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Dynamics of cosmological phase crossover during Bose–Einstein condensation of dark matter in Tsallis cosmology

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Abstract During the cosmic evolution process, as the temperature of a cosmological boson gas falls below a certain threshold, a Bose–Einstein condensation process can occur at various points throughout the cosmic history of the Universe. In this model, dark matter, conceptualized as a non-relativistic, Newtonian gravitational condensate is governed by the Gross–Pitaevskii–Poisson system. In our present study, we investigate the Bose–Einstein condensation process of bosonic DM by treating it as an approximate first-order phase transition within a modified cosmological framework, known as Tsallis cosmology. We examine the evolution of relevant physical quantities characterizing the evolution dynamics of the Universe, including energy density, temperature, redshift, scale factor, Hubble parameter, and dimensionless deceleration parameter before, during, and following the Bose–Einstein condensation phase transition takes place. Additionally, we especially investigate the specific era of the evolution of the Universe characterized by a mixture of *normal* and condensate phases of dark matter. We analyze the behavior of temporal evolution of an important time-dependent parameter, called the condensate dark matter fraction throughout the condensation process and find the time duration of condensation of dark matter in the Tsallis cosmological model. We see that the presence of Bose–Einstein condensate dark matter in the framework of Tsallis-modified cosmology significantly alters the cosmological evolution of the Universe as compared to the standard model of cosmology. We also find for a typical value of Tsallis non-extensive parameter $\beta = 0.35$, the model could explain an accelerated Universe without invoking any additional energy component and solve the age problem of our Universe.

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Contents

1 Introduction

The groundbreaking discovery [1–4] that a black hole (BH) acts as a thermodynamic object made a paradigm shift in understanding general relativity and its association with quantum field theory. In the early 1970s, two great physicists Bekenstein [1,2] and Hawking [3,4] showed that BHs emit radiation like a black body, with characteristic horizon entropy and corresponding temperature

$$\mathbb{S}_{BH} = \frac{A_{hor}}{4} \quad \text{and} \quad T_{hor} = \frac{\kappa_s}{2\pi} \quad \text{(in geometrized units)}, \tag{1}$$

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where A_{hor} and κ_s are the area and surface gravity of the BH horizon respectively. Gibbons and Hawking [5] were the first people, who extended the idea of the close relationship between thermodynamics and the event horizon of ordinary BHs to more general cosmological models. They showed that cosmological event horizon with a repulsive cosmological constant term (Λ) has a close resemblance with BH event horizon, and it obeys almost identical zeroth, first and second laws of classical BH thermodynamics [6].

Verlinde [7] proposed a new idea to understand the nature of gravity in comparison to the preexist compelling Einstein's geometric description of gravity. He proclaimed gravity is an emergent phenomenon rather than a fundamental force. Based on the holographic principle [8–10] and the law of equipartition of energy [11–13] and motivated by AdS/CFT correspondence [14–18], he was able to derive Newtonian gravity law, Poisson equation, and Einstein's field equations by identifying gravity as an entropic force produced by the change of entropy linked with the information on the holographic screen [7]. Meanwhile, Padmanabhan [19] suggested a different outlook toward the emergence of spacetime dynamics. He claimed that in the context of cosmology, one can treat time as a distinct entity from space, and cosmic space emerges with the progression of cosmic time. He also argued that the difference between the number of degrees of freedom in the surface boundary and the bulk region in an expanding flat de-Sitter Universe drives the accelerated expansion, and one can reproduce the Friedmann equation in standard cosmology [19]. A lot of studies have been accomplished in order to unveil the profound interrelation between geometry and thermodynamics [20–29], thereby manifesting the same in the cosmological arena [30–42].

It is to be noted that the entropic cosmology, which emerges from the thermodynamic point of view of BHs, the entopic nature of force, and the holographic principle is self-sufficient to build a cosmological model without introducing new fields or dark energy (DE) components and without transcending canonical general theory of relativity [43]. In the traditional approach of GR, the Einstein field equations are obtained from an action functional containing the Hilbert–Einstein term $\mathscr{R}$ (Ricci scalar) plus the term containing matter fields $\mathscr{L}_m$ (matter Lagrangian) i.e. considering only the volume term but ignoring the surface term $\mathscr{K}$. The inclusion of the surface term in the action [43,44]

$$\mathscr{A} = \int_V (\mathscr{R} + \mathscr{L}_m) + \frac{1}{8\pi} \oint_{\partial V} \mathscr{K} \qquad \text{(schematic)}, \tag{2}$$

leads to the modified Einstein equation[1]

$$R_{ab} - \frac{1}{2} g_{ab} = \frac{8\pi G}{c^4} T_{ab} + \textit{ surface terms}, \tag{3}$$

where R_{ab} and T_{ab} are components of the Ricci tensor and the energy-momentum tensor respectively. The surface term in the aforesaid action is responsible for the thermodynamic characteristics of the horizon and yields the field equations beyond that in Einstein's general theory of gravity. This modification is equivalent to the theory coming out from the horizon entropy which is not proportional to its area. The reason for considering the surface term in the usual Einstein–Hilbert action is imprinted in the holographic principle, which postulates an elementary correspondence between the volume and surface degrees of freedom [45,46]. Recently, Easson et al. published two papers on two accelerating periods of the Universe, viz. the early inflationary Universe [47] and the late time accelerated Universe [44] by taking into account the surface term. They suggested the reason behind both periods of acceleration is an emergent entropic force, which arises due to the storage of information on the cosmological horizon holographically. However, for Bekenstein–Hawking entropy-area relation [1–3,48]

$$\mathbb{S}_H = \frac{k_B c^3}{4\hbar G} A_H, \tag{4}$$

i.e. when the entropy of the horizon $\mathbb{S}_H$ is proportional to horizon area A_H, one obtains classical GR results. In this context, Jacobson [49] derived the Raychaudhuri equation and Einstein field equation by exploiting the equivalence principle along with Clausius relation $\delta Q = T d\mathbb{S}$ of thermodynamics, where δQ is the energy flux, $\mathbb{S}$ is the entropy and T is the Unruh temperature measured by an accelerated observer staying just inside the Universe's horizon. Besides, Cai et al. in [50], showed that one can derive standard Friedmann equations of a Friedmann–Robertson–Walker (FLRW) Universe by employing the equipartition theorem together with the holographic principle and Unruh temperature.

Many works have been done by modifying Bekenstein–Hawking area law in higher-order curvature theories (see [40, 51,52] and references therein). Two possible modifications that take place through the incorporation of quantum effects are power-law and logarithmic corrections. The power-law modifications occur when there is an entanglement between quantum fields inside and outside the horizon [53–55]. The origin of logarithmic modifications resides in loop quantum

[1] Throughout this article, the Latin indices such as $a, b, ...i, j, ...$, etc., span across the values 0, 1, 2 and 3. Here, the index 0 denotes the time dimension, while the indices 1, 2 and 3 correspond to the conventional spatial dimensions. The speed of light c, the Boltzmann constant k_B, gravitational constant G, and the Planck constant $\hbar$ are denoted in standard notation.

gravity and arises from thermal fluctuations at equilibrium and quantum fluctuations [56–60].

Another kind of correction to the area law of BHs originates from the fact that the thermodynamics for D-dimensional non-standard systems cannot be associated with the additive Boltzmann–Gibbs entropy S_{BG} (or for quantum mechanical systems referred to as the von Neumann entropy), but with generalized non-additive entropies. The BG entropy is incapable of illustrating systems with divergent partition functions, viz. gravitational systems [61–63]. The conventional area law of BH, given by Bekenstein and Hawking has a fundamental shortcoming as the BH entropy violates the thermodynamical extensivity, pointed out in [61]. In this context, it is to be noted that if a thermodynamical system is to be physically classified as a $(D-1)$ dimensional, then one can associate the additive entropy S_{BG} as its thermodynamical entropy. While, on the other hand, if it is D dimensional, then S_{BG} cannot be regarded as its thermodynamical entropy, but rather a non-additive entropic functional is necessary to sport that role [61,64]. The violation of such thermodynamical law in connection to the area law is somehow overlooked or not taken seriously. In fact, it was not expected to maintain the thermodynamical extensivity of such complex systems at that time. However, there exist a few mathematical and scientific facts that point out such a viewpoint as anomalous in character. In order to find an approach to solve the paradox of non-standard complex systems (such as BHs, strongly entangled systems, and systems satisfying area law in general), when standard additive BG-von Neumann entropy is not proportional to its volume, one must introduce a generalized entropy which is non-additive in nature [61,64]. The requirement of non-additive generalization in the definition of entropy is therefore essential to retain the entropic extensivity of a thermodynamical system. In this context, Hanel and Thurner [65,66] established the relevance of using non-additive generalized entropic forms to ensure the extensivity of entropy by exploiting the Khinchine axioms and surface-dominant statistics on complex systems. Tsallis entropy is an extension of BG entropy [61,67,68], which is suggested to rectify the thermodynamic conundrum. In connection to this, Tsallis and Cirto [61,67] put forward a microscopic mathematical expression of BH entropy that shows non-extensive statistics. In this circumstance, the area entropy relation of BHs gets modified to $\mathbb{S}_H \propto A_H^{\beta}$, where β in the exponent signifies the non-extensive Tsallis parameter, quantifying the degree of non-extensivity in the system. Throughout the years, the Tsallis entropy has been getting attention from researchers for its applications in various fields. It has produced impressive results in different complex systems so far, such as in sectors of BHs [61,67], holographic DE [69–72] and DM [73], neutrinos [74,75], background radiation [76], self-gravitating stellar systems [77,78], thermodynamic gravity [79–81], polymer chains [82], and low-dimensional dissipative systems [62]. Teimoori et. al. [83] focused on a slow-roll type inflationary scenario by reconstructing a $f(\mathscr{R})$ gravity model in equivalence with Tsallis entropy-based cosmology. They derived the inflationary observables, e.g. the tensor-to-scalar ratio and the scalar spectral index from the study of power spectra of scalar and tensor perturbations. They found an improved observational consistency with the Planck 2018 CMB data [84] in the framework of Tsallis entropy-based inflation. In a study reported in [85], the authors investigated the consequences of Tsallis entropy-based cosmology on forming light elements in the early Universe and checked its viability as an effective modified cosmology. It is also to be noted that, the Tsallis cosmology resolves the observed discrepancy between the present bound on DM relic abundance and the current IceCube high-energy neutrino data [86]. By employing modifications in Friedmann equations based on Tsallis entropy, authors have accommodated these differences keeping the Tsallis scaling exponent at around 1.57 [86]. Basilakos et. al. [87] presented a way to alleviate both H_0 and σ_8 tensions at the same time by applying a thermodynamics-gravity conjecture using non-additive Tsallis entropy in place of the standard Hawking–Bekenstein one. They showed for a particular choice of Tsallis exponent ($\lesssim 1$), one can obtain an effective phantom DE equation of state solving H_0 tension, and an increased friction term in the matter-perturbation evolution equation and a relatively small effective Newton's constant solving σ_8 tension. In the context of large-scale structure formation, it is shown that perturbations grow faster in Tsallis-modified Universe in comparison to standard Friedmann one [88]. In [89], it has been claimed that the age problem of the Universe cannot be solved within the framework of the standard model of cosmology without accounting for the cosmological constant or DE. Later Sheykhi in [90] showed a resolution of the age problem as well as late-time acceleration of the Universe in the framework of Tsallis cosmology without considering any form of DE. Thus, from the primordial stage to the late time phase of our Universe, Tsallis entropy-modified cosmology could be one of the possible alternatives to standard FLRW cosmology.

The cosmological concordance ΛCDM model, recognized as the standard model of modern cosmology, comprises baryonic matter, cold dark matter (CDM), and DE. This model has been extraordinarily successful in explaining various cosmological phenomena, from the accelerating expansion of late-time Universe [91,92] to the statistical properties and power spectrum of cosmic microwave background (CMB) anisotropy [93]. Additionally, it effectively characterizes the features of large-scale cosmological structures [94,95] and the observed abundances of light nuclei (e.g. hydrogen and helium) [96–99] as well. Despite its impressive successes and straightforwardness of this framework, the ΛCDM model is currently under rigorous investigation, as discussed in references [100–104], due to significant theoretical and observational challenges. These challenges incorporate the cosmological constant problem, which connects to the dissimilarity between the observationally measured value of the cosmological constant and theoretical expectations from quantum field theory [105–107], and the late-time coincidence problem, which indicates to the baffling observation that the energy densities of DM and DE are of the same order of magnitude at a recent redshift $z \approx 0.55$, despite their different evolutionary paths over cosmic time [108,109]. Besides, there are other important anomalies such as the CMB anisotropy anomalies [84,110], Hubble tension [104,111,112], and Baryon Acoustic Oscillations (BAO) curiosities [113–115], etc (see also [116]). At small scales (less than a few hundred $kpcs$), several predictions of the ΛCDM model diverge from observations on several occasions [117–119]. Particularly while observing galaxies, observations reveal many shortcomings that the ΛCDM model faces in explaining structures on relatively smaller scales, specifically those below 1 Mpc roughly [120–122]. The 'core-cusp problem' [123–126] highlights a notable discrepancy, which arises within the DM halo density profile in low-mass galaxies, as portrayed by cosmological N-body simulations, which is a vital method in physical cosmology for determining the speculations of the ΛCDM model. This density profile is traditionally represented as having a cuspy nature theoretically [127–130]. In contrast, the observed density profile of low surface brightness galaxies usually exhibits a core configuration [131–134]. The 'missing satellites problem' [135–138], highlights a significant overabundance of small-scale structures. This issue arises from the huge disparity between the forecasted number of substructures within DM halos, as predicted by precise collisionless cosmological N-body simulations, and the exact number of satellite galaxies observed within the Local Group. Specifically, the ΛCDM model predicts a larger number of satellites (frequently in the thousands) in comparison to the relatively smaller number (around fifty) of observed dwarf galaxies [137,139]. These two small-scale problems are significant in connection to structure formation. The limitations imposed by the core-cusp issue and the excess of satellite galaxies can be handled by considering alternative strategies. In 2000, Spergel and Steinhardt [140] suggested a solution to the problems of the ΛCDM model by considering weakly interacting CDM particles if they are self-interacting with a large scattering cross-section but negligible dissipation or annihilation. Hu et al. [141] suggested that DM consists of extremely lightweight and free scalar particles in the same year. These scalar particles, with masses around $10^{-22}\, eV$, would form a cold Bose-Einstein condensate (BEC) referred to as 'fuzzy cold dark matter' (FCDM). The intrinsic wave nature of ultralight DM can prevent the formation of kpc-scale cusps within DM halos, and reduce the number of low-mass halos as well. In 2015, Suárez and Chavanis [142] assumed the self-interacting complex scalar field as a DM model (SFDM model) governed by the Klein–Gordon–Einstein (KGE) equations. The SFDM model has an intrinsic small-scale finite Jeans cut-off length related to quantum mechanics, which could solve the missing satellite problem. The appearance of the quantum potential (arising from Heisenberg's uncertainty principle in the case of a non-interacting scalar field (SF)), and the pressure due to scattering among particles in the case of a self-interacting SF, act as a barrier against gravitational collapse on small scales. Hence, this phenomenon results in central density cores instead of cusps. A considerable amount of studies have been conducted on the possibility of bosonic structures. Considering a bosonic complex SF with quadratic and quartic self-coupling in 1990, Press et al. [143] examined small-scale and large-scale systems with soft-bosonic particles, where baryons are gravitationally coupled. They confirmed that Heisenberg's uncertainty principle prevents soft-bosonic matter from falling into clusters of galaxies. In 1992, Frieman et al. [144] depicted an essential physical phenomenon 'late-time cosmological phase transition' involving pseudo-Nambu–Goldstone bosons of ultralow-mass, based on particle physics. By analyzing the cosmological evolution of the bosonic field, they specified regions of parameter space and concluded that it could make a significant contribution to the energy density of the Universe. In 1994, Sin [145] investigated the quantum mechanical galactic DM halo formation with pseudo-Nambu–Goldstone bosons that emerged in the late-time cosmic phase transition. This work was followed by a study on galaxy rotation curves in another article [146] by using Landau–Ginzberg type theory. It is presumed that DM is composed of scalar boson-type particles. Consequently, there is a possibility that DM could exist in BEC form at some point in cosmic history, where the mass and scattering length of particles are considered to be free parameters to represent the system. Notably, at very low temperatures, all particles of a dilute quantum Bose gas start to occupy the same quantum ground state, forming a momentum-space condensate. The BEC process was first predicted for an ideal gas in 1924 by S. N. Bose [147] and later extended by Einstein [148]. The empirical realization of BEC for trapped dilute Bose gases (e.g. atomic vapor of rubidium and lithium) is reported in [149–151]. The assertion that DM could be in a BEC state was first proposed in [152] and later reassessed in different contexts in various articles [141,153–156]. In the cosmological circumstances, the concept of BEC structure formation is not new (see articles [157–166] for details). In 2007, Böhmer and Harko [167] performed an in-depth investigation on gravitationally trapped BEC DM halos, designed by the Gross–Pitaevskii–Poisson (GPP) system. In 2009, Sikivie and Yang [168] showed that self-interacting axions could change their phase to BEC after thermalization. They provide a comparative discussion between CDM and axion BEC density perturbations. In 2011, Harko [166] investigated the cosmological evolution of density contrast for non-relativistic DM employing the pseudo-Newtonian approach. Chavanis [157] studied the growth of density perturbations in self-gravitating BEC DM in an expanding Universe, both with and without special relativistic effects in 2012. Also, in the same year, Kian and Ling [169] analyzed the growth of inhomogeneities in BEC SFDM in the backgrounds of both Newtonian and general relativity, emphasizing the differences between the BEC model and standard ΛCDM cosmology. By employing a gauge-invariant general relativistic formalism, Freitas and Gonçalves [164] investigated the growth of density perturbations during the phase transition from *normal* DM to BEC DM in 2013. In 2015, Suárez and Chavanis [142] explored the hydrodynamic representation of SFDM through the Klein–Gordon–Einstein (KGE) equations in a weak gravitational field. They also discussed the growing and oscillatory modes of density contrast for non-relativistic SF or BEC. In 2020, Crăciun and Harko [170] established that the data from Spitzer Photometry and Accurate Rotation Curves (SPARC), which incorporates a database of 173 galaxies, aligns excellently with the theoretical predictions for the slowly rotating BEC model. In our previous work [171] in 2024, we studied the temporal evolution of cosmological density perturbations of the BEC DM and discussed the possible corrections to standard cosmology. Therefore, the theoretical modeling of DM as BEC is well established in previous studies, and observationally it has proven successful over the standard ΛCDM model.

In this article, we consider our Universe to be made of visible baryonic matter, radiation, and DM. A substantial category of DM candidates in particle physics is non-relativistic, non-baryonic, and weakly-interacting massive particles (WIMPs). There are various other hypothetical contenders of DM like axions, standard model neutrinos, supersymmetric candidates (e.g. neutralinos, axinos, gravitinos, etc), and sterile neutrinos, etc [172–174]. Here in our work, we suppose DM to be comprised of quantum scalar bosonic particles that went through a phase transition from their *normal* form to BEC form as the temperature of the Universe fell below the critical temperature T_{crit} during the evolution of the Universe. The ground state of the system became macroscopically populated by the particles belonging to the same quantum state forming BEC. The phase transition did not happen instantaneously, but both phases existed together until the whole conversion took place (i.e. smooth phase transition). Throughout the phase crossover, and after, the dynamics of the Universe was not as usual. The dynamics of cosmological evolution in the presence of BEC DM in standard cosmology [159] and loop quantum cosmology [163] have been reported so far. In [175], the non-linear clustering of BEC DM is studied in Thomas–Fermi approximation by considering both abrupt and smooth first-order phase transitions. Motivated by the above works [159,163,175] and the success of the BEC model and Tsallis cosmology over ΛCDM standard cosmology, in this article we aim to investigate the BE condensation process of DM in the framework of Tsallis cosmology, thereby assuring solutions to small-scale anomalies and late-time acceleration of the Universe without considering any form of DE. By considering a first-order phase transition of DM in a Tsallis-modified Universe, we look into the important cosmological parameters including temperature, redshift, energy density, and scale factor before, during, and after the phase transition. We also analyze the nature of temporal evolution of a relevant time-dependent parameter, called the condensate DM fraction for the duration of the BE condensation process and find the condensation time of DM under the influence of Tsallis cosmology.

The present manuscript is organized as follows. In Sect. 2, at first we briefly present the derivation of the modified Friedmann equation in Tsallis cosmology. After that, we outline the properties of *normal* DM and BEC DM along with the physical process involving phase transition. In the next Sect. 3, we briefly discuss relevant cosmological parameters at the critical point of phase transition. Along with this, the evolution of the condensation process and the cosmological dynamics of the pre and post-condensation phase of DM are analyzed in the framework of Tsallis cosmology. Finally in Sect. 4, we briefly summarize our work and make some concluding remarks.

2 *Normal* and Bose–Einstein condensate phases in Tsallis cosmology

In this section, we review the physical process involving the condensation of DM from its *normal* to BEC form in the framework of Tsallis-modified cosmology in general with their properties. First, in the following subsection, we briefly discuss the modified Friedmann equation in Tsallis cosmology.

2.1 Friedmann equation in Tsallis cosmology

We assume the background of our Universe is expanding in a spatially isotropic and homogeneous way, following the Friedman–Lemitre–Robertson–Walker (FLRW) metric [176]

$$g_{ab} = diag\left(-c^2,\ \frac{S^2}{1-\kappa r^2},\ S^2r^2,\ S^2r^2\sin^2\theta\right), \qquad (5)$$

where the scale factor $S = S(t)$ describes the cosmological expansion. The spatial curvature κ takes the value $+1, 0, -1$ for closed, flat, and open Universes respectively. We consider our Universe to be bounded by a physical boundary, called an apparent horizon with a radius [41]

$$R_H = \frac{c}{\sqrt{H^2 + \kappa c^2/S^2}}, \tag{6}$$

satisfying the first and second laws of thermodynamics. Here, $H \stackrel{\text{def}}{=} \dot{S}/S$ is the time-dependent Hubble parameter, measuring the rate of expansion of the Universe. For simplicity, the energy and matter contents of the Universe are taken to be in the form of a perfect fluid with covariant energy–momentum tensor components [176]

$$\mathscr{T}_{ab} = \left(\rho + \frac{P}{c^2}\right) u_a u_b + P g_{ab}, \tag{7}$$

where ρ and P are the matter density and pressure of fluid respectively. u_a is the four-velocity 1-form components of the fluid-like constituents of the Universe, satisfying the condition $u_a u^b = 1$. In curved space-time, $\nabla_a \mathscr{T}^{ab} = 0$, describes the way the external gravitational field influences the fluid materials, leading to the conservation of mass and energy as [176]

$$\dot{\rho} + 3H\left(\rho + \frac{P}{c^2}\right) = 0. \tag{8}$$

It is also assumed that a similar form of the first law of thermodynamics applies to the boundary of the Universe and takes the form [23]

$$dE = T_H d\mathbb{S}_H + \mathscr{W} dV, \tag{9}$$

where $\mathscr{W}$ is an invariant of contravariant energy–momentum tensor $\mathscr{T}$, called work density and defined as [2] $\mathscr{W} = -\frac{1}{2}\, trace\mathscr{T}$ [177,178]. Here, T_H and $\mathbb{S}_H$ are the temperature (Hawking temperature) and entropy associated with the apparent horizon of area A_H, which can be defined as [23,79,179]

$$T_H = \frac{\hbar c}{2\pi k_B R_H}, \tag{10}$$

and [61]

$$\mathbb{S}_H = \frac{k_B}{4L_{pl}^{2\beta}} A_H^{\beta} \tag{11}$$

respectively. Here $L_{pl} = \sqrt{\hbar G/c^3}$ is the Planck's length. Meanwhile, Eq. (11) refers to the modification to area law due to Tsallis entropy, where β is called the Tsallis parameter (or non-extensive parameter) measuring the degree of non-extensivity in the system. For $\beta = 1$, the relation (11) reduces to the Bekenstein-Hawking area law (4). In order to restore the desired extensivity of the horizon entropy one must relate the non-extensive parameter with the spatial dimension (D) of the system as $\beta = D/(D-1)$ for $D > 1$ [61]. For example, we note (3 + 1) dimensional world with $D = 3$, $\beta = 3/2$ leads to extensivity.

We assume our perfect fluid-like Universe is a three-dimensional sphere of radius R_H and volume $V_H = \frac{4}{3}\pi R_H^3$, containing total energy $E = \rho c^2 V_H$, and the entropy associated with the apparent horizon of area $A_H = 4\pi R_H^2$ is given in terms of Tsallis entropy in (11). Combining all these expressions into the first law of Thermodynamics (9) leads to the modified Friedmann equation in Tsallis cosmology [79,90]

$$\left(H^2 + \frac{\kappa c^2}{S^2}\right)^{2-\beta} = \frac{(4-2\beta)(4\pi)^{2-\beta} c^{5-2\beta} L_{pl}^{2\beta}}{3\hbar\beta}\rho. \tag{12}$$

This imposes a restriction on the maximum value of Tsallis parameter β, $\beta < 2$. The standard Friedmann equation in Einstein's gravity can be recovered in the limit $\beta \to 1$.

As the cosmological evolution of matter contents of the Universe is studied in the framework of Tsallis cosmology, we do not have to include the DE component Λ [44,47,79]. Therefore, we assume that matter density ρ consists of radiation matter density ρ_{rad} with pressure $P_{rad} = \rho_{rad}c^2/3$, pressureless ($P_{bar} = 0$) baryonic matter density ρ_{bar} and DM density ρ_χ with pressure P_χ only. From Eq. (8) the cosmological evolution of radiation and baryonic matter densities are given by $\rho_{rad} = \rho_{rad,0}/(S/S_0)^4$ and $\rho_{bar} = \rho_{bar,0}/(S/S_0)^3$ respectively and for the DM we consider a general form of density as $\rho_\chi = \rho_{\chi,0}/f_\chi(S/S_0)$. Here $\rho_{rad,0}$, $\rho_{bar,0}$, and $\rho_{\chi,0}$ are the matter densities of radiation, baryon, and DM respectively corresponding to present-day scale factor $S = S_0$. Depending on DM matter types an arbitrary function $f_\chi(S/S_0)$ of the scale factor can be chosen. Now, we introduce modified critical density

$$\rho_{cr,0}^{mod} = \frac{3\hbar\beta H_0^{4-2\beta}}{(4-2\beta)(4\pi)^{2-\beta} c^{5-2\beta} L_{pl}^{2\beta}} = \frac{1}{(2-\beta)}\left(\frac{H_0 L_{pl}}{2\sqrt{\pi} c}\right)^{2-2\beta} \rho_{cr,0} \tag{13}$$

in Tsallis cosmology, where $H_0 = H(S = S_0)$ is the Hubble parameter and $\rho_{cr,0} = \frac{3\hbar H_0^2}{8\pi c^3 L_{pl}^2}$ is the critical density in standard Friedmann cosmology at present. Therefore, the Friedmann Eq. (12) for a flat Tsallis-modified Universe reduces to

$$\left(\frac{H}{H_0}\right)^{4-2\beta} = \left[\frac{\Omega_{rad,0}^{mod}}{(S/S_0)^4} + \frac{\Omega_{bar,0}^{mod}}{(S/S_0)^3} + \frac{\Omega_{\chi,0}^{mod}}{f_\chi(S/S_0)}\right]. \tag{14}$$

Here we have defined modified dimensionless density parameter of $\alpha = rad,\ bar,$ and χ

$$\Omega_{\alpha,0}^{mod} = \frac{\rho_{\alpha,0}}{\rho_{cr,0}^{mod}} = (2-\beta)\left(\frac{2\sqrt{\pi}c}{H_0 L_{pl}}\right)^{2-2\beta} \Omega_{\alpha,0}, \tag{15}$$

[2] The $trace$ refers to the two-dimensional trace defined perpendicular to the spheres of symmetry. Therefore, in our system $\mathscr{W} = -\frac{1}{2}(\mathscr{T}^{00}g_{00} + \mathscr{T}^{11}g_{11}) = \frac{1}{2}(\rho c^2 - P)$.

where $\Omega_{\alpha,0} = \rho_{\alpha,0}/\rho_{cr,0}$ is the density parameter in the standard Friedmann Universe obeying $\Omega_{rad,0}+\Omega_{bar,0}+\Omega_{\chi,0} = 1-\Omega_\Lambda$. Here Ω_Λ stands for the DE density parameter in standard cosmology.

2.2 *Normal* dark matter

It is assumed that DM is composed of bosonic particles that follow Bose–Einstein (BE) statistics. In the early times of the Universe, the bosonic particles of mass m_χ and temperature T were in thermal equilibrium with hot relativistic plasma and then decoupled from surrounding plasma at a decoupling temperature T_{dec}. We further assume that the bosonic DM particles formed an isotropic gas, and were in kinetic equilibrium among themselves [159,163,180–182]. Therefore, the spatial particle number density is given by

$$n_\chi = \frac{g}{(2\pi\hbar)^3}\int f_{BE}(\vec{p})\, d^3p, \tag{16}$$

where g is the spin degeneracy, and f_{BE} be the phase-space occupancy (distribution function) of BE distribution is defined as

$$f_{BE}(\vec{p}) = \frac{1}{\exp{(E-\mu)/k_BT} - 1}. \tag{17}$$

The momentum $\vec{p}$ and the energy E of the particle are related by a relation $E^2 = (\vec{p}\cdot\vec{p})c^2 + m_\chi^2c^4$. Bosonic DM that decoupled from the existing plasma in the early Universe got its momentum redshifted according to $\vec{p} = \vec{p}_{dec}S_{dec}/S$, where $\vec{p}_{dec}$ and S_{dec} are the momenta of particles and scale factor of the Universe respectively during decoupling. Further the particle number density n_χ changes as $n_\chi \propto S^{-3}$ with the expansion of the Universe [159,163,180–182]. The phase-space BE distribution function $f_{BE}(\vec{p})$ at a time t is linked with the same at the time of decoupling t_{dec} by $f(\vec{p},t) = f(\vec{p}S/S_{dec}, t_{dec})$, and keeps on maintaining equilibrium in both ultra-relativistic ($m_\chi c^2 << k_BT_{dec}$) and non-relativistic ($m_\chi c^2 >> k_BT_{dec}$) regimes. In ultra-relativistic decoupling ($E \approx pc$, $\mu = \mu_{dec}S_{dec}/S$, and $T = T_{dec}S_{dec}/S$), the phase-space occupancy can be derived as

$$f_{BE}^{UR}(\vec{p}) = \frac{1}{\exp{(pc-\mu)/k_BT} - 1}. \tag{18}$$

Moreover, for non-relativistic decoupling ($E-\mu \approx p^2/2m_\chi - \mu_{kin}$, $\mu_{kin} \equiv \mu - m_\chi c^2 = \mu_{kin,dec}(S/S_{dec})^2$, and $T = T_{dec}(S_{dec}/S)^2$), the phase-space occupancy reads

$$f_{BE}^{NR}(\vec{p}) = \frac{1}{\exp{(p^2/2m_\chi - \mu_{kin})/k_BT} - 1}. \tag{19}$$

As we consider DM to be a bosonic non-relativistic species, therefore to obtain an equation of state, we find the pressure of an isotropic momenta distribution of bosons

$$P = \frac{g}{(2\pi\hbar)^3}\int \frac{p^2}{3E} f_{BE}(\vec{p})\, d^3p$$

in the non-relativistic regime as [159,163,180–182]

$$P_\chi = \frac{g}{(2\pi\hbar)^3}\int \frac{p^2}{3E} f_{BE}^{NR}(\vec{p})\, 4\pi p^2 dp = \sigma_\chi^2\rho_\chi c^2, \tag{20}$$

where $\sigma_\chi = \sqrt{\langle\vec{v}_\chi^2\rangle/3c^2}$ is the one-dimensional velocity dispersion, and $\langle\vec{v}_\chi^2\rangle$ is the average squared velocity v_χ of non-relativistic DM particles with energy $E \approx m_\chi c^2$, momentum $p \approx m_\chi v_\chi$, and density $\rho_\chi = m_\chi n_\chi$. From the conservation law in Eq. (8) for the DM

$$\dot{\rho}_\chi + 3H(1+\sigma_\chi^2)\rho_\chi = 0, \tag{21}$$

we obtain the general solution

$$\rho_\chi = \frac{\rho_{\chi,0}}{(S/S_0)^{3(1+\sigma_\chi^2)}}. \tag{22}$$

Here $\rho_{\chi,0}$ is the present-day DM density as mentioned earlier. Therefore, the dynamics of the Tsallis-modified Universe which is given in modified Friedmann Eq. (14) reduces to

$$\left(\frac{H}{H_0}\right)^{4-2\beta} = \left[\frac{\Omega_{rad,0}^{mod}}{(S/S_0)^4} + \frac{\Omega_{bar,0}^{mod}}{(S/S_0)^3} + \frac{\Omega_{\chi,0}^{mod}}{(S/S_0)^{3(1+\sigma_\chi^2)}}\right]. \tag{23}$$

2.3 Cosmological phase transition from *normal* to Bose–Einstein condensate dark matter

So far in this article, we are considering bosonic DM particles with mass m_χ and temperature T in the early Universe. These particles were initially in thermal equilibrium and decoupled from the remaining hot plasma at a temperature of T_{dec}. They can be considered a *normal* form of DM. As time advanced, the Universe gradually cooled down to its critical temperature, $T_{crit} \approx 2\pi\hbar^2\rho_\chi^{2/3}/m_\chi^{5/3}k_B$, initiating the transformation of *normal* DM into BEC DM via a phase transition process. However, this phase transition did not happen all of a sudden; instead, a mixed phase co-existed until all the *normal* DM had fully converted into the BE condensate form [159,171]. In this context, we should note that the determination of order characterizing the BEC phase transition exhibits some degree of uncertainty. In [183,184], authors argue that the BEC phase transition is categorized by a spontaneous global $U(1)$ symmetry-breaking process, where the condensation factor serves as an order parameter, indicating a second-order phase transition. Nonetheless, several mean-field theoretical models, including Hartree–Fock, Popov, Yukalov–Yukalova, and many-body t-matrix, have not suggested a second-order phase transition regarding BEC

[185]. An extensive analysis of the thermodynamic instability of a confined ideal Bose gas with a finite number of particles recognizes a discontinuous phase transition, indicated by a pure mathematical singularity validating first-order phase transition process [186]. Reinforcing this perspective, Harko [159] insists on a first-order phase transition accurately representing BEC dynamics. In this particular context, it is worth discussing that first-order cosmological phase transitions in the early Universe are described by nucleation and collision of true vacuum bubbles in a false vacuum background. The collisions among the walls of the expanding bubbles, or the surrounding plasma shells produce gravitational waves. Such scenarios are first-order inflation, spontaneous symmetry breaking in grand unified theory (GUT), and spontaneous electroweak symmetry breaking [187–189]. Apart from that, a similar phase transition in the dark sectors could also be possible and studied in texts [190–194]. Over the years, it has been a popular research topic as it predicts the production of detectable gravitational wave signals providing a significant amount of latent heat needs to be released and a notable fraction of energy in the primordial plasma takes part during the first-order phase transition [195]. The latent heat that a first-order phase change in a plasma produces could be disseminated to the degrees of freedom associated with the bubble wall. This causes the plasma to produce acoustic waves, which in turn produce gravitational waves when the transition is accomplished [189,196]. As the *normal* to BEC DM phase transition is said to be first-order, we naturally expect an outflow of latent heat during the phase crossover which causes gravitational wave production either detectable or undetectable. Furthermore, the release of latent heat and the dynamics of bubble nucleation could drastically influence the non-linear clustering mechanism of structure formation during the phase transition [175]. A theoretical treatment of cosmological gravitational waves during phase transition from non-condensate to condensate DM in standard Friedmann cosmology is analyzed in [164].

2.4 Bose–Einstein condensate dark matter

BEC relies on the wave properties inherent to quantum particles. Each particle is associated with a de Broglie wavelength, denoted as $\lambda_{dB} = \sqrt{2\pi\hbar^2/m_\chi k_B T}$. Upon cooling of particles to critical temperature T_{crit}, their de Broglie wavelengths begin to increase, eventually surpassing the interparticle separation, as observed in [159,164]. At sufficiently low temperatures, these wavelengths converge, leading to a macroscopic population in the system's ground state. As the temperature approaches absolute zero ($T = 0$), a coherent state emerges, forming a pure BEC. BEC DM is posited to comprise a weakly interacting ultra-cold dilute gas, wherein solely low-energy two-body collisions characterized by the s-wave scattering length (l_s) are pertinent. The dynamics of this system can be effectively explained by the time-dependent Gross–Pitaevskii (GP) equation at $T = 0$, wherein particle interaction is approximated through the mean-field approach. Furthermore, the dynamics of BEC DM confined in gravitational traps, consisting of N bosonic particles with mass m_χ and experiencing non-linear short-range interactions, are delineated by the GPP system [157,159,164,171,197–200]

$$-\frac{\hbar^2}{2m_\chi}\vec{\nabla}_r^2\Psi(\vec{r},t) + m_\chi\left[\Phi_{grav}(\vec{r},t) + \xi(\rho_\chi(\vec{r},t))\right]\Psi(\vec{r},t) = i\hbar\left.\frac{\partial\Psi(\vec{r},t)}{\partial t}\right|_r, \tag{24}$$

$$\vec{\nabla}_r^2\Phi_{grav}(\mathbf{r},t) = 4\pi G\rho_\chi(\mathbf{r},t), \tag{25}$$

where $\rho_\chi(\vec{r},t) = Nm_\chi|\Psi(\vec{r},t)|^2$ represents the density of BEC DM, $\Phi_{grav}(\vec{r},t)$ denotes the gravitational trapping potential and G stands for the universal gravitational constant. $\Psi(\vec{r},t)$ in Eq. (24) signifies the macroscopic wavefunction of the condensate. Moreover, the non-linear effective potential term ξ in GP Eq. (24) is expressed as follows [199]

$$\xi(\rho_\chi(\vec{r},t)) = \lambda_2\rho_\chi(\vec{r},t) + \lambda_3\rho_\chi^2(\vec{r},t), \tag{26}$$

The linear term in density ρ_χ in Eq. (26) appears due to the two-body interparticle interaction, represented by the coupling constant $\lambda_2 = 4\pi\hbar^2 l_s/m_\chi^3$ as mentioned in [157,200]. It is worth mentioning that encounters among quantum particles at low energies are generally dominated by the s-wave scattering length l_s. In dilute BEC clouds, due to the smallness of the scattering length in comparison to the interparticle separation only two-body interactions are predominant. The numerical value of scattering length l_s can be either positive or negative, indicating repulsive or attractive boson-boson interparticle interactions, respectively [201]. For instance, in laboratory systems, the interactions can be either repulsive, as observed with Rb^{87} atoms ($l_s = 5.45$ nm) [202], or attractive, as with Li^7 atoms ($l_s = -1.45$ nm) [203]. In this study, we shall concentrate exclusively on $l_s > 0$ cases. The effect of l_s on the mass-radius relation of astrophysical BEC structures has been explored in [204,205]. Besides, the quadratic term in density ρ_χ in Eq. (26) appears for three-body interparticle interaction, with a coupling constant λ_3 that becomes insignificant at low densities. Therefore, in the conventional treatment of BEC, the contribution of this term is often disregarded ($\lambda_3 \approx 0$) [206,207]. Employing the Madelung transformation, the wave function $\Psi(\vec{r},t)$ can be represented as [157,208]

$$\Psi(\vec{r},t) = \sqrt{\rho_\chi(\vec{r},t)/Nm_\chi}\exp[i\mathscr{A}(\vec{r},t)/\hbar]. \tag{27}$$

Here, $\mathscr{A}(\vec{r},t)$ in the phase has a dimension of an action. By substituting this transformation into Eq. (25) and subsequently separating the imaginary and real components, we get [157]

$$\left.\frac{\partial \rho_\chi(\vec{r},t)}{\partial t}\right|_r + \vec{\nabla}_r.\{\rho_\chi(\vec{r},t)\vec{u}(\vec{r},t)\} = 0, \tag{28}$$

$$\left.\frac{\partial \vec{u}(\vec{r},t)}{\partial t}\right|_r + \{\vec{u}(\vec{r},t).\vec{\nabla}_r\}\vec{u}(\vec{r},t) = -\frac{\vec{\nabla}_r P_\chi(\vec{r},t)}{\rho_\chi(\vec{r},t)} -\vec{\nabla}_r\Phi_{grav}(\vec{r},t) - \frac{\vec{\nabla}_r\Phi_{quant}(\vec{r},t)}{m_\chi}, \tag{29}$$

respectively. In deriving Eqs. (28–29), we exploit identifications of the pressure gradient, the velocity field regarding the quantum fluid

$$\vec{\nabla}_r P_\chi(\vec{r},t) = \rho_\chi(\vec{r},t)\vec{\nabla}_r\xi, \tag{30a}$$

$$\vec{u}(\vec{r},t) = \frac{\vec{\nabla}_r\mathscr{A}(\vec{r},t)}{m_\chi}, \tag{30b}$$

respectively, and the quantum potential that emerged due to Heisenberg's uncertainty principle

$$\Phi_{quant}(\vec{r},t) = -\frac{\hbar^2}{2m_\chi}\frac{\vec{\nabla}_r^2\sqrt{\rho_\chi(\vec{r},t)}}{\sqrt{\rho_\chi(\vec{r},t)}}. \tag{31}$$

In the derivation of Eq. (29), $\mathscr{A}(\vec{r},t)$ is assumed to be non-singular. As a result of this assumption, according to the identification in (30b), the condition for irrotational flow, $\vec{\nabla}_r \times \vec{u} = 0$, holds [201]. The significance of the quantum pressure term becomes meaningful in the vicinity of the condensate boundary, especially when the number of particles within the gravitationally bound BEC reaches an adequately large magnitude. Consequently, the quantum stress term within the equation of motion for the condensate can often be ignored, leading to the Thomas–Fermi approximation. This approximation achieves accuracy as the kinetic energy term of particles becomes insignificant and the number of particles within the BEC approaches infinity, by that means representing a classical limit of the theory [159,163,201]. Equations (28) and (29) are usually referred to as the *continuity equation* and the *quantum Euler equation*, respectively. This suggests that BEC DM can be conceptualized as a non-relativistic Newtonian fluid with quantum effects. In the conventional treatment of BEC, the equation of state is defined by [157,167,199]

$$P_\chi(\vec{r},t) = \xi\rho_\chi - \int \xi\, d\rho_\chi = \frac{2\pi l_s\hbar^2}{m_\chi^3}\rho_\chi^2(\vec{r},t) = \frac{\lambda_2}{2}\rho_\chi^2(\vec{r},t). \tag{32}$$

According to Eq. (32), the continuity equation in (8) is now reads

$$\dot{\rho}_\chi + 3H\left(1 + \frac{\lambda_2}{2c^2}\rho_\chi\right)\rho_\chi = 0, \tag{33}$$

with the solution

$$\rho_\chi = \frac{C_\chi}{\left[\left(\frac{S}{S_0}\right)^3 - \left(\frac{\lambda_2}{2c^2}\right)C_\chi\right]}, \tag{34}$$

where C_χ is the arbitrary integration constant. Utilizing the condition that $\rho_\chi = \rho_{\chi,0}$ at $S = S_0$, it is easy to derive the expression for the density of the condensate as follows

$$\rho_\chi = \left(\frac{2c^2}{\lambda_2}\right)\frac{\rho_{0\chi}}{\left[\left(\frac{S}{S_0}\right)^3 - \rho_{0\chi}\right]}, \tag{35}$$

where we symbolize

$$\rho_{0\chi} = \frac{\rho_{\chi,0}\lambda_2/2c^2}{1+\rho_{\chi,0}\lambda_2/2c^2} = \frac{\Omega^{mod}_{\chi,0}\,\rho^{mod}_{cr,0}\lambda_2/2c^2}{1+\Omega^{mod}_{\chi,0}\,\rho^{mod}_{cr,0}\lambda_2/2c^2} = \frac{\Omega_{\chi,0}\rho_{cr,0}\lambda_2/2c^2}{1+\Omega_{\chi,0}\rho_{cr,0}\lambda_2/2c^2}, \tag{36}$$

upon using relations (13) and (15).

3 Dynamics of dark matter phases in Tsallis cosmology

In this section, the cosmological dynamics of the condensation process of DM in the framework of Tsallis-modified cosmology is discussed. Furthermore, we also take note of the behavior of DM in the post-condensation phase.

3.1 Cosmological parameters at the critical point of phase transition

In general, the chemical potential μ within a physical system depends on both temperature T and particle density $n = N/V$, i.e $\mu = \mu(n, T)$, where N represents the total particle count in the volume V. The Helmholtz free energy $\mathscr{F} = \mathscr{F}(N, V, T)$ showcases an extensivity property, enabling two interchangeable expressions: $\mathscr{F} = Vf(n, T)$ or $\mathscr{F} = N\tilde{f}(1/n, T)$, where $f = n\tilde{f}$. Consequently, both expressions fetch the same physical insights. Making use of the free energy (Helmholtz), we obtain the chemical potential $\mu(n, T) = (\partial f/\partial n)_T$ and the pressure $p(n, T) = n^2(\partial\tilde{f}/\partial n)_T$. In this way, both chemical potential and pressure encapsulate equivalent necessary details. According to the laws of thermodynamics, both the chemical potential and pressure must possess single values. Therefore, for any given values of n and T, there should exist only one value for μ and p [185].

In the realm of cosmological BE DM condensation, a shift from ordinary DM to BEC DM occurred through a first-order phase transition. Throughout such a phase transition, the temperature and pressure are kept constant at their critical values, denoted as $T = T_{crit}$ and $P = P_{crit}$ respectively. A crucial thermodynamic requisite known as 'the continuity of pressure' must be maintained exactly at the point of phase transition ($T = T_{crit}$, $P = P_{crit}$). This criterion decides the critical transition density [159,171]

$$\rho_\chi^{crit} = \frac{\sigma_\chi^2 m_\chi^3}{2\pi\hbar^2 l_s} c^2, \tag{37}$$

pointing the crossover from the *normal* state to the BEC state. We can see from (37) that, the numerical value of transition density hinges on three unspecified parameters: the mass of the DM particle m_χ, the scattering length l_s, and the velocity dispersion σ_χ of the DM particles. Assuming a standard DM particle mass m_χ on the scale of $1eV$ (where $1eV = 1.78266 \times 10^{-33} g$), a typical scattering length l_s around $10^{-10} cm$, and an average square velocity $\langle \vec{v}_\chi^2 \rangle$ approximately $81 \times 10^{14} cm^2/s^2$, the critical transition density can be expressed as [159]

$$\rho_\chi^{crit} = 3.85705 \times 10^{-21} \times \left(\frac{\sigma_\chi^2}{3 \times 10^{-6}}\right) \times \left(\frac{m_\chi}{10^{-33} g}\right)^3 \times \left(\frac{l_s}{10^{-10} cm}\right)^{-1} g/cm^3. \tag{38}$$

The critical temperature on the onset of BE condensation is determined as [159]

$$\begin{aligned} T_{crit} &\approx \frac{2\pi\hbar^2}{\zeta(3/2)^{2/3} m_\chi^{5/3} k_B} (\rho_\chi^{crit})^{2/3} \\ &= \frac{(2\pi\hbar^2)^{1/3} m_\chi^{1/3} c^{4/3} (\sigma_\chi)^{2/3}}{\zeta(3/2)^{2/3} k_B l_s^{2/3}} \\ &= 6.56504 \times 10^3 \times \left(\frac{m_\chi}{10^{-33} g}\right)^{1/3} \\ &\quad \times \left(\frac{\sigma_\chi^2}{3 \times 10^{-6}}\right)^{2/3} \times \left(\frac{l_s}{10^{-10} cm}\right)^{-2/3} K, \end{aligned} \tag{39}$$

where $\zeta\ (3/2) \approx 2.612375$ is the Riemann zeta function $\zeta(k)$ for $k = 3/2$.

The critical pressure of the DM fluid at the point of condensation can be evaluated as [159]

$$P_{crit} = c^2 \sigma_\chi^2 \rho_\chi^{crit} = 1.03996 \times 10^{-5} \times \left(\frac{m_\chi}{10^{-33} g}\right)^3 \times \left(\frac{\sigma_\chi^2}{3 \times 10^{-6}}\right)^2 \times \left(\frac{l_s}{10^{-10} cm}\right)^{-1} dyne/cm^2 \tag{40}$$

The critical value of the scale factor and redshift of the Universe at the starting point of phase transition can be calculated as

$$\begin{aligned} \frac{S_{crit}}{S_0} &= \left(\frac{2\pi\hbar^2 l_s \Omega_{\chi,0}^{mod} \rho_{cr,0}^{mod}}{\sigma_\chi^2 m_\chi^3 c^2}\right)^{\frac{1}{3(1+\sigma_\chi^2)}} \\ &= \left(\frac{2\pi\hbar^2 l_s \Omega_{\chi,0} \rho_{cr,0}}{\sigma_\chi^2 m_\chi^3 c^2}\right)^{\frac{1}{3(1+\sigma_\chi^2)}} \\ &= (1.6386 \times 10^{-104})^{\frac{1}{3(1+\sigma_\chi^2)}} \times \left(\frac{m_\chi}{10^{-33} g}\right)^{\frac{-1}{(1+\sigma_\chi^2)}} \\ &\quad \times \left(\frac{\sigma_\chi^2}{3 \times 10^{-6}}\right)^{\frac{-1}{3(1+\sigma_\chi^2)}} \times \left(\frac{l_s}{10^{-10} cm}\right)^{\frac{1}{3(1+\sigma_\chi^2)}} \end{aligned} \tag{41a}$$

and

$$\begin{aligned} z_{crit} &= -1 + \left(\frac{2\pi\hbar^2 l_s \Omega_{\chi,0}^{mod} \rho_{cr,0}^{mod}}{\sigma_\chi^2 m_\chi^3 c^2}\right)^{\frac{-1}{3(1+\sigma_\chi^2)}} \\ &= -1 + \left(\frac{2\pi\hbar^2 l_s \Omega_{\chi,0} \rho_{cr,0}}{\sigma_\chi^2 m_\chi^3 c^2}\right)^{\frac{-1}{3(1+\sigma_\chi^2)}} \\ &= -1 + (1.6386 \times 10^{-104})^{\frac{-1}{3(1+\sigma_\chi^2)}} \times \left(\frac{m_\chi}{10^{-33} g}\right)^{\frac{1}{(1+\sigma_\chi^2)}} \\ &\quad \times \left(\frac{\sigma_\chi^2}{3 \times 10^{-6}}\right)^{\frac{1}{3(1+\sigma_\chi^2)}} \times \left(\frac{l_s}{10^{-10} cm}\right)^{\frac{-1}{3(1+\sigma_\chi^2)}}, \end{aligned} \tag{41b}$$

respectively [159]. In deriving the above Eqs. (38–41), we have used the values of the parameters and constants: $H_0 = 72\, km\, s^{-1} Mpc^{-1} = 2.33 \times 10^{-18} s^{-1}$ [209], $c = 2.99792 \times 10^{10}\, cm\, s^{-1}$ [210], $k_B = 1.38065 \times 10^{-16} erg\, K$ [210], $\hbar = 1.05457 \times 10^{-27}\, erg\, sec$ [210], $\Omega_{\chi,0} \approx 0.228$ [159], and $\rho_{cr,0} = 9.24 \times 10^{-30}\, g\, cm^{-3}$ [159].

The study in [165] demonstrated that a cosmic BEC comes into existence provided the boson's mass $m_\chi < 1.87\, eV$. Therefore, taking $m_\chi = 0.55073\, eV$ within the mass constraint and assuming $\sigma_\chi^2 = 3 \times 10^{-6}$ and $l_s = 10^{-10}\, cm$, we calculate the critical redshift $z_{crit} = 1200$. The time-redshift relationship [211]

$$t \approx \frac{28}{1 + (1+z)^2} Gyrs. \tag{42}$$

leads to the determination of the critical time $t_{crit} \approx 6.12166 \times 10^{11}\, s$ associated to the critical redshift $z_{crit} = 1200$. In this context, it is worth noticing that, the critical density, the critical temperature, the critical pressure, the critical scale factor, the critical redshift, and the time corresponding to the critical redshift are highly sensitive to free parameters like the mass of the DM particle, scattering length, and velocity dispersion and insensitive to Tsallis parameter. A slight change in these free parameters leads to a drastic change in the estimated value of critical redshift or the starting point of BE condensation of DM. As there is no observational data

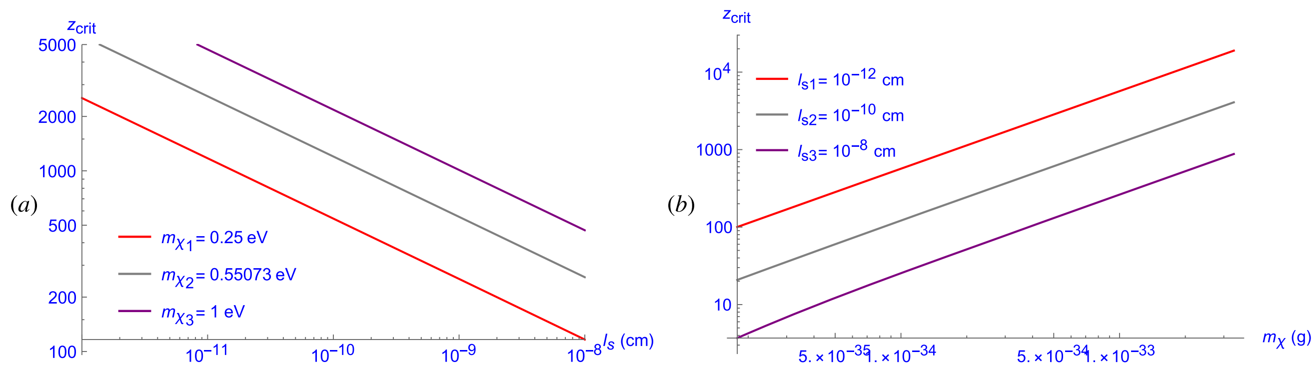

Fig. 1 Log-Log plots of critical redshift z_{crit} versus (a) scattering length $l_s(cm)$ and (b) mass $m_\chi(g)$ from Eq. (41b) for various masses and scattering lengths respectively

regarding the specific starting time of condensation, typical free parameters have been chosen in this article. Therefore constraining the free parameters is quite impossible. The obtained values in our case are similar to [159,163,171]. In an article [175], authors presented the dependence of the transition redshift z_{crit} on BEC model parameters m_χ and l_s and took a wide range of values of model parameters in order to discuss the possible cosmological consequences in detail. We also plot the variation of z_{crit} with l_s and m_χ for various masses and scattering lengths respectively in Fig. 1, and observe that z_{crit} falls with the increase of l_s and rises with the increase of m_χ.

3.2 Cosmological evolution of dark matter during Bose-Einstein condensation phase in Tsallis cosmology

In the course of a first-order phase transition, the temperature (T) and pressure (P) remain constant at their critical values, $T = T_{crit}$ and $P = P_{crit}$, respectively. Similarly, the entropy $(\mathbb{S})$ and enthalpy $(\mathscr{H})$ maintain their values as conserved quantities, with $\mathbb{S} = sS^3$ and $\mathscr{H} = (\rho + P)S^3$ respectively, where s is the entropy density. As the phase transition initiates, the density of DM, denoted by $\rho_\chi(t)$, decreases from its critical value $\rho_\chi^{crit}(T_{crit}) \equiv \rho_\chi^{nor}$ (when all DM exists in a *normal*, non-condensate form at $t = t_{crit}$) to $\rho_\chi(T_{crit}) \equiv \rho_\chi^{bec}$, corresponds to complete conversion into a condensate state at $t = t_{bec}$ [159]. It has proven advantageous to define and find a quantity $f(t)$, the *condensate DM fraction* in the BE condensation phase as

$$f(t) = \frac{\rho_\chi(t) - \rho_\chi^{nor}}{\rho_\chi^{bec} - \rho_\chi^{nor}}, \tag{43}$$

that measures the change in relative density of DM as a function of cosmic time during phase transition in the framework of Tsallis cosmology. Denoting $n_\chi = \frac{\rho_\chi^{bec} - \rho_\chi^{nor}}{\rho_\chi^{nor}}$ (or $= \frac{\Omega_\chi^{bec} - \Omega_\chi^{nor}}{\Omega_\chi^{nor}}$, where $\Omega_\chi^{bec} = \frac{\rho_\chi^{bec}}{\rho_{cr,0}}$ and $\Omega_\chi^{nor} = \frac{\rho_\chi^{nor}}{\rho_{cr,0}}$ are the density parameter of DM at $t = t_{bec}$ and $t = t_{crit}$ respectively), we write the DM density during the BE condensation process

$$\rho_\chi(t) = \rho_\chi^{nor}[1 + n_\chi f(t)]. \tag{44}$$

At the onset of the BE condensation process, $f(t_{crit}) = 0$, with t_{crit} representing the time marking the initiation of the phase transition, and $\rho_\chi(t_{crit}) \equiv \rho_\chi^{nor}$. At the time when the condensation concluded, $f(t_{bec}) = 1$, with t_{bec} signifying the moment when the phase transition completed, $\rho_\chi(t_{bec}) \equiv \rho_\chi^{bec}$. Beyond t_{bec}, the Universe underwent a transition into the phase of BE condensed DM.

Using conservation law (8) during condensation phase, one can obtain [159]

$$H(t) = \frac{\dot{S}(t)}{S(t)} = -\frac{1}{3}\frac{r\dot{f}(t)}{1 + rf(t)}, \tag{45}$$

where

$$r = \frac{n_\chi}{1 + (P_{crit}/\rho_\chi^{nor}c^2)} = \frac{n_\chi}{1 + \sigma_\chi^2} = \frac{(\Omega_\chi^{bec}/\Omega_\chi^{nor}) - 1}{1 + \sigma_\chi^2}. \tag{46}$$

In writing (45) we have made use of Eq. (44). For $\rho_\chi^{bec} < \rho_\chi^{nor}$, typically $r < 0$, where $r \in (-1, 0)$, and $n_\chi < 0$, respectively. Equation (45) can be readily solved and yields the expression of the scale factor of the Universe during the BE condensation phase as

$$S(t) = S_{crit}[1 + rf(t)]^{-1/3}, \quad t \in (t_{crit}, t_{bec}), \tag{47}$$

where $S_{crit} = S(t = t_{crit})$. In deriving Eq. (47), we have exploited the initial condition $f(t_{crit}) = 0$. At the ending point of the phase transition, as $f(t_{bec}) = 1$, the scale factor of the Universe has the following value

$$S_{bec} = S(t = t_{bec}) = S_{crit}(1 + r)^{-1/3}. \tag{48}$$

From Eqs. (44) and (46), we can find the expression of the DM density parameter during phase transition

$$\Omega_\chi(t) = \frac{\rho_\chi(t)}{\rho_{cr,0}} = \Omega_\chi^{nor}[1 + (1 + \sigma_\chi^2)rf(t)]. \tag{49}$$

As $f(t=t_{bec})=1$, we get

$$\Omega_\chi^{bec}=\Omega_\chi(t=t_{bec})=\Omega_\chi^{nor}[1+(1+\sigma_\chi^2)r]. \tag{50}$$

Considering the mixture of *normal* and condensate DM during phase transition as a single fluid, from Eqs. (40) and (44) we get the equation of the state of mixed DM[3]

$$\omega_{mix}(t)=\frac{P_{crit}}{\rho_\chi(t)c^2}=\frac{\sigma_\chi^2}{1+(1+\sigma_\chi^2)rf(t)}, \tag{51}$$

for $t_{crit}\leq t\leq t_{bec}$.

Therefore, from Eqs. (49) and (51) one can write

$$\Omega_\chi(t)=\Omega_\chi^{nor}\frac{\sigma_\chi^2}{\omega_{mix}(t)}. \tag{52}$$

This above relation tells how the DM density parameter changes with the equation of state parameter during phase crossover. Just at the finishing of phase transition (or beginning of post-BEC phase) at $t=t_{bec}$, the DM equation of state parameter $\omega_{bec}=\omega_{mix}(t=t_{bec})$ reads

$$\omega_{bec}=\frac{\sigma_\chi^2}{1+(1+\sigma_\chi^2)r}. \tag{53}$$

Now from Eqs. (50) and (53) we get

$$\Omega_\chi^{bec}=\Omega_\chi^{nor}\frac{\sigma_\chi^2}{\omega_{bec}}. \tag{54}$$

As the quantities Ω_χ^{nor} and Ω_χ^{bec} are dependent on uncertain (to the best of our knowledge) model parameters $(m_\chi, l_s, r, \sigma_\chi)$ on a cosmological scale, an estimate of the numerical values of these quantities is not trustworthy. Based on our considered typical model parameters $m_\chi=0.55073\,eV$, $l_s=10^{-10}cm$, $r=-0.9135$, $\sigma_\chi=3\times10^{-6}$, the estimated values are $\Omega_\chi^{nor}=3.95007\times10^8$ and $\Omega_\chi^{bec}=3.4167\times10^7$. One can calculate the values of the quantities mentioned above from the reference [159] that are of similar order estimated by us.

As we have assumed the Universe is filled with baryons, radiation, and the aforementioned two forms of DM, it is interesting to see that, BE condensation alters the collective expansion rate of the Universe. Consequently, throughout the phase transition, the evolution of pressureless baryonic matter and radiation density follows from the conservation law [159]

$$\rho_{bar}=\frac{\rho_{bar,0}}{(S_{crit}/S_0)^3}[1+rf(t)],\quad t\in(t_{crit},t_{bec}), \tag{55}$$

and

$$\rho_{rad}=\frac{\rho_{rad,0}}{(S_{crit}/S_0)^4}[1+rf(t)]^{4/3},\quad t\in(t_{crit},t_{bec}), \tag{56}$$

respectively. Therefore, the time evolution of the condensate matter fraction $f(t)$ during the BE condensation process is delineated by the equation

$$\frac{df(t)}{dt}=-3H_0\left(\frac{1}{r}+f(t)\right)\times\left[\frac{\Omega_{bar,0}^{mod}}{(S_{crit}/S_0)^3}(1+rf(t))+\frac{\Omega_{rad,0}^{mod}}{(S_{crit}/S_0)^4}(1+rf(t))^{4/3}+\Omega_{\chi,nor}^{mod}(1+n_\chi f(t))\right]^{\frac{1}{4-2\beta}}. \tag{57}$$

Here, we have designated $\Omega_{\chi,nor}^{mod}=\rho_\chi^{nor}/\rho_{cr,0}^{mod}$. In deriving Eq. (57), we have used a set of Eqs. (14, 44, 45, 55, and 56). The change in condensate matter fraction $f(t)$ during BE condensation in (57) is different from that in [159,163] because of the modification in Friedmann equation due to Tsallis cosmology. Given that $P_{crit}/c^2\rho_\chi^{nor}=\sigma_\chi^2<<1$ as per Eq. (46), it implies that $r\approx n_\chi$, an approximation we employ subsequently. Again, if the energy density contribution from radiation to the total energy density of the Universe can be disregarded, Eq. (57) can be precisely integrated to yield

$$f(t)=\frac{1}{r}\left[1+\frac{3H_0}{4-2\beta}(\Omega_{tr}^{mod})^{\frac{1}{4-2\beta}}(t-t_{crit})\right]^{-(4-2\beta)}-\frac{1}{r}, \tag{58}$$

where Ω_{tr}^{mod} is denoted as

$$\Omega_{tr}^{mod}=\frac{\Omega_{bar,0}^{mod}}{(S_{crit}/S_0)^3}+\Omega_{\chi,nor}^{mod}. \tag{59}$$

Substituting the functional form of $f(t)$ from (58) in Eqs. (47) and (45), we find the expressions of the scale factor and Hubble parameter in terms of cosmic time during the phase transition

$$S(t)=S_{crit}\left[1+\frac{3H_0}{4-2\beta}(\Omega_{tr}^{mod})^{\frac{1}{4-2\beta}}(t-t_{crit})\right]^{\frac{4-2\beta}{3}}, \tag{60}$$

and

$$H(t)=\frac{(4-2\beta)H_0}{(4-2\beta)(\Omega_{tr}^{mod})^{\frac{-1}{4-2\beta}}+3H_0(t-t_{crit})} \tag{61}$$

respectively. We have shown the nature of the scale factor and Hubble parameter for different values of β in Figs. 2 and 3 respectively. The variation of the Hubble parameter for CDM in standard cosmology during the period of BE condensation is also shown in Fig. 3. Using Eqs. (51) and (58), we can show the Hubble parameter during phase transition in Eq. (61) reduces to

$$H(t)=H_0\left[\frac{\Omega_{tr}^{mod}\sigma_\chi^2}{\omega_{mix}(t)}\right]^{\frac{1}{4-2\beta}}. \tag{62}$$

[3] Note: During the first-order phase transition pressure is kept constant at $P=P_{crit}$.

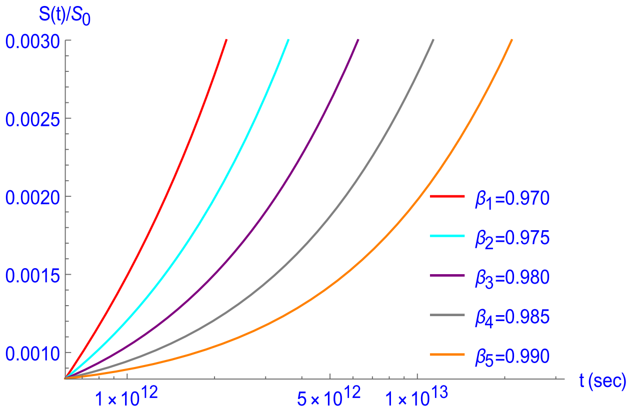

Fig. 2 Log-linear plot of normalized scale factor versus cosmic time t (seconds) for *normal*-BEC co-exist DM Universe from Eq. (60) with different values of β, keeping $r = -0.9135$

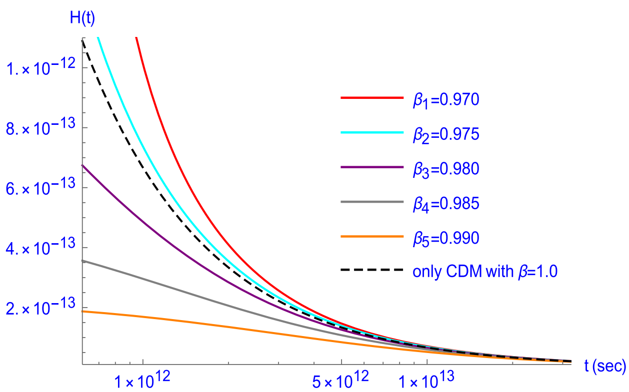

Fig. 3 Log-linear plot of Hubble parameter versus cosmic time t (seconds) during condensation with different values of β, keeping $r = -0.9135$ for *normal*-BEC co-exist DM Universe (solid lines) from Eq. (61) and standard matter-dominated (CDM) Universe (dashed line)

Equation (62) describes the change in Hubble parameter with the change of equation of state parameter during the phase transition. The dimensionless deceleration parameter $q(t) \stackrel{\text{def}}{=} -S\ddot{S}/\dot{S}^2$ during condensation can be calculated from Eq. (60) as

$$q(t) = \frac{2\beta - 1}{4 - 2\beta}, \tag{63}$$

which is independent of cosmic time and the equation of state parameter. We see Eq. (58) is a modified version of DM condensate fraction, which reduces to

$$f(t) = \frac{1}{r}\left[1 + \frac{3}{2}H_0\sqrt{\Omega_{tr}}(t - t_{crit})\right]^{-2} - \frac{1}{r} \tag{64}$$

for $\beta = 1$, presented in [159]. Here we identify $\Omega_{tr} = (\Omega_{tr}^{mod})|_{\beta=1}$. From Eq. (58), we plot $f(t)$ vs t with different values of β, keeping $r = -0.9135$ (see Fig. 4), with different values of r, keeping $\beta = 0.99$ (see Fig. 5) and with different values of β, corresponding to $r = -0.1$, and $r = -0.9$ (see Fig. 6) respectively.

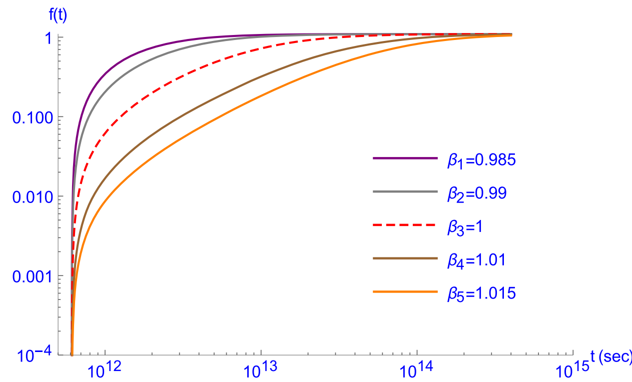

Fig. 4 Log-log plot of BE condensed DM fraction $f(t)$ versus cosmic time t (seconds) from Eq. (58) with different values of β, keeping $r = -0.9135$ during BE condensation

To find the time dependence of the equation of state parameter $\omega_{mix}(t)$ of DM during the phase of co-exist, we substitute Eq. (58) into Eq. (51) and obtain

$$\omega_{mix}(t) = \frac{\sigma_\chi^2}{\left[1 + \frac{3H_0}{4-2\beta}(\Omega_{tr}^{mod})^{\frac{1}{4-2\beta}}(t - t_{crit})\right]^{-(4-2\beta)}}. \tag{65}$$

Now, using Eq. (65) in Eq. (52), we see the DM density parameter throughout the phase transition varies as

$$\Omega_\chi(t) = \Omega_\chi^{nor}\left[1 + \frac{3H_0}{4 - 2\beta}(\Omega_{tr}^{mod})^{\frac{1}{4-2\beta}}(t - t_{crit})\right]^{-(4-2\beta)}. \tag{66}$$

We show the plots of temporal dependence of the parameters $\omega_{mix}(t)$ and $\Omega_\chi(t)$ in Figs. 7 and 8 accordingly. We see that the equation of state increases, and the density parameter of

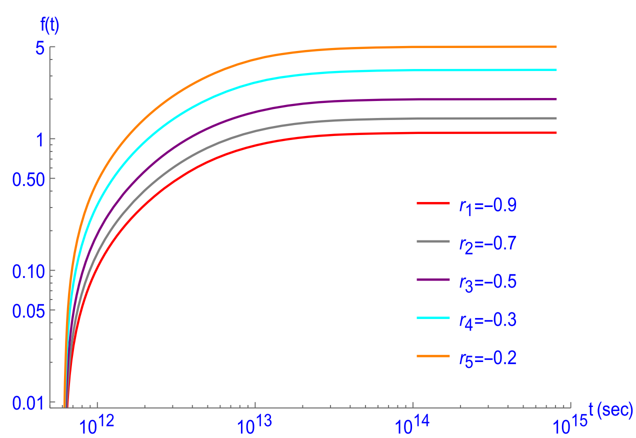

Fig. 5 Log-log plot of BE condensed DM fraction $f(t)$ versus cosmic time t (seconds) from Eq. (58) with different values of r, keeping $\beta = 0.99$ during BE condensation

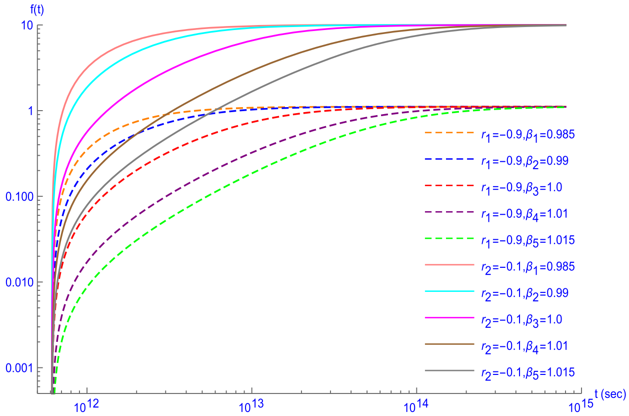

Fig. 6 Log-log plot of BE condensed DM fraction $f(t)$ versus cosmic time t (seconds) from Eq. (58) with different values of β, corresponding to $r = -0.1$ (solid lines) and $r = -0.9$ (dashed lines) during BE condensation

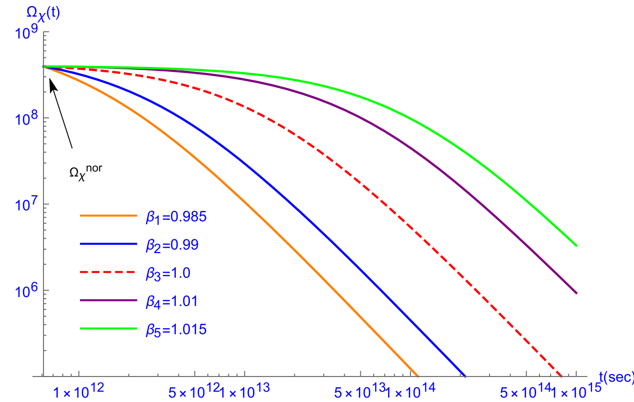

Fig. 8 Log-log plot of DM density parameter versus cosmic time t (seconds) from Eq. (66) during condensation with different values of β, for *normal*-BEC co-exist DM Universe

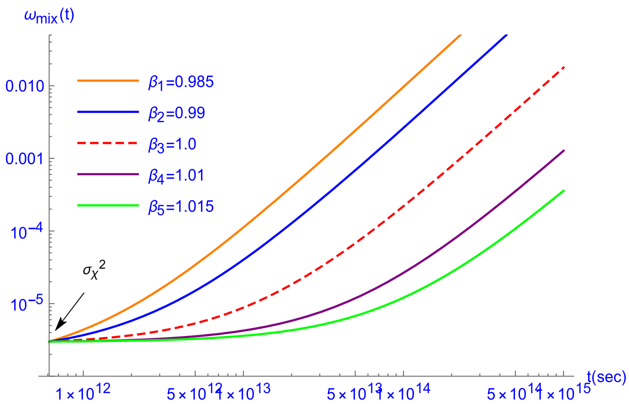

Fig. 7 Log-log plot of the equation of state parameter versus cosmic time t (seconds) from Eq. (65) during condensation with different values of β for *normal*-BEC co-exist DM Universe

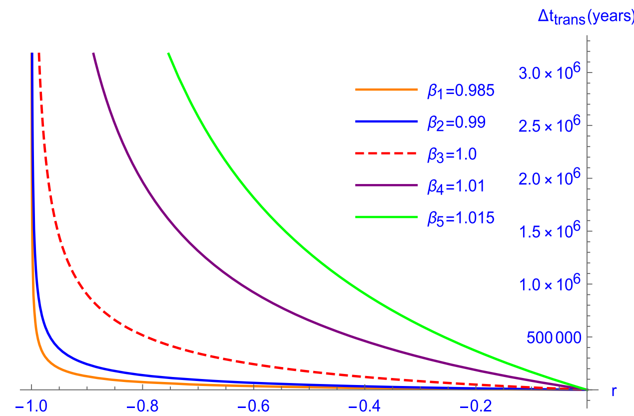

Fig. 9 Plot of condensation time (years) versus r from Eq. (67) with different values of β

DM decreases with cosmic time from its initial value during the phase transition.

The time duration required to transform all the ordinary (*normal*) DM into the BE condensed phase, is provided by

$$\Delta t_{trans} = t_{bec} - t_{crit} = \frac{4-2\beta}{3H_0(\Omega_{tr}^{mod})^{\frac{1}{4-2\beta}}}\left[(1+r)^{-\frac{1}{4-2\beta}} - 1\right], \tag{67}$$

where we have used the fact that $f(t = t_{bec}) = 1$. For a Tsallis non-extensive parameter $\beta = 1$, we obtain the transition time in standard Friedmann Universe $\Delta t_{trans} = 3.15768 \times 10^{13} s$ ($\approx 10^6 years$) [159,171], while a slide departure from standard Friedmann Universe, say for $\beta = 1.03$, the transition time becomes $\Delta t_{trans} = 1.91626 \times 10^{15} s$ ($\approx 6 \times 10^7 years$). In evaluating the value of Δt_{trans}, apart from the values of parameters and constants mentioned in Sect. 3.1, we have also used the values $\Omega_{bar,0} = 0.045$ [159], $L_{pl} = 1.616255 \times 10^{-33} cm$ [210] and taken parameter $r = -0.9135$ that lies within the mentioned constraint. In this context, it is worth mentioning that a bound on $\beta \gtrsim 0.99984$, the DM relic abundance can be matched with the observation using the DM freeze-out mechanism [85]. The variation of condensation time with a parameter r for different Tsallis parameters is shown in Fig. 9 for our assumed standard values of m_χ and l_s. This shows Δt_{trans} decreases with increasing r parameter. We also show plots for BE condensation time Δt_{trans} versus scattering length l_s with three different masses for Tsallis parameter: (a) $\beta = 0.99$, (b) $\beta = 1.0$, and (c) $\beta = 1.01$ in Fig. 10. In Fig. 11, Δt_{trans} versus DM particle mass m_χ with three different scattering lengths for Tsallis parameter: (a) $\beta = 0.99$, (b) $\beta = 1.0$, and (c) $\beta = 1.01$ is shown. From Figs. 10 and 11, we observe Δt_{trans} increases with the increment of l_s but decreases with the increment of m_χ.

So far in the previous Sect. 3.2, we have discussed the evolution dynamics of the condensate DM fraction during the BE condensation phase in the framework of Tsallis cosmology. In the next Sect. 3.3, we see how the cosmological dynam-

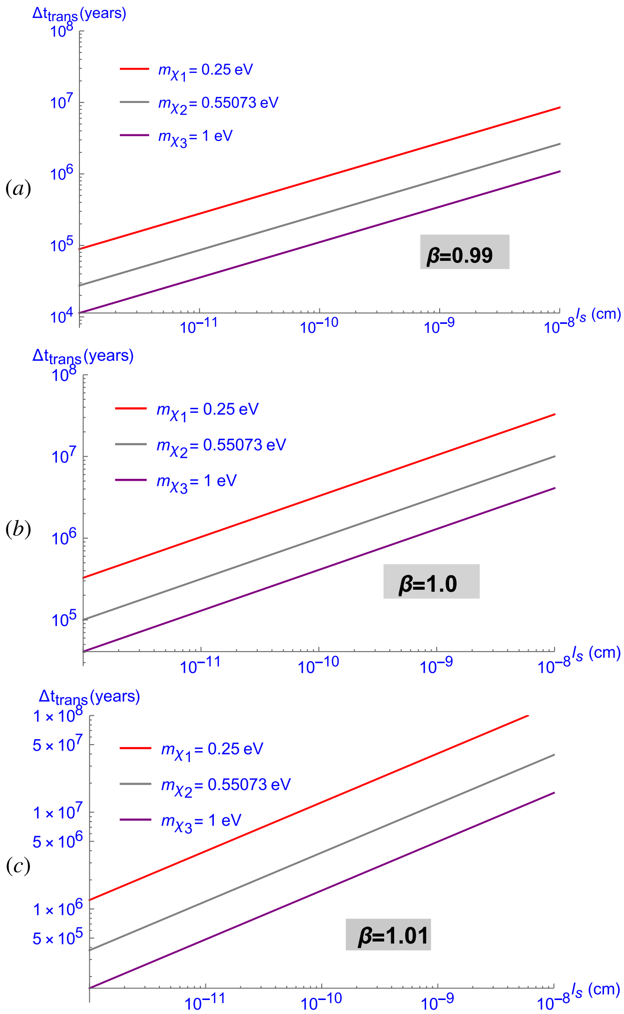

Fig. 10 Log-Log plots for BE condensation time $\Delta\ t_{trans}$ (years) versus scattering length $l_s(cm)$ from Eq. (67) with three different masses for Tsallis parameter: (**a**) $\beta = 0.99$, (**b**) $\beta = 1.0$, and (**c**) $\beta = 1.01$

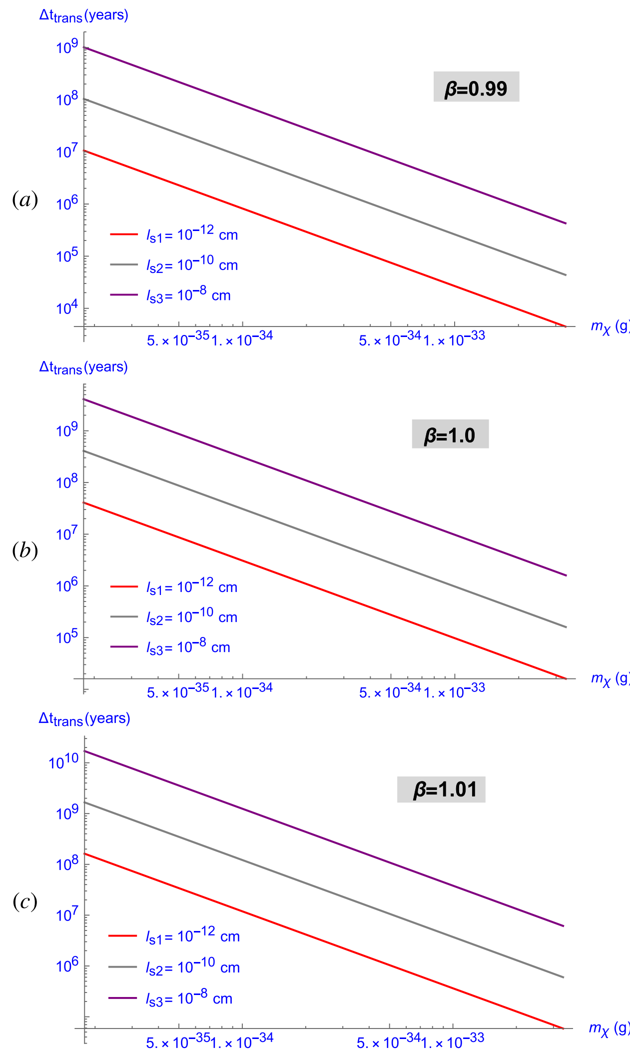

Fig. 11 Log-Log plots for BE condensation time $\Delta\ t_{trans}$ (years) versus DM particle mass $m_\chi(g)$ from Eq. (67) with three different scattering lengths for Tsallis parameter: (**a**) $\beta = 0.99$, (**b**) $\beta = 1.0$, and (**c**) $\beta = 1.01$

ics of the Tsallis-modified Universe changes in the presence of BEC DM after the BE condensation phase is over and compare our results over the standard one.

3.3 The post-condensation phase of dark matter in Tsallis cosmology

During the condensation phase, the cosmological dynamics is influenced by both the co-existed *normal* as well as BEC DM. After the transition to the BEC phase is complete, the post-condensation phase commences, initiating at time $t = t_{bec}$ with $S_{bec} = S_{crit}(1+r)^{-1/3}$. Here the cosmological dynamics is solely influenced by BEC DM. In the following, we elaborately discuss the cosmological dynamics in the post-condensation phase within the framework of Tsallis cosmology.

Referring to Eq. (35), the cosmological density of DM can be derived in the following manner

$$\rho_\chi^{bec} = \frac{2c^2}{\lambda_2} \frac{(1+r)\rho_{0\chi}}{(S_{crit}/S_0)^3 - (1+r)\rho_{0\chi}}. \tag{68}$$

Here we see, that the requirement for the density to be positive imposes a constraint that $(S_{crit}/S_0) > (1+r)^{1/3}\rho_{0\chi}^{1/3}$ on the BEC model parameters.

The equation that governs the temporal progression of the scale factor for DM within the BEC phase is expressed as

$$\frac{1}{S}\frac{dS}{dt} = H_0 \frac{(\Omega_{BE}^{mod})^{\frac{1}{4-2\beta}}}{[(S/S_0)^3 - \rho_{0\chi}]^{\frac{1}{4-2\beta}}}, \tag{69}$$

where Ω_{BE}^{mod} stands for

$$\Omega_{BE}^{mod} = \frac{\Omega_{\chi,0}^{mod}}{1+\rho_{\chi,0}\lambda_2/2c^2}. \tag{70}$$

On integrating Eq. (69), we get the relationship between cosmic time t and scale factor S during the condensation process in Tsallis cosmology as

$$(\Omega_{BE}^{mod})^{\frac{1}{4-2\beta}} H_0(t-\tau) = \frac{2}{3}(2-\beta)\left(\frac{S}{S_0}\right)^{\frac{3}{4-2\beta}} \times {}_2F_1\left(\frac{-1}{4-2\beta}, \frac{-1}{4-2\beta}; \frac{3-2\beta}{4-2\beta}; \left(\frac{S_0}{S}\right)^3 \rho_{0\chi}\right), \tag{71}$$

where τ is an arbitrary integration constant, and a function of type ${}_2F_1(a,b;c;z)$ represents a hypergeometric function with a, b and c are parameters, and z is variable. Setting $S = S_{bec}$ at $t = t_{bec}$, we find the integration constant as

$$\tau = t_{bec} - \frac{2}{3H_0}(\Omega_{BE}^{mod})^{\frac{-1}{4-2\beta}}(2-\beta)\left(\frac{S_{bec}}{S_0}\right)^{\frac{3}{4-2\beta}} \times {}_2F_1\left(\frac{-1}{4-2\beta}, \frac{-1}{4-2\beta}; \frac{3-2\beta}{4-2\beta}; \left(\frac{S_0}{S_{bec}}\right)^3 \rho_{0\chi}\right). \tag{72}$$

For a Universe containing baryonic matter with minimal pressure, radiation, and BEC DM, the differential equation governs the temporal changes in the scale factor for $t \geq t_{bec}$ reads

$$\frac{1}{S}\frac{dS}{dt} = H_0\left[\frac{\Omega_{rad,0}^{mod}}{(S/S_0)^4} + \frac{\Omega_{bar,0}^{mod}}{(S/S_0)^3} + \frac{\Omega_{BE}^{mod}}{(S/S_0)^3 - \rho_{0\chi}}\right]^{\frac{1}{4-2\beta}}. \tag{73}$$

Right after the phase transition, the emergence of a BEC could profoundly alter the cosmological dynamics of the Universe. As indicated by Eq. (73), when the scale factor $S/S_0 \rightarrow \rho_{0\chi}^{1/3}$ for a certain period, the evolution of the Universe is dominated by the condensed DM. Under such a circumstance, the energy density of BEC DM (See Eq. 35) becomes significantly high, overpowering all other cosmological energy components. For the condition $S/S_0 \rightarrow \rho_{0\chi}^{1/3}$, we get

$${}_2F_1\left(\frac{-1}{4-2\beta}, \frac{-1}{4-2\beta}; \frac{3-2\beta}{4-2\beta}; \left(\frac{S_0}{S}\right)^3 \rho_{0\chi}\right) \longrightarrow \frac{\Gamma\left[\frac{9-4\beta}{4-2\beta}\right] \times \Gamma\left[\frac{-1}{4-2\beta}\right]}{2\beta-5}, \tag{74}$$

a finite value. Therefore the scale factor $S(t)$ in the BEC DM-dominated Tsallis Universe can be estimated as

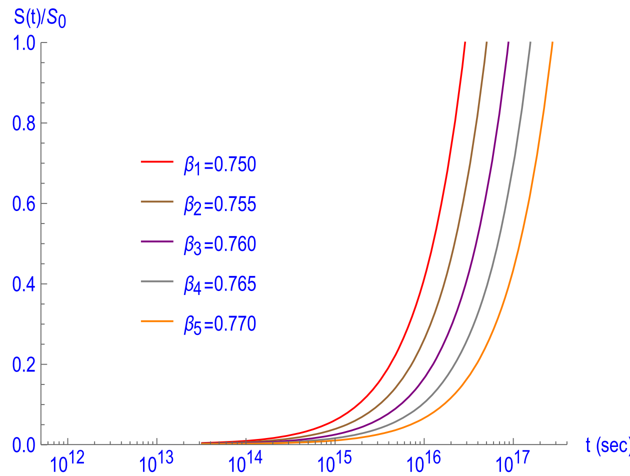

Fig. 12 Log-linear plot of normalized scale factor versus cosmic time t (seconds) with different values of β from Eq. (75), keeping $\rho_{0\chi} = 10^{-11}$ and $r = -0.9135$ in post-BEC phase

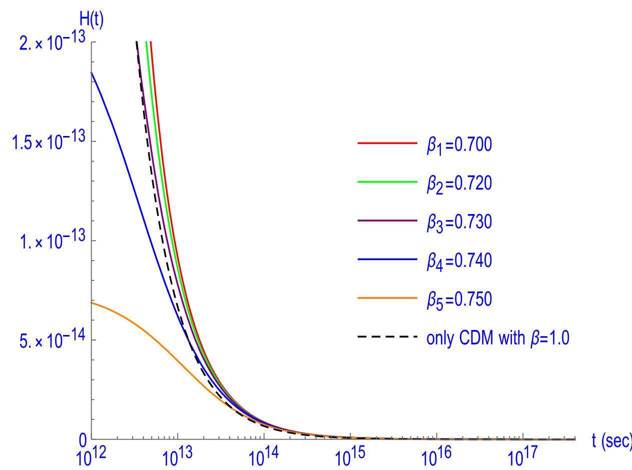

Fig. 13 Log-linear plot of Hubble parameter versus cosmic time t (seconds) in post-BEC phase with different values of β, keeping $\rho_{0\chi} = 10^{-11}$ and $r = -0.9135$ for BEC DM dominated Universe (solid lines) from Eq. (76) and standard matter-dominated (CDM) Universe (dashed line)

$$\frac{S(t)}{S_0} \approx \left[\frac{3(2\beta-5)(\Omega_{BE}^{mod})^{\frac{1}{4-2\beta}} H_0}{2(2-\beta)\times\Gamma\left[\frac{9-4\beta}{4-2\beta}\right]\times\Gamma\left[\frac{-1}{4-2\beta}\right]}\right]^{\frac{4-2\beta}{3}} (t-\tau)^{\frac{4-2\beta}{3}}. \tag{75}$$

In Fig. 12 we depict normalized $S(t)$ versus t from Eq. (75) with different values of β, keeping $\rho_{0\chi} = 10^{-11}$ and $r = -0.9135$. In plotting Fig. 12 we have used the expression of τ in (72). Throughout this phase, the Hubble parameter $H(t)$ and the dimensionless deceleration parameter $q(t) \stackrel{\text{def}}{=} -S\ddot{S}/\dot{S}^2$ can be easily determined from (75) and written as

$$H(t) \approx \frac{4-2\beta}{3(t-\tau)}, \tag{76}$$

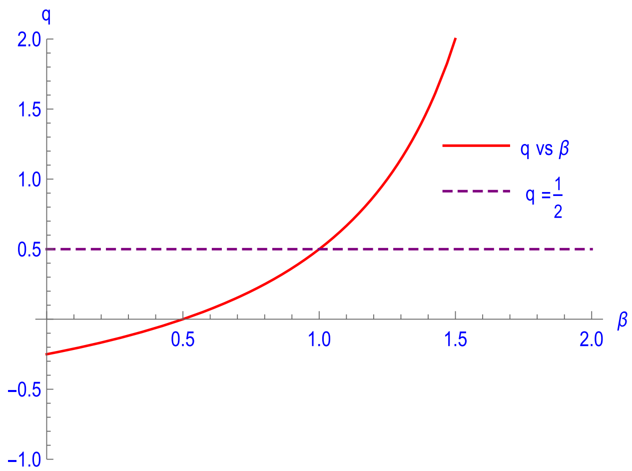

Fig. 14 Plot of deceleration parameter q versus Tsallis parameter β in post-BEC phase for BEC DM-dominated Universe in Tsallis cosmology (solid line) from Eq. (77) and matter-dominated Universe in standard cosmology (or the BEC DM-dominated era in standard Friedmann Universe) (dashed line)

and

$$q(t) \approx \frac{2\beta - 1}{4 - 2\beta} \tag{77}$$

respectively. Here, the deceleration parameter in the post-BEC phase takes the same form as for the condensation period shown in Eq. (63). Accordingly, for $\beta = 1$, we get

$$S(t) \propto \frac{1}{(t - \tau')^{\frac{2}{3}}}, \tag{78}$$

$$H(t) \approx \frac{2}{3(t - \tau')}, \tag{79}$$

and

$$q(t) \approx \frac{1}{2}, \tag{80}$$

which resembles the results in the matter-domination era in standard cosmology [212,213] and the BEC DM-dominated era in standard Friedmann Universe [159]. Here in Eqs. (78) and (79) we identify $\tau' = \tau|_{\beta=1}$. The variation of Hubble parameter with cosmic time in the post-BEC phase in Tsallis cosmology and standard matter-dominated Universe is plotted in Fig. 13. The variation of the deceleration parameter with the Tsallis parameter is shown in Fig. 14. From the expression of the deceleration parameter in Eq. (77) and Fig. 14, we observe that for $\beta < 1/2$, we get $q(t) < 0$, confirming the accelerated expansion of the Universe without assuming any form of DE. In order to get a rough estimate of the age of the Universe t_0, considering a BEC DM-dominated Universe in Tsallis cosmology, we look at the expression of the Hubble parameter in Eq. (76), and get

$$t_0 \approx \frac{2(2 - \beta)}{3H_0} + \tau, \tag{81}$$

where $H_0 = H(t = t_0)$ is the Hubble parameter at present. Once again, assuming τ to be small compared to the remaining part, we write Eq. (81) as

$$t_0 \approx (2 - \beta) \times t_{m,0}, \tag{82}$$

where $t_{m,0} = 2/(3H_0)$ is the age of the Universe in standard cosmology considering only matter-dominated Universe. In an accelerated Universe, we have $\beta < 1/2$, that implies

$$t_0 > \frac{3}{2} t_{m,0}. \tag{83}$$

Let's examine how the aforementioned relationship (83) could address the issue of age in conventional cosmology. The Hubble parameter at present is typically expressed as

$$H_0 = 100h\ km\, s^{-1}\, Mpc^{-1}, \tag{84}$$

where h denotes the uncertainty regarding the value of H_0. The Hubble Key Project's observations constrain the value of h to 0.72 ± 0.08 [209]. Consequently, the Hubble time is $t_H = 1/H_0 = 9.78 \times 10^9\, h^{-1}\, years$. Utilizing this h value, the age of the Universe in standard cosmology falls within the range $8.2\, Gyr < t_{m,0} < 10.2\, Gyr$ [89]. According to Eq. (83), the age of the Universe in Tsallis cosmology exceeds by more than $3/2$ times the age of the Universe in standard cosmology, and this holds true when selecting a value of Tsallis parameter in an accelerated Universe, i.e. $\beta < 1/2$. As an illustration, by choosing the Tsallis parameter $\beta = 0.35$ typically, Eq. (82) yields $t_0 \approx 1.65\, t_{m,0}$. Hence, the age of the Universe in Tsallis cosmology ranges approximately between $13.53\, Gyr$ and $16.83\, Gyr$ surpassing the age of the oldest globular clusters. In this context, we should note some information regarding the age of globular clusters, e.g. Carretta et al. in [214] assessed the age of globular clusters in the Milky Way to be $12.9 \pm 2.9\, Gyr$, whereas Jimenez et al. in [215] estimated $13.5 \pm 2\, Gyr$. Using the white dwarf cooling sequence method, Hansen et al. [216] have estimated the age of the globular cluster M4 to be $12.7 \pm 0.7\, Gyr$. In most instances, the ages of globular clusters exceed $11\, Gyr$, suggesting that the cosmic age estimated in standard cosmology is incompatible with the ages of the oldest globular clusters. It has been argued that within the framework of the standard model of cosmology, this issue cannot be resolved without accounting for the cosmological constant or DE [89,90]. Nevertheless, we have shown that the issue of cosmic age can be solved automatically for an accelerated expanding Universe in the framework of Tsallis cosmology. Hence, the challenge of the cosmic age is effectively mitigated by considering a BEC DM-dominated accelerated expanding Universe in Tsallis cosmology, without requiring the introduction of additional energy components. This study of the age of the Universe in the framework of Tsallis cosmology resembles the study in [79], except for the consideration of BEC DM in place of matter (dust).

4 Discussions and conclusions

We have discussed in this text that at the early time of the Universe DM of *normal* form was in equilibrium and decoupled from leftover plasma at a temperature T_{dec}. With the evolution of the Universe, *normal* DM underwent an approximate first-order phase transition to the BEC phase when the temperature fell below the critical temperature T_{crit}. Throughout the condensation both phases co-exist, and using an important thermodynamic criterion "*the continuity of pressure*" at the transition point leads to the determination of thermodynamical quantities of DM. The explicit numerical values of model parameters $(m_\chi, l_s, r, \sigma_\chi)$ taken in our study are highly uncertain. A small adjustment in BEC parameters could significantly affect the cosmological dynamics. Due to uncertainty in the model parameters, estimating the exact cosmological observations from theoretical prediction is very hard. In this manuscript, we have investigated the consequences of modified area law in the framework of Tsallis cosmology (therefore the entropic origin of gravity) on the evolution and phase transition process of DM. The findings of our paper are summarized as follows:

- In Sect. 3.1, assuming typical values of condensation parameters, we find critical redshift $z_{crit} = 1200$ and the corresponding critical time $t_{crit} \approx 6.12166 \times 10^{11}\ s$ when the condensation begins. As we have found z_{crit} and t_{crit} are independent of β, the obtained results are similar to [159,163,171]. We plot the variation of z_{crit} with l_s and m_χ for various masses and scattering lengths respectively in Fig. 1, and observe that z_{crit} falls with increasing of l_s and rises with increasing m_χ.
- In the Sect. 3.2, an important quantity $f(t)$, the condensate DM fraction is defined, and its temporal evolution equation in Tsallis cosmology is obtained. Then we solve for $f(t)$ and find the expression for the scale factor and Hubble parameter for the period of phase transition. Subsequently, we plot the temporal behavior of the normalized scale factor, Hubble parameter, and the condensate DM fraction in Figs. 2, 3, 4, 5, and 6 by choosing suitable values of BEC parameters and Tsallis parameter. We find that the scale factor and Hubble parameter both decrease with the increasing β values. The variation of the Hubble parameter in a standard matter-dominated Universe is also plotted in Fig. 3. We see the curves corresponding to the Hubble parameter in Tsallis cosmology converge to the curve corresponding to the Hubble parameter in standard cosmology at a later time of condensation. We show the plots of temporal dependence of the parameters DM equation of state parameter $\omega_{mix}(t)$ and density parameter $\Omega_\chi(t)$ in Figs. 7 and 8 during the BE condensation respectively. It is seen that the equation of state increases, and the density parameter of DM decreases with cosmic time from its initial value during the phase transition. Based on our considered standard model parameters, the estimated values of Ω_χ^{nor} and Ω_χ^{bec} are 3.95007×10^8 and $\Omega_\chi^{bec} = 3.4167 \times 10^7$ respectively. Again we observe the rate of change of the function $f(t)$ increases with the decrement of β for a particular value of r, and with the increment of r for a particular choice of β. We also find the time duration of the condensation process $\Delta\ t_{trans}$ in the framework of Tsallis cosmology and observe its dependence with Tsallis parameter β. In standard cosmology ($\beta = 1$), $\Delta\ t_{trans}$ turns out to be approximately $10^6\ years$ while a small departure from standard cosmology, say for $\beta = 1.03$, $\Delta\ t_{trans}$ becomes $6 \times 10^7\ years$. The variation of $\Delta\ t_{trans}$ with parameters r, l_s and m_χ considering different β values are plotted in Figs. 9, 10, and 11 accordingly. We note $\Delta\ t_{trans}$ increases with increasing of l_s but decreases with the increment of r and m_χ.
- Afterwards in Sect. 3.3, the post-condensation phase in Tsallis cosmology is analyzed. We show the variation of normalized scale factor with cosmic time in a Tsallis-modified BEC Universe in Fig. 12. We also find the expression for the Hubble parameter and deceleration parameter in the post-BEC phase and plot in Figs. 13 and 14 respectively. We see the scale factor and Hubble parameter both increase with the decreasing β values. The variation of the Hubble parameter for the standard matter-dominated Universe is also shown in Fig. 13. We see the graphs corresponding to the Hubble parameter in Tsallis cosmology coincide with the graph corresponding to the Hubble parameter in standard cosmology in the late time. From Fig. 14, we observe for a particular bound of the Tsallis parameter $\beta < 1/2$, the Universe undergoes accelerated expansion even without assuming any additional components of energy. Apart from that, we have also estimated the age of the Universe by considering typical values of the Tsallis parameter, confirming good agreement with cosmological observations. These two results validate the viability of the Tsallis cosmological model.

Apart from our work, there have been several works published in articles in the field of cosmology [60,81,85], particle physics [217,218], and entanglement measurements [219,220], where recent bounds on the Tsallis parameter breach the notion of extensivity. According to Tsallis and Cirto [61], as the dimension of a system is connected to the extensivity of entropy, then the violation of extensivity could put a question mark on the dimensionality of the system.

Finally, we assert that a perfect understanding of numerical values of parameters related to BEC cosmology could appreciably support interpreting cosmological observations precisely. More works are needed in this direction in order to survey the theoretical speculations of the BEC model in the framework of Tsallis cosmology, together with inspecting the potential existence of cosmological BEC DM. Furthermore, this condensed DM may have a significant influence on the structure formation of the Universe. Hence, comprehending the post-condensation phase and its associated cosmological parameters within the Tsallis cosmological model is of paramount importance and deserves further investigation.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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