and propagation of thermal energy inside the system. It controls the flow of radiation in fluid. The non-vanishing value of time relaxation parameter τ (a positive definite quantity having different physical meaning) serves as a cardinal parameter in this equation which defines the time interval in which the system returns to its steady state. Consequently, studying transient regimes, e.g., the evolution between two steady states, the role of τ cannot be ignored. The heat transport equation for propagation of thermal perturbations is given as

$$\tau h^{\alpha}_{\beta}q^{\beta}_{;\mu}u^{\mu}+q^{\alpha}=-Kh^{\alpha\beta}(\mathbb{T}_{,\beta}+\mathbb{T}a_{\beta})-\frac{1}{2}K\mathbb{T}^{2}\left(\frac{\tau u^{\mu}}{K\mathbb{T}^{2}}\right)_{;\mu}q^{\alpha}, \tag{1.5.9}$$

where K is thermal conductivity parameter and $\mathbb{T}$ represents temperature parameter which measures the quantity of heat (degree of hotness and coldness).

1.6 Structure Scalars and Evolution Equations

The scalar functions associated with fluid contents are termed as structure scalars. Bel [40] introduced these quantities through orthogonal splitting of the Riemann tensor. These scalars in turn are used to reduce complexity of analyzing various astrophysical scenarios to describe structure and evolution of self-gravitating systems. The triplet of tensors are given as follows [23]

$$X_{\mu\nu}=\frac{1}{2}\eta^{\alpha\beta}_{\mu\gamma}R^{*}_{\alpha\beta\nu\delta}u^{\gamma}u^{\delta},\quad Y_{\mu\nu}=R_{\mu\gamma\nu\delta}u^{\gamma}u^{\delta},\quad Z_{\mu\nu}=\frac{1}{2}\eta_{\mu\gamma\alpha\beta}R^{\alpha\beta}_{\nu\delta}u^{\gamma}u^{\delta},\qquad(1.6.1)$$

where $R^{*}_{\mu\nu\gamma\delta}=\frac{1}{2}\eta_{\alpha\beta\gamma\delta}R^{\alpha\beta}_{\mu\nu}$. The tensors $X_{\mu\nu}$ and $Y_{\mu\nu}$ are further written in the combination of their trace and traceless parts as

$$X_{\mu\nu}=\frac{1}{3}X_{T}h_{\mu\nu}+X_{TF}(u_{\mu}u_{\nu}-\frac{1}{3}h_{\mu\nu}),\quad Y_{\mu\nu}=\frac{1}{3}Y_{T}h_{\mu\nu}+Y_{TF}(u_{\mu}u_{\nu}-\frac{1}{3}h_{\mu\nu}),\quad(1.6.2)$$

where $X_{T}=X^{\mu}_{\mu}$, $Y_{T}=Y^{\mu}_{\mu}$ are the trace parts of the corresponding tensors and X_{TF}, Y_{TF} denote traceless parts of the respective tensors. The scalar function associated to the tensor $Z_{\mu\nu}$ is defined as $Z=\sqrt{Z_{\mu\nu}Z^{\mu\nu}}$. The quantities X_{T}, X_{TF}, Y_{T}, Y_{TF} and Z are known as structure scalars (scalar functions). These scalars further consist of dynamical variables and have individual physical meanings for the description of several features of relativistic self-gravitating fluids. Some particular physical aspects of these scalars are given as follows.

- X_{T} is directly linked with the energy density of fluid.
- The trace-free part X_{TF} controls density inhomogeneity (irregularity) in the fluid configuration.
- The trace part Y_{T} describes the mass for a system in equilibrium.
- The trace-free part Y_{TF} measures the effects of inhomogeneous density and anisotropic pressure on the mass function.
- The scalar Z describes the dissipative flux.

An equation (usually expressed in differential form) which explores various phases of an evolving system by linking structure scalars is known as evolution equation.

There is a complete set of evolution equations evoking the phenomenon of stellar evolution. This set of equations comprises dynamical (conservation) equations, Raychaudhuri equation for expansion parameter and evolution equations for shear as well as the Weyl tensor. The dynamical equations evaluate conservation of total energy of an evolving system obtained through Bianchi identities $R_{\mu\nu\alpha\beta;\gamma}+R_{\mu\nu\gamma\alpha;\beta}+R_{\mu\nu\beta\gamma;\alpha}=0$. Raychaudhuri equation describes the evolution (expansion or contraction) equation of expansion scalar which measures volume of the system. This equation is derived from Ricci identity for four velocity

$$u_{\mu;\nu;\alpha}-u_{\mu;\alpha;\nu}=R^{\beta}_{\mu\nu\alpha}u_{\beta}, \tag{1.6.3}$$

where $u_{\mu;\nu}=a_{\mu}u_{\nu}+\sigma_{\mu\nu}+\frac{1}{3}\vartheta h_{\mu\nu}$. The evolution equation of shear tensor demonstrates stellar deformation appearing during different phases of the evolution. The evolution equations for the Weyl tensor are used to analyze factors causing density inhomogeneity.

Chapter 2

Wormhole Solutions

In this chapter, we explore static spherically symmetric wormhole solutions under the dark effects of $f(G)$ gravity. For this purpose, we construct field equations by taking the effective energy-momentum tensor which contributes combined effects of matter and dark sources. Firstly, we explore possible existence of wormhole in galactic halo region either by assuming a viable $f(G)$ model (1.1.6) to construct shape function or deduce $f(G)$ model by specifying the shape function. We also investigate stability of the resulting wormhole solutions. Secondly, we study wormhole solutions with conformal symmetry by taking two types of shape function and power-law $f(G)$ model (1.1.7). In both cases, we examine the behavior of WEC to deduce the nature of matter threading the wormholes.

This chapter is organized in three sections. In section **2.1**, we develop field equations for static spherically symmetric wormhole spacetime in the presence of dark sources. Section **2.2** investigates possible existence of wormhole in galactic halo region. In section **2.3**, we discuss wormhole solutions by imposing conformal symmetry. Results of this chapter have been compiled in the form of two papers which have been published separately [41, 42].

2.1 Field Equations

For wormhole geometry (1.2.1), we matter distribution given as

$$T_{\mu\nu} = (\rho + p_t)u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t)v_\mu v_\nu, \tag{2.1.1}$$

where p_r and p_t are radial and tangential components of pressure. The four velocity u_μ and unit spacelike vector v_μ satisfy the relations $u_\mu u^\mu = -1$, $v_\mu v^\mu = 1$ and $u_\mu v^\mu = 0$. The energy-momentum tensor can also be written as $T_{\mu\nu} = diag(-\rho(r), p_r(r), p_t(r), p_t(r))$. Equation (1.1.3) yields the corresponding field equations as

$$\frac{b'}{r^2} + \frac{4}{r^5}[(b - rb')(3b - 2r)f'_G + 2rb(r - b)f''_G] - Gf_G + f = \rho, \tag{2.1.2}$$

$$2\frac{\mathcal{R}'}{r}(1 - \frac{b}{r}) - \frac{b}{r^3} + \frac{4\mathcal{R}'}{r^3}(1 - \frac{b}{r})(3b - 2r)f'_G + Gf_G - f = p_r, \tag{2.1.3}$$

$$\begin{aligned}&(1 - \frac{b}{r})[\mathcal{R}'' + \frac{\mathcal{R}'}{r} + \mathcal{R}'^2 - \frac{b'r - b}{2r(r - b)}(\mathcal{R}' + \frac{1}{r})] - \frac{2}{r^3}(1 - \frac{b}{r})[2r(\mathcal{R}'' + \mathcal{R}'^2)\\ &\times(r - b) - 3\mathcal{R}'(b'r - b)]f'_G - \frac{4\mathcal{R}'}{r}(1 - \frac{b}{r})^2 f''_G + Gf_G - f = p_t.\end{aligned} \tag{2.1.4}$$

The GB invariant (0.0.1) takes the form

$$G = \frac{4}{r^5}[\mathcal{R}'(3b - 2r)(b'r - b) - 2r^2 b(\mathcal{R}'^2 + \mathcal{R}'')(1 - \frac{b}{r})]. \tag{2.1.5}$$

In order to check the possibility that wormhole might form without exotic matter, we examine NEC by taking effective energy density and radial pressure from Eqs.(2.1.2) and (2.1.3) as

$$\overset{(\mathrm{eff})}{\rho} + \overset{(\mathrm{eff})}{p_r} = \frac{b'r - b}{r^3} + \frac{2\mathcal{R}'}{r}(1 - \frac{b}{r}) < 0.$$

This implies that

$$\frac{2\mathcal{R}'}{r} < \frac{b - rb'}{2r^2}(1 - \frac{b}{r})^{-1},$$

whose right hand side remains positive leading to the violation of NEC.

2.2 Wormhole in Galactic Halo Region

For a traversable wormhole, there should be no horizon and even no singularity which is possible only if the redshift function $\mathcal{R}(r)$ is finite for all values of r. Thus we assume redshift function as $e^{2\mathcal{R}(r)} = C_1 r^h$ [43], where $h = 2(v^\phi)^2$, v^ϕ is the rotational velocity for flat rotational curves observed in galactic halo. This rotational velocity is nearly constant through these rotational profiles and we let $C_1 = (\frac{1}{r_c})^h$, the integrating constant. Consequently, the field equations (2.1.2)-(2.1.4) and GB invariant (2.1.5) become

$$\frac{b'}{r^2} + \frac{4}{r^5}[(b - rb')(3b - 2r)f'_G + 2rb(r - b)]f''_G - Gf_G + f = \rho, \tag{2.2.1}$$

$$\frac{h}{r^2}(1 - \frac{b}{r}) - \frac{b}{r^3} + \frac{2h}{r^4}(1 - \frac{b}{r})(3b - 2r)f'_G + Gf_G - f = p_r, \tag{2.2.2}$$

$$(1 - \frac{b}{r})[(\frac{h}{2r})^2 - \frac{(h+2)(b'r - b)}{4r^2(r - b)}] - \frac{2h}{r^4}(1 - \frac{b}{r})[(h - 1) \times(r - b) - \frac{3}{2}(b'r - b)]f'_G - \frac{2h}{r^2}(1 - \frac{b}{r})^2 f''_G + Gf_G - f = p_t, \tag{2.2.3}$$

$$G = \frac{4}{r^5}[\frac{h}{2r}(3b - 2r)(b'r - b) - \frac{h(h-2)b}{2}(1 - \frac{b}{r})]. \tag{2.2.4}$$

These are the general expressions of matter contents in terms of $b(r)$ and specific $f(G)$ model to thread the wormhole. Using Eq.(1.3.1) in (2.2.1), we obtain

$$\frac{b'}{r^2} + \frac{4}{r^5}[(b - rb')(3b - 2r)f'_G + 2rb(r - b)]f''_G - Gf_G + f = \frac{kr_c^2}{r(r + r_c)}, \tag{2.2.5}$$

which consists of two unknowns $f(G)$ and $b(r)$. In order to discuss the wormhole structure in galactic halo, we impose constraints on matter contents by examining the validity of WEC. For this purpose, we adopt strategy of specifying (i) an arbitrary $f(G)$ model to construct $b(r)$ and (ii) an expression of $b(r)$ to deduce $f(G)$ model.

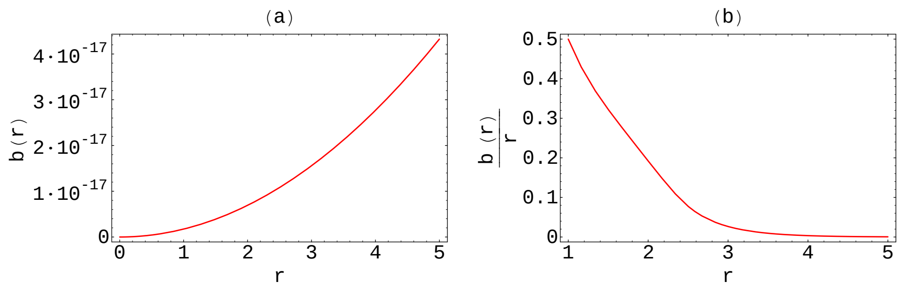

Figure 2.1: Plots of $b(r)$ and $\frac{b(r)}{r}$ versus r. (a) displays the shape function $b(r)$ and (b) shows $\frac{b(r)}{r}$ versus r.

2.2.1 Wormholes for a Viable $f(G)$ Model

Using Eq.(1.1.6) in (2.2.5), we obtain differential equation for $b(r)$ as

$$\frac{b'}{r^2}+\frac{4}{r^5}[\alpha G^{l-2}G'((b-rb')(3b-2r))(l(l-1)+\beta(l+m)(l+m-1)G^m) \\ +2r^2b(1-\frac{b}{r})(\alpha l(l-1)G^{l-2}((l-2)G^{-1}G'^2+G'')+\alpha\beta(l+m)(l+m-1) \\ \times G^{l+m-2}((l+m-2)G^{-1}G'^2+G''))]-\alpha G^l(l-1+\beta(l+m-1)G^m) \\ =\frac{kr_c^3}{r(r+r_c)^2}, \tag{2.2.6}$$

where G is given in Eq.(2.2.4). This equation cannot be solved analytically, so we solve it numerically for $b(r)$. To check the behavior of shape function, we use graphical analysis by incorporating the values of parameters as $\alpha = 0.001,\ \beta = 1,\ k = 0.05,\ r_c = 10,\ l = 0.0002$ and $m = -0.05$ along with initial conditions $b(1) = 0.5$ and $b'(1) = 0.6$. These are arbitrary constants which give us reasonable results. Any other combination of these parameters change the results. Figure **2.1(a)** shows increasing behavior of $b(r)$ versus r while Figure **2.1(b)** indicates that $\frac{b}{r}\rightarrow 0$ as $r\rightarrow\infty$ leading to an asymptotically flat universe. Also, it meets the condition $1-\frac{b}{r}>0$. The throat radius of wormhole is located at $r = r_0$ for which the plot of $b(r) - r$ crosses the radial axis. Figure **2.2(a)** shows that the plot of $b(r) - r$ meets the radial axis at

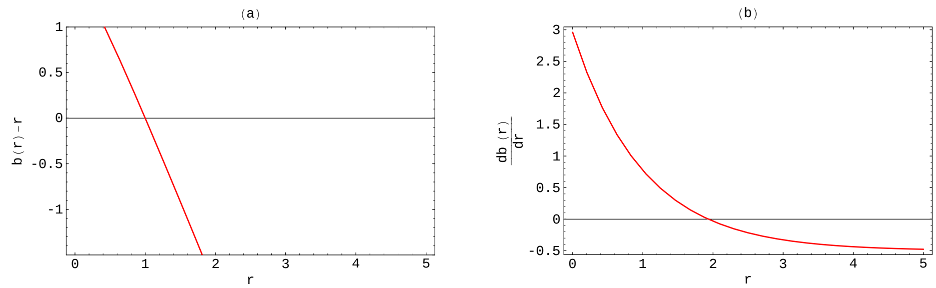

Figure 2.2: Plot (a) shows the behavior of $b(r)-r$ and (b) represents $\frac{db(r)}{dr}$ versus r.

$r_0 \approx 1.1$ which is the throat radius. Also, $\frac{b}{r} < 1$ for $r_0 \approx 1.1$ satisfying the essential condition of shape function. The graph of $\frac{db}{dr}$ is shown in Figure **2.2(b)** which depicts $\frac{db}{dr}(1.1) \approx 0.49 < 1$ (i.e., $\frac{db}{dr}(r_0) < 1$) and hence the flaring-out condition is fulfilled.

We analyze the nature of matter that threads a wormhole via WEC for which the energy density and pressure components are given as follows

$$\begin{aligned}
\rho &= \frac{b'}{r^2} + \frac{4}{r^5}[\alpha G^{l-2}G'((b-rb')(3b-2r))(l(l-1)+\beta(l+m)(l+m-1)G^m)\\
&+2r^2 b(1-\frac{b}{r})(\alpha l(l-1)G^{l-2}((l-2)G^{-1}G'^2+G'')+\alpha\beta(l+m)(l+m-1)\\
&\times G^{l+m-2}((l+m-2)G^{-1}G'^2+G''))] - \alpha G^l(l-1+\beta(l+m-1)G^m),\\
p_r &= \frac{h}{r^2}(1-\frac{b}{r}) - \frac{b}{r^3} + \frac{2h\alpha}{r^4}G^{l-2}(1-\frac{b}{r})(3b-2r)(l(l-1)+\beta(l+m)(l+m-1)\\
&\times \; G^m)G' + \alpha G^l(l-1+\beta(l+m-1)G^m),\\
p_t &= (1-\frac{b}{r})[(\frac{h}{2r})^2 - \frac{(h+2)(b'r-b)}{4r^2(r-b)}] - \frac{2h\alpha}{r^4}(1-\frac{b}{r})[(h-1)(r-b)-\frac{3}{2}(b'r-b)]\\
&\times \; G^{l-2}(l(l-1)+\beta(l+m)(l+m-1)G^m)G' - \frac{2h}{r^2}(1-\frac{b}{r})^2(\alpha l(l-1)((l-2)\\
&\times \; G^{-1}G'^2+G'')G^{l-2}+\alpha\beta(l+m)(l+m-1)G^{l+m-2}((l+m-2)G^{-1}G'^2+G''))\\
&+ \; \alpha G^l(l-1+\beta(l+m-1)G^m).
\end{aligned}$$

The graphs of WEC ($\rho,\ \rho+p_r$ and $\rho+p_t$) against r (2.2.6) are shown in Figure **2.3** using the same parametric values. Figure **2.3(a)** indicates the behavior of energy

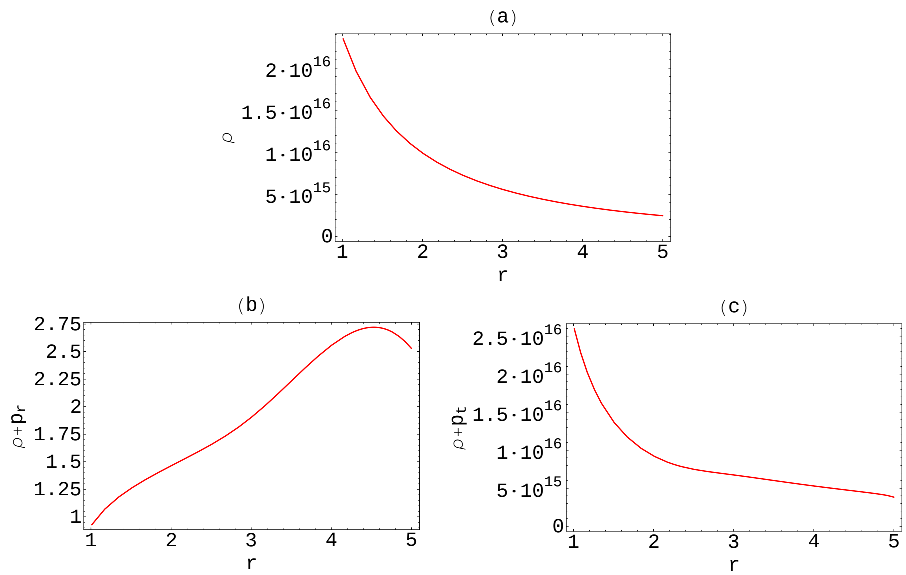

Figure 2.3: Plots of WEC with specific $f(G)$ model, (a) represents ρ versus r, (b) represents $\rho+p_r$ versus r and (c) shows $\rho+p_t$ versus r.

density which is decreasing but remains positive for the whole range of r. The plot in Figure **2.3(b)** represents initially increasing then decreasing but positive behavior of $\rho+p_r$ for all values of r. The profile of $\rho+p_t$ in Figure **2.3(c)** shows almost the same behavior as Figure **2.3(a)**. Thus, the model and resulting shape function obey WEC in the galactic halo region and hence accommodate the wormhole geometry with ordinary matter.

2.2.2 Wormholes for a Particular Shape Function

Here we assume a specific form of the shape function $b(r)$ and construct $f(G)$ model. We consider the following particular form of the shape function as [44]

$$b(r)=r_t\left(\frac{r}{r_t}\right)^{\gamma},\quad \gamma<1 \tag{2.2.7}$$

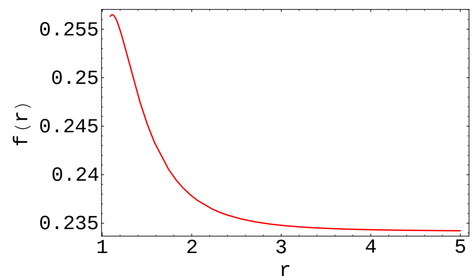

Figure 2.4: Plot of $f(r)$ against r.

where γ is an arbitrary constant and r_t is the throat radius. The expression for GB invariant becomes

$$G=\frac{4}{r^5}(\frac{r}{r_t})^{\gamma}[\frac{hr_t}{2r}(3r_t(\frac{r}{r_t})^{\gamma}-2r)(\gamma-1)-\frac{h(h-2)r_t}{2}(1-(\frac{r}{r_t})^{\gamma-1})].$$

Using this equation and (2.2.7) in Eq.(2.2.5), we obtain the following differential equation for $f(G)$ in terms of r as

$$\frac{\gamma}{r^2}(\frac{r}{r_t})^{\gamma-1}+\frac{4}{r^5}(\frac{r}{r_t})^{\gamma}[(3r_t(\frac{r}{r_t})^{\gamma}-2r)(r_t-r\gamma(\frac{r}{r_t})^{\gamma})\frac{G'f''-f'G''}{(G')^2}-(1-(\frac{r}{r_t})^{\gamma-1})$$
$$\times 2r^2r_t(\frac{G'f''-f'G''}{(G')^2})']-G\frac{f'}{G'}+f=\frac{kr_c^3}{r(r+r_c)^2}.$$

We solve this equation numerically by choosing the values $r_t=0.35$ and $\gamma=0.02$ with initial conditions $f(1)=0.25,\ f'(1)=0.15$. The function obtained from the above equation is based on the Navarro-Frenk-White energy density profile and should be sufficient to motivate researchers to look for wormholes in galactic halos observationally. Figure **2.4** shows that the function $f(r)$ is positively decreasing against r.

To thread the wormhole solutions by normal matter, this function should satisfy WEC. The graphs of $\rho,\ \rho+p_r$ and $\rho+p_t$ are given in Figure **2.5** by taking the same values of parameters. The graph of energy density in Figure **2.5(a)** shows decreasing

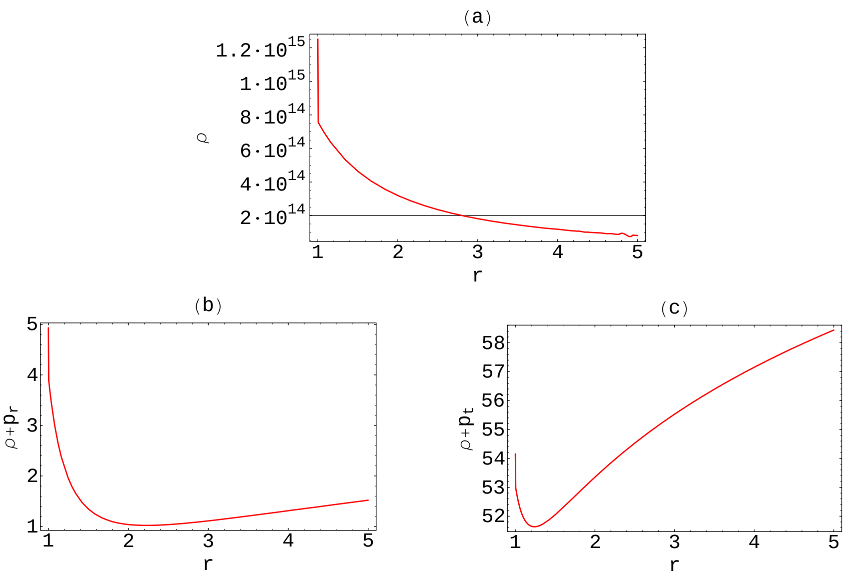

Figure 2.5: Plots of WEC (ρ, $\rho+p_r$ and $\rho+p_t$) versus r for specific shape function $b(r)$.

but positive behavior of ρ. The behavior of $\rho+p_r$ in Figure **2.5(b)** initially decreases with positive values and then continuously increases after $r \approx 2.35$ while Figure **2.5(c)** depicts the behavior of $\rho+p_t$ which is the same as that of $\rho+p_r$, thus WEC is satisfied. This shows that physically acceptable wormholes exist in galactic halo threaded by normal matter for all values of r.

2.2.3 Equilibrium Condition

Here, we check the equilibrium state of wormhole solutions for both cases. For this purpose, we consider the generalized Tolman-Oppenheimer-Volkov equation [45] in effective manner as

$$\frac{d\overset{(\text{eff})}{p_r}}{dr}+\frac{\mu'}{2}(\overset{(\text{eff})}{\rho}+\overset{(\text{eff})}{p_r})+\frac{2}{r}(\overset{(\text{eff})}{p_r}-\overset{(\text{eff})}{p_t})=0,$$

for the metric $ds^2=diag(e^{\mu(r)},-e^{\nu(r)},-r^2,-r^2\sin^2\theta)$. We rewrite this equation as

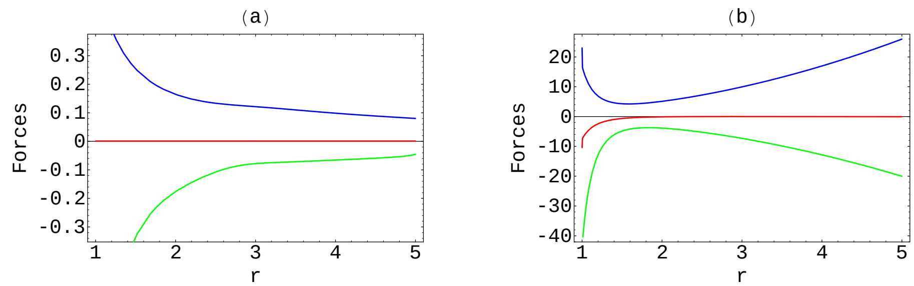

Figure 2.6: Plots of three forces gravitational (red), hydrostatic (green) and anisotropic (blue) acting on a fluid inside the galactic halo for both cases. (a) shows the equilibrium state of wormholes for f(G) model and (b) represents the equilibrium state for shape function.

$$-\frac{\overset{(\text{eff})}{\mathbb{M}}(\overset{(\text{eff})}{\rho}+\overset{(\text{eff})}{p_r})}{r^2}e^{\frac{\mu-\nu}{2}}-\frac{d\overset{(\text{eff})}{p_r}}{dr}+\frac{2}{r}(\overset{(\text{eff})}{p_t}-\overset{(\text{eff})}{p_r})=0, \tag{2.2.8}$$

where $\overset{(\text{eff})}{\mathbb{M}}=\frac{\mu' r^2}{2}e^{\nu-\mu}$ is the effective gravitational mass. Equation (2.2.8) describes equilibrium state for wormhole supported by gravitational and hydrostatic forces plus anisotropic force (force due to anisotropy in matter distribution). These forces, respectively, are defined as

$$F_G=-\frac{\overset{(\text{eff})}{\mathbb{M}}(\overset{(\text{eff})}{\rho}+\overset{(\text{eff})}{p_r})}{r^2}e^{\frac{\mu-\nu}{2}},\quad F_H=-\frac{d\overset{(\text{eff})}{p_r}}{dr},\quad F_A=\frac{2}{r}(\overset{(\text{eff})}{p_t}-\overset{(\text{eff})}{p_r}).$$

For a wormhole to be in equilibrium, these forces should satisfy the relation

$$F_G+F_H+F_A=0.$$

Figure **2.6** shows three forces F_G, F_H and F_A for both cases taking the same values of parameters. Both graphs indicate that the equilibrium state of wormhole solutions can be attained through the combined effect of these forces. We can see that gravitational force is much smaller (becomes zero) than the other two forces while hydrostatic and anisotropic forces are opposite to each other. This balances the system and makes the wormhole solutions in equilibrium state.

2.3 Conformally Symmetric Traversable Wormholes

Herrera and his collaborators [46] considered static symmetry vector ξ and found singular solutions for isotropic and anisotropic fluids at the center of stars. To overcome this shortfall, Maartens and Maharaj [47] assumed static φ (conformal factor) but non-static ξ and obtained singularity free solutions at the center of stars. Here, we shall follow this later approach. It is mentioned here that singular solutions at the center of stars are not problematic for wormhole geometries as there is no center for wormholes. Thus we also consider phantom wormholes with static conformal symmetry. The non-static conformal vector field is [47]

$$\xi = \mathfrak{a}(t,r)\partial_t + \mathfrak{b}(t,r)\partial_r, \tag{2.3.1}$$

and static conformal factor is $\varphi = \varphi(r)$. Equation (1.4.3) for (1.2.1) yields

$$\begin{aligned} \mathfrak{a} &= a_1 + \frac{\mathfrak{c}t}{2}, \quad \mathfrak{b} = \frac{a_2 r}{2}\sqrt{1-\frac{b(r)}{r}}, & (2.3.2)\\ \varphi(r) &= a_2\sqrt{1-\frac{b(r)}{r}}, & (2.3.3)\\ e^{2\mathcal{R}(r)} &= a_3 r^2 \exp\left[-\frac{2\mathfrak{c}}{a_2}\int\frac{1}{r\sqrt{1-\frac{b(r)}{r}}}dr\right], & (2.3.4)\end{aligned}$$

where a_1, a_2, a_3 and $\mathfrak{c}$ are constants of integration. Using Eq.(2.3.2) in (2.3.1), we have

$$\xi = \left(a_1 + \frac{\mathfrak{c}t}{2}\right)\partial_t + \frac{a_2 r}{2}\sqrt{1-\frac{b(r)}{r}}\,\partial_r. \tag{2.3.5}$$

Without any loss of generality, we may assign $a_1 = 0$ and $a_2 = 1$ [47] so that

$$\begin{aligned} \xi &= \left(\frac{\mathfrak{c}t}{2}\right)\partial_t + \frac{r}{2}\sqrt{1-\frac{b(r)}{r}}\,\partial_r, & (2.3.6)\\ b(r) &= r(1-\varphi^2(r)), & (2.3.7)\end{aligned}$$

$$\mathcal{R}(r) = \ln[a_3 r] - \mathfrak{c}\int \frac{1}{r\sqrt{1-\frac{b}{r}}}dr. \tag{2.3.8}$$

Interestingly, the conformal factor in Eq.(2.3.7) reduces to zero at throat (i.e., $\varphi(r_0) = 0$ as $b(r_0) = r_0$ at throat). Consequently, the field equations (2.1.2)-(2.1.4) and GB invariant (2.1.5) in terms of conformal factor can be written as

$$\begin{aligned}
\rho &= \frac{1}{r^2}(1-\varphi^2-2r\varphi\varphi') - \frac{2\varphi^2}{r^3}[(4r\mathfrak{c}\varphi'-\varphi^2+\mathfrak{c}^2+6r\varphi\varphi'-6r\mathfrak{c}\varphi')]f'_G - \frac{8\varphi^3}{r^5} \\
&\times (\varphi-\mathfrak{c})f''_G - Gf_G + f, &(2.3.9)\\
p_r &= \frac{1}{r^2}(3\varphi^2-2\mathfrak{c}\varphi-1) + \frac{4}{r^3}\varphi^2(\varphi-\mathfrak{c})(1-3\varphi^2)f'_G + Gf_G - f, &(2.3.10)\\
p_t &= \frac{1}{r^2}(2\varphi^2-2\mathfrak{c}\varphi+2r\varphi\varphi'+\mathfrak{c}^2) - \frac{4\varphi^2}{r^3}[3r\varphi\varphi'-\mathfrak{c}r\varphi'-\mathfrak{c}\varphi+\mathfrak{c}^2]f'_G - \frac{4\varphi}{r^2} \\
&\times (\varphi-\mathfrak{c})f''_G + Gf_G - f, &(2.3.11)\\
G &= \frac{8}{r^4}[r\varphi\varphi'(\mathfrak{c}\varphi-1)-(\mathfrak{c}-\varphi)(3r\varphi^2\varphi'-\mathfrak{c}\varphi+\mathfrak{c})]. &(2.3.12)
\end{aligned}$$

Here we discuss some specific wormhole solutions by considering a particular $f(G)$ model (1.1.7) with parameters as $\alpha = 0.1,\ \mathfrak{c} = 1.5,\ \varpi = 0.1$ and initial conditions $b(0.5) = 1,\ b'(0.5) = 0.1$ for graphical analysis.

2.3.1 Some Specific Wormhole Solutions

We investigate wormhole solutions by assuming two types of shape functions and a particular equation of state to deduce shape function. First, we consider two types of shape functions [14].

i. $b(r) = r_0$

The interesting feature of this constant shape function is that the energy density for matter vanishes and we are left only with energy density of dark sources. Equation

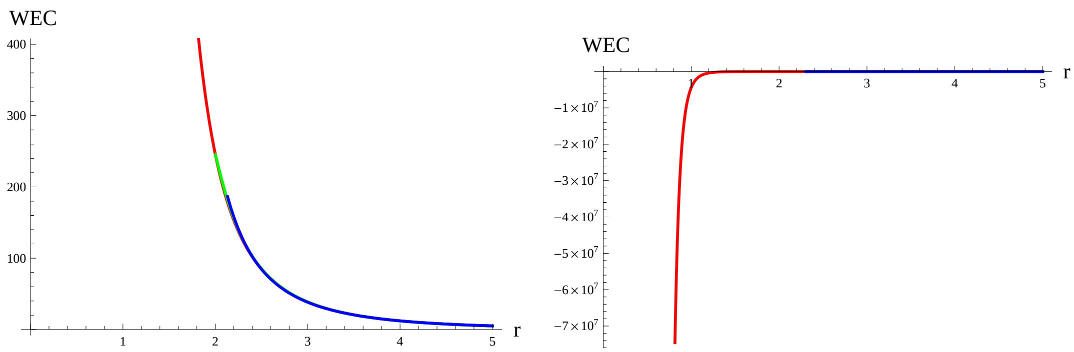

Figure 2.7: Plots of WEC ρ (red), $\rho+p_r$ (green) and $\rho+p_t$ (blue) for constant shape function $b(r)=r_0$ taking $r_0=2$, $n=2$ (left) and $n=3$ (right).

(2.3.4) leads to

$$e^{2\mathcal{R}(r)}=a_3^2r^2\left(2r-r_0+2r\sqrt{1-\frac{r_0}{r}}\right)^{-2\mathfrak{c}}.$$

The matter variables and GB invariant become

$$\begin{aligned}
\rho &= \frac{4}{r^3}[r_0(3r_0-2r)f'_G+2rr_0(r-r_0)f''_G-Gf_G+f,\\
p_r &= \frac{2r-3r_0}{r^3}-\frac{2\mathfrak{c}}{r^2}\sqrt{1-\frac{r_0}{r}}+\frac{4}{r^3}(1-\frac{r_0}{r})(\frac{1}{r}-\frac{\mathfrak{c}}{\sqrt{r(r-r_0)}})(3r_0-2r)f'_G\\
&+ Gf_G-f,\\
p_t &= \frac{1}{r^2}(2+\mathfrak{c}^2-\frac{r_0}{r}-2\mathfrak{c}\sqrt{1-\frac{r_0}{r}})-\frac{2}{r^3}(1-\frac{r_0}{r})[\frac{r-r_0}{(r(r-r_0))^{\frac{3}{2}}}(r+2\mathfrak{c}\sqrt{r(r-r_0)})\\
&+ 3r_0(\frac{1}{r}-\frac{\mathfrak{c}}{\sqrt{r(r-r_0)}})]f'_G+Gf_G-f,\\
G &= \frac{8}{r^4}[\frac{r_0}{2r}(\mathfrak{c}\sqrt{1-\frac{r_0}{r}}-1)-(\frac{3r_0}{2r}\sqrt{1-\frac{r_0}{r}}+\mathfrak{c}\frac{r_0}{r})(\mathfrak{c}-\sqrt{1-\frac{r_0}{r}})].
\end{aligned}$$

We examine the behavior of WEC by plotting their graphs which indicate that these conditions are satisfied for even values of n but violated for odd range of n. Here, we give one graph for $n=2$ (Figure **2.7** (left), exemplifying the even range of n) and one graph for $n=3$ (Figure **2.7** (right) indicating the odd range of n) for the respective case. The graph on left side shows the same decreasing but positive behavior for

ρ (red), $\rho + p_r$ (green) and $\rho + p_t$ (blue) representing the validity of WEC. Thus the even values of n lead to physically realistic wormholes (i.e., wormholes threaded by normal matter). The graph for odd values of n on the right side of Figure **2.7** shows increasing but negative behavior for ρ (red), $\rho + p_r$ (green) and $\rho + p_t$ (blue) indicating the violation of WEC. This demonstrates that wormholes are supported by exotic matter. In fact, here the effective energy-momentum tensor is responsible for this violation with odd range of n.

ii. $b(r) = \frac{r_0^2}{r}$

For this shape function, Eq.(2.3.4) gives

$$e^{2\mathcal{R}(r)} = a_3^2 r^2 (r + \sqrt{r^2 - r_0^2}).$$

The field equations for this solution take the form

$$\begin{aligned}
\rho &= -\frac{2r_0^2}{r^4} + \frac{4}{r^5}[\frac{2r_0^3}{r^2}(3 - 2r)f'_G + 2rr_0(r^2 - r_0^2)f''_G] - Gf_G + f, \\
p_r &= \frac{1}{r^2}(2 - \frac{3r_0^2}{r^2} - \frac{2\mathfrak{c}\sqrt{r^2 - r_0^2}}{r}) + \frac{4}{r^6}(\frac{1}{r} - \frac{\mathfrak{c}}{\sqrt{r^2 - r_0^2}})(r^2 - r_0^2)(3r_0^2 - 2r^2)f'_G \\
&+ Gf_G - f, \\
p_t &= \frac{1}{r^2}(2 + \mathfrak{c}^2 - \frac{2\mathfrak{c}\sqrt{r^2 - r_0^2}}{r}) - \frac{4}{r^6}\sqrt{r^2 - r_0^2}(r_0^2 - r^2 + \mathfrak{c}r\sqrt{r^2 - r_0^2})f'_G \\
&- \frac{4}{r^3}(r - r_0^2)^2(\frac{1}{r} - \frac{\mathfrak{c}}{\sqrt{r^2 - r_0^2}})f''_G + Gf_G - f
\end{aligned}$$

with GB invariant

$$G = \frac{8}{r^4}[\frac{r_0^2}{r^2}(\frac{\mathfrak{c}\sqrt{r^2 - r_0^2}}{r} - 1) - (\mathfrak{c} - \frac{\sqrt{r^2 - r_0^2}}{r})(\frac{r_0^2\sqrt{r^2 - r_0^2}}{r^3} - \frac{\mathfrak{c}(r^2 - r_0^2)}{r^2} + \mathfrak{c})].$$

Similar to the first case, WEC are valid only for even values of n. Figure **2.8** (left) shows the behavior of WEC ($\rho,\ \rho + p_r,\ \rho + p_t$) for $n = 2$ (exemplifying the even

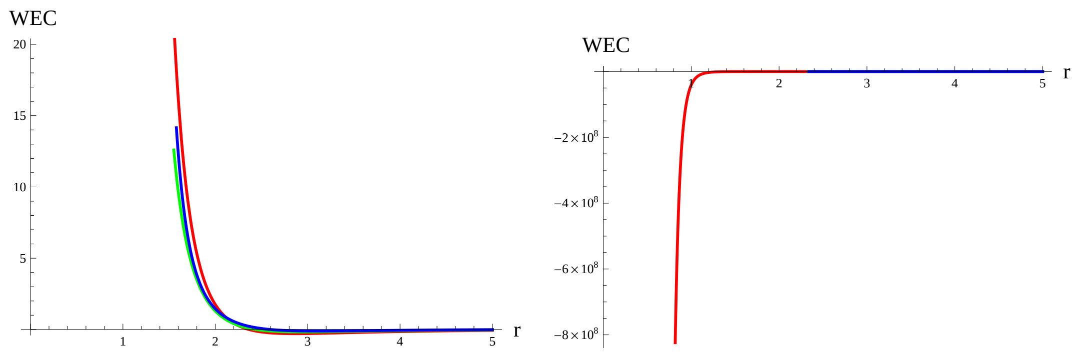

Figure 2.8: Plots of WEC ρ (red), $\rho + p_r$ (green) and $\rho + p_t$ (blue) for specific shape function $b(r) = \frac{r_0^2}{r}$ using $n = 2$ (left) and $n = 3$ (right).

values of n). This shows the same behavior for $\rho,\ \rho + p_r,\ \rho + p_t$ slightly down to the negative side but then goes along r-axis (zero) for $r \geq 4$. Hence physically realistic wormholes are threaded by normal matter for $r \geq 4$ with even values of n. The plot of WEC on the right side of Figure **2.8** shows negatively increasing behavior for $n = 3$ (illustrating the odd range of n) which accomplishes the violation of energy conditions leading to the wormholes threaded by exotic matter.

Figures **2.7** and **2.8** represent the behavior of energy conditions for shape functions $b(r) = r_0$ and $b(r) = \frac{r_0^2}{r}$, respectively along with model (1.1.7) for which G is defined in Eq.(2.3.12). The model parameter n can take the values $1, 2, 3, 4, 5 \cdots$ to draw the graphs of WEC. On substituting the successive values of n, the WEC shows negative behavior for $n = 1, 3, 5 \cdots$ depicting the violation of energy conditions. This violation arises due to the effective energy-momentum tensor which logically leads to the wormholes threaded by exotic matter. On the other hand, for $n = 2, 4, 6, \cdots$, WEC stays on positive side representing the validity of energy conditions leading to physically realistic wormholes (threaded by normal matter). The graphical analysis clearly shows that when n is even, matter terms become dominant over GB (DE)

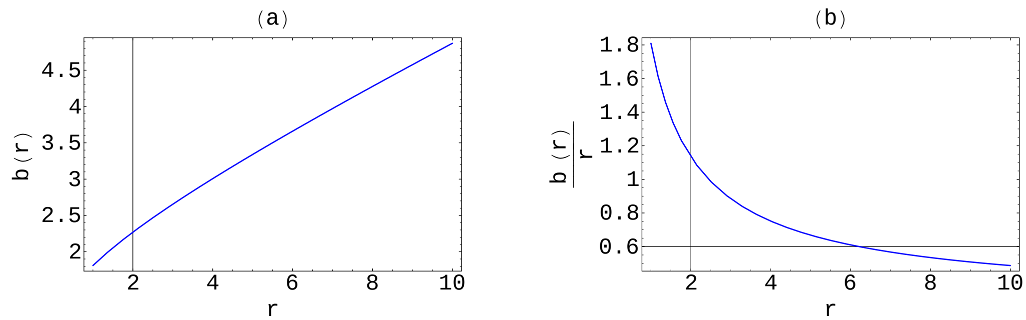

Figure 2.9: Plots of (a) $b(r)$ and (b) $\frac{b(r)}{r}$ versus r with $n = 2$ when EoS is $\rho = \varpi(p_t - p_r)$.

terms. This leads to physically realistic state where the usage of exotic matter has been minimized and wormholes could exist which can be threaded by normal matter. When n is odd, DE terms become dominant over matter terms, i.e., the effective energy-momentum tensor becomes pivotal which recommends exotic matter to thread unrealistic wormholes.

iii. Specific Equation of State

An interesting EoS of the form $\rho = \varpi(p_t - p_r)$ (ϖ is an arbitrary constant) was first time studied by Böhmer et al. [14] which they used to evaluate shape function. Using Eqs.(2.1.2)-(2.1.4) in this EoS, we obtain

$$\frac{b'}{r^2} - \varpi(2\frac{\mathcal{R}'}{r}(1-\frac{b}{r}) - \frac{b}{r^3} - (1-\frac{b}{r})[\mathcal{R}'' + \frac{\mathcal{R}'}{r} + \mathcal{R}'^2 - \frac{b'r-b}{2r(r-b)}(\mathcal{R}' + \frac{1}{r})]) + [\frac{4}{r^5}$$
$$\times(3b-2r)(b-rb') - \varpi(\frac{4\mathcal{R}'}{r^3}(1-\frac{b}{r})(3b-2r)) + \frac{2}{r^3}(1-\frac{b}{r})(2r(\mathcal{R}''+\mathcal{R}'^2)(r-b)$$
$$-3\mathcal{R}'(b'r-b))]f'_G + [\frac{4}{r^5}2rb(r-b) + \varpi\frac{4\mathcal{R}'}{r}(1-\frac{b}{r})^2]f''_G - Gf_G + f = 0,$$

for which R is given in Eq.(2.3.8). This is a differential equation in terms of shape function. We solve it numerically by inserting the values as stated above. The behavior of shape function is shown in Figure **(2.9a)** which is an increasing function

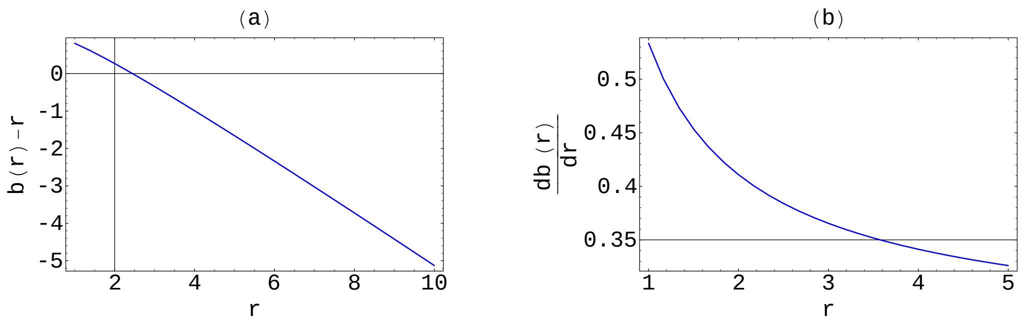

Figure 2.10: Plots of (a) $b(r)-r$ and (b) $\frac{db(r)}{dr}$ versus r taking $n=2$ for EoS $\rho=\varpi(p_t-p_r)$.

while Figure **(2.9b)** indicates that $\frac{b}{r}\rightarrow 0$ as $r\rightarrow\infty$ leading to an asymptotically flat universe. The wormhole throat is located at $r=r_0$ where the graph $b-r$ crosses the radial axis and $b(r)<r$ for $r>r_0$, i.e., the graph of $b-r$ is decreasingly cuts the radial axis. According to this definition, Figure **(2.10a)** suggests throat radius $r_0\approx 2.1$ for which $\frac{db}{dr}\approx 0.38<1$ as shown in Figure **(2.10b)**. This function satisfies the flaring-out condition and hence called the shape function for wormhole geometry.

2.3.2 Phantom Wormholes

Here we discuss traversable wormhole by using another interesting EoS, $p_r=\omega\rho$ in phantom regime ($\omega<-1$) [48]. Using Eqs.(2.1.2) and (2.1.3) with this EoS, we obtain

$$\frac{1}{r^3}(\omega rb'+b)-\frac{2\mathcal{R}'}{r}(1-\frac{b}{r})+\frac{4}{r^5}(3b-2r)[\mathcal{R}'r^2(1-\frac{b}{r})-\omega(b-rb')]f'_G+\frac{8\omega}{r^4}\times(r-b)f''_G-(Gf_G-f)(\omega+1)=0, \quad (2.3.13)$$

yielding a differential equation in terms of redshift and shape function. We solve this equation for static and non-static conformal symmetries.

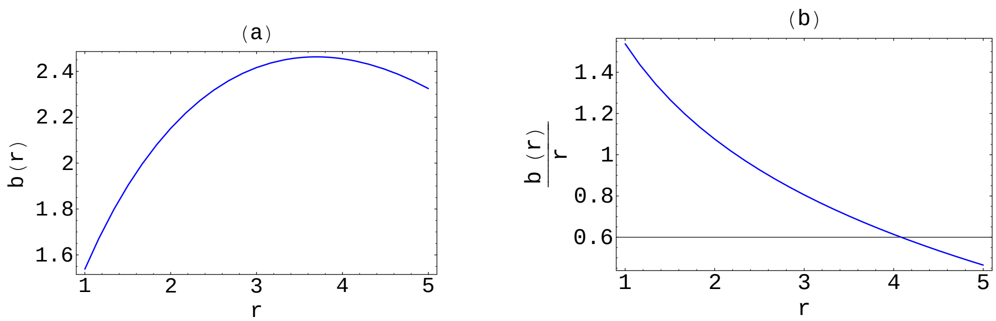

Figure 2.11: Plots of (a) $b(r)$ and (b) $\frac{b(r)}{r}$ versus r choosing $n = 2$ for phantom wormholes with static conformal symmetry.

i. Static Phantom Wormholes

Static conformal symmetry implies the dependence of Eq.(2.3.1) only on radial coordinate r which further implies that $\mathfrak{c} = 0$ in Eqs.(2.3.6) and (2.3.8). With this choice of $\mathfrak{c}$, Eq.(2.3.8) implies that

$$R(r) = \ln(a_3 r).$$

Inserting this value in Eq.(2.3.13), it follows that

$$\frac{1}{r^3}(\omega r b' + b) - \frac{2}{r^2}(1 - \frac{b}{r}) + \frac{4}{r^5}(3b - 2r)[r(1 + \omega b') - b(1 + \omega)]f'_G + \frac{8\omega}{r^4} \times (r - b)f''_G - (Gf_G - f)(\omega + 1) = 0, \tag{2.3.14}$$

which can be solved numerically for shape function $b(r)$. The numerical solution for shape function (with $\omega = -3$) is shown in Figure **(2.11a)** which indicates increasing behavior while Figure **(2.11b)** represents $\frac{b}{r} \to 0$ as $r \to \infty$ which leads to asymptotically flat universe. Figure **(2.12a)** represents that $b - r$ cuts the radial axis at $r_0 \approx 2.1$ which is the throat radius and $\frac{db}{dr}|_{r_0=(2.1)} \approx 0.36 < 1$ (Figure **(2.12b)**. Hence, the flaring-out condition is satisfied and gives the shape function for wormholes.

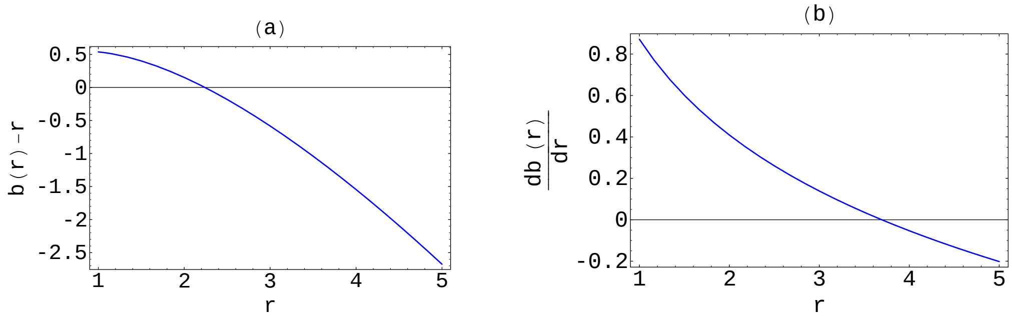

Figure 2.12: Plots of (a) $b(r)-r$ and (b) $\frac{db(r)}{dr}$ versus r with $n=2$ for phantom wormhole with static conformal symmetry.

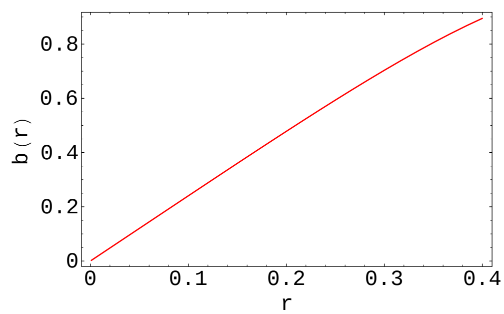

Figure 2.13: Plot of $b(r)$ versus r with $n=2$ for phantom wormhole with non-static conformal symmetry.

ii. Non-static Phantom Wormholes

In this case, the energy density (2.1.2) and radial pressure (2.1.3) along with EoS lead to the following equation

$$\frac{1}{r^2}(\omega+1)-\frac{(\omega+3)}{r^2}\sqrt{1-\frac{b}{r}}+\frac{2\mathfrak{c}}{r^2}\sqrt{1-\frac{b}{r}}-\frac{\omega(b-rb')}{r^3}+[-(1+\omega r^4)$$
$$\times(1-\frac{b}{r})\frac{12(b-rb')}{r^3}+8\omega r(b-rb')-\frac{4(1+2\omega r^4)}{r^3}(1-\frac{b}{r})^2+\frac{4}{r^3}(1-\frac{b}{r})$$
$$\times(2\omega r^6+2\mathfrak{c}r-3r+3r^2+6)+\frac{4\mathfrak{c}(1-2r)}{r^3}(1-\frac{b}{r})^{\frac{3}{2}}+\frac{4\mathfrak{c}(r-r^2-6)}{r^3}$$
$$\times\sqrt{1-\frac{b}{r}}]f'_{_G}-(Gf_{_G}+f)(\omega-1)=0 \tag{2.3.15}$$

for shape function. Its numerical solution is shown in Figure **2.13** which depicts constant behavior and does not satisfy the flaring-out condition, hence no wormhole exists for this case.

Chapter 3

Spherically Symmetric Self-Gravitating Fluid Models and Structure Scalars

In this chapter, we explore self-gravitating spherically symmetric fluid models and derive the set of scalar functions (structure scalars) in $f(G)$ gravity. We develop a set of equations governing the structure and evolution of the system. Using these equations, we discuss some particular fluid models according to different dynamical conditions. We construct structure scalars and rewrite the set of governing equations in terms of these scalar functions. We also identify inhomogeneity factor and derive static inhomogeneous anisotropic spherical solutions with the help of structure scalars. The results of this chapter have been published in two papers [49, 50].

This chapter comprises two sections. Section **2.1** provides a comprehensive description of spherically symmetric self-gravitating fluid models through evolution equations. Section **2.2** evaluates structure scalars for the system to determine density inhomogeneity factors and examines anisotropic inhomogeneous spheres in the presence of dark sources.

3.1 Spherically Symmetric Self-Gravitating Fluid Models

This section formulates a set of equations which governs the evolution of self-gravitating system through the Weyl tensor, shear tensor, expansion scalar, pressure anisotropy, density inhomogeneity as well as heat dissipation in the presence of dark sources.

3.1.1 Field Equations and Dynamical Quantities

We first formulate the field equations and then evaluate dynamical quantities. The general line element for non-static spherical configuration is given as

$$ds^2 = -A^2dt^2 + B^2dr^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1.1}$$

where $A = A(t,r),\ B = B(t,r)$ and $C = C(t,r)$ are functions of temporal (t) and radial (r) coordinates. For matter distribution, the energy-momentum tensor is defined as

$$\overset{\text{(m)}}{T}_{\mu\nu} = \rho u_\mu u_\nu + p_{_t} h_{\mu\nu} + (p_{_r} - p_{_t})v_\mu v_\nu + q(v_\mu u_\nu + u_\mu v_\nu) + \epsilon l_\mu l_\nu, \tag{3.1.2}$$

where q and ϵ are dissipation (heat-flux) and radiation density, respectively. The quantities $u^\mu,\ v^\mu$ (unit four-vector) and l^μ (null four-vector) are defined as

$$u^\mu = A^{-1}\delta^\mu_0,\quad v^\mu = B^{-1}\delta^\mu_1,\quad l^\mu = A^{-1}\delta^\mu_0 + B^{-1}\delta^\mu_1,$$

satisfying the relations

$$u^\mu u_\mu = -1,\quad v^\mu v_\mu = 1,\quad v^\mu u_\mu = 0,\quad l^\mu u_\mu = -1,\quad l^\mu l_\mu = 0,\quad h_{\mu\nu}u^\mu = 0.$$

We can write Eq.(3.1.2) as

$$\overset{\text{(m)}}{T}_{\mu\nu} = \tilde{\rho} u_\mu u_\nu + p_{_t} h_{\mu\nu} + \Pi v_\mu v_\nu + \tilde{q}_\mu u_\nu + \tilde{q}_\nu u_\mu, \tag{3.1.3}$$

where $\tilde{\rho}=\rho+\epsilon,\ \Pi=\tilde{p}_{r}+p_{t},\ \tilde{p}_{r}=p_{r}+\epsilon,\ \tilde{q}_{\mu}=\tilde{q}V_{\mu},\ \tilde{q}=q+\epsilon.$ The corresponding field equations are

$$
\begin{aligned}
\tilde{\rho} &= \frac{\dot{C}}{A^{2}C}\left(2\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{1}{B^{2}}\left(2\frac{C''}{C}+\left(\frac{C'}{C}\right)^{2}-2\frac{B'C'}{BC}-\left(\frac{B}{C}\right)^{2}\right)\\
&+ \frac{4}{A^{2}BC^{2}}\left(\dot{B}-2\frac{\dot{C}C''}{B}-\frac{\dot{B}C'^{2}}{B^{2}}+3\frac{\dot{B}\dot{C}^{2}}{A^{2}}+2\frac{B'C'\dot{C}}{B^{2}}\right)\dot{f}_{G}+\frac{4}{A^{2}B^{3}C^{2}}\\
&\times \left(A^{2}B'+B'\dot{C}^{2}-2\dot{B}\dot{C}C'+2\frac{A^{2}C'C''}{B}-3\frac{A^{2}B'C'^{2}}{B^{2}}\right)f'_{G}+\frac{4}{A^{2}B^{3}C^{2}}\\
&\times \left(\frac{A^{2}C'^{2}}{B^{2}}-A^{2}-2\dot{C}\right)f''_{G}-Gf_{G}+f,\\
\tilde{q} &= \frac{2}{AB}\left(\frac{\dot{B}C'}{BC}+\frac{A'\dot{C}}{AC}-\frac{\dot{C}'}{C}\right)+\frac{4}{ABC^{2}}\left(1+\left(\frac{\dot{C}}{A}\right)^{2}-\left(\frac{C'}{B}\right)^{2}\right)\dot{f}'_{G}\\
&+ \frac{4}{A^{2}BC^{2}}\left(-A'+2\frac{\dot{C}\dot{C}'}{A}+\frac{A'C'^{2}}{B^{2}}-3\frac{A'\dot{C}^{2}}{A^{2}}-2\frac{\dot{B}C'\dot{C}}{AB}\right)\dot{f}_{G}+\frac{4}{AB^{2}C^{2}}\\
&\times \left(-\dot{B}-2\frac{\dot{C}'C'}{B}+3\frac{\dot{B}C'^{2}}{B^{2}}-\frac{\dot{B}\dot{C}^{2}}{A^{2}}+2\frac{A'C'\dot{C}}{AB}\right)f'_{G},\\
\tilde{p}_{r} &= \frac{1}{A^{2}}\left(\frac{\dot{C}}{C}\left(2\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)-2\frac{\ddot{C}}{C}\right)+\frac{C'}{B^{2}C}\left(2\frac{A'}{A}+\frac{C'}{C}\right)-\frac{1}{C^{2}}+\frac{4}{A^{2}C^{2}}\\
&\times \left(\left(\frac{C'}{B}\right)^{2}-\left(\frac{\dot{C}}{A}\right)^{2}-1\right)\ddot{f}_{G}+\frac{4}{A^{3}C^{2}}\left(\dot{A}-2\frac{\dot{C}\ddot{C}}{A}+2\frac{A'C'\dot{C}}{B^{3}}-\frac{\dot{A}C'^{2}}{B^{2}}\right.\\
&+ \left.3\frac{\dot{A}\dot{C}^{2}}{A^{2}}\right)\dot{f}_{G}+\frac{4}{AB^{2}C^{2}}(A'+\frac{1}{A^{2}}-2\frac{\dot{A}\dot{C}C'}{A^{2}}-3\frac{A'C'^{2}}{B^{2}}+2\frac{\ddot{C}C'}{A})f'_{G}\\
&+ Gf_{G}-f,\\
p_{t} &= \frac{1}{A^{2}}\left(\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{\dot{B}\dot{C}}{BC}\right)+\frac{1}{B^{2}}\left(\frac{A''}{A}+\frac{C''}{C}-\frac{A'B'}{AB}\right.\\
&+ \left.\frac{C'}{C}\left(\frac{A'}{A}-\frac{B'}{B}\right)\right)+\frac{4}{A^{3}BC}\left(\frac{1}{B}\left(2\dot{C}'A'-\dot{A}C''+A''\dot{C}\right)-\frac{1}{A}\right.\\
&\times \left(\dot{B}\ddot{C}+\dot{C}\ddot{B}\right)-\frac{1}{B^{2}}\left(A'B'\dot{C}+A'\dot{B}C'-\dot{A}B'C'\right)+3\frac{\dot{A}\dot{B}\dot{C}}{A^{2}}-2
\end{aligned}
$$

$$\begin{aligned}
&\times \left.\frac{A'^2\dot{C}}{AB}\right)\dot{f}_{_G}+\frac{4}{AB^3C}\left(\frac{1}{A}\left(2\dot{B}\dot{C}'-B'\ddot{C}+\ddot{B}C'\right)-\frac{1}{B}\left(A'C''+A''C'\right)\right.\\
&- \left.\frac{1}{A^2}\left(\dot{A}\dot{B}C'+A'\dot{B}\dot{C}-\dot{A}B'\dot{C}\right)+3\frac{A'B'C'}{B^2}-2\frac{\dot{B}^2C'}{AB}\right)f'_{_G}+\frac{8}{A^2B^2C}\\
&\times \left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-1\right)\dot{f}'_{_G}-\frac{4}{AB^2C}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\ddot{C}\right)f''_{_G}-\frac{4}{A^2BC}\\
&\times \left(\frac{\dot{B}\dot{C}}{A^2}+\frac{B'C'}{A}\right)\ddot{f}_{_G}+Gf_{_G}-f.
\end{aligned}$$

The expression for GB invariant is calculated as

$$\begin{aligned}
G &= \frac{8}{ABC}\left[\left(\frac{A'B'}{B^2C}+\frac{\ddot{B}}{AC}\right)\left(1+\frac{\dot{C}^2}{A^2}\right)+\left(\frac{\dot{A}\dot{B}}{A^2C}+\frac{A''}{BC}\right)\left(\frac{C'^2}{B^2}-1\right)\right.\\
&- \frac{1}{AC}\left(\frac{A''\dot{C}^2}{AB}+\ddot{B}C'^2\right)+2\{\frac{C''}{BC}\left(\frac{A'C'}{B^2}-\frac{\ddot{C}}{A}\right)+\frac{B'C'}{AB^2C}\left(\ddot{C}-\frac{\dot{A}\dot{C}}{A}\right)\\
&+ \frac{\dot{C}}{A^3C}\left(\ddot{C}\dot{B}+\frac{\dot{C}A'^2}{B}\right)+\frac{\dot{C}}{A^2BC}\left(\dot{A}C''+\frac{A'\dot{B}C'}{B}\right)+\frac{1}{ABC}\left(\dot{C}'^2\right.\\
&+ \left.\left.\frac{\dot{B}^2C'^2}{B^2}\right)\}-\frac{3}{C}\left(\frac{A'B'C'^2}{B^4}+\frac{\dot{A}\dot{B}\dot{C}^2}{A^4}\right)-\frac{4\dot{C}'}{ABC}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}\right)\right].
\end{aligned}$$

Here dot and prime denote partial derivatives with respect to t and r, respectively.

The four acceleration (1.5.1) and expansion parameter (1.5.2) are defined as

$$a_\mu=av_\mu,\quad a_{(1)}=\frac{A'}{A},\quad a^2=\left(\frac{A'}{AB}\right)^2,\quad \vartheta=\frac{1}{A}\left(\frac{\dot{B}}{B}+2\frac{\dot{C}}{C}\right).$$

Another form of shear tensor (1.5.3), its trace as well as non-zero components are

$$\sigma_{\mu\nu}=\sigma\left(v\mu v_\nu-\frac{1}{3}h_{\mu\nu}\right),\quad \sigma=\frac{1}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right),\quad \sigma_{11}=\frac{2}{3}\sigma B^2,\quad \sigma_{22}=-\frac{1}{3}\sigma C^2$$

and $\sigma_{33}=\sin^2\theta\sigma_{22}$. The magnetic part of the Weyl tensor (1.5.7) vanishes for spherical symmetry. The non-zero components of the Weyl tensor and electric part (1.5.8) in terms of four unit vector as well as projection tensor are given as

$$\mathbb{E}_{\mu\nu}=\varepsilon(v_\mu v_\nu-\frac{1}{3}h_{\mu\nu}),\quad \mathbb{E}_{11}=\frac{2}{3}\varepsilon B^2,\quad \mathbb{E}_{22}=-\frac{1}{3}\varepsilon C^2,\quad \mathbb{E}_{33}=\mathbb{E}_{22}\sin^2\theta,\quad (3.1.4)$$

where

$$\varepsilon=\frac{1}{2}\left(\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}-(\frac{\dot{C}}{C}-\frac{\dot{B}}{B})\frac{\dot{C}}{C}\right)+\frac{1}{2B^2}\left(-\frac{C''}{C}+(\frac{C'}{C}+\frac{B'}{B})\frac{C'}{C}\right)-\frac{1}{2C^2} \tag{3.1.5}$$

is a scalar quantity of the electric part. The Misner-Sharp mass function calculates the total energy of spherically symmetric system within the radius $r=C$ given as

$$\mathcal{M}=\frac{1}{2}C^3R^{23}_{23}=\frac{1}{2}C\left(\left(\frac{\dot{C}}{A}\right)^2-\left(\frac{C'}{B}\right)^2+1\right). \tag{3.1.6}$$

Now we calculate the variation of this mass function of radiating fluid inside the sphere. For this purpose, we introduce two useful derivative operators with respect to radial and proper time coordinates as

$$D_{_T}=\frac{1}{A}\frac{\partial}{\partial t},\quad D_{_C}=\frac{1}{C}\frac{\partial}{\partial r}$$

and the relativistic velocity of the collapsing fluid turns out to be

$$\mathcal{U}=D_{_T}C=\frac{1}{A}\frac{\partial C}{\partial t}=\frac{\dot{C}}{A}. \tag{3.1.7}$$

Combining Eqs.(3.1.6) and (3.1.7), we obtain

$$E=\frac{C'}{B}=\left(1+\mathcal{U}^2-\frac{2}{C}\mathcal{M}\right)^{\frac{1}{2}}. \tag{3.1.8}$$

Using Eqs.(1.1.4), (3.1.7) and (3.1.8), it follows that

$$D_{_T}\mathcal{M} = -\frac{C^2}{2}\left\{\left(\tilde{p}_{_r}+\frac{1}{B^2}\overset{\text{(GB)}}{T}_{_{11}}\right)\mathcal{U}+\left(\tilde{q}-\frac{1}{AB}\overset{\text{(GB)}}{T}_{_{01}}\right)E\right\}, \tag{3.1.9}$$

$$D_{_C}\mathcal{M} = \frac{C^2}{2}\left\{\left(\tilde{\rho}+\frac{1}{A^2}\overset{\text{(GB)}}{T}_{_{00}}\right)+\left(\tilde{q}-\frac{1}{AB}\overset{\text{(GB)}}{T}_{_{01}}\right)\frac{\mathcal{U}}{E}\right\}. \tag{3.1.10}$$

These equations represent the variation of mass inside the spherical surface of evolving fluid. Equation (3.1.9) represents the effects of radial pressure, dissipation, relativistic velocity and GB curvature terms on the proper derivative of mass function within spherically bounded region, while Eq.(3.1.10) indicates the combined effect of pressure, dissipation, relativistic velocity and extra GB curvature terms on the variation of mass distribution in radial direction. Equation (3.1.10) yields

$$\mathcal{M}'=\frac{C^2C'}{2}\left\{\left(\tilde{\rho}+\frac{1}{A^2}\overset{(\mathrm{GB})}{T_{00}}\right)+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\}.$$

A particular combination of radiating energy density, dissipation and $f(G)$ correction terms through mass function can be obtained using the above equation as

$$3\frac{\mathcal{M}}{C^3}=\frac{3}{2C^3}\int_0^r C^2C'\left\{\left(\tilde{\rho}+\frac{1}{A^2}\overset{(\mathrm{GB})}{T_{00}}\right)+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\}dr. \qquad (3.1.11)$$

3.1.2 Evolution Equations

Here, we construct dynamical equations for dissipative spherically distributed self-gravitating fluid for the model (1.1.7). The Riemann curvature tensor can be defined as

$$\frac{1}{2}R^{\mu}_{\alpha\beta\nu}u_{\mu}=a_{\alpha;[\nu}u_{\beta]}+a_{\alpha}u_{[\beta;\nu]}+\sigma_{\alpha[\beta;\nu]}+\frac{1}{3}\{\vartheta_{[\nu}h_{\beta]\alpha}+\vartheta h_{\alpha[\beta;\nu]}\}. \qquad (3.1.12)$$

In the following, we formulate evolution equations like Raychaudhuri equation, propagation equation of shear, constraint equation, evolution equations for the Weyl tensor and dynamical equations.

(i) Raychaudhuri Equation

This equation describes the evolution of expansion and is obtained by contracting Eq.(3.1.12) with u^{β} and then indices ν as well as α. It turns out to be

$$\vartheta_{;\mu}u^{\mu}+\frac{1}{3}\vartheta^2+\sigma_{\mu\nu}\sigma^{\mu\nu}-a^{\mu}_{;\mu}=-u_{\mu}u^{\nu}R^{\mu}_{\nu}.$$

This equation along with Eq.(1.1.4) gives

$$\vartheta_{;\mu}u^{\mu}+\frac{1}{3}\vartheta^{2}+\frac{2}{3}\sigma^{2}-a^{\mu}_{;\mu}=-\frac{1}{2}(\tilde{\rho}+3\tilde{p}_{r}). \tag{3.1.13}$$

We note that Raychaudhuri equation for $f(G)$ gravity is the same as for GR [20], i.e., GB terms have no contribution in Raychaudhuri equation. This is the evolution equation for expansion and hence measures the expansion rate of self-gravitating relativistic fluid for GR as well as $f(G)$ cosmology.

(ii) Propagation Equation of Shear

This equation is obtained by contracting Eq.(3.1.12) with $u^{\beta}h^{\alpha}_{\gamma}h^{\nu}_{\delta}$ as follows

$$\begin{aligned} u^{\beta}u_{\mu}R^{\mu}_{\alpha\beta\nu}h^{\alpha}_{\gamma}h^{\nu}_{\delta} &= h^{\alpha}_{\gamma}h^{\nu}_{\delta}(a_{\alpha;\nu}-\sigma_{\alpha\nu;\beta}u^{\beta})-a_{\gamma}a_{\delta}-u^{\beta}_{;\nu}h^{\nu}_{\delta}(\sigma_{\gamma\beta}+\frac{1}{3}\vartheta h_{\gamma\beta}) \\ &- \frac{1}{3}\vartheta_{,\alpha}u^{\alpha}h_{\gamma\delta}. \end{aligned} \tag{3.1.14}$$

Again contracting with $v^{\gamma},\ v^{\delta}$ and using Eq.(1.1.4), we obtain

$$\begin{aligned} u^{\beta}u_{\mu}R^{\mu}_{\alpha\beta\nu}h^{\alpha}_{\gamma}h^{\nu}_{\delta}v^{\gamma}v^{\delta} &= \varepsilon-\frac{1}{2}\Pi+2n\alpha[-R^{\phi}_{\rho\sigma\mu}v^{\mu}v_{\phi}-R_{\rho\mu}v^{\mu}v_{\sigma}+g_{\sigma\rho}R_{\alpha\mu} \\ &\times\ v^{\alpha}v^{\mu}-R_{\alpha\sigma}v_{\rho}v^{\alpha}+\frac{1}{2}Rv_{\rho}V_{\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n}. \end{aligned} \tag{3.1.15}$$

This is the propagation equation of shear in $f(G)$ gravity with some other dynamical variables which yields the effects of GB terms in the shearing motion of evolving self-gravitating spherical objects.

(iii) Constraint Equation

We obtain this equation from Eq.(3.1.12) by contracting α and ν and then contracting with $h^{\alpha\beta}v_{\alpha}$ as

$$R^{\mu}_{\beta}u_{\mu}h^{\alpha\beta}=h^{\alpha}_{\beta}(\sigma^{\beta\gamma}_{;\gamma}-\frac{2}{3}\vartheta^{;\beta})+\sigma^{\alpha\beta}a_{\beta},$$

which gives

$$h^{\alpha}_{\beta}(\sigma^{\beta\gamma}_{;\gamma}-\frac{2}{3}\vartheta^{;\beta})+\sigma^{\alpha\beta}a_{\beta} = -\tilde{q}v^{\alpha}+2n\alpha[-u_{\mu}h^{\alpha\beta}R^{\mu}_{\rho\sigma\beta}+g_{\sigma\rho}u_{\mu}h^{\alpha\beta}R^{\mu}_{\beta} + u_{\phi}h^{\alpha}_{\rho}R^{\phi}_{\sigma}+\frac{1}{8}Rh^{\alpha}_{\rho}u_{\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n}. \quad (3.1.16)$$

This equation directly relates shear tensor, expansion scalar, heat flux and modified terms of GB gravity.

(iv) Evolution Equations for the Weyl Tensor

Bainchi identities can also be written as

$$C^{\eta}_{\alpha\beta\gamma;\eta}=R_{\gamma[\alpha;\beta]}-\frac{1}{6}g_{\gamma[\alpha}R_{,\beta]}. \quad (3.1.17)$$

This is the relation between Weyl and Ricci tensors which can also be written in terms of the Weyl and effective energy-momentum tensors by using Eq.(1.1.4) as

$$C^{\eta}_{\alpha\beta\gamma;\eta}=\overset{(\mathrm{eff})}{T}_{\gamma[\alpha;\beta]}-\frac{1}{6}g_{\gamma[\alpha}\overset{(\mathrm{eff})}{T}_{,\beta]}. \quad (3.1.18)$$

We can write

$$u^{\beta}C^{\eta}_{\alpha\beta\gamma;\eta}+u^{\beta}_{;\eta}C^{\eta}_{\alpha\beta\gamma}=\vartheta\mathbb{E}_{\alpha\gamma}+u^{\mu}\mathbb{E}_{\alpha\gamma;\mu}-u_{\gamma;\eta}\mathbb{E}^{\eta}_{\alpha}-u_{\gamma}\mathbb{E}^{\eta}_{\alpha;\eta}, \quad (3.1.19)$$

where $u^{\beta}C_{\alpha\beta\gamma\delta}=\mathbb{E}_{\alpha\gamma}u_{\delta}-\mathbb{E}_{\alpha\delta}u_{\gamma}$. After contraction with $h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}v^{\mu}v^{\nu}$, Eq.(3.1.19) gives

$$h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}v^{\mu}v^{\nu}C^{\eta}_{\alpha\beta\gamma;\eta}=\frac{4}{3}\vartheta\mathbb{E}_{\mu\nu}v^{\mu}v^{\nu}-u_{\nu;\eta}\mathbb{E}^{\eta}_{\mu}v^{\mu}v^{\nu}+u^{\beta}\mathbb{E}_{\alpha\gamma;\beta}h^{\alpha}_{\mu}h^{\gamma}_{\nu}v^{\mu}v^{\nu} +h_{\mu\nu}\sigma^{\gamma\beta}\mathbb{E}_{\gamma\beta}v^{\mu}v^{\nu}-\sigma^{\gamma\nu}\mathbb{E}^{\gamma}_{\mu}v^{\mu}v^{\nu}-\sigma_{\gamma\mu}\mathbb{E}^{\gamma}_{\nu}v^{\mu}v^{\nu}. \quad (3.1.20)$$

Furthermore, the effective energy-momentum tensor provides

$$h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}\overset{(\mathrm{eff})}{T}_{\gamma\alpha;\beta} = h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}\overset{(\mathrm{m})}{T}_{\gamma\alpha;\beta}+h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}\overset{(\mathrm{GB})}{T}_{\gamma\alpha;\beta}$$

$$
\begin{aligned}
&= (p_t)_{,\beta}h_{\mu\nu}u^{\beta} + (\Pi v_{\gamma}v_{\alpha})_{;\beta}h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta} + \tilde{q}_{\nu}a_{\mu} + \tilde{q}_{\mu}a_{\nu} \\
&+ 8n\alpha\left[h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}R_{\gamma\rho\alpha\sigma;\beta} + h^{\alpha}_{\mu}h_{\nu\sigma}u^{\beta}R_{\rho\alpha;\beta} - h_{\mu\nu}u^{\beta}\right. \\
&\times \left. R_{\rho\sigma;\beta} - h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}g_{\sigma\rho}R_{\gamma\alpha;\beta} + h_{\mu\rho}h^{\gamma}_{\nu}u^{\beta}R_{\gamma\alpha;\beta}\right] \\
&\times \nabla^{\rho}\nabla^{\sigma}G^{n}, \qquad (3.1.21)
\end{aligned}
$$

$$
\begin{aligned}
h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}\overset{\text{(eff)}}{T}_{\gamma\beta;\alpha} &= h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}\overset{\text{(m)}}{T}_{\gamma\beta;\alpha} + h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}\overset{\text{(GB)}}{T}_{\gamma\beta;\alpha} \\
&= (p_t - \tilde{\rho})(\sigma_{\mu\nu} + \frac{1}{3}\vartheta h_{\mu\nu}) + \Pi_{;\beta}u^{\beta}v_{\gamma}v_{\beta;\alpha}h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta} \\
&- \tilde{q}_{,\alpha}h^{\alpha}_{\mu}v_{\nu} + 8n\alpha\left[h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}R_{\gamma\rho\beta\sigma;\alpha} + h^{\alpha}_{\mu}h_{\nu\sigma}u^{\beta}R_{\rho\beta;\alpha}\right. \\
&- \left. h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}g_{\sigma\rho}R_{\gamma\beta;\alpha} + h^{\alpha}_{\mu}h^{\gamma}_{\nu}u_{\rho}R_{\gamma\sigma;\alpha}\right]\nabla^{\rho}\nabla^{\sigma}G^{n}, \qquad (3.1.22)
\end{aligned}
$$

$$
\begin{aligned}
h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}g_{\gamma[\alpha}\overset{\text{(eff)}}{T}_{;\beta]} &= \frac{1}{2}\left[h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}g_{\gamma\alpha}\overset{\text{(eff)}}{T}_{;\beta} - h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}g_{\gamma\beta}\overset{\text{(eff)}}{T}_{;\alpha}\right] \\
&= \frac{1}{2}(\Pi + 3p_t - \tilde{\rho})_{,\beta}u^{\beta}h_{\mu\nu} - n\alpha h_{\mu\nu}[R_{\rho\sigma} + g_{\sigma\rho}R]_{;\beta}u^{\beta} \\
&\times \nabla^{\rho}\nabla^{\sigma}G^{n}. \qquad (3.1.23)
\end{aligned}
$$

Feeding back Eqs.(3.1.20)-(3.1.23) into (3.1.18), we have

$$
\begin{aligned}
&\vartheta\left(\frac{1}{3}(\tilde{\rho} + \tilde{p}_r)h_{\mu\nu} + \mathbb{E}_{\mu\nu}\right) + u^{\beta}(\mathbb{E}_{\alpha\gamma} - \Pi_{\alpha\gamma})_{;\beta}h^{\alpha}_{\mu}h^{\gamma}_{\nu} + (\varepsilon + \tilde{\rho} + \tilde{p}_r)\sigma_{\mu\nu} \\
&+\frac{4}{3}\tilde{\rho}_{,\beta}u^{\beta}h_{\mu\nu} - \Pi_{\beta\nu}(\sigma^{\beta}_{\mu} - \frac{1}{3}\vartheta h^{\beta}_{\mu}) - \tilde{q}_{\mu}a_{\nu} - \tilde{q}_{\nu}a_{\mu} - 4n\alpha h^{\alpha}_{\mu}h^{\gamma}_{\nu}u^{\beta}[R_{\gamma\rho[\alpha\sigma;\beta]} \\
&+g_{\sigma\rho}R_{\gamma[\alpha;\beta]}]\nabla^{\rho}\nabla^{\sigma}G^{n} - n\alpha[h^{\alpha}_{\mu}h_{\nu\sigma}u^{\beta}R_{\rho[\alpha;\beta]} - h_{\mu\nu}u^{\beta}R_{\rho\sigma;\beta} + h_{\mu\rho}h^{\gamma}_{\nu}u^{\beta} \\
&\times R_{\gamma\alpha;\beta} - h^{\alpha}_{\mu}h^{\gamma}_{\nu}u_{\rho}R_{\gamma\sigma;\alpha}]\nabla^{\rho}\nabla^{\sigma}G^{n} = 0. \qquad (3.1.24)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&h^{\alpha}_{\mu}\mathbb{E}^{\lambda}_{\alpha;\lambda} + a^{\lambda}\mathbb{E}_{\mu\lambda} + \tilde{q}_{\gamma}(\sigma^{\gamma}_{\mu} + \frac{1}{3}\vartheta h^{\gamma}_{\mu}) + \frac{1}{3}h^{\beta}_{\mu}(-2\tilde{\rho} + 2\tilde{p}_r + p_t)_{,\beta} - a^{\gamma}\Pi_{\mu\gamma} \\
&+a_{\mu}(-\tilde{\rho} + \tilde{p}_r) + u^{\beta}h^{\alpha}_{\mu}\tilde{q}_{\alpha;\beta} + 4n\alpha u^{\gamma}u^{\beta}\left[R_{\gamma\rho\alpha\sigma;\beta} - R_{\gamma\rho\beta\sigma;\alpha} + g_{\sigma\rho}R_{\gamma\beta;\alpha}\right. \\
&\left.-2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta}\right]\nabla^{\rho}\nabla^{\sigma}G^{n} + 4n\alpha u^{\beta}[2u_{[\sigma}R_{\alpha]\rho;\beta} + g_{\rho[\sigma}u_{\alpha]}R_{;\beta}]\nabla^{\rho}\nabla^{\sigma}G^{n} \\
&+4n\alpha u^{\gamma}u^{\beta}[R_{\gamma\rho\alpha\sigma} - g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta} + u^{\beta}[\frac{1}{2}n\alpha R u_{[\alpha}g_{\sigma]\rho} + u_{[\sigma}
\end{aligned}
$$

$$\times R_{\alpha]\rho}]\nabla^\rho\nabla^\sigma(G^n)_{,\beta}+\frac{1}{2}\alpha(n-1)u_\alpha u^\beta(G^n)_{,\beta}-4n\alpha[R_{\rho\sigma;\alpha}+R_{\rho\beta;\alpha}u_\sigma$$
$$\times u^\beta+u_\gamma u_\rho R_{\gamma\sigma;\alpha}-\frac{1}{2}h_{\sigma\rho}R_{,\alpha}]\nabla^\rho\nabla^\sigma G^n-4n\alpha[u^\gamma u^\beta R_{\gamma\rho\beta\sigma}+u^\beta u_\sigma R_{\rho\beta}$$
$$+R_{\rho\sigma}-u^\gamma u^\beta g_{\sigma\rho}R_{\gamma\beta}+u^\gamma u_\rho R_{\gamma\sigma}-\frac{1}{2}h_{\sigma\rho}R]\nabla^\rho\nabla^\sigma(G^n)_{,\alpha}+\frac{1}{2}\alpha(n-1)$$
$$\times(G^n)_{,\alpha}-\frac{4}{3}n\alpha[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})]_{,\beta}\nabla^\rho\nabla^\sigma G^n-8n\alpha[R_{\rho\sigma}$$
$$+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})]\nabla^\rho\nabla^\sigma(G^n)_{,\beta}+\frac{1}{3}\alpha(n-1)(G^n)_{,\beta}=0. \qquad (3.1.25)$$

Equations (3.1.24) and (3.1.25) are the evolution equations for the Weyl tensor. These represent the relationship between the Weyl tensor, dynamical variables (heat flux, anisotropic parameter, density, shear and projection tensors etc) and modified terms due to $f(G)$ gravity.

(v) Dynamical Equations

These equations describe the conservation of total energy of the evolving star obtained through Bianch identities as

$$(\overset{(\mathrm{m})}{T^{\mu\nu}}+\overset{(\mathrm{GB})}{T^{\mu\nu}})_{;\nu}u_\mu=0,\quad(\overset{(\mathrm{m})}{T^{\mu\nu}}+\overset{(\mathrm{GB})}{T^{\mu\nu}})_{;\nu}v_\mu=0, \qquad (3.1.26)$$

where the first equation represents equation of continuity while the second is the equation of motion. Both equations yield

$$\tilde{\rho}_{,\mu}u^\mu+\vartheta(\tilde{\rho}+\tilde{p}_r)-\frac{2}{3}(\sigma+\vartheta)\Pi+\tilde{q}_{,\mu}v^\mu+2\tilde{q}\left(\frac{C'}{BC}+a\right)+8n\alpha[R^{\mu\nu}_{\rho\sigma}$$
$$-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}u_\mu\nabla^\rho\nabla^\sigma G^n+8n\alpha[(u_\sigma R^\nu_\rho-u^\nu R_{\rho\sigma})_{,\nu}+\Gamma^\mu_{\nu\gamma}(R^\nu_\rho\delta^\gamma_\sigma-R_{\rho\sigma}g^{\gamma\nu})$$
$$\times u_\mu+\Gamma^\nu_{\nu\gamma}(u_\sigma R^\gamma_\rho-u^\gamma R_{\rho\sigma})]\nabla^\rho\nabla^\sigma G^n=0, \qquad (3.1.27)$$
$$\tilde{p}_{r,\mu}v^\mu+a(\tilde{\rho}+\tilde{p}_r)+2\Pi\frac{C'}{BC}+\tilde{q}_{,\mu}v^\mu+\frac{2}{3}\tilde{q}(\sigma+2\vartheta)+8n\alpha[-R^{\mu\nu}g_{\sigma\rho}$$
$$+R^{\mu\nu}_{\rho\sigma}]_{;\nu}v_\mu\nabla^\rho\nabla^\sigma G^n+8n\alpha[(v_\sigma R^\nu_\rho-v^\nu R_{\rho\sigma}+R^\mu_\sigma\delta^\nu_\rho)_{,\nu}+\Gamma^\mu_{\nu\gamma}(R^\nu_\rho\delta^\gamma_\sigma+R^\gamma_\sigma$$
$$\times\delta^\nu_\rho)v_\mu+\Gamma^\nu_{\nu\gamma}(v_\sigma R^\gamma_\rho+v^\gamma R_{\rho\sigma})]\nabla^\rho\nabla^\sigma G^n=0, \qquad (3.1.28)$$

representing the effects of GB terms in the evolution of energy density and pressure.

3.1.3 Some Self-Gravitating Fluid Models

In this section, we study governing equations for some specific self-gravitating fluid models under the influence of $f(G)$ gravity and give their comparison with GR [20].

(i) Geodesic Non-dissipative Isotropic Fluids

If the fluid particles are moving along geodesics, then $a^\mu = 0$ and the fluid is geodesic for which $g_{00} = constant$. Thus for locally isotropic (radial and tangential pressures are same, i.e., $\Pi = 0$), non-dissipative (vanishing heat flux and radiation density, i.e., $q = \epsilon = 0$) and geodesic fluids, we obtain the governing equations under the effects of $f(G)$ gravity as follows

$$u^\beta u_\mu R^\mu_{\alpha\beta\nu} h^\alpha_\gamma h^\nu_\delta v^\gamma v^\delta = \varepsilon + 2n\alpha[-R^\phi_{\rho\sigma\mu} v^\mu v_\phi - R_{\rho\mu} v^\mu v_\sigma + g_{\sigma\rho} R_{\alpha\mu} v^\alpha \times v^\mu - R_{\alpha\sigma} v_\rho v^\alpha + \frac{1}{2} R v_\rho v_\sigma] \nabla^\delta \nabla^\sigma G^n, \tag{3.1.29}$$

$$\vartheta \left(\frac{1}{3}(\rho + p_r) h_{\mu\nu} + \mathbb{E}_{\mu\nu}\right) + u^\beta (\mathbb{E}_{\alpha\gamma})_{;\beta} h^\alpha_\mu h^\gamma_\nu + (\varepsilon + \rho + p_r)\sigma_{\mu\nu} + \frac{4}{3}\rho_{,\beta} u^\beta \times h_{\mu\nu} - 4n\alpha h^\alpha_\mu h^\gamma_\nu u^\beta [R_{\gamma\rho[\alpha\sigma;\beta]} + g_{\sigma\rho} R_{\gamma[\alpha;\beta]}] \nabla^\rho \nabla^\sigma G^n - n\alpha [h^\alpha_\mu h_{\nu\sigma} R_{\rho[\alpha;\beta]} \times u^\beta - h_{\mu\nu} u^\beta R_{\rho\sigma;\beta} + h_{\mu\rho} h^\gamma_\nu U^\beta R_{\gamma\alpha;\beta} - h^\alpha_\mu h^\gamma_\nu u_\rho R_{\gamma\sigma;\alpha}] \nabla^\rho \nabla^\sigma G^n = 0, \tag{3.1.30}$$

$$\frac{1}{3} h^\beta_\mu (-2\rho)_{,\beta} + 4n\alpha u^\gamma u^\beta \left[R_{\gamma\rho\alpha\sigma;\beta} - R_{\gamma\rho\beta\sigma;\alpha} - 2(R_{\gamma[\alpha} g_{\sigma]\rho})_{;\beta} + g_{\sigma\rho} R_{\gamma\beta;\alpha}\right] \nabla^\rho \nabla^\sigma G^n + 4n\alpha u^\beta [2u_{[\sigma} R_{\alpha]\rho;\beta} + g_{\rho[\sigma} u_{\alpha]} R_{;\beta}] \nabla^\rho \nabla^\sigma \times G^n + 4n\alpha u^\gamma u^\beta [R_{\gamma\rho\alpha\sigma} - g_{\rho[\alpha} R_{\sigma]\gamma}] \nabla^\gamma \nabla^\sigma (G^n)_{,\beta} + u^\beta [u_{[\sigma} R_{\alpha]\rho} + \frac{1}{2} n\alpha \times R u_{[\alpha} g_{\sigma]\rho}] \nabla^\rho \nabla^\sigma (G^n)_{,\beta} + \frac{1}{2}\alpha(n-1) u_\alpha u^\beta (G^n)_{,\beta} - 4[R_{\rho\sigma;\alpha} + R_{\rho\beta;\alpha} u_\sigma \times u^\beta + u_\gamma u_\rho R_{\gamma\sigma;\alpha} - \frac{1}{2} n\alpha h_{\sigma\rho} R_{;\alpha}] \nabla^\gamma \nabla^\sigma G^n - 4n\alpha [u^\gamma u^\beta R_{\gamma\rho\beta\sigma} + u^\beta u_\sigma$$

$$\times R_{\rho\beta}+R_{\rho\sigma}-u^{\gamma}u^{\beta}g_{\sigma\rho}R_{\gamma\beta}+u^{\gamma}u_{\rho}R_{\gamma\sigma}-\frac{1}{2}h_{\sigma\rho}R]\nabla^{\gamma}\nabla^{\sigma}(G^{n})_{,\alpha}+\frac{1}{2}\alpha$$
$$\times(n-1)(G^{n})_{,\alpha}-\frac{4}{3}n\alpha[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})]_{,\beta}\nabla^{\rho}\nabla^{\sigma}G^{n}-8n$$
$$\times\alpha[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta}+\frac{1}{3}\alpha(n-1)$$
$$\times(G^{n})_{,\beta}=0, \tag{3.1.31}$$
$$(p_{r})_{,\mu}V^{\mu}+8n\alpha[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}v_{\mu}\nabla^{\rho}\nabla^{\sigma}G^{n}+8n\alpha[(v_{\sigma}R^{\nu}_{\rho}-v^{\nu}R_{\rho\sigma}$$
$$+R^{\mu}_{\sigma}\delta^{\nu}_{\rho})_{,\nu}+\Gamma^{\mu}_{\nu\gamma}(R^{\nu}_{\rho}\delta^{\gamma}_{\sigma}+R^{\gamma}_{\sigma}\delta^{\nu}_{\rho})v_{\mu}+\Gamma^{\nu}_{\nu\gamma}(V_{\sigma}R^{\gamma}_{\rho}+v^{\gamma}R_{\rho\sigma})]\nabla^{\rho}\nabla^{\sigma}$$
$$\times G^{n}=0. \tag{3.1.32}$$

This case represents dust fluid model (pressureless fluid) in GR while the conformally flatness condition ($\varepsilon=0$) implies shear-free condition ($\sigma=0$) and vice versa [20]. In the scenario of $f(G)$ gravity, the equation of motion (3.1.32) depends upon the gradient of pressure and extra curvature terms (GB terms). The choice $f(G)=$ *constant* (i.e., $f_{G}=0$) corresponds to the cosmological constant and the standard results can be imitated. For this type of fluid model, pressure gradient vanishes which consequently gives $p_{r}=constant$. In this case, matter particles will exert equal pressure at each point of evolving relativistic spherically distributed self-gravitating fluid. Thus geodesic fluids with isotropy and non-dissipation exert constant (non-zero) pressure (no dust) in $f(G)$ cosmology. For constant to be zero, this yields dust. Equations (3.1.29) and (3.1.30) represent that conformally flatness and shear-free conditions rely on GB terms. Thus conformally flatness condition does not imply shear-free condition. Equation (3.1.31) indicates that energy density inhomogeneity depends upon GB terms as well as the Weyl tensor. If the Weyl tensor vanishes, then GB terms are totally responsible for energy density inhomogeneity.

(ii) Geodesic Non-dissipative Anisotropic Fluids

Here we take geodesic fluid with locally anisotropy (different radial and tangential pressures, i.e., $\Pi \neq 0$) and non-dissipation for which the governing equations are as follows

$$u^\beta u_\mu R^\mu_{\alpha\beta\nu} h^\alpha_\gamma h^\nu_\delta v^\gamma v^\delta = \varepsilon - \frac{1}{2}\Pi + 2n\alpha[-R^\phi_{\rho\sigma\mu} v^\mu v_\phi - R_{\rho\mu} v^\mu v_\sigma + g_{\sigma\rho} \times R_{\alpha\mu} v^\alpha v^\mu - R_{\alpha\sigma} v_\rho v^\alpha + \frac{1}{2} R v_\rho v_\sigma] \nabla^\delta \nabla^\sigma G^n, \tag{3.1.33}$$

$$\vartheta\left(\frac{1}{3}(\rho + p_r) h_{\mu\nu} + \mathbb{E}_{\mu\nu}\right) + u^\beta(\mathbb{E}_{\alpha\gamma} - \Pi_{\alpha\gamma})_{;\beta} h^\alpha_\mu h^\gamma_\nu + (\varepsilon + \rho + p_r)\sigma_{\mu\nu} + \frac{4}{3}\rho_{,\beta} u^\beta - \Pi_{\beta\nu}(\sigma^\beta_\mu - \frac{1}{3}\vartheta h^\beta_\mu) h_{\mu\nu} - 4n\alpha h^\alpha_\mu h^\gamma_\nu u^\beta [R_{\gamma\rho[\alpha\sigma;\beta]} + g_{\sigma\rho} R_{\gamma[\alpha;\beta]}] \times \nabla^\rho \nabla^\sigma G^n - n\alpha[h^\alpha_\mu h_{\nu\sigma} R_{\rho[\alpha;\beta]} u^\beta - h_{\mu\nu} u^\beta R_{\rho\sigma;\beta} + h_{\mu\rho} \times h^\gamma_\nu U^\beta R_{\gamma\alpha;\beta} - h^\alpha_\mu h^\gamma_\nu u_\rho R_{\gamma\sigma;\alpha}] \nabla^\rho \nabla^\sigma G^n = 0, \tag{3.1.34}$$

$$\frac{-2}{3} h^\beta_\mu(\rho)_{,\beta} + \frac{1}{3} h^\beta_\mu(\Pi)_{,\beta} + 4n\alpha u^\gamma u^\beta \left[R_{\gamma\rho\alpha\sigma;\beta} - R_{\gamma\rho\beta\sigma;\alpha} - 2(R_{\gamma[\alpha} g_{\sigma]\rho})_{;\beta} + g_{\sigma\rho} R_{\gamma\beta;\alpha}\right] \nabla^\rho \nabla^\sigma G^n + 4n\alpha u^\beta [2u_{[\sigma} R_{\alpha]\rho;\beta} + g_{\rho[\sigma} u_{\alpha]} R_{;\beta}] \nabla^\rho \nabla^\sigma \times G^n + 4n\alpha u^\gamma u^\beta [R_{\gamma\rho\alpha\sigma} - g_{\rho[\alpha} R_{\sigma]\gamma}] \nabla^\gamma \nabla^\sigma (G^n)_{,\beta} + u^\beta [u_{[\sigma} R_{\alpha]\rho} + \frac{1}{2} n\alpha \times R u_{[\alpha} g_{\sigma]\rho}] \nabla^\rho \nabla^\sigma (G^n)_{,\beta} + \frac{1}{2}\alpha(n-1) u_\alpha u^\beta (G^n)_{,\beta} - 4[R_{\rho\sigma;\alpha} + R_{\rho\beta;\alpha} u_\sigma \times u^\beta + u_\gamma u_\rho R_{\gamma\sigma;\alpha} - \frac{1}{2} n\alpha h_{\sigma\rho} R_{;\alpha}] \nabla^\gamma \nabla^\sigma G^n - 4n\alpha[u^\gamma u^\beta R_{\gamma\rho\beta\sigma} + u^\beta u_\sigma \times R_{\rho\beta} + R_{\rho\sigma} - u^\gamma u^\beta g_{\sigma\rho} R_{\gamma\beta} + u^\gamma u_\rho R_{\gamma\sigma} - \frac{1}{2} h_{\sigma\rho} R] \nabla^\gamma \nabla^\sigma (G^n)_{,\alpha} + \frac{1}{2}\alpha \times (n-1)(G^n)_{,\alpha} - \frac{4}{3} n\alpha [R_{\rho\sigma} + R g_{\sigma\rho} - \frac{1}{2} R(4g_{\sigma\rho} - \delta_{\rho\sigma})]_{,\beta} \nabla^\rho \nabla^\sigma G^n - 8n \times \alpha [R_{\rho\sigma} + R g_{\sigma\rho} - \frac{1}{2} R(4g_{\sigma\rho} - \delta_{\rho\sigma})] \nabla^\rho \nabla^\sigma (G^n)_{,\beta} + \frac{1}{3}\alpha(n-1) \times (G^n)_{,\beta} = 0, \tag{3.1.35}$$

$$\rho_{,\mu} u^\mu + \vartheta(\rho + p_r) - \frac{2}{3}(\sigma + \vartheta)\Pi + 8n\alpha[R^{\mu\nu}_{\rho\sigma} - R^{\mu\nu} g_{\sigma\rho}]_{;\nu} u_\mu \nabla^\rho \nabla^\sigma G^n + 8n\alpha[(u_\sigma R^\nu_\rho - u^\nu R_{\rho\sigma})_{,\nu} + \Gamma^\mu_{\nu\gamma}(R^\nu_\rho \delta^\gamma_\sigma - R_{\rho\sigma} g^{\gamma\nu}) u_\mu + \Gamma^\nu_{\nu\gamma}(u_\sigma R^\gamma_\rho - u^\gamma$$

$$\times R_{\rho\sigma})]\nabla^{\rho}\nabla^{\sigma}G^{n}=0, \tag{3.1.36}$$

$$(p_{r})_{,\mu}v^{\mu}+2\Pi\frac{C'}{CB}+8n\alpha[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}v_{\mu}\nabla^{\rho}\nabla^{\sigma}G^{n}+8n\alpha[(v_{\sigma}R^{\nu}_{\rho}$$
$$-v^{\nu}R_{\rho\sigma}+R^{\mu}_{\sigma}\delta^{\nu}_{\rho})_{,\nu}+\Gamma^{\mu}_{\nu\gamma}(R^{\nu}_{\rho}\delta^{\gamma}_{\sigma}+R^{\gamma}_{\sigma}\delta^{\nu}_{\rho})v_{\mu}+\Gamma^{\nu}_{\nu\gamma}(v_{\sigma}R^{\gamma}_{\rho}+V^{\gamma}R_{\rho\sigma})]$$
$$\times\nabla^{\rho}\nabla^{\sigma}G^{n}=0. \tag{3.1.37}$$

The geometry for this type of fluid revolves around a physical quantity of matter, i.e., pressure anisotropy. In GR, pressure gradient, shear-free and conformally flatness conditions are linked with pressure anisotropy while density inhomogeneity depends on the Weyl tensor as well as pressure anisotropy. The $f(G)$ theory affects these relations by the inclusion of GB terms. In the absence of Weyl tensor, Eq.(3.1.34) shows that density inhomogeneity is caused by pressure anisotropy as well as GB terms.

(iii) Non-geodesic Non-dissipative Isotropic Fluids

In this case, we obtain the following set of governing equations

$$h^{\alpha}_{\beta}(\sigma^{\beta\gamma}_{;\gamma}-\frac{2}{3}\vartheta^{;\alpha})+\sigma^{\alpha\beta}a_{\beta}=2n\alpha[-u_{\mu}h^{\alpha\beta}R^{\mu}_{\rho\sigma\beta}+g_{\sigma\rho}u_{\mu}h^{\alpha\beta}R^{\mu}_{\beta}+u_{\phi}h^{\alpha}_{\rho}R^{\phi}_{\sigma}$$
$$+\frac{1}{8}Rh^{\alpha}_{\rho}u_{\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n}, \tag{3.1.38}$$
$$\frac{1}{3}h^{\beta}_{\alpha}(\varepsilon)_{,\beta}+\frac{2}{3}h^{\beta}_{\alpha}\rho_{,\beta}+4n\alpha u^{\gamma}u^{\beta}\left[R_{\gamma\rho\alpha\sigma;\beta}-R_{\gamma\rho\beta\sigma;\alpha}+g_{\sigma\rho}R_{\gamma\beta;\alpha}\right.$$
$$\left.-2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta}\right]\nabla^{\rho}\nabla^{\sigma}G^{n}+4n\alpha u^{\beta}[2u_{[\sigma}R_{\alpha]\rho;\beta}+g_{\rho[\sigma}u_{\alpha]}R_{,\beta}]\nabla^{\rho}\nabla^{\sigma}G^{n}$$
$$+4n\alpha u^{\gamma}u^{\beta}[R_{\gamma\rho\alpha\sigma}-g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta}+n\alpha u^{\beta}[u_{[\sigma}R_{\alpha]\rho}+\frac{1}{2}Ru_{[\alpha}$$
$$\times g_{\sigma]\rho}]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta}+\frac{1}{2}\alpha(n-1)u_{\alpha}u^{\beta}(G^{n})_{,\beta}-4n\alpha[u_{\sigma}u^{\beta}R_{\rho\beta;\alpha}+u_{\gamma}$$
$$\times u_{\rho}R_{\gamma\sigma;\alpha}+R_{\rho\sigma;\alpha}-\frac{1}{2}h_{\sigma\rho}R_{;\alpha}]\nabla^{\gamma}\nabla^{\sigma}G^{n}-4n\alpha[u^{\gamma}u^{\beta}R_{\gamma\rho\beta\sigma}+u^{\beta}u_{\sigma}R_{\rho\beta}$$
$$+R_{\rho\sigma}-u^{\gamma}u^{\beta}g_{\sigma\rho}R_{\gamma\beta}+u^{\gamma}u_{\rho}R_{\gamma\sigma}-\frac{1}{2}h_{\sigma\rho}R]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\alpha}+\frac{1}{2}\alpha(n-1)$$

$$\times(G^n)_{,\alpha}-\frac{4}{3}n\alpha[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})]_{,\beta}\nabla^\rho\nabla^\sigma G^n-8n\alpha[R_{\rho\sigma}$$
$$+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})]\nabla^\rho\nabla^\sigma(G^n)_{,\beta}+\frac{1}{3}\alpha(n-1)(G^n)_{,\beta}=0, \qquad (3.1.39)$$

In GR, shear-free condition provides expansion free condition for such fluid while conformal flatness condition implies irregularities (inhomogeneity) in energy density and vice-versa. For modified GB gravity, we see from Eq.(3.1.39) that conformal flatness condition depends upon inhomogeneity of energy density as well as GB terms. If the fluid is conformally flat, the dependence of energy density inhomogeneity depends on GB terms, hence GB terms are responsible for density inhomogeneity. Equation (3.1.38) indicates that shear-free condition does not imply expansion-free fluid due to GB terms.

(iv) Non-geodesic Non-dissipative Anisotropic Fluids

Here we take non-dissipative ($q=\epsilon=0$) and anisotropic ($\Pi\neq 0$) fluids. For this case, we have

$$\frac{1}{3}h^\beta_\alpha(\varepsilon-\Pi)_{,\beta}+\frac{2}{3}h^\beta_\alpha\rho_{,\beta}+4n\alpha u^\gamma u^\beta\,[R_{\gamma\rho\alpha\sigma;\beta}-R_{\gamma\rho\beta\sigma;\alpha}+g_{\sigma\rho}R_{\gamma\beta;\alpha}$$
$$-2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta}]\,\nabla^\rho\nabla^\sigma G^n+4n\alpha u^\beta[2u_{[\sigma}R_{\alpha]\rho;\beta}+g_{\rho[\sigma}u_{\alpha]}R_{,\beta}]\nabla^\rho\nabla^\sigma G^n+4$$
$$\times n\alpha u^\gamma u^\beta[R_{\gamma\rho\alpha\sigma}-g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^\rho\nabla^\sigma(G^n)_{,\beta}+n\alpha u^\beta[u_{[\sigma}R_{\alpha]\rho}+\frac{1}{2}Ru_{[\alpha}g_{\sigma]\rho}]$$
$$\times\nabla^\rho\nabla^\sigma(G^n)_{,\beta}+\frac{1}{2}\alpha(n-1)u_\alpha u^\beta(G^n)_{,\beta}-4n\alpha[u_\sigma u^\beta R_{\rho\beta;\alpha}+R_{\rho\sigma;\alpha}+u_\gamma$$
$$\times u_\rho R_{\gamma\sigma;\alpha}-\frac{1}{2}h_{\sigma\rho}R_{;\alpha}]\nabla^\gamma\nabla^\sigma G^n-4n\alpha[u^\gamma u^\beta R_{\gamma\rho\beta\sigma}+u^\beta u_\sigma R_{\rho\beta}+R_{\rho\sigma}$$
$$-u^\gamma u^\beta g_{\sigma\rho}R_{\gamma\beta}+u^\gamma u_\rho R_{\gamma\sigma}-\frac{1}{2}h_{\sigma\rho}R]\nabla^\rho\nabla^\sigma(G^n)_{,\alpha}+\frac{1}{2}\alpha(n-1)(G^n)_{,\alpha}-\frac{4}{3}$$
$$\times n\alpha[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})]_{,\beta}\nabla^\rho\nabla^\sigma G^n-8n\alpha[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R$$
$$\times(4g_{\sigma\rho}-\delta_{\rho\sigma})]\nabla^\rho\nabla^\sigma(G^n)_{,\beta}+\frac{1}{3}\alpha(n-1)(G^n)_{,\beta}=0. \qquad (3.1.40)$$

In GR, this relates energy density inhomogeneity, the Weyl tensor and anisotropy. This equation clearly shows that energy density inhomogeneity is linked with the Weyl tensor, anisotropy and GB terms.

(v) Non-geodesic Dissipative Anisotropic Fluids

This is a more general case with the presence of dissipation $q \neq 0$ ($\epsilon = 0$ for simplicity) and anisotropy ($\Pi \neq 0$). From Eq.(3.1.25), we have

$$\begin{aligned}
& h^{\alpha}_{\mu}\mathbb{E}^{\lambda}_{\alpha;\lambda} + a^{\lambda}\mathbb{E}_{\mu\lambda} + q_{\gamma}(\sigma^{\gamma}_{\mu} + \frac{1}{3}\vartheta h^{\gamma}_{\mu}) + \frac{1}{3}h^{\beta}_{\mu}(-2\rho + 3p_{r} + \Pi)_{,\beta} + a_{\mu}(-\rho + p_{r}) \\
& -a^{\gamma}\Pi_{\mu\gamma} + u^{\beta}h^{\alpha}_{\mu}\tilde{q}_{\alpha;\beta} + 4n\alpha u^{\gamma}u^{\beta}\left[R_{\gamma\rho\alpha\sigma;\beta} - R_{\gamma\rho\beta\sigma;\alpha} - 2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta} + g_{\sigma\rho}\right. \\
& \left.\times R_{\gamma\beta;\alpha}\right]\nabla^{\rho}\nabla^{\sigma}G^{n} + 4n\alpha u^{\beta}[2u_{[\sigma}R_{\alpha]\rho;\beta} + g_{\rho[\sigma}u_{\alpha]}R_{,\beta}]\nabla^{\rho}\nabla^{\sigma}G^{n} + 4n\alpha u^{\gamma}u^{\beta} \\
& \times[R_{\gamma\rho\alpha\sigma} - g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta} + n\alpha u^{\beta}[u_{[\sigma}R_{\alpha]\rho} + \frac{1}{2}Ru_{[\alpha}g_{\sigma]\rho}]\nabla^{\rho}\nabla^{\sigma} \\
& (G^{n})_{,\beta} + \frac{1}{2}\alpha(n-1)u_{\alpha}u^{\beta}(G^{n})_{,\alpha} - 4n\alpha[u_{\sigma}u^{\beta}R_{\rho\beta;\alpha} + R_{\rho\sigma;\alpha} + u_{\gamma}u_{\rho}R_{\gamma\sigma;\alpha} \\
& -\frac{1}{2}h_{\sigma\rho}R_{;\alpha}]\nabla^{\rho}\nabla^{\sigma}G^{n} - 4n\alpha[u^{\gamma}u^{\beta}R_{\gamma\rho\beta\sigma} + u^{\beta}u_{\sigma}R_{\rho\beta} + R_{\rho\sigma} - u^{\gamma}u^{\beta}g_{\sigma\rho} \\
& \times R_{\gamma\beta} + u^{\gamma}u_{\rho}R_{\gamma\sigma} - \frac{1}{2}h_{\sigma\rho}R]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\alpha} + \frac{1}{2}\alpha n - 1(G^{n})_{,\alpha} - \frac{4}{3}n\alpha[R_{\rho\sigma} \\
& +Rg_{\sigma\rho} - \frac{1}{2}R(4g_{\sigma\rho} - \delta_{\rho\sigma})]_{,\beta}\nabla^{\rho}\nabla^{\sigma}G^{n} - 8n\alpha[R_{\rho\sigma} + Rg_{\sigma\rho} - \frac{1}{2}R(4g_{\sigma\rho} \\
& -\delta_{\rho\sigma})]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\alpha} + \frac{1}{3}\alpha(n-1)(G^{n})_{,\beta} = 0. \qquad (3.1.41)
\end{aligned}$$

This equation represents the link of dark source (GB) terms with the Weyl tensor and other dynamical quantities. This represents that the tidal force that an object feels while moving along a geodesic is affected by dynamical quantities as well as GB terms. This also shows that inhomogeneity of energy density as well as dark source terms are not affected by the absence of shear and expansion parameters. This indicates that inhomogeneity of energy density not only depend upon dark source terms but also on other dynamical variables, so its homogeneity does not alter inhomogeneity of energy density. The transport equation (1.5.9) gives

$$q=\frac{\tau[(p_r)_{,\alpha}v^{\alpha}+(\rho+p_r)a+2\Pi\frac{C'}{BC}+GB]-K[\mathbb{T}_{\alpha}u^{\alpha}+a\mathbb{T}]}{1+\frac{1}{2}\tau[\frac{1}{3}(2\sigma-5\vartheta)+\frac{1}{\tau}u^{\alpha}\tau_{,\alpha}-\frac{1}{K}u^{\alpha}K_{,\alpha}-\frac{2}{\mathbb{T}}u^{\alpha}\mathbb{T}_{,\alpha}]},\tag{3.1.42}$$

where

$$\begin{aligned} GB &= 8n\alpha[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}v_{\mu}\nabla^{\rho}\nabla^{\sigma}G^{n}+8n\alpha[(v_{\sigma}R^{\nu}_{\rho}-v^{\nu}R_{\rho\sigma}+R^{\mu}_{\sigma}\\ &\times\ \delta^{\nu}_{\rho})_{,\nu}+\Gamma^{\mu}_{\nu\gamma}(R^{\nu}_{\rho}\delta^{\gamma}_{\sigma}+R^{\gamma}_{\sigma}\delta^{\nu}_{\rho})v_{\mu}+\Gamma^{\nu}_{\nu\gamma}(v_{\sigma}R^{\gamma}_{\rho}+v^{\gamma}R_{\rho\sigma})]\nabla^{\rho}\nabla^{\sigma}G^{n}. \end{aligned}$$

Inserting Eq.(3.1.42) in (3.1.41), we obtain a relation between density inhomogeneity and thermodynamics variables.

3.2 Structure Scalars for Spherically Symmetric Self-Gravitating System

In this section, we investigate spherically symmetric self-gravitating system by constructing structure scalars in the background of $f(G)$ gravity. We then rewrite the set of governing equations in terms of these scalars. We also obtain inhomogeneity factors and static inhomogeneous anisotropic spherical solutions by using these structure scalars.

3.2.1 Modified Structure Scalars

Here we develop structure scalars by splitting the Reimman curvature tensor orthogonally. From Eq.(1.5.7), we can write

$$R^{\mu}_{\rho\sigma\nu}=C^{\mu}_{\rho\sigma\nu}+\frac{1}{2}R^{\mu}_{\sigma}g_{\rho\nu}-\frac{1}{2}R_{\rho\sigma}\delta^{\mu}_{\nu}+\frac{1}{2}R_{\rho\nu}\delta^{\mu}_{\sigma}-\frac{1}{2}R^{\mu}_{\nu}g_{\rho\sigma}-\frac{1}{6}R(\delta^{\mu}_{\sigma}g_{\rho\nu}+\delta^{\mu}_{\nu}g_{\rho\sigma}).$$

This equation can be written for Eq.(1.1.4) as

$$R^{\mu\gamma}_{\nu\delta} = C^{\mu\gamma}_{\nu\delta} + 2\overset{\text{(eff)}}{T}{}^{[\mu}{}_{[\nu}\delta^{\gamma]}_{\delta]} + \overset{\text{(eff)}}{T}\left(\frac{1}{3}\delta^{\mu}_{[\nu}\delta^{\gamma}_{\delta]} - \delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}\right). \quad (3.2.1)$$

We decompose the Riemann curvature tensor into five parts as

$$R^{\mu\gamma}_{\nu\delta} = \overset{\text{(I)}}{R}{}^{\mu\gamma}{}_{\nu\delta} + \overset{\text{(II)}}{R}{}^{\mu\gamma}{}_{\nu\delta} + \overset{\text{(III)}}{R}{}^{\mu\gamma}{}_{\nu\delta} + \overset{\text{(IV)}}{R}{}^{\mu\gamma}{}_{\nu\delta} + \overset{\text{(V)}}{R}{}^{\mu\gamma}{}_{\nu\delta},$$

which are defined using Eq.(3.2.1) as

$$\overset{\text{(I)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = 2\left(\tilde{\rho}u^{[\mu}u_{[\nu}\delta^{\gamma]}_{\delta]} + p_t h^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}\right) + (-\tilde{\rho} + 3p_t + \Pi)\left(\frac{1}{3}\delta^{\mu}_{[\nu}\delta^{\gamma}_{\delta]} - \delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}\right), \quad (3.2.2)$$

$$\overset{\text{(II)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = 2\left(\Pi v^{[\mu}v_{[\nu}\delta^{\gamma]}_{\delta]} + \tilde{q}v^{[\mu}u_{[\nu}\delta^{\gamma]}_{\delta]} + \tilde{q}u^{[\mu}v_{[\nu}\delta^{\gamma]}_{\delta]}\right), \quad (3.2.3)$$

$$\overset{\text{(III)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = 4u^{[\mu}u_{[\nu}\mathbb{E}^{\gamma]}_{\delta]} - \varepsilon^{\mu\gamma}_{\kappa}\varepsilon_{\nu\delta\eta}\mathbb{E}^{\kappa\eta}, \quad (3.2.4)$$

$$\begin{aligned}\overset{\text{(IV)}}{R}{}^{\mu\gamma}{}_{\nu\delta} &= 8\Big[R^{[\mu}_{\rho\sigma[\nu}\delta^{\gamma]}_{\delta]} + R_{\rho[\nu}\delta^{[\mu}_{\sigma}\delta^{\gamma]}_{\delta]} - R_{\rho\sigma}\delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]} - \delta_{\sigma\lambda}\delta^{\lambda}_{\rho}R^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]} + g_{\rho[\nu}R^{[\mu}_{\sigma}\delta^{\gamma]}_{\delta]} \\ &+ \frac{1}{2}R\left(\delta_{\sigma\lambda}\delta^{\lambda}_{\rho}\delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]} - \delta^{[\mu}_{\sigma}g_{\rho[\nu}\delta^{\gamma]}_{\delta]}\right)\Big]\nabla^{\rho}\nabla^{\sigma}f_G + (Gf_G - f)\delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}, \quad (3.2.5)\end{aligned}$$

$$\overset{\text{(V)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = (8[2R_{\rho\sigma} - Rg_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}f_G + Gf_G - f)\left(\frac{1}{3}\delta^{\mu}_{[\nu}\delta^{\gamma}_{\delta]} - \delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}\right). \quad (3.2.6)$$

Using Eqs.(3.2.2)-(3.2.6) in (1.6.1) along with (1.1.7), we obtain

$$\begin{aligned}X_{\mu\nu} &= \overset{\text{(m)}}{X}_{\mu\nu} + \overset{\text{(GB)}}{X}_{\mu\nu} \\ &= p_t h_{\mu\nu} + \frac{1}{2}\Pi(h_{\mu\nu} - v_{\mu}v_{\nu}) - \frac{1}{3}h_{\mu\nu}(-\tilde{\rho} + 3p_t + \Pi) - \mathbb{E}_{\mu\nu} + n\alpha[5R_{\rho\sigma} \\ &\times \delta_{\mu\nu} - 2R_{\mu\rho\sigma\nu} + 10R^{\kappa}_{\sigma}u_{\mu}u_{\kappa}g_{\rho\nu} - 2R^{\eta}_{\rho\sigma\nu}u_{\mu}u_{\eta} - 2R^{\kappa}_{\lambda}u_{\kappa}u^{\lambda}g_{\sigma\rho}\delta_{\mu\nu} - R^{\kappa}_{\sigma} \\ &\times u_{\mu}u_{\kappa}\delta_{\rho\nu} - R_{\rho\nu}u_{\mu}u_{\sigma} + 9R_{\mu\nu}g_{\sigma\rho} + 2R_{\mu\lambda}u_{\nu}u^{\lambda}g_{\sigma\rho} + \frac{1}{2}R(3g_{\sigma\rho}g_{\mu\nu} + g_{\rho\nu} \\ &\times \delta_{\mu\sigma} + 2g_{\sigma\rho}u_{\mu}u_{\nu})]\nabla^{\rho}\nabla^{\sigma}G^{n-1} + \alpha(n-1)G^{n}(4h_{\mu\nu} + 4\delta_{\mu\nu} + u_{\mu}u_{\nu}) \\ &- \frac{1}{3}n\alpha\left(8[2R_{\rho\sigma} - Rg_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1} + \alpha(n-1)G^{n}\right)(g_{\mu\nu} + \delta_{\mu\nu}), \quad (3.2.7)\\ Y_{\mu\nu} &= \overset{\text{(m)}}{Y}_{\mu\nu} + \overset{\text{(GB)}}{Y}_{\mu\nu}\end{aligned}$$

$$= \frac{1}{2}h_{\mu\nu}\left(\tilde{\rho} - p_{_t} - \Pi v_\mu v_\nu + \frac{2}{3}(-\tilde{\rho} + 3p_{_t} + \Pi)\right) + \mathbb{E}_{\mu\nu}, \tag{3.2.8}$$

where the notation (m) and (GB) in superscript respectively show matter and GB parts of the relevant tensor. We note from Eq.(3.2.8) that the variable $Y_{\mu\nu}$ does not contain GB terms. Hence GB terms do not affect this variable.

We can write these tensors as the combination of their trace and traceless parts in the following way. From Eqs.(3.2.7) and (3.2.8), we have

$$X_{_T} \equiv Tr(X) = \overset{(m)}{X}_{_T} + \overset{(GB)}{X}_{_T}, \quad Y_{_T} \equiv Tr(Y) = \overset{(m)}{Y}_{_T},$$

where

$$\overset{(m)}{X}_{_T} = \tilde{\rho}, \tag{3.2.9}$$

$$\overset{(GB)}{X}_{_T} = -\frac{1}{3}\alpha[n(74R_{\rho\sigma} + 6u^\nu u_\eta R^\eta_{\rho\sigma\nu} + 18u_\kappa u^\lambda R^\kappa_\lambda g_{\sigma\rho} + 3u_\sigma u_\mu R^\mu_\rho + \frac{1}{2} \times R(\delta_{\rho\sigma} - 25g_{\sigma\rho}))\nabla^\rho\nabla^\sigma G^{n-1} - 73(n-1)G^n], \tag{3.2.10}$$

$$\overset{(m)}{Y}_{_T} = \frac{1}{2}(\tilde{\rho} + 3\tilde{p}_{_r} - 2\Pi). \tag{3.2.11}$$

Moreover, their corresponding traceless parts are found as

$$X_{\langle\mu\nu\rangle} = \overset{(m)}{X}_{_{TF}}(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}) + \overset{(GB)}{X}_{_{TF}}(-\frac{1}{3}h_{\mu\nu}),$$

$$Y_{\langle\mu\nu\rangle} = \overset{(m)}{Y}_{_{TF}}(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}),$$

where TF in subscript represents traceless part of the relevant tensor and

$$\overset{(m)}{X}_{_{TF}} = -(\varepsilon - \frac{1}{2}\Pi), \tag{3.2.12}$$

$$\overset{(GB)}{X}_{_{TF}} = \frac{1}{2}n\alpha(12R^\kappa_\lambda u_\kappa u^\lambda g_{\sigma\rho} - 55Rg_{\sigma\rho})\nabla^\rho\nabla^\sigma G^{n-1} - 22\alpha(n-1)G^n, \tag{3.2.13}$$

$$\overset{(m)}{Y}_{_{TF}} = \varepsilon + \frac{1}{2}\Pi. \tag{3.2.14}$$

The six quantities $\overset{(\mathrm{m})}{X}_{T}$, $\overset{(\mathrm{GB})}{X}_{T}$, $\overset{(\mathrm{m})}{X}_{TF}$, $\overset{(\mathrm{GB})}{X}_{TF}$, $\overset{(\mathrm{m})}{Y}_{T}$ and $\overset{(\mathrm{m})}{Y}_{TF}$ are scalar functions (structure scalars) in $f(G)$ gravity.

Now we briefly discuss physical aspects of these structure scalars in this gravity that have a direct correspondence with the dynamics of spherical systems. We note that $\overset{(\mathrm{m})}{X}_{T}$ evidently represents the energy density. The scalar variables $\overset{(\mathrm{m})}{X}_{TF}$ and $\overset{(\mathrm{m})}{Y}_{TF}$ play a crucial role in the physical interpretation of fluid. Both scalars combine the Weyl tensor with pressure anisotropy. Their sum indicates local anisotropy while their difference provides the effects of tidal force. With local isotropy ($\Pi = 0$), they behave like

$$\overset{(\mathrm{m})}{X}_{TF} = -\overset{(\mathrm{m})}{Y}_{TF}$$

while in the absence of the Weyl tensor (conformally flat condition), both are same

$$\overset{(\mathrm{m})}{X}_{TF} = \overset{(\mathrm{m})}{Y}_{TF}.$$

We observe that local isotropy and conformally flatness condition contradict each other. This means that spherical system is either isotropic or conformally flat or it preserves local anisotropy as well as conformal flat condition. Furthermore, the combination

$$-\frac{1}{2}\overset{(\mathrm{m})}{X}_{T} + \overset{(\mathrm{m})}{X}_{TF} + \overset{(\mathrm{m})}{Y}_{T} + \overset{(\mathrm{m})}{Y}_{TF} = \frac{3}{2}\tilde{p}_{r}$$

describes the radial pressure and the sum $\overset{(\mathrm{m})}{X}_{T} + \overset{(\mathrm{GB})}{X}_{T}$ demonstrates that matter energy density is connected with dark source terms. The combination $\overset{(\mathrm{m})}{X}_{TF} + \overset{(\mathrm{m})}{Y}_{TF} + \overset{(\mathrm{GB})}{X}_{TF}$ makes spherical system conformally flat and indicates that pressure anisotropy is controlled by dark source terms due to $f(G)$ gravity. The unification of scalars as $\overset{(\mathrm{m})}{Y}_{TF} - \overset{(\mathrm{m})}{X}_{TF} + \overset{(\mathrm{GB})}{X}_{TF}$ shows that conformal flatness of fluid is controlled by GB terms.

3.2.2 Evolution Equations in terms of Structure Scalars

A set of governing equations can be deduced in terms of kinematical variables to describe the self-gravitating system. Here we rewrite this set of equations in terms of modified structure scalars in the realm of $f(G)$ gravity.

- **Raychaudhuri equation**: Equation (3.1.13) defines Raychaudhuri equation for expansion of self-gravitating relativistic system which in terms of scalar function becomes

$$\vartheta_{;\mu}u^{\mu}+\frac{1}{3}\vartheta^2+\frac{2}{3}\sigma^2-a^{\mu}_{;\mu}=-\overset{(\mathrm{m})}{Y}_{T}. \qquad (3.2.15)$$

  We observe from this relation that GB terms have no contribution and $\overset{(\mathrm{m})}{Y}_{T}$ has an extreme role in measuring the expansion rate of self-gravitating fluid in GR as well as $f(G)$ cosmology.

- **Propagation equation of shear**: This equation describes the shear evolution of self-gravitating system given in Eq.(3.1.15) which in terms of modified scalar variables reduces to

$$a_{,\mu}v^{\mu}+a^2-\sigma_{,\mu}u^{\mu}-\frac{1}{3}\sigma^2-\frac{2}{3}\sigma\vartheta-a\frac{C'}{BC}=-\overset{(\mathrm{m})}{Y}_{TF}+\frac{2}{67}n\alpha\overset{(\mathrm{GB})}{X}_{TF}+22\alpha(n-1)G^{n} \qquad (3.2.16)$$

  showing the importance of GB correction terms in the shearing motion of the evolving self-gravitating system.

- **Constraint equation**: A direct relation among shear tensor, expansion scalar, heat flux and dark source terms due to $f(G)$ gravity is obtained through constraint equation (3.1.16) which in terms of scalar variables becomes

$$(\vartheta+\frac{1}{2}\sigma)_{,\alpha}v^{\alpha} = -\frac{3C'}{2BC}\sigma-\frac{3}{2B}\tilde{q}-\frac{1}{64}(R\delta_{\rho\sigma})\nabla^{\rho}\nabla^{\sigma}f_{G}+\frac{9}{444}\overset{(\mathrm{GB})}{X}_{TF}$$

$$- \quad \frac{3}{32}\overset{(\mathrm{GB})}{X}_{T} - \frac{7407}{929}\alpha(n-1)G^{n}. \tag{3.2.17}$$

- **Dynamical equations:** These equations describe the conservation of total energy of the evolving star given in Eqs.(3.1.27) and (3.1.28) and in terms of scalar variables, these become

$$\tilde{\rho}_{,\mu}u^{\mu} + \frac{1}{3}(\overset{(\mathrm{m})}{X}_{T} + \overset{(\mathrm{m})}{Y}_{T} - \overset{(\mathrm{m})}{X}_{TF} - \overset{(\mathrm{m})}{Y}_{TF})\vartheta + \frac{2}{3}(\overset{(\mathrm{m})}{X}_{TF} + \overset{(\mathrm{m})}{Y}_{TF}) + 2\tilde{q}\left(\frac{C'}{BC} + a\right)$$

$$+\tilde{q}_{,\mu}v^{\mu} - \frac{1}{58}(\overset{(\mathrm{GB})}{X}_{TF})_{,\nu}u^{\nu} - \frac{11}{29}\alpha(n-1)(G^{n})_{,\nu}u^{\nu} = 0, \tag{3.2.18}$$

$$\tilde{p}_{r,\mu}v^{\mu} + a(\overset{(\mathrm{m})}{X}_{T} + \overset{(\mathrm{m})}{Y}_{T} - \overset{(\mathrm{m})}{X}_{TF} - \overset{(\mathrm{m})}{Y}_{TF}) + (\overset{(\mathrm{m})}{X}_{TF} + \overset{(\mathrm{m})}{Y}_{TF})\frac{2C'}{BC} + \tilde{q}_{,\mu}v^{\mu} + \frac{2}{3}\tilde{q}$$

$$\times(\sigma + 2\vartheta) - \frac{1}{58}(\overset{(\mathrm{GB})}{X}_{TF})_{,\nu}v^{\nu} - \frac{11}{29}\alpha(n-1)(G^{n})_{,\nu}v^{\nu} = 0, \tag{3.2.19}$$

which show that the rate of change of energy density and radial pressure depend on scalar functions of matter and dark source terms.

- **Evolution equations for the Weyl tensor:** These equations represent the relationship between the Weyl tensor, dynamical variables and extra dark source terms as (3.1.24) and (3.1.25). In terms of scalar functions, these become

$$\frac{1}{3}\left((-\frac{1}{2}\overset{(\mathrm{m})}{X}_{T} + \overset{(\mathrm{m})}{X}_{TF} + \overset{(\mathrm{m})}{Y}_{T} + \overset{(\mathrm{m})}{Y}_{TF}) + \frac{9}{2C^{3}}\mathcal{M}\right)(\vartheta + \frac{1}{2}\sigma) + (\frac{1}{2}\overset{(\mathrm{m})}{X}_{T} - \overset{(\mathrm{m})}{X}_{TF})$$

$$+\frac{3C'}{2BC}\tilde{q} + \frac{1}{4}n\alpha[v_{\rho}v_{\sigma}(R_{,\beta}u^{\beta} - R_{,\alpha}v^{\alpha})]\nabla^{\rho}\nabla^{\sigma}G^{n-1} + \frac{1}{4}n\alpha(Rv_{\rho}v_{\sigma})\nabla^{\rho}\nabla^{\sigma}$$

$$\times((G^{n-1})_{,\beta}u^{\beta} - (G^{n-1})_{,\alpha}v^{\alpha}) + \frac{1}{4}n\alpha(Ru_{\rho}u_{\sigma})_{,\beta}u^{\beta}\nabla^{\rho}\nabla^{\sigma}G^{n-1} + \frac{9}{32}n\alpha$$

$$\times(Ru_{\rho}u_{\sigma})\nabla^{\rho}\nabla^{\sigma}(G^{n-1})_{,\beta}u^{\beta} + (\frac{7}{8}\overset{(\mathrm{GB})}{X}_{T} + \frac{9}{2^{6}\cdot 87}\overset{(\mathrm{GB})}{X}_{TF})_{,\beta}u^{\beta} - \frac{1975}{24}(n-1)$$

$$\times\alpha(G^{n})_{,\beta}u^{\beta} = 0, \tag{3.2.20}$$

$$(\frac{1}{2}\tilde{\rho})_{,\alpha}v^{\alpha} - (\overset{(\mathrm{m})}{X}_{TF})_{,\alpha}v^{\alpha} - \frac{3C'}{BC}\overset{(\mathrm{m})}{X}_{TF} - \tilde{q}(\vartheta + \frac{1}{2}\sigma) - \frac{5}{116}(\overset{(\mathrm{GB})}{X}_{TF})_{,\alpha}v^{\alpha} - \frac{61}{58}$$

$$\times\alpha(n-1)(G^{n})_{,\alpha}v^{\alpha} = 0. \tag{3.2.21}$$

The above two equations relate the effects of tidal force with fluid parameters as well as GB terms. Equation (3.2.21) shows the dependence of density inhomogeneity on two scalars $\overset{(m)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, dissipation and $f(G)$ model.

Now, we figure out density inhomogeneity factor on the surface of spherical system under the influence of $f(G)$ gravity. In a collapsing system, the surface of celestial object suffers density inhomogeneity caused by some specific quantities or specific relations of dynamical and geometrical variables. The vanishing of these quantities or relations assures density homogeneity [23]. The dissipation and scalar variable of matter X_{TF} lead to density inhomogeneity for spherical system in GR. It is evident from Eq.(3.2.21) that if we neglect dissipation and matter variable $\overset{(m)}{X}_{TF}$, we obtain

$$(\frac{1}{2}\tilde{\rho})_{,\alpha}v^{\alpha}-\frac{5}{116}(\overset{(GB)}{X}_{TF})_{,\alpha}v^{\alpha}-\frac{61}{58}\alpha(n-1)(G^{n})_{,\alpha}v^{\alpha}=0.$$

We are left with density inhomogeneity and GB terms which suggest that density inhomogeneity is controlled by GB terms. Furthermore, if $\overset{(GB)}{X}_{TF}=0$ and $f(G)=$ *constant*, we have

$$(\frac{1}{2}\tilde{\rho})_{,\alpha}=0, \tag{3.2.22}$$

which reveals that $f(G)$ model remains homogeneous and constant during the evolution. However, a viable and realistic $f(G)$ model cannot be a constant and hence scalar functions do not vanish which leads to inhomogeneous (irregular) distribution of fluid.

3.2.3 The Weyl tensor with Mass Function and GB Terms

This relation can be obtained by using Eqs.(1.1.4), (1.5.7) and (3.1.6) as

$$\frac{3\mathcal{M}}{C^{3}}=\tilde{\rho}-\Pi-\varepsilon-\frac{1}{3}n\alpha[2Rg_{\rho\sigma}+3R\delta_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\frac{1}{6}\alpha(n-1)G^{n}.$$

In terms of scalar functions, it turns out to be

$$\frac{3\mathcal{M}}{C^3}=\frac{1}{2}\overset{(\mathrm{m})}{X}_{T}-\overset{(\mathrm{m})}{X}_{TF}+\frac{1}{87}\overset{(\mathrm{GB})}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\frac{219}{522}\alpha(n-1)G^{n}. \quad (3.2.23)$$

3.2.4 Anisotropic Inhomogeneous Spherical Models

In this section, we restrict ourselves to the static case and modify the line element (3.1.1) in terms of scalar functions. The resulting line element can yield static spherical solutions with inhomogeneity and anisotropy in $f(G)$ gravity. We consider $C=r$ for static configuration for which $\vartheta=\sigma=0$. Three possible alternative forms are given as follows.

(i) First Alternative Form

It can directly be seen from Eq.(3.1.6) that Misner-Sharp mass function for static case becomes

$$\frac{2}{r}\mathcal{M}=1-\frac{1}{B^2}, \quad (3.2.24)$$

which gives

$$\begin{aligned} B &= \left(1-\frac{2r^2}{3}\left(\frac{1}{2}\overset{(\mathrm{m})}{X}_{T}-\overset{(\mathrm{m})}{X}_{TF}+\frac{1}{87}\overset{(\mathrm{GB})}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1}\right.\right.\\ &+ \left.\left.\frac{219}{522}\alpha(n-1)G^{n}\right)\right)^{-\frac{1}{2}}. \quad (3.2.25)\end{aligned}$$

Next, using Eqs.(3.1.13) and (3.1.15) with (1.5.7) for static case, we obtain

$$\frac{A'}{AB}=\frac{1}{3}Br[\overset{(\mathrm{m})}{Y}_{T}+\overset{(\mathrm{m})}{Y}_{TF}-\frac{2}{67}\overset{(\mathrm{GB})}{Y}_{TF}-22\alpha(n-1)G^{n}], \quad (3.2.26)$$

which after integration gives

$$A = \lambda_1\exp[\int\frac{r}{3}\left(1-\frac{2r^2}{3}\left(\frac{1}{2}\overset{(\mathrm{m})}{X}_{T}-\overset{(\mathrm{m})}{X}_{TF}+\frac{1}{87}\overset{(\mathrm{GB})}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1}\right.\right.$$

$$+\quad \frac{219}{522}\alpha(n-1)G^{n}\Big)\Big)^{-1}(\overset{\text{(m)}}{Y}_{T}+\overset{\text{(m)}}{Y}_{TF}-\frac{2}{67}\overset{\text{(GB)}}{X}_{TF}-22\alpha(n-1)G^{n})dr],\quad (3.2.27)$$

where λ_1 is a constant of integration. In this case, the line element (3.1.1) becomes for Eqs.(3.2.25) and (3.2.27) as

$$\begin{aligned} ds^2 &= -[\lambda_1\exp[\int\frac{r}{3}\left(1-\frac{2r^2}{3}\left(\frac{1}{2}\overset{\text{(m)}}{X}_{T}-\overset{\text{(m)}}{X}_{TF}+\frac{1}{87}\overset{\text{(GB)}}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}\right.\right.\\ &\times \left.\left. G^{n-1}+\frac{219}{522}\alpha(n-1)G^{n}\right)\right)^{-1}(\overset{\text{(m)}}{Y}_{T}+\overset{\text{(m)}}{Y}_{TF}-\frac{2}{67}\overset{\text{(GB)}}{X}_{TF}-22\alpha(n-1)\\ &\times G^{n})dr]]^2dt^2+\left(1-\frac{2r^2}{3}\left(\frac{1}{2}\overset{\text{(m)}}{X}_{T}-\overset{\text{(m)}}{X}_{TF}+\frac{1}{87}\overset{\text{(GB)}}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}\right.\right.\\ &\times \left.\left. G^{n-1}+\frac{219}{522}\alpha(n-1)G^{n}\right)\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \end{aligned}$$

A singularity seems to appear if

$$\frac{2r^2}{3}(\frac{1}{2}\overset{\text{(m)}}{X}_{T}-\overset{\text{(m)}}{X}_{TF}+\frac{1}{87}\overset{\text{(GB)}}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\frac{219}{522}\alpha(n-1)G^{n})=1.$$

By inserting the values of five scalar functions $\overset{\text{(m)}}{X}_{T}$, $\overset{\text{(m)}}{X}_{TF}$, $\overset{\text{(GB)}}{X}_{TF}$, $\overset{\text{(m)}}{Y}_{T}$ and $\overset{\text{(m)}}{Y}_{TF}$, we can obtain all possible inhomogeneous static anisotropic spherical solutions.

(ii) Second Alternative Form

Using Eqs.(3.1.11), (3.2.9) and (3.2.23), we obtain the relation

$$\frac{3}{r^3}\mathcal{M}=\frac{\mathcal{M}'}{r^2}-\overset{\text{(m)}}{X}_{TF}+\frac{377}{87(29)}\overset{\text{(GB)}}{X}_{TF},$$

which implies after integration that

$$\mathcal{M}=r^3\left(\int(\overset{\text{(m)}}{X}_{TF}-\frac{377}{87(29)}\overset{\text{(GB)}}{X}_{TF})dr+\lambda_2\right),$$

where λ_2 is another integration constant. Using this equation in (3.2.24), we have

$$B=\left(1-2r^2\left(\int(\overset{\text{(m)}}{X}_{TF}-\frac{377}{87(29)}\overset{\text{(GB)}}{X}_{TF})dr+\lambda_2\right)\right)^{-\frac{1}{2}}.\qquad (3.2.28)$$

Combining Eqs.(3.2.26) and (3.2.28), it follows that

$$\begin{aligned} A &= \lambda_3 \exp[\frac{1}{3}\int\left(1-2r^2\left(\int(\overset{(m)}{X}_{TF}-\frac{377}{87(29)}\overset{(GB)}{X}_{TF})dr+\lambda_2\right)\right)^{-1} r[\overset{(m)}{Y}_{T}+\overset{(m)}{Y}_{TF} \\ &- \frac{2}{67}\overset{(GB)}{X}_{TF}-22\alpha(n-1)G^n]dr], \end{aligned} \tag{3.2.29}$$

λ_3 is another constant of integration. The line element (3.1.1) takes the following form for Eqs.(3.2.28) and (3.2.29) as

$$\begin{aligned} ds^2 &= -[\lambda_3 \exp[\frac{1}{3}\int r\left(1-2r^2\left(\int(\overset{(m)}{X}_{TF}-\frac{377}{87(29)}\overset{(GB)}{X}_{TF})dr+\lambda_2\right)\right)^{-1}[\overset{(m)}{Y}_{T} \\ &+ \overset{(m)}{Y}_{TF}-\frac{2}{67}\overset{(GB)}{X}_{TF}-22\alpha(n-1)G^n]dr]]^2dt^2+\left(1-2r^2\left(\int(\overset{(m)}{X}_{TF}\right.\right. \\ &- \left.\left.\frac{377}{87(29)}\overset{(GB)}{X}_{TF})dr+\lambda_2\right)\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \end{aligned}$$

In this case, singularity can appear if $2r^2\left(\int(\overset{(m)}{X}_{TF}-\frac{377}{87(29)}\overset{(GB)}{X}_{TF})dr+\lambda_2\right)=1$ and all spherical inhomogeneous static anisotropic solutions can be obtained by inserting the values of five scalars $\overset{(m)}{X}_{T}$, $\overset{(m)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, $\overset{(m)}{Y}_{T}$ and $\overset{(m)}{Y}_{TF}$.

(iii) Third Alternative Form

Equation (3.1.5) reduces to

$$\varepsilon=\frac{1}{2B^2r}\left(\frac{1}{r}+\frac{B'}{B}\right)-\frac{1}{2r^2}. \tag{3.2.30}$$

Alternatively, Eq.(3.2.30) can be rearranged as follows

$$B'+\frac{1}{r}B=B^3\left(\frac{1}{r}+2r(\overset{(m)}{Y}_{TF}-\overset{(m)}{X}_{TF})\right),$$

where $\varepsilon=\overset{(M)}{Y}_{TF}-\overset{(m)}{X}_{TF}$. This is the Bernoulli's differential equation which gives

$$B=\left(-4r^2\int\frac{1}{r}(\overset{(m)}{Y}_{TF}-\overset{(m)}{X}_{TF})dr+1+\lambda_4r^2\right)^{-\frac{1}{2}}. \tag{3.2.31}$$

Using Eq.(3.2.31) in (3.2.26), it follows that

$$A = \frac{\lambda_5}{3}\exp[\int r\left(-4r^2\int\frac{1}{r}(\overset{(\mathrm{m})}{Y}_{TF}-\overset{(\mathrm{m})}{X}_{TF})dr+1+\lambda_4 r^2\right)^{-1}[\overset{(\mathrm{m})}{Y}_{T}+\overset{(\mathrm{m})}{Y}_{TF} - \frac{2}{67}\overset{(\mathrm{GB})}{X}_{TF}-22\alpha(n-1)G^n]dr]. \quad (3.2.32)$$

The corresponding form of the line element (3.1.1) becomes

$$\begin{aligned} ds^2 &= -[\frac{\lambda_5}{3}\exp[\int r\left(-4r^2\int\frac{1}{r}(\overset{(\mathrm{m})}{Y}_{TF}-\overset{(\mathrm{m})}{X}_{TF})dr+1+\lambda_4 r^2\right)^{-1}[\overset{(\mathrm{m})}{Y}_{T}+\overset{(\mathrm{m})}{Y}_{TF} \\ &- \frac{2}{67}\overset{(\mathrm{GB})}{X}_{TF}-22\alpha(n-1)G^n]dr]^2dt^2+\left(-4r^2\int\frac{1}{r}(\overset{(\mathrm{m})}{Y}_{TF}-\overset{(\mathrm{m})}{X}_{TF})dr+1\right. \\ &+ \left.\lambda_4 r^2\right)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2), \end{aligned}$$

where singularity might appear if $-4r^2\int\frac{1}{r}(\overset{(\mathrm{m})}{Y}_{TF}-\overset{(\mathrm{m})}{X}_{TF})dr+1+\lambda_4 r^2=0$. In this alternative form of static inhomogeneous sphere with anisotropy, all solutions depend upon four scalars $\overset{(\mathrm{m})}{X}_{TF}$, $\overset{(\mathrm{GB})}{X}_{TF}$, $\overset{(\mathrm{m})}{Y}_{T}$ and $\overset{(\mathrm{m})}{Y}_{TF}$.

Chapter 4

Dynamics of Axially Symmetric System and Structure Scalars

This chapter explores dynamics of axially symmetric collapsing fluid under the dark effects of $f(G)$ gravity. It also investigates the effects of shear-free condition on the dynamics of the system. The results of this chapter have been divided into two parts. The first part with shear stress has been published in [51] and the other with shear-free condition has been published in [52]. The format of this chapter is as follows. The next section discusses evolution of axially symmetric system in the presence of shear stress while section **4.2** explores the effects of shear-free condition on this evolution.

4.1 Evolution of Axially Symmetric Systems

In this section, we study evolution of dissipative axially symmetric collapsing fluid by developing a set of governing equations in terms of structure scalars.

4.1.1 Axial System and Kinematical Variables

The line element for axial and reflection symmetric system is [53]

$$ds^2=-A^2(t,r,\theta)dt^2+B^2(t,r,\theta)(dr^2+r^2d\theta^2)+2E(t,r,\theta)dtd\theta+C^2(t,r,\theta)d\phi^2. \quad (4.1.1)$$

The energy distribution of respective fluid observed by an observer with four velocity u^μ ($u^\mu=(A^{-1},0,0,0)$ and $u_\mu=(-A,0,\frac{E}{A},0)$) can be represented by the energy-momentum tensor given by

$$\overset{(\mathrm{eff})}{T}_{\mu\nu}=\overset{(\mathrm{m})}{T}_{\mu\nu}+\overset{(\mathrm{GB})}{T}_{\mu\nu}=(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p})u_\mu u_\nu+\overset{(\mathrm{eff})}{p}g_{\mu\nu}+\overset{(\mathrm{eff})}{\Pi}_{\mu\nu}+\overset{(\mathrm{eff})}{q}_\mu u_\nu+\overset{(\mathrm{eff})}{q}_\nu u_\mu, \quad (4.1.2)$$

where the effective energy density, isotropic pressure, anisotropic tensor and heat flux, respectively are defined as

$$\overset{(\mathrm{eff})}{\rho}=\overset{(\mathrm{m})}{\rho}_{\mu\nu}+\overset{(\mathrm{GB})}{\rho}_{\mu\nu}, \qquad \overset{(\mathrm{eff})}{p}=\overset{(\mathrm{m})}{p}_{\mu\nu}+\overset{(\mathrm{GB})}{p}_{\mu\nu},$$
$$\overset{(\mathrm{eff})}{\Pi}_{\mu\nu}=\overset{(\mathrm{m})}{\Pi}_{\mu\nu}+\overset{(\mathrm{GB})}{\Pi}_{\mu\nu}, \qquad \overset{(\mathrm{eff})}{q}_{\mu}=\overset{(\mathrm{m})}{q}_{\mu\nu}+\overset{(\mathrm{GB})}{q}_{\mu\nu}.$$

We obtain these effective quantities from Eq.(4.1.2) as

$$\begin{aligned}
\overset{(\mathrm{eff})}{\rho} &= \overset{(\mathrm{eff})}{T}u^\mu u^\nu=\overset{(\mathrm{m})}{T}_{\mu\nu}u^\mu u^\nu+\frac{3}{2}[Rg_{\rho\sigma}]\nabla^\rho\nabla^\sigma f_G-Gf_G+f, && (4.1.3)\\
\overset{(\mathrm{eff})}{q}_{\mu} &= -\overset{(\mathrm{eff})}{\rho}u_\mu-\overset{(\mathrm{eff})}{T}_{\mu\nu}u^\nu=-\overset{(\mathrm{eff})}{\rho}u_\mu-\overset{(\mathrm{m})}{T}_{\mu\nu}u^\nu-\frac{1}{2}[Rg_{\rho\sigma}u_\mu]\nabla^\rho\nabla^\sigma f_G\\
&+ (Gf_G-f)u_\mu, && (4.1.4)\\
\overset{(\mathrm{eff})}{p} &= \frac{1}{3}h^{\mu\nu}\overset{(\mathrm{eff})}{T}_{\mu\nu}=\frac{1}{3}h^{\mu\nu}\overset{(\mathrm{m})}{T}_{\mu\nu}+4[\frac{11}{8}Rg_{\rho\sigma}+Ru_\rho u_\sigma-h_{\rho\sigma}]\nabla^\rho\nabla^\sigma f_G\\
&+ 3(Gf_G-f), && (4.1.5)\\
\overset{(\mathrm{eff})}{\Pi}_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu(\overset{(\mathrm{eff})}{T}_{\alpha\beta}-\overset{(\mathrm{eff})}{p}h_{\alpha\beta})=h^\alpha_\mu h^\beta_\nu(\overset{(\mathrm{m})}{T}_{\alpha\beta}-\overset{(\mathrm{m})}{p}h_{\alpha\beta})+8[\frac{5}{8}Rg_{\sigma\rho}\delta_{\mu\nu}\\
&+ \frac{9}{8}Rg_{\sigma\rho}u_\mu u_\nu+\frac{1}{2}Rg_{\nu\rho}h_{\mu\sigma}+\frac{1}{2}Rh_{\mu\sigma}u_\rho u_\nu+\frac{1}{2}Ru_\sigma u_\rho u_\mu u_\nu-Rh_{\mu\sigma}h_{\rho\nu}]\\
&\times \nabla^\rho\nabla^\sigma f_G+(Gf_G-f)(\delta_{\mu\nu}+u_\mu u_\nu)+4h_{\mu\nu}[\frac{11}{8}Rg_{\rho\sigma}+Ru_\rho u_\sigma-h_{\rho\sigma}]
\end{aligned}$$

$$\times \quad \nabla^{\rho}\nabla^{\sigma}f_{_G}+3h_{\mu\nu}(Gf_{_G}-f). \tag{4.1.6}$$

The spacelike unit four-vectors are defined as

$$v_{\mu}=B\delta^{1}_{\mu}, \quad s_{\mu}=\frac{1}{A}(A^2B^2r^2+E^2)^{\frac{1}{2}}\delta^{2}_{\mu}, \quad k_{\mu}=C\delta^{3}_{\mu},$$

which satisfy the relations

$$u^{\mu}u_{\mu}=-v^{\mu}v_{\mu}=-s^{\mu}s_{\mu}=-k^{\mu}k_{\mu}=-1,$$
$$u^{\mu}v_{\mu}=u^{\mu}s_{\mu}=u^{\mu}k_{\mu}=v^{\mu}s_{\mu}=v^{\mu}k_{\mu}=k^{\mu}s_{\mu}=0.$$

The pressure anisotropic parameter has significant effects in controlling hydrostatic equilibrium. The physical phenomena like mixture of two fluids and phase transition cause pressure anisotropy in the stellar models [54]. Some other prominent sources for pressure anisotropy are the magnetic field present in the compact objects (such as neutron stars and white dwarfs), magnetized strange quark stars, magnetic field acting on a Fermi gas, viscosity present in neutron stars as well as in highly densed matter [55]. For the sake of convenience, we convert the anisotropic tensor (4.1.6) in terms of scalar quantities as follows

$$\overset{(\mathrm{eff})}{\Pi}_{\mu\nu}=\frac{1}{3}(2\overset{(\mathrm{eff})}{\Pi}_{1}+\overset{(\mathrm{eff})}{\Pi}_{2})(v_{\mu}v_{\nu}-\frac{1}{3}h_{\mu\nu})+\frac{1}{3}(2\overset{(\mathrm{eff})}{\Pi}_{2}+\overset{(\mathrm{eff})}{\Pi}_{1})(s_{\mu}s_{\nu}-\frac{1}{3}h_{\mu\nu})+2\overset{(\mathrm{eff})}{\Pi}_{vs}v_{(\mu}s_{\nu)}, \tag{4.1.7}$$

where

$$\overset{(\mathrm{eff})}{\Pi}_{vs} = v^{\mu}s^{\nu}\overset{(\mathrm{eff})}{T}_{\mu\nu}, \quad \overset{(\mathrm{eff})}{\Pi}_{1}=(2v^{\mu}v^{\nu}-s^{\mu}s^{\nu}-k^{\mu}k^{\nu})\overset{(\mathrm{eff})}{T}_{\mu\nu}, \tag{4.1.8}$$
$$\overset{(\mathrm{eff})}{\Pi}_{2} = (2s^{\mu}s^{\nu}-k^{\mu}k^{\nu}-v^{\mu}v^{\nu})\overset{(\mathrm{eff})}{T}_{\mu\nu}. \tag{4.1.9}$$

Equations (4.1.8) and (4.1.9) indicate that anisotropy scalars $\overset{(\mathrm{eff})}{\Pi}_{vs}$, $\overset{(\mathrm{eff})}{\Pi}_{1}$ and $\overset{(\mathrm{eff})}{\Pi}_{2}$ depend on matter as well as dark sources. Hence the inhomogeneous distribution of dark sources generate pressure anisotropy in collapsing fluid. Dissipation of heat flux (due to emission of photons or neutrinos which are massless particle) during collapse cannot be overemphasized. Indeed, it is a characteristic process during stellar evolution. Dissipation due to neutrino emission of gravitational binding energy leads to the formation of neutron stars or black holes [56]. The field equations along with $q^{\mu}u_{\mu}=0$ give $T_{03}=0$ which implies that

$$\overset{(\mathrm{eff})}{q}_{\mu}=\overset{(\mathrm{eff})}{q}_{1}v_{\mu}+\overset{(\mathrm{eff})}{q}_{2}s_{\mu}. \tag{4.1.10}$$

Here

$$\overset{(\mathrm{eff})}{q}_{1}=\overset{(\mathrm{eff})}{q}_{\mu}v^{\mu}=\overset{(\mathrm{eff})}{T}_{\mu\nu}u^{\nu}v^{\mu},\quad \overset{(\mathrm{eff})}{q}_{2}=\overset{(\mathrm{eff})}{q}_{\mu}s^{\mu}=\overset{(\mathrm{eff})}{T}_{\mu\nu}u^{\nu}s^{\mu}. \tag{4.1.11}$$

These equations indicate that inhomogeneous distribution of matter and dark sources generate heat dissipation.

The characteristics of self-gravitating collapsing fluid depend upon the behavior of kinematical variables. The four acceleration (1.5.1) is given as

$$a_{\mu}=a_{1}v_{\mu}+a_{2}s_{\mu}=\left(0,\frac{A'}{A},\left(-\frac{\dot{A}}{A}+\frac{\dot{E}}{E}\right)\frac{E}{A^{2}}+\frac{A^{\theta}}{A},0\right), \tag{4.1.12}$$

where θ indicates derivative with respect to theta coordinate. The expansion scalar (1.5.2) is

$$\vartheta=\frac{A^{2}B^{2}}{A^{2}B^{2}r^{2}+E^{2}}\left[\left(2\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)r^{2}+\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}+\frac{\dot{E}}{E}+\frac{\dot{C}}{C}\right)\frac{E^{2}}{A^{2}B^{2}}\right]. \tag{4.1.13}$$

The non-zero components of the shear tensor (1.5.3) are

$$\sigma_{11}=\left[\left(C\dot{B}B-\frac{\dot{C}}{C}\right)A^{2}B^{2}r^{2}-\left(\frac{2\dot{B}}{B}+\frac{\dot{A}}{A}-\frac{\dot{E}}{E}-\frac{\dot{C}}{C}\right)E^{2}\right]$$

$$\times \quad \frac{B^2}{3A(A^2B^2r^2+E^2)}, \tag{4.1.14}$$

$$\sigma_{22} = \left[\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)A^2B^2r^2-\left(2\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{2\dot{E}}{E}+\frac{\dot{C}}{C}\right)E^2\right]\frac{1}{3A^2}, \tag{4.1.15}$$

$$\sigma_{33} = \left[2\left(-\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)A^2B^2r^2+\left(\frac{2\dot{C}}{C}-\frac{\dot{B}}{B}-\frac{\dot{E}}{E}+\frac{\dot{A}}{A}\right)E^2\right]$$

$$\times \quad \frac{C^2}{3A(A^2B^2r^2+E^2)}. \tag{4.1.16}$$

The alternative form of shear tensor (1.5.3) in terms of two scalar functions $\sigma_1,\ \sigma_2$ is

$$\sigma_{\mu\nu} = \frac{1}{3}(2\sigma_1+\sigma_2)(v_\mu v_\nu-\frac{1}{3}h_{\mu\nu})+\frac{1}{3}(2\sigma_2+\sigma_1)(s_\mu s_\nu-\frac{1}{3}h_{\mu\nu}). \tag{4.1.17}$$

Equations (4.1.14)-(4.1.16) imply that

$$2\sigma_1+\sigma_2 = \left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\frac{3}{A}, \tag{4.1.18}$$

$$2\sigma_2+\sigma_1 = \left[\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)AB^2r^2-\left(\frac{\dot{A}}{A}-\frac{\dot{E}}{E}-\frac{\dot{C}}{C}\right)\frac{E^2}{A}\right]\frac{3}{(A^2B^2r^2+E^2)}. \tag{4.1.19}$$

The vorticity scalar function Ω is given as

$$\Omega_{\mu\nu} = \Omega(s_\mu v_\nu - s_\nu v_\mu), \quad \Omega = \frac{E(\frac{E'}{E}-\frac{2A'}{A})}{2B(A^2B^2r^2+E^2)^{\frac{1}{2}}}. \tag{4.1.20}$$

Equations (1.5.4) and (4.1.20) yield $w_\mu = -\Omega k_\mu$. It can be seen from Eq.(4.1.20) that $\Omega = 0$ if and only if $E = 0$, i.e., the system becomes vorticity free (spinless) if and only if reflection symmetric term (E) is zero. The expressions of kinematical variables show that they are metric (geometric terms) dependent, there is no contribution from energy terms.

The Weyl tensor (1.5.7) and Eq.(1.1.4) yield a link between the Weyl tensor and effective energy terms through Riemann/Ricci tensors and Ricci scalar. In this way, the Weyl tensor is also associated with the dynamics of dark sources and defines the effects of tidal forces due to gravitational as well as repulsive gravitational forces. There are three non-zero components for electric part while two for the magnetic part. These components can be written in terms of scalar functions as

$$\begin{aligned}\overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\nu} &= \frac{1}{3}(2\overset{(\mathrm{eff})}{\varepsilon}_I + \overset{(\mathrm{eff})}{\varepsilon}_2)(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}) + \frac{1}{3}(2\overset{(\mathrm{eff})}{\varepsilon}_2 + \overset{(\mathrm{eff})}{\varepsilon}_1)(s_\mu s_\nu - \frac{1}{3}h_{\mu\nu}) \\ &+ \varepsilon_{vs}(v_\mu s_\nu + v_\nu s_\mu), \qquad (4.1.21)\\ \overset{(\mathrm{eff})}{M}_{\mu\nu} &= \overset{(\mathrm{eff})}{M}_1(k_\mu v_\nu + k_\nu v_\mu) + \overset{(\mathrm{eff})}{M}_2(k_\mu s_\nu + k_\nu s_\mu). \qquad (4.1.22)\end{aligned}$$

4.1.2 Structure Scalars and Evolution Equations

Here we construct a set of structure scalars (scalar functions) that help to write down the set of governing equations in simpler form. We decompose the Riemann tensor into energy terms with the help of Eqs.(1.1.4) and (1.5.8) as

$$R^{\mu\nu}_{\alpha\beta} = \overset{(\mathrm{F})}{R}{}^{\mu\nu}_{\alpha\beta} + \overset{(\mathrm{Q})}{R}{}^{\mu\nu}_{\alpha\beta} + \overset{(\mathrm{E})}{R}{}^{\mu\nu}_{\alpha\beta} + \overset{(\mathrm{M})}{R}{}^{\mu\nu}_{\alpha\beta},$$

where

$$\begin{aligned}\overset{(\mathrm{F})}{R}{}^{\mu\nu}_{\alpha\beta} &= \frac{2}{3}(\overset{(\mathrm{eff})}{\rho} + 3\overset{(\mathrm{eff})}{p})u^{[\mu}u_{[\alpha}h^{\nu]}_{\beta]} + \frac{2}{3}\overset{(\mathrm{eff})}{\rho}h^{\mu}_{[\alpha}h^{\nu}_{\beta]}, \qquad (4.1.23)\\ \overset{(\mathrm{Q})}{R}{}^{\mu\nu}_{\alpha\beta} &= -2u^{[\mu}h^{\nu]}_{[\alpha}\overset{(\mathrm{eff})}{q}{}_{\beta]}, \qquad (4.1.24)\\ \overset{(\mathrm{E})}{R}{}^{\mu\nu}_{\alpha\beta} &= 4u^{[\mu}u_{[\alpha}\mathbb{E}^{(eff)\nu]}_{\beta]} + 4h^{[\mu}_{[\alpha}\mathbb{E}^{(eff)\nu]}_{\beta]}, \qquad (4.1.25)\\ \overset{(\mathrm{M})}{R}{}^{\mu\nu}_{\alpha\beta} &= -2\epsilon^{\mu\nu\gamma}u_{[\alpha}\overset{(\mathrm{eff})}{M}{}_{\beta]\gamma} - 2\epsilon_{\alpha\beta\gamma}u^{[\mu}\overset{(\mathrm{eff})}{M}{}^{\nu]\gamma}, \qquad (4.1.26)\end{aligned}$$

$\epsilon_{\mu\nu\gamma} = \eta_{\beta\mu\nu\gamma}u^\beta$, the notations in top F (density and pressure), Q (heat dissipation), $\mathbb{E}$ (electric part of the Weyl tensor) and M (magnetic part of the Weyl tensor) show decomposed parts of the Riemann curvature tensor relating to various aspects of fluid and carry the effects of dark sources. Using Eqs.(1.5.8) and (4.1.23)-(4.1.26), we obtain

$$\overset{(\mathrm{eff})}{Y}_{\mu\nu} = \frac{1}{3}\overset{(\mathrm{eff})}{Y_T}h_{\mu\nu} + \frac{1}{3}(2\overset{(\mathrm{eff})}{Y}_{TF1} + \overset{(\mathrm{eff})}{Y}_{TF2})(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}) + \frac{1}{3}(2\overset{(\mathrm{eff})}{Y}_{TF2} + \overset{(\mathrm{eff})}{Y}_{TF1})$$
$$\times \quad (s_\mu s_\nu - \frac{1}{3}h_{\mu\nu}) + \overset{(\mathrm{eff})}{Y}_{vs}(v_\mu s_\nu + v_\nu s_\mu).$$

Here the subscript T stands for the trace part while the components with notation $TF1$, $TF2$ and vs in subscript represent trace-free parts of the corresponding tensor. The scalar quantities

$$\overset{(\mathrm{eff})}{Y_T} = \frac{1}{2}(\overset{(\mathrm{eff})}{\rho} + 3\overset{(\mathrm{eff})}{p}), \quad \overset{(\mathrm{eff})}{Y}_{TF1} = \overset{(\mathrm{eff})}{\varepsilon}_1 - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_1, \quad \overset{(\mathrm{eff})}{Y}_{TF2} = \overset{(\mathrm{eff})}{\varepsilon}_2 - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_2, \quad \overset{(\mathrm{eff})}{Y}_{vs} = \overset{(\mathrm{eff})}{\varepsilon}_{vs} - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_{vs},$$

are the set of scalar functions associated with the tensor $\overset{(\mathrm{eff})}{Y}_{\mu\nu}$. Likewise, the set of scalar functions associated with the tensor $\overset{(\mathrm{eff})}{X}_{\mu\nu}$ are given by

$$\overset{(\mathrm{eff})}{X_T} = \overset{(\mathrm{eff})}{\rho}, \quad \overset{(\mathrm{eff})}{X}_{TF1} = -\overset{(\mathrm{eff})}{\varepsilon}_1 - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_1, \quad \overset{(\mathrm{eff})}{X}_{TF2} = -\overset{(\mathrm{eff})}{\varepsilon}_2 - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_2, \quad \overset{(\mathrm{eff})}{X}_{vs} = -\overset{(\mathrm{eff})}{\varepsilon}_{vs} - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_{vs},$$

Inserting Eqs.(1.5.8), (4.1.23)-(4.1.26) in the expression of $\overset{(\mathrm{eff})}{Z}_{\mu\nu}$, we obtain

$$\overset{(\mathrm{eff})}{Z}_{\mu\nu} = \overset{(\mathrm{eff})}{M}_{\mu\nu} + \frac{1}{2}\overset{(\mathrm{eff})}{q^{\delta}}_{\mu\nu\delta},$$

or equivalently,

$$\overset{(\mathrm{eff})}{Z}_{\mu\nu} = \overset{(\mathrm{eff})}{Z}_1 v_\mu k_\nu + \overset{(\mathrm{eff})}{Z}_2 v_\nu k_\mu + \overset{(\mathrm{eff})}{Z}_3 s_\mu k_\nu + \overset{(\mathrm{eff})}{Z}_4 s_\nu k_\mu.$$

Here

$$\overset{(\mathrm{eff})}{Z}_1 = (\overset{(\mathrm{eff})}{M}_1 - \frac{1}{2}\overset{(\mathrm{eff})}{q}_2), \quad \overset{(\mathrm{eff})}{Z}_2 = (\overset{(\mathrm{eff})}{M}_1 + \frac{1}{2}\overset{(\mathrm{eff})}{q}_2),$$
$$\overset{(\mathrm{eff})}{Z}_3 = (\overset{(\mathrm{eff})}{M}_2 - \frac{1}{2}\overset{(\mathrm{eff})}{q}_1), \quad \overset{(\mathrm{eff})}{Z}_4 = (\overset{(\mathrm{eff})}{M}_2 + \frac{1}{2}\overset{(\mathrm{eff})}{q}_1),$$

are the structure scalars related to $\overset{(\mathrm{eff})}{Z}_{\mu\nu}$.

We briefly discuss dynamical aspects of scalar quantities under the dark effects of $f(G)$ gravity. The set of scalar functions describe contribution of matter as well as dark sources in the evolution of axially symmetric self-gravitating systems. The scalar $\overset{(\text{eff})}{X}_T$ expresses total energy density of matter and dark sources of the system. Three scalars $\overset{(\text{eff})}{X}_{TF1}$, $\overset{(\text{eff})}{X}_{TF2}$ as well as $\overset{(\text{eff})}{X}_{vs}$ indicate combine effects of anisotropy and electric part of the Weyl tensor in one and the same direction. Another scalar $\overset{(\text{eff})}{Y}_T$ represents sum of energy density and pressure (total energy) of the system. The scalars $\overset{(\text{eff})}{Y}_{TF1}$, $\overset{(\text{eff})}{Y}_{TF2}$ as well as $\overset{(\text{eff})}{Y}_{vs}$ provide combine effects of anisotropy and electric part of the Weyl tensor in opposite directions. The set of scalar functions $\overset{(\text{eff})}{Z}_1$, $\overset{(\text{eff})}{Z}_2$, $\overset{(\text{eff})}{Z}_3$, $\overset{(\text{eff})}{Z}_4$ represent various combinations of heat dissipation and magnetic parts of the Weyl tensor. We note that matter as well as dark sources take part in the dynamics of any axial system. If we neglect matter contributions, then dark sources become responsible for the dynamics of the system.

By contracting Eq.(1.5.9) with s_α and using Eqs.(4.1.12) and (4.1.20), we obtain

$$\frac{\tau}{A}(\dot{q}_2^{(eff)} + A\overset{(\text{eff})}{q_1}\Omega) + \overset{(\text{eff})}{q_2} = \frac{K}{A}\left(\frac{-E\dot{\mathbb{T}} + A^2\mathbb{T}^\theta}{(A^2B^2r^2 + E^2)} - A\mathbb{T}a_2\right) - \frac{K\mathbb{T}^2 q_2}{2}\left(\frac{\tau u^\mu}{K\mathbb{T}^2}\right)_{;\mu}. \tag{4.1.27}$$

Similarly, the contraction of Eq.(1.5.9) with v_α gives

$$\frac{\tau}{A}(\dot{q}_1^{(eff)} - A\overset{(\text{eff})}{q_2}\Omega) + \overset{(\text{eff})}{q_1} = -\frac{K}{A}\left(\mathbb{T}' + B\mathbb{T}a_1\right) - \frac{K\mathbb{T}^2 q_1}{2}\left(\frac{\tau u^\mu}{K\mathbb{T}^2}\right)_{;\mu}. \tag{4.1.28}$$

These two equations describe the effective thermal energy transport (thermal transport due to matter as well as dark sources) in the presence of vorticity (spinning configuration). The set of evolution equations depending upon dynamical variables give the dynamics of collapsing stellar configuration. In the following, we formulate these equations in the presence of dark sources under the effects of $f(G)$ gravity.

(i) Conservation Equations and Thermodynamical Aspect of the System

From the conservation law, $\overset{\text{(eff)}}{T^{\mu}}_{\nu;\mu} = 0$, we obtain two conservation equations in terms of scalar quantities as

$$\overset{\text{(eff)}}{X}_{T;\mu}u^{\mu} + 2\vartheta(\overset{\text{(eff)}}{Y_T} - \overset{\text{(eff)}}{p}) + q^{\mu}_{;\mu} + q^{\mu}a_{\mu} + \frac{1}{9}[(2\sigma_1 + \sigma_2)\Pi_1 + (\sigma_1 + 2\sigma_2)\Pi_2] = 0, \quad (4.1.29)$$

$$2a_{\mu}(\overset{\text{(eff)}}{Y_T} - \overset{\text{(eff)}}{p}) + h^{\nu}_{\mu}(\frac{1}{3}(2\overset{\text{(eff)}}{Y_T} - \overset{\text{(eff)}}{X_T})_{;\nu} + \overset{\text{(eff)}}{\Pi^{\alpha}}_{\nu;\alpha} + \overset{\text{(eff)}}{q}_{\nu;\alpha}u^{\alpha})$$
$$+(\frac{4}{3}\vartheta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu})\overset{\text{(eff)}}{q^{\nu}} = 0. \quad (4.1.30)$$

The first is the continuity equation and the second is named as the Euler equation. In order to explore thermodynamical effects, we use Eqs.(1.5.9)-(4.1.28) and (4.1.30). The combination of Eqs.(1.5.9) and (4.1.30) leads to effective inertial mass while Eqs.(4.1.27) and (4.1.28) provide the relation between thermodynamics and vorticity. Just analogous to inertial mass defined in classical dynamics (Newton's second law), a similar concept also exists in instantaneous rest-frame in relativistic theory. Since in instantaneous rest-frame, the acceleration is parallel (proportional) to the applied force, so the inertial mass behaves as a factor of proportionality among them [57]. However, in some cases when there is no interaction between particles, this factor of proportionality does not represent mass of the particles. In such a case, this factor of proportionality is referred as to effective inertial mass.

The value of effective inertial mass of a particle moving through a solid body (like crystal) may differ from the value calculated for the same particle moving in free space under the same force [58]. In the present case, the combination of Eqs.(1.5.9) and (4.1.30) gives

$$\left[2(\overset{\text{(eff)}}{Y_T} - \overset{\text{(eff)}}{p}) - \frac{K\mathbb{T}}{\tau}\right]a_{\mu} = -h^{\nu}_{\mu}\Pi^{(eff)\alpha}_{\nu;\alpha} - \frac{1}{3}h^{\nu}_{\mu}(2\overset{\text{(eff)}}{Y_T} - \overset{\text{(eff)}}{X_T})_{,\nu} - (\sigma_{\mu\nu}$$

$$+\omega_{\mu\nu})\overset{(\text{eff})}{q}^{\nu} + \frac{K}{\tau}h^{\nu}_{\mu}\mathbb{T}_{,\nu} + \left[\frac{1}{\tau} - \frac{5}{6}\vartheta + \frac{1}{2}D_t\left(ln(\frac{\tau}{K\mathbb{T}^2})\right)\right]q_{\mu}, \quad (4.1.31)$$

where $D_t f = f_{,\nu}u^{\nu}$. The expression on the right hand side contains some extra terms other than dissipative terms which represent hydrodynamic force acting upon fluid and dark sources. The factor multiplying with four acceleration on the left side depicts effective inertial mass density given as

$$2(\overset{(\text{eff})}{Y_T} - \overset{(\text{eff})}{p})(1 - \overset{(\text{eff})}{\Psi}),$$

with $\overset{(\text{eff})}{\Psi} = \frac{K\mathbb{T}}{2\tau(\overset{(\text{eff})}{Y_T} - \overset{(\text{eff})}{p})}$. When the system deviates from thermal equilibrium state, the effective inertial mass density of dissipative fluid is diminished by a factor $(1 - \overset{(\text{eff})}{\Psi})$. It disappears for $\overset{(\text{eff})}{\Psi} = 1$ or even becomes negative for $\overset{(\text{eff})}{\Psi} > 1$. It can be observed that the factor $\overset{(\text{eff})}{\Psi}$ depends upon the values of temperature, the scalar $\overset{(\text{eff})}{Y_T}$ as well as pressure in the presence of matter and dark sources. It is mentioned here that in GR, the effective inertial mass density of the dissipative fluid is given by $(\overset{(\text{m})}{\rho} + \overset{(\text{m})}{p})$ which is reduced by a factor $\Psi = \frac{K\mathbb{T}}{\tau(\overset{(\text{m})}{\rho} + \overset{(\text{m})}{p})}$. For $f(G)$ gravity, the generalized effective inertial mass density becomes

$$2(\overset{(\text{eff})}{Y_T} - \overset{(\text{eff})}{p}) = 2(\overset{(\text{m})}{Y_T} - \overset{(\text{m})}{p}) + [7Rg_{\rho\sigma} + 4Ru_{\rho}u_{\sigma} - 4h_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}f_{G} + 2(Gf_{G} - f),$$

which is the sum of matter and dark source terms. In the absence of dark sources, this reduces to $2(\overset{(\text{eff})}{Y_T} - \overset{(\text{eff})}{p}) = 2(\overset{(\text{m})}{Y_T} - \overset{(\text{m})}{p})$. Thus the dark source terms affect thermodynamics of dissipative collapsing system.

Now we check the relationship between thermodynamics and vorticity given in Eqs.(4.1.27) and (4.1.28). For this purpose, we consider the system is in thermodynamic equilibrium in θ direction and assume $\overset{(\text{eff})}{q_2} = 0$ with constant (corresponding) temperature. Under these considerations, Eq.(4.1.27) gives

$$(\overset{(\text{eff})}{q_2})_{,t} = -A\Omega\overset{(\text{eff})}{q_1}$$

showing the spinning configuration (vorticity) or heat flux of matter in r-direction which controls the vanishing of time propagation of meridional flow (thermal equilibrium in θ direction). Inversely, under the same type of assumption in Eq.(4.1.28), $(\overset{(\mathrm{eff})}{q_1})_{,t} = A\Omega\overset{(\mathrm{eff})}{q_2}$ shows that time propagation of the vanishing of radial heat flux at initial time will depend upon the values of vorticity and meridional heat flux. From the above discussion, we note that dark sources along with matter terms take part in time propagation of thermal equilibrium in either direction (r and θ). In GR, it depends only on matter and vorticity.

(ii) Ricci Evolutionary Equations and Kinematic Variables

We discuss the evolution of kinematical quantities and effects of dark sources through Ricci evolutionary equations. The time propagation equation of expansion scalar is derived by contracting Ricci identities for four velocity given by

$$\vartheta_{;\mu}u^{\mu} + \frac{1}{3}\vartheta^2 + 2(\sigma^2 - \Omega^2) - a^{\mu}_{;\mu} + \overset{(\mathrm{eff})}{Y_T} = 0, \tag{4.1.32}$$

where $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$. We note from this equation that for axial system, the evolution of expansion parameter depends upon shear, voticity and the scalar $\overset{(\mathrm{eff})}{Y_T}$ in geodesic as well as non-geodesic cases. In the absence of matter, dark sources control the evolution of expansion scalar.

The propagation equation of shear tensor is obtained by contracting Ricci identities with four velocity as well as combination of projection tensor and unit four-vectors given by

$$\begin{aligned} &h^{\alpha}_{\mu}h^{\beta}_{\nu}\sigma_{\alpha\beta;\gamma}u^{\gamma} + \sigma^{\alpha}_{\mu}\sigma_{\nu\alpha} + \frac{2}{3}\vartheta\sigma_{\mu\nu} - \frac{1}{3}(2\sigma^2 + \Omega^2 - a^{\gamma}_{;\gamma})h_{\mu\nu} + w_{\mu}w_{\nu} - a_{\mu}a_{\nu} \\ &-h^{\alpha}_{(\mu}h^{\beta}_{\nu)}a_{\alpha;\beta} + \overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\nu} - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_{\mu\nu} = 0. \end{aligned}$$

Additionally, contracting this equation with **ss**, **vv** and **vs**, respectively, it follows that

$$\sigma_{2,\gamma}u^{\gamma}+\frac{1}{3}\sigma_{2}(\sigma_{2}+2\vartheta)-(2\sigma^{2}+\Omega^{2}-a^{\gamma}_{;\gamma})-3(s^{\alpha}s^{\beta}a_{\alpha;\beta}+a_{2}^{2})+\overset{(\mathrm{eff})}{Y}_{TF2}=0, \quad (4.1.33)$$

$$\sigma_{1,\gamma}u^{\gamma}+\frac{1}{3}\sigma_{1}(\sigma_{1}+2\vartheta)-(2\sigma^{2}+\Omega^{2}-a^{\gamma}_{;\gamma})-3(v^{\alpha}v^{\beta}a_{\beta;\alpha}+a_{1})+\overset{(\mathrm{eff})}{Y}_{TF1}=0, \quad (4.1.34)$$

$$\frac{1}{3}(\sigma_{1}-\sigma_{2})\Omega-a_{1}a_{2}-v^{(\alpha}s^{\beta)}a_{\alpha;\beta}+\overset{(\mathrm{eff})}{Y}_{vs}=0. \quad (4.1.35)$$

Equations (4.1.33) and (4.1.34) demonstrate that the evolution of shear depends on vorticity and scalars $\overset{(\mathrm{eff})}{Y}_{TF1}$, $\overset{(\mathrm{eff})}{Y}_{TF2}$. If we consider geodesic fluid with vorticity-free condition, then expansion parameter and scalars $\overset{(\mathrm{eff})}{Y}_{TF1}$, $\overset{(\mathrm{eff})}{Y}_{TF2}$ will control the evolution of shear scalars. Since $\overset{(\mathrm{eff})}{Y}_{TF1}$ and $\overset{(\mathrm{eff})}{Y}_{TF2}$ contain pressure anisotropy and electric part of the Weyl tensor, therefore these scalars will affect the shearing collapsing fluid. Equation (4.1.35) in geodesic case with vorticity-free condition gives $\overset{(\mathrm{eff})}{Y}_{vs}=0$.

Constraint equations are obtained as

$$h^{\nu}_{\mu}\left(\frac{2}{3}\vartheta_{;\nu}-\sigma^{\alpha}_{\nu;\alpha}+\Omega^{\alpha}_{\nu;\alpha}\right)+(\sigma_{\mu\nu}+\Omega_{\mu\nu})\,a^{\nu}=\overset{(\mathrm{eff})}{q}_{\mu}, \quad (4.1.36)$$

$$2w_{(\mu}a_{\nu)}+h^{\alpha}_{(\mu}h_{\nu)\beta}(\sigma_{\alpha\gamma}+\Omega_{\alpha\gamma);\delta}\eta^{\beta\kappa\delta\gamma}u_{\kappa}=\overset{(\mathrm{eff})}{M}_{\mu\nu}. \quad (4.1.37)$$

The contraction of Eq.(4.1.36) with **v** and **s** gives the following scalar equations

$$\frac{2}{3B}\vartheta'-\Omega_{;\alpha}s^{\alpha}+\Omega(s_{\nu;\beta}v^{\beta}v^{\nu}-s^{\beta}_{;\beta})+\frac{1}{3}a_{I}\sigma_{1}-a_{2}\Omega-\frac{1}{3}\sigma_{1;\alpha}v^{\alpha}$$
$$-\frac{1}{3}(2\sigma_{1}+\sigma_{2})(v^{\alpha}_{;\alpha}-\frac{a_{1}}{3})-\frac{1}{3}(\sigma_{1}+2\sigma_{2})(s_{\nu;\alpha}s^{\alpha}s^{\nu}-\frac{a_{1}}{3})=\overset{(\mathrm{eff})}{q}_{1}, \quad (4.1.38)$$

$$\frac{1}{3}(E^{2}+A^{2}B^{2}r^{2})^{\frac{1}{2}}\left(2A\vartheta^{\theta}+\frac{2E}{A}\dot{\vartheta}\right)+\frac{\sigma_{2}a_{2}}{3}+\omega_{;\alpha}v^{\alpha}+\Omega(v^{\alpha}_{;\alpha}$$
$$+s^{\beta}v^{\nu}s_{\nu;\beta})+\Omega a_{I}-\frac{1}{3}\sigma_{2;\alpha}s^{\alpha}+\frac{1}{3}(\sigma_{2}+2\sigma_{1})(s_{\nu;\alpha}s^{\alpha}s^{\nu}-\frac{a_{2}}{3})$$
$$-\frac{1}{3}(\sigma_{1}+2\sigma_{2})(s^{\alpha}_{;\alpha}-\frac{a_{2}}{3})=\overset{(\mathrm{eff})}{q}_{2}. \quad (4.1.39)$$

Similarly, contraction of Eq.(4.1.37) with **vk** and **sk** provide

$$-\Omega a_{1}-\frac{1}{2}\left(v^{\alpha}k_{\beta}+k^{\alpha}v_{\beta}\right)\left(\sigma_{\alpha\delta}+\Omega_{\alpha\delta}\right)_{;\gamma}\epsilon^{\beta\gamma\delta}=\overset{(\mathrm{eff})}{M}_{1}, \qquad (4.1.40)$$

$$-\Omega a_{2}-\frac{1}{2}\left(s^{\alpha}k_{\beta}+k^{\alpha}s_{\beta}\right)\left(\sigma_{\alpha\delta}+\Omega_{\alpha\delta}\right)_{;\gamma}\epsilon^{\beta\gamma\delta}=\overset{(\mathrm{eff})}{M}_{2}. \qquad (4.1.41)$$

Equation (4.1.38) demonstrates that effective dissipation scalar rules the propagation of vorticity in the case of shear and expansion-free geodesic fluid. In this way, dark sources affect the vorticity of fluid. Equations (4.1.38)-(4.1.41) exhibit the relations between $\overset{(\mathrm{eff})}{M}_{1}$, $\overset{(\mathrm{eff})}{M}_{2}$, $\overset{(\mathrm{eff})}{q}_{1}$, $\overset{(\mathrm{eff})}{q}_{2}$, shear and vorticity. Thus these relations also affect the evolution of expansion scalar. The time propagation equation for the vorticity tensor can be derived from the Ricci identity by contracting it with the combination of projection tensor and four velocity vector as [27]

$$h^{\alpha}_{\mu}h^{\beta}_{\nu}\Omega_{\alpha\beta;\gamma}u^{\gamma}+\frac{2}{3}\vartheta\sigma_{\mu\nu}+2\sigma_{\alpha[\mu}\Omega^{\alpha}_{\nu]}-h^{\alpha}_{[\mu}h^{\beta}_{\nu]}a_{\mu;\nu}=0. \qquad (4.1.42)$$

Contraction of the above equation with **vs** yields

$$\Omega_{,\alpha}u^{\alpha}+\frac{1}{3}(2\vartheta+\sigma_{1}+\sigma_{2})\Omega+v^{[\mu}s^{\nu]}a_{\mu;\nu}=0, \qquad (4.1.43)$$

which expresses the evolution of vorticity. We note that it does not depend on dark term even in the presence of dark sources due to $f(G)$ gravity. Hence for general fluid, vorticity is not affected by dark sources.

(iii) Bianchi Evolutionary Equations and Density Inhomogeneity

The Weyl tensor usually narrates effects of gravity due to tidal force in the universe. In our case, it describes both attractive (gravity) as well as repulsive gravitational effects due to the coupling of tidal force with dark sources. Evolution equations calculated for various components of the Weyl tensor establish relations among structur scalars, dark sources and tidal force. These equations evaluate inhomogeneity (irregularity) factors in the system. The collection of dynamical variables which causes density inhomogeneity is known as density inhomogeneity factor. The vanishing of such combination of dynamical variables is the necessary and sufficient for homogeneity of energy density ($h^{\nu}_{\mu}\overset{(\mathrm{eff})}{\rho}_{,\nu}=0$). The Weyl equations are derived by using Eqs.(1.1.4) and (1.5.7) in Bianchi identities given as

$$\begin{aligned}
&h^{\alpha}_{(\mu}h^{\beta}_{\nu)}\overset{(\mathrm{eff})}{\mathbb{E}}_{\alpha\beta;\gamma}u^{\gamma}+\vartheta\overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\nu}+h_{\mu\nu}\overset{(\mathrm{eff})}{\mathbb{E}}_{\alpha\beta}\sigma^{\alpha\beta}-3\overset{(\mathrm{eff})}{\mathbb{E}}_{\alpha(\mu}\sigma^{\alpha}_{\nu)}+h^{\alpha}_{(\mu}\eta^{\gamma\delta\kappa}_{\nu)}u_{\delta}\overset{(\mathrm{eff})}{M}_{\gamma\alpha;\kappa}\\
&-\overset{(\mathrm{eff})}{\mathbb{E}}_{\delta(\mu}\Omega^{\delta}_{\nu)}-2\overset{(\mathrm{eff})}{M^{\alpha}}_{(\mu}\eta_{\nu)\delta\alpha\kappa}u^{\delta}a^{\kappa}=\frac{1}{2}(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p})\sigma_{\mu\nu}-\frac{1}{6}\vartheta\overset{(\mathrm{eff})}{\Pi}_{\mu\nu}-\frac{1}{2}h^{\alpha}_{(\mu}h^{\beta}_{\nu)}\\
&\times\overset{(\mathrm{eff})}{\Pi}_{\alpha\beta;\delta}u^{\delta}-\frac{1}{2}\sigma_{\alpha(\mu}\overset{(\mathrm{eff})}{\Pi^{\alpha}}_{\nu)}-\frac{1}{2}\Omega^{\alpha}_{(\mu}\overset{(\mathrm{eff})}{\Pi}_{\nu)\alpha}-a_{(\mu}\overset{(\mathrm{eff})}{q}_{\nu)}+\frac{1}{6}(\overset{(\mathrm{eff})}{\Pi}_{\alpha\beta}\sigma^{\alpha\beta}+a_{\alpha}\overset{(\mathrm{eff})}{q^{\alpha}}\\
&+\overset{(\mathrm{eff})}{q^{\alpha}}_{;\alpha})h_{\mu\nu}-\frac{1}{2}h^{\alpha}_{(\mu}h^{\beta}_{\nu)}\overset{(\mathrm{eff})}{q}_{\beta;\alpha}, \qquad (4.1.44)\\
&h^{\alpha}_{\mu}h^{\beta\nu}\overset{(\mathrm{eff})}{\mathbb{E}}_{\alpha\beta;\nu}-\eta^{\delta\beta\kappa}_{\mu}u_{\delta}\sigma^{\gamma}_{\beta}\overset{(\mathrm{eff})}{M}_{\kappa\gamma}+3\overset{(\mathrm{eff})}{M}_{\mu\nu}w^{\nu}=\frac{1}{3}h^{\nu}_{\mu}\overset{(\mathrm{eff})}{\rho}_{;\nu}-\frac{1}{2}h^{\nu}_{\mu}h^{\alpha\beta}\overset{(\mathrm{eff})}{\Pi}_{\nu\beta;\alpha}\\
&-\frac{1}{2}\left(\frac{2}{3}\vartheta h^{\nu}_{\mu}-\sigma^{\nu}_{\mu}+3\Omega^{\nu}_{\mu}\right)\overset{(\mathrm{eff})}{q}_{\nu}, \qquad (4.1.45)\\
&(\sigma_{\mu\delta}\overset{(\mathrm{eff})}{\mathbb{E}^{\delta}}_{\nu}+3\Omega_{\mu\delta}\overset{(\mathrm{eff})}{\mathbb{E}^{\delta}}_{\nu})\epsilon^{\mu\nu}_{\kappa}+a^{\beta}\overset{(\mathrm{eff})}{M}_{\beta\kappa}-\overset{(\mathrm{eff})}{M^{\beta\gamma}}_{;\gamma}h_{\beta\kappa}=\frac{1}{2}(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p})\Omega_{\mu\nu}\epsilon^{\mu\nu}_{\kappa}\\
&+\frac{1}{2}\left[\overset{(\mathrm{eff})}{q}_{\mu;\nu}+\overset{(\mathrm{eff})}{\Pi}_{\beta\nu}(\sigma^{\beta}_{\nu}+\Omega^{\beta}_{\nu})\right]\epsilon^{\mu\nu}_{\kappa}, \qquad (4.1.46)\\
&2a_{\nu}\overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\kappa}\epsilon^{\mu\nu}_{\gamma}-\overset{(\mathrm{eff})}{\mathbb{E}}_{\beta\nu;\delta}h^{\beta}_{\kappa}\epsilon^{\delta\nu}_{\gamma}+\overset{(\mathrm{eff})}{\mathbb{E}^{\delta}}_{\nu;\delta}\epsilon^{\nu}_{\gamma\kappa}+\frac{2}{3}\vartheta\overset{(\mathrm{eff})}{M}_{\kappa\gamma}+\overset{(\mathrm{eff})}{M^{\alpha}}_{\beta;\delta}u^{\delta}h^{\beta}_{\kappa}h_{\alpha\gamma}-\overset{(\mathrm{eff})}{M^{\delta}}_{\gamma}\\
&\times(\sigma_{\delta\kappa}+\Omega_{\delta\kappa})+(\sigma_{\nu\delta}+\Omega_{\nu\delta})\overset{(\mathrm{eff})}{M^{\alpha}}_{\mu}\epsilon^{\delta}_{\kappa\alpha}\epsilon^{\mu\nu}_{\gamma}+\frac{1}{3}\vartheta\overset{(\mathrm{eff})}{M^{\alpha}}_{\mu}\epsilon^{\delta}_{\kappa\alpha}\epsilon^{\mu}_{\gamma\delta}=\frac{1}{6}\overset{(\mathrm{eff})}{\rho}_{;\nu}\epsilon^{\nu}_{\gamma\kappa}\\
&+\frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_{\mu\beta;\nu}h^{\beta}_{\kappa}\epsilon^{\mu\nu}_{\gamma}+\frac{1}{2}\left[\overset{(\mathrm{eff})}{q}_{\kappa}\Omega_{\mu\nu}+\overset{(\mathrm{eff})}{q}_{\mu}(\sigma_{\kappa\nu}+\Omega_{\kappa\nu}+\frac{1}{3}\vartheta h_{\kappa\nu})\right]\epsilon^{\mu\nu}_{\gamma}. \qquad (4.1.47)
\end{aligned}$$

The two Weyl equations are obtained by contracting Eq.(4.1.45) with $\mathbf{v}$, $\mathbf{s}$ and using scalar functions as

$$\begin{aligned}
&-\frac{1}{3}\overset{(\mathrm{eff})}{X}_{TF1,\nu}v^{\nu}-\overset{(\mathrm{eff})}{X}_{vs,\nu}s^{\nu}-\frac{1}{3}(2\overset{(\mathrm{eff})}{X}_{TF1}+\overset{(\mathrm{eff})}{X}_{TF2})(v^{\nu}_{;\nu}-a_{\beta}v^{\beta})-\frac{1}{3}s_{\alpha;\nu}s^{\nu}v^{\alpha}\\
&\times(\overset{(\mathrm{eff})}{X}_{TF1}+2\overset{(\mathrm{eff})}{X}_{TF2})-\overset{(\mathrm{eff})}{X}_{vs}(s_{\alpha;\nu}v^{\alpha}v^{\nu}+s^{\nu}_{;\nu}-a_{\nu}s^{\nu})-\frac{1}{3}M_{2}(\sigma_{1}+2\sigma_{2})-3\Omega\overset{(\mathrm{eff})}{M}_{1}
\end{aligned}$$

$$= \frac{1}{3}\overset{(\mathrm{eff})}{\rho}_{;\nu}v^{\nu} - \frac{1}{6}\overset{(\mathrm{eff})}{q_{_1}}(2\vartheta - \sigma_{_1}) + 4\Omega\overset{(\mathrm{eff})}{q_{_2}}, \tag{4.1.48}$$

$$\frac{1}{3}\overset{(\mathrm{eff})}{X}_{_{TF2\nu}}s^{\nu} - \overset{(\mathrm{eff})}{X}_{_{vs,\nu}}v^{\nu} - \frac{1}{3}(\overset{(\mathrm{eff})}{X}_{_{TF1}} + 2\overset{(\mathrm{eff})}{X}_{_{TF2}})(s^{\nu}_{;\nu} - a_{\nu}s^{\nu}) - \frac{1}{3}(2\overset{(\mathrm{eff})}{X}_{_{TF1}} + \overset{(\mathrm{eff})}{X}_{_{TF2}})$$
$$\times v_{\alpha;\nu}s^{\alpha}v^{\nu} - \overset{(\mathrm{eff})}{X}_{_{vs}}(v_{\alpha;\nu}s^{\alpha}s^{\nu} + v^{\nu}_{;\nu} - a_{\nu}v^{\nu}) + \frac{1}{3}\overset{(\mathrm{eff})}{M_{_1}}(2\sigma_{_1} + \sigma_{_2}) - 3\Omega\overset{(\mathrm{eff})}{M_{_2}}$$
$$= \frac{1}{3}\overset{(\mathrm{eff})}{\rho}_{;\nu}s^{\nu} - 4\Omega\overset{(\mathrm{eff})}{q_{_1}} - \frac{\overset{(\mathrm{eff})}{q_2}}{6}(2\vartheta - \sigma_{_2}), \tag{4.1.49}$$

where the scalars $\overset{(\mathrm{eff})}{X}_{_{TF1}}$, $\overset{(\mathrm{eff})}{X}_{_{TF2}}$, $\overset{(\mathrm{eff})}{X}_{vs}$, $\overset{(\mathrm{eff})}{Z_{_1}}$, $\overset{(\mathrm{eff})}{Z_{_2}}$, $\overset{(\mathrm{eff})}{Z_{_3}}$ and $\overset{(\mathrm{eff})}{Z_{_4}}$ generate energy density inhomogeneity.

In order to investigate the role of dark sources in energy density inhomogeneity, we suppose that matter contribution is almost absent which leaves only dark source terms. This indicates dark sources as agent of energy density inhomogeneity. In the absence of $f(G)$ terms, magnetic part of the Weyl tensor produces irregularity in the respective region. Hence, the inhomogeneous distribution of dark sources causes heat dissipation and repulsive gravitational effects in the interstellar region to produce energy density inhomogeneity.

4.2 Shear-Free Axial System

In this section, we study the effects of shear-free condition on axial system in the presence of dark sources (for $f(G)$ model (1.1.7)) for non-geodesic (as well as geodesic) dissipative (as well as non-dissipative) cases.

4.2.1 Non-Geodesic Condition

We assume that the evolution of non-geodesic dissipative fluid is shear-free ($\sigma_{\alpha\beta} = 0 = \sigma_1 = \sigma_2$). Under this assumption, Eqs.(4.1.18) and (4.1.19) yield

$$C(t,r,\theta) = B(t,r,\theta)R_1(r,\theta), \quad E(t,r,\theta) = A(t,r,\theta)B(t,r,\theta)R_2(r,\theta), \qquad (4.2.1)$$

where $R_1(r,\theta)$, $R_2(r,\theta)$ are functions of integration with respect to temporal coordinate t. These functions must satisfy $R_1(0,\theta) = R_2(0,\theta) = 0$ to be compatible with regular condition at the origin. Equation (4.1.37) provides

$$2w_{<\alpha}a_{\beta>} + \nabla_{<\alpha}w_{\beta>} = \overset{(\text{eff})}{M}_{\alpha\beta}. \qquad (4.2.2)$$

Here $\nabla_\alpha w_\beta = h^\mu_\alpha w_{\beta;\mu}$ and angled brackets indicate symmetric and trace-free part. The above equation implies that under shear-free condition, the effective magnetic part of the Weyl tensor depends upon the rotational parameter and for $w_\alpha = 0$, we have $\overset{(\text{eff})}{M}_{\alpha\beta} = 0$. Inversely, $\overset{(\text{eff})}{M}_{\alpha\beta} = 0$ in Eq.(4.2.2) yields

$$\nabla_\alpha w^\alpha = -2a_\alpha w^\alpha. \qquad (4.2.3)$$

However, Eqs.(1.5.3) and (1.5.4) in the shear-free condition yield the identity

$$\nabla_\alpha w^\alpha = a_\alpha w^\alpha. \qquad (4.2.4)$$

Equations (4.2.3) and (4.2.4) imply that $w_\alpha = 0$ which further gives $\Omega_{\alpha\beta} = 0$ (from Eq.(1.5.4)). In both cases, we obtain $\overset{(\text{eff})}{M}_{\alpha\beta} = 0$ implying that

$$\overset{(\text{eff})}{M}_1 = \overset{(\text{eff})}{M}_2 = 0 \quad \Leftrightarrow \quad \Omega = 0, \qquad (4.2.5)$$

which provides that necessary and sufficient condition for irrotational shear-free fluid is the vanishing of magnetic part of the Weyl tensor. For shear-free irrotational fluid, Eqs.(4.1.38) and (4.1.39) give heat dissipation scalars as

$$\overset{(\mathrm{eff})}{q_{1}} = \frac{\vartheta_{,r}}{3B}, \quad \overset{(\mathrm{eff})}{q_{2}} = \frac{\vartheta_{,\theta}}{3Br}. \tag{4.2.6}$$

This shows the behavior of expansion scalar which depends upon heat dissipation. In the absence of dissipation, Eq.(4.2.6) implies that the expansion scalar depends upon temporal coordinate only, i.e., $\vartheta = \vartheta(t)$ (becomes homogeneous) but dissipation due to dark sources does not vanish. Hence the expansion of axial system remains inhomogeneous under the dark effects of $f(G)$ gravity.

4.2.2 Geodesic Condition

Here we restrict our system to be geodesic, i.e., a system with vanishing four acceleration. Consequently, Eq.(4.1.12) along with Eqs.(4.1.13) and (4.1.24) gives

$$A = R_{3}(t,\theta), \quad R_{2}\vartheta B = R_{4}(t,\theta), \tag{4.2.7}$$

where $R_{3}(t,\theta)$ and $R_{4}(t,\theta)$ are arbitrary functions of integration representing null contributions from a_{1} and a_{2}, respectively. Under regularity conditions, we have $\Omega(t,0,\theta) = E(t,0,\theta) = 0$, i.e., vanishing of the coefficient of cross term results irrotational fluid. The condition $E(t,0,\theta) = 0$ further gives $R_{2}(t,0,\theta) = 0$ and consequently, from Eq.(4.2.7), we obtain $R_{4}(t,0,\theta) = 0$. As a result, we have either $\Omega = 0$ or $\vartheta = 0$. A similar result can be found from Eq.(4.1.43) which reads for the underlying case as

$$h^{\alpha}_{\mu}h^{\beta}_{\nu}\Omega_{\alpha\beta;\gamma}u^{\gamma} = -\frac{2}{3}\vartheta\Omega_{\mu\nu}, \quad or \quad \Omega_{,\alpha}u^{\alpha} = -\frac{2}{3}\vartheta\Omega. \tag{4.2.8}$$

This equation along with Eqs.(4.1.13), (4.1.18) and (4.1.19) provides $\vartheta\Omega = 0$ which indicates that shear-free geodesic fluid yields either $\Omega = 0$ or $\vartheta = 0$. In the following, we analyze both cases separately.

(i) Vorticity-Free Expanding Fluid

First we consider the case with $\Omega_{\mu\nu} = 0$ but $\vartheta \neq 0$ which implies that $A = R_3(t),\ E = 0$. After reparametrization of time coordinate, the line element (4.1.1) becomes

$$ds^2 = -dt^2 + B^2(t,r,\theta)\left[dr^2 + r^2 d\theta^2 + F^2(r,\theta)d\phi^2\right]. \qquad (4.2.9)$$

This represents restricted class of axially symmetric cosmic structure. The continuity and Euler equations ((4.1.29) and (4.1.30), respectively) reduce to

$$\overset{(\text{eff})}{\rho}_{;\alpha}u^{\alpha} + (\overset{(\text{eff})}{\rho} + \overset{(\text{eff})}{p})\vartheta + \overset{(\text{eff})}{q}{}^{\alpha}_{;\alpha} = 0, \qquad (4.2.10)$$

$$h^{\beta}_{\alpha}(\overset{(\text{eff})}{p}_{,\beta} + \overset{(\text{eff})}{\Pi}{}^{\mu}_{\beta;\mu} + \overset{(\text{eff})}{q}_{\beta;\mu}u^{\mu}) + \frac{4}{3}\vartheta\overset{(\text{eff})}{q}_{\alpha} = 0, \qquad (4.2.11)$$

while the heat transport equation (1.5.9) gives

$$\tau h^{\mu}_{\nu}\overset{(\text{eff})}{q}{}^{\nu}_{;\alpha}u^{\alpha} + \overset{(\text{eff})}{q}{}^{\mu} = -Kh^{\mu\nu}(\mathbb{T}_{,\nu}) - \frac{1}{2}K\mathbb{T}^2\left(\frac{\tau u^{\alpha}}{K\mathbb{T}^2}\right)_{;\alpha}\overset{(\text{eff})}{q}{}^{\mu}. \qquad (4.2.12)$$

The combination of Eqs.(4.2.11) and (4.2.12) yields

$$h^{\beta}_{\alpha}\overset{(\text{eff})}{\Pi}{}^{\mu}_{\beta;\mu} + \nabla_{\alpha}\overset{(\text{eff})}{p} + \frac{K}{\tau}\nabla_{\alpha}\mathbb{T} - \left[\frac{1}{\tau} + \frac{1}{2}D_t\left(\ln(\frac{\tau}{K\mathbb{T}^2})\right) - \frac{5}{6}\vartheta\right]\overset{(\text{eff})}{q}_{\alpha} = 0, \qquad (4.2.13)$$

which gives a link between pressure gradient, pressure anisotropy and thermodynamic quantities. This suggests that any acceptable EoS for the system under consideration is restricted by thermodynamic quantities through the heat transport equation. Equations (4.1.33)-(4.1.35) give

$$\overset{(\text{eff})}{Y}_{TF1} = \overset{(\text{eff})}{Y}_{TF2} = \overset{(\text{eff})}{Y}_{vs} = 0. \qquad (4.2.14)$$

The vanishing of this set of scalar functions associated with $\overset{(\text{eff})}{Y}_{\mu\nu}$ provide the relations

$$\overset{(\text{eff})}{\varepsilon}_{1} = \frac{1}{2}\overset{(\text{eff})}{\Pi}_{1}, \quad \overset{(\text{eff})}{\varepsilon}_{2} = \frac{1}{2}\overset{(\text{eff})}{\Pi}_{2}, \quad \overset{(\text{eff})}{\varepsilon}_{vs} = \frac{1}{2}\overset{(\text{eff})}{\Pi}_{vs}, \qquad (4.2.15)$$

and accordingly the tensor $\overset{(\mathrm{eff})}{X}_{\mu\nu}$ reduces to

$$\overset{(\mathrm{eff})}{X}_{TF1}=-2\overset{(\mathrm{eff})}{\varepsilon}_{1},\quad \overset{(\mathrm{eff})}{X}_{TF2}=-2\overset{(\mathrm{eff})}{\varepsilon}_{2},\quad \overset{(\mathrm{eff})}{X}_{vs}=-2\overset{(\mathrm{eff})}{\varepsilon}_{vs}. \tag{4.2.16}$$

Now we turn our attention to non-dissipative fluid in the respective case, i.e., $\overset{(\mathrm{eff})}{q}_{1}=\overset{(\mathrm{eff})}{q}_{2}=0$. Under this condition, Eqs.(4.1.13), (4.2.6), (4.2.10), (4.2.14) yield homogenous parameters given by

$$\begin{aligned}
B(t,r,\theta) &= \alpha(t)b(r,\theta),\quad \overset{(\mathrm{eff})}{\rho}=\overset{(\mathrm{eff})}{\rho}(t),\quad \overset{(\mathrm{eff})}{p}=\overset{(\mathrm{eff})}{p}(t),\\
\overset{(\mathrm{eff})}{\Pi}_{1} &= \overset{(\mathrm{eff})}{\Pi}_{1}(t),\quad \overset{(\mathrm{eff})}{\Pi}_{2}=\overset{(\mathrm{eff})}{\Pi}_{2}(t),\quad \overset{(\mathrm{eff})}{\Pi}_{vs}=\overset{(\mathrm{eff})}{\Pi}_{vs}(t),\\
\overset{(\mathrm{eff})}{\varepsilon}_{1} &= \overset{(\mathrm{eff})}{\varepsilon}_{1}(t),\quad \overset{(\mathrm{eff})}{\varepsilon}_{2}=\overset{(\mathrm{eff})}{\varepsilon}_{2}(t),\quad \overset{(\mathrm{eff})}{\varepsilon}_{vs}=\overset{(\mathrm{eff})}{\varepsilon}_{vs}(t).
\end{aligned} \tag{4.2.17}$$

This is possible only if our $f(G)$ models become constant ($n=0$) which shows homogeneous distribution of dark sources. In our case, we cannot choose $n=0$ and therefore the above parameters remain inhomogeneous. We examine the non-dissipative shear-free geodesic evolution and homogeneous distribution of dark sources. The differential equations of the Weyl tensor (4.1.45)-(4.1.47) reduce to

$$-\frac{1}{3}\left(\overset{(\mathrm{eff})}{X}_{TF1}-\overset{(\mathrm{eff})}{\rho}\right)_{,t}+\frac{1}{3}\overset{(\mathrm{eff})}{\varepsilon}_{TF1}\vartheta=\frac{-1}{3}\left(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p}+\frac{1}{3}\overset{(\mathrm{eff})}{\Pi}_{2}\right)\vartheta, \tag{4.2.18}$$

$$-\dot{X}^{(eff)}_{vs}-\vartheta\overset{(\mathrm{eff})}{X}_{vs}=\frac{1}{3}\overset{(\mathrm{eff})}{\Pi}_{vs}\vartheta, \tag{4.2.19}$$

$$\frac{1}{3}(-\overset{(\mathrm{eff})}{X}_{TF2}+\overset{(\mathrm{eff})}{\rho})_{,t}+\frac{\vartheta}{3}\overset{(\mathrm{eff})}{\varepsilon}_{2}=-\frac{1}{3}\left(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p}+\frac{1}{3}\overset{(\mathrm{eff})}{\Pi}_{2}\right)\vartheta. \tag{4.2.20}$$

Using (4.2.16) and (4.2.17), the above set of equations can be integrated to

$$\overset{(\mathrm{eff})}{\varepsilon}_{1}=c_{1}\exp(-\frac{2}{3}\int\vartheta dt),\quad \overset{(\mathrm{eff})}{\varepsilon}_{2}=c_{2}\exp(-\frac{2}{3}\int\vartheta dt),\quad \overset{(\mathrm{eff})}{\varepsilon}_{vs}=c_{3}\exp(-\frac{2}{3}\int\vartheta dt),$$

where c_1, c_2 and c_3 are constants of integration. Equation (4.1.13) along with

Eq.(4.1.24) gives $\vartheta = 3\frac{\dot{B}}{B}$ and hence the expressions in the above equation are calculated as

$$\overset{(\mathrm{eff})}{\varepsilon_1} = \frac{c_1}{B^2}, \quad \overset{(\mathrm{eff})}{\varepsilon_2} = \frac{c_2}{B^2}, \quad \overset{(\mathrm{eff})}{\varepsilon_{vs}} = \frac{c_3}{B^2}.$$

This suggests that B in (4.2.17) reduces to $B = \beta(t)$ and the line element represents FRW spacetime for $\vartheta > 0$. This is consistent with GR [35] in the case of homogeneous distribution of dark sources. Otherwise, the axial system preserves its symmetry under dissipation-less case in the presence of dark sources.

(ii) Expansion-Free Rotating Fluid

Now we consider $\Omega \neq 0$ but $\vartheta = 0$, i.e., expansion-free but rotating fluid. The assumption $\Omega \neq 0$ indicates that the metric (1.2.1) remains non-diagonal while the expansion scalar (4.1.13) with assumption $\vartheta = 0$ indicates that the system becomes time independent. Equations (4.1.32)-(4.1.34) yield the relations

$$\overset{(\mathrm{eff})}{Y_{T}} = 2\overset{(\mathrm{eff})}{Y_{TF1}} = 2\overset{(\mathrm{eff})}{Y_{TF2}} = 2\Omega^2, \tag{4.2.21}$$

which generates a relationship between rotation parameter and the scalars associated with tensor $\overset{(\mathrm{eff})}{Y_{\mu\nu}}$ while the scalar Y_{vs} vanishes in this case. One of the conservation equations (4.1.29) along with Eq.(4.1.3) reduces to

$$\overset{(\mathrm{eff})}{X_{T;\mu}}u^{\mu} - (\overset{(\mathrm{m})\mu}{T_{\nu}}u^{\nu} - 2n\alpha[Rg_{\rho\sigma}u^{\mu}]\nabla^{\rho}\nabla^{\sigma}G^{n-1} + 2\alpha(n-1)G^{n}u^{\mu})_{;\mu} = 0,$$

which indicates that the only factor which controls the evolution of energy density is dissipation from matter as well as dark sources. In the absence of matter, dark sources control the evolution of energy density and if α is zero, then there is no evolution for energy density. Equation (4.1.36) produces a connection between dissipation and vorticity as

$$h^{\nu}_{\mu}\Omega^{\alpha}_{\nu;\alpha}u^{\mu} = \overset{(\mathrm{m})}{q}_{\mu}u^{\mu} + 2n\alpha[Rg_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1} - 2\alpha(n-1)G^{n}.$$

In dissipation-less case ($\overset{(\mathrm{m})}{q}_{\mu} = \alpha = 0$), we obtain a system with zero rotation ($\Omega = 0$) which contradicts our considered case. We can say that such a system remains dissipative as α cannot be zero (although matter can be neglected). Hence due to inhomogeneous distribution of dark sources, the system must be dissipative while from Eq.(4.1.39), we obtain

$$(\Omega BR_{1})' = \overset{(\mathrm{eff})}{q_{2}}B^{2}R_{1} \quad or \quad \Omega = \frac{1}{BR_{1}}\int \overset{(\mathrm{eff})}{q_{2}}B^{2}R_{1}dr + R_{5(\theta)}, \tag{4.2.22}$$

which gives $\Omega = 0$ in non-dissipative case. Using Eq.(4.2.22) in (4.1.32), we have

$$\overset{(\mathrm{m})}{\rho} = (\frac{2}{BR_{1}}\int \overset{(\mathrm{eff})}{q_{2}}B^{2}R_{1}dr + R_{5(\theta)})^{2} - \frac{267n\alpha}{2}[Rg_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1} + 8\alpha(n-1)G^{n} - 3\overset{(\mathrm{m})}{p}.$$

For non-dissipative case and $\alpha = 0$ (absence of dark sources), we obtain EoS $\rho = -3p$ which is consistent with GR [35]. Hence in the presence of dark sources, all geodesic shear/expansion-free fluids can be rotational without dissipation under the dark effects of $f(G)$ gravity. This generalizes the results of GR where such kind of fluids must be dissipative. If we apply regularity conditions on the first expression of Eq.(4.2.22), no such model ($\vartheta = 0$) exists even in the presence of dark sources.

Chapter 5

Concluding Remarks

In this thesis, we have studied static spherically symmetric wormhole solutions in galactic halo region and by imposing an extra symmetry, i.e., CKVs. We have examined the effects of DE in the phenomenon of stellar evolution which has been characterized by the set of governing equations and structure scalars in the framework of modified Gauss-Bonnet gravity. In the following, we summarize and briefly discuss the main results of this thesis.

It is well-known that the violation of NEC is the basic ingredient for the static traversable wormhole in GR. To reduce the wormhole dependence on exotic matter, the study of viable realistic models is an important task in modified theories. In chapter **TWO**, we have investigated static spherically symmetric wormhole solutions in galactic halo region and using CKVs technique in modified Gauss-Bonnet gravity. We have discussed wormhole solutions for galactic halo either by considering a viable $f(G)$ model or a particular shape function. For a viable $f(G)$ model with Navarro-Frenk-White density profile, the graphical behavior shows that it satisfies all conditions related to the wormhole geometry and WEC is also satisfied. In the second case, we have taken a particular form of the shape function and constructed $f(G)$

model. We have found that physically acceptable wormhole solution exists for all values of r in galactic halo region. It is concluded that normal matter satisfies WEC in this modified gravity which leads the modified theories to minimize the dependence of wormhole geometry on exotic matter.

We have also considered a systematic approach to study wormhole solutions by assuming static spherically symmetric metric with conformal symmetry by choosing two types of shape function and also formulated the shape function for a particular as well as phantom EoS. For specific choice of shape functions, we have found physically realistic wormholes are threaded by normal matter. The graphical behavior shows that these functions satisfy all conditions related to the wormhole geometry and hence provides wormholes for the whole range of r. For phantom wormholes with non-static conformal symmetry, the function does not obey the assumptions related to wormhole physics and hence there does not exist any wormhole in this case.

In chapter **THREE**, we have explored spherically symmetric self-gravitating fluid models in this gravity. For locally isotropic and non-dissipative geodesic fluid, due to extra curvature terms from dark sources, fluid is not dust and conformally flatness condition does not imply shear-free condition while density inhomogeneity is controlled by the Weyl tensor as well as GB terms. If we include pressure anisotropy in the fluid, all results revolve around this physical variable. For non-geodesic fluids with local anisotropy and non-dissipation, dark sources affect energy density inhomogeneity factor, the Weyl tensor and anisotropy due to GB terms. For non-geodesic fluids with local isotropy and non-dissipation, shear-free and expansion-free conditions are not linked with each other and density inhomogeneity is found from conformally flatness condition as well as dark sources. In the absence of conformally flatness condition, density inhomogeneity is assured by GB terms. In a more general case, anisotropic fluid with dissipation, the evolution of self-gravitating fluid depends upon GB terms along with dynamical quantities. We have also obtained a relation between density inhomogeneity and thermodynamics variables plus GB terms through transport equation.

This chapter also investigates dynamics of self-gravitating spherically distributed fluid in terms of structure scalars $\overset{(m)}{X}_{T}$, $\overset{(GB)}{X}_{T}$, $\overset{(m)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, $\overset{(m)}{Y}_{T}$ and $\overset{(m)}{Y}_{TF}$ for matter as well as GB terms to deduce all governing equations in terms of these scalars. The evolution equation for the Weyl tensor indicates that energy density inhomogeneity is caused by $\overset{(m)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, dissipation and $f(G)$ model. It is concluded that spherical systems should necessarily be inhomogeneous in this gravity. Finally, we have constructed three forms of the line elements for inhomogeneous anisotropic spheres in terms of scalar functions which lead to further physical relevance of scalar functions. We have found that the evolution of spherically symmetric self-gravitating fluid not only depends upon dynamical variables but also on GB terms. The choice $f(G) = constant$ or $n = 0$ corresponds to the cosmological constant and the standard results can be recovered. For this choice of the model, our results are consistent with standard results [20] otherwise GB terms affect the results which deviate from GR. It is worthwhile to mention here that our constructed self-gravitating fluid models are more general as compared to GR.

In chapter **FOUR**, we have explored the evolution of axially and reflection symmetric dissipative collapsing fluid with and without shear stress in this gravity. The Weyl tensor in the presence of dark sources deals with gravitational (attractive) as well as repulsive forces. We have constructed a set of 12 scalar functions which further consist of energy terms, electric and magnetic parts of the Weyl tensor. It is found that inertial mass of the system is reduced by a factor depending upon the thermal effects as well as dark sources. The coupling of heat flux with vorticity controls thermal equilibrium in r and θ directions of the fluid flow. We have seen that kinematical variables remain unchanged under the influence of dark sources but their evolution is affected. Evolution of expansion scalar is determined by shear, vorticity and $\overset{(\mathrm{eff})}{Y_T}$ in geodesic as well as non-geodesic case. The two scalars $\overset{(\mathrm{eff})}{Y}_{TF1}$ and $\overset{(\mathrm{eff})}{Y}_{TF2}$ control the evolution of propagation equation of shear while the evolution of vorticity does not depend upon dark sources but its absence in geodesic case leaves $\overset{(\mathrm{eff})}{Y}_{vs} = 0$. From the evolution equations for the Weyl tensor, we have found that the set of seven scalars $\overset{(\mathrm{eff})}{X}_{TF1}$, $\overset{(\mathrm{eff})}{X}_{TF2}$, $\overset{(\mathrm{eff})}{X}_{vs}$, $\overset{(\mathrm{eff})}{Z}_1$, $\overset{(\mathrm{eff})}{Z}_2$, $\overset{(\mathrm{eff})}{Z}_3$ and $\overset{(\mathrm{eff})}{Z}_4$ are the inhomogeneity factors. The presence of dark sources in these scalars thus affect the collapsing configuration.

For shear-free axial system, with non-geodesic dissipative fluid, we have found that rotation or spinning of the system is linked with magnetic part of the Weyl tensor. The vanishing of magnetic part turns out to be the necessary and sufficient condition for irrotational evolution even in the presence of dark terms. For shear-free geodesic fluid, we have taken $\Omega = 0,\ \vartheta \neq 0$ and $\Omega \neq 0,\ \vartheta = 0$. In the first case, non-dissipative fluid with homogeneous distribution of dark sources turns the axial system to FRW universe model which is consistent with GR result. However, dissipation due to dark sources is not negligible, hence this correspondence in our case cannot be possible. In the second case, evolution of energy density is controlled by dissipation from matter as well as dark sources. We conclude that all geodesic shear/expansion-free non-dissipative fluids can be rotational under the dark effects of $f(G)$ gravity which generalizes the results of GR where such kind of fluids must be dissipative. Using regularity conditions, we have found that models with zero expansion do not exist even in the presence of dark sources.

It would be interesting

- To explore the role of charge in wormhole solutions with non-commutative geometrical background in this gravity.

- To explore wormhole solutions by considering a general spherically symmetric wormhole spacetime for various shape functions or $f(G)$ models (and vice-versa).

- For different viable $f(G)$ models and shape functions, possible existence of realistic wormhole solutions can be explored in galactic halo region.

The dynamical analysis of self-gravitating systems in modified theories appear as a compelling candidate to describe stellar evolution in the presence of dark sources. It would be worthwhile

- To explore dynamics of self-gravitating systems with expansion-free condition.

- To investigate role of charge in the evolution of self-gravitating fluids.

Appendix A

List of Publications

The contents of this thesis are based on the following research papers published or submitted in journals of International repute. These papers are also attached herewith.

1. Sharif, M. and **Fatima, H.I.**: *Wormhole Solutions for $f(G)$ Gravity in Galactic Halo Region*
   Astrophys. Space. Sci **361**(2016)127.

2. Sharif, M. and **Fatima, H.I.**: *Conformally Symmetric Traversable Wormholes in $f(G)$ Gravity*
   Gen. Relativ. Gravit. **48**(2016)148.

3. Sharif, M. and **Fatima, H.I.**: *Effects of $f(G)$ Gravity on the Dynamics of Self-Gravitating Fluids*
   Eur. Phys. J. Plus **131**(2016)265.

4. Sharif, M. and **Fatima, H.I.**: *Structure Scalars and Evolution Equations in $f(G)$ Cosmology*
   Gen. Relativ. Gravit. **49**(2017)1.

5. Sharif, M. and **Fatima, H.I.**: *Evolution of Axially Symmetric Systems and $f(G)$ Gravity*
   Int. J. Mod. Phys. D **26**(2017)1750109.

6. Sharif, M. and **Fatima, H.I.**: *Shear-Free Axial System and $f(G)$ Gravity*
   Eur. Phys. J. Plus **132**(2017)300.

Also, the following papers related to this thesis have been published, accepted or submitted for publication.

1. Sharif, M. and **Fatima, H.I.**: *Energy conditions for Bianchi type I universe in $f(G)$ gravity*
   Astrophys. Space Sci. **353**(2014)259.

2. Sharif, M. and **Fatima, H.I.**: *Thermodynamics with Corrected Entropies in $f(G)$ Gravity*
   Astrophys. Space Sci. **354**(2014)2124.

3. Sharif, M. and **Fatima, H.I.**: *Non-commutative Wormhole Solutions in $f(G)$ Gravity*
   Mod. Phys. Lett. A **30**(2015)1550142.

4. Sharif, M. and **Fatima, H.I.**:*Built in Inflation in $f(G)$ Gravity*
   Int. J. Mod. Phys. D **25**(2016)1650011.

5. Sharif, M. and **Fatima, H.I.**: *Noether Symmetries in $f(G)$ Gravity*
   J. Exp. Theor. Phys. **149**(2016)122.

6. Sharif, M. and **Fatima, H.I.**: *Static Spherically Symmetric Solutions in $f(G)$ Gravity*
   Int. J. Mod. Phys. D **25**(2016)1650083.

7. Sharif, M. and **Fatima, H.I.**: *Dynamics of Stellar Filaments in $f(G)$ Gravity*
   Eur. Phys. J. Plus **132**(2017)127.

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ORIGINAL ARTICLE

Wormhole solutions for $f(G)$ gravity in galactic halo region

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Abstract In this paper, we study static spherically symmetric wormhole solutions in galactic halo region. Two observational results, Navarro–Frenk–White energy density profile in standard cosmological model and the observed flat rotational curves, are used to discuss traversable wormholes supported by galactic halo in modified Gauss–Bonnet gravity. We explore these solutions either by considering a viable $f(G)$ model to construct shape function or by specifying the shape function to deduce $f(G)$ model. We explore energy conditions and find physically acceptable wormhole solutions threaded by normal matter for all values of r. Finally, we investigate stability of the resulting wormhole solutions.

Keywords Galactic halo · Wormhole solutions · $f(G)$ gravity

1 Introduction

It is well-known that dark matter (DM) exists in baryonic and non-baryonic forms in the universe. Baryonic DM is the matter which can be detected through the emission of photons in all directions. Such matter emits light and is less than 4 percent of the over all matter of the universe. It is argued that the remaining baryonic DM is in the nonluminous form such as MACHOs (massive astrophysical compact halo objects) which may fulfill the required percentage of baryonic matter in the universe. However, the second major constituent of the universe is non-baryonic DM which covers approximately 23 percent of the universe matter. Astronomers cannot detect such matter directly because it cannot emit or absorb/scatter light of any wavelength. The observed flat circular curves of neutral hydrogen clouds in the galactic halo as well as in the outer region of galaxies cannot be explained by luminous (ordinary) matter. This led to the hypothesis that galaxies even cluster of galaxies are filled with nonluminous (dark) matter which is considered as the result of flat rotating curves. The presence of nonluminous DM in galactic halo can only be detected through its gravitational impression on luminous DM (Faber and Visser 2006). Some candidates of nonbaryonic DM include global monopoles (Nukamendi et al. 2000), noncommutative geometry (Rahaman et al. 2012), supersymmetry (Jungman et al. 1996), scalar fields (Fay 2004) and modified theories (Böhmer et al. 2008).

When spacelike sections of metric are embedded in Euclidean space, two asymptotically flat connected regions are exhibited. These asymptotically flat connected regions define a wormhole and the connecting path is termed as bridge or tunnel through which an observer can traverse easily. The supporting matter yielding such geometry is the exotic matter for which energy density becomes negative. This behavior of energy density leads to violation of null energy condition (NEC). The study of wormhole solutions in general relativity (GR) as well as modified theories of gravity has attracted many researchers. Morris and Thorne (1988) did the pioneer work by developing some properties of spacetime metric to hold such solutions.

To discuss wormholes in these bizarre astrophysical phenomena, we rely on Navarro–Frenk–White energy density

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profile (Navarro et al. 1996)

$$\rho = \frac{k}{\frac{r}{r_c}(1 + \frac{r}{r_c})^2}, \quad (1)$$

where k and r_c represent characteristic density and characteristic scale radius, respectively. This density profile was found from N-body numerical simulations to explore dark galactic halos (galaxies or cluster of galaxies) in standard cosmological model. The structure of these dark galactic halos may be differentiated by the traversable wormhole geometry along with density profile as well as flat galactic circular curves. Rahaman et al. (2014) discussed some characteristics of these galactic halos which support wormhole geometry and checked the equilibrium condition for this geometry. They also proposed its existence through scattered scalar waves.

The violation of null energy condition (NEC) is a necessary tool for the existence of wormhole in GR (González-Díaz 2004; Lobo 2005), but this violation no longer holds in modified (higher order derivative) theories of gravity. In these theories, the effective energy-momentum tensor is responsible for this violation (Bhawal and Kar 1992; Furey and DeBenedictis 2005). Thus it would be interesting to investigate wormhole geometry in galactic halos and check the validity of this energy condition for modified Gauss–Bonnet theory of gravity. The $f(G)$ theory is the modification of GR obtained by including an arbitrary function of the Gauss–Bonnet quadratic invariant, G in the Einstein–Hilbert action (Nojiri et al. 2006). The motivation for this theory comes from string theory by low energy effective scale (Cognola et al. 2006, 2007). This theory has been analyzed for the cosmic expansion of the universe (Easson 2005) to avoid four types of finite future singularities (Nojiri et al. 2005), solar system tests (Nojiri et al. 2007), thermodynamics (Sadjadi 2011; Chatterjee and Parikh 2014; Sharif and Fatima 2014b) and many other phenomena. Recently, Sharif and Ikram (2015) explored traversable wormholes by considering power-law function $f(G) = \alpha G^n$ as well as redshift function. They investigated these solutions for traceless, barotropic as well as isotropic fluids and found that physically acceptable solutions exist according to different powers of G. We have studied noncommutative wormhole geometry for this theory and found wormhole solutions threaded by normal matter (Sharif and Fatima 2015).

Sharif and Rani explored wormhole for galactic halo region (2014a), dynamical (2013a), noncommutative (2013b) and charged noncommutative (2014b) wormhole solutions in $f(T)$ gravity. They concluded that WEC is satisfied for specific range of radial coordinate r and normal matter gives some physically acceptable solutions. Mehdizadeh et al. (2015) explored traversable wormholes for Einstein–Gauss–Bonnet gravity satisfying WEC. Sharif and Zahra (2013) studied wormholes taking isotropic, barotropic as well as anisotropic fluid in $f(R)$ gravity and concluded that WEC is satisfied for barotropic fluid but is violated for isotropic as well as anisotropic fluids. Lobo and Oliveira (2009) investigated wormhole solutions in $f(R)$ gravity by choosing a specific form of shape function and examined energy conditions.

In this paper, we study wormhole solutions in galactic halo region for $f(G)$ gravity. The paper is organized as follows. The next section provides basics of $f(G)$ gravity and energy conditions. Section 3 is devoted to construct wormhole geometry and field equations. Section 4 provides solutions for specific $f(G)$ model as well as $b(r)$ function. We also check the behavior of WEC for both cases. In Sect. 5, we discuss equilibrium state for our constructed solutions. Finally, we conclude our results in the last section.

2 $f(G)$ gravity and energy conditions

In this section, we briefly overview $f(G)$ gravity and energy constraints. The action for $f(G)$ gravity is given by Li et al. (2007)

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\big[R + f(G)\big] + S_M, \quad (2)$$

where $\kappa^2 = 8\pi\mathcal{G}$ is the coupling constant, R, f and S_M are Ricci scalar, an arbitrary function of Gauss–Bonnet invariant G and matter action, respectively. The Gauss–Bonnet invariant is defined as

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}, \quad (3)$$

where $R_{\mu\nu}$ and $R_{\mu\nu\sigma\rho}$ are the Ricci and Riemann tensors, respectively. Varying the action (2) with respect to $g_{\mu\nu}$, we obtain the modified field equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 8\Big[R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} \\ - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho} \\ + \frac{1}{2}R(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})\Big]\nabla^\rho\nabla^\sigma f_G \\ + (Gf_G - f)g_{\mu\nu} = \kappa^2\mathcal{T}_{\mu\nu}, \end{aligned} \quad (4)$$

where subscript G represents derivative of f with respect to G and $\mathcal{T}_{\mu\nu}$ denotes the energy-momentum tensor. For anisotropic distribution of matter, $\mathcal{T}_{\mu\nu}$ is given as

$$\mathcal{T}_{\mu\nu} = (\rho + p_t)\upsilon_\mu\upsilon_\nu - p_t g_{\mu\nu} + (p_r - p_t)\eta_\mu\eta_\nu,$$

where p_r, p_t denote radial and tangential pressures, υ^μ, η^μ are four-velocity and spacelike four-vector satisfying the relations $\upsilon^\mu\upsilon_\mu = 1$, $\eta^\mu\eta_\mu = -1$ and $\upsilon^\mu\eta_\mu = 0$. The energy-momentum tensor can also be written as $\mathcal{T}_{\mu\nu} = \text{diag}(\rho(r), -p_r(r), -p_t(r), -p_t(r))$.

Energy conditions are fundamental tools which are used to work out various important results about black holes in different physical scenarios. In GR, the violation of these conditions is the necessary tool for the existence of wormholes. These conditions arise from the relationship of Raychaudhuri equation with expansion scalar (Raychaudhuri 1979). Using the condition of attractive nature of gravity of hypersurface orthogonal congruences (i.e., rotation associated to congruence defined by null vector is zero) in these equations, yield $R_{\mu\nu}v^{\mu}v^{\nu} \geq 0$ and $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ for null v^{μ} and timelike k^{μ} vectors. Replacing the Ricci tensor by the energy-momentum tensor in these inequalities, we obtain energy conditions. These are defined as $(\rho + p \geq 0)$, $(\rho \geq 0, \rho + p \geq 0)$, $(\rho + p \geq 0, \rho + 3p \geq 0)$ and $(\rho \geq 0, \rho \pm p \geq 0)$ NEC, WEC, strong energy condition (SEC) and dominant energy condition (DEC), respectively. These are purely geometrical conditions and can be used in any alternative theory of gravity (Liu and Reboucas 2012). In the case of any other alternative theory of gravitation, the Ricci tensor is replaced by modified energy-momentum tensor, i.e., $T^{(\text{eff})}_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ ($T^{(\text{eff})}_{\mu\nu} = T^{(d)}_{\mu\nu}$ (dark matter terms) + $T^{(m)}_{\mu\nu}$ (normal matter terms)), which means that we include modified energy density and pressure in these conditions. We also impose $T^{(m)}_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ for normal matter. Taking into account that Raychaudhuri equation fits for any theory of gravitation, we will maintain its physical motivation, i.e., the focussing of geodesic congruences along with the attractive nature of the gravitational interaction to deduce the energy conditions in the scenario of $f(G)$ gravity (García 2011). For convenience, the field equations (4) can also be written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T^{\text{eff}}_{\mu\nu},$$

where

$$T^{\text{eff}}_{\mu\nu} = \kappa^2 T_{\mu\nu} - 8\Big[R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho} + \frac{1}{2}R(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})\Big]\nabla^{\rho}\nabla^{\sigma}f_G + (Gf_G - f)g_{\mu\nu}.$$

In this context, the energy conditions are

$$\text{NEC}: \quad \rho^{\text{eff}} + p_i^{\text{eff}} \geq 0,$$

$$\text{WEC}: \quad \rho^{\text{eff}} \geq 0, \qquad \rho^{\text{eff}} + p_i^{\text{eff}} \geq 0,$$

$$\text{SEC}: \quad \rho^{\text{eff}} + p_i^{\text{eff}} \geq 0, \qquad \rho^{\text{eff}} + 3p_i^{\text{eff}} \geq 0,$$

$$\text{DEC}: \quad \rho^{\text{eff}} \geq 0, \qquad \rho^{\text{eff}} \pm p_i^{\text{eff}} \geq 0,$$

where $i = r, t$ (radial and tangential components).

3 Wormhole geometry and field equations

A wormhole provides a feasible way to connect two distant patches of the universe. The flat rotating curves due to neutral hydrogen clouds in the outer region of rotating galaxies indicate the existence of DM. In such galaxies, the outer neutral hydrogen clouds are treated as test particles. These test particles move in circular orbits which can be described by static spherically symmetric spacetime. Morris and Thorne (1988) found the metric for traversable wormholes which is static and spherically symmetric given by

$$ds^2 = e^{2h(r)}dt^2 - \left(\frac{dr^2}{1 - \frac{b(r)}{r}} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right), \quad (5)$$

where $h(r)$ and $b(r)$ are arbitrary functions of radial coordinate r related to gravitational redshift and shape of wormhole known as redshift and shape functions, respectively. The coordinate r behaves non-monotonically as it declines from infinity to the least value r_0 (wormhole throat radius) and then goes back from wormhole throat to infinity. The corresponding field equations are

$$\frac{b'}{r^2} - 4\big[(b'r - b)(3b - r) - 2rb(r - b)\big]f'_G + Gf_G - f = \kappa^2\rho, \quad (6)$$

$$2\frac{h'}{r}\left(1 - \frac{b}{r}\right) - \frac{b}{r^3} - \frac{4}{r^3}\left(1 - \frac{b}{r}\right)\Big[2r^3\left(1 - \frac{b}{r}\right)(h'^2 + h'') - (2 + rh')(b'r - b) + 6bh'\Big]f'_G - Gf_G + f = \kappa^2 p_r, \quad (7)$$

$$\left(1 - \frac{b}{r}\right)\left[h'' + \frac{h'}{r} + h'^2 - \frac{b'r - b}{2r(r - b)}\left(h' + \frac{1}{r}\right)\right] + \frac{8}{2r^3}\left(1 - \frac{b}{r}\right)\Big[2r^2\left(1 - \frac{b}{r}\right)(h' - h'^2 + h'') - 3h'(b'r - b)\Big]f'_G - Gf_G + f = \kappa^2 p_t, \quad (8)$$

and Gauss–Bonnet invariant takes the form

$$G = \frac{4}{r^5}\left[h'(3b - 2r)(b'r - b) - 2r^2b(h'^2 + h'')\left(1 - \frac{b}{r}\right)\right]. \quad (9)$$

Here prime denotes radial derivative. For a traversable wormhole, there should be no horizon and even no singularity which is possible only if the redshift function $h(r)$ is finite for all values of r. Thus we assume redshift function as

$e^{2h(r)} = Cr^n$ (Kar and Sahdev 1995), where $n = 2(v^\phi)^2$, v^ϕ is the rotational velocity for flat rotational curves observed in galactic halo. This rotational velocity is nearly constant through these rotational profiles and we let $C = (\frac{1}{r_c})^n$, the integrating constant. Consequently, the field equations (6)–(8) and Gauss–Bonnet invariant (9) become

$$\frac{b'}{r^2} - 4\big[(b'r-b)(3b-r) - 2rb(r-b)\big]f'_G + Gf_G - f = \kappa^2\rho, \quad (10)$$

$$2\frac{n}{r^2}\left(1-\frac{b}{r}\right) - \frac{b}{r^3} - \frac{4}{r^3}\left(1-\frac{b}{r}\right)\left[\frac{rn(n-2)}{2}\left(1-\frac{b}{r}\right) - \frac{n+4}{2}(b'r-b) + \frac{3nb}{r}\right]f'_G - Gf_G + f = \kappa^2 p_r, \quad (11)$$

$$\left(1-\frac{b}{r}\right)\left[\left(\frac{n}{2r}\right)^2 - \frac{(n+2)(b'r-b)}{4r^2(r-b)}\right] - \frac{8}{2r^3}\left(1-\frac{b}{r}\right)\left[\frac{n^2}{2}\left(1-\frac{b}{r}\right) + \frac{3n}{2r}(b'r-b)\right]f'_G - Gf_G + f = \kappa^2 p_t, \quad (12)$$

$$G = \frac{4}{r^5}\left[\frac{n}{2r}(3b-2r)(b'r-b) - \frac{n(n-2)b}{2}\left(1-\frac{b}{r}\right)\right]. \quad (13)$$

These are the general expressions of matter contents in terms of $b(r)$ and specific $f(G)$ model to thread the wormhole. For the above choice of redshift function, the resulting shape function constitutes traversable wormhole if it obeys flaring-out conditions at wormhole throat radius $r = r_0$, i.e., if it satisfies the conditions $\frac{b-rb'}{b^2} > 0$ (Morris and Thorne 1988) and $b(r_0) = r_0$ (the location of throat radius), which implies the relation $b'(r_0) < 1$ with $\frac{b(r)}{r} \to 0$ as $r \to \infty$. To examine the nature of matter threading the wormhole solutions, it is worthwhile to impose condition on matter contents by checking the validity of WEC. In GR, the violation of NEC leads to wormhole solutions. In alternative theories of gravity, the effective energy-momentum tensor is responsible for this violation but normal matter satisfies the energy conditions for wormhole geometry, e.g., energy conditions are satisfied in D-dimensional Einstein–Gauss–Bonnet gravity (Bhawal and Kar 1992). Using Eqs. (10) and (11), the expression for NEC (radial component only) is obtained as

$$\rho^{\mathrm{eff}} + p_r^{\mathrm{eff}} = \frac{b'r-b}{r^3} + \frac{2n}{r^2}\left(1-\frac{b}{r}\right) < 0,$$

from which we deduce

$$\frac{2n}{r^2} < \frac{rb'-b}{2r^2}\left(1-\frac{b}{r}\right)^{-1},$$

whose right hand side remains positive and leads to the violation of NEC. Using Eq. (1) in (10), we obtain

$$\frac{b'}{r^2} - 4\big[(b'r-b)(3b-r) - 2rb(r-b)\big]f'_G + Gf_G - f = \kappa^2\frac{kr_c^2}{r(r+r_c)}, \quad (14)$$

which consists of two unknowns $f(G)$ and $b(r)$. In order to discuss the wormhole structure in galactic halo, we impose constraints on matter contents by examining the validity of WEC.

4 Wormhole solutions

In this section, we discuss wormhole solutions in galactic halo by finding the solution of Eq. (14). For this purpose, we adopt the strategy of specifying (i) an arbitrary $f(G)$ model to construct $b(r)$ and (ii) an expression of $b(r)$ to deduce $f(G)$. We discuss wormholes by checking the behavior of WEC.

4.1 Wormholes for a viable $f(G)$ model

First we take a viable $f(G)$ model as (Bamba et al. 2010; Sharif and Fatima 2014a)

$$f(G) = \alpha G^l\left(1 + \beta G^m\right), \quad (15)$$

where α, β, l and m are arbitrary constants with $l > 0$. This model is useful to cure four types of finite-time future singularities (Nojiri and Odintsov 2008). It is also consistent with local tests and cosmological bounds (Nojiri and Odintsov 2007). Using Eq.(15) in (14), we obtain differential equation for $b(r)$ as

$$\frac{b'}{r^2} - 4\alpha G^{l-2}G'\left[(rb'-b)(3b-r) - 2r^2b\left(1-\frac{b}{r}\right)\right](l(l-1) + \beta(l+m)(l+m-1)G^m + \alpha G^l\left(l+1+\beta(l+m+1)G^m\right) = \kappa^2\frac{kr_c^3}{r(r+r_c)^2}, \quad (16)$$

where G is given in Eq. (13). This equation cannot be solved analytically, so we solve it numerically for $b(r)$. To check the behavior of shape function, we use graphical technique by incorporating the values of parameters as $\kappa^2 = 6.074 \times 10^{-25}$ kpc/kg, $\alpha = 0.001$, $\beta = 1$, $k = 0.05$, $r_c = 10$, $l = 0.0002$ and $m = -0.05$ along with initial conditions $b(1) = 0.5$ and $b'(1) = 0.6$. These are arbitrary constants which give us reasonable results. Any other combination of these parameters change the results. The plot in

Fig. 1 Plots of $b(r)$ and $\frac{b(r)}{r}$ versus r. **a** displays the shape function $b(r)$ and **b** shows $\frac{b(r)}{r}$ versus r

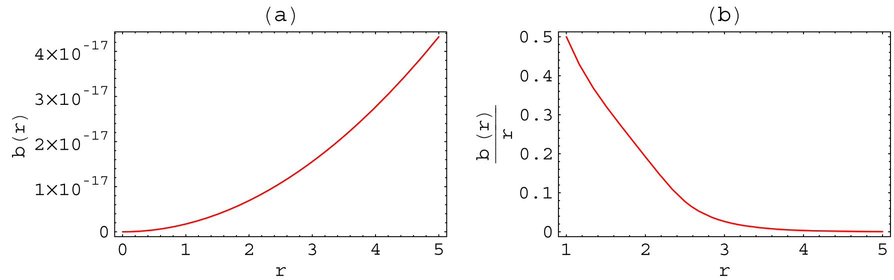
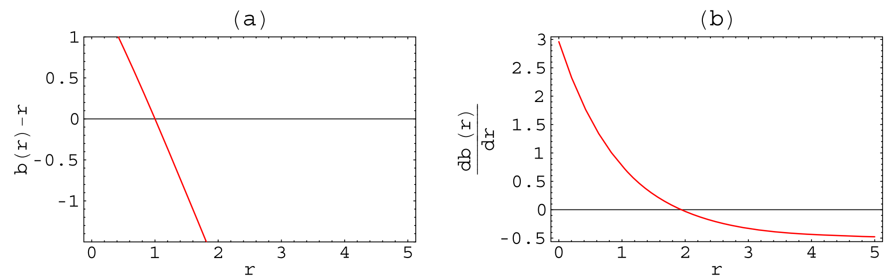


Fig. 2 Plot **a** shows the behavior of $b(r)-r$ and **b** represents $\frac{db(r)}{dr}$ versus r



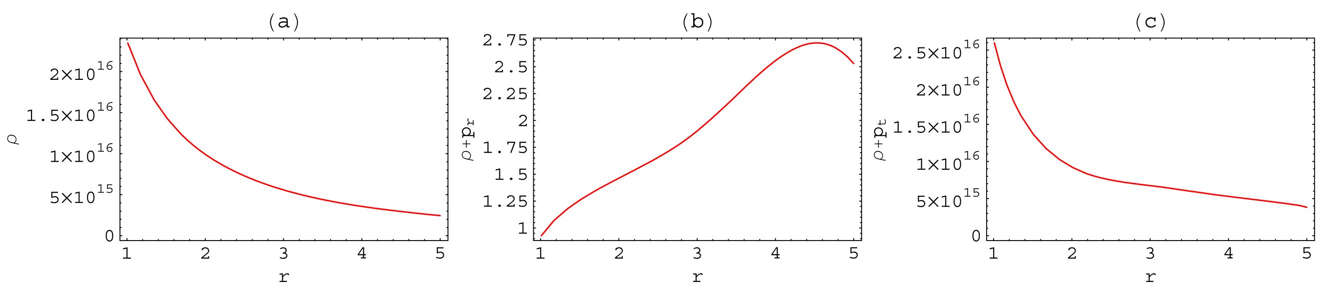


Fig. 3 Plots of WEC with specific $f(G)$ model, **a** represents ρ versus r, **b** represents $\rho+p_r$ versus r and **c** shows $\rho+p_t$ versus r

Fig. 1(a) shows increasing behavior of $b(r)$ versus r. The plot of $\frac{b}{r}$ in Fig. 1(b) indicates that $\frac{b}{r}\rightarrow 0$ as $r\rightarrow\infty$ leading to an asymptotically flat universe. Also, it meets the condition $1-\frac{b}{r}>0$. The throat radius of wormhole is located at $r=r_0$ for which $b(r)-r$ crosses the radial axis and $b(r)-r<0$ for $r>r_0$ (i.e., $b(r)-r$ decreasingly cuts the radial axis). Figure 2(a) shows that the plot of $b(r)-r$ meets the radial axis at $r_0\approx 1.1$ which is the throat radius. Also, $\frac{b}{r}<1$ for $r_0\approx 1.1$ satisfying the essential condition of shape function. The graph of $\frac{db}{dr}$ is shown in Fig. 2(b) which depicts $\frac{db}{dr}(1.1)\approx 0.49<1$ (i.e., $\frac{db}{dr}(r_0)<1$) and hence the flare-out conditions are fulfilled.

We analyze the nature of matter that threads a wormhole using energy conditions. We explicitly check WEC for which the expressions of matter contents are given as follows

$$\begin{aligned}\kappa^2\rho &= \frac{b'}{r^2}-4\alpha G^{l-2}G'\bigg[\Big(rb'-b\Big)(3b-r)\\ &\quad -2r^2b\Big(1-\frac{b}{r}\Big)\bigg]\Big(l(l-1)+\beta(l+m)\\ &\quad\times(l+m-1)G^m\Big)+\alpha G^l\Big(l+1+\beta(l+m+1)G^m\Big)\\ &\quad+\alpha G^l(l-1)-\alpha\beta(l+m-1)G^{l+m},\end{aligned}\tag{17}$$

$$\begin{aligned}\kappa^2 p_r &= \frac{n}{r^2}\Big(1-\frac{b}{r}\Big)-\frac{b}{r^3}-\frac{4}{r^3}\Big(1-\frac{b}{r}\Big)\bigg[\frac{rn(n-1)}{2}\\ &\quad\times\Big(1-\frac{b}{r}\Big)-\frac{4+n}{2}(b'r-b)+\frac{3nb}{r}\bigg]\\ &\quad\times\alpha G^{l-2}G'\Big(l(l-1)+\beta(l+m)(l+m-1)G^m\Big)\\ &\quad-\alpha G^l(l-1)+\alpha\beta(l+m-1)G^{l+m},\end{aligned}\tag{18}$$

$$\begin{aligned}\kappa^2 p_t &= \Big(1-\frac{b}{r}\Big)\bigg[\Big(\frac{n}{2r}\Big)^2-\frac{(l+2)(b'r-b)}{4r^2}(r-b)\bigg]\\ &\quad-\frac{8}{2r^3}\Big(1-\frac{b}{r}\Big)\bigg[\frac{n^2}{2}\Big(1-\frac{b}{r}\Big)+\frac{3n}{2r}(b'r-b)\bigg]\\ &\quad\times\alpha G^{l-2}G'\Big(l(l-1)+\beta(l+m)(l+m-1)G^m\Big)\\ &\quad-\alpha G^l(l-1)+\alpha\beta(l+m-1)G^{l+m}.\end{aligned}\tag{19}$$

The graphs of WEC (ρ, $\rho+p_r$ and $\rho+p_t$) against r are shown in Fig. 3 using the same parametric values. Figure 3(a) shows the behavior of energy density which is decreasing but remains positive for the whole range of r. The plot in Fig. 3(b) represents initially increasing then decreasing but positive behavior for $\rho + p_r$ for all values of r. The profile of $\rho + p_t$ in Fig. 3(c) indicates almost the same behavior as Fig. 3(a). Thus, the model and resulting shape function obey WEC in the galactic halo region and hence accommodate the wormhole geometry with ordinary matter.

4.2 Wormholes for a particular shape function

Here we assume a specific form of the shape function $b(r)$ and construct $f(G)$. We consider the following particular form of the shape function as (Sharif and Jawad 2014)

$$b(r) = r_t\left(\frac{r}{r_t}\right)^{\gamma}, \tag{20}$$

with γ as an arbitrary constant and r_t is the throat radius. It satisfies the flaring-out conditions if $b'(r_t) < 1$ implying that $b'(r_t) = \gamma < 1$ and the condition $b(r_t) = r_t$. The asymptotically flat universe is attained if $\frac{b}{r} = r_t^{1-\gamma} r^{\gamma-1}$ approaches to zero as r tends to infinity. The expression for Gauss–Bonnet invariant becomes

$$G = \frac{4}{r^5}\left(\frac{r}{r_t}\right)^{\gamma}\left[\frac{nr_t}{2r}\left(3r_t\left(\frac{r}{r_t}\right)^{\gamma} - 2r\right)(\gamma-1) - \frac{n(n-2)r_t}{2}\left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right)\right]. \tag{21}$$

Using Eqs. (20) and (14), we obtain the following differential equation for $f(G)$ in terms of r as

$$\frac{\gamma}{r^2}\left(\frac{r}{r_t}\right)^{\gamma-1} - 4r_t\left(\frac{r}{r_t}\right)^{\gamma}\left[\left(3r_t\left(\frac{r}{r_t}\right)^{\gamma} - r\right)(\gamma-1) - 2r^2\left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right)\right]\frac{G'f''-f'G''}{(G')^2} + G\frac{f'}{G'} - f = \kappa^2\frac{kr_c^3}{r(r+r_c)^2}. \tag{22}$$

We solve this equation numerically by choosing the values $r_t = 0.35$ and $\gamma = 0.02$ with initial conditions $f(1) = 0.25$, $f'(1) = 0.15$. The function obtained from the above equation is based on the Navarro–Frenk–White energy density profile and should be sufficient to motivate researchers to look for wormholes in galactic halos observationally. Figure 4 shows that the function $f(r)$ is positively decreasing against r. To thread the wormhole solutions by normal matter, this function should satisfy WEC. The expressions of matter contents are given as

$$\kappa^2\rho = \frac{\gamma}{r^2}\left(\frac{r}{r_t}\right)^{\gamma-1} - 4r_t\left(\frac{r}{r_t}\right)^{\gamma}\left[\left(3r_t\left(\frac{r}{r_t}\right)^{\gamma} - r\right)(\gamma-1) - 2r^2\left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right)\right]\frac{G'f''-f'G''}{(G')^2} + G\frac{f'}{G'} - f, \tag{23}$$

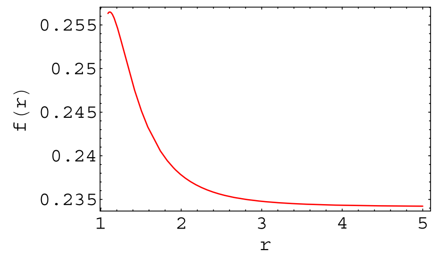


Fig. 4 Plot of $f(r)$ against r

$$\kappa^2 p_r = \frac{n}{r^2} - \left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right) - \frac{1}{r^2}\left(\frac{r}{r_t}\right)^{\gamma-1} - \frac{4}{r^3}\left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right)\left[\frac{n(n-1)r}{2}\left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right) - r_t\left(\frac{r}{r_t}\right)^{\gamma}\left\{(\gamma-1)\left(\frac{n+4}{2}\right) + \frac{3nr_t}{r}\right\}\right] \times \frac{G'f''-f'G''}{(G')^2} - G\frac{f'}{G'} + f, \tag{24}$$

$$\kappa^2 p_t = \left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right)\left[\left(\frac{n}{2r}\right)^2 - \frac{(n+2)r(\frac{r}{r_t})^{\gamma-1}(\gamma-1)}{4r(r-r_t(\frac{r}{r_t})^{\gamma})}\right] - \frac{8}{2r^3}\left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right) \times \left[\frac{n^2}{2}\left(1-\left(\frac{r}{r_t}\right)^{\gamma-1}\right) + \frac{3n}{2}\left(\frac{r}{r_t}\right)^{\gamma-1}(\gamma-1)\right] \times \frac{G'f''-f'G''}{(G')^2} - G\frac{f'}{G'} + f. \tag{25}$$

The graphs of ρ, $\rho + p_r$ and $\rho + p_t$ are given in Fig. 5 by taking the same values of parameters. The graph of energy density in Fig. 5(a) shows decreasing but positive behavior of ρ. The behavior of $\rho + p_r$ in Fig. 5(b) is initially decreases with positive values and then continuously increases after $r \approx 2.35$ while Fig. 5(c) depicts the behavior of $\rho + p_t$ which is same as of $\rho + p_r$, thus WEC is satisfied. This shows that physically acceptable wormholes exist in galactic halo threaded by normal matter for all values of r.

5 Equilibrium condition

Here, we check the equilibrium state of wormhole solutions for both cases. For this purpose, we consider the generalized Tolman–Oppenheimer–Volkov (TOV) equation (Ra-

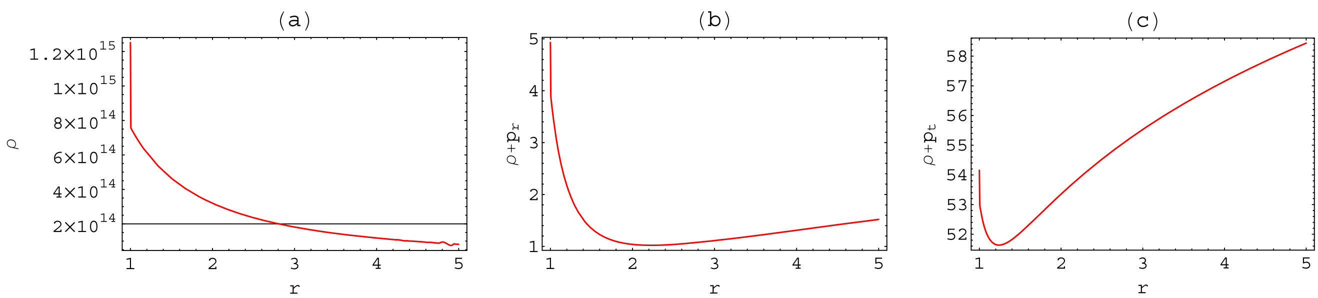
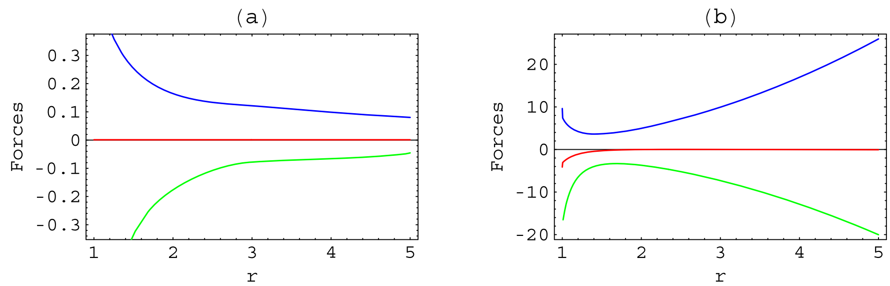


Fig. 5 Plots of WEC (ρ, $\rho + p_r$ and $\rho + p_t$) versus r for specific shape function $b(r)$

Fig. 6 Plots of three forces gravitational (*red*), hydrostatic (*green*) and anisotropic (*blue*) acting on a fluid inside the galactic halo for both cases. **a** shows the equilibrium state of wormholes for $f(G)$ model and **b** represents the equilibrium state for shape function



haman et al. 2014)

$$\frac{dp_r}{dr} + \frac{\mu'}{2}(\rho + p_r) + \frac{2}{r}(p_r - p_t) = 0,$$

for the metric $ds^2 = \mathrm{diag}(e^{\mu(r)}, -e^{\nu(r)}, -r^2, -r^2 \sin^2\theta)$. We rewrite the TOV equation for anisotropic distribution of mass in galactic halo region (as it is suggested by De León 1993 for anisotropic matter distribution) in galactic halo as

$$-\frac{M^{\mathrm{eff}}(\rho + p_r)}{r^2} e^{\frac{\mu-\nu}{2}} - \frac{dp_r}{dr} + \frac{2}{r}(p_t - p_r) = 0, \tag{26}$$

where the effective gravitational mass M^{eff} inside the galactic halo measured from wormhole throat to any radius r is defined as $M^{\mathrm{eff}} = \frac{\mu' r^2}{2} e^{\nu-\mu}$. Equation (26) describes equilibrium state for wormhole supported by gravitational and hydrostatic forces plus anisotropic force (force due to anisotropy in matter distribution). These forces, respectively are defined as

$$F_G = -\frac{M^{\mathrm{eff}}(\rho + p_r)}{r^2} e^{\frac{\mu-\nu}{2}}, \qquad F_H = -\frac{dp_r}{dr},$$
$$F_A = \frac{2}{r}(p_t - p_r). \tag{27}$$

For a wormhole to be in equilibrium, these forces should satisfy the relation

$$F_G + F_H + F_A = 0.$$

Figure 6 shows three forces F_G, F_H and F_A for both cases taking the same values of parameters. Both graphs indicate that the equilibrium state of wormhole solutions can be attained through the combined effect of these forces. We can see that gravitational force is much smaller (becomes zero) than the other two forces while hydrostatic and anisotropic forces are opposite to each other. This balances the system and makes the wormhole solutions in equilibrium state.

6 Conclusions

It is well-known that the violation of NEC is the basic ingredient for the static traversable wormhole in GR. To reduce the wormhole dependence on exotic matter, the study of viable realistic models is an important task in modified theories. Navarro–Frenk–White density function for galactic halo has some important characteristics which can generate traversable wormholes. In this analysis, we have investigated static spherically symmetric wormhole solutions in galactic halo region for $f(G)$ gravity. We have discussed these solutions either by considering a viable $f(G)$ model or a particular shape function. To examine the nature of matter threading the wormhole solutions, the energy conditions have been imposed on matter contents. The equilibrium state of the solutions have also been examined through generalized TOV equation. The results of the analysis are summarized as follows.

For a viable $f(G)$ model with Navarro–Frenk–White density profile for galactic halo, we have obtained the shape function numerically by taking particular values of model constants. The graphical behavior shows that it satisfies all conditions related to the wormhole geometry and WEC is also satisfied. In the second case, we choose a particular form of the shape function and construct $f(G)$ model. We find that WEC is satisfied showing that physically acceptable wormhole solution exists for all values of r in galactic halo region. It is concluded that normal matter satisfies WEC in this modified gravity which leads the modified theories to minimize the dependence of wormhole geometry on exotic matter. Finally, we have checked the equilibrium state of wormhole solutions. It is found that systems initially show unbalanced state then become stable after some time due to the canceling effect of equal but opposite forces (balancing forces).

We have also studied wormhole solutions for $f(G)$ gravity by taking noncommutative geometry (Sharif and Fatima 2015). We have constructed these solutions (i) by assuming a viable $f(G)$ model to construct the shape function and (ii) by specifying the shape function to deduce $f(G)$ model. We have found physically acceptable wormhole solutions threaded by normal matter (for all values of r) in the first case while the second case provides physical solution for higher values of r. Sharif and Ikram (2015) have explored traversable wormholes by considering power-law function $f(G) = \alpha G^n$ as well as redshift function. They have investigated these solutions for traceless, barotropic as well as isotropic fluids and have found that physically acceptable solutions exist for different powers of G. In the present work, we have found wormhole solutions in galactic halo region for $f(G)$ gravity using two observational results, Navarro–Frenk–White energy density profile in standard cosmological model and the observed flat rotational curves. We have established physically acceptable wormhole solutions threaded by normal matter for all values of r and have investigated their stability.

Conflict of interest The authors have no conflict of interest.

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RESEARCH ARTICLE

Conformally symmetric traversable wormholes in $f(G)$ gravity

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Abstract We discuss non-static conformally symmetric traversable wormholes for spherically symmetric spacetime using the model $f(G) = \alpha G^n$, where $n > 0$ and α is an arbitrary constant. We investigate wormhole solutions by taking two types of shape function and found that physically realistic wormholes exist only for even values of n. We also check the validity of flare-out condition, required for wormhole construction, for the shape functions deduced from two types of equation of state. It is found that this condition is satisfied by these functions in all cases except phantom case with non-static conformal symmetry.

Keywords Wormhole solutions · Conformal symmetry · $f(G)$ gravity

1 Introduction

Traversable wormholes are spacetime shortcuts or tunnels allowing free passage of observers in either two different regions of the same spacetime (inter-galactic wormhole) or two different spacetimes (inter-universe wormhole). This free passage is the hypothetical backbone of time travel and travel to any other parallel world. Paging through history, wormhole physics can be traced back to 1916 when Flamm analyzed a newly described Schwarzschild solution [1]. He found that this solution represents

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wormhole. Later, wormhole-type solutions were presented by Einstein and Rosen in 1935 as an elementary particle model known as Einstein–Rosen bridge [2]. This is simply conceivable as a bridge connecting two identical sheets. In the modern age, Morris and Thorne [3] brought back the idea of wormhole geometry in 1988 which rekindled the researchers to study traversable wormholes. They derived static spherically symmetric wormhole solutions from the Einstein field equations and proposed that the wormholes might be the objects of nature like stars and black holes.

Conventionally, wormhole solutions adopt the reverse philosophy while solving the Einstein field equations, first constructing the spacetime metric, then deducing the energy-momentum tensor which violates the null energy condition (NEC). This is the weakest of the energy conditions whose violation indicates the violation of other energy conditions. The existence of wormholes and violation of energy conditions based on controversial-nature-type matter, i.e., exotic matter. This is the matter with strong negative energy density. The search for a realistic model that supports the energy conditions or to minimize the utilization of exotic matter has a significant role in the wormhole history. For this purpose, different directions are adopted such as, dynamical wormhole solutions [4], scalar field models [5], non-minimal coupling of curvature matter [6], generalized Chaplygin gas [7], brane wormholes [8] as well as modified theories of gravity such as $f(R)$, $f(T)$, $f(G)$ and Brans–Dicke theories of gravity etc. These theories and models may cure the violation of these conditions and lead to the wormholes threaded by normal matter. In fact in these theories, the effective energy-momentum tensor is responsible for the violation of energy conditions (extra terms or modified terms take part in the violation) while normal matter satisfies these conditions.

Sharif and Rani explored wormhole solutions for the galactic halo region and with non-commutative geometry [9,10] in generalized teleparallel gravity. They also studied the role of charge in non-commutative wormhole solutions and searched for the dynamical wormholes [4,11]. Mehdizadeh [12] found wormholes threaded by normal matter for Einstein–Gauss–Bonnet gravity. Lobo and Oliveira [6] examined wormhole solutions for some particular shape functions as well as specific equation of state (EoS) in $f(R)$ gravity and found that these solutions satisfy energy conditions. Sharif and Zahra [13] investigated wormholes by considering isotropic and anisotropic fluids plus barotropic EoS.

The modified Gauss–Bonnet gravity ($f(G)$ theory of gravity) is the consequence of the inclusion of an arbitrary function $f(G)$ in the Einstein–Hilbert action [14], where G is the Gauss–Bonnet quadratic invariant. This theory expeditiously interprets recent cosmic accelerated expansion, transition from decelerating to accelerating phases and passes solar system tests. Also, it efficiently narrates thermodynamics [15,16] and cures four types of finite time future singularities [17]. Myrzakulov et al. [18] explored dark energy as well as inflationary era in this gravity. We have studied energy conditions [19], built-in inflation [20] as well as Noether symmetries [21] in this theory.

It is always difficult to find exact solutions of the Einstein field equations unless some certain symmetry restrictions are imposed on spacetime geometry. These restrictions are expressed in terms of isometries (Killing vectors) possessed by spacetime metric. Various symmetries arising from geometrical viewpoint are known as collineations defined by

$$\mathcal{L}_{\xi}\Phi = \Theta, \tag{1}$$

where $\mathcal{L}$ is the Lie derivative, ξ is collineation (symmetry) vector, Φ is tensor field can be $g_{\mu\nu}$, $R_{\mu\nu}$, $R^{\eta}_{\mu\nu\sigma}$, $\Gamma^{\sigma}_{\mu\nu}$ and Θ is the tensor with same index symmetries as Φ. One can deduce the known collineations by substituting particular forms of Φ and Θ. Amongst them, the conformal Killing vectors (CKVs) are the best for deeper insight into spacetime geometry which are obtained by replacing $\Phi = g_{\mu\nu}$ and $\Theta = \varphi g_{\mu\nu}$ [φ is an arbitrary function (conformal factor)] in Eq. (1). This provides inheritance symmetry which helps to find exact solutions from highly non-linear field equations [22]. There is a lot of literature available about CKVs in general relativity [23–26].

Böhmer et al. [27] discussed traversable wormholes using conformal symmetry. Sharif and Ikram [28] investigated traversable wormholes by considering redshift function in $f(G)$ gravity. They used traceless, barotropic as well as isotropic fluids and found physically acceptable wormhole solutions. We have studied wormholes with noncommutative geometry as well as galactic halo region in this theory and found wormholes threaded by normal matter [29,30]. However, no remarkable work has been done using CKVs in modified theories of gravity. This mathematical technique may be fruitful in these theories for deducing exact solutions from highly non-linear partial differential equations. It would therefore be interesting to discuss wormhole solutions using CKVs in $f(G)$ gravity.

This work investigates traversable wormholes admitting non-static CKVs in $f(G)$ gravity. The paper is arranged as follows. Next section briefly reviews this gravity and non-static conformal symmetry. Section 3 provides wormhole geometry and energy conditions. In Sect. 4, we explore wormhole solutions by considering two types of shape function and also deduce shape function from a particular EoS. In Sect. 5, we explore phantom wormholes with static as well as non-static conformal symmetries. Section 6 carries concluding remarks.

2 $f(G)$ gravity and non-static conformal symmetry

In this section, we briefly review $f(G)$ gravity as well as non-static conformal symmetry. The action for $f(G)$ gravity is given by [31]

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[R + f(G)] + S_M, \tag{2}$$

where κ, R, $f(G)$ are the coupling constant, the Ricci scalar, arbitrary function of G, respectively and S_M is the matter action. The Gauss–Bonnet invariant is defined as

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}. \tag{3}$$

Here $R_{\mu\nu}$ and $R_{\mu\nu\sigma\rho}$ are the Ricci and Riemann tensors, respectively. Varying the action (2) with respect to $g_{\mu\nu}$, we obtain the modified field equations

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+8\Big[R_{\mu\rho\nu\sigma}+R_{\rho\nu}g_{\sigma\mu}-R_{\rho\sigma}g_{\nu\mu}-R_{\mu\nu}g_{\sigma\rho}+R_{\mu\sigma}g_{\nu\rho}$$
$$+\frac{1}{2}R\left(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho}\right)\Big]\nabla^{\rho}\nabla^{\sigma}f_G+\left(Gf_G-f\right)g_{\mu\nu}=\kappa^2T_{\mu\nu}, \quad (4)$$

where f_G denotes derivative of f with respect to G and $T_{\mu\nu}$ is the energy-momentum tensor. For anisotropic fluid, it is given as

$$T_{\mu\nu}=(\rho+p_t)u_\mu u_\nu-p_t g_{\mu\nu}+(p_r-p_t)\eta_\mu\eta_\nu,$$

where $p_r,\ p_t$ denote radial and tangential pressures, $u^\mu,\ \eta^\mu$ are four-velocity and spacelike four-vector satisfying the relations $u^\mu u_\mu = 1,\ \eta^\mu\eta_\mu = -1$ and $u^\mu\eta_\mu = 0$. The energy-momentum tensor can also be written as $T_{\mu\nu} = \mathrm{diag}(\rho(r), -p_r(r), -p_t(r), -p_t(r))$.

In the analysis, we shall use a systematic approach in order to derive solutions which was introduced by Maartens and Maharaj [32], where they used static spherically symmetric spacetime possessing non-static conformal symmetry. It is to be noted that neither ξ nor φ need to be static even though one can take a static metric [27]. Suppose that the vector field ξ generates conformal symmetry so that $g_{\mu\nu}$ is conformally mapped onto itself along ξ, then from Eq. (1), we obtain

$$\mathcal{L}_\xi g_{\mu\nu}=\varphi g_{\mu\nu}, \quad (5)$$

For $\varphi = 0$, this equation yields Killing vectors, $\varphi = constant$ (real) gives homotheties (homothetic vector field) and the general choice $\varphi = \varphi(t, X)$ produces CKVs. Equation (5) can also be written as

$$g_{\mu\nu,\alpha}\xi^\alpha+g_{\alpha\nu}\xi^\alpha_{,\mu}+g_{\mu\alpha}\xi^\alpha_\nu=\varphi g_{\mu\nu}. \quad (6)$$

Herrera et al. [33,34] considered static ξ and found singular solutions for isotropic and anisotropic fluids at the center of stars. To overcome this shortfall, Maartens and Maharaj [32] assumed static φ but non-static ξ and obtained singularity free solutions at the center of stars. In this paper, we shall follow this later approach. It is mentioned here that singular solutions at the center of stars are not problematic for wormhole geometries as there is no center for wormholes. Thus we also consider phantom wormholes with static conformal symmetry. The non-static conformal vector field is [32]

$$\xi=\mathfrak{a}(t,r)\partial_t+\mathfrak{b}(t,r)\partial_r, \quad (7)$$

and static conformal factor is $\varphi = \varphi(r)$.

3 Wormhole geometry and energy conditions

This section studies wormhole geometry and gives overview of energy conditions. The static spherically symmetric spacetime representing wormhole is given by [3]

$$ds^2 = e^{2R(r)}dt^2 - \left(\frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{8}$$

where $R(r)$ and $b(r)$ are known as redshift and shape functions, respectively. The radial coordinate r bears non-monotonical behavior as it goes down from infinity to the lowest value of r (wormhole throat r_0) and then returns from throat to infinity. For a traversable wormhole, various conditions are required to be satisfied [3]. Firstly, there should be no event horizon which is only possible if the redshift function is finite for all values of r (condition on redshift function). Secondly, the flaring-out condition, $\frac{rb'-b}{b^2} < 0$, should be fulfilled (condition on the shape function) which can be written as $b'(r) < 1$ at throat (due to the condition $b(r_0) = r_0$). Also, the conditions $\frac{b}{r} < 1$ and $b(r_0) = r_0$ are satisfied at throat. To check out all these conditions, first we write down the field Eqs. (4) for (8) as follows

$$\frac{b'}{r^2} - 4\left[(b'r-b)(3b-r) - 2rb(r-b)\right] f'_G + Gf_G - f = \kappa^2\rho, \tag{9}$$

$$2\frac{R'}{r}\left(1-\frac{b}{r}\right) - \frac{b}{r^3} - \frac{4}{r^3}\left(1-\frac{b}{r}\right)\left[2r^3\left(1-\frac{b}{r}\right)\left(R'^2+R''\right) - \left(2+rR'\right)\left(b'r-b\right) + 6bR'\right] f'_G - Gf_G + f = \kappa^2 p_r, \tag{10}$$

$$\left(1-\frac{b}{r}\right)\left[R'' + \frac{R'}{r} + R'^2 - \frac{b'r-b}{2r(r-b)}\left(R'+\frac{1}{r}\right)\right] + \frac{8}{2r^3}\left(1-\frac{b}{r}\right)\left[2r^2\left(1-\frac{b}{r}\right) \times (R' - R'^2 + R'') - 3R'\left(b'r-b\right)\right] f'_G - Gf_G + f = \kappa^2 p_t. \tag{11}$$

The Gauss–Bonnet invariant takes the form

$$G = \frac{4}{r^5}\left[R'(3b-2r)\left(b'r-b\right) - 2r^2 b\left(R'^2+R''\right)\left(1-\frac{b}{r}\right)\right]. \tag{12}$$

Making use of Eq. (8) in (6), we obtain

$$\mathfrak{a} = a_1 + \frac{\mathfrak{c}t}{2}, \quad \mathfrak{b} = \frac{a_2 r}{2}\sqrt{1-\frac{b(r)}{r}}, \tag{13}$$

$$\varphi(r) = a_2\sqrt{1-\frac{b(r)}{r}}, \tag{14}$$

$$e^{2R(r)} = a_3 r^2 \exp\left[-\frac{2\mathfrak{c}}{a_2}\int \frac{1}{r\sqrt{1-\frac{b(r)}{r}}}dr\right], \tag{15}$$

where a_1, a_2, a_3 and $\mathfrak{c}$ are constants of integration. Using Eq. (13) in (7), we have

$$\xi = \left(a_1 + \frac{\mathfrak{c}t}{2}\right)\partial_t + \frac{a_2 r}{2}\sqrt{1-\frac{b(r)}{r}}\,\partial_r. \tag{16}$$

Without any loss of generality, we may assign $a_1 = 0$ and $a_2 = 1$ [32] so that

$$\xi = \left(\frac{\mathfrak{c}t}{2}\right)\partial_t + \frac{r}{2}\sqrt{1-\frac{b(r)}{r}}\,\partial_r, \tag{17}$$

$$b(r) = r(1-\varphi^2(r)), \tag{18}$$

$$R(r) = \ln[a_3 r] - \mathfrak{c}\int\frac{1}{r\sqrt{1-\frac{b}{r}}}dr. \tag{19}$$

Interestingly, the conformal factor in Eq. (18) reduces to zero at throat (i.e., $\varphi(r_0) = 0$ as $b(r_0) = r_0$ at throat). Consequently, the field Eqs. (9)–(11) and Gauss–Bonnet invariant (12) in terms of conformal factor can be written as

$$\kappa^2\rho = \frac{1}{r^2}\left(1-\varphi^2-2r\varphi\varphi'\right) - 8r^3\varphi\left[\varphi'\left(3\varphi^2-2\right)+\varphi\left(\varphi^2-1\right)\right]f'_G + Gf_G - f(G), \tag{20}$$

$$\kappa^2 p_r = \frac{1}{r^2}\left(3\varphi^2-2\mathfrak{c}\varphi-1\right) - \frac{4}{r^3}\left[-\varphi^4+\mathfrak{c}\varphi^3(1-2r)+2\mathfrak{c}^2r\varphi^2+\varphi(3+\mathfrak{c}\varphi)\times\left(r^2-r+6\right)+6r^2\varphi^3\varphi'\right]f'_G - Gf_G + f(G), \tag{21}$$

$$\kappa^2 p_t = \frac{1}{r^2}\left(2\varphi^2-2\mathfrak{c}\varphi+2r\varphi\varphi'+\mathfrak{c}^2\right) + \frac{8}{r^3}\varphi^2\left[\varphi^2(r-2)+\mathfrak{c}\varphi(3-8r)-\mathfrak{c}^2 + r\varphi'(\mathfrak{c}+4)\right]f'_G - Gf_G + f(G), \tag{22}$$

$$G = \frac{8}{r^4}\left[r\varphi\varphi'(\mathfrak{c}\varphi-1)-(\mathfrak{c}-\varphi)\left(3r\varphi^2\varphi'-\mathfrak{c}\varphi+\mathfrak{c}\right)\right]. \tag{23}$$

Energy conditions are used to work out various important results in different physical scenarios. In general relativity, the violation of these conditions is the necessary tool for the construction of wormholes. These conditions arise from the relationship of Raychaudhuri's equation with expansion scalar [35]. Using the condition of attractive nature of gravity of hypersurface orthogonal congruences (i.e., rotation associated to congruence defined by null vector is zero) in these equations, it follows that $R_{\mu\nu}v^\mu v^\nu \geq 0$ and $R_{\mu\nu}k^\mu k^\nu \geq 0$ for null v^μ and timelike k^μ vectors. Replacing the Ricci tensor by the energy-momentum tensor in these inequalities, we obtain energy conditions as $(\rho + p \geq 0)$, $(\rho \geq 0,\ \rho + p \geq 0)$, $(\rho + p \geq 0,\ \rho + 3p \geq 0)$ and $(\rho \geq 0,\ \rho \pm p \geq 0)$ as NEC, weak energy condition (WEC), strong energy conditions (SEC) and dominant energy conditions (DEC), respectively. These are purely geometrical conditions and can be used in any alternative theory of gravity [36,37]. In modified theories of gravitation, the Ricci tensor is replaced by modified energy-momentum tensor, i.e., $T^{(eff)}_{\mu\nu}k^\mu k^\nu \geq 0$ ($T^{(eff)}_{\mu\nu} = T^{(d)}_{\mu\nu}$ (dark matter terms) $+\, T^{(m)}_{\mu\nu}$ (normal matter terms)), which means that we include modified energy density and pressure in these conditions. We also impose $T^{(m)}_{\mu\nu}k^\mu k^\nu \geq 0$ for normal matter. In the scenario of $f(G)$ gravity [38], the energy conditions are

$$\text{NEC} \ : \rho^{eff} + p_i^{eff} \geq 0,$$

$$\begin{aligned}
\text{WEC} &: \rho^{eff} \geq 0, \quad \rho^{eff} + p_i^{eff} \geq 0, \\
\text{SEC} &: \rho^{eff} + p_i^{eff} \geq 0, \quad \rho^{eff} + 3p_i^{eff} \geq 0, \\
\text{DEC} &: \rho^{eff} \geq 0, \quad \rho^{eff} \pm p_i^{eff} \geq 0,
\end{aligned}$$

where $i = r,\ t$ (radial and tangential components).

The WEC are calculated from Eqs. (20)–(22) as

$$\begin{aligned}
\rho + p_r =& \frac{1}{r^2}\left(2\varphi^2 - 2\mathfrak{c}\varphi - 2r\varphi\varphi'\right) + \Bigg[4\varphi^4\left(\frac{1}{r^3} - 2r^3\right) + \frac{4\mathfrak{c}\varphi^3}{r^2}\left(2 - \frac{1}{r}\right) \\
& + 4\varphi^2\left(2r^3 + \frac{3}{r^2} - \frac{3}{r} - \frac{\mathfrak{c}}{r^3} - \frac{2\mathfrak{c}^2}{r^2}\right) + \frac{4\mathfrak{c}\varphi}{r}\left(1 - \frac{1}{r} - \frac{6}{r^2}\right) \\
& + 24\varphi^3\varphi'\left(r^3 - \frac{1}{r}\right) + 16r^3\varphi\varphi'\Bigg] f'_G \geq 0, \qquad (24) \\
\rho + p_t =& \frac{1}{r^2}(1 + \mathfrak{c}^2 + \varphi^2 - 2\mathfrak{c}\varphi) + \frac{8\varphi}{r^3}\Big[\varphi^3(r-2) + \mathfrak{c}\varphi^2(3 - 8r) - \mathfrak{c}^2\varphi^2 \\
& + r\varphi\varphi'(\mathfrak{c} + 4) - r^6\varphi'(3\varphi^2 - 2) - r^6\varphi(\varphi^2 - 1)\Big] f'_G \geq 0. \qquad (25)
\end{aligned}$$

Here we discuss some specific wormhole solutions by considering a particular $f(G)$ model

$$f(G) = \alpha G^n, \qquad (26)$$

where α is any constant and $n > 0$ [39]. In small curvature regime (phantom phase) with $n < \frac{1}{2}$, $f(G)$ term dominates over Einstein term. For $n < 0$, it corresponds to non-phantom phase (as curvature term dominates). The universe starts with large curvature (non-phantom phase) but it turns out to be small (phantom phase) gradually. Thus, the transition of non-phantom to phantom phase can naturally occur in this model.

In order to study graphical analysis, we take the parameters as $\kappa^2 = 1,\ \alpha = 0.1,\ r_0 = 2,\ n = 2,\ \mathfrak{c} = 1.5,\ \varepsilon = 0.1$ and initial conditions $b(0.5) = 1,\ b'(0.5) = 0.1$.

4 Some specific wormhole solutions

In this section, we investigate wormhole solutions by assuming two types of shape function and a particular equation of state to deduce shape function.

4.1 Specific shape function

First, we consider two types of shape function.

i. $b(r) = r_0$

The interesting feature of this constant shape function is that the energy density for matter vanishes and we are left only with the energy density of dark sources. Equation (15) leads to

$$e^{2R(r)} = a_3^2 r^2 \left(2r - r_0 + 2r\sqrt{1 - \frac{r_0}{r}}\right)^{-2\mathfrak{c}}.$$

The matter variables and Gauss–Bonnet invariant become

$$\rho = 4r_0\left(3r_0 - r + 2r^2 - 2rr_0\right) f'_G + Gf_G - f,$$

$$p_r = \frac{2r - 3r_0}{r^3} - \frac{2\mathfrak{c}}{r^2}\sqrt{1 - \frac{r_0}{r}} - \frac{4}{r^3}\left[\left(2\mathfrak{c}^2 + 3r\right)(r - r_0) + \left(1 - \frac{r_0}{r}\right)^{\frac{3}{2}} \right.$$
$$\times(\mathfrak{c} - 1 - 2\mathfrak{c}r) + \mathfrak{c}\sqrt{1 - \frac{r_0}{r}}\left(r - 6 - \mathfrak{c}r^2\right) + 6r_0\left(1 - \frac{1}{r}\right) + 6 - 3r$$
$$\left.-\frac{3r_0^2}{r}\right] f'_G - Gf_G + f(G),$$

$$p_t = \frac{1}{r^2}\left(2 + \mathfrak{c}^2 - \frac{r_0}{r} - 2\mathfrak{c}\sqrt{1 - \frac{r_0}{r}}\right) + \frac{8}{r^3}[(r - 2)\left(1 - \frac{r_0}{r}\right)^2 + \mathfrak{c}(3 - 8r)$$
$$\times\left(1 - \frac{r_0}{r}\right)^{\frac{3}{2}} - \mathfrak{c}^2\left(1 - \frac{r_0}{r}\right) + \frac{r_0}{2r}(\mathfrak{c} + 4)]f'_G - Gf_G + f(G),$$

$$G = \frac{8}{r^4}\left[\frac{r_0}{2r}\left(\mathfrak{c}\sqrt{1 - \frac{r_0}{r}} - 1\right) - \left(\frac{3r_0}{2r}\sqrt{1 - \frac{r_0}{r}} + \mathfrak{c}\frac{r_0}{r}\right)\left(\mathfrak{c} - \sqrt{1 - \frac{r_0}{r}}\right)\right].$$

We examine the behavior of WEC by plotting their graphs which indicate that these conditions are satisfied for even values of n but violated for odd range of n. Here, we give one graph for $n = 2$ (Fig. 1 (left), exemplifying the even range of n) and one graph for $n = 3$ (Fig. 1 (right) indicating the odd range of n) for the respective case. The graph on left side shows the same decreasing but positive behavior for ρ (red), $\rho + p_r$ (green) and $\rho + p_t$ (blue) representing the validity of WEC. Thus the even values of n lead to physically realistic wormholes (i.e., wormholes threaded by normal matter). The graph for odd values of n on the right side of Fig. 1 shows increasing but negative behavior for ρ (red), $\rho + p_r$ (green) and $\rho + p_t$ (blue) indicating the violation of WEC. This demonstrates that wormholes are supported by exotic matter. In fact, here the effective energy-momentum tensor is responsible for this violation with odd range of n.

ii. $b(r) = \frac{r_0^2}{r}$

For this shape function, Eq. (15) gives

$$e^{2R(r)} = a_3^2 r^2\left(r + \sqrt{r^2 - r_0^2}\right).$$

The field equations for this solution take the form

$$\rho = -\frac{2r_0^2}{r^4} - 8r_0^2\left[\left(1 - \frac{3r_0^2}{r^2}\right) - \frac{\sqrt{r^2 - r_0^2}}{r}\right] f'_G + Gf_G - f(G),$$

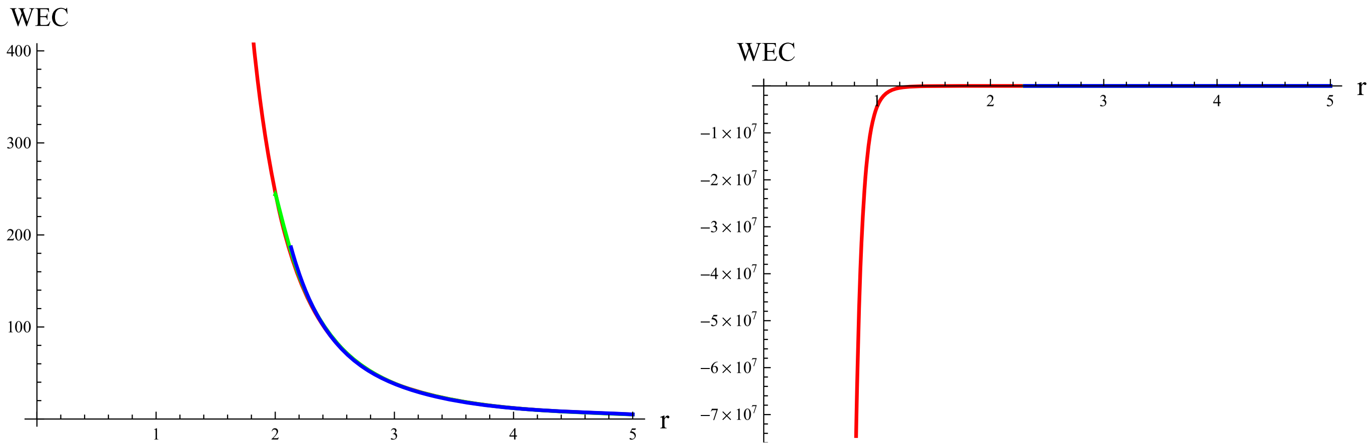

Fig. 1 Plots of WEC ρ (*red*), $\rho + p_r$ (*green*) and $\rho + p_t$ (*blue*) for constant shape function $b(r) = r_0$ taking $n = 2$ (*left*) and $n = 3$ (*right*) (color figure online)

$$p_r = \frac{1}{r^2}\left(2 - \frac{3r_0^2}{r^2} - \frac{2\mathfrak{c}\sqrt{r^2 - r_0^2}}{r}\right) - \frac{4\left(r^2 - r_0^2\right)}{r^5}\left[r\left(2\mathfrak{c}^2 - 3 + 3r\right)\right.$$
$$\left. + \mathfrak{c}\sqrt{r^2 - r_0^2}\left(\frac{1}{r} - 2\right) + \frac{\mathfrak{c}r}{\sqrt{r^2 - r_0^2}}\left(r - r^2 - 6\right) + \frac{5r^2 + 7r_0^2}{r^2}\right] f_G'$$
$$- Gf_G + f(G),$$
$$p_t = \frac{1}{r^2}\left(2 + \mathfrak{c}^2 - \frac{2\mathfrak{c}\sqrt{r^2 - r_0^2}}{r}\right) + \frac{8\left(r^2 - r_0^2\right)}{r^5}\left[\frac{\left(r^2 - r_0^2\right)(r - 2)}{r^2}\right.$$
$$\left. + \frac{\mathfrak{c}(3 - 8r)\sqrt{r^2 - r_0^2}}{r} - c^2 + \frac{r_0^2(\mathfrak{c} + 4)}{r\sqrt{r^2 - r_0^2}}\right] f_G' - Gf_G + f(G)$$

with Gauss–Bonnet invariant

$$G = \frac{8}{r^4}\left[\frac{r_0^2}{r^2}\left(\frac{\mathfrak{c}\sqrt{r^2 - r_0^2}}{r} - 1\right)\right.$$
$$\left. - \left(\mathfrak{c} - \frac{\sqrt{r^2 - r_0^2}}{r}\right)\left(\frac{r_0^2\sqrt{r^2 - r_0^2}}{r^3} - \frac{\mathfrak{c}\left(r^2 - r_0^2\right)}{r^2} + \mathfrak{c}\right)\right].$$

Similar to the first case, WEC are valid only for even values of n. Figure 2 (left) shows the behavior of WEC ($\rho,\ \rho + p_r,\ \rho + p_t$) for $n = 2$ (exemplifying the even values of n). This shows the same behavior for $\rho,\ \rho + p_r,\ \rho + p_t$ slightly down to the negative side but then goes along r-axis (zero) for $r \geq 4$. Hence physically realistic wormholes are threaded by normal matter for $r \geq 4$ with even values of n. The plot of WEC on right side of Fig. 2 shows negatively increasing behavior for $n = 3$ (illustrating the odd range of n) which accomplishes the violation of energy conditions leading to the wormholes threaded by exotic matter.

Figures 1 and 2 represent the behavior of energy conditions for shape functions $b(r) = r_0$ and $b(r) = \frac{r_0^2}{r}$, respectively along with model (26) for which G is defined in Eq. (23). The model parameter n can take the values $1, 2, 3, 4, 5 \ldots$ to draw the graphs of WEC. On substituting the successive values of n, the WEC shows negative behavior for $n = 1, 3, 5 \ldots$ depicting the violation of energy conditions. This violation arises due to the effective energy-momentum tensor which logically leads to the wormholes threaded by exotic matter. On other hand, for $n = 2, 4, 6, \ldots$, WEC stays on positive side representing the validity of energy conditions leading to physically realistic wormholes (threaded by normal matter).

This graphical analysis clearly shows that when n is even, matter terms become dominant over Gauss–Bonnet (dark energy) terms. This leads to physically realistic state where the usage of exotic matter has been minimized and wormholes could exist

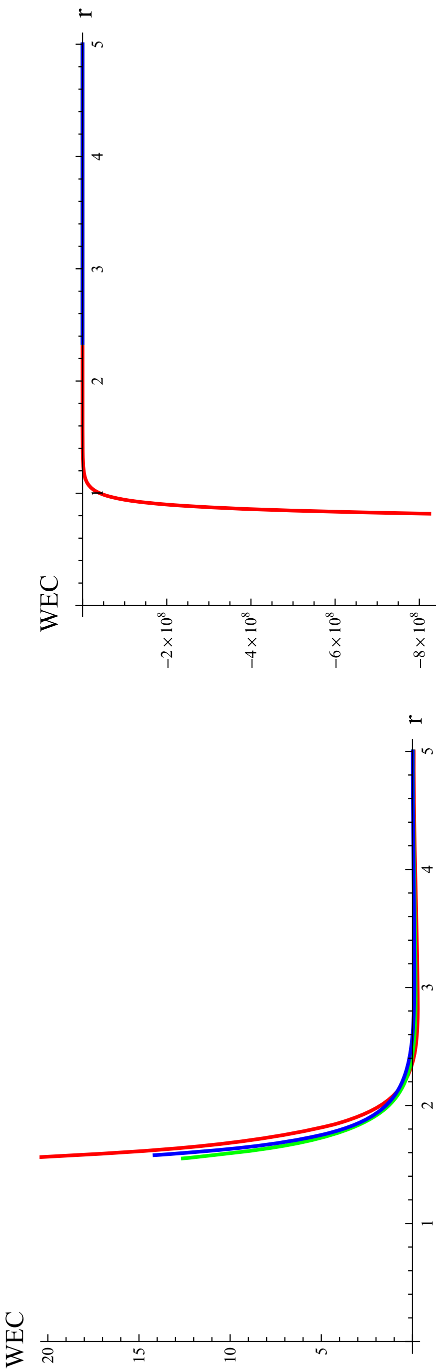

Fig. 2 Plots of WEC ρ (*red*), $\rho + p_r$ (*green*) and $\rho + p_t$ (*blue*) for specific shape function $b(r) = \frac{r_0^2}{r}$ using $n = 2$ (*left*) and $n = 3$ (*right*) (color figure online)

which can be threaded by normal matter. When n is odd, dark energy terms become dominant over matter terms, i.e., the effective energy-momentum tensor becomes pivotal which recommends exotic matter to thread unrealistic wormholes.

4.2 Specific equation of state

An interesting EoS of the form $\rho = \varepsilon(p_t - p_r)$ was first time studied by Böhmer et al. [27] which they used to evaluate shape function. Using Eqs. (9)–(11) along with Eqs. (14) and (15) in this EoS, we obtain

$$\begin{aligned}
&\frac{1}{r^2}\left(1-\varepsilon-\varepsilon\mathfrak{c}^2\right)+\frac{\varepsilon-1}{r^2}\left(1-\frac{b}{r}\right)-\varepsilon\frac{b-rb'}{r^2}\Bigg\{-\frac{4\mathfrak{c}\varepsilon}{r}\sqrt{1-\frac{b}{r}}\\
&\quad\times\left(-\frac{3}{r^2}+\frac{1}{r}-1\right)-\frac{4}{r}\left(1-\frac{b}{r}\right)\left(\frac{6\varepsilon}{r^2}+\frac{\varepsilon\left(2\mathfrak{c}^2-3\right)}{r}+3\varepsilon-2r^4\right)-\frac{4\varepsilon}{r^2}\\
&\quad\times\left(1-\frac{b}{r}\right)^{\frac{3}{2}}\left(\frac{\mathfrak{c}-16\mathfrak{c}^2-1}{r}-2\mathfrak{c}\right)+\frac{8}{r^2}\left(\frac{2\varepsilon}{r}-\varepsilon-r^5\right)\left(1+\frac{b^2}{r^2}-\frac{2b}{r}\right)\\
&\quad-\frac{4}{r^3}\sqrt{1-\frac{b}{r}}\left(b-rb'\right)\left(\frac{\varepsilon\left(\mathfrak{c}+4\right)}{r}+3\varepsilon\sqrt{1-\frac{b}{r}}+3r^4\sqrt{1-\frac{b}{r}}\right)\\
&\quad+8r(b-rb')\Bigg\}f'_G+Gf_G-f(G)=0, \qquad (27)
\end{aligned}$$

which is a differential equation in terms of shape function. We solve it numerically by inserting the values as stated above. The behavior of shape function is shown in Fig. (3I) which is an increasing function while Fig. (3II) indicates that $\frac{b}{r}\rightarrow 0$ as $r\rightarrow\infty$ leading to an asymptotically flat universe. The wormhole throat is located at $r = r_0$ where the graph $b - r$ crosses the radial axis and $b(r) < r$ for $r > r_0$, i.e., the graph of $b - r$ is decreasingly cuts the radial axis. According to this definition, Fig. (4I) suggests the throat radius $r_0 \approx 2.1$ for which $\frac{db}{dr} \approx 0.38 < 1$ as shown in Fig. (4II). Consequently, this function satisfies the flaring-out condition and hence called the shape function for wormhole geometries.

5 Phantom wormholes

Here we discuss traversable wormhole by using another interesting EoS, $p_r = \omega\rho$ in phantom regime ($\omega < -1$) which has extensively been studied in literature [40,41]. Using Eqs. (9) and (10) with this EoS, we obtain

$$\begin{aligned}
&\frac{1}{r^3}\left(\omega rb'+b\right)-\frac{2R'}{r}\left(1-\frac{b}{r}\right)+\Bigg[\frac{24bR'}{r^3}\left(1-\frac{b}{r}\right)-\frac{4\left(rb'-b\right)}{r^3}\Bigg(\omega r^3\left(3b-r\right)\\
&\quad+\left(2+rR'\right)\left(1-\frac{b}{r}\right)\Bigg)+2\left(1-\frac{b}{r}\right)\left(\omega r^2b+4\left(1-\frac{b}{r}\right)\left(R'^2+R''\right)\right)\Bigg]f'_G\\
&\quad+\left(Gf_G-f(G)\right)(\omega+1)=0, \qquad (28)
\end{aligned}$$

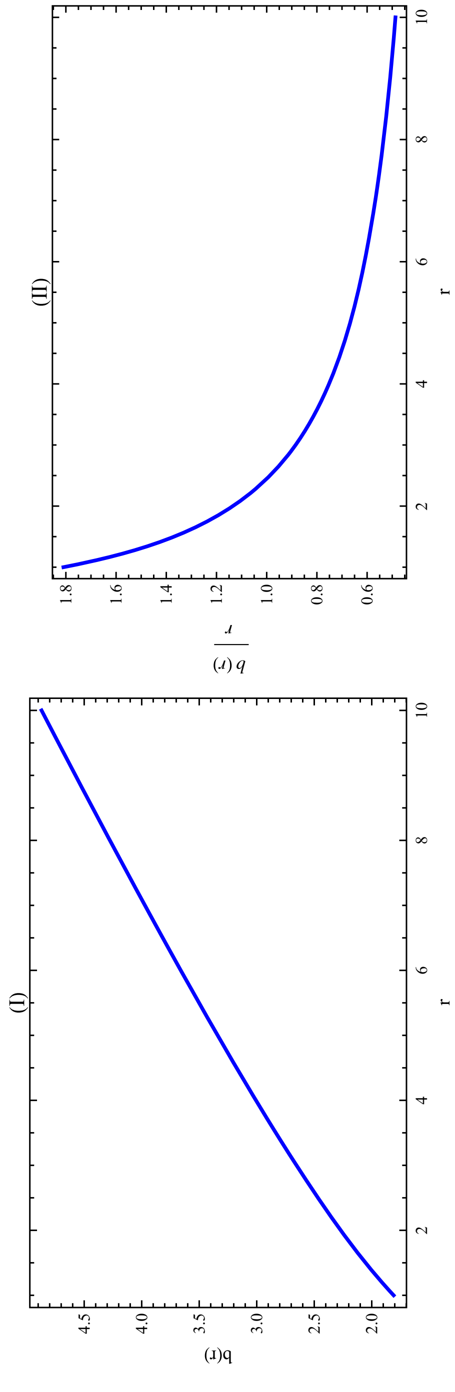

Fig. 3 Plots of (I) $b(r)$ and (II) $\frac{b(r)}{r}$ versus r with $n = 2$ when EoS is $\rho = \varepsilon(p_t - p_r)$

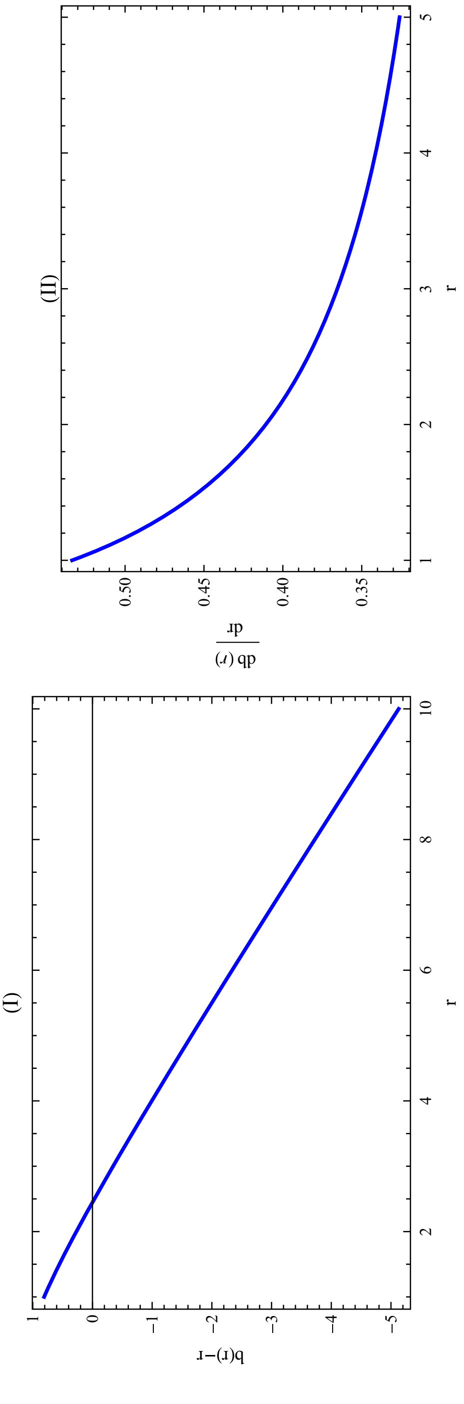

Fig. 4 Plots of (I) $b(r) - r$ and (II) $\frac{db(r)}{dr}$ versus r taking $n = 2$ for EoS $\rho = \varepsilon(p_t - p_r)$

yielding a differential equation in terms of redshift and shape function. We solve this equation for static and non-static conformal symmetries.

5.1 Static conformal symmetric phantom wormholes

Static conformal symmetry implies the dependence of Eq. (7) only on radial coordinate r which further implies that $\mathfrak{c} = 0$ in Eqs. (17) and (19). With this choice of $\mathfrak{c}$, Eq. (19) implies that

$$R(r) = \ln(a_3 r).$$

Inserting this value in Eq. (28), it follows that

$$\frac{1}{r^3}(\omega r b' + b) - \frac{2}{r^2}\left(1-\frac{b}{r}\right) + \left[\frac{24b}{r^4}\left(1-\frac{b}{r}\right) - \frac{4\left(rb'-b\right)}{r^3}\left(\omega r^3\left(3b-r\right) + 3\left(1-\frac{b}{r}\right)\right) + 2\omega r^2 b\left(1-\frac{b}{r}\right)\right] f'_G + (Gf_G - f(G))(\omega+1) = 0, \quad (29)$$

which can be solved numerically for shape function $b(r)$. The numerical solution for shape function (with $\omega = -3$) is shown in Fig. (5I) which indicates increasing behavior while Fig. (5II) represents $\frac{b}{r} \to 0$ as $r \to \infty$ which leads to asymptotically flat universe. Figure (6I) represents that $b - r$ cuts the radial axis at $r_0 \approx 2.1$ which is the throat radius and $\frac{db}{dr}|_{r_0=(2.1)} \approx 0.36 < 1$ (Fig. 6II). Hence, the flaring-out condition is satisfied and gives the shape function for wormholes.

5.2 Non-static conformal symmetric phantom wormholes

For non-static conformal symmetric phantom wormholes, the energy density (20) and radial pressure (21) along with EoS lead to the following equation

$$\frac{1}{r^2}(\omega+1) - \frac{(\omega+3)}{r^2}\sqrt{1-\frac{b}{r}} + \frac{2\mathfrak{c}}{r^2}\sqrt{1-\frac{b}{r}} - \frac{\omega}{r}\frac{b-rb'}{r^2} + \left[-\left(1+\omega r^4\right) \times\left(1-\frac{b}{r}\right)\frac{12\left(b-rb'\right)}{r^3} + 8\omega r\left(b-rb'\right) - \frac{4\left(1+2\omega r^4\right)}{r^3}\left(1-\frac{b}{r}\right)^2 + \frac{4}{r^3}\left(1-\frac{b}{r}\right)\times\left(2\omega r^6 + 2\mathfrak{c}r - 3r + 3r^2 + 6\right) + \frac{4\mathfrak{c}\left(1-2r\right)}{r^3}\left(1-\frac{b}{r}\right)^{\frac{3}{2}} + \frac{4\mathfrak{c}\left(r-r^2-6\right)}{r^3}\times\sqrt{1-\frac{b}{r}}\right] f'_G + (Gf_G + f(G))(\omega-1) = 0 \quad (30)$$

for shape function. Its numerical solution is shown in Fig. 7 which depicts constant behavior for very short range of r $(0.4996 \leq r \leq 0.5)$. This function does not satisfy the flaring-out condition and hence no wormhole exists for this case.

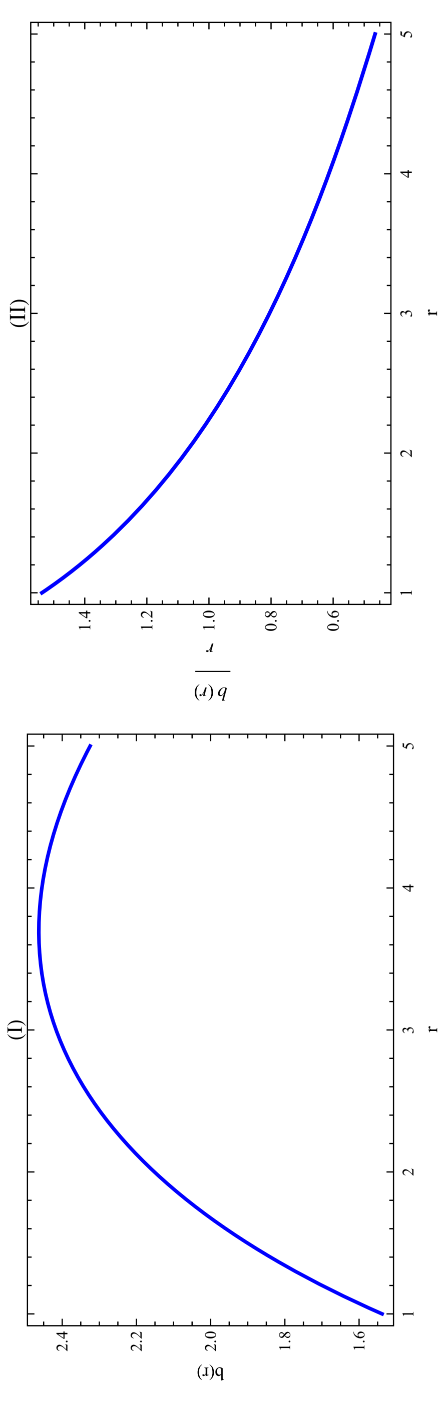

Fig. 5 Plots of (I) $b(r)$ and (II) $\frac{b(r)}{r}$ versus r choosing $n = 2$ for phantom wormholes with static conformal symmetry

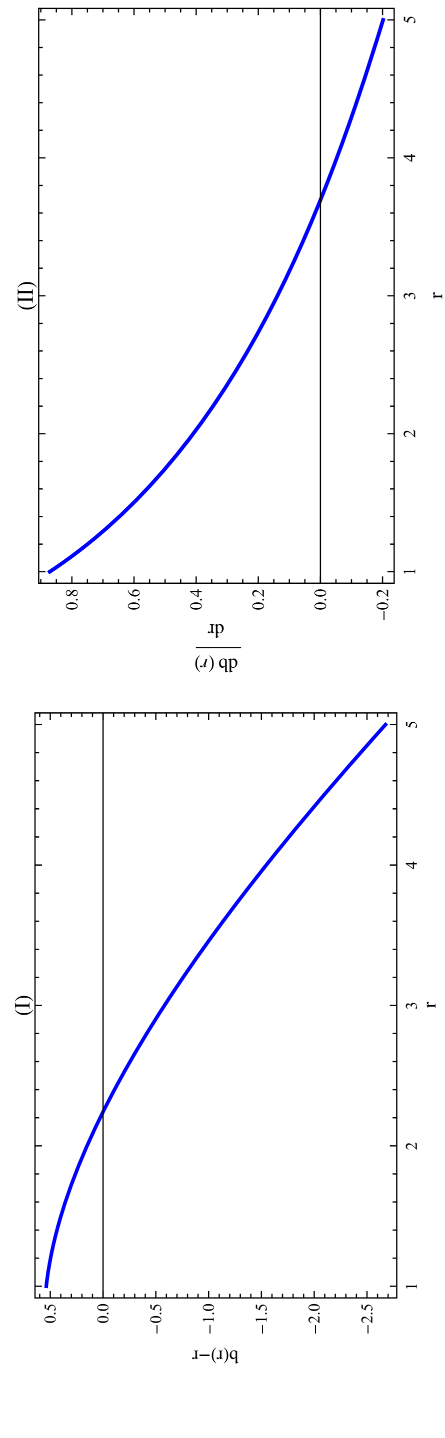

Fig. 6 Plots of (I) $b(r) - r$ and (II) $\frac{db(r)}{dr}$ versus r with $n = 2$ for phantom wormhole with static conformal symmetry

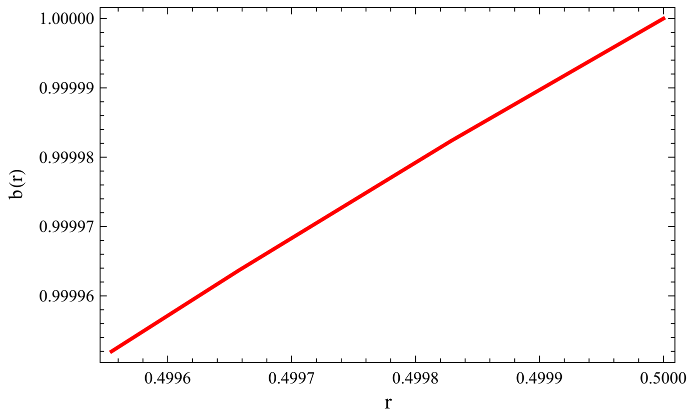

Fig. 7 Plot of $b(r)$ versus r with $n = 2$ for phantom wormhole with non-static conformal symmetry

6 Concluding remarks

It has always been interesting to formulate wormhole solutions, i.e., by first considering spacetime metric and then deducing the energy-momentum tensor which may violate the energy conditions. A hypothetical type of matter is affiliated with this violation, namely, exotic matter. To minimize the dependence of wormholes on exotic matter, the exploration of viable realistic models is an important issue. In this paper, we have considered a systematic approach to study wormhole solutions by assuming static spherically symmetric metric with non-static conformal symmetry in the background of $f(G)$ gravity. We have explored wormhole solutions by choosing two types of shape function and also formulated the shape function for a particular as well as phantom EoS. We have also discussed a particular case with static conformal symmetry in the context of phantom wormhole. The results are summarized as follows.

For specific choice of shape functions, we have found that WEC (exemplifying all energy conditions) are satisfied only for the even values of n while violated for odd range. The validity of energy conditions confirms the existence of physically realistic wormholes threaded by normal matter which leads the modified theories to minimize the dependence of wormhole geometry on exotic matter. The behavior of the shape function are shown graphically for a particular as well as phantom EoS. The graphical behavior shows that these functions satisfy all conditions related to the wormhole geometry and hence provides wormholes for the whole range of r. For phantom wormholes with non-static conformal symmetry, the function does not obey the assumptions related to wormhole physics and hence there does not exist any wormhole in this case.

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Effects of f(G) gravity on the dynamics of self-gravitating fluids

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Abstract. We study the dynamics of self-gravitating fluids bounded by spherically symmetric surface in the background of $f(G)$ gravity. The link between physical and geometrical variables, such as anisotropy, density inhomogeneity, dissipation, the Weyl tensor, expansion scalar, shear tensor and modified (Gauss-Bonnet) curvature terms, is given. We also investigate some particular fluid models according to various dynamical conditions. It is found that our results are consistent with general relativity for constant $f(G)$ model (regular distribution of dark energy in the universe). Any other choice of the model leads to irregular distribution of dark energy and deviates from general relativity.

1 Introduction

Gravitational collapse and its resulting remanent, such as stars, black holes, planets and galaxies, have attracted many researchers to explore their dynamics. The most fascinating model to study the dynamics of these celestial objects are the self-gravitating fluid models. These are based on some dynamical variables (physical and geometrical quantities) like Weyl tensor, expansion scalar, anisotropy, shear tensor, density inhomogeneity (irregularity of energy density) as well as dissipation. Mena and Tavakol [1] studied the evolution of a self-gravitating system through these quantities. The Weyl tensor describes the tidal force, which makes the gravitating fluid more inhomogeneous during evolution as time proceeds. The effects of this tensor on energy density inhomogeneity has been studied during the evolution of self-gravitating fluid [2,3]. Herrera [4] investigated stability of spherical self-gravitating fluid and found physical relevance of the Weyl tensor with such fluids. The significant roles of local anisotropy of pressure [5–7] as well as shear tensor [8–11] in the evolution of self-gravitating fluid models have largely been studied. Herrera *et al.* [12] analyzed dissipative spherically symmetric self-gravitating system using all dynamical quantities. Herrera [13] studied contributions of the Weyl tensor and dissipation for axial fluid model.

Stimulating results from type Ia supernovae, cosmic microwave background radiations and large-scale structures, revolutionized the field of gravitational physics and cosmology, which indicate that our universe is expanding at an accelerating rate. This strange behavior of the universe is linked with a mysterious repulsive force known as dark energy (DE) possessing highly negative pressure. In order to explain the mysterious nature of DE, modification in Einstein-Hilbert action has been proposed leading to different modified theories like $f(R)$ (R is the Ricci scalar), $f(R,T)$ (T is the trace of energy-momentum tensor), Brans-Dicke and Gauss-Bonnet theories of gravity, etc. These theories are consistent with GR in weak gravitational field regime but may disagree in strong field regime. Gravitational collapse is the phenomenon of strong gravitational field regime and hence can be described by modified theories of gravitation.

The modified Gauss-Bonnet gravity ($f(G)$ theory of gravity) is the consequence of the inclusion of an arbitrary function $f(G)$ in the Einstein-Hilbert action [14], where G is the Gauss-Bonnet quadratic invariant. This theory efficiently interprets recent cosmic accelerated expansion, transition from decelerated to accelerated phases and passes solar system tests. Also, it efficiently narrates thermodynamics [15,16] and cures four types of finite time future singularities [17]. Myrzakulov *et al.* [18] explored DE as well as inflationary era in this gravity. We have studied energy conditions [19], built-in inflation [20] as well as Noether symmetries [21] in this theory.

Sharif and Manzoor [22,23] investigated self-gravitating fluid models with spherical as well as cylindrical symmetries in Brans-Dicke theory and originated a set of equations governing the dynamics of dissipative anisotropic fluids. Sharif

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and Nasir [24] discussed the evolution of axial dissipative fluid in $f(R)$ gravity through structure scalars in terms of dynamical quantities. Sharif and Yousaf [25] analyzed the dynamics of spherical self-gravitating fluid in terms of structure scalars for a particular $f(R)$ model.

In this paper, we discuss the evolution of dissipative anisotropic shearing spherical matter configuration in $f(G)$ gravity. We construct a set of governing equations for the dynamics of fluid with spherically symmetric spacetime in $f(G)$ gravity. The scheme of the paper is as follows. In the next section, we formulate the modified field equations and dynamical quantities. In sect. 3, we derive the evolution equations for a viable $f(G)$ model. Section 4 describes the dynamics for different fluid models. The last section concludes the results.

2 Field equations and dynamical quantities

In this section, we first formulate the field equations for $f(G)$ gravity and then evaluate dynamical quantities. The general line element for non-static spherical configuration is given as

$$\mathrm{d}s^2 = -A^2\mathrm{d}t^2 + B^2\mathrm{d}r^2 + C^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{1}$$

where $A = A(t,r)$, $B = B(t,r)$ and $C = C(t,r)$ are functions of comoving coordinates t and r. The action for $f(G)$ gravity is given by [26]

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x[R + f(G)]\sqrt{-g} + S_M, \tag{2}$$

where κ, R, $f(G)$ are the coupling constant, the Ricci scalar, arbitrary function of G, respectively, and S_M is the matter action. The Gauss-Bonnet term is

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho},$$

where $R_{\mu\nu\sigma\rho}$, $R_{\mu\nu}$ are the Riemann and Ricci tensors, respectively. Varying the action (2) with respect to the metric tensor, we obtain

$$\mathcal{G}_{\mu\nu} = \kappa^2 \overset{(\mathrm{eff})}{T}_{\mu\nu} = \kappa^2\left[\overset{(\mathrm{M})}{T}_{\mu\nu} + \overset{(\mathrm{GB})}{T}_{\mu\nu}\right], \tag{3}$$

where $\overset{(\mathrm{M})}{T}_{\mu\nu}$ and $\overset{(\mathrm{BG})}{T}_{\mu\nu}$ are the energy-momentum tensors for matter and Gauss-Bonnet (GB) terms, respectively, and

$$\begin{aligned}\overset{(\mathrm{GB})}{T}_{\mu\nu} = 8\Big[&R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho}\\ &+ \frac{1}{2}R(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})\Big]\nabla^\rho\nabla^\sigma f_G + (Gf_G - f)g_{\mu\nu},\end{aligned} \tag{4}$$

where f_G denotes derivative of f with respect to G.

For matter distribution, the energy-momentum tensor is defined as

$$\overset{(\mathrm{M})}{T}_{\mu\nu} = \rho U_\mu U_\nu + p_t h_{\mu\nu} + (p_r - p_t)V_\mu V_\nu + q(V_\mu U_\nu + U_\mu V_\nu) + \epsilon l_\mu l_\nu, \tag{5}$$

where ρ, p_r, p_t, q and ϵ are energy density, radial pressure, tangential pressure, dissipation (heat-flux) and radiation density, respectively. The quantities U^μ (4-velocity vector), V^μ (unit 4-vector in radial direction), $h^{\mu\nu}$ (projection tensor) and l^μ (null 4-vector) are defined as

$$U^\mu = A^{-1}\delta^\mu_0, \qquad V^\mu = B^{-1}\delta^\mu_1, \qquad h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu, \qquad l^\mu = A^{-1}\delta^\mu_0 + B^{-1}\delta^\mu_1,$$

satisfying the relations

$$U^\mu U_\mu = -1, \qquad V^\mu V_\mu = 1, \qquad V^\mu U_\mu = 0, \quad l^\mu U_\mu = -1, \qquad l^\mu l_\mu = 0, \qquad h_{\mu\nu}U^\mu = 0.$$

We can write eq. (5) as

$$\overset{(\mathrm{M})}{T}_{\mu\nu} = \tilde{\rho}U_\mu U_\nu + p_t h_{\mu\nu} + \Pi V_\mu V_\nu + \tilde{q}_\mu U_\nu + \tilde{q}_\nu U_\mu, \tag{6}$$

where $\tilde{\rho} = \rho + \epsilon$, $\mathit{\Pi} = \tilde{p}_r + p_t$, $\tilde{p}_r = p_r + \epsilon$, $\tilde{q}_\mu = \tilde{q}V_\mu$, $\tilde{q} = q + \epsilon$. The corresponding field equations are

$$\begin{aligned}
\tilde{\rho} &= \frac{\dot{C}}{A^2C}\left(2\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{1}{B^2}\left(2\frac{C''}{C}+\left(\frac{C'}{C}\right)^2-2\frac{B'C'}{BC}-\left(\frac{B}{C}\right)^2\right)\\
&+\frac{4}{A^2BC^2}\left(\dot{B}-2\frac{\dot{C}C''}{B}-\frac{\dot{B}C'^2}{B^2}+3\frac{\dot{B}\dot{C}^2}{A^2}+2\frac{B'C'\dot{C}}{B^2}\right)\dot{f}_G+\frac{4}{A^2B^3C^2}\\
&\times\left(A^2B'+B'\dot{C}^2-2\dot{B}\dot{C}C'+2\frac{A^2C'C''}{B}-3\frac{A^2B'C'^2}{B^2}\right)f'_G+\frac{4}{A^2B^3C^2}\\
&\times\left(\frac{A^2C'^2}{B^2}-A^2-2\dot{C}\right)f''_G-Gf_G+f,
\end{aligned} \tag{7}$$

$$\begin{aligned}
\tilde{q} &= \frac{2}{AB}\left(\frac{\dot{B}C'}{BC}+\frac{A'\dot{C}}{AC}-\frac{\dot{C}'}{C}\right)+\frac{4}{ABC^2}\left(1+\left(\frac{\dot{C}}{A}\right)^2-\left(\frac{C'}{B}\right)^2\right)\dot{f}'_G\\
&+\frac{4}{A^2BC^2}\left(-A'+2\frac{\dot{C}\dot{C}'}{A}+\frac{A'C'^2}{B^2}-3\frac{A'\dot{C}^2}{A^2}-2\frac{\dot{B}C'\dot{C}}{AB}\right)\dot{f}_G+\frac{4}{AB^2C^2}\\
&\times\left(-\dot{B}-2\frac{\dot{C}'C'}{B}+3\frac{\dot{B}C'^2}{B^2}-\frac{\dot{B}\dot{C}^2}{A^2}+2\frac{A'C'\dot{C}}{AB}f'_G\right),
\end{aligned} \tag{8}$$

$$\begin{aligned}
\tilde{p}_r &= \frac{1}{A^2}\left(\frac{\dot{C}}{C}\left(2\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)-2\frac{\ddot{C}}{C}\right)+\frac{C'}{B^2C}\left(2\frac{A'}{A}+\frac{C'}{C}\right)-\frac{1}{C^2}+\frac{4}{A^2C^2}\\
&\times\left(\left(\frac{C'}{B}\right)^2-\left(\frac{\dot{C}}{A}\right)^2-1\right)\ddot{f}_G+\frac{4}{A^3C^2}\left(\dot{A}-2\frac{\dot{C}\ddot{C}}{A}+2\frac{A'C'\dot{C}}{B^3}-\frac{\dot{A}C'^2}{B^2}\right.\\
&\left.+3\frac{\dot{A}\dot{C}^2}{A^2}\right)\dot{f}_G+\frac{4}{AB^2C^2}\left(A'+\frac{1}{A^2}-2\frac{\dot{A}\dot{C}C'}{A^2}-3\frac{A'C'^2}{B^2}+2\frac{\ddot{C}C'}{A}\right)f'_G\\
&+Gf_G-f,
\end{aligned} \tag{9}$$

$$\begin{aligned}
p_t &= \frac{1}{A^2}\left(\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{\dot{B}\dot{C}}{BC}\right)+\frac{1}{B^2}\left(\frac{A''}{A}+\frac{C''}{C}-\frac{A'B'}{AB}\right.\\
&\left.+\frac{C'}{C}\left(\frac{A'}{A}-\frac{B'}{B}\right)\right)+\frac{4}{A^3BC}\left(\frac{1}{B}\left(2\dot{C}'A'-\dot{A}C''+A''\dot{C}\right)-\frac{1}{A}\right.\\
&\times\left(\dot{B}\ddot{C}+\dot{C}\ddot{B}\right)-\frac{1}{B^2}\left(A'B'\dot{C}+A'\dot{B}C'-\dot{A}B'C'\right)+3\frac{\dot{A}\dot{B}\dot{C}}{A^2}-2\\
&\times\left.\frac{A'^2\dot{C}}{AB}\right)\dot{f}_G+\frac{4}{AB^3C}\left(\frac{1}{A}\left(2\dot{B}\dot{C}'-B'\ddot{C}+\ddot{B}C'\right)-\frac{1}{B}\left(A'C''+A''C'\right)\right.\\
&\left.-\frac{1}{A^2}\left(\dot{A}\dot{B}C'+A'\dot{B}\dot{C}-\dot{A}B'\dot{C}\right)+3\frac{A'B'C'}{B^2}-2\frac{\dot{B}^2C'}{AB}\right)f'_G+\frac{8}{A^2B^2C}\\
&\times\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}-1\right)\dot{f}'_G-\frac{4}{AB^2C}\left(\frac{A'C'}{B^2}+\frac{\dot{A}\dot{C}}{A^2}-\ddot{C}\right)f''_G-\frac{4}{A^2BC}\\
&\times\left(\frac{\dot{B}\dot{C}}{A^2}+\frac{B'C'}{A}\right)\ddot{f}_G+Gf_G-f.
\end{aligned} \tag{10}$$

The expression for Gauss-Bonnet invariant is calculated as

$$G = \frac{8}{ABC}\left[\left(\frac{A'B'}{B^2C}+\frac{\ddot{B}}{AC}\right)\left(1+\frac{\dot{C}^2}{A^2}\right)+\left(\frac{\dot{A}\dot{B}}{A^2C}+\frac{A''}{BC}\right)\left(\frac{C'^2}{B^2}-1\right)\right.$$
$$-\frac{1}{AC}\left(\frac{A''\dot{C}^2}{AB}+\ddot{B}C'^2\right)+2\left\{\frac{C''}{BC}\left(\frac{A'C'}{B^2}-\frac{\ddot{C}}{A}\right)+\frac{B'C'}{AB^2C}\left(\ddot{C}-\frac{\dot{A}\dot{C}}{A}\right)\right.$$
$$\left.+\frac{\dot{C}}{A^3C}\left(\ddot{C}\dot{B}+\frac{\dot{C}A'^2}{B}\right)+\frac{\dot{C}}{A^2BC}\left(\dot{A}C''+\frac{A'\dot{B}C'}{B}\right)+\frac{1}{ABC}\left(\dot{C}'^2+\frac{\dot{B}^2C'^2}{B^2}\right)\right\}$$
$$\left.-\frac{3}{C}\left(\frac{A'B'C'^2}{B^4}+\frac{\dot{A}\dot{B}\dot{C}^2}{A^4}\right)-\frac{4\dot{C}'}{ABC}\left(\frac{A'\dot{C}}{A}+\frac{\dot{B}C'}{B}\right)\right]. \tag{11}$$

Here dot and prime denote partial derivatives with respect to t and r, respectively, and we have assumed the unit system $\kappa^2=\frac{8\pi G}{c}=1$ (G is the gravitational constant and c is the speed of light).

The four acceleration a_μ (which defines the effects of gravitational as well as inertial forces on fluid) is defined as

$$a_\mu = U_{\mu;\nu}U^\nu, \qquad a_\mu = aV_\mu, \qquad a_{(1)}=\frac{A'}{A}, \qquad a^2=a^\mu a_\mu=\left(\frac{A'}{AB}\right)^2.$$

The volume expansion of fluid can be measured by expansion parameter ϑ as

$$\vartheta = U^\mu_{;\mu}=\frac{1}{A}\left(\frac{\dot{B}}{B}+2\frac{\dot{C}}{C}\right).$$

The shear tensor $\sigma_{\mu\nu}$ is used to evaluate distortion appearing in the fluid due to motion defined as

$$\sigma_{\mu\nu}=U_{(\mu;\nu)}+a_{(\mu}U_{\nu)}-\frac{1}{3}\vartheta h_{\mu\nu}.$$

Its alternative form is given as

$$\sigma_{\mu\nu}=\sigma\left(V_\mu V_\nu-\frac{1}{3}h_{\mu\nu}\right),$$

which provides

$$\sigma_{\mu\nu}\sigma^{\mu\nu}=\frac{2}{3}\sigma^2, \qquad \sigma=\frac{1}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)$$

and its non-zero components are

$$\sigma_{11}=\frac{2}{3}\sigma B^2, \qquad \sigma_{22}=\frac{1}{\sin^2\theta}, \qquad \sigma_{33}=-\frac{1}{3}\sigma C^2.$$

The Weyl tensor is described as a combination of the Riemann tensor, Ricci tensor and Ricci scalar. It narrates the effects of tidal force borne by the object while moving along geodesics (in the region without matter) and is given by

$$C^\mu_{\rho\sigma\nu}=R^\mu_{\rho\sigma\nu}-\frac{1}{2}R^\mu_\sigma g_{\rho\nu}+\frac{1}{2}R_{\rho\sigma}\delta^\mu_\nu-\frac{1}{2}R_{\rho\nu}\delta^\mu_\sigma+\frac{1}{2}R^\mu_\nu g_{\rho\sigma}+\frac{1}{6}R(\delta^\mu_\sigma g_{\rho\nu}-\delta^\mu_\nu g_{\rho\sigma}). \tag{12}$$

This tensor can be written as magnetic $M_{\mu\nu}$ and electric parts $E_{\mu\nu}$. The magnetic part of this tensor vanishes for spherical symmetry while the electric part is given by

$$E_{\mu\nu}=C_{\mu\alpha\nu\beta}U^\alpha U^\beta, \tag{13}$$

where

$$C_{\mu\alpha\nu\beta}=(g_{\mu\alpha\kappa\eta}g_{\nu\beta\gamma\delta}-\varepsilon_{\mu\alpha\kappa\eta}\varepsilon_{\nu\beta\gamma\delta})U^\kappa U^\gamma E^{\eta\delta},$$

with $g_{\mu\alpha\kappa\eta}=g_{\mu\kappa}g_{\alpha\eta}-g_{\mu\eta}g_{\alpha\kappa}$ and $\varepsilon_{\mu\alpha\kappa\eta}$ is the Levi-Civita tensor. The electric part in terms of 4-unit vector and projection tensor can be written as

$$E_{\mu\nu}=\varepsilon\left(V_\mu V_\nu-\frac{1}{3}h_{\mu\nu}\right), \tag{14}$$

where $\varepsilon=\frac{1}{2}(\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}-(\frac{\dot{C}}{C}-\frac{\dot{B}}{B})\frac{\dot{C}}{C})+\frac{1}{2B^2}(-\frac{C''}{C}+(\frac{C'}{C}+\frac{B'}{B})\frac{C'}{C})-\frac{1}{2C^2}$ is the Weyl scalar and the non-zero components of the electric part are

$$E_{11}=\frac{2}{3}\varepsilon B^2,\qquad E_{22}=-\frac{1}{3}\varepsilon C^2,\qquad E_{33}=E_{22}\sin^2\theta.$$

The Misner-Sharp mass function calculates the total energy of spherically symmetric system within the radius $r=C$ given as

$$\mathcal{M}=\frac{1}{2}C^3R^{23}_{23}=\frac{1}{2}C\left(\left(\frac{\dot{C}}{A}\right)^2-\left(\frac{C'}{B}\right)^2+1\right). \tag{15}$$

Now we calculate the variation of this mass function of radiating fluid inside the sphere. For this purpose, we introduce two useful derivative operators with respect to radial and proper time coordinates as

$$D_T=\frac{1}{A}\frac{\partial}{\partial t},\qquad D_C=\frac{1}{C}\frac{\partial}{\partial r}$$

and the relativistic velocity of the collapsing fluid turns out to be

$$\mathcal{U}=D_TC=\frac{1}{A}\frac{\partial C}{\partial t}=\frac{\dot{C}}{A}. \tag{16}$$

Combining eqs. (15) and (16), we obtain

$$E=\frac{C'}{B}=\left(1+\mathcal{U}^2-\frac{2}{C}\mathcal{M}\right)^{\frac{1}{2}}. \tag{17}$$

Using eqs. (3), (16) and (17), it follows that

$$D_T\mathcal{M}=-\frac{C^2}{2}\left\{\left(\tilde{p}_r+\frac{1}{B^2}\overset{(\mathrm{GB})}{T_{11}}\right)\mathcal{U}+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T_{01}}\right)E\right\}, \tag{18}$$

$$D_C\mathcal{M}=\frac{C^2}{2}\left\{\left(\tilde{\rho}+\frac{1}{A^2}\overset{(\mathrm{GB})}{T_{00}}\right)+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\}. \tag{19}$$

These equations represent the variation of mass inside the spherical surface of evolving fluid. Equation (18) represents the effects of radial pressure, dissipation, relativistic velocity and GB curvature terms on the proper derivative of mass function within spherically bounded region, while eq. (19) indicates the combined effect of pressure, dissipation, relativistic velocity and extra GB curvature terms on the variation of mass distribution in radial direction. Equation (19) yields

$$\mathcal{M}'=\frac{C^2C'}{2}\left\{\left(\tilde{\rho}+\frac{1}{A^2}\overset{(\mathrm{GB})}{T_{00}}\right)+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\},$$

which further implies that

$$\mathcal{M}=\frac{1}{2}\int_0^r C^2C'\left\{\left(\tilde{\rho}+\frac{1}{A^2}\overset{(\mathrm{GB})}{T_{00}}\right)+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\}\mathrm{d}r. \tag{20}$$

A particular combination of radiating energy density, dissipation and $f(G)$ correction terms through mass function can be attained using eq. (20) as

$$3\frac{\mathcal{M}}{C^3}=\frac{3}{2C^3}\int_0^r C^2C'\left\{\left(\tilde{\rho}+\frac{1}{A^2}\overset{(\mathrm{GB})}{T_{00}}\right)+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\}\mathrm{d}r. \tag{21}$$

We have taken a regular matter distribution at $r=0$ (center), *i.e.*, $\mathcal{M}(t,0)=C(t,0)=0$.

3 f(G) model and the evolution equations

In this section, we construct the dynamical equations for dissipative spherically distributed self-gravitating fluid by assuming the following $f(G)$ model

$$f(G) = \alpha G^n, \tag{22}$$

where α is any constant and $n > 0$ [27]. A modified gravity model should avoid ghosts (*i.e.*, instabilities such as Dolgov-Kawasaki as well as Ostrogradski's instability) to preserve precise cosmological dynamics. Ghost may appear while dealing with modified theories due to the repulsive nature of DE. This model (22) is viable if it satisfies the following conditions [28]:

– $f(G)$ and all of its derivatives ($f_G, f_{GG}, f_{GGG}\ldots$) are regular;
– $f_{GG} > 0$, $\forall\ G$ and $f_{GG} \to 0$ as $|G| \to +\infty$.

The condition $\dot{f}_G > 0$ is required to avoid ghost. This model satisfies all the required conditions to be viable. The model parameter n has some effects on $R + f(G)$ cosmology. If $n < \frac{1}{2}$, the GB terms become dominant over curvature term (Einstein term) in the weak-field regime, where the curvature term is negligible and we are in the non-phantom phase. For $n < 0$, the big-rip singularity seems to occur. Near this singularity, the curvature becomes dominant, *i.e.*, Einstein term becomes dominant as compared to GB terms. To avoid this singularity, GB terms can be neglected and we arrive at phantom era. Thus, the transition of non-phantom to phantom phase can naturally occur in this model. When $0 < n < \frac{1}{2}$ in strong gravitational field regime, GB terms can be neglected and we are left with the Einstein gravity (deceleration). For late times, the GB terms may become dominant as compared with the matter Lagrangian density (acceleration). Thus transition from decelerated to accelerated universe can occur.

3.1 Evolution equations

Ricci identities are used to define curvature by using four-velocity, four acceleration, expansion scalar, shear and projection tensors. The iterated Ricci identities are obtained by taking one extra derivative, such as

$$U_{\mu;\nu;\alpha} - U_{\mu;\alpha;\nu} = R^{\beta}_{\mu\nu\alpha} U_\beta,$$

where $U_{\mu;\nu} = a_\mu U_\nu + \sigma_{\mu\nu} + \frac{1}{3}\vartheta h_{\mu\nu}$. The Riemann curvature tensor can be defined as

$$\frac{1}{2} R^{\mu}_{\alpha\beta\nu} U_\mu = a_{\alpha;[\nu} U_{\beta]} + a_\alpha U_{[\beta;\nu]} + \sigma_{\alpha[\beta;\nu]} + \frac{1}{3}\left\{\vartheta_{[\nu} h_{\beta]\alpha} + \vartheta h_{\alpha[\beta;\nu]}\right\}. \tag{23}$$

In the following, we formulate evolution equations like Raychaudhuri equation, constraints equation, propagation equation of shear (using eq. (23)), evolution equations for the Weyl tensor and dynamical equations.

3.1.1 Raychaudhuri equation

This equation describes the evolution of expansion and is obtained by contracting eq. (23) with U^β and then with indices ν and α. It turns out to be

$$\vartheta_{;\mu} U^\mu + \frac{1}{3}\vartheta^2 + \sigma_{\mu\nu}\sigma^{\mu\nu} - a^{\mu}_{;\mu} = -U_\mu U^\nu R^{\mu}_{\nu}. \tag{24}$$

This equation along with eq. (3) gives

$$\vartheta_{;\mu} U^\mu + \frac{1}{3}\vartheta^2 + \frac{2}{3}\sigma^2 - a^{\mu}_{;\mu} = -\frac{1}{2}(\tilde{\rho} + 3\tilde{p}_r). \tag{25}$$

We note that the Raychaudhuri equation for $f(G)$ gravity is the same as for GR [12], *i.e.*, GB terms have no contribution in Raychaudhuri equation. This is the evolution equation for expansion, therefore, measures the expansion rate of self-gravitating relativistic fluid for GR as well as in $f(G)$ cosmology.

3.1.2 Propagation equation of shear

This equation is obtained by contracting eq. (23) with $U^\beta h^\alpha_\gamma h^\nu_\delta$ as follows:

$$U^\beta U_\mu R^\mu_{\alpha\beta\nu} h^\alpha_\gamma h^\nu_\delta = h^\alpha_\gamma h^\nu_\delta \left(a_{\alpha;\nu} - \sigma_{\alpha\nu;\beta} U^\beta\right) - a_\gamma a_\delta - U^\beta_{;\nu} h^\nu_\delta \left(\sigma_{\gamma\beta} + \frac{1}{3}\vartheta h_{\gamma\beta}\right) - \frac{1}{3}\vartheta_{,\alpha} U^\alpha h_{\gamma\delta}. \tag{26}$$

Alternatively, this relation can be derived from eq. (12) as

$$R^\mu_{\alpha\beta\nu} = C^\mu_{\alpha\beta\nu} + \frac{1}{2}R^\mu_\beta g_{\alpha\nu} - \frac{1}{2}R_{\alpha\beta}\delta^\mu_\nu + \frac{1}{2}R_{\alpha\nu}\delta^\mu_\beta - \frac{1}{2}R^\mu_\nu g_{\alpha\beta} - \frac{1}{6}R(\delta^\mu_\beta g_{\alpha\nu} - \delta^\mu_\nu g_{\alpha\beta}),$$

which, after contraction with $U^\beta U_\mu h^\alpha_\gamma h^\nu_\delta$, gives eq. (26). Again contracting with V^γ, V^δ and using eq. (3), we obtain

$$U^\beta U_\mu R^\mu_{\alpha\beta\nu} h^\alpha_\gamma h^\nu_\delta V^\gamma V^\delta = \varepsilon - \frac{1}{2}\Pi + 2n\alpha\Big[- R^\phi_{\rho\sigma\mu} V^\mu V_\phi - R_{\rho\mu} V^\mu V_\sigma + g_{\sigma\rho} R_{\alpha\mu} \times V^\alpha V^\mu - R_{\alpha\sigma} V_\rho V^\alpha + \frac{1}{2} R V_\rho V_\sigma\Big] \nabla^\rho \nabla^\sigma G^n. \tag{27}$$

This is the propagation equation of the shear in $f(G)$ gravity with some other dynamical variables which yields the effects of GB terms in the shearing motion of evolving self-gravitating spherical objects.

3.1.3 Constraint equation

We obtain this equation from eq. (23) by contracting α and ν and then contracting with $h^{\alpha\beta}V_\alpha$ as

$$R^\mu_\beta U_\mu h^{\alpha\beta} = h^\alpha_\beta \left(\sigma^{\beta\gamma}_{;\gamma} - \frac{2}{3}\vartheta^{;\beta}\right) + \sigma^{\alpha\beta} a_\beta,$$

which gives

$$h^\alpha_\beta \left(\sigma^{\beta\gamma}_{;\gamma} - \frac{2}{3}\vartheta^{;\beta}\right) + \sigma^{\alpha\beta} a_\beta = -\tilde{q} V^\alpha + 2n\alpha\Big[- U_\mu h^{\alpha\beta} R^\mu_{\rho\sigma\beta} + g_{\sigma\rho} U_\mu h^{\alpha\beta} R^\mu_\beta + U_\phi h^\alpha_\rho R^\phi_\sigma + \frac{1}{8} R h^\alpha_\rho U_\sigma\Big] \nabla^\rho \nabla^\sigma G^n. \tag{28}$$

This equation directly relates shear tensor, expansion scalar, heat flux and modified terms of the Gauss-Bonnet gravity.

3.2 Evolution equations for the Weyl tensor

Bianchi identities ($R_{\alpha\beta\gamma\delta;\eta} + R_{\alpha\beta\eta\gamma;\delta} + R_{\alpha\beta\delta\eta;\gamma} = 0$) can also be written as

$$C^\eta_{\alpha\beta\gamma;\eta} = R_{\gamma[\alpha;\beta]} - \frac{1}{6} g_{\gamma[\alpha} R_{,\beta]}. \tag{29}$$

This is the relation between Weyl and Ricci tensors given by Kundt and Trümper [29]. This can also be written in terms of the Weyl and effective energy-momentum tensors by using eq. (3) as

$$C^\eta_{\alpha\beta\gamma;\eta} = \overset{(\mathrm{eff})}{T}_{\gamma[\alpha;\beta]} - \frac{1}{6} g_{\gamma[\alpha} \overset{(\mathrm{eff})}{T}_{,\beta]}, \tag{30}$$

where $\overset{(\mathrm{eff})}{T} = \overset{(\mathrm{M})}{T} + \overset{(\mathrm{GB})}{T}$. From eq. (2), we can write

$$U^\beta C^\eta_{\alpha\beta\gamma;\eta} + U^\beta_{;\eta} C^\eta_{\alpha\beta\gamma} = \vartheta E_{\alpha\gamma} + U^\mu E_{\alpha\gamma;\mu} - U_{\gamma;\eta} E^\eta_\alpha - U_\gamma E^\eta_{\alpha;\eta}, \tag{31}$$

where $U^\beta C_{\alpha\beta\gamma\delta} = E_{\alpha\gamma} U_\delta - E_{\alpha\delta} U_\gamma$. After contraction with $h^\alpha_\mu h^\gamma_\nu U^\beta V^\mu V^\nu$, eq. (31) gives

$$h^\alpha_\mu h^\gamma_\nu U^\beta V^\mu V^\nu C^\eta_{\alpha\beta\gamma;\eta} = \frac{4}{3}\vartheta E_{\mu\nu} V^\mu V^\nu - U_{\nu;\eta} E^\eta_\mu V^\mu V^\nu + U^\beta E_{\alpha\gamma;\beta} h^\alpha_\mu h^\gamma_\nu V^\mu V^\nu + h_{\mu\nu} \sigma^{\gamma\beta} E_{\gamma\beta} V^\mu V^\nu - \sigma^{\gamma\nu} E^\gamma_\mu V^\mu V^\nu - \sigma_{\gamma\mu} E^\gamma_\nu V^\mu V^\nu. \tag{32}$$

Furthermore, the effective energy-momentum tensor provides

$$h^\alpha_\mu h^\gamma_\nu U^\beta \overset{(\mathrm{eff})}{T}_{\gamma\alpha;\beta} = h^\alpha_\mu h^\gamma_\nu U^\beta \overset{(\mathrm{M})}{T}_{\gamma\alpha;\beta} + h^\alpha_\mu h^\gamma_\nu U^\beta \overset{(\mathrm{GB})}{T}_{\gamma\alpha;\beta}$$
$$= (p_t)_{,\beta} h_{\mu\nu} U^\beta + (\Pi V_\gamma V_\alpha)_{;\beta} h^\alpha_\mu h^\gamma_\nu U^\beta + \tilde{q}_\nu a_\mu + \tilde{q}_\mu a_\nu$$
$$+ 8n\alpha \left[h^\alpha_\mu h^\gamma_\nu U^\beta R_{\gamma\rho\alpha\sigma;\beta} + h^\alpha_\mu h_{\nu\sigma} U^\beta R_{\rho\alpha;\beta} - h_{\mu\nu} U^\beta\right.$$
$$\left.\times\ R_{\rho\sigma;\beta} - h^\alpha_\mu h^\gamma_\nu U^\beta g_{\sigma\rho} R_{\gamma\alpha;\beta} + h_{\mu\rho} h^\gamma_\nu U^\beta R_{\gamma\alpha;\beta}\right] \nabla^\rho \nabla^\sigma G^n, \tag{33}$$

$$h^\alpha_\mu h^\gamma_\nu U^\beta \overset{(\mathrm{eff})}{T}_{\gamma\beta;\alpha} = h^\alpha_\mu h^\gamma_\nu U^\beta \overset{(\mathrm{M})}{T}_{\gamma\beta;\alpha} + h^\alpha_\mu h^\gamma_\nu U^\beta \overset{(\mathrm{GB})}{T}_{\gamma\beta;\alpha}$$
$$= (p_t - \tilde{\rho}) \left(\sigma_{\mu\nu} + \frac{1}{3}\vartheta h_{\mu\nu}\right) + \Pi_{;\beta} U^\beta V_\gamma V_{\beta;\alpha} h^\alpha_\mu h^\gamma_\nu U^\beta$$
$$- \tilde{q}_{,\alpha} h^\alpha_\mu V_\nu + 8n\alpha \left[h^\alpha_\mu h^\gamma_\nu U^\beta R_{\gamma\rho\beta\sigma;\alpha} + h^\alpha_\mu h_{\nu\sigma} U^\beta R_{\rho\beta;\alpha}\right.$$
$$\left.-\ h^\alpha_\mu h^\gamma_\nu U^\beta g_{\sigma\rho} R_{\gamma\beta;\alpha} + h^\alpha_\mu h^\gamma_\nu U_\rho R_{\gamma\sigma;\alpha}\right] \nabla^\rho \nabla^\sigma G^n, \tag{34}$$

$$h^\alpha_\mu h^\gamma_\nu U^\beta g_{\gamma[\alpha} \overset{(\mathrm{eff})}{T}_{;\beta]} = \frac{1}{2}\left[h^\alpha_\mu h^\gamma_\nu U^\beta g_{\gamma\alpha} \overset{(\mathrm{eff})}{T}_{;\beta} - h^\alpha_\mu h^\gamma_\nu U^\beta g_{\gamma\beta} \overset{(\mathrm{eff})}{T}_{;\alpha}\right]$$
$$= \frac{1}{2}(\Pi + 3p_t - \tilde{\rho})_{,\beta} U^\beta h_{\mu\nu} - n\alpha h_{\mu\nu}[R_{\rho\sigma} + g_{\sigma\rho} R]_{;\beta} U^\beta \nabla^\rho \nabla^\sigma G^n. \tag{35}$$

Feeding back eqs. (32)–(35) into eq. (30), we have

$$\vartheta\left(\frac{1}{3}(\tilde{\rho} + \tilde{p}_r) h_{\mu\nu} + E_{\mu\nu}\right) + U^\beta (E_{\alpha\gamma} - \Pi_{\alpha\gamma})_{;\beta} h^\alpha_\mu h^\gamma_\nu + (\varepsilon + \tilde{\rho} + \tilde{p}_r)\sigma_{\mu\nu}$$
$$+ \frac{4}{3}\tilde{\rho}_{,\beta} U^\beta h_{\mu\nu} - \Pi_{\beta\nu}\left(\sigma^\beta_\mu - \frac{1}{3}\vartheta h^\beta_\mu\right) - \tilde{q}_\mu a_\nu - \tilde{q}_\nu a_\mu - 4n\alpha h^\alpha_\mu h^\gamma_\nu U^\beta [R_{\gamma\rho[\alpha\sigma;\beta]}$$
$$+\ g_{\sigma\rho} R_{\gamma[\alpha;\beta]}]\nabla^\rho\nabla^\sigma G^n - n\alpha[h^\alpha_\mu h_{\nu\sigma} U^\beta R_{\rho[\alpha;\beta]} - h_{\mu\nu} U^\beta R_{\rho\sigma;\beta} + h_{\mu\rho} h^\gamma_\nu U^\beta$$
$$\times R_{\gamma\alpha;\beta} - h^\alpha_\mu h^\gamma_\nu U_\rho R_{\gamma\sigma;\alpha}]\nabla^\rho\nabla^\sigma G^n = 0. \tag{36}$$

Similarly, contracting eq. (31) with U^γ, it follows that

$$U^\gamma U^\beta C^\eta_{\alpha\beta\gamma;\eta} = -E^\eta_{\alpha;\eta} - a^\eta E_{\alpha\eta} - \sigma^\beta_\eta E^\eta_\beta U_\alpha. \tag{37}$$

Also, we have

$$U^\gamma U^\beta \overset{(\mathrm{eff})}{T}_{\gamma\alpha;\beta} = U^\gamma U^\beta \overset{(\mathrm{M})}{T}_{\gamma\alpha;\beta} + U^\gamma U^\beta \overset{(\mathrm{GB})}{T}_{\gamma\alpha;\beta}$$
$$= U^\beta (U_\alpha \tilde{\rho_{,\beta}} + \tilde{q}_{\alpha;\beta}) + a_\alpha(\tilde{\rho} + \tilde{p_r}) - a^\gamma(\tilde{q}_\gamma U_\alpha + \Pi_{\alpha\gamma})$$
$$+\ 8n\alpha U^\gamma U^\beta \left[R_{\gamma\rho\alpha\sigma} - 2R_{r[\alpha} g_{\sigma]\rho}\right]_{;\beta} \nabla^\rho\nabla^\sigma G^n + 8n\alpha U^\beta$$
$$\times [2U_{[\sigma} R_{\alpha]\rho;\beta} + g_{\rho[\sigma} U_{\alpha]} R_{;\beta}]\nabla^\rho\nabla^\sigma G^n + 8n\alpha U^\gamma U^\beta [R_{\gamma\rho\alpha\sigma}$$
$$- g_{\rho[\alpha} R_{\sigma]\gamma}]\nabla^\rho\nabla^\sigma (G^n)_{,\beta} + n\alpha U^\beta \left[U_{[\sigma} R_{\alpha]\rho} + \frac{1}{2} R U_{[\alpha} g_{\sigma]\rho}\right]$$
$$\times \nabla^{rho}\nabla^\sigma (G^n)_{,\beta} + \alpha(n-1) U_\alpha U^\beta (G^n)_{,\beta}, \tag{38}$$

$$U^\gamma U^\beta \overset{(\mathrm{eff})}{T}_{\gamma\beta;\alpha} = U^\gamma U^\beta \overset{(\mathrm{M})}{T}_{\gamma\beta;\alpha} + U^\gamma U^\beta \overset{(\mathrm{GB})}{T}_{\gamma\beta;\alpha}$$
$$= \tilde{\rho}_{,\alpha} - 2\tilde{q}_\gamma \left(\sigma^\gamma_\alpha + a^\gamma U_\alpha + \frac{1}{3}\vartheta h^\gamma_\alpha\right) + 8n\alpha U^\gamma U^\beta [R_{\gamma\rho\beta\sigma}$$
$$- g_{\sigma\rho} R_{\gamma\beta}]_{;\alpha} \nabla^\rho\nabla^\sigma G^n + 8n\alpha \Bigg[U_\sigma U^\beta R_{\rho\beta;\alpha} + R_{\rho\sigma;\alpha}$$
$$+\ U_\gamma U_\rho R_{\gamma\sigma;\alpha} - \frac{1}{2} h_{\sigma\rho} R_{;\alpha}\Bigg] \nabla^\rho\nabla^\sigma G^n + 8n\alpha \Bigg[U^\gamma U^\beta R_{\gamma\rho\beta\sigma}$$
$$+ U^\beta U_\sigma R_{\rho\beta} + R_{\rho\sigma} - U^\gamma U^\beta g_{\sigma\rho} R_{\gamma\beta} + U^\gamma U_\rho R_{\gamma\sigma} - \frac{1}{2} h_{\sigma\rho}$$
$$\times R\Bigg] \nabla^\rho\nabla^\sigma (G^n)_{,\alpha} - \alpha(n-1)(G^n)_{,\alpha}, \tag{39}$$

$$U^\gamma U^\beta g_{\gamma[\alpha} \overset{(\mathrm{eff})}{T}{}_{;\beta]} = -\frac{1}{2}h^\beta_\alpha \overset{(\mathrm{eff})}{T}{}_{,\beta} = -\frac{1}{2}h^\beta_\alpha\left(\overset{(\mathrm{M})}{T}{}_{,\beta} + \overset{(\mathrm{GB})}{T}{}_{,\beta}\right)$$
$$= -\frac{1}{2}h^\beta_\alpha(-\tilde{\rho}+2\tilde{p}_r+p_t)_{,\beta} - 8n\alpha\bigg[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}$$
$$-\delta_{\rho\sigma})\bigg]_{,\beta}\nabla^\rho\nabla^\sigma G^n - 8n\alpha\left[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\right]$$
$$\times\nabla^\rho\nabla^\sigma(G^n)_{,\beta} + 4\alpha(n-1)(G^n)_{,\beta}. \tag{40}$$

Substituting eqs. (37)–(40) back into eq. (30), we obtain

$$h^\alpha_\mu E^\lambda_{\alpha;\lambda} + a^\lambda E_{\mu\lambda} + \tilde{q}_\gamma\left(\sigma^\gamma_\mu + \frac{1}{3}\vartheta h^\gamma_\mu\right) + \frac{1}{3}h^\beta_\mu(-2\tilde{\rho}+2\tilde{p}_r+p_t)_{,\beta} - a^\gamma\Pi_{\mu\gamma}$$
$$+ a_\mu(-\tilde{\rho}+\tilde{p}_r) + U^\beta h^\alpha_\mu\tilde{q}_{\alpha;\beta} + 4n\alpha U^\gamma U^\beta\left[R_{\gamma\rho\alpha\sigma;\beta} - R_{\gamma\rho\beta\sigma;\alpha} + g_{\sigma\rho}R_{\gamma\beta;\alpha}\right.$$
$$\left.-2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta}\right]\nabla^\rho\nabla^\sigma G^n + 4n\alpha U^\beta[2U_{[\sigma}R_{\alpha]\rho;\beta} + g_{\rho[\sigma}U_{\alpha]}R_{;\beta}]\nabla^\rho\nabla^\sigma G^n$$
$$+ 4n\alpha U^\gamma U^\beta[R_{\gamma\rho\alpha\sigma} - g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^\rho\nabla^\sigma(G^n)_{,\beta} + U^\beta\bigg[\frac{1}{2}n\alpha RU_{[\alpha}g_{\sigma]\rho} + U_{[\sigma}$$
$$\times R_{\alpha]\rho}\bigg]\nabla^\rho\nabla^\sigma(G^n)_{,\beta} + \frac{1}{2}\alpha(n-1)U_\alpha U^\beta(G^n)_{,\beta} - 4n\alpha\bigg[R_{\rho\sigma;\alpha} + R_{\rho\beta;\alpha}U_\sigma$$
$$\times U^\beta + U_\gamma U_\rho R_{\gamma\sigma;\alpha} - \frac{1}{2}h_{\sigma\rho}R_{,\alpha}\bigg]\nabla^\rho\nabla^\sigma G^n - 4n\alpha\bigg[U^\gamma U^\beta R_{\gamma\rho\beta\sigma} + U^\beta U_\sigma R_{\rho\beta}$$
$$+ R_{\rho\sigma} - U^\gamma U^\beta g_{\sigma\rho}R_{\gamma\beta} + U^\gamma U_\rho R_{\gamma\sigma} - \frac{1}{2}h_{\sigma\rho}R\bigg]\nabla^\rho\nabla^\sigma(G^n)_{,\alpha} + \frac{1}{2}\alpha(n-1)$$
$$\times(G^n)_{,\alpha} - \frac{4}{3}n\alpha\left[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\right]_{,\beta}\nabla^\rho\nabla^\sigma G^n - 8n\alpha\bigg[R_{\rho\sigma}$$
$$+ Rg_{\sigma\rho} - \frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\bigg]\nabla^\rho\nabla^\sigma(G^n)_{,\beta} + \frac{1}{3}\alpha(n-1)(G^n)_{,\beta} = 0. \tag{41}$$

Equations (36) and (41) are the evolution equations for the Weyl tensor. These represent the relationship between the Weyl tensor, dynamical variables (heat flux, anisotropic parameter, density, shear and projection tensors etc) and modified terms due to $f(G)$ gravity.

3.3 Dynamical equations

These equations describe the conservation of total energy of the evolving star obtained through Bianchi identities as

$$\left(\overset{(\mathrm{M})}{T^{\mu\nu}} + \overset{(\mathrm{GB})}{T^{\mu\nu}}\right)_{;\nu}U_\mu = 0, \qquad \left(\overset{(\mathrm{M})}{T^{\mu\nu}} + \overset{(\mathrm{GB})}{T^{\mu\nu}}\right)_{;\nu}V_\mu = 0, \tag{42}$$

where the first equation represents equation of continuity while the second is the equation of motion. Both equations yield

$$\tilde{\rho}_{,\mu}U^\mu + \vartheta(\tilde{\rho}+\tilde{p}_r) - \frac{2}{3}(\sigma+\vartheta)\Pi + \tilde{q}_{,\mu}V^\mu + 2\tilde{q}\left(\frac{C'}{BC}+a\right) + 8n\alpha[R^{\mu\nu}_{\rho\sigma}$$
$$- R^{\mu\nu}g_{\sigma\rho}]_{;\nu}U_\mu\nabla^\rho\nabla^\sigma G^n + 8n\alpha[(U_\sigma R^\nu_\rho - U^\nu R_{\rho\sigma})_{,\nu} + \Gamma^\mu_{\nu\gamma}(R^\nu_\rho\delta^\gamma_\sigma - R_{\rho\sigma}g^{\gamma\nu})$$
$$\times U_\mu + \Gamma^\nu_{\nu\gamma}(U_\sigma R^\gamma_\rho - U^\gamma R_{\rho\sigma})]\nabla^\rho\nabla^\sigma G^n = 0, \tag{43}$$

$$\tilde{p}_{r,\mu}V^\mu + a(\tilde{\rho}+\tilde{p}_r) + 2\Pi\frac{C'}{BC} + \tilde{q}_{,\mu}V^\mu + \frac{2}{3}\tilde{q}(\sigma+2\vartheta) + 8n\alpha[-R^{\mu\nu}g_{\sigma\rho}$$
$$+ R^{\mu\nu}_{\rho\sigma}]_{;\nu}V_\mu\nabla^\rho\nabla^\sigma G^n + 8n\alpha[(V_\sigma R^\nu_\rho - V^\nu R_{\rho\sigma} + R^\mu_\sigma\delta^\nu_\rho)_{,\nu} + \Gamma^\mu_{\nu\gamma}(R^\nu_\rho\delta^\gamma_\sigma + R^\gamma_\sigma$$
$$\times\delta^\nu_\rho)V_\mu + \Gamma^\nu_{\nu\gamma}(V_\sigma R^\gamma_\rho + V^\gamma R_{\rho\sigma})]\nabla^\rho\nabla^\sigma G^n = 0, \tag{44}$$

representing the effects of GB terms in the evolution of energy density and pressure.

4 Some self-gravitating fluid models

In this section, we study governing equations for some specific self-gravitating fluid models under the influence of $f(G)$ gravity and give their comparison with GR [12]. The set of governing equations is given by eqs. (25), (27), (28), (36), (41), (43) and (44).

4.1 Geodesic non-dissipative isotropic fluids

If the fluid particles are moving along geodesics, then $a^\mu=0$ and the fluid is geodesic for which $g_{00}=\text{const.}$ Thus for locally isotropic (radial and tangential pressures are same, *i.e.*, $\Pi=0$), non-dissipative (vanishing heat flux and radiation density, *i.e.*, $q=\epsilon=0$) and geodesic fluids, we obtain the governing equations under the effects of $f(G)$ gravity as follows:

$$\vartheta_{;\mu}U^\mu+\frac{1}{3}\vartheta^2+\frac{2}{3}\sigma^2=-\frac{1}{2}(\rho+3p_r), \tag{45}$$

$$U^\beta U_\mu R^\mu_{\alpha\beta\nu}h^\alpha_\gamma h^\nu_\delta V^\gamma V^\delta=\varepsilon+2n\alpha\bigg[-R^\phi_{\rho\sigma\mu}V^\mu V_\phi-R_{\rho\mu}V^\mu V_\sigma+g_{\sigma\rho}R_{\alpha\mu}V^\alpha \times V^\mu-R_{\alpha\sigma}V_\rho V^\alpha+\frac{1}{2}RV_\rho V_\sigma\bigg]\nabla^\delta\nabla^\sigma G^n, \tag{46}$$

$$h^\alpha_\beta\left(\sigma^{\beta\gamma}_{;\gamma}-\frac{2}{3}\vartheta^{;\beta}\right)+\sigma^{\alpha\beta}a_\beta=2n\alpha\bigg[-U_\mu h^{\alpha\beta}R^\mu_{\rho\sigma\beta}+g_{\sigma\rho}U_\mu h^{\alpha\beta}R^\mu_\beta+U_\phi h^\alpha_\rho R^\phi_\sigma +\frac{1}{8}Rh^\alpha_\rho U_\sigma\bigg]\nabla^\rho\nabla^\sigma G^n, \tag{47}$$

$$\vartheta\left(\frac{1}{3}(\rho+p_r)h_{\mu\nu}+E_{\mu\nu}\right)+U^\beta(E_{\alpha\gamma})_{;\beta}h^\alpha_\mu h^\gamma_\nu+(\varepsilon+\rho+p_r)\sigma_{\mu\nu}+\frac{4}{3}\rho_{,\beta}U^\beta \times h_{\mu\nu}-4n\alpha h^\alpha_\mu h^\gamma_\nu U^\beta[R_{\gamma\rho[\alpha\sigma;\beta]}+g_{\sigma\rho}R_{\gamma[\alpha;\beta]}]\nabla^\rho\nabla^\sigma G^n-n\alpha[h^\alpha_\mu h_{\nu\sigma}R_{\rho[\alpha;\beta]} \times U^\beta-h_{\mu\nu}U^\beta R_{\rho\sigma;\beta}+h_{\mu\rho}h^\gamma_\nu U^\beta R_{\gamma\alpha;\beta}-h^\alpha_\mu h^\gamma_\nu U_\rho R_{\gamma\sigma;\alpha}]\nabla^\rho\nabla^\sigma G^n=0, \tag{48}$$

$$\begin{aligned}&\frac{1}{3}h^\beta_\mu(-2\rho)_{,\beta}+4n\alpha U^\gamma U^\beta\left[R_{\gamma\rho\alpha\sigma;\beta}-R_{\gamma\rho\beta\sigma;\alpha}-2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta}\right.\\&\left.+g_{\sigma\rho}R_{\gamma\beta;\alpha}\right]\nabla^\rho\nabla^\sigma G^n+4n\alpha U^\beta[2U_{[\sigma}R_{\alpha]\rho;\beta}+g_{\rho[\sigma}U_{\alpha]}R_{;\beta}]\nabla^\rho\nabla^\sigma\\&\times G^n+4n\alpha U^\gamma U^\beta[R_{\gamma\rho\alpha\sigma}-g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^\gamma\nabla^\sigma(G^n)_{,\beta}+U^\beta\bigg[U_{[\sigma}R_{\alpha]\rho}+\frac{1}{2}n\alpha\\&\times RU_{[\alpha}g_{\sigma]\rho}\bigg]\nabla^\rho\nabla^\sigma(G^n)_{,\beta}+\frac{1}{2}\alpha(n-1)U_\alpha U^\beta(G^n)_{,\beta}-4\bigg[R_{\rho\sigma;\alpha}+R_{\rho\beta;\alpha}U_\sigma\\&\times U^\beta+U_\gamma U_\rho R_{\gamma\sigma;\alpha}-\frac{1}{2}n\alpha h_{\sigma\rho}R_{;\alpha}\bigg]\nabla^\gamma\nabla^\sigma G^n-4n\alpha\bigg[U^\gamma U^\beta R_{\gamma\rho\beta\sigma}+U^\beta U_\sigma\\&\times R_{\rho\beta}+R_{\rho\sigma}-U^\gamma U^\beta g_{\sigma\rho}R_{\gamma\beta}+U^\gamma U_\rho R_{\gamma\sigma}-\frac{1}{2}h_{\sigma\rho}R\bigg]\nabla^\gamma\nabla^\sigma(G^n)_{,\alpha}+\frac{1}{2}\alpha\\&\times(n-1)(G^n)_{,\alpha}-\frac{4}{3}n\alpha\left[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\right]_{,\beta}\nabla^\rho\nabla^\sigma G^n-8n\\&\times\alpha\left[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\right]\nabla^\rho\nabla^\sigma(G^n)_{,\beta}+\frac{1}{3}\alpha(n-1)\\&\times(G^n)_{,\beta}=0,\end{aligned} \tag{49}$$

$$\begin{aligned}&\rho_{,\mu}U^\mu+\vartheta(\rho+p_r)+8n\alpha[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}U_\mu\nabla^\rho\nabla^\sigma G^n+8n\alpha[(U_\sigma R^\nu_\rho\\&-U^\nu R_{\rho\sigma})_{,\nu}+\Gamma^\mu_{\nu\gamma}(R^\nu_\rho\delta^\gamma_\sigma-R_{\rho\sigma}g^{\gamma\nu})U_\mu+\Gamma^\nu_{\nu\gamma}(U_\sigma R^\gamma_\rho-U^\gamma R_{\rho\sigma})]\nabla^\rho\nabla^\sigma\\&\times G^n=0,\end{aligned} \tag{50}$$

$$\begin{aligned}&(p_r)_{,\mu}V^\mu+8n\alpha[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}V_\mu\nabla^\rho\nabla^\sigma G^n+8n\alpha[(V_\sigma R^\nu_\rho-V^\nu R_{\rho\sigma}\\&+R^\mu_\sigma\delta^\nu_\rho)_{,\nu}+\Gamma^\mu_{\nu\gamma}(R^\nu_\rho\delta^\gamma_\sigma+R^\gamma_\sigma\delta^\nu_\rho)V_\mu+\Gamma^\nu_{\nu\gamma}(V_\sigma R^\gamma_\rho+V^\gamma R_{\rho\sigma})]\nabla^\rho\nabla^\sigma\\&\times G^n=0.\end{aligned} \tag{51}$$

This represents dust fluid model (pressureless fluid) in GR while the conformally flatness condition ($\varepsilon = 0$) implies shear free condition ($\sigma = 0$) and vice versa [12]. In the scenario of $f(G)$ gravity, the equation of motion (51) depends upon the gradient of pressure and extra curvature terms (GB terms). The choice $f(G) = \text{const.}$ (*i.e.*, $f_G = 0$) corresponds to the cosmological constant and the standard results can be imitated. For this type of fluid model, pressure gradient vanishes in eq. (51) which consequently gives $p_r = \text{const}$. In this case, matter particles will exert equal pressure at each point of evolving relativistic spherically distributed self-gravitating fluid. Thus geodesic fluids with isotropy and non-dissipation exert constant (non-zero) pressure (no dust) in $f(G)$ cosmology. For constant to be zero, this yields dust. Equations (46) and (48) represent that the conformally flatness condition and shear free condition rely on GB terms. Thus conformally flatness condition does not imply shear free condition. Equation (49) indicates that energy density inhomogeneity depends upon GB terms as well as the Weyl tensor. If the Weyl tensor vanishes, then GB terms are totally responsible for energy density inhomogeneity.

4.2 Geodesic non-dissipative anisotropic fluids

Here we take geodesic fluid with locally anisotropy (different radial and tangential pressures, *i.e.*, $\Pi \neq 0$) and non-dissipation for which the governing equations are as follows:

$$\begin{aligned}
&U^\beta U_\mu R^\mu_{\alpha\beta\nu} h^\alpha_\gamma h^\nu_\delta V^\gamma V^\delta = \varepsilon - \frac{1}{2}\Pi + 2n\alpha\bigg[- R^\phi_{\rho\sigma\mu} V^\mu V_\phi - R_{\rho\mu} V^\mu V_\sigma + g_{\sigma\rho} \\
&\times R_{\alpha\mu} V^\alpha V^\mu - R_{\alpha\sigma} V_\rho V^\alpha + \frac{1}{2} R V_\rho V_\sigma \bigg] \nabla^\delta \nabla^\sigma G^n,
\end{aligned} \tag{52}$$

$$\begin{aligned}
&\vartheta\left(\frac{1}{3}(\rho + p_r) h_{\mu\nu} + E_{\mu\nu}\right) + U^\beta (E_{\alpha\gamma} - \Pi_{\alpha\gamma})_{;\beta} h^\alpha_\mu h^\gamma_\nu + (\varepsilon + \rho + p_r)\sigma_{\mu\nu} \\
&+ \frac{4}{3}\rho_{,\beta} U^\beta - \Pi_{\beta\nu}\left(\sigma^\beta_\mu - \frac{1}{3}\vartheta h^\beta_\mu\right) h_{\mu\nu} - 4n\alpha h^\alpha_\mu h^\gamma_\nu U^\beta [R_{\gamma\rho[\alpha\sigma;\beta]} + g_{\sigma\rho} R_{\gamma[\alpha;\beta]}] \\
&\times \nabla^\rho \nabla^\sigma G^n - n\alpha [h^\alpha_\mu h_{\nu\sigma} R_{\rho[\alpha;\beta]} U^\beta - h_{\mu\nu} U^\beta R_{\rho\sigma;\beta} + h_{\mu\rho} \\
&\times h^\gamma_\nu U^\beta R_{\gamma\alpha;\beta} - h^\alpha_\mu h^\gamma_\nu U_\rho R_{\gamma\sigma;\alpha}] \nabla^\rho \nabla^\sigma G^n = 0,
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\frac{-2}{3} h^\beta_\mu (\rho)_{,\beta} + \frac{1}{3} h^\beta_\mu (\Pi)_{,\beta} + 4n\alpha U^\gamma U^\beta \left[R_{\gamma\rho\alpha\sigma;\beta} - R_{\gamma\rho\beta\sigma;\alpha} - 2(R_{\gamma[\alpha} g_{\sigma]\rho})_{;\beta}\right. \\
&\left.+ g_{\sigma\rho} R_{\gamma\beta;\alpha}\right] \nabla^\rho \nabla^\sigma G^n + 4n\alpha U^\beta [2U_{[\sigma} R_{\alpha]\rho;\beta} + g_{\rho[\sigma} U_{\alpha]} R_{;\beta}] \nabla^\rho \nabla^\sigma \\
&\times G^n + 4n\alpha U^\gamma U^\beta [R_{\gamma\rho\alpha\sigma} - g_{\rho[\alpha} R_{\sigma]\gamma}] \nabla^\gamma \nabla^\sigma (G^n)_{,\beta} + U^\beta \bigg[U_{[\sigma} R_{\alpha]\rho} + \frac{1}{2} n\alpha \\
&\times R U_{[\alpha} g_{\sigma]\rho}\bigg] \nabla^\rho \nabla^\sigma (G^n)_{,\beta} + \frac{1}{2}\alpha(n-1) U_\alpha U^\beta (G^n)_{,\beta} - 4\bigg[R_{\rho\sigma;\alpha} + R_{\rho\beta;\alpha} U_\sigma \\
&\times U^\beta + U_\gamma U_\rho R_{\gamma\sigma;\alpha} - \frac{1}{2} n\alpha h_{\sigma\rho} R_{;\alpha}\bigg] \nabla^\gamma \nabla^\sigma G^n - 4n\alpha \bigg[U^\gamma U^\beta R_{\gamma\rho\beta\sigma} + U^\beta U_\sigma \\
&\times R_{\rho\beta} + R_{\rho\sigma} - U^\gamma U^\beta g_{\sigma\rho} R_{\gamma\beta} + U^\gamma U_\rho R_{\gamma\sigma} - \frac{1}{2} h_{\sigma\rho} R\bigg] \nabla^\gamma \nabla^\sigma (G^n)_{,\alpha} + \frac{1}{2}\alpha \\
&\times (n-1)(G^n)_{,\alpha} - \frac{4}{3} n\alpha \left[R_{\rho\sigma} + R g_{\sigma\rho} - \frac{1}{2} R(4g_{\sigma\rho} - \delta_{\rho\sigma})\right]_{,\beta} \nabla^\rho \nabla^\sigma G^n - 8n \\
&\times \alpha \left[R_{\rho\sigma} + R g_{\sigma\rho} - \frac{1}{2} R(4g_{\sigma\rho} - \delta_{\rho\sigma})\right] \nabla^\rho \nabla^\sigma (G^n)_{,\beta} + \frac{1}{3}\alpha(n-1) \\
&\times (G^n)_{,\beta} = 0,
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\rho_{,\mu} U^\mu + \vartheta(\rho + p_r) - \frac{2}{3}(\sigma + \vartheta)\Pi + 8n\alpha [R^{\mu\nu}_{\rho\sigma} - R^{\mu\nu} g_{\sigma\rho}]_{;\nu} U_\mu \nabla^\rho \nabla^\sigma G^n \\
&+ 8n\alpha [(U_\sigma R^\nu_\rho - U^\nu R_{\rho\sigma})_{,\nu} + \Gamma^\mu_{\nu\gamma} (R^\nu_\rho \delta^\gamma_\sigma - R_{\rho\sigma} g^{\gamma\nu}) U_\mu + \Gamma^\nu_{\nu\gamma} (U_\sigma R^\gamma_\rho - U^\gamma \\
&\times R_{\rho\sigma})] \nabla^\rho \nabla^\sigma G^n = 0,
\end{aligned} \tag{55}$$

$$\begin{aligned}
&(p_r)_{,\mu} V^\mu + 2\Pi \frac{C'}{CB} + 8n\alpha [R^{\mu\nu}_{\rho\sigma} - R^{\mu\nu} g_{\sigma\rho}]_{;\nu} V_\mu \nabla^\rho \nabla^\sigma G^n + 8n\alpha [(V_\sigma R^\nu_\rho \\
&- V^\nu R_{\rho\sigma} + R^\mu_\sigma \delta^\nu_\rho)_{,\nu} + \Gamma^\mu_{\nu\gamma} (R^\nu_\rho \delta^\gamma_\sigma + R^\gamma_\sigma \delta^\nu_\rho) V_\mu + \Gamma^\nu_{\nu\gamma} (V_\sigma R^\gamma_\rho + V^\gamma R_{\rho\sigma})] \\
&\times \nabla^\rho \nabla^\sigma G^n = 0.
\end{aligned} \tag{56}$$

The geometry for this type of fluid revolves around a physical quantity of matter, *i.e.*, pressure anisotropy. In GR, pressure gradient, shear free and conformally flatness conditions are linked with pressure anisotropy while density inhomogeneity depends on the Weyl tensor as well as pressure anisotropy. The $f(G)$ theory affects these relations by the inclusion of GB terms. In the absence of the Weyl tensor, eq. (53) shows that density inhomogeneity is caused by pressure anisotropy as well as GB terms.

4.3 Non-geodesic non-dissipative isotropic fluids

In this case, we obtain the following set of governing equations:

$$\vartheta_{;\mu}U^{\mu}+\frac{1}{3}\vartheta^2+\frac{2}{3}\sigma^2-a^{\mu}_{;\mu}=-\frac{1}{2}(\rho+3p_r), \tag{57}$$

$$U^{\beta}U_{\mu}R^{\mu}_{\alpha\beta\nu}h^{\alpha}_{\gamma}h^{\nu}_{\delta}V^{\gamma}V^{\delta}=\varepsilon+2n\alpha\bigg[-R^{\phi}_{\rho\sigma\mu}V^{\mu}V_{\phi}-R_{\rho\mu}V^{\mu}V_{\sigma}+g_{\sigma\rho}R_{\alpha\mu}V^{\alpha}\times V^{\mu}-R_{\alpha\sigma}V_{\rho}V^{\alpha}+\frac{1}{2}RV_{\rho}V_{\sigma}\bigg]\nabla^{\rho}\nabla^{\sigma}G^n, \tag{58}$$

$$h^{\alpha}_{\beta}\left(\sigma^{\beta\gamma}_{;\gamma}-\frac{2}{3}\vartheta^{;\alpha}\right)+\sigma^{\alpha\beta}a_{\beta}=2n\alpha\bigg[-U_{\mu}h^{\alpha\beta}R^{\mu}_{\rho\sigma\beta}+g_{\sigma\rho}U_{\mu}h^{\alpha\beta}R^{\mu}_{\beta}+U_{\phi}h^{\alpha}_{\rho}R^{\phi}_{\sigma}+\frac{1}{8}Rh^{\alpha}_{\rho}U_{\sigma}\bigg]\nabla^{\rho}\nabla^{\sigma}G^n, \tag{59}$$

$$\vartheta\left(E_{\mu\nu}+\frac{1}{3}(\rho+p_r)h_{\mu\nu}\right)+U^{\beta}(E_{\alpha\gamma})_{;\beta}h^{\alpha}_{\mu}h^{\gamma}_{\nu}+(\varepsilon+\rho+p_r)\sigma_{\mu\nu}+\frac{4}{3}\rho_{;\beta}U^{\beta}\times h_{\mu\nu}-4n\alpha h^{\alpha}_{\mu}h^{\gamma}_{\nu}U^{\beta}[R_{\gamma\rho[\alpha\sigma;\beta]}+g_{\sigma\rho}R_{\gamma[\alpha;\beta]}]\nabla^{\rho}\nabla^{\sigma}G^n-n\alpha[h^{\alpha}_{\mu}h_{\nu\sigma}U^{\beta}R_{\rho[\alpha;\beta]}-h_{\mu\nu}U^{\beta}R_{\rho\sigma;\beta}+h_{\mu\rho}h^{\gamma}_{\nu}U^{\beta}R_{\gamma\alpha;\beta}-h^{\alpha}_{\mu}h^{\gamma}_{\nu}U_{\rho}R_{\gamma\sigma;\alpha}]\nabla^{\rho}\nabla^{\sigma}G^n=0, \tag{60}$$

$$\begin{aligned}&\frac{1}{3}h^{\beta}_{\alpha}(\varepsilon)_{,\beta}+\frac{2}{3}h^{\beta}_{\alpha}\rho_{,\beta}+4n\alpha U^{\gamma}U^{\beta}\left[R_{\gamma\rho\alpha\sigma;\beta}-R_{\gamma\rho\beta\sigma;\alpha}+g_{\sigma\rho}R_{\gamma\beta;\alpha}\right.\\&\left.-2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta}\right]\nabla^{\rho}\nabla^{\sigma}G^n+4n\alpha U^{\beta}[2U_{[\sigma}R_{\alpha]\rho;\beta}+g_{\rho[\sigma}U_{\alpha]}R_{,\beta}]\nabla^{\rho}\nabla^{\sigma}G^n\\&+4n\alpha U^{\gamma}U^{\beta}[R_{\gamma\rho\alpha\sigma}-g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^{\rho}\nabla^{\sigma}(G^n)_{,\beta}+n\alpha U^{\beta}\bigg[U_{[\sigma}R_{\alpha]\rho}+\frac{1}{2}RU_{[\alpha}\\&\times g_{\sigma]\rho}\bigg]\nabla^{\rho}\nabla^{\sigma}(G^n)_{,\beta}+\frac{1}{2}\alpha(n-1)U_{\alpha}U^{\beta}(G^n)_{,\beta}-4n\alpha\bigg[U_{\sigma}U^{\beta}R_{\rho\beta;\alpha}+U_{\gamma}\\&\times U_{\rho}R_{\gamma\sigma;\alpha}+R_{\rho\sigma;\alpha}-\frac{1}{2}h_{\sigma\rho}R_{;\alpha}\bigg]\nabla^{\gamma}\nabla^{\sigma}G^n-4n\alpha\bigg[U^{\gamma}U^{\beta}R_{\gamma\rho\beta\sigma}+U^{\beta}U_{\sigma}R_{\rho\beta}\\&+R_{\rho\sigma}-U^{\gamma}U^{\beta}g_{\sigma\rho}R_{\gamma\beta}+U^{\gamma}U_{\rho}R_{\gamma\sigma}-\frac{1}{2}h_{\sigma\rho}R\bigg]\nabla^{\rho}\nabla^{\sigma}(G^n)_{,\alpha}+\frac{1}{2}\alpha(n-1)\\&\times(G^n)_{,\alpha}-\frac{4}{3}n\alpha\left[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\right]_{,\beta}\nabla^{\rho}\nabla^{\sigma}G^n-8n\alpha\bigg[R_{\rho\sigma}\\&+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\bigg]\nabla^{\rho}\nabla^{\sigma}(G^n)_{,\beta}+\frac{1}{3}\alpha(n-1)(G^n)_{,\beta}=0,\end{aligned} \tag{61}$$

$$\rho_{,\mu}U^{\mu}+\vartheta(\rho+p_r)+8n\alpha[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}U_{\mu}\nabla^{\rho}\nabla^{\sigma}G^n+8n\alpha[(U_{\sigma}R^{\nu}_{\rho}-U^{\nu}R_{\rho\sigma})_{,\nu}+\Gamma^{\mu}_{\nu\gamma}(R^{\nu}_{\rho}\delta^{\gamma}_{\sigma}-R_{\rho\sigma}g^{\gamma\nu})U_{\mu}+\Gamma^{\nu}_{\nu\gamma}(U_{\sigma}R^{\gamma}_{\rho}-U^{\gamma}R_{\rho\sigma})]\times\nabla^{\rho}\nabla^{\sigma}G^n=0, \tag{62}$$

$$(p_r)_{,\mu}V^{\mu}+a(\rho+p_r)+8n\alpha[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}V_{\mu}\nabla^{\rho}\nabla^{\sigma}G^n+8n\alpha[(V_{\sigma}R^{\nu}_{\rho}-V^{\nu}R_{\rho\sigma}+R^{\mu}_{\sigma}\delta^{\nu}_{\rho})_{,\nu}+\Gamma^{\mu}_{\nu\gamma}(R^{\nu}_{\rho}\delta^{\gamma}_{\sigma}+R^{\gamma}_{\sigma}\delta^{\nu}_{\rho})V_{\mu}+\Gamma^{\nu}_{\nu\gamma}(V_{\sigma}R^{\gamma}_{\rho}+V^{\gamma}R_{\rho\sigma})]\times\nabla^{\rho}\nabla^{\sigma}G^n=0. \tag{63}$$

In GR, shear free condition provides expansion free condition for such fluid while conformal flatness condition implies irregularities (inhomogeneity) in energy density and vice versa. For modified Gauss-Bonnet gravity, we see from eq. (61) that conformal flatness condition depends upon inhomogeneity of energy density as well as GB terms. If the fluid is conformally flat, the dependence of energy density inhomogeneity depends on GB terms, hence GB terms are responsible for density inhomogeneity. Equation (59) indicates that shear free condition does not imply expansion free fluid due to GB terms.

4.4 Non-geodesic non-dissipative anisotropic fluids

Here we take non-dissipative ($q=\epsilon=0$) and anisotropic ($\Pi\neq 0$) fluids. For this case, we have

$$\begin{aligned}
&\frac{1}{3}h^{\beta}_{\alpha}(\varepsilon-\Pi)_{,\beta}+\frac{2}{3}h^{\beta}_{\alpha}\rho_{,\beta}+4n\alpha U^{\gamma}U^{\beta}\left[R_{\gamma\rho\alpha\sigma;\beta}-R_{\gamma\rho\beta\sigma;\alpha}+g_{\sigma\rho}R_{\gamma\beta;\alpha}\right.\\
&\left.-2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta}\right]\nabla^{\rho}\nabla^{\sigma}G^{n}+4n\alpha U^{\beta}[2U_{[\sigma}R_{\alpha]\rho;\beta}+g_{\rho[\sigma}U_{\alpha]}R_{,\beta}]\nabla^{\rho}\nabla^{\sigma}G^{n}+4\\
&\times n\alpha U^{\gamma}U^{\beta}[R_{\gamma\rho\alpha\sigma}-g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta}+n\alpha U^{\beta}\left[U_{[\sigma}R_{\alpha]\rho}+\frac{1}{2}RU_{[\alpha}g_{\sigma]\rho}\right]\\
&\times\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta}+\frac{1}{2}\alpha(n-1)U_{\alpha}U^{\beta}(G^{n})_{,\beta}-4n\alpha\Big[U_{\sigma}U^{\beta}R_{\rho\beta;\alpha}+R_{\rho\sigma;\alpha}+U_{\gamma}\\
&\times U_{\rho}R_{\gamma\sigma;\alpha}-\frac{1}{2}h_{\sigma\rho}R_{;\alpha}\Big]\nabla^{\gamma}\nabla^{\sigma}G^{n}-4n\alpha\Big[U^{\gamma}U^{\beta}R_{\gamma\rho\beta\sigma}+U^{\beta}U_{\sigma}R_{\rho\beta}+R_{\rho\sigma}\\
&-U^{\gamma}U^{\beta}g_{\sigma\rho}R_{\gamma\beta}+U^{\gamma}U_{\rho}R_{\gamma\sigma}-\frac{1}{2}h_{\sigma\rho}R\Big]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\alpha}+\frac{1}{2}\alpha(n-1)(G^{n})_{,\alpha}-\frac{4}{3}\\
&\times n\alpha\left[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\right]_{,\beta}\nabla^{\rho}\nabla^{\sigma}G^{n}-8n\alpha\Big[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R\\
&\times(4g_{\sigma\rho}-\delta_{\rho\sigma})\Big]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta}+\frac{1}{3}\alpha(n-1)(G^{n})_{,\beta}=0.
\end{aligned}\tag{64}$$

In GR, this relates energy density inhomogeneity, the Weyl tensor and anisotropy. Here, eq. (64) clearly shows that energy density inhomogeneity is linked with the Weyl tensor, anisotropy and GB terms.

4.5 Non-geodesic dissipative anisotropic fluids

This is more general case with the presence of dissipation $q\neq 0$ ($\epsilon=0$ for simplicity) and anisotropy ($\Pi\neq 0$). From eq. (41), we have

$$\begin{aligned}
&h^{\alpha}_{\mu}E^{\lambda}_{\alpha;\lambda}+a^{\lambda}E_{\mu\lambda}+q_{\gamma}\left(\sigma^{\gamma}_{\mu}+\frac{1}{3}\vartheta h^{\gamma}_{\mu}\right)+\frac{1}{3}h^{\beta}_{\mu}(-2\rho+3p_{r}+\Pi)_{,\beta}+a_{\mu}(-\rho+p_{r})\\
&-a^{\gamma}\Pi_{\mu\gamma}+U^{\beta}h^{\alpha}_{\mu}\tilde{q}_{\alpha;\beta}+4n\alpha U^{\gamma}U^{\beta}\left[R_{\gamma\rho\alpha\sigma;\beta}-R_{\gamma\rho\beta\sigma;\alpha}-2(R_{\gamma[\alpha}g_{\sigma]\rho})_{;\beta}+g_{\sigma\rho}\right.\\
&\left.\times\ R_{\gamma\beta;\alpha}\right]\nabla^{\rho}\nabla^{\sigma}G^{n}+4n\alpha U^{\beta}[2U_{[\sigma}R_{\alpha]\rho;\beta}+g_{\rho[\sigma}U_{\alpha]}R_{,\beta}]\nabla^{\rho}\nabla^{\sigma}G^{n}+4n\alpha U^{\gamma}U^{\beta}\\
&\times[R_{\gamma\rho\alpha\sigma}-g_{\rho[\alpha}R_{\sigma]\gamma}]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\beta}+n\alpha U^{\beta}\left[U_{[\sigma}R_{\alpha]\rho}+\frac{1}{2}RU_{[\alpha}g_{\sigma]\rho}\right]\nabla^{\rho}\nabla^{\sigma}\\
&(G^{n})_{,\beta}+\frac{1}{2}\alpha(n-1)U_{\alpha}U^{\beta}(G^{n})_{,\alpha}-4n\alpha\Big[U_{\sigma}U^{\beta}R_{\rho\beta;\alpha}+R_{\rho\sigma;\alpha}+U_{\gamma}U_{\rho}R_{\gamma\sigma;\alpha}\\
&-\frac{1}{2}h_{\sigma\rho}R_{;\alpha}\Big]\nabla^{\rho}\nabla^{\sigma}G^{n}-4n\alpha\Big[U^{\gamma}U^{\beta}R_{\gamma\rho\beta\sigma}+U^{\beta}U_{\sigma}R_{\rho\beta}+R_{\rho\sigma}-U^{\gamma}U^{\beta}g_{\sigma\rho}\\
&\times R_{\gamma\beta}+U^{\gamma}U_{\rho}R_{\gamma\sigma}-\frac{1}{2}h_{\sigma\rho}R\Big]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\alpha}+\frac{1}{2}\alpha n-1(G^{n})_{,\alpha}-\frac{4}{3}n\alpha\Big[R_{\rho\sigma}\\
&+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}-\delta_{\rho\sigma})\Big]_{,\beta}\nabla^{\rho}\nabla^{\sigma}G^{n}-8n\alpha\Big[R_{\rho\sigma}+Rg_{\sigma\rho}-\frac{1}{2}R(4g_{\sigma\rho}\\
&-\delta_{\rho\sigma})\Big]\nabla^{\rho}\nabla^{\sigma}(G^{n})_{,\alpha}+\frac{1}{3}\alpha(n-1)(G^{n})_{,\beta}=0.
\end{aligned}\tag{65}$$

This equation represents the link of dark source (GB) terms with the Weyl tensor and other dynamical quantities. This represents that the tidal force that an object feels while moving along a geodesic is affected by dynamical quantities as well as GB terms. This also shows that the inhomogeneity of energy density as well as dark source terms do not disturb due to the absence of shear and expansion parameters. This equation indicates that the inhomogeneity of energy density does not depend upon dark source terms so its homogeneity does not alter the inhomogeneity of energy density.

The transport equation [30] is used to discuss the flow of heat inside the density inhomogeneity given by

$$\tau h^{\mu}_{\nu}U^{\alpha}q^{\nu}_{,\alpha}+q^{\mu}-K\left(h^{\mu\nu}(T_{,\nu}+Ta_{\nu})-\frac{1}{2}q^{\mu}\left(\frac{\tau U^{\nu}}{KT^{2}}\right)_{;\nu}\right)=0, \tag{66}$$

where K, T, and τ are the thermal conductivity, temperature and time relaxation. Using eq. (44) in (66), we have

$$q=\frac{\tau[(p_r)_{,\alpha}V^{\alpha}+(\rho+p_r)a+2\Pi\frac{C'}{BC}+GB]-K[T_{\alpha}U^{\alpha}+aT]}{1+\frac{1}{2}\tau[\frac{1}{3}(2\sigma-5\vartheta)+\frac{1}{\tau}U^{\alpha}\tau_{,\alpha}-\frac{1}{K}U^{\alpha}K_{,\alpha}-\frac{2}{T}U^{\alpha}T_{,\alpha}]}\,, \tag{67}$$

where

$$\begin{aligned}GB&=8n\alpha[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}]_{;\nu}V_{\mu}\nabla^{\rho}\nabla^{\sigma}G^{n}+8n\alpha[(V_{\sigma}R^{\nu}_{\rho}-V^{\nu}R_{\rho\sigma}+R^{\mu}_{\sigma}\\&\times\delta^{\nu}_{\rho})_{,\nu}+\Gamma^{\mu}_{\nu\gamma}(R^{\nu}_{\rho}\delta^{\gamma}_{\sigma}+R^{\gamma}_{\sigma}\delta^{\nu}_{\rho})V_{\mu}+\Gamma^{\nu}_{\nu\gamma}(V_{\sigma}R^{\gamma}_{\rho}+V^{\gamma}R_{\rho\sigma})]\nabla^{\rho}\nabla^{\sigma}G^{n}.\end{aligned}$$

Inserting eq. (67) in (65), we obtain a relation between density inhomogeneity and thermodynamics variables.

5 Conclusions

In this paper, we have generated a set of governing equations for the dynamics of evolving fluid bounded by spherically symmetric surface under the influence of $f(G)$ gravity. These equations express a connection between dynamical variables and dark source (Gauss-Bonnet curvature) terms as well as their roles for evolving self-gravitating fluids. We have discussed different fluid models for various dynamical conditions and found that the role of dark source terms is crucial for gravitational effects during the evolution of self-gravitating fluids. In the following, we summarize our results and compare with GR [12].

For a geodesic fluid with locally isotropic and non-dissipative, GR gives dust where density inhomogeneity depends upon the Weyl tensor and shear free condition implies conformal flatness. In $f(G)$ gravity, such relations are not imitated due to extra curvature terms from dark source. Fluid is not dust and the conformally flatness condition does not imply shear free condition, while density inhomogeneity is controlled by the Weyl tensor as well as GB terms. If we include pressure anisotropy in the fluid, all results revolve around this physical variable. For non-geodesic fluids with local anisotropy and non-dissipation, GR gives the combination of energy density inhomogeneity, the Weyl tensor and anisotropy while $f(G)$ theory affects this combination by including GB terms.

For non-geodesic fluids with local isotropy and non-dissipation in GR, shear free and expansion free conditions are linked with each other and density inhomogeneity is attained from conformally flatness condition. In our case, these results are affected by GB terms and in the absence of conformally flatness condition, density inhomogeneity is assured by GB terms. In the more general case, anisotropic fluid with dissipation, GR describes the evolution of self-gravitating fluid in terms of dynamical variables. In the present case, there is one extra dependence of the evolution of self-gravitating fluid on GB terms along with dynamical quantities. We have obtained a relation between density inhomogeneity and thermodynamics variables plus GB terms through transport equation.

We have found that the evolution of self-gravitating fluid not only depends upon dynamical variables but also on GB terms. The choice $f(G)=$ const. (*i.e.*, $f_G=0$) or $n=0$ corresponds to the cosmological constant and the standard results can be recovered. For this choice of the model, our results are consistent with standard results [12] otherwise GB terms affect the results which deviate from GR. It is worthwhile to mention here that our constructed self-gravitating fluid models are more general as compared to GR.

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RESEARCH ARTICLE

Structure scalars and evolution equations in $f(G)$ cosmology

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Abstract In this paper, we study the dynamics of self-gravitating fluid using structure scalars for spherical geometry in the context of $f(G)$ cosmology. We construct structure scalars through orthogonal splitting of the Riemann tensor and deduce a complete set of equations governing the evolution of dissipative anisotropic fluid in terms of these scalars. We explore different causes of density inhomogeneity which turns out to be a necessary condition for viable models. It is explicitly shown that anisotropic inhomogeneous static spherically symmetric solutions can be expressed in terms of these scalar functions.

Keywords $f(G)$ gravity · Dissipative fluid · Structure scalars

1 Introduction

Gravitational collapse has a key role in the formation of universe structure. The initial smooth distribution of matter collapses with the passage of time and forms pockets of higher density leading to hierarchy of compressed celestial structures like galaxies, cluster of galaxies, stars, black holes and planets. Observations from redshift surveys such as 2 dF (two degree field) galaxy redshift survey (with 3.9 m equatorially mounted telescope) and Sloan digital sky survey (with 2.5 m wide angle optical telescope)

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[1,2] describe these objects as self-gravitating systems. These are characterized by kinematical quantities (physical and geometrical dynamical variables) like expansion scalar, the Weyl tensor, density inhomogeneity, anisotropy, dissipation (heat flux) as well as shear tensor.

Gravitational collapse initiates in highly inhomogeneous state which can be described through the relations of energy density inhomogeneity with the Weyl tensor or other kinematical quantities [3–5]. The Weyl tensor illustrates tidal force which makes fluid more inhomogeneous during the evolution. Herrera [6] discussed stability of spherically distributed self-gravitating fluid and found physical relevance of the Weyl tensor with such fluids. The pressure anisotropy occurs in phenomena like mixture of two fluids, phase transition, slow rotation and solid core. The significant roles of local anisotropy of pressure [7–9] as well as shear tensor [10–13] in the evolution of self-gravitating fluid models have largely been studied. Since gravitational collapse is a highly dissipative process, so the relevance of dissipation in self-gravitating objects has attracted many people [14–16]. Herrera et al. [17] developed structure scalars of dissipative anisotropic spherically as well as axially symmetric [18] self-gravitating systems by using dynamical quantities.

Stimulating observations from type Ia supernovae, large scale structures and cosmic microwave background radiations indicate the cosmic expansion at an accelerating rate. The phenomenon of cosmic expansion is linked with dark energy (DE) which is a mysterious repulsive force possessing highly negative pressure. To study the mystical effects of DE, the Einstein–Hilbert action has been generalized leading to various alternative (modified) theories of gravitation as $f(R)$, $f(R,T)$, where R is the Ricci scalar and T is the trace of energy-momentum tensor, Brans–Dicke and Gauss–Bonnet etc. The modified Lagrangian significantly helps to analyze the mystical effects of DE. These theories are consistent with (general relativity) GR in the regime of weak field but may vary in strong field. Gravitational collapse is categorized as the phenomenon of strong field, hence modified theories are the best candidates to explain this phenomenon.

The $f(G)$ or modified Gauss–Bonnet theory of gravity is obtained by adding an arbitrary function ($f(G)$) of the Gauss–Bonnet quadratic invariant G in the action [19]. This theory efficiently elucidates accelerated expansion of the universe, transition from deceleration to accelerating phase of the universe and also satisfies solar system tests. Furthermore, it is useful in explaining thermodynamics [20,21] and protects all possible four types of future singularities [22]. Myrzakulov et al. [23] explored this theory to study DE as well as inflationary era. We have studied energy conditions [24], wormhole solutions [25,26], built-in inflation [27], Noether symmetries [28] as well as spherical solution with conformal symmetry [29] in this theory.

The description of collapsing fluids in stellar interiors remains a debateable issue in GR as well as modified theories. For this purpose, a set of scalars have been introduced in [30,31] that have individual physical meaning and appeared to be well applicable for the description of relativistic fluids. In this context, Sharif and Manzoor [32–34] explored self-gravitating fluid models in Brans–Dicke theory using spherical as well as cylindrical symmetries. They deduced a set of governing equations in terms of structure scalars to study the dynamics of anisotropic dissipative fluids. There is a large body of literature [35–40] available about structure scalars in GR and $f(R)$ gravity.

In this paper, we construct a set of scalars in terms of dynamical quantities to discuss the evolution of dissipative anisotropic shearing spherical matter configuration in $f(G)$ gravity. The paper has the following format. The next section formulates the modified field equations and dynamical variables. In Sect. 3, we construct structure scalars for a particular $f(G)$ model. Section 4 contains the set of governing equations in terms of scalars. We also explore density inhomogeneity factors. In Sect. 5, we write down all three possible static anisotropic spherical solutions in terms of scalars. Finally, Sect. 6 concludes the results.

2 Modified field equations and dynamical variables

In this section, we first formulate the field equations for $f(G)$ gravity taking non-static spherically symmetric metric and then evaluate dynamical variables. The most general non-static line element for spherically symmetric configuration is

$$ds^2 = -A^2dt^2 + B^2dr^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where the metric coefficients $A = A(t, r),\ B = B(t, r)$ and $C = C(t, r)$ are functions of co-moving coordinates t and r. The generalized GR action for $f(G)$ gravity is given by [41]

$$S = \frac{1}{2\kappa^2}\int d^4x[R + f(G)]\sqrt{-g} + S_M, \tag{2}$$

where $\kappa,\ R,\ f(G)$ are the coupling constant, the Ricci scalar, arbitrary function of G, respectively, and S_M is the matter action. We assume the unit system $\kappa^2 = \frac{8\pi G}{c} = 1$ (G is the gravitational constant and c is the speed of light). The Gauss–Bonnet quadratic invariant term is defined as

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho},$$

where $R_{\mu\nu\sigma\rho},\ R_{\mu\nu}$ are the Riemann and Ricci tensors, respectively. Varying the $f(G)$ action (2) with respect to the metric tensor $g_{\mu\nu}$, we obtain

$$\mathcal{G}_{\mu\nu} = \overset{\text{(eff)}}{T}_{\mu\nu} = \overset{\text{(M)}}{T}_{\mu\nu} + \overset{\text{(GB)}}{T}_{\mu\nu}, \tag{3}$$

where $\mathcal{G}_{\mu\nu}$ is the Einstein tensor and $\overset{\text{(M)}}{T}_{\mu\nu}/\overset{\text{(BG)}}{T}_{\mu\nu}$ are the energy-momentum tensors for matter/Gauss–Bonnet (GB) terms, respectively and

$$\overset{\text{(GB)}}{T}_{\mu\nu} = 8\Big[R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho} + \frac{1}{2}R(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})\Big]\nabla^\rho\nabla^\sigma f_G + (Gf_G - f)g_{\mu\nu}, \tag{4}$$

where subscript G denotes derivative of f with respect to Gauss–Bonnet invariant G. This is the energy-momentum tensor sharing the gravitational contribution coming from $f(G)$ extra degrees of freedom.

We assume a more complex problem in which non-static spherical geometry is coupled with locally anisotropic shearing fluid configurations with dissipation. The energy-momentum tensor for this type of fluid is defined as

$$\overset{(M)}{T}_{\mu\nu}=\rho u_\mu u_\nu+p_t h_{\mu\nu}+(p_r-p_t)v_\mu v_\nu+q(v_\mu u_\nu+u_\mu v_\nu)+\epsilon l_\mu l_\nu, \tag{5}$$

where $\rho,\ p_r,\ p_t,\ q$ and ϵ are energy density, radial pressure, tangential pressure, dissipation and radiation density, respectively. The scalar quantity q corresponds to a heat conducting vector q_μ. The quantities u^μ (4-velocity vector), v^μ (unit 4-vector in radial direction), $h^{\mu\nu}$ (projection tensor) and l^μ (null 4-vector) are defined as

$$u^\mu=A^{-1}\delta^\mu_0,\ \ v^\mu=B^{-1}\delta^\mu_1,\ \ h_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu,\ \ l^\mu=A^{-1}\delta^\mu_0+B^{-1}\delta^\mu_1,$$

obeying the relations

$$u^\mu u_\mu=-1,\ \ v^\mu v_\mu=1,\ \ v^\mu u_\mu=0,\ \ l^\mu u_\mu=-1,\ \ l^\mu l_\mu=0,\ \ h_{\mu\nu}u^\mu=0.$$

The effects of gravitational as well as inertial forces on the fluid can be described by 4-acceleration as

$$a_\mu=u_{\mu;\nu}u^\nu,\quad a_\mu=av_\mu,\quad a_{(1)}=\frac{A'}{A},\quad a^2=a^\mu a_\mu=\left(\frac{A'}{AB}\right)^2,$$

where prime denotes partial derivative with respect to radial coordinate r. The volume expansion and contraction of fluid can be measured by kinematical scalar variable ϑ (expansion parameter). Mathematically, it is defined as $\vartheta=u^\mu_{;\mu}$

$$\vartheta=\frac{1}{A}\left(\frac{\dot{B}}{B}+2\frac{\dot{C}}{C}\right),$$

where dot represents the temporal partial derivative. The shear tensor is used to evaluate distortion appearing in the fluid due to motion defined as

$$\sigma_{\mu\nu}=u_{(\mu;\nu)}+a_{(\mu}u_{\nu)}-\frac{1}{3}\vartheta h_{\mu\nu}.$$

Its alternative form and the non-zero components are given as

$$\sigma_{\mu\nu}=\sigma\left(v_\mu v_\nu-\frac{1}{3}h_{\mu\nu}\right),\quad \sigma_{11}=\frac{2}{3}\sigma B^2,\quad \sigma_{22}=\frac{1}{\sin^2\theta}\sigma_{33}=-\frac{1}{3}\sigma C^2.$$

Also,

$$\sigma_{\mu\nu}\sigma^{\mu\nu}=\frac{2}{3}\sigma^2,\quad \sigma=\frac{1}{A}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right).$$

We can write Eq. (5) as

$$\overset{(M)}{T}_{\mu\nu}=\tilde{\rho}u_\mu u_\nu+p_t h_{\mu\nu}+\Pi v_\mu v_\nu+\tilde{q}_\mu u_\nu+\tilde{q}_\nu u_\mu, \tag{6}$$

where $\tilde{\rho}=\rho+\epsilon,\ \Pi=\tilde{p}_r-p_t,\ \tilde{p}_r=p_r+\epsilon,\ \tilde{q}_\mu=\tilde{q}v_\mu,\ \tilde{q}=q+\epsilon.$

The corresponding field equations are

$$\begin{aligned}\tilde{\rho}=&\frac{\dot{C}}{A^2C}\left(2\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{1}{B^2}\left(2\frac{C''}{C}+\left(\frac{C'}{C}\right)^2-2\frac{B'C'}{BC}-\left(\frac{B}{C}\right)^2\right)\\&+\frac{4}{A^2BC^2}\left(\dot{B}-2\frac{\dot{C}C''}{B}-\frac{\dot{B}C'^2}{B^2}+3\frac{\dot{B}\dot{C}^2}{A^2}+2\frac{B'C'\dot{C}}{B^2}\right)\dot{f}_G+\frac{4}{A^2B^3C^2}\\&\times\left(A^2B'+B'\dot{C}^2-2\dot{B}\dot{C}C'+2\frac{A^2C'C''}{B}-3\frac{A^2B'C'^2}{B^2}\right)f'_G+\frac{4}{A^2B^3C^2}\\&\times\left(\frac{A^2C'^2}{B^2}-A^2-2\dot{C}\right)f''_G-Gf_G+f,\end{aligned} \tag{7}$$

$$\begin{aligned}\tilde{q}=&\frac{2}{AB}\left(\frac{\dot{B}C'}{BC}+\frac{A'\dot{C}}{AC}-\frac{\dot{C}'}{C}\right)+\frac{4}{ABC^2}\left(1+\left(\frac{\dot{C}}{A}\right)^2-\left(\frac{C'}{B}\right)^2\right)\dot{f}'_G\\&+\frac{4}{A^2BC^2}\left(-A'+2\frac{\dot{C}\dot{C}'}{A}+\frac{A'C'^2}{B^2}-3\frac{A'\dot{C}^2}{A^2}-2\frac{\dot{B}C'\dot{C}}{AB}\right)\dot{f}_G+\frac{4}{AB^2C^2}\\&\times\left(-\dot{B}-2\frac{\dot{C}'C'}{B}+3\frac{\dot{B}C'^2}{B^2}-\frac{\dot{B}\dot{C}^2}{A^2}+2\frac{A'C'\dot{C}}{AB}f'_G\right),\end{aligned} \tag{8}$$

$$\begin{aligned}\tilde{p}_r=&\frac{1}{A^2}\left(\frac{\dot{C}}{C}\left(2\frac{\dot{A}}{A}-\frac{\dot{C}}{C}\right)-2\frac{\ddot{C}}{C}\right)+\frac{C'}{B^2C}\left(2\frac{A'}{A}+\frac{C'}{C}\right)-\frac{1}{C^2}+\frac{4}{A^2C^2}\\&\times\left(\left(\frac{C'}{B}\right)^2-\left(\frac{\dot{C}}{A}\right)^2-1\right)\ddot{f}_G+\frac{4}{A^3C^2}\left(\dot{A}-2\frac{\dot{C}\ddot{C}}{A}+2\frac{A'C'\dot{C}}{B^3}-\frac{\dot{A}C'^2}{B^2}\right.\\&\left.+\ 3\frac{\dot{A}\dot{C}^2}{A^2}\right)\dot{f}_G+\frac{4}{AB^2C^2}\left(A'+\frac{1}{A^2}-2\frac{\dot{A}\dot{C}C'}{A^2}-3\frac{A'C'^2}{B^2}+2\frac{\ddot{C}C'}{A}\right)f'_G\\&+Gf_G-f,\end{aligned} \tag{9}$$

$$\begin{aligned}p_t=&\frac{1}{A^2}\left(\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{\ddot{B}}{B}-\frac{\ddot{C}}{C}-\frac{\dot{B}\dot{C}}{BC}\right)+\frac{1}{B^2}\left(\frac{A''}{A}+\frac{C''}{C}-\frac{A'B'}{AB}\right.\\&\left.+\frac{C'}{C}\left(\frac{A'}{A}-\frac{B'}{B}\right)\right)+\frac{4}{A^3BC}\left(\frac{1}{B}\left(2\dot{C}'A'-\dot{A}C''+A''\dot{C}\right)-\frac{1}{A}\right.\\&\times\left(\dot{B}\ddot{C}+\dot{C}\ddot{B}\right)-\frac{1}{B^2}\left(A'B'\dot{C}+A'\dot{B}C'-\dot{A}B'C'\right)+3\frac{\dot{A}\dot{B}\dot{C}}{A^2}-2\end{aligned}$$

$$\times \frac{A'^2\dot{C}}{AB}\Bigg)\dot{f}_G + \frac{4}{AB^3C}\left(\frac{1}{A}\left(2\dot{B}\dot{C}' - B'\ddot{C} + \ddot{B}C'\right) - \frac{1}{B}\left(A'C'' + A''C'\right)\right.$$
$$\left. - \frac{1}{A^2}\left(\dot{A}\dot{B}C' + A'\dot{B}\dot{C} - \dot{A}B'\dot{C}\right) + 3\frac{A'B'C'}{B^2} - 2\frac{\dot{B}^2C'}{AB}\right)f'_G + \frac{8}{A^2B^2C}$$
$$\times\left(\frac{A'\dot{C}}{A} + \frac{\dot{B}C'}{B} - 1\right)\dot{f}'_G - \frac{4}{AB^2C}\left(\frac{A'C'}{B^2} + \frac{\dot{A}\dot{C}}{A^2} - \ddot{C}\right)f''_G - \frac{4}{A^2BC}$$
$$\times\left(\frac{\dot{B}\dot{C}}{A^2} + \frac{B'C'}{A}\right)\ddot{f}_G + Gf_G - f. \tag{10}$$

The expression for Gauss–Bonnet invariant is calculated as

$$G = \frac{8}{ABC}\left[\left(\frac{A'B'}{B^2C} + \frac{\ddot{B}}{AC}\right)\left(1 + \frac{\dot{C}^2}{A^2}\right) + \left(\frac{\dot{A}\dot{B}}{A^2C} + \frac{A''}{BC}\right)\left(\frac{C'^2}{B^2} - 1\right)\right.$$
$$- \frac{1}{AC}\left(\frac{A''\dot{C}^2}{AB} + \ddot{B}C'^2\right) + 2\left\{\frac{C''}{BC}\left(\frac{A'C'}{B^2} - \frac{\ddot{C}}{A}\right) + \frac{B'C'}{AB^2C}\left(\ddot{C} - \frac{\dot{A}\dot{C}}{A}\right)\right.$$
$$+ \frac{\dot{C}}{A^3C}\left(\ddot{C}\dot{B} + \frac{\dot{C}A'^2}{B}\right) + \frac{\dot{C}}{A^2BC}\left(\dot{A}C'' + \frac{A'\dot{B}C'}{B}\right) + \frac{1}{ABC}\left(\dot{C}'^2\right.$$
$$\left.\left. + \frac{\dot{B}^2C'^2}{B^2}\right)\right\} - \frac{3}{C}\left(\frac{A'B'C'^2}{B^4} + \frac{\dot{A}\dot{B}\dot{C}^2}{A^4}\right) - \frac{4\dot{C}'}{ABC}\left(\frac{A'\dot{C}}{A} + \frac{\dot{B}C'}{B}\right)\Bigg]. \tag{11}$$

The Weyl tensor or the Weyl curvature tensor is described as a combination of the Riemann tensor, Ricci tensor and Ricci scalar. It explains how an object is distorted due to the effects of tidal force while moving on geodesics. Its expression is given by

$$C^{\mu}_{\rho\sigma\nu} = R^{\mu}_{\rho\sigma\nu} - \frac{1}{2}R^{\mu}_{\sigma}g_{\rho\nu} + \frac{1}{2}R_{\rho\sigma}\delta^{\mu}_{\nu} - \frac{1}{2}R_{\rho\nu}\delta^{\mu}_{\sigma} + \frac{1}{2}R^{\mu}_{\nu}g_{\rho\sigma} + \frac{1}{6}R\left(\delta^{\mu}_{\sigma}g_{\rho\nu} - \delta^{\mu}_{\nu}g_{\rho\sigma}\right). \tag{12}$$

This tensor can be decomposed into two parts, magnetic $M_{\mu\nu}$ and electric $E_{\mu\nu}$ parts. The magnetic part vanishes for spherical symmetry while the electric part is defined as

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta}, \tag{13}$$

where

$$C_{\mu\alpha\nu\beta} = \left(g_{\mu\alpha\kappa\eta}g_{\nu\beta\gamma\delta} - \varepsilon_{\mu\alpha\kappa\eta}\varepsilon_{\nu\beta\gamma\delta}\right)u^{\kappa}u^{\gamma}E^{\eta\delta}$$

with $g_{\mu\alpha\kappa\eta} = g_{\mu\kappa}g_{\alpha\eta} - g_{\mu\eta}g_{\alpha\kappa}$ and $\varepsilon_{\mu\alpha\kappa\eta}$ is the Levi-Civita tensor. The electric part can be written in terms of 4-unit vector and projection tensor as

$$E_{\mu\nu} = \varepsilon\left(v_{\mu}v_{\nu} - \frac{1}{3}h_{\mu\nu}\right), \tag{14}$$

where

$$\varepsilon=\frac{1}{2}\left(\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}-\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\frac{\dot{C}}{C}\right)+\frac{1}{2B^2}\left(-\frac{C''}{C}+\left(\frac{C'}{C}+\frac{B'}{B}\right)\frac{C'}{C}\right)-\frac{1}{2C^2} \tag{15}$$

is the Weyl scalar and the non-zero components of electric part are

$$E_{11}=\frac{2}{3}\varepsilon B^2,\quad E_{22}=-\frac{1}{3}\varepsilon C^2,\quad E_{33}=E_{22}\sin^2\theta.$$

The Misner-Sharp mass function calculates the total energy of spherically symmetric system within the radius $r=C$ given as

$$\mathcal{M}=\frac{1}{2}C^3R^{23}_{23}=\frac{1}{2}C\left(\left(\frac{\dot{C}}{A}\right)^2-\left(\frac{C'}{B}\right)^2+1\right). \tag{16}$$

Now we calculate the variation of this mass function of radiating fluid inside the sphere. For this purpose, we introduce two useful derivative operators with respect to radial and proper time coordinates as

$$D_T=\frac{1}{A}\frac{\partial}{\partial t},\quad D_C=\frac{1}{C}\frac{\partial}{\partial r} \tag{17}$$

and the relativistic velocity of the collapsing fluid turns out to be

$$\mathcal{U}=D_TC=\frac{1}{A}\frac{\partial C}{\partial t}=\frac{\dot{C}}{A}. \tag{18}$$

Combining Eqs. (16) and (18), we obtain

$$E=\frac{C'}{B}=\left(1+\mathcal{U}^2-\frac{2}{C}\mathcal{M}\right)^{\frac{1}{2}}. \tag{19}$$

Using Eqs. (3), (16) and (17), it follows that

$$D_T\mathcal{M}=-\frac{C^2}{2}\left\{\left(\tilde{p}_r+\frac{1}{B^2}\overset{(\mathrm{GB})}{T}_{11}\right)\mathcal{U}+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T}_{01}\right)E\right\}, \tag{20}$$

$$D_C\mathcal{M}=\frac{C^2}{2}\left\{\left(\tilde{\rho}+\frac{1}{A^2}\overset{(\mathrm{GB})}{T}_{00}\right)+\left(\tilde{q}-\frac{1}{AB}\overset{(\mathrm{GB})}{T}_{01}\right)\frac{\mathcal{U}}{E}\right\}. \tag{21}$$

These equations represent the variation of mass inside the spherical surface of evolving fluid. Equation (20) depicts the combined role of pressure, dissipation, relativistic velocity and GB curvature terms on the derivative of mass function within the evolving spherical matter distribution. Equation (21) shows the role of energy density, dissipation, relativistic velocity and extra GB curvature terms on the variation of mass distribution in radial direction within surface of radius C. Equation (21) yields

$$\mathcal{M}' = \frac{C^2C'}{2}\left\{\left(\tilde{\rho} + \frac{1}{A^2}\overset{(GB)}{T_{00}}\right) + \left(\tilde{q} - \frac{1}{AB}\overset{(GB)}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\},$$

which further implies that

$$\mathcal{M} = \frac{1}{2}\int_0^r C^2C'\left\{\left(\tilde{\rho} + \frac{1}{A^2}\overset{(GB)}{T_{00}}\right) + \left(\tilde{q} - \frac{1}{AB}\overset{(GB)}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\}dr. \tag{22}$$

A particular combination of radiating energy density, dissipation and $f(G)$ correction terms through mass function can be attained using Eq. (22) as

$$3\frac{\mathcal{M}}{C^3} = \frac{3}{2C^3}\int_0^r C^2C'\left\{\left(\tilde{\rho} + \frac{1}{A^2}\overset{(GB)}{T_{00}}\right) + \left(\tilde{q} - \frac{1}{AB}\overset{(GB)}{T_{01}}\right)\frac{\mathcal{U}}{E}\right\}dr. \tag{23}$$

We have taken a regular matter distribution at $r = 0$ (center), i.e., $\mathcal{M}(t, 0) = C(t, 0) = 0$.

3 $f(G)$ model and modified structure scalars

In this section, we first consider a viable $f(G)$ model and then develop structure scalars by splitting the Reimman curvature tensor orthogonally. We assume the following $f(G)$ model

$$f(G) = \alpha G^n, \tag{24}$$

where α is any constant and $n > 0$ [42]. This model is viable if it satisfies the following conditions [43]

- $f(G)$ and all its derivatives (f_G, f_{GG}, f_{GGG}...) are regular,
- $f_{GG} > 0$, $\forall$ G and $f_{GG} \rightarrow 0$ as $|G| \rightarrow +\infty$.

The condition $\dot{f}_G > 0$ is required to avoid ghost. This model satisfies all the required conditions for a viable model. The model parameter n has some effects on $R + f(G)$ cosmology. If $n < \frac{1}{2}$, GB terms become dominant over curvature term (Einstein term) in weak field regime, i.e., curvature term becomes negligible in this stage leading to non-phantom phase. For $n < 0$, the big-rip singularity might seem to occur. Near this singularity, the curvature becomes dominant, i.e., Einstein term becomes dominant as compared to GB terms. To avoid this singularity, GB terms can be neglected and we arrive at phantom era. Thus, the transition of non-phantom to phantom phase can naturally occur in this model. If $0 < n < \frac{1}{2}$ in strong gravitational field regime, GB terms can be neglected and we are left with Einstein gravity (deceleration). For late times, the GB terms may become dominant as compared to matter Lagrangian density (acceleration), thus transition from decelerated to accelerated universe can occur.

In order to obtain scalar functions, we use decomposition of the Riemann curvature tensor orthogonally developed by Bel [30]. Here we are using this procedure with some changes [31] and present couple of tensors as follows

$$X_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}_{\mu\gamma} R^{*}_{\alpha\beta\nu\delta} u^{\gamma} u^{\delta}, \tag{25}$$

$$Y_{\mu\nu} = R_{\mu\gamma\nu\delta} u^{\gamma} u^{\delta}, \tag{26}$$

where $R^{*}_{\mu\nu\gamma\delta} = \frac{1}{2}\eta_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu}$. From Eq. (12), we can write

$$R^{\mu}_{\rho\sigma\nu} = C^{\mu}_{\rho\sigma\nu} + \frac{1}{2}R^{\mu}_{\sigma} g_{\rho\nu} - \frac{1}{2}R_{\rho\sigma}\delta^{\mu}_{\nu} + \frac{1}{2}R_{\rho\nu}\delta^{\mu}_{\sigma} - \frac{1}{2}R^{\mu}_{\nu} g_{\rho\sigma} - \frac{1}{6}R\left(\delta^{\mu}_{\sigma} g_{\rho\nu} + \delta^{\mu}_{\nu} g_{\rho\sigma}\right). \tag{27}$$

This equation can be written for Eq. (3) as

$$R^{\mu\gamma}_{\nu\delta} = C^{\mu\gamma}_{\nu\delta} + 2\overset{\text{(eff)}}{T}{}^{[\mu}{}_{[\nu}\delta^{\gamma]}_{\delta]} + \overset{\text{(eff)}}{T}\left(\frac{1}{3}\delta^{\mu}_{[\nu}\delta^{\gamma}_{\delta]} - \delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}\right). \tag{28}$$

We decompose the Riemann curvature tensor orthogonally into five parts as

$$R^{\mu\gamma}_{\nu\delta} = \overset{\text{(I)}}{R}{}^{\mu\gamma}{}_{\nu\delta} + \overset{\text{(II)}}{R}{}^{\mu\gamma}{}_{\nu\delta} + \overset{\text{(III)}}{R}{}^{\mu\gamma}{}_{\nu\delta} + \overset{\text{(IV)}}{R}{}^{\mu\gamma}{}_{\nu\delta} + \overset{\text{(V)}}{R}{}^{\mu\gamma}{}_{\nu\delta},$$

which are defined using Eq. (28) as

$$\overset{\text{(I)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = 2\left(\tilde{\rho} u^{[\mu} u_{[\nu}\delta^{\gamma]}_{\delta]} + p_t h^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}\right) + (-\tilde{\rho} + 3p_t + \Pi)\left(\frac{1}{3}\delta^{\mu}_{[\nu}\delta^{\gamma}_{\delta]} - \delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}\right), \tag{29}$$

$$\overset{\text{(II)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = 2\left(\Pi v^{[\mu} v_{[\nu}\delta^{\gamma]}_{\delta]} + \tilde{q} v^{[\mu} u_{[\nu}\delta^{\gamma]}_{\delta]} + \tilde{q} u^{[\mu} v_{[\nu}\delta^{\gamma]}_{\delta]}\right), \tag{30}$$

$$\overset{\text{(III)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = 4u^{[\mu} u_{[\nu} E^{\gamma]}_{\delta]} - \varepsilon^{\mu\gamma}_{\kappa}\varepsilon_{\nu\delta\eta} E^{\kappa\eta}, \tag{31}$$

$$\overset{\text{(IV)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = 8\left[R^{[\mu}_{\rho\sigma[\nu}\delta^{\gamma]}_{\delta]} + R_{\rho[\nu}\delta^{[\mu}_{\sigma}\delta^{\gamma]}_{\delta]} - R_{\rho\sigma}\delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]} - \delta_{\sigma\lambda}\delta^{\lambda}_{\rho} R^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]} + g_{\rho[\nu} R^{[\mu}_{\sigma}\delta^{\gamma]}_{\delta]} + \frac{1}{2}R\left(\delta_{\sigma\lambda}\delta^{\lambda}_{\rho}\delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]} - \delta^{[\mu}_{\sigma} g_{\rho[\nu}\delta^{\gamma]}_{\delta]}\right)\right]\nabla^{\rho}\nabla^{\sigma} f_G + (Gf_G - f)\delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}, \tag{32}$$

$$\overset{\text{(V)}}{R}{}^{\mu\gamma}{}_{\nu\delta} = \left(8[2R_{\rho\sigma} - Rg_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma} f_G + Gf_G - f\right)\left(\frac{1}{3}\delta^{\mu}_{[\nu}\delta^{\gamma}_{\delta]} - \delta^{[\mu}_{[\nu}\delta^{\gamma]}_{\delta]}\right). \tag{33}$$

Using Eqs. (29)–(33) in Eqs. (25) and (26) along with Eq. (24), we obtain

$$X_{\mu\nu} = \overset{\text{(M)}}{X}_{\mu\nu} + \overset{\text{(GB)}}{X}_{\mu\nu}$$
$$= p_t h_{\mu\nu} + \frac{1}{2}\Pi(h_{\mu\nu} - v_{\mu}v_{\nu}) - \frac{1}{3}h_{\mu\nu}(-\tilde{\rho} + 3p_t + \Pi) - E_{\mu\nu} + n\alpha\left[5R_{\rho\sigma}\right.$$

$$\times\delta_{\mu\nu}-2R_{\mu\rho\sigma\nu}+10R^{\kappa}_{\sigma}u_{\mu}u_{\kappa}g_{\rho\nu}-2R^{\eta}_{\rho\sigma\nu}u_{\mu}u_{\eta}-2R^{\kappa}_{\lambda}u_{\kappa}u^{\lambda}g_{\sigma\rho}\delta_{\mu\nu}-R^{\kappa}_{\sigma}$$
$$\times u_{\mu}u_{\kappa}\delta_{\rho\nu}-R_{\rho\nu}u_{\mu}u_{\sigma}+9R_{\mu\nu}g_{\sigma\rho}+2R_{\mu\lambda}u_{\nu}u^{\lambda}g_{\sigma\rho}+\frac{1}{2}R\left(3g_{\sigma\rho}g_{\mu\nu}+g_{\rho\nu}\right.$$
$$\left.\times\delta_{\mu\sigma}+2g_{\sigma\rho}u_{\mu}u_{\nu}\right)\big]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\alpha(n-1)G^{n}\left(4h_{\mu\nu}+4\delta_{\mu\nu}+u_{\mu}u_{\nu}\right)$$
$$-\frac{1}{3}n\alpha\left(8\left[2R_{\rho\sigma}-Rg_{\rho\sigma}\right]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\alpha(n-1)G^{n}\right)\left(g_{\mu\nu}+\delta_{\mu\nu}\right), \quad (34)$$

$$Y_{\mu\nu}=\overset{(M)}{Y}_{\mu\nu}+\overset{(GB)}{Y}_{\mu\nu}$$
$$=\frac{1}{2}h_{\mu\nu}\left(\tilde{\rho}-p_{t}-\Pi v_{\mu}v_{\nu}+\frac{2}{3}(-\tilde{\rho}+3p_{t}+\Pi)\right)+E_{\mu\nu}, \quad (35)$$

where the notation (M) and (GB) in superscript respectively show matter and Gauss–Bonnet parts of the relevant tensor. We note from Eq. (35) that the variable $Y_{\mu\nu}$ does not contain GB terms. Hence GB terms do not affect this variable.

We can write these tensors as the combination of their trace and traceless parts in the following way

$$X_{\mu\nu}=\frac{1}{3}Tr(X)h_{\mu\nu}+X_{\langle\mu\nu\rangle},$$
$$Y_{\mu\nu}=\frac{1}{3}Tr(Y)h_{\mu\nu}+Y_{\langle\mu\nu\rangle},$$

where $Tr(X)=X^{\mu}_{\mu}$ is trace of $X_{\mu\nu}$ and $X_{\langle\mu\nu\rangle}$ is traceless part of the tensor defined as

$$X_{\langle\mu\nu\rangle}=h^{\alpha}_{\mu}h^{\beta}_{\nu}(X_{\alpha\beta}-\frac{1}{3}Tr(X)h_{\alpha\beta}).$$

From Eqs. (34) and (35), we have

$$X_{T}\equiv Tr(X)=\overset{(M)}{X}_{T}+\overset{(GB)}{X}_{T},$$
$$Y_{T}\equiv Tr(Y)=\overset{(M)}{Y}_{T}.$$

Here

$$\overset{(M)}{X}_{T}=\tilde{\rho}, \quad (36)$$
$$\overset{(GB)}{X}_{T}=-\frac{1}{3}\alpha\left[n(74R_{\rho\sigma}+6u^{\nu}u_{\eta}R^{\eta}_{\rho\sigma\nu}+18u_{\kappa}u^{\lambda}R^{\kappa}_{\lambda}g_{\sigma\rho}+3u_{\sigma}u_{\mu}R^{\mu}_{\rho}+\frac{1}{2}\right.$$
$$\left.\times R(\delta_{\rho\sigma}-25g_{\sigma\rho}))\nabla^{\rho}\nabla^{\sigma}G^{n-1}-73(n-1)G^{n}\right], \quad (37)$$
$$\overset{(M)}{Y}_{T}=\frac{1}{2}(\tilde{\rho}+3\tilde{p}_{r}-2\Pi). \quad (38)$$

Moreover, their corresponding traceless parts are found as

$$X_{\langle\mu\nu\rangle}=\overset{(M)}{X}_{TF}\left(v_{\mu}v_{\nu}-\frac{1}{3}h_{\mu\nu}\right)+\overset{(GB)}{X}_{TF}\left(-\frac{1}{3}h_{\mu\nu}\right),$$

$$Y_{\langle\mu\nu\rangle} = \overset{(M)}{Y}_{TF}\left(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}\right),$$

where TF in subscript represents traceless part of the relevant tensor and

$$\overset{(M)}{X}_{TF} = -\left(\varepsilon - \frac{1}{2}\Pi\right), \tag{39}$$

$$\overset{(GB)}{X}_{TF} = \frac{1}{2}n\alpha\left(12R^\kappa_\lambda u_\kappa u^\lambda g_{\sigma\rho} - 55Rg_{\sigma\rho}\right)\nabla^\rho\nabla^\sigma G^{n-1} - 22\alpha(n-1)G^n, \tag{40}$$

$$\overset{(M)}{Y}_{TF} = \varepsilon + \frac{1}{2}\Pi. \tag{41}$$

The six quantities $\overset{(M)}{X}_T$, $\overset{(GB)}{X}_T$, $\overset{(M)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, $\overset{(M)}{Y}_T$ and $\overset{(M)}{Y}_{TF}$ are scalar functions (structure scalars) in $f(G)$ gravity.

Now we briefly discuss physical aspects of these structure scalars under the influence of $f(G)$ gravity that have a direct correspondence with the dynamics of relativistic spherical systems. We note that $\overset{(M)}{X}_T$ evidently represents the energy density. The scalar variables $\overset{(M)}{X}_{TF}$ and $\overset{(M)}{Y}_{TF}$ play a crucial role in the physical interpretation of fluid. Both scalars combine the Weyl tensor with pressure anisotropy. Their sum indicates local anisotropy while their difference provides the effects of tidal force. With local isotropy (when anisotropic parameter is assumed to be zero, i.e., $\Pi = 0$), they behave like

$$\overset{(M)}{X}_{TF} = -\overset{(M)}{Y}_{TF}$$

while in the absence of the Weyl tensor (conformally flat condition), both are same

$$\overset{(M)}{X}_{TF} = \overset{(M)}{Y}_{TF}.$$

We observe that local isotropy and conformally flatness condition contradict each other. This means that spherical system is either isotropic or conformally flat or it preserves local anisotropy as well as conformal flat condition. Furthermore, the combination

$$-\frac{1}{2}\overset{(M)}{X}_T + \overset{(M)}{X}_{TF} + \overset{(M)}{Y}_T + \overset{(M)}{Y}_{TF} = \frac{3}{2}\tilde{p}_r$$

describes the radial pressure and the sum $\overset{(M)}{X}_T + \overset{(GB)}{X}_T$ demonstrates that matter energy density is connected with dark source terms. The combination $\overset{(M)}{X}_{TF} + \overset{(M)}{Y}_{TF} + \overset{(GB)}{X}_{TF}$ makes the relativistic spherical system conformally flat and indicates that pressure anisotropy is controlled by dark source terms arising due to $f(G)$ gravity. The unification of scalars as $\overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF} + \overset{(GB)}{X}_{TF}$ shows that conformal flatness of fluid is controlled by GB terms.

4 Evolution equations

A set of governing equations can be deduced in terms of kinematical variables to describe the self-gravitating system. In the gravitational system of $f(G)$ theory, this set has been derived in [44]. Here we rewrite this set of equations in terms of modified structure scalars in the realm of $f(G)$ gravity.

4.1 Raychaudhuri equation

This equation describes the expansion rate of self-gravitating relativistic system and is given as

$$\vartheta_{;\mu}u^{\mu}+\frac{1}{3}\vartheta^{2}+\frac{2}{3}\sigma^{2}-a^{\mu}_{;\mu}=-\frac{1}{2}\left(\tilde{\rho}+3\tilde{p}_{r}-2\Pi\right),$$

which in terms of scalar function becomes

$$\vartheta_{;\mu}u^{\mu}+\frac{1}{3}\vartheta^{2}+\frac{2}{3}\sigma^{2}-a^{\mu}_{;\mu}=-\overset{\text{(M)}}{Y}_{T}. \tag{42}$$

We observe from this relation that GB terms have no contribution and $\overset{\text{(M)}}{Y}_{T}$ has an extreme role in measuring the expansion rate of self-gravitating relativistic fluid in GR as well as $f(G)$ cosmology.

4.2 Propagation equation of shear

This equation describes the shear evolution of self-gravitating system and is of the form

$$\begin{aligned}a_{,\mu}v^{\mu}+a^{2}-\sigma_{,\mu}u^{\mu}-\frac{1}{3}\sigma^{2}-\frac{2}{3}\sigma\vartheta-a\frac{C'}{BC}&=\varepsilon-\frac{1}{2}\Pi+2n\alpha\left[-R^{\phi}_{\rho\sigma\mu}v^{\mu}v_{\phi}\right.\\&\left.-R_{\rho\mu}v^{\mu}v_{\sigma}+g_{\sigma\rho}R_{\alpha\mu}v^{\alpha}v^{\mu}-R_{\alpha\sigma}v_{\rho}v^{\alpha}+\frac{1}{2}Rv_{\rho}v_{\sigma}\right]\nabla^{\rho}\nabla^{\sigma}G^{n},\end{aligned}$$

which in terms of modified scalar variables reduces to

$$\begin{aligned}a_{,\mu}v^{\mu}+a^{2}-\sigma_{,\mu}u^{\mu}-\frac{1}{3}\sigma^{2}-\frac{2}{3}\sigma\vartheta-a\frac{C'}{BC}&=-\overset{\text{(M)}}{Y}_{TF}+\frac{2}{67}n\alpha\overset{\text{(GB)}}{X}_{TF}\\&+22\alpha(n-1)G^{n}\end{aligned} \tag{43}$$

showing the importance of GB correction terms in the shearing motion of the evolving self-gravitating system.

4.3 Constraint equation

A direct relation among shear tensor, expansion scalar, heat flux and dark source terms due to $f(G)$ gravity is obtained through constraint equation defined as

$$\left(\vartheta+\frac{1}{2}\sigma\right)_{,\alpha}v^{\alpha}=-\frac{3C'}{2BC}\sigma-\frac{3}{2B}\tilde{q}+2n\alpha\bigg[-U_{\mu}h^{\alpha\beta}R^{\mu}_{\rho\sigma\beta}+g_{\sigma\rho}U_{\mu}h^{\alpha\beta}R^{\mu}_{\beta}+U_{\phi}h^{\alpha}_{\rho}R^{\phi}_{\sigma}+\frac{1}{8}Rh^{\alpha}_{\rho}U_{\sigma}\bigg]\nabla^{\rho}\nabla^{\sigma}G^{n}.$$

In terms of scalar variables, this equation becomes

$$\left(\vartheta+\frac{1}{2}\sigma\right)_{,\alpha}v^{\alpha}=-\frac{3C'}{2BC}\sigma-\frac{3}{2B}\tilde{q}-\frac{1}{64}(R\delta_{\rho\sigma})\nabla^{\rho}\nabla^{\sigma}f_{G}+\frac{9}{444}\overset{\text{(GB)}}{X}_{TF}-\frac{3}{32}\overset{\text{(GB)}}{X}_{T}-\frac{7407}{929}\alpha(n-1)G^{n}. \tag{44}$$

4.4 Dynamical equations

These equations describe the conservation of total energy of the evolving star and are obtained as

$$\tilde{\rho}_{,\mu}u^{\mu}+\vartheta(\tilde{\rho}+\tilde{p}_{r})-\frac{2}{3}(\sigma+\vartheta)\Pi+\tilde{q}_{,\mu}v^{\mu}+2\tilde{q}\left(\frac{C'}{BC}+a\right)+8n\alpha\left[R^{\mu\nu}_{\rho\sigma}-R^{\mu\nu}g_{\sigma\rho}\right]_{;\nu}u_{\mu}\nabla^{\rho}\nabla^{\sigma}G^{n}+8n\alpha\Big[\left(u_{\sigma}R^{\nu}_{\rho}-u^{\nu}R_{\rho\sigma}\right)_{,\nu}+\Gamma^{\mu}_{\nu\gamma}\left(R^{\nu}_{\rho}\delta^{\gamma}_{\sigma}-R_{\rho\sigma}g^{\gamma\nu}\right)\times u_{\mu}+\Gamma^{\nu}_{\nu\gamma}\left(u_{\sigma}R^{\gamma}_{\rho}-u^{\gamma}R_{\rho\sigma}\right)\Big]\nabla^{\rho}\nabla^{\sigma}G^{n}=0,$$

$$\tilde{p}_{r,\mu}v^{\mu}+a(\tilde{\rho}+\tilde{p}_{r})+2\Pi\frac{C'}{BC}+\tilde{q}_{,\mu}v^{\mu}+\frac{2}{3}\tilde{q}(\sigma+2\vartheta)+8n\alpha\left[-R^{\mu\nu}g_{\sigma\rho}+R^{\mu\nu}_{\rho\sigma}\right]_{;\nu}v_{\mu}\nabla^{\rho}\nabla^{\sigma}G^{n}+8n\alpha\Big[\left(v_{\sigma}R^{\nu}_{\rho}-v^{\nu}R_{\rho\sigma}+R^{\mu}_{\sigma}\delta^{\nu}_{\rho}\right)_{,\nu}+\Gamma^{\mu}_{\nu\gamma}\left(R^{\nu}_{\rho}\delta^{\gamma}_{\sigma}+R^{\gamma}_{\sigma}\times\delta^{\nu}_{\rho}\right)v_{\mu}+\Gamma^{\nu}_{\nu\gamma}\left(v_{\sigma}R^{\gamma}_{\rho}+v^{\gamma}R_{\rho\sigma}\right)\Big]\nabla^{\rho}\nabla^{\sigma}G^{n}=0.$$

In terms of scalar variables, these become

$$\tilde{\rho}_{,\mu}u^{\mu}+\frac{1}{3}\left(\overset{\text{(M)}}{X}_{T}+\overset{\text{(M)}}{Y}_{T}-\overset{\text{(M)}}{X}_{TF}-\overset{\text{(M)}}{Y}_{TF}\right)\vartheta+\frac{2}{3}\left(\overset{\text{(M)}}{X}_{TF}+\overset{\text{(M)}}{Y}_{TF}\right)+2\tilde{q}\left(\frac{C'}{BC}+a\right)+\tilde{q}_{,\mu}v^{\mu}-\frac{1}{58}\left(\overset{\text{(GB)}}{X}_{TF}\right)_{,\nu}u^{\nu}-\frac{11}{29}\alpha(n-1)(G^{n})_{,\nu}u^{\nu}=0, \tag{45}$$

$$\tilde{p}_{r,\mu}v^{\mu}+a\left(\overset{\text{(M)}}{X}_{T}+\overset{\text{(M)}}{Y}_{T}-\overset{\text{(M)}}{X}_{TF}-\overset{\text{(M)}}{Y}_{TF}\right)+\left(\overset{\text{(M)}}{X}_{TF}+\overset{\text{(M)}}{Y}_{TF}\right)\frac{2C'}{BC}+\tilde{q}_{,\mu}v^{\mu}+\frac{2}{3}\tilde{q}\times(\sigma+2\vartheta)-\frac{1}{58}\left(\overset{\text{(GB)}}{X}_{TF}\right)_{,\nu}v^{\nu}-\frac{11}{29}\alpha(n-1)(G^{n})_{,\nu}v^{\nu}=0, \tag{46}$$

which show that the rate of change of energy density and radial pressure depend on scalar functions of matter and dark source terms.

4.5 Relation of the Weyl tensor with mass function and GB terms

This relation can be obtained by using Eqs. (3), (12) and (16) as

$$\frac{3\mathcal{M}}{C^3}=\tilde{\rho}-\Pi-\varepsilon-\frac{1}{3}n\alpha\left[2Rg_{\rho\sigma}+3R\delta_{\rho\sigma}\right]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\frac{1}{6}\alpha(n-1)G^{n}.$$

In terms of scalar functions, it turns out to be

$$\frac{3\mathcal{M}}{C^3}=\frac{1}{2}\overset{(M)}{X}_{T}-\overset{(M)}{X}_{TF}+\frac{1}{87}\overset{(GB)}{X}_{TF}-n\alpha\left[R\delta_{\rho\sigma}\right]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\frac{219}{522}\alpha(n-1)G^{n}. \tag{47}$$

4.6 Evolution equations for the Weyl tensor

These evolution equations represent the relationship between the Weyl tensor, dynamical variables (heat flux, anisotropic parameter, density, shear and projection tensors etc) and extra dark source terms as

$$\begin{aligned}&\frac{1}{2}\left(\tilde{p}_r+\frac{3}{C^3}\mathcal{M}\right)\left(\vartheta+\frac{1}{2}\sigma\right)+\left(\varepsilon-\frac{1}{2}(\Pi-\tilde{\rho}\right)+\frac{3C'}{2BC}\tilde{q}+\frac{1}{4}n\alpha\left[v_{\rho}v_{\sigma}\left(R_{,\beta}u^{\beta}\right.\right.\\&\quad\left.\left.-R_{,\alpha}v^{\alpha}\right)\right]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\frac{1}{4}n\alpha(Rv_{\rho}v_{\sigma})\nabla^{\rho}\nabla^{\sigma}\left((G^{n-1})_{,\beta}u^{\beta}-(G^{n-1})_{,\alpha}v^{\alpha}\right)\\&\quad-\frac{1}{48}n\alpha\left(R\delta_{\rho\sigma}-8Rg_{\rho\sigma}\right)_{;\beta}u^{\beta}\nabla^{\rho}\nabla^{\sigma}G^{n-1}-\frac{1}{6}\alpha(n-1)(G^{n})_{,\beta}u^{\beta}+\frac{1}{48}n\alpha\\&\quad\times(R\delta_{\rho\sigma})\nabla^{\rho}\nabla^{\sigma}(G^{n-1})_{,\beta}u^{\beta}=0,\\&\left(\varepsilon+\frac{1}{2}(\tilde{\rho}-\Pi)\right)_{,\alpha}v^{\alpha}-\frac{3C'}{BC}\left(\frac{1}{2}\Pi-\varepsilon\right)-\tilde{q}\left(\vartheta+\frac{1}{2}\sigma\right)-\frac{1}{4}n\alpha\left[(Rg_{\rho\sigma})\nabla^{\rho}\nabla^{\sigma}\right.\\&\quad\left.\times G^{n-1}\right]_{;\alpha}v^{\alpha}-n\alpha\left[(Rg_{\rho\sigma})\nabla^{\rho}\nabla^{\sigma}G^{n-1}\right]_{,\alpha}v^{\alpha}-2\alpha(n-1)(G^{n})_{,\alpha}u^{\alpha}=0.\end{aligned}$$

In terms of scalar functions, these become

$$\begin{aligned}&\frac{1}{3}\left(\left(-\frac{1}{2}\overset{(M)}{X}_{T}+\overset{(M)}{X}_{TF}+\overset{(M)}{Y}_{T}+\overset{(M)}{Y}_{TF}\right)+\frac{9}{2C^3}\mathcal{M}\right)\left(\vartheta+\frac{1}{2}\sigma\right)+\left(\frac{1}{2}\overset{(M)}{X}_{T}-\overset{(M)}{X}_{TF}\right)\\&\quad+\frac{3C'}{2BC}\tilde{q}+\frac{1}{4}n\alpha\left[v_{\rho}v_{\sigma}(R_{,\beta}u^{\beta}-R_{,\alpha}v^{\alpha})\right]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\frac{1}{4}n\alpha(Rv_{\rho}v_{\sigma})\nabla^{\rho}\nabla^{\sigma}\\&\quad\times\left((G^{n-1})_{,\beta}u^{\beta}-(G^{n-1})_{,\alpha}v^{\alpha}\right)+\frac{1}{4}n\alpha(Ru_{\rho}u_{\sigma})_{,\beta}u^{\beta}\nabla^{\rho}\nabla^{\sigma}G^{n-1}+\frac{9}{32}n\alpha\\&\quad\times(Ru_{\rho}u_{\sigma})\nabla^{\rho}\nabla^{\sigma}(G^{n-1})_{,\beta}u^{\beta}+\left(\frac{7}{8}\overset{(GB)}{X}_{T}+\frac{9}{2^6\cdot 87}\overset{(GB)}{X}_{TF}\right)_{,\beta}u^{\beta}-\frac{1975}{24}(n-1)\\&\quad\times\alpha(G^{n})_{,\beta}u^{\beta}=0,\end{aligned} \tag{48}$$

$$\left(\frac{1}{2}\tilde{\rho}\right)_{,\alpha} v^{\alpha} - \left(\overset{(\mathrm{M})}{X}_{TF}\right)_{,\alpha} v^{\alpha} - \frac{3C'}{BC}\overset{(\mathrm{M})}{X}_{TF} - \tilde{q}\left(\vartheta + \frac{1}{2}\sigma\right) - \frac{5}{116}\left(\overset{(\mathrm{GB})}{X}_{TF}\right)_{,\alpha} v^{\alpha} - \frac{61}{58} \times \alpha(n-1)(G^{n})_{,\alpha} v^{\alpha} = 0. \tag{49}$$

The above two equations relate the effects of tidal force with fluid parameters as well as GB terms. Equation (49) shows the dependence of density inhomogeneity on two scalars $\overset{(\mathrm{M})}{X}_{TF}$, $\overset{(\mathrm{GB})}{X}_{TF}$, dissipation and $f(G)$ model.

Now, we figure out density inhomogeneity factors on the surface of spherical system under the influence of $f(G)$ gravity. In a collapsing system, the surface of celestial object suffers density inhomogeneity caused by some specific quantities or specific relations of dynamical and geometrical variables. The vanishing of these quantities or relations assures density homogeneity [31]. The dissipation and scalar variable of matter X_{TF} lead to density inhomogeneity for spherical system in GR. It is evident from Eq. (49) that if we neglect dissipation and matter variable $\overset{(\mathrm{M})}{X}_{TF}$, we obtain

$$\left(\frac{1}{2}\tilde{\rho}\right)_{,\alpha} v^{\alpha} - \frac{5}{116}\left(\overset{(\mathrm{GB})}{X}_{TF}\right)_{,\alpha} v^{\alpha} - \frac{61}{58}\alpha(n-1)(G^{n})_{,\alpha} v^{\alpha} = 0.$$

We are left with density inhomogeneity and GB terms which suggests that density inhomogeneity is controlled by GB terms. Furthermore, if $\overset{(\mathrm{GB})}{X}_{TF} = 0$ and $f(G) = constant$ (or $n = 0$ for our model), we have

$$\left(\frac{1}{2}\tilde{\rho}\right)_{,\alpha} = 0, \tag{50}$$

which reveals that $f(G)$ model remains homogeneous and constant during the evolution. However, a viable and realistic $f(G)$ model cannot be a constant and hence scalar functions do not vanish which leads to inhomogeneous (irregular) distribution of fluid.

5 Anisotropic inhomogeneous spherical models

In this section, we restrict ourselves to the static case and modify the line element (1) in terms of scalar functions. The resulting line element can yield static spherical solutions with inhomogeneity and anisotropy in $f(G)$ gravity. We consider $C = r$ for static configuration for which $\vartheta = \sigma = 0$. Three possible alternative forms are given as follows.

5.1 First alternative form

It can directly be seen from Eq. (16) that the Misner-Sharp mass function for static case becomes

$$\frac{2}{r}\mathcal{M} = 1 - \frac{1}{B^{2}}, \tag{51}$$

which gives

$$B=\left(1-\frac{2r^2}{3}\left(\frac{1}{2}\overset{(M)}{X}_T-\overset{(M)}{X}_{TF}+\frac{1}{87}\overset{(GB)}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^\rho\nabla^\sigma G^{n-1}+\frac{219}{522}\right.\right.$$
$$\left.\left.\times\ \alpha(n-1)G^n\right)\right)^{-\frac{1}{2}}. \tag{52}$$

Next, using Eqs. (42) and (43) with (12) for static case, we obtain

$$\frac{A'}{AB}=\frac{1}{3}Br\left[\overset{(M)}{Y}_T+\overset{(M)}{Y}_{TF}-\frac{2}{67}\overset{(GB)}{Y}_{TF}-22\alpha(n-1)G^n\right], \tag{53}$$

which after integration gives

$$A=\lambda_1 exp\left[\int\frac{r}{3}\left(1-\frac{2r^2}{3}\left(\frac{1}{2}\overset{(M)}{X}_T-\overset{(M)}{X}_{TF}+\frac{1}{87}\overset{(GB)}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^\rho\nabla^\sigma G^{n-1}\right.\right.\right.$$
$$\left.\left.\left.+\ \frac{219}{522}\alpha(n-1)G^n\right)\right)^{-1}\left(\overset{(M)}{Y}_T+\overset{(M)}{Y}_{TF}-\frac{2}{67}\overset{(GB)}{X}_{TF}-22\alpha(n-1)G^n\right)dr\right] \tag{54}$$

with λ_1 as a constant of integration. In this case, the line element (1) becomes for Eqs. (52) and (54) as

$$ds^2=-\left[\lambda_1\exp\left[\int\frac{r}{3}\left(1-\frac{2r^2}{3}\left(\frac{1}{2}\overset{(M)}{X}_T-\overset{(M)}{X}_{TF}+\frac{1}{87}\overset{(GB)}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^\rho\nabla^\sigma\right.\right.\right.\right.$$
$$\times\ G^{n-1}+\frac{219}{522}\alpha(n-1)G^n\Big)\Big)^{-1}\Big(\overset{(M)}{Y}_T+\overset{(M)}{Y}_{TF}-\frac{2}{67}\overset{(GB)}{X}_{TF}-22\alpha(n-1)$$
$$\times G^n\Big)dr\Big]\Big]^2dt^2+\Big(1-\frac{2r^2}{3}\Big(\frac{1}{2}\overset{(M)}{X}_T-\overset{(M)}{X}_{TF}+\frac{1}{87}\overset{(GB)}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^\rho\nabla^\sigma$$
$$\times\ G^{n-1}+\frac{219}{522}\alpha(n-1)G^n\Big)\Big)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right).$$

A singularity seems to appear if

$$\frac{2r^2}{3}\left(\frac{1}{2}\overset{(M)}{X}_T-\overset{(M)}{X}_{TF}+\frac{1}{87}\overset{(GB)}{X}_{TF}-n\alpha[R\delta_{\rho\sigma}]\nabla^\rho\nabla^\sigma G^{n-1}+\frac{219}{522}\alpha(n-1)G^n\right)=1.$$

By inserting the values of five scalar functions $\overset{(M)}{X}_T$, $\overset{(M)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, $\overset{(M)}{Y}_T$ and $\overset{(M)}{Y}_{TF}$, we can obtain all possible inhomogeneous static anisotropic spherical solutions.

5.2 Second alternative form

Using Eqs. (23), (36) and (47), we obtain the relation

$$\frac{3}{r^3}\mathcal{M}=\frac{\mathcal{M}'}{r^2}-\overset{(M)}{X}_{TF}+\frac{377}{87(29)}\overset{(GB)}{X}_{TF},$$

which implies after integration that

$$\mathcal{M}=r^3\left(\int\left(\overset{(M)}{X}_{TF}-\frac{377}{87(29)}\overset{(GB)}{X}_{TF}\right)dr+\lambda_2\right),$$

where λ_2 is another constant of integration. Using this equation in (51), we have

$$B=\left(1-2r^2\left(\int\left(\overset{(M)}{X}_{TF}-\frac{377}{87(29)}\overset{(GB)}{X}_{TF}\right)dr+\lambda_2\right)\right)^{-\frac{1}{2}}. \tag{55}$$

Now combining Eqs. (53) and (55), it follows that

$$A=\lambda_3 exp\left[\frac{1}{3}\int\left(1-2r^2\left(\int\left(\overset{(M)}{X}_{TF}-\frac{377}{87(29)}\overset{(GB)}{X}_{TF}\right)dr+\lambda_2\right)\right)^{-1}r\left[\overset{(M)}{Y}_{T}+\overset{(M)}{Y}_{TF}\right.\right.$$
$$\left.\left.-\frac{2}{67}\overset{(GB)}{X}_{TF}-22\alpha(n-1)G^n\right]dr\right]. \tag{56}$$

Here λ_3 is another constant of integration. The line element (1) takes the following form for Eqs. (55) and (56) as

$$ds^2=-\left[\lambda_3 exp\left[\frac{1}{3}\int r\left(1-2r^2\left(\int\left(\overset{(M)}{X}_{TF}-\frac{377}{87(29)}\overset{(GB)}{X}_{TF}\right)dr+\lambda_2\right)\right)^{-1}\left[\overset{(M)}{Y}_{T}\right.\right.\right.$$
$$\left.\left.\left.+\overset{(M)}{Y}_{TF}-\frac{2}{67}\overset{(GB)}{X}_{TF}-22\alpha(n-1)G^n\right]dr\right]\right]^2dt^2+\left(1-2r^2\left(\int\left(\overset{(M)}{X}_{TF}\right.\right.\right.$$
$$\left.\left.\left.-\frac{377}{87(29)}\overset{(GB)}{X}_{TF}\right)dr+\lambda_2\right)\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right).$$

In this case, singularity can appear if $2r^2\left(\int(\overset{(M)}{X}_{TF}-\frac{377}{87(29)}\overset{(GB)}{X}_{TF})dr+\lambda_2\right)=1$ and all spherical inhomogeneous static anisotropic solutions can be obtained by inserting the values of five scalars $\overset{(M)}{X}_{T}$, $\overset{(M)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, $\overset{(M)}{Y}_{T}$ and $\overset{(M)}{Y}_{TF}$.

5.3 Third alternative form

Equation (15) reduces to

$$\varepsilon = \frac{1}{2B^2 r}\left(\frac{1}{r} + \frac{B'}{B}\right) - \frac{1}{2r^2}. \tag{57}$$

Alternatively, Eq. (57) can be rearranged as follows

$$B' + \frac{1}{r}B = B^3\left(\frac{1}{r} + 2r\left(\overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF}\right)\right),$$

where $\varepsilon = \overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF}$. This is Bernoulli's differential equation which gives

$$B = \left(-4r^2\int\frac{1}{r}\left(\overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF}\right)dr + 1 + \lambda_4 r^2\right)^{-\frac{1}{2}}. \tag{58}$$

Using Eqs. (58) in (53), it follows that

$$A = \frac{\lambda_5}{3}exp\left[\int r\left(-4r^2\int\frac{1}{r}\left(\overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF}\right)dr + 1 + \lambda_4 r^2\right)^{-1}\left[\overset{(M)}{Y}_{T} + \overset{(M)}{Y}_{TF}\right.\right.$$
$$\left.\left. -\frac{2}{67}\overset{(GB)}{X}_{TF} - 22\alpha(n-1)G^n\right]dr\right], \tag{59}$$

The alternate form of the line element (1) becomes

$$ds^2 = -\left[\frac{\lambda_5}{3}exp\left[\int r\left(-4r^2\int\frac{1}{r}\left(\overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF}\right)dr + 1 + \lambda_4 r^2\right)^{-1}\left[\overset{(M)}{Y}_{T} + \overset{(M)}{Y}_{TF}\right.\right.\right.$$
$$\left.\left.\left. -\frac{2}{67}\overset{(GB)}{X}_{TF} - 22\alpha(n-1)G^n\right]dr\right]^2 dt^2 + \left(-4r^2\int\frac{1}{r}\left(\overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF}\right)dr + 1\right.$$
$$\left. + \lambda_4 r^2\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where singularity might appear if $-4r^2\int\frac{1}{r}(\overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF})dr + 1 + \lambda_4 r^2 = 0$. In this alternative form of static inhomogeneous sphere with anisotropy, all solutions depend upon four scalars $\overset{(M)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, $\overset{(M)}{Y}_{T}$ and $\overset{(M)}{Y}_{TF}$.

6 Conclusions

This paper is devoted to investigate the dynamics of self-gravitating spherically distributed fluid in terms of structure scalars in $f(G)$ cosmology. We have constructed structure scalars $\overset{(M)}{X}_T$, $\overset{(GB)}{X}_T$, $\overset{(M)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, $\overset{(M)}{Y}_T$ and $\overset{(M)}{Y}_{TF}$ for matter as well as GB terms through orthogonal splitting of the Riemann tensor and deduced all governing equations in terms of these scalars. Furthermore, we have investigated the causes of density inhomogeneity and also showed that inhomogeneous static spherically symmetric solutions with anisotropy can be expressed in terms of these scalar functions. The physical effects of these scalars and results are summarized as follows.

We have found that the scalar $\overset{(M)}{X}_T$ represents energy density while $\overset{(M)}{X}_{TF}$ and $\overset{(M)}{Y}_{TF}$ control conformal flatness as well as the effects of anisotropy of the fluid. The sum of four scalars ($\overset{(M)}{X}_T$, $\overset{(M)}{X}_{TF}$, $\overset{(M)}{Y}_T$ and $\overset{(M)}{Y}_{TF}$) gives radial pressure. In the absence of anisotropy, the two scalars $\overset{(M)}{X}_{TF}$, and $\overset{(M)}{Y}_{TF}$ show that isotropic fluid with spherical symmetry is not conformally flat. In the case of dust fluid, the scalars $\overset{(M)}{X}_T$, and $\overset{(M)}{Y}_T$ have the same behavior, i.e., $\overset{(M)}{X}_T = \overset{(M)}{Y}_T$ but $\overset{(M)}{X}_{TF} = -\overset{(M)}{Y}_{TF}$. The scalar functions from dark side along with matter scalars have some physical effects as the sum $\overset{(M)}{X}_T + \overset{(GB)}{X}_T$ indicate that energy density is linked with dark source terms. The combination $\overset{(M)}{X}_{TF} + \overset{(M)}{Y}_{TF} + \overset{(GB)}{X}_{TF}$ gives relativistic conformally flat spherical system and represents that pressure anisotropy is driven by dark source terms. The unification of scalars as $\overset{(M)}{Y}_{TF} - \overset{(M)}{X}_{TF} + \overset{(GB)}{X}_{TF}$ suggests that conformal flat condition is controlled by GB terms. In GR, the structure scalar X_T gives energy density, Y_{TF} signifies the role of anisotropy on Tolman mass and X_{TF}/Y_{TF} both describe the effects of anisotropy for general fluid [31]. The evolution equation for the Weyl tensor indicates that energy density inhomogeneity is caused by $\overset{(M)}{X}_{TF}$, $\overset{(GB)}{X}_{TF}$, dissipation and constant $f(G)$ model. In GR, the density inhomogeneity is caused by X_{TF} and dissipation only. It is concluded that spherical systems should necessarily be inhomogeneous in this gravity. Finally, we have constructed three alternate forms of the line elements for inhomogeneous anisotropic spheres in terms of scalar functions which lead to further physical relevance of scalar functions.

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Evolution of axially symmetric systems and $f(G)$ gravity

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This paper explores evolution of dissipative axially symmetric collapsing fluid under the dark effects of $f(G)$ gravity. We formulate the dynamical variables and study the effects of dark sources in pressure anisotropy as well as heat dissipation. The structure scalars (scalar functions) as well as their role in the dynamics of source are investigated. Finally, we develop heat transport equation to examine the thermodynamic aspect and a set of equations governing the evolution of dynamical variables. It is concluded that dark sources affect thermodynamics of the system, evolution of kinematical quantities as well as density inhomogeneity.

Keywords: Structure scalars; Self-gravitating systems; $f(G)$ gravity.

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1. Introduction

Gravitational collapse is the source of energy behind structure formation in the universe where over the time, once vastly distributed matter collapses to high density pockets. This ultimately leads to the hierarchy of all cosmological structures like galaxies, black holes and all types of stars. Loss of hydrostatic equilibrium of a self-gravitating body results collapse under its own gravity leading to the situation where the dynamics of gravitational force dominates all other forces. The hydrostatic time scale is usually quite short among various stellar evolutionary phases. For Sun, it turns out to be about 27 min, 4.5 sec for a white dwarf and for a neutron star, weighing one solar mass and 10 km radius, it is only 10^{-4} s.[1] Other more intense dynamical phases (like gravitational collapse) persist for time scale of the order of magnitude of the hydrostatic time length.[2] This obviously mandates the exploration of the factors indicating the deviation from equilibrium state.

The dynamical variables like local pressure anisotropy, heat dissipation, expansion scalar, shear, vorticity, density inhomogeneity as well as the Weyl tensors are considered as the essential tools for the dynamics of stellar configurations. Herrera and his companions[3,4] developed a new strategy for investigating the dynamics of stellar configurations by constructing evolution equations in terms of dynamical variables and scalar functions. Scalar functions (structure scalars) consist of various combinations of energy terms which help to simplify the complications during the dynamical analysis of the system under consideration. In general relativity (GR), such studies do not explain the influence of dark sources on the dynamics of self-gravitating systems.

The $f(G)$ or modified Gauss–Bonnet (GB) theory of gravity is obtained by adding an arbitrary function ($f(G)$) of the GB quadratic invariant G in the action.[5] This theory efficiently elucidates accelerated expansion of the universe, transition from deceleration to accelerating phases of the universe and also satisfies solar system tests. Furthermore, it is useful in explaining thermodynamics[6] and protects all possible four types of finite time future singularities.[7] Myrzakulov *et al.*[8] explored this theory to study dark energy (DE) as well as inflationary era. We have studied energy conditions, wormhole solutions, built-in inflation, Noether symmetries, spherical wormhole solutions with conformal symmetry as well as dynamics of self-gravitating fluids in this theory.[9]

Modified theories are consistent with GR in weak field regime but may deviate in strong field. Gravitational collapse is classified as the phenomenon of strong field, hence modified theories can be the best candidates to explain this phenomenon. These theories may remodel the collapse process as well as improve its dynamics which may unveil fascinating results related to structure formation of the universe. In this regard, Sharif and Manzoor[10] investigated self-gravitating systems with spherical as well as cylindrical symmetries through structure scalars in the background of self-interacting Brans–Dicke gravity. Sharif and his collaborators[11] analyzed the evolution of self-gravitating systems under the dark effects of $f(R)$ gravity. Recently, we have constructed scalar functions and evolution equations to study the dynamics of spherical as well as cylindrical symmetric self-gravitating systems in $f(G)$ gravity.[12]

It is believed that astrophysical objects are endowed with angular momentum, e.g. stellar compact objects (like white dwarfs, neutron stars) are in rotational motion and can deviate from spherical symmetry incidentally which gives rise to axially symmetry. Thus, the dynamical analysis for self-gravitating fluids with such symmetry would be interesting. Herrera and Varela[13] investigated effects of axially symmetric perturbations of matter variables by considering only perfect fluids. The assumption of perfect fluid seems to be a stringent restriction for axially symmetric sources even in static case.[14] Sharif and Bhatti[15] explored instability regions for axial and reflection symmetric systems with anisotropic matter configurations. Sharif and Manzoor[16] examined the dynamics and stability for axial and reflection symmetric model in self-interacting Brans–Dicke gravity. They also studied the effects of dark source terms on dissipative axially symmetric collapsing fluid.[17]

In this paper, we investigate the effects of dark sources by taking a general source comprising dissipation and all nonvanishing stresses consistent with axial symmetry in modified GB gravity. The paper has the following format. In the next section, we provide formalism of gravity for axial system with general source. Section 3 yields nonzero structure scalars corresponding to the system. Section 4 leads to heat transport equation as well as some dynamical aspects through the set of governing equations. Finally, we summarize our results in the last section.

2. $f(G)$ Gravity and Axial System

The generalized GR action for $f(G)$ gravity is given by[5]

$$S = \frac{1}{2\kappa^2}\int d^4x[R + f(G)]\sqrt{-g} + S_{_M}, \tag{1}$$

where κ, R, $f(G)$ are the coupling constant, the Ricci scalar, arbitrary function of G, respectively, and $S_{_M}$ is the matter action. We assume the unit system $\kappa^2 = \frac{8\pi G}{c} = 1$ (G is the gravitational constant and c is the speed of light). The GB quadratic invariant term is

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho},$$

where $R_{\mu\nu}$, $R_{\mu\nu\sigma\rho}$ are the Ricci and Riemann tensors, respectively. Varying the action (1) with respect to $g_{\mu\nu}$, we have

$$\mathcal{G}_{\mu\nu} = \overset{(\mathrm{eff})}{T}_{\mu\nu} = \overset{(\mathrm{M})}{T}_{\mu\nu} + \overset{(\mathrm{GB})}{T}_{\mu\nu}, \tag{2}$$

where notation (eff) (short for effective) denotes combined effects of matter and dark sources (GB terms), $\mathcal{G}_{\mu\nu}$ is the Einstein tensor, $\overset{(\mathrm{M})}{T}_{\mu\nu}/\overset{(\mathrm{GB})}{T}_{\mu\nu}$ are the energy–momentum tensors for matter/GB terms, respectively, and

$$\begin{aligned}\overset{(\mathrm{GB})}{T}_{\mu\nu} = 8[&R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho}\\ &+ \frac{1}{2}R(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})]\nabla^\rho\nabla^\sigma f_{_G} + (Gf_{_G} - f)g_{\mu\nu},\end{aligned} \tag{3}$$

G in subscript denotes derivative of f with respect to GB invariant. This is the energy–momentum tensor contributing the gravitational effects due to extra dark source terms.

The line element for axially and reflection symmetric system is[4]

$$\begin{aligned}ds^2 = &-A^2(t,r,\theta)dt^2 + B^2(t,r,\theta)(dr^2 + r^2d\theta^2)\\ &+ 2E(t,r,\theta)dtd\theta + C^2(t,r,\theta)d\phi^2.\end{aligned} \tag{4}$$

The energy distribution of respective fluid observed by an observer with four-velocity u^μ ($u^\mu = (A^{-1},0,0,0)$ and $u_\mu = (-A,0,\frac{E}{A},0)$) can be represented by the energy–momentum tensor given by

$$\overset{(\mathrm{eff})}{T}_{\mu\nu} = \overset{(\mathrm{m})}{T}_{\mu\nu} + \overset{(\mathrm{GB})}{T}_{\mu\nu} = (\overset{(\mathrm{eff})}{\rho} + \overset{(\mathrm{eff})}{p})u_\mu u_\nu + \overset{(\mathrm{eff})}{p} g_{\mu\nu} + \overset{(\mathrm{eff})}{\Pi}_{\mu\nu} + \overset{(\mathrm{eff})}{q}_\mu u_\nu + \overset{(\mathrm{eff})}{q}_\nu u_\mu, \quad (5)$$

where effective energy density, isotropic pressure, anisotropic tensor and heat flux, respectively are defined as

$$\overset{(\mathrm{eff})}{\rho} = \overset{(\mathrm{m})}{\rho}_{\mu\nu} + \overset{(\mathrm{GB})}{\rho}_{\mu\nu}, \quad \overset{(\mathrm{eff})}{p} = \overset{(\mathrm{m})}{p}_{\mu\nu} + \overset{(\mathrm{GB})}{p}_{\mu\nu},$$

$$\overset{(\mathrm{eff})}{\Pi}_{\mu\nu} = \overset{(\mathrm{m})}{\Pi}_{\mu\nu} + \overset{(\mathrm{GB})}{\Pi}_{\mu\nu}, \quad \overset{(\mathrm{eff})}{q}_{\mu} = \overset{(\mathrm{m})}{q}_{\mu\nu} + \overset{(\mathrm{GB})}{q}_{\mu\nu}.$$

We obtain these effective quantities from Eq. (5) as

$$\overset{(\mathrm{eff})}{\rho} = \overset{(\mathrm{eff})}{T}_{\mu\nu}u^\mu u^\nu = \overset{(\mathrm{m})}{T}_{\mu\nu}u^\mu u^\nu + \frac{3}{2}[Rg_{\rho\sigma}]\nabla^\rho\nabla^\sigma f_G - Gf_G + f, \quad (6)$$

$$\overset{(\mathrm{eff})}{q}_{\mu} = -\overset{(\mathrm{eff})}{\rho} u_\mu - \overset{(\mathrm{eff})}{T}_{\mu\nu}u^\nu = -\overset{(\mathrm{eff})}{\rho} u_\mu - \overset{(\mathrm{m})}{T}_{\mu\nu}u^\nu - \frac{1}{2}[Rg_{\rho\sigma}u_\mu]\nabla^\rho\nabla^\sigma f_G + (Gf_G - f)u_\mu, \quad (7)$$

$$\overset{(\mathrm{eff})}{p} = \frac{1}{3}h^{\mu\nu}\overset{(\mathrm{eff})}{T}_{\mu\nu} = \frac{1}{3}h^{\mu\nu}\overset{(\mathrm{m})}{T}_{\mu\nu} + 4\left[\frac{11}{8}Rg_{\rho\sigma} + Ru_\rho u_\sigma - h_{\rho\sigma}\right]\nabla^\rho\nabla^\sigma f_G + 3(Gf_G - f), \quad (8)$$

$$\begin{aligned}\overset{(\mathrm{eff})}{\Pi}_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu(\overset{(\mathrm{eff})}{T}_{\alpha\beta} - \overset{(\mathrm{eff})}{p} h_{\alpha\beta}) = h^\alpha_\mu h^\beta_\nu(\overset{(\mathrm{m})}{T}_{\alpha\beta} - \overset{(\mathrm{m})}{p} h_{\alpha\beta}) \\ &+ 8\left[\frac{5}{8}Rg_{\sigma\rho}\delta_{\mu\nu} + \frac{9}{8}Rg_{\sigma\rho}u_\mu u_\nu + \frac{1}{2}Rg_{\nu\rho}h_{\mu\sigma} \right.\\ &\left. + \frac{1}{2}Rh_{\mu\sigma}u_\rho u_\nu + \frac{1}{2}Ru_\sigma u_\rho u_\mu u_\nu - Rh_{\mu\sigma}h_{\rho\nu}\right] \\ &\times \nabla^\rho\nabla^\sigma f_G + (Gf_G - f)(\delta_{\mu\nu} + u_\mu u_\nu) \\ &+ 4h_{\mu\nu}\left[\frac{11}{8}Rg_{\rho\sigma} + Ru_\rho u_\sigma - h_{\rho\sigma}\right] \\ &\times \nabla^\rho\nabla^\sigma f_G + 3h_{\mu\nu}(Gf_G - f),\end{aligned} \quad (9)$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection tensor. The spacelike unit 4-vectors are defined as

$$v_\mu = B\delta^1_\mu, \quad s_\mu = \frac{1}{A}(A^2B^2r^2 + E^2)^{\frac{1}{2}}\delta^2_\mu, \quad k_\mu = C\delta^3_\mu,$$

which satisfy the relations

$$u^\mu u_\mu = -v^\mu v_\mu = -s^\mu s_\mu = -k^\mu k_\mu = -1,$$

$$u^\mu v_\mu = u^\mu s_\mu = u^\mu k_\mu = v^\mu s_\mu = v^\mu k_\mu = k^\mu s_\mu = 0.$$

The pressure anisotropic parameter has significant effects in controlling hydrostatic equilibrium. The physical phenomena like mixture of two fluids and phase transition cause pressure anisotropy in the stellar models.[18] Some other prominent sources for pressure anisotropy are the magnetic field present in the compact objects (such as neutron stars and white dwarfs), magnetized strange quark stars, magnetic field acting on a Fermi gas, viscosity present in neutron stars as well as in highly densed matter.[19] For the sake of convenience, we convert the anisotropic tensor (9) in terms of scalar quantities as follows:

$$\overset{(\mathrm{eff})}{\Pi}_{\mu\nu} = \frac{1}{3}(2\overset{(\mathrm{eff})}{\Pi}_{1} + \overset{(\mathrm{eff})}{\Pi}_{2})\left(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}\right) + \frac{1}{3}(2\overset{(\mathrm{eff})}{\Pi}_{2} + \overset{(\mathrm{eff})}{\Pi}_{1})\left(s_\mu s_\nu - \frac{1}{3}h_{\mu\nu}\right) + 2\overset{(\mathrm{eff})}{\Pi}_{vs} v_{(\mu}s_{\nu)}, \tag{10}$$

where

$$\overset{(\mathrm{eff})}{\Pi}_{vs} = v^\mu s^\nu \overset{(\mathrm{eff})}{T}_{\mu\nu}, \quad \overset{(\mathrm{eff})}{\Pi}_{1} = (2v^\mu v^\nu - s^\mu s^\nu - k^\mu k^\nu)\overset{(\mathrm{eff})}{T}_{\mu\nu}, \tag{11}$$

$$\overset{(\mathrm{eff})}{\Pi}_{2} = (2s^\mu s^\nu - k^\mu k^\nu - v^\mu v^\nu)\overset{(\mathrm{eff})}{T}_{\mu\nu}. \tag{12}$$

Equations (11) and (12) indicate that anisotropy scalars $\overset{(\mathrm{eff})}{\Pi}_{vs}$, $\overset{(\mathrm{eff})}{\Pi}_{1}$ and $\overset{(\mathrm{eff})}{\Pi}_{2}$ depend on matter as well as dark sources. Hence the inhomogeneous distribution of dark sources generate pressure anisotropy in collapsing fluid.

Dissipation of heat flux (due to emission of photons or neutrinos which are massless particle) during collapse cannot be overemphasized. Indeed, it is a characteristic process during stellar evolution. Dissipation due to neutrino emission of gravitational binding energy leads to the formation of neutron stars or black holes.[20] The field equations along with $q^\mu u_\mu = 0$ gives $T_{03} = 0$ which implies that

$$\overset{(\mathrm{eff})}{q}_{\mu} = \overset{(\mathrm{eff})}{q}_{1} v_\mu + \overset{(\mathrm{eff})}{q}_{2} s_\mu, \tag{13}$$

where

$$\overset{(\mathrm{eff})}{q}_{1} = \overset{(\mathrm{eff})}{q}_{\mu} v^\mu = \overset{(\mathrm{eff})}{T}_{\mu\nu} u^\nu v^\mu, \quad \overset{(\mathrm{eff})}{q}_{2} = \overset{(\mathrm{eff})}{q}_{\mu} s^\mu = \overset{(\mathrm{eff})}{T}_{\mu\nu} u^\nu s^\mu. \tag{14}$$

These equations indicate that inhomogeneous distribution of dark sources generate heat dissipation. Using Eq. (14), we can express (13) in contravariant and covariant forms as

$$q^\mu_{(\mathrm{eff})} = \left(\frac{\overset{(\mathrm{eff})}{T}_{\mu\nu} u^\nu s^\mu E}{A(A^2B^2r^2 + E^2)^{\frac{1}{2}}}, \frac{\overset{(\mathrm{eff})}{T}_{\mu\nu} u^\nu v^\mu}{B}, \frac{A\overset{(\mathrm{eff})}{T}_{\mu\nu} u^\nu s^\mu}{(A^2B^2r^2 + E^2)^{\frac{1}{2}}}, 0\right), \tag{15}$$

$$\overset{(\mathrm{eff})}{q}_{\mu} = \left(0, B\overset{(\mathrm{eff})}{T}_{\mu\nu} u^\nu v^\mu, \frac{(A^2B^2r^2 + E^2)^{\frac{1}{2}}\overset{(\mathrm{eff})}{T}_{\mu\nu} u^\nu s^\mu}{A}, 0\right). \tag{16}$$

2.1. *Kinematical variables and the Weyl tensor*

The characteristics of self-gravitating collapsing fluid depend upon the behavior of kinematical variables. The 4-acceleration is given as

$$a_\mu = u^\nu u_{\mu;\nu} = a_1 v_\mu + a_2 s_\mu = \left(0, \frac{A'}{A}, \left(-\frac{\dot{A}}{A} + \frac{\dot{E}}{E}\right)\frac{E}{A^2} + \frac{A^\theta}{A}, 0\right), \qquad (17)$$

where prime represents partial derivative with respect to radial coordinate, dot is the temporal and θ indicates derivative with respect to theta coordinate. The expansion scalar controls the volume expansion of the fluid and is defined as

$$\vartheta = u^\mu_{;\mu} = \frac{A^2B^2}{A^2B^2r^2 + E^2}\left[\left(2\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)r^2 + \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{E}}{E} + \frac{\dot{C}}{C}\right)\frac{E^2}{A^2B^2}\right]. \qquad (18)$$

The shear tensor measures distortion appearing in the fluid motion given as

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{3}\vartheta h_{\mu\nu}.$$

The nonzero components of the shear tensor are

$$\sigma_{11} = \left[\left(C\dot{B}B - \frac{\dot{C}}{C}\right)A^2B^2r^2 - \left(\frac{2\dot{B}}{B} + \frac{\dot{A}}{A} - \frac{\dot{E}}{E} - \frac{\dot{C}}{C}\right)E^2\right] \times \frac{B^2}{3A(A^2B^2r^2 + E^2)}, \qquad (19)$$

$$\sigma_{22} = \left[\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)A^2B^2r^2 - \left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{2\dot{E}}{E} + \frac{\dot{C}}{C}\right)E^2\right]\frac{1}{3A^2}, \qquad (20)$$

$$\sigma_{33} = \left[2\left(-\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)A^2B^2r^2 + \left(\frac{2\dot{C}}{C} - \frac{\dot{B}}{B} - \frac{\dot{E}}{E} + \frac{\dot{A}}{A}\right)E^2\right] \times \frac{C^2}{3A(A^2B^2r^2 + E^2)}. \qquad (21)$$

The alternative form of shear tensor in terms of two scalar functions σ_1, σ_2 are derived as

$$\sigma_{\mu\nu} = \frac{1}{3}(2\sigma_1 + \sigma_2)\left(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}\right) + \frac{1}{3}(2\sigma_2 + \sigma_1)\left(s_\mu s_\nu - \frac{1}{3}h_{\mu\nu}\right). \qquad (22)$$

Equations (19)–(21) imply that

$$2\sigma_1 + \sigma_2 = \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)\frac{3}{A}, \qquad (23)$$

$$2\sigma_2 + \sigma_1 = \left[\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)AB^2r^2 - \left(\frac{\dot{A}}{A} - \frac{\dot{E}}{E} - \frac{\dot{C}}{C}\right)\frac{E^2}{A}\right]\frac{3}{(A^2B^2r^2 + E^2)}. \qquad (24)$$

The local spinning of the system is defined by the vorticity vector as

$$w_\mu = \frac{1}{2}\eta_{\mu\nu\alpha\beta}u^{\nu;\alpha}u^\beta = \frac{1}{2}\eta_{\mu\nu\alpha\beta}\Omega^{\nu\alpha}u^\beta, \tag{25}$$

where $\eta_{\mu\nu\alpha\beta}$ and $\Omega_{\mu\nu} = u_{[\mu;\nu]} + a_{[\mu}u_{\nu]}$ define Levi-Civita and vorticity tensors, respectively. Another form of vorticity tensor in terms of vorticity scalar function Ω is given as

$$\Omega_{\mu\nu} = \Omega(s_\mu v_\nu - s_\nu v_\mu), \quad \Omega = \frac{E\left(\frac{E'}{E} - \frac{2A'}{A}\right)}{2B(A^2B^2r^2 + E^2)^{\frac{1}{2}}}. \tag{26}$$

Equations (25) and (26) yield $w_\mu = -\Omega k_\mu$. It can be seen from Eq. (26) that $\Omega = 0$ if and only if $E = 0$, i.e. the system becomes vorticity free (spinless) if and only if reflection symmetric term (E) is zero. The expressions of kinematical variables show that they are metric (geometric terms) dependent, there is no contribution from energy terms.

The Weyl tensor describes the tidal force effects given as

$$\begin{aligned} C^\mu_{\alpha\beta\nu} &= R^\mu_{\alpha\beta\nu} - \frac{1}{2}R^\mu_\beta g_{\alpha\nu} + \frac{1}{2}R_{\alpha\mu}\delta^\nu_\beta - \frac{1}{2}R_{\alpha\mu}\delta^\nu_\beta \\ &\quad + \frac{1}{2}R^\mu_\nu g_{\alpha\beta} + \frac{1}{6}R(\delta^\mu_\beta g_{\alpha\nu} - g_{\alpha\beta}\delta^\mu_\nu). \end{aligned} \tag{27}$$

Equations (2) and (27) yield a link between the Weyl tensor and effective energy terms through Riemann/Ricci tensors and Ricci scalar. In this way, the Weyl tensor is also associated with the dynamics of dark sources and defines effects of tidal forces due to gravitational as well as repulsive gravitational forces. The Weyl tensor is further divided into electric and magnetic parts as

$$\overset{(\text{eff})}{\mathbb{E}}_{\mu\nu} = C_{\mu\alpha\nu\beta}u^\mu u^\nu, \quad \overset{(\text{eff})}{M}_{\mu\nu} = \frac{1}{2}\eta_{\mu\alpha\delta\gamma}C^{\delta\gamma}_{\nu\lambda}u^\alpha u^\lambda. \tag{28}$$

There are three nonzero components of electric part while two for the magnetic part. These elements of the Weyl tensor can be written in terms of scalar functions as

$$\begin{aligned} \overset{(\text{eff})}{\mathbb{E}}_{\mu\nu} &= \frac{1}{3}(2\overset{(\text{eff})}{\varepsilon}_I + \overset{(\text{eff})}{\varepsilon}_2)\left(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}\right) \\ &\quad + \frac{1}{3}(2\overset{(\text{eff})}{\varepsilon}_2 + \overset{(\text{eff})}{\varepsilon}_1)\left(s_\mu s_\nu - \frac{1}{3}h_{\mu\nu}\right) + \varepsilon_{vs}(v_\mu s_\nu + v_\nu s_\mu), \end{aligned} \tag{29}$$

$$\overset{(\text{eff})}{M}_{\mu\nu} = \overset{(\text{eff})}{M}_1(k_\mu v_\nu + k_\nu v_\mu) + \overset{(\text{eff})}{M}_2(k_\mu s_\nu + k_\nu s_\mu). \tag{30}$$

3. Structure Scalars

Here we construct a set of structure scalars (scalar functions) through orthogonal splitting of the Riemann tensor.[21] These scalars help to write down the set of governing equations in simpler form. We decompose the Riemann tensor into energy terms with the help of Eqs. (2) and (28) as

$$R^{\mu\nu}_{\alpha\beta} = \overset{(F)}{R}{}^{\mu\nu}_{\alpha\beta} + \overset{(Q)}{R}{}^{\mu\nu}_{\alpha\beta} + \overset{(E)}{R}{}^{\mu\nu}_{\alpha\beta} + \overset{(M)}{R}{}^{\mu\nu}_{\alpha\beta},$$

where

$$\overset{(F)}{R}{}^{\mu\nu}_{\alpha\beta} = \frac{2}{3}(\overset{(\mathrm{eff})}{\rho} + 3\overset{(\mathrm{eff})}{p})u^{[\mu}u_{[\alpha}h^{\nu]}_{\beta]} + \frac{2}{3}\overset{(\mathrm{eff})}{\rho}h^{\mu}_{[\alpha}h^{\nu}_{\beta]}, \tag{31}$$

$$\overset{(Q)}{R}{}^{\mu\nu}_{\alpha\beta} = -2u^{[\mu}h^{\nu]}_{[\alpha}\overset{(\mathrm{eff})}{q}{}_{\beta]}, \tag{32}$$

$$\overset{(E)}{R}{}^{\mu\nu}_{\alpha\beta} = 4u^{[\mu}u_{[\alpha}\mathbb{E}^{(\mathrm{eff})\nu]}_{\beta]} + 4h^{[\mu}_{[\alpha}\mathbb{E}^{(\mathrm{eff})\nu]}_{\beta}, \tag{33}$$

$$\overset{(M)}{R}{}^{\mu\nu}_{\alpha\beta} = -2\epsilon^{\mu\nu\gamma}u_{[\alpha}\overset{(\mathrm{eff})}{M}{}_{\beta]\gamma} - 2\epsilon_{\alpha\beta\gamma}u^{[\mu}\overset{(\mathrm{eff})}{M}{}^{\nu]\gamma}, \tag{34}$$

$\epsilon_{\mu\nu\gamma} = \eta_{\beta\mu\nu\gamma}u^{\beta}$, the notations in top F (density and pressure), Q (heat dissipation), $\mathbb{E}$ (electric part of the Weyl tensor) and M (magnetic part of the Weyl tensor) show decomposed parts of the Riemann curvature tensor relating to various aspects of fluid and carry the effects of dark sources. The triplets of the tensors are given as

$$\overset{(\mathrm{eff})}{Y}{}_{\mu\nu} = R_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta}, \quad \overset{(\mathrm{eff})}{X}{}_{\mu\nu} = \frac{1}{2}\eta^{\delta\lambda}_{\mu\alpha}R^{*}_{\delta\lambda\nu\beta}u^{\alpha}u^{\beta}, \quad \overset{(\mathrm{eff})}{Z}{}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\lambda\delta}R^{\delta\lambda}_{\beta\nu}u^{\beta}.$$

where $R^{*}_{\mu\nu\alpha\beta} = \frac{1}{2}\eta_{\delta\lambda\alpha\beta}R^{\delta\lambda}_{\mu\nu}$ is the dual form of the Riemann tensor. Using Eqs. (28) and (31)–(34), we obtain

$$\overset{(\mathrm{eff})}{Y}{}_{\mu\nu} = \frac{1}{3}\overset{(\mathrm{eff})}{Y}_T h_{\mu\nu} + \frac{1}{3}(2\overset{(\mathrm{eff})}{Y}_{TF1} + \overset{(\mathrm{eff})}{Y}_{TF2})\left(v_{\mu}v_{\nu} - \frac{1}{3}h_{\mu\nu}\right) + \frac{1}{3}(2\overset{(\mathrm{eff})}{Y}_{TF2} + \overset{(\mathrm{eff})}{Y}_{TF1})$$
$$\times\left(s_{\mu}s_{\nu} - \frac{1}{3}h_{\mu\nu}\right) + \overset{(\mathrm{eff})}{Y}_{vs}(v_{\mu}s_{\nu} + v_{\nu}s_{\mu}).$$

Here the subscript T stands for the trace part while the components with notations $TF1$, $TF2$ and vs in subscript represent trace free parts of the corresponding tensor. The scalar quantities

$$\overset{(\mathrm{eff})}{Y}_T = \frac{1}{2}(\overset{(\mathrm{eff})}{\rho} + 3\overset{(\mathrm{eff})}{p}), \quad \overset{(\mathrm{eff})}{Y}_{TF1} = \overset{(\mathrm{eff})}{\varepsilon}_1 - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_1,$$

$$\overset{(\mathrm{eff})}{Y}_{TF2} = \overset{(\mathrm{eff})}{\varepsilon}_2 - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_2, \quad \overset{(\mathrm{eff})}{Y}_{vs} = \overset{(\mathrm{eff})}{\varepsilon}_{vs} - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_{vs},$$

are the set of scalar functions associated with the tensor $\overset{(\mathrm{eff})}{Y}{}_{\mu\nu}$. Likewise, the set of scalar functions associated with the tensor $\overset{(\mathrm{eff})}{X}{}_{\mu\nu}$ are given by

$$\overset{(\mathrm{eff})}{X}_T = \overset{(\mathrm{eff})}{\rho}, \quad \overset{(\mathrm{eff})}{X}_{TF1} = -\overset{(\mathrm{eff})}{\varepsilon}_1 - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_1,$$

$$\overset{(\mathrm{eff})}{X}_{TF2} = -\overset{(\mathrm{eff})}{\varepsilon}_2 - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_2, \quad \overset{(\mathrm{eff})}{X}_{vs} = -\overset{(\mathrm{eff})}{\varepsilon}_{vs} - \frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_{vs},$$

Inserting Eqs. (28) and (31)–(34) in the expression of $\overset{(\mathrm{eff})}{Z}_{\mu\nu}$, we obtain

$$\overset{(\mathrm{eff})}{Z}_{\mu\nu} = \overset{(\mathrm{eff})}{M}_{\mu\nu} + \frac{1}{2}\overset{(\mathrm{eff})}{q}{}^{\delta}{}_{\mu\nu\delta},$$

or equivalently,

$$\overset{(\mathrm{eff})}{Z}_{\mu\nu} = \overset{(\mathrm{eff})}{Z}_1 v_\mu k_\nu + \overset{(\mathrm{eff})}{Z}_2 v_\nu k_\mu + \overset{(\mathrm{eff})}{Z}_3 s_\mu k_\nu + \overset{(\mathrm{eff})}{Z}_4 s_\nu k_\mu.$$

Here

$$\overset{(\mathrm{eff})}{Z}_1 = \left(\overset{(\mathrm{eff})}{M}_1 - \frac{1}{2}\overset{(\mathrm{eff})}{q}_2\right), \quad \overset{(\mathrm{eff})}{Z}_2 = \left(\overset{(\mathrm{eff})}{M}_1 + \frac{1}{2}\overset{(\mathrm{eff})}{q}_2\right),$$

$$\overset{(\mathrm{eff})}{Z}_3 = \left(\overset{(\mathrm{eff})}{M}_2 - \frac{1}{2}\overset{(\mathrm{eff})}{q}_1\right), \quad \overset{(\mathrm{eff})}{Z}_4 = \left(\overset{(\mathrm{eff})}{M}_2 + \frac{1}{2}\overset{(\mathrm{eff})}{q}_1\right),$$

are the structure scalars related to $\overset{(\mathrm{eff})}{Z}_{\mu\nu}$.

Now we briefly discuss dynamical aspects of scalar quantities under the dark effects of $f(G)$ gravity. The set of scalar functions describe contribution of matter as well as dark sources in the evolution of axially symmetric self-gravitating systems. The scalar $\overset{(\mathrm{eff})}{X}_T$ expresses the total energy density of matter and dark sources of the system. Three scalars $\overset{(\mathrm{eff})}{X}_{TF1}$, $\overset{(\mathrm{eff})}{X}_{TF2}$ as well as $\overset{(\mathrm{eff})}{X}_{vs}$ indicate combine effects of anisotropy and electric part of the Weyl tensor in one and the same direction. Another scalar $\overset{(\mathrm{eff})}{Y}_T$ represents sum of energy density and pressure (total energy) of the system. The scalar $\overset{(\mathrm{eff})}{Y}_{TF1}$, $\overset{(\mathrm{eff})}{Y}_{TF2}$ as well as $\overset{(\mathrm{eff})}{Y}_{vs}$ provide combine effects of anisotropy and electric part of the Weyl tensor in opposite directions. The set of scalar functions $\overset{(\mathrm{eff})}{Z}_1$, $\overset{(\mathrm{eff})}{Z}_2$, $\overset{(\mathrm{eff})}{Z}_3$, $\overset{(\mathrm{eff})}{Z}_4$ represent various combinations of heat dissipation and magnetic parts of the Weyl tensor. We note that matter as well as dark sources take part in the dynamics of any axial symmetric system. If we neglect matter contributions, then dark sources become responsible for the dynamics of the system.

4. Evolution of Axially Symmetric Dissipative Fluid

Here we discuss heat transport equation and explore the evolution of axially/reflection symmetric dissipative fluid through the set of evolution equations associated with $f(G)$ gravity.

4.1. *Heat transport equation*

The heat transport equation elucidates dissipation process and propagation of thermal energy inside the system.[22] The nonvanishing value of time relaxation parameter τ (a positive definite quantity having different physical meaning) serves as a cardinal parameter in this equation which defines the time interval in which the system returns to its steady state. Consequently, studying transient regimes, e.g. the evolution between two steady states, the role of τ cannot be ignored. The heat transport equation for propagation of thermal perturbations under the effects of $f(G)$ gravity is given as

$$\tau h^{\alpha}_{\beta} q^{\beta}_{;\mu} u^{\mu} + \overset{(\mathrm{eff})}{q^{\alpha}} = -K h^{\alpha\beta}(\mathbb{T}_{,\beta} + \mathbb{T} a_{\beta}) - \frac{1}{2} K \mathbb{T}^2 \left(\frac{\tau u^{\mu}}{K\mathbb{T}^2}\right)_{;\mu} \overset{(\mathrm{eff})}{q^{\alpha}}, \tag{35}$$

where K and $\mathbb{T}$ represent thermal conductivity and temperature, respectively. By contracting the above equation with s_{α} and using Eqs. (17) and (26), we obtain

$$\frac{\tau}{A}(\dot{q}_2^{(\mathrm{eff})} + A \overset{(\mathrm{eff})}{q_1} \Omega) + \overset{(\mathrm{eff})}{q_2} = \frac{K}{A}\left(\frac{-E\dot{\mathbb{T}} + A^2 \mathbb{T}^{\theta}}{(A^2B^2r^2 + E^2)} - A\mathbb{T}a_2\right) - \frac{K\mathbb{T}^2 q_2}{2}\left(\frac{\tau u^{\mu}}{K\mathbb{T}^2}\right)_{;\mu}. \tag{36}$$

Similarly, the contraction of Eq. (35) with v_{α} gives

$$\frac{\tau}{A}(\dot{q}_1^{(\mathrm{eff})} - A \overset{(\mathrm{eff})}{q_2} \Omega) + \overset{(\mathrm{eff})}{q_1} = -\frac{K}{A}(\mathbb{T}' + B\mathbb{T}a_1) - \frac{K\mathbb{T}^2 q_1}{2}\left(\frac{\tau u^{\mu}}{K\mathbb{T}^2}\right)_{;\mu}. \tag{37}$$

These two equations describe the effective thermal energy transport (thermal transport due to matter as well as dark sources) in the presence of vorticity (spinning configuration).

4.2. *Evolution of the system*

The set of evolution equations depending upon dynamical variables give the dynamics of collapsing stellar configuration. We have obtained this set of equations for spherical as well as cylindrical symmetric configurations.[12] Depending upon the problem under consideration (for axially symmetric configurations), these equations are derived through the contraction of Bianchi and Ricci identities which are then converted into scalar form by using structure scalars and projecting them with all possible combinations of 4-vectors ($\mathbf{u}, \mathbf{v}, \mathbf{s}$ and $\mathbf{k}$).[23] In the following, we formulate these equations in the presence of dark sources under the effects of $f(G)$ gravity.

4.2.1. *Conservation equations*

From the conservation law, $\overset{(\mathrm{eff})}{T^{\mu}}_{\nu;\mu} = 0$, we obtain two conservation equations in terms of scalar quantities as

$$\overset{(\mathrm{eff})}{X}_{T;\mu} u^{\mu} + 2\vartheta(\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{p}) + q^{\mu}_{;\mu} + q^{\mu} a_{\mu} + \frac{1}{9}[(2\sigma_1 + \sigma_2)\Pi_1 + (\sigma_1 + 2\sigma_2)\Pi_2] = 0, \tag{38}$$

$$2a_\mu(\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{p}) + h^\nu_\mu\left(\frac{1}{3}(2\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{X_T})_{;\nu} + \overset{(\mathrm{eff})}{\Pi^\alpha}{}_{\nu;\alpha} + \overset{(\mathrm{eff})}{q}{}_{\nu;\alpha}u^\alpha\right)$$

$$+\left(\frac{4}{3}\vartheta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}\right)\overset{(\mathrm{eff})}{q^\nu} = 0. \qquad (39)$$

The first is the continuity equation which gives the evolution of energy density in the presence of expansion scalar, heat dissipation, shear, pressure anisotropy as well as the scalar $\overset{(\mathrm{eff})}{Y_T}$ and the second is named as the Euler equation. In order to explore thermodynamical effects, we use Eqs. (35)–(37) and (39). The combination of Eqs. (35) and (39) leads to effective inertial mass while Eqs. (36) and (37) provide the relation between thermodynamics and vorticity. Just analogous to inertial mass defined in classical dynamics (Newton's second law), a similar concept also exists in instantaneous rest-frame in relativistic theory. Since in instantaneous rest-frame, the acceleration is parallel (proportional) to the applied force, so the inertial mass behaves as a factor of proportionality among them.[24] However, in some cases, when there is no interaction between particles, this factor of proportionality does not represent mass of the particles. In such a case, this factor is referred to effective inertial mass.

The value of effective inertial mass of a particle moving through a solid body (like crystal) may differ from the value calculated for the same particle moving in free space under the same force.[25] In the present case, the combination of Eqs. (35) and (39) gives

$$\left[2(\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{p}) - \frac{K\mathbb{T}}{\tau}\right]a_\mu = -h^\nu_\mu\Pi^{(\mathrm{eff})\alpha}_{\nu;\alpha} - \frac{1}{3}h^\nu_\mu(2\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{X_T})_{,\nu} - (\sigma_{\mu\nu} + \omega_{\mu\nu})\overset{(\mathrm{eff})}{q^\nu}$$

$$+\frac{K}{\tau}h^\nu_\mu\mathbb{T}_{,\nu} + \left[\frac{1}{\tau} - \frac{5}{6}\vartheta + \frac{1}{2}D_t\left(ln\left(\frac{\tau}{K\mathbb{T}^2}\right)\right)\right]q_\mu, \qquad (40)$$

where $D_t f = f_{,\nu}u^\nu$. The expression on the right hand side contains some extra terms other than dissipative terms which represent hydrodynamic force acting upon fluid and dark sources. The factor multiplying with 4-acceleration on the left side depicts effective inertial mass density given as

$$2(\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{p})(1 - \overset{(\mathrm{eff})}{\Psi})$$

with $\overset{(\mathrm{eff})}{\Psi} = \frac{K\mathbb{T}}{2\tau(\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{p})}$. When the system deviates from thermal equilibrium state, the effective inertial mass density of dissipative fluid is diminished by a factor $(1 - \overset{(\mathrm{eff})}{\Psi})$. It disappears for $\overset{(\mathrm{eff})}{\Psi} = 1$ or even becomes negative for $\overset{(\mathrm{eff})}{\Psi} > 1$. It can be observed that the factor $\overset{(\mathrm{eff})}{\Psi}$ depends upon the values of temperature, the scalar $\overset{(\mathrm{eff})}{Y_T}$ as well as pressure in the presence of matter and dark sources. It is mentioned here that in GR, the effective inertial mass density of the dissipative fluid is given by $(\overset{(\mathrm{m})}{\rho} + \overset{(\mathrm{m})}{p})$ which is reduced by a factor $\Psi = \frac{K\mathbb{T}}{\tau(\overset{(\mathrm{m})}{\rho} + \overset{(\mathrm{m})}{p})}$. In $f(G)$ gravity, the generalized effective inertial mass density becomes

$$2(\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{p}) = 2(\overset{(\mathrm{m})}{Y_T} - \overset{(\mathrm{m})}{p}) + [7Rg_{\rho\sigma} + 4Ru_\rho u_\sigma - 4h_{\rho\sigma}] \times \nabla^\rho\nabla^\sigma f_G + 2(Gf_G - f),$$

which is the sum of matter and dark source terms. In the absence of dark sources, this reduces to $2(\overset{(\mathrm{eff})}{Y_T} - \overset{(\mathrm{eff})}{p}) = 2(\overset{(\mathrm{m})}{Y_T} - \overset{(\mathrm{m})}{p})$. Thus the dark source terms affect thermodynamics of dissipative collapsing system.

Now we check the relationship between thermodynamics and vorticity given in Eqs. (36) and (37). For this purpose, we consider the system is in thermodynamic equilibrium in θ direction and we assume $\overset{(\mathrm{eff})}{q_2} = 0$ with constant (corresponding) temperature. Under these considerations, Eq. (36) gives

$$\dot{q}_2^{(\mathrm{eff})} = -A\Omega \overset{(\mathrm{eff})}{q_1}$$

showing the spinning configuration (vorticity) or heat flux of matter in r-direction which controls the vanishing of time propagation of meridional flow (thermal equilibrium in θ direction). Inversely, under the same type of assumption in Eq. (37), $(\overset{(\mathrm{eff})}{q_1})_{,t} = A\Omega \overset{(\mathrm{eff})}{q_2}$ shows that time propagation of the vanishing of radial heat flux at initial time will depend upon the values of vorticity and meridional heat flux. From the above discussion, we note that dark sources along with matter terms take part in time propagation of thermal equilibrium in either direction (r and θ). In GR, it depends only on matter and vorticity.

4.2.2. *Ricci evolutionary equations and kinematic variables*

Here we discuss the evolution of kinematical quantities and effects of dark sources through Ricci evolutionary equations (propagation equations of expansion, shear/vorticity tensors and constraint equations). The time propagation equation of expansion scalar (ϑ) is derived by contracting Ricci identities for 4-velocity vector u_μ.[23] It is given in terms of scalars as

$$\vartheta_{;\mu}u^\mu + \frac{1}{3}\vartheta^2 + 2(\sigma^2 - \Omega^2) - a^\mu_{;\mu} + \overset{(\mathrm{eff})}{Y_T} = 0, \tag{41}$$

where $\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu}$. We note from this equation that for axial symmetric system, the evolution of expansion parameter depends upon shear velocity and the scalar $\overset{(\mathrm{eff})}{Y_T}$ in geodesic as well as nongeodesic cases. In the absence of matter, dark sources control the evolution of expansion scalar.

The propagation equation of shear tensor is obtained by contracting Ricci identities with 4-velocity as well as the combination of projection tensor and unit 4-vectors given by

$$h^{\alpha}_{\mu}h^{\beta}_{\nu}\sigma_{\alpha\beta;\gamma}u^{\gamma}+\sigma^{\alpha}_{\mu}\sigma_{\nu\alpha}+\frac{2}{3}\vartheta\sigma_{\mu\nu}-\frac{1}{3}(2\sigma^{2}+\Omega^{2}-a^{\gamma}_{;\gamma})h_{\mu\nu}$$
$$+w_{\mu}w_{\nu}-a_{\mu}a_{\nu}-h^{\alpha}_{(\mu}h^{\beta}_{\nu)}a_{\alpha;\beta}+\overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\nu}-\frac{1}{2}\overset{(\mathrm{eff})}{\Pi}_{\mu\nu}=0.$$

Additionally, contracting this equation with **ss**, **vv** and **vs**, respectively, it follows that

$$\sigma_{2,\gamma}u^{\gamma}+\frac{1}{3}\sigma_{2}(\sigma_{2}+2\vartheta)-(2\sigma^{2}+\Omega^{2}-a^{\gamma}_{;\gamma})-3(s^{\alpha}s^{\beta}a_{\alpha;\beta}+a_{2}^{2})+\overset{(\mathrm{eff})}{Y}_{TF2}=0, \quad (42)$$

$$\sigma_{1,\gamma}u^{\gamma}+\frac{1}{3}\sigma_{1}(\sigma_{1}+2\vartheta)-(2\sigma^{2}+\Omega^{2}-a^{\gamma}_{;\gamma})-3(v^{\alpha}v^{\beta}a_{\beta;\alpha}+a_{1})+\overset{(\mathrm{eff})}{Y}_{TF1}=0, \quad (43)$$

$$\frac{1}{3}(\sigma_{1}-\sigma_{2})\Omega-a_{1}a_{2}-v^{(\alpha}s^{\beta)}a_{\alpha;\beta}+\overset{(\mathrm{eff})}{Y}_{vs}=0. \quad (44)$$

Equations (42) and (43) demonstrate that the evolution of shear depends on vorticity and scalars $\overset{(\mathrm{eff})}{Y}_{TF1}$, $\overset{(\mathrm{eff})}{Y}_{TF2}$. If we consider geodesic fluid with vorticity free condition, then expansion parameter and scalars $\overset{(\mathrm{eff})}{Y}_{TF1}$, $\overset{(\mathrm{eff})}{Y}_{TF2}$ will control the evolution of shear scalars. Since $\overset{(\mathrm{eff})}{Y}_{TF1}$ and $\overset{(\mathrm{eff})}{Y}_{TF2}$ contain pressure anisotropy and electric part of the Weyl tensor, therefore these scalars will affect the shearing collapsing fluid. Equation (44) in geodesic case with vorticity free condition gives $\overset{(\mathrm{eff})}{Y}_{vs}=0$.

Constraint equations are obtained as

$$h^{\nu}_{\mu}\left(\frac{2}{3}\vartheta_{;\nu}-\sigma^{\alpha}_{\nu;\alpha}+\Omega^{\alpha}_{\nu;\alpha}\right)+(\sigma_{\mu\nu}+\Omega_{\mu\nu})\,a^{\nu}=\overset{(\mathrm{eff})}{q}_{\mu}, \quad (45)$$

$$2w_{(\mu}a_{\nu)}+h^{\alpha}_{(\mu}h_{\nu)\beta}(\sigma_{\alpha\gamma}+\Omega_{\alpha\gamma)};_{\delta}\eta^{\beta\kappa\delta\gamma}u_{\kappa}=\overset{(\mathrm{eff})}{M}_{\mu\nu}. \quad (46)$$

The contraction of Eq. (45) with **v** and **s** gives the following scalar equations

$$\frac{2}{3B}\vartheta'-\Omega_{;\alpha}s^{\alpha}+\Omega(s_{\nu;\beta}v^{\beta}v^{\nu}-s^{\beta}_{;\beta})+\frac{1}{3}a_{I}\sigma_{1}-a_{2}\Omega-\frac{1}{3}\sigma_{1;\alpha}v^{\alpha}$$
$$-\frac{1}{3}(2\sigma_{1}+\sigma_{2})\left(v^{\alpha}_{;\alpha}-\frac{a_{1}}{3}\right)-\frac{1}{3}(\sigma_{1}+2\sigma_{2})\left(s_{\nu;\alpha}s^{\alpha}s^{\nu}-\frac{a_{1}}{3}\right)=\overset{(\mathrm{eff})}{q}_{1}, \quad (47)$$

$$\frac{1}{3}(E^{2}+A^{2}B^{2}r^{2})^{\frac{1}{2}}\left(2A\vartheta^{\theta}+\frac{2E}{A}\dot{\vartheta}\right)+\frac{\sigma_{2}a_{2}}{3}+\omega_{;\alpha}v^{\alpha}+\Omega(v^{\alpha}_{;\alpha}$$
$$+s^{\beta}v^{\nu}s_{\nu;\beta})+\Omega a_{I}-\frac{1}{3}\sigma_{2;\alpha}s^{\alpha}+\frac{1}{3}(\sigma_{2}+2\sigma_{1})\left(s_{\nu;\alpha}s^{\alpha}s^{\nu}-\frac{a_{2}}{3}\right)$$
$$-\frac{1}{3}(\sigma_{1}+2\sigma_{2})\left(s^{\alpha}_{;\alpha}-\frac{a_{2}}{3}\right)=\overset{(\mathrm{eff})}{q}_{2}. \quad (48)$$

Similarly, contraction of Eq. (46) with **vk** and **sk** provide

$$-\Omega a_1 - \frac{1}{2}(v^\alpha k_\beta + k^\alpha v_\beta)(\sigma_{\alpha\delta} + \Omega_{\alpha\delta})_{;\gamma}\epsilon^{\beta\gamma\delta} = \overset{(\text{eff})}{M_1}, \tag{49}$$

$$-\Omega a_2 - \frac{1}{2}(s^\alpha k_\beta + k^\alpha s_\beta)(\sigma_{\alpha\delta} + \Omega_{\alpha\delta})_{;\gamma}\epsilon^{\beta\gamma\delta} = \overset{(\text{eff})}{M_2}. \tag{50}$$

Equation (47) demonstrates that effective dissipation scalar rules the propagation of vorticity in the case of shear and expansion free geodesic fluid. In this way, dark sources affect the vorticity of fluid. Equations (47)–(50) exhibit the relations between $\overset{(\text{eff})}{M_1}$, $\overset{(\text{eff})}{M_2}$, $\overset{(\text{eff})}{q_1}$, $\overset{(\text{eff})}{q_2}$, shear and vorticity. Thus these relations also affect the evolution of expansion scalar.

The time propagation equation for the vorticity tensor $\Omega_{\mu\nu}$ can be derived from Ricci identity by contracting it with the combination of projection tensor and 4-velocity vector as[14]

$$h^\alpha_\mu h^\beta_\nu \Omega_{\alpha\beta;\gamma}u^\gamma + \frac{2}{3}\vartheta\sigma_{\mu\nu} + 2\sigma_{\alpha[\mu}\Omega^\alpha_{\nu]} - h^\alpha_{[\mu}h^\beta_{\nu]}a_{\mu;\nu} = 0. \tag{51}$$

Contraction of the above equation with **vs** yields

$$\Omega_{,\alpha}u^\alpha + \frac{1}{3}(2\vartheta + \sigma_1 + \sigma_2)\Omega + v^{[\mu}s^{\nu]}a_{\mu;\nu} = 0, \tag{52}$$

which expresses the evolution of vorticity. We note that it does not depend on dark term even in the presence of dark sources due to $f(G)$ gravity. Hence for general fluid, vorticity is not affected by dark sources.

4.2.3. *Bianchi evolutionary equations and density inhomogeneity*

Bianchi evolutionary equations or evolution equations for the Weyl tensor are given in Appendix A. The Weyl tensor usually narrates the effects of gravity due to tidal force in the universe. In our case, it describes both attractive (gravity) as well as dark source (repulsive) effects due to the coupling of tidal force with dark sources. Evolution equations for various components of the Weyl tensor establish relations among structural scalars, dark sources and tidal force. These equations evaluate inhomogeneity (irregularity) factors in the system. The collection of dynamical variables which causes density inhomogeneity is known as density inhomogeneity factor. The vanishing of such combination of dynamical variables is the necessary and sufficient for homogeneity of energy density ($h^\nu_\mu \overset{(\text{eff})}{\rho}_{,\nu} = 0$). The two Weyl equations are obtained by contracting Eq. (A.2) with **v**, **s** and using scalar functions

$$\begin{aligned} &-\frac{1}{3}\overset{(\text{eff})}{X}_{TF1,\nu}v^\nu - \overset{(\text{eff})}{X}_{vs,\nu}s^\nu - \frac{1}{3}(2\overset{(\text{eff})}{X}_{TF1} + \overset{(\text{eff})}{X}_{TF2})(v^\nu_{;\nu} - a_\beta v^\beta) - \frac{1}{3}s_{\alpha;\nu}s^\nu v^\alpha \\ &\quad \times (\overset{(\text{eff})}{X}_{TF1} + 2\overset{(\text{eff})}{X}_{TF2}) - \overset{(\text{eff})}{X}_{vs}(s_{\alpha;\nu}v^\alpha v^\nu + s^\nu_{;\nu} - a_\nu s^\nu) - \frac{1}{3}\overset{(\text{eff})}{M_2}(\sigma_1 + 2\sigma_2) - 3\Omega\overset{(\text{eff})}{M_1} \\ &= \frac{1}{3}\overset{(\text{eff})}{\rho}_{;\nu}v^\nu - \frac{1}{6}\overset{(\text{eff})}{q_1}(2\vartheta - \sigma_1) + 4\Omega\overset{(\text{eff})}{q_2}, \end{aligned} \tag{53}$$

$$\frac{1}{3}\overset{(\text{eff})}{X}_{_{TF2\nu}} s^{\nu} - \overset{(\text{eff})}{X}_{_{vs,\nu}} v^{\nu} - \frac{1}{3}(\overset{(\text{eff})}{X}_{_{TF1}} + 2\overset{(\text{eff})}{X}_{_{TF2}})(s^{\nu}_{;\nu} - a_{\nu}s^{\nu}) - \frac{1}{3}(2\overset{(\text{eff})}{X}_{_{TF1}} + \overset{(\text{eff})}{X}_{_{TF2}})$$
$$\times v_{\alpha;\nu}s^{\alpha}v^{\nu} - \overset{(\text{eff})}{X}_{_{vs}}(v_{\alpha;\nu}s^{\alpha}s^{\nu} + v^{\nu}_{;\nu} - a_{\nu}v^{\nu}) + \frac{1}{3}\overset{(\text{eff})}{M}_{_1}(2\sigma_{_1} + \sigma_{_2}) - 3\Omega\overset{(\text{eff})}{M}_{_2}$$
$$= \frac{1}{3}\overset{(\text{eff})}{\rho}_{;\nu}s^{\nu} - 4\Omega\overset{(\text{eff})}{q}_{_1} - \frac{\overset{(\text{eff})}{q}_{_2}}{6}(2\vartheta - \sigma_{_2}), \tag{54}$$

where the scalars $\overset{(\text{eff})}{X}_{_{TF1}}$, $\overset{(\text{eff})}{X}_{_{TF2}}$, $\overset{(\text{eff})}{X}_{vs}$, $\overset{(\text{eff})}{Z}_{_1}$, $\overset{(\text{eff})}{Z}_{_2}$, $\overset{(\text{eff})}{Z}_{_3}$ and $\overset{(\text{eff})}{X}_{_4}$ generate energy density inhomogeneity. In order to investigate the role of dark sources in energy density inhomogeneity, we suppose that matter contribution is almost absent which leaves only dark source terms. This indicates dark sources as agent of energy density inhomogeneity. In the absence of $f(G)$ terms, magnetic part of the Weyl tensor produces irregularity in the respective region. Thus the inhomogeneous distribution of dark sources leads to heat dissipation and dark source effects in the interstellar region to produce energy density inhomogeneity. Since the magnetic part of the Weyl tensor as well as the dark source terms are not zero, therefore, the axial system remains inhomogeneous in this gravity.

5. Final Remarks

In this paper, we have explored the evolution of axially and reflection symmetric dissipative collapsing fluid under the influence of dark sources due to $f(G)$ gravity. For this purpose, we have considered the modified gravity coupled with dissipative anisotropic fluid and constructed structure scalars through orthogonal splitting of the Riemann tensor. We have formulated the set of evolution equations in terms of these scalars reflecting the physical meanings and their role in the dynamics of the system. We have found that the inhomogeneous distributions of dark sources generate pressure anisotropy and heat dissipation while the kinematical variables are not disturbed under influence of dark sources. The summary of results is given as follows.

We have evaluated tidal effects through the Weyl tensor in the presence of dark sources which deals with gravitational (attractive) effects of tidal force. This represents the effects for both gravitational as well as repulsive forces. We have constructed a set of 12 scalar functions which further consist of energy terms (energy density, pressure, dissipation, anisotropy), electric and magnetic parts of the Weyl tensor. These scalars simplify the complexities of the dynamical analysis of the system. We have also formulated heat transport equation and a set of evolution equations which demonstrate effects of dark sources in thermodynamics as well as dynamics of dissipative collapsing fluid. We have studied thermodynamical aspects of the system through the combination of transport equation and the Euler equation of motion. It is found from Eq. (39) that inertial mass of the system is reduced by a factor depending upon the thermal effects as well as dark sources. The coupling of heat flux with vorticity controls thermal equilibrium in r and θ directions of the fluid flow.

We have seen that kinematical variables remain unchanged under the influence of dark sources but their evolution is effected by it. Evolution of expansion scalar is determined by shear, vorticity and $\overset{(\mathrm{eff})}{Y_T}$ in geodesic as well as nongeodesic case. The two scalars $\overset{(\mathrm{eff})}{Y}_{_{TF1}}$ and $\overset{(\mathrm{eff})}{Y}_{_{TF2}}$ control the evolution of propagation equation of shear while the evolution of vorticity does not depend upon dark sources but its absence in geodesic case leaves $\overset{(\mathrm{eff})}{Y}_{_{vs}}=0$. From the evolution equations of the Weyl tensor, we have found that the set of seven scalars $\overset{(\mathrm{eff})}{X}_{_{TF1}}$, $\overset{(\mathrm{eff})}{X}_{_{TF2}}$, $\overset{(\mathrm{eff})}{X}_{vs}$, $\overset{(\mathrm{eff})}{Z}_{_1}$, $\overset{(\mathrm{eff})}{Z}_{_2}$, $\overset{(\mathrm{eff})}{Z}_{_3}$ and $\overset{(\mathrm{eff})}{X}_{_4}$ are the inhomogeneity factors. The presence of dark sources in these scalars also produce pressure anisotropy, heat dissipation and repulsive tidal forces which generate density irregularity in the system. In GR, all the collapsing factors depend only upon matter contribution but in our case, dark sources also play crucial role. Thus the presence of dark sources not only explain the cosmic expansion but also affect the collapsing configuration.

Appendix A. Bianchi Evolutionary Equations

These equations are derived by using Eqs. (2) and (27) in Bianchi identities $(R_{\mu\nu\gamma\delta;\beta}+R_{\mu\nu\beta\kappa;\delta}+R_{\mu\nu\delta\beta;\kappa}=0)$ given as

$$\begin{aligned}
&h^{\alpha}_{(\mu}h^{\beta}_{\nu)}\overset{(\mathrm{eff})}{\mathbb{E}}_{\alpha\beta;\gamma}u^{\gamma}+\vartheta\overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\nu}+h_{\mu\nu}\overset{(\mathrm{eff})}{\mathbb{E}}_{\alpha\beta}\sigma^{\alpha\beta}-3\overset{(\mathrm{eff})}{\mathbb{E}}_{\alpha(\mu}\sigma^{\alpha}_{\nu)}+h^{\alpha}_{(\mu}\eta^{\gamma\delta\kappa}_{\nu)}u_{\delta}\overset{(\mathrm{eff})}{M}_{\gamma\alpha;\kappa}\\
&\quad-\overset{(\mathrm{eff})}{\mathbb{E}}_{\delta(\mu}\Omega^{\delta}_{\nu)}-2\overset{(\mathrm{eff})}{M^{\alpha}}_{(\mu}\eta_{\nu)\delta\alpha\kappa}u^{\delta}a^{\kappa}=\frac{1}{2}(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p})\sigma_{\mu\nu}-\frac{1}{6}\vartheta\overset{(\mathrm{eff})}{\Pi}_{\mu\nu}\\
&\quad-\frac{1}{2}h^{\alpha}_{(\mu}h^{\beta}_{\nu)}\overset{(\mathrm{eff})}{\Pi}_{\alpha\beta;\delta}u^{\delta}-\frac{1}{2}\sigma_{\alpha(\mu}\overset{(\mathrm{eff})}{\Pi^{\alpha}}_{\nu)}-\frac{1}{2}\Omega^{\alpha}_{(\mu}\overset{(\mathrm{eff})}{\Pi}_{\nu)\alpha}-a_{(\mu}\overset{(\mathrm{eff})}{q}_{\nu)}\\
&\quad+\frac{1}{6}(\overset{(\mathrm{eff})}{\Pi}_{\alpha\beta}\sigma^{\alpha\beta}+a_{\alpha}\overset{(\mathrm{eff})}{q^{\alpha}}+\overset{(\mathrm{eff})}{q^{\alpha}}_{;\alpha})h_{\mu\nu}-\frac{1}{2}h^{\alpha}_{(\mu}h^{\beta}_{\nu)}\overset{(\mathrm{eff})}{q}_{\beta;\alpha},
\end{aligned}\tag{A.1}$$

$$\begin{aligned}
&h^{\alpha}_{\mu}h^{\beta\nu}\overset{(\mathrm{eff})}{\mathbb{E}}_{\alpha\beta;\nu}-\eta^{\delta\beta\kappa}_{\mu}u_{\delta}\sigma^{\gamma}_{\beta}\overset{(\mathrm{eff})}{M}_{\kappa\gamma}+3\overset{(\mathrm{eff})}{M}_{\mu\nu}w^{\nu}\\
&\quad=\frac{1}{3}h^{\nu}_{\mu}\overset{(\mathrm{eff})}{\rho}_{;\nu}-\frac{1}{2}h^{\nu}_{\mu}h^{\alpha\beta}\overset{(\mathrm{eff})}{\Pi}_{\nu\beta;\alpha}-\frac{1}{2}\left(\frac{2}{3}\vartheta h^{\nu}_{\mu}-\sigma^{\nu}_{\mu}+3\Omega^{\nu}_{\mu}\right)\overset{(\mathrm{eff})}{q}_{\nu},
\end{aligned}\tag{A.2}$$

$$\begin{aligned}
&(\sigma_{\mu\delta}\overset{(\mathrm{eff})}{\mathbb{E}^{\delta}}_{\nu}+3\Omega_{\mu\delta}\overset{(\mathrm{eff})}{\mathbb{E}^{\delta}}_{\nu})\epsilon^{\mu\nu}_{\kappa}+a^{\beta}\overset{(\mathrm{eff})}{M}_{\beta\kappa}-\overset{(\mathrm{eff})}{M^{\beta\gamma}}_{;\gamma}h_{\beta\kappa}\\
&\quad=\frac{1}{2}(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p})\Omega_{\mu\nu}\epsilon^{\mu\nu}_{\kappa}+\frac{1}{2}\left[\overset{(\mathrm{eff})}{q}_{\mu;\nu}+\overset{(\mathrm{eff})}{\Pi}_{\beta\nu}(\sigma^{\beta}_{\nu}+\Omega^{\beta}_{\nu})\right]\epsilon^{\mu\nu}_{\kappa},
\end{aligned}\tag{A.3}$$

$$2a_\nu \overset{(\text{eff})}{\mathbb{E}}_{\mu\kappa}\epsilon_\gamma^{\mu\nu} - \overset{(\text{eff})}{\mathbb{E}}_{\beta\nu;\delta}h_\kappa^\beta\epsilon_\gamma^{\delta\nu} + \overset{(\text{eff})}{\mathbb{E}^\delta}_{\nu;\delta}\epsilon_{\gamma\kappa}^\nu + \frac{2}{3}\vartheta \overset{(\text{eff})}{M}_{\kappa\gamma} + \overset{(\text{eff})}{M^\alpha}_{\beta;\delta}u^\delta h_\kappa^\beta h_{\alpha\gamma}$$
$$- \overset{(\text{eff})}{M^\delta}_\gamma(\sigma_{\delta\kappa} + \Omega_{\delta\kappa}) + (\sigma_{\nu\delta} + \Omega_{\nu\delta})\overset{(\text{eff})}{M^\alpha}_\mu\epsilon_{\kappa\alpha}^\delta\epsilon_\gamma^{\mu\nu} + \frac{1}{3}\vartheta\overset{(\text{eff})}{M^\alpha}_\mu\epsilon_{\kappa\alpha}^\delta\epsilon_{\gamma\delta}^\mu$$
$$= \frac{1}{6}\overset{(\text{eff})}{\rho}_{;\nu}\epsilon_{\gamma\kappa}^\nu + \frac{1}{2}\overset{(\text{eff})}{\Pi}_{\mu\beta;\nu}h_\kappa^\beta\epsilon_\gamma^{\mu\nu}$$
$$+ \frac{1}{2}\left[\overset{(\text{eff})}{q}_\kappa\Omega_{\mu\nu} + \overset{(\text{eff})}{q}_\mu\left(\sigma_{\kappa\nu} + \Omega_{\kappa\nu} + \frac{1}{3}\vartheta h_{\kappa\nu}\right)\right]\epsilon_\gamma^{\mu\nu}. \quad \text{(A.4)}$$

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Shear-free axial system and f(G) gravity

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Abstract. This paper investigates the influences of dark sources on the evolution of shear-free axially symmetric dissipative fluid in $f(G)$ gravity. Matter contents (energy terms of the system), corresponding dynamical variables as well as scalar functions are derived in the presence of dark sources. The dark terms appear as one of the sources producing pressure anisotropy and heat dissipation. We then study non-geodesic and geodesic fluids with and without dissipation under shear-free condition. The non-geodesic (non-dissipative) fluid gives inhomogeneous expansion while the geodesic fluid makes the system either vorticity-free or expansion-free. The vorticity-free non-dissipative geodesic fluid reduces the axial system to FRW model in the presence of homogeneous distribution of dark sources while the expansion-free geodesic fluid does not exist even in the presence of dark sources.

1 Introduction

On the basis of recent cosmological observations, it is believed that there is an obscure type of energy possessing repulsive force pushing various cosmic objects far away from each other against their gravitational force. This strange energy is termed as dark energy and is considered as the pivotal constituent for cosmic accelerated expansion. In order to study puzzling effects of dark energy and cosmic accelerated expansion, the Einstein-Hilbert action has been altered leading to various modified (alternative) theories of gravity. Modified Gauss-Bonnet (GB) gravity or $f(G)$ gravity is one of the modified versions of general relativity (GR). This is obtained by adding an arbitrary function of GB quadratic invariant G ($f(G)$) in the action [1], where $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ (R, $R_{\mu\nu}$, $R_{\mu\nu\sigma\rho}$ are the Ricci scalar, Ricci and the Riemann tensors, respectively).

This theory efficiently elucidates cosmic accelerated expansion and transition phases of the universe from deceleration to acceleration [2]. The solar system experiments are essential constrains on modified gravity that measure the deviation of these theories from GR. A viable gravity model must satisfy these constraints and $f(G)$ theory passes all solar system tests [3]. Furthermore, it explains black hole thermodynamics [4] and avoids all possible four types of finite time future singularities [5]. This theory has also been explored to study dark energy as well as inflationary era [6]. We have investigated energy conditions, wormhole solutions, built-in inflation, Noether symmetries, spherical wormhole solutions with conformal symmetry as well as dynamics of self-gravitating fluids [7–14].

Stellar objects undergo various phases during their evolution due to different kinematical factors including the shear, which measures distortion in configuration preserving the volume. The role of the shear tensor during the evolution of stellar objects and the consequences emerging from its vanishing have attracted many researchers. Glass [15] observed that the shear-free condition leaves a perfect fluid irrotational if and only if the magnetic part of the Weyl tensor vanishes. Collins and Wainwright [16] showed that the class of shear-free evolving irrotational perfect fluids having equation of state $p = p(\rho)$ is either an FRW model or a special case of plane symmetric models. Tomimura and Nunes [17] investigated a radiating collapse with heat flow of shear-free geodesic fluid. Herrera *et al.* [18] investigated the stability of spherically symmetric anisotropic matter distribution under this condition. Herrera and his collaborators [19,20] provided a comprehensive analysis of shear-free rotating fluid in the presence of pressure anisotropy and heat dissipation.

Modified theories are consistent with GR in the weak field regime but may deviate in the strong field. Gravitational collapse is classified as the phenomenon of strong field, hence modified theories can be the best candidates to explain

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this phenomenon. These theories may remodel the collapse process as well as improve its dynamics which may unveil fascinating results related to the structure formation of the universe. In this regard, Sofuoğlu and Mutuş [21] analyzed stellar system with perfect fluids under shear-free condition and found two $f(R)$ models which incorporate rotation as well as expansion. Sharif and his collaborators [22,23] studied the influences of $f(R)$ gravity on the evolution of spherical as well as axial system with shear-free fluids. Jawad and Rani [24] discussed the dynamical instability of spherically symmetric collapsing star under shear-free condition in generalized teleparallel gravity.

It is believed that astrophysical objects are endowed with angular momentum, *e.g.*, stellar compact objects (like white dwarfs or neutron stars) are in rotational motion and can deviate from spherical symmetry, which gives rise to axial symmetry. Thus, the dynamical analysis of self-gravitating fluids with this symmetry would be interesting. Herrera and Varela [25] investigated the effects of axially symmetric perturbations of matter variables by considering only perfect fluids. The assumption of perfect fluid seems to be a stringent restriction for axially symmetric sources even in the static case [26,27]. Sharif and Bhatti [28] explored instability regions for axial and reflection symmetric systems with anisotropic matter configurations. Sharif and Manzoor [29] examined the dynamics and stability for axial and reflection symmetric model in self-interacting Brans-Dicke gravity. They also studied the effects of dark source terms on a dissipative axially symmetric collapsing fluid [30].

In this paper, we investigate the effects of dark sources by taking a general source comprising dissipation and all non-vanishing stresses consistent with axial symmetry in the modified Gauss-Bonnet gravity. The paper has the following format. In the next section, we provide formalism of gravity for axial system with general source. Section 3 yields non-zero structure scalars corresponding to the system. Section 4 explores shear-free axial model by considering geodesic as well as non-geodesic fluids. Finally, we summarize our results in the last section.

2 f(G) Gravity and axial system

The action for $f(G)$ gravity is given by [1]

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x[R+f(G)]\sqrt{-g}+S_M, \tag{1}$$

where κ is the coupling constant and S_M is the matter action. We assume the unit system $\kappa^2 = \frac{8\pi\mathfrak{g}}{c} = 1$ ($\mathfrak{g}$ is the gravitational constant and c is the speed of light). Varying the action (1) with respect to $g_{\mu\nu}$, we obtain

$$\mathcal{G}_{\mu\nu} = \overset{(\mathrm{eff})}{T}_{\mu\nu} = \overset{(\mathrm{m})}{T}_{\mu\nu} + \overset{(\mathrm{GB})}{T}_{\mu\nu}, \tag{2}$$

where the notation (eff) (shorten for effective) denotes the combined effects of matter and dark sources (GB terms), $\mathcal{G}_{\mu\nu}$ is the Einstein tensor and $\overset{(\mathrm{m})}{T}_{\mu\nu}/\overset{(\mathrm{GB})}{T}_{\mu\nu}$ are the energy-momentum tensors for matter/GB terms, respectively,

$$\overset{(\mathrm{GB})}{T}_{\mu\nu} = 8[R_{\mu\rho\nu\sigma}+R_{\rho\nu}g_{\sigma\mu}-R_{\rho\sigma}g_{\nu\mu}-R_{\mu\nu}g_{\sigma\rho}+R_{\mu\sigma}g_{\nu\rho} + \frac{1}{2}R(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho})]\nabla^\rho\nabla^\sigma f_G + (Gf_G - f)g_{\mu\nu}, \tag{3}$$

where G in subscript denotes derivative of f with respect to GB invariant. This is the energy-momentum tensor contributing the gravitational effects due to $f(G)$ extra dark source terms. We assume the $f(G)$ model [31],

$$f(G) = \alpha G^n, \tag{4}$$

where α is any constant and $n > 0$. For the viability of this model, it must satisfy the conditions that $f(G)$ and all its derivatives $(f_G, f_{GG}, f_{GGG}, \dots)$ are regular and $f_{GG} > 0$, $\forall G$ [3]. The model parameter n has some significant effects on $R+f(G)$ cosmology. For $n < 0$, this model describes transition from non-phantom to phantom phases while $0 < n < \frac{1}{2}$ gives transition from decelerated to accelerated universe [31].

The line element for axially and reflection symmetric system is [20]

$$\mathrm{d}s^2 = -A^2(t,r,\theta)\mathrm{d}t^2 + B^2(t,r,\theta)(\mathrm{d}r^2+r^2\mathrm{d}\theta^2) + 2E(t,r,\theta)\mathrm{d}t\,\mathrm{d}\theta + C^2(t,r,\theta)\mathrm{d}\phi^2. \tag{5}$$

The energy distribution of respective fluid observed by an observer with four-velocity u^μ ($u^\mu = (A^{-1},0,0,0)$ and $u_\mu = (-A,0,\frac{E}{A},0)$) can be represented by the energy-momentum tensor given by

$$\overset{(\mathrm{eff})}{T}_{\mu\nu} = \overset{(\mathrm{m})}{T}_{\mu\nu} + \overset{(\mathrm{GB})}{T}_{\mu\nu} = \left(\overset{(\mathrm{eff})}{\rho} + \overset{(\mathrm{eff})}{p}\right)u_\mu u_\nu + \overset{(\mathrm{eff})}{p}g_{\mu\nu} + \overset{(\mathrm{eff})}{\Pi}_{\mu\nu} + \overset{(\mathrm{eff})}{q}_\mu u_\nu + \overset{(\mathrm{eff})}{q}_\nu u_\mu, \tag{6}$$

where effective energy density, isotropic pressure, anisotropic tensor and heat flux, respectively are defined as

$$\overset{(\text{eff})}{\rho} = \overset{(\text{m})}{\rho} + \overset{(\text{GB})}{\rho}, \qquad \overset{(\text{eff})}{p} = \overset{(\text{m})}{p} + \overset{(\text{GB})}{p},$$
$$\overset{(\text{eff})}{\Pi}_{\mu\nu} = \overset{(\text{m})}{\Pi}_{\mu\nu} + \overset{(\text{GB})}{\Pi}_{\mu\nu}, \qquad \overset{(\text{eff})}{q}_{\mu} = \overset{(\text{m})}{q}_{\mu} + \overset{(\text{GB})}{q}_{\mu}.$$

We obtain these effective quantities from eq. (6) using eqs. (3) and (4) as

$$\overset{(\text{eff})}{\rho} = \overset{(\text{eff})}{T}_{\mu\nu}u^\mu u^\nu = \overset{(\text{m})}{T}_{\mu\nu}u^\mu u^\nu + \frac{3n\alpha}{2}[Rg_{\rho\sigma}]\nabla^\rho\nabla^\sigma G^n - \alpha(n-1)G^n, \tag{7}$$

$$\overset{(\text{eff})}{q}_{\mu} = -\overset{(\text{eff})}{\rho}u_\mu - \overset{(\text{eff})}{T}_{\mu\nu}u^\nu = -2n\alpha[Rg_{\rho\sigma}u_\mu]\nabla^\rho\nabla^\sigma G^n + 2\alpha(n-1)G^n u_\mu, \tag{8}$$

$$\overset{(\text{eff})}{p} = \frac{1}{3}h^{\mu\nu}\overset{(\text{eff})}{T}_{\mu\nu} = \frac{1}{3}h^{\mu\nu}\overset{(\text{m})}{T}_{\mu\nu} + 4n\alpha\left[\frac{11}{8}Rg_{\rho\sigma} + Ru_\rho u_\sigma - h_{\rho\sigma}\right]\nabla^\rho\nabla^\sigma G^n + 3\alpha(n-1)G^n, \tag{9}$$

$$\begin{aligned}\overset{(\text{eff})}{\Pi}_{\mu\nu} &= h^\alpha_\mu h^\beta_\nu\left(\overset{(\text{eff})}{T}_{\alpha\beta} - \overset{(\text{eff})}{p}h_{\alpha\beta}\right) = h^\alpha_\mu h^\beta_\nu\left(\overset{(\text{m})}{T}_{\alpha\beta} - \overset{(\text{m})}{p}h_{\alpha\beta}\right) + 8n\alpha\left[\frac{5}{8}Rg_{\sigma\rho}\delta_{\mu\nu} + \frac{9}{8}\right.\\ &\quad\left.\times Rg_{\sigma\rho}u_\mu u_\nu + \frac{1}{2}Rg_{\nu\rho}h_{\mu\sigma} + \frac{1}{2}Rh_{\mu\sigma}u_\rho u_\nu + \frac{1}{2}Ru_\sigma u_\rho u_\mu u_\nu - Rh_{\mu\sigma}h_{\rho\nu}\right]\\ &\quad\times\nabla^\rho\nabla^\sigma G^n + \alpha(n-1)G^n(\delta_{\mu\nu} + u_\mu u_\nu) + \left[\frac{11}{8}Rg_{\rho\sigma} + Ru_\rho u_\sigma - h_{\rho\sigma}\right]\\ &\quad\times 4n\alpha\nabla^\rho\nabla^\sigma G^n h_{\mu\nu} + 3\alpha(n-1)h_{\mu\nu}G^n,\end{aligned} \tag{10}$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection tensor. If we put $n = 0$, the $f(G)$ model becomes constant and all these energy terms reduce to matter part only. The constant $f(G)$ model (*i.e.*, $f_G = 0$) corresponds to the cosmological constant and standard results are imitated. The spacelike unit four-vectors are defined as

$$v_\mu = B\delta^1_\mu, \qquad s_\mu = \frac{1}{A}(A^2B^2r^2 + E^2)^{\frac{1}{2}}\delta^2_\mu, \qquad k_\mu = C\delta^3_\mu,$$

which satisfy the relations

$$u^\mu u_\mu = -v^\mu v_\mu = -s^\mu s_\mu = -k^\mu k_\mu = -1,$$
$$u^\mu v_\mu = u^\mu s_\mu = u^\mu k_\mu = v^\mu s_\mu = v^\mu k_\mu = k^\mu s_\mu = 0.$$

Pressure anisotropy cannot be ignored during collapse process which has significant effects in controlling hydrostatic equilibrium. Some prominent sources for pressure anisotropy are the magnetic field present in the compact objects, magnetized strange quark stars, magnetic field acting on a Fermi gas, viscosity present in neutron stars as well as in highly densed matter [32–35]. For the sake of convenience, we convert anisotropic tensor (10) in terms of scalar quantities as follows:

$$\overset{(\text{eff})}{\Pi}_{\mu\nu} = \frac{1}{3}\left(2\overset{(\text{eff})}{\Pi_1} + \overset{(\text{eff})}{\Pi_2}\right)\left(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}\right) + \frac{1}{3}\left(2\overset{(\text{eff})}{\Pi_2} + \overset{(\text{eff})}{\Pi_1}\right)\left(s_\mu s_\nu - \frac{1}{3}h_{\mu\nu}\right) + 2\overset{(\text{eff})}{\Pi}_{vs}v_{(\mu}s_{\nu)}, \tag{11}$$

where

$$\overset{(\text{eff})}{\Pi}_{vs} = v^\mu s^\nu\overset{(\text{eff})}{T}_{\mu\nu}, \qquad \overset{(\text{eff})}{\Pi_1} = (2v^\mu v^\nu - s^\mu s^\nu - k^\mu k^\nu)\overset{(\text{eff})}{T}_{\mu\nu}, \tag{12}$$

$$\overset{(\text{eff})}{\Pi_2} = (2s^\mu s^\nu - k^\mu k^\nu - v^\mu v^\nu)\overset{(\text{eff})}{T}_{\mu\nu}. \tag{13}$$

These indicate that the anisotropy scalars, $\overset{(\text{eff})}{\Pi}_{vs}$, $\overset{(\text{eff})}{\Pi_1}$ and $\overset{(\text{eff})}{\Pi_2}$, depend on matter as well as dark sources. Hence the inhomogeneous distribution of dark sources generate pressure anisotropy in collapsing fluid.

Dissipation of heat flux (due to emission of photons or neutrinos which are massless particles) during collapse cannot be overemphasized. Indeed, it is a characteristic process during stellar evolution. Dissipation due to neutrino emission of gravitational binding energy leads to formation of neutron stars or black holes [36]. The field equations along with the condition $q^\mu u_\mu = 0$ give $T_{03} = 0$ implying that

$$\overset{(\text{eff})}{q}_{\mu} = \overset{(\text{eff})}{q_1}v_\mu + \overset{(\text{eff})}{q_2}s_\mu, \tag{14}$$

where

$$\overset{(\mathrm{eff})}{q_1} = \overset{(\mathrm{eff})}{q_\mu} v^\mu = \overset{(\mathrm{eff})}{T}{}_{\mu\nu} u^\nu v^\mu, \qquad \overset{(\mathrm{eff})}{q_2} = \overset{(\mathrm{eff})}{q_\mu} s^\mu = \overset{(\mathrm{eff})}{T}{}_{\mu\nu} u^\nu s^\mu. \tag{15}$$

These equations indicate that inhomogeneous distribution of dark sources generate heat dissipation.

2.1 Kinematical variables and the Weyl tensor

The characteristics of self-gravitating collapsing fluid depend upon the behavior of kinematical variables including four-acceleration, expansion scalar, shear tensor and a vorticity vector. The four-acceleration gives combined effects of gravitational and inertial forces given as

$$a_\mu = u^\nu u_{\mu;\nu} = a_1 v_\mu + a_2 s_\mu, \tag{16}$$

where

$$a_1 = \frac{A'}{AB}, \qquad a_2 = \frac{A}{(A^2B^2r^2 + E^2)^{\frac{1}{2}}} \left[\frac{E}{A^2}\left(-\frac{\dot{A}}{A} + \frac{\dot{E}}{E}\right) + \frac{A^\theta}{A}\right], \tag{17}$$

prime represents partial derivative with respect to radial coordinate, dot is the temporal and θ indicates derivative with respect to theta coordinate. The expansion scalar controls the volume expansion of the fluid and is defined as

$$\vartheta = u^\mu_{;\mu} = \frac{A^2B^2}{A^2B^2r^2 + E^2}\left[\left(2\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) r^2 + \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{E}}{E} + \frac{\dot{C}}{C}\right)\frac{E^2}{A^2B^2}\right]. \tag{18}$$

The shear tensor measures distortion appearing in the fluid motion given as

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{3}\vartheta h_{\mu\nu}. \tag{19}$$

The alternative form of shear tensor in terms of two scalar functions σ_1, σ_2 is

$$\sigma_{\mu\nu} = \frac{1}{3}(2\sigma_1 + \sigma_2)\left(v_\mu v_\nu - \frac{1}{3}h_{\mu\nu}\right) + \frac{1}{3}(2\sigma_2 + \sigma_1)\left(s_\mu s_\nu - \frac{1}{3}h_{\mu\nu}\right). \tag{20}$$

The non-zero components of shear tensor are derived in appendix A. Equations (A.1)–(A.3) imply that

$$2\sigma_1 + \sigma_2 = \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)\frac{3}{A}, \tag{21}$$

$$2\sigma_2 + \sigma_1 = \left[\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) AB^2r^2 - \left(\frac{\dot{A}}{A} - \frac{\dot{E}}{E} - \frac{\dot{C}}{C}\right)\frac{E^2}{A}\right]\frac{3}{(A^2B^2r^2 + E^2)}. \tag{22}$$

The local spinning of the system is defined by the vorticity vector as

$$w_\mu = \frac{1}{2}\eta_{\mu\nu\alpha\beta} u^{\nu;\alpha} u^\beta = \frac{1}{2}\eta_{\mu\nu\alpha\beta}\Omega^{\nu\alpha}u^\beta, \tag{23}$$

where $\eta_{\mu\nu\alpha\beta}$ and $\Omega_{\mu\nu} = u_{[\mu;\nu]} + a_{[\mu}u_{\nu]}$ define the Levi-Civita and vorticity tensors, respectively. Another form of vorticity tensor in terms of vorticity scalar function Ω is given as

$$\Omega_{\mu\nu} = \Omega(s_\mu v_\nu - s_\nu v_\mu), \qquad \Omega = \frac{E(\frac{E'}{E} - \frac{2A'}{A})}{2B(A^2B^2r^2 + E^2)^{\frac{1}{2}}}. \tag{24}$$

Equations (23) and (24) yield $w_\mu = -\Omega k_\mu$. Equation (24) implies that the system becomes spinless ($\Omega = 0$) if and only if $E = 0$. From the expressions of kinematical variables, we observe that they are totally depending upon geometrical terms and there is zero contribution from matter/dark sources. This implies that dark sources do not affect kinematical variables.

The Weyl tensor describes the tidal force effects upon the system as

$$C^\mu_{\alpha\beta\nu} = R^\mu_{\alpha\beta\nu} - \frac{1}{2}R^\mu_\beta g_{\alpha\nu} + \frac{1}{2}R_{\alpha\mu}\delta^\nu_\beta - \frac{1}{2}R_{\alpha\mu}\delta^\nu_\beta + \frac{1}{2}R^\mu_\nu g_{\alpha\beta} + \frac{1}{6}R(\delta^\mu_\beta g_{\alpha\nu} - g_{\alpha\beta}\delta^\mu_\nu). \tag{25}$$

Equations (2) and (25) generate a link between the Weyl tensor and effective energy terms through Riemann/Ricci tensors and Ricci scalar (which involve in modified field equations). In this way, the Weyl tensor is also associated with dynamics of dark sources and defines effects of tidal forces due to gravitational as well as repulsive forces. The Weyl tensor is further divided into electric and magnetic parts as

$$\overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\nu}=C_{\mu\alpha\nu\beta}u^{\mu}u^{\nu},$$
$$\overset{(\mathrm{eff})}{M}_{\mu\nu}=\frac{1}{2}\eta_{\mu\alpha\delta\gamma}C^{\delta\gamma}_{\nu\lambda}u^{\alpha}u^{\lambda}. \tag{26}$$

There are three non-zero components for electric part while two for the magnetic part. These elements of the Weyl tensor can be written in terms of scalar functions as

$$\overset{(\mathrm{eff})}{\mathbb{E}}_{\mu\nu}=\frac{1}{3}\left(2\overset{(\mathrm{eff})}{\varepsilon_1}+\overset{(\mathrm{eff})}{\varepsilon_2}\right)\left(v_\mu v_\nu-\frac{1}{3}h_{\mu\nu}\right)+\frac{1}{3}\left(2\overset{(\mathrm{eff})}{\varepsilon_2}+\overset{(\mathrm{eff})}{\varepsilon_1}\right)\left(s_\mu s_\nu-\frac{1}{3}h_{\mu\nu}\right)+\varepsilon_{vs}(v_\mu s_\nu+v_\nu s_\mu), \tag{27}$$
$$\overset{(\mathrm{eff})}{M}_{\mu\nu}=\overset{(\mathrm{eff})}{M_1}(k_\mu v_\nu+k_\nu v_\mu)+\overset{(\mathrm{eff})}{M_2}(k_\mu s_\nu+k_\nu s_\mu). \tag{28}$$

3 Modified structure scalars

In this section, we calculate structure scalars by orthogonal splitting of the Riemman tensor [37]. We decompose the Riemann tensor into energy terms with the help of eqs. (2) and (26) as

$$R^{\mu\nu}_{\alpha\beta}=\overset{(\mathrm{F})}{R^{\mu\nu}_{\alpha\beta}}+\overset{(\mathrm{Q})}{R^{\mu\nu}_{\alpha\beta}}+\overset{(\mathrm{E})}{R^{\mu\nu}_{\alpha\beta}}+\overset{(\mathrm{M})}{R^{\mu\nu}_{\alpha\beta}},$$

where

$$\overset{(\mathrm{F})}{R^{\mu\nu}_{\alpha\beta}}=\frac{2}{3}\left(\overset{(\mathrm{eff})}{\rho}+3\overset{(\mathrm{eff})}{p}\right)u^{[\mu}u_{[\alpha}h^{\nu]}_{\beta]}+\frac{2}{3}\overset{(\mathrm{eff})}{\rho}h^{\mu}_{[\alpha}h^{\nu}_{\beta]}, \tag{29}$$
$$\overset{(\mathrm{Q})}{R^{\mu\nu}_{\alpha\beta}}=-2u^{[\mu}h^{\nu]}_{[\alpha}\overset{(\mathrm{eff})}{q}_{\beta]}, \tag{30}$$
$$\overset{(\mathrm{E})}{R^{\mu\nu}_{\alpha\beta}}=4u^{[\mu}u_{[\alpha}\mathbb{E}^{(\mathrm{eff})\nu]}_{\beta]}+4h^{[\mu}_{[\alpha}\mathbb{E}^{(\mathrm{eff})\nu]}_{\beta]}, \tag{31}$$
$$\overset{(\mathrm{M})}{R^{\mu\nu}_{\alpha\beta}}=-2\epsilon^{\mu\nu\gamma}u_{[\alpha}\overset{(\mathrm{eff})}{M}_{\beta]\gamma}-2\epsilon_{\alpha\beta\gamma}u^{[\mu}\overset{(\mathrm{eff})}{M}{}^{\nu]\gamma}, \tag{32}$$

$\epsilon_{\mu\nu\gamma}=\eta_{\beta\mu\nu\gamma}u^{\beta}$, the notations in top F (density and pressure), Q (heat dissipation), $\mathbb{E}$ (electric part of the Weyl tensor) and M (magnetic part of the Weyl tensor) show decomposed parts of the Riemann curvature tensor relating to various aspects of fluid and carry the effects of dark sources. The triplets of the tensors are given as

$$\overset{(\mathrm{eff})}{Y}_{\mu\nu}=R_{\mu\alpha\nu\beta}u^{\alpha}u^{\beta},$$
$$\overset{(\mathrm{eff})}{X}_{\mu\nu}=\frac{1}{2}\eta^{\delta\lambda}_{\mu\alpha}R^{*}_{\delta\lambda\nu\beta}u^{\alpha}u^{\beta},$$
$$\overset{(\mathrm{eff})}{Z}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\lambda\delta}R^{\delta\lambda}_{\beta\nu}u^{\beta},$$

where $R^{*}_{\mu\nu\alpha\beta}=\frac{1}{2}\eta_{\delta\lambda\alpha\beta}R^{\delta\lambda}_{\mu\nu}$ is the dual form of the Riemann tensor. Using eqs. (26) and (29)–(32), we obtain

$$\overset{(\mathrm{eff})}{Y}_{\mu\nu}=\frac{1}{3}\overset{(\mathrm{eff})}{Y_T}h_{\mu\nu}+\frac{1}{3}\left(2\overset{(\mathrm{eff})}{Y_{TF1}}+\overset{(\mathrm{eff})}{Y_{TF2}}\right)\left(v_\mu v_\nu-\frac{1}{3}h_{\mu\nu}\right)+\frac{1}{3}\left(2\overset{(\mathrm{eff})}{Y_{TF2}}+\overset{(\mathrm{eff})}{Y_{TF1}}\right)$$
$$\times\left(s_\mu s_\nu-\frac{1}{3}h_{\mu\nu}\right)+\overset{(\mathrm{eff})}{Y_{vs}}(v_\mu s_\nu+v_\nu s_\mu),\qquad \overset{(\mathrm{eff})}{Z_{\mu\nu}}=\overset{(\mathrm{eff})}{M_{\mu\nu}}+\frac{1}{2}\overset{(\mathrm{eff})}{q^{\delta}}_{\mu\nu\delta}.$$

Here the subscript T stands for the trace part while the components with notations $TF1$, $TF2$ and vs in subscript represent the trace-free parts of the corresponding tensor. The scalar quantities corresponding to $\overset{(\mathrm{eff})}{X}_{\mu\nu}$, $\overset{(\mathrm{eff})}{Y}_{\mu\nu}$ and

$\overset{(\text{eff})}{Z}_{\mu\nu}$ are, respectively, given as

$$\overset{(\text{eff})}{X}_T = \overset{(\text{eff})}{\rho}, \qquad \overset{(\text{eff})}{X}_{TF1} = -\overset{(\text{eff})}{\varepsilon}_1 - \frac{1}{2}\overset{(\text{eff})}{\Pi}_1, \qquad \overset{(\text{eff})}{X}_{TF2} = -\overset{(\text{eff})}{\varepsilon}_2 - \frac{1}{2}\overset{(\text{eff})}{\Pi}_2, \qquad \overset{(\text{eff})}{X}_{vs} = -\overset{(\text{eff})}{\varepsilon}_{vs} - \frac{1}{2}\overset{(\text{eff})}{\Pi}_{vs},$$
$$\overset{(\text{eff})}{Y}_T = \frac{1}{2}\left(\overset{(\text{eff})}{\rho} + 3\overset{(\text{eff})}{p}\right), \qquad \overset{(\text{eff})}{Y}_{TF1} = \overset{(\text{eff})}{\varepsilon}_1 - \frac{1}{2}\overset{(\text{eff})}{\Pi}_1, \qquad \overset{(\text{eff})}{Y}_{TF2} = \overset{(\text{eff})}{\varepsilon}_2 - \frac{1}{2}\overset{(\text{eff})}{\Pi}_2,$$
$$\overset{(\text{eff})}{Y}_{vs} = \overset{(\text{eff})}{\varepsilon}_{vs} - \frac{1}{2}\overset{(\text{eff})}{\Pi}_{vs}, \qquad \overset{(\text{eff})}{Z}_1 = \left(\overset{(\text{eff})}{M}_1 - \frac{1}{2}\overset{(\text{eff})}{q}_2\right), \qquad \overset{(\text{eff})}{Z}_2 = \left(\overset{(\text{eff})}{M}_1 + \frac{1}{2}\overset{(\text{eff})}{q}_2\right),$$
$$\overset{(\text{eff})}{Z}_3 = \left(\overset{(\text{eff})}{M}_2 - \frac{1}{2}\overset{(\text{eff})}{q}_1\right), \qquad \overset{(\text{eff})}{Z}_4 = \left(\overset{(\text{eff})}{M}_2 + \frac{1}{2}\overset{(\text{eff})}{q}_1\right),$$

This is a set of 12 scalar functions which describe the evolution of axially symmetric self-gravitating systems in the presence of matter as well as dark sources. The scalar $\overset{(\text{eff})}{X}_T$ expresses the total energy density of the system due to matter and dark sources. Three scalars $\overset{(\text{eff})}{X}_{TF1}$, $\overset{(\text{eff})}{X}_{TF2}$ as well as $\overset{(\text{eff})}{X}_{vs}$ give combine effects of anisotropy and electric part of the Weyl tensor in one and the same direction. Another scalar $\overset{(\text{eff})}{Y}_T$ represents sum of energy density and pressure (total energy) of the system. The set of scalars $\overset{(\text{eff})}{Y}_{TF1}$, $\overset{(\text{eff})}{Y}_{TF2}$, $\overset{(\text{eff})}{Y}_{vs}$ provide combine effects of anisotropy and electric part of the Weyl tensor in opposite directions. The scalar functions $\overset{(\text{eff})}{Z}_1$, $\overset{(\text{eff})}{Z}_2$, $\overset{(\text{eff})}{Z}_3$, $\overset{(\text{eff})}{Z}_4$ represent various combinations of heat dissipation and magnetic parts of the Weyl tensor. We note that matter as well as dark sources take part in the dynamics of any axial symmetric system. If we neglect matter contributions, then dark sources become responsible for the dynamics of the system.

4 Shear-free axial system

Here we study the effects of shear-free condition on axial system in the presence of dark sources for non-geodesic (as well as geodesic) dissipative (as well as non-dissipative) cases. For this purpose, we have developed a set of governing equations corresponding to the system and heat transport equation in appendix A.

4.1 Non-geodesic condition

We assume that the evolution of non-geodesic dissipative fluid is shear-free ($\sigma_{\alpha\beta} = 0 = \sigma_1 = \sigma_2$). Under this assumption, eqs. (21) and (22) yield

$$C(t,r,\theta) = B(t,r,\theta)R_1(r,\theta), \qquad E(t,r,\theta) = A(t,r,\theta)B(t,r,\theta)R_2(r,\theta), \tag{33}$$

where $R_1(r,\theta)$, $R_2(r,\theta)$ are functions of integration with respect to temporal coordinate t. These functions must satisfy $R_1(0,\theta) = R_2(0,\theta) = 0$ to be compatible with regular condition at the origin. Equation (A.13) provides

$$2w_{<\alpha}a_{\beta>} + \nabla_{<\alpha}w_{\beta>} = \overset{(\text{eff})}{M}_{\alpha\beta}. \tag{34}$$

Here $\nabla_\alpha w_\beta = h^\mu_\alpha w_{\beta;\mu}$ and angled brackets indicate symmetric and trace-free part. The above equation implies that under shear-free condition, the effective magnetic part of the Weyl tensor depends upon the rotational function and for $w_\alpha = 0$, we have $\overset{(\text{eff})}{M}_{\alpha\beta} = 0$. Inversely, $\overset{(\text{eff})}{M}_{\alpha\beta} = 0$ in eq. (31) yields

$$\nabla_\alpha w^\alpha = -2a_\alpha w^\alpha. \tag{35}$$

However, eqs. (19) and (23) in the shear-free condition yield the identity

$$\nabla_\alpha w^\alpha = a_\alpha w^\alpha. \tag{36}$$

Equations (35) and (36) imply that $w_\alpha = 0$ which further gives $\Omega_{\alpha\beta} = 0$ (from eq. (23)). In both cases, we obtain $\overset{(\text{eff})}{M}_{\alpha\beta} = 0$ implying that

$$\overset{(\text{eff})}{M}_1 = \overset{(\text{eff})}{M}_2 = 0 \Leftrightarrow \Omega = 0, \tag{37}$$

which provides that necessary and sufficient condition for irrotational shear-free fluid is the vanishing of magnetic part of the Weyl tensor. For shear-free irrotational fluid, eqs. (A.14) and (A.15) give heat dissipation scalars as

$$\overset{(\mathrm{eff})}{q_1} = \frac{2\vartheta_{,r}}{3B}, \qquad \overset{(\mathrm{eff})}{q_2} = \frac{\vartheta_{,\theta}}{3Br}. \tag{38}$$

This shows the behavior of expansion scalar which depends upon heat dissipation. In the absence of dissipation, eq. (38) implies that the expansion scalar depends upon temporal coordinate only, *i.e.*, $\vartheta = \vartheta(t)$ (becomes homogeneous) but dissipation due to dark sources does not vanish. Hence the expansion of axial system remains inhomogeneous under the influence of $f(G)$ gravity.

4.2 Geodesic condition

Here we restrict our system to be geodesic, *i.e.*, a system with vanishing four-acceleration. Consequently, eqs. (17) and (18) provide

$$A = R_3(t,\theta), \qquad R_2\vartheta B = R_4(t,\theta), \tag{39}$$

where $R_3(t,\theta)$ and $R_4(t,\theta)$ are arbitrary functions of integration representing null contributions from a_1 and a_2, respectively. Under regularity conditions, we have $\Omega(t,0,\theta) = E(t,0,\theta) = 0$ (eq. (33)), *i.e.*, the vanishing of the coefficient of cross term results irrotational fluid. The condition $E(t,0,\theta) = 0$ further gives $R_2(t,0,\theta) = 0$ and, consequently, from eq. (39), we obtain $R_4(t,0,\theta) = 0$. As a result, we have either $\Omega = 0$ or $\vartheta = 0$. A similar result can be found from eq. (A.16) which reads, for the underlying case,

$$h^\alpha_\mu h^\beta_\nu \Omega_{\alpha\beta;\gamma}u^\gamma = -\frac{2}{3}\vartheta\Omega_{\mu\nu}, \quad \text{or} \quad \Omega_{,\alpha}u^\alpha = -\frac{2}{3}\vartheta\Omega. \tag{40}$$

This equation along with eqs. (18), (21) and (22) provides $\vartheta\Omega = 0$ which indicates that shear-free geodesic fluid yields either $\Omega = 0$ or $\vartheta = 0$. In the following, we analyze both cases separately.

Vorticity-free expanding fluid

First we consider the case with $\Omega_{\mu\nu} = 0$ but $\vartheta \neq 0$ which implies that $A = R_3(t)$, $E = 0$. After reparametrization of time coordinate, the line element (5) becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + B^2(t,r,\theta)\left[\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + F^2(r,\theta)\mathrm{d}\phi^2\right]. \tag{41}$$

This represents restricted class of axially symmetric cosmic structure. The continuity and Euler equations (eqs. (A.6) and (A.7), respectively) reduce to

$$\overset{(\mathrm{eff})}{\rho}_{;\alpha}u^\alpha + \left(\overset{(\mathrm{eff})}{\rho} + \overset{(\mathrm{eff})}{p}\right)\vartheta + \overset{(\mathrm{eff})}{q}{}^\alpha_{;\alpha} = 0, \tag{42}$$

$$h^\beta_\alpha\left(\overset{(\mathrm{eff})}{p}_{,\beta} + \overset{(\mathrm{eff})}{\Pi}{}^\mu_{\beta;\mu} + \overset{(\mathrm{eff})}{q}_{\beta;\mu}u^\mu\right) + \frac{4}{3}\vartheta\overset{(\mathrm{eff})}{q}_\alpha = 0, \tag{43}$$

while the heat transport equation (A.4) gives

$$\tau h^\mu_\nu \overset{(\mathrm{eff})}{q}{}^\nu_{;\alpha}u^\alpha + \overset{(\mathrm{eff})}{q}{}^\mu = -Kh^{\mu\nu}(\mathbb{T}_{,\nu}) - \frac{1}{2}K\mathbb{T}^2\left(\frac{\tau u^\alpha}{K\mathbb{T}^2}\right)_{;\alpha}\overset{(\mathrm{eff})}{q}{}^\mu. \tag{44}$$

The combination of eqs. (43) and (44) yields

$$h^\beta_\alpha\overset{(\mathrm{eff})}{\Pi}{}^\mu_{\beta;\mu} + \nabla_\alpha\overset{(\mathrm{eff})}{p} + \frac{K}{\tau}\nabla_\alpha\mathbb{T} - \left[\frac{1}{\tau} + \frac{1}{2}D_t\left(\ln\left(\frac{\tau}{K\mathbb{T}^2}\right)\right) - \frac{5}{6}\vartheta\right]\overset{(\mathrm{eff})}{q}_\alpha = 0, \tag{45}$$

which gives a link between pressure gradient, pressure anisotropy and thermodynamic quantities. This suggests that any acceptable equation of state for the system under consideration is restricted by thermodynamic quantities through the heat transport equation. Equations (A.9)–(A.11) give

$$\overset{(\mathrm{eff})}{Y}_{TF1} = \overset{(\mathrm{eff})}{Y}_{TF2} = \overset{(\mathrm{eff})}{Y}_{vs} = 0. \tag{46}$$

The vanishing of this set of scalar functions associated with the tensor $\overset{(\text{eff})}{Y}_{\mu\nu}$ provide the relations

$$\overset{(\text{eff})}{\varepsilon_1} = \frac{1}{2}\overset{(\text{eff})}{\Pi_1}, \qquad \overset{(\text{eff})}{\varepsilon_2} = \frac{1}{2}\overset{(\text{eff})}{\Pi_2}, \qquad \overset{(\text{eff})}{\varepsilon_{vs}} = \frac{1}{2}\overset{(\text{eff})}{\Pi_{vs}} \tag{47}$$

and, accordingly, the tensor $\overset{(\text{eff})}{X}_{\mu\nu}$ reduces to

$$\begin{aligned} \overset{(\text{eff})}{X_{TF1}} &= -2\overset{(\text{eff})}{\varepsilon_1}, \\ \overset{(\text{eff})}{X_{TF2}} &= -2\overset{(\text{eff})}{\varepsilon_2}, \\ \overset{(\text{eff})}{X_{vs}} &= -2\overset{(\text{eff})}{\varepsilon_{vs}}. \end{aligned} \tag{48}$$

Now we turn our attention to non-dissipative fluid in the respective case, *i.e.*, $\overset{(\text{eff})}{q_1} = \overset{(\text{eff})}{q_2} = 0$. Under this condition, eqs. (18), (38), (42), (46) yield homogenous parameters given by

$$\begin{aligned} B(t,r,\theta) = \alpha(t)b(r,\theta), \qquad &\overset{(\text{eff})}{\rho} = \overset{(\text{eff})}{\rho}(t), \qquad \overset{(\text{eff})}{p} = \overset{(\text{eff})}{p}(t), \\ \overset{(\text{eff})}{\Pi_1} = \overset{(\text{eff})}{\Pi_1}(t), \qquad &\overset{(\text{eff})}{\Pi_2} = \overset{(\text{eff})}{\Pi_2}(t), \qquad \overset{(\text{eff})}{\Pi_{vs}} = \overset{(\text{eff})}{\Pi_{vs}}(t), \\ \overset{(\text{eff})}{\varepsilon_1} = \overset{(\text{eff})}{\varepsilon_1}(t), \qquad &\overset{(\text{eff})}{\varepsilon_2} = \overset{(\text{eff})}{\varepsilon_2}(t), \qquad \overset{(\text{eff})}{\varepsilon_{vs}} = \overset{(\text{eff})}{\varepsilon_{vs}}(t). \end{aligned} \tag{49}$$

This is possible only if our $f(G)$ models become constant ($n = 0$) which shows homogeneous distribution of dark sources. In our case, we cannot choose $n = 0$ and therefore the above quantities remain inhomogeneous. We examine the non-dissipative shear-free geodesic evolution and homogeneous distribution of dark sources. The differential equations of the Weyl tensor (A.17)–(A.19) reduce to

$$-\frac{1}{3}\left(\overset{(\text{eff})}{X_{TF1}} - \overset{(\text{eff})}{\rho}\right)_{,t} + \frac{1}{3}\overset{(\text{eff})}{\varepsilon_{TF1}}\vartheta = \frac{-1}{3}\left(\overset{(\text{eff})}{\rho} + \overset{(\text{eff})}{p} + \frac{1}{3}\overset{(\text{eff})}{\Pi_2}\right)\vartheta, \tag{50}$$

$$-\dot{X}_{vs}^{(\text{eff})} - \vartheta\overset{(\text{eff})}{X_{vs}} = \frac{1}{3}\overset{(\text{eff})}{\Pi}_{vs}\vartheta, \tag{51}$$

$$\frac{1}{3}\left(-\overset{(\text{eff})}{X_{TF2}} + \overset{(\text{eff})}{\rho}\right)_{,t} + \frac{\vartheta}{3}\overset{(\text{eff})}{\varepsilon_2} = -\frac{1}{3}\left(\overset{(\text{eff})}{\rho} + \overset{(\text{eff})}{p} + \frac{1}{3}\overset{(\text{eff})}{\Pi_2}\right)\vartheta. \tag{52}$$

Using (48) and (49), the above set of equations can be integrated to

$$\overset{(\text{eff})}{\varepsilon_1} = c_1\exp\left(-\frac{2}{3}\int\vartheta\mathrm{d}t\right), \qquad \overset{(\text{eff})}{\varepsilon_2} = c_2\exp\left(-\frac{2}{3}\int\vartheta\mathrm{d}t\right), \qquad \overset{(\text{eff})}{\varepsilon_{vs}} = c_3\exp\left(-\frac{2}{3}\int\vartheta\mathrm{d}t\right),$$

where c_1, c_2 and c_3 are constants of integration. Equation (18) along with eq. (30) gives $\vartheta = 3\frac{\dot{B}}{B}$ and hence the expressions in the above equation are calculated as

$$\overset{(\text{eff})}{\varepsilon_1} = \frac{c_1}{B^2}, \qquad \overset{(\text{eff})}{\varepsilon_2} = \frac{c_2}{B^2}, \qquad \overset{(\text{eff})}{\varepsilon_{vs}} = \frac{c_3}{B^2}.$$

This suggests that B in (49) reduces to $B = \beta(t)$ and the line element represents FRW spacetime for $\vartheta > 0$. This is consistent with GR [20] in the case of homogeneous distribution of dark sources. Otherwise, the axial system (41) preserves its symmetry under dissipation-less case in the presence of dark sources.

Expansion-free rotating fluid

Now we consider $\Omega \neq 0$ but $\vartheta = 0$, *i.e.*, expansion-free but rotating fluid. The assumption $\Omega \neq 0$ indicates that the metric (5) remains non-diagonal while the expansion scalar (18) with assumption $\vartheta = 0$ indicates that the system becomes time-independent. Equations (A.8)–(A.10) yield the relations

$$\overset{(\text{eff})}{Y_T} = 2\overset{(\text{eff})}{Y_{TF1}} = 2\overset{(\text{eff})}{Y_{TF2}} = 2\Omega^2, \tag{53}$$

which generates a relationship between rotation parameter and the scalars associated with tensor $\overset{(\mathrm{eff})}{Y}_{\mu\nu}$ while the scalar Y_{vs} vanishes in this case. One of the conservation equations (A.6) along with eq. (8) reduces to

$$\overset{(\mathrm{eff})}{X}_{T;\mu}u^{\mu}-\left(\overset{(\mathrm{m})}{T}{}^{\mu}{}_{\nu}u^{\nu}-2n\alpha[Rg_{\rho\sigma}u^{\mu}]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+2\alpha(n-1)G^{n}u^{\mu}\right)_{;\mu}=0,$$

which indicates that the only factor which controls the evolution of energy density is dissipation from matter as well as dark sources. In the absence of matter, dark sources control the evolution of energy density and if α is zero, then there is no evolution for energy density. Equation (A.12) produces a connection between dissipation and vorticity as

$$h^{\nu}_{\mu}\Omega^{\alpha}_{\nu;\alpha}u^{\mu}=\overset{(\mathrm{m})}{q}{}_{\mu}u^{\mu}+2n\alpha[Rg_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1}-2\alpha(n-1)G^{n}.$$

In the dissipation-less case ($\overset{(\mathrm{m})}{q}{}_{\mu}=\alpha=0$), we obtain a system with zero rotation ($\Omega=0$) which contradicts our considered case. We can say that such a system remains dissipative as α cannot be zero (although matter can be neglected). Hence due to inhomogeneous distribution of dark sources, the system must be dissipative while from eq. (A.15), we obtain

$$(\Omega BR_1)'=\overset{(\mathrm{eff})}{q_2}B^2R_1\quad\text{or}\quad\Omega=\frac{1}{BR_1}\int\overset{(\mathrm{eff})}{q_2}B^2R_1\mathrm{d}r+R_{5(\theta)},\tag{54}$$

which gives $\Omega=0$, in the non-dissipative case. Using eq. (54) in (A.8), we obtain

$$\overset{(\mathrm{m})}{\rho}=\left(\frac{2}{BR_1}\int\overset{(\mathrm{eff})}{q_2}B^2R_1\mathrm{d}r+R_{5(\theta)}\right)^2-\frac{267n\alpha}{2}[Rg_{\rho\sigma}]\nabla^{\rho}\nabla^{\sigma}G^{n-1}+8\alpha(n-1)G^{n}-3\overset{(\mathrm{m})}{p}.$$

For the non-dissipative case and $\alpha=0$ (absence of dark sources), we obtain equation of state $\rho=-3p$, which is consistent with GR [20]. Hence, in the presence of dark sources, all geodesic shear/expansion-free fluids can be rotational without dissipation under the dark effects of $f(G)$ gravity. This generalizes the results of GR where such kind of fluids must be dissipative. If we apply regularity conditions on the first expression of eq. (54), no such model ($\vartheta=0$) exists even in the presence of dark sources.

5 Final remarks

In this paper, we have explored the evolution of shear-free axially symmetric configuration in the presence of dark sources. For this purpose, We have chosen a power-law $f(G)$ model as a dark energy candidate and formulated energy terms (matter contents) of the model, corresponding dynamical variables as well as non-zero structure scalars. The kinematical variables do not depend upon dark source terms while pressure anisotropy and heat dissipation are affected by dark sources. Geodesic as well as non-geodesic fluid models with and without dissipation have been considered to discuss consequences of shear-free condition. The results can be summarized as follows.

The Weyl tensor usually provides tidal gravitational effects. We have observed that this tensor elaborates both gravitational as well as repulsive force due to the presence of dark sources. We have obtained 12 non-zero structure scalars consisting of dark/matter terms, electric and magnetic parts of the Weyl tensor which describe the evolution of axial system in the presence of dark sources. In GR, these scalars depend upon matter contents only. For the shear-free axial system, we have considered non-geodesic as well as geodesic fluids. In the case of a non-geodesic fluid with dissipation, we have found that rotation or spinning of the system is linked with magnetic part of the Weyl tensor. The vanishing of magnetic part turns out to be the necessary and sufficient condition for irrotational evolution even in the presence of dark terms. We have noticed that the behavior of expansion of the system is linked with the presence or absence of dissipation. In the absence of dissipation, the expansion scalar becomes homogeneous (as in GR). In our case, dissipation due to dark sources does not vanish and makes the expansion of axial system inhomogeneous. Hence, shear effects of an evolving configuration control the behavior of radiating system in the presence of dark sources.

For shear-free geodesic fluid, we have taken $\Omega=0$, $\vartheta\neq0$ and $\Omega\neq0$, $\vartheta=0$. In the first case, the non-dissipative fluid with homogeneous distribution of dark sources turns the axial system to FRW universe model which is consistent with GR result. However, dissipation due to dark sources is not negligible, hence this correspondence in our case cannot be possible. In the second case, evolution of energy density is controlled by dissipation from matter as well as dark sources. In the absence of matter, dark sources control the evolution of energy density. Without dissipation and homogeneous distribution of dark sources, we have concluded that all geodesic shear/expansion-free fluids can be rotational under the dark effects of $f(G)$ gravity which generalizes the results of GR where such kind of fluids must be dissipative. Using regularity conditions, we have found that models with zero expansion do not exist even in the presence of dark sources.

Appendix A.

The non-zero components of the shear tensor are given by

$$\sigma_{11}=\left[\left(C\dot{B}B-\frac{\dot{C}}{C}\right)A^2B^2r^2-\left(\frac{2\dot{B}}{B}+\frac{\dot{A}}{A}-\frac{\dot{E}}{E}-\frac{\dot{C}}{C}\right)E^2\right]\times\frac{B^2}{3A(A^2B^2r^2+E^2)}, \tag{A.1}$$

$$\sigma_{22}=\left[\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)A^2B^2r^2-\left(2\frac{\dot{A}}{A}+\frac{\dot{B}}{B}-\frac{2\dot{E}}{E}+\frac{\dot{C}}{C}\right)E^2\right]\frac{1}{3A^2}, \tag{A.2}$$

$$\sigma_{33}=\left[2\left(-\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)A^2B^2r^2+\left(\frac{2\dot{C}}{C}-\frac{\dot{B}}{B}-\frac{\dot{E}}{E}+\frac{\dot{A}}{A}\right)E^2\right]\times\frac{C^2}{3A(A^2B^2r^2+E^2)}. \tag{A.3}$$

The heat transport equation

The heat transport equation is given by

$$\tau h^\alpha_\beta q^\beta_{;\mu}u^\mu+\overset{(\mathrm{eff})}{q^\alpha}=-Kh^{\alpha\beta}(\mathbb{T}_{,\beta}+\mathbb{T}a_\beta)-\frac{1}{2}K\mathbb{T}^2\left(\frac{\tau u^\mu}{K\mathbb{T}^2}\right)_{;\mu}\overset{(\mathrm{eff})}{q^\alpha}, \tag{A.4}$$

$$\frac{\tau}{A}\left(\dot{\overset{(\mathrm{eff})}{q_2}}+A\overset{(\mathrm{eff})}{q_1}\Omega\right)+\overset{(\mathrm{eff})}{q_2}=\frac{K}{A}\left(\frac{-E\dot{\mathbb{T}}+A^2\mathbb{T}^\theta}{(A^2B^2r^2+E^2)}-A\mathbb{T}a_2\right)-\frac{K\mathbb{T}^2q_2}{2}\left(\frac{\tau u^\mu}{K\mathbb{T}^2}\right)_{;\mu}. \tag{A.5}$$

Conservation equations

From the conservation law, $\overset{(\mathrm{eff})}{T^\mu}{}_{\nu;\mu}=0$, we obtain two conservation equations in terms of scalar quantities as

$$\overset{(\mathrm{eff})}{X}_{T;\mu}u^\mu+2\vartheta\left(\overset{(\mathrm{eff})}{Y_T}-\overset{(\mathrm{eff})}{p}\right)+q^\mu_{;\mu}+q^\mu a_\mu+\frac{1}{9}[(2\sigma_1+\sigma_2)\Pi_1+(\sigma_1+2\sigma_2)\times\Pi_2]=0, \tag{A.6}$$

$$2a_\mu\left(\overset{(\mathrm{eff})}{Y_T}-\overset{(\mathrm{eff})}{p}\right)+h^\nu_\mu\left(\frac{1}{3}\left(2\overset{(\mathrm{eff})}{Y_T}-\overset{(\mathrm{eff})}{X_T}\right)_{;\nu}+\overset{(\mathrm{eff})}{\Pi^\alpha}{}_{\nu;\alpha}+\overset{(\mathrm{eff})}{q}{}_{\nu;\alpha}u^\alpha\right)+\left(\frac{4}{3}\vartheta h_{\mu\nu}+\sigma_{\mu\nu}+\omega_{\mu\nu}\right)\overset{(\mathrm{eff})}{q^\nu}=0. \tag{A.7}$$

Here the first equation is the continuity equation and the second is named as the Euler equation obtained for $f(G)$ gravity.

Ricci evolutionary equations

The Ricci evolutionary equations are propagation equations of expansion, shear/vorticity tensors and constraint equations. The time propagation equation of expansion scalar is derived by contracting Ricci identities for four-velocity vector. It is given in terms of scalars as

$$\vartheta_{;\mu}u^\mu+\frac{1}{3}\vartheta^2+2(\sigma^2-\Omega^2)-a^\mu_{;\mu}+\overset{(\mathrm{eff})}{Y_T}=0,\quad \sigma^2=\sigma_{\mu\nu}\sigma^{\mu\nu}. \tag{A.8}$$

The propagation equations of shear tensor are given by

$$\sigma_{2,\gamma}u^\gamma+\frac{1}{3}\sigma_2(\sigma_2+2\vartheta)-(2\sigma^2+\Omega^2-a^\gamma_{;\gamma})-3(s^\alpha s^\beta a_{\alpha;\beta}+a_2^2)+\overset{(\mathrm{eff})}{Y}_{TF2}=0, \tag{A.9}$$

$$\sigma_{1,\gamma}u^\gamma+\frac{1}{3}\sigma_1(\sigma_1+2\vartheta)-(2\sigma^2+\Omega^2-a^\gamma_{;\gamma})-3(v^\alpha v^\beta a_{\beta;\alpha}+a_1)+\overset{(\mathrm{eff})}{Y}_{TF1}=0, \tag{A.10}$$

$$\frac{1}{3}(\sigma_1-\sigma_2)\Omega-a_1a_2-v^{(\alpha}s^{\beta)}a_{\alpha;\beta}+\overset{(\mathrm{eff})}{Y}_{vs}=0. \tag{A.11}$$

Constraint equations are obtained as

$$h^{\nu}_{\mu}\left(\frac{2}{3}\vartheta_{;\nu}-\sigma^{\alpha}_{\nu;\alpha}+\Omega^{\alpha}_{\nu;\alpha}\right)+\left(\sigma_{\mu\nu}+\Omega_{\mu\nu}\right)a^{\nu}=\overset{(\mathrm{eff})}{q}_{\mu}, \tag{A.12}$$

$$2w_{(\mu}a_{\nu)}+h^{\alpha}_{(\mu}h_{\nu)\beta}(\sigma_{\alpha\gamma}+\Omega_{\alpha\gamma});_{\delta}\eta^{\beta\kappa\delta\gamma}u_{\kappa}=\overset{(\mathrm{eff})}{M}_{\mu\nu}, \tag{A.13}$$

$$\frac{2}{3B}\vartheta'-\Omega_{;\alpha}s^{\alpha}+\Omega\left(s_{\nu;\beta}v^{\beta}v^{\nu}-s^{\beta}_{;\beta}\right)+\frac{1}{3}a_{I}\sigma_1-a_2\Omega-\frac{1}{3}\sigma_{1;\alpha}v^{\alpha}$$
$$-\frac{1}{3}(2\sigma_1+\sigma_2)\left(v^{\alpha}_{;\alpha}-\frac{a_1}{3}\right)-\frac{1}{3}(\sigma_1+2\sigma_2)\left(s_{\nu;\alpha}s^{\alpha}s^{\nu}-\frac{a_1}{3}\right)=\overset{(\mathrm{eff})}{q_1}, \tag{A.14}$$

$$\frac{1}{3}(E^2+A^2B^2r^2)^{-\frac{1}{2}}\left(2A\vartheta^{\theta}+\frac{2E}{A}\dot{\vartheta}\right)+\frac{\sigma_2a_2}{3}+\omega_{;\alpha}v^{\alpha}+\Omega(v^{\alpha}_{;\alpha}+s^{\beta}v^{\nu}s_{\nu;\beta})$$
$$+\Omega a_I-\frac{1}{3}\sigma_{2;\alpha}s^{\alpha}+\frac{1}{3}(\sigma_2+2\sigma_1)\left(s_{\nu;\alpha}s^{\alpha}s^{\nu}-\frac{a_2}{3}\right)$$
$$-\frac{1}{3}(\sigma_1+2\sigma_2)\left(s^{\alpha}_{;\alpha}-\frac{a_2}{3}\right)=\overset{(\mathrm{eff})}{q_2}. \tag{A.15}$$

The time propagation equation for the vorticity tensor $\Omega_{\mu\nu}$ can be derived from Ricci identity as

$$h^{\alpha}_{\mu}h^{\beta}_{\nu}\Omega_{\alpha\beta;\gamma}u^{\gamma}+\frac{2}{3}\vartheta\Omega_{\mu\nu}+2\sigma_{\alpha[\mu}\Omega^{\alpha}_{\nu]}-h^{\alpha}_{[\mu}h^{\beta}_{\nu]}a_{\mu;\nu}=0. \tag{A.16}$$

Bianchi evolutionary equations

Bianchi evolutionary equations or evolution equations for the Weyl tensor are

$$-\frac{1}{3}\left(\overset{(\mathrm{eff})}{X}_{TF1}-\frac{1}{2}\overset{(\mathrm{eff})}{\rho}\right)_{,\delta}u^{\delta}+\frac{1}{9}\overset{(\mathrm{eff})}{\varepsilon_1}(3\vartheta-\sigma_1+\sigma_2)+\frac{1}{2}(\sigma_1+2\sigma_2)\overset{(\mathrm{eff})}{\varepsilon_2}=-v_{\nu}\epsilon^{\nu\gamma\kappa}$$
$$\times\left[\overset{(\mathrm{eff})}{M}_{1,\kappa}k_{\gamma}+\overset{(\mathrm{eff})}{M_1}k_{\gamma;\kappa}+\overset{(\mathrm{eff})}{M_2}(k_{\mu;\kappa}s_{\gamma}v^{\mu}+s_{\mu;\kappa}k_{\gamma}v^{\mu})\right]+\Omega\overset{(\mathrm{eff})}{X}_{vs}=2a_2\overset{(\mathrm{eff})}{M_1}$$
$$-\frac{1}{6}\left(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p}+\frac{1}{3}\overset{(\mathrm{eff})}{\Pi_1}\right)(\sigma_1+\vartheta)-a_1\overset{(\mathrm{eff})}{q_1}-\frac{1}{2B}q_I'^{(\mathrm{eff})}-\frac{A\overset{(\mathrm{eff})}{q_2}}{(A^2B^2r^2+E^2)^{\frac{1}{2}}}\times\left(\frac{E\dot{B}}{A^2B}+\frac{B^{\theta}}{B}\right), \tag{A.17}$$

$$-\overset{(\mathrm{eff})}{X}_{vs,\delta}u^{\delta}+\frac{1}{6}\Omega\left(\overset{(\mathrm{eff})}{X}_{TF2}-\overset{(\mathrm{eff})}{X}_{TF1}\right)-\frac{1}{2}\overset{(\mathrm{eff})}{X}_{vs}(2\vartheta-\sigma_2-\sigma_1)+a_1\overset{(\mathrm{eff})}{M_1}-a_2\overset{(\mathrm{eff})}{M_2}$$
$$-\frac{1}{2}\left[\left(\overset{(\mathrm{eff})}{M}_{1;\kappa}k_{\gamma}+\overset{(\mathrm{eff})}{M_1}(k_{\gamma;\kappa}+k_{\mu;\kappa}v_{\gamma}v^{\mu})+\overset{(\mathrm{eff})}{M_2}k_{\gamma}s_{\mu;\kappa}v^{\mu}\right)\epsilon^{\beta\gamma\kappa}s_{\beta}-\left(\overset{(\mathrm{eff})}{M_1}v^{\nu}k_{\gamma}+\overset{(\mathrm{eff})}{M_2}k^{\mu}s_{\gamma}\right)s_{\mu;\kappa}\epsilon^{\beta\gamma\kappa}v_{\beta}\right]$$
$$-\frac{1}{2}\left(\overset{(\mathrm{eff})}{M}_{1;\kappa}k_{\gamma}+\overset{(\mathrm{eff})}{M_2}k_{\gamma;\kappa}\right)\epsilon^{\beta\gamma\kappa}v_{\beta}=\frac{1}{3}\overset{(\mathrm{eff})}{\Pi}_{vs}(\vartheta-\sigma_1-\sigma_2)$$
$$-\frac{1}{2}a_2\overset{(\mathrm{eff})}{q_1}-\frac{1}{4}(v^{\mu}s^{\nu}+v^{\nu}s^{\mu})\overset{(\mathrm{eff})}{q}_{\nu;\mu}-\frac{1}{2}a_1\overset{(\mathrm{eff})}{q_2}, \tag{A.18}$$

$$\frac{1}{3}\left(-\overset{(\mathrm{eff})}{X}_{TF2}+\frac{1}{2}\overset{(\mathrm{eff})}{\rho}\right)_{;\delta}u^{\delta}+\frac{1}{9}\overset{(\mathrm{eff})}{\varepsilon_2}(3\vartheta-\sigma_2+\sigma_1)+\frac{1}{9}(\sigma_2+2\sigma_1)\overset{(\mathrm{eff})}{\varepsilon_1}-\Omega\overset{(\mathrm{eff})}{X}_{vs}$$
$$-\left[\overset{(\mathrm{eff})}{M}_{2;\kappa}-\overset{(\mathrm{eff})}{M_1}(k_{\gamma}s_{\mu;\kappa}v^{\mu}+s_{\mu;\kappa}v^{\mu}v_{\gamma})+\overset{(\mathrm{eff})}{M_2}k_{\gamma;\kappa}\right]\epsilon^{\nu\gamma\kappa}s_{\nu}+2a_1\overset{(\mathrm{eff})}{M_2}=$$
$$-\frac{1}{6}\left(\overset{(\mathrm{eff})}{\rho}+\overset{(\mathrm{eff})}{p}+\frac{1}{3}\overset{(\mathrm{eff})}{\Pi_2}\right)(\sigma_2+\vartheta)-a_2\overset{(\mathrm{eff})}{q_2}-\frac{1}{2}s^{\mu}s_{\nu}\overset{(\mathrm{eff})}{q}{}^{\nu}_{;\mu}. \tag{A.19}$$

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