On the Green and Wald formalism

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Backreaction in the cosmological context is a longstanding problem that is especially
important in the present era of precise cosmology. The standard model of a homogeneous
background plus density perturbations is most probably oversimplified and is expected
to fail to fully account for the near-future observations of sub-percent precision. From a
theoretical point of view, the problem of backreaction is very complicated and deserves
careful examination. Recently, Green and Wald claimed in a series of papers to have
developed a formalism to properly describe the influence of density inhomogeneities on
average properties of the Universe, i.e., the backreaction effect. A brief discussion of this
framework is presented, focussing on its drawbacks and on misconceptions that have
arisen during the "backreaction debate".

Keywords: Inhomogeneous cosmology; backreaction.

1. A brief history of the "backreaction debate"

The subject of backreaction in cosmology has quite a long history, given that modern
cosmology itself is a young discipline. In general, backreaction can be understood
as the influence of density inhomogeneities on average properties of the Universe
that are usually described by the Friedman–Lemaître–Robertson–Walker (FLRW)
metric, and in turn, their influence on light propagation and observables. Several
approaches have been developed to estimate the magnitude of backreaction, with
a wide range of results, depending on the method used. The smallest effects are
found in the perturbative regime, and the biggest for scalar averaging and for exact
solutions. Disregarding the issue of the magnitude of backreaction, most of the
methods show that this effect may act like dark energy, i.e., produce an effective
large-scale expansion in addition to that of a strictly FLRW model.

Here, we focus on a particular framework proposed to address the problem of
backreaction, the Green and Wald formalism, which reaches the opposite conclu-
sions, i.e., that backreaction is trace-free and cannot mimic a dark energy com-
ponent. This claim contradicts the results obtained by several different methods,
and thus became a subject of debate which we refer to here as the "backreaction
debate".

Key "backreaction debate" articles, for (+) and against (−) backreaction *poten-
tially* being able to mimic dark energy, include the following:

- Ishibashi and Wald: *"Can the acceleration of our Universe be explained by
  the effects of inhomogeneities?"*[1]—the authors claim that backreaction is

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*BFR: during invited lectureship; JJO: during long-term visit.

negligible based on the smallness of the derivatives of metric deviations.

- The Green and Wald formalism: *"New framework for analyzing the effects of small scale inhomogeneities in cosmology"*[2]—the authors claim to have developed a formalism for a mathematically rigorous treatment of backreaction, with surprising results: the only effect that the density inhomogeneities can have on the background dynamics is via gravitational radiation.

- Examples by Green and Wald: *"Examples of backreaction of small-scale inhomogeneities in cosmology"*[3]—the authors present two examples of their framework: a vacuum space–time, and a metric with an associated stress–energy tensor that violates the weak energy condition.

+ Rebuttal by Buchert et al.: *"Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?"*[4]—this paper presents a critical analysis of the Green and Wald formalism;

- Response to the rebuttal, by Green and Wald: *"Comments on Backreaction"*[5]—the authors reassert their main results from previous articles[a].

There is an important difference between the first and second of these articles. In Ref. 1, the authors use the arguments from standard perturbation theory, i.e., that not only does the metric of the real Universe barely deviate from the FLRW metric, but its derivatives are also small. The same applies to stress–energy tensor perturbations, and the perturbed Einstein equation reads

$$\delta G_{ab} = 8\pi \delta T_{ab} \,. \tag{1}$$

This formulation has as an obvious limitation in describing our Universe—we know that density perturbations are much greater than one in amplitude at recent epochs, on length scales well above those of "black holes and neutron stars",[2] implying— through the Einstein equation—the importance of curvature (i.e., second derivatives of the metric). In addition, as noted by Green and Wald in Ref. 2, there is no particular reason why the metric derivatives should also be small. In Ref. 2, the metric first derivatives are allowed to be large but finite, and no constraints are placed on the second derivatives (in practice, however, for the formalism to give a non-zero result, specific constraints have to be put on the second derivatives[4]). Furthermore, no constraints are made on the stress-energy tensor. All of these are desired features of a good backreaction model; Green and Wald's effort to take these into account deserves credit. Unfortunately, in our opinion, the authors failed to provide a physically valid general statement, as we outline below.

2. The Green and Wald formalism

In this section we briefly describe the Green and Wald formalism (see Ref. 2 for details). The formalism is based on the existence of a λ-dependent family of metrics $g_{ab}(\lambda, x)$ on an arbitrary background manifold (M) that are close to a background

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[a]Commented on in the final version of the rebuttal paper.

metric $g_{ab}(0, x)$, and which converge to the latter point-wise as the scalar parameter λ approaches zero. This assumption is followed by another four concerning the behaviour of the metric and its derivatives:

(i) For all $\lambda > 0$ the metric $g_{ab}(\lambda, x)$ satisfies:

$$G_{ab}(g(\lambda, x)) + \Lambda g_{ab}(\lambda, x) = 8\pi T_{ab}(\lambda), \tag{2}$$

where $T_{ab}(\lambda)$ obeys the weak energy condition.

(ii) There exists a smooth function $C_1(x)$ on $(M, g(0))$ such that:

$$|h_{ab}(\lambda, x)| \leq \lambda C_1(x) \; ; \; h_{ab}(\lambda, x) = g_{ab}(\lambda, x) - g_{ab}(0, x). \tag{3}$$

(iii) There exists a smooth function $C_2(x)$ on $(M, g(0))$ such that $|\nabla_c h_{ab}(\lambda, x)| \leq C_2(x)$, i.e. the derivatives do not have to be small.

(iv) There exists a smooth tensor field μ_{abcdef} on $(M, g(0))$ such that[b]:

$$\underset{\lambda \searrow 0}{\text{w–lim}}[\nabla_a h_{cd}(\lambda, x)\nabla_b h_{ef}(\lambda, x)] = \mu_{abcdef}. \tag{4}$$

The last assumption uses the notion of the weak limit, w–lim: we say that an arbitrary tensor field $A_{a_1 \ldots a_n}(\lambda)$ converges weakly to $B_{a_1 \ldots a_n}$, i.e., $\text{w–lim}_{\lambda \searrow 0} A_{a_1 \ldots a_n}(\lambda) = B_{a_1 \ldots a_n}$, if $\forall f^{a_1 \ldots a_n}$ of compact support,

$$\lim_{\lambda \searrow 0} \int f^{a_1 \ldots a_n} A_{a_1 \ldots a_n}(\lambda) = \int f^{a_1 \ldots a_n} B_{a_1 \ldots a_n}. \tag{5}$$

The integration is performed over a 4D region of space–time fixed in the background manifold. With these five assumptions, the Einstein equation for the background metric is derived by comparing and manipulating the curvature terms. The Green and Wald equations for the background metric $g_{ab}(0, x)$ then read:

$$\underset{\lambda \searrow 0}{\text{w–lim}} \left[G_{ab} \left(g_{ab}(0, x) \right) \right] + \Lambda g_{ab}(0, x) = 8\pi \underset{\lambda \searrow 0}{\text{w–lim}} \left[T_{ab}(\lambda) + t_{ab}(\lambda) \right], \tag{6}$$

where:

$$t_{ab}(\lambda) := 2\nabla_{[a}C^e_{e]b} - 2C^f_{b[a}C^e_{e]f} - g_{ab}(\lambda)g^{cd}(\lambda)\nabla_{[c}C^e_{e]d} + g_{ab}(\lambda)g^{cd}(\lambda)C^f_{d[c}C^e_{e]f}, \tag{7}$$

and

$$C^c_{ab} = \frac{1}{2}g^{cd}(\lambda)\left[\nabla_a g_{bd}(\lambda) + \nabla_b g_{ad}(\lambda) - \nabla_d g_{ab}(\lambda)\right].$$

We can now give names to the terms in (6):

$$G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab} = 8\pi T^{(0)}_{ab} + 8\pi t^{(0)}_{ab}, \tag{8}$$

where according to Green and Wald, $T^{(0)}_{ab}$, defined

$$\underset{\lambda \searrow 0}{\text{w–lim}} T_{ab}(\lambda) = T^{(0)}_{ab}, \tag{9}$$

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[b]Eq. (9) of Buchert et al.[4] has a typo here: $|\cdot|$ should read $[\cdot]$.

represents a stress–energy tensor averaged over small-scale inhomogeneities. Green and Wald examined an "effective" stress-energy tensor $t_{ab}^{(0)}$, inferring that:

- $t_{ab}^{(0)}$ is trace-less, i.e. $t^{(0)a}{}_a = 0$;
- $t_{ab}^{(0)}$ obeys the weak energy condition i.e. $t_{ab}^{(0)} t^a t^b \geq 0$.

To put it in words: $t_{ab}^{(0)}$ cannot mimic dark energy.

3. Discussion

This eventually led to the rebuttal paper by Buchert et al.[4] Shortly after this appeared on the *ArXiv* preprint server, Green and Wald published a response in which they uphold their previous statements. Additionally, they clarify the domain of applicability of their formalism in relation to popular approaches to backreaction. In particular, Green and Wald state that their formalism does not apply to situations when:

- the actual metric (e.g., at recent epochs) is far from FLRW; or
- one wishes to construct an effective metric (or other effective quantities) through some averaging procedure

This, in principle, ends the debate about whether backreaction has been excluded as a dark energy candidate: the Green and Wald formalism does not apply to the main body of backreaction research; backreaction remains a viable dark energy candidate. We briefly outline some characteristics that the formalism lacks.

3.1. *Backreaction without backreaction*

Let us introduce a more precise definition of backreaction (setting $\Lambda = 0$ for simplicity). Assume that we know the inhomogeneous metric describing the real Universe and that we want to derive its averaged dynamical behaviour on a certain scale. We do this by applying a procedure such that by smoothing over larger and larger scales of inhomogeneities, we end up with a background metric (on a scale that we accept as homogeneous). However, to construct the Einstein tensor, and thus to describe the averaged dynamics, we need not only the metric but also the metric derivatives and products of derivatives. In general, averaging and differentiating or taking products of derivatives are non-commuting operations.[6] Thus, while it is trivially true that, provided that an averaging procedure $\langle . \rangle$ is properly defined, $\langle G_{ab}(g_{ab}) \rangle = 8\pi \langle T_{ab} \rangle$, we cannot expect that

$$G_{ab}(\langle g_{ab} \rangle) = 8\pi \langle T_{ab} \rangle. \tag{10}$$

Backreaction (τ_{ab}) is then the term compensating the discrepancies coming from this non-commutativity:

$$G_{ab}(\langle g_{ab} \rangle) = 8\pi \left(\langle T_{ab} \rangle + \tau_{ab} \right). \tag{11}$$

This term should, in principle, be present at each intermediate scale between a small scale and the homogeneous scale.

Let us take a look at (2), i.e. Green and Wald's assumption (i), which becomes

$$G_{ab}(g(\lambda, x)) = 8\pi T_{ab}(\lambda), \tag{12}$$

which we can treat, from the lack of any other options, as a definition of the λ-dependent family of stress–energy tensors. Suppose that for some $\lambda > 0$, $T_{ab}(\lambda)$ is smoother than the inhomogeneous, unsmoothed stress-energy tensor $T_{ab}(1)$, thanks to an averaging procedure. Then a backreaction term $\tau_{ab}(\lambda)$ must appear in the λ-dependent Einstein equation, due to the non-commutativity of averaging (11). But this contradicts (12). Hence, in the Green and Wald formalism, the only averaging allowed for $\lambda > 0$ is exactly commutative averaging ($\tau_{ab}(\lambda) = 0 \ \forall \lambda > 0$), and a backreaction term $t_{ab}(0)$ suddenly blips on at $\lambda = 0$, discontinuously.[4] Thus, it is difficult to find a physically meaningful interpretation of this limiting procedure.

3.2. *Averaging without averaging*

There is no overall agreement on a physically meaningful way to average the Einstein equation. One can quite quickly run into conceptual problems trying to perform averaging of fields on manifolds in a unique and gauge-independent way. In their formalism, Green and Wald bypass these difficulties by being very general, and yet they are able to extract some crucial features of averaged equations. The role of the averaging operator is played by the weak limit specified up to some arbitrary well-behaving test tensor field used to contract the averaged quantities. According to Green and Wald, the action of the weak limit can be interpreted as follows:[2] *"Roughly speaking, the weak limit performs a local space-time average of $A_{a_1 \ldots a_n}(\lambda)$ before letting $\lambda \to 0$"*, where $A_{a_1 \ldots a_n}(\lambda)$ is a one-parameter family of tensor fields. For this to be non-trivial, the limit operator and the integral cannot commute: $\lim_{\lambda \to 0} \int \neq \int \lim_{\lambda \to 0}$, since otherwise, the averaging would be performed on the background value of the tensor field and the integration would become redundant for the homogeneous background.

However, for a fixed domain of integration, fulfilling this requirement of non-commutativity requires breaking one of the dominant convergence theorem assumptions (e.g., Figure 1). Indeed, in the examples provided by Green and Wald, the derivatives and the products of derivatives, while being continuous in the space–time coordinates, do not have pointwise limits as $\lambda \searrow 0$.

Fig. 1. Schematic illustration of density $\rho := \lambda^{-1} \sin(x/\lambda) = \mathrm{d}^2/\mathrm{d}x^2\, h$, where $h := -\lambda \sin(x/\lambda)$. As $\lambda \searrow 0$, h approaches zero pointwise, but ρ is pointwise unbounded.

Figure 1 shows the behaviour of a toy model $\rho(x) = \mathrm{d}^2/\mathrm{d}x^2\, h(x)$ (analogous to the usual density ρ that relates to the second derivatives of the metric via the Einstein equation), where h approaches zero pointwise, while ρ is λ-discontinuous (it does not have a pointwise limit). Instead
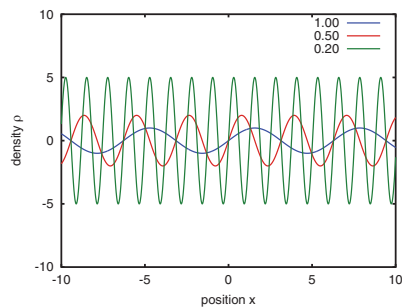
of successive coarse-graining, the density profile approaches homogeneity by a *fine-graining* process, which is the opposite of what we would expect from an averaging procedure. (In contrast, GW's procedure *does* have features likely to be relevant to gravitational radiation.)

Moreover, integrating by parts, we have (e.g. for $A_{a_1...a_n} := h_{ab}$):

$$\lim_{\lambda \to 0} \int (\nabla_b A_{a_1...a_n}(\lambda)) f^{b,a_1...a_n} =$$

$$\lim_{\lambda \to 0} A_{a_1...a_n}(\lambda) f^{b,a_1...a_n} - \int \lim_{\lambda \to 0} A_{a_1...a_n}(\lambda) \nabla_b f^{b,a_1...a_n} = 0. \qquad (13)$$

Thus, the apparently non-local integration of derivatives in GW's procedure reduces to pointwise operations; it misses generic features of averaging: non-locality and scale-dependence.

Conclusions

The Green and Wald formalism is not mathematically general—see Ref. 4 for the hidden assumptions—and its physical interpretation is far from obvious. An unaware reader might be tempted to think that mathematically general arguments against common approaches to backreaction (e.g., exact inhomogeneous cosmological solutions or the Buchert equations) were presented, but this is not the case.

Acknowledgments

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