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Aspects of M-theory and Quantum Information

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Declaration

I herewith certify that all material in this dissertation which is not my own work has been properly acknowledged.

Leron Borsten

Abstract

As the frontiers of physics steadily progress into the 21st century we should bear in mind that the conceptual edifice of 20th-century physics has at its foundations two mutually incompatible theories; quantum mechanics and Einstein's general theory of relativity. While general relativity refuses to succumb to quantum rule, black holes are raising quandaries that strike at the very heart of quantum theory. M-theory is a compelling candidate theory of quantum gravity. Living in eleven dimensions it encompasses and connects the five possible 10-dimensional superstring theories. However, M-theory is fundamentally non-perturbative and consequently remains largely mysterious, offering up only disparate corners of its full structure. The physics of black holes has occupied centre stage in uncovering its non-perturbative structure.

The dawn of the 21st-century has also played witness to the birth of the information age and with it the world of quantum information science. At its heart lies the phenomenon of quantum entanglement. Entanglement has applications in the emerging technologies of quantum computing and quantum cryptography, and has been used to realize quantum teleportation experimentally. The longest standing open problem in quantum information is the proper characterisation of multipartite entanglement. It is of utmost importance from both a foundational and a technological perspective.

In 2006 the entropy formula for a particular 8-charge black hole appearing in M-theory was found to be given by the 'hyperdeterminant', a quantity introduced by the mathematician Cayley in 1845. Remarkably, the hyperdeterminant also measures the degree of tripartite entanglement shared by three qubits, the basic units of quantum information. It turned out that the different possible types of three-qubit entanglement corresponded directly to the different possible subclasses of this particular black hole. This initial observation provided a link relating various black holes and quantum information systems. Since then, we have been examining this two-way dictionary between black holes and qubits and have used our knowledge of M-theory to discover new things about multipartite entanglement and quantum information theory and, vice-versa, to garner new insights into black holes and M-theory. There is now a growing dictionary, which translates a variety of phenomena in one language to those in the other.

Developing these fascinating relationships, exploiting them to better understand both M-theory and quantum entanglement is the goal of this thesis. In particular, we adopt the elegant mathematics of octonions, Jordan algebras and the Freudenthal triple system as our guiding framework. In the course of this investigation we will see how these fascinating algebraic structures can be used to quantify entanglement and define new black hole dualities.

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Welcome, O life! I go to encounter for the millionth time the reality of experience and to forge in the smithy of my soul the uncreated conscience of my race

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CHAPTER 1

Introduction

1. Overview

It ever so often happens that two seemingly disparate areas of theoretical physics share the same mathematics. This may be indicative of a genuine physical duality, in which case one would expect a profound leap in our understanding to follow. Even if this rather grand hope is not realised, the study of such mathematical relations has the potential to reveal unexpected insights on both sides of the equation.

This interdisciplinary investigation centres on an unexpected example of such a duality. The two subjects in question are:

1. Bekenstein-Hawking black hole entropy in M-theory.
2. Qubit entanglement in quantum information theory.

As the frontiers of physics steadily progress into the 21st century we should bear in mind that the conceptual edifice of 20th-century physics has at its foundations two mutually incompatible theories: quantum mechanics and Einstein's general theory of relativity. While general relativity refuses to succumb to quantum rule, black holes are raising quandaries that strike at the very heart of quantum theory. Without incorporating gravity and quantum theory into a single consistent framework, such paradoxes will continue to haunt us.

M-theory is a promising candidate theory of quantum gravity. Living in eleven dimensions, it encompasses and connects the five consistent 10-dimensional superstring theories, as well as 11-dimensional supergravity and, as such, has the potential to unify the fundamental forces. However, M-theory is fundamentally non-perturbative and consequently remains largely mysterious, offering up only disparate corners of its full structure. The physics of black holes has occupied centre stage, offering unique insights into the non-perturbative structure of M-theory. Whatever final formulation M-theory eventually takes, understanding its black hole solutions will play an essential role in its evolution.

The dawn of the 21st-century has also played witness to the birth of the (quantum) information age and with it the world of quantum information (QI) science. At its heart lies the phenomenon of quantum entanglement. The quantum states of two or more objects have to be described with reference to each other, even though the individual objects may be spatially separated. This leads to classically unexplainable, but experimentally observable, quantum correlations between the spatially separated systems. Quantum entanglement has applications in the emerging technologies of quantum computing and quantum cryptography, and has been used to realise quantum teleportation experimentally. The longest standing open problem in quantum information is the proper characterisation of multipartite entanglement. It is of utmost importance from both a foundational and a technological perspective.

In 2006 the entropy formula for a particular 8-charge black hole appearing in M-theory, specifically the STU model [1–3], was found to be given by the "hyperdeterminant", a quantity introduced by the mathematician Cayley in 1845 [4, 5]. Remarkably, the hyperdeterminant also measures the degree of tripartite entanglement shared by three qubits, the basic units of quantum information[1] [6,7]. It turned out that the physically distinct forms of 3-qubit entanglement correspond directly to the different possible subclasses of the STU black hole [8]. These initial observations provided a link relating various black holes and QI systems. Since then, we have been examining this two-way dictionary between black holes and qubits and have used our knowledge of M-theory to discover new things about multipartite entanglement and quantum information theory and, vice-versa, to garner new insights into black holes and M-theory. This correspondence has grown to relate numerous aspects of both disciplines [4,8–29]. In particular, the present author is included amongst the co-authors of [16–21, 28, 29]. One of the co-authors, namely Dahanayake, is submitting their thesis [30] simultaneous with this thesis. Since the introductory material in both theses draws on [18], we should make clear that the original content in this thesis will focus on [19, 20, 28] while that in Dahanayake's will focus on [17, 21, 29].

Part I of this thesis forms a review of these developments, including a brief survey of the work in [30]. For example:

- The entropy S of the 8-charge STU black hole is related to the tripartite entanglement of three qubits (Alice, Bob and Charlie) as measured by the (unnormalised) 3-tangle τ_{ABC} [6] [4]

$$S = \tfrac{\pi}{2}\sqrt{\tau_{ABC}}. \tag{1.1}$$

  Note, τ_{ABC} is given by the magnitude of Cayley's *hyperdeterminant* [5, 31].

- The classification of three-qubit entanglements is related to the classification of $\mathcal{N} = 2$ supersymmetric STU black holes [8] shown in Table 1.1 and explained in more detail in section 1.2. One important distinction is that the black hole charges are real. Consequently the GHZ class splits into two pieces on the black hole side in the sense that there are two possible real forms for the GHZ stabilizer, which coincide in the complex case.

- The attractor mechanism on the black hole side is related to optimal local distillation protocols on the QI side [14, 26].

- Dimensionally reducing the STU model in a time-like direction, the black hole solutions can be used to classify the entanglement classes of the much harder 4-qubit system [29]. Interestingly, the principal difference between the $D = 4$ black holes, which have real charges, and the

[1] One can only assume that Cayley had anticipated both quantum information and M-theory.

Class	S_A	S_B	S_C	Det a	Black hole	SUSY
A-B-C	0	0	0	0	small	1/2
A-BC	0	>0	>0	0	small	1/2
B-CA	>0	0	>0	0	small	1/2
C-AB	>0	>0	0	0	small	1/2
W	>0	>0	>0	0	small	1/2
GHZ	>0	>0	>0	<0	large	1/2
GHZ	>0	>0	>0	>0	large	0

Table 1.1.: The values of the local entropies S_A, S_B, and S_C and the hyperdeterminant $\mathrm{Det}\, a$ (defined in section 2.6.1 and section 2.6.2) are used to partition three-qubit states into entanglement classes. The entropy/entanglement correspondence relates these to $D = 4, \mathcal{N} = 2, STU$ model black holes.

Class	S_A	S_B	S_C	Det a	Black hole	SUSY
A-B-C	0	0	0	0	small	1/2
A-BC	0	>0	>0	0	small	1/4
B-CA	>0	0	>0	0	small	1/4
C-AB	>0	>0	0	0	small	1/4
W	>0	>0	>0	0	small	1/8
GHZ	>0	>0	>0	<0	large	1/8
GHZ	>0	>0	>0	>0	large	0

Table 1.2.: As in Table 1.1 entanglement measures are used to classify states, but this time concerning the tripartite entanglement of seven qubit states. The correspondence relates these to the $D = 4, \mathcal{N} = 8$ black holes. Note, the amount of supersymmetry preserved corresponds to the degree of entanglement.

qubits, which have complex charges, disappears - the time-like reduced STU model classifies the entanglement of four *complex* qubits.

- The 56-charge black hole of $\mathcal{N} = 8$ supergravity admits a quantum information theoretic interpretation in a 7-fold direct sum of the 3-qubit Hilbert space [10, 11]. This follows from decomposition $E_{7(7)} \supset [\mathrm{SL}(2)]^7$, which describes a tripartite entanglement of seven qubits. The entanglement measure is given by Cartan's quartic $E_{7(7)}$ invariant.

- The classification of tripartite entanglements of seven qubits is related to the classification of $(\mathcal{N} = 8, D = 4)$ supersymmetric black holes [17] shown in Table 1.2 and explained in more detail in section 4.

- A *microscopic* interpretation of the STU black hole entropy and 3-qubit entanglement correspondence [17]. The microscopic string-theoretic interpretation of the black hole charges may be described by configurations of intersecting D3-branes wrapped around the six compact dimensions. The 3-qubit basis vectors $|ABC\rangle$, $(A, B, C = 0$ or 1) are associated with the eight wrapping cycles, where $|0\rangle$ corresponds to xo and $|1\rangle$ to ox in Table 4.2. By allowing one of the D3-branes to intersect at an angle, the most general real 3-qubit state, parameterised by four real numbers and an angle, is identified with the most general STU black hole, described by four D3-branes intersecting at an angle.

- A supersymmetric generalisation of the qubit, the *superqubit* [21]. This was based on the minimal supersymmetric extension of $\mathrm{SL}(2)$, which is given by the supergroup $\mathrm{Osp}(2|1)$ [32]. We studied the entanglement classification of small superqubit systems. In the case of two superqubits the entanglement measure is given by the *supedeterminant*. Remarkably Cayley's hyperdeterminant may be supersymmetrised [33]. The *superhyperdeterminant* provides the entanglement measure of three superqubits.

- For $D = 5$ supergravity, the entropy of a 9-charge $\mathcal{N} = 8$ black hole is given by the entanglement of two qutrits, as measured by the 2-tangle τ_{AB} [12],

$$S = 2\pi\sqrt{|\det a_{AB}|}, \tag{1.2}$$

where

$$\tau_{AB} = 27|\det a_{AB}|^2. \tag{1.3}$$

- The full 27 charge $\mathcal{N} = 8, D = 5$ black hole may be interpreted in terms of a Hilbert space consisting of three copies of the two-qutrit Hilbert space [13]. It relies on the decomposition $E_{6(6)} \supset [\mathrm{SL}(3)]^3$ and admits the interpretation of a bipartite entanglement of three qutrits, with the entanglement measure given by Cartan's cubic $E_{6(6)}$ invariant.

- The classification of the bipartite entanglements of three qutrits is related to the classification of $\mathcal{N} = 8, D = 5$ supersymmetric black holes [17] shown in Table 1.3.

Class	C_2	τ_{AB}	Black hole	SUSY
A-B	0	0	small	1/2
Rank 2 Bell	>0	0	small	1/4
Rank 3 Bell	>0	>0	large	1/8

Table 1.3.: The $D = 5$ analogue of Table 1.1 and Table 1.3 relates two-qutrit entanglements and their corresponding $D = 5, \mathcal{N} = 8$ black holes.

- The two-qutrit basis vectors $|AB\rangle$, $(A, B = 0$ or 1 or 2) can be associated with the nine wrapping cycles of intersecting M2-brane on a T^6, where $|0\rangle$ corresponds to xoo, $|1\rangle$ to oxo and $|2\rangle$ to oox. Just as the most general real 2-qutrit state can be parameterised by three real numbers, the most general black hole can be described by three intersecting M2-branes [17].

Developing these fascinating relationships and exploiting them to better understand both M-theory and quantum information are the principal aims of this thesis. However, the interdisciplinary nature of this project has naturally led to a rather diverse set of topics [16–21, 28, 29]. We do not attempt to cover them all, but rather take certain algebraic structures as our guiding motif. In particular, the work of this thesis is best characterised as: The application of the octonions, Jordan algebras and the Freudenthal triple system (FTS) to M-theory, quantum information and their interrelation.

These developments form the main body of this thesis and the content of Part II. The key aspects are summarised here:

Algebras and black holes

- The extremal black p-brane solutions of supergravity have played, and continue to play, a key role in unravelling the non-perturbative aspects of M-theory. Evidently, one would like to understand the structure of these solutions. In particular, one would like to know how such solutions are interrelated by U-duality, the global symmetry of the equations of motion. The electric/magnetic charge vectors of the asymptotically flat p-brane solutions form irreducible U-duality representations.

  In many relevant cases the macroscopic leading-order black p-brane entropy is a function of these charges only, a result of the attractor mechanism [34–37]. Consequently, an important question is whether two *a priori* distinct black p-brane charge configurations are in fact related by U-duality. Mathematically this amounts to determining the distinct charge vector orbits under U-duality.

- It has been well know for sometime now that a class of $D = 5$ and $D = 4$ supergravity theories are most naturally described by Jordan algebras and their associated Freudenthal triple systems, respectively [38–40]. In particular, the U-duality symmetries and their action on the black hole charges are described by the symmetries of the Jordan algebras and FTS. The black hole entropies are proportional to the square root of the cubic and quartic norms of the Jordan algebras and FTS, respectively. We recast the known U-duality orbits in the framework of these algebras and their *ranks*. Moreover, we obtain the hitherto unknown orbits and their representative states for the reducible sequence of $D = 4, \mathcal{N} = 2$ supergravity coupled to $n + 1$ Abelian vector multiplets.

Octonions and the tripartite entanglement of seven qubits

- In [10, 11] it was observed that the tripartite entanglement of seven qubits is described by the Fano plane [41], which is also the multiplication table of the imaginary octonions.

  This remarkable fact is supported by the $\mathbb{O}$-graded algebraic structure of E_7 [42,43], compatible with its action on the minimal 56-dimensional representation. Explicitly,

$$\begin{gathered}\mathfrak{e}_7 = \times_l\, \mathfrak{sl}(A_l)e_0 \oplus \bigoplus_{1\leq i\leq 7} \left[\bigotimes_{i\notin l} A_l\right] e_i,\\ \mathbf{56} = \bigoplus_{1\leq i\leq 7} \left[\bigotimes_{i\in l} A_l\right] e_i,\end{gathered} \tag{1.4}$$

  where

$$i \in \{1, \dots, 7\} \tag{1.5}$$

  are the seven points of the Fano plane,

$$l \in \{124, 235, 346, 457, 561, 672, 713\} \tag{1.6}$$

  are the seven lines, and we have attached to each line a two-dimensional vector space A_l. The real and imaginary octonions are denoted as e_0 and e_i, respectively. There is a quaternionic analogue of this construction where we consider just three of the seven lines. This describes the $\mathcal{N} = 4$ subsector.

  Strings can carry two kinds of charge: NS-NS coming from right and left moving bosonic modes and R-R coming from right and left moving fermionic modes (NS = Neveu-Schwarz and R

= Ramond). When the U-duality group $E_{7(7)}$ is decomposed under the $\mathrm{SL}(2)$ S-duality and $\mathrm{SO}(6,6)$ T-duality

$$\begin{aligned} E_{7(7)} &\supset \mathrm{SL}(2) \times \mathrm{SO}(6,6), \\ \mathbf{56} &\to (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}), \end{aligned} \tag{1.7}$$

the first term describes the $\mathcal{N}=4$ subsector with 24 NS-NS charges and the second term describes the 32 R-R charges. In terms of the seven lines of the Fano plane of Figure 7.1, the STU charges correspond to the single line 124 (which describes the imaginary complex number), the NS-NS charges correspond to the three lines $124, 561, 713$ (which describe the three imaginary quaternions) and the R-R to the four lines $235, 346, 457, 672$ (which each describe the four complementary imaginary octonions).

- Inspired by this observation we show that the precise dictionary relating the 56 black hole charges to the 56-dimensional tripartite entanglement of seven qubits is detemined by the imaginary octonions.

- These observations led us in [18] to speculate on the possible relations between the $STU, \mathcal{N}=4$ and $\mathcal{N}=8$ theories. In particular, noting that $24 = 8 \times 3$ we found that these 24 NS-NS charges may be interpreted as the eight charges of the STU model defined over the three imaginary quaternions. Accordingly, the $\mathcal{N}=4$ Cartan invariant with $\mathrm{SL}(2) \times \mathrm{SO}(6,6)$ symmetry, may be written as Cayley's hyperdeterminant defined over the imaginary quaternions, provided we adopt a suitable operator ordering. Noting that $56 = 8 \times 7$, it is tempting to employ a similar construction replacing imaginary quaternions by imaginary octonions to describe the full 56 charges, including the 32 R-R. However, this is very much work in progress and what final formulation it may take is certainly not clear. Accordingly, we do not treat this topic in the present work.

Freudenthal classification of qubit entanglement

- Exploiting the correspondence between the STU black hole and three qubits we use the Freudenthal triple system to classify the entanglement classes. In particular, the four FTS ranks given the four entanglement classes [19]. The advantage of this formulation over the conventional classification [44] is that it is manifestly covariant under the equivalence group of *stochastic local operations and classical communication* (SLOCC). Moreover, it facilitates the determination of the orbit cosets.

- Inspired by this classification of three qubits, we introduce an n-qubit generalisation of the FTS. The basic idea is to reorganise the n-qubit state into its permutation related subsets. The SLOCC transformations are expressed as the natural generalisations of the FTS operations. Similarly, the FTS invariants and triple product extend to the n-qubit system. Perhaps, the term n-qubit Freudenthal *triple* system is somewhat misleading since, for example, there is no triple product in the 4-qubit case. There is, however, an analogous *quintic* product. We making some preliminary attempts at using this framework to classify n-qubit entanglement. This is a work in progress and there are number of open questions, most importantly how the ranks of the FTS are to be generalised.

Discrete black hole charge orbits in $\mathcal{N} = 8$ supergravity

- The continuous U-duality charge orbits of the maximally supersymmetric theories, valid in the limit of large charges, have been known for sometime now [45, 46]. However, in the full quantum theory the U-duality group is broken to a discrete subgroup, a consequence of the Dirac-Zwanziger-Schwinger charge quantization conditions [47]. Consequently, the U-duality orbits are furnished with an increased level structural complexity. We address this issue for the $\mathcal{N} = 8$ theories in six , five and four dimesions by exploiting the mathematical framework of *integral Jordan algebras*, the *integral Freudenthal triple system* and, in particular, the work of Krutelevich [28, 48].

- The analysis of the discrete U-duality orbits relies on two key ingredients. First, to use the discrete symmetries of the integral Jordan algebras and FTS to bring the charge vectors into a *diagonally reduced canonical form*. Second, to construct from the algebraic operations of the Jordan algebras and FTS new arithmetic invariants that are absent in the continuous theory. They are given by the greatest common divisor $(\gcd)$ of irreducible representations built out of powers of the charge vectors. Ideally these invariants then uniquely determine the canonical form of a given state.

- The charge vector of the dyonic black string in $D = 6$ is $SO(5, 5; \mathbb{Z})$ related to a two-charge reduced canonical form uniquely specified by a set of two arithmetic U-duality invariants.

- Similarly, the black hole (string) charge vectors in $D = 5$ are $E_{6(6)}(\mathbb{Z})$ equivalent to a three-charge canonical form, again uniquely fixed by a set of three arithmetic U-duality invariants.

- The charge vector of the dyonic black hole in $D = 4$ is $E_{7(7)}(\mathbb{Z})$ related to a five-charge reduced canonical form. This canonical form implies that the R-R charges may always be transformed away, as required for the validity of the manifestly $E_{7(7)}(\mathbb{Z})$ invariant dyon degeneracy formula of type II string theory on T^6 derived in [49].

- However, the canonical form is not uniquely determined by the known arithmetic invariants. While black holes preserving more than 1/8 of the supersymmetries may be fully classified by known arithmetic $E_{7(7)}(\mathbb{Z})$ invariants, 1/8-BPS and non-BPS black holes yield increasingly subtle orbit structures, which remain to be properly understood. However, for the very special subclass of *projective* black holes a complete classification is known. All projective black holes are $E_{7(7)}(\mathbb{Z})$ related to a four or five charge canonical form determined uniquely by the set of known arithmetic U-duality invariants. Moreover, $E_{7(7)}(\mathbb{Z})$ acts transitively on the charge vectors of projective black holes with a given leading-order entropy.

- In all cases the black hole/string entropy is quantized.

- Following Bhargava's work on higher Gauss composition [50], it is clear that the discrete orbits of large black holes (non-vanishing leading order entropy) appearing in the STU model are in one-to-one correspondence with the equivalence classes of balanced triples of ideal classes in a quadratic ring. One hope is that this correspondence may lead to a more transparent group theoretic classification of the orbits. This ought to provide important information for the $\mathcal{N} = 8$ case.

Freudenthal duality

- We introduce a new duality, distinct from U-duality, which acts on the $D = 4$ black hole charges and leaves the leading order entropy invariant. This duality is defined in terms of the Freudenthal triple product and quartic norm, hence the name "Freudenthal duality" (F-duality).
- The integral lattice of quantized charges is only preserved under F-duality if the quartic norm is square. Consequently, only a special subset of all black holes admit a consistent F-dual.
- Since some, but not all, of the arithmetic U-duality invariant are preserved under F-duality the question of higher order corrections to the leading order entropy remains an open question.
- We furthermore introduce an analogous "Jordan duality" (J-duality) which acts on the $D = 5$ black hole/string charges and leaves the leading order entropy invariant. It is defined in terms of the quadratic adjoint and cubic norm of the Jordan algebra. It should be thought as mapping holes to strings, but it is not clear how this is realised at the geometrical level.
- We show that all J-dual black holes/strings that preserve the greatest common divisor of the charges are U-duality related.
- The 4D/5D lift [51] associates a rotating 5D black hole to a non-rotating 4D black hole. We show that two black holes related by F-duality in 4D are related by J-duality when lifted to 5D.

2. Structure of thesis

This thesis is a game of two halves. In Part I we summarise the main features of qubits, black holes and their relationship, which includes a brief review of the work appearing [30]. In Part II, we develop these ideas in the context of the octonions, Jordan algebras and the FTS. This second half forms the main body of this thesis.

Part I We begin in chapter 2 with a brief, elementary introduction to quantum information and entanglement. We assume as little knowledge as possible given that we approach this project from within the string theory community. In section 1 we introduce the basic concepts of QI, paying particular attention to qubits and entanglement. In section 2 we develop further our understanding of entanglement classification and especially the paradigm of SLOCC. We present some examples, most notably the entanglement classification of three qubits. The 3-tangle, which measures the genuine tripartite entanglement shared by three qubits is introduced in terms of the all-important Cayley's hyperdeterminant.

Chapter 3 contains a rather concise introduction to M-theory, U-duality and extremal black holes. The role of U-duality and its implications for black hole entropy is emphasized. We also describe the importance of the *attractor mechanism*, which fixes the black hole entropy as a function of the electric/magnetic charges. In section 4 the STU model is treated in some detail as an example, but also in anticipation of its significance in the black hole qubit correspondence, which is the subject of chapter 4.

In section 1 we describe how the STU black hole entropy is related to the 3-tangle. The classification of $\mathcal{N} = 2$ black holes is matched to the 3-qubit entanglement classes. We briefly discuss the higher-order corrections to the black hole entropy formula and their possible QI interpretation. Finally, we introduce Lévay's work on the black hole attractor mechanism as SLOCC.

Having summarised the $STU/3$-qubit correspondence we consider its generalisation to the maximally supersymmetric $\mathcal{N} = 8$ theory in section 2. The black holes now carry 56 charges transforming irreducibly under $E_{7(7)}$, the U-duality group of $\mathcal{N} = 8$ supergravity. Cayley's Hyperdeterminant, invariant under $[\mathrm{SL}(2)]^3$, is promoted to Cartan's quartic $E_{7(7)}$ invariant. The black holes are identified with the tripartite entanglement of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George) on the QI side.

We conclude the first half of this thesis, in section 5, with a survey of some additional developments [30]. In particular, we describe: (1) the microscopic interpretation of black holes as qubits (2) the classification of 4-qubit entanglement from STU black holes in three dimensions (3) the *superqubit*, a supersymmetric generalisation of the qubit.

Part II Having now reviewed in Part I the relevant aspects of quantum information, entanglement M-theory, black hole entropy and their relationship, we may now begin in earnest our main investigation. The basic proposal is to take certain algebraic structures, identified by their natural occurrence in supergravity, and apply them to develop our understanding of both entanglement and black holes. In particular, we will make extensive use of the the division algebras, most notably the octonions, Jordan algebras and the Freudenthal triple system.

We start off in chapter 5 with a general review of the required algebras and their symmetries. This is largely well known material, extensively covered in the mathematics literature, our primary references being [52–69]. However, there are some new results scattered throughout, in particular the canonical forms of the Freudenthal triple system $\mathfrak{F}^{2,n}$, which facilitates the U-duality orbit classification of black holes in $D = 4, \mathcal{N} = 2$ Maxwell-Einstein supergravity. Our basic tools include a set of FTS automorphisms, which are used to find canonical forms for the black hole charges, and the FTS Lie action, which is used to find the corresponding orbits. An explicit example, for the exceptional magic supergravity, is presented in Appendix C.

This brings us neatly onto the subject of chapter 6, which provides an overview of Jordan algebra and the FTS in supergravity. The part played by these algebras in supergravity was known already in 1983 [38–40, 70]. Indeed, they have already been used to classify black hole solutions in a number of cases [45, 71–73], most importantly in $D = 4, 5, \mathcal{N} = 8$ supergravity. However, we devote some time to the subject of U-duality orbits from the purely Jordan algebraic and FTS perspective, partly to setup later sections, but primarily to re-derive the know result in our current formulation and to complete the picture for the cases that have yet to be treated in this manner. These include the small orbits of the $\mathcal{N} = 2$ Maxwell-Einstein theories, such as the STU model.

This essentially completes our introduction to algebras and their role in black hole physics. Building on these results, in chapter 7 we revisit the correspondence between $\mathcal{N} = 8$ supergravity and the tripartite entanglement of seven qubits. In [10,11,18] it was observed that the tripartite entanglement of seven qubits is neatly described by a mathematical gadget known as the Fano plane. The Fano plane describes the multiplication table of the octonions and we show that the dictionary relating the $\mathcal{N} = 8$ black hole charges to the tripartite entanglement of seven qubits, generalising the $STU/3$-qubit case, is determined precisely by the imaginary octonions. We also determine the dictionary in the Freudenthal picture.

In the course of this work we establish just how the STU model sits inside the Freudenthal triple system of $\mathcal{N} = 8$ supergravity, which quite naturally leads us on, through the STU/3-qubit correspondence, to the topic of chapter 8: the FTS classification of 3-qubit entanglement. In section 1 we identify the FTS, and its underlying cubic Jordan algebra, that describes the system of three qubits. The automorphism group of the FTS is the SLOCC-equivalence group and the FTS ranks automatically distinguish the entanglement class. This has an advantage over the conventional classification [44] in that it is manifestly SLOCC-covariant, which lays the ground work for possible n-qubit generalisations. Indeed, in section 2 we make some preliminary attempts to generalise the 3-qubit FTS to an n-qubit FTS with partial success. While we establish the correct mathematical framework, we do not quite get as far as a full 4-qubit entanglement classification, this is a work in progress. We do, however, obtain a subset of the desired results, using the more conventional covariant approach, which we hope to use as a signpost along the way.

The closing chapter 9 on integral structures, while still using our favorite algebras, is to some degree a departure from the preceding work. We use the integral Jordan algebras and FTS, which are defined over the integral split-octonions, to study the quantized black holes and strings of $\mathcal{N} = 8$ supergravity in six, five and four dimensions. In section 2 we study the discrete U-duality orbits, which carry a rich mathematical structure. These results are interesting in themselves, but, more importantly, provide important tools with which to study the newly defined Freudenthal duality, the subject of section 3. This new transformation relates pairs of black holes with matching lowest order entropy. However, only certain black holes admit a consistent F-dual since the preservation of the charge lattice requires the FTS quartic norm to be a perfect square. The issue of higher-order corrections to the entropy remains open as some, but not all, of the discrete U-duality invariants are Freudenthal invariant.

Finally, in chapter 10 we list some unsolved problems and directions for future research.

Part I.

BLACK HOLES AND QUANTUM INFORMATION

CHAPTER 2

Quantum information and entanglement

Quantum mechanics is a strange beast. Before contemplating why exactly, we should perhaps first consider what it actually means for a theory of physical reality to be "strange". Notions of reality have been borne over the ages through our dialectic relationship with the material world, first as common place experience and then later as its scientifically precise articulation, experiment. Historically, these experiences have predominantly been confined to a rather limited range of scales. That is, our physical intuition has grown up in the context of what is to be understood as only an effective description of reality, valid on scales relatively close to those probed by evolutionary beings such as ourselves. Of course, the scientific endeavor is relentlessly advancing these boundaries, from Hook and the Flee to Hubble and the Universe, and with each frontier of scale crossed our picture of reality has altered. Nevertheless, the way in which we reason about these developments is dominated by the weight of history. With this in mind we may conclude that there really is no such thing as a "strange" theory. For why should we expect the underlying reality to adhere to our sense of normality which is grounded in what is, after all, only an effective description. We ought only require our more fundamental theories consistently reproduce our familiar experience in appropriate limits. It would be prejudice, indeed strange, to expect our "classical" understanding of reality to persist on all scales. For example, while the special theory of relativity is at first sight certainly counter-intuitive, it is actually only strange in that we must relinquish a reasoning established on the basis of a Newtonian understanding of space and time. Re-calibrating our basic intuition can be a difficult task, as Kelvin attests [74]:

> "I am never content until I have constructed a mechanical model of the object that I am studying. If I succeed in making one, I understand; otherwise I do not. Hence I cannot grasp the electromagnetic theory of light." - Kelvin

This is not to say that electromagnetism is wrong or even strange, only that it is not easily visualized in terms of material objects on the human scale.

However, quantum theory is next level weird. It is *genuinely* strange, previous comments notwithstanding. It challenges the very notion of what we mean by "reality"[1].

[1]One might argue that what we conventionally mean by "reality" has also grown up in the context of our essentially classical understanding of physics implying quantum theory is also only first order strange.

Of course, what is meant by reality is, and has always been, a philosophically perilous line of inquiry. Then there is the secondary question of how any particular notion of reality is to be put into practice, i.e. as a physical theory. In the context of classical physics, however, we may make a fairly uncontentious first approximation. The fundamental substance (reality) of objects in classical physics is constituted by their observable properties; "things" in themselves are defined as the bearer of said properties, which are in turn determinable from the object itself [75,76]. Essential to such a prescription of reality is the crucial assumption that all physical attributes do indeed possess definite, if not definitely known, values at all times [75–78]. These ideas form the crux of what may be called a realist philosophy of nature. This realist stance is naturally embodied by the mathematical (set-theoretic) formalism underpinning classical theory. Moreover, it implies that propositions are handled with the "classical" Boolean logic [76,78].

A further requirement that one might demand of a sensible description of reality is *locality* (or perhaps more precisely in this context *separability*). Indeed, this concept was very dear to Einstein's heart, as he describes to Max Born [79],

> "I just want to explain what I mean when I say that we should try to hold on to physical reality. We all of us have some idea of what the basic axioms of physics will turn out to be... whatever we regard as existing (real) should somehow be localised in time and space. That is, the real in part of space A should (in theory) somehow 'exist' independently of what is thought of as real in space B. When a system in physics extends over the parts of space A *and* B, then that which exists in B should somehow exist independently of that which exists in A. That which really exists in B should therefore not depend on what kind of measurement is carried out in part of space A; it should also be independent of whether or not any measurement at all is carried out in space A." - Einstein

The union of these ideas is typically referred to as *local realism*. The pinnacle of classical physics, Einstein's general theory of relativity, is a complete local realist theory. The orthodox Copenhagen interpretation of quantum physics rejects local realism in all its parts.

As for realism, it is not possible to speak of a physical system as actually "possessing" values for all their physical observables at a given time, and that these values are intrinsic and independent of any measurement setup used to reveal them. This is not merely a philosophical whim, it is a mathematical consequence of the Kochen-Specker theorem [80]. See [76,78] for introductory discussions. Rather, one is restricted to making probabilistic counter-factual statements, in which the act of measurement plays a fundamental role, "if measured, an observable A will have a value a with a probabilty p".

We also lose our grip on locality due to the quintessentially quantum phenomenon of *entanglement*, the main subject of this introduction to quantum information. Quantum entanglement is a phenomenon in which the quantum states of two or more objects must be described with reference to each other, even though the individual objects may be spatially separated [81–86]. This leads to classically unexplainable, but experimentally observable, quantum correlations between the spatially separated systems. A "spooky" action at a distance, as Einstein called it.

Given the startling nature of these two statements it comes as no surprise that over the years many a physicist and philosopher has taken exception, not least of all Einstein. This led to the now famous Bohr-Einstein dialog, with Einstein fighting the corner of local realism and Bohr that of the Copenhagen interpretation. Its culmination was the seminal 1935 work by Einstein, Podolsky, and Rosen (EPR) [81]. They correctly concluded that, assuming local realism, the quantum mechanical wave function cannot be a complete description of reality. They speculated on the existence of a more fundamental underlying (classical) theory that towed the line of local realism. Subsequent attempts to develop such a hypothetical theory typically assumed that the mathematical representation of physical states, the wave function, is completed by a set of "hidden variables". The previously *fundamental* probabilistic nature of quantum mechanics would then admit a pedestrian classical ignorance interpretation. However, through an ingenious development of the Aharonov-Bohm [87] spin based version of the EPR setup, Bell demonstrated that there are no local realist hidden variable models that can correctly reproduce the predictions of quantum mechanics [83]. The quantum mechanical predictions where experimentally verified in 1982 by Alain Aspect *et al* [88]. We will return to this remarkable subject in section 1.2 as it is essential to the proper understanding of entanglement. However, to this day many, scientists feel uncomfortable, for important reasons such as the measurement problem and the trouble with quantum cosmology, with the operationalist interpretation of quantum mechanics and the debate rages on.

Rather than stewing in such potentially solipsistic meditations, let us turn all this existential angst on its head, take quantum theory at face value, and ask not what we can do for quantum theory, but what quantum theory can do for us? Of course, first and foremost, is quantum theory's tremendous predictive power, in particular the unprecedented successes of quantum field theory. This is, after all, the single most compelling reason to accept the peculiarities of quantum theory in the first place. However, the humble domain of finite, non-relativistic quantum mechanics also has a lot to offer. What can we learn from and how can we use the elementary properties of quantum theory, such as the superposition principle and entanglement. This is the standpoint of many quantum information scientists, to embrace the basic mathematical formalism in an effort not only to better understand quantum theory, but also to go beyond what can be achieved in the field of classical information theory. In the following section we will introduce the basic ideas of quantum information theory with a particular emphasis on the central role of entanglement. There are a number of good introductions to the topics of QI and entanglement. We found [89–95] very useful and have used them throughout the reminder of this and the subsequent chapter.

1. A brief introduction to quantum information

What is quantum information theory? First, there is the subject of quantum information and computation as the study of information processing systems which rely on the fundamental properties of quantum mechanics. From this perspective a central motivation is to challenge the strong form of the Church-Turing thesis:

Any algorithmic process can be simulated efficiently using a probabilistic Turing machine.

That is, there is an expectation that quantum mechanics could be used to perform computational tasks beyond the capability of any, even idealistic, purely classical device[2] as embodied by the universal Turing machine. There is a certain poetic element to this idea; just as the conventional microchip will meet its fundamental limit, set by the appearance of quantum noise at the atomic scale, the very same quantum phenomena can be used to develop new, superior, modes of computation.

[2]Classical not in the sense that it does not utilize any quantum physics, such as semiconductor components, but that its algorithmic structures are classical, and, as such could be performed by any classical device whether it be made of the most advanced microchips or just sticks and twigs, geologically slow as it might be in that case.

However, QI is not purely, or even primarily, a technological enterprise. There is a great expectation, which has already proved itself well-founded, that by adopting an information theoretic stance we are able to develop our understanding of quantum theory at a fundamental level, to clarify and sharpen our intuition about the features of quantum mechanics which may otherwise seem incomprehensible. This may take the form of developing the tools to better understand standard predictions or, to scrutinize, and potentially alter, the very foundations of the subject.

A key player in these developments has been entanglement, which forms the focus of this introduction in anticipation of its surprising connection to stringy black holes, the subject of chapter 4.

Entanglement is both one of the most startling and exploitable characteristics of quantum mechanics and, as such, it is central to our fundamental understanding and as a quantum computing tool.

Entanglement may be thought of as a quantum information *resource* in the same sense as entropy or energy are classical resources. However, its properties profoundly differ from the properties of those familiar concepts. We have, at best, an incomplete description. In order to fully understand entanglement we would like to be able to precisely describe its creation and transformation, to classify the distinct types of entanglement, to quantitatively measure it, to utilize it as a resource and to illustrate, precisely, how it differs from classical resources at a fundamental level.

1.1. Qubits

Quantum information theory derives its computational efficiency from the quantum mechanical generalisation of digital information coding. Digital information is conventionally represented by a bit string of "0"s and "1"s. The physical realisation of bits takes many forms. In real world technologies current, voltage or light pulses are predominately used. For example, a "0" is represented by no light pulse while "1" is represented by a short pulse.

One could imagine the length of the pulse being gradually reduced to a single photon - a genuine quantum mechanical object. In this case we could represent bits not by whether a photon was sent or not, but instead by its polarisation: horizontal for "0" and vertical for "1". However, unlike the light pulse, which was either there or not, a single photon can exist in a superposition of polarisation states. This is the idea of the quantum bit, or *qubit*; a quantum superposition of the binary digits "0" and "1". The particular physical realisation (there are many: photon polarisations, quantum dots, trapped ions, mode splitters, to name but a few) of the qubit is not important, any two state quantum system will do. Hence, qubits are simply denoted abstractly as elements of the 2-dimensional Hilbert space $\mathbb{C}^2$ equipped with the conventional norm, where the two basis states are labelled $|0\rangle$ and $|1\rangle$. An n-qubit bit string lives in the n-fold tensor product of $\mathbb{C}^2$.

One qubit: The 1-qubit system (Alice) is described by the state $|\Psi\rangle \in \mathbb{C}^2$,

$$|\Psi\rangle = a_A|A\rangle, \qquad \text{where} \qquad A = 0, 1 \tag{2.1}$$

so

$$|\Psi\rangle = a_0|0\rangle + a_1|1\rangle. \tag{2.2}$$

The density matrix ρ, defined by

$$\rho = |\Psi\rangle\langle\Psi|, \qquad \rho_{A_1A_2} = a_{A_1}a^*_{A_2}, \tag{2.3}$$

obeys

$$\operatorname{tr}\rho = \langle\Psi|\Psi\rangle. \tag{2.4}$$

Two qubits: The 2-qubit system (Alice and Bob) is described by the state $|\Psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$,

$$|\Psi\rangle = a_{AB}|AB\rangle, \qquad \text{where} \qquad A, B = 0, 1 \tag{2.5}$$

so

$$|\Psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle. \tag{2.6}$$

We defined the partially reduced density matrices

$$\begin{aligned} \rho_A &= \mathrm{Tr}_B|\Psi\rangle\langle\Psi|, \\ \rho_B &= \mathrm{Tr}_A|\Psi\rangle\langle\Psi|, \end{aligned} \tag{2.7}$$

or

$$\begin{aligned} (\rho_A)_{A_1A_2} &= \delta^{B_1B_2} a_{A_1B_1} a^*_{A_2B_2} = (\rho_A)_{A_2A_1}, \\ (\rho_B)_{B_1B_2} &= \delta^{A_1A_2} a_{A_1B_1} a^*_{A_2B_2} = (\rho_B)_{B_2B_1}. \end{aligned} \tag{2.8}$$

Explicitly,

$$\begin{aligned} \rho_A &= \begin{pmatrix} |a_0|^2 + |a_1|^2 & a_0a_2^* + a_1a_3^* \\ a_2a_0^* + a_3a_1^* & |a_2|^2 + |a_3|^2 \end{pmatrix}, \\ \rho_B &= \begin{pmatrix} |a_0|^2 + |a_2|^2 & a_0a_1^* + a_2a_3^* \\ a_1a_0^* + a_3a_2^* & |a_1|^2 + |a_3|^2 \end{pmatrix}. \end{aligned} \tag{2.9}$$

Note,

$$\operatorname{tr}\rho_A = \operatorname{tr}\rho_A = \langle\Psi|\Psi\rangle. \tag{2.10}$$

n **qubits:** The n-qubit system (Alice$_1, \ldots,$ Alice$_n$) is described by the state $|\Psi\rangle \in \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$,

$$|\Psi\rangle = a_{A_1\ldots A_n}|A_1\ldots A_n\rangle, \qquad \text{where} \qquad A_1, \ldots, A_n = 0, 1 \tag{2.11}$$

so

$$|\Psi\rangle = a_{00\ldots00}|00\ldots00\rangle + a_{00\ldots01}|00\ldots01\rangle + \ldots a_{11\ldots10}|11\ldots10\rangle + a_{11\ldots11}|11\ldots11\rangle. \tag{2.12}$$

For any subset of qubits $X = \{A_i, A_j \ldots\} \subset \{A_1, \ldots, A_n\}$ we define the partially reduced density matrices

$$\begin{aligned} \rho_X &= \mathrm{Tr}_{\overline{X}}|\Psi\rangle\langle\Psi|, \\ \rho_{\overline{X}} &= \mathrm{Tr}_X|\Psi\rangle\langle\Psi|, \end{aligned} \tag{2.13}$$

where $\overline{X}$ denotes the set theoretic complement. Note,

$$\mathrm{Tr}_X\,\rho_{\overline{X}} = \mathrm{Tr}_{\overline{X}}\,\rho_X = \langle\Psi|\Psi\rangle. \tag{2.14}$$

1.2. Entanglement and the Bell inequality

Already in 1935 both EPR and Schrödinger had identified the phenomenon of quantum entanglement as central to quantum theory and were, at the very least, uncomfortable with its implications. While EPR had mathematically challenged the completeness of quantum theory on the basis of entanglement, it remained a conceptual matter, apparently not accessible to experiment. All this changed in 1965 when Bell introduced his famous inequality. In one fell swoop, entanglement had been elevated from a conceptual puzzle to an experimental observable confronting the very assumptions of local realism. Moreover, Bell had laid the foundations of entanglement as a quantum information theoretic resource. For example, in 1991 Ekert [96] used the Bell inequality as the basis for a new secure means of communication.

Essentially, if the state of a composite system cannot be written as a tensor product of its constituent subsystems then it is entangled. As a consequence, measurements on one component can affect the measurement results on another component, even if they are space-like separated. Consider, for example, the 2-qubit Bell state,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2.15}$$

If Alice measures 0, Bob must also measure 0 and if Alice measures 1, Bob must also measure 1. Moreover, on tracing out either Alice or Bob we are left with a totally mixed state - while having maximal knowledge of the composite system we have minimal knowledge of it constituent pieces.

However, to understand properly the fundamental implications of entanglement we must first establish more precisely the philosophical stance of the quantum skeptic. Recall, he is a principled local realist. However, with good reason we have been rather vague about this concept until now. Like EPR we make no attempt to give a comprehensive definition, but rather adopt their set of minimal conditions (which are phrased in the context of a 2-particle experiment but generalise in an obvious way) [81]:

1. *Reality*: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

2. *Locality*: "Since at the time of measurement the two systems no longer interact, no real change can take place in the second system in consequence of anything that may be done to the first system"

3. *Completeness*: "Every element of the Physical Reality must have a counterpart in the physical theory"

However, while a staunch realist, our quantum skeptic does not doubt the experimental results and, moreover, is in full agreement that they are correctly predicted by the quantum formalism. Where he diverges from the quantum converts is that he maintains there is, in fact, a deeper theory which will give the correct experimental prognosis while adhering to the above principles of local realism. This theory would have well defined single measurement predictions; the probabilistic counterfactual statements, "*if measured* such and such observable will have this or that value *with probability* so and so", of quantum theory are simply an artifact of its incompleteness. The complete theory would depend on some additional physical elements currently out of reach. Perhaps our theoretical framework is deficient or our current experimental techniques inadequate. Whatever the case, these “hidden variables” would complete our theory providing a local and realistic description of nature.

However, as we shall see, quantum theory shows its doubters no mercy. The Bell inequalities [83] were the final nail in the coffin of local realism. The quantum mechanical violation of the Bell inequalities was verified in 1982 by the now famous *Orsay experiment* [88]. Here is the basic setup (the full experiment will be described below). An entangled photon pair is produced by some suitable atomic process. The emitted photons travel in opposite directions which define our z-axis. Eventually they each meet an analyser, denoted A and B, measuring their polarisations in perpendicular x- and y-directions. See Figure 2.1.

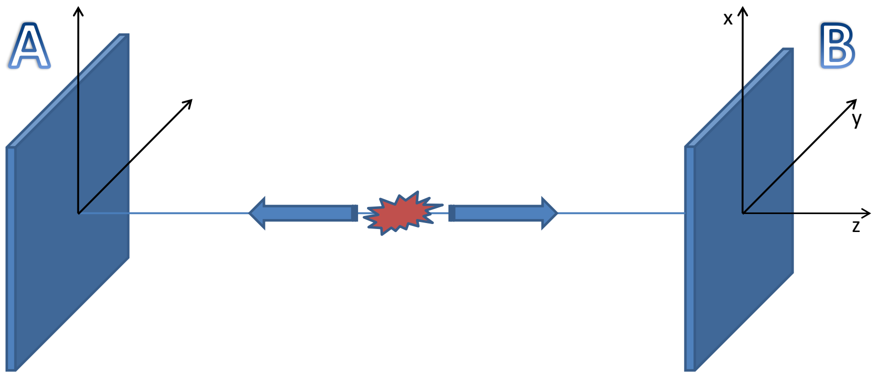


Figure 2.1.: Entangled photon pair are measured by the horizontal/vertical polarisation analysers, A and B.

Experimentally one finds that only the pairs (h, h) and (v, v) are observed, where h and v denote horizontal and vertical polarisations respectively. Quantum mechanically this 2-photon system is represented by the Bell state,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|hh\rangle + |vv\rangle). \tag{2.16}$$

So, the photon pair is totally correlated. EPR used these perfect correlations and the assumptions of *Reality*, *Locality* and *Completeness* to argue as follows: since we can predict with certainty the result at A by merely noting the result at B, by *Locality* and *Reality* there is an element of reality corresponding to the polarisation of the photon at A. This is true regardless of which specific $x-y$ axises we choose to perform the experiment with. Hence, there is an element of reality corresponding to the polarisation of the photon at A in all possible $x - y$ orientations. However, there is no photon quantum state for which its polarisation in all directions has a definite value. Hence, by the *Completeness* criterion, quantum mechanics is incomplete.

Are such definite correlations enough to convince our quantum skeptic? Not at all, the EPR argument has demonstrated that quantum mechanics cannot be considered complete under the assumptions of local realism - it says nothing about the possible existence or not of a complete local realist theory. We must work a little harder before we declare experimental victory over classical reality. This was Bell's great insight - to derive from EPR's criteria something which could be used to experimentally check the phenomenological viability of local realism.

Bell had to consider two points. First, the experimental setup had to be tuned to best reveal its true quantum nature. Second, he had to formulate a sufficiently generic hidden variable model to be tested against the subsequent results.

As for the experiment we will use the Clauser-Horne-Shimony-Holt (CHSH) [97] variant of Bell's original arrangement, as this is the setup that was experimentally reproduced in [88]. The analysers are now allowed to be individually rotated about the z-axis into four possible orientations differing in steps of a rotation angle θ. See Figure 2.2. The experiment is performed with four

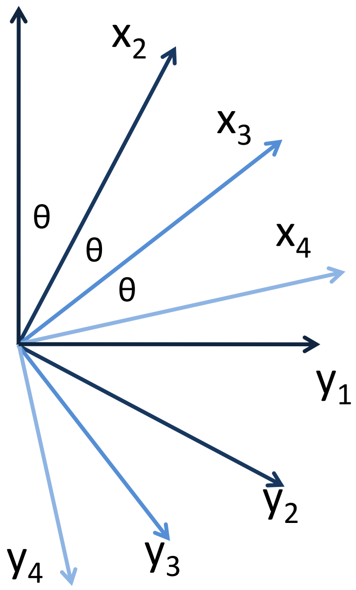


Figure 2.2.: Each polarisation analyser may be rotated about the z-axis to measure the polarisation along one of four sets of x-y axises differing by steps of θ.

out of the sixteen possible configurations, denoted $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_1, x_4\}$, where the first (second) slot specifies the orientation of analyser A (B). Each of the four configurations has four possible measurement outcomes. For example, the possible measurement pairs for $\{x_2, x_3\}$ are $(h_2, h_3), (h_2, v_3), (v_2, h_3), (v_2, v_3)$. We define the functions σ_A^i such that

$$\sigma_A^i = \begin{cases} +1 & \text{for measurement outcome} \quad h_i \\ -1 & \text{for measurement outcome} \quad v_i \end{cases} \tag{2.17}$$

where h_i and v_i are the measurement outcomes at analyser A for orientation $i \in \{1, \ldots, 4\}$. The functions σ_B^i is defined similarly for analyser B. The experiment is run for many 2-photon pairs and the results are used to calculate the correlation coefficient,

$$S(\theta) := E(3,2) + E(3,4) + E(1,2) - E(1,4), \tag{2.18}$$

where

$$E(i,j) := \overline{\sigma_A^i \sigma_B^j}. \tag{2.19}$$

It is experimentally observed that the maximum value is $S_{\max}(\theta_0) = 2.697 \pm 0.015$, where $\theta_0 = 22.5°$ [88]. Note, thus far we have made no reference to any particular physical theory, we have only been dealing with experimental data.

There are now two questions: 1) is quantum theory consistent with these results? 2) is there a local hidden variables model consistent with these results? We are assuming from the outset that quantum mechanics correctly predicts the experimental data and a standard quantum mechanical calculation shows this is indeed the case [88]. So, let us now address the question of hidden variables.

As tradition dictates our hidden variables will be denoted λ. The full state of our hypothetical complete theory might be written $|\Psi, \lambda\rangle$. The particular form of λ is not of importance. Our outcome functions $\sigma_A^i(\lambda)$ and $\sigma_B^j(\lambda)$ now depend on the hidden variables. The essential assumption of locality is that the result $\sigma_A^i(\lambda)$ at analyser A does not depend on the setting j at analyser B and vice-versa. The characteristics of the photon source are captured by a probability density $\rho(\lambda)$. A 2-photon state with parameter $\lambda \in [\lambda, \lambda + \mathrm{d}\lambda]$ is produced with probability $\rho(\lambda)\mathrm{d}\lambda$. As Bell noted, this setup subsumes the case in which σ_A and σ_B depend on two distinct sets of hidden variables.

The expectation value of a given configuration $\{x_i, x_j\}$ is given by,

$$E(i,j) = \int \sigma_A^i(\lambda)\sigma_B^j(\lambda)\rho(\lambda)\mathrm{d}\lambda. \tag{2.20}$$

Hence, the hidden variable correlation coefficient is given by,

$$S_{\mathrm{HV}}(\theta) = \int [\sigma_A^3(\lambda)\sigma_B^2(\lambda) + \sigma_A^3(\lambda)\sigma_B^4(\lambda) + \sigma_A^1(\lambda)\sigma_B^2(\lambda) - \sigma_A^1(\lambda)\sigma_B^4(\lambda)]\rho(\lambda)\mathrm{d}\lambda. \tag{2.21}$$

Since $\int \rho(\lambda)\mathrm{d}\lambda = 1$ and $\sigma = \pm 1$ it is clear that,

$$S_{\mathrm{HV}}(\theta) \leq 2. \tag{2.22}$$

This is the famous (generalised) Bell inequality [97]. The correlation coefficient for a local realistic hidden variables model cannot exceed 2. The Orsay experiment obtained a maximum of 2.697 ± 0.015.

Reality is dead. Long live Reality!

1.3. Entanglement dependent quantum information

This experiment and its subsequent refinements have established entanglement as a now accepted feature of nature. We claimed that the quantum information theorists regards entanglement as one of his key resources. Let us spend a moment justifying this claim. The literature on this subject is vast and we can only give the briefest of glimpses.

1.3.1. Sheep yes, qubits no

A single quantum cannot be cloned [98]. What would happen were quantum copying possible? Wootters and Zurek argue as follows [98]. If it were possible to copy a quantum state then one could ascertain the precise state of the original. This would facilitate super-luminal communication. Consider a Bell pair consisting of two photons. Once one photon has been measured the other will be in an eigenstate of the same polarisation. If the second photon could be copied and, hence, have its state precisely determined, the measurement basis used at the first photon would be known, allowing for a faster-than-light transmission of information encoded in the choice of measurement basis.

The logic negating this possibility is surprisingly simple and essentially relies only on the linearity of quantum mechanics. Suppose we do have a quantum copying device. We denote the copier's basic pre-copy state by $|A_\bullet\rangle$. We use use it to replicate a qubit state $|\Psi\rangle$. Mathematically,

$$|A_\bullet\rangle \otimes |\Psi\rangle \otimes |\Phi\rangle \to |A_\Psi\rangle \otimes |\Psi\rangle \otimes |\Psi\rangle, \tag{2.23}$$

where the arrow denotes some unitary evolution and $|A_\Psi\rangle$ is the post-copying state of the device, which may or may not actually depend on $|\Psi\rangle$. Now consider the case where $|\Psi\rangle = |0\rangle$,

$$|A_\bullet\rangle \otimes |0\rangle \otimes |\Phi\rangle \to |A_0\rangle \otimes |0\rangle \otimes |0\rangle. \tag{2.24}$$

However, for a state $|\Psi\rangle = a_0|0\rangle + a_1|1\rangle$ we find, by linearity,

$$|A_\bullet\rangle \otimes (a_0|0\rangle + a_1|1\rangle) \otimes |\Phi\rangle \to a_0|A_0\rangle \otimes |0\rangle \otimes |0\rangle + a_1|A_1\rangle \otimes |1\rangle \otimes |1\rangle. \tag{2.25}$$

If the device states $|A_0\rangle, |A_1\rangle$ differ then the qubits will emerge in a mixed state, if $|A_0\rangle, |A_1\rangle$ are the same then the qubits will emerge as a Bell-pair. In either case we do not obtain the desired copy state,

$$(a_0|0\rangle + a_1|1\rangle) \otimes (a_0|0\rangle + a_1|1\rangle). \tag{2.26}$$

1.3.2. Quantum cryptography

The art of cryptography as a long history, from protecting Mesopotamian bakery recipes to modern internet security. The central ingredient in almost all contemporary cryptographic schemes is the *key*. To send a message, the plain text and a key are fed as inputs to the encryption algorithm. The receiver then feeds the encrypted message and his key into the decrypting algorithm. The cryptogram and encryption/decryption algorithms may be known by anyone. The security depends entirely on the secrecy of the keys. Historically, a *symmetric-key* algorithm has been the principal method employed. In this case there is essentially one key used to both encrypt and decrypt. Alice invents a key and secretly shares it with her intended correspondent, Bob. When the day comes Alice uses her key to encode her message and sends it to Bob who is then able to decrypt it using the same key. The tricky part is the sharing of the key. Back in the day, Alice and Bod could exchange keys face-to-face in some clandestine moonlit clearing, deep in the local forest. However, despite their best efforts, Amazon.co.uk have not been able to make this particular key distribution protocol workable in todays consumer driven society.

The problem of symmetric-key distribution was circumvented in the mid 70's with the invention of *asymmetric-key* cryptography. In this case Alice creates two keys. One *public* key which she shares with the world, one *private* key which she keeps entirely to herself. Now anyone can send Alice a message using the public key, which only she can decode using the private key. Obviously, the public and private keys must be mathematically related. The crucial point is that the private key cannot *feasibly* be deduced from the public key. A much used example is the Rivest, Shamir and Adleman (RSA) scheme which relies on the fact that, while multiplication is easy, factorisation is not

- there is no known efficient classical factorisation algorithm. There is, however, an efficient quantum algorithm [99]. The practical realisation of a quantum computer would mean abandoning RSA.

There is a third way. The symmetric-key protocol would be completely secure if both Alice and Bob could check whether their shared secret key had been intercepted during transmission. Any intercepted keys would be discarded. Alice and Bob would only proceed when they are both sure their key is truly secret. This is the approach taken by quantum cryptography. Any interception method must involve some measurement procedure performed by the eavesdropper. Classically, this can, in principle, be done without disturbing the message. Quantum mechanically, this is not so, which suggests the possibility of a totally secure quantum key distribution method. The first entanglement based quantum cryptography scheme was introduced by Ekert [96]. There is a source of Bell pairs from which one qubit in each pair is sent to Alice and the other to Bob. So, Alice and Bob receive a string of qubits which they measure in some basis independently chosen at random from a set of previously agreed upon possibilities. Once complete, they publicly share their choice of basis for each measurement. For those instances in which they happen to choose the same basis their results will be perfectly correlated and can be used to form a bit string, which is then used as the private key. The remaining results can be used to check the Bell inequalities, which, only if no-one has tried to measure the qubits, will be violated. If the Bell inequalities are not violated they can be sure that someone was listening. This scheme was experimentally implemented in 1999 [100,101].

1.3.3. Quantum teleportation

Can Alice send an unknown qubit state to Bob? No-cloning implies that this is not possible using only classical channels. However, Bennett *et al.* showed that a qubit can be *teleported* using only two bits of classical information together with the correlations of a Bell-pair [102]. The use of "teleportation" emphasizes the fact that, once Bob has received the qubit, Alice is left with no trace of the teleported state.

The teleportation protocol proceeds as follows. Alice has a qubit in an unknown state $|\Psi\rangle$ she wishes to send to Bob. Alice and Bob share a Bell-pair. The total composite system is represented by the state,

$$|\Psi\rangle = |\Psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{2.27}$$

Note, the first two slots in the tensor product (the unknown state and one half of the Bell-pair) are owned by Alice, the third slot is Bob's half of the Bell-pair. This state may be rewritten using the Bell basis,

$$|\Psi\rangle = \frac{1}{2}(|\Phi^+\rangle|\Psi\rangle + |\Phi^-\rangle\sigma_3|\Psi\rangle + |\Psi^+\rangle\sigma_1|\Psi\rangle + |\Psi^-\rangle i\sigma_2|\Psi\rangle), \tag{2.28}$$

where,

$$\begin{aligned} \Psi^\pm &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \\ \Phi^\pm &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \end{aligned} \tag{2.29}$$

and σ_i are the Pauli matrices. Alice then simply measures her pair of qubits in the Bell basis and communicates the result to Bob, which requires two classical bits of information. Finally, Bob performs a simple unitary rotation according to Alice's result and is left with precisely $|\Psi\rangle$ as required.

This procedure, or variations of it, is now key to many quantum communication protocols [93]. The first actual quantum teleporation was perform in a pioneering experiment by Bouwmeester *et al.* in 1997 [103].

2. Entanglement classification

It ought to be clear by now that entanglement has striking foundational implications as well as important technological applications. Consequently, a proper understanding of entanglement is central to quantum information science as well as quantum theory in general. How can we manipulate entanglement? Are there different forms of entanglement? If so how are they distinguished and how do we quantify them?

2.1. Bell inequalities without the inequality

Let us now address the first question: are there qualitatively different forms of entanglement?

It is argued here that this is indeed the case. The crucial observation is that by generalising the EPR setup to a *tripartite* system the principles of local realism are unequivocally shattered by a *single* run of the (thought) experiment. This remarkable experiment was introduced by Greenberger, Horne and Zeilinger (GHZ) [104]. The very special tripartite state upon which their argument relies has come to be known as the GHZ-state. The quantum nature of entanglement is so dramatically demonstrated that no appeal to inequalities is necessary. This seems to us sufficient justification for the claim that the GHZ-state possesses a qualitatively greater degree of entanglement compared to the Bell-state[3].

Actually, as emphasized by GHZ [104] this is the first instance that directly confronts the particular case when predictions can be made with certainty, as described in EPR's *Reality* condition. The requirement of certainty is not covered by Bell's analysis which is necessarily statistical. Only when both analyzers are aligned does the result on one photon fix with certainty the result on the other. For this particular sub-arrangement one is able to construct a local realist theory which correctly reproduces the quantum mechanical results [83]. Can one always find classical models for these subcases of experimental certainty? In a sense this is the most physically intriguing question. Cases in which one can with certainty predict the value of a physical property one might expect, more so than in any other case, that there really is an actual physical object possessing that specific property with that precise value. The one run certitude of the GHZ elaboration does away with any such possibility.

GHZ originally considered a four-particle system [104], while the experimental test was performed with three entangled photons [106]. Here, we present a simplified version involving three spin-1/2 particles introduced by Mermin [85]. Imagine the three spin-1/2 particles leaving a common source in the entangled state,

$$|\Psi\rangle_{\text{GHZ}} = \frac{1}{\sqrt{2}}(|\uparrow_z\uparrow_z\uparrow_z\rangle + |\downarrow_z\downarrow_z\downarrow_z\rangle), \tag{2.30}$$

where, $\uparrow_z$ and $\downarrow_z$ denote spin-up or -down along the z-axis as defined by the flight of the particles.

This is the famous GHZ-state. Each particle eventually meets a Stern-Gerlach device, denoted A, B, C, which may be individually orientated to measure spin along the x- or y-axis. See Figure 2.3. The experiment is performed with the measuring devices randomly set in one of four configurations (out of the eight possible). These are the three configurations where two devices measure along the

[3]For an interesting discussion on the subtleties of this matter see [105]

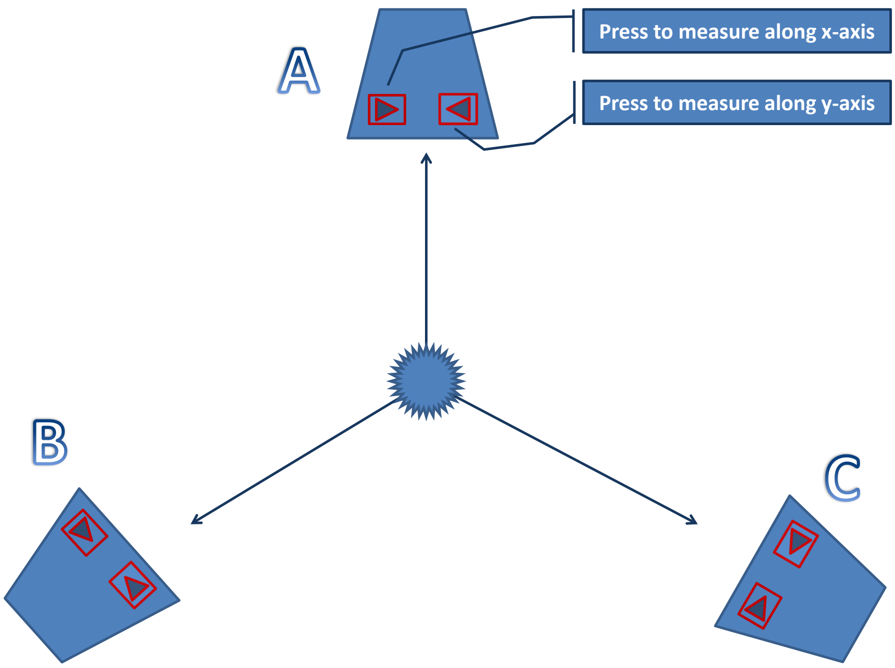


Figure 2.3.: Three Stern-Gerlach device, denoted A, B, C, which can be individually set to measure spin along their respective x- or y-axises.

y-axis and one along the x-axis, denoted xyy, yxy, yyx, and the single configuration where all three devices measure along the x-axis, denoted xxx.

Let us consider the case with device A set to measure along the x-axis and devices B, C along the y-axis. Using

$$|\uparrow_z\rangle = \frac{1}{\sqrt{2}}(|\uparrow_x\rangle + |\downarrow_x\rangle), \quad |\downarrow_z\rangle = -\frac{1}{\sqrt{2}}(|\uparrow_x\rangle - |\downarrow_x\rangle) \tag{2.31}$$

and

$$|\uparrow_z\rangle = \frac{1}{\sqrt{2}}(|\uparrow_y\rangle + |\downarrow_y\rangle), \quad |\downarrow_z\rangle = -\frac{i}{\sqrt{2}}(|\uparrow_y\rangle - |\downarrow_y\rangle) \tag{2.32}$$

we can write (2.30) in the measurement eigenbasis,

$$|\Psi\rangle_{\text{GHZ}} = \frac{1}{2}(|\uparrow_x\uparrow_y\uparrow_y\rangle + |\downarrow_x\downarrow_y\uparrow_y\rangle + |\downarrow_x\uparrow_y\downarrow_y\rangle + |\uparrow_x\downarrow_y\downarrow_y\rangle), \tag{2.33}$$

from which it is clear that, with the detectors in the xyy orientation, only an odd number of particles will be measured as spin-up. By symmetry, this is also true for both the yxy and yyx orientations. On the other hand, for the final configuration with all three devices set to measure along the x-axis, (2.30) in the measurement eigenbasis is given by,

$$|\Psi\rangle_{\text{GHZ}} = \frac{1}{2}(|\downarrow_x\downarrow_x\downarrow_x\rangle + |\downarrow_x\uparrow_x\uparrow_x\rangle + |\uparrow_x\downarrow_x\uparrow_x\rangle + |\uparrow_x\uparrow_x\downarrow_x\rangle), \tag{2.34}$$

so that only an even number of particles are found to be spin-up. Note, in all cases we can with certainty predict the measurement result of any one particle by simply noting the outcomes of the other two particles.

The question now, as before, is not whether quantum mechanics gets it right (we are assuming this from the outset), but whether there is a local realist theory that can reproduce these (hypothetical) experimental observations.

Mermin [85] gave a particularly clear reasoning as to why there is no such theory. Following EPR Mermin made the crucial inference that since there are no connections between the detectors, the highly coordinated experimental results must derive from the common source of the particles. The information determining the spin observed at A, which must be consistent with the spins observed at B and C, must be carried by the particle triggering A. This argument can be applied equally to any of the three particles. Moreover, since the particles have no knowledge of which direction the detectors are set to measure and the setting can change during flight, it would seem essential that each particle carries information specifying what spin will be measured for either of the detector settings.

In summary, we ought to be able to label the particles with the spin that will be observed given the detector is set to measure along the x- or y-axis. For example, the particle triggering detector A could be labeled $(\uparrow_x, \downarrow_y)_A$ where the first (second) slot denotes the spin that will be measured with the detector set to measure along the x-axis (y-axis). For the detector orientation xyy a possible labeling of the three particles is given by,

$$(\uparrow_x, \downarrow_y)_A(\uparrow_x, \uparrow_y)_B(\uparrow_x, \uparrow_y)_C \tag{2.35}$$

as all three detectors would register spin-up consistent with the experimental observation that only an odd number of spin-up particles are observed for xyy. However, this labeling is inconsistent with the yxy set-up as we would observe two spin-up results. A little careful thought shows that there are only eight labelings simultaneously consistent with xyy, yxy and yyx,

$$\begin{aligned}
&(\uparrow_x, \uparrow_y)_A(\uparrow_x, \uparrow_y)_B(\uparrow_x, \uparrow_y)_C\\
&(\uparrow_x, \uparrow_y)_A(\downarrow_x, \downarrow_y)_B(\downarrow_x, \downarrow_y)_C\\
&(\downarrow_x, \downarrow_y)_A(\uparrow_x, \uparrow_y)_B(\downarrow_x, \downarrow_y)_C\\
&(\downarrow_x, \downarrow_y)_A(\downarrow_x, \downarrow_y)_B(\uparrow_x, \uparrow_y)_C\\
&(\uparrow_x, \downarrow_y)_A(\downarrow_x, \uparrow_y)_B(\downarrow_x, \uparrow_y)_C\\
&(\downarrow_x, \uparrow_y)_A(\uparrow_x, \downarrow_y)_B(\downarrow_x, \uparrow_y)_C\\
&(\downarrow_x, \uparrow_y)_A(\downarrow_x, \uparrow_y)_B(\uparrow_x, \downarrow_y)_C\\
&(\uparrow_x, \downarrow_y)_A(\uparrow_x, \downarrow_y)_B(\uparrow_x, \downarrow_y)_C.
\end{aligned} \tag{2.36}$$

However, a cursory glance reveals that all above labelings yield an odd number of spin-up measurements for xxx contradicting the observation, quantum mechanically derived from (2.34), that only an *even* number of spin-up states are ever recorded for this configuration.

Hence, a *single* run of the experiment in the xxx configuration suffices to put to bed the seemingly unavoidable idea, implied by local realism, that the particles carry labels. The fact that such a simple one run violation of local realism can be performed with the GHZ state, but not the Bell state, justifies our claim that there are qualitatively different forms of entanglement.

2.2. The SLOCC paradigm

Let us reconsider what we mean by entanglement. Conventionally, the state of a composite system is said to be entangled if it cannot be written as a tensor product of states of the constituent subsystems. However, this particular definition is perhaps insufficient to really capture the various subtleties of entanglement. For example, there are two totally non-separable 3-qubit states that have distinct entanglement properties, as we shall see later. Is there a more illuminating notion of entanglement? Let us take our cues from experiment. We do not actually observe the tensor product structure, even though it underpins our theoretical understanding. What we do observe are correlations between spatially separated systems that admit no classical explanation. This motivates the more general and quantum information theoretic notion of entanglement as correlations between constituent pieces of a composite system that are of a quantum origin [92].

The question now is, how does one differentiate between classical correlations and those correlations which may be attributed to genuine quantum phenomena. A quantum information theoretic perspective provides a precise solution: classical correlations are *defined* as those which may be generated by *Local Operations and Classical Communication* (LOCC) [92,107,108]. Any classical correlation may be experimentally established using LOCC. Conversely, all correlations unobtainable via LOCC are regarded as *bona fide* quantum entanglement.

The LOCC paradigm is quite intuitive. Heuristically, given a composite quantum system with its components spread among different laboratories around the world we allow each experimenter to perform any quantum operation or measurement on their component locally in their lab. These are the local operations (LO), which clearly cannot establish any correlations, classical or quantum. The local operations may be supplemented by classical communication (CC): the experimenters may communicate any information (experimental procedures, measurement results, family history) they see fit via a classical channel (carrier pigeon, smoke signals, even e-mail). Any number of LO and CC rounds may be performed. It seems eminently reasonable to expect any classical correlation may be set-up in this manner. However, no quantum correlations could have been generated - all information exchanged between the separated parties at any point was intrinsically classical.

If, after some LOCC protocol, we are left with a multipartite state which may be used to perform some classically forbidden task, such as a Bell inequality violation, then the state properties facilitating this operation are not a result of LOCC - they must correspond to quantum correlations that were *already present* in the initial state before commencing the LOCC protocol [92].

LOCC cannot create entanglement.

Allowing for multiple rounds of classical communication implies that LOCC protocols are not strictly local and actually have a rather complicated mathematical structure. Since the experimenters may classically communicate either before or after any round of local operations, any subsequent rounds may be conditional on previous outcomes. Due to this additional complexity, there is no simple universal characterisation of LOCC operations [92]. However, by focusing instead on LOCC/SLOCC *equivalence* (described in the subsequent sections) we need not make use of general LOCC operations, for which a good summary is given in [93].

2.3. Entanglement measures

In the context of LOCC we can begin in earnest to consider the first essential question. How do we quantify the amount of entanglement contained in a given system? A *measure* of entanglement is a map from the state space to $\mathbb{R}$ which quantifies some aspect of entanglement. There are quite a number of entanglement measures in the literature. For reviews of such measures see, for example, [92, 93, 109]. On the whole these are physically motivated by their relevance to some quantum information theoretic task. For example, the *entanglement cost* of a state ρ is defined as the rate at which one can generate copies of ρ from a set of 2-qubit Bell states [107].

However, one would also like to have a more axiomatic scheme which identifies the minimal properties that any decent entanglement measure must possess. The axiomatic approach was originally initiated in [110]. They argued that any sensible entanglement measure E must fulfill the following criteria:

1. $E(\rho) = 0$ if and only if ρ is separable.
2. $E(\rho)$ is invariant under local unitaries.
3. $E(\rho)$ is monotonically decreasing under LOCC protocols. That is,

$$E(\Lambda(\rho)) \leq E(\rho), \tag{2.37}$$

where Λ represents an arbitrary LOCC operation.

Monotonicity under LOCC operations is now often considered as *the* basic requirement [93, 111]. Indeed, invariance under local unitaries is actually implied by monotonicity. The motivation is quite clear; LOCC cannot create entanglement, hence any good measure of entanglement should not increase under LOCC operations. Any map satisfying the monotonicity requirement is referred to as an *entanglement monotone*.

There are several extra postulates that one might consider:

1. The stronger monotonicity condition that $E(\rho)$ decreases on *average* under LOCC. That is,

$$\sum_i p_i E(\rho_i) \leq E(\rho), \tag{2.38}$$

where the LOCC protocol maps ρ to ρ_i with probability p_i.

2. Convexity,

$$E(\sum_i p_i \rho_i) \leq \sum_i p_i E(\rho_i). \tag{2.39}$$

3. Additivity,

$$E(\rho_1 \otimes \rho_2) = E(\rho_1) + E(\rho_2). \tag{2.40}$$

More on entanglement measures may be found in [92, 112–120].

2.4. Stochastic LOCC equivalence

As emphasised LOCC cannot create entanglement. Consequently, from a quantum information theoretic perspective, any two states which may be interrelated using LOCC ought to be physically equivalent with respect to their entanglement properties. This motivates the concept of *Stocastic* LOCC equivalence, introduced in [44, 108]:

Definition 1 (SLOCC-equivalence:)**.** *Two states lie in the same SLOCC-equivalence class if and only if they may be transformed into one another with some* non-zero probability *using LOCC operations.*

The crucial observation is that since LOCC cannot create entanglement any two SLOCC-equivalent states must necessarily possess the same entanglement, irrespective of the particular measure used. It is this property which make the SLOCC paradigm so suited to the task of classifying entanglement. It is also operationally motivated by the fact that any set of SLOCC-equivalent entangled states may be used to probabilistically perform the very same non-classical, entanglement dependent, operations.

The stronger concept of LOCC-equivalence is defined analogously:

Definition 2 (LOCC-equivalence:)**.** *Two states lie in the same LOCC-equivalence class if and only if they may be transformed into one another with* certainty *using LOCC operations.*

Clearly, LOCC-equivalence implies SLOCC-equivalence. From here-on-in we will make the drastic simplification of only considering pure states (which is difficult enough).

For an n-qudit system, two pure states,

$$|\Psi\rangle = a_{A_1\dots A_n}|A_1\dots A_n\rangle, \qquad |\Phi\rangle = b_{B_1\dots B_n}|B_1\dots B_n\rangle, \tag{2.41}$$

are SLOCC-equivalent if and only if they are related by an element of

$$\mathrm{SL}_1(d,\mathbb{C})\times\mathrm{SL}_2(d,\mathbb{C})\times\dots\mathrm{SL}_n(d,\mathbb{C}), \tag{2.42}$$

under which $a_{A_1\dots A_n}$ transforms in the fundamental representation [44]. That is,

$$|\Psi\rangle \sim_{\mathrm{SLOCC}} |\Phi\rangle \tag{2.43}$$

if and only if there exist $d\times d$ matrices $M^i\in\mathrm{SL}_i(d,\mathbb{C})$ such that,

$$a_{A_1A_2\dots A_n} = M^1{}_{A_1}{}^{B_1}M^2{}_{A_2}{}^{B_2}\dots M^n{}_{A_n}{}^{B_n}b_{B_1\dots B_n}. \tag{2.44}$$

The set of transformations relating equivalent states will be referred to as the *SLOCC-equivalence group*. It may be thought of as a gauge group with respect to entanglement in the sense that it mods out the physically redundant (local or classical) information.

These SLOCC transformations partition the Hilbert space into equivalence classes or *orbits*. For the n-qudit system the space of SLOCC-equivalence classes is given by [44],

$$\frac{\mathbb{C}^d\otimes\mathbb{C}^d\dots\otimes\mathbb{C}^d}{\mathrm{SL}_1(d,\mathbb{C})\times\mathrm{SL}_2(d,\mathbb{C})\times\dots\mathrm{SL}_n(d,\mathbb{C})}. \tag{2.45}$$

It is the space of physically distinct entanglement classes and, hence, its structure determines the classification of entanglement under the SLOCC. Consequently, it will play an essential role in our understanding of entanglement.

For the n-qubit system we have,

$$\frac{\mathbb{C}^2\otimes\mathbb{C}^2\dots\otimes\mathbb{C}^2}{\mathrm{SL}_1(2,\mathbb{C})\times\mathrm{SL}_2(2,\mathbb{C})\times\dots\mathrm{SL}_n(2,\mathbb{C})}. \tag{2.46}$$

and, in this case, the lower bound on the number of real continuous variables needed to parameterise the space of orbits is $2(2^n-1)-6n$.

For the stricter case of LOCC-equivalence it was shown in [108] that two states of a composite system are LOCC equivalent if and only if they may be transformed into one another using the group of *local unitaries* (LU), unitary transformations which factorise into separate transformations on the component parts.

For a n-qudit system the LOCC-equivalence group (up to a global phase) is given by [113,121],

$$\mathrm{SU}_1(d,\mathbb{C})\times\mathrm{SU}_2(d,\mathbb{C})\times\ldots\mathrm{SU}_n(d,\mathbb{C}). \tag{2.47}$$

Hence, the space of orbits is given by [113,121],

$$\frac{\mathbb{C}^d\otimes\mathbb{C}^d\ldots\otimes\mathbb{C}^d}{\mathrm{U}(1)\times\mathrm{SU}_1(d,\mathbb{C})\times\mathrm{SU}_2(d,\mathbb{C})\times\ldots\mathrm{SU}_n(d,\mathbb{C})}. \tag{2.48}$$

For the n-qubit system we have,

$$\frac{\mathbb{C}^2\otimes\mathbb{C}^2\ldots\otimes\mathbb{C}^2}{\mathrm{U}(1)\times\mathrm{SU}_1(2,\mathbb{C})\times\mathrm{SU}_2(2,\mathbb{C})\times\ldots\mathrm{SU}_n(2,\mathbb{C})} \tag{2.49}$$

and, in this case, the number of real continuous variables needed to parameterise the space of orbits is precisely $2^{n+1}-(3n+1)$ [113,121].

Given that we wish classify the physically distinct forms of entanglement, but do not really care whether it is Alice and Bob or Alice and Charlie who shares the entanglement, one could also to consider the SLOCC-equivalence group plus permutations of the qubits. We denote the SLOCC-equivalence semi-direct product the permutations by SLOCC*. In this case the entanglement measure must be permutation invariant.

The challenge now is to characterise the space (2.45) of SLOCC-equivalence classes. One possibility is the "covariant" approach which distinguishes the orbits by the vanishing or not of SLOCC-equivalence group covariants/invariants built out of the state vector. This philosophy is adopted for the 3-qubit case in chapter 8, which makes use of the Freudenthal triple system. Typically, one expects that the space of equivalence classes will contain several discrete (0-dimensional) pieces which are distinguished by a set of algebraically independent covariants. For example, for n qubits the 0-dimensional equivalence class of totally separable states is distinguished by the vanishing of *all* covariants. On the other hand, there will also be some finite dimensional pieces parametrised by some set of of algebraically independent invariants. For instance, the 1-dimensional space of Bell states is parametrised by the *concurrence* [122], which is both SLOCC invariant and a good entanglement measure. In fact, any SLOCC invariant so constructed is a good entanglement measure [116]. More on this in the following sections and chapter 8.

2.5. Two qubit entanglement

2.5.1. Generic bipartite systems

For bipartite states,

$$|\Psi\rangle = a_{AB}|AB\rangle, \tag{2.50}$$

where $A=0,\ldots m$ and $B=0,\ldots n$ (without loss of generality we assume $n\leq m$), one can always answer the question of whether a state is entangled or not. It is separable if and only if a_{AB} is rank one.

The SLOCC classification is particularly simple in this case. The set of local unitaries is contained in SLOCC and consequently, using the Schmidt decomposition, any state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_{i=1}^{n_\Psi} \sqrt{\alpha_i}|ii\rangle, \quad \alpha_i > 0, \tag{2.51}$$

$n_\Psi \leq n$, where n is the dimension of the smaller of the two sub-systems. The Schmidt number n_Ψ is given by the rank of either one of the reduced density matrices (2.9). Since their rank cannot be changed using $\mathrm{SL}(m,\mathbb{C}) \times \mathrm{SL}(n,\mathbb{C})$ there are n entanglement classes under SLOCC [44]. A state of a given rank may be transformed into any state of a lower rank with some non-zero probability using non-invertible SLOCC operations. No SLOCC operation can increase the rank of a reduced density matrix. Hence, the SLOCC classification is stratified: the higher the rank the stronger the entanglement [44].

2.5.2. Two qubits

For a two-qubit system there are then only two SLOCC classes: entangled and separable corresponding to rank 2 or rank 1 reduced density matrices, respectively. The bipartite entanglement is measured by the concurrence [122],

$$C_{AB} := 2\sqrt{\det \rho_A} = 2\sqrt{\det \rho_B}. \tag{2.52}$$

Note that

$$|\det a|^2 = \tfrac{1}{4}\varepsilon^{A_1A_2}\varepsilon^{B_1B_2}a_{A_1B_1}a_{A_2B_2}\varepsilon^{A_3A_4}\varepsilon^{B_3B_4}a^*_{A_3B_3}a^*_{A_4B_4}. \tag{2.53}$$

Using the identity

$$\varepsilon^{A_1A_2}\varepsilon^{A_3A_4} = \delta^{A_1A_3}\delta^{A_2A_4} - \delta^{A_1A_4}\delta^{A_2A_3}, \tag{2.54}$$

we have

$$\begin{aligned}
|\det a|^2 &= \tfrac{1}{4}(\delta^{A_1A_3}\delta^{A_2A_4} - \delta^{A_1A_4}\delta^{A_2A_3})\varepsilon^{B_1B_2}a_{A_1B_1}a_{A_2B_2}\varepsilon^{B_3B_4}a^*_{A_3B_3}a^*_{A_4B_4} \\
&= \tfrac{1}{4}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}(\rho_{B_1B_3}\rho_{B_2B_4} - \rho_{B_1B_4}\rho_{B_2B_3}) \\
&= \tfrac{1}{2}\varepsilon^{B_1B_2}\varepsilon^{B_3B_4}\rho_{B_1B_3}\rho_{B_2B_4} \\
&= \det \rho_B = \det \rho_A.
\end{aligned} \tag{2.55}$$

Hence,

$$C_{AB} = 2\,|\det a|, \tag{2.56}$$

which relates the concurrence to the so-called 2-tangle, introduced in [122],

$$\tau_{AB} := 4\,|\det a| = 4(|a_0|^2|a_3|^2 + |a_1|^2|a_2|^2 - (a_0a_2^*a_3a_1^* + a_1a_3^*a_2a_0^*)). \tag{2.57}$$

Recall that the eigenvalues of a 2×2 matrix obey the characteristic equation

$$\det\rho - \mathrm{tr}\ \rho\lambda + \lambda^2 = 0. \tag{2.58}$$

Hence the eigenvalues of ρ_A are

$$\lambda_0 = \tfrac{1}{2}[\mathrm{tr}\,\rho + \sqrt{(\mathrm{tr}\,\rho)^2 - 4\det\rho}], \tag{2.59a}$$

and

$$\lambda_1 = \tfrac{1}{2}[\operatorname{tr}\rho - \sqrt{(\operatorname{tr}\rho)^2 - 4\det\rho}], \tag{2.59b}$$

obeying

$$\begin{aligned} \lambda_0 + \lambda_1 &= \operatorname{tr}\rho, \\ \lambda_0\lambda_1 &= \det\rho. \end{aligned} \tag{2.59c}$$

so for an entangled state, with non-zero concurrence, ρ_A has rank 2 but only rank 1 for a product state as required by the SLOCC classification.

2.5.3. Bell states

An example of a totally separable state is

$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |01\rangle), \tag{2.60}$$

since when Alice measures 0, Bob can measure either 0 or 1 with equal probability. Here

$$\tau_{AB} = 0. \tag{2.61}$$

An example of a maximally entangled state is the Bell state:

$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{2.62}$$

for which

$$\tau_{AB} = 1. \tag{2.63}$$

2.6. Three qubits

The case of three qubits (Alice, Bob, Charlie) is particularly interesting [44,113–115,121,123–126] since it provides the simplest example of inequivalently entangled states. It is by now well understood that there are seven entanglement classes: (0) Null, (1) Separable A-B-C, (2a) Biseparable A-BC, (2b) Biseparable B-CA, (2c) Biseparable C-AB, (3) W and (4) GHZ.

In the case of three qubits, the group of SLOCC transformations is $\mathrm{SL}_A(2,\mathbb{C})\times\mathrm{SL}_B(2,\mathbb{C})\times\mathrm{SL}_C(2,\mathbb{C})$. Tensors transforming under the Alice, Bob or Charlie $\mathrm{SL}(2,\mathbb{C})$ carry indices A_1, A_2..., B_1, B_2... or C_1, C_2..., respectively, so a_{ABC} transforms as a $(\mathbf{2},\mathbf{2},\mathbf{2})$.

The first SLOCC classification of three qubit entanglement, presented in section 2.6.3, was performed in [44] using a subset of the algebraically independent local unitary invariants, which are described in the following section, and the so-called 3-tangle [6]. The 3-tangle is the unique SLOCC invariant and measures the genuine tripartite entanglement between three qubits. It is given by Cayley's hyperdeterminant [5,31], which is described in section 2.6.2.

2.6.1. Local unitary invariants

The number of parameters needed to describe unnormalised LOCC (local unitary) inequivalent states is given by the dimension of the space of orbits [121],

$$\frac{\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2}{\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)}, \tag{2.64}$$

namely $16 - 10 = 6$. This is also equivalent to the number of algebraically independent LU invariants [124], which are presented below.

1 The norm squared:

$$|\Psi|^2 = \langle \Psi | \Psi \rangle. \tag{2.65}$$

2A, 2B, 2C The local entropies:

$$S_A = 4 \det \rho_A, \qquad S_B = 4 \det \rho_B, \qquad S_C = 4 \det \rho_C, \tag{2.66}$$

where ρ_A, ρ_B, ρ_C are the doubly reduced density matrices:

$$\rho_A = \mathrm{Tr}_{BC} |\Psi\rangle\langle\Psi|, \qquad \rho_B = \mathrm{Tr}_{CA} |\Psi\rangle\langle\Psi|, \qquad \rho_C = \mathrm{Tr}_{AB} |\Psi\rangle\langle\Psi|. \tag{2.67}$$

3 The Kempe invariant [115]:

$$\begin{aligned} K &= \mathrm{tr}(\rho_A \otimes \rho_B \rho_{AB}) - \mathrm{tr}(\rho_A^3) - \mathrm{tr}(\rho_B^3) \\ &= \mathrm{tr}(\rho_B \otimes \rho_C \rho_{BC}) - \mathrm{tr}(\rho_B^3) - \mathrm{tr}(\rho_C^3) \\ &= \mathrm{tr}(\rho_C \otimes \rho_A \rho_{CA}) - \mathrm{tr}(\rho_C^3) - \mathrm{tr}(\rho_A^3), \end{aligned} \tag{2.68}$$

where $\rho_{AB}, \rho_{BC}, \rho_{CA}$ are the singly reduced density matrices:

$$\rho_{AB} = \mathrm{Tr}_C |\Psi\rangle\langle\Psi|, \qquad \rho_{BC} = \mathrm{Tr}_A |\Psi\rangle\langle\Psi|, \qquad \rho_{CA} = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|. \tag{2.69}$$

4 The 3-tangle [6]:

$$\tau_{ABC} = 4 |\operatorname{Det} a_{ABC}| \tag{2.70}$$

where $|\Psi\rangle = a_{ABC}|ABC\rangle$ and $\operatorname{Det} a_{ABC}$ is Cayley's hyperdeterminant [5,31]:

$$\operatorname{Det} a_{ABC} := -\tfrac{1}{2} \epsilon^{A_1A_2} \epsilon^{B_1B_2} \epsilon^{A_3A_4} \epsilon^{B_3B_4} \epsilon^{C_1C_4} \epsilon^{C_2C_3} a_{A_1B_1C_1} a_{A_2B_2C_2} a_{A_3B_3C_3} a_{A_4B_4C_4}. \tag{2.71}$$

Here ϵ is the $\mathrm{SL}(2, \mathbb{C})$–invariant alternating tensor

$$\epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.72}$$

Hence, under local unitary operations the most general state may be written as a six real parameter generating solution [127]. For subsequent comparison with the STU black hole we restrict our attention to states with *real* coefficients a_{ABC}. In this case, one can show that there are five algebraically independent LU invariants [127]: $\operatorname{Det} a$, S_A, S_B, S_C and the norm $\langle \Psi | \Psi \rangle$, corresponding to the dimension of

$$\frac{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2}{SO(2) \times SO(2) \times SO(2)}, \tag{2.73}$$

namely $8 - 3 = 5$. Hence, the most general real three-qubit state can be described by just five parameters [127], conveniently taken as four real numbers N_0, N_1, N_2, N_3 and an angle θ:

$$\begin{aligned}|\Psi\rangle = &-N_3 \cos^2\theta|001\rangle - N_2|010\rangle + N_3 \sin\theta\cos\theta|011\rangle \\ &- N_1|100\rangle - N_3 \sin\theta\cos\theta|101\rangle + (N_0 + N_3 \sin^2\theta)|111\rangle.\end{aligned} \tag{2.74}$$

2.6.2. Cayley's hyperdeterminant

Cayley described a multi-indexed array, such as a_{ABC}, as a *hypermatrix*. The present $2 \times 2 \times 2$ case may be graphically represented as a cube, as in Figure 2.4. In 1845 he generalised the determinant of

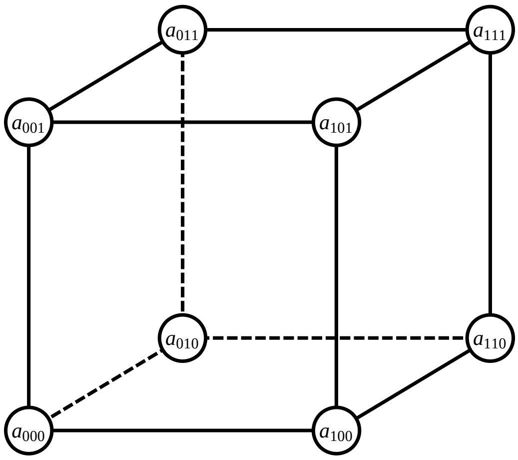


Figure 2.4.: The 3-index quantity a_{ABC} is an example of a hypermatrix, here depicted as a cube. In 1845 Cayley generalised the determinant of a 2×2 matrix to the hyperdeterminant of a $2 \times 2 \times 2$ hypermatrix.

a 2×2 matrix to the *hyperdeterminant* of a $2 \times 2 \times 2$ hypermatrix a_{ABC} [5]

$$\begin{aligned}\operatorname{Det} a := &-\tfrac{1}{2}\, \epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{C_1C_4}\epsilon^{C_2C_3} \\ &\times a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4}\end{aligned} \tag{2.75a}$$

$$\begin{aligned}= &\; a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{100}^2a_{011}^2 \\ &- 2\,(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\ &\quad + a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} \\ &\quad + a_{001}a_{100}a_{011}a_{110} + a_{010}a_{100}a_{011}a_{101}) \\ &+ 4\,(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111})\end{aligned} \tag{2.75b}$$

$$\begin{aligned}= &\; a_0^2a_7^2 + a_1^2a_6^2 + a_2^2a_5^2 + a_3^2a_4^2 \\ &- 2\,(a_0a_1a_6a_7 + a_0a_2a_5a_7 + a_0a_4a_3a_7 \\ &\quad + a_1a_2a_5a_6 + a_1a_3a_4a_6 + a_2a_3a_4a_5) \\ &+ 4\,(a_0a_3a_5a_6 + a_1a_2a_4a_7),\end{aligned} \tag{2.75c}$$

where we have made the binary conversion 0, 1, 2, 3, 4, 5, 6, 7 for 000, 001, 010, 011, 100, 101, 110, 111. (The cyclic permutations are 0, 4, 1, 5, 2, 6, 3, 7 and 0, 2, 4, 6, 1, 3, 5, 7. See Table 2.1.) Crucially,

	Binary			Decimal	
ABC	CAB	BCA	ABC	CAB	BCA
000	000	000	0	0	0
001	100	010	1	4	2
010	001	100	2	1	4
011	101	110	3	5	6
100	010	001	4	2	1
101	110	011	5	6	3
110	011	101	6	3	5
111	111	111	7	7	7

Table 2.1.: Three cyclic permutations of the binary notation. The hyperdeterminant, (2.75a), is invariant under this triality.

the hyperdeterminant is invariant under $[\mathrm{SL}(2)]^3$ and under a triality that interchanges A, B and C, irrespective of whether the a_{ABC} are complex, real or integer.

One way to understand this triality is to think of having three different metrics (Alice, Bob and Charlie) [18]

$$\begin{aligned}\gamma^A(a)_{A_1A_2} &= \epsilon^{B_1B_2}\epsilon^{C_1C_2}a_{A_1B_1C_1}a_{A_2B_2C_2},\\ \gamma^B(a)_{B_1B_2} &= \epsilon^{C_1C_2}\epsilon^{A_1A_2}a_{A_1B_1C_1}a_{A_2B_2C_2},\\ \gamma^C(a)_{C_1C_2} &= \epsilon^{A_1A_2}\epsilon^{B_1B_2}a_{A_1B_1C_1}a_{A_2B_2C_2}.\end{aligned} \tag{2.76}$$

which rotate into each other under the A-B-C triality. Explicitly,

$$\begin{aligned}\gamma^A(a) &= \begin{pmatrix} 2(a_0a_3 - a_1a_2) & a_0a_7 - a_1a_6 + a_4a_3 - a_5a_2 \\ a_0a_7 - a_1a_6 + a_4a_3 - a_5a_2 & 2(a_4a_7 - a_5a_6)\end{pmatrix},\\ \gamma^B(a) &= \begin{pmatrix} 2(a_0a_5 - a_4a_1) & a_0a_7 - a_4a_3 + a_2a_5 - a_6a_1 \\ a_0a_7 - a_4a_3 + a_2a_5 - a_6a_1 & 2(a_2a_7 - a_6a_3)\end{pmatrix},\\ \gamma^C(a) &= \begin{pmatrix} 2(a_0a_6 - a_2a_4) & a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 \\ a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 & 2(a_1a_7 - a_3a_5)\end{pmatrix}.\end{aligned} \tag{2.77}$$

All are equivalent, however, since

$$\det\gamma^A(a) = \det\gamma^B(a) = \det\gamma^C(a) = -\operatorname{Det} a. \tag{2.78}$$

We may also isolate a single qubit to make the isomorphisms $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\cong\mathrm{SO}(2,2)$ (or $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\cong\mathrm{SO}(4,\mathbb{C})$ in the complex case) manifest. For example, selecting Charlie, the components a_{AB0} and a_{AB1} form two 4-vectors as follows,

$$\begin{aligned} a_0 &= \tfrac{1}{\sqrt{2}}(P^0 - P^2) & a_1 &= -\tfrac{1}{\sqrt{2}}(Q_0 + Q_2)\\ a_2 &= \tfrac{1}{\sqrt{2}}(P^1 - P^3) & a_3 &= -\tfrac{1}{\sqrt{2}}(Q_3 + Q_1)\\ a_4 &= \tfrac{1}{\sqrt{2}}(P^1 + P^3) & a_5 &= \tfrac{1}{\sqrt{2}}(Q_3 - Q_1)\\ a_6 &= -\tfrac{1}{\sqrt{2}}(P^0 + P^2) & a_7 &= \tfrac{1}{\sqrt{2}}(Q_0 - Q_2),\end{aligned} \tag{2.79}$$

or inversely,

$$
\begin{aligned}
P^0 &= \tfrac{1}{\sqrt{2}}(a_0 - a_6) & Q_0 &= \tfrac{1}{\sqrt{2}}(a_7 - a_1)\\
P^1 &= \tfrac{1}{\sqrt{2}}(a_4 + a_2) & Q_1 &= -\tfrac{1}{\sqrt{2}}(a_5 + a_3)\\
P^2 &= -\tfrac{1}{\sqrt{2}}(a_0 + a_6) & Q_2 &= -\tfrac{1}{\sqrt{2}}(a_7 + a_1)\\
P^3 &= \tfrac{1}{\sqrt{2}}(a_4 - a_2) & Q^3 &= \tfrac{1}{\sqrt{2}}(a_5 - a_3),
\end{aligned}
\tag{2.80}
$$

then

$$
\begin{aligned}
2(-a_0a_6 + a_2a_4) &= P^2 = {P^0}^2 + {P^1}^2 - {P^2}^2 - {P^3}^2,\\
2(-a_1a_7 + a_3a_5) &= Q^2 = {Q_0}^2 + {Q_1}^2 - {Q_2}^2 - {Q_3}^2,\\
a_0a_7 - a_2a_5 + a_1a_6 - a_3a_4 &= P\cdot Q = P^0Q_0 + P^1Q_1 + P^2Q_2 + P^3Q_3,
\end{aligned}
\tag{2.81}
$$

and

$$
\gamma^3 = \begin{pmatrix} -P^2 & P\cdot Q \\ P\cdot Q & -Q^2 \end{pmatrix}. \tag{2.82}
$$

Hence

$$
\operatorname{Det} a = -P^2Q^2 + (P\cdot Q)^2. \tag{2.83}
$$

This basis makes the $\mathrm{SO}(2,2)_{AB} \times \mathrm{SL}(2,\mathbb{R})_C$ symmetry clear, P and Q together transform as an $\mathrm{SL}(2,\mathbb{R})_C$ doublet while individually they transform as vectors of $\mathrm{SO}(2,2)_{AB}$. In two component spinor notation

$$
\begin{aligned}
P^{AB} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} P^0 - P^2 & P^1 - P^3 \\ P^1 + P^3 & -P^0 - P^2 \end{pmatrix},\\
Q^{AB} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} -Q_0 - Q_2 & -Q_1 - Q_3 \\ -Q_1 + Q_3 & Q_0 - Q_2 \end{pmatrix},
\end{aligned}
\tag{2.84}
$$

we have

$$
a_{ABC} = \begin{pmatrix} P^{AB} \\ Q^{AB} \end{pmatrix}, \tag{2.85}
$$

and

$$
\gamma^C = \epsilon^{A_1A_2}\epsilon^{B_1B_2}\begin{pmatrix} P_{A_1B_1}P_{A_2B_2} & P_{A_1B_1}Q_{A_2B_2} \\ Q_{A_1B_1}P_{A_2B_2} & Q_{A_1B_1}Q_{A_2B_2} \end{pmatrix}. \tag{2.86}
$$

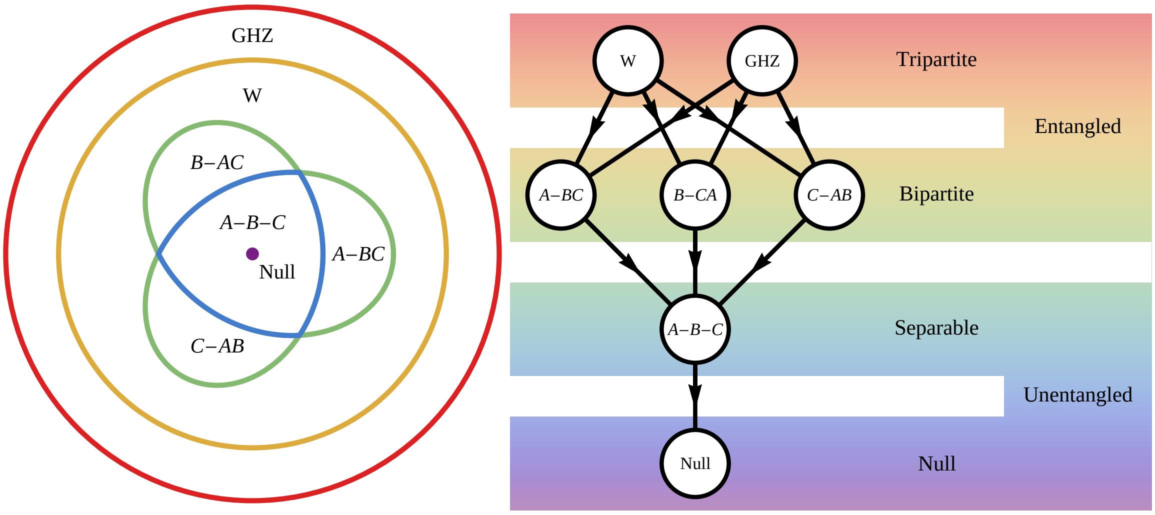
This is manifestly invariant under $\mathrm{SO}(2,2)_{AB}$ and transforms as a $\mathbf{3}$ under $\mathrm{SL}(2)_C$. Group theoretically this corresponds to the $(\mathbf{1},\mathbf{1},\mathbf{3})$ in the tensor product,

$$
\begin{aligned}
(\mathbf{2},\mathbf{2},\mathbf{2}) \times (\mathbf{2},\mathbf{2},\mathbf{2}) = \ & (\mathbf{1},\mathbf{1},\mathbf{1})\\
&+ (\mathbf{1},\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{1})\\
&+ (\mathbf{3},\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{3},\mathbf{3})\\
&+ (\mathbf{3},\mathbf{3},\mathbf{3}).
\end{aligned}
\tag{2.87}
$$

Similarly, γ^A and γ^B correspond to the $(\mathbf{3},\mathbf{1},\mathbf{1})$ and $(\mathbf{1},\mathbf{3},\mathbf{1})$ respectively.

2.6.3. Entanglement classification

Dür et al. [44] used simple arguments concerning the conservation of ranks of reduced density matrices to show that there are only six types of 3-qubit equivalence classes (or seven if we count the



(a) Onion structure (b) Hierarchy

Figure 2.5.: (a) Onion-like classification of SLOCC orbits. (b) Stratification. The arrows are non-invertible SLOCC transformations between classes that generate the entanglement hierarchy. The partial order defined by the arrows is transitive, so we may omit e.g. GHZ $\to$ A-B-C and A-BC $\to$ Null arrows for clarity.

Table 2.2.: The values of the local entropies S_A, S_B, and S_C and the hyperdeterminant Det a are used to partition three-qubit states into entanglement classes.

Class	Representative	Condition				
		Ψ	S_A	S_B	S_C	Det a
Null	0	$=0$	$=0$	$=0$	$=0$	$=0$
A-B-C	$\|000\rangle$	$\neq 0$	$=0$	$=0$	$=0$	$=0$
A-BC	$\|010\rangle + \|001\rangle$	$\neq 0$	$=0$	$\neq 0$	$\neq 0$	$=0$
B-CA	$\|100\rangle + \|001\rangle$	$\neq 0$	$\neq 0$	$=0$	$\neq 0$	$=0$
C-AB	$\|010\rangle + \|100\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$=0$	$=0$
W	$\|100\rangle + \|010\rangle + \|001\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$=0$
GHZ	$\|000\rangle + \|111\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$

null state); only two of which show *genuine* tripartite entanglement. They are as follows:

Null: The trivial zero entanglement orbit corresponding to vanishing states,

$$\text{Null}: \quad 0. \tag{2.88}$$

Separable: Another zero entanglement orbit for completely factorisable product states,

$$A\text{-}B\text{-}C: \quad |000\rangle. \tag{2.89}$$

Biseparable: Three classes of bipartite entanglement,

$$\begin{aligned} A\text{-}BC: &\quad |010\rangle + |001\rangle, \\ B\text{-}CA: &\quad |100\rangle + |001\rangle, \\ C\text{-}AB: &\quad |010\rangle + |100\rangle. \end{aligned} \tag{2.90}$$

Note, these three classes are identified under SLOCC*.

W: Three-way entangled states that do not maximally violate Bell-type inequalities in the same way as the GHZ class. However, they are robust in the sense that tracing out a subsystem generically results in a bipartite mixed state that is maximally entangled under a number of criteria [44],

$$\mathrm{W}: \quad |100\rangle + |010\rangle + |001\rangle. \tag{2.91}$$

GHZ: Genuinely tripartite entangled Greenberger-Horne-Zeilinger [104] states. These maximally violate Bell-type inequalities but, in contrast to class W, are fragile under the tracing out of a subsystem since the resultant state is completely unentangled,

$$\mathrm{GHZ}: \quad |000\rangle + |111\rangle. \tag{2.92}$$

These classes and their representative states are summarised in Table 2.2. They are characterised [44] by the vanishing or not of the invariants listed in the table. Note that the Kempe invariant is redundant in this SLOCC classification. A visual representation of these SLOCC orbits is provided by the onion-like classification [31] of Figure 1a.

These SLOCC equivalence classes are then stratified by *non-invertible* SLOCC operations into an entanglement hierarchy [44] as depicted in Figure 1b. Note that no SLOCC operations (invertible or not) relate the GHZ and W classes; they are genuinely distinct classes of tripartite entanglement. However, from either the GHZ class or W class one may use non-invertible SLOCC transformations to descend to one of the biseparable or separable classes and hence we have a hierarchical entanglement structure. For more on three qubit entanglement see [126, 128–132].

CHAPTER 3

M-theory and black holes

The dawn of the 20th century was a magnificent time for physics. Perhaps in no other period have so many profoundly important concepts emerged concurrently. The atomic foundation of matter, the unification of space and time, the uncertainty principle, the fundamentally probabilistic nature of reality - these are not just some of the greatest achievements of modern times, they are milestones in the history of humanity. It is sobering to think that these reasoned, liberating, scientific revolutions occurred at a time when the world at large was suffering the brutalities of supreme dogma. That for me is the beauty of science, it does not deal with absolute truth, which is to dance with the devil, but asks with humility what we can know about nature in-spite of our intrinsic fallibility[1].

As the dust settled and the rubble cleared, in Europe and around the world, two conceptual pillars were left standing. The foundations of physics today: general relativity and quantum theory. Each of which has independently revolutionised our basic understanding and heralded an age of unprecedented predictive power. Yet, these developments have largely evolved along separate lines. While general relativity deals with large scale structures of our universe, quantum theory reigns over the microscopic world. However, at high enough energy scales both quantum and gravitational effects become simultaneously significant and one would anticipate a consistent theory of quantum gravity to exist. Our understanding of the fundamental laws of nature is surely lacking until we have a consistent framework which marries quantum theory and gravity.

Direct attempts at quantizing gravity using perturbative quantum field theory are plagued by uncontrollable infinities. Order by order in perturbation theory the ultraviolet divergences, corresponding to radiative corrections to gravitation, become increasingly severe. The standard renormalisation methods fail, as can be seen by simple power counting. A careful analysis shows there can be no miracle cure [133].

There are many possible avenues one might explore in hope of finding a theory of quantum gravity that goes beyond these naïve first attempts. There are non-perturbative approaches such as Loop quantum gravity and causal dynamical triangulations (CDT) [134]. The CDT approach first discretises spacetime into locally flat pieces, so as to facilitate a non-perturbative path integral formulation, and then finally takes the continuum limit. By contrast, casual set theory postulates that spacetime is fundamentally discrete and that its causal structure is of primary importance. See [135] and reference

[1]A tribute to J. Bronowski.

therein. The quantum dynamics of the casual set spacetime may take the form of a "sum-over-causal-sets", which could require a reformulation of quantum theory itself [136]. Similarly, there are those who consider the current formulation of quantum theory fundamentally ill-suited to the concept of quantum gravity and that one must first remedy this situation before tackling the issue of quantum gravity head on [77].

Another very promising approach, the subject of this chapter, is string theory. In some sense, string theory is rather conservative. It assumes that our present formulation of quantum mechanics is correct and begins by simply[2] quantizing propagating 1-dimensional loops. It is not *a priori* clear that this has anything to do with quantum gravity, but it turns out that the quantum string incorporates perturbative quantum gravity.

What is rather intriguing, is that, despite the simplicity of the basic premise, string theory seems to almost have a life of its own. It is as if it has, hand-in-hand with those who have developed it, guided its own path, revealing along the way many profound and unexpected features. The result is that it has evolved into what is now known as M-theory, a fundamentally non-perturbative theory of higher dimensional branes, that one could have scarcely anticipated at the outset.

1. The road to M-theory

This journey begins with a relatively simple conceptual leap: to relinquish the long-cherished and most profitable notion that the fundamental constituents of matter are (mathematically) point like. While this possibility had certainly been previously entertained [137], it only really found its feet with the advent of *string theory*, the theory of propagating 1-dimensional extended objects. Originally formulated, in its bosonic guise, as a theory of the strong nuclear force in 1968 [138], it was quickly superseded by quantum chromodynamics (QCD). See the introductory sections of [139] for a brief summary of these developments. However, just as the stringy theory of the strong force was being eclipsed by QCD it emerged from the shadows, re-conceived as a candidate theory of quantum gravity, with, moreover, grand aspirations of unification. The vibrational modes of the string are to be thought of as representing different particles. In particular, the closed string spectrum included a massless spin two particle, which coupled to the other modes in the manner of general relativity. Was this the correct realisation of the graviton? Furthermore, quantum mechanical consistency required that strings propagate in a 26-dimensional spacetime, an undesirable feature in the context of strong interactions, but a welcome addition in the setting of Kaluza-Klein unification.

While all this was certainly an enticing prospect, it was not without its own serious faults. Principally, fermions were entirely absent, a real set back for any theory purporting to describe matter! Moreover, the bosonic closed string spectrum contains a tachyonic mode, which, to this day, has not been resolved.

These potentially fatal flaws were surmounted during what would come to be called *the first string revolution*. The *a priori* unrelated discovery of supersymmetry [140] in 1975 allowed for the consistent inclusion of world-sheet fermions resulting in the Ramond-Neveu-Schwarz (RNS) string. Following on from the work of Gliozzi, Scherk and Olive, in which they showed that the RNS string could be freed of the tachyon, Green and Schwarz discovered a formulation with spacetime supersymmetry. The *superstring* was born and with it a great new hope. Superstrings propagate in ten spacetime di-

[2] Simple in the conceptual, not technical, sense.

mensions for quantum mechanical consistency and it turns out there are five anomaly-free theories: the $E_8 \times E_8$ and $\mathrm{SO}(32)$ heterotic strings, the $\mathrm{SO}(32)$ type I, and the type IIA and type IIB strings. It was shown that the superstring is 1-loop finite and, while there is as yet no complete universally accepted proof, it is expected to be finite at all orders. Superstring theory would appear to be the first example of a perturbatively finite theory of quantum gravity. Moreover, via Kaluza-Klein compactification on some Calabi-Yau manifold [141], the heterotic superstring seemed capable, in principle at least, of correctly accounting for the standard model.

Meanwhile, in (not entirely) separate developments there had been a section of the quantum gravity community looking to 11-dimensional supergravity for answers. Not only is eleven dimensions the upper limit in which one can formulate a consistent theory of supergravity [142], it is where it takes its most elegant form [143]. Moreover, it is *unique* (at the two derivative level, assuming diffeomorphism invariance, local Lorentz invariance, local supersymmetry and Abelian gauge invariance). Initially 11-dimensional supergravity was seen as more of a tool for obtaining, via dimensional reduction, the various extended supergravity theories existing in four dimensions [144]. In particular, the maximally extended $\mathcal{N} = 8$ theory was thus first derived [145]. However, it was not long before the extra seven dimensions started to be taken seriously. In 1981 Witten considered the possibility of obtaining a realistic Kaluza-Klein model of particle physics starting from 11-dimensional supergravity concluding that, remarkably, seven extra dimensions are in fact the minimum required [146]. See [147] for a review of the Kaluza-Klein program in the context of supergravity.

A further important 11-dimensional development followed from the 1986 discovery, by Hughes, Liu and Polchinski, that, in $D = 6$, not only strings but also *membranes* could be consistently made supersymmetric [148]. The following year Bergshoeff, Sezgin and Townsend described their supermembrane propagating in an 11-dimensional supergravity background [149]. The eleven dimensional supermembrane was subsequently derived as a solution to $D = 11$ supergravity preserving one half of the supersymmetries [150]. An important distinction was made in this work. The supermembrane of Hughes, Liu and Polchinski was a *solitonic* solution of a $D = 6$ guage theory. However, the supermembrane solution to $D = 11$ supergravity is singular and carries Noether charges so should not be considered solitonic, but rather *elementary* or *fundamental*. The proper solitonic solution of 11-dimensional supergravity was later found to be the superfivebrane [151].

Despite these clear successes, the world of eleven dimensions went, to some degree at least, unnoticed by the 10-dimension universe of superstrings, which was making substantial progress in its own right. Interest in 11-dimensional supergravity was also eroded by the realisation that a 4-dimensional theory with the requisite chiral fermions could not be obtained from a non-chiral theory in $D = 11$ using the traditional Kaluza-Klein approach [152].

In fact, the separate worlds of superstrings and 11-dimensional supergravity would appear to be *curiously* at odds with one-another [153]. First, strings demanded ten dimensions while supersymmetry had chosen eleven. Second, once one had contradicted the orthodoxy of point particles in favour of strings, why stop there, especially in light of the emerging work on supersymmetric p-branes. Finally, one might have had certain misgivings about the superstring itself. There were open questions which called for a non-perturbative solution, such as the paradoxes of quantum black holes. Could the non-perturbative solitonic p-branes play a role? What about the existence of five superstring theories, could they be consistently incorporated into a single framework? Does eleven dimensions have anything to do with this?

With hindsight, these apparent discrepancies make an eventual synthesis seem inevitable. Indeed, substantial evidence had been steadily accumulating in favour of some kind of integration [144]. In 1987 it was shown that the perturbative type IIA string could be obtained from $D = 11$ supergravity with $\mathbb{R}^{1,9} \times S^1$ topology by wrapping a membrane around the circle and taking the zero radius limit [154]. It was later conjectured that this correspondence goes beyond the perturbative regime. The previously unwanted Kaluza-Klein modes, associated with the non-zero radius, are to be identified as charged extreme black holes states of the $D = 10$ theory [155–157]. There was also good evidence that compactifying a further dimension of the $D = 11$ theory on a second circle yields the type IIB string on a single circle, its non-perturbative S-duality is then given by the modular group of the $D = 11$ torus [158]. The heterotic string was thrown into the mix by compactifying on not a circle but a line segment [159]. Moveover, a number of surprising dualities relating these various theories had been discovered [2, 47, 160–163], creating a web of relations between the five string theories and 11-dimensional supergravity. For the many more intriguing aspects of these remarkable developments the reader is referred to [144].

All this progress culminated in the landmark work by Witten [164], which collected the five superstring theories together with the work on eleven dimensions under a unified framework, which he rather mysteriously dubbed "M-theory". The five superstrings and 11-dimensional supergravity are to be thought of as merely corners of the M-theory. However, the basic objects were no longer strings, but, rather, the eleven dimensional membrane and fivebrane, henceforth denoted as the M2-brane and M5-brane, respectively, so as to emphasize their new found significance. The second string revolution was well under way and the 11-dimensional M-branes were leading the charge.

Subsequently, it was realised that certain p-brane solutions, carrying R-R charge, of the type IIA/B theories could be interpreted as surfaces on which open strings, obeying Dirichlet boundary conditions, can end [165]. These *Dp-branes* have complemented the theory of black holes as intersecting black-branes wrapped around the six (seven) compact dimensions of string theory (M-theory) [155,157,166–173], providing a much sought after framework for a quantum mechanical understanding of black hole entropy. Indeed, in 1996 Strominger and Vafa reproduced the Bekenstein-Hawking black hole entropy formula by counting D-brane quantum states [174].

Despite these substantial developments it is fair to say M-theory is a work in progress. Black holes and, more generally, black p-branes offer unique insights into its deep structure and will clearly play an important role in its development. In the words of 't Hooft [175],

> "If we wish to know how to do non-perturbative gravity, it is the black holes that we must study." - 't Hooft.

2. U-duality

With a slight abuse of terminology we will use "U-duality" in two senses: (1) the global symmetries of the supergravity equations of motion (2) the duality symmetry of the full string/M-theory. Of course, we say slight since these two concepts are intimately related. Indeed, the U-dualities of supergravity are conjectured to be precisely the U-dualities of M-theory [47, 160].

Let us begin with the global symmetries of supergravity. The U-dualities of the classical maximally supersymmetric theories, which are obtain by the toroidal compactification of 11-dimensional supergravity, grow as one descends in dimension. The full symmetry group is only manifest once the field strengths have been suitably dualised [145]. The sequence of U-duality groups G and their maximal compact subgroups H is summarised in Table 3.1. The field strengths of these theories transform

D	scalars	vectors	G	H
10A	1	1	$\mathrm{SO}(1,1,\mathbb{R})$	$-$
10B	2	0	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2,\mathbb{R})$
9	3	3	$\mathrm{SL}(2,\mathbb{R}) \times SO(1,1,\mathbb{R})$	$\mathrm{SO}(2,\mathbb{R})$
8	7	6	$\mathrm{SL}(2,\mathbb{R}) \times SL(3,\mathbb{R})$	$\mathrm{SO}(2,\mathbb{R}) \times SO(3,\mathbb{R})$
7	14	10	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5,\mathbb{R})$
6	25	16	$\mathrm{SO}(5,5,\mathbb{R})$	$\mathrm{SO}(5,\mathbb{R}) \times SO(5,\mathbb{R})$
5	42	27	$E_{6(6)}(\mathbb{R})$	$\mathrm{Usp}(8)$
4	70	28	$E_{7(7)}(\mathbb{R})$	$\mathrm{SU}(8)$
3	128	-	$E_{8(8)}(\mathbb{R})$	$\mathrm{SO}(16,\mathbb{R})$

Table 3.1.: The symmetry groups (G) of the low energy supergravity theories with 32 supercharges in different dimensions (D) and their maximal compact subgroups (H).

as linear irreducible representations of G, as described in Table 3.2. On the other hand, the scalars parametrise the homogeneous coset spaces G/H. For example, in $D = 4$ the 28 gauge potentials and their duals transform as the fundamental $\mathbf{56}$ of $E_{7(7)}(\mathbb{R})$, while the $70 = 133 - 63$ scalars live in $E_{7(7)(\mathbb{R})}/\,\mathrm{SU}(8)$. Consequently, black holes can carry 28 electric charges q^{Λ} and 28 magnetic charges p_{Λ}, where $\Lambda = 1, \ldots 28$. Together, these charges transform under $E_{7(7)}(\mathbb{R})$ as a $\mathbf{56}$.

However, in the quantum theory these charges are quantized as a consequence of the Dirac-Schwinger-Zwanziger quantization condition. For two dyons, $(p^{\Lambda}, q_{\Lambda})$ and $(p'^{\Lambda}, q'_{\Lambda})$,

$$p^{\Lambda} q'_{\Lambda} - p'^{\Lambda} q_{\Lambda} \in \mathbb{Z}. \tag{3.1}$$

Hence, $E_{7(7)}(\mathbb{R})$ is broken to a discrete subgroup preserving the charge lattice, $E_{7(7)}(\mathbb{Z}) = E_{7(7)}(\mathbb{R}) \cap \mathrm{Sp}(56, \mathbb{Z})$ [47].

These discrete subgroups are believed to be the non-perturbative symmetries of M-theory or type II string theory on a k-torus, where $k = 11 - D$ or $k = 10 - D$, respectively. U-duality subsumes the superstring S- and T-dualities. It is the "unity of superstring dualities" [47].

String theory may be formulated as a world-sheet sigma-model, for which the background spacetime plays the role of target space [139, 178]. Typically, two distinct backgrounds yield different quantum string theories. However, there are instances for which two backgrounds result in identical quantum string theories. In such a case, no experimental string test can distinguish which background we are living in - they must be physically identified. The transformations which take you between indistinguishable backgrounds form a discrete group. A well known example is that of T-duality. See, for example, [178, 179]. Consider a type II superstring theory compactified on a circle. The quantum theory of a type IIA string on circle with radius R is identical to that of the type IIB theory on a circle of radius α'/R, where α' is the tension of the string. Note, while T-duality is non-perturbative in α' it is perturbatively valid order by order in the string coupling g_s. From the perspective of open strings and D-branes, Dirichlet boundary conditions go into Neumann boundary conditions, and vice versa [165].

More generally one could consider such transformations on the k circles of T^k. For the low-energy

D	G	Form Valence									
		1	2	3	4	5	6	7	8	9	10
10A	$\mathbb{R}^+$	**1**	**1**	**1**		**1**	**1**	**1**	**1**	**1**	**1** **1**
10B	$SL(2,\mathbb{R})$		**2**		**1**		**2**		**3**		**4** **2**
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	**2** **1**	**2**	**1**	**1**	**2**	**2** **1**	**3** **1**	**3** **2**	**4** **2** **2**	
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$	$(\mathbf{8},\mathbf{1})$ $(\mathbf{1},\mathbf{3})$	$(\mathbf{6},\mathbf{2})$ $(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{15},\mathbf{1})$ $(\mathbf{3},\mathbf{3})$ $(\mathbf{3},\mathbf{1})$ $(\mathbf{3},\mathbf{1})$		
7	$SL(5,\mathbb{R})$	$\overline{\mathbf{10}}$	**5**	$\overline{\mathbf{5}}$	**10**	**24**	$\overline{\mathbf{40}}$ $\overline{\mathbf{15}}$	**70** **45** **5**			
6	$SO(5,5)$	**16**	**10**	$\overline{\mathbf{16}}$	**45**	**144**	**320** $\overline{\mathbf{126}}$ **10**				
5	$E_{6(+6)}$	**27**	$\overline{\mathbf{27}}$	**78**	**351**	$\overline{\mathbf{1{,}728}}$ $\overline{\mathbf{27}}$					
4	$E_{7(+7)}$	**56**	**133**	**912**	**8,645** **133**						
3	$E_{8(+8)}$	**248**	**3,875** **1**	**147,250** **3,875** **248**							

Table 3.2.: The representations of the U-duality group G of all the forms of maximal supergravities in any dimension [176,177]. The $(D-2)$-forms dual to the scalars always belong to the adjoint representation. The scalars, parameterising the coset G/H, are not included in the table.

effective field theory the moduli space of the torus,

$$\frac{\mathrm{SO}(k,k;\mathbb{R})}{\mathrm{SO}(k,\mathbb{R})\times\mathrm{SO}(k,\mathbb{R})} \tag{3.2}$$

transforms naturally under the isometry group $\mathrm{SO}(k,k;\mathbb{R})$. However, only the discrete subgroup $\mathrm{SO}(k,k;\mathbb{Z})$ transforms between equivalent string theories. This is the T-duality group of the type II string compactified on a k-torus.

Let us turn our attention to the conjectured superstring S-duality [161, 162, 180–184]. S-duality is a non-perturbative symmetry which maps a theory at strong coupling to the same theory at weak coupling. To illustrate this let us consider the type IIB string. The massless spectrum includes two scalars, the dilaton ϕ and the axion C. The low energy effective action may be written using the complex scalar combination,

$$\tau = C + ie^{-\phi}, \tag{3.3}$$

and equations of motion are invariant under an $\mathrm{SL}(2,\mathbb{R})$ which acts fractionally on τ,

$$\tau \to \frac{a\tau+b}{c\tau+d}, \tag{3.4}$$

where $ad-bc=1$. If $a,b,c,d\in\mathbb{Z}$ then this is a symmetry of the full string theory. The weak/strong nature of this transformation is best seen by considering the special case with $C=0$ and letting $-b=c=1$ and $d=a=0$ so that,

$$\tau \to \frac{-1}{\tau}. \tag{3.5}$$

Since $e^{\phi}=g_{\mathrm{s}}$, we relate two theories with inverse couplings.

The field strengths transform linearly as a doublet of $\mathrm{SL}(2,\mathbb{Z})$. Specifically, the NS-NS 2-form B-field, which couples electrically to the fundamental string, and the R-R 2-form C_2, which couples magnetically to the D-string, are rotated into one another. S-duality mixes electric and magnetic charges. The quantized charges of the F-string/D-string bound states requires the discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$ of the classical $\mathrm{SL}(2,\mathbb{R})$.

Returning to the type II string on T^6, the combined S- and T-duality group is given by,

$$\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SO}(6,6;\mathbb{Z}), \tag{3.6}$$

which, we note, is a maximal subgroup of $E_{7(7)}(\mathbb{Z})$. Indeed, these S- and T-dualities are combined into the U-duality group $E_{7(7)}(\mathbb{Z})$, which mixes the sigma model and string coupling constants, α' and g_{s} respectively.

The significance of these U-duality symmetries in the current context of black holes and qubits is that the black hole entropy must be U-duality invariant [49, 185–187]. Thus, our entanglement measures will be built from U-duality invariants.

3. Black holes in supergravity

It is believed that black hole physics provides a testing ground for our attempts at a theory of quantum gravity, such as M-theory. This rather challenging expectation derives largely from their intriguing thermodynamic properties, which were described in the 1970s by Bekenstein and Bardeen, Carter and Hawking [188,189]. It was not long before Hawking demonstrated that black holes quantum mechanically radiate as if they are indeed black bodies [190]. The entropy of a black hole is given by its horizon area, as described in the famous Bekenstein-Hawking black hole entropy formula,

$$S_{\rm BH} = \frac{A}{4\hbar G_4}, \tag{3.7}$$

where G_4 is the four dimensional Newton constant. This fascinating equality, which relates the thermodynamic quantity $S_{\rm BH}$ to the geometric property A, has motivated much theoretical work over the decades. It raises an important question: what are the quantum microstates underlying the statistical interpretation of $S_{\rm BH}$? Any candidate theory of quantum gravity ought to provide an answer. The non-pertubative nature of black holes leaves the peturbative superstring lacking in this respect. However, non-perturbative developments using D-branes have provided a partial answer for the special subclass of *extremal* black holes.

The simplest example is given by the Reissner–Nordström solution, which describes a static, isotropic black hole of mass M and electric charge Q. The solution has two horizons,

$$\rho_\pm = M \pm \sqrt{M^2 - Q^2}, \tag{3.8}$$

where ρ is a radial coordinate. The cosmic censorship conjecture, i.e. there are no naked singularities in nature, implies the bound,

$$M \,\geq\, |Q|. \tag{3.9}$$

Extremal Reissner–Nordström black holes saturate this bound, in which case the two horizons coincide and the Hawking temperature,

$$T_{\rm H} = \frac{\hbar\kappa_S}{2\pi} = \frac{\hbar\sqrt{M^2 - Q^2}}{2\pi[2M(M + \sqrt{M^2 - Q^2}) - Q^2]}, \tag{3.10}$$

where κ_S is the surface gravity, vanishes. The extremal Reissner–Nordström black hole interpolates between a conformally flat $AdS_2 \times S^2$ geometry near the horizon and Minkowski spacetime at radial infinity, as can be seen from the line element,

$$ds^2 = dt^2 \left(1 + \frac{Q}{r}\right)^{-2} - \left(1 + \frac{Q}{r}\right)^2 \left(dr^2 + r^2\, d\Omega^2\right) \tag{3.11}$$

where $\rho = r + M$.

The Reissner–Nordström solution provides an example of an extremal black hole in the context of classical general relativity. However, supergravity incorporates gravitation and therefore its classical solutions include black holes, but with the addition of both vectors and scalars appearing in the delicate proportions demanded by supersymmetry. In particular, we are interested in extended supergavities for which the scalars parametrise a homogeneous space G/H, where G is the U-duality group and H its maximal compact subgroup. This includes all $\mathcal{N} \geq 4$ theories, but also some $\mathcal{N} = 2$ theories [39]. Note, for $\mathcal{N} \leq 4$ supergravity can couple to matter multiplets, in which case $H = H_{\rm Aut} \times H_{\rm matter}$, where $H_{\rm Aut}$ is the automorphism group of the supersymmetry algebra and $H_{\rm matter}$ is the isotropy group of the matter multiplets. For $\mathcal{N} > 4$ $H = H_{\rm Aut}$. In four dimensions the bosonic Lagrangian of such a theory may be written [191],

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}R + \frac{1}{2}G_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b - \frac{1}{4}m_{\Lambda\Sigma}(\phi)F^{\Lambda\mu\nu}F^\Sigma_{\mu\nu} - \frac{1}{4}a_{\Lambda\Sigma}(\phi)\star F^{\Lambda\mu\nu}F^\Sigma_{\mu\nu}\right) \tag{3.12}$$

where $\Lambda, \Sigma = 1, \ldots k$ and G_{ab} is the metric of the scalar manifold G/H. The U-duality group is necessarily a subgroup of $\mathrm{Sp}(2k, \mathbb{R})$ [192]. It acts on ϕ through the isometries of the scalar manifold G/H. The black hole solutions carry "bare" charges defined by,

$$p^\Lambda = \frac{1}{4\pi}\int_{S^2} F^\Lambda, \qquad q_\Lambda = \frac{1}{4\pi}\int_{S^2} m_{\Lambda\Sigma}\star F^\Sigma + a_{\Lambda\Sigma}F^\Sigma. \tag{3.13}$$

The charges may be combined into a $2k$-vector,

$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \tag{3.14}$$

which transformation linearly under the U-duality group, $Q \mapsto MQ$ for $M \in G$.

Remarkably, assuming the regularity of the scalars and geometry at the black hole horizon, the area is completely fixed by the so-called *black hole potential* [191],

$$\frac{A}{4\pi} = \mathcal{V}_{BH}(p, q, \phi^i_h), \tag{3.15}$$

where ϕ^i_h are the scalars evaluated at the horizon and

$$\mathcal{V}_{\mathrm{BH}}(p, q, \phi^i) = \frac{1}{2}Q^T\begin{pmatrix} m + am^{-1}a & am \\ m^{-1}a & m^{-1}\end{pmatrix}Q. \tag{3.16}$$

The black hole potential may also be written in terms of the central charges Z_{AB} and matter charges Z_I (which are scalar dressings of the bare charges Q),

$$\mathcal{V}_{\mathrm{BH}}(p, q, \phi^i) = Z_{AB}\overline{Z}^{AB} + Z_I\overline{Z}^I \tag{3.17}$$

where A, B and I transform under H_{Aut} and H_{matter}, respectively.

The reader at this stage might be worried that the entropy, which, if it is to have a microscopic interpretation, should only depend on discrete quantities such as the charges, is a function of the continuous scalar fields.

However, these extremal black holes enjoy a remarkable property. While the dynamics depends on the scalars, the event horizon losses all information of them. Independent of their asymptotic values the scalars flow to a fixed point as one approaches the horizon. This phenomenon goes by the name of the *attractor mechanism* [34, 36, 37, 191]. This was originally thought to be a consequence of supersymmetry, but in fact only requires the assumption of regularity near the horizon, which implies [191],

$$\left.\frac{\partial\mathcal{V}_{\mathrm{BH}}}{\partial\phi^i}\right|_h = 0. \tag{3.18}$$

Consequently, the scalars are fixed at the horizon in terms of the bare charges Q and they drop out of the entropy formula,

$$S_{\text{BH}} = \pi \mathcal{V}_{BH}(p, q). \tag{3.19}$$

Recall, the charges transform as the fundamental representation of the U-duality group. However, the metric (in the Einstein frame) is a singlet under G and hence geometric quantities such as the horizon area ought to be invariant. The entropy must be given by a U-duality invariant expression built out of p and q [193]. This is indeed the case [168, 193]. One finds, for the class of theories considered here, that the entropy is actually given by a quartic U-duality invariant I_4,

$$S_{\text{BH}} = \pi \sqrt{|I_4|}. \tag{3.20}$$

See, for example, [194] and the reference therein.

A black hole solution which preserves some degree of supersymmetry is said to be BPS (Bogomolnyi-Prasad-Sommefield) and non-BPS otherwise. A black hole may preserve at most 1/2 of the supersymmetries. Note, all BPS black holes are extremal but extremal black holes may be either BPS or non-BPS. Black holes with $I_4 < 0$ are always non-BPS.

4. The STU model

The low energy limit of the STU model is described by $\mathcal{N} = 2$ supergravity coupled to three Abelian vector multiplets and so has three complex scalars S, T and U [1, 2, 195, 196]. It admits three *triality* related stringy interpretations: (1) A consistent truncation of the $\mathcal{N} = 4$ Heterotic string on a T^6. (2) Type IIA on $K3 \times T^2$. (3) Type IIB on the mirror $K3 \times T^2$.

In the Heterotic case, S, T, U correspond to the dilaton/axion, complex Kähler form and complex structure fields respectively. The strong/weak S-duality is given by $\text{SL}(2, \mathbb{Z})_S$ while the T-duality is given by $\text{SO}(2, 2) \cong \text{SL}(2, \mathbb{Z})_T \times \text{SL}(2, \mathbb{Z})_U$. The Heterotic theory is related to the type IIA theory by a string/string duality which interchanges the roles of S and T [47, 161, 163]. This theory is in turn related to the type IIB string, but with the T and U now exchanged.

The black hole solutions carry four electric and four magnetic charges [2, 168], coming from the graviphoton and the three vectors multiplets. The black hole entropy is a quartic $[\text{SL}(2, \mathbb{Z})]^3$ invariant built out of the eight charges [3]. In fact, it is given by the square root of Cayley's hyperdeterminant.

Starting from the heterotic string, the bosonic action [2] is given by

$$\begin{aligned} I_{STU} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} e^{-\phi} \Big[& R + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{12} g^{\mu\lambda} g^{\nu\tau} g^{\rho\sigma} H_{\mu\nu\rho} H_{\lambda\tau\sigma} \\ & + \tfrac{1}{4} \operatorname{tr}(\partial \mathcal{M}_T{}^{-1} \partial \mathcal{M}_T) + \tfrac{1}{4} \operatorname{tr}(\partial \mathcal{M}_U{}^{-1} \partial \mathcal{M}_U) \\ & - \tfrac{1}{4} F_{S\mu\nu}{}^{\mathsf{T}} (\mathcal{M}_T \times \mathcal{M}_U) F_S{}^{\mu\nu} \Big]. \end{aligned} \tag{3.21}$$

where we have defined the matrices $\mathcal{M}_S, \mathcal{M}_T$ and $\mathcal{M}_U$,

$$\mathcal{M}_S = \frac{1}{S_2} \begin{pmatrix} 1 & S_1 \\ S_1 & |S|^2 \end{pmatrix}, \qquad S = S_1 + iS_2. \tag{3.22}$$

The metric $g_{\mu\nu}$ is related to the four-dimensional canonical Einstein metric $g^c_{\mu\nu}$ by $g_{\mu\nu} = e^\phi g^c{}_{\mu\nu}$. The

3-form field strength is given by

$$H_{\mu\nu\rho} = 3\left(\partial_{[\mu}B_{\nu\rho]} - \tfrac{1}{2}A^{\mathsf{T}}_{S[\mu}(\epsilon_T \times \epsilon_U)F_{S\nu\rho]}\right). \tag{3.23}$$

This action is manifestly invariant under T-duality and U-duality. The 2-form vector field strengths transform as a $(\mathbf{2},\mathbf{2})$ under T-duality and U-duality, while the scalars transform fractionally:

$$F_{S\mu\nu} \to ({\omega_T}^{-1} \times {\omega_U}^{-1})F_{S\mu\nu}, \qquad \mathcal{M}_{T/U} \to {\omega_{T/U}}^{\mathsf{T}}\,\mathcal{M}_{T/U}\,\omega_{T/U}, \tag{3.24}$$

where ω are $\mathrm{SL}(2,\mathbb{Z})$ matrices. The dilaton, metric and B-field are singlets.

The equations of motion and Bianchi identities are also invariant under S-duality,

$$\begin{pmatrix} {F_{S\mu\nu}}^a \\ {\widetilde{F}_{S\mu\nu}}{}^a \end{pmatrix} \to {\omega_S}^{-1}\begin{pmatrix} {F_{S\mu\nu}}^a \\ {\widetilde{F}_{S\mu\nu}}{}^a \end{pmatrix}, \tag{3.25}$$

where

$${\widetilde{F}_{S\mu\nu}}{}^a = -S_2[({\mathcal{M}_T}^{-1} \times {\mathcal{M}_U}^{-1})(\epsilon_T \times \epsilon_U)]^a{}_b \star {F_{S\mu\nu}}^b - S_1{F_{S\mu\nu}}^a. \tag{3.26}$$

In total, the action is $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ and $T \leftrightarrow U$ invariant and the equations of motion are $\mathrm{SL}(2,\mathbb{Z})_S \times \mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ and $S \leftrightarrow T \leftrightarrow U$ invariant. Equivalently, we could have started from either the type IIA or the type IIB action by cyclically permuting S, T and U [2].

Finally, one may consider a formulation [3] in which the S, T and U fields enter democratically through the prepotential,

$$F = d_{ijk}\frac{X^iX^jX^k}{X^0}, \qquad i,j,k = 1,2,3, \qquad d_{ijk} = \frac{1}{3!}|\epsilon_{ijk}| \tag{3.27}$$

which defines a holomorphic section (X^Λ, F_Λ) where $F_\Lambda = \frac{\partial F}{\partial X^\Lambda}$ and $\Lambda = 0,\dots,3$. The special STU coordinates are given by

$$z^i = \frac{X^i}{X^0}, \quad \text{where} \quad (z^1 = S,\ z^2 = T,\ z^3 = U) \text{ and } X^0 = 1. \tag{3.28}$$

The central charge is given by

$$Z(p,q,z) = e^{K/2}(X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) \tag{3.29}$$

where $K = -i\log\{(z^1-\bar{z}^1)(z^2-\bar{z}^2)(z^3-\bar{z}^3)\}$ is the Kähler potential. For BPS states the charges may be expressed in terms of the stabilization equations [3],

$$\begin{aligned} p^\Lambda =& ie^{K/2}(\bar{Z}X^\Lambda - Z\bar{X}^\Lambda) \\ q_\Lambda =& ie^{K/2}(\bar{Z}F^\Lambda - Z\bar{F}_\Lambda). \end{aligned} \tag{3.30}$$

The electric and magnetic charges have $O(2,2)$ scalar products

$$\begin{aligned} p^2 &= (p^0)^2 + (p^1)^2 - (p^2)^2 - (p^3)^2, \\ q^2 &= (q_0)^2 + (q_1)^2 - (q_2)^2 - (q_3)^2, \\ p\cdot q &= (p^0q_0) + (p^1q_1) + (p^2q_2) + (p^3q_3). \end{aligned} \tag{3.31}$$

The frozen complex scalars at the horizon are functions of the conserved charges [3],

$$z^i = \frac{(p \cdot q - 2p^i q_i) - i\sqrt{D}}{2(3d_{ijk}p^j p^k - p^0 q_i)} \tag{3.32}$$

where

$$\begin{aligned} D(p^\Lambda, q_\Lambda) = &-(p \cdot q)^2 + 4\big((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2)\big) \\ &- 4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3. \end{aligned} \tag{3.33}$$

Therefore the Kähler potential is given by

$$e^{-K} = \frac{D^{\frac{3}{2}}}{\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3}, \qquad \mathcal{P}_i = 3d_{ijk}p^j p^k - p^0 q_i. \tag{3.34}$$

Note, $\mathcal{P}_i$ is the charge vector of the $D = 5$ rotating black hole one obtains using the $4D/5D$ M-theory lift [51], and $\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$ is the contribution to the entropy coming from the charges.

The Kähler potential is only defined when $D > 0$ for BPS states. Using the stabilization equations (3.30) one obtains

$$ie^{K/2} Z = \frac{(p^0 z^1 - p^1)}{(\bar{z}^1 - z^1)}. \tag{3.35}$$

Combined with (3.32) this yields the rather neat result

$$Z\overline{Z} = \big(D(p, q)\big)^{1/2}. \tag{3.36}$$

The matter charges $Z_i = D_i Z$, where $i = S, T, U$ and D_i is the covariant derivative on the scalar manifold, are zero for BPS states [3, 34, 36, 37]. Hence, the BPS black hole entropy may be written in terms of the electric and magnetic charges,

$$S_{\text{BH}} = \pi\sqrt{D(p, q)} \tag{3.37}$$

For both BPS and non-BPS extremal black holes the equation becomes [196],

$$S_{\text{BH}} = \pi\sqrt{|D(p, q)|}. \tag{3.38}$$

The function $D(p^\Lambda, q_\Lambda)$ is symmetric under transformations: $p^1 \leftrightarrow p^2 \leftrightarrow p^3$ and $q_1 \leftrightarrow q_2 \leftrightarrow q_3$.

CHAPTER 4

Black holes and qubits

1. The STU model and tripartite entanglement of three qubits

1.1. Entropy/entanglement correspondence

The eight STU charges may be represented by the cube shown in Figure 4.1. The black hole/qubit

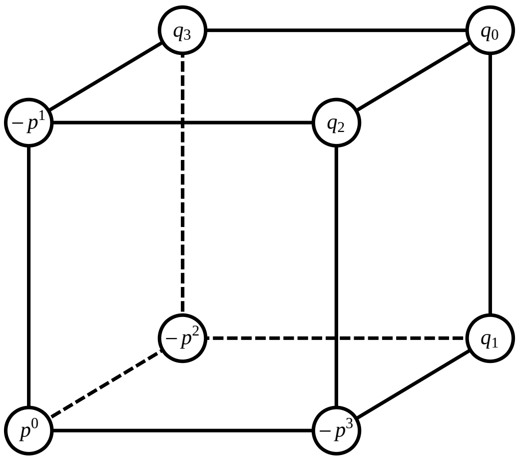


Figure 4.1.: The vertices of the hypermatrix cube from Figure 2.4 are transformed under the dictionary (4.1) to electric and magnetic charges of the STU black hole.

correspondence now comes about by identifying [4] the black hole charge hypermatrix (4.1) with the 3-qubit hypermatrix: [1]

$$\begin{aligned} &\big(p^0,\ \ p^1\ ,\ \ p^2\ ,\ \ p^3\ , q_0, q_1, q_2, q_3\big) \\ =\ &\big(a_0, -a_1, -a_2, -a_4, a_7, a_6, a_5, a_3\big). \end{aligned} \tag{4.1}$$

We then recognise from (2.75a) that

$$D(p^\Lambda, q_\Lambda) = -\operatorname{Det} a, \tag{4.2}$$

[1]Note that this is the convention of [8]; in [4] the sign of a_0, a_3, a_4, a_7 is flipped, which gives the same answer.

and hence, as advertised in the Introduction, the STU black hole entropy and Alice-Bob-Charlie 3-tangle are related by

$$S = \frac{\pi}{2}\sqrt{\tau_{ABC}}. \tag{4.3}$$

Thus Cayley's hyperdeterminant provides an interesting connection, at least at the level of mathematics, between string theory and quantum entanglement. However, even at the mathematical level there are some important distinctions. Primarily, the black hole charges take values on an integral lattice. Thus, on the black hole side a_{ABC} are integers [2–4, 8] and hence the symmetry group is $[\mathrm{SL}(2,\mathbb{Z})]^3$ rather than $[\mathrm{SL}(2,\mathbb{C})]^3$. As suggested by by Lévay [9], one could restrict to the theory of real qubits, called *rebits*, for which the a_{ABC} are real [197, 198]. Secondly, the black hole charges are not normalised. For the classification of entanglement this is not too great an impedant though.

For rebits there are three reality classes which can be characterised by the hyperdeterminant

$$\mathrm{Det}\, a < 0, \tag{4.4a}$$
$$\mathrm{Det}\, a = 0, \tag{4.4b}$$
$$\mathrm{Det}\, a > 0. \tag{4.4c}$$

Case (4.4a) corresponds to the non-separable or GHZ class [104], for example,

$$|\Psi\rangle = \tfrac{1}{2}(-|000\rangle + |011\rangle + |101\rangle + |110\rangle). \tag{4.5}$$

Case (4.4b) corresponds to the separable (A-B-C, A-BC, B-CA, C-AB) and W classes, for example

$$|\Psi\rangle = \tfrac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \tag{4.6}$$

These examples both correspond to 1/2-BPS black holes solutions [4]. The GHZ example (4.4a) has non-zero horizon area and entropy, since it has non-vanishing hyperdeterminant, and is referred to as a "large" black hole. Accordingly, case (4.4b) corresponds to a "small" black hole, with vanishing horizon area. These small black holes may acquire a non-zero entropy via higher derivative stringy contributions to the effective actions (α' corrections) or quantum loop effects (g_s corrections). In section 1.3, we consider the QI interpretation of these effects proposed in [8].

Case (4.4c) is also GHZ, but non-BPS on the black hole side. For example,

$$|\Psi\rangle = \tfrac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle), \tag{4.7}$$

which is just (4.5) with a sign flip. However, their SLOCC stabilisers are given by different real forms of $[\mathrm{SO}(2,\mathbb{C})]^2$ as in Table 8.3. This distinction disappears for complex qubits. A second example of a non-BPS black hole is given by the canonical GHZ state

$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}(|111\rangle + |000\rangle). \tag{4.8}$$

1.2. Classification of $\mathcal{N} = 2$ black holes and three-qubit states

Comparing some examples of $\mathcal{N} = 2$ black holes with examples of 3-qubit states, following [8], we see that the black holes are classified according to the entanglement classes.

A-B-C states and singly charged black holes: A black hole with just one charge, say q_0 as in Figure 4.2, has vanishing entropy and corresponds to a completely separable A-B-C state

$$|\Psi\rangle = q_0|111\rangle, \tag{4.9}$$

with

$$\begin{aligned} S_A = S_B = S_C = 0, \\ \tau_{ABC} = 0. \end{aligned} \tag{4.10}$$

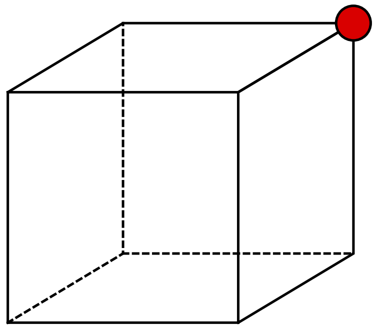


Figure 4.2.: The hypermatrix cube of Figure 4.1 is restricted to correspond to the state (4.9) by retaining only a single nonzero entry at the q_0 vertex, denoted by the red disc. The state is completely separable and accordingly, the entropy vanishes for this cube.

Microscopically, in for example type IIB string theory, a singly charged supersymmetric black hole solutions [155] may be thought of as a single D3-brane wrapping the 6-torus. A single charge black hole has vanishing entropy. However, it is possible to build 2-, 3- and 4-charge states by carefully intersecting additional D3-branes, which correspond to 2-, 3- and 4-particle bound states at threshold [2, 155, 199]. For black holes preserving some supersymmetry only the 4-particle states may have nonzero entropy.

One could also consider a black hole with two charges, say q_0 and q_1 as in Figure 4.3, having vanishing entropy and corresponding to another completely separable state

$$|\Psi\rangle = q_0|111\rangle + q_1|110\rangle \tag{4.11}$$

also satisfying (4.10).

A-BC states and doubly charged black holes: A black hole with just two charges, say q_0 and p^1 as in Figure 4.4, has vanishing entropy and corresponds to a bipartite entangled state

$$|\Psi\rangle = q_0|111\rangle - p^1|001\rangle, \tag{4.12}$$

with

$$\begin{aligned} S_A = S_B = 4(q_0 p^1)^2, \\ S_C = 0, \\ \operatorname{Det} a = 0. \end{aligned} \tag{4.13}$$

W states and triply charged black holes: A black hole with just three charges, say q_0, p^1 and p^2 as

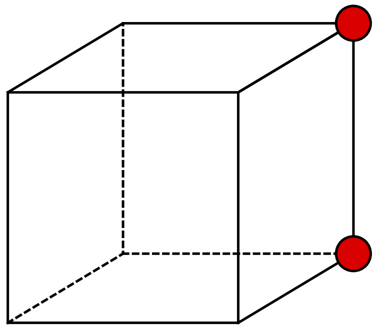


Figure 4.3.: The hypermatrix cube of Figure 4.1 is restricted to correspond to the state (4.11) by retaining nonzero entries at the q_0 and q_1 vertices, denoted by the red discs. Despite having two nonzero vertices the cube's entropy vanishes since the state is again completely separable.

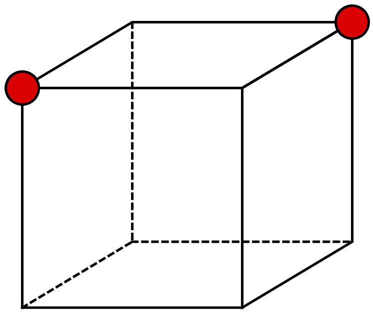


Figure 4.4.: The hypermatrix cube of Figure 4.1 is restricted to correspond to the state (4.12) by retaining nonzero entries at the q_0 and p^1 vertices, denoted by the red discs. In this case the state is bi-separable, so the entropy vanishes once more.

in Figure 4.5, has vanishing entropy and corresponds to a W state

$$|\Psi\rangle = q_0|111\rangle - p^1|001\rangle - p^2|010\rangle, \tag{4.14}$$

with

$$\begin{aligned} S_A &= 4(q_0)^2((p^1)^2 + (p^2)^2), \\ S_B &= 4(p^1)^2((q_0)^2 + (p^2)^2), \\ S_C &= 4(p^2)^2((q_0)^2 + (p^1)^2), \\ \text{Det}\, a &= 0. \end{aligned} \tag{4.15}$$

GHZ and 4-charge BPS and non-BPS black holes: A black hole with just four charges, say q_0, p^1,p^2 and p^3 as in Figure 4.6, has non-vanishing entropy and corresponds to a GHZ state

$$|\Psi\rangle = q_0|111\rangle - p^1|001\rangle - p^2|010\rangle - p^3|100\rangle, \tag{4.16}$$

with

$$\begin{aligned} S_A &= 4((p^3)^2 + (q_0)^2)((p^1)^2 + (p^2)^2), \\ S_B &= 4((p^1)^2 + (p^3)^2)((q_0)^2 + (p^2)^2), \\ S_C &= 4((p^2)^2 + (p^3)^2)((q_0)^2 + (p^1)^2), \\ \text{Det}\, a &= -4q_0p^1p^2p^3. \end{aligned} \tag{4.17}$$

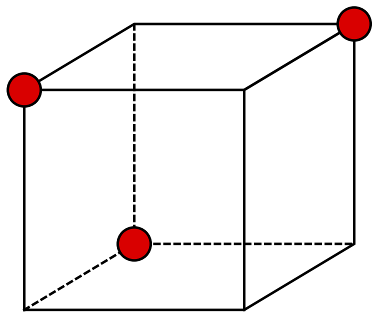


Figure 4.5.: The hypermatrix cube of Figure 4.1 is restricted to correspond to the state (4.14) by retaining nonzero entries at the q_0, p^1 and p^2 vertices, denoted by the red discs. The entropy vanishes this time since the cube corresponds to a W state.

This is a large BPS black hole if $\mathrm{Det}\, a < 0$ and a large non-BPS black hole if $\mathrm{Det}\, a > 0$.

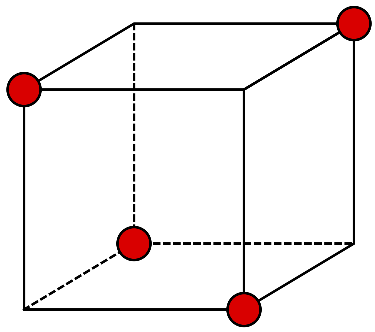


Figure 4.6.: The hypermatrix cube of Figure 4.1 is restricted to correspond to the state (4.16) by retaining nonzero entries at the q_0, p^1, p^2 and p^3 vertices, denoted by the red discs. This is a GHZ state exhibiting genuine tripartite entanglement and accordingly the cube has nonzero entropy. While the previous cubes corresponded to small BPS black holes, this black hole is large and BPS/non-BPS for $\mathrm{Det}\, a < 0/\mathrm{Det}\, a > 0$.

GHZ and 2-charge non-BPS black holes: A black hole with just two charges Figure 4.7, say q_0 and p^0 as in Figure 4.7, has non-vanishing entropy and corresponds to a GHZ state

$$|\Psi\rangle = q_0|111\rangle + p^0|000\rangle, \tag{4.18}$$

with

$$S_A = S_B = S_C = 4(p^0)^2(q_0)^2, \tag{4.19}$$

and

$$\mathrm{Det}\, a = (p^0)^2(q_0)^2. \tag{4.20}$$

Since $\mathrm{Det}\, a > 0$, this describes a non-BPS large black hole [200, 201].

1.3. Higher order corrections

The small black holes have a singular horizon with vanishing area and entropy at the classical level, but may acquire nonvanishing area and entropy due to quantum corrections, characterised by higher derivatives in the supergravity lagrangian. One can interpret this as consequence of the quantum stretching of the horizon conjectured by Susskind [202] and Sen [157, 166]. See also [203–208].

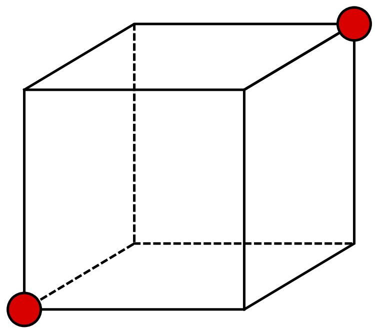


Figure 4.7.: The hypermatrix cube of Figure 4.1 is restricted to correspond to the state (4.18) by retaining nonzero entries at the q_0 and p^0 vertices, denoted by the red discs. Despite having only two nonzero charges the cube has nonvanishing entropy since the state is GHZ. The corresponding black hole is large and always non-BPS.

Kallosh and Linde [8] have noted that this quantum entropy also admits an interpretation in terms of qubit entanglement measures. They propose a general formula that correctly reduces to the known special cases. It is given by

$$S_{\text{total}} = \frac{\pi}{2}\sqrt{\tau_{ABC} + \frac{4c_2}{3}(C_{AB} + C_{BC} + C_{CA}) + \frac{8K^2}{3}|\Psi|}, \tag{4.21}$$

where $C^2_{AB} = \frac{1}{2}(S_A + S_B - S_C - \tau_{ABC})$ is the concurrence [6] and c_2, K are constants that depend upon the compactification.

For completely separable states with only one nonzero charge, this reduces to

$$S = K\sqrt{\tfrac{2}{3}|\Psi|} = \pi K\sqrt{\tfrac{2}{3}|q_0|}. \tag{4.22}$$

For the bipartite and W-states at large values of the charges, the concurrences are much greater than $|\Psi|$ and the formula reduces to

$$S_{\text{total}} = \pi\sqrt{\frac{c_2}{3}(C_{AB} + C_{BC} + C_{CA})} \tag{4.23}$$

$$= 4\pi\sqrt{|q_0(p^1 + p^2)|}. \tag{4.24}$$

Finally, for the GHZ states the (unnormalised) 3-tangle is much greater than the concurrences and we regain

$$S_{\text{BH}} = \tfrac{\pi}{2}\sqrt{\tau_{ABC}}. \tag{4.25}$$

As with the lowest order black hole/qubit correspondence, there is as yet no underlying physical explanation for these mathematical coincidences at higher orders.

1.4. Attractors and SLOCC

The STU/3-qubit correspondence defines a rather special state. The 3-qubit state is unnormalized and the coefficients are given by the integer valued black hole charges via (4.1). However, following Lévay [14], we may define a more general complex state, which depends not only on the charges but also the complex moduli S, T and U

$$|\Psi'\rangle = e^{-3i\pi/4}e^{K/2}\begin{pmatrix}\bar{U} & -1\\ -U & 1\end{pmatrix}\otimes\begin{pmatrix}\bar{T} & -1\\ -T & 1\end{pmatrix}\otimes\begin{pmatrix}\bar{S} & -1\\ -S & 1\end{pmatrix}|\Psi\rangle. \tag{4.26}$$

This is an $[\mathrm{SL}(2,\mathbb{C})]^3$ transformation of the original state and therefore leaves the value of the 3-tangle invariant. Note, while $|\Psi\rangle$ is a 3-qubit state with eight complex amplitudes, it is $[\mathrm{SU}(2)]^3$ equivalent to a state with real amplitudes. We may now describe the all important black hole potential, $\mathcal{V}_{BH}$, in terms of our transformed 3-qubit state $|\Psi\rangle$

$$\mathcal{V}_{BH} = \frac{1}{2}||\Psi||^2 \tag{4.27}$$

where the norm is the usual scalar product on a complex vector space.

The state $|\Psi\rangle$ is a function of the scalars and consequently transforms under the action of the covariant derivatives of the scalar manifold. To understand the quantum information theoretic interpretation of this action it is convenient to use the flat covariant derivatives defined by $D_{\hat{i}} = e^i_{\hat{i}} D_i$. Here $e^i_{\hat{i}}$ is the vielbien of the scalar manifold. Lévay showed that [14]

$$\begin{aligned}
D_{\hat{S}}|\Psi\rangle &= i(I\otimes I\otimes X_+)|\Psi\rangle, & D_{\hat{\bar{S}}}|\Psi\rangle &= i(I\otimes I\otimes X_-)|\Psi\rangle\\
D_{\hat{T}}|\Psi\rangle &= i(I\otimes X_+\otimes I)|\Psi\rangle, & D_{\hat{\bar{T}}}|\Psi\rangle &= i(I\otimes X_-\otimes I)|\Psi\rangle\\
D_{\hat{U}}|\Psi\rangle &= i(X_+\otimes I\otimes I)|\Psi\rangle, & D_{\hat{\bar{U}}}|\Psi\rangle &= i(X_-\otimes I\otimes I)|\Psi\rangle
\end{aligned} \tag{4.28}$$

where $X_\pm$ is the raising (lowering) operator

$$X_+|0\rangle = |1\rangle, \quad X_+|1\rangle = 0, \quad X_-|1\rangle = |0\rangle, \quad X_-|0\rangle = 0. \tag{4.29}$$

These are in fact the *projective error* operators from the field quantum error correction [14].

Moreover, Lévay demonstrated that the extremization of the central charge at the horizon, $D_i Z(z,p,q) = 0$, required for BPS solutions, guarantees that the only non-zero components of $|\Psi\rangle$ are $|000\rangle$ and $|111\rangle$. That is, for BPS states, the attractor mechanism brings an arbitrary state, $|\Psi\rangle$, defined at radial infinity, into a state belonging to the GHZ-class. Lévay argues that this is equivalent to the entanglement distillation procedure utilized by quantum information theorists to determine the optimal set of SLOCC transformations for taking a general 3-qubit state into a GHZ state [14,26].

2. $\mathcal{N} = 8$ supergravity and the tripartite entanglement of seven qubits

Despite its apparent successes, this investigation thus far has been limited to a very specific example, namely STU blacks holes and 3-qubit states. What about other supergravity theories and qubit systems? Interestingly, there are three related theories for which there exists a quartic invariant akin to Cayley's hyperdeterminant whose square root yields the corresponding black hole entropy:

1. $\mathcal{N} = 2$ supergravity coupled to $n+1$ vector multiplets where the symmetry is $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SO}(2,n;\mathbb{Z})$ and the black holes carry charges belonging to the $(2, n+2)$ representation ($n+2$ electric plus $n+2$ magnetic).

2. $\mathcal{N} = 4$ supergravity coupled to n vector multiplets where the symmetry is $\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SO}(6,n;\mathbb{Z})$

where the black holes carry charges belonging to the $(2, 6+n)$ representation ($n+6$ electric plus $n+6$ magnetic).

3. $\mathcal{N} = 8$ supergravity where the symmetry is the non-compact exceptional group $E_{7(7)}(\mathbb{Z})$ and the black holes carry charges belonging to the fundamental 56-dimensional representation (28 electric plus 28 magnetic). See Table 3.2

Perhaps the most intriguing option is offered by $\mathcal{N} = 8$ supergravity. This was considered in [10] by Duff and Ferrara.

2.1. $\mathcal{N} = 8$ supergravity and black holes

The black hole solutions of $\mathcal{N} = 8$ supergravity depend on the $28 + 28$ electric/magnetic charges which form the fundamental **56** of $E_{7(7)}$ [145]. The entropy is given by [193]

$$S = \pi\sqrt{|I_4|}, \tag{4.30}$$

where I_4 is the unique Cremmer-Julia quartic $E_{7(7)}$ invariant [145]

$$I_4 = \operatorname{tr}(\bar{Z}Z)^2 - \tfrac{1}{4}(\operatorname{tr}\bar{Z}Z)^2 + 4\left(\operatorname{Pf} Z + \operatorname{Pf}\bar{Z}\right), \tag{4.31}$$

and Z_{AB} is the central charge matrix. The relation between the entropy of stringy black holes and the Cartan-Cremmer-Julia $E_{7(7)}$ invariant was established in [193]. Alternatively, it may be written in terms of the quantized charges using the Cartan form [8,209,210]

$$I_4 = -\operatorname{tr}(xy)^2 + \tfrac{1}{4}(\operatorname{tr} xy)^2 - 4\left(\operatorname{Pf} x + \operatorname{Pf} y\right). \tag{4.32}$$

where the charges, x^{IJ} and y_{IJ}, are 8×8 antisymmetric matrices and Pf is the Pfaffian. The charges may be related to branes wrapping cycles of the T^7 [211]. The exact relation between the Cartan invariant in (4.32) and Cremmer-Julia invariant [145] in (4.31) was established in [211,212],

$$Z_{AB} = -\tfrac{1}{4\sqrt{2}}(x^{IJ} + iy_{IJ})(\gamma^{IJ})_{AB}, \tag{4.33}$$

and

$$x^{IJ} + iy_{IJ} = -\tfrac{\sqrt{2}}{4}Z_{AB}(\gamma^{AB})_{IJ}. \tag{4.34}$$

The matrices $(\gamma^{IJ})_{AB}$ form the $\mathrm{SO}(8)$ algebra. Here (I, J) are the 8 vector indices and (A, B) are the 8 spinor indices. The $(\gamma^{IJ})_{AB}$ matrices can be considered also as $(\gamma^{AB})_{IJ}$ matrices due to equivalence of the vector and spinor representations of the $\mathrm{SO}(8)$ Lie algebra. The quartic invariant I_4 of $E_{7(7)}$ is also related to the octonionic Jordan algebra $\mathfrak{J}_3^{\mathbb{O}}$ [45] as described in chapter 6.

Given that the exceptional Lie groups are anything but common place in the world of QI, we might expect that establishing an entanglement correspondence in this case would require some exotic arrangement of qudits. However, the work of Ferrara, Kallosh and Linde [8,210] suggested that we may need only to appeal to some generalisation of the, by now, familiar 3-tangle. This is due to the fact that we may bring Z_{AB} into a canonical form using an $\mathrm{SU}(8)$ transformation,

$$Z_{AB} = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.35}$$

where $z_i = \rho_i e^{i\varphi_i}$ are complex. In this way the number of entries is reduced from 56 to 8. In a systematic treatment in [71], the meaning of these parameters was clarified. From the four complex values of $z_i = \rho_i e^{i\varphi_i}$ one can remove three phases by an $\mathrm{SU}(8)$ rotation, but the overall phase cannot be removed; it is related to an extra parameter in the class of black hole solutions [170, 213]. In this basis, the quartic invariant takes the form [193]

$$\begin{aligned} I_4 = \sum_i |z_i|^4 - 2\sum_{i<j} |z_i|^2 |z_j|^2 + 4\left(z_1 z_2 z_3 z_4 + \bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4\right) \\ = \quad & (\rho_1 + \rho_2 + \rho_3 + \rho_4) \\ & \times (\rho_1 + \rho_2 - \rho_3 - \rho_4) \\ & \times (\rho_1 - \rho_2 + \rho_3 - \rho_4) \\ & \times (\rho_1 - \rho_2 - \rho_3 + \rho_4) \\ & + \; 8\rho_1\rho_2\rho_3\rho_4 \left(\cos\varphi - 1\right). \end{aligned} \tag{4.36}$$

This 5-parameter solution is called a generating solution for other black holes in $\mathcal{N} = 8$ supergravity/M-theory [214, 215].

In terms of the charges the canonical form (4.35) is given by [8, 210],

$$(x^{IJ} + iy_{IJ})_{\text{can}} = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_4 & 0 \end{pmatrix}. \tag{4.37}$$

Writing λ_α in terms of the parameters x^{IJ} and y_{IJ}, the matrix a_{ABD} and the black hole charges p^i and q_k [4],

$$\begin{aligned} \lambda_1 &= x^{01} + iy_{01} = a_{111} + ia_{000} = q_0 + ip^0, \\ \lambda_2 &= x^{23} + iy_{23} = a_{100} + ia_{011} = -p^3 + q_3, \\ \lambda_3 &= x^{45} + iy_{45} = -a_{010} - ia_{101} = p^2 - iq_2, \\ \lambda_4 &= x^{56} + iy_{56} = -a_{001} - ia_{110} = p^1 - iq_1. \end{aligned} \tag{4.38}$$

we once again recover

$$I_4 = -\,\mathrm{Det}\, a. \tag{4.39}$$

We have identified the STU model invariant within the $\mathcal{N} = 8$ invariant and, hence, a connection to 3-qubit entanglement. Consequently, we expect some relationship to 3-qubit entanglement when considering the full $\mathcal{N} = 8$ generalization.

3. E_7 and the tripartite entanglement of seven qubits

How do we determine the precise qubit system corresponding to 56-charge black holes of $\mathcal{N} = 8$ supergravity? The tripartite entanglement of three qubits is given by Cayley's hyperdeterminant, which follows from the SLOCC-equivalence group $[\mathrm{SL}(2,\mathbb{C})]^3$. However, a quantum information theoretic interpretation in the $\mathcal{N} = 8$ theory cannot be naïvely obtained by adding more qubits. Recall, the n-qubit SLOCC-equivalence group is $[\mathrm{SL}(2,\mathbb{C})]^n$, which dramatically differs from E_7 and leaves little hope of reproducing the subtleties of $\mathcal{N} = 8$ black holes. Duff and Ferrara [10] approached this conundrum by considering the decomposition of the fundamental 56-dimensional representation of $E_{7(7)}$ under seven factors of $\mathrm{SL}(2)$, the maximum number contained as a subgroup,

$$E_{7(7)}(\mathbb{Z}) \supset [SL(2,\mathbb{Z})]^7, \tag{4.40}$$

and

$$E_7(\mathbb{C}) \supset [SL(2,\mathbb{C})]^7. \tag{4.41}$$

We shall now show that the corresponding system in quantum information theory is that of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George). However, the larger symmetry requires that they at most undergo a tripartite entanglement of a very specific kind. The entanglement measure will be given by the quartic Cartan $E_7(\mathbb{C})$ invariant [45,145,193,209].

The crucial ingredient is the observation that E_7 contains seven copies of the single qubit SLOCC group $\mathrm{SL}(2)$ and that the $\mathbf{56}$ decomposes in a very particular way. We begin by considering the maximal subgroup $\mathrm{SL}(2)_A \times \mathrm{SO}(6,6)$,

$$\begin{aligned} E_{7(7)} &\supset \mathrm{SL}(2)_A \times \mathrm{SO}(6,6), \\ \mathbf{56} &\to (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}). \end{aligned} \tag{4.42}$$

Further decomposing $\mathrm{SO}(6,6)$ in (4.42)

$$\begin{aligned} \mathrm{SL}(2)_A \times \mathrm{SO}(6,6) &\supset \mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_D \times \mathrm{SO}(4,4), \\ (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}) &\to (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{8_v}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8_s}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{8_c}) + (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}). \end{aligned} \tag{4.43}$$

Further decomposing $\mathrm{SO}(4,4)$,

$$\begin{gathered} \mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_D \times \mathrm{SO}(4,4) \\ \supset \mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_D \times \mathrm{SO}(2,2) \times \mathrm{SO}(2,2) \\ (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{8_v}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8_s}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{8_c}) \\ \to (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{4}) \\ + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2}). \end{gathered} \tag{4.44}$$

Finally, further decomposing each $\mathrm{SO}(2,2)$

$$
\begin{aligned}
&\mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_D \times \mathrm{SO}(2,2) \times \mathrm{SO}(2,2) \\
&\supset \mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_D \times \mathrm{SL}(2)_C \times \mathrm{SL}(2)_G \times \mathrm{SL}(2)_F \times \mathrm{SL}(2)_E \\
&(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{4},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{4}) \\
&+ (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2}) \\
&\to (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) \\
&+ (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}).
\end{aligned}
\tag{4.45}
$$

In summary,

$$
E_{7(7)} \supset \mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_C \times \mathrm{SL}(2)_D \times \mathrm{SL}(2)_E \times \mathrm{SL}(2)_F \times \mathrm{SL}(2)_G, \tag{4.46}
$$

and the $\mathbf{56}$ decomposes as

$$
\begin{aligned}
\mathbf{56} \to \quad & (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) \\
&+ (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) \\
&+ (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \\
&+ (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}) \\
&+ (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \\
&+ (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) \\
&+ (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}).
\end{aligned}
\tag{4.47}
$$

An analogous decomposition holds for

$$
E_7(\mathbb{C}) \supset [\mathrm{SL}(2,\mathbb{C})]^7. \tag{4.48}
$$

Notice that we find seven copies of the $(\mathbf{2},\mathbf{2},\mathbf{2})$ appearing in the STU model. This translates into seven copies of the 3-qubit Hilbert space:

$$
\begin{aligned}
|\Psi\rangle_{56} = \quad & a_{ABD}|ABD\rangle \\
&+ b_{BCE}|BCE\rangle \\
&+ c_{CDF}|CDF\rangle \\
&+ d_{DEG}|DEG\rangle \\
&+ e_{EFA}|EFA\rangle \\
&+ f_{FGB}|FGB\rangle \\
&+ g_{GAC}|GAC\rangle.
\end{aligned}
\tag{4.49}
$$

Note that:

1. Any pair of states has an individual in common
2. Each individual is excluded from four out of the seven states
3. Two given individuals are excluded from two out of the seven states

4. Three given individuals are never excluded

So we have seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred and George) but where Alice has tripartite entanglement not only with Bob/Dave but also with Emma/Fred and also Charlie/George, and similarly for the other six individuals. So, in fact, each person has tripartite entanglement with each of the remaining three couples.

The entanglement may be represented by a heptagon as in Figure 4.8 with seven vertices A, B, C, D, E, F, G, and seven triangles

$$ABD, BCE, CDF, DEG, EFA, FGB, GAC.$$

Each of the seven states transforms as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ under three of the $\mathrm{SL}(2)$'s and are singlets under the

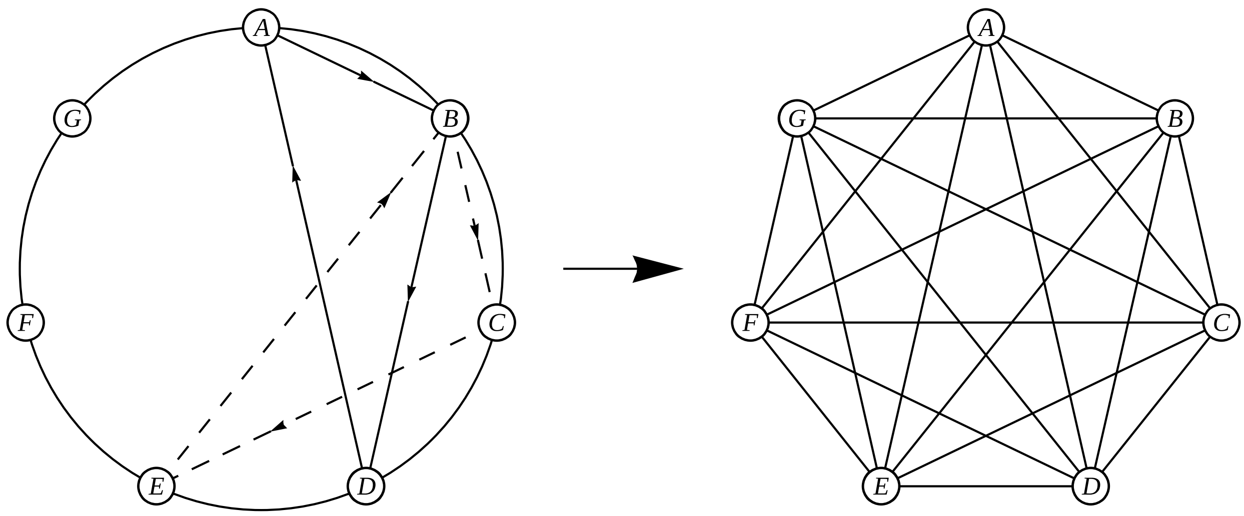


Figure 4.8.: The E_7 entanglement diagram corresponding to the decomposition (4.47) and the state (4.49). Each of the seven vertices A, B, C, D, E, F, G represents a qubit and each of the seven triangles $ABD, BCE, CDF, DEG, EFA, FGB, GAC$ describes a tripartite entanglement. As discussed in section 1 the oriented triangles correspond to quaternionic cycles in the multiplication table of imaginary octonions.

remaining four. On restricting to just three of the seven qubits the entanglement ought to be given by Cayley's hyperdeterminant. Hence, we expect seven copies of Cayley's hyperdeterminant and consequently the entanglement measure must be quartic polynomial of the seven a, b, c, d, e, f, g. Taken together however, we see from (4.47) that they transform as a complex **56** of $E_7(\mathbb{C})$ and, hence, the entanglement measure must be invariant under $E_7(\mathbb{C})$. The unique possibility is the Cartan invariant I_4, and so the entanglement is given by

$$\tau_{ABCDEFG} = 4|I_4|. \tag{4.50}$$

It may be written as the sum of seven terms each of which is invariant under $[\mathrm{SL}(2)]^3$ plus cross terms. To see this, denote a **2** in one of the seven entries in (4.47) by A, B, C, D, E, F, G. So we may rewrite (4.47) as

$$\mathbf{56} = (ABD) + (BCE) + (CDF) + (DEG) + (EFA) + (FGB) + (GAC), \tag{4.51}$$

or symbolically

$$\mathbf{56} = a + b + c + d + e + f + g. \tag{4.52}$$

Then I_4 is the singlet in $\mathbf{56} \times \mathbf{56} \times \mathbf{56} \times \mathbf{56}$:

$$\begin{aligned}
I_4 = {} & a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4 \\
& + 2\Big[a^2b^2 + a^2c^2 + a^2d^2 + a^2e^2 + a^2f^2 + a^2g^2 \\
& \quad + b^2c^2 + b^2d^2 + b^2e^2 + b^2f^2 + b^2g^2 \\
& \quad + c^2d^2 + c^2e^2 + c^2f^2 + c^2g^2 \\
& \quad + d^2e^2 + d^2f^2 + d^2g^2 \\
& \quad + e^2f^2 + e^2g^2 \\
& \quad + f^2g^2\Big] \\
& + 8\left[abce + bcdf + cdeg + defa + efgb + fgac + gabd\right],
\end{aligned} \tag{4.53}$$

where products like

$$\begin{aligned}
a^4 &= (ABD)(ABD)(ABD)(ABD) \\
&= \tfrac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{D_2D_3} \\
&\quad \times a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4},
\end{aligned} \tag{4.54}$$

exclude four individuals (here Charlie, Emma, Fred, and George), products like

$$\begin{aligned}
a^2b^2 &= (ABD)(ABD)(BCE)(BCE) \\
&= \tfrac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_3}\epsilon^{D_1D_2}\epsilon^{B_2B_4}\epsilon^{C_3C_4}\epsilon^{E_3E_4} \\
&\quad \times a_{A_1B_1D_1}a_{A_2B_2D_2}b_{B_3C_3E_3}b_{B_4C_4E_4},
\end{aligned} \tag{4.55}$$

exclude two individuals (here Fred and George), and products like

$$\begin{aligned}
abce &= (ABD)(BCE)(CDF)(EFA) \\
&= \tfrac{1}{2}\epsilon^{A_1A_4}\epsilon^{B_1B_2}\epsilon^{C_2C_3}\epsilon^{D_1D_3}\epsilon^{E_2E_4}\epsilon^{F_3F_4} \\
&\quad \times a_{A_1B_1D_1}b_{B_2C_2E_2}c_{C_3D_3F_3}e_{E_4F_4A_4},
\end{aligned} \tag{4.56}$$

exclude one individual (here George). These results may be verified using the dictionary between a, b, c, d, e, f, g and the x and y as discussed in chapter 7. Note that a^4 is just minus Cayley's hyperdeterminant.

4. Classification of $\mathcal{N} = 8$ black holes and seven-qubit states

In the $\mathcal{N} = 8$ theory, "large" and "small" black holes are classified by the sign of I_4:

$$I_4 > 0, \tag{4.57a}$$

$$I_4 = 0, \tag{4.57b}$$

$$I_4 < 0. \tag{4.57c}$$

Once again, non-zero I_4 corresponds to large black holes, which are BPS for $I_4 > 0$ and non-BPS for $I_4 < 0$, and vanishing I_4 corresponds to small black holes. However, in contrast to $\mathcal{N} = 2$, case (4.57a) requires that only $1/8$ of the supersymmetry is preserved, while we may have $1/8$, $1/4$ or $1/2$ for case (4.57b).

The large black hole solutions can be found [210] by solving the $\mathcal{N} = 8$ classical attractor equations [34]. The charge orbits [45,46,71] for the black holes depend on the number of unbroken supersymmetries or the number of vanishing eigenvalues as in Table 4.1.

Class	Orbit	s_1	s_2	s_3	s_4	I_4	Black hole	SUSY
A-B-C	$E_{7(7)}/(E_{6(6)} \ltimes \mathbb{R}^{27})$	> 0	0	0	0	0	small	$1/2$
A-BC	$E_{7(7)}/(O(5,6) \ltimes \mathbb{R}^{32} \times \mathbb{R})$	> 0	> 0	0	0	0	small	$1/4$
W	$E_{7(7)}/(F_{4(4)} \ltimes \mathbb{R}^{26})$	> 0	> 0	> 0	0	0	small	$1/8$
GHZ	$E_{7(7)}/E_{6(2)}$	> 0	> 0	> 0	> 0	> 0	large	$1/8$
GHZ	$E_{7(7)}/E_{6(6)}$	> 0	> 0	> 0	< 0	< 0	large	0

Table 4.1.: Classification of $D = 4, N = 8$ black holes. The distinct charge orbits are determined by the number of non-vanishing eigenvalues and I_4, as well as the number of preserved supersymmetries. The entanglement classes assigned by interpreting the eight parameter canonical form as a three qubit system.

5. Further developments

In this section we briefly review the further developments of the PhD thesis of Duminda Dahanayake [30]. In particular, we describe: (1) the microscopic interpretation of black holes as qubits [17] (2) the classification of 4-qubit entanglement from STU black holes in three dimensions [29] (3) the *superqubit*, a supersymmetric generalisation of the qubit [21].

5.1. Microscopic interpretation

In section 1.1 we established a correspondence between the tripartite entanglement measure of three qubits and the macroscopic entropy of the 4-dimensional 8-charge STU black hole of supergravity. In this section we briefly review the microscopic interpretation proposed in [17,30].

The microscopic string-theoretic interpretation of the charges is given by configurations of intersecting D3-branes, wrapping around the six compact dimensions T^6. The 3-qubit basis vectors $|ABC\rangle$ are associated with the corresponding eight wrapping cycles. In particular, one can relate a well-known fact of quantum information theory, that the most general real three-qubit state up to local unitaries can be parameterised by four real numbers and an angle, to a well-known fact of string theory, that the most general STU black hole can be described by four D3-branes intersecting at an angle.

The microscopic analysis is not unique since there are many ways of embedding the STU model in string/M-theory, but a useful example from our point of view is that of four D3-branes of Type IIB wrapping the (579), (568), (478), (469) cycles of T^6 with wrapping numbers N_0, N_1, N_2, N_3 and intersecting over a string [171]. The wrapped circles are denoted by crosses and the unwrapped circles by noughts as shown in Table 4.2. This picture is consistent with the interpretation of the 4-charge black hole as bound state at threshold of four 1-charge black holes [2,155,199]. The fifth parameter

θ is obtained [216, 217] by allowing the N_3 brane to intersect at an angle which induces additional effective charges on the $(579), (569), (479)$ cycles. The microscopic calculation of the entropy consists of taking the logarithm of the number of microstates and yields the same result as the macroscopic one [218].

To make the black hole/qubit correspondence we associate the three T^2 with the $\mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_C$ of the three qubits Alice, Bob, and Charlie. The eight different cycles then yield eight different basis vectors $|ABC\rangle$ as in the last column of Table 4.2, where $|0\rangle$ corresponds to xo and $|1\rangle$ to ox. To wrap or not to wrap; that is the qubit. We see immediately that we reproduce the five parameter three-qubit state $|\Psi\rangle$ of (2.74):

$$\begin{aligned}|\Psi\rangle = &-N_3\cos^2\theta|001\rangle - N_2|010\rangle + N_3\sin\theta\cos\theta|011\rangle \\ &- N_1|100\rangle - N_3\sin\theta\cos\theta|101\rangle + (N_0 + N_3\sin^2\theta)|111\rangle.\end{aligned} \tag{4.58}$$

Note the GHZ state describes four D3-branes intersecting over a string.

4	5	6	7	8	9	macro charges	micro charges	$\|ABC\rangle$
x	o	x	o	x	o	p^0	0	$\|000\rangle$
o	x	o	x	x	o	q_1	0	$\|110\rangle$
o	x	x	o	o	x	q_2	$-N_3\sin\theta\cos\theta$	$\|101\rangle$
x	o	o	x	o	x	q_3	$N_3\sin\theta\cos\theta$	$\|011\rangle$
o	x	o	x	o	x	q_0	$N_0 + N_3\sin^2\theta$	$\|111\rangle$
x	o	x	o	o	x	$-p^1$	$-N_3\cos^2\theta$	$\|001\rangle$
x	o	o	x	x	o	$-p^2$	$-N_2$	$\|010\rangle$
o	x	x	o	x	o	$-p^3$	$-N_1$	$\|100\rangle$

Table 4.2.: Three qubit interpretation of the 8-charge $D = 4$ black hole from four D3-branes wrapping around the lower four cycles of T^6 with wrapping numbers N_0, N_1, N_2, N_3 and then allowing N_3 to intersect at an angle θ.

5.2. Supersymmetric quantum information

In [21, 30] a supersymmetric generalization of the qubit, the *superqubit*, was put forward. They proceeded by extending the n-qubit SLOCC equivalence group $[\mathrm{SL}(2,\mathbb{C})]^n$ and the LOCC equivalence group $[\mathrm{SU}(2)]^n$ to the supergroups $[\mathrm{OSp}(1|2)]^n$ and $[\mathrm{uOSp}(1|2)]^n$, respectively. A single superqubit forms a 3-dimensional representation of $\mathrm{OSp}(1|2)$ consisting of two commuting "bosonic" components and one anticommuting "fermionic" component. For $n = 2$ and $n = 3$ we introduce the appropriate supersymmetric generalizations of the conventional entanglement measures. In particular, super-Bell and super-GHZ states are characterized, respectively, by nonvanishing superdeterminant (distinct from the Berezinian) and superhyperdeterminant[2].

This mathematical construction seems a very natural one. Moreover, from a physical point of view, it makes contact with various condensed-matter systems. For example, the three-dimensional representation of $\mathrm{Osp}(1|2)$ is encountered in the supersymmetric t-J model where it describes spinons and holons on a one-dimensional lattice [219–223]. It also shows up in the quantum Hall effect [224] and Affleck-Kennedy-Lieb-Tasaki models of superconductivity [225]

[2]This work was in part inspired by the construction of the superhyperdeterminant in [33].

5.3. 4-qubit entanglement and the STU model in $D = 3$

For the STU model we found that the classification of the black hole solutions corresponded to the well known entanglement classification of three qubits. However, we can go one better and derive the much harder 4-qubit entanglement classification [29,30]. We consider $D = 4$ supergravity theories in which the moduli parameterize a symmetric space of the form $M_4 = G_4/H_4$, where G_4 is the global U-duality group and H_4 is its maximal compact subgroup. After a further time-like reduction to $D = 3$ the moduli space becomes a pseudo-Riemannian symmetric space $M_3^* = G_3/H_3^*$, where G_3 is the $D = 3$ duality group and H_3^* is a non-compact form of the maximal compact subgroup H_3. One finds that geodesic motion on M_3^* corresponds to stationary solutions of the $D = 4$ theory [27,226–230]. These geodesics are parameterized by the Lie algebra valued matrix of Noether charges Q and the problem of classifying the spherically symmetric extremal (non-extremal) black hole solutions consists of classifying the nilpotent (semisimple) orbits of Q (Nilpotent means $Q^n = 0$ for some sufficiently large n.)

In the case of the STU model the $D = 3$ moduli space G_3/H_3^* is $\mathrm{SO}(4,4)/[\mathrm{SL}(2,\mathbb{R})]^4$ (a para-quaternionic manifold), which yields the Lie algebra decomposition

$$\mathfrak{so}(4,4) \cong [\mathfrak{sl}(2,\mathbb{R})]^4 \oplus (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}). \tag{4.59}$$

The relevance of (4.59) to four qubits was pointed out in [18] and recently spelled out more clearly by Lévay [27] who relates four qubits to $D = 4$ STU black holes. The Kostant-Sekiguchi correspondence [231] then implies that the nilpotent orbits of $\mathrm{SO}(4,4)$ acting on the adjoint representation $\mathbf{28}$ are in one-to-one correspondence with the nilpotent orbits of $[\mathrm{SL}(2,)]^4$ acting on the fundamental representation $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$ and hence with the classification of four-qubit entanglement. Note furthermore that it is the complex qubits that appear automatically, thereby relaxing the restriction to real qubits.

We found that there are 31 entanglement families which reduce to nine up to permutations of the four qubits. The nine agrees with [232,233] while the 31 is new.

Part II.

ALGEBRAIC PERSPECTIVES

CHAPTER 5

Algebras

The octonions and Jordan algebras have never quite managed to find their *raison d'etre* in physics. An interesting idea here [234], another there [235], but nothing to cement them in the pages of scientific history as yet. On the other hand they enjoy a significant role in various branches of mathematics. In particular, there is an extensive body of work detailing their applications to the exceptional Lie groups, which now play an almost ubiquitous role in supergravity and M-theory. So, under the wing of exceptional groups, the octonions and Jordan algebras do have their day after all.

In this chapter we introduce the necessary basics of, first, composition algebras and in particular the octonions, then, second, Jordan algebras and finally, third, the Freudenthal triple system. For the most part these are standard results readily available in the literature. See, for example, [56, 68, 236] and the references therein.

1. Composition algebras

The quaternions $\mathbb{H}$ were the unexpected prize of Hamilton's efforts to generalise the complex doublet, and its applications in 2-dimensional geometry, to a triplet of numbers that would play the same role in three dimensions. Of course, the original triplet was doomed from the outset, as we now know that the division algebras only exist in even dimensions. Upon hearing of Hamilton's magical discovery Graves, a close friend, asked [236],

> "If with your alchemy you can make three pounds of gold, why should you stop there?"
> - Graves

Graves decided on some prospecting of his own and came up good, happening upon a rare nugget indeed, the octonions $\mathbb{O}$. The 8-dimensional octonions are the largest division algebra, completing the now famous sequence, $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, but they are not the end of the story. In particular, the exceptional groups make equal use of their split signature cousins, which contain zero divisors. Jacobson in 1958 brought all these concepts together under the umbrella of *compsition algebras* [52].

In the following we introduce the basic theory of composition algebras and some of their structural properties.

Much of the following will rely on a quadratic norm and its associated bilinear form:

Definition 3 (Quadratic norm)**.** *A* quadratic norm *on a vector space V over a field $\mathbb{F}$ is a map $\mathbf{n} : V \to \mathbb{F}$ such that:*

1. $\mathbf{n}(\lambda a) = \lambda^2 \mathbf{n}(a), \qquad \lambda \in \mathbb{F}, a \in V,$
2. $\langle a, b\rangle := \mathbf{n}(a+b) - \mathbf{n}(a) - \mathbf{n}(b)$ *is bilinear.*

The quadratic norm is said to be non-degenerate if,

$$\langle a, b\rangle = 0 \quad \forall b \quad \Rightarrow a = 0. \tag{5.1}$$

Definition 4 (Composition algebra)**.** *An algebra $\mathbb{A}$ defined over a field $\mathbb{F}$ with identity element e, is said to be* composition *if it has a non-degenerate quadratic form $\mathbf{n} : \mathbb{A} \to \mathbb{F}$ such that,*

$$\mathbf{n}(ab) = \mathbf{n}(a)\mathbf{n}(b), \quad \forall a, b \in \mathbb{A}, \tag{5.2}$$

where we denote multiplicative product of the algebra by juxtaposition.

Every composition algebra satisfies the quadratic equation,

$$a^2 - \langle a, e\rangle a + \mathbf{n}(a)e = 0. \tag{5.3}$$

Conjugation may be defined using the bilinear form, $\overline{a} := \langle a, e\rangle e - a$, and satisfies,

1. $a\overline{a} = \overline{a}a = \mathbf{n}(a)e,$
2. $\overline{ab} = \overline{b}\overline{a},$
3. $\overline{\overline{a}} = a,$
4. $\overline{a+b} = \overline{a} + \overline{b},$
5. $\mathbf{n}(\overline{a}) = \mathbf{n}(a),$
6. $\langle \overline{a}, \overline{b}\rangle = \langle a, b\rangle.$

We denote respectively the commutator and associator by $[a, b]$ and $[a, b, c]$,

$$\begin{aligned} [a, b] &= ab - ba, \\ [a, b, c] &= (ab)c - a(bc). \end{aligned} \tag{5.4}$$

An algebra is said to be associative if the associator vanishes and commutative if the commutator vanishes. If the associator is an alternating function of its arguments then the algebra is said to be *alternative*. All composition algebras are alternative.

By considering $\mathbb{F} \subset \mathbb{A}$ as the scalar multiples of the identity we may decompose $\mathbb{A}$ into its "real" and "imaginary" parts $\mathbb{A} = \mathbb{F} \oplus \mathbb{A}'$, where $\mathbb{A}' \subset \mathbb{A}$ is the subspace orthogonal to $\mathbb{F}$. An arbitrary element $a \in \mathbb{A}$ may be written $a = \Re(a) + \Im(a)$, where $\Re(a) \in \mathbb{F}$ and $\Im(a) \in \mathbb{A}'$.

Definition 5 (Division algebra)**.** *A composition algebra $\mathbb{A}$ is said to be* division *if it contains no zero divisors,*

$$ab = 0 \Rightarrow \quad a = 0 \quad \textit{or} \quad b = 0.$$

Hurwitz's celebrated theorem states that there are only four division algebras [237]: the reals, complexes, quaternions and octonions denoted respectively by $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. These algebras are obtained by the Cayley-Dickson process. With a slight modification one can also generate their split signature cousins $\mathbb{C}_s, \mathbb{H}_s$ and $\mathbb{O}_s$ [57].

1.1. Octonions

We recall some useful facts about the octonions:

- Octonions, $a, b, c \in \mathbb{R}^8$, are an 8-dimensional real vector space, with basis elements $e_0, \ldots, e_7$, where e_0 is the identity.
- Conjugation sends $e_0 \mapsto e_0$ and $e_i \mapsto -e_i$ for $i = 1, \ldots 7$.
- Non-commutative.
- Non-associative.

If we denote the imaginary octonions by e_i where $i = 1, \ldots 7$, the structure constants C_{ijk} are defined by

$$e_i e_j = -\delta_{ij} + C_{ijk} e_k, \tag{5.5}$$

and satisfy

$$C_{pmk} C_{qkn} + C_{qmk} C_{pkn} = \delta_{pm}\delta_{qn} + \delta_{pn}\delta_{qm} - 2\delta_{pq}\delta_{mn}. \tag{5.6}$$

Following [238] we define the "Jacobian" C_{ijkl}

$$\begin{aligned} &[e_i, [e_j, e_k]] + [e_k, [e_i, e_j]] + [e_j, [e_k, e_i]] \\ &= 4(C_{jkm}C_{imn} + C_{ijm}C_{kmn} + C_{kim}C_{jmn})e_n \\ &\equiv 3C_{ijkl}e_l, \end{aligned} \tag{5.7}$$

where C_{ijkl} satisfies

$$C_{ijkl} = \tfrac{1}{6}\epsilon_{ijklmnp} C_{mnp}, \tag{5.8}$$

and

$$C_{ijkl} = -C_{mij}C_{mkl} - \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}. \tag{5.9}$$

The norm of an arbitrary octonion $a = a_I e_I$, where $I = (0, i)$, is

$$\mathbf{n}(a) = (a_0)^2 + (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2 + (a_5)^2 + (a_6)^2 + (a_7)^2, \tag{5.10}$$

or, in the split case,

$$\mathbf{n}(a) = (a_0)^2 + (a_1)^2 + (a_2)^2 + (a_3)^2 - (a_4)^2 - (a_5)^2 - (a_6)^2 - (a_7)^2. \tag{5.11}$$

Let us now construct an example octonionic multiplication table. Rather than use the Cayley-Dickson method, we adopt an equivalent procedure, more suited to defining a particular basis, presented in [236].

Starting with an algebra, $\mathbb{A}$, we adjoin to it an orthogonal imaginary basis element, i, which must satisfy the following conditions

$$i^2 = -1, \qquad (\text{or } \ i^2 = 1 \ \text{ in the split case})$$

and

$$a(ib) = i(\bar{a}b), \qquad (ai)b = (a\bar{b})i, \qquad (ia)(b\bar{i}) = \overline{ab} \tag{5.12}$$

for all $a, b \in \mathbb{A}$. We begin with the complex numbers, which we write as $a = a^0 e_0 + a^1 e_1$ ($e_0 = 1$ and $e_1 = \sqrt{-1}$). To construct the quaternions we introduce a second imaginary unit, e_2, and define the product $e_1 e_2 = e_4$. Using the second identity in (5.12) with $a = 1$ and $b = e_1$ we obtain

$$e_2 e_1 = -e_1 e_2 = -e_4$$

so they are non-commutative. Now we may compute the remaining products

$$e_1 e_4 = -e_1(e_2 e_1) = -e_2$$

where we have used the first identity in (5.12), and

$$e_2 e_4 = e_2(e_1 e_2) = e_4.$$

It is not hard to verify that in general we have

$$e_i e_j = -\delta_{ij} + C_{ijk} e_k \tag{5.13}$$

where i, j and k run over 1, 2 and 4 and C_{ijk} is totally antisymmetric with $C_{124} = 1$.

Let us now build the octonions by introducing a fourth imaginary element, e_3 and defining the products

$$e_1 e_3 = e_7, \quad e_2 e_3 = e_5, \quad e_4 e_3 = -e_6. \tag{5.14}$$

Each triple appearing in the above products is isomorphic to the quaternions, there is no reason to not have begun the process with, say, e_7. Finally, we need to compute the remaining three products

$$e_1 e_5 = e_1(e_2 e_3) = e_3(e_1 e_2) = e_3 e_4 = e_6 \tag{5.15}$$

where we have made use of the first identity. Following a similar process for the remaining two products we find the nonzero structure constants are:

$$C_{124} = C_{235} = C_{346} = C_{457} = C_{561} = C_{672} = C_{713} = 1 \tag{5.16}$$

where the remaining non-zero constants are determined by the total antisymmetry of C_{ijk}[1]. Those of you paying close attention may have noticed that, in this basis, the structure constants describe the tripartite entanglement of seven qubits.

[1]Note, in the split case C_{ijk} is not necessarily antisymmetric on the last two indices.

2. Jordan algebras

2.1. Preliminaries

The advent of the matrix mechanics formulation of quantum theory was shortly followed by a number of investigations into its possible algebraic generalisations. These forays were principally motivated by a certain perception, prevalent at the time, that the irrefutable successes of quantum mechanics as applied to the atom would not be readily extended to the relativistic domain [239]. Chief amongst these attempts was the use of Jordan algebras as a suitable representation of physical observables [240,241]. The intention was to place emphasis on observables, as opposed to states, and in doing so abstract the essential characteristics of the set of Hermitian operators, displacing the Hilbert space from its central position in the mathematical foundations of the theory [240–242]. We briefly review the Jordan formulation of quantum mechanics in Appendix A.

Their applications to quantum theory, in the end, proved limited. However, they are interesting objects in their own right and an expansive literature on the subject has developed over the years. See [58, 68] for comprehensive historical accounts. Their intimate relationship with the exceptional Lie groups is of central importance in their applications to string theory and supergravity.

Definition 6 (Jordan algebra)**.** *A vector space, defined over a ground field* $\mathbb{F}$*, equipped with a product satisfying,*

$$x \circ y = y \circ x; \qquad x^2 \circ (x \circ y) = x \circ (x^2 \circ y) \qquad \forall x, y \in \mathfrak{J}, \tag{5.17}$$

is a Jordan algebra $\mathfrak{J}$.

An obvious example is given by the set of real matrices with Jordan product defined as $x \circ y = \frac{1}{2}(xy + yx)$. More generally, this definition of the Jordan product may be used to construct a Jordan algebra starting from any associative algebra.

Any algebra is said to be *formally real* if,

$$x^2 + y^2 + z^2 \ldots = 0 \qquad \Longrightarrow \qquad x = y = z = \ldots = 0. \tag{5.18}$$

Assuming that a given Jordan algebra is formally real it can be shown that the Jordan identity (5.17) is equivalent to power associativity [239],

$$x^m \circ x^n = x^{(m+n)}. \tag{5.19}$$

This is significant when considering the application of Jordan algebras to quantum mechanics providing some physical motivation for the Jordan identity as will be discussed in Appendix A.

The full classification of all formally real Jordan algebras was completed in [239]. There are four infinite sequences of simple Jordan algebras and one exceptional case. (A Jordan algebra is simple if it contains no proper ideals. All Jordan algebras may be decomposed into a direct sum of simple Jordan algebras.) Three of the infinite sequences are given by the sets $\mathfrak{J}_n^{\mathbb{A}}$ of $n \times n$ Hermitian matrices defined over the three associative division algebras, $\mathbb{A} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The Jordan product in these cases is simply given by $x \circ y = \frac{1}{2}(xy + yx)$, where xy denotes conventional matrix multiplication. The fourth is given by $\mathbb{R} \oplus \Gamma_n$, where Γ_n is a n-dimensional real vector space with Euclidean norm. The one exceptional simple Jordan algebra is given by $\mathfrak{J}_3^{\mathbb{O}}$, the set of 3×3 Hermitian matrices defined over the octonions.

However, we will generally be concerned with the larger class of *cubic* Jordan algebras which need not be formally real. For example, $\mathfrak{J}_3^{\mathbb{O}_s}$, the set of 3×3 Hermitian matrices defined over the split octonions, is not formally real but nonetheless underpins $\mathcal{N} = 8$ supergravity.

2.2. Cubic Jordan algebras

A large class of Jordan may be derived following the Freudenthal-Springer-Tits construction [54, 60]. They are defined in terms of a quadratic or cubic norm and the resulting algebras are respectively referred to as quadratic Jordan algebras and cubic Jordan algebras.

Both cases are relevant in various supergravity theories [39, 40, 70]. In particular, the quadratic Jordan algebras are relevant to the $D = 6, \mathcal{N} = 8$ theory and the cubic Jordan algebras are relevant to the $D = 5, \mathcal{N} = 8$ theory.

In the following we present in detail the cubic case as it facilitates the definition of the FTS required for the black holes in $D = 4$. The entirely analogous quadratic construction may be found in section B.1. Our primary cubic Jordan algebra references are [48,68,69,243], which are used throughout the following.

2.2.1. Definitions

Let V be a vector space over a field $\mathbb{F}$ equipped with a cubic norm.

Definition 7 (Cubic norm)**.** *A* cubic norm *is a homogeneous map of degree three*

$$N_3 : V \to \mathbb{F}, \quad \textit{s.t.} \quad N_3(\alpha A) = \alpha^3 N_3(A), \quad \forall \alpha \in \mathbb{F}, A \in V \tag{5.20}$$

such that its linearization,

$$N_3(A, B, C) := \frac{1}{6}(N_3(A+B+C)-N_3(A+B)-N_3(A+C)-N_3(B+C)+N_3(A)+N_3(B)+N_3(C)) \tag{5.21}$$

is trilinear.

If V further contains a base point $N_3(c) = 1, c \in V$ one may define the following four maps,

1. The trace,
$$\mathrm{Tr}(A) = 3N_3(c, c, A), \tag{5.22a}$$
2. A quadratic map,
$$S(A) = 3N_3(A, A, c), \tag{5.22b}$$
3. A bilinear map,
$$S(A, B) = 6N_3(A, B, c), \tag{5.22c}$$
4. A trace bilinear form,
$$\mathrm{Tr}(A, B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B) - S(A, B). \tag{5.22d}$$

A cubic Jordan algebra $\mathfrak{J}$ with multiplicative identity $\mathbb{1} = c$ may be derived from any such vector space if N is *Jordan cubic*, that is:

1. The trace bilinear form (5.22d) is non-degenerate.

2. The quadratic adjoint map, $\sharp\colon \mathfrak{J} \to \mathfrak{J}$, uniquely defined by $\mathrm{Tr}(A^\sharp, B) = 3N_3(A, A, B)$, satisfies

$$(A^\sharp)^\sharp = N_3(A)A, \qquad \forall A \in \mathfrak{J}. \tag{5.23}$$

The Jordan product is then defined using,

$$A \circ B = \tfrac{1}{2}\big(A \times B + \mathrm{Tr}(A)B + \mathrm{Tr}(B)A - S(A, B)\mathbb{1}\big), \tag{5.24}$$

where, $A \times B$ is the linearization of the quadratic adjoint,

$$A \times B = (A + B)^\sharp - A^\sharp - B^\sharp. \tag{5.25}$$

Finally, the Jordan triple product is defined as

$$\{A, B, C\} = (A \circ B) \circ C + A \circ (B \circ C) - (A \circ C) \circ B. \tag{5.26}$$

Definition 8 (Irreducible idempotent)**.** *An element E satisfying*

$$E \circ E = E, \qquad \mathrm{Tr}(E) = 1 \tag{5.27}$$

is said to be an irreducible idempotent. *Cubic Jordan algebras have three irreducible idempotents.*

2.2.2. Symmetries

There are many good references on the symmetries associated with Jordan algebras and, in particular, the exceptional Lie groups. Here we have used [56, 61, 64, 66] and in particular [48, 69, 243].

Definition 9 (Automorphism group $\mathrm{Aut}(\mathfrak{J})$)**.** *Invertible $\mathbb{F}$-linear transformations σ preserving the Jordan product,*

$$\begin{aligned}\mathrm{Aut}(\mathfrak{J}) :=& \{\sigma \in \mathrm{Iso}_\mathbb{R}(\mathfrak{J}) | \sigma(A \circ B) = \sigma A \circ \sigma B\}\\ =& \{\sigma \in \mathrm{Iso}_\mathbb{R}(\mathfrak{J}) | \sigma(A \times B) = \sigma A \times \sigma B\}\\ =& \{\sigma \in \mathrm{Iso}_\mathbb{R}(\mathfrak{J}) | N_3(\sigma A) = N_3(A), \sigma\mathbb{1} = \mathbb{1}\}.\end{aligned} \tag{5.28}$$

The corresponding Lie algebra is given by the set of derivations $\mathfrak{der}(\mathfrak{J})$*,*

$$\begin{aligned}\mathfrak{Aut}(\mathfrak{J}) = \mathfrak{der}(\mathfrak{J}) =& \{\delta \in \mathrm{Hom}_\mathbb{R}(\mathfrak{J}) | \delta(A \circ B) = \delta A \circ B + A \circ \delta B\}\\ =& \{\delta \in \mathrm{Hom}_\mathbb{R}(\mathfrak{J}) | \delta A \times B + A \times \delta B = 0\}\\ =& \{\delta \in \mathrm{Hom}_\mathbb{R}(\mathfrak{J}) | N_3(\delta A, A, A) = 0, \delta\mathbb{1} = 0\}.\end{aligned} \tag{5.29}$$

Definition 10 (Reduced structure group $\mathrm{Str}_0(\mathfrak{J})$)**.** *Invertible $\mathbb{F}$-linear transformations σ preserving the cubic norm,*

$$\begin{aligned}\mathrm{Str}_0(\mathfrak{J}) :=& \{\tau \in \mathrm{Iso}_\mathbb{R}(\mathfrak{J}) | N_3(\tau A) = N_3(A)\}\\ =& \{\tau \in \mathrm{Iso}_\mathbb{R}(\mathfrak{J}) | {}^t\tau^{-1}(A \times B) = \tau A \times \tau B\}.\end{aligned} \tag{5.30}$$

The corresponding Lie algebra $\mathfrak{Str}_0(\mathfrak{J})$ is given by,

$$\begin{aligned}\mathfrak{Str}_0(\mathfrak{J}) =& \{\phi \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{J}) | N_3(\phi A, A, A) = 0\} \\ =& \{\phi \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{J}) | {}^t\phi(A \times B) = \phi A \times B + A \times \phi B\}.\end{aligned} \tag{5.31}$$

$\mathfrak{Str}_0(\mathfrak{J})$ may be decomposed with respect to the automorphism algebra,

$$\mathfrak{Str}_0(\mathfrak{J}) = L'(\mathfrak{J}) \oplus \mathfrak{der}(\mathfrak{J}), \tag{5.32}$$

where $L'(\mathfrak{J})$ denotes the set of left Jordan products by traceless elements, $L_A(B) = A \circ B$ where $\mathrm{Tr}(A) = 0$.

Definition 11 (Structure group $\mathrm{Str}(\mathfrak{J})$)**.** *Invertible $\mathbb{F}$-linear transformations σ preserving the cubic norm up to a fixed scalar factor,*

$$\mathrm{Str}(\mathfrak{J}) := \{\sigma \in \mathrm{Iso}_{\mathbb{R}}(\mathfrak{J}) | N_3(\sigma A) = \lambda N_3(A), \lambda \in \mathbb{F}\}. \tag{5.33}$$

The corresponding Lie algebra $\mathfrak{Str}(\mathfrak{J})$ is given by,

$$\mathfrak{Str}(\mathfrak{J}) = L(\mathfrak{J}) \oplus \mathfrak{der}(\mathfrak{J}), \tag{5.34}$$

where $L(\mathfrak{J})$ denotes the set of left Jordan products $L_A(B) = A \circ B$.

The following chain of embeddings clearly holds,

$$\mathrm{Aut}\,\mathfrak{J} \subset \mathrm{Str}_0(\mathfrak{J}) \subset \mathrm{Str}(\mathfrak{J}). \tag{5.35}$$

A cubic Jordan algebra element may be assigned a $\mathrm{Str}(\mathfrak{J})$ invariant *rank* [53].

Definition 12 (Cubic Jordan algebra rank)**.** *An element $A \in \mathfrak{J}$ has a rank given by:*

$$\begin{aligned}&\mathrm{Rank}A = 1 \Leftrightarrow A^\sharp = 0; \\ &\mathrm{Rank}A = 2 \Leftrightarrow N_3(A) = 0,\ A^\sharp \neq 0; \\ &\mathrm{Rank}A = 3 \Leftrightarrow N_3(A) \neq 0.\end{aligned} \tag{5.36}$$

2.3. Cubic Jordan algebras: Examples

2.3.1. Magic Jordan algebras

We denote by $\mathfrak{J}_3^{\mathbb{A}}$ the cubic Jordan algebra of 3×3 Hermitian matrices with entries in a composition algebra $\mathbb{A}$ defined over the field $\mathbb{F}$. We will assume $\mathbb{F} = \mathbb{R}$ here.

An arbitrary element may be written as,

$$X = \begin{pmatrix} \alpha & c & \bar{b} \\ \bar{c} & \beta & a \\ b & \bar{a} & \gamma \end{pmatrix}, \quad \text{where} \quad \alpha, \beta, \gamma \in \mathbb{R} \quad \text{and} \quad a, b, c \in \mathbb{A}. \tag{5.37}$$

The cubic norm (5.20) is defined as,

$$N_3(X) = \alpha\beta\gamma - \alpha a\bar{a} - \beta b\bar{b} - \gamma c\bar{c} + (ab)c + \bar{c}(\bar{b}\bar{c}), \tag{5.38}$$

which, for associative $\mathbb{A}$ coincides with the matrix determinant.

The Jordan product (5.24) is given by

$$X \circ Y = \frac{1}{2}(XY + YX), \qquad X, Y \in \mathfrak{J}_3^{\mathbb{A}}, \tag{5.39}$$

where juxtaposition denotes the conventional matrix product. Evidently, $E = \text{diag}(1, 1, 1)$ is a base point and the corresponding Jordan algebra maps are given by,

$$\begin{aligned} \text{Tr}(X) &= \text{tr}(X), \\ \text{Tr}(X, Y) &= \text{tr}(X \circ Y), \end{aligned} \tag{5.40a}$$

where tr is the conventional matrix trace. The quadratic adjoint (5.23) is given by,

$$X^\sharp = \begin{pmatrix} \beta\gamma - |a|^2 & \bar{b}\bar{a} - \gamma c & ca - \beta\bar{b} \\ ab - \gamma\bar{c} & \alpha\gamma - |b|^2 & \bar{c}\bar{b} - \alpha a \\ \overline{ac} - \beta b & bz - \alpha\bar{a} & \beta\alpha - |c|^2 \end{pmatrix}. \tag{5.41}$$

The irreducible idempotents are given by,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.42}$$

For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ the reduced structure groups are $\text{Str}_0(\mathfrak{J}_3^{\mathbb{A}}) = \text{SL}(3, \mathbb{R}), \text{SL}(3, \mathbb{C}), \text{SU}^\star(6), E_{6(-26)}$ under which $A \in \mathfrak{J}_3^{\mathbb{A}}$ transforms as a $\mathbf{6}, \mathbf{9}, \mathbf{15}, \mathbf{27}$. For $\mathfrak{J}_3^{\mathbb{O}}$ the trace bilinear form and cubic norm are the singlets in $\mathbf{27} \times \mathbf{27}'$ and $\mathbf{27} \times \mathbf{27} \times \mathbf{27}$ (or $\mathbf{27}' \times \mathbf{27}' \times \mathbf{27}'$), respectively. The quadratic adjoint is the $\mathbf{27}$ in $\mathbf{27}' \times \mathbf{27}'$ (or the $\mathbf{27}'$ in $\mathbf{27} \times \mathbf{27}$).

Theorem 13 (Shukuzawa, 2006 [69]). *Every element $A \in \mathfrak{J}_3^{\mathbb{O}}$ of a given rank is $\text{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ related one of the following canonical forms:*

1. *Rank 1*
   a) $A_{1a} = (1, 0, 0) = E_1$
   b) $A_{1b} = (-1, 0, 0) = -E_1$
2. *Rank 2*
   a) $A_{2a} = (1, 1, 0) = E_1 + E_2$
   b) $A_{2b} = (-1, 1, 0) = -E_1 + E_2$
   c) $A_{2c} = (-1, -1, 0) = -E_1 - E_2$
3. *Rank 3*
   a) $A_{3a} = (1, 1, k) = E_1 + E_2 + kE_3$

b) $A_{3b} = (-1,-1,k) = -E_1 - E_2 + kE_3$

Holds for the subcases $\mathfrak{J}_3^{\mathbb{R}}, \mathfrak{J}_3^{\mathbb{C}}, \mathfrak{J}_3^{\mathbb{H}}$.

For the split-octonionic case $\mathfrak{J}_3^{\mathbb{O}_s}$ the reduced structure group is given by $\mathrm{Str}_0(\mathfrak{J}_3^{\mathbb{O}_s}) = E_{6(6)}$, the maximally non-compact form of E_6, under which $A \in \mathfrak{J}_3^{\mathbb{O}_s}$ transforms as a $\mathbf{27}$. The canonical forms simplify in this case.

Theorem 14 (Krutelevich, 2002 [67]). *Every element* $A \in \mathfrak{J}_3^{\mathbb{O}_s}$ *of a given rank is* $\mathrm{Str}_0(\mathfrak{J}_3^{\mathbb{O}_s})$ *related one of the following canonical forms:*

1. *Rank 1*

   a) $A_1 = (1,0,0) = E_1$

2. *Rank 2*

   a) $A_2 = (1,1,0) = E_1 + E_2$

3. *Rank 3*

   a) $A_3 = (1,1,k) = E_1 + E_2 + kE_3$

Holds for the subcases $\mathfrak{J}_3^{\mathbb{R}_s}, \mathfrak{J}_3^{\mathbb{C}_s}, \mathfrak{J}_3^{\mathbb{H}_s}$.

2.3.2. Lorentzian spin factors

The Lorentzian spin factors $\mathfrak{J}_{1,n-1} := \mathbb{R} \oplus \Gamma_{1,n-1}$ are defined by the cubic norm,

$$N(A) = aa_\mu a^\mu = a(a_0^2 - a_i a_i), \quad \text{where} \quad a \in \mathbb{R} \quad \text{and} \quad a_\mu \in \mathbb{R}^{1,n-1} \tag{5.43}$$

for elements $A \in \mathfrak{J}_{1,n-1}$,

$$A = (a; a_\mu) = (a; a_0, a_i). \tag{5.44}$$

For notational convenience we will often only write the first three components $(a; a_0, a_1)$ if $a_i = 0$ for $i > 1$. The linearisation of the cubic norm is given by,

$$N(A,B,C) = \frac{1}{3}(ab_\mu c^\mu + ca_\mu b^\mu + bc_\mu a^\mu). \tag{5.45}$$

Evidently, $E = (1,1,0)$ is a base point and the corresponding Jordan algebra maps are given by,

$$\begin{aligned} \mathrm{Tr}(A) &= a + 2a_0, \\ S(A) &= 2aa_0 + a_\mu a^\mu, \\ S(A,B) &= 2(ab_0 + ba_0 + a_\mu b^\mu), \\ \mathrm{Tr}(A,B) &= ab + 2(a_0 b_0 + a_i b_i). \end{aligned} \tag{5.46a}$$

Using the required identity $\mathrm{Tr}(A^\#, B) = 3N(A,A,B)$ one obtains the quadratic adjoint,

$$A^\# = (a_\mu a^\mu; aa^\mu). \tag{5.47}$$

Note the raised index. Its linearisation $A \times B = (A+B)^\# - A^\# - B^\#$ yields

$$A \times B = (2a_\mu b^\mu; ba^\mu + ab^\mu). \tag{5.48}$$

It is not difficult to verify that

$$A^{\#\#} = N(A)A \tag{5.49}$$

so that the quadratic adjoint is indeed Jordan cubic. Hence, we obtain a well defined Jordan algebra with Jordan product defined by

$$A \circ B = \tfrac{1}{2}(A \times B + \mathrm{Tr}(A)B + \mathrm{Tr}(B)A - S(A, B)c) \tag{5.50}$$

which gives

$$A \circ B = (ab; a_0 b_0 + a_i b_i, a_0 b_i + b_0 a_i). \tag{5.51}$$

The three irreducible idempotents are

$$E_1 = (1; 0), \quad E_2 = (0; \tfrac{1}{2}, \tfrac{1}{2}), \quad E_3 = (0; \tfrac{1}{2}, -\tfrac{1}{2}). \tag{5.52}$$

The reduced structure group $\mathrm{Str}_0(\mathfrak{J}_{1,\mathfrak{n}-1})$ is given by $\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)$ under which A transforms as,

$$(a; a_\mu) \mapsto (e^{2\lambda}a; e^{-\lambda}\Lambda_\mu^\nu a_\nu), \quad \text{where} \quad \lambda \in \mathbb{R}, \Lambda \in \mathrm{SO}(1, n-1). \tag{5.53}$$

Theorem 15. *Every element $A = (a; a_\mu) \in \mathfrak{J}_{1,n-1}$ of a given rank is $\mathrm{Str}_0(\mathfrak{J}_{1,n-1})$ related one of the following canonical forms:*

1. *Rank 1*
   a) $A_{1a} = (1; 0) = E_1$
   b) $A_{1b} = (-1; 0) = -E_1$
   c) $A_{1c} = (0; \tfrac{1}{2}, \tfrac{1}{2}) = E_2$
2. *Rank 2*
   a) $A_{2a} = (0; 1, 0) = E_1 + E_2$
   b) $A_{2b} = (0; 0, 1) = E_1 - E_2$
   c) $A_{2c} = (1; \tfrac{1}{2}, \tfrac{1}{2}) = E_1 + E_2$
   d) $A_{2d} = (-1; \tfrac{1}{2}, \tfrac{1}{2}) = -E_1 + E_2$
3. *Rank 3*
   a) $A_{3a} = (1; \tfrac{1}{2}(1+k), \tfrac{1}{2}(1-k)) = E_1 + E_2 + kE_3$
   b) $A_{3b} = (-1; \tfrac{1}{2}(1+k), \tfrac{1}{2}(1-k)) = -E_1 + E_2 + kE_3$

Note, if one restricts to the identity component of $\mathrm{SO}(1, n-1)$ the orbits A_{1c}, A_{2c} and A_{2d} each split into two cases, $A^{\pm}_{1c}$, $A^{\pm}_{2c}$ and $A^{\pm}_{2d}$, corresponding to the future and past light cones. Similarly, A_{2a} splits into two disconnected components, $A^{\pm}_{2a}$, corresponding to the future and past hyperboloids. For $k > 0$ the orbits A_{3a} and A_{3b} also split into disconnected future and past hyperboloids, $A^{\pm}_{3a}$ and $A^{\pm}_{3b}$.

Proof. $\mathrm{Rank}A = 1 \Rightarrow$

$$A^\sharp = (a_\mu a^\mu, a a^\mu) = 0, \qquad (a; a_\mu) \neq 0. \tag{5.54}$$

This corresponds to two cases: (i) $a_\mu = 0, a \neq 0$ or (ii) $a = 0, a_\mu a^\mu = 0, a_\mu \neq 0$. In case (i) we have

$$A = (a; 0) \mapsto (e^{2\lambda}a; 0) = (\pm 1; 0). \tag{5.55}$$

In case (ii) we have

$$A = (0; a_\mu) \mapsto (0; \Lambda_\mu^\nu a_\nu) = (0; \tfrac{1}{2}, \tfrac{1}{2}). \tag{5.56}$$

Rank$A = 2 \Rightarrow$

$$N(A) = aa_\mu a^\mu = 0, \qquad A^\sharp = (a_\mu a^\mu; aa^\mu) \neq 0. \tag{5.57}$$

This corresponds to two cases: (i) $a = 0, a_\mu a^\mu \neq 0$ or (ii) $a_\mu a^\mu = 0, a \neq 0, a_\mu \neq 0$. In case (i) we have

$$A = (0; a_\mu) \mapsto (0; e^{-\lambda}\Lambda_\mu^\nu a_\nu) = (0; 1, 0) \quad \text{or} \quad (0; 0, 1). \tag{5.58}$$

In case (ii) we have

$$A = (a; a_\mu) \mapsto (e^{2\lambda}a; e^{-\lambda}\Lambda_\mu^\nu a_\nu) = (\pm 1; \tfrac{1}{2}, \tfrac{1}{2}). \tag{5.59}$$

Rank$A = 3 \Rightarrow$

$$N(A) = aa_\mu a^\mu \neq 0. \tag{5.60}$$

Hence,

$$A = (a; a_\mu) \mapsto (e^{2\lambda}a; e^{-\lambda}\Lambda_\mu^\nu a_\nu) = (\pm 1; \tfrac{1}{2}(1+k), \tfrac{1}{2}(1-k)). \tag{5.61}$$

□

3. The Freudenthal triple system

The Freudenthal triple system [59,244,245] provides a natural representation of the dyonic black hole charge vectors for a broad class of 4-dimensional supergravity theories. Namely, $\mathcal{N} = 8$, $\mathcal{N} = 4, 2$ coupled to Abelian vector mulitiplets and the magic $\mathcal{N} = 2$ theories. In the following we present the axiomatic definition of the Freudenthal triple system which is manifestly covariant with respect to the 4-dimensional U-duality group G_4. Subsequently, we present a particular realization in terms of Jordan algebras. This is equivalent to decomposing G_4 with respect to 5-dimensional U-duality group G_5, which is modeled by the corresponding Jordan algebra. Consequently, this particular realization is manifestly covariant with respect to G_5. We use [48, 59, 61, 63, 69, 243] throughout the following.

3.1. Axiomatic definition

A Freudenthal triple system is axiomatically defined [59] as a finite dimensional vector space $\mathfrak{F}$ over a field $\mathbb{F}$ not of characteristic 2 or 3 such that:

1. $\mathfrak{F}$ possesses a non-degenerate antisymmetric bilinear form $\{x, y\}$.
2. $\mathfrak{F}$ possesses a symmetric four-linear form $q(x, y, z, w)$ which is not identically zero.
3. If the ternary product $T(x, y, z)$ is defined on $\mathfrak{F}$ by $\{T(x, y, z), w\} = q(x, y, z, w)$ then

$$3\{T(x, x, y), T(y, y, y)\} = \{x, y\}q(x, y, y, y).$$

It was shown in [63] that every simple reduced FTS is isomorphic to some FTS defined over a cubic Jordan algebra, as described in the following section.

3.2. Definition over a Jordan Algebra

Given a cubic Jordan algebra[2] $\mathfrak{J}$ defined over $\mathbb{R}$, there exists a corresponding FTS,

$$\mathfrak{F}(\mathfrak{J}) = \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{J} \oplus \mathfrak{J}. \tag{5.62}$$

An arbitrary element $x \in \mathfrak{F}(\mathfrak{J})$ may be written as a "2×2 matrix",

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{R} \quad \text{and} \quad A, B \in \mathfrak{J}. \tag{5.63}$$

The FTS comes equipped with a non-degenerate bilinear antisymmetric quadratic form, a quartic form and a trilinear triple product:

1. Quadratic form $\{\bullet, \bullet\}$: $\mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \to \mathbb{R}$

$$\{x, y\} = \alpha\delta - \beta\gamma + \mathrm{Tr}(A, D) - \mathrm{Tr}(B, C), \qquad \text{where} \qquad x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix},\ y = \begin{pmatrix} \gamma & C \\ D & \delta \end{pmatrix}. \tag{5.64a}$$

2. Quartic form $\Delta : \mathfrak{F}(\mathfrak{J}) \to \mathbb{R}$

$$\Delta(x) = -(\alpha\beta - \mathrm{Tr}(A, B))^2 - 4[\alpha N(A) + \beta N(B) - \mathrm{Tr}(A^\sharp, B^\sharp)]. \tag{5.64b}$$

3. Triple product $T : \mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \times \mathfrak{F}(\mathfrak{J}) \to \mathfrak{F}(\mathfrak{J})$ which is uniquely defined by

$$\{T(x, y, w), z\} = 2\Delta(x, y, w, z), \tag{5.64c}$$

where $\Delta(x, y, w, z)$ is the full linearization of $\Delta(x)$ normalized such that $\Delta(x, x, x, x) = \Delta(x)$. For future convenience we present here an explicit form for $T(x) = T(x, x, x)$:

$$T(x) = \begin{pmatrix} T_\alpha & T_A \\ T_B & T_\beta \end{pmatrix} = 2 \begin{pmatrix} -\alpha\kappa(x) - N(B) & -(\beta B^\sharp - B \times A^\sharp) + \kappa(x)A \\ (\alpha A^\sharp - A \times B^\sharp) - \kappa(x)B & \beta\kappa(x) + N(A) \end{pmatrix}, \tag{5.64d}$$

where we have defined the $\mathrm{Str}_0(\mathfrak{J})$ singlet,

$$\kappa(x) := \tfrac{1}{2}[\alpha\beta - \mathrm{Tr}(X, Y)]. \tag{5.64e}$$

Note that all the necessary definitions, such as the cubic and trace bilinear forms, are inherited from the underlying Jordan algebra $\mathfrak{J}$. For notational convenience let,

$$\mathfrak{F}^{\mathbb{A}_{(s)}} = \mathfrak{F}(\mathfrak{J}_3^{\mathbb{A}_{(s)}}), \quad \mathfrak{F}^{2,n} = \mathfrak{F}(\mathfrak{J}_{1,n-1}), \quad \mathfrak{F}^{6,n} = \mathfrak{F}(\mathfrak{J}_{5,n-1}). \tag{5.65}$$

For $\phi \in \mathrm{Str}_0(\mathfrak{J}), X, Y \in \mathfrak{J}, \nu \in \mathbb{R}$ we define $\Phi : \mathfrak{Str}_0(\mathfrak{J}) \oplus \mathfrak{J} \oplus \mathfrak{J} \oplus \mathbb{R} \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{F})$ by,

$$\Phi(\phi, X, Y, \nu) \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} = \begin{pmatrix} \alpha\nu + (Y, B) & \phi A - \frac{1}{3}\nu A + 2Y \times B + \beta X \\ -{}^t\phi B + \frac{1}{3}\nu B + 2X \times A + \alpha Y & -\beta\nu + (X, A) \end{pmatrix}. \tag{5.66}$$

[2] We can more generally consider $\mathfrak{J}$ over a field $\mathbb{F}$ and will do so for the 3-qubit FTS.

Definition 16 (Freudenthal product). *For* $x = (\alpha, \beta, A, B),\;\; y = (\delta, \gamma, C, D)$ *define the Freudenthal product*

$$\wedge : \mathfrak{F} \times \mathfrak{F} \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{F})$$

by,

$$x \wedge y = \Phi(\phi, X, Y, \nu), \quad \text{where} \quad \begin{cases} \phi &= -(A \vee D + B \vee C) \\ X &= -\frac{1}{2}(B \times D - \alpha C - \delta A) \\ Y &= \frac{1}{2}(A \times C - \beta D - \gamma B) \\ \nu &= \frac{1}{4}(\mathrm{Tr}(A, D) + \mathrm{Tr}(C, B) - 3(\alpha\gamma + \beta\delta)) \end{cases} \tag{5.67}$$

where $A \vee B \in \mathfrak{Str}_0(\mathfrak{J})$ *is defined by* $(A \vee B)C = \frac{1}{2}\mathrm{Tr}(B, C)A + \frac{1}{6}\mathrm{Tr}(A, B)C - \frac{1}{2}B \times (A \times C)$. *Note,* $(x \wedge x)x = T(x)$.

3.3. Symmetries

Definition 17 (The automorphism group $\mathrm{Aut}(\mathfrak{F})$). *The* automorphism *group is defined in [59] as the set of invertible* $\mathbb{R}$*-linear transformations preserving the quartic and quadratic forms* (5.68a). *Alternatively, it is defined in [69, 243] as the set of invertible* $\mathbb{R}$*-linear transformations preserving the Freudenthal product* (5.68b).

$$\begin{aligned} \mathrm{Aut}(\mathfrak{F}) \;\; &:= \{\sigma \in \mathrm{Iso}_{\mathbb{R}}(\mathfrak{F}) | \{\sigma x, \sigma y\} = \{x, y\},\; \Delta(\sigma x) = \Delta(x)\} && (5.68a) \\ &= \{\sigma \in \mathrm{Iso}_{\mathbb{R}}(\mathfrak{F}) | \sigma(x \wedge y)\sigma^{-1} = \sigma x \wedge \sigma y\}, && (5.68b) \end{aligned}$$

The equivalence of these alternative definitions follows from:

Lemma 18. *The conditions used in* (5.68a) *and* (5.68b) *are equivalent,*

$$\{\sigma x, \sigma y\} = \{x, y\},\; \Delta(\sigma x) = \Delta(x) \Leftrightarrow \sigma(x \wedge y)\sigma^{-1} = \sigma x \wedge \sigma y \tag{5.69}$$

Proof. We first show that (5.68b) implies (5.68a). To show that,

$$\sigma(x \wedge y)\sigma^{-1} = \sigma x \wedge \sigma y \Rightarrow \{\sigma x, \sigma y\} = \{x, y\} \tag{5.70}$$

we use the identity [243],

$$(x \wedge y)x - (x \wedge x)y + \frac{3}{8}\{x, y\}x = 0, \tag{5.71}$$

from which it follows,

$$\begin{aligned} &\Rightarrow (\sigma x \wedge \sigma y)\sigma x - (\sigma x \wedge \sigma x)\sigma y + \frac{3}{8}\{\sigma x, \sigma y\}\sigma x = 0 \\ &\Rightarrow \sigma(x \wedge y)x - \sigma(x \wedge x)y + \frac{3}{8}\{\sigma x, \sigma y\}\sigma x = 0 \\ &\Rightarrow \sigma(-\frac{3}{8}\{x, y\}x) + \frac{3}{8}\{\sigma x, \sigma y\}\sigma x = 0 \\ &\Rightarrow \frac{3}{8}(\{\sigma x, \sigma y\} - \{x, y\})\sigma x = 0 \\ &\Rightarrow \{\sigma x, \sigma y\} = \{x, y\}. \end{aligned} \tag{5.72}$$

Since $(x \wedge x)x = T(x)$, $\{\sigma x, \sigma y\} = \{x, y\} \Rightarrow \Delta(\sigma x) = \Delta(x)$.

We now show that (5.68a) implies (5.68b). From (5.68a) we have that,

$$\begin{aligned}
\{(\sigma x \wedge \sigma x)\sigma x, \sigma y\} &= \{(x \wedge x)x, y\} \\
&= \{\sigma(x \wedge x)\sigma^{-1}\sigma x, \sigma y\} \\
\Rightarrow \{[(\sigma x \wedge \sigma x) - \sigma(x \wedge x)\sigma^{-1}]\sigma x, \sigma y\} &= 0 \qquad \forall y \\
\Rightarrow [(\sigma x \wedge \sigma x) - \sigma(x \wedge x)\sigma^{-1}]\sigma x &= 0 \\
\Rightarrow (\sigma x \wedge \sigma x) &= \sigma(x \wedge x)\sigma^{-1}.
\end{aligned} \tag{5.73}$$

It then follows from (5.71) that,

$$\begin{aligned}
(\sigma x \wedge \sigma y)\sigma x &= \sigma[(x \wedge x)y - \frac{3}{8}\{x, y\}x] \\
&= \sigma(x \wedge y)x \\
&= \sigma(x \wedge y)\sigma^{-1}\sigma x \\
\Rightarrow \{[(\sigma x \wedge \sigma x) - \sigma(x \wedge y)\sigma^{-1}]\sigma x &= 0 \\
\Rightarrow (\sigma x \wedge \sigma y) &= \sigma(x \wedge y)\sigma^{-1}.
\end{aligned} \tag{5.74}$$

□

Lemma 19 (Brown, 1969). *The following transformations generate elements of* $\mathrm{Aut}(\mathfrak{F})$*:*

$$\begin{aligned}
\varphi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha + (B, C) + (A, C^\sharp) + \beta N(C) & A + \beta C \\ B + A \times C + \beta C^\sharp & \beta \end{pmatrix}; \\
\psi(D) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha & A + B \times D + \alpha D^\sharp \\ B + \alpha D & \beta + (A, D) + (B, D^\sharp) + \alpha N(D) \end{pmatrix}; \\
\widehat{\tau} : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \lambda\alpha & \tau A \\ {}^t\tau^{-1}B & \lambda^{-1}\beta \end{pmatrix};
\end{aligned} \tag{5.75}$$

where $C, D \in \mathfrak{J}$ *and* $\tau \in \mathrm{Str}(\mathfrak{J})$ *s.t.* $N(\tau A) = \lambda N(A)$.

For convenience we define [48],

$$\mathcal{Z} : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} -\beta & -B \\ A & \alpha \end{pmatrix}, \tag{5.76}$$

where $\mathcal{Z} = \phi(-\mathbb{1})\psi(\mathbb{1})\phi(-\mathbb{1})$.

The Freudenthal triple systems, defined over various Jordan algebras, and their associated automorphism groups are summarized in Table 5.1. This table covers a number supergravities of interest: $\mathcal{N} = 2, 4$ coupled to $n + 1$ or n vector multiplets respectively, $\mathcal{N} = 2$ *STU* ($n = 2$), magic $\mathcal{N} = 2$ and $\mathcal{N} = 8$.

Lemma 20. *The Lie algebra* $\mathfrak{Aut}(\mathfrak{F})$ *is given by*

$$\mathfrak{Aut}(\mathfrak{F}) = \{\phi \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{F}) | \Delta(\phi x, x, x, x) = 0, \{\phi x, y\} + \{x, \phi y\} = 0, \ \forall x, y \in \mathfrak{F}\}. \tag{5.77}$$

Table 5.1.: The automorphism group $\mathrm{Aut}(\mathfrak{F}(\mathfrak{J}))$ and the dimension of its representation $\dim \mathfrak{F}(\mathfrak{J})$ given by the Freudenthal construction defined over the cubic Jordan algebra $\mathfrak{J}$ with dimension $\dim \mathfrak{J}$ and reduced structure group $\mathrm{Str}_0(\mathfrak{J})$ [48, 59, 69, 243].

Jordan algebra $\mathfrak{J}$	$\mathrm{Str}_0(\mathfrak{J})$	$\dim \mathfrak{J}$	$\mathrm{Aut}(\mathfrak{F}(\mathfrak{J}))$	$\dim \mathfrak{F}(\mathfrak{J})$
$\mathbb{R}$	$-$	1	$\mathrm{SL}(2,\mathbb{R})$	4
$\mathbb{R}\oplus\mathbb{R}$	$\mathrm{SO}(1,1)$	2	$\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	6
$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	$\mathrm{SO}(1,1)\times\mathrm{SO}(1,1)$	3	$\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	8
$\mathbb{R}\oplus\Gamma_{1,n-1}$	$\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)$	$n+1$	$\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)$	$2(n+2)$
$\mathbb{R}\oplus\Gamma_{5,n-1}$	$\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)$	$n+5$	$\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)$	$2(n+6)$
$\mathfrak{J}_3^{\mathbb{R}}$	$\mathrm{SL}(3,\mathbb{R})$	6	$\mathrm{Sp}(6,\mathbb{R})$	14
$\mathfrak{J}_3^{\mathbb{C}}$	$\mathrm{SL}(3,\mathbb{C})$	9	$\mathrm{SU}(3,3)$	20
$\mathfrak{J}_3^{\mathbb{H}}$	$\mathrm{SU}^\star(6)$	15	$\mathrm{SO}^\star(12)$	32
$\mathfrak{J}_3^{\mathbb{O}}$	$E_{6(-26)}$	27	$E_{7(-25)}$	56
$\mathfrak{J}_3^{\mathbb{O}_s}$	$E_{6(6)}$	27	$E_{7(7)}$	56

Proof. If $\phi \in \mathrm{Hom}_R(\mathfrak{F})$ satisfies $\Delta(e^{t\phi}x, e^{t\phi}x, e^{t\phi}x, e^{t\phi}x) = \Delta(x,x,x,x)$, where $t \in \mathbb{R}$, differentiating with respect to t and then setting $t = 0$ one obtains $\Delta(\phi x, x, x, x) = 0$. Similarly, if $\{e^{t\phi}x, e^{t\phi}y\} = \{x,y\}$ then $\{\phi x, y\} + \{x, \phi y\} = 0$. Conversely, assuming $\{\phi x, y\} + \{x, \phi y\} = 0$ for all $x, y \in \mathfrak{F}$ let $\sigma = e^{t\phi}$. Then,

$$\begin{aligned}\{e^{t\phi}x, e^{t\phi}y\} &= \{(1 + t\phi + \tfrac{1}{2}t^2\phi^2 + \ldots)x, (1 + t\phi + \tfrac{1}{2}t^2\phi^2 + \ldots)y\} \\ &= \{x,y\} + t(\{\phi x, y\} + \{x, \phi y\}) \\ &\quad + t^2(\tfrac{1}{2}\{\phi x, \phi y\} + \tfrac{1}{2}\{\phi^2 x, y\} + \tfrac{1}{2}\{\phi x, \phi y\} + \tfrac{1}{2}\{x, \phi^2 y\}) + \ldots \\ &= \{x,y\}.\end{aligned} \tag{5.78}$$

Similarly, assuming $\Delta(\phi x, x, x, x) = 0$ and letting $\sigma = e^{t\phi}$ then $\Delta(\sigma x) = \Delta(x)$. □

Theorem 21 (Imai and Yokota, 1980)**.**

$$\mathfrak{Aut}(\mathfrak{F}) = \{\Phi(\phi, X, Y, \nu) \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{F}) | \phi \in \mathfrak{Str}_0(\mathfrak{J}), X, Y \in \mathfrak{J}, \nu \in \mathbb{R}\}. \tag{5.79}$$

where the Lie bracket

$$[\Phi(\phi_1, X_1, Y_1, \nu_1), \Phi(\phi_2, X_2, Y_2, \nu_2)] = \Phi(\phi, X, Y, \nu) \tag{5.80}$$

is given by

$$\begin{aligned}\phi &= [\phi_1, \phi_2] + 2(X_1 \vee Y_2 - X_2 \vee Y_1) \\ X &= (\phi_1 + \frac{2}{3}\nu_1)X_2 - (\phi_2 + \frac{2}{3}\nu_2)X_1 \\ Y &= (\phi_2 + \frac{2}{3}\nu_2)Y_1 - ({}^t\phi_1 + \frac{2}{3}\nu_1)Y_2 \\ \nu &= \mathrm{Tr}(X_1, Y_2) - \mathrm{Tr}(Y_1, X_2).\end{aligned} \tag{5.81}$$

Remark 22. *The subset of generators not in* $\mathfrak{Str}(\mathfrak{J})$ *are given by*

$$\widehat{\Phi}(X, Y) := \Phi(0, X, Y, 0). \tag{5.82}$$

The Hermitian conjugate is defined by

$$\widehat{\Phi}^{\dagger}(X,Y) = \widehat{\Phi}(Y,X). \tag{5.83}$$

Hermitian (resp. anti-Hermitian) generators are non-compact (resp. compact) [72].

Lemma 23 (Krutelevich 2004)**.** *Every non-zero element of $\mathfrak{F}(\mathfrak{J})$, where $\mathfrak{J}$ is one of $\mathfrak{J}_3^{\mathbb{A}}, \mathfrak{J}_3^{\mathbb{A}_s}, \mathfrak{J}_{1,n-1}, \mathfrak{J}_{5,n-1}$, can be brought into the form*

$$\begin{pmatrix} 1 & A \\ 0 & \beta \end{pmatrix} \tag{5.84}$$

by $\mathrm{Aut}(\mathfrak{F})$.

Proof. The proof presented by Krutelevich [48] also holds in these cases since it only requires that $\mathfrak{J}$ is spanned by its rank 1 elements and that the bilinear trace form is non-degenerate. □

Lemma 24 (Krutelevich 2004)**.** *An element of $\mathfrak{F}(\mathfrak{J})$, where $\mathfrak{J}$ is one of $\mathfrak{J}_3^{\mathbb{A}}, \mathfrak{J}_3^{\mathbb{A}_s}, \mathfrak{J}_{1,n-1}, \mathfrak{J}_{5,n-1}$, of the form*

$$\begin{pmatrix} \alpha & a_i E_i \\ 0 & \beta \end{pmatrix} \tag{5.85}$$

is $\mathrm{Aut}(\mathfrak{F})$ *related to:*

1.
$$\begin{pmatrix} \alpha & (a_1 + \beta c - \frac{a_2 a_3 c^2}{\alpha})E_1 + a_2 E_2 + a_3 E_3 \\ 0 & \beta - \frac{2a_2 a_3 c}{\alpha} \end{pmatrix}. \tag{5.86}$$

2.
$$\begin{pmatrix} \alpha & a_1 E_1 + (a_2 + \beta c - \frac{a_1 a_3 c^2}{\alpha})E_2 + a_3 E_3 \\ 0 & \beta - \frac{2a_1 a_3 c}{\alpha} \end{pmatrix}. \tag{5.87}$$

3.
$$\begin{pmatrix} \alpha & a_1 E_1 + a_2 E_2 + (a_3 + \beta c - \frac{a_1 a_2 c^2}{\alpha})E_3 \\ 0 & \beta - \frac{2a_1 a_2 c}{\alpha} \end{pmatrix}. \tag{5.88}$$

Proof. The proof used in [48] may be seen to hold in all cases by direct calculation. □

Definition 25 (FTS ranks [48])**.** *An FTS element may be assigned an* $\mathrm{Aut}(\mathfrak{F})$ *invariant* rank*:*

$$\begin{aligned}
\mathrm{Rank}x = 1 &\Leftrightarrow \Upsilon(x,x,y) = 0\ \forall y,\ x \neq 0;\\
\mathrm{Rank}x = 2 &\Leftrightarrow T(x) = 0,\ \exists y \text{ s.t. } \Upsilon(x,x,y) \neq 0;\\
\mathrm{Rank}x = 3 &\Leftrightarrow \Delta(x) = 0,\ T(x) \neq 0;\\
\mathrm{Rank}x = 4 &\Leftrightarrow \Delta(x) \neq 0,
\end{aligned} \tag{5.89}$$

where we have defined $\Upsilon(x,x,y) := 3T(x,x,y) + \{x,y\}x$.

For an element in the reduced form (5.84) the rank conditions simplify:

$$\begin{aligned}
\mathrm{Rank}x = 1 &\Leftrightarrow A = 0,\ \beta = 0;\\
\mathrm{Rank}x = 2 &\Leftrightarrow A^{\sharp} = 0,\ \beta = 0,\ A \neq 0;\\
\mathrm{Rank}x = 3 &\Leftrightarrow 4N(A) = -\beta^2,\ A^{\sharp} \neq 0;\\
\mathrm{Rank}x = 4 &\Leftrightarrow 4N(A) \neq -\beta^2.
\end{aligned} \tag{5.90}$$

In order to distinguish orbits of the same rank we will use the following quadratic form introduced by Shukuzawa [69].

Definition 26 (FTS quadratic form)**.** *Define, for a non-zero constant element $y \in \mathfrak{F}$, the real quadratic form,*

$$\begin{aligned} B_y: \quad \mathfrak{F} \times \mathfrak{F} &\rightarrow \mathbb{R} \\ (x, x) &\mapsto B_y(x) := \{(x \wedge x)y, y\}. \end{aligned} \tag{5.91}$$

Lemma 27 (Shukuzawa, 2006)**.** *If $y' = \sigma y$ for $y \neq 0$ and $\sigma \in \mathrm{Aut}(\mathfrak{F})$ then*

$$B_y(x) = B_{y'}(x'), \quad \textit{where} \quad x' = \sigma x \in \mathfrak{F}.$$

3.4. FTS: Examples

3.4.1. Magic Freudenthal triple systems

The Freudenthal triple system defined over the the Jordan algebra $\mathfrak{J}_3^{\mathbb{O}}$ has automorphism group $\mathrm{Aut}(\mathfrak{F}^{\mathbb{O}_s}) = E_{7(-25)}$. Elements $x \in \mathfrak{F}^{\mathbb{O}_s}$ have $1+1+27+27$ components transforming as the fundamental $\mathbf{56}$ of $E_{7(-25)}$. The Lie algebra, given by $\Phi(\phi, X, Y, \nu)$ is $(78+27+27+1)$-dimensional. Under, $\mathrm{Aut}(\mathfrak{F}^{\mathbb{O}}) \supset \mathrm{Str}(\mathfrak{J}_3^{\mathbb{O}})$, where $\mathrm{Str}(\mathfrak{J}_3^{\mathbb{O}}) = \mathrm{SO}(1,1) \times E_{6(-26)}$ the $\mathbf{56}$ breaks as,

$$\mathbf{56} \rightarrow \mathbf{1}_3 + \mathbf{1}_{-3} + \mathbf{27}_1 + \mathbf{27'}_{-1}, \tag{5.92}$$

where, for an element $x = (\alpha, \beta, A, B)$, α and A correspond to the $\mathbf{1}_{-3}$ and $\mathbf{27}_1$ respectively. The quartic invariant corresponds to the singlet in $\mathbf{56} \times \mathbf{56} \times \mathbf{56} \times \mathbf{56}$ which decomposes as,

$$(\mathbf{1}_3\mathbf{1}_{-3} + \mathbf{27}_1\mathbf{27'}_{-1})^2 + \mathbf{1}_3\mathbf{27'}_{-1}\mathbf{27'}_{-1}\mathbf{27'}_{-1} + \mathbf{1}_{-3}\mathbf{27}_1\mathbf{27}_1\mathbf{27}_1 + (\mathbf{27}_1\mathbf{27}_1)(\mathbf{27'}_{-1}\mathbf{27'}_{-1}). \tag{5.93}$$

The antisymmetric bilinear is the singlet in $\mathbf{56} \times_a \mathbf{56}$, while the Freudenthal product is the $\mathbf{133}$ in $\mathbf{56} \times_s \mathbf{56}$. The triple product is the $\mathbf{56}$ in $(\mathbf{56} \times \mathbf{56} \times \mathbf{56})_s$.

The automorphism group may be used to bring an arbitrary element of a given norm into a canonical form.

Theorem 28 (Shukuzawa, 2006)**.** *Every element $x \in \mathfrak{F}^{\mathbb{O}}$ of a given rank is $\mathrm{Aut}(\mathfrak{F}^{\mathbb{O}})$ related one of the following canonical forms:*

1. *Rank 1*
   a) $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
2. *Rank 2*
   a) $x_{2a} = \begin{pmatrix} 1 & (1,0,0) \\ 0 & 0 \end{pmatrix}$
   b) $x_{2b} = \begin{pmatrix} 1 & (-1,0,0) \\ 0 & 0 \end{pmatrix}$
3. *Rank 3*
   a) $x_{3a} = \begin{pmatrix} 1 & (1,1,0) \\ 0 & 0 \end{pmatrix}$

b) $x_{3b} = \begin{pmatrix} 1 & (-1,-1,0) \\ 0 & 0 \end{pmatrix}$

4. *Rank 4*

   a) $x_{4a} = k \begin{pmatrix} 1 & (-1,-1,-1) \\ 0 & 0 \end{pmatrix}$

   b) $x_{4b} = k \begin{pmatrix} 1 & (1,1,-1) \\ 0 & 0 \end{pmatrix}$

   c) $x_{4c} = k \begin{pmatrix} 1 & (1,1,1) \\ 0 & 0 \end{pmatrix}$

   where $k > 0$.

Holds for $\mathfrak{F}^{\mathbb{R}}, \mathfrak{F}^{\mathbb{C}}, \mathfrak{F}^{\mathbb{H}}$. *Note, the rank 4 cases presented here differ from those originally obtained in [69], but their equivalence is easily checked.*

The maximally split Freudenthal triple system defined over the the Jordan algebra $\mathfrak{J}_3^{\mathbb{O}_s}$ has automorphism group $\text{Aut}(\mathfrak{F}^{\mathbb{O}_s}) = E_{7(7)}$, the maximally non-compact form of E_7. The automorphism group may be used to bring an arbitrary element of a given norm into a canonical form.

Theorem 29 (Krutelevich, 2004)**.** *Every element* $x \in \mathfrak{F}^{\mathbb{A}_s}$ *of a given rank is* $\text{Aut}(\mathfrak{F}^{\mathbb{A}_s})$ *related one of the following canonical forms:*

1. *Rank 1*

   a) $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2. *Rank 2*

   a) $x_{2a} = \begin{pmatrix} 1 & (1,0,0) \\ 0 & 0 \end{pmatrix}$

3. *Rank 3*

   a) $x_{3a} = \begin{pmatrix} 1 & (1,1,0) \\ 0 & 0 \end{pmatrix}$

4. *Rank 4*

   a) $x_{4a} = k \begin{pmatrix} 1 & (-1,-1,-1) \\ 0 & 0 \end{pmatrix}$

   b) $x_{4b} = k \begin{pmatrix} 1 & (1,1,1) \\ 0 & 0 \end{pmatrix}$

   where $k > 0$.

Note, the rank 4 cases presented here differ from those originally obtained in [48], but their equivalence is easily checked.

3.4.2. Lorentzian Freudenthal triple systems

The Freudenthal triple system defined over the the Lorentzian spin factors $\mathfrak{J}_{1,n-1}$ has automorphism group $\mathrm{Aut}(\mathfrak{F}^{2,n}) = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2,n)$. Elements $x \in \mathfrak{F}^{\mathbb{O}_s}$ have $1+1+n+n$ components transforming as the $(\mathbf{2}, \mathbf{2}+\mathbf{n})$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2,n)$. The Lie algebra, given by $\Phi(\phi, X, Y, \nu)$ has $n(n-1)/2 + 1 + (n+1) + (n+1) + 1 = (n+2)(n+1)/2 + 3$ generators.

The automorphism group may be used to bring an arbitrary element of a given norm into a canonical form.

Theorem 30. *Every element $x \in \mathfrak{F}^{2,n}$ of a given rank is $\mathrm{Aut}(\mathfrak{F}^{2,n})$ related one of the following canonical forms:*

1. *Rank 1*

   a) $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2. *Rank 2*

   a) $x_{2a} = \begin{pmatrix} 1 & (1;0,0) \\ 0 & 0 \end{pmatrix}$

   b) $x_{2b} = \begin{pmatrix} 1 & (-1;0,0) \\ 0 & 0 \end{pmatrix}$

   c) $x_{2c} = \begin{pmatrix} 1 & (0;\frac{1}{2},\frac{1}{2}) \\ 0 & 0 \end{pmatrix}$

3. *Rank 3*

   a) $x_{3a} = \begin{pmatrix} 1 & (0;1,0) \\ 0 & 0 \end{pmatrix}$

   b) $x_{3b} = \begin{pmatrix} 1 & (0;0,1) \\ 0 & 0 \end{pmatrix}$

4. *Rank 4*

   a) $x_{4a} = k\begin{pmatrix} 1 & (-1;1,0) \\ 0 & 0 \end{pmatrix}$

   b) $x_{4b} = k\begin{pmatrix} 1 & (1;0,1) \\ 0 & 0 \end{pmatrix}$

   c) $x_{4c} = k\begin{pmatrix} 1 & (-1;0,1) \\ 0 & 0 \end{pmatrix}$

   where $k > 0$.

Proof. We begin by transforming to the generic canonical form

$$\begin{pmatrix} 1 & A \\ 0 & \beta \end{pmatrix}$$

and proceed case by case according to rank.

Rank 1: Rank$x = 1 \Rightarrow A = 0, \beta = 0$ so that every rank 1 element is $\mathrm{Aut}(\mathfrak{F}^{2,n})$ related to

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.94}$$

Rank 2: Rank$x = 2 \Rightarrow A^\sharp = 0, \beta = 0, A \neq 0$ so that every rank 2 element is $\mathrm{Aut}(\mathfrak{F}^{2,n})$ related to

$$x = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix}. \tag{5.95}$$

where A is a rank 1 Jordan algebra element. A may be brought into canonical form via $\hat{\tau}$ where $\tau \in \mathrm{Str}_0(\mathfrak{J}_{1,n-1})$,

$$\widehat{\tau} : \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \tau A \\ 0 & 0 \end{pmatrix},$$

so that x may be brought into three forms corresponding to the three rank 1 canonical forms for A,

$$x_{2a} = \begin{pmatrix} 1 & A_{1a} \\ 0 & 0 \end{pmatrix}; \qquad x_{2b} = \begin{pmatrix} 1 & A_{1b} \\ 0 & 0 \end{pmatrix}; \qquad x_{2c} = \begin{pmatrix} 1 & A_{1c} \\ 0 & 0 \end{pmatrix}. \tag{5.96}$$

These are in fact unrelated as can be seen by computing the quadratic forms,

$$\begin{aligned} B_{x_{2a}}(y) &= -c_\mu c^\mu + \gamma d; \\ B_{x_{2b}}(y) &= c_\mu c^\mu - \gamma d; \\ B_{x_{2c}}(y) &= -cc_0 - cc_1 + \gamma d_0 + \gamma d_1, \end{aligned} \tag{5.97}$$

where

$$y = \begin{pmatrix} \delta & C \\ D & \gamma \end{pmatrix}. \tag{5.98}$$

Diagonalizing (5.97) one can verify that the three forms have distinct signatures and hence, by Sylvester's Law of Inertia, x_{2a}, x_{2b} and x_{2c} lie in distinct orbits.

Rank 3: Rank$x = 3 \Rightarrow N(A) = -\frac{\beta^2}{4}, A^\sharp \neq 0$. If $\beta \neq 0$ then A is $\mathrm{Str}_0(\mathfrak{J}_{1,n-1})$ related to

$$(\pm 1; \frac{1}{2}(1 \mp \frac{\beta^2}{4}), \frac{1}{2}(1 \pm \frac{\beta^2}{4}), 0, \ldots) = \pm E_1 + E_2 \mp \frac{\beta^2}{4} E_3 \tag{5.99}$$

so that by an application of $\hat{\tau}$

$$x = \begin{pmatrix} 1 & \pm E_1 + E_2 \mp \frac{\beta^2}{4} E_3 \\ 0 & \beta \end{pmatrix}. \tag{5.100}$$

Then by Theorem 24 with $c = \mp\frac{\beta^2}{4}$ x may be brought into the form,

$$\begin{pmatrix} 1 & \pm E_1 + E_2 \\ 0 & 0 \end{pmatrix}. \tag{5.101}$$

Hence, we may assume from the outset that

$$x = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix} \tag{5.102}$$

where A is a rank 2 Jordan algebra element. Via an application of $\hat{\tau}$, where $\tau \in \mathrm{Str}_0(\mathfrak{J}_{1,n-1})$, x may be brought into one of four forms corresponding to the four rank 2 canonical forms for A,

$$x_{3a} = \begin{pmatrix} 1 & A_{2a} \\ 0 & 0 \end{pmatrix}; \qquad x_{3b} = \begin{pmatrix} 1 & A_{2b} \\ 0 & 0 \end{pmatrix}; \qquad x_{3c} = \begin{pmatrix} 1 & A_{2c} \\ 0 & 0 \end{pmatrix}; \qquad x_{3d} = \begin{pmatrix} 1 & A_{2d} \\ 0 & 0 \end{pmatrix}. \tag{5.103}$$

We are able to show x_{3a} and x_{3b} are $\mathrm{Aut}(\mathfrak{F}^{2,n})$ related to x_{3d} and x_{3c} respectively. The proof proceeds by an application of $\varphi(\tilde{C})\psi(D)\varphi(C)$ with $\tilde{C}^\sharp = D^\sharp = C^\sharp = 0$ which yields,

$$\begin{pmatrix} 1 & A_{2a} \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 + (A_{2a} \times C + D, \tilde{C}) & A_{2a} + (A_{2a} \times C) \times D + (A_{2a}, D)\tilde{C} \\ A_{2a} \times C + D + (A_{2a} + (A_{2a} \times C) \times D) \times \tilde{C} & (A_{2a}, D) \end{pmatrix}.$$

Assuming $(A_{2a}, D) = 1$ and $\tilde{C} = -(A_{2a} + (A_{2a} \times C) \times D)$ we obtain

$$\begin{pmatrix} 1 - (A_{2a} \times C + D, A_{2a} + (A_{2a} \times C) \times D) & 0 \\ A_{2a} \times C + D & 1 \end{pmatrix}. \tag{5.104}$$

This is achieved by the choice $C = (0; -\frac{1}{2}, -\frac{1}{2}, 0, \ldots)$ and $D = (0; \frac{1}{2}, \frac{1}{2}, 0, \ldots)$. This yields $\tilde{C} = (0; -\frac{1}{2}, \frac{1}{2}, 0, \ldots)$ and $(A_{2a} \times C + D, \tilde{C}) = 1$ so that,

$$\begin{pmatrix} 0 & 0 \\ (-1; \frac{1}{2}, \frac{1}{2}, 0, \ldots) & 1 \end{pmatrix},$$

which, after three applications of $\varphi(-\mathbb{1})\psi(\mathbb{1})\varphi(-\mathbb{1})$, is the desired form,

$$\begin{pmatrix} 1 & A_{2d} \\ 0 & 0 \end{pmatrix}. \tag{5.105}$$

Similary, x_{2b} is $\mathrm{Aut}(\mathfrak{F}^{2,n})$ related to x_{2c}.

The remaining two possiblilities are unrelated as can be seen by computing the quadratic forms,

$$\begin{aligned} B_{x_{3c}}(y) &= -\delta c_0 - \delta c_1 - cc_0 - cc_1 - c_\mu c^\mu + \gamma d + \gamma d_0 + \gamma d_1 + dd_0 + dd_1; \\ B_{x_{3d}}(y) &= \delta c_0 + \delta c_1 - cc_0 - cc_1 + c_\mu c^\mu - \gamma d + \gamma d_0 + \gamma d_1 - dd_0 - dd_1. \end{aligned} \tag{5.106}$$

Diagonalizing (5.106) one can verify that the two forms have distinct signatures and hence, by Sylvester's Law of Inertia, x_{2a} and x_{2b} lie in distinct orbits.

Rank 4: $\mathrm{Rank}x = 4 \Rightarrow \Delta(x) = -N(A) - \frac{\beta^2}{4} \neq 0$. By Theorem 24 we may assume from the outset that

$$x = \begin{pmatrix} 1 & A \\ 0 & 0 \end{pmatrix} \tag{5.107}$$

where A is a rank 3 Jordan algebra element. Via an application of $\hat{\tau}$, where $\tau \in \mathrm{Str}_0(\mathfrak{J}_{1,n-1})$, x may be brought into one of two forms corresponding to the two rank 3 canonical forms for A,

$$x_{4a} = \begin{pmatrix} 1 & (1; \frac{1}{2}(1-k), \frac{1}{2}(1+k), 0, \dots) \\ 0 & 0 \end{pmatrix}, \qquad x_{4b} = \begin{pmatrix} 1 & (-1; \frac{1}{2}(1+k), \frac{1}{2}(1-k), 0, \dots) \\ 0 & 0 \end{pmatrix}, \quad (5.108)$$

where we have chosen our conventions such that $\Delta(x_{4a}) = \Delta(x_{4b}) = k$.

To determine under what conditions x_{4a} and x_{4a} are related we again use the quadratic forms,

$$\begin{aligned} B_{x_{4a}}(y) =& \delta kc - \delta c_0 + \delta kc_0 - cc_0 + kcc_0 - \delta c_1 - \delta kc_1 - cc_1 - kcc_1 - c_\mu c^\mu \\ &+ \gamma d + \gamma d_0 - \gamma kd_0 + dd_0 - kdd_0 - kd_\mu d^\mu + \gamma d_1 + \gamma kd_1 + dd_1 + kdd_1; \\ B_{x_{4b}}(y) =& -\delta kc + \delta c_0 + \delta kc_0 - cc_0 - kcc_0 + \delta c_1 - \delta kc_1 - cc_1 + kcc_1 - c_\mu c^\mu \\ &- \gamma d + \gamma d_0 + \gamma kd_0 - dd_0 - kdd_0 - kd_\mu d^\mu + \gamma d_1 - \gamma kd_1 - dd_1 + kdd_1. \end{aligned} \quad (5.109)$$

Diagonalizing (5.109) leads to quite complicated expressions for the two metrics. However, they only differ in three components, $(1, -\frac{k}{2}, k)$ and $(-1, -k, -\frac{k}{2})$, from which we see that for $k > 0$ the metrics have different signatures. Hence, for $k > 0$ x_{4a} and x_{4b} lie in distinct orbits by Sylvester's Law of Inertia. On the other hand for $k < 0$ the signatures match and using a similar argument to the one used in the rank 3 case with $\varphi(\tilde{C})\psi(D)\varphi(C)$ and $\tilde{C}^\sharp = D^\sharp = C^\sharp = 0$ we may verify they are indeed related. □

CHAPTER 6

Algebraic black p-branes

The extremal black p-brane solutions of supergravity have played, and continue to play, a key role in unravelling the non-perturbative aspects of M-theory. Evidently, understanding the structure of these solutions is of utmost importance. In particular, one would like to know how such solutions are interrelated by U-duality. The electric/magnetic charge vectors of the asymptotically flat p-brane solutions form irreducible U-duality representations as in Table 6.7, which is the sub-table of Table 3.2 satisfying $p \leq D-4$ as required for asymptotic flatness.

Table 6.1.: Asymptotically flat p-brane U-duality representations

D	G	0	1	2	3	4	5	6
10A	$\mathbb{R}^+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$		$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
10B	$\mathrm{SL}(2,\mathbb{R})$		$\mathbf{2}$		$\mathbf{1}$		$\mathbf{2}$	
9	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}^+$	$\mathbf{2+1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2+1}$	
8	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$(\mathbf{3'},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3'},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$		
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathbf{10'}$	$\mathbf{5}$	$\mathbf{5'}$	$\mathbf{10}$			
6	$\mathrm{SO}(5,5;\mathbb{R})$	$\mathbf{16}$	$\mathbf{10}$	$\mathbf{16'}$				
5	$E_{6(6)}(\mathbb{R})$	$\mathbf{27}$	$\mathbf{27'}$					
4	$E_{7(7)}(\mathbb{R})$	$\mathbf{56}$						

In many relevant cases the macroscopic leading-order black p-brane entropy is a function of these charges only, a result of the attractor mechanism [34–37]. Consequently, an important question is whether two *a priori* distinct black p-brane charge configurations are in fact related by U-duality. Mathematically this amounts to determining the distinct charge vector orbits under U-duality. In the classical limit the answer for a large class of theories has been known for some time now [45, 46, 72, 185, 246]. For the maximally supersymmetric theories, obtained by the toroidal compactification of $D = 11, \mathcal{N} = 1$ supergravity, a complete classification of all orbits in all dimensions $D \geq 4$ is known [45, 46]. In the following sections we examine a subset of these theories and, in particular, their U-duality orbits from the perspective of the Jordan algebras and Freudenthal triple systems introduced in chapter 5.

It is well known that there are a number of supergravity theories in five dimensions that are most naturally characterised by Jordan algebras [38–40,45,71–73,246–249]. The scale manifold geometry is fixed by the cubic norm and, for the symmetric theories, is given by $\mathrm{Str}_0(\mathfrak{J})/\mathrm{Aut}(\mathfrak{J})$. Furthermore, the

U-duality group is nothing but $\mathrm{Str}_0(\mathfrak{J})$. The black hole (string) charges transform as the fundamental of $\mathrm{Str}_0(\mathfrak{J})$ and their leading-order Bekenstein-Hawking entropy is given by the cubic norm,

$$S_{D=5,\mathrm{BH}} = \pi\sqrt{|N_3|}. \tag{6.1}$$

Reducing to four dimensions the resulting theories are then described by the corresponding Freudenthal triple system [18,28,38–40,45,71,250–252]. The simplest way of understanding this phenomenon is that, reducing on a circle, we obtain a single extra 1-form gauge potential from the metric (the $D = 4$ graviphoton). In four dimensions the black holes carry one electric and one magnetic charge for each gauge potential and its dual. Hence, we have $1 + 1$ electric/magnetic graviphoton charges, which are singlets under the 5-dimensional U-duality, and $\dim\mathfrak{J} + \dim\mathfrak{J}$ electric/magnetic charges coming from the $\dim\mathfrak{J}$ gauge potentials already present in $D = 5$, giving a "2×2" FTS matrix of charges. The scalar manifolds are now given by $\mathrm{Aut}(\mathfrak{F}(\mathfrak{J}))/\,\mathrm{Str}(\mathfrak{J})'$, where $\mathrm{Str}(\mathfrak{J})'$ is a compact form of $\mathrm{Str}(\mathfrak{J})$. Predictably, the U-duality group is precisely $\mathrm{Aut}(\mathfrak{F})$, the black hole charges transforming as the fundamental. The leading-order Bekenstein-Hawking entropy is given by the quartic norm,

$$S_{D=4,\mathrm{BH}} = \pi\sqrt{|\Delta|}. \tag{6.2}$$

1. Jordan algebras and black holes (strings) in $D = 5$

Jordan algebras make their first appearance in $D = 5, \mathcal{N} = 2$ Maxwell-Einstein supergravity [38–40,246,249]. There are essentially two cases. First, there is the infinite sequence of reducible $\mathcal{N} = 2$ theories coupled to n vector multiplets [38]. These correspond to the Lorentzian spin factors $\mathfrak{J}_{1,n-1} = \mathbb{R} \oplus \Gamma_{1,n-1}$. The term reducible derives from the factorisability of the scalar manifold, which is , in turn, a consequence of the cubic norm being a product of a scalar and a quadratic form. The infinite sequence of reducible $\mathcal{N} = 4$ supergravities coupled to n vector multiplets admits an analogous treatment based on $\mathfrak{J}_{5,n-1} = \mathbb{R} \oplus \Gamma_{5,n-1}$. Second, if the cubic norm does not factorise then the scalar manifold is not necessarily symmetric and a complete classification is not known. However, under the assumption of symmetry there are only four possibilities. These are the so-called "magic" supergravities [38–40]. Their scalar manifolds and U-dualities are naturally described by the sequence of magic Jordan algebras of 3×3 Hermitian matrices defined over the division algebras. They are "magical" in the sense that their U-dualities are given by the magic square of Freudenthal, Rozenfeld and Tits [253–255]. Of them, only the exceptional octonionic theory cannot be obtained via a truncation of the maximally supersymmetric $\mathcal{N} = 8$ theory, which is described by the Jordan algebra of 3×3 Hermitian matrices defined over the *split*-octonions. One way of understanding this is that $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ may be obtained as subalgebras of the split-octonions, whereas $\mathbb{O}$ clearly cannot be. The final Jordan algebraic theory we will consider appears in the $D = 6$ supergravity. While, in general Jordan algebras are perhaps less relevant from a $D = 6$ perspective, there are some theories that are quite naturally characterised by Jordan algebras based on a *quadratic* norm. In particular, the maximally supersymmetric $\mathcal{N} = 8$ theory, which we will briefly discuss in anticipation of its role in chapter 9. The black string charges may be represented as elements of the Jordan algebra of 2×2 Hermitian matrices defined over the split-octonions and have an entropy given by the quadratic norm. Interestingly, the black hole/membrane charges may be represented as split-octonionic 2-vectors, but this particular avenue is left for future work.

1.1. Jordan algebras and $\mathcal{N}=2$ Maxwell-Einstein supergravities

1.1.1. Reducible supergravities and Lorentzian spin factors

The $n+1$ electric black hole charges may be represented as elements $Q=(q;q_\mu)$, where $q\in\mathbb{R}$ and $q_\mu, \mu=0,1,\ldots,n-1$ is a vector of $\mathrm{SO}(1,n-1)$, of the $(n+1)$-dimensional reducible Jordan algebra $\mathfrak{J}_{1,n-1}$. Note, we have adopted the $(1,n-1)$ convention to emphasize the relation to the corresponding $D=4$ theory, whereas in [249] the $(1,n)$ convention was used. For notational convenience we will often only write the first three components $(q;q_0,q_1)$ if $q_i=0$ for $i>1$. See section 2.2 and section 2.3.2 for details. The reducible $D=5, \mathcal{N}=2$ U-duality groups $G_5^{1,n-1}$ are given by the reduced structure group $\mathrm{Str}_0(\mathfrak{J}_{1,n-1})$. $G_5^{1,n-1}=\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)$ under which Q transforms as,

$$(q;q_\mu)\mapsto(e^{2\lambda}q;e^{-\lambda}\Lambda_\mu^\nu q_\nu),\quad \text{where}\quad \lambda\in\mathbb{R},\Lambda\in\mathrm{SO}(1,n-1). \tag{6.3}$$

The cubic norm is given by,

$$I_3(Q)=N_3(Q)=qq_\mu q^\mu. \tag{6.4}$$

The black hole entropy is given by [246],

$$S_{D=5,\mathrm{BH}}=\sqrt{|I_3(Q)|}. \tag{6.5}$$

The U-duality charge orbits may be classified according to the U-duality invariant Jordan rank of the charge vector. Rank 1 and 2 vectors are referred to as critical and light-like, respectively. It follows directly from Theorem 15 that:

Proposition 31 ($\mathcal{N}=2$ reducible canonical forms)**.** *Every black hole charge vector* $Q=(q;q_\mu)\in\mathfrak{J}_{1,n-1}$ *of a given rank is* $\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)$ *related to one of the following canonical forms:*

1. *Rank 1*
   - *a)* $Q_{1a}=(1;0)=E_1$
   - *b)* $Q_{1b}=(-1;0)=-E_1$
   - *c)* $Q_{1c}=(0;\frac{1}{2},\frac{1}{2})=E_2$
2. *Rank 2*
   - *a)* $Q_{2a}=(0;1,0)=E_1+E_2$
   - *b)* $Q_{2b}=(0;0,1)=E_1-E_2$
   - *c)* $Q_{2c}=(1;\frac{1}{2},\frac{1}{2})=E_1+E_2$
   - *d)* $Q_{2d}=(-1;\frac{1}{2},\frac{1}{2})=-E_1+E_2$
3. *Rank 3*
   - *a)* $Q_{3a}=(1;\frac{1}{2}(1+k),\frac{1}{2}(1-k))=E_1+E_2+kE_3$
   - *b)* $Q_{3b}=(-1;\frac{1}{2}(1+k),\frac{1}{2}(1-k))=-E_1+E_2+kE_3$

Note, if one restricts to the identity component of $\mathrm{SO}(1,n-1)$ *the orbits* Q_{1c}*,* Q_{2c} *and* Q_{2d} *each split into two cases,* $Q^\pm_{1c}$*,* $Q^\pm_{2c}$ *and* $Q^\pm_{2d}$*, corresponding to the future and past light cones. Similarly,* Q_{2a} *splits into two disconnected components,* $Q^\pm_{2a}$*, corresponding to the future and past hyperboloids. For* $k>0$ *the orbits* Q_{3a} *and* Q_{3b} *also split into disconnected future and past hyperboloids,* $Q^\pm_{3a}$ *and* $Q^\pm_{3b}$*.*

The orbit stabilizers are easily deduced from the canonical forms. However, the physically distinct solutions are more subtle and depend also on the central charge. The various solutions were carefully derived by different methods in [246] for the large black holes and [249] for the remaining small orbits. The rank, supersymmetry and orbit stabilizers are summarized in Table 6.2.

Table 6.2.: Stability groups of the $D = 5, \mathcal{N} = 2$ Jordan symmetric sequence

Rank	Black hole	Susy	Stabilizer
$1a$	small critical	$1/2$	$\mathrm{SO}(1, n-1)$
$1c$	small critical	$1/2$	$\mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2}$
$2a$	small light-like	$1/2$	$\mathrm{SO}(n-1)$
$2b$	small light-like	0	$\mathrm{SO}(1, n-2)$
$2c^+$	small light-like	$1/2$	$\mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2}$
$2c^-$	small light-like	0	$\mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2}$
$2d^-$	small light-like	$1/2$	$\mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2}$
$2d^+$	small light-like	0	$\mathrm{SO}(n-2) \ltimes \mathbb{R}^{n-2}$
$3a^+(k > 0)$	large	$1/2$	$\mathrm{SO}(n-1)$
$3b^-(k > 0)$	large	$1/2$	$\mathrm{SO}(n-1)$
$3a^-(k > 0)$	large	0 $(Z_H \neq 0)$	$\mathrm{SO}(n-1)$
$3b^+(k > 0)$	large	0 $(Z_H \neq 0)$	$\mathrm{SO}(n-1)$
$3ab(k < 0)$	large	0 $(Z_H = 0)$	$\mathrm{SO}(1, n-2)$

1.1.2. Magic supergravities and Hermitian matrices over the division algebras

The $3 + 3 \dim \mathbb{A}$ electric black hole charges may be represented as elements

$$Q = \begin{pmatrix} q_1 & Q_s & \overline{Q_c} \\ \overline{Q_s} & q_2 & Q_v \\ Q_c & \overline{Q}_v & q_3 \end{pmatrix}, \quad \text{where} \quad q_1, q_2, q_3 \in \mathbb{R} \quad \text{and} \quad Q_{v,s,c} \in \mathbb{A} \tag{6.6}$$

of the $(3+3 \dim \mathbb{A})$-dimensional Jordan algebra $\mathfrak{J}_3^{\mathbb{A}}$ of 3×3 Hermitian matrices over the division algebra $\mathbb{A}$. See section 2.2 and section 2.3.1 for details. The magic $D = 5, \mathcal{N} = 2$ U-duality groups $G_5^{\mathbb{A}}$ are given by the reduced structure group $\mathrm{Str}_0(\mathfrak{J}_3^{\mathbb{A}})$. See section 2.2.2 for details. For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ the U-duality $G_5^{\mathbb{A}}$ is $\mathrm{SL}(3, \mathbb{R}), \mathrm{SL}(3, \mathbb{C}), \mathrm{SU}^\star(6), E_{6(-26)}$ under which $Q \in \mathfrak{J}_3^{\mathbb{A}}$ transforms as a $\mathbf{6}, \mathbf{9}, \mathbf{15}, \mathbf{27}$. The cubic norm is given by,

$$I_3(Q) = N(Q) = q_1q_2q_3 - q_1Q_v\overline{Q_v} - q_2Q_c\overline{Q_c} - q_3Q_s\overline{Q_s} + (Q_vQ_c)Q_s + \overline{Q_s}(\overline{Q_cQ_v}). \tag{6.7}$$

The black hole entropy is given by [37],

$$S_{D=5,\mathrm{BH}} = \pi\sqrt{|I_3(Q)|}. \tag{6.8}$$

The U-duality charge orbits may be classified according to the U-duality invariant Jordan rank of the charge vector. Rank 1 and 2 vectors are referred to as critical and light-like, respectively. It follows directly from Theorem 13 that:

Proposition 32 ($\mathcal{N} = 2$ magic canonical forms)**.** *Every black hole charge vector $Q \in \mathfrak{J}_3^{\mathbb{A}}$ of a given rank is U-duality related to one of the following canonical forms:*

1. *Rank 1*
   a) $Q_{1a} = \mathrm{diag}(1, 0, 0) = E_1$
   b) $Q_{1b} = \mathrm{diag}(-1, 0, 0) = -E_1$
2. *Rank 2*
   a) $Q_{2a} = \mathrm{diag}(1, 1, 0) = E_1 + E_2$
   b) $Q_{2b} = \mathrm{diag}(-1, 1, 0) = -E_1 + E_2$
   c) $Q_{2c} = \mathrm{diag}(-1, -1, 0) = -E_1 - E_2$
3. *Rank 3*
   a) $Q_{3a} = \mathrm{diag}(1, 1, k) = E_1 + E_2 + kE_3$
   b) $Q_{3b} = \mathrm{diag}(-1, -1, k) = -E_1 - E_2 + kE_3$

Note, the orbits generated by the conical forms Q_{1a} and Q_{1b} are isomorphic as are those generated by Q_{2a} and Q_{2c}. The ranks, supersymmetry and orbit stabilizer, which were derived in [45] and [249] for large and small black holes respectively, are summarized in Table 6.3. The $\mathcal{N} = 6$ interpretation of the $\mathfrak{J}_3^{\mathbb{H}}$ charge orbits was given in [249].

Table 6.3.: Stability groups of the magic $D = 5, \mathcal{N} = 2$ supergravities

Rank	Black hole	Susy	$\mathfrak{J}_3^{\mathbb{O}}$	$\mathfrak{J}_3^{\mathbb{H}}$
1	small critical	1/2	$\mathrm{SO}(1,9) \ltimes \mathbb{R}^{16}$	$[\mathrm{SO}(1,5) \times \mathrm{SO}(3)] \ltimes \mathbb{R}^{(4,2)}$
$2a$	small light-like	0	$\mathrm{SO}(1,8) \ltimes \mathbb{R}^{16}$	$[\mathrm{SO}(1,4) \times \mathrm{SO}(3)] \ltimes \mathbb{R}^{(4,2)}$
$2b$	small light-like	1/2	$\mathrm{SO}(9) \ltimes \mathbb{R}^{16}$	$[\mathrm{SO}(5) \times \mathrm{SO}(3)] \ltimes \mathbb{R}^{(4,2)}$
$3a(k>0)$	large	1/2	$F_{4(-52)}$	$\mathrm{Usp}(6)$
$3b(k>0)$	large	0 ($Z_H \neq 0$)	$F_{4(-20)}$	$\mathrm{Usp}(2,4)$

Rank	Black hole	Susy	$\mathfrak{J}_3^{\mathbb{C}}$	$\mathfrak{J}_3^{\mathbb{R}}$
1	small critical	1/2	$[\mathrm{SO}(1,3) \times \mathrm{SO}(2)] \ltimes \mathbb{R}^{2,2}$	$\mathrm{SO}(1,2) \ltimes \mathbb{R}^2$
$2a$	small light-like	0	$[\mathrm{SO}(1,2) \times \mathrm{SO}(2)] \ltimes \mathbb{R}^{2,2}$	$\mathrm{SO}(1,1) \ltimes \mathbb{R}^2$
$2b$	small light-like	1/2	$[\mathrm{SO}(3) \times \mathrm{SO}(2)] \ltimes \mathbb{R}^{2,2}$	$\mathrm{SO}(2) \ltimes \mathbb{R}^2$
$3a(k>0)$	large	1/2	$\mathrm{SU}(3)$	$\mathrm{SO}(3)$
$3b(k>0)$	large	0 ($Z_H \neq 0$)	$\mathrm{SU}(1,2)$	$\mathrm{SO}(1,2)$

1.2. Jordan algebras and $\mathcal{N} = 8$ supergravity

1.2.1. $D = 5, \mathcal{N} = 8$ supergravity and 3×3 Hermitian matrices over the split-octonions

The 27 electric black hole charges Q transform as the fundamental $\mathbf{27}$ of the continuous U-duality group $E_{6(6)}$. Under $\mathrm{SO}(1,1;\mathbb{R}) \times \mathrm{SO}(5,5;\mathbb{R})$ the $\mathbf{27}$ breaks as

$$\mathbf{27} \to \mathbf{1}_4 + \mathbf{10}_{-2} + \mathbf{16}_1 \tag{6.9}$$

where the singlet may be identified as the graviphoton charge descending from $D = 6$, the $\mathbf{10}$ as the remaining NS-NS sector charges and the $\mathbf{16}$ as the R-R sector charges [71]. Further decomposing under $\mathrm{SO}(4,4;\mathbb{R})$ one obtains

$$\mathbf{27} \to \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{8}_v + \mathbf{8}_s + \mathbf{8}_c. \tag{6.10}$$

In this basis the charges Q may be conveniently represented as an element Q of the cubic Jordan algebra $\mathfrak{J}_3^{\mathbb{O}_s}$ of *split*-octonionic 3×3 Hermitian matrices,

$$Q = \begin{pmatrix} q_1 & Q_s & \overline{Q_c} \\ \overline{Q_s} & q_2 & Q_v \\ Q_c & \overline{Q}_v & q_3 \end{pmatrix}, \quad \text{where} \quad q_1, q_2, q_3 \in \mathbb{R} \quad \text{and} \quad Q_{v,s,c} \in \mathbb{O}_s. \tag{6.11}$$

The cubic norm N_3 (5.37) is then given by the determinant like object,

$$N_3(Q) = q_1 q_2 q_3 - q_1 Q_v \overline{Q_v} - q_2 Q_c \overline{Q_c} - q_3 Q_s \overline{Q_s} + (Q_v Q_c) Q_s + \overline{Q_s}(\overline{Q_c Q_v}). \tag{6.12}$$

The set of invertible linear transformations leaving the cubic norm and trace bilinear form invariant is nothing but the $D = 5$ U-duality group $E_{6(6)}(\mathbb{R})$.

The black hole entropy is simply given by the cubic norm,

$$S_{\text{BS}} = \pi \sqrt{|N_3(Q)|}. \tag{6.13}$$

In this case there are three U-duality orbits, 1/2-BPS and 1/4-BPS “small” orbits and a single 1/8-BPS “large” orbit [45]. These orbits may distinguished by the Jordan rank of Q,

$$\begin{array}{lcl} \text{Rank 1} & Q \neq 0, Q^\sharp = 0 & \text{1/2-BPS}, \\ \text{Rank 2} & Q^\sharp \neq 0, N_3(Q) = 0 & \text{1/4-BPS}, \\ \text{Rank 3} & N_3(Q) \neq 0 & \text{1/8-BPS}, \end{array} \tag{6.14}$$

It follows directly from Theorem 14 that:

Proposition 33 ($\mathcal{N} = 8$ canonical forms)**.** *Every black hole charge vector $Q \in \mathfrak{J}_3^{\mathbb{O}_s}$ of a given rank is U-duality related to one of the following canonical forms:*

1. *Rank 1*
   a) $Q_1 = \operatorname{diag}(1, 0, 0) = E_1$
2. *Rank 2*
   a) $Q_2 = \operatorname{diag}(1, 1, 0) = E_1 + E_2$
3. *Rank 3*
   a) $Q_3 = \operatorname{diag}(1, 1, k) = E_1 + E_2 + kE_3$

The orbits with their rank conditions, dimensions and representative states are summarized in Table 6.4. The 27 magnetic black string charges P form the contragradient $\mathbf{27}'$ of $E_{6(6)}(\mathbb{R})$. The orbit classification is identical to the black hole case.

1.2.2. $D = 6, \mathcal{N} = 8$ supergravity and 2×2 Hermitian matrices over the split-octonions

The $5 + 5$ electric/magnetic black string charges form an $\mathrm{SO}(5, 5; \mathbb{R})$ vector $\mathcal{Q}$. Under $\mathrm{SO}(1, 1; \mathbb{R}) \times \mathrm{SO}(4, 4; \mathbb{R})$ the vector breaks as

$$\mathbf{10} \rightarrow \mathbf{1}_2 + \mathbf{1}_{-2} + \mathbf{8}_{v0}, \tag{6.15}$$

Table 6.4.: $D = 5$ black hole orbits, their corresponding rank conditions, dimensions and SUSY [45].

Rank	Rank/orbit conditions		Representative state	Orbit	dim	SUSY
	non-vanishing	vanishing				
1	Q	$Q^\sharp$	$\mathrm{diag}(1,0,0)$	$\dfrac{E_{6(6)}(\mathbb{R})}{O(5,5;\mathbb{R}) \ltimes \mathbb{R}^{16}}$	17	1/2
2	$Q^\sharp$	$N_3(Q)$	$\mathrm{diag}(1,1,0)$	$\dfrac{E_{6(6)}(\mathbb{R})}{O(5,4;\mathbb{R}) \ltimes \mathbb{R}^{16}}$	26	1/4
3	$N_3(Q)$	–	$\mathrm{diag}(1,1,k)$	$\dfrac{E_{6(6)}(\mathbb{R})}{F_{4(4)}(\mathbb{R})}$	26	1/8

where the singlets lie in the NS-NS sector and correspond to a fundamental string and an NS5-brane, while the $\mathbf{8}_v$ is made up of R-R charges. In this basis the charges $\mathcal{Q}$ may be conveniently represented as an element $\mathcal{Q}$ of the Jordan algebra $\mathfrak{J}_2^{\mathbb{O}_s}$ of *split*-octonionic 2×2 Hermitian matrices,

$$\mathcal{Q} = \begin{pmatrix} p^0 & Q_v \\ \overline{Q}_v & q_0 \end{pmatrix}, \quad \text{where} \quad q^0, p_0 \in \mathbb{R} \quad \text{and} \quad Q_v \in \mathbb{O}_s. \tag{6.16}$$

The set of linear invertible transformations leaving the quadratic norm, $N_2(\mathcal{Q}) = \det(\mathcal{Q})$, invariant is the $D = 6$ U-duality group $\mathrm{SO}(5,5;\mathbb{R})$. The black string entropy is proportional to the quadratic norm,

$$S_{D=6,\text{BS}} \sim |N_2(\mathcal{Q})|. \tag{6.17}$$

See *e.g.* [45, 256, 257], and Refs. therein.

There are two U-duality orbits, one 1/2-BPS "small" orbit and one 1/4-BPS "large" orbit [46, 71, 257]. These orbits may distinguished by the Jordan rank of $\mathcal{Q}$ as detailed in section B.1,

$$\begin{array}{lcl} \text{Rank 1} & \mathcal{Q} \neq 0, N_2(\mathcal{Q}) = 0 & \text{1/2-BPS,} \\ \text{Rank 2} & N_2(\mathcal{Q}) \neq 0 & \text{1/4-BPS,} \end{array} \tag{6.18}$$

It follows directly from Theorem 14 (or as a corollary of (9.26)) that:

Proposition 34 ($\mathcal{N} = 8$ canonical forms)**.** *Every black hole charge vector $Q \in \mathfrak{J}_2^{\mathbb{O}_s}$ of a given rank is U-duality related to one of the following canonical forms:*

1. *Rank 1*

   a) $Q_{1a} = \mathrm{diag}(1,0) = E_1$

2. *Rank 2*

   a) $Q_{2a} = \mathrm{diag}(1,k) = E_1 + kE_2$

The orbits, their rank conditions, dimensions and representative states are summarized in Table 6.5.

2. Freudenthal triple systems and black holes in $D = 4$

Dimensionally reducing the $D = 5, \mathcal{N} = 2$ Maxwell-Einstein theories described by $\mathfrak{J}$, we obtain the $D = 4$ Maxwell-Einstein theories described by $\mathfrak{F}(\mathfrak{J})$ [38–40, 246, 249]. The infinite sequence of reducible $\mathcal{N} = 2$ theories coupled to $n + 1$ vector multiplets corresponds to $\mathfrak{F}^{2,n} = \mathfrak{F}(\mathfrak{J}_{1,n-1})$. Like the

Table 6.5.: $D=6$ black string orbits, their corresponding rank conditions, dimensions and SUSY [46].

Rank	Rank/orbit conditions non-vanishing	Rank/orbit conditions vanishing	Representative state	Orbit	dim	SUSY
1	$\mathcal{Q}$	$N_2(\mathcal{Q})$	$\text{diag}(1,0)$	$\frac{\text{SO}(5,5;\mathbb{R})}{\text{SO}(4,4;\mathbb{R})\rtimes\mathbb{R}^8}$	9	1/2
2	$N_2(\mathcal{Q})$	–	$\text{diag}(1,k)$	$\frac{\text{SO}(5,5;\mathbb{R})}{\text{SO}(5,4;\mathbb{R})}$	9	1/4

parent $D=5$ theory, the scalar manifold is factorisable. The infinite sequence of reducible $\mathcal{N}=4$ supergravities coupled to n vector multiplets admits an analogous treatment based on $\mathfrak{F}^{6,n}=\mathfrak{F}(\mathfrak{J}_{5,n-1})$. The magic supergravities $\mathfrak{F}^{\mathbb{A}}=\mathfrak{F}(\mathfrak{J}_3^{\mathbb{A}})$ again have U-dualities given by the magic square [40]. Only the exceptional octonionic theory cannot be obtained via a truncation of the maximally supersymmetric $\mathcal{N}=8$ theory, which is described by $\mathfrak{F}^{\mathbb{O}_s}=\mathfrak{F}(\mathfrak{J}_3^{\mathbb{O}_s})$.

2.1. The FTS and $\mathcal{N}=2$ Maxwell-Einstein supergravities

2.1.1. Reducible supergravities and the reducible FTS

The $(n+2)+(n+2)$ electric+magnetic black hole charges may be represented as elements

$$x=\begin{pmatrix}-q_0 & P\\ Q & p^0\end{pmatrix},\quad \text{where}\quad p^0,q^0\in\mathbb{R}\quad\text{and}\quad Q,P\in\mathfrak{J}_{1,n-1} \tag{6.19}$$

of the Freudenthal triple system $\mathfrak{F}^{2,n}:=\mathfrak{F}(\mathfrak{J}_{1,n-1})$. The reducible $D=4$, $\mathcal{N}=2$ U-duality groups $G_4^{2,n}$ are given by the automorphism group $\text{Aut}(\mathfrak{F}^{2,n})$. $\text{Aut}(\mathfrak{F}^{2,n})=\text{SL}(2,\mathbb{R})\times\text{SO}(2,n)$ under which $x\in\mathfrak{F}^{2,n}$ transforms as a $(\mathbf{2},\mathbf{2}+\mathbf{n})$. The black hole entropy is given by,

$$S_{D=4,\text{BH}}=\pi\sqrt{|\Delta(x)|}. \tag{6.20}$$

The U-duality charge orbits are classified according to the $G_4^{2,n}$-invariant FTS rank of the charge vector. It follows directly from Theorem 30 that:

Theorem 35. *Every black hole charge vector $x\in\mathfrak{F}^{2,n}$ of a given rank is U-duality related to one of the following canonical forms:*

1. *Rank 1*

   a) $x_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$

2. *Rank 2*

   a) $x_{2a}=\begin{pmatrix}1&(1;0,0)\\0&0\end{pmatrix}$

   b) $x_{2b}=\begin{pmatrix}1&(-1;0,0)\\0&0\end{pmatrix}$

   c) $x_{2c}=\begin{pmatrix}1&(0;\frac{1}{2},\frac{1}{2})\\0&0\end{pmatrix}$

3. *Rank 3*

   a) $x_{3a} = \begin{pmatrix} 1 & (0;1,0) \\ 0 & 0 \end{pmatrix}$

   b) $x_{3b} = \begin{pmatrix} 1 & (0;0,1) \\ 0 & 0 \end{pmatrix}$

4. *Rank 4*

   a) $x_{4a} = k \begin{pmatrix} 1 & (-1;1,0) \\ 0 & 0 \end{pmatrix}$

   b) $x_{4b} = k \begin{pmatrix} 1 & (1;0,1) \\ 0 & 0 \end{pmatrix}$

   c) $x_{4c} = k \begin{pmatrix} 1 & (-1;0,1) \\ 0 & 0 \end{pmatrix}$

   where $k > 0$.

The orbits are summarized in Table 6.6, with a comparison to the classification of [249]. Note, the STU model corresponds to $n = 2$. In this case, the rank 2 orbits become identified and have identical cosets as do the rank 3 orbits. This is related to the triality symmetry possessed by the STU model [2], that is absent for $n \neq 2$.

Table 6.6.: Stability groups of the $D = 4, \mathcal{N} = 2$ reducible sequence.

Rank	[249]	Black hole	Susy	Stabilizer
1	**A.3**	small doubly critical	1/2	$[\mathrm{SO}(1,1) \times \mathrm{SO}(1,n-1)] \ltimes (\mathbb{R} \times \mathbb{R}^n)$
$2a$	**A.2**	small critical	0	$\mathrm{SO}(2,n-1) \times \mathbb{R}$
$2b$	**A.1**	small critical	1/2	$\mathrm{SO}(1,n) \times \mathbb{R}$
$2c$	**B**	small critical	0	$\mathrm{SO}(2,1) \times \mathrm{SO}(n-2) \times \mathbb{R}$
$3a$	**C.1**	small light-like	1/2	$\mathrm{SO}(n-1) \ltimes \mathbb{R}^{n-1} \times \mathbb{R}$
$3b$	**C.2**	small light-like	0	$\mathrm{SO}(1,n-2) \ltimes \mathbb{R}^{n-1} \times \mathbb{R}$
$4a$	α	large time-like	1/2	$\mathrm{SO}(2) \times \mathrm{SO}(n)$
$4b$	γ	large time-like	0, $Z = 0$	$\mathrm{SO}(2) \times \mathrm{SO}(2,n-2)$
$4c$	β	large space-like	0, $Z \neq 0$	$\mathrm{SO}(1,1) \times \mathrm{SO}(1,n-1)$

2.1.2. Magic supergravities and the magic FTS

The $(4+3\dim\mathbb{A})+(4+3\dim\mathbb{A})$ electric+magnetic black hole charges may be represented as elements

$$x = \begin{pmatrix} -q_0 & P \\ Q & p^0 \end{pmatrix}, \quad \text{where} \quad p^0, q^0 \in \mathbb{R} \quad \text{and} \quad Q, P \in \mathfrak{J}_3^{\mathbb{A}}. \tag{6.21}$$

of the Freudenthal triple system $\mathfrak{F}^{\mathbb{A}} := \mathfrak{F}(\mathfrak{J}_3^{\mathbb{A}})$. The magic $D = 4$, $\mathcal{N} = 2$ U-duality groups $G_4^{\mathbb{A}}$ are given by the automorphism group $\mathrm{Aut}(\mathfrak{F}^{\mathbb{A}})$. For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ the U-duality group $G_4^{\mathbb{A}}$ is $\mathrm{Sp}(6,\mathbb{R}), \mathrm{SU}(3,3), \mathrm{SO}^\star(12), E_{7(-25)}$ under which $x \in \mathfrak{F}^{\mathbb{A}}$ transforms as a $\mathbf{14}, \mathbf{20}, \mathbf{32}, \mathbf{56}$. The black hole entropy is given by,

$$S_{D=4,\mathrm{BH}} = \pi\sqrt{\Delta(x)|}. \tag{6.22}$$

The U-duality charge orbits are classified according to the $G_4^{\mathbb{A}}$-invariant FTS rank of the charge vector. It follows directly from Theorem 28 that:

Proposition 36. *Every black hole charge vector $x \in \mathfrak{F}^{\mathbb{A}}$ of a given rank is $G_4^{\mathbb{A}}$ related one of the following canonical forms:*

1. *Rank 1*

   a) $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2. *Rank 2*

   a) $x_{2a} = \begin{pmatrix} 1 & (1,0,0) \\ 0 & 0 \end{pmatrix}$

   b) $x_{2b} = \begin{pmatrix} 1 & (-1,0,0) \\ 0 & 0 \end{pmatrix}$

3. *Rank 3*

   a) $x_{3a} = \begin{pmatrix} 1 & (1,1,0) \\ 0 & 0 \end{pmatrix}$

   b) $x_{3b} = \begin{pmatrix} 1 & (-1,-1,0) \\ 0 & 0 \end{pmatrix}$

4. *Rank 4*

   a) $x_{4a} = k \begin{pmatrix} 1 & (-1,-1,-1) \\ 0 & 0 \end{pmatrix}$

   b) $x_{4b} = k \begin{pmatrix} 1 & (1,1,-1) \\ 0 & 0 \end{pmatrix}$

   c) $x_{4c} = k \begin{pmatrix} 1 & (1,1,1) \\ 0 & 0 \end{pmatrix}$

   where $k > 0$.

The orbit stabilizers are summarized in Table 6.7. The details for $\mathfrak{F}^{\mathbb{O}}$ are presented as an example in Appendix C.

2.1.3. $\mathcal{N} = 8$ black holes and U-duality orbits of $E_{7(7)}(\mathbb{R})$

The 28+28 electric/magnetic black hole charges x transform as the fundamental **56** of the continuous U-duality group $E_{7(7)}(\mathbb{R})$. Under $\mathrm{SO}(1,1;\mathbb{R}) \times E_{6(6)}(\mathbb{R})$ the **56** breaks as

$$\mathbf{56} \to \mathbf{1}_3 + \mathbf{1}_{-3} + \mathbf{27}_1 + \mathbf{27'}_{-1} \tag{6.23}$$

where the singlets may be identified as the graviphoton charge and its electromagnetic dual descending from $D = 5$. In this basis the charges x may be conveniently represented as an element x of the Freudenthal triple system $\mathfrak{F}(\mathfrak{J}_3^{s})$,

$$x = \begin{pmatrix} -q_0 & P \\ Q & p^0 \end{pmatrix}, \quad \text{where} \quad q_0, p^0 \in \mathbb{R} \quad \text{and} \quad Q, P \in \mathfrak{J}_3^{s}. \tag{6.24}$$

Table 6.7.: Stability groups of the magic $D = 4$ supergravities

Rank	Black hole	Susy	$\mathfrak{F}^{\mathbb{O}}$	$\mathfrak{F}^{\mathbb{H}}$
1	small doubly critical	1/2	$E_{6(-26)} \ltimes \mathbb{R}^{27}$	$\mathrm{SU}^{\star}(6) \ltimes \mathbb{R}^{15}$
$2a$	small critical	0	$\mathrm{SO}(2,9) \ltimes \mathbb{R}^{32} \times \mathbb{R}$	$[\mathrm{SO}(2,5) \times \mathrm{SO}(3)] \ltimes \mathbb{R}^{8,2} \times \mathbb{R}$
$2b$	small critical	1/2	$\mathrm{SO}(1,10) \ltimes \mathbb{R}^{32} \times \mathbb{R}$	$[\mathrm{SO}(1,6) \times \mathrm{SO}(3)] \ltimes \mathbb{R}^{8,2} \times \mathbb{R}$
$3a$	small light-like	0	$F_{4(-20)} \ltimes \mathbb{R}^{26}$	$\mathrm{Usp}(6) \ltimes \mathbb{R}^{14}$
$3b$	small light-like	1/2	$F_{4(-52)} \ltimes \mathbb{R}^{26}$	$\mathrm{Usp}(4,2) \ltimes \mathbb{R}^{14}$
$4a$	large time-like	1/2	$E_{6(-78)}$	$\mathrm{SU}(6)$
$4b$	large time-like	0 ($Z = 0$)	$E_{6(-14)}$	$\mathrm{SU}(4,2)$
$4c$	large space-like	0 ($Z \neq 0$)	$E_{6(-26)}$	$\mathrm{SU}^{\star}(6)$
Rank	BH	Susy	$\mathfrak{F}^{\mathbb{C}}$	$\mathfrak{F}^{\mathbb{R}}$
1	small doubly critical	1/2	$\mathrm{SL}(3,\mathbb{C}) \ltimes \mathbb{R}^{9}$	$\mathrm{SL}(3,\mathbb{R}) \ltimes \mathbb{R}^{6}$
$2a$	small critical	0	$[\mathrm{SO}(2,3) \times \mathrm{SO}(2)] \ltimes \mathbb{R}^{4,2} \times \mathbb{R}$	$\mathrm{SO}(2,2) \ltimes \mathbb{R}^{4} \times \mathbb{R}$
$2b$	small critical	1/2	$[\mathrm{SO}(1,4) \times \mathrm{SO}(2)] \ltimes \mathbb{R}^{4,2} \times \mathbb{R}$	$\mathrm{SO}(1,3) \ltimes \mathbb{R}^{4} \times \mathbb{R}$
$3a$	small light-like	0	$\mathrm{SU}(1,2) \ltimes \mathbb{R}^{8}$	$\mathrm{SU}(1,1) \ltimes \mathbb{R}^{5}$
$3b$	small light-like	1/2	$\mathrm{SU}(3) \ltimes \mathbb{R}^{8}$	$\mathrm{SU}(2) \ltimes \mathbb{R}^{5}$
$4a$	large time-like	1/2	$\mathrm{SU}(3) \times \mathrm{SU}(3)$	$\mathrm{SU}(3)$
$4b$	large time-like	0 ($Z = 0$)	$\mathrm{SU}(1,2) \times \mathrm{SU}(1,2)$	$\mathrm{SU}(1,2)$
$4c$	large space-like	0 ($Z \neq 0$)	$\mathrm{SL}(3,\mathbb{C})$	$\mathrm{SL}(3,\mathbb{R})$

Here, p^0, q_0 are the graviphotons and P, Q are the magnetic/electric $\mathbf{27}'$ and $\mathbf{27}$ respectively.

The set of invertible linear transformations leaving the quartic norm and the antisymmetric bilinear form (5.64a) invariant is nothing but the $D = 4$ U-duality group $E_{7(7)}(\mathbb{R})$. The black hole entropy is simply given by the quartic norm [45, 193],

$$S_{D=4,\mathrm{BH}} = \pi\sqrt{|\Delta(x)|}. \tag{6.25}$$

In this case there are five U-duality orbits, three 1/2-BPS, 1/4-BPS, 1/8-BPS "small" orbits and two "large" orbits, one 1/8-BPS and one non-BPS depending on the sign of the unique quartic $E_{7(7)}(\mathbb{Z})$ invariant [45]. These orbits may be distinguished by the FTS rank of x,

$$\begin{array}{lcl}
\text{Rank 1} & x \neq 0, 3T(x,x,y) + x\{x,y\} = 0\,\forall y & \text{1/2-BPS,} \\
\text{Rank 2} & \exists y \text{ s.t. } 3T(x,x,y) + x\{x,y\} \neq 0, T(x,x,x) = 0 & \text{1/4-BPS,} \\
\text{Rank 3} & T(x,x,x) \neq 0, \Delta(x) = 0 & \text{1/8-BPS,} \\
\text{Rank 4} & \Delta(x) > 0 & \text{1/8-BPS,} \\
\text{Rank 4} & \Delta(x) < 0 & \text{non-BPS,}
\end{array} \tag{6.26}$$

The orbits [45] with their rank conditions, dimensions and representative states are summarized in Table 6.8.

Table 6.8.: $D=4$ black hole orbits, their corresponding rank conditions, dimensions and SUSY.

Rank	Rank conditions $\neq 0(\exists\, y$ s.t.)	Rank conditions $=0(\forall\, y)$	Rep state	Orbit	dim	Susy
1	x	$\Upsilon(x,x,y)$	$\begin{pmatrix}1 & (0,0,0)\\ 0 & 0\end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{E_{6(6)}(\mathbb{R})\ltimes\mathbb{R}^{27}}$	28	1/2
2	$\Upsilon(x,x,y)$	$T(x,x,x)$	$\begin{pmatrix}1 & (1,0,0)\\ 0 & 0\end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{\mathrm{O}(6,5;\mathbb{R})\ltimes\mathbb{R}^{32}\times\mathbb{R}}$	45	1/4
3	$T(x,x,x)$	$\Delta(x)$	$\begin{pmatrix}0 & (1,1,1)\\ 0 & 0\end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{F_{4(4)}(\mathbb{R})\ltimes\mathbb{R}^{26}}$	55	1/8
4	$\Delta(x)>0$	–	$\begin{pmatrix}1 & (1,1,k)\\ 0 & 0\end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{E_{6(2)}(\mathbb{R})}$	55	1/8
4	$\Delta(x)<0$	–	$\begin{pmatrix}1 & (0,0,0)\\ 0 & k\end{pmatrix}$	$\dfrac{E_{7(7)}(\mathbb{R})}{E_{6(6)}(\mathbb{R})}$	55	0

2.1.4. STU subsectors in the FTS

The STU model may be obtained as a consistent truncation of the full $\mathcal{N}=8$ theory. This corresponds to the simple case where we put $P_{s,v,c}, Q_{s,v,c}$ all to zero, then

$$x=\begin{pmatrix}-q_0 & P(p^i)\\ Q(q_i) & p^0\end{pmatrix}, \tag{6.27}$$

where

$$P(p^i)=\mathrm{diag}(p^1,p^2,p^3), \qquad Q(q_i)=\mathrm{diag}(q_1,q_2,q_3). \tag{6.28}$$

In this case,

$$N_3(P)=p^1p^2p^3, \qquad N_3(Q)=q_1q_2q_3, \tag{6.29}$$

and

$$P^\#=\mathrm{diag}(p^2p^3,p^1p^3,p^1p^2), \qquad Q^\#=\mathrm{diag}(q_2q_3,q_1q_3,q_1q_2), \tag{6.30}$$

and Δ becomes

$$\Delta=-(p\cdot q)^2+4\left((p^1q_1)(p^2q_2)+(p^1q_1)(p^3q_3)+(p^3q_3)(p^2q_2)\right)-4p^0q_1q_2q_3+4q_0p^1p^2p^3. \tag{6.31}$$

If we make the identifications (4.1), we recover Cayley's hyperdeterminant. Combined with (2.79) we obtain the transformation between P,Q and p,q:

$$\begin{bmatrix}p^0\\p^1\\p^2\\p^3\\q_0\\q_1\\q_2\\q_3\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}P^0-P^2\\Q_0+Q_2\\P^3-P^1\\-P^3-P^1\\Q_0-Q_2\\-P^0-P^2\\Q_3-Q_1\\-Q_3-Q_1\end{bmatrix}. \tag{6.32}$$

This transformation gives us the relations:

$$\begin{aligned} P^2 &= 2(p^2p^3 - p^0q_1), \\ P\cdot Q &= p\cdot q - 2p^1q_1, \\ Q^2 &= 2(p^1q_0 + q_2q_3), \end{aligned} \tag{6.33}$$

hence we find

$$\Delta = P^2Q^2 - (P\cdot Q)^2, \tag{6.34}$$

which is manifestly invariant under $\mathrm{SL}(2)\times\mathrm{SO}(2,2)$.

CHAPTER 7

$\mathcal{N} = 8$ supergravity and the tripartite entanglement of seven qubits: revisited

In section 3 of chapter 4 we introduced the correspondence between $\mathcal{N} = 8$ supergravity and the tripartite entanglement of seven qubits. In the course of this discussion we found that the seven qubit system had a very special structure which followed from the decomposition of E_7 under $[\mathrm{SL}(2)]^7$. We have since met the octonions and their split cousins, which play an important role in the Jordan algebraic formulation of supergravity. In light of these observations we examine more closely the role of the octonions and Jordan algebras in the correspondence between $\mathcal{N} = 8$ black holes and the tripartite entanglement of seven qubits.

In particular, as was shown by Duff and Ferrara in [10], we will see that the special seven qubit state has the structure of the *Fano plane*, which describes the multiplication table of the imaginary octonions. Consequently, truncating the $\mathcal{N} = 8$ theory to its $\mathcal{N} = 4$ and $\mathcal{N} = 2$ subsectors corresponds to truncating the imaginary octonions to the imaginary quaternions and complexes respectively [10, 18].

We should emphasize that the appearance of the octonions via the Fano plane is qualitatively distinct from their appearance in the FTS formulation. On the one hand we have the *imaginary* octonions while on the other we have the *full* split-octonions. Nevertheless, we may construct a dictionary relating the three formulations, charges (Cartan basis), qubits (Fano basis) and $E_{6(6)}$-covariant (Freudenthal basis), which forms the final section of this chapter. In particular, we show that the dictionary relating the 56 black hole charges (x, y) to the 56 state vector coefficients a, b, c, d, e, f, g is determined precisely by the imaginary octonions.

1. The Fano plane, octonions and the tripartite entanglement of seven qubits

Our special state,

$$
\begin{aligned}
|\Psi\rangle_{56} = \quad & a_{ABD}|ABD\rangle \\
& + b_{BCE}|BCE\rangle \\
& + c_{CDF}|CDF\rangle \\
& + d_{DEG}|DEG\rangle \\
& + e_{EFA}|EFA\rangle \\
& + f_{FGB}|FGB\rangle \\
& + g_{GAC}|GAC\rangle,
\end{aligned} \tag{7.1}
$$

has some very distinctive structure. Observe,

1. Two distinct qubits appear together in one and only one tripartite entanglement.
2. Any two tripartite entanglements have at least one qubit in common.
3. Every qubit belongs to three distinct tripartite entanglements.

On replacing the words qubit and tripartite entanglement with the words point and line, respectively, it becomes apparent that the state describes the projective plane of order 2. This is know as the Fano plane, which is depicted in Figure 7.1.

That the Fano plane corresponds to the multiplication table of the imaginary octonions may be seen in Table 7.1. The non-vanishing independent components of the octonionic structure constants

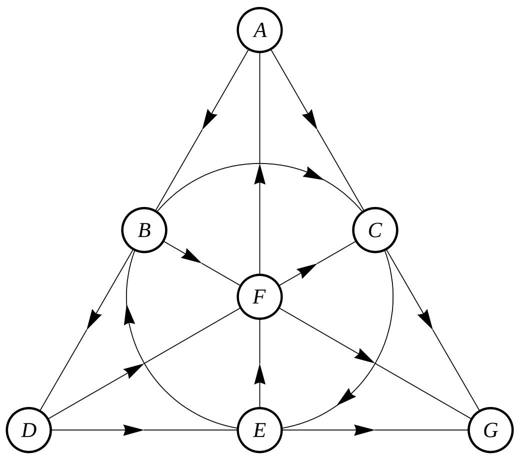


Figure 7.1.: The Fano plane is a projective plane with seven points and seven lines (the circle counts as a line). We may associate it to the state (4.49) by interpreting the points as the seven qubits A-G and the lines as the seven tripartite entanglements.

C_{ijk} and their duals C_{lmno} are then given in Table 7.2.

We may define a dual Fano plane by associating the lines to points and points to lines as in Figure 7.2.

Another way to understand the appearance of the dual Fano plane is to recognise the seven rows in (4.47) as the lines of the Fano plane and the seven columns as vertices as in Table 7.3. The dual

	A	B	C	D	E	F	G
A		D	G	$-B$	F	$-E$	$-C$
B	$-D$		E	A	$-C$	G	$-F$
C	$-G$	$-E$		F	B	$-D$	A
D	B	$-A$	$-F$		G	C	$-E$
E	$-F$	C	$-B$	$-G$		A	D
F	E	$-G$	D	$-C$	$-A$		B
G	C	F	$-A$	E	$-D$	$-B$	

Table 7.1.: Following the oriented lines of Figure 7.1 allows one to construct the Fano multiplication table, by identifying the qubits with the imaginary basis octonions. The minus signs arise when the Fano lines are followed counter to their orientation.

i	j	k	l	m	n	o
1	2	4	3	5	6	7
2	3	5	4	6	7	1
3	4	6	5	7	1	2
4	5	7	6	1	2	3
5	6	1	7	2	3	4
6	7	2	1	3	4	5
7	1	3	2	4	5	6

Table 7.2.: From Table 7.1 one can read off which components of the Fano structure constants C_{ijk} are nonzero. Using (5.8) one then obtains the associator coefficients C_{lmno}.

Fano plane corresponds to the transpose of this matrix and leads to a dual state

$$\begin{aligned} |\widetilde{\Psi}\rangle_{56} = \quad & A_{aeg}|a\,e\,g\rangle \\ & + B_{bfa}|b\,f\,a\rangle \\ & + C_{cgb}|c\,g\,b\rangle \\ & + D_{dac}|d\,a\,c\rangle \\ & + E_{ebd}|e\,b\,d\rangle \\ & + F_{fce}|f\,c\,e\rangle \\ & + G_{gdf}|g\,d\,f\rangle. \end{aligned} \tag{7.2}$$

The dual Fano plane corresponds to the multiplication table of the imaginary octonions given in Table 7.4. The non-vanishing independent components of the octonionic structure constants c_{ijk} and their duals c_{lmno} are then given by Table 7.5. Another way to understand the appearance of the dual Fano plane is to recognise the seven rows in (4.47) as the lines of the Fano plane and the seven columns as vertices as in Table 7.3.

1.1. Subsectors

Having understood the role of the Fano plane in the $\mathcal{N}=8$ tripartite entanglement of seven qubits, we now turn our attention to understanding the QI interpretation of its $\mathcal{N}=4$ truncation [10, 18], for which the black holes carry $12+12$ electric/magnetic charges transforming as a $(\mathbf{2},\mathbf{12})$ under the $\mathrm{SL}(2)\times\mathrm{SO}(6,6)$ U-duality. As we shall see this corresponds to keeping a single line of the dual Fano plane Figure 7.2, which represents one of the seven possible reductions to the imaginary

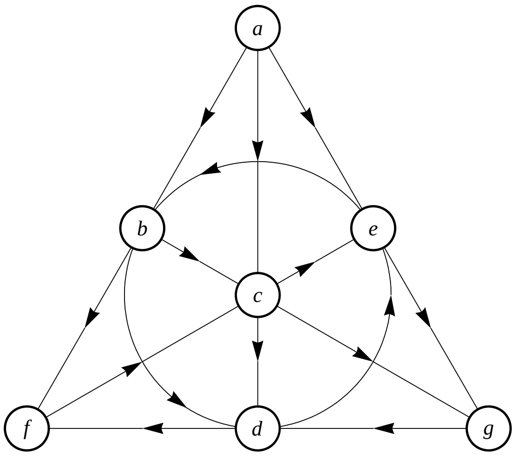


Figure 7.2.: Like the Fano plane of Figure 7.1 the dual Fano plane has seven points and seven lines, but this time the plane is associated with the dual state (7.2) interpreting the points as the seven tripartite entanglements and the lines as the seven qubits.

	A	B	C	D	E	F	G
a	2	2	1	2	1	1	1
b	1	2	2	1	2	1	1
c	1	1	2	2	1	2	1
d	1	1	1	2	2	1	2
e	2	1	1	1	2	2	1
f	1	2	1	1	1	2	2
g	2	1	2	1	1	1	2

Table 7.3.: The seven terms in decomposition (4.47) may be written in a grid such that Fano lines and vertices are rows and columns.

quaternions. We could also make the complementary truncation of the dual Fano plane, throwing away one line and keeping the remaining quadrangle. This is not actually a truncation of the $\mathcal{N} = 8$ theory, but rather a particular black hole configuration with only R-R charges turned on. A quadrangle corresponds to the four imaginary octonions orthogonal to the quaternion subalgebra picked out by the complementary line and, hence, is not a closed subalgebra. Genuine consistent truncations correspond to closed subalgebras of the octonions. Finally, picking a single point of the dual Fano plane, i.e. an imaginary complex number, we recover the familiar $\mathcal{N} = 2$ case with $\mathrm{SL}(2) \times \mathrm{SO}(2,2) \cong [\mathrm{SL}(2)]^3$.

First we recall the decomposition (4.42) of the fundamental 56-dimensional representation of $E_{7(7)}$ under its maximal subgroup,

$$\begin{aligned} E_{7(7)} &\supset \mathrm{SL}(2)_A \times \mathrm{SO}(6,6), \\ \mathbf{56} &\to (\mathbf{2}, \mathbf{12}) + \ (\mathbf{1}, \mathbf{32}). \end{aligned} \tag{7.3}$$

The $\mathcal{N} = 4$ subsector consists of just the 24 states belonging to the $(\mathbf{2}, \mathbf{12})$. Throwing away the $(\mathbf{1}, \mathbf{32})$ and continuing the decomposition under the remaining six factors of $\mathrm{SL}(2)$ we find,

$$(\mathbf{2}, \mathbf{12}) \to (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}) + (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{2}) \tag{7.4}$$

	a	b	c	d	e	f	g
a		f	d	$-c$	g	$-b$	$-e$
b	$-f$		g	e	$-d$	a	$-c$
c	$-d$	$-g$		a	f	$-e$	b
d	c	$-e$	$-a$		b	g	$-f$
e	$-g$	d	$-f$	$-b$		c	a
f	b	$-a$	e	$-g$	$-c$		d
g	e	c	$-b$	f	$-a$	$-d$	

Table 7.4.: In direct analogy with Figure 7.1 and Table 7.1, the oriented dual Fano plane of Figure 7.2 is used to write an octonionic multiplication table.

i	j	k	l	m	n	o
1	2	6	3	4	5	7
2	3	7	4	5	6	1
3	4	1	5	6	7	2
4	5	2	6	7	1	3
5	6	3	7	1	2	4
6	7	4	1	2	3	5
7	1	5	2	3	4	6

Table 7.5.: Dual Fano structure constants c_{ijk} and c_{lmno} are obtained from Table 7.4 in the same manner that Table 7.2 is obtained from Table 7.1.

which corresponds to the state

$$|\Psi\rangle = a_{ABD}|ABD\rangle + e_{EFA}|EFA\rangle + g_{GAC}|GAC\rangle. \tag{7.5}$$

So only Alice talks to all the others. This is described by just those three lines passing through A in the Fano plane or the aeg line in the dual Fano plane. Then the equation analogous to (4.51) is

$$(\mathbf{2}, \mathbf{12}) = (ABD) + (EFA) + (GAC) = a + e + g, \tag{7.6}$$

and the corresponding quartic invariant, I_4, reduces to the singlet in $(\mathbf{2}, \mathbf{12}) \times (\mathbf{2}, \mathbf{12}) \times (\mathbf{2}, \mathbf{12}) \times (\mathbf{2}, \mathbf{12})$

$$I_4 \sim a^4 + e^4 + g^4 + 2[e^2g^2 + g^2a^2 + a^2e^2]. \tag{7.7}$$

If we identify the 24 numbers $(a_{ABD}, e_{EFA}, g_{GAC})$ with (P^μ, Q_ν) with $\mu, \nu = 0, \ldots, 11$ in a way analogous to (2.79) this becomes the $\mathrm{SL}(2) \times \mathrm{SO}(6,6)$ invariant [2,170,213]

$$I_4 = P^2Q^2 - (P \cdot Q)^2. \tag{7.8}$$

So

$$I_4 = I_{aeg} \equiv \det(\gamma^1(a) + \gamma^2(g) + \gamma^3(e)). \tag{7.9}$$

This reduction from $\mathcal{N} = 8$ to $\mathcal{N} = 4$ corresponds to a reduction from the imaginary octonions of Table 7.4 to the imaginary quaternions of Table 7.6. This suggest that I_4 of (7.8) may be written as Cayley's hyperdeterminant over the imaginary quaternions, and this is indeed the case, as shown in [18]. From a stringy point of view, this subsector describes just the NS-NS charges.

	a	e	g
a		g	$-e$
e	$-g$	a	
g	e	$-a$	

Table 7.6.: The aeg quaternionic multiplication table is obtained by selecting the aeg quaternionic cycle from Table 7.4.

A different subsector which excludes Alice is obtained by keeping just the $(\mathbf{1},\mathbf{32})$ in (4.42), This is described by just those four lines not passing through A in the Fano plane or the $bcdf$ quadrangle in the dual Fano plane. From a stringy point of view, this subsector describes just the R-R charges. Following the same logic as before we find,

$$(\mathbf{1},\mathbf{32}) \to (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) \tag{7.10}$$

or symbolically

$$(\mathbf{1},\mathbf{32}) = (BCE) + (CDF) + (DEG) + (FGB) = b + c + d + f. \tag{7.11}$$

The quartic invariant, I_4, reduces to the singlet in $(\mathbf{1},\mathbf{32}) \times (\mathbf{1},\mathbf{32}) \times (\mathbf{1},\mathbf{32}) \times (\mathbf{1},\mathbf{32})$

$$I_4 \sim b^4 + c^4 + d^4 + f^4 + 2[b^2c^2 + c^2d^2 + d^2e^2 + d^2f^2 + c^2f^2 + f^2b^2] + 8bcdf. \tag{7.12}$$

This does not correspond to any $\mathcal{N} = 4$ black hole but rather to an $\mathcal{N} = 8$ black hole with only the R-R charges switched on.

For $\mathcal{N} = 2$, as may be seen from (4.43), we only have the $[\mathrm{SL}(2)]^3$ subgroup of the STU model where there are only eight states

$$|\Psi\rangle = a_{ABD}|ABD\rangle. \tag{7.13}$$

This is described by just the ABD line in the Fano plane or the a vertex in the dual Fano plane. This is simply the usual tripartite entanglement, for which

$$(\mathbf{2},\mathbf{2},\mathbf{2}) = (ABD) = a, \tag{7.14}$$

and the corresponding quartic invariant

$$I_4 \sim a^4, \tag{7.15}$$

is just Cayley's hyperdeterminant

$$I_4 = -\operatorname{Det} a. \tag{7.16}$$

1.2. Discrete symmetry of the fano plane

It ought to be clear by now that the Fano plane plays a central role in the seven qubit interpretation of $\mathcal{N} = 8$ black holes. It is therefore natural ask how the symmetries of the Fano plane are manifested in the 56 dimensional seven qubit state.

Let $V_0 = V(n+1,\mathbb{F})/\{0\}$ be a $n+1$-dimensional vector space, defined over a field $\mathbb{F}$, with the additive identity 0 removed. Note, $\mathbb{F}$ may be a finite field with characteristic q, in which case we denote it by $\mathbb{F}_q$. The n-dimensional projective space over $\mathbb{F}$, which we write as $PG(n, \mathbb{F})$, is the space of equivalence classes defined by the relation, $x \sim y$ iff $x = \alpha y$, where $\alpha \in \mathbb{F}/\{0\}$ and $x, y \in V_0$. The set of projectivities[1] of $PG(n, \mathbb{F}_q)$ is the *projective general linear group* $PGL(n+1, \mathbb{F}_q)$, i.e. the group of non-singular linear transformations on V_0 up to an overall multiplicative factor (see for example, [41]).

The Fano plane is the projective plane over the finite field of order two, $PG(2, \mathbb{F}_2)$. It is the smallest example of a projective plane. In this case the projective general linear group, $PGL(3, \mathbb{F}_2)$, is isomorphic to the projective *special* linear group $PSL(3, \mathbb{F}_2)$[2], the set of determinant one projectivities. In fact, in this particular instance, we have a second useful isomorphism, $PSL(3, \mathbb{F}_2) \simeq PSL(2, \mathbb{F}_7)$. $PSL(2, \mathbb{F}_7)$ is the second smallest finite non-abelian simple group, after the alternating group $\mathcal{A}_5$, with 168 elements. It has many guises, but, perhaps most significantly, it is the automorphism group of the Klein quartic. Further, it is the only finite simple subgroup of $\mathrm{SU}(3)$ and consequently, in light of the recently measured neutrino mixing patterns, it has been receiving increasing attention as a candidate finite non-abelian flavour group [258, 259].

$PSL(2, \mathbb{F}_7)$ admits a two generator presentation [259, 260],

$$\langle\, s, t \mid s^2 = t^3 = (st)^7 = [s, t]^4 = e \,\rangle, \tag{7.17}$$

where the commutator, $[s, t]$, is defined as $s^{-1}t^{-1}st$.

It has six conjugacy classes and, therefore six irreps, as summarised in Table 7.7. It has a convenient

	$C_1^{[1]}$	$21C_2^{[2]}(s)$	$56C_3^{[3]}(t)$	$42C_4^{[4]}([s,t])$	$24C_5^{[7]}(st)$	$24C_6^{[7]}(st^2)$
$\chi^{[\mathbf{1}]}$	1	1	1	1	1	1
$\chi^{[\mathbf{3}]}$	3	-1	0	1	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$
$\chi^{[\bar{\mathbf{3}}]}$	3	-1	0	1	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$
$\chi^{[\mathbf{6}]}$	6	2	0	0	-1	-1
$\chi^{[\mathbf{7}]}$	7	-1	1	-1	0	0
$\chi^{[\mathbf{8}]}$	8	0	-1	0	1	1
$\chi^{[\mathbf{56}]}$	56	8	2	0	0	0

Table 7.7.: Character table for $PSL(2, \mathbb{F}_7)$. The number in square brackets on each conjugacy class C_α corresponds to the order of the elements in that class. A simple representative for each class is given in the parentheses. Note that geometrical interpretation of the **6** is given by its action on the Fano plane [258]. The final row corresponds to the compound characters of the reducible 56 dimensional representation described here.

action defined on the Fano plane given by the permutation of its points. For example, we may consider the permutations,

$$s_{\text{fano}} = (AC)(B)(DE)(F)(G), \qquad t_{\text{fano}} = (ADB)(C)(EFG), \tag{7.18}$$

which are automorphisms of the un-oriented Fano plane. This yields a 7 dimensional real representation for which it is easily verified that (7.17) is satisfied. This representation is reducible,

$$\mathbf{7}^{\text{fano}} \to \mathbf{1} + \mathbf{6}, \tag{7.19}$$

[1]A projectivity is a linear bijection $PG(n, \mathbb{F}) \to PG(n, \mathbb{F})$ that preserves incidence.

[2]More generally, $PGL(n, \mathbb{F}_q) \simeq PSL(n, \mathbb{F}_q)$ iff $\gcd(n+1, q) = 1$.

as can be checked from its characters.

These permutations also interchange the lines of the Fano plane and, consequently, may be considered as permutations of the points of the *dual* Fano plane,

$$s_{\text{dualfano}} = (ab)(ce)(d)(f)(g), \qquad t_{\text{dualfano}} = (a)(bcg)(efd). \tag{7.20}$$

As described in [18], the Fano plane representation may be used to build a 56-dimensional representation of the $PSL(2, \mathbb{F}_7)$ generators, denoted s_{56} and t_{56}, acting on our particular tripartite entangled 7-qubit state, which leaves quartic entanglement measure, I_4, invariant. The structure of the generators is depicted in Figure 7.3. $PSL(2, \mathbb{F}_7)$ has six classes and therefore six irreps, as

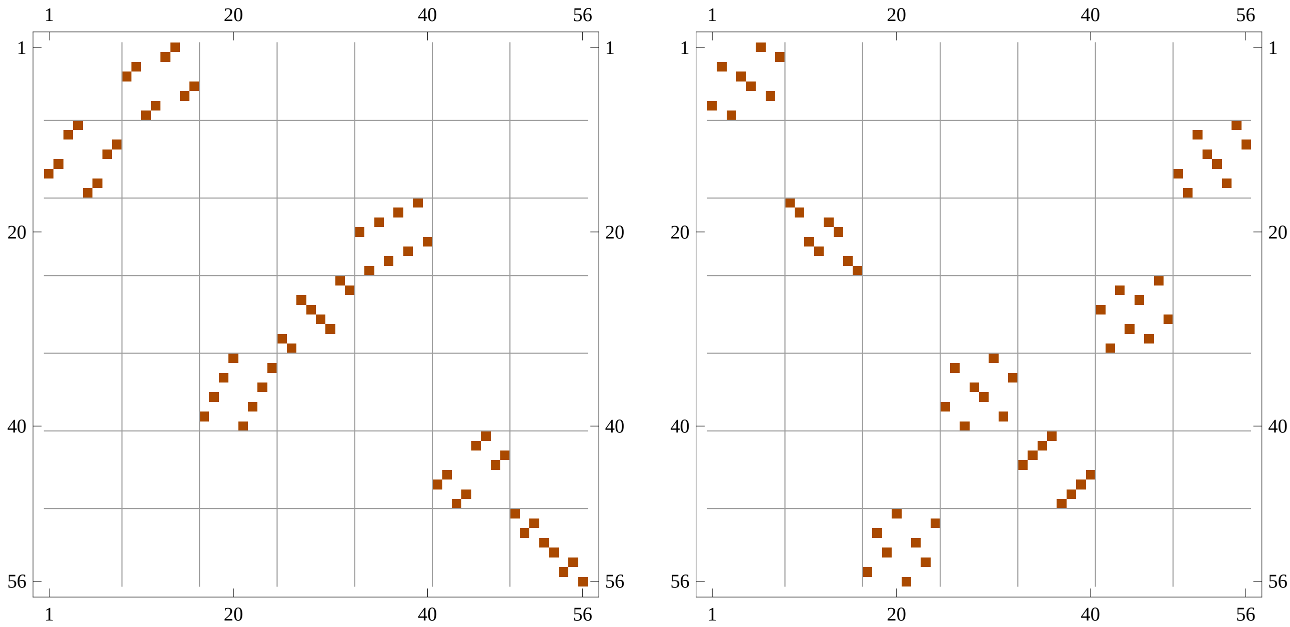


Figure 7.3.: Graphical representation of the generators of the 56 dimensional representation of $PSL(2, \mathbb{F}_7)$. These are the 56×56 matrices s_{56} (left) and t_{56} (right). The gridlines partition the matrices into 8×8 blocks that transform a single letter, while the filled squares correspond to the non-zero unit entries. Clearly each letter octet is transformed into another, without mixing between octets. One can read off, for example, that s_{56} converts a's to b's and *vice versa*, while t_{56} transforms the a's amongst themselves.

summarised in Table 7.7 [258]. Note, the final row corresponds to the compound characters of the 56-dimensional reducible representation. To determine how it decomposes, we may use,

$$a_\mu = \frac{1}{g} \sum_\alpha g_\alpha \chi_\alpha^{[\mu]*} \chi_\alpha^{[\mathbf{56}]}, \tag{7.21}$$

where a_μ counts the number of times irrep μ appears in the $\mathbf{56}$. Note, g and g_α are the dimensions of the group and conjugacy class, C_α, respectively. Using Table 7.7 one finds,

$$a_\mu = \{2, 0, 0, 4, 2, 2\}, \tag{7.22}$$

and, hence, we have

$$\mathbf{56} = \mathbf{1} + \mathbf{1} + \mathbf{6} + \mathbf{6} + \mathbf{6} + \mathbf{6} + \mathbf{7} + \mathbf{7} + \mathbf{8} + \mathbf{8}. \tag{7.23}$$

This is consistent with the breaking of the fundamental $\mathbf{56}$ of E_7 under $PSL(2,\mathbb{F}_7)$, which goes as[3],

$$\mathbf{56} \to \mathbf{1}+\mathbf{1}+\mathbf{6}+\mathbf{6}+\mathbf{6}+\mathbf{6}+\mathbf{7}+\mathbf{7}+\mathbf{8}+\mathbf{8}. \tag{7.24}$$

In [15] the $PSL(2,\mathbb{F}_7)$ symmetry of the $\mathcal{N}=8$ black hole entropy has been related, via a special set of 63 3-qubit operators, to the generalised hexagon of order two using the dictionary constructed in section 1.3.1. They further suggested that the full $G_2(2)$ symmetry of the hexagon may be preserved by the black hole entropy.

In [261] in was shown that the U-duality subgroup which maps purely electric and magnetic black hole solutions amongst themselves is the Weyl group of $E_{7(7)}$. This is analogous to the $\mathbb{Z}_2$ subgroup of the $\mathrm{U}(1)$ electric-magnetic duality group in Maxwell theory. Clearly, $PSL(2,\mathbb{F}_7)$ preserves some more structure than the Weyl group. In the seven qubit interpretation it does not mix the tripartite entanglements, whereas the full $E_{7(7)}$ would. It is not immediately clear that there is an obvious black hole interpretation. However, it is tempting to speculate that there might be some connection to the supersymmetric configuration of seven intersecting D-branes, which itself has a Fano plane structure [262].

1.3. Three descriptions

We have now seen three distinct formulations of the fundamental $\mathbf{56}$ of E_7 and their associated quartic invariants:

1. Cartan basis:

$$\begin{aligned} E_7 &\supset \mathrm{SO}(8), \\ \mathbf{56} &\to \mathbf{28}+\mathbf{28}, \\ I_4(x,y) &= -\operatorname{tr}(xy)^2+\tfrac{1}{4}(\operatorname{tr} xy)^2-4(\operatorname{Pf} x+\operatorname{Pf} y). \end{aligned} \tag{7.25}$$

   where x^{IJ} and y_{IJ} are antisymmetric 8×8 matrices and Pf is the Pfaffian.

2. Freudenthal/Jordan basis

$$\begin{aligned} E_7 &\supset E_6, \\ \mathbf{56} &\to \mathbf{1}+\mathbf{27}+\mathbf{1}+\mathbf{27}', \\ I_4(x) &= -(p^0q_0-\operatorname{Tr}(P,Q))^2-4[p^0N(Q)-q_0N(P)-\operatorname{Tr}(P^\sharp,Q^\sharp)]. \end{aligned} \tag{7.26}$$

   where A is a member of the cubic Jordan algebra $\mathfrak{J}_3^{\mathfrak{O}_s}$.

[3] We are grateful to Christopher Luhn for this point.

3. Fano basis

$$E_7 \supset \mathrm{SL}(2)^7, \tag{7.27a}$$

$$\begin{aligned}
\mathbf{56} \rightarrow &(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) \\
&+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) \\
&+(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \\
&+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}) \\
&+(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \\
&+(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) \\
&+(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}),
\end{aligned} \tag{7.27b}$$

$$\begin{aligned}
I_4 = {} & a^4+b^4+c^4+d^4+e^4+f^4+g^4 \\
&+2\Big[a^2b^2+a^2c^2+a^2d^2+a^2e^2+a^2f^2+a^2g^2 \\
&\quad +b^2c^2+b^2d^2+b^2e^2+b^2f^2+b^2g^2 \\
&\quad +c^2d^2+c^2e^2+c^2f^2+c^2g^2 \\
&\quad +d^2e^2+d^2f^2+d^2g^2 \\
&\quad +e^2f^2+e^2g^2 \\
&\quad +f^2g^2\Big] \\
&+8\left[abce+bcdf+cdeg+defa+efgb+fgac+gabd\right],
\end{aligned} \tag{7.27c}$$

where

$$\begin{aligned}
a^4 = {} & \tfrac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{D_2D_3} \\
& \times a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4},
\end{aligned} \tag{7.27d}$$

etc;

$$\begin{aligned}
a^2b^2 = {} & \tfrac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_3}\epsilon^{D_1D_2}\epsilon^{B_2B_4}\epsilon^{C_3C_4}\epsilon^{E_3E_4} \\
& \times a_{A_1B_1D_1}a_{A_2B_2D_2}b_{B_3C_3E_3}b_{B_4C_4E_4},
\end{aligned} \tag{7.27e}$$

etc;

$$\begin{aligned}
abce = {} & \tfrac{1}{2}\epsilon^{A_1A_4}\epsilon^{B_1B_2}\epsilon^{C_2C_3}\epsilon^{D_1D_3}\epsilon^{E_2E_4}\epsilon^{F_3F_4} \\
& \times a_{A_1B_1D_1}b_{B_2C_2E_2}c_{C_3D_3F_3}e_{E_4F_4A_4},
\end{aligned} \tag{7.27f}$$

etc.

Black holes are more conveniently described in either the Cartan or Freudenthal bases, whereas the Fano basis is tailored to the qubits. Hence it is important to find the three dictionaries that relate these three descriptions. As we shall see the structures of these dictionaries are themselves interesting, with the octonions once again playing an intriguing role.

1.3.1. Cartan-Fano dictionary

The dual Fano plane structure constants of Table 7.5 define antisymmetric matrices x^{IJ} and y_{IJ} according to the dictionary of Table 7.8.

The 56 state vector coefficients, a_{ABD} through g_{GAC}, are arranged in x^{IJ} and y_{IJ} according to the octonionic multiplication table of the dual Fano plane, compare Table 7.4 with the matrices (7.28a) and (7.28b) below. This uniquely determines the rows of Table 7.8.

The positions of the binary indices in x^{IJ} and y_{IJ} are specified by the columns of Table 7.8. The first column describes the position of 111 in x^{IJ} (and 000 in y_{IJ}). Note, the first column consists of all pairs $i0$, where $i = 1, \dots, 7$, i.e. the first row and column of x^{IJ} and y_{IJ}. To understand the structure of the remaining three columns let us consider a specific example given by considering Alice's qubit A. For each row in Table 7.8 one can form a triple ijk from the pair $i0$, appearing in the first column, and any one of the remaining pairs jk in that row. We note that 715 is the unique triple common to rows a_{ABD}, e_{EFA} and g_{GCA}, the subsector defined by the common qubit A. Then, in each case the non-trivial pair jk sits in the column labelled by the position of the common qubit. In our example this is A. Therefore the pair 57 belonging to row a_{ABD} sits in the 100 column where the position of A in ABD corresponds to the position of 1 in 100 or, equivalently, the position of 0 in 011. Similarly, 71 sits in the column labelled 001 because A is last in e_{EFA}. Finally, 15 sits in the column labelled 010 because A is second in g_{GAC}. Repeating this procedure for the remaining six qubits, B through G, uniquely determines all the columns of Table 7.8. This procedure may be followed to construct the dictionary based on any octonionic basis.

x^{IJ}	111	010	001	100
y_{IJ}	000	101	110	011
a_{ABD}	10	26	34	57
b_{BCE}	20	37	45	61
c_{CDF}	30	41	56	72
d_{DEG}	40	52	67	13
e_{EFA}	50	63	71	24
f_{FGB}	60	74	12	35
g_{GAC}	70	15	23	46

Table 7.8.: The Cartan basis dictionary. The binary triples denote the indices on the 56 state vector coefficients, while the pairs give positions within the x^{IJ}, y_{IJ} matrices. These are the positive elements of x^{IJ} and y_{IJ}, the remaining elements being fixed by antisymmetry.

$$x^{IJ} = \begin{pmatrix} 0 & -a_{111} & -b_{111} & -c_{111} & -d_{111} & -e_{111} & -f_{111} & -g_{111} \\ a_{111} & 0 & f_{001} & d_{100} & -c_{010} & g_{010} & -b_{100} & -e_{001} \\ b_{111} & -f_{001} & 0 & g_{001} & e_{100} & -d_{010} & a_{010} & -c_{100} \\ c_{111} & -d_{100} & -g_{001} & 0 & a_{001} & f_{100} & -e_{010} & b_{010} \\ d_{111} & c_{010} & -e_{100} & -a_{001} & 0 & b_{001} & g_{100} & -f_{010} \\ e_{111} & -g_{010} & d_{010} & -f_{100} & -b_{001} & 0 & c_{001} & a_{100} \\ f_{111} & b_{100} & -a_{010} & e_{010} & -g_{100} & -c_{001} & 0 & d_{001} \\ g_{111} & e_{001} & c_{100} & -b_{010} & f_{010} & -a_{100} & -d_{001} & 0 \end{pmatrix}, \tag{7.28a}$$

$$y_{IJ} = \begin{pmatrix} 0 & -a_{000} & -b_{000} & -c_{000} & -d_{000} & -e_{000} & -f_{000} & -g_{000} \\ a_{000} & 0 & f_{110} & d_{011} & -c_{101} & g_{101} & -b_{011} & -e_{110} \\ b_{000} & -f_{110} & 0 & g_{110} & e_{011} & -d_{101} & a_{101} & -c_{011} \\ c_{000} & -d_{011} & -g_{110} & 0 & a_{110} & f_{011} & -e_{101} & b_{101} \\ d_{000} & c_{101} & -e_{011} & -a_{110} & 0 & b_{110} & g_{011} & -f_{101} \\ e_{000} & -g_{101} & d_{101} & -f_{011} & -b_{110} & 0 & c_{110} & a_{011} \\ f_{000} & b_{011} & -a_{101} & e_{101} & -g_{011} & -c_{110} & 0 & d_{110} \\ g_{000} & e_{110} & c_{011} & -b_{101} & f_{101} & -a_{011} & -d_{110} & 0 \end{pmatrix}. \tag{7.28b}$$

We can summarise the dictionary by writing $I = (0, i), i \in \{1, \ldots, 7\}, (a^0, a^1, \ldots, a^7) = (0, a, \ldots, g)$ and

$$x^{IJ} = \begin{cases} a^I_{111} & J = 0, \\ c_{IJK} a^K_{010} & |I - J| = 3 \text{ or } 4, \\ c_{IJK} a^K_{001} & |I - J| = 1 \text{ or } 6, \\ c_{IJK} a^K_{100} & |I - J| = 2 \text{ or } 5, \end{cases} \qquad y_{IJ} = \begin{cases} a^I_{000} & J = 0, \\ c_{IJK} a^K_{101} & |I - J| = 3 \text{ or } 4, \\ c_{IJK} a^K_{110} & |I - J| = 1 \text{ or } 6, \\ c_{IJK} a^K_{011} & |I - J| = 2 \text{ or } 5. \end{cases} \tag{7.29a}$$

Here we have extended c^i_{jk} to c^i_{JK} by setting $c^i_{JK} = c^i_{jk}$ whenever $J(= j)$ and $K(= k)$ are not equal to 0, while defining c^i_{JK} to be zero whenever J or K is equal to 0.

The 8×8 gamma matrices γ^i_{IJ} in seven dimensions, which satisfy the Clifford algebra

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\mathbb{1}, \tag{7.30}$$

can be written in terms of the octonionic structure constants. The hermitian (purely imaginary and antisymmetric) gamma matrices in seven dimensions can then be chosen as

$$\gamma^i_{IJ} = i\left(c^i{}_{IJ} \pm \delta_{iI}\delta_{J0} \mp \delta_{iJ}\delta_{I0}\right), \tag{7.31}$$

where the signs are correlated. The antisymmetric products of gamma matrices are defined as usual, with unit weight,

$$\gamma^{ij\cdots k} = \gamma^{[i}\gamma^j \cdots \gamma^{k]}. \tag{7.32}$$

The antisymmetric self-dual and anti-self-dual tensors $c^{\pm}_{IJKL}$, $(I, J, \ldots = 0, 1, 2, \ldots, 7)$ in eight dimensions will be defined as:

$$c^{\pm}_{ijkl} = c_{ijkl}, \quad \text{and} \quad c^{\pm}_{ijk0} = \pm c_{ijk}. \tag{7.33}$$

With the above choices of gamma matrices one finds

$$\begin{aligned} \gamma^{ij}_{IJ} &= c_{ijIJ} + \delta^i_I\delta^j_J - \delta^i_J\delta^j_I \pm c_I{}^{ij}\delta_{J0} \mp c_J{}^{ij}\delta_{I0} \\ &= c^{\pm}_{ijIJ} + \delta^i_I\delta^j_J - \delta^i_J\delta^j_I. \end{aligned} \tag{7.34}$$

Note that $(-i\gamma^i)_{IJ}$ do not form an 8×8 representation of the octonions;

$$\begin{aligned} (-i\gamma^i)_{IK}(-i\gamma^j)_{KJ} &= -\delta^{ij}\delta_{IJ} - \gamma^{ij}_{IJ} \\ &= -\delta^{ij}\delta_{IJ} - c_{ijab} - \delta^i_I\delta^j_J + \delta^i_J\delta^j_I \mp c_I{}^{ij}\delta_{J0} \pm c_J{}^{ij}\delta_{I0}, \end{aligned} \tag{7.35}$$

which is to be compared with

$$e_i e_j = -\delta_{ij} + c_{ijk}e_k. \tag{7.36}$$

Whereas

$$c_{ijk}(-i\gamma^k)_{IJ} = c_{ijk}(c^i{}_{IJ} \pm \delta_{iI}\delta_{J0} \mp \delta_{iJ}\delta_{I0}). \tag{7.37}$$

Accordingly one can rewrite (7.29a) as

$$x^{JK} = -i\gamma^{iJK} \times \begin{cases} a^i_{111} & K=0, \\ a^i_{010} & |J-K|=3, \\ a^i_{001} & |J-K|=1, \\ a^i_{100} & |J-K|=2, \end{cases} \qquad y_{JK} = -i\gamma^{iJK} \times \begin{cases} a^i_{000} & K=0, \\ a^i_{101} & |J-K|=3, \\ a^i_{110} & |J-K|=1, \\ a^i_{011} & |J-K|=2. \end{cases} \tag{7.38}$$

This dictionary has been established empirically. While it has been verified for a number of distinct Fano planes (octonionic bases), we do not have a proof, as such, that it holds for all possibilities. However, it seems unlikely that it does not. Moreover, given that the automorphism group of the octonions, which maps one dictionary into another, is $G_2 \subset E_7$, we would expect it to leave I_4 invariant. So, if it works for one set of octonions it should work for all.

1.3.2. Freudenthal-Fano dictionary

Let us now construct the analogous dictionary relating the 56 charges in the Freudenthal basis with the 56 state vector coefficients specifying the tripartite entanglement of seven qubits. See also [16]. It is instructive to consider the chain of group decompositions $E_{7(7)} \to E_{6(6)} \to \mathrm{SO}(4,4)$ under which the $\mathbf{56}$ decomposes as

$$\mathbf{56} \to \mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{8_s}+\mathbf{8_c}+\mathbf{8_v}+\mathbf{8_s}+\mathbf{8_c}+\mathbf{8_v}. \tag{7.39}$$

Combining this with the STU embedding in the FTS (6.27), it is clear that the eight state vector coefficients, a_{ABD}, are associated with the eight singlets appearing in (7.39). Now, consider the subgroup containing the three copies of $\mathrm{SL}(2)$ associated with the tripartite entanglement of qubits A, B and D,

$$E_{7(7)} \supset \mathrm{SL}(2)_A \times \mathrm{SL}(2)_B \times \mathrm{SL}(2)_D \times \mathrm{SO}(4,4), \tag{7.40}$$

under which,

$$\mathbf{56} \to (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{8_v}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8_s}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{8_c}). \tag{7.41}$$

We note that qubit A transforms as doublet with the $\mathbf{8}_v$. This suggests that we associate the subsector defined by the common qubit A, namely $a_{ABD}|ABD\rangle + e_{EFA}|EFA\rangle + g_{GAC}|GAC\rangle$, with the $\mathbf{8_v}$. This is one of the consistent $\mathcal{N}=4$ truncations of the full $\mathcal{N}=8$ theory, the 24 black hole charges transforming as a $(\mathbf{2},\mathbf{12})$ of $\mathrm{SL}(2)_A \times \mathrm{SO}(6,6)$ [10]. Repeating this analysis for qubit B leads us to identify the b_{BCE} and f_{FGB} with the $\mathbf{8_s}$, while considering D we identify c_{CDF} and d_{DEG} with the $\mathbf{8_c}$.

To specify more precisely the dictionary between (e_{EFA}, g_{GAC}) and (P_v, Q_v), (b_{BCE}, f_{FGB}) and (P_s, Q_s), and finally, (c_{CDF}, d_{DEG}) and (P_c, Q_c), we begin by noting that the eight charges of the STU model may be arranged in a cube as depicted in Figure 4.1. Following [50], the cube may be partitioned into a pair of 2×2 matrices, (M_i, N_i) in three independent ways. These are given by the three possible slicings of the cube along its planes of symmetry,

$$M_1 = \begin{pmatrix} -p^3 & q_2 \\ q_1 & q_0 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} p^0 & -p^1 \\ -p^2 & q_3 \end{pmatrix}, \tag{7.42a}$$

$$M_2 = \begin{pmatrix} -p^2 & q_3 \\ q_1 & q_0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} p^0 & -p^1 \\ -p^3 & q_2 \end{pmatrix}, \tag{7.42b}$$

$$M_3 = \begin{pmatrix} -p^1 & q_3 \\ q_2 & q_0 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} p^0 & -p^2 \\ -p^3 & q_1 \end{pmatrix}. \tag{7.42c}$$

For any element,

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{SL}(2)_i, \qquad 1 \le i \le 3, \tag{7.43}$$

the action on the cube is given by

$$(M_i, N_i) \mapsto (rM_i + sN_i, tM_i + uN_i). \tag{7.44}$$

The individual actions of the three $\mathrm{SL}(2)_i$ all commute and, therefore, this provides a natural representation of $[\mathrm{SL}(2)]^3$. Define the three binary quadratic forms, one for each slicing,

$$f_i(x, y) = \det(M_i x + N_i y), \qquad 1 \le i \le 3. \tag{7.45}$$

Explicitly

$$\begin{aligned} f_1 &= -(q_2q_1 + p^3q_0)x^2 + (p \cdot q - 2p^3q_3)xy - (p^2p^1 - p^0q_3)y^2, \\ f_2 &= -(q_1q_3 + p^2q_0)x^2 + (p \cdot q - 2p^2q_2)xy - (p^1p^3 - p^0q_2)y^2, \\ f_3 &= -(q_3q_2 + p^1q_0)x^2 + (p \cdot q - 2p^1q_1)xy - (p^3p^2 - p^0q_1)y^2. \end{aligned} \tag{7.46}$$

These quadratic forms may also be systematically derived using transvectants as presented in [18]. Each one is invariant under two of the three factors in $[\mathrm{SL}(2)]^3$. For example, f_1 is invariant under the subgroup $\{\mathrm{id}_1\} \times \mathrm{SL}_2(2) \times \mathrm{SL}_3(2) \subset [\mathrm{SL}(2)]^3$. Taking the determinant of the Hessian, $H(f_i) = \gamma^i(a)$ yields Cayley's hyperdeterminant

$$\det H(f_i) = \det \begin{pmatrix} (f_i)_{xx} & (f_i)_{xy} \\ (f_i)_{yx} & (f_i)_{yy} \end{pmatrix} = \det \gamma^i(a) = -\operatorname{Det} a_{ABC}, \qquad 1 \le i \le 3. \tag{7.47}$$

Now, consider keeping (p^i, q_i) and only one of (P_s, Q_s), (P_v, Q_v) or (P_c, Q_c) and computing I_4 from the FTS. Recall, this gives us the entropy of one of the three $\mathcal{N} = 4$ subsectors as defined by one of the three qubits, A, B or D. Keeping only (p^i, q_i, P_v, Q_v), the $\mathcal{N} = 4$ subsector defined by the common qubit A, one finds

$$I_4 = 4(p^2p^1 - p^0q_3)(q_2q_1 + p^3q_0) - 4(p.q - 2p^3q_3)^2 + 4{P_v}^2{Q_v}^2 - (P^v \cdot Q_v)^2 \tag{7.48}$$

$$- 4(p^2p^1 - p^0q_3){Q_v}^2 - 4(q_2q_1 + p^3q_0){P_v}^2 - 4(p \cdot q - 2p^3q_3)P_v \cdot Q_v, \tag{7.49}$$

where ${P_v}^2 = P_v\bar{P}_v$ and $2P_v \cdot Q_v = (P_v\bar{Q}_v + Q_v\bar{P}_v)$. The terms involving (p^i, q_i) correspond to $\gamma^1(a)$, which is correctly associated with qubit A,

$$2(p^1p^2 - p^0q_3) = -\gamma^1(a)_{00}, \qquad 2(q_1q_2 + p^3q_0) = -\gamma^1(a)_{11}, \qquad p \cdot q - 2p^3q_3 = \gamma^1(a)_{01}. \tag{7.50}$$

This agrees with the conclusions drawn from the decomposition given in (7.41). Keeping either (P^s, Q_s) or (P^c, Q_c) instead would have resulted in a different associated slicing of the cube Figure 4.1 and, hence, matrix $\gamma^i(a)$. We then identify (g_{GAC}, e_{EFA}) with (P_v, Q_v) such that

$$P_v{}^2 = \gamma^2(g)_{00} + \gamma^3(e)_{00}, \qquad Q_v{}^2 = \gamma^2(g)_{11} + \gamma^3(e)_{11}, \qquad P_v \cdot Q_v = \gamma^2(g)_{01} + \gamma^3(e)_{01}, \tag{7.51}$$

where, for example, the index on $\gamma^2(g)$ is determined by the position of the common qubit A in the corresponding tripartite subsystem, GAC. Computing I_4 one finds

$$I_4 = \det(\gamma^1(a) + \gamma^3(e) + \gamma^2(g)) \sim -\operatorname{Det} a - \operatorname{Det} e - \operatorname{Det} g + 2(a^2g^2 + a^2e^2 + e^2g^2), \tag{7.52}$$

where products like

$$a^2e^2 = \tfrac{1}{2}\epsilon^{A_1A_4}\epsilon^{B_1B_2}\epsilon^{D_1D_2}\epsilon^{E_3E_4}\epsilon^{F_3F_4}\epsilon^{A_2A_3}a_{A_1B_1D_1}a_{A_2B_2D_2}e_{E_3F_3A_3}e_{E_4F_4A_4}, \tag{7.53}$$

describe the entanglement between two tripartite subsystems connected by a common qubit, in this case A. This may be repeated for the remaining two cases, keeping (P_s, Q_s) or (P_c, Q_c), associated with the common qubits B and D respectively, to construct the whole dictionary[4]. For each of the seven possible $\mathcal{N} = 4$ subsectors one obtains the appropriate result, analogous to (7.52), as presented in (7.56).

We are now able to select any particular subsector of the full $\mathcal{N} = 8$ theory by choosing the appropriate components of the FTS and systematically determine the corresponding qubit system and its measure of entanglement.

In summary the dictionary is

$$\begin{aligned}
\big(p^0, p^1, p^2, p^3, q_0, q_1, q_2, q_3\big) &= \big(a_0, -a_1, -a_2, -a_4, a_7, a_6, a_5, a_3\big),\\
P_v = (g_{G0C}, e_{EF0}), &\quad Q_v = (g_{G1C}, e_{EF1}),\\
P_s = (f_{FG0}, b_{0CE}), &\quad Q_s = (f_{FG1}, b_{1CE}),\\
P_c = (d_{0EG}, c_{C0F}), &\quad Q_c = (d_{1EG}, c_{C1F}),
\end{aligned} \tag{7.54}$$

where explicitly

$$\begin{aligned}
\begin{pmatrix} d_0 & d_1 \\ d_2 & d_3 \end{pmatrix} &= \begin{pmatrix} P_c^0 - P_c^4 & P_c^1 - P_c^5 \\ -P_c^1 - P_c^5 & P_c^0 + P_c^4 \end{pmatrix} & \begin{pmatrix} d_4 & d_5 \\ d_6 & d_7 \end{pmatrix} &= \begin{pmatrix} Q_c^0 - Q_c^4 & Q_c^1 - Q_c^5 \\ -Q_c^1 - Q_c^5 & Q_c^0 + Q_c^4 \end{pmatrix}\\
\begin{pmatrix} c_0 & c_1 \\ c_4 & c_5 \end{pmatrix} &= \begin{pmatrix} -P_c^6 - P_c^2 & -P_c^3 - P_c^7 \\ P_c^3 - P_c^7 & P_c^6 - P_c^2 \end{pmatrix} & \begin{pmatrix} c_2 & c_3 \\ c_6 & c_7 \end{pmatrix} &= \begin{pmatrix} -Q_c^6 - Q_c^2 & -Q_c^3 - Q_c^7 \\ Q_c^3 - Q_c^7 & Q_c^6 - Q_c^2 \end{pmatrix}\\
\begin{pmatrix} f_0 & f_2 \\ f_4 & f_6 \end{pmatrix} &= \begin{pmatrix} P_s^3 - P_s^7 & P_s^6 + P_s^2 \\ P_s^6 - P_s^2 & P_s^3 + P_s^7 \end{pmatrix} & \begin{pmatrix} f_1 & f_3 \\ f_5 & f_7 \end{pmatrix} &= \begin{pmatrix} Q_s^3 - Q_s^7 & Q_s^6 + Q_s^2 \\ Q_s^6 - Q_s^2 & Q_s^3 + Q_s^7 \end{pmatrix}\\
\begin{pmatrix} b_0 & b_1 \\ b_2 & b_3 \end{pmatrix} &= \begin{pmatrix} P_s^1 - P_s^5 & P_s^4 - P_s^0 \\ P_s^4 + P_s^0 & P_s^1 + P_s^5 \end{pmatrix} & \begin{pmatrix} b_4 & b_5 \\ b_6 & b_7 \end{pmatrix} &= \begin{pmatrix} Q_s^1 - Q_s^5 & Q_s^4 - Q_s^0 \\ Q_s^4 + Q_s^0 & Q_s^1 + Q_s^5 \end{pmatrix}\\
\begin{pmatrix} g_0 & g_1 \\ g_4 & g_5 \end{pmatrix} &= \begin{pmatrix} P_v^0 + P_v^4 & P_v^5 + P_v^1 \\ P_v^5 - P_v^1 & P_v^0 - P_v^4 \end{pmatrix} & \begin{pmatrix} g_2 & g_3 \\ g_6 & g_7 \end{pmatrix} &= \begin{pmatrix} Q_v^0 + Q_v^4 & Q_v^5 + Q_v^1 \\ Q_v^5 - Q_v^1 & Q_v^0 - Q_v^4 \end{pmatrix}\\
\begin{pmatrix} e_0 & e_2 \\ e_4 & e_6 \end{pmatrix} &= \begin{pmatrix} P_v^2 + P_v^6 & P_v^7 + P_v^3 \\ P_v^7 - P_v^3 & P_v^2 - P_v^6 \end{pmatrix} & \begin{pmatrix} e_1 & e_3 \\ e_5 & e_7 \end{pmatrix} &= \begin{pmatrix} Q_v^2 + Q_v^6 & Q_v^7 + Q_v^3 \\ Q_v^7 - Q_v^3 & Q_v^2 - Q_v^6 \end{pmatrix}.
\end{aligned} \tag{7.55}$$

[4]Note, determining the precise form of the full dictionary and verifying that it does indeed give the stated results was done explicitly using *Mathematica*.

The $\mathcal{N} = 4$ subsector invariant under $\mathrm{SL}(2)_X \times \mathrm{SO}(6,6)$ is given by

$$
\begin{aligned}
I_{dac} &= \det(\gamma^1(d) + \gamma^3(a) + \gamma^2(c)), \quad X = D \\
I_{ebd} &= \det(\gamma^1(e) + \gamma^3(b) + \gamma^2(d)), \quad X = E \\
I_{fce} &= \det(\gamma^1(f) + \gamma^3(c) + \gamma^2(e)), \quad X = F \\
I_{gdf} &= \det(\gamma^1(g) + \gamma^3(d) + \gamma^2(f)), \quad X = G \\
I_{aeg} &= \det(\gamma^1(a) + \gamma^3(e) + \gamma^2(g)), \quad X = A \\
I_{bfa} &= \det(\gamma^1(b) + \gamma^3(f) + \gamma^2(a)), \quad X = B \\
I_{cgb} &= \det(\gamma^1(c) + \gamma^3(g) + \gamma^2(b)), \quad X = C.
\end{aligned}
\tag{7.56}
$$

Let us conclude with some remarks on the status of these dictionaries. First, while the Cartan-Fano dictionary was precisely specified by our rules, we did not really understand their basic origin. However, it was subsequently shown that it is intimately related to the geometry of the *split Cayley hexagon of order two*, which is in turn related to the 63 real generalized Pauli matrices of 3-qubits [15]. The significance of these structures on the black hole side is unclear. We leave this for future work. Similarly, we do not properly understand the Fano-Freudenthal dictionary. In fact, it would have not been possible to construct in its entirety or check without the rather impressive Mathematica package "Black Holes, Qubits and Octonions" developed by Duminda Dayahanake. It would be desirable to have a complete dictionary expressed in a mathematically concise manner, but we also leave this for future work.

CHAPTER 8

The FTS classification of qubit entanglement

The STU model has played a rather ubiquitous role in the developments described so far. We draw your attention to two such instances: (1) The classification STU black holes corresponded to the entanglement classification of three qubits. (2) The STU model admits an FTS interpretation, as we discovered via its embedding in the $\mathcal{N}=8$ theory.

Consequently, we would expect the elegant mathematics of Jordan algebras and Freudenthal triple systems to naturally capture the entanglement classification of three qubits. As we shall see in section 1, the FTS ranks, in a succinct algebraic manner, do indeed yield the correct classification. The entanglement classes correspond to FTS ranks 0, 1, 2a, 2b, 2c, 3 and 4, or, for SLOCC*, simply 0, 1, 2, 3, 4. In fact, we would argue that this is perhaps the most natural classification scheme. This is not only a matter of aesthetics. The classification of [44] based on the local entropies $S_{A,B,C}$ (see section 2.6 of section 2) did not make the $[\mathrm{SL}(2,\mathbb{C})]^3$ symmetry manifest. While the hyperdeteminant is SLOCC-covariant, the local entropies are not; they are natural objects for a classification based on local unitaries not SLOCC. This observation is important from the perspective of generalising to n qubits. Three qubits is the only non-trivial data point we have for a full SLOCC classification. If we seek to generalise this result, we should first formulate it in terms of the objects that will generalise, i.e. SLOCC-covariants. The FTS formulation is manifestly SLOCC-covariant since the automorphism group coincides with the SLOCC-equivalence group.

More speculatively, by studying the FTS classification, one might hope to identify those algebraic features which would usefully carry over to an n-qubit generalisation. Some preliminary ideas in this direction are presented in section 2.

1. The FTS classification of three qubit entanglement

1.1. The STU Freudenthal triple system

In chapter 6 we saw that the STU model is given by the $n=2$ point of the sequence $\mathfrak{F}(\mathbb{R}\oplus\Gamma_{1,n-1})$, which corresponds to sequence of reducible $\mathcal{N}=2$ supergravities coupled to $n+1$ vector multiplets. For arbitrary n the automorphism group is given by $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n;\mathbb{R})$, which, for the STU model, picks out one $\mathrm{SL}(2,\mathbb{R})$ as special, obscuring its triality invariance [2]. However, a simple rotation allows to rewrite the Jordan algebra $\mathbb{R}\oplus\Gamma_{1,1}$ as an equivalent Jordan algebra, $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$, which is specific to the STU model. The resulting FTS treats all three factors of $\mathrm{SL}(2,\mathbb{R})$ on an equal footing. Recall, for $\mathbb{R}\oplus\Gamma_{1,1}$ elements are written,

$$A = (a; a_\mu), \tag{8.1}$$

where a_μ is an $\mathrm{SO}(1,1;\mathbb{R})$ vector. The cubic norm and the (counter-intuitive) Jordan product are given by,

$$N_3(A) = aa^\mu a_\mu, \qquad A\circ B = (ab; a_0b_0 + a_1b_1, a_0b_1 + b_0a_1). \tag{8.2}$$

Letting

$$A_1 = a, \qquad A_2 = a_0 + a_1, \qquad A_3 = a_0 - a_1 \tag{8.3}$$

we find

$$N_3(A) = A_1A_2A_3, \qquad A\circ B = (A_1B_1, A_2B_2, A_3B_3). \tag{8.4}$$

This motivates:

Definition 37 (STU cubic Jordan algebra)**.** *We define the STU cubic Jordan algebra, denoted $\mathfrak{J}_{STU}$, as the real vector space $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ with elements,*

$$A = (A_1, A_2, A_3), \tag{8.5}$$

and cubic norm,

$$N_3(A) = A_1A_2A_3. \tag{8.6}$$

Using the cubic Jordan algebra construction (5.22), one finds

$$\mathrm{Tr}(A,B) = A_1B_1 + A_2B_2 + A_3B_3, \tag{8.7}$$

Then, using $\mathrm{Tr}(A^\sharp, B) = 3N(A,A,B)$, the quadratic adjoint is given by

$$A^\sharp = (A_2A_3, A_1A_3, A_1A_2), \tag{8.8}$$

and therefore

$$\begin{aligned}(A^\sharp)^\sharp &= (A_1A_2A_3A_1, A_1A_2A_3A_2, A_1A_2A_3A_3)\\ &= N(A)A.\end{aligned} \tag{8.9}$$

It is not hard to check $\mathrm{Tr}(A,B)$ is non-degenerate and so N_3 is Jordan cubic. Hence, we have a *bona fide* cubic Jordan algebra $\mathfrak{J}_{STU} = \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ with product given by

$$A\circ B = (A_1B_1, A_2B_2, A_3B_3). \tag{8.10}$$

The structure and reduced structure groups are given by $[\mathrm{SO}(2,\mathbb{R})]^3$ and $[\mathrm{SO}(2,\mathbb{R})]^2$ respectively.

The 3-qubit cubic Jordan algebra is defined by simply promoting $\mathbb{R}$ to $\mathbb{C}$,

Definition 38 (3-qubit cubic Jordan algebra)**.** *We define the 3-qubit cubic Jordan algebra, denoted $\mathfrak{J}_{ABC}$, as the complex vector space $\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$ with elements:*

$$A = (A_1, A_2, A_3), \tag{8.11}$$

and cubic norm,

$$N_3(A) = A_1 A_2 A_3. \tag{8.12}$$

1.2. The 3-qubit Freudenthal triple system

Definition 39 (3-qubit Freudenthal triple system)**.** *We define the 3-qubit Freudenthal triple system, denoted $\mathfrak{F}_{ABC}$, as the complex vector space,*

$$\mathfrak{F}_{ABC} := \mathbb{C} \oplus \mathbb{C} \oplus \mathfrak{J}_{ABC} \oplus \mathfrak{J}_{ABC}, \tag{8.13}$$

with elements

$$\begin{pmatrix} \alpha & A = (A_1, A_2, A_3) \\ B = (B_1, B_2, B_3) & \beta \end{pmatrix} \tag{8.14}$$

and antisymmetric bilinear form, quartic norm and triple product as defined in (5.64)*.*

We identify the eight complex components of $\mathfrak{F}_{ABC}$ with the three qubit wavefunction $|\psi\rangle = a_{ABC}|ABC\rangle$,

$$\begin{pmatrix} \alpha & (A_1, A_2, A_3) \\ (B_1, B_2, B_3) & \beta \end{pmatrix} \leftrightarrow \begin{pmatrix} a_{111} & (a_{001}, a_{010}, a_{100}) \\ (a_{110}, a_{101}, a_{011}) & a_{000} \end{pmatrix} \tag{8.15}$$

so that

$$|\Psi\rangle = a_{ABC}|ABC\rangle \quad \leftrightarrow \quad \Psi := \begin{pmatrix} a_{111} & (a_{001}, a_{010}, a_{100}) \\ (a_{110}, a_{101}, a_{011}) & a_{000} \end{pmatrix}. \tag{8.16}$$

Using (B.14) one finds that the quartic norm $\Delta(\Psi)$ is related to Cayley's hyperdeterminant by

$$\begin{aligned} \Delta(\Psi) &= \{T(\Psi, \Psi, \Psi), \Psi\} \\ &= 2 \det \gamma^A = 2 \det \gamma^B = 2 \det \gamma^C \\ &= -2 \operatorname{Det} a_{ABC}, \end{aligned} \tag{8.17}$$

The triple product maps a state Ψ, which transforms as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $[\mathrm{SL}(2, \mathbb{C})]^3$, to another state $T(\Psi, \Psi, \Psi)$, cubic in the state vector coefficients, also transforming as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$. Explicitly, $T(\Psi, \Psi, \Psi)$ may be written as

$$T(\Psi, \Psi, \Psi) = T_{ABC}|ABC\rangle \tag{8.18}$$

where T_{ABC} takes one of three equivalent forms

$$\begin{aligned} T_{A_3B_1C_1} &= \epsilon^{A_1A_2} a_{A_1B_1C_1} (\gamma^A)_{A_2A_3} \\ T_{A_1B_3C_1} &= \epsilon^{B_1B_2} a_{A_1B_1C_1} (\gamma^B)_{B_2B_3} \\ T_{A_1B_1C_3} &= \epsilon^{C_1C_2} a_{A_1B_1C_1} (\gamma^C)_{C_2C_3}. \end{aligned} \tag{8.19}$$

The γ's are related to the local entropies of section 2.6 by

$$S_A = 4\Big[\operatorname{tr} \gamma^{B\dagger}\gamma^B + \operatorname{tr} \gamma^{C\dagger}\gamma^C\Big], \quad \operatorname{tr} \gamma^{A\dagger}\gamma^A = \tfrac{1}{8}\big[S_B + S_C - S_A\big] \tag{8.20}$$

and their cyclic permutations. This permits us to link T to the norm, local entropies and the Kempe

Class	Rank	FTS rank condition	
		vanishing	non-vanishing
Null	0	Ψ	–
A-B-C	1	$3T(\Psi,\Psi,\Phi)+\{\Psi,\Phi\}\Psi$	Ψ
A-BC	2a	$T(\Psi,\Psi,\Psi)$	γ^A
B-CA	2b	$T(\Psi,\Psi,\Psi)$	γ^B
C-AB	2c	$T(\Psi,\Psi,\Psi)$	γ^C
W	3	$\Delta(\Psi)$	$T(\Psi,\Psi,\Psi)$
GHZ	4	–	$\Delta(\Psi)$

Table 8.1.: The entanglement classification of three qubits as according to the FTS rank system.

invariant of section 2.6:

$$\langle T|T\rangle = \tfrac{2}{3}(K-|\psi|^6)+\tfrac{1}{16}|\psi|^2(S_A+S_B+S_C). \tag{8.21}$$

Having couched the 3-qubit system within the FTS framework we may assign an abstract FTS rank (5.89) to an arbitrary state Ψ:

$$\begin{aligned}
&\text{Rank}\Psi = 1 \Leftrightarrow \Upsilon(\Psi,\Psi,\Phi)=0,\ \Psi\neq 0;\\
&\text{Rank}\Psi = 2 \Leftrightarrow T(\Psi)=0,\ \Upsilon(\Psi,\Psi,\Phi)\neq 0;\\
&\text{Rank}\Psi = 3 \Leftrightarrow \Delta(\Psi)=0,\ T(\Psi)\neq 0;\\
&\text{Rank}\Psi = 4 \Leftrightarrow \Delta(\Psi)\neq 0.
\end{aligned} \tag{8.22}$$

Strictly speaking, the automorphism group $\text{Aut}(\mathfrak{F}_{ABC})$ is not simply $\text{SL}(2,\mathbb{C})\times\text{SL}(2,\mathbb{C})\times\text{SL}(2,\mathbb{C})$ but includes a semi-direct product with the interchange triality $A\leftrightarrow B\leftrightarrow C$. The rank conditions are invariant under this triality. Hence the ranks naturally provide an SLOCC* classification. However, as we shall demonstrate, the set of rank 2 states may be subdivided into three distinct classes which are inter-related by this triality. In the next section we show that these rank conditions give the correct entanglement classification of three qubits as in Table 8.1.

1.3. The FTS rank entanglement classes

1.3.1. Rank 1 and the class of separable states

A non-zero state Ψ is rank 1 if and only if

$$\Upsilon(\Psi,\Psi,\Phi) := 3T(\Psi,\Psi,\Phi)+\{\Psi,\Phi\}\Psi = 0, \quad \forall\,\Phi. \tag{8.23}$$

The weaker condition $T(\Psi,\Psi,\Psi)=0$ implies that there is at most one non-vanishing γ since,

$$\begin{aligned}
(\gamma^A)_{A_1A_2}(\gamma^C)_{C_1C_2} &= \epsilon^{B_1B_2}\epsilon^{Z_1Z_2}a_{A_1B_1Z_1}a_{A_2B_2Z_2}(\gamma^C)_{C_1C_2}\\
&= \epsilon^{B_2B_1}a_{A_1B_1C_1}T_{A_2B_2C_2}+\epsilon^{B_1B_2}a_{A_2B_2C_1}T_{A_1B_1C_2},
\end{aligned} \tag{8.24}$$

and similarly for $(\gamma^B)_{B_1B_2}(\gamma^A)_{A_1A_2}$ and $(\gamma^C)_{C_1C_2}(\gamma^B)_{B_1B_2}$. In component form Υ is given by,

$$-\Upsilon_{A_1B_1C_1} = \epsilon^{A_2A_3}b_{A_3B_1C_1}(\gamma^A)_{A_1A_2}+\epsilon^{B_2B_3}b_{A_1B_3C_1}(\gamma^B)_{B_1B_2}+\epsilon^{C_2C_3}b_{A_1B_1C_3}(\gamma^C)_{C_1C_2} \tag{8.25}$$

where

$$|\phi\rangle = b_{ABC}|ABC\rangle \quad \leftrightarrow \quad \Phi = \begin{pmatrix} b_{111} & (b_{001}, b_{010}, b_{100}) \\ (b_{110}, b_{101}, b_{011}) & b_{000} \end{pmatrix}. \tag{8.26}$$

Hence, (8.23) implies all three gammas must vanish. Using (8.20) it is then clear that all three local entropies vanish.

Conversely, $S_A = S_B = S_C = 0$ implies that each of the three γ's vanish and the rank 1 condition is satisfied. Hence FTS rank 1 is equivalent to the class of separable states as in Table 8.1.

1.3.2. Rank 2 and the class of biseparable states

A nonzero state Ψ is rank 2 or less if and only if $T(\Psi,\Psi,\Psi) = 0$, which implies there is at most one non-vanishing γ. To be rank > 1 there must exist some Φ such that $3T(\Psi,\Psi,\Phi) + \{\Psi,\Phi\}\Psi \neq 0$, which implies there is at least one non-vanishing γ. Hence, rank 2 states have precisely one nonzero γ.

Using (8.20) it is clear that the choices $\gamma^A \neq 0$ or $\gamma^B \neq 0$ or $\gamma^C \neq 0$ give $S_A = 0, S_{B,C} \neq 0$ or $S_B = 0, S_{C,A} \neq 0$ or $S_C = 0, S_{A,B,} \neq 0$, respectively. These are precisely the conditions for the biseparable class A-BC or B-CA or C-AB presented in Table 2.2.

Conversely, using (8.20) and the fact that the local entropies and $\mathrm{tr}(\gamma^\dagger\gamma)$ are positive semidefinite, we find that all states in the biseparable class are rank 2, the particular subdivision being given by the corresponding non-zero γ. Hence FTS rank 2 is equivalent to the class of biseparable states as in Table 8.1.

1.3.3. Rank 3 and the class of W-states

A non-zero state Ψ is rank 3 if $\Delta(\Psi) = -2\,\mathrm{Det}\,a = 0$ but $T(\Psi,\Psi,\Psi) \neq 0$. From (8.19) all three γ's are then non-zero but from (8.17) all have vanishing determinant. In this case (8.20) implies that all three local entropies are non-zero but $\mathrm{Det}\,a = 0$. So all rank 3 Ψ belong to the W-class.

Conversely, from (8.20) it is clear that no two γ's may simultaneously vanish when all three S's > 0. We saw in section 1.3.1 that $T(\Psi,\Psi,\Psi) = 0$ implied at least two of the γ's vanish. Consequently, for all W-states $T(\Psi,\Psi,\Psi) \neq 0$ and, therefore, all W-states are rank 3. Hence FTS rank 3 is equivalent to the class of W-states as in Table 8.1.

1.3.4. Rank 4 and the class of GHZ-states

The rank 4 condition is given by $\Delta(\Psi) \neq 0$ and, since for the 3-qubit FTS $\Delta(\Psi) = -2\,\mathrm{Det}\,a$, we immediately see that the set of rank 4 states is equivalent to the GHZ class of genuine tripartite entanglement as in Table 8.1.

Note, $\mathrm{Aut}(\mathfrak{F}_{ABC})$ acts transitively only on rank 4 states with the same value of $\Delta(\Psi)$ as in the standard treatment. The GHZ class really corresponds to a 1-dimensional space of orbits parametrised by Δ.

In summary, we have demonstrated that each rank corresponds to one of the entanglement classes described in section 2 section 2.6.

Class	FTS Rank	Orbits	dim	Projective orbits	dim
Separable	1	$\frac{[\mathrm{SL}(2,\mathbb{C})]^3}{[\mathrm{SO}(2,\mathbb{C})]^2 \ltimes \mathbb{C}^3}$	4	$\frac{[\mathrm{SL}(2,\mathbb{C})]^3}{[\mathrm{SO}(2,\mathbb{C}) \ltimes \mathbb{C}]^3}$	3
Biseparable	2	$\frac{[\mathrm{SL}(2,\mathbb{C})]^3}{O(3,\mathbb{C}) \times \mathbb{C}}$	5	$\frac{[\mathrm{SL}(2,\mathbb{C})]^3}{O(3,\mathbb{C}) \times (\mathrm{SO}(2,\mathbb{C}) \ltimes \mathbb{C})}$	4
W	3	$\frac{[\mathrm{SL}(2,\mathbb{C})]^3}{\mathbb{C}^2}$	7	$\frac{[\mathrm{SL}(2,\mathbb{C})]^3}{\mathrm{SO}(2,\mathbb{C}) \ltimes \mathbb{C}^2}$	6
GHZ	4	$\frac{[\mathrm{SL}(2,\mathbb{C})]^3}{[\mathrm{SO}(2,\mathbb{C})]^2}$	7	$\frac{[\mathrm{SL}(2,\mathbb{C})]^3}{[\mathrm{SO}(2,\mathbb{C})]^2}$	7

Table 8.2.: Coset spaces of the orbits of the 3-qubit state space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ under the action of the SLOCC group $[\mathrm{SL}(2,\mathbb{C})]^3$.

1.4. SLOCC orbits

We now turn our attention to the coset parametrisation of the entanglement classes. The coset space of each orbit ($i = 1,2,3,4$) is given by G/H_i where $G = [\mathrm{SL}(2,\mathbb{C})]^3$ is the SLOCC group and $H_i \subset [\mathrm{SL}(2,\mathbb{C})]^3$ is the stability subgroup leaving the representative state of the ith orbit invariant. We proceed by considering the infinitesimal action of $\mathrm{Aut}(\mathfrak{F})$ on the representative states of each class. The subalgebra annihilating the representative state gives, upon exponentiation, the stability group H. Note, $\mathfrak{Der}(\mathfrak{J}_{ABC})$ is empty due to the associativity of $\mathfrak{J}_{ABC}$. Consequently, $\mathfrak{Str}(\mathfrak{J}_{ABC}) = L_{\mathbb{1}}\mathbb{C} \oplus \mathfrak{Str}_0(\mathfrak{J}_{ABC})$ has complex dimension 3, while $\mathfrak{Str}_0(\mathfrak{J}_{ABC})$ is now simply $L_{\mathfrak{J}'}$ and has complex dimension 2. Recall, $\mathfrak{Str}(\mathfrak{J}_{ABC})$ and $\mathfrak{Str}_0(\mathfrak{J}_{ABC})$ generate $[\mathrm{SO}(2,\mathbb{C})]^3$ and $[\mathrm{SO}(2,\mathbb{C})]^2$, respectively the structure and reduced structure groups of $\mathfrak{J}_{\mathbb{C}}$.

The results are summarised in Table 8.2. To be clear, in the preceding analysis we have regarded the three-qubit state as a point in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the philosophy adopted in, for example, [113,121,124]. We could have equally well considered the projective Hilbert space regarding states as rays in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, that is, identifying states related by a global complex scalar factor, as was done in [7,31,126]. The coset spaces obtained in this case are also presented in Table 8.2, the dimensions of which agree with the results of [31,128]. Note that the three-qubit separable projective coset is just a direct product of three individual qubit cosets $\mathrm{SL}(2,\mathbb{C})/\mathrm{SO}(2,\mathbb{C}) \ltimes \mathbb{C}$. Furthermore, the biseparable projective coset is just the direct product of the two entangled qubits coset $[\mathrm{SL}(2,\mathbb{C})]^2/\mathrm{O}(3,\mathbb{C})$ and an individual qubit coset. As noted in [9,123], the case of real qubits or "rebits" is qualitatively different from the complex case. An interesting observation is that on restricting to real states the GHZ class actually has two distinct orbits, characterised by the sign of $\Delta(\Psi)$. This difference shows up in the cosets in the different possible real forms of $[\mathrm{SO}(2,\mathbb{C})]^2$. For positive $\Delta(\Psi)$ there are two disconnected orbits, both with $[\mathrm{SL}(2,\mathbb{R})]^3/[\mathrm{U}(1)]^2$ cosets, while for negative $\Delta(\Psi)$ there is one orbit $[\mathrm{SL}(2,\mathbb{R})]^3/[\mathrm{SO}(1,1,\mathbb{R})]^2$. In which of the two positive $\Delta(\Psi)$ orbits a given state lies is determined by the sign of the eigenvalues of the three γ's, as shown in Table 8.3. This phenomenon also has its counterpart in the black-hole context [18,45,71,72,196,263], where the two disconnected $\Delta(\Psi) > 0$ orbits are given by $1/2$-BPS black holes and non-BPS black holes with vanishing central charge respectively [72].

2. Generalising to an n-qubit FTS

The FTS classification of 3-qubit entanglement raises the question of generalisation. Are there other QI systems amenable to the FTS? More ambitiously, is it possible to treat an arbitrary number of

Class	FTS Rank	$\Delta(\Psi)$	Orbits	dim
Separable	1	$=0$	$\dfrac{[\mathrm{SL}(2,\mathbb{R})]^3}{[\mathrm{SO}(1,1)]^2 \ltimes \mathbb{R}^3}$	4
Biseparable	2	$=0$	$\dfrac{[\mathrm{SL}(2,\mathbb{R})]^3}{O(2,1)\times\mathbb{R}}$	5
W	3	$=0$	$\dfrac{[\mathrm{SL}(2,\mathbb{R})]^3}{\mathbb{R}^2}$	7
GHZ	4	<0	$\dfrac{[\mathrm{SL}(2,\mathbb{R})]^3}{[\mathrm{SO}(1,1)]^2}$	7
GHZ	4	>0	$\dfrac{[\mathrm{SL}(2,\mathbb{R})]^3}{[U(1)]^2}$	7
GHZ	4	>0	$\dfrac{[\mathrm{SL}(2,\mathbb{R})]^3}{[U(1)]^2}$	7

Table 8.3.: Coset spaces of the orbits of the real case $\mathfrak{J}_{\mathbb{R}} = \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ under $[\mathrm{SL}(2,\mathbb{R})]^3$.

qubits? Perhaps the most natural option is to do as was done for the $\mathcal{N}=8$ case and see whether the various FTS's appearing in the context of supergravity admit a QI interpretation. This would have the obvious additional payoff of further developing the black hole/QI correspondence. However, an alternative route is to identify and generalise those structural aspects of the FTS that made it so suitable for the 3-qubit classification. This is the approach taken here.

2.1. Three qubits and the FTS re-examined

2.1.1. The n-qubit state

Recall, the Jordan algebra formulation of the FTS corresponded to decomposing the representation carried by the FTS under the $\mathrm{Str}_0(\mathfrak{J}) \subset \mathrm{Aut}(\mathfrak{F})$. In the case of three qubits we found the state split into the direct sum of four pieces,

$$\begin{aligned}\alpha &= a_{111},\\ \beta &= a_{000},\\ A &= (a_{100}, a_{010}, a_{001}),\\ B &= (a_{011}, a_{101}, a_{110}),\end{aligned} \tag{8.27}$$

where α,β are $\mathrm{Str}_0(\mathfrak{J})$ singlets. This leads us to the first important observation: α,β,A,B are the closed subsets under the 3-qubit permutation group S_3. Indeed, for the n-qubit SLOCC* entanglement classification, it is only natural to work with S_n closed subsets as the basic building blocks. This is the first generalisation we will make. The 2^n state vector coefficients will be collected into the $n+1$ closed subsets of S_n. It will prove convenient to represent these subsets using $n+1$ totally symmetric tensors with ranks ranging from 0 to n,

$$\mathcal{A}_n := \{A_0, A_{i_1}, A_{i_1 i_2}, \ldots, A_{i_1 i_2 \ldots i_n}\}, \qquad \text{where} \qquad i_k = 1,2,\ldots,n, \tag{8.28}$$

which are vanishing on any diagonal, i.e. $A_{i_1 i_2\ldots i_n}=0$ if any two indices are the same. The counting of components goes like p-forms, correctly yielding a total of 2^n independent numbers. In fact, why not use p-forms and bring to bear on entanglement all the powerful associated geometrical machinery. While there is an obvious isomorphism to p-forms, they cannot play any deep role. The simplest way to see this is that they would actually imply an enhancement of the symmetry from $[\mathrm{SL}(2,\mathbb{C})]^n$ to $\mathrm{SL}(n,\mathbb{C})$.

So, for three qubits we have (with a slight abuse of notation for A_{ijk} and B),

$$\begin{aligned} A_0 &= & a_{000} &= \beta, \\ A_i &= & \begin{pmatrix} a_{100} \\ a_{010} \\ a_{001} \end{pmatrix} &= A, \\ A_{ij} &= & \begin{pmatrix} 0 & a_{110} & a_{101} \\ a_{110} & 0 & a_{011} \\ a_{101} & a_{011} & 0 \end{pmatrix} &= B, \\ A_{ijk} &= & a_{111} &= \alpha. \end{aligned} \tag{8.29}$$

Note, numbering the qubits from left to right, the values of the indices on the symmetric tensors determine which indices on its corresponding state vector coefficient take the value 1. For example, $A_1 = a_{100}, A_2 = a_{010}, A_3 = a_{001}$ and $A_{12} = a_{110}, A_{13} = a_{101}$ and so on[1].

2.1.2. The n-qubit algebra

The second feature we might hope to generalise is the set of cubic Jordan algebra maps, $A\times B, \mathrm{Tr}(A,B)$, $N_3(A)$, see section 2.2, which played such a key role in the construction of the various covariants and invariants. Recall, group theoretically these maps correspond to picking out certain irreps appearing in the tensor product of the $\mathrm{Str}_0(\mathfrak{J})$-representation carried by $A, B \in \mathfrak{J}$. For example, in the case $\mathrm{Str}_0(\mathfrak{J}_3^{\mathbb{O}}) = E_{6(-26)}$, with A and B transforming as the $\mathbf{27}$, $A\times B$ is the $\mathbf{27}'$ in $\mathbf{27}\times\mathbf{27} = \mathbf{27}'_s + \mathbf{351}_a + \widetilde{\mathbf{351}}_s$. Similarly, for A and B transforming as the $\mathbf{27}$ and $\mathbf{27}'$ respectively, $\mathrm{Tr}(A,B)$ is the singlet in $\mathbf{27}\times\mathbf{27}' = \mathbf{1}+\mathbf{78}+\mathbf{650}$. Finally, $N_3(A)$ is the singlet in $\mathbf{27}\times\mathbf{27}\times\mathbf{27}$ or, equivalently, in $\mathbf{27}'\times\mathbf{27}'\times\mathbf{27}'$. Hence, each of the cubic Jordan algebra maps, in this case, may be written using the irreducible $E_{6(-26)}$ invariant tensors, d_{ijk} and d^{ijk}, where a downstairs (upstairs) $i = 1, 2, \dots, 27$ transforms as a $\mathbf{27}$ ($\mathbf{27}'$). That is,

$$\begin{aligned} (A^\sharp)^i &= \frac{1}{2!} d^{ijk} A_j A_k, \\ (B^\sharp)_i &= \frac{1}{2!} d_{ijk} B^j B^k, \\ (A\times B)^i &= d^{ijk} A_j B_k, \\ (A\times B)_i &= d_{ijk} A^j B^k, \\ \mathrm{Tr}(A,B) &= A_i B^i, \\ N_3(A) &= \frac{1}{3!} d^{ijk} A_i A_j A_k, \\ N_3(B) &= \frac{1}{3!} d_{ijk} B^i B^j B^k. \end{aligned} \tag{8.30}$$

[1]We are grateful to Duminda Dahanayake for spotting this rule.

For the sake of clarity, we will often drop the combinatorial factors in the following. For three qubits, the invariant tensors were simply

$$\begin{aligned} d_{ijk} &= |\epsilon_{ijk}|, \\ d^{ijk} &= |\epsilon^{ijk}|, \end{aligned} \tag{8.31}$$

which naturally suggests the n-qubit generalisation,

$$\begin{aligned} d_{i_1\dots i_n} &:= |\epsilon_{i_1\dots i_n}|, \\ d^{i_1\dots i_n} &:= |\epsilon^{i_1\dots i_n}|. \end{aligned} \tag{8.32}$$

This allows us to dualise a rank p tensor as follows,

$$\begin{aligned} A^{i_1 i_2\dots i_{n-p}} &:= d^{i_1 i_2\dots i_{n-p} i_{n-p+1}\dots i_n} A_{i_{n-p+1}\dots i_n}, \\ A_{i_1 i_2\dots i_{n-p}} &:= d_{i_1 i_2\dots i_{n-p} i_{n-p+1}\dots i_n} A^{i_{n-p+1}\dots i_n}. \end{aligned} \tag{8.33}$$

For an n-qubit state, rank p pairs $A_{i_1 i_2\dots i_p}, A^{i_1 i_2\dots i_p}$ are precisely bit-flip related. For example, for three qubits, $A_i = (a_{100}, a_{010}, a_{001})$ and $A^i = (a_{011}, a_{101}, a_{110})$. This is crucial for building S_n invariants.

Equipped with $d_{i_1\dots i_n}, d^{i_1\dots i_n}$ the n-qubit space $\mathcal{A}_n$ of symmetric tensors may be endowed with a sort of algebraic structure. The space $\mathcal{A}_n$ is closed so long as we compose the tensors by contracting with $d_{i_1\dots i_n}$ and $d^{i_1\dots i_n}$. Consider the 4-qubit example,

$$\mathcal{A}_4 := \{A_0, A_i, A_{ij}, A_{ijk}, A_{ijkl}\}. \tag{8.34}$$

For rank one tensors, we may define in analogy to the cubic Jordan algebra maps,

$$\begin{aligned} (A^\flat)^i &:= d^{ijkl} A_j A_k A_l, \\ (A, B, C)^i &:= d^{ijkl} A_j B_k C_l, \\ (A, B, C)_i &:= d_{ijkl} A^j B^k C^l, \\ \operatorname{Tr}(A, B) &:= A_i B^i, \\ N_4(A) &:= d^{ijkl} A_i A_j A_k A_l, \\ N_4(B) &:= d_{ijkl} B^i B^j B^k. \end{aligned} \tag{8.35}$$

Note, the quadratic adjoint $A^\sharp$ and cubic norm $N_3(A)$ are replaced, in this case, by a *cubic* adjoint $A^\flat$ and *quartic* norm $N_4(A)$, which satisfy $(A^\flat)^\flat = N_4(A)^2 X$ in analogy to the Jordan cubic condition (5.23). In fact, the quartic norm may used to define a *quartic* Jordan algebra as described in section B.2. However, in the current context the tensors may be composed in all possible ways allowed by contracting with $d_{i_1\dots i_n}$ and $d^{i_1\dots i_n}$. This is the n-qubit pseudo-algebraic generalisation of the cubic Jordan algebra appearing in the 3-qubit FTS.

2.1.3. The n-qubit SLOCC transformations

The final ingredient is the n-qubit generalisation of the FTS transformations (5.75). For three qubits in the our current notation (5.75) is given by,

$$\begin{aligned}
\phi(C):\begin{pmatrix}A_0\\A_i\\A^i\\A^0\end{pmatrix}&\mapsto\begin{pmatrix}A_0&&&&&&\\C_iA_0&+&A_i&&&&\\d^{ijk}C_jC_kA_0&+&d^{ijk}C_jA_k&+&A^i&&\\d^{ijk}C_iC_jC_kA_0&+&d^{ijk}C_iC_jA_k&+&C_iA^i&+&A^0\end{pmatrix}\\
\psi(D):\begin{pmatrix}A_0\\A_i\\A^i\\A^0\end{pmatrix}&\mapsto\begin{pmatrix}d_{ijk}D^iD^jD^kA^0&+&d_{ijk}D^iD^jA^k&+&D^iA_i&+&A_0\\d_{ijk}D^jD^kA^0&+&d_{ijk}D^jA^k&+&A_i&&\\D^iA^0&+&A^i&&&&\\A^0&&&&&&\end{pmatrix}.
\end{aligned}\tag{8.36}$$

where $C=(C_1,C_2,C_3)$ and $D=(D^1,D^2,D^3)$. In terms of conventional $\mathrm{SL}(2,\mathbb{C})$ matrices,

$$\begin{aligned}
\phi(C_i)\quad&\longleftrightarrow\quad\begin{pmatrix}1&C_1\\0&-1\end{pmatrix}\otimes\begin{pmatrix}1&C_2\\0&-1\end{pmatrix}\otimes\begin{pmatrix}1&C_3\\0&-1\end{pmatrix},\\
\psi(D^i)\quad&\longleftrightarrow\quad\begin{pmatrix}1&0\\D^1&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\D^2&-1\end{pmatrix}\otimes\begin{pmatrix}1&0\\D^3&-1\end{pmatrix},
\end{aligned}\tag{8.37}$$

which are generated by the Lie algebra elements,

$$\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}.\tag{8.38}$$

Note, C_i and $D_{ij}=d_{ijk}D^k$ are rank 1 and $n-1$ tensors. Finally, the transformation $\hat{\tau}$, where $\tau\in\mathrm{Str}(\mathfrak{J}_{ABC})$, is given by,

$$\hat{\tau}(M):\begin{pmatrix}A_0\\A_i\\A_{ij}\\A_{ijk}\end{pmatrix}\mapsto\begin{pmatrix}d_{lmn}M^lM^mM^nA_0\\N_id_{imn}M^mM^nA_i\\N_iN_jd_{ijn}M^nA_{ij}\\N_iN_jN_kd_{ijk}A_{ijk}\end{pmatrix},\tag{8.39}$$

where $M^i=N_i^{-1}$. In terms of conventional $\mathrm{SL}(2,\mathbb{C})$ matrices,

$$\begin{pmatrix}M_1&0\\0&M_1^{-1}\end{pmatrix}\otimes\begin{pmatrix}M_2&0\\0&M_2^{-1}\end{pmatrix}\otimes\begin{pmatrix}M_3&0\\0&M_3^{-1}\end{pmatrix}\tag{8.40}$$

which are generated by the Lie algebra element,

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\tag{8.41}$$

This suggests the obvious extension to n qubits,

$$\begin{array}{lcl}\phi(C_i) & \longleftrightarrow & \begin{pmatrix}1 & C_1\\ 0 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & C_2\\ 0 & -1\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}1 & C_n\\ 0 & -1\end{pmatrix},\\ \psi(D^i) & \longleftrightarrow & \begin{pmatrix}1 & 0\\ D^1 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ D^2 & -1\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}1 & 0\\ D^n & -1\end{pmatrix},\\ \hat{\tau}(M) & \longleftrightarrow & \begin{pmatrix}M_1 & 0\\ 0 & M_1^{-1}\end{pmatrix}\otimes\begin{pmatrix}M_2 & 0\\ 0 & M_2^{-1}\end{pmatrix}\otimes\ldots\otimes\begin{pmatrix}M_n & 0\\ 0 & M_n^{-1}\end{pmatrix}\end{array} \tag{8.42}$$

These transformations are most naturally written in terms of $\mathcal{A}_n$. Under $\phi(C_i)$ a rank p tensor $A_{i_1 i_2 \ldots i_p}$ transforms into the sum of all $A_{i_1 i_2 \ldots i_q}$ with $q \leq p$, contracted with the necessary powers of C_i and $d_{i_1 \ldots i_n}, d^{i_1 \ldots i_n}$ to give rank p. As an example, take four qubits,

$$\begin{array}{lcll}A_0 & \mapsto & d_{ijkl}d^{ijkl} & [A_0],\\ A_i & \mapsto & d_{ijkl}d^{jklm} & [A_m + C_m A_0],\\ A_{ij} & \mapsto & d_{ijkl}d^{klmn} & [A_{mn} + C_m A_n + C_m C_n A_0],\\ A_{ijk} & \mapsto & d_{ijkl}d^{lmnp} & [A_{mnp} + C_m A_{np} + C_m C_n A_p + C_m C_n C_p A_0],\\ A_{ijkl} & \mapsto & d_{ijkl}d^{mnpq} & [A_{mnpq} + C_m A_{npq} + C_m C_n A_{pq} + C_m C_n C_p A_q + C_m C_n C_p C_q A_0],\end{array} \tag{8.43}$$

Through a judicious choice of dualisations we get the simplified form,

$$\begin{array}{lcl}A^{ijkl} & \mapsto & [A^{ijkl}].\\ A^{ijk} & \mapsto & [A^{ijk} + C_l A^{ijkl}],\\ A^{ij} & \mapsto & [A^{ij} + C_k A^{ijk} + C_k C_l A^{ijkl}],\\ A^{i} & \mapsto & [A^{i} + C_j A^{ij} + C_j C_k A^{ijk} + C_j C_k C_l A^{ijkl}],\\ A^{0} & \mapsto & [A^{0} + C_i A^{i} + C_i C_j A^{ij} + C_i C_j C_k A^{ijk} + C_i C_j C_k C_l A^{ijkl}],\end{array} \tag{8.44}$$

which makes the n-qubit generalisation quite clear. Similarly, under $\psi(D^i)$ a rank p tensor $A_{i_1 i_2 \ldots i_p}$ transforms into the sum of all $A_{i_1 i_2 \ldots i_q}$ with $q \geq p$, contracted with the necessary powers of $D^i = d^{ijkl}D_{jkl}$ and $d_{i_1 \ldots i_n}, d^{i_1 \ldots i_n}$ to give rank p. For four qubits, with the appropriate dualisations in place, one finds,

$$\begin{array}{lcl}A_0 & \mapsto & [A_0 + D^i A_i + D^i D^j A_{ij} + D^i D^j D^k A_{ijk} + D^i D^j D^k D^l A_{ijkl}],\\ A_i & \mapsto & [A_i + D^j A_{ij} + D^j D^k A_{ijk} + D^j D^k D^l A_{ijkl}],\\ A_{ij} & \mapsto & [A_{ij} + D^k A_{ijk} + D^k D^l A_{ijkl}],\\ A_{ijk} & \mapsto & [A_{ijk} + D^l A_{ijkl}],\\ A_{ijkl} & \mapsto & [A_{ijkl}].\end{array} \tag{8.45}$$

Finally, $\hat{\tau}$ generalises to four or more qubits in the obvious manner,

$$\begin{array}{lcl}A_0 & \mapsto & [d_{mnpq}M^m M^n M^p M^q A_0],\\ A_i & \mapsto & [N_i d_{inpq} M^n M^p M^q A_i],\\ A_{ij} & \mapsto & [N_i N_j d_{ijpq} M^p M^q A_{ij}],\\ A_{ijk} & \mapsto & [N_i N_j N_k d_{ijkq} M^q A_{ijk}],\\ A_{ijkl} & \mapsto & [N_i N_j N_k N_l d_{ijkl} A_{ijkl}].\end{array} \tag{8.46}$$

Adopting the notational convention $A_{[p]}$ ($A^{[p]}$) for a rank p tensor with downstairs (upstairs) indices, the n-qubit transformations ϕ, ψ may be summarised as follows,

$$\begin{array}{rcll} \phi(C_{[1]}): & A^{[p]} & \mapsto & \sum_{k=p}^{n} C_{[1]}^{(k-p)} A^{[k]}, \\ \psi(D^{[1]}): & A_{[p]} & \mapsto & \sum_{k=p}^{n} D^{[1](k-p)} A_{[k]}. \end{array} \tag{8.47}$$

One useful observation that follows from this analysis is that one can always assume $A_0 = 1, A^i = 0$ under SLOCC.

However, we have yet to develop a general scheme for writing invariants in the FTS basis. One example, though, defined for n qubits, is given by

$$\begin{aligned} (\Psi, \Phi) &= \frac{1}{0!}A_0 B^0 - \frac{1}{1!}A_i B^i + \frac{1}{2!}A_{ij}B^{ij} - \frac{1}{3!}A_{ijk}B^{ijk} + \frac{1}{4!}A_{ijkl}B^{ijkl} \cdots \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{k!} A_{[k]} B^{[k]}. \end{aligned} \tag{8.48}$$

This is symmetric (antisymmetric) for even (odd) n. It is simply the determinant in the 2-qubit case and the antisymmetric bilinear form of the FTS in the 3-qubit case.

2.2. Examples

2.2.1. Two qubits

The 2-qubit state corresponds to,

$$|\psi\rangle \leftrightarrow \Psi = \{A_0, A_i, A_{ij}\}, \qquad i, j = 1, \ldots, 2 \tag{8.49}$$

where

$$\{A_0 = a_{00}, \quad A_i = \begin{pmatrix} a_{10} \\ a_{01} \end{pmatrix}, \quad A_{ij} = \begin{pmatrix} 0 & a_{11} \\ a_{11} & 0 \end{pmatrix}\}, \tag{8.50}$$

or

$$\{A_0 = a_{00}, \quad A_i = \begin{pmatrix} a_{10} \\ a_{01} \end{pmatrix}, \quad A^0 = a_{11}\}. \tag{8.51}$$

The SLOCC generating transformations are given by,

$$\begin{aligned} \phi(C): \begin{pmatrix} A_0 \\ A_i \\ A^0 \end{pmatrix} &\mapsto \begin{pmatrix} A_0 & & & & \\ C_i A_0 & + & A_i & & \\ d^{ij}C_iC_jA_0 & + & d^{ij}C_iA_j & + & A^0 \end{pmatrix} \\ \psi(D): \begin{pmatrix} A_0 \\ A_i \\ A^0 \end{pmatrix} &\mapsto \begin{pmatrix} d_{ij}D^iD^jA^0 & + & D^iA_i & + & A_0 \\ d_{ij}D^iA^0 & + & A_i & & \\ A^0 & & & & \end{pmatrix}. \end{aligned} \tag{8.52}$$

The 2-tangle $\tau_{AB} = \det a$ is given by,

$$\det a = \frac{1}{2}d^{ij}[A_0 A_{ij} - A_i A_j] = A_0 A^0 - \frac{1}{2}A_i A^i. \tag{8.53}$$

Lemma 40. *Every state Ψ is SLOCC-equivalent to the reduced canonical form:*

$$\{1, 0, k\}, \longleftrightarrow |00\rangle + k|11\rangle \tag{8.54}$$

where $k = \det a$.

Proof. First show that, using (8.52), we may always assume A_i is non-zero. If $A_i \neq 0$ then our job is done. If $A_i = 0$ then we may assume that A_0 non-zero using τ if necessary. Now apply $\phi(C)$ to get a non-zero A_i. Apply $\psi(D)$ with $d_{ij}D^iD^j = 0$ so that $A_0 \mapsto A_0 + D^iA_i$. Choose D s.t. $A_0 + D^iA_i = 1$. Finally, apply $\phi(C)$ with $C_i = -A_i$. □

2.2.2. Three qubits

The 3-qubit state corresponds to,

$$|\psi\rangle \leftrightarrow \Psi = \{A_0, A_i, A_{ij}, A_{ijk}\}, \qquad i, j, k = 1, \ldots, 3 \tag{8.55}$$

where (after dualisation)

$$\{A_0 = a_{000}, \quad A_i = \begin{pmatrix} a_{100} \\ a_{010} \\ a_{001} \end{pmatrix}, \quad A^i = \begin{pmatrix} a_{011} \\ a_{101} \\ a_{110} \end{pmatrix}, \quad A^0 = a_{111}\}. \tag{8.56}$$

The SLOCC generating transformations are given in (8.70). The 3-tangle $\tau_{ABC} = 4|\operatorname{Det} a|$ is given by,

$$\begin{aligned} \operatorname{Det} a &= [(A_0A^0 - A_iA^i)]^2 + 4[A_0d_{ijk}A^iA^jA^k + A^0d^{ijk}A_iA_jA_k - d^{ijk}A_jA_kd_{ilm}A^lA^m] \\ &= A_0A^0A_0A^0 - 2A_0A^0A_iA^i + A_iA^iA_jA^j + A_0A^iA^jA_{ij} + A^0A_iA_jA^{ij} - A_iA^{jk}A_{jk}A^k. \end{aligned} \tag{8.57}$$

Lemma 41. *Every state Ψ is SLOCC-equivalent to the reduced canonical form:*

$$\{0, A_i, 0, 1\} \longleftrightarrow |111\rangle + a_{011}|011\rangle + a_{101}|101\rangle + a_{110}|110\rangle \tag{8.58}$$

where $\operatorname{Det} a = a_{011}a_{101}a_{110}$.

Proof. First show that, using (8.70), we may always assume A^i is non-zero. If $A^i \neq 0$ then our job is done. Assume $A^i = 0$. If $A_i \neq 0$ use $\mathcal{Z}$. If $A_i = 0$ then we may assume that A^0 non-zero, using $\mathcal{Z}$ if necessary. Then apply $\psi(D)$ to get a non-zero A^i, as required.

Next, apply $\phi(C)$ with $d^{ij}C_iC_j = 0$ so that $A^0 \mapsto A^0 + C_iA^i$. Choose C s.t. $A^0 + C_iA^i = 1$, which is clearly always possible. Then apply $\psi(D)$ with $D^i = -A^i$. We are then left with,

$$\{A'_0, A'_i, 0, 1\}. \tag{8.59}$$

Hence, we may assume from the outset that our state is in the form

$$\{A_0, A_i, 0, 1\}. \tag{8.60}$$

Now we show that we may further assume $A_0 = 0$. If $A_0 = 0$ our job is done. Assume $A_0 \neq 0$. There are then three subcases to consider: (1) $A_i = 0$ (2) $A_i \neq 0, d^{ijk}A_jA_k = 0$ (3) $d^{ijk}A_jA_k \neq 0$.

(1) Apply $\psi(D)$ with $d_{ijk}D^iD^jD^k = -A_0$ so that we are left with

$$\{0, A'_i, A'^i, 1\}. \tag{8.61}$$

Applying $\phi(C)$ with $d^{ijk}C_jA'_k = -A'^i$ we obtain,

$$\{0, A''_i, 0, A'^0\}. \tag{8.62}$$

Using $\hat{\tau}$ we obtain the required form,

$$\{0, A'''_i, 0, 1\}. \tag{8.63}$$

(2) Assume, with out loss of generality, $A_1 \neq 0, A_2 = A_3 = 0$. Apply ϕC with $C_1 = C_3 = 0$,

$$\phi(C) : \{A_0, A_i, 0, 1\} \mapsto \{A_0, A_i + A_0C_i, d^{ijk}C_jA_k, 1\}. \tag{8.64}$$

Follow with $\psi(D)$ with $D^i = -d^{ijk}C_jA_k$, which yields,

$$\{A_0 + \underbrace{(A_i + A_0C_i)d^{ijk}C_jA_k}_{=0}, A_i + A_0C_i - \underbrace{d_{ijk}d^{jlm}C_lA_md^{knp}C_nA_p}_{=0}, \underbrace{d^{ijk}C_jA_k - d^{ijk}C_jA_k}_{=0}, 1\}. \tag{8.65}$$

so that we are left with $\{A_0, A_i + A_0C_i, 0, 1\}$. Since $d^{ijk}C_iA_i \neq 0$, we find ourselves in case (3).

(3) Without loss of generality we may assume $A_2, A_3 \neq 0$. Apply $\phi(C)$ with $C_2 = C_3 = 0$ followed by $\psi(D)$ with $D^i = -d^{ijk}C_jA_k$,

$$\phi(C) : \{A_0, A_i, 0, 1\} \mapsto \{A_0 - 2A_2A_3C_1, A'_i, 0, 1\}, \tag{8.66}$$

where $A'_1 = (A_1 + A^0C_1 - A_2A_3(C_1)^2), A'_2 = A_2, A'_3 = A_3$. Let $C_1 = 1/(2A_2A_3)$ to obtain the required form,

$$\{0, A'_i, 0, 1\}. \tag{8.67}$$

□

2.2.3. Four qubits

The 4-qubit state corresponds to,

$$|\psi\rangle \leftrightarrow \Psi = \{A_0, A_i, A_{ij}, A_{ijk}, A_{ijkl}\}, \qquad i, j, k, l = 1, \ldots, 4 \tag{8.68}$$

where (after dualisation)

$$\{A_0 = a_{0000}, A_i = \begin{pmatrix} a_{1000} \\ a_{0100} \\ a_{0010} \\ a_{0001} \end{pmatrix}, A^{ij} = \begin{pmatrix} 0 & a_{0011} & a_{0101} & a_{0110} \\ a_{0011} & 0 & a_{1001} & a_{1010} \\ a_{0101} & a_{1001} & 0 & a_{1100} \\ a_{0110} & a_{1010} & a_{1100} & 0 \end{pmatrix}, A^i = \begin{pmatrix} a_{0111} \\ a_{1011} \\ a_{1101} \\ a_{1110} \end{pmatrix}, A^0 = a_{111}\}. \tag{8.69}$$

The SLOCC generating transformations are given by,

$$\phi(C):\begin{pmatrix}A_0\\A_i\\A^{ij}\\A^i\\A^0\end{pmatrix}\mapsto\begin{pmatrix}A_0\\C_iA_0+A_i\\d^{ijkl}C_kC_lA_0+d^{ijkl}C_kA_l+A^{ij}\\d^{ijkl}C_jC_kC_lA_0+d^{ijkl}C_jC_kA_l+d^{ijkl}C_jA_{ijkl}+A^i\\d^{ijkl}C_iC_jC_kC_lA_0+d^{ijkl}C_iC_jC_kA_l+d^{ijkl}C_iC_jA_{kl}+d^{ijkl}C_iA_{jkl}+A^0\end{pmatrix}$$

$$\psi(D):\begin{pmatrix}A_0\\A_i\\A_{ij}\\A^i\\A^0\end{pmatrix}\mapsto\begin{pmatrix}d_{ijkl}D^iD^jD^kD^lA^0+d_{ijkl}D^iD^jD^kA^l+d_{ijkl}D^iD^jA^{kl}+d_{ijkl}D^iA^{jkl}+A_0\\d_{ijkl}D^jD^kD^lA^0+d_{ijkl}D^jD^kA^l+d_{ijkl}D^jA^{kl}+A_i\\d_{ijkl}D^kD^lA^0+d_{ijkl}D^kA^l+A_{ij}\\D^iA^0+A^i\\A^0\end{pmatrix}. \tag{8.70}$$

There are four algebraically independent 4-qubit SLOCC*-invariants [233,264] of order two, six, eight and twelve. The order two invariant H is given by,

$$H=A_0A^0-A_iA^i+\frac{1}{2!}A_{ij}A^{ij}. \tag{8.71}$$

This is just the $n=4$ example of (8.48),

$$\begin{aligned}(\Psi,\Phi)&=\frac{1}{0!}A_0B^0-\frac{1}{1!}A_iB^i+\frac{1}{2!}A_{ij}B^{ij}-\frac{1}{3!}A_{ijk}B^{ijk}+\frac{1}{4!}A_{ijkl}B^{ijkl}\\&=\sum_{k=0}^{4}\frac{(-1)^k}{k!}A_{[k]}B^{[k]},\end{aligned} \tag{8.72}$$

where $\Phi=\{B_0,B_i,B_{ij},B_{ijk},B_{ijkl}\}$. There is also a *quintic* product $Q(\Psi)$ which can be used to obtain the order six invariant,

$$\Gamma=(\Psi,Q(\Psi)). \tag{8.73}$$

We have explicit forms for these invariants, derived from conventional $\mathrm{SL}(2,\mathbb{C})$ theory. However, their expression in the FTS basis is rather unwieldy. A more systematic treatment of the n-qubit FTS invariants is required.

2.3. Further work

Several features of our proposed n-qubit generalisation of the FTS need development. Principally, we are lacking a systematic generalisation of the FTS rank, a vital tool in the entanglement classification of three qubits. Similarly, it is not clear how to systematically generate SLOCC invariants without first appealing to the conventional $[\mathrm{SL}(2,\mathbb{C})]^n$ theory and then working backwards. Both these issues must be addressed before we can really assess the utility of the n-qubit FTS.

In an effort to establish how many ranks the 4-qubit case ought to have we have found the covariant classification of the degenerate classes, as in Table 8.4., which suggests five ranks for the degenerate classes at least. The total number of ranks is not clear without a full classification. The classes are distinguished by the following SLOCC covariants/invariants[2]:

[2]There is some redundancy in this set as they are not all algebraically independent. However, the entanglement classifi-

Order 2 One invariant,

$$H = a^{ABCD} a_{ABCD}, \tag{8.74}$$

where the indices have been raised using the SL(2)-invariant tensor, $V^A = \epsilon^{AA'} V_{A'}$.

Six covariants,

$$\begin{aligned}
(\gamma^{AB})_{A_1B_1A_2B_2} &= a_{(A_1B_1}{}^{CD} a_{A_2B_2)CD} \\
(\gamma^{CD})_{C_1D_1C_2D_2} &= a^{AB}{}_{(C_1D_1} a_{ABC_2D_2)} \\
(\gamma^{BC})_{B_1C_1B_2C_2} &= a^{A}{}_{(B_1C_1}{}^{D} a_{AB_2C_2D)} \\
(\gamma^{AD})_{A_1D_1A_2D_2} &= a_{(A_1}{}^{BC}{}_{D_1} a_{A_2BCD_2)} \\
(\gamma^{AC})_{A_1C_1A_2C_2} &= a_{(A_1}{}^{B}{}_{C_1}{}^{D} a_{A_2BC_2D)} \\
(\gamma^{BD})_{B_1D_1B_2D_2} &= a^{A}{}_{(B_1}{}^{C}{}_{D_1} a_{AB_2CD_2)}
\end{aligned} \tag{8.75}$$

which transform as a $(\mathbf{3},\mathbf{3},\mathbf{1},\mathbf{1}), (\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{3}), (\mathbf{1},\mathbf{3},\mathbf{3},\mathbf{1}), (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{3}), (\mathbf{3},\mathbf{1},\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{3})$ respectively.

Order 3 Four covariants

$$\begin{aligned}
(t^A)_{A_1A_2A_3BCD} &= a_{(A_1|}{}^{B'}{}_{CD} (\gamma^{AB})_{|A_2B'A_3)B}, \\
(t^B)_{AB_1B_2B_3CD} &= a_{A(B_1}{}^{C'}{}_{|D|} (\gamma^{BC})_{B_2C'B_3)C}, \\
(t^C)_{ABC_1C_2C_3D} &= a^{A'}{}_{B(C_1|D|} (\gamma^{AC})_{A'C_2|A|C_3)}, \\
(t^D)_{ABCD_1D_2D_3} &= a_{A}{}^{B'}{}_{C(D_1} (\gamma^{BD})_{B'D_2|B|D_3)},
\end{aligned} \tag{8.76}$$

transforming as a $(\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{2}), (\mathbf{2},\mathbf{4},\mathbf{2},\mathbf{2}), (\mathbf{2},\mathbf{2},\mathbf{4},\mathbf{2})$ and $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{4})$ respectively. Three covariants

$$\begin{aligned}
T^1_{ABCD} &= a_{AB}{}^{C'D'} (\gamma^{CD})_{C'D'CD} = a^{A'B'}{}_{CD} (\gamma^{AB})_{A'B'AB}, \\
T^2_{ABCD} &= a_{A}{}^{B'}{}_{C}{}^{D'} (\gamma^{BD})_{B'D'BD} = a^{A'}{}_{B}{}^{C'}{}_{D} (\gamma^{AC})_{A'C'AC}, \\
T^3_{ABCD} &= a^{A'}{}_{BC}{}^{D'} (\gamma^{AD})_{A'D'AD} = a_{A}{}^{B'C'}{}_{D} (\gamma^{BC})_{B'C'BC}.
\end{aligned} \tag{8.77}$$

all transforming as a $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$.

Order 4 Four covariants

$$\begin{aligned}
(P^A)_{A_1A_2A_3A_4} &= a_{(A_1}{}^{BCD} (t^A)_{A_2A_3A_4)BCD}, \\
(P^B)_{B_1B_2B_3B_4} &= a^{A}{}_{(B_1}{}^{CD} (t^B)_{AB_2B_3B_4)CD}, \\
(P^C)_{C_1C_2C_3C_4} &= a^{AB}{}_{(C_1}{}^{D} (t^C)_{ABC_2C_3C_4D)}, \\
(P^D)_{D_1D_2D_3D_4} &= a^{ABC}{}_{(D_1} (t^D)_{ABCD_2D_3D_4)},
\end{aligned} \tag{8.78}$$

transforming as a $(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{1}), (\mathbf{1},\mathbf{5},\mathbf{1},\mathbf{1}), (\mathbf{1},\mathbf{1},\mathbf{5},\mathbf{1})$, and $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{5})$ respectively. Three quartic

cation of the degenerate classes is not affected.

invariants,

$$
\begin{aligned}
H_1 &= (\gamma^{AB})^{ABAB}(\gamma^{AB})_{ABAB} = (\gamma^{CD})^{CDCD}(\gamma^{CD})_{CDCD}, \\
H_2 &= (\gamma^{AC})^{ACAC}(\gamma^{AC})_{ACAC} = (\gamma^{BD})^{BDBD}(\gamma^{BD})_{BDBD}, \\
H_3 &= (\gamma^{AD})^{ADAD}(\gamma^{AD})_{ADAD} = (\gamma^{BC})^{BCBC}(\gamma^{BC})_{BCBC}.
\end{aligned} \tag{8.79}
$$

Briand *et al* [264] define a quartic invariant by writing a_{ABCD} as a 4×4 matrix $L_{\alpha\beta}$ where $\alpha = AB$ and $\beta = CD$ and taking its determinant

$$L = \det L_{\alpha\beta}. \tag{8.80}$$

To connect this to our quartic invariants use the identity,

$$\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} = \epsilon^{A_1A_3}\epsilon^{A_2A_4}\epsilon^{B_1B_2}\epsilon^{B_3B_4} - \epsilon^{A_1A_2}\epsilon^{A_3A_4}\epsilon^{B_1B_3}\epsilon^{B_2B_4}, \tag{8.81}$$

which yields,

$$L = H_2 - H_3. \tag{8.82}$$

Two further invariants, M and N, may be defined similarly by splitting the $ABCD$ indices in the two remaining distinct ways. These invariants trivially satisfy

$$L + M + N = 0. \tag{8.83}$$

Table 8.4.: 4-qubit entanglement classification of degenerate classes

Class		Vanishing	Non-vanishing
Separable	A-B-C-D	γ	a
Triseparable	A-B-CD	$\gamma^{AB}, \gamma^{BC}, \gamma^{AC}, \gamma^{BD}, \gamma^{AD}$	γ^{CD}
	AB-C-D	$\gamma^{BC}, \gamma^{AC}, \gamma^{BD}, \gamma^{AD}, \gamma^{CD}$	γ^{AB}
	A-BC-D	$\gamma^{AC}, \gamma^{BD}, \gamma^{AD}, \gamma^{CD}, \gamma^{AB}$	γ^{BC}
	AC-B-D	$\gamma^{AB}, \gamma^{BD}, \gamma^{AD}, \gamma^{CD}, \gamma^{BC}$	γ^{AC}
	AD-B-C	$\gamma^{AB}, \gamma^{BD}, \gamma^{AC}, \gamma^{CD}, \gamma^{BC}$	γ^{AD}
	A-BD-C	$\gamma^{AB}, \gamma^{BC}, \gamma^{CD}, \gamma^{AD}, \gamma^{AC}$	γ^{BD}
Biseparable	A-GHZ	H_i	t^A, P^A
	B-GHZ	H_i	t^B, P^B
	C-GHZ	H_i	t^C, P^C
	D-GHZ	H_i	t^D, P^D
	A-W	H_i, P^A	t^A
	B-W	H_i, P^B	t^B
	C-W	H_i, P^C	t^C
	D-W	H_i, P^D	t^D
	AB-CD	L, T^2, T^3	H, T^1, M, N
	AD-BC	M, T^2, T^1	H, T^3, L, N
	AC-BD	N, T^1, T^3	H, T^2, M, L

CHAPTER 9

Integral structures

Much of our discussion has centred around U-duality and in particular the representations carried by the black p-brane charges. However, this has been a purely classical discussion; the charges have been treated as real valued continuous parameters. It is time to face reality[1]. The charges actually take their values on a lattice due to the Dirac-Schwinger-Zwansiger charge quantization conditions. Consequently, the continuous U-dualities are broken to discrete subgroups [47].

The U-duality orbits are furnished with an increased level of structural complexity, which, in some cases, is of particular mathematical significance [48,50]. However, the question of discrete U-duality orbits is not only interesting in its own right, it is also of physical importance with implications for a number of topics including the stringy origins of microscopic black hole entropy [49,265–272]. Moreover, following a conjecture of finiteness for $D = 4, \mathcal{N} = 8$ supergravity [273], it has recently been observed that some of the orbits of $E_{7(7)}(\mathbb{Z})$ should play an important role in counting microstates of this theory [272], even if it may differ from its superstring or M-theory completion [274].

In section 2 we address this issue in the context of $\mathcal{N} = 8$ supergravity in four, five and six dimensions. To this end we exploit the mathematical framework of *integral Jordan algebras* and the *integral Freudenthal triple system*, which have at their basis the ring of *integral split-octonions* [48,67,275,276], all of which are introduced in section 1. To a large extent this work is a continuation of the analysis used in studying the recently introduced black hole *Freudenthal duality*, which is the subject of section 3, the concluding part of this chapter.

1. Integral algebras and their symmetries

1.1. The integral split-octonions

The split-octonions are an 8-dimensional (non-division) composition algebra. They are both non-commutative and non-associative but are alternative. They may be generated from the split-quaternions via the Cayley-Dickson process. The split-quaternions are a 4-dimensional (non-division) composi-

[1]Perhaps this is a bad choice word given the poor phenomenological properties of $\mathcal{N} = 8$ supergravity.

tion algebra. The three imaginary units i, j, k obey the following multiplication rules:

$$\begin{aligned} &i^2 = 1, && j^2 = 1, && k^2 = -1, \\ &ij = -ji = k, && ik = -ki = j, && jk = -kj = i. \end{aligned} \tag{9.1}$$

There is a convenient matrix representation of this algebra given by,

$$\begin{aligned} 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & i &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ j &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & k &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{9.2}$$

such that an arbitrary quaternion $a \in \mathbb{H}$ may be written,

$$a = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \qquad a_{ij} \in \mathbb{R}. \tag{9.3}$$

The norm, real part and conjugation are given by,

$$\mathbf{n}(a) = \det(a), \quad 2\Re(a) = \operatorname{tr}(a), \quad \bar{a} = -ja^{\mathrm{T}}j, \tag{9.4}$$

where $^{\mathrm{T}}$ denotes the matrix transpose.

The split-octonions $\mathbb{O}_s$ may then be defined by introducing a forth imaginary unit ν,

$$a + b\nu, \qquad a, b \in \mathfrak{H}_s. \tag{9.5}$$

The octonionic multiplication rules are defined as per the Cayley-Dickson process,

$$(a + b\nu)(c + d\nu) := (ac - \bar{d}b) + (da + b\bar{c})\nu. \tag{9.6}$$

The norm, real part and conjugation are given by,

$$\mathbf{n}(a + b\nu) = \det(a) + \det(b), \quad 2\Re(a + b\nu) = \operatorname{tr}(a), \quad \overline{a + b\nu} = \bar{a} - b\nu. \tag{9.7}$$

The ring of integral split-quaternions $\mathfrak{H}_s$ is defined as the ring of 2×2 matrices with entries in $\mathbb{Z}$. The norm and trace have integral values and the ring is closed under conjugation.

The ring of integral split-octonions $\mathfrak{O}_s$ is then defined in the obvious manner and we may write an arbitrary integral split-octonions as,

$$a + b\nu, \qquad a, b \in \mathfrak{H}_s. \tag{9.8}$$

The norm, the trace and conjugation (9.7) are well defined functions taking their values in $\mathbb{Z}$ and, moreover, $\mathfrak{O}_s$ is a maximal order [277, 278].

1.2. Integral Jordan algebras

Until now we have been working with Jordan algebras as originally axiomatised in 1934 [239]. These are nowadays referred to as *linear* Jordan algebras so as to distinguish them from the modern formulation, first axiomatised in [55]. The modern axioms were motivated by the breakdown of the linear theory for scalar fields of characteristic two, or the absence of 1/2. The Jordan product $x \circ y = \frac{1}{2}(xy+yx)$ is replaced by a new basic operation $U_x y = xyx$. Physically, Hermitian matrices are also closed under xyx. These are referred to as *quadratic* Jordan algebras reflecting the fact that the operation U_x is quadratic in x, but we will refer to them as U-Jordan algebras so as to avoid confusion with the quadratic Jordan algebras defined by a quadratic norm.

Definition 42 (U-Jordan algebras)**.** *A module (i.e. a vector space defined over a ring) equipped with an operation U_x,*

$$\begin{array}{lcl} U_1 & = & \mathbb{1}, \\ U_x V_{y,x} & = & V_{x,y} U_x, \\ U_{U_x y} & = & U_x U_y U_x, \end{array} \tag{9.9}$$

*where $V_{x,y}(z) = (U_{x+z} - U_x - U_z)y$, is a U-*Jordan algebra.

The presence of 1/2, the linear and quadratic formulations are categorically equivalent. The U-Jordan generalisation allows the definition of integral Jordan algebras, which are the subject of the following sections.

1.2.1. 2×2 Hermitian matrices

Definition 43 (Integral Jordan algebra $\mathfrak{J}_2^{\mathfrak{A}}$)**.** 2×2 *Hermitian matrices defined over a ring of integral composition algebras $\mathfrak{A}$. An arbitrary element may be written as,*

$$A = \begin{pmatrix} \alpha & a \\ \overline{a} & \beta \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{Z} \text{ and } a \in \mathfrak{A}. \tag{9.10}$$

$\mathfrak{J}_2^{\mathfrak{A}}$ is not a linear Jordan algebra as it is not closed under the Jordan product. It is, however, a well defined U-Jordan algebra [68]. Crucially, the quadratic norm and trace form take values in $\mathbb{Z}$.

Definition 44 (The discrete reduced structure group $\mathrm{Str}_0(\mathfrak{J}_2^{\mathfrak{A}})$)**.** *Invertible $\mathbb{Z}$-linear transformations leaving the quadratic norm invariant,*

$$\mathrm{Str}_0(\mathfrak{J}_2^{\mathfrak{A}}) := \{\tau \in \mathrm{Iso}_{\mathbb{Z}}(\mathfrak{J}_2^{\mathfrak{A}}) | N_2(\tau A) = N_2(A)\}. \tag{9.11}$$

An important subset of $\mathrm{Str}_0(\mathfrak{J}_2^{\mathfrak{A}})$ transformation or generated by $\sigma_{st}(b)$ with $s \neq t$,

$$\sigma_{st}(b) : A \mapsto (\mathbb{1} + bE_{st})A(\mathbb{1} + \overline{b}E_{ts}), \tag{9.12}$$

where $b \in \mathfrak{A}$ and E_{st} is a 2×2 matrix with a single non-zero unit entry in the st position. Explicitly,

$$\begin{aligned} \sigma_{12}(b) : A &\mapsto \begin{pmatrix} \alpha + \mathrm{tr}(b\overline{a}) + \beta\mathbf{n}(b) & a + \beta b \\ \overline{a} + \beta\overline{b} & \beta \end{pmatrix}, \\ \sigma_{21}(b) : A &\mapsto \begin{pmatrix} \alpha & a + \alpha\overline{b} \\ \overline{a} + \alpha b & \beta + \mathrm{tr}(ba) + \alpha\mathbf{n}(b) \end{pmatrix}. \end{aligned} \tag{9.13}$$

Definition 45 (Arithmetic $\mathrm{Str}_0(\mathfrak{J}_2^{\mathfrak{A}})$-invariants)**.** *For an element $A \in \mathfrak{J}_2^{\mathfrak{A}}$, an integer α* divides *A, denoted $\alpha|A$, if $A = \alpha A'$, where $A' \in \mathfrak{J}_2^{\mathfrak{A}}$. Arithmetic invariants are defined by the* greatest common divisor *(gcd):*

$$\begin{aligned} b_1(A) &:= \gcd(A), \\ b_2(A) &:= N_2(A). \end{aligned} \tag{9.14}$$

For the integral Jordan algebra of 2×2 Hermitian matrices defined over the ring of integral split-octonions $\mathrm{Str}_0(\mathfrak{J}_2^{\mathfrak{O}_s})$ is a model, in sense of [275], for $\mathrm{SO}(5,5;\mathbb{Z})$.

1.2.2. 3×3 Hermitian matrices

Definition 46 (Integral Jordan algebra $\mathfrak{J}_3^{\mathfrak{A}}$)**.** *$3 \times 3$ Hermitian matrices defined over a ring of integral composition algebras $\mathfrak{A}$. An arbitrary element may be written as,*

$$A = \begin{pmatrix} \alpha & a & \bar{b} \\ \bar{a} & \beta & c \\ b & \bar{c} & \gamma \end{pmatrix}, \quad \text{where } \alpha, \beta, \gamma \in \mathbb{Z} \text{ and } a, b, c \in \mathfrak{A}. \tag{9.15}$$

$\mathfrak{J}_3^{\mathfrak{A}}$ is not a linear Jordan algebra as it is not closed under the Jordan product. It is, however, a well defined U-Jordan algebra [68]. Crucially, the cubic norm and trace form take values in $\mathbb{Z}$ and $\mathfrak{J}_3^{\mathfrak{A}}$ is closed under the quadratic adjoint map.

Definition 47 (The discrete reduced structure group $\mathrm{Str}_0(\mathfrak{J}_3^{\mathfrak{A}})$)**.** *Invertible $\mathbb{Z}$-linear transformations leaving the cubic norm invariant,*

$$\mathrm{Str}_0(\mathfrak{J}_3^{\mathfrak{A}}) := \{\tau \in \mathrm{Iso}_{\mathbb{Z}}(\mathfrak{J}_3^{\mathfrak{A}}) | N_3(\tau A) = N_3(A)\}. \tag{9.16}$$

An important subset of $\mathrm{Str}_0(\mathfrak{J}_3^{\mathfrak{A}})$ transformations are generated by $\sigma_{st}(b)$,

$$\sigma_{st}(b) : A \mapsto (\mathbb{1} + bE_{st})A(\mathbb{1} + \bar{b}E_{ts}), \tag{9.17}$$

where $b \in \mathfrak{A}$ and E_{st} is a 3×3 matrix with a single non-zero unit entry in the st position.

Definition 48 (Arithmetic $\mathrm{Str}_0(\mathfrak{J}_3^{\mathfrak{A}})$-invariants)**.**

$$\begin{aligned} c_1(A) &:= \gcd(A), \\ c_2(A) &:= \gcd(A^\sharp), \\ c_3(A) &:= N_3(A). \end{aligned} \tag{9.18}$$

For the integral Jordan algebra of 3×3 Hermitian matrices defined over the ring of integral split-octonions $\mathrm{Str}_0(\mathfrak{J}_3^{\mathfrak{O}_s})$ is a model, in sense of [275], for $E_{6(6)}(\mathbb{Z})$.

1.3. Integral Freudenthal triple system

Definition 49 (Integral Freudenthal triple system $\mathfrak{F}^{\mathfrak{A}}$)**.**

$$\mathfrak{F}^{\mathfrak{A}} = \mathfrak{F}(\mathfrak{J}_3^{\mathfrak{A}}) := \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{J}_3^{\mathfrak{A}} \oplus \mathfrak{J}_3^{\mathfrak{A}}, \tag{9.19}$$

Table 9.1.: The discrete automorphism group $\text{Aut}(\mathfrak{F}(\mathfrak{J}))$ and the dimension of its representation $\dim \mathfrak{F}(\mathfrak{J})$ given by the Freudenthal construction defined over the integral cubic Jordan algebra $\mathfrak{J}$ with dimension $\dim \mathfrak{J}$ and discrete reduced structure group $\text{Str}_0(\mathfrak{J})$.

Jordan algebra $\mathfrak{J}$	$\text{Str}_0(\mathfrak{J})$	$\dim \mathfrak{J}$	$\text{Aut}(\mathfrak{F}(\mathfrak{J}))$	$\dim \mathfrak{F}(\mathfrak{J})$
$\mathbb{Z}$	$-$	1	$\text{SL}(2,\mathbb{Z})$	4
$\mathbb{Z}\oplus\mathbb{Z}$	$\text{SO}(1,1;\mathbb{Z})$	2	$\text{SL}(2,\mathbb{Z})\times\text{SL}(2,\mathbb{Z})$	6
$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$	$\text{SO}(1,1;\mathbb{Z})\times\text{SO}(1,1;\mathbb{Z})$	3	$\text{SL}(2,\mathbb{Z})\times\text{SL}(2,\mathbb{Z})\times\text{SL}(2,\mathbb{Z})$	8
$\mathbb{Z}\oplus\Gamma_{1,n-1}$	$\text{SO}(1,1)\times\text{SO}(1,n-1)$	$n+1$	$\text{SL}(2,\mathbb{Z})\times\text{SO}(2,n;\mathbb{Z})$	$2(n+2)$
$\mathbb{Z}\oplus\Gamma_{5,n-1}$	$\text{SO}(1,1)\times\text{SO}(5,n-1)$	$n+5$	$\text{SL}(2,\mathbb{Z})\times\text{SO}(6,n;\mathbb{Z})$	$2(n+6)$
$\mathfrak{J}_3^{\mathfrak{R}}$	$\text{SL}(3,\mathbb{Z})$	6	$\text{Sp}(6,\mathbb{Z})$	14
$\mathfrak{J}_3^{\mathfrak{C}}$	$\text{SL}(3,\mathfrak{C})$	9	$\text{SU}(3,3;\mathbb{Z})$	20
$\mathfrak{J}_3^{\mathfrak{H}}$	$\text{SU}^\star(6)(\mathbb{Z})$	15	$\text{SO}^\star(12,\mathbb{Z})$	32
$\mathfrak{J}_3^{\mathfrak{O}}$	$E_{6(-26)}(\mathbb{Z})$	27	$E_{7(-25)}(\mathbb{Z})$	56
$\mathfrak{J}_3^{\mathfrak{O}_s}$	$E_{6(6)}(\mathbb{Z})$	27	$E_{7(7)}(\mathbb{Z})$	56

where the antisymmetric bilinear form (5.64a), *quartic norm* (B.14) *and triple product* (5.64c) *are defined as before. An arbitrary element may be written,*

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \text{ where } \alpha, \beta \in \mathbb{Z} \text{ and } A, B \in \mathfrak{J}_3^{\mathfrak{A}}. \tag{9.20}$$

The quartic norm and antisymmetric bilinear form are both integer valued and, hence, the triple product is well defined. The quartic norm is either $4n$ or $4n+1$ for some $n \in \mathbb{Z}$.

Definition 50 (The discrete automorphism group $\text{Aut}(\mathfrak{F}^{\mathfrak{A}})$)**.** *Invertible $\mathbb{Z}$-linear transformations leaving the quartic norm and antisymmetric bilinear form invariant,*

$$\text{Aut}(\mathfrak{F}^{\mathfrak{A}}) := \{\sigma \in \text{Iso}_{\mathbb{Z}}(\mathfrak{F}^{\mathfrak{A}}) | \{\sigma x, \sigma y\} = \{x, y\},\ \Delta(\sigma x, \sigma y, \sigma z, \sigma w) = \Delta(x, y, z, w)\}. \tag{9.21}$$

Lemma 51 (Krutelevic, 2004)**.** *The following elementary transformations generate the group* $\text{Aut}(\mathfrak{F}^{\mathfrak{A}})$*:*

$$\begin{aligned}
\varphi(C) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha + (B,C) + (A,C^\sharp) + \beta N(C) & A + \beta C \\ B + A \times C + \beta C^\sharp & \beta \end{pmatrix}; \\
\psi(D) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha & A + B \times D + \alpha D^\sharp \\ B + \alpha D & \beta + (A,D) + (B,D^\sharp) + \alpha N(D) \end{pmatrix}; \\
\widehat{\tau} : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} &\mapsto \begin{pmatrix} \alpha & \tau A \\ {}^t\tau^{-1} B & \beta \end{pmatrix};
\end{aligned} \tag{9.22}$$

where $C, D \in \mathfrak{J}_3^{\mathfrak{A}}$ and $\tau \in \text{Str}_0(\mathfrak{J}_3^{\mathfrak{A}})$.

For the integral Jordan algebra of 3×3 Hermitian matrices defined over the ring of integral split-octonions $\text{Aut}(\mathfrak{F}^{\mathfrak{O}_s})$ is a model for $E_{7(7)}(\mathbb{Z})$ [48], the U-duality group of type II string theory compactified on a 6-torus. The elements of $\mathfrak{F}^{\mathfrak{O}_s}$ form a 56-dimensional representation of $E_{7(7)}(\mathbb{Z})$ corresponding to $D = 4$ the black hole charges. A summary of integral Jordan algebras, FTSs, and their associated symmetry groups is given in (9.1).

Definition 52 (Arithmetic $\mathrm{Aut}(\mathfrak{F}(\mathfrak{J}_3^{\mathfrak{A}}))$-invariants)**.**

$$\begin{aligned}
&d_1(x) := \gcd(x),\\
&d_2(x) := \gcd(3T(x,x,y) + \{x,y\}x),\ \forall y\\
&d_2'(x) := \gcd(\mathcal{P}(x),\ \mathcal{Q}(x),\ \mathcal{R}(x))\\
&d_3(x) := \gcd(T(x,x,x)),\\
&d_4(x) := \Delta(x),\\
&d_4'(x) := \gcd(x \wedge T(x)).
\end{aligned} \tag{9.23}$$

where $\wedge$ denotes the antisymmetric tensor product. $\mathcal{P}(x) = Y^\sharp - \alpha X$ and $\mathcal{Q}(x) = X^\sharp - \beta Y$ are the charge combinations appearing in the 4D/5D lift [20, 48, 51] and $\mathcal{R}(x) : \mathfrak{J}_3 \to \mathfrak{J}_3$ is a Jordan algebra endomorphism given by [48]

$$\mathcal{R}(x)(Z) = 2\kappa(x)Z + 2\{X,Y,Z\}, \tag{9.24}$$

where $\{X,Y,Z\}$ is the Jordan triple product (5.26)*.*

Taken together, $(\mathcal{P}(x),\ \mathcal{Q}(x),\ \mathcal{R}(x))$ form the adjoint representation of the 4-dimensional U-duality: $\mathbf{133}$ in the case of $E_{7(7)}(\mathbb{Z})$. Under the 5-dimensional U-duality, they transform as the fundamental, contragredient fundamental and adjoint representations, respectively: $\mathbf{27}$, $\mathbf{27}'$ and $\mathbf{1} + \mathbf{78}$ in the case of $E_{6(6)}(\mathbb{Z})$. After subtracting the symplectic trace, $x \wedge T(x)$ transforms as the $\mathbf{1539}$ in $\mathbf{56} \times_a \mathbf{56}$.

2. Integral U-duality orbits

We have been until now using various algebras and their relation to certain relevant Lie groups, the U-dualites of supergravity or the SLOCC equivalences of QI, to study the properties of black holes and entanglement. However, as mentioned in passing several times, the black hole charges are in fact quantized and correspondingly the U-dualities are broken to discrete (typically not finite) subgroups. One may understand this from the M-theoretic perspective where the continuous U-duality of the low energy effective supergravity theories actually correspond to the discrete symmetries of the full string/M-theory. Alternatively, without invoking M-theory the U-dualities of supergravity are simply broken by the Dirac-Schwinger-Zwanziger quantization of the black hole charges. Either way, the mathematical structure of the U-duality orbits and black hole entropy becomes far richer and more challenging. Focusing on the maximally supersymmetric theories in six, five and four dimensions our proposal, following on from the work in [48, 67, 276], is this. Each of these theories in the classical limit has a natural and physically well motivated description in terms of Jordan algebras and the FTS, which fit inside one another when dimensionally reducing from six through to four dimensions. Consequently, there must exist some embedding of the lattice of quantized charges into the Jordan algebras and FTS. Moreover, since the D-dimensional U-duality does not act on spacetime it survives dimensional reduction. Hence, it must be a subgroup of the $(D-1)$-dimensional U-duality and the D-dimensional charge lattice must be a sublattice of the $(D-1)$-dimensional charge lattice. The most natural way, ceratinly the only apparent way, of consistently achieving this is by invoking the the integral split-octonions and using the integral Jordan and FTS structures they induce.

The analysis of the discrete U-duality orbits in [48, 67] relies on two key ingredients. First, to use the discrete symmetries of the integral Jordan algebras and FTS to bring the charge vectors into a *diagonally reduced canonical form*. Second, to construct from the algebraic operations of the Jordan algebras and FTS new arithmetic invariants that are absent in the continuous theory. These are essentially the $\gcd$ of irreducible representations built out of powers of the charge vectors. Note, as emphasized in [257], the $\gcd$ of a U-duality representation, built out of the relevant basic charge vector representation, is only well defined if that representation is non-vanishing. In practice this means first computing which class of orbits as defined by the continuous analysis a given state lies in. This, in turn, determines the subset of the arithmetic invariants that are well defined for this particular state. It is this subset that is then to be used in specifying the particular discrete orbit to which the state belongs, the remaining arithmetic invariants being ill-defined and contentless. Ideally these invariants then uniquely determine the canonical form of a given state. We find that:

- The charge vector of the dyonic black string in $D = 6$ is $SO(5,5;\mathbb{Z})$ related to a two-charge reduced canonical form uniquely specified by a set of two arithmetic U-duality invariants.
- Similarly, the black hole (string) charge vectors in $D = 5$ are $E_{6(6)}(\mathbb{Z})$ equivalent to a three-charge canonical form, again uniquely fixed by a set of three arithmetic U-duality invariants.
- The charge vector of the dyonic black hole in $D = 4$ is $E_{7(7)}(\mathbb{Z})$ related to a five-charge reduced canonical. This canonical form implies that the R-R charges may always be transformed away as required for the validity of the manifestly $E_{7(7)}(\mathbb{Z})$ invariant dyon degeneracy formula of type II string theory on T^6derived in [49].
- However, the canonical form is not uniquely determined by the known arithmetic invariants. While black holes preserving more than 1/8 of the supersymmetries may be fully classified by known arithmetic $E_{7(7)}(\mathbb{Z})$ invariants, 1/8-BPS and non-BPS black holes yield increasingly subtle orbit structures, which remain to be properly understood. However, for the very special subclass of *projective* black holes a complete classification is known. All projective black holes are $E_{7(7)}(\mathbb{Z})$ related to a four or five charge canonical form determined uniquely by the set of known arithmetic U-duality invariants. Moreover, $E_{7(7)}(\mathbb{Z})$ acts transitively on the charge vectors of projective black holes with a given leading-order entropy.

2.1. $D = 6, \mathcal{N} = 8$ black string charge orbits

For quantized charges the continuous U-duality is broken to an infinite discrete subgroup, which for $D = 6$ is given by $\mathrm{SO}(5,5;\mathbb{Z}) \subset \mathrm{SO}(5,5;\mathbb{R})$ [47]. Let $\mathrm{SO}(5,5;\mathbb{Z})$ be defined by $\mathrm{Str}_0(\mathfrak{J}_2^{\mathfrak{O}_s})$. The quantized black string charge vector may be represented by a Jordan algebra element,

$$\mathcal{Q} = \begin{pmatrix} p^0 & Q_v \\ \overline{Q}_v & q_0 \end{pmatrix}, \quad \text{where} \quad q^0, p_0 \in \mathbb{Z} \quad \text{and} \quad Q_v \in \mathfrak{O}_s. \tag{9.25}$$

The first important observation is that the charge conditions defining the orbits in the continuous theory are manifestly invariant under the discrete subgroup $\mathrm{SO}(5,5;\mathbb{Z})$ and, hence, those states unrelated by U-duality in the classical theory remain unrelated in the quantum theory. There are two disjoint *classes* of orbits, one 1/2-BPS and one 1/4-BPS, corresponding to the two orbits of the continuous case. However, each of these classes is broken up into a countably infinite set of discrete orbits. To classify these orbits we use $\mathrm{SO}(5,5;\mathbb{Z})$ to bring an arbitrary charge vector into a *diagonal reduced* canonical form.

Proposition 53 (Black string canonical form). *Every black string charge vector $\mathcal{Q} \in \mathfrak{J}_2^{\mathfrak{O}_s}$ is U-duality equivalent to a diagonally reduced canonical form,*

$$\mathcal{Q}_{can} = k \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}, \quad \textit{where} \quad k > 0, |l| \geq 0. \tag{9.26}$$

Proof. Consider an arbitrary element

$$A = \begin{pmatrix} \alpha & a \\ \overline{a} & \beta \end{pmatrix}. \tag{9.27}$$

We start by diagonalising A. First show that we may assume $\alpha \neq 0$. If $\alpha \neq 0$ our job is done so assume $\alpha = 0$. If $a = 0$ then $\beta \neq 0$ and A is already diagonal and the reduced canonical form follows from,

$$\sigma_{12}(1)\sigma_{21}(-1)\sigma_{12}(1) : \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix} \mapsto \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}. \tag{9.28}$$

If $a \neq 0$ then there exists a $b \in \mathfrak{O}_s$ such that $\mathrm{tr}(b\overline{a}) = a_i \in \mathbb{Z}$, where a_i is some component of a, and $\mathbf{n}(b) = 0$, so that, after an application of $\sigma_{12}(b)$ we may always assume $\alpha \neq 0$ from the outset.

Assuming $\alpha, a \neq 0$ we proceed as follows. We can reduce the components of a modulo $|\alpha|$ by applying $\sigma_{21}(b)$ with appropriately chosen b. Following the reduction, either $a = 0$ and we are done, or a has at least one non-zero component which must lie in the interval $[0, \dots |\alpha| - 1]$. In the latter case, there exists a zero-norm b such that $\sigma_{12}(b) : \alpha \mapsto \alpha' = \alpha + \mathrm{tr}(b\overline{a})$ and $\alpha' \in [0, \dots, |\alpha| - 1]$. We repeat until $a = 0$.

Hence, we may assume A is diagonal,

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \tag{9.29}$$

and take the final steps towards the reduced canonical form. Let $\delta = \gcd(\alpha, \beta)$. Then there are integers m, n such that $\delta = \alpha m + \beta n$. Use $\sigma_{12}(1)\sigma_{21}((m-1)\overline{b})\sigma_{12}((n-1)b)$ where $\mathrm{tr}(b) = 1$ and $|b| = 0$. We find,

$$\sigma_{12}(1)\sigma_{21}((m-1)\overline{b})\sigma_{12}((n-1)b) : \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mapsto \begin{pmatrix} \delta & \delta c \\ \delta\overline{c} & \beta \end{pmatrix}, \tag{9.30}$$

where $c = \frac{\beta + (\delta - \alpha - \beta)b}{\delta}$. Now apply $\sigma_{12}(-\overline{c})$ to obtain the desired result,

$$\sigma_{12}(-\overline{c}) : \begin{pmatrix} \delta & \delta c \\ \delta\overline{c} & \beta \end{pmatrix} \mapsto \begin{pmatrix} \delta & 0 \\ 0 & \beta - \delta\mathbf{n}(c) \end{pmatrix}. \tag{9.31}$$

□

The split $\mathrm{SO}(5,5;\mathbb{Z})$ case treated here relies on the existence of a zero norm octonion. The non-split case $\mathfrak{J}_2^{\mathfrak{O}}$ corresponds to $\mathrm{SO}(1,9;\mathbb{Z})$ and the above proof does not hold. This conforms with the classic results of [279], which agree with our $\mathrm{SO}(5,5;\mathbb{Z})$ canonical form, but are only valid for orthogonal groups of signature $\leq r - 4$, where r is the rank of the group. Similarly, it was shown in [276] that for the non-split case $\mathfrak{J}_3^{\mathfrak{O}}$ not all elements are diagonalisable. This suggests that the integral U-duality orbits of the exceptional magic supergravities will have a far more complicated structure than the maximally supersymmetric theories.

The canonical form is uniquely determined by the set of arithmetic invariants (9.14) since

$$\begin{aligned} b_1(\mathcal{Q}_{\text{can}}) &= k, \\ b_2(\mathcal{Q}_{\text{can}}) &= k^2 l, \end{aligned} \tag{9.32}$$

so that for arbitrary $\mathcal{Q}$ one obtains $k = b_1(\mathcal{Q})$ and $l = k^{-2} b_2(\mathcal{Q})$. Hence, black strings with distinct canonical forms are unrelated by U-duality. The canonical forms allow for a full classification of the black string charge configurations.

$D = 6, \mathcal{N} = 8$ **Black string orbit classification**

1. The complete set of distinct 1/2-BPS charge vector orbits is given by,

$$\Big\{ \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad k > 0 \Big\}. \tag{9.33}$$

2. The complete set of distinct 1/4-BPS charge vector orbits is given by,

$$\Big\{ \begin{pmatrix} k & 0 \\ 0 & kl \end{pmatrix}, \quad \text{where} \quad k, |l| > 0 \Big\}. \tag{9.34}$$

2.2. $D = 5, \mathcal{N} = 8$ black hole charge orbits

For quantized charges the continuous U-duality is broken to an infinite discrete subgroup, which for $D = 5$ is given by $E_{6(6)}(\mathbb{Z}) \subset E_{6(6)}(\mathbb{R})$ [47]. Let $E_{6(6)}(\mathbb{Z})$ be defined by $\text{Str}_0(\mathfrak{J}_3^{\mathfrak{O}_s})$. The quantized black hole charge vector may be represented by a Jordan algebra element,

$$Q = \begin{pmatrix} q_1 & Q_s & \overline{Q_c} \\ \overline{Q_s} & q_2 & Q_v \\ Q_c & \overline{Q_v} & q_3 \end{pmatrix}, \quad \text{where} \quad q_1, q_2, q_3 \in \mathbb{Z} \quad \text{and} \quad Q_{v,s,c} \in \mathfrak{O}_s. \tag{9.35}$$

The Dirac-Schwinger quantisation condition for an electric black hole and a magnetic string with charges Q, P in the Jordan language is given by

$$\text{Tr}(Q, P) \in \mathbb{Z}. \tag{9.36}$$

As was the case in $D = 6$, the charge conditions defining the orbits in the continuous theory are manifestly invariant under the discrete subgroup $E_{6(6)}(\mathbb{Z})$ and, hence, those states unrelated by U-duality in the classical theory remain unrelated in the quantum theory. There are three disjoint *classes* of orbits, one 1/2-BPS, one 1/4-BPS and one 1/8-BPS, corresponding to the three continuous orbits. However, each of these classes is broken up into a countably infinite set of discrete orbits [20]. To classify these orbits we use the main theorem of [67] to bring an arbitrary charge vector into a diagonal reduced canonical form.

Proposition 54 (Black hole canonical form)**.** *Every black hole charge vector* $\mathcal{Q} \in \mathfrak{J}_3^{\mathfrak{O}_s}$ *is U-duality equiva-*

lent to a diagonally reduced canonical form,

$$Q_{can} = k \begin{pmatrix} 1 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & lm \end{pmatrix}, \quad \textit{where} \quad k > 0, l \geq 0. \tag{9.37}$$

The canonical form is uniquely determined by (9.18) since

$$\begin{aligned} c_1(Q_{\text{can}}) &= k, \\ c_2(Q_{\text{can}}) &= k^2 l, \\ c_3(Q_{\text{can}}) &= k^3 l^2 m. \end{aligned} \tag{9.38}$$

so that for arbitrary Q one obtains $k = c_1(Q)$, $l = k^{-2}c_2(Q)$ and $m = k^{-3}l^{-2}c_3(Q)$ [67].

$D = 5, \mathcal{N} = 8$ **Black hole orbit classification**

1. The complete set of distinct 1/2-BPS charge vector orbits is given by,

$$\Big\{ \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad k > 0 \Big\}. \tag{9.39}$$

2. The complete set of distinct 1/4-BPS charge vector orbits is given by,

$$\Big\{ \begin{pmatrix} k & 0 & 0 \\ 0 & kl & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad k, l > 0 \Big\}. \tag{9.40}$$

3. The complete set of distinct 1/8-BPS charge vector orbits is given by,

$$\Big\{ \begin{pmatrix} k & 0 & 0 \\ 0 & kl & 0 \\ 0 & 0 & klm \end{pmatrix}, \quad \text{where} \quad k, l, |m| > 0 \Big\}. \tag{9.41}$$

2.3. $D = 4, \mathcal{N} = 8$ black hole charge orbits

For quantized charges the continuous U-duality is broken to an infinite discrete subgroup, which for $D = 4$ is given by $E_{7(7)}(\mathbb{Z}) \subset E_{7(7)}(\mathbb{R})$ [47]. Let $E_{7(7)}(\mathbb{Z})$ be defined by $\text{Aut}(\mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}_s}))$. The quantized black hole charge vector may be represented by an FTS element,

$$x = \begin{pmatrix} -q_0 & P \\ Q & p^0 \end{pmatrix}, \quad \text{where} \quad q_0, p^0 \in \mathbb{Z} \quad \text{and} \quad Q, P \in \mathfrak{J}_3^{\mathfrak{O}_s}. \tag{9.42}$$

Note, the quartic norm and, thus, the entropy squared are quantized. In fact, $\Delta(x)$ is equal to either $4n$ or $4n + 1$ for some $n \in \mathbb{Z}$. The Dirac-Schwinger-Zwanziger quantisation condition relating two black holes with charges x and x' within the FTS language is given by

$$\{x, x'\} \in \mathbb{Z}. \tag{9.43}$$

Like the previous examples in $D = 5, 6$, the charge conditions defining the orbits in the continuous theory are manifestly invariant under the discrete subgroup $E_{7(7)}(\mathbb{Z})$ and, hence, those states unrelated by U-duality in the classical theory remain unrelated in the quantum theory. There are five disjoint *classes* of orbits corresponding to the five continuous orbits. Three of which are the small 1/2-BPS, 1/4-BPS and 1/8-BPS classes, with vanishing $\Delta(x)$. There are two large classes of orbits, one 1/8-BPS and one non-BPS as determined by the sign of $\Delta(x)$. However, each of these classes is broken up into a countably infinite set of discrete orbits. To classify these orbits Krutelevich used $E_{7(7)}(\mathbb{Z})$ to bring an arbitrary charge vector into a diagonal reduced canonical form [48]. However, unlike the previous case this canonical form is *not* uniquely defined. A partial classification of the orbits is achieved via the set of arithmetic invariants (9.23).

Proposition 55 (Black hole canonical form)**.** *Every black hole charge vector $x \in \mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}^s})$ is U-duality equivalent to a diagonally reduced canonical form,*

$$x_{can} = \alpha \begin{pmatrix} 1 & k\,\mathrm{diag}(1, l, lm) \\ 0 & j \end{pmatrix}, \quad \textit{where} \quad \alpha > 0. \tag{9.44}$$

Note, $k\,\mathrm{diag}(1, l, lm)$ is the $D = 5$ diagonally reduced canonical form (9.37)*. From here-on-in we will often use (a, b, c) to mean* $\mathrm{diag}(a, b, c)$.

In [49] a manifestly U-duality invariant dyon spectrum in type II on T^6 was derived under the assumption that $d_2'(x) = 1$ and that the R-R charges may always be U-dualised away. The canonical form implies that the latter is *always* true.

While the $D = 4$ canonical form for a generic charge vector is not uniquely determined by the discrete invariants (9.23) it is uniquely specified for the subclass of black holes preserving more than 1/8 of the supersymmetries, i.e. rank 1 and rank 2 charge vectors [48]. In this case the canonical form is simplified.

Proposition 56 (>1/8-BPS black hole canonical form)**.** *The charge vector $x \in \mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}^s})$ of every black hole preserving more than 1/8 of the supersymmetries is U-duality equivalent to a diagonally reduced canonical form,*

$$x_{>1/8\text{-}BPS\ can} = \alpha \begin{pmatrix} 1 & k(1, 0, 0) \\ 0 & 0 \end{pmatrix}, \quad \textit{where} \quad \alpha, k > 0. \tag{9.45}$$

The simplified canonical form is uniquely determined by the two well defined arithmetic invariants from (9.23), since

$$\begin{aligned} d_1(Q_{\text{can}}) &= \alpha, \\ d_2(Q_{\text{can}}) &= 2\alpha^2 k, \end{aligned} \tag{9.46}$$

so that for arbitrary rank 1 or 2 x one obtains $\alpha = d_1(x)$ and $k = (\sqrt{2}\alpha)^{-2} d_2(x)$ [48]. This facilitates the orbit classification for such states as is described below.

$D=4,\mathcal{N}=8$ **Black hole orbit classification for >1/8-BPS**

1. The complete set of distinct 1/2-BPS charge vector orbits is given by,

$$\begin{pmatrix} \{\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \alpha > 0\}. \tag{9.47}$$

2. The complete set of distinct 1/4-BPS charge vector orbits is given by,

$$\{\alpha \begin{pmatrix} 1 & k(1,0,0) \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad \alpha, k > 0\}. \tag{9.48}$$

2.3.1. Projective black holes

For black holes preserving less than 1/4 of the supersymmetries the analysis becomes increasingly complex and the orbit classification for generic charge vectors is not known. However, for a subclass of such black holes, satisfying particular arithmetic conditions, the orbit classification is known. These black holes are referred to as *projective*.

A black hole charge vector x is said to be projective if its U-duality orbit contains a diagonal reduced element (9.58) satisfying [20,48],

$$\begin{aligned} \gcd(\alpha k, \alpha j, (\alpha k l)^2 m) =& 1; \\ \gcd(\alpha k l, \alpha j, (\alpha k)^2 l m) =& 1; \\ \gcd(\alpha k l m, \alpha j, (\alpha k)^2 l) =& 1. \end{aligned} \tag{9.49}$$

One immediately sees that projectivity implies $\alpha = 1$ in the canonical form (9.58) and therefore $\gcd(x) = 1$. Black holes satisfying $\gcd(x) = 1$ are conventionally referred to as *primitive*.

While the general treatment of orbits in $D = 4$ is lacking, the orbit representatives of projective black holes have been fully classified in [20,48]. This classification relies on Krutelevich's main theorem [48].

Proposition 57 (Black hole canonical form)**.** *Every projective black hole charge vector $x \in \mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}_s})$ is U-duality equivalent to a diagonally reduced canonical form,*

$$\begin{pmatrix} 1 & (1,1,m) \\ 0 & j \end{pmatrix}, \quad \textit{where} \quad j \in \{0,1\}. \tag{9.50}$$

The values of m and j are uniquely determined by $\Delta(x)$.

Further,

- $E_{7(7)}(\mathbb{Z})$ acts transitively on projective elements of a given norm $\Delta(x)$.
- If $\Delta(x)$ is a squarefree[2] integer equal to 1 (mod 4) or if $\Delta(x) = 4n$, where n is squarefree and equal to 2 or 3 (mod 4), then x is projective and hence U-duality acts transitively.

[2] An integer is squarefree if its prime decomposition contains no repetition.

In the projective case all black holes with the same quartic norm and hence lowest order entropy are U-duality related.

As already emphasized the generic case of not necessarily projective black holes is not fully understood.

2.3.2. Note on the dyon degeneracy formulae for type II string theory on a T^6

The string theoretic completion of $\mathcal{N} = 8$ supergravity, given by type II string theory on a T^6, is conjectured to respect not only the S- and T-dualities, but also the full $E_{7(7)}(\mathbb{Z})$ U-duality. Accordingly, one would expect the 1/8-BPS dyon spectrum to be manifestly $E_{7(7)}(\mathbb{Z})$ invariant. In principle, one needs to take care of both the dyon charges and the scalar fields under U-duality. However, just as the attractor mechanism freed the Bekenstein-Hawking entropy of any scalar dependence, one can use the helicity trace index B_{14} [280], which is invariant under smooth variations of the scalar fields, to count the degeneracy of states for a given set of charges. There is one potential pitfall, this index can potentially jump as the scalars cross "walls of marginal stability" on which the 1/8 BPS dyon may decay into a pair of 1/2-BPS dyons. This is not a concern here since it is know that such walls are absent for states with $\Delta > 0$ [271].

In [49] a manifestly U-duality invariant dyon degeneracy (index) formula was derived for 1/8-BPS dyons under the assumptions:

1. Up to U-duality there are no R-R charges.
2. $d_2'(x) = 1$.

The canonical form (9.58) implies that the first condition is always satisfied. The degeneracy index, $d(x) \equiv (-1)^{\Delta(x)} B_{14}$, is given by,

$$d(x) = \sum_{s\in\mathbb{Z}, 2s|d_4'(x)} s\,\hat{c}(\Delta(x)/s^2)\,, \tag{9.51}$$

where

$$-\,\vartheta_1(z|\tau)^2\,\eta(\tau)^{-6} \equiv \sum_{k,l} \hat{c}(4k - l^2)\,e^{2\pi i(k\tau + lz)} \tag{9.52}$$

and $\vartheta_1(z|\tau)$ and $\eta(\tau)$ are respectively the odd Jacobi theta function and the Dedekind eta function [265,281]. This formula is evidently $E_{7(7)}(\mathbb{Z})$ invariant.

For 1/8-BPS states that have vanishing Δ, i.e. rank 3, the formula reduces to [272],

$$d(x)_{\text{Rank 3}} = \sum_{s\in\mathbb{Z}, 2s|d_4'(x)} 2s. \tag{9.53}$$

For 1/4- and 1/2-BPS states the arithmetic invariant $d_4'(x)$ is no longer well defined and one would expect separately invariant formulae, which were indeed obtained in [272].

It would be interesting to see if one can relax the condition that $d_2'(x) = 1$. Alternatively, one might anticipate a particularly simple form under the stronger assumption of projectivity, which is the most interesting from a number theoretic perspective [50]. In either case, the orbit classification will be of use.

2.4. Conclusion

We have summarized our current understanding of the black hole/string charge vector orbits under the discrete U-dualities of $\mathcal{N} = 8$ supergravity in six, five and four dimensions. The discrete orbits of both the black strings in $D = 6$ and the black holes/strings in $D = 5$ [67] admit a complete classification. Two distinct technical elements made this analysis tractable. First, the discrete U-duality groups, $\mathrm{SO}(5,5;\mathbb{Z})$ in $D = 6$ and $E_{6(6)}(\mathbb{Z})$ in $D = 5$, may be modeled, in the sense of [275], by the integral exceptional quadratic and cubic norm Jordan algebras, respectively. These explicit representations, which both fundamentally rely upon the ring of integral split-octonions, yielded diagonally reduced canonical forms for the charge vectors, from which the orbit representatives could, in principle, be obtained. Second, a complete list of independent arithmetic invariants, typically given by the gcd of irreps built out of the basic charge vector representations, is known. These invariants are sufficient to uniquely fix the canonical form for a given charge vector. These two features together allow for the complete classification of the discrete orbits.

- $D = 6$: the black string charge vector $\mathcal{Q} \in \mathfrak{J}_2^{\mathfrak{O}_s}$ is $\mathrm{SO}(5,5;\mathbb{Z}) := \mathrm{Str}_0(\mathfrak{J}_2^{\mathfrak{O}_s})$ equivalent to a two charge diagonally reduced canonical form,

$$\mathcal{Q}_{\text{can}} = \begin{pmatrix} k & 0 \\ 0 & kl \end{pmatrix}, \quad k > 0, \tag{9.54}$$

  which is uniquely determined by the two following arithmetic invariants,

$$\begin{aligned} b_1(\mathcal{Q}) &:= \gcd(\mathcal{Q}), \\ b_2(\mathcal{Q}) &:= N_2(\mathcal{Q}) = \det \mathcal{Q}. \end{aligned} \tag{9.55}$$

- $D = 5$: the black hole (string) charge vector $Q \in \mathfrak{J}_3^{\mathfrak{O}_s}$ is $E_{6(6)}(\mathbb{Z}) := \mathrm{Str}_0(\mathfrak{J}_3^{\mathfrak{O}_s})$ equivalent to a three charge diagonally reduced canonical form,

$$Q_{\text{can}} = \begin{pmatrix} k & 0 & 0 \\ 0 & kl & 0 \\ 0 & 0 & klm \end{pmatrix}, \quad k > 0, l \geq 0, \tag{9.56}$$

  which is uniquely determined by the three following arithmetic invariants,

$$\begin{aligned} c_1(Q) &:= \gcd(Q), \\ c_2(Q) &:= Q^\sharp, \\ c_3(Q) &:= N_3(Q). \end{aligned} \tag{9.57}$$

The analogous treatment of the 4-dimensional black hole is not so transparent. The integral FTS does indeed provide an elegant and natural representation of the discrete U-duality group $E_{7(7)}(\mathbb{Z})$, which again yields a diagonally reduced canonical charge vector.

- Every black hole charge vector $x \in \mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}_s})$ is U-duality equivalent to a diagonally reduced canonical form,

$$x_{\text{can}} = \alpha \begin{pmatrix} 1 & k\,\mathrm{diag}(1,l,lm) \\ 0 & j \end{pmatrix}, \quad \text{where} \quad \alpha > 0. \tag{9.58}$$

- The black hole entropy is quantized since $\Delta = 4n, 4n + 1$.
- The R-R charges can always be U-dualised away.

However, this canonical form is not uniquely determined by the known set of arithmetic U-duality invariants. The complete classification is known for two subcases: 1) Black holes preserving more than 1/8 of the supersymmetries 2) Black holes satisfying the projectivity condition.

- $D = 4 > 1/8$-BPS: the black hole charge vector $x \in \mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}_s})$ is $E_{7(7)}(\mathbb{Z}) := \mathrm{Aut}(\mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}_s}))$ equivalent to a two charge diagonally reduced canonical form,

$$x_{\text{can}} = \alpha \begin{pmatrix} 1 & (k, 0, 0) \\ 0 & 0 \end{pmatrix}, \quad \alpha > 0, \tag{9.59}$$

  which is uniquely determined by the two following arithmetic invariants,

$$\begin{aligned} d_1(x) &:= \gcd(x), \\ d_2(x) &:= \gcd(3T(x, x, y) + \{x, y\}x), \quad \forall y \in \mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}_s}). \end{aligned} \tag{9.60}$$

- $D = 4$ projective: the black hole charge vector $x \in \mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}_s})$ is $E_{7(7)}(\mathbb{Z}) := \mathrm{Aut}(\mathfrak{F}(\mathfrak{J}_3^{\mathfrak{O}_s}))$ equivalent to a four or five charge diagonally reduced canonical form,

$$x_{\text{proj can}} = \begin{pmatrix} 1 & (1, 1, m) \\ 0 & j \end{pmatrix}, \quad \text{where} \quad j \in \{0, 1\}. \tag{9.61}$$

  The values of m and j are uniquely determined by the quartic $E_{7(7)}(\mathbb{R})$ invariant, $\Delta(x)$.

Evidently, there are a number of open questions. Chiefly, is it possible that the full space of 4-dimensional orbits could be resolved if the complete list of independent arithmetic invariants was known? For example, we have thus far used gcd of the $\mathbf{133}$ appearing in $\mathbf{56} \times_s \mathbf{56}$. What about the $\mathbf{1463}$? Following [16, 18], we may truncate to the eight charges of the STU model [1, 2, 195, 196], which transform as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$. Using this truncation, the $\mathbf{1463}$ in $\mathbf{56} \times_s \mathbf{56}$ reduces to the $(\mathbf{3}, \mathbf{3}, \mathbf{3})$ in $(\mathbf{2}, \mathbf{2}, \mathbf{2}) \times_s (\mathbf{2}, \mathbf{2}, \mathbf{2})$. Computing the gcd of this $(\mathbf{3}, \mathbf{3}, \mathbf{3})$ gives the square of $d_1(x_{\text{can}})$ and, therefore, adds no additional information. To proceed further, it would serve us well to have a full classification of the independent $E_{7(7)}(\mathbb{Z})$ arithmetic invariants.

It would be interesting to see, armed with a complete list of invariants and orbit classification, if the dyon degeneracy formula of [49] could be generalised to all black holes.

Finally, returning to the maximally supersymmetric theory in six dimensions, the black hole and membrane charges transform as the spinors $\mathbf{16}$ and $\mathbf{16}'$ of $\mathrm{SO}(5, 5; \mathbb{R})$, both of which may be represented as a pair of split-octonions [65]. An integral structure could then be induced, as it was for the string, by using the integral split-octonions, again providing a natural framework with which to study the discrete U-duality orbits of $\mathrm{SO}(5, 5; \mathbb{Z})$. We leave this for future work.

We close this section with some rather intriguing observations regarding the integral STU model:

Definition 58 (Integral STU Freudenthal triple system $\mathfrak{F}^{\mathbb{Z}}_{STU}$).

$$\mathfrak{F}^{\mathbb{Z}}_{STU} = \mathfrak{F}(\mathfrak{J}^{\mathbb{Z}}_{STU}) := \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{J}^{\mathbb{Z}}_{STU} \oplus \mathfrak{J}^{\mathbb{Z}}_{STU}, \tag{9.62}$$

where $\mathfrak{J}^{\mathbb{Z}}_{STU} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. *The automorphism group is*

$$\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z}). \tag{9.63}$$

This structure has a direct relationship to Bhargava's recent work generalising Gauss's composition law for binary quadratic forms, almost precisely 200 years on from their introduction [50]. Essentially, the set of U-duality equivalence classes of projective STU black holes with hyperdeterminant $D \neq 0$ has a unique group composition law. The resulting group is isomorphic to $\mathrm{Cl}^+(S) \times \mathrm{Cl}^+(S)$, where $\mathrm{Cl}^+(S)$ denotes the narrow class group of the quadratic ring S with discriminant D. This relies on the crucial observation that a ring with a finite rank as a $\mathbb{Z}$-module must have a discriminant equal to $4n$ or $4n + 1$ for $n \in \mathbb{Z}$. This is precisely the quantization of the integral FTS quartic norm.

Furthermore, the U-duality orbits of the large black holes, which we have not been able to classify, are in one-to-one correspondence with pairs $(S, (I_1, I_2, I_3))$ (up to isomorphism), where *very roughly* S is a quadratic ring with non-zero discriminant and (I_1, I_2, I_3) is an equivalence class of triples of ideals satisfying, $I_1 I_2 I_3 \subseteq S$ and $N(I_1)N(I_2)N(I_3) = 1$ [50]. There is some classification at least!

Actually, this correspondence is surprisingly concrete. Given a black hole with charges a_{ABC} the quadratic ring S is determined by the hyperdeterminant, which gives the discriminant, and the six bases for the ideal classes I_i are determined by a system of eight equations involving a_{ACB}. Conversely, given any pair S and (I_1, I_2, I_3) the corresponding black hole a_{ABC} is directly obtained from the assumption that $I_1 I_2 I_3 \subseteq S$.

Can this construction be extend to the other FTS's considered here. It seems plausible given that the quartic norm in each case can be related to the hyperdeterminant. Indeed, Bhargava has already treated the case of 3×3 Hermitian matrices over the integral split-complexes. Whether or not these remarkable observations have any real significance for stringy black hole physics is not clear, but certainly merits some further investigation.

3. Freudenthal duality

3.1. Introduction

In recent work [20] we introduced a new duality of black holes: the *Freudenthal duality* $x \to \tilde{x}$, for which $\tilde{\tilde{x}} = -x$. Although distinct from U-duality it nevertheless leaves $\Delta(x)$ invariant. However, the requirement that $\tilde{x}$ be integer restricts us to the subset of black holes for which $\Delta(x)$ is necessarily a perfect square. Despite the non-polynomial nature of the F-dual, it scales linearly in the sense that $\tilde{x}(nx) = n\tilde{x}(x)$.

The U-duality integral invariants $\{x, y\}$ and $\Delta(x, y, z, w)$ are not generally invariant under F-duality but $\{\tilde{x}, x\}$, $\Delta(x)$, and hence the lowest-order black hole entropy, are invariant. However, higher order corrections may also depend on discrete U-duality invariants involving the various gcds [49, 267, 268, 270, 271]. Under F-duality certain discrete U-duality invariants are conserved while others are not necessarily, as is discussed in section 3.3. For example, the product $d_1(x)d_3(x)$ is invariant but $d_1(x)$ and $d_3(x)$ separately need not be. The F-dual of a primitive black hole need not itself be primitive.

Similar remarks apply to the quantised charges A of five dimensional black strings and the quantised charges B of five dimensional black holes. We introduced an analogous *Jordan dual* $A^\star$, with $N(A)$ necessarily a perfect cube, for which $A^{\star\star} = A$ and which leaves $N(A)$ invariant. Despite its non-polynomial nature J-dual scales linearly in the sense that $A^\star(nA) = nA^\star(A)$.

The U-duality integral invariants $\mathrm{Tr}(X, Y)$ and $N(X, Y, Z)$ are not generally invariant under Jordan duality but $\mathrm{Tr}(X^\star, X)$, $N(X)$ and hence the lowest-order black hole and black string entropy, are invariant. However, higher order corrections may also depend on arithmetic U-duality invariants. Under J-duality certain discrete U-duality invariants are conserved while others are not necessarily, as is discussed in section 3.7. For example, the product $d_1(A)d_2(A)$ is invariant but $d_1(A)$ and $d_2(A)$ separately need not be. As for the F-dual, the J-dual of a primitive black hole/string need not itself be primitive.

The 4D/5D lift [51] associates a rotating 5D black hole to a non-rotating 4D black hole. In section 3.9 we show that two black holes related by F-duality in 4D are related by J-duality when lifted to 5D.

3.2. The 4D Freudenthal dual

Definition 59 (The Freudenthal dual)**.** *Given a black hole with charges x, we define its Freudenthal dual $\tilde{x}$ by*

$$\tilde{x} := T(x)|\Delta(x)|^{-1/2}. \tag{9.64}$$

As described in section 2.2, the FTS divides black holes into five distinct ranks or orbits. F-duality (9.64) is initially defined for large rank 4 black holes for which both T and Δ are nonzero. Small black holes are discussed in [20].

The invariance of $\Delta(x)$ follows by noting that

$$2\Delta(x) = \{T(x), x\} \tag{9.65}$$

where $T(x) = T(x, x, x)$ obeys

$$T(T(x)) = -\Delta^2(x)x \tag{9.66}$$

and hence

$$\Delta(T(x)) = \Delta(x)^3 \tag{9.67}$$

So

$$\Delta(\tilde{x}) = \Delta(T(x))\Delta(x)^{-2} = \Delta(x). \tag{9.68}$$

Moreover

$$\tilde{\tilde{x}} = T(\tilde{x})|\Delta(x)|^{-1/2} = T(T(x))\Delta(x)^{-2} = -x. \tag{9.69}$$

In the case of two black holes related by Freudenthal duality, the Dirac-Schwinger-Zwanziger quantisation condition (9.43) becomes

$$\{\tilde{x}, x\} = \{T(x), x\}|\Delta(x)|^{-1/2} = 2\,\mathrm{sgn}(\Delta)|\Delta(x)|^{1/2} \tag{9.70}$$

which is also invariant.

As noted in section 3.1, for a valid dual charge vector $\tilde{x}$, we require that $|\Delta(x)|$ is a perfect square. So we may write

$$|\Delta(x)| = \tfrac{1}{4}\{\tilde{x}, x\}^2 \tag{9.71}$$

with

$$\{\tilde{x}, x\} = \tilde{\alpha}\beta - \tilde{\beta}\alpha + \mathrm{Tr}(\tilde{A}, B) - \mathrm{Tr}(\tilde{B}, A), \tag{9.72}$$

This is a necessary, but not sufficient condition because we further require that

$$d_4(x) = \left[\frac{d_3(x)}{d_1(\tilde{x})}\right]^2 = \left[\frac{d_3(\tilde{x})}{d_1(x)}\right]^2 = d_4(\tilde{x}). \tag{9.73}$$

Since F-duality requires that $\Delta(x)$ is a perfect square, the squarefree condition discussed in section 2.3.1 does not apply to the subset of black holes admitting an F-dual, which may or may not be projective:

> *Non-projective black holes related by an F-duality not conserving d_1 provide examples of configurations with the same quartic norm and hence lowest order entropy that are definitely not U-duality related,*

but more surprisingly,

> *Non-projective black holes related by an F-duality conserving d_1 provide examples of configurations with the same quartic norm, and same arithmetic invariants* (9.23)*, that are apparently not U-duality related.*

The U-duality integral invariants $\{x, y\}$ and $\Delta(x, y, z, w)$ are not generally invariant under Freudenthal duality while $\{\tilde{x}, x\}$, $\Delta(x)$, and hence the lowest-order black hole entropy, are invariant. However, higher order corrections to the black hole entropy depend on some of the discrete U-duality invariants, to which we now turn.

3.3. The action of F-duality on arithmetic U-duality invariants

The first important observation we make is:

Remark 60. *F-duality commutes with U-duality*

$$\widetilde{\sigma(x)} = \sigma(\tilde{x}). \tag{9.74}$$

This follows directly from,

$$T(\sigma(x), \sigma(y), \sigma(z)) = \sigma(T(x, y, z)), \qquad \forall\, \sigma \in \operatorname{Aut} \mathfrak{F}(\mathfrak{J}). \tag{9.75}$$

We shall see that of the discrete U-duality invariants listed in (9.23), not only $d_4(x)$ but also $d_2(x)$, $d'_2(x)$ and $d'_4(x)$ are F-dual invariant. However, $d_1 = \gcd(x)$ and $d_3 = \gcd(T(x))$ need not be.

Proposition 61. *The arithmetic U-duality invariant*

$$d'_4(x) := \gcd(x \wedge T(x)) \tag{9.76}$$

is preserved under F-duality.

Proof. The invariance of $d'_4(x)$ follows from (9.66) which implies

$$\begin{aligned} \tilde{x} \wedge T(\tilde{x}) &= T(x)|\Delta|^{-1/2} \wedge T(T(x)|\Delta|^{-1/2}) \\ &= -|\Delta|^{-2} T(x) \wedge \Delta^2 x \\ &= \operatorname{sgn}(\Delta) x \wedge T(x) \end{aligned} \tag{9.77}$$

and, hence, $d'_4(x) = d'_4(\tilde{x})$. □

Proposition 62. *The arithmetic U-duality invariant*

$$d'_2(x) := \gcd(\mathcal{P}(x), \mathcal{Q}(x), \mathcal{R}_x) \tag{9.78}$$

is preserved under F-duality.

Proof. To establish the invariance of $d'_2(x)$ we examine

$$\begin{aligned} \mathcal{P}(x) &= B^\sharp - \alpha A \\ \mathcal{Q}(x) &= A^\sharp - \beta A \\ \mathcal{R}_x(C) &= 2\kappa(x)C + 2\{A, B, C\} \end{aligned} \tag{9.79}$$

in turn. First, for the black string magnetic charge $\mathcal{P}$ we find from (5.64d)

$$|\Delta|\tilde{\mathcal{P}} = 4\big\{[-\alpha A^\sharp + A \times B^\sharp + \kappa(x)B]^\sharp - (\alpha\kappa(x) + N_3(B))[\beta B^\sharp - B \times A^\sharp - \kappa(x)A]\big\}. \tag{9.80}$$

Using $(X+Y)^\sharp = X \times Y + X^\sharp + Y^\sharp$ the first term of (9.80) gives

$$\begin{aligned} [-\alpha A^\sharp + A \times B^\sharp + \kappa(x)B]^\sharp = &- \alpha(A \times B^\sharp) \times A^\sharp + \kappa(x)(A \times B^\sharp) \times B \\ &- \alpha\kappa(x)A^\sharp \times B + (A \times B^\sharp)^\sharp + \alpha^2 N_3(A)A + \kappa(x)^2 B^\sharp. \end{aligned} \tag{9.81}$$

This may be further simplified using the identities

$$\begin{aligned} X^\sharp \times (X \times Y) &= N_3(X)Y + \mathrm{Tr}(X^\sharp, Y)X \\ (X \times Y)^\sharp &= \mathrm{Tr}(X^\sharp, Y)Y + \mathrm{Tr}(Y^\sharp, X)X - X^\sharp \times Y^\sharp, \end{aligned} \tag{9.82}$$

which follow from the quadratic adjoint definition and the requirement that $(X^\sharp)^\sharp = N_3(X)X$. These identities yield

$$\begin{aligned} (A \times B^\sharp) \times A^\sharp &= N_3(A)B^\sharp + \mathrm{Tr}(A^\sharp, B^\sharp)A \\ (A \times B^\sharp) \times B &= N_3(B)A + \mathrm{Tr}(A, B)B^\sharp \\ (A \times B^\sharp)^\sharp &= \mathrm{Tr}(A^\sharp, B^\sharp)B^\sharp + \mathrm{Tr}(A, B)N_3(B)A - N_3(B)A^\sharp \times B. \end{aligned} \tag{9.83}$$

Using the above to simplify (9.81) and then substituting into (9.85) gives, after collecting terms,

$$|\Delta|\tilde{\mathcal{P}} = 4[\alpha^2\beta - \alpha\,\mathrm{Tr}(A, B) - \alpha\kappa(x)]A^\sharp \times B + \Delta(B^\sharp - \alpha A). \tag{9.84}$$

The first term vanishes identically so that

$$\tilde{\mathcal{P}} = \mathrm{sgn}(\Delta)(B^\sharp - \alpha A) = \mathrm{sgn}(\Delta)\mathcal{P}. \tag{9.85}$$

A similar treatment goes through for $\mathcal{Q}$:

$$\tilde{\mathcal{Q}} = \mathrm{sgn}(\Delta)(A^\sharp - \beta A) = \mathrm{sgn}(\Delta)\mathcal{Q}. \tag{9.86}$$

Finally, in order to demonstrate the invariance of $\mathcal{R}_x$ we exploit the fact that since U-duality commutes with F-duality we may assume x to be in reduced form (9.58) so that

$$\mathcal{R}_x(C) = \alpha\beta\, C. \tag{9.87}$$

For reduced x the dual is given by

$$|\Delta|^{1/2}\tilde{x} = \begin{pmatrix} -\alpha^2\beta & \alpha\beta A \\ 2\alpha A^\sharp & \alpha\beta^2 + 2N_3(A) \end{pmatrix}, \tag{9.88}$$

where $\Delta = -\alpha^2\beta^2 - 4\alpha N_3(A)$. On substituting in for $\mathcal{R}_x(C)$ one finds

$$\begin{aligned} |\Delta|\mathcal{R}(\tilde{x})(C) &= \alpha\beta\big(- [\alpha^2\beta^2 + 8N_3(A)]C + 4\alpha\{A, A^\sharp, C\}\big) \\ &= \Delta\alpha\beta\, C, \end{aligned} \tag{9.89}$$

where we have used $\{X, X^\sharp, Y\} = N_3(X)Y$ [68] in the final step. Hence, $\mathcal{R}$ is also invariant up to a sign.

$$\tilde{\mathcal{R}} = \operatorname{sgn}(\Delta)\alpha\beta C = \operatorname{sgn}(\Delta)\mathcal{R}. \tag{9.90}$$

This clearly establishes the invariance of $d_2'(x)$ under F-duality. □

Proposition 63. *The arithmetic U-duality invariant*

$$d_2(x) := \gcd(3\,T(x,x,y) + \{x,y\}\,x) \qquad \forall y \tag{9.91}$$

is preserved under F-duality.

Proof. To prove the invariance of $d_2(x)$ we first rephrase the problem using the fact that an integer n divides $3\,T(x,x,y) + \{x,y\}\,x$ for all y if and only if it divides the following five expressions [48]:

$$2\,\mathcal{P},\ 2\,\mathcal{Q},\ 3\alpha\beta - \operatorname{Tr}(A,B),\ \mathcal{R}_x,\ \mathcal{R}_{x'}, \tag{9.92}$$

where

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad x' = \begin{pmatrix} \beta & B \\ A & \alpha \end{pmatrix}. \tag{9.93}$$

Hence, we are only further required to establish the invariance of $3\alpha\beta - \operatorname{Tr}(A,B)$. The proof goes along much the same lines as before to obtain

$$3\tilde{\alpha}\tilde{\beta} - \operatorname{Tr}(\tilde{A},\tilde{B}) = \operatorname{sgn}(\Delta)[3\alpha\beta - \operatorname{Tr}(A,B)]. \tag{9.94}$$

□

Finally, recall that restricting to the STU subsector $d_2(x)$ takes the reduced form

$$d_2(x) = \gcd(\gamma^A, \gamma^B, \gamma^C). \tag{9.95}$$

In this case the proof of F-dual invariance is simplified since each γ is individually invariant, up to a sign, under F-duality.

As for $d_1(x)$ and $d_3(x)$, it follows from (9.73) that their product is invariant

$$d_1(x)d_3(x) = d_1(\tilde{x})d_3(\tilde{x}) \tag{9.96}$$

but separately they need not be. Another way to state this is that the F-dual of a primitive black hole may not itself be primitive. To see this, recall that by definition

$$x = d_1(x)x_0, \tag{9.97}$$

where x_0 is primitive with $d_1(x_0) = 1$. Hence

$$T(x) = d_1(x)^3 T(x_0) \tag{9.98}$$

and

$$\Delta(x) = d_1(x)^4 \Delta(x_0). \tag{9.99}$$

So

$$\tilde{x} = d_1(x)T(x_0)|\Delta(x_0)|^{-1/2} = d_1(x)\tilde{x}_0 \tag{9.100}$$

and

$$d_1(\tilde{x}) = d_1(x)d_1(\tilde{x}_0). \tag{9.101}$$

Hence $d_1(x)$ is invariant if $d_1(\tilde{x}_0) = d_1(x_0) \equiv 1$, which is not necessarily so.

Typically, the literature on exact 4D black hole degeneracies [49,174,206,247,265,267–271,281–286] deals only with primitive black holes $d_1(x) = 1$. We are not required to impose this condition and generically do not do so.

In [20], we provide examples which preserve $d_1(x)$ and examples that do not. If desired, however, one might restrict the subset of black holes admitting an F-dual even further by demanding that $d_1(x)$, and hence $d_3(x)$, be conserved.

3.4. F-dual in canonical basis

D. Dahanayake developed a general treatment of the F-dual in the canonical basis. See [20] for more details and explicit examples. Recall that we may write any black hole in the diagonally reduced canonical form (9.58),

$$x = \alpha \begin{pmatrix} 1 & k(1, l, lm) \\ 0 & j \end{pmatrix}, \tag{9.102}$$

where $\alpha > 0, k, l \geq 0$, and $\alpha, j, k, l, m \in \mathbb{Z}$. The quartic norm of this element is

$$\Delta(x) = -(j^2 + 4k^3l^2m)\alpha^4. \tag{9.103}$$

For x to be a rank 4 we must impose

$$j \neq 0 \vee klm \neq 0 \tag{9.104}$$

where $\vee$ here denotes logical disjunction. Note that in order for the charge vector to be BPS we need $\text{sgn}(j^2 + 4k^3l^2m) = -1$ and hence $\text{sgn}(m) = -1$ is a necessary condition. Using (9.103) and the general form for $T(x)$, we find that the general F-dual is

$$\tilde{x} = \alpha|j^2 + 4k^3l^2m|^{-1/2} \begin{pmatrix} -j & jk(1,l,lm) \\ 2k^2l(lm,m,1) & j^2 + 2k^3l^2m \end{pmatrix}. \tag{9.105}$$

In order that $\tilde{x}$ be integer, we need to impose the following three constraints:

$$|j^2 + 4k^3l^2m|^{1/2} = n_0 \in \mathbb{N}, \tag{9.106a}$$

$$\alpha j/n_0 = n_1 \in \mathbb{Z}, \tag{9.106b}$$

$$2k^2l\alpha/n_0 = n_2 \in \mathbb{N}_0, \tag{9.106c}$$

where $\operatorname{sgn} n_1 = \operatorname{sgn} j$. Equation (9.106a) forces Δ to be a perfect square, (9.106b) then ensures that the $\tilde{\alpha}$ component of $\tilde{x}$ lies in $\mathbb{Z}$, and (9.106c) guarantees that the $\tilde{B}$ component is integral. These conditions are also sufficient to make the $\tilde{A}$ and $\tilde{\beta}$ components integer valued. The dual system then becomes

$$\tilde{x} = \begin{pmatrix} -n_1 & n_1k(1,l,lm) \\ n_2(lm,m,1) & n_1j + n_2klm \end{pmatrix}. \tag{9.107}$$

The utility of this form is that all valid dual charge vectors can be specified, modulo a sign, by their j, k, l, m, n_1 and n_2 values. Clearly if both n_1 and n_2 vanish the entire system vanishes, failing to preserve rank. However, n_1 and n_2 can vanish separately and still leave a rank 4 system. This is to be expected since F-dual preserves Δ so that (9.107) must also satisfy (9.104), telling us that one of n_1, n_2 must be nonzero, given the definitions (9.106). As a sanity check we may evaluate the quartic form for (9.107) to discover that we require

$$jn_1 \neq 0 \vee klmn_1 \neq 0 \vee klmn_2 \neq 0 \tag{9.108}$$

for the dual system to be a large black hole. Satisfyingly, (9.108) is equal to its logical conjunction with (9.104). Furthermore, we find

$$\begin{aligned} d_1(\tilde{x}) &= \gcd(n_1, n_2), \\ d_2(\tilde{x}) &= \gcd(2kn_1^2 + 2mn_2^2, 2n_1\left(jn_1 + 2klmn_2\right), -k^2ln_1^2 + jn_1n_2 + klmn_2^2) \\ &= \alpha^2 \gcd(j, 2k), \\ d_2'(\tilde{x}) &= \gcd(kn_1^2 + mn_2^2, n_1\left(jn_1 + 2klmn_2\right), -k^2ln_1^2 + jn_1n_2 + klmn_2^2) \\ &= \alpha^2 \gcd(j, k), \\ d_3(\tilde{x}) &= \alpha^3 n_0, \\ d_4(\tilde{x}) &= d_4(x), \\ d_5(\tilde{x}) &= d_5(x). \end{aligned} \tag{9.109}$$

As expected, (9.73) is satisfied and

$$\begin{aligned} d_1(\tilde{x})d_3(\tilde{x}) &= \alpha^3 n_0 \gcd(n_1, n_2) \\ &= \alpha^4 \gcd(j, 2k^2l) \\ &= d_1(x)d_3(x). \end{aligned} \tag{9.110}$$

3.5. The NS-NS sector

3.5.1. P, Q notation

Under the decomposition of the $\mathcal{N} = 8$ U-duality group $E_{7(7)}(\mathbb{Z})$ to the S-duality group $\mathrm{SL}(2,\mathbb{Z})$ and the T-duality group $\mathrm{SO}(6,6;\mathbb{Z})$

$$E_{7(7)}(\mathbb{Z}) \supset \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SO}(6,6;\mathbb{Z}) \tag{9.111}$$

the $\mathbf{56}$ decomposes as

$$\mathbf{56} \to (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}). \tag{9.112}$$

The $(\mathbf{2},\mathbf{12})$ is identified as the NS-NS sector whereas the $(\mathbf{1},\mathbf{32})$ is associated with the R-R charges. Since any $\mathcal{N} = 8$ charge vector x is U-dual to a diagonal reduced form (9.58), the R-R charges can always be transformed away for a generic black hole[3] and we are free to consider those black holes with only NS-NS charges present. We write the 12 electric and 12 magnetic charges as Q and P respectively. In this case the quartic norm takes the simple, manifestly $\mathrm{SL}(2,\mathbb{Z})\times\mathrm{SO}(6,6;\mathbb{Z})$ invariant form

$$\Delta(P,Q) = P^2Q^2 - (P\cdot Q)^2. \tag{9.113}$$

Applying the trilinear map to x in this sector one finds[4]

$$\begin{pmatrix} T_P \\ T_Q \end{pmatrix} = \begin{pmatrix} P\cdot Q & -P^2 \\ Q^2 & -P\cdot Q \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}, \tag{9.114}$$

where T_P and T_Q denote the new P and Q components. The Freudenthal dual then becomes

$$\begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} = \frac{1}{\sqrt{|\Delta|}} \begin{pmatrix} T_P \\ T_Q \end{pmatrix} = \frac{1}{\sqrt{|\Delta|}} \begin{pmatrix} P\cdot Q & -P^2 \\ Q^2 & -P\cdot Q \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}. \tag{9.115}$$

While we have been focusing here on the NS-NS sector of the $\mathcal{N} = 8$ theory, the same formulae (9.113), (9.114), (9.115) also apply to the toroidal compactification of the heterotic string with $\mathcal{N} = 4$ supersymmetry and $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SO}(6,22;\mathbb{Z})$ U-duality. The relevant Jordan algebra is $\mathbb{Z} \oplus \Gamma_{5,21}$ [247,252] and P and Q are now 28-vectors[5].

In this case we may introduce a further discrete U-duality invariant, the *torsion* [266]:

$$r(P,Q) = \gcd(P_{\mu\nu}), \tag{9.116}$$

where

$$P_{\mu\nu} = P_\mu Q_\nu - P_\nu Q_\mu. \tag{9.117}$$

For primitive P and Q, the complete set of independent T-duality invariants was determined in [268]. It consists of the three familiar invariants P^2, Q^2 and $P\cdot Q$, the torsion $r(P,Q)$ and two further interdependent discrete invariants u_1 and u_2 which are constructed below. If P and Q are not

[3] Answering in the affirmative the question posed in [49]: Can one always assume that a $D = 4$, $\mathcal{N} = 8$ black hole is U-duality related to a configuration with only NS-NS charges present?

[4] This form of the trilinear map also appears in [49].

[5] This case is mentioned in the mathematical literature e.g. [48] but it is not clear how many of the results of section 2 continue to apply. See however [268]. We leave this question for future work

individually primitive there are two additional T-duality invariants given by $\gcd(P)$ and $\gcd(Q)$[6]. Assume P and Q individually primitive and let a, b be two charge vectors satisfying

$$a \cdot Q = 1, \qquad b \cdot P = 1. \tag{9.118}$$

Define

$$\begin{aligned} u_1 &= a \cdot P \mod r(P,Q), \\ u_2 &= b \cdot Q \mod r(P,Q). \end{aligned} \tag{9.119}$$

It was shown in [268] that u_1, u_2, so defined, are independent of the choice of a, b, are T-duality invariant and that u_2 is uniquely determined by u_1 (and vice versa). Any two such dyons are T-duality related if and only if all five invariants have identical values.

Let us consider the action of F-duality on these T-duality invariants. P^2, Q^2 and $P \cdot Q$ are invariant up to a sign determined by the quartic norm,

$$\begin{aligned} \tilde{P}^2 &= \operatorname{sgn}(\Delta) P^2 \\ \tilde{Q}^2 &= \operatorname{sgn}(\Delta) Q^2 \\ \tilde{P} \cdot \tilde{Q} &= \operatorname{sgn}(\Delta) P \cdot Q. \end{aligned} \tag{9.120}$$

Moreover

$$P \cdot \tilde{Q} = -\tilde{P} \cdot Q = \operatorname{sgn}(\Delta)|\Delta|^{1/2}, \tag{9.121}$$

and the quantisation rule is

$$P \cdot \tilde{Q} - \tilde{P} \cdot Q = \operatorname{sgn}(\Delta) 2|\Delta|^{1/2}. \tag{9.122}$$

Note also that

$$\tilde{P}_{\mu\nu} = \operatorname{sgn}(\Delta) P_{\mu\nu} \tag{9.123}$$

and therefore the torsion is also invariant under F-duality.

Clearly, when $d_1(P,Q)$ and $d_3(P,Q)$ are not conserved under F-duality, then neither u_1 nor $\gcd(P), \gcd(Q)$ are preserved . However, in cases when $\gcd(P) = 1$ and $\gcd(Q) = 1$ are in fact conserved under F-duality it is not difficult to verify that $u_1(P,Q)$ is also preserved.

Consequently, two $1/4$-BPS ($\Delta > 0$) F-dual states are T-dual if and only if both $\gcd(P) = 1$ and $\gcd(Q) = 1$ are preserved. On the other hand, non-BPS ($\Delta < 0$) F-dual states cannot be T-duality related. Moreover, since $d_1(P,Q)$ is not necessarily invariant under F-duality, $\gcd(P)$ and $\gcd(Q)$ are not generically invariant.

It is worth emphasising that the F-duality (9.115) is not generically an $\mathrm{SL}(2,\mathbb{Z})$ S-duality, but in certain specific circumstances with Δ positive the two may coincide.

3.5.2. F-dual in Sen basis

Although the canonical basis of section 3.4 is most convenient for our purposes, it is also useful to re-express our results in the basis used by Sen and collaborators [49, 268, 270], which may be more

[6] For the heterotic string we have a complete set of T-duality invariants which uniquely determine the black hole charges up to T-duality. This contrasts with the $\mathcal{N} = 8$ case and its U-duality invariants.

familiar to the black hole community:

$$P = \begin{pmatrix} Q_1 \\ J \\ Q_5 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ n \\ 0 \\ 1 \end{pmatrix}, \tag{9.124}$$

with $Q_5|J, Q_1$. Here, n represents an NS 5-brane winding charge, Q_1 a fundamental string winding charge, while J and Q_5 are units of KK monopole charge associated with two distinct circles of the T^6. In FTS language we have,

$$\begin{pmatrix} \alpha & (A_1, A_2, A_3) \\ (B_1, B_2, B_3) & \beta \end{pmatrix} = \begin{pmatrix} -1 & (n, Q_1, Q_5) \\ (0,0,0) & J \end{pmatrix}. \tag{9.125}$$

We see immediately that x is chosen to be primitive and that we must impose

$$Q_5 \neq 0 \wedge (J \neq 0 \vee nQ_1 \neq 0), \tag{9.126}$$

for x to be a valid rank 4 charge vector. Using the metric

$$\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \tag{9.127}$$

we have

$$Q^2 = 2n, \quad P^2 = 2Q_1Q_5, \quad P \cdot Q = J, \tag{9.128}$$

$$\begin{pmatrix} T_P \\ T_Q \end{pmatrix} = \begin{pmatrix} J & -2Q_1Q_5 \\ 2n & -J \end{pmatrix}, \begin{pmatrix} P \\ Q \end{pmatrix}, \tag{9.129}$$

and

$$\Delta = 4nQ_1Q_5 - J^2. \tag{9.130}$$

The Freudenthal dual is then given by

$$\tilde{P} = |4nQ_1Q_5 - J^2|^{-1/2} \begin{pmatrix} JQ_1 \\ J^2 - 2nQ_1Q_5 \\ JQ_5 \\ -2Q_1Q_5 \end{pmatrix}, \quad \tilde{Q} = |4nQ_1Q_5 - J^2|^{-1/2} \begin{pmatrix} 2nQ_1 \\ nJ \\ 2nQ_5 \\ -J \end{pmatrix}. \tag{9.131}$$

3.6. The 5D Jordan dual

Definition 64 (The Jordan dual)**.** *Given a black string with charges A or black hole with charges B, we define its Jordan dual by*

$$A^\star = A^\sharp N(A)^{-1/3}, \qquad B^\star = B^\sharp N(B)^{-1/3}, \tag{9.132}$$

where we take the real root as implied by the notation.

As described in section 2.2, the Jordan algebra divides black holes and strings into four distinct ranks or orbits. J-duality is initially defined for large rank 3 strings for which both $A^\sharp$ and $N(A)$ are nonzero and large rank 3 holes for which both $B^\sharp$ and $N(B)$ are nonzero. Small black holes and strings are discussed in [20]. An alternative definition of the Jordan dual is presented in Appendix D.

The invariance of $N(A)$ follows by noting that

$$\mathrm{Tr}(A^\sharp, A) = 3N(A), \tag{9.133}$$

where $A^\sharp$ obeys

$$(A^\sharp)^\sharp = N(A)A, \tag{9.134}$$

and hence

$$N(A^\sharp) = N(A)^2. \tag{9.135}$$

So

$$N(A^\star) = N(A^\sharp N(A)^{-1/3}) = N(A). \tag{9.136}$$

Moreover

$$A^{\star\star} = (A^\sharp N(A^\sharp)^{-1/3})^\sharp N(A^\star)^{-1/3} = A. \tag{9.137}$$

Similar results hold for B.

In the case of a black hole and black string related by Jordan duality, the Dirac-Schwinger-Zwanziger quantisation condition (9.36) is given by

$$\mathrm{Tr}(A^\star, A) = 3N(A)^{2/3}, \tag{9.138}$$

which is also invariant. Note the factor of 3.

As noted in section 3.1, for a valid dual $A^\star$, we require that $N(A)$ is a perfect cube. This is a necessary, but not sufficient condition because we further require that

$$d_3(A) = \left[\frac{d_2(A)}{d_1(A^\star)}\right]^3 = \left[\frac{d_2(A^\star)}{d_1(A^)}\right]^3 = d_3(A^\star). \tag{9.139}$$

In the 5D case the canonical reduced diagonal form of (9.37) is unique in the sense that it is unambiguously determined by the U-duality invariants $d_1(A)$, $d_2(A)$ and $N(A)$.

Black holes related by a J-duality not conserving $d_1(A)$ provide examples of configurations with the same cubic norm and hence lowest order entropy that are not U-duality related.

The U-duality integral invariants $\mathrm{Tr}(X, Y)$ and $N(X, Y, Z)$ are not generally invariant under Jordan duality while $\mathrm{Tr}(A^\star, A)$ and $N(A)$, and hence the lowest-order black hole entropy are. However, higher order corrections to the black hole entropy depend on some of the discrete U-duality invariants, to which we now turn.

3.7. The action of J-duality on discrete U-duality invariants

J-duality commutes with U-duality in the sense that $A^\star$ transforms contragredient to A. This follows from the property that a linear transformation s belongs to the norm preserving group if and only if

$$\tau(A) \times \tau(B) =^t \tau^{-1}(A \times B), \tag{9.140}$$

which implies

$$(\tau(A))^\star =^t \tau^{-1}(A^\star). \tag{9.141}$$

As we shall see in the following section, of the discrete invariants listed in (9.18), only the cubic norm $d_3(A)$ is generically preserved under J-duality.

3.8. Smith diagonal form and its dual

We have already seen in section 2.2 that we may write the most general black string charge configuration, up to U-duality, as

$$A = k(1, l, lm), \tag{9.142}$$

where $k, l \geq 0$. In this case

$$A^\sharp = k^2 l(lm, m, 1), \tag{9.143}$$

and

$$N_3(A) = k^3 l^2 m. \tag{9.144}$$

So the Jordan dual black string is given by

$$A^\star = k(l/m)^{1/3}(lm, m, 1). \tag{9.145}$$

Hence, we require $k^3 l = n^3|m|$, $n \in \mathbb{N}$. A generic J-duality pair $A, A^\star$ are given by

$$A = k(1, l, lm), \quad A^\star = n(lm, m, 1), \tag{9.146}$$

so that,

$$\begin{aligned} d_1(A) &= k & d_1(A^\star) &= n \\ d_2(A) &= k^2 l & d_2(A^\star) &= n^2|m| \\ d_3(A) &= k^3 l^2 |m| & d_3(A^\star) &= n^3 m^2 l. \end{aligned} \tag{9.147}$$

So $d_3(A)$ is conserved as expected and so is the product $d_1(A)d_2(A)$ but not $d_1(A)$ and $d_2(A)$ separately, except when $n = k$.

The similar form of A and $A^\star$ when $n = k$ suggests they may be related. In fact they must be related by a U-duality because they have the same d_1, d_2 and d_3.

Note that $N_3(A^\star)^2$ is a perfect cube

$$N_3(A^\star)^2 = (nklm)^3, \tag{9.148}$$

which also implies that $N_3(A)$ is a perfect cube, consistent with the claim in section 3.1.

3.9. Freudenthal/Jordan duality and the 4D/5D lift

3.9.1. The 4D/5D lift

Recent work [51] has established a simple correspondence relating the entropy of 4D BPS black holes in type IIA theory compactified on a Calabi-Yau Y to the entropy of spinning 5D BPS black holes in M-theory compactified on $Y \times TN_\beta$, where TN_β is a Euclidean 4-dimensional Taub-NUT space with NUT charge β. Using this 4D/5D lift the electric black hole charge $\mathcal{Q}$ and spin $\mathcal{J}_\beta$ may be identified with the dyonic charges of the 4D black hole giving a precise relationship between the leading order entropy formulae. This relationship has then been used to count the 4D BPS black hole degeneracies in $\mathcal{N}=8$ string theory [281] exploiting the known results from the analysis of 5-dimensional black holes [49, 174, 265, 267, 271, 281, 283].

This correspondence between the $D=5$ black hole changes $\mathcal{Q}$ and $\mathcal{J}_\beta$ and the $D=4$ electric/magnetic black hole charges is neatly captured in terms of the FTS.

Remark 65 (4D/5D lift: FTS→Jordan algebra)**.** *Under the 4D/5D lift the black string magnetic charge $\mathcal{P}$ and black hole electric charge $\mathcal{Q}$ are defined in terms of the 4D black hole charges as*

$$\mathcal{P}=B^\sharp-\alpha A, \qquad \mathcal{Q}=A^\sharp-\beta B. \tag{9.149}$$

Their corresponding angular momenta are given by

$$\mathcal{J}_\alpha=-\tfrac{1}{2}T_\alpha=\alpha\kappa(x)+N_3(B), \qquad \mathcal{J}_\beta=-\tfrac{1}{2}T_\beta=-\beta\kappa(x)-N_3(A). \tag{9.150}$$

The respective entropies of the rotating 5D black string and black hole are given by

$$S_{5(\text{black string})}=2\pi\sqrt{|N_3(\mathcal{P})-\mathcal{J}_\alpha{}^2|}, \qquad S_{5(\text{black hole})}=2\pi\sqrt{|N_3(\mathcal{Q})-\mathcal{J}_\beta{}^2|}. \tag{9.151}$$

Lemma 66. *With the black string/hole charges and momenta defined as above, the 4D quartic invariant $\Delta(x)$ is given by,*

$$\frac{4}{\alpha^2}\{N_3(\mathcal{P})-\mathcal{J}_\alpha{}^2\}=\frac{4}{\beta^2}\{N_3(\mathcal{Q})-\mathcal{J}_\beta{}^2\}. \tag{9.152}$$

Proof. For strictly Jordan algebraic proof we begin by using the identity $\operatorname{Tr}(X,X^\sharp)=3N_3(X)$ to write

$$3N_3(\alpha A-B^\sharp)=\operatorname{Tr}(\alpha A-B^\sharp,(\alpha A-B^\sharp)^\sharp). \tag{9.153}$$

Then, using $(X+Y)^\sharp=X\times Y+X^\sharp+Y^\sharp$, we have

$$\begin{aligned}3N_3(\alpha A-B^\sharp)&=\operatorname{Tr}(\alpha A-B^\sharp,(-\alpha A)\times B^\sharp+\alpha^2A^\sharp+N_3(B)B)\\&=\operatorname{Tr}(\alpha A-B^\sharp,(-\alpha A)\times B^\sharp)+3\alpha^3N_3(A)\\&\quad-\alpha^2\operatorname{Tr}(A^\sharp,B^\sharp)+\alpha\operatorname{Tr}(A,B)N_3(B)-3N_3(B)^2.\end{aligned} \tag{9.154}$$

Finally, using $\operatorname{Tr}(X,Y\times Z)=6N_3(X,Y,Z)$, which may be derived from the definition of the quadratic adjoint $\operatorname{Tr}(X^\sharp,Y)=3N_3(X,X,Y)$, we see that

$$\begin{aligned}\operatorname{Tr}(\alpha A-B^\sharp,(-\alpha A)\times B^\sharp)&=6N_3(\alpha A-B^\sharp,-\alpha A,B^\sharp)\\&=2[N_3(\alpha A-B^\sharp)+N_3(B^\sharp)-N_3(\alpha A)].\end{aligned} \tag{9.155}$$

Hence, on substituting back into (9.154) one finds

$$\begin{aligned}N_3(\alpha A-B^\sharp)=&\ \alpha^3N_3(A)-\alpha^2\operatorname{Tr}(A^\sharp,B^\sharp)\\&+\alpha\operatorname{Tr}(A,B)N_3(B)-N_3(B)^2,\end{aligned} \tag{9.156}$$

so that

$$\begin{aligned}\Delta(x) &= -\frac{4}{\alpha^2}\big\{[\alpha\kappa + N_3(B)]^2 + \alpha^3 N_3(A) - \alpha^2 \operatorname{Tr}(A^\sharp, B^\sharp) + \alpha \operatorname{Tr}(A,B) N_3(B) - N_3(B)^2\big\} \\ &= \quad \frac{4}{\alpha^2}\{N_3(B^\sharp - \alpha A) - [\alpha\kappa + N_3(B)]^2\},\end{aligned} \tag{9.157}$$

as required. Had we started with $N_3(\mathcal{Q})$ we would have obtained the analogous black hole equation. □

Hence, we find that the 4D and 5D entropies are related as follows,

$$S_4 = \frac{1}{\alpha} S_{5(\text{black string})} = \frac{1}{\beta} S_{5(\text{black hole})}. \tag{9.158}$$

3.9.2. F-duality→J-duality under the 4D/5D lift

We recall that a black hole can be put into reduced form:

$$x = \begin{pmatrix} \alpha & A \\ 0 & \beta \end{pmatrix}. \tag{9.159}$$

We now show that for these five parameter black holes the lift of the Freudenthal dual is related to the Jordan dual. For the black hole in (9.159) we have

$$\Delta(x) = -\alpha^2\beta^2 - 4\alpha N_3(A), \quad T(x) = \begin{pmatrix} -\alpha^2\beta & \alpha\beta A \\ 2\alpha A^\sharp & \alpha\beta^2 + 2N_3(A) \end{pmatrix}. \tag{9.160}$$

The 5D black hole/string charges are given by

$$\mathcal{P}(x) = -\alpha A, \quad \mathcal{Q}(x) = A^\sharp, \tag{9.161}$$

with the following norms

$$N_3(\mathcal{P}(x)) = -\alpha^3 N_3(A), \quad N_3(\mathcal{Q}(x)) = N_3(A)^2, \tag{9.162}$$

and angular momenta

$$\begin{aligned}\mathcal{J}_\alpha &= -\tfrac{1}{2}T_\alpha = \quad \tfrac{1}{2}\alpha^2\beta, \\ \mathcal{J}_\beta &= -\tfrac{1}{2}T_\beta = -\tfrac{1}{2}\alpha\beta^2 - N_3(A).\end{aligned} \tag{9.163}$$

The Freudenthal dual of x is given by

$$\tilde{x} = \begin{pmatrix} \tilde{\alpha} & \tilde{A} \\ \tilde{B} & \tilde{\beta} \end{pmatrix} = \frac{1}{|\Delta|^{1/2}} \begin{pmatrix} -\alpha^2\beta & \alpha\beta A \\ 2\alpha A^\sharp & \alpha\beta^2 + 2N_3(A) \end{pmatrix} \tag{9.164}$$

and

$$\tilde{A}^\sharp = \frac{\alpha^2\beta^2 A^\sharp}{|\Delta|}, \qquad \tilde{B}^\sharp = \frac{4\alpha^2 N_3(A) A}{|\Delta|}. \tag{9.165}$$

Hence, we find the following $\mathcal{P}(\tilde{x})$ and $\mathcal{Q}(\tilde{x})$

$$\begin{aligned}\mathcal{P}(\tilde{x}) &= \tilde{B}^\sharp - \tilde{\alpha}\tilde{A}\\ &= \frac{4\alpha^2 N_3(A)A}{|\Delta|} - \frac{-\alpha^2\beta}{|\Delta|^{1/2}}\cdot\frac{\alpha\beta A}{|\Delta|^{1/2}}\\ &= \alpha\frac{(\alpha^2\beta^2 + 4\alpha N_3(A)}{|\Delta|}A\\ &= -\operatorname{sgn}(\Delta)\alpha A,\end{aligned} \tag{9.166}$$

and

$$\begin{aligned}\mathcal{Q}(\tilde{x}) &= \tilde{A}^\sharp - \tilde{\beta}\tilde{B}\\ &= \frac{\alpha^2\beta^2 A^\sharp}{|\Delta|} - \frac{\alpha\beta^2 + 2N_3(A)}{|\Delta|^{1/2}}\cdot\frac{2\alpha A^\sharp}{|\Delta|^{1/2}}\\ &= -\frac{(\alpha^2\beta^2 + 4\alpha N_3(A)}{|\Delta|}A^\sharp\\ &= \operatorname{sgn}(\Delta)A^\sharp.\end{aligned} \tag{9.167}$$

Hence

$$\mathcal{P}(\tilde{x}) = \operatorname{sgn}(\Delta)\mathcal{P}(x), \qquad \mathcal{Q}(\tilde{x}) = \operatorname{sgn}(\Delta)\mathcal{Q}(x), \tag{9.168}$$

as expected from (9.85) and (9.86). Similarly we find

$$\mathcal{J}_\alpha(\tilde{x}) = |\Delta|^{1/2}\alpha, \qquad \mathcal{J}_\beta(\tilde{x}) = |\Delta|^{1/2}\beta, \tag{9.169}$$

so that

$$\Delta(\tilde{x}) = |\Delta|\mathcal{J}_\alpha{}^{-2}\left(4\operatorname{sgn}(\Delta)N(\mathcal{P}) - |\Delta|\alpha^2\right) = \Delta(x). \tag{9.170}$$

Now, if we take the Jordan duals of $\mathcal{P}(x)$ and $\mathcal{Q}(x)$, we have

$$\mathcal{P}^\star(x) = \frac{\mathcal{P}(x)^\sharp}{N_3(\mathcal{P}(x))^{1/3}}, \qquad \mathcal{Q}^\star(x) = \frac{\mathcal{Q}(x)^\sharp}{N_3(\mathcal{Q}(x))^{1/3}}. \tag{9.171}$$

We can calculate $\mathcal{P}^\sharp$ and $\mathcal{Q}^\sharp$ from (9.161), for which we get $\mathcal{P}^\sharp = \alpha^2 A^\sharp$ and $\mathcal{Q}^\sharp = N_3(A)A$, we already know $N_3(\mathcal{P})$ and $N_3(\mathcal{Q})$ from (9.162), so that we now have

$$\begin{aligned}\mathcal{P}^\star(x) &= \frac{\alpha^2 A^\sharp}{(-\alpha^3 N_3(A))^{1/3}} & \mathcal{Q}^\star(x) &= \frac{N_3(A)A}{N_3(A)^{2/3}}\\ &= -\frac{\alpha}{N_3(A)^{1/3}}A^\sharp, & &= N_3(A)^{1/3}A,\end{aligned} \tag{9.172}$$

with norms

$$N_3(\mathcal{P}^\star) = -\alpha^3 N_3(A) \qquad\qquad N_3(\mathcal{Q}^\star) = N_3(A)^2. \tag{9.173}$$

Putting all this together, we find

$$\begin{aligned}\hat{\mathcal{P}}^\star(x) &= \hat{\mathcal{Q}}(\tilde{x}),\\ \hat{\mathcal{Q}}^\star(x) &= \hat{\mathcal{P}}(\tilde{x}),\end{aligned} \tag{9.174}$$

where the hat denotes an element with the unit norm;

$$\hat{X} = \frac{X}{N_3(x)^{1/3}}, \qquad\qquad N_3(\hat{X}) = 1. \tag{9.175}$$

Thus we have established

$$\begin{array}{ccc} \text{4D black hole } x & \xrightarrow{\text{4D/5D lift}} & \text{5D black string } A \sim \tilde{B}^\star \\ {\scriptstyle\text{Freudenthal dual}}\Big\downarrow & & \Big\downarrow{\scriptstyle\text{Jordan dual}} \\ \text{dual 4D black hole } \tilde{x} & \xrightarrow[\text{4D/5D lift}]{} & \text{dual 5D black hole } \tilde{B} \sim A^\star \end{array} \tag{9.176}$$

3.10. Conclusions

3.10.1. $\mathcal{N} = 8$:

In the subcases where $d_1(x)$ is conserved, F-duality $x \to \tilde{x}$ preserves all the U-duality invariants (9.23). The degeneracy formula for the class of black holes considered in [49] depends explicitly on only $\Delta(x)$ and $d_5(x)$ and therefore the exact entropy in this case is F-dual invariant. The more general case remains an open question since we are not aware of a general U-duality invariant expression for dyon degeneracies.

In the projective case, this result is somewhat trivial because all black holes are U-duality related and so, in particular, the F-dual $\tilde{x}$ is U-dual equivalent to x.

In the non-projective case, this result seems non-trivial because we are not aware of any argument that would indicate that the F-dual $\tilde{x}$ is U-dual equivalent to x. For example,

$$\begin{aligned} x &= \alpha \begin{pmatrix} 1 & (0,0,0) \\ (0,0,0) & j \end{pmatrix}, \\ \tilde{x} &= \alpha \begin{pmatrix} 1 & (0,0,0) \\ (0,0,0) & -j \end{pmatrix}. \end{aligned} \tag{9.177}$$

Without a complete orbit classification the U-equivalence, or not, of F-dual black holes is a difficult question to answer in general. Even with a full orbit classification the invariance of the higher-order corrections to the entropy would remain unsettled as we cannot be sure on which invariants they depend. Could there be black holes with the same precision entropy that are not U-duality related but are F-duality related?

In the subcases where $d_1(x)$ is not conserved, we can be absolutely sure that the F-dual $\tilde{x}$ is not U-dual equivalent to x. In this case, however, we do not know whether F-duality leaves higher order corrections invariant because all the treatments of higher-order corrections we are aware of are restricted to $d_1(x) = 1$.

These 4D conclusions, and the simpler 5D ones, are summarised in Table 9.2.

$\mathcal{N} = 4$, heterotic:

F-duality $x \to \tilde{x}$ leaves invariant Δ and (up to a sign) P^2, Q^2 and $P \cdot Q$. Moreover, the discrete torsion $r(P, Q)$ is invariant. This result seems non-trivial because we are not aware of any argument that would indicate that the F-dual $\tilde{x}$ is T-dual equivalent to x. In the cases where P^2, Q^2 and $P \cdot Q$

Table 9.2.: Are F or J duals related by U-duality?

Duality	d_1 conserved ?		U-dual ?
F-dual	Yes	Projective	Yes
		Non-projective	?
F-dual	No		No
J-dual	Yes		Yes
J-dual	No		No

flip sign, we can be absolutely sure that the F-dual $\tilde{x}$ is not T-dual equivalent to x. This corresponds specifically to non-BPS black holes and, hence, the conjectured counting formula for all 1/4-BPS dyons is not applicable. However, it is perhaps encouraging that torsion is left invariant as it plays a central role in the current $\mathcal{N} = 4$ dyon degeneracy calculations [269].

In the subcases where $d_1(x)$ is not conserved, we can be absolutely sure that the F-dual $\tilde{x}$ is not U-dual equivalent to x. In this case, however, we do not know whether F-duality leaves higher order corrections invariant because all the treatments of higher-order corrections we are aware of are restricted to $d_1(x) = 1$. This restriction is typically imposed to avoid complications arising from the possibility that dyons with $d_1(x) > 1$ may decay into single particle states. The consequences of this phenomenon for F-dual black holes remains an open question.

$\mathcal{N} = 2$**, magic:**

The magic $\mathcal{N} = 2$ black holes may require a separate analysis since the diagonally reduced form, central to our present treatment, is not necessarily applicable in these instances. In particular, for the octonionic $\mathcal{N} = 2$ example (as opposed to the *split*-octonionic $\mathcal{N} = 8$ case) it is well know that there are integral Jordan algebra elements that cannot be diagonalised [67, 275, 276].

Further work

For the time being the microscopic stringy interpretation of F-duality remains unclear. In part, this is due to the F-duality action only being defined on the black hole charges and not the component fields of the lowest order action.

The 4D/5D lift is also in some sense unsatisfactory in its current form, in part, because the 5D angular momentum is a J-dual singlet, but transforms under F-duality.

Finally, having specified the necessary and sufficient conditions (9.106) required for a well defined F-dual charge vector, one might ask how this space of black holes is mathematically characterised and whether it has a broader significance.

CHAPTER 10

Conclusions

We have given an overview of the intriguing correspondence relating black hole entropy and entanglement. Along the way many subjects have been touched upon:

- The relationship between black hole entropy in the STU model and the entanglement of three qubits.
- The generalisation to $\mathcal{N} = 8$ supergravity which admits a QI interpretation, via the Fano plane, in terms of the tripartite entanglement of seven qubits.
- The microscopic interpretation of black holes as qubits in terms of wrapped D-branes.
- The classification of 4-qubit entanglement from the STU model in three dimensions.

However, we have concentrated on the role of the octonions, Jordan algebras and the Freudenthal triple system in M-theory, QI and their interrelation. In particular:

- We have elucidated the role of the imaginary octonions in relating the tripartite entanglement of seven qubit to the black holes of $\mathcal{N} = 8$ supergravity.
- We provided an elegant and manifestly SLOCC-covariant classification of 3-qubit entanglement using the Freudenthal triple system of the STU model. We, moreover, generalised this FTS to an arbitrary number of qubits.
- We introduced a new Freudenthal duality of 4-dimensional black holes with quantized charges. This relied on the analysis of the discrete U-duality orbits for which we used the integral Freudenthal triple system. Three important consequences are that (1) the leading order black hole entropy is quantized (2) every black hole state is U-duality related to one with no R-R charges (3) the 1/2-, 1/4- and 1/8-BPS dyon degeneracy formulae are defined separately by the arithmetic U-duality invariants.

There are clearly many open questions and possible lines for future research. Before considering the more general context of black holes and qubits at large, let us first discuss the possible directions stemming from the algebraic perspective we have adopted here. For example:

- The appearance of the Fano plane and, in particular, the fact that truncating to its lines and points corresponded to truncating $\mathcal{N} = 8$ supergravity to its NS-NS and STU subsectors, respectively, led us to speculate in [18] that these theories may be intimately related by the division algebras in a manner not yet appearing in the literature. This would appear to be true, however, while the algebras are correctly identified, the candidate theories to be related have changed. This is, however, still rather speculative and further work needs to be done before we can be sure of these claims.

  We also demonstrated that the symmetries of the Fano plane form a subset of the Weyl transformations which do not mix electric and magnetic charges. This is the $\mathcal{N} = 8$ analog of the $\mathbb{Z}_2$ electric-magnetic duality of Maxwell theory. However, the significance of the Fano transformations remains mysterious. One would anticipate that they are related to the supersymmetric configurations of seven intersecting D-branes, which are themselves determined by the Fano plane. A second interesting, but seemingly unrelated question also arises from the Fano plane symmetries. We have recently shown that the Fano symmetries may be used to construct new Joyce manifolds. While still work in progress, such manifolds potentially have interesting phenomenological applications.

- Several features of our proposed n-qubit generalisation of the FTS need development. Principally, we are lacking a systematic generalisation of the FTS rank, a vital tool in the entanglement classification of three qubits. Similarly, it is not clear how to systematically generate SLOCC invariants without first appealing to the conventional $[\mathrm{SL}(2, \mathbb{C})]^n$ theory and then working backwards. Both these issues must be addressed before we can really assess the utility of the n-qubit FTS.

  Ultimately, one would like to have a general (perhaps inductive) classification of an arbitrary number of qubits. It is our feeling that this would be best approached from the perspective of algebraic geometry. However, it is hard to get a handle on the problem, since we currently only have one non-trivial data point, the entanglement of three qubits. It is our hope that the FTS formulation will be of use in both the short term goal of classifying small numbers of qubits, as well as the more ambitious task of determining a general entanglement classification scheme.

- While Freudenthal duality is non-trivial, it is, in a certain sense, "low-level". While it has a non-trivial action on the black hole charges, we do not know if and how it should act on the other fields of the theory. It is also not clear whether it is a symmetry of the entropy in the full theory, or whether it is only valid at the low-energy effective description of two derivative supergravity. This brings us to a related set of open questions.

- In order to determine whether F-duality preserves the $\mathcal{N} = 8$ black hole entropy exactly, one needs to know the full manifestly $E_{7(7)}(\mathbb{Z})$ invariant dyon degeneracy formulae. It has recently been observed that some of the orbits of $E_{7(7)}(\mathbb{Z})$ should play an important role in these counting formulae [272]. However, we do not yet have a general classification of these orbits for 1/8- or non-BPS states. It is our contention that such a classification is possible in the context of the integral FTS. The basic idea is to use the diagonally reduced 5-charge canonical form together with the complete set of algebraically independent arithmetic U-duality invariants to show that any state is uniquely determined up to U-duality by the said invariants. For example, we have shown that the $\gcd$ of the charges, the quadratic adjoint and the cubic norm are the complete set of algebraically independent invariants for $D = 5, \mathcal{N} = 8$ and that they do indeed uniquely specify the state up to U-duality.

The first task is the construction of the invariants evaluated at the canonical form. This becomes increasingly complicated as one includes more powers of the charges. Even at quadratic order one needs to consider the 133- and 1463-dimensional representations. However, one feasible approach is to use the truncation to the STU model for which the explicit forms of the invariants are tractable. This is already work in progress.

Such a classification might facilitate the derivation of a more general dyon degeneracy formula than the one presented in [49], which requires $d'_2 = 1$.

A second interesting spin-off is the possibility that such a classification has applications to theory of higher composition laws. See [50].

There are many more possibilities besides these few, but let us rather conclude with some future directions in the more general context. One might consider, for example:

Wrapped branes: Although there are no black holes with non-zero entropy in $D \leq 6$ dimensions, there are black strings and other intersecting brane configurations with entropies given by U-duality invariants. Do they have a qubit interpretation? In fact, we can relate supersymmetric configurations of intersecting wrapped M-branes to systems of one through sixteen qubits by compacitifying in steps from $D = 11$ down to $D = 3$ (or from $D = 10$ to $D = 2$ using D-branes). However, one loses the attractive entropy-entanglement correspondence as these configurations are not necessarily asymptotically flat. There is, however, good evidence that the supersymmetric configurations will correspond to representative states of the entanglement classes. This question cannot be answered without the covariant classifications (analogous to the FTS classification of three qubits) for more than three qubits.

3D black holes: It is natural to ask whether the techniques, based on nilpotent orbits and the Kostant-Sekiguchi theorem, which provided the classification of 4-qubit entanglement from $D = 3$ STU black holes, could be used for other QI systems. Compactifying the $\mathcal{N} = 8$ theory to $D = 3$, we ought to obtain the entanglement classification of the "four-way entanglement of eight qubits" [18],

$$\begin{aligned} |\Psi\rangle_{224} = \quad & a_{HABD}|HAB\bullet D\bullet\bullet\bullet\rangle + \tilde{a}_{CEFG}|\bullet\bullet\bullet C\bullet EFG\rangle \\ &+ b_{HBCE}|H\bullet BC\bullet E\bullet\bullet\rangle + \tilde{b}_{DFGA}|\bullet A\bullet\bullet D\bullet FG\rangle \\ &+ c_{HCDF}|H\bullet\bullet CD\bullet F\bullet\rangle + \tilde{c}_{EGAB}|\bullet AB\bullet\bullet E\bullet G\rangle \\ &+ d_{HDEG}|H\bullet\bullet\bullet DE\bullet G\rangle + \tilde{d}_{FABC}|\bullet ABC\bullet\bullet F\bullet\rangle \\ &+ e_{HEFA}|HA\bullet\bullet\bullet EF\bullet\rangle + \tilde{e}_{GBCD}|\bullet\bullet BCD\bullet\bullet\bullet\rangle \\ &+ f_{HFGB}|H\bullet B\bullet\bullet\bullet FG\rangle + \tilde{f}_{ACDE}|\bullet A\bullet CDE\bullet\bullet\rangle \\ &+ g_{HGAC}|HA\bullet C\bullet\bullet\bullet G\rangle + \tilde{g}_{BDEF}|\bullet\bullet B\bullet DEF\bullet\rangle. \end{aligned} \tag{10.1}$$

The 112 are associated with quadrangles of the Fano plane and the other 112 with the quadrangles of the dual Fano plane. The relevance of this state derives from the fact that it may be assigned to the coset $E_{8(8)}/[\mathrm{SL}(2,\mathbb{R})]^8$ [18] so that Kostant-Sekiguchi theorem applies.

Superqubits: In addition to its mathematical naturalness, the superqubit has some potential relevance to certain condensed matter systems, such as the supersymmetric t-J model [219–223], which deserves further investigation. A physical realisation of the superqubit would certainly be remarkable. It would also be interesting to see just how much of conventional QI could be cast in the supersymmetric framework.

We hope that the reader is convinced that this mathematical duality relating black holes and qubits, while lacking a physical basis, is interesting, fruitful and may yet provide crucial insights into the deep structure of both M-theory and entanglement.

APPENDIX A

The Jordan algebra formulation of quantum mechanics

Taken alone, it is not entirely clear that the Jordan identity (5.17) ought to be of any fundamental importance when representing the algebra of observables. To better understand its physical relevance let us consider what is expected of such a representation starting from the conventional matrix theory. We follow closely the presentation given in [287].

Observables are represented by Hermitian matrices, possibly of infinite dimension. Central to their particular suitability in this role is that their eigenvalues are real, that distinct eigenvalues correspond to distinct orthogonal eigenvectors and that they are formally real in the sense of (5.18). These properties, coupled with the spectral decomposition theorem, imply that for any polynomial function F and observable A,

$$F(A) = \sum_m F(a_m)P_m, \quad \text{for} \quad A = \sum_m a_m P_m, \tag{A.1}$$

where P_m and a_m are the projectors and their associated eigenvalues, respectively, appearing in the spectral decomposition of A. The physical significance of this statement is that, if a is the value of some observable A, energy say, for some system in a given state, then one would expect that $F(a)$ would be the value of $F(A)$, energy squared say, for the same state. However, this relies crucially on the power associativity (5.19) of Hermitian matrices; $A^mA^n = A^{(m+n)}$ ensures that $F(A)$ is defined unambiguously.

Now, given a commutative, formally real algebra, the Jordan identity then *follows* from the physically well motivated assumption of power associativity [239]. Equally, given the same initial assumptions, the Jordan identity implies power associativity. However, Hermitian matrices do not in general commute. Consequently, the matrix product of two Hermitian matrices does not necessarily yield a third Hermitian matrix. They do not form a closed algebra under standard matrix multiplication. These considerations, in part, motivate the definition of the Jordan product on Hermitian matrices,

$$A \circ B = \tfrac{1}{2}(AB + BA), \tag{A.2}$$

which is by definition commutative (but nonassociative) and closed with respect to Hermiticity. The algebra of Hermitian matrices, with multiplicative composition defined by (A.2), is closed, formally real, commutative, power associative and, consequently, satisfies the Jordan identity (5.17). In this particular case, these properties simply amount to a set of identities. However, in seeking generalisations, one could make the paradigm shift and take the Jordan identity as primary, the Jordan product emerging as a secondary consequence. That is, we start from a Jordan algebra and require that, in addition, it be formally real and contain an identity element so as to ensure its suitability as an algebra of observables [287]. In summary, we axiomatise the algebra of observables $\mathfrak{A}$:

1. $x \circ y = y \circ x$,
2. $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$,
3. $x^2 + y^2 + z^2 + \cdots = 0 \qquad \Longrightarrow \qquad x = y = z = \cdots = 0$,
4. $\exists\, \mathbb{1} \in \mathfrak{A}$ s.t. $\mathbb{1} \circ x = x \circ \mathbb{1} = x \quad \forall x \in \mathfrak{A}$.

Having, to some degree at least, set an axiomatic foundation for the algebra of quantum observables, it is natural to ask what generalisations, beyond the orthodox framework, this allows for. This question was essentially answered by the classification of all simple formally real Jordan algebras [239] (see section 2.1). However, before commenting on the possible alternatives let us address a more immediate issue; how the Jordan formulation captures the statistics of standard quantum mechanics, an obvious minimal prerequisite.

Having articulated the space of quantum observables in terms of Jordan algebras let us now turn our attention to the representation of states and the problem of time evolution. States in quantum theory are represented by rays in a Hilbert space. However, given an orthonormal basis, $\{|m\rangle\}$ the projection operators,

$$P_m = |m\rangle\langle m|, \tag{A.3}$$

which satisfy,

$$\operatorname{tr} P_m = 1, \tag{A.4}$$

and

$$P_m^2 = P_m, \tag{A.5}$$

correspond to an equivalent representation. Any normalised pure state $|\psi\rangle$ may be expressed as a projector, $P_\psi = |\psi\rangle\langle\psi|$, satisfying (A.4) and (A.5). The expectation value of an observable, represented by a Hermitian matrix A, is then given by,

$$\langle A\rangle_\psi = \operatorname{tr}(P_\psi A). \tag{A.6}$$

More generally, any state, pure or mixed, may be represented as a Hermitian, trace one, positive semi-definite density matrix ρ in which case,

$$\langle A\rangle_\rho = \operatorname{tr}(\rho A). \tag{A.7}$$

To reproduce these results using the Jordan framework it is necessary to introduce a trace form [287, 288],

$$\operatorname{tr} \mathbb{1} = \nu, \quad \operatorname{tr} b_i = 0, \tag{A.8}$$

where $\nu \in \mathbb{Z}$ and the set $\{b_i\}$ forms a basis for the Jordan algebra such that any element a may be written as $a = a^i b_i$, $a^i \in \mathbb{R}$. This defines a positive definite bilinear inner product,

$$\langle b_i, b_j \rangle = \tfrac{1}{\nu} \operatorname{tr}(b_i \circ b_j) = \delta_{ij}, \tag{A.9}$$

since one can always choose a basis such that $b_i \circ b_j = \delta_{ij}\mathbb{1} + f_{ijk} b_k$. Any two idempotents[1], P_1 and P_2, are orthogonal, $P_1 \circ P_2 = 0$, if they are orthogonal with respect to the inner product (A.9). A general idempotent E is said to be *irreducible* or *primitive* if it cannot be decomposed as the sum of two orthogonal idempotents. The highest possible number of orthogonal primitive idempotents is the *degree* of the algebra and is equal to the ν appearing in the trace form if one normalises $\operatorname{tr} E = 1$, c.f. (A.4). Any complete set $\{E_i\}$ of orthogonal primitive idempotents satisfies,

$$\sum_{i=1}^{\nu} E_i = \mathbb{1}, \tag{A.10}$$

constituting a resolution of the identity. Then, in analogy with the spectral decomposition theorem, any element a in the algebra can be expressed as a linear sum of primitive idempotents,

$$a = \sum_{i=1}^{\nu} a^i E_i(a), \tag{A.11}$$

where the maximal set $\{E_i(a)\}$ depends on the particular element a under question.

It is now possible to represent an arbitrary state ρ in the Jordan formulation,

$$\rho = \sum_{i=1}^{\nu} p_i E_i, \quad \text{where} \quad \sum_{i=1}^{\nu} p_i = 1 \quad \text{and} \quad p_i \in [0, 1]. \tag{A.12}$$

This clearly relates to the density matrix formalism of conventional quantum mechanics: ρ is a positive semi-definite, trace one element that satisfies $\rho^2 \leq \rho$ with equality holding only for pure states i.e. when ρ is a primitive idempotent [287]. The expectation value of an observable a in the Jordan algebra with respect to a state ρ is then given by,

$$\langle a \rangle_\rho = \operatorname{tr}(a \circ \rho). \tag{A.13}$$

Hence, this Jordan algebra formulation is essentially equivalent to the density matrix picture of the conventional quantum mechanics [287].

Let us now consider time evolution. Here the Jordan formalism does depart, to some extent, from the standard density matrix picture. If we assume that the affine structure of a general density matrix is preserved by time evolution then,

$$\partial_t \rho(t) = -i[H, \rho(t)], \tag{A.14}$$

where H is the Hamiltonian. An important feature of (A.14) is that it maps pure states into pure

[1] Any algebra element P is said to be idempotent if $P^2 = P$.

states. Given two Hermitian matrices, A and B, we have,

$$\begin{aligned}\partial_t(AB) &= -i[H, AB] \\ &= -i([H,A]B + A[H,B]) \\ &= (\partial_t A)B + A(\partial_t B).\end{aligned} \tag{A.15}$$

The differential evolution operator acts as a derivation as one would expect. Requiring this condition in the Jordan formulation, so that for any two elements x and y

$$\partial_t(x \circ y) = D(x \circ y) = x \circ D(y) + D(x) \circ y = x \circ \partial_t(y) + \partial_t(x) \circ y, \tag{A.16}$$

the differential evolution operator D acts as a derivation of the Jordan algebra. The set of derivations generates the automorphism group of the algebra. Hence, the corresponding time translation operator, $T_{t_1 \to t_2}$, taking an element, x_{t_1}, at time t_1, to the corresponding element, x_{t_2}, at time t_2, preserves the Jordan product. If

$$x_{t_1} \circ y_{t_1} = z_{t_1}, \tag{A.17}$$

then

$$T_{t_1 \to t_2}(x_{t_1}) \circ T_{t_1 \to t_2}(y_{t_1}) = T_{t_1 \to t_2}(z_{t_1}) \quad \text{or, equivalently} \quad x_{t_2} \circ y_{t_2} = z_{t_2}. \tag{A.18}$$

Consequently, $T_{t_1 \to t_2}$ takes pure states, represented by primitive idempotents E_{t_1}, into pure states, as can be seen from,

$$\begin{aligned}E_{t_1} \circ E_{t_1} = E_{t_1} &\implies T_{t_1 \to t_2}(E_{t_1}) \circ T_{t_1 \to t_2}(E_{t_1}) = T_{t_1 \to t_2}(E_{t_1}) \\ &\implies E_{t_2} \circ E_{t_2} = E_{t_2}.\end{aligned} \tag{A.19}$$

Remarkably, any derivation $D(z)$ can be expressed as,

$$D(z) = D_{x,y}(z) = x \circ (z \circ y) - (x \circ z) \circ y, \tag{A.20}$$

where x and y are any two traceless elements. Recalling the definition of the associator, this implies that the evolution of any (mixed or pure) state is given by,

$$\partial_t \rho = [x, \rho, y]. \tag{A.21}$$

It would seem that, in the Jordan formalism, the associator plays a role equivalent to that of the commutator in the standard picture. Let us consider the case closest to conventional quantum mechanics, the Jordan algebra of $n \times n$ Hermitian matrices defined over $\mathbb{C}$. In this case (A.21) reduces to,

$$\partial_t \rho = [[x, y], \rho]. \tag{A.22}$$

If $H = i[x, y] + \lambda \mathbb{1}$ then $[[x, y], \rho] = -i[H, \rho]$ and (A.22) reproduces the conventional unitary evolution equation (A.14). It is as if the Jordan formulation is, in words of [287], the "square root" of standard theory.

In light of this relationship between the Jordan formulation and the density matrix formalism, the simple Jordan algebras of $n \times n$ Hermitian matrices over the associative division algebras seem to offer an obvious generalisation. However, it would seem that they simply amount to conventional quantum theory defined over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ [289]. This leaves the exceptional octonionic example. In this case a Hilbert space formulation is not possible due to the nonassociative nature of the octonions. It has, however, been shown that the usual axioms of quantum theory may be satisfied by taking a more abstract "propositional" approach [234, 287, 288, 290, 291].

APPENDIX **B**

More on Jordan algebras

B.1. Quadratic Jordan algebras

A *quadratic* form[1] N_2 on a vector space V defined over a field $\mathbb{F}$ is a homogeneous mapping from V to $\mathbb{F}$ of degree 2,

$$N_2 : V \to \mathbb{F} \quad \text{s.t.} \quad N_2(\alpha A) = \alpha^2 N_2(A) \quad \forall \alpha \in \mathbb{F}, A \in V, \tag{B.1}$$

such that its linearization,

$$N_2(A, B) := N_2(A + B) - N_2(A) - N_2(B) \tag{B.2}$$

is bilinear. A *base point* is then defined as an element $c \in V$ satisfying $N_2(c) = 1$. Given a space equipped with a quadratic form and possessing a base point we can define the *trace* form,

$$\mathrm{Tr}(A) := N_2(A, c). \tag{B.3}$$

A quadratic Jordan algebra $\mathfrak{J}_2$ may be derived from such a space by setting the identity $\mathbb{1} = c$ and defining the Jordan product as,

$$A \circ B := \frac{1}{2}(\mathrm{Tr}(A)B + \mathrm{Tr}(B)A - N_2(A, B)\mathbb{1}). \tag{B.4}$$

On setting $A = B$ one obtains

$$A^2 - \mathrm{Tr}(A)A + N_2(A)\mathbb{1} = 0, \qquad \forall A \in \mathfrak{J}_2 \tag{B.5}$$

and $\mathfrak{J}_2$ is said to be of *degree 2* [68]. Moreover, on taking the trace of (B.5) one finds,

$$N_2(A) = \frac{1}{2}[\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2)], \tag{B.6}$$

which is suggestively the form of the determinant of a 2×2 matrix written in terms of the trace of powers and powers of the trace.

[1] We avoid using the conventional notation Q for the quadratic form due to the plethora of Q's representing electric charges.

There are three groups of particular importance associated with such quadratic Jordan algebras:

1. The *automorphism* group $\mathrm{Aut}(\mathfrak{J}_2)$ defined by the set of invertible $\mathbb{F}$-linear transformations τ preserving the Jordan product,

$$\tau(A \circ B) = \tau(A) \circ \tau(B). \tag{B.7}$$

   The corresponding Lie algebra is given by the set of *derivations* $\mathfrak{der}(\mathfrak{J}_2)$,

$$D(A \circ B) = D(A) \circ B + A \circ D(B), \qquad \forall D \in \mathfrak{der}(\mathfrak{J}_2). \tag{B.8}$$

2. The *structure* group $\mathrm{Str}(\mathfrak{J}_2)$ defined by the set of invertible $\mathbb{F}$-linear transformations τ preserving the quadratic norm up to a scalar factor,

$$N_2(\tau(A)) = \alpha N_2(A), \qquad \alpha \in \mathbb{F}. \tag{B.9}$$

   The corresponding Lie algebra $\mathfrak{Str}(\mathfrak{J}_2)$ is given by,

$$\mathfrak{Str}(\mathfrak{J}_2) = L(\mathfrak{J}_2) \oplus \mathfrak{der}(\mathfrak{J}_2), \tag{B.10}$$

   where $L(\mathfrak{J}_2)$ denotes the set of left Jordan products $L_A(B) = A \circ B$.

3. The *reduced structure* group $\mathrm{Str}_0(\mathfrak{J}_2)$ defined by the set of invertible $\mathbb{F}$-linear transformations τ preserving the quadratic norm,

$$N_2(\tau(A)) = N_2(A). \tag{B.11}$$

   The corresponding Lie algebra $\mathfrak{Str}_0(\mathfrak{J}_2)$ is given by factoring out scalar multiples of the identity in $L(\mathfrak{J}_2)$,

$$\mathfrak{Str}_0(\mathfrak{J}_2) = L'(\mathfrak{J}_2) \oplus \mathfrak{der}(\mathfrak{J}_2), \tag{B.12}$$

   where $L'(\mathfrak{J}_2)$ denotes the set of left Jordan products by traceless elements, $L_A(B) = A \circ B$ where $\mathrm{Tr}(A) = 0$.

A $\mathrm{Str}_0(\mathfrak{J}_2)$ invariant rank may be assigned to elements in $\mathfrak{J}_2$,

$$\begin{aligned} &\mathit{Rank} A = 1 \Leftrightarrow N_2(A) = 0,\ A \neq 0;\\ &\mathit{Rank} A = 2 \Leftrightarrow N_2(A) \neq 0. \end{aligned} \tag{B.13}$$

B.2. Quartic Jordan algebras

Let V be a vector space equipped with a quartic norm, i.e. a homogeneous map of degree 4

$$N_4 : V \to \mathbb{F}, \quad \text{s.t.} \quad N_4(\alpha X) = \lambda^4 N_4(X), \quad \forall \alpha \in \mathbb{F}, X \in V \tag{B.14}$$

such that its linearization,

$$\begin{aligned}24N_4(X,Y,Z,W) :=& N_4(X+Y+Z+W)\\ &- N_4(X+Y+Z) - N_4(X+Y+W)\\ &- N_4(X+Z+W) - N_4(Y+Z+W)\\ &+ N_4(X+Y) + N_4(X+Z) + N_4(X+W)\\ &+ N_4(Y+Z) + N_4(Y+W) + N_4(Z+W)\\ &- N_4(X) - N_4(Y) - N_4(Z) - N_4(W)\end{aligned} \tag{B.15}$$

is quadrilinear. If V further contains a base point $N_4(c) = 1, c \in V$ one may define the following seven maps,

1. The trace,
$$\operatorname{Tr}(X) = 4N_4(c,c,c,X), \tag{B.16a}$$

2. A quadratic map,
$$S(X) = 6N_4(X,X,c,c), \tag{B.16b}$$

3. A bilinear map,
$$S(X,Y) = 12N_4(X,Y,c,c), \tag{B.16c}$$

4. A trace bilinear form,
$$\operatorname{Tr}(X,Y) = \operatorname{Tr}(X)\operatorname{Tr}(Y) - S(X,Y). \tag{B.16d}$$

5. A cubic map,
$$T(X) = 4N_4(X,X,X,c), \tag{B.16e}$$

6. A trilinear map,
$$T(X,Y,Z) = 12N_4(X,Y,Z,c), \tag{B.16f}$$

7. A trace trilinear form,

$$\begin{aligned}\operatorname{Tr}(X,Y,Z) =& \operatorname{Tr}(X)\operatorname{Tr}(Y)\operatorname{Tr}(Z)\\ &- 6S(X,Y)\operatorname{Tr}(Z) - 6S(X,Z)\operatorname{Tr}(Y) - 6S(Y,Z)\operatorname{Tr}(X)\\ &+ T(X,Y,Z).\end{aligned} \tag{B.16g}$$

A quartic Jordan algebra $\mathfrak{J}$ with multiplicative identity $\mathbb{1} = c$ may be derived from any such vector space if N_4 is *Jordan quartic*, that is:

1. The trace bilinear form (B.16g) is non-degenerate.

2. The quartic adjoint map, $\sharp\colon \mathfrak{J} \to \mathfrak{J}$, uniquely defined by $\operatorname{Tr}(X^\sharp, Y) = 4N_3(X,X,X,Y)$, satisfies

$$(X^\sharp)^\sharp = N_4(X)^2 X, \qquad \forall X \in \mathfrak{J}. \tag{B.17}$$

(X, Y, Z) is the linearization of the cubic adjoint,

$$(X, Y, Z) = (X + Y + Z)^\sharp - (X + Y)^\sharp - (X + Z)^\sharp - (Y + Z)^\sharp + X^\sharp + Y^\sharp + Z^\sharp. \tag{B.18}$$

The Jordan product is then defined using,

$$X \circ Y = \tfrac{1}{2}\big(X \times Y + \mathrm{Tr}(X)Y + \mathrm{Tr}(Y)X - S(X, Y)\mathbb{1}\big), \tag{B.19}$$

where, $X \times Y$ is the linearization of the quadratic adjoint,

$$X \times Y = (X, Y, c). \tag{B.20}$$

Finally, the Jordan triple product is defined as

$$\{X, Y, Z\} = (X \circ Y) \circ Z + X \circ (Y \circ Z) - (X \circ Z) \circ Y. \tag{B.21}$$

APPENDIX C

Magic supergravity stabilizers

In the following we examine the $\mathfrak{F}(\mathfrak{J}_3^{\mathbb{O}})$ case as an example. To determine the orbits space we will use the infinitesimal Lie algebra action to determine the Lie sub-algebras annihilating the the canonical forms presented in Theorem 28. A word of warning. This analysis is done at the level of the Lie algebra and hence does not take proper care of the global properties of the orbits. Since orbits are notoriously badly behaved these results are not strictly mathematically rigorous.

For all canonical forms one obtains

$$\Phi(x_{\text{can}}) = \begin{pmatrix} \nu & \phi A_{\text{can}} - \frac{1}{3}\nu A_{\text{can}} \\ X \times A_{\text{can}} + Y & \mathrm{Tr}(Y, A_{\text{can}}) \end{pmatrix}, \quad \text{where} \quad x_{\text{can}} = \begin{pmatrix} 1 & A_{\text{can}} \\ 0 & 0 \end{pmatrix}, \tag{C.1}$$

so we may set the dilatation generator ν to zero throughout.

Rank 1: $A_{\text{can}} = 0$

$$\Phi(x_1) = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \tag{C.2}$$

$\Rightarrow Y = 0$ while X and ϕ are unconstrained. Hence, the stability group is

$$H_1 = E_{6(-26)} \ltimes \mathbb{R}^{27}, \tag{C.3}$$

where $E_{6(-26)}$ is generated by ϕ and the 27 translations are generated by X.

Rank 2a: $A_{\text{can}} = (1, 0, 0)$

$$\Phi(x_{2a}) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} + Y & \mathrm{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{C.4}$$

From the $D = 5$ analysis we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\text{can}} = 0$ has 36 compact, 9 non-compact semi-simple generators and 16 translational generators giving $\mathfrak{so}(1, 9) \oplus \mathbb{R}^{16}$. For the remaining $27 + 27$ generators we obtain the following constraints:

1.

$$\mathrm{Tr}(Y, A_{\text{can}}) = 0 \Rightarrow y_{11} = 0. \tag{C.5}$$

2.

$$X \times A_{\text{can}} + Y = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_{33} & -x_{23} \\ 0 & -\overline{x}_{23} & x_{22} \end{pmatrix} = \begin{pmatrix} 0 & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & -y_{22} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & -y_{33} \end{pmatrix} \tag{C.6}$$

This gives 1 compact and 9 non-compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{C.7}$$

where, writing $x_{22} = x + y$ and $x_{33} = x - y$,

$$\tilde{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x+y & x_{23} \\ 0 & \overline{x}_{23} & x-y \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -x+y & x_{23} \\ 0 & \overline{x}_{23} & -x-y \end{pmatrix}. \tag{C.8}$$

These, together with the 36 compact and 9 non-compact generators from $\mathfrak{so}(1,9) \subset \mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ gives a total of 37 compact generators and 18 non-compact semi-simple generators producing $\mathfrak{so}(2,9)$, where we have used the fact that
$SO(m,n)$ has $[m(m-1)+n(n-1)]/2$ compact and mn non-compact generators.

The other $1+16$ components of X generate translations,

$$X^{'} = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X^{''} = \begin{pmatrix} 0 & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & 0 & 0 \\ x_{13} & 0 & 0 \end{pmatrix}, \tag{C.9}$$

where $X^{'}$ commutes with $\mathfrak{so}(2,9)$. The remaining $16+16$ translational generators transform as the spinor of $\mathfrak{so}(2,9)$. Hence, the stability group is

$$H_{2a} = \mathrm{SO}(2,9) \ltimes \mathbb{R}^{32} \times \mathbb{R}. \tag{C.10}$$

Rank 2b: $A_{\text{can}} = (-1,0,0)$

$$\Phi(x_1) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} - Y & \mathrm{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{C.11}$$

The analysis goes through as above but with the sign of $\tilde{Y}$ flipped. This gives a total of 45 compact and 10 non-compact semi-simple generators giving $\mathfrak{so}(1,10)$. Hence, the stability group is

$$H_{2b} = \mathrm{SO}(1,10) \ltimes \mathbb{R}^{32} \times \mathbb{R}. \tag{C.12}$$

Rank 3a: $A_{\text{can}} = (1,1,0)$

$$\Phi(x_{3a}) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} + Y & \mathrm{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{C.13}$$

From the $D=5$ analysis we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\text{can}} = 0$ has 36 compact semi-simple generators and 16 translational generators giving $\mathfrak{so}(9) \oplus \mathbb{R}^{16}$. For the remaining

$27+27$ generators we obtain the following constraints:

1.

$$\mathrm{Tr}(Y, A_{\text{can}}) = 0 \Rightarrow y_{11} = -y_{22}. \tag{C.14}$$

2.

$$\begin{aligned} X \times A_{\text{can}} + Y = 0 \Rightarrow \begin{pmatrix} x_{33} & 0 & -\overline{x}_{13} \\ 0 & x_{33} & -x_{23} \\ -x_{13} & -\overline{x}_{23} & x_{11}+x_{22} \end{pmatrix} &= \begin{pmatrix} -y_{11} & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & y_{11} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & -y_{33} \end{pmatrix} \\ \Rightarrow x_{33} = y_{11} = 0. \end{aligned} \tag{C.15}$$

This gives 16 non-compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{C.16}$$

where,

$$\tilde{X} = \tilde{Y} = \begin{pmatrix} 0 & 0 & \overline{x}_{13} \\ 0 & 0 & x_{23} \\ x_{13} & \overline{x}_{23} & 0 \end{pmatrix}. \tag{C.17}$$

These, together with the 36 semi-simple generators from $\mathfrak{so}(9) \subset \mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ gives a total of 36 compact generators and 16 non-compact generators producing $F_{4(-20)}$, which is a non-compact form of $\mathrm{Aut}(\mathfrak{J}_3^{\mathbb{O}})$.

The remaining 10 components of X generate translations which, together with the 16 preserved translational generators of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$, transform as the fundamental **26** of $F_{4(-20)}$.

Hence, the stability group is

$$H_{3a} = F_{4(-20)} \ltimes \mathbb{R}^{26}. \tag{C.18}$$

Rank 3b: $A_{\text{can}} = (-1, -1, 0)$

$$\Phi(R_1) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} - Y & \mathrm{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{C.19}$$

The analysis goes through as above but with the sign of $\tilde{Y}$ flipped so that the 16 previously non-compact semi-simple generators become compact giving the compact form $F_{4(-52)} = \mathrm{Aut}(\mathfrak{J}_3^{\mathbb{O}})$. Hence, the stability group is

$$H_{3a} = F_{4(-52)} \ltimes \mathbb{R}^{26}. \tag{C.20}$$

Rank 4a: $A_{\text{can}} = (-1, -1, -1)$

$$\Phi(x_{4a}) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} + Y & \mathrm{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{C.21}$$

From the $D=5$ analysis we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\text{can}} = 0$ has 52 compact semi-simple generators giving $F_{4(-52)}$. For the remaining $27+27$ generators we obtain the following constraints:

1.

$$\mathrm{Tr}(Y, A_{\text{can}}) = 0 \Rightarrow y_{11} + y_{22} + y_{33} = 0. \tag{C.22}$$

2.

$$X \times A_{\text{can}} + Y = 0 \Rightarrow \begin{pmatrix} x_{11} & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & x_{11} + x_{22} \end{pmatrix} = \begin{pmatrix} -y_{11} & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & -y_{22} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & -(y_{11} + y_{22}) \end{pmatrix}. \tag{C.23}$$

This gives 26 compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{C.24}$$

where

$$\tilde{X} = \begin{pmatrix} x_{11} & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & x_{11} + x_{22} \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} -x_{11} & -x_{12} & -\overline{x}_{13} \\ -\overline{x}_{12} & -x_{22} & -x_{23} \\ -x_{13} & -\overline{x}_{23} & -(x_{11} + x_{22}) \end{pmatrix}. \tag{C.25}$$

These, together with the 52 compact semi-simple generators from $F_{4(-52)}$ gives a total of 78 compact generators producing $E_{6(-78)}$.

Hence, the stability group is

$$H_{4a} = E_{6(-78)}. \tag{C.26}$$

Rank 4b: $A_{\text{can}} = (1, 1, -1)$

$$\Phi(x_{4b}) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} + Y & \mathrm{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{C.27}$$

From the $D = 5$ analysis we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\text{can}} = 0$ has 36 compact and 16 non-compact semi-simple generators giving $F_{4(-20)}$. For the remaining $27 + 27$ generators we obtain the following constraints:

1.

$$\mathrm{Tr}(Y, A_{\text{can}}) = 0 \Rightarrow y_{11} + y_{22} = y_{33}. \tag{C.28}$$

2.

$$X \times A_{\text{can}} + Y = 0 \Rightarrow \begin{pmatrix} x_{11} & x_{12} & -\overline{x}_{13} \\ \overline{x}_{12} & x_{22} & -x_{23} \\ -x_{13} & -\overline{x}_{23} & x_{11} + x_{22} \end{pmatrix} = \begin{pmatrix} -y_{11} & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & -y_{22} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & -(y_{11} + y_{22}) \end{pmatrix}. \tag{C.29}$$

This gives 10 compact and 16 non-compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{C.30}$$

where

$$\tilde{X} = \begin{pmatrix} x_{11} & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & x_{11} + x_{22} \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} -x_{11} & -x_{12} & \overline{x}_{13} \\ -\overline{x}_{12} & -x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & -(x_{11} + x_{22}) \end{pmatrix}. \tag{C.31}$$

These, together with the 36 compact and 16 non-compact semi-simple generators from $F_{4(-20)}$ gives a total of 46 compact generators and 32 non-compact generators producing $E_{6(-14)}$.

Hence, the stability group is

$$H_{4b} = E_{6(-14)}. \tag{C.32}$$

Rank 4c: $A_{\text{can}} = (1, 1, 1)$

$$\Phi(x_{4c}) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} + Y & \mathrm{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{C.33}$$

From the $D = 5$ analysis we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\text{can}} = 0$ has 52 compact semi-simple generators giving $F_{4(-52)} = \mathrm{Aut}(\mathfrak{J}_3^{\mathbb{O}})$. For the remaining $27 + 27$ generators we obtain the following constraints:

1.

$$\mathrm{Tr}(Y, A_{\text{can}}) = 0 \Rightarrow y_{11} + y_{22} + y_{33} = 0. \tag{C.34}$$

2.

$$X \times A_{\text{can}} + Y = 0 \Rightarrow \begin{pmatrix} -x_{11} & -x_{12} & -\overline{x}_{13} \\ -\overline{x}_{12} & -x_{22} & -x_{23} \\ -x_{13} & -\overline{x}_{23} & x_{11} + x_{22} \end{pmatrix} = \begin{pmatrix} -y_{11} & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & -y_{22} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & y_{11} + y_{22} \end{pmatrix}. \tag{C.35}$$

This gives 26 non-compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{C.36}$$

where

$$\tilde{X} = \tilde{Y} = \begin{pmatrix} x_{11} & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & -(x_{11} + x_{22}) \end{pmatrix}. \tag{C.37}$$

These, together with the 52 compact semi-simple generators from $F_{4(-52)}$ gives a total of 52 compact generators and 26 non-compact generators producing $E_{6(-26)} = \mathrm{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$.

Hence, the stability group is

$$H_{4c} = E_{6(-26)}. \tag{C.38}$$

APPENDIX D

Alternative Jordan dual formulation

Recall from section 3.6 that we defined the Jordan dual $A^\star$ of A as

$$A \to A^\star = \frac{A^\sharp}{N(A)^{1/3}}, \tag{D.1}$$

part of the motivation for this definition is that the entropy is preserved under J-Duality:

$$\begin{aligned} N(A^\star) &= N\left(\frac{A^\sharp}{N(A)^{1/3}}\right) = \frac{1}{N(A)}N(A^\sharp) \\ &= \frac{N(A)^2}{N(A)} = N(A). \end{aligned} \tag{D.2}$$

However, we note that, while A belongs to the fundamental representation eg $\mathbf{27}$ of E_6 and describes a black string, $A^\star$ belongs to the contragredient representation eg $\mathbf{27}'$ of E_6 and corresponds to a black hole (the $\sharp$ map is a map between the two representations).

An alternative definition which maps $\mathbf{27}$ to $\mathbf{27}$ and $\mathbf{27}'$ to $\mathbf{27}'$ begins with a black string/hole pair. To lowest order, the extremal non-rotating black string and black hole entropies are given respectively by

$$S_5 = (\pi\sqrt{|N(A)|}, \pi\sqrt{|N(B)|}), \tag{D.3}$$

where $N(A) = N(A, A, A)$. Large BPS and small BPS correspond to $N(A) \neq 0$, and $N(A) = 0$, respectively. The Dirac-Schwinger quantisation condition relating a black string/hole pair with charges (A, B) to one with charges (A', B') in the Jordan language is given by

$$\operatorname{Tr}(A, B') - \operatorname{Tr}(B, A') \in \mathbb{Z}. \tag{D.4}$$

The alternative *Jordan dual* or J-dual, defined for "large" black strings and holes by

$$(A^\star, B^\star) = \pm\left(\frac{B^\sharp}{N(B)^{1/3}}, \frac{A^\sharp}{N(A)^{1/3}}\right), \tag{D.5}$$

for which

$$(A^{\star\star}, B^{\star\star}) = (A, B). \tag{D.6}$$

In the case of a black string and a black hole related by J-duality

$$\mathrm{Tr}(B^\star, A) - \mathrm{Tr}(A^\star, B) = 3(N(A)^{2/3} - N(B)^{2/3}). \tag{D.7}$$

Note the factor of three. Hence, for a valid dual $(A^\star, B^\star)$ we require that $N(A)^2$ and $N(B)^2$ are perfect cubes. This is a necessary, but not sufficient condition because we also require that $A^\star$ and $B^\star$ are themselves integer. This restricts us to that subset of black strings and holes for which

$$d_3(\tilde{B}) = \left[\frac{d_2(B)}{d_1(A^\star)}\right]^3, \qquad d_3(\tilde{A}) = \left[\frac{d_2(A)}{d_1(B^\star)}\right]^3, \tag{D.8}$$

where $d_1(A) = \mathrm{gcd}(A)$, $d_2(A) = \mathrm{gcd}(A^\sharp)$ and $d_3(A) = |N(A)|$. Then

$$\begin{aligned} &\mathrm{Tr}(B^\star, A) - \mathrm{Tr}(A^\star, B) \\ &= 3\left\{\left[\frac{d_2(B)}{d_1(A^\star)}\right]^2 - \left[\frac{d_2(A)}{d_1(B^\star)}\right]^2\right\}. \end{aligned} \tag{D.9}$$

The U-duality integral invariants $\mathrm{Tr}(A, B)$ and $N(A, B, C)$ are not generally invariant under Jordan duality but $\mathrm{Tr}(B^\star, A) - \mathrm{Tr}(A^\star, B)$, $N(A)$ and $N(B)$ and hence the lowest-order black string and black hole entropy, are invariant under this alternative J-duality but only up to an A-B interchange:

$$(N(A^\star), N(B^\star)) = (N(B), N(A)). \tag{D.10}$$

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