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Amplitude Dependence of the Tune Shift

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Recent studies in the Tevatron have measured the tune shift as a function of the displacement from a closed orbit. The measured values of tune shift are much smaller than one would expect from the measured distribution of the normal octupole moments in the Tevatron.[1] We have performed tracking studies (using Tevlat) to see if, and under what conditions, the observed results could be obtained.

The particle to be tracked was given an initial displacement (x_o, y_o) and followed through the Tevatron lattice for 4096 turns. The phase space coordinates (x,x'), (y,y') we recorded as the particle passed the initial location in the lattice. Thus, 4 sequences of values $\{x_n\}$, $\{x'_n\}$, $\{y_n\}$, $\{y'_n\}$ were recorded where n represents the turn number. The sequences $\{x_n\}$ and $\{y_n\}$ were Fourier analyzed to determine the tunes of the motion. These tunes are then compared with the tune corresponding to zero initial displacement ($x_o = y_o = 0$). The tune shift is defined as the tune with non-zero initial displacement minus the tune with zero initial displacement.

In a simple model where there is no linear coupling it is possible in a straight forward way to calculate the tune shift due to any distribution of multiples in the lattice (Appendix I). Because our interest lies in understanding the failure to observe the effect expected from the octupole moments we consider only the octupole moments of the dipoles. This is done to facilitate comparison of the tracking results with the calculated tune shift. The comparison is found in Table I and is quite good; the tracking reproduces the tune shift that one can calculate.[2]

We introduce coupling by including the skew quadrupole circuit R_o into the tracking. The value of R_o can be varied and the smear[3] calculated by tracking. The tune shift can be computed as a function of R_o. Figure 2 shows the tune shift as a function of R_o when x_o = 3 mm. The only non-linear elements are the normal octupole components of the dipoles. It is clear that the amplitude dependent tune shift decreases as the coupling increases. Figure 2 also shows the tune shift as a function of smear.

In order to understand the tracking results we construct a simple model (also described in Appendix I) in which we include linear coupling so that ε_H and ε_V vary but $\varepsilon_H + \varepsilon_V$ is constant. The result of this coupling is to reduce the average value of ε_H but more importantly to introduce a contribution of ε_V to the tune shift ν_x. Since the distribution of octupole moments in the Tevatron is such that the shift in ν_x due to ε_H and ε_V are comparable in magnitude but opposite in sign a coupling such that $\langle\varepsilon_V\rangle$ is comparable to $\langle\varepsilon_H\rangle$ can significantly reduce the coefficient of the x^2 dependence of the tune shift. Numerical calculations of the size of the effect are compared with the tracking calculations are found in Table II and are in fair agreement. We therefore, conclude that linear coupling can significantly reduce the coefficient of x_o^2 of the tune shift. Therefore, it could be very useful in the interpretation of the experimental data to measure the linear coupling at the same time that the tune shift is measured.[4] In Appendix I also a relation between the strength of the coupling and the "smear" is calculated.

Appendix I

In this Appendix we calculate:

A. The dependence of the tune shift on the initial coordinate of a particle in the presence of a distribution of octupole moments in an accelerator.

B. The effect of a linear coupling on the tune shift.

C. The "smear" as a function of the linear coupling.

A. Our discussion is simply a rewriting of that of E.J.N. Wilson in CERN 77-13 pp 111-138.

Consider the motion of a particle in an accelerator with only linear elements and without coupling. It is well known that the motion is given by

$$u = a\cos(\phi+\delta) \qquad u = (x \text{ or } y) \qquad \text{I.1}$$

$$a = \sqrt{\epsilon\beta} \qquad \epsilon = \left[\gamma u^2 + 2\alpha\beta u u' + \beta u'^2\right] \qquad \text{I.2}$$

$$u' = -\sqrt{\frac{\epsilon}{\beta}}\left(\alpha\cos(\phi+\delta) + \sin(\phi+\delta)\right) \qquad \text{I.3}$$

$$\alpha = -\frac{1}{2}\beta'$$

$$v \equiv \beta u' + \alpha u = -a\sin(\phi+\delta) \qquad \text{I.4}$$

$$u^2 + v^2 = \epsilon\beta$$

Now we consider a kick $\Delta u'$ ($\Delta u = 0$)

$$\Delta u = 0 = \Delta a \cos(\phi+\delta) - a \sin(\phi+\delta)\Delta\phi$$

$$\Delta v = \beta \Delta u' = -\Delta a \sin(\phi+\delta) - a\cos(\phi+\delta)\Delta\phi$$

$$\Delta a = a \frac{\sin(\phi+\delta)}{\cos(\phi+\delta)} \Delta\phi$$

$$\Delta\phi = -\beta \frac{\Delta u'}{a} \cos(\phi+\delta) \qquad \text{I. 5}$$

Next we assume that Δu is generated by a normal octupole field

$$B_y = B_o \ell\, b_3 (x^3 - 3xy^2) \qquad \Delta x' = -\frac{B_o \ell b_3}{(B\rho)} (x^3 - 3xy^2)$$

$$\Delta\phi_x = \beta_x \left(\frac{B_o \ell}{(B\rho)}\right) b_3 \left[a_x^2 \cos^4(\phi_x+\delta_x) - 3a_y^2 \cos^2(\phi_x+\delta_x)\cos^2(\phi_y+\delta_y)\right]$$

$$B_x = B_o \ell\, b_3 (3x^2y - y^3) \qquad \Delta y' = +\frac{B_o \ell b_3}{(B\rho)} (3x^2y - y^3)$$

$$\Delta\phi_y = -\beta_y \left(\frac{B_o \ell}{(B\rho)}\right) b_3 \left[3a_x^2 \cos^2(\phi_x+\delta_x)\cos^2(\phi_y+\delta_y) - a_y^2 \cos^4(\phi_y+\delta_y)\right)$$

We next average $\Delta\phi_x$ and $\Delta\phi_y$ over many turns (i.e. over ϕ) to get the average $\Delta\phi$ per turn.

$$\Delta\phi_x = \frac{B_o \ell}{(B\rho)} b_3 \left[\frac{3}{8}\beta_x^2 \epsilon_H - \frac{3}{4}\beta_x\beta_y \epsilon_V\right]$$

$$\Delta\phi_y = \frac{B_o \ell}{(B\rho)} b_3 \left[-\frac{3}{4}\beta_x\beta_y \epsilon_H + \frac{3}{8}\beta_y^2 \epsilon_V\right]$$

where $a_x^2 = \beta_x \epsilon_H \quad a_y^2 = \beta_y \epsilon_V$

We next sum over all the N magnets in the accelerator

$$\sum_{i=1}^{N} \Delta\phi_x^i = \frac{B_0 \ell}{(B\rho)} \sum b_{3i} \left(\frac{3}{8}\beta_{xi}^2 \epsilon_H - \frac{3}{4}\beta_{xi}\beta_{yi}\epsilon_V\right)$$

$$\sum_{i=1}^{N} \Delta\phi_y^i = \frac{B_0 \ell}{(B\rho)} \sum b_{3i} \left(-\frac{3}{4}\beta_{xi}\beta_{yi}\epsilon_H + \frac{3}{8}\beta_{yi}^2 \epsilon_V\right)$$

$$A \equiv \sum_{i=1}^{N} \beta_{xi}^2 b_{3i}/N ; \quad B \equiv \sum_{i=1}^{N} \beta_{xi}\beta_{yi} b_{3i}/N \quad C \equiv \sum_{i=1}^{N} \beta_{yi}^2 b_{3i}/N$$

$$\frac{N B_0 \ell}{(B\rho)} = 2\pi$$

$$\left.\begin{aligned} \Delta\nu_x &= \frac{1}{2\pi} \sum_{i=1}^{N} \Delta\phi_x^i = \frac{3}{8} A \epsilon_H - \frac{3}{4} B \epsilon_V \\ \Delta\nu_y &= \frac{1}{2\pi} \sum_{i=1}^{N} \Delta\phi_y^i = -\frac{3}{4} B \epsilon_V + \frac{3}{8} C \epsilon_V \end{aligned}\right\} \quad I.7$$

For the dipole magnets only used in the calculations performed using Tevlat we find

$A = -6808 \ m^{-1}$

$B = -4586 \ m^{-1}$

$C = -4231 \ m^{-1}$

$$\begin{aligned} \Delta\nu_x &= -2553\ \varepsilon_H + 3440\ \varepsilon_V \\ \Delta\nu_y &= 3440\ \varepsilon_H - 1587\ \varepsilon_V \end{aligned} \quad I.8$$

We both start the tracking with $x_0' = y_0' = 0$ and $\gamma_x = 1.6776 \times 10^{-2}$, $\gamma_y = 1.6721 \times 10^{-2}\ m^{-1}$.

$$\begin{aligned} \Delta\nu_x &= -42.83 x_0^2 + 57.51 y_0^2 \\ \Delta\nu_y &= +57.70 x_0^2 - 26.54 y_0^2 \end{aligned} \quad I.9$$

B. To incorporate the effect of linear coupling we recognize that ϵ_H and ϵ_V are no longer constants and we write

$$\begin{aligned} \epsilon_H &= \epsilon_{Ho}(1-k^2\sin^2\theta) \\ \epsilon_V &= \epsilon_{Ho}\, k^2 \sin^2\Theta \end{aligned} \qquad \text{I.10}$$

where we are assuming that initially $\epsilon_{VO} = 0$. k^2 is a measure of the strength of the coupling and Θ represents the phase of the coupling between the x and y motion.

We substitute these expressions I.10 into the equations I.8 and find

$$\Delta\nu_x = [-2553\,(1-k^2\sin^2\theta) + 3440\,k^2\sin^2\theta]\,\epsilon_{Ho}$$

$$\Delta\nu_y = [3440\,(1-k^2\sin^2\theta) - 1587\,k^2\sin^2\theta]\,\epsilon_{Ho}$$

$$\Delta\nu_x = [-2553 + 5993\,k^2\sin^2\theta]\,\epsilon_{Ho}$$

$$\Delta\nu_y = [3440 - 5027\,k^2\sin^2\theta]\,\epsilon_{Ho}$$

averaging over Θ we get the results

$$\begin{aligned} \Delta\nu_x &= (-42.83 + 50.27\,k^2)\,X_0^2 \\ \Delta\nu_y &= (57.70 - 84.33\,k^2)\,X_0^2 \end{aligned} \qquad \text{I.11}$$

$$\epsilon_{Ho} = \gamma_x X_0^2$$

C. We use a modified definition of the smear from that defined by the SSC Aperture Working Group. Our definition follows with the smear denoted by R.

$$\beta_x = \beta_y = \beta \quad (\text{which is true in our case})$$

Let $a \equiv \beta^{1/2}(\gamma_x x^2 + 2\alpha_x \beta x x' + \beta x'^2)^{1/2} = \sqrt{\epsilon_H \beta}$

$$b \equiv \beta^{1/2}(\gamma_y y^2 + 2\alpha_y \beta y y' + \beta y'^2)^{1/2} = \sqrt{\epsilon_V \beta}$$

We track the particle for many turns and at the end of turns i compute a_i and b_i.

Then when we have completed the tracking $\bar{a}$, $\sigma_a^2 = \overline{a^2} - \bar{a}^2$ and $\bar{b}$, σ_b^2 are calculated from the individual a_i and b_i

$$R \equiv \frac{\sqrt{\sigma_a^2 + \sigma_b^2}}{\sqrt{\bar{a}^2 + \bar{b}^2}}$$

For a linear system without coupling R=0. With linear coupling

$$a = \sqrt{\beta \epsilon_{H0} (1 - k^2 \sin^2\theta)}$$

$$b = \sqrt{\beta \epsilon_{H0} k^2 \sin^2\theta}$$

Averaging over Θ

$$\bar{a} = \sqrt{\beta \epsilon_{H0}} \left(1 - \frac{k^2}{4} - \frac{3}{64} k^4\right) \qquad \overline{a^2} = \beta \epsilon_{H0} (1 - k^2/2)$$

$$\sigma_a^2 = \beta \epsilon_{H0} \frac{k^4}{32}$$

$$\bar{b} = \sqrt{\beta \epsilon_{H0}}\, k \times \frac{2}{\pi} \qquad \overline{b^2} = \beta \epsilon_{H0} \frac{k^2}{2}$$

$$\sigma_b^2 = \beta \epsilon_{H0} k^2 \left(\frac{1}{2} - \left(\frac{2}{\pi}\right)^2\right) = 9.47 \times 10^{-2} \beta \epsilon_{H0} k^2$$

$$\sqrt{\sigma_a^2 + \sigma_b^2} \approx \sqrt{9.47 \times 10^{-2} \beta \epsilon_{H0} k^2}$$

$$\sqrt{\bar{a}^2 + \bar{b}^2} \approx \sqrt{\beta \epsilon_{H0} (1 - 9.47 \times 10^{-2} k^2)}$$

$$R^2 \approx 9.47 \times 10^{-2} k^2$$

$$R = 0.3077 k$$

$$k = 3.25 R$$

Substituting I.12 into I.11 we find

$$\Delta\nu_x = [-42.83 + 530.75 R^2] x_0^2$$
$$\Delta\nu_y = [57.70 - 890.35 R^2] x_0^2 \qquad \text{I.13}$$

FOOTNOTES

1. Memo of Sho Ohnuma (January 16, 1986) to Rol Johnson

2. If we include in the tracking the contribution of the octupole moments of the quadrupoles they increase the tune shift by ~18%.

   The skew quadrupole moments of the magnets is next included in the tracking while the skew quadrupole circuit (whose strength is given by R_o in amps) is adjusted to minimize the smear[3]. The tune shift as a function of x_o (the initial displacement of the particle, $y_o = 0.0$) shows an x^2 dependence (Fig. 1)

   $$\Delta\nu_x = Cx^2$$

   and that C has the same value it had before the introduction of the skew quadrupole components of the magnets. We therefore, conclude that when the skew quadrupole current is adjusted to minimize the smear that the distribution of skew quadrupole moments in the lattice does not effect the x^2 dependence of the tune shift.

3. The smear is defined in the report of the Aperture Workshop Group (1985) of the SSC Central Design Group.

Our method of estimating the smear is to compute after each turn in the tracking the following two quantities

$$a_i = \sqrt{x_i^2 + (\beta_x x_i' + \alpha_x x_i)^2}$$

$$b_i = \sqrt{y_i^2 + (\beta_y y_i' + \alpha_y y_i)^2}$$

Where x_i, y_i, x_i', y_i' are the phase space coordinates of the particle after tracking for i turns. β_x, β_y, α_x, α_y are the values of the lattice functions at the point at which we start the tracking and at which we calculate a_i and b_i.

From the sequences $\{a_i\}$, $\{b_i\}$ we calculate a, b, σ_a and σ_b.

The smear is then estimated by

$$R = \frac{\sqrt{\sigma_a^2 + \sigma_b^2}}{\sqrt{a^2 + b^2}}$$

4. The original motivation for these calculations were measurements of the amplitude dependence of the tune shift. The data are collected in Table III and plotted

in Figure 3. The preceding calculation and discussion, based as they are, on a simplified model of the Tevatron do not quantitatively confront the experimental data. Accordingly Tevlat was run with all the multipole moments included, using the MTF values measured at 2000 A which corresponds to the 400 GeV energy at which the data were taken. The current in the skew quadrupole circuit was adjusted to given minimum smear and the particle was tracked from E17. The value of the tune shift $\Delta\nu_x$ resulting from the tracking in Tevlat is $\Delta\nu_x = -4.38\times10^{-3}$ for an initial amplitude of 5.4 mm. The measured value is $\Delta\nu_{xm} = +7\times10^{-5}\pm10^{-4}$. The agreement is obviously very poor. If the skew quadrupole current is deliberately moved away from its optimum value so that the smear is ~0.3 we find from the tracking for x_o = 5.4 mm $\Delta\nu_x = -4\times10^{-4}$ which is in approximate agreement with the measurement. I therefore would conclude that if the actual smear was small (i.e. small linear coupling) when the measurments were made then the current model of the Tevatron (which means Tevlat plus the MTF data) is unable to explain the measurements. On the other hand if the smear was in fact large then the measurements may be in agreement with the Tevlat calculations.

TABLE I Comparison of the Calculated Tune Shift with the Tracking Results - No Linear Coupling

x_o(mm)	$\Delta\nu_x$(tracking)x10^4	$\Delta\nu_x$(calculated) x 10^4	%diff
2	-1.725	-1.713	.7
4	-6.896	-6.853	.6
6	-15.515	-15.419	.6
8	-27.568	-27.411	.6

TABLE II Comparison of the Calculated Tune Shift with the Tracking Results - Linear Coupling Model x_o = 3 mm

R(smear)x	$\Delta\nu_x$(tracking)x10^4	$\Delta\nu_x$(model)x10^4	%diff
0	-3.88	-3.85	.7
4.55x10^{-2}	-3.71	-3.76	1.3
9.27x10^{-2}	-3.20	-3.44	6.3
1.43x10^{-1}	-2.43	-2.87	15.3
2.02x10^{-1}	-1.52	-1.90	25.

TABLE III

kick	(mm)	ν_h	ν_v
0	0	0.36146	0.46292
2	3.2	0.36150	0.46293
3	5.4	0.36153	0.46300
4	7.4	0.36158	0.46290

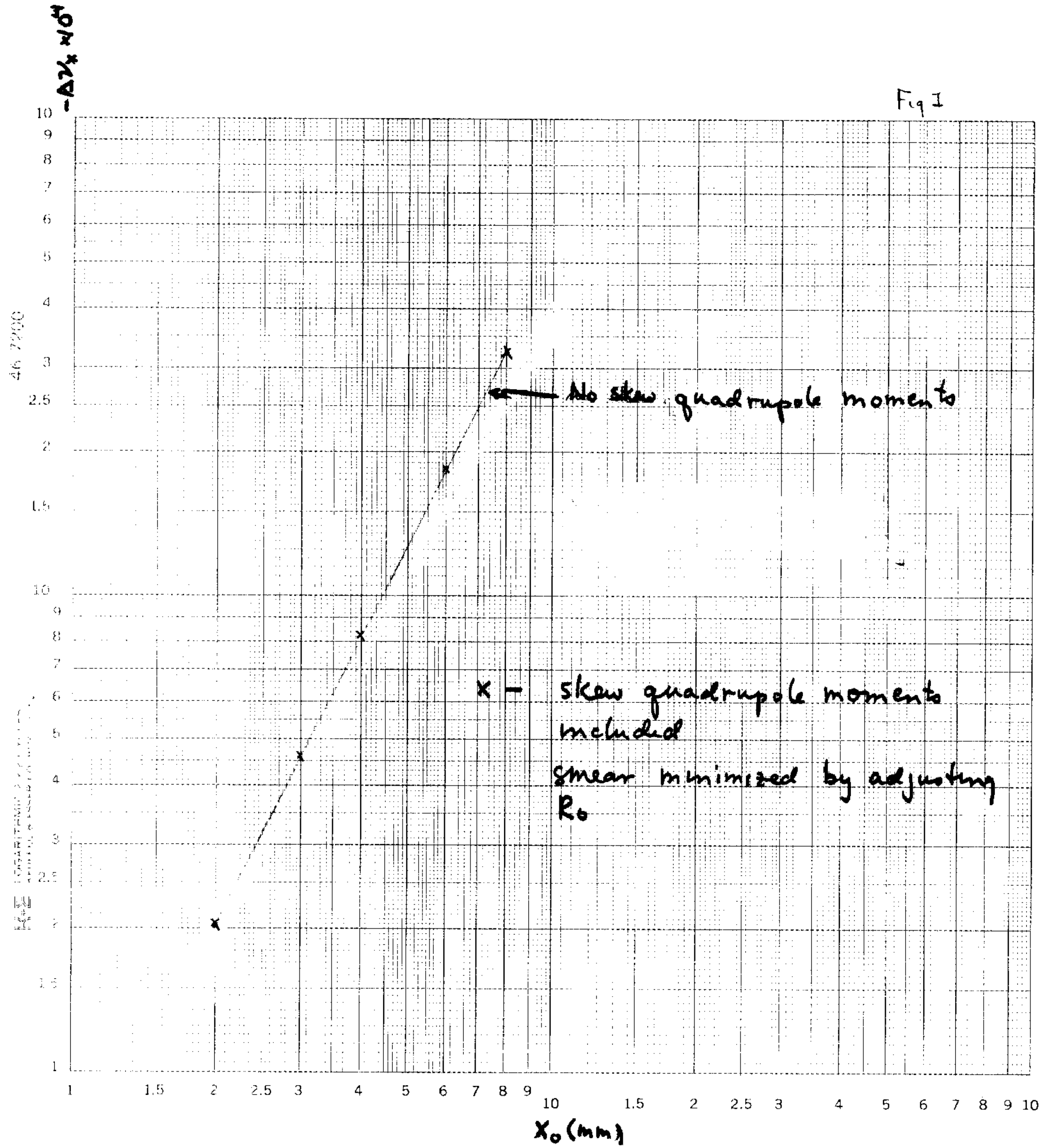


Fig I
$-\Delta\nu_x \times 10^{-4}$
No skew quadrupole moments
× – skew quadrupole moments included
smear minimized by adjusting R_0
X_0 (mm)
10
9
8
7
6
5
4
3
2.5
2
1.5
1
1.5
2
2.5
3
4
5
6
7
8
9
10

fig 2

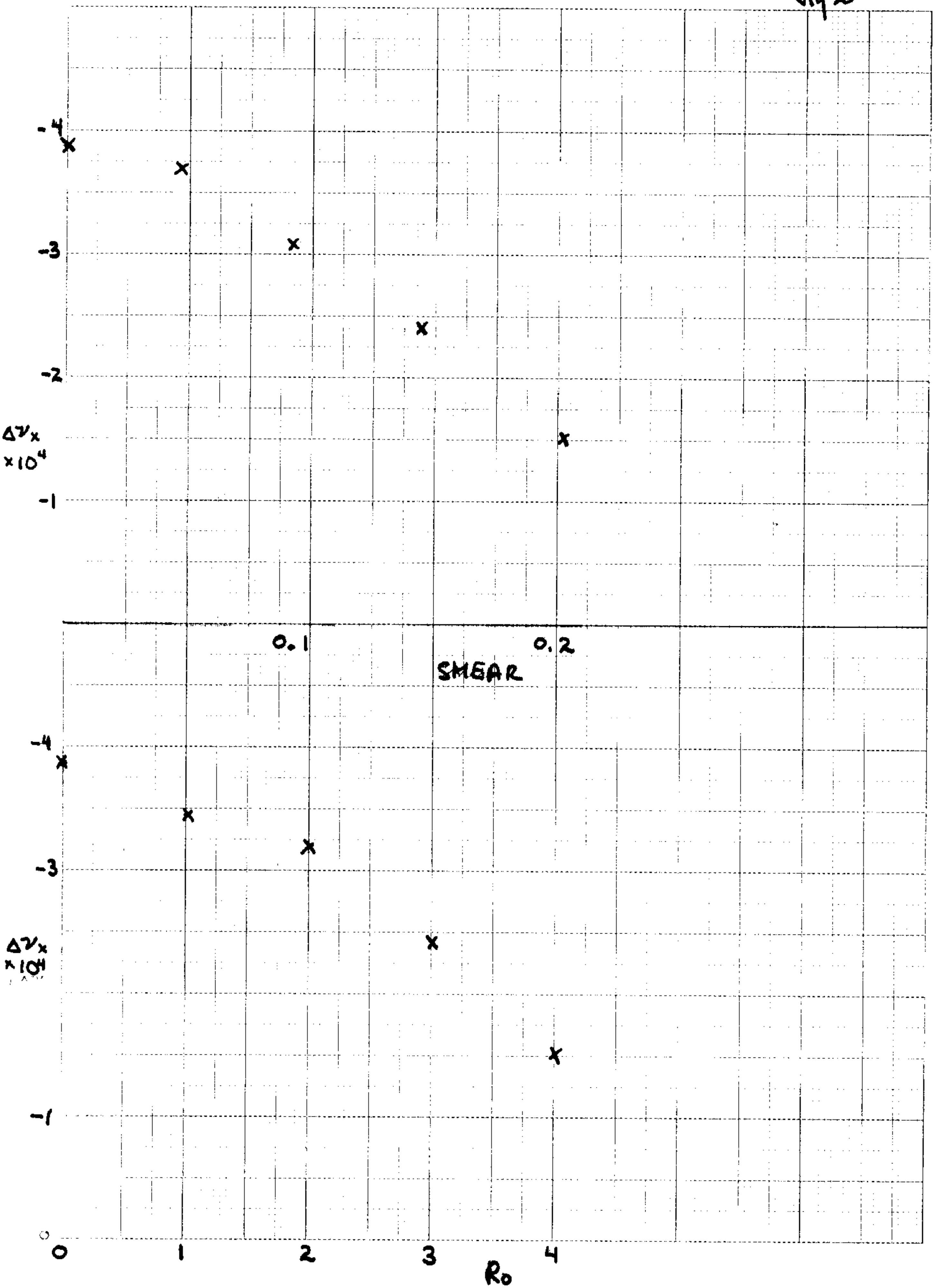
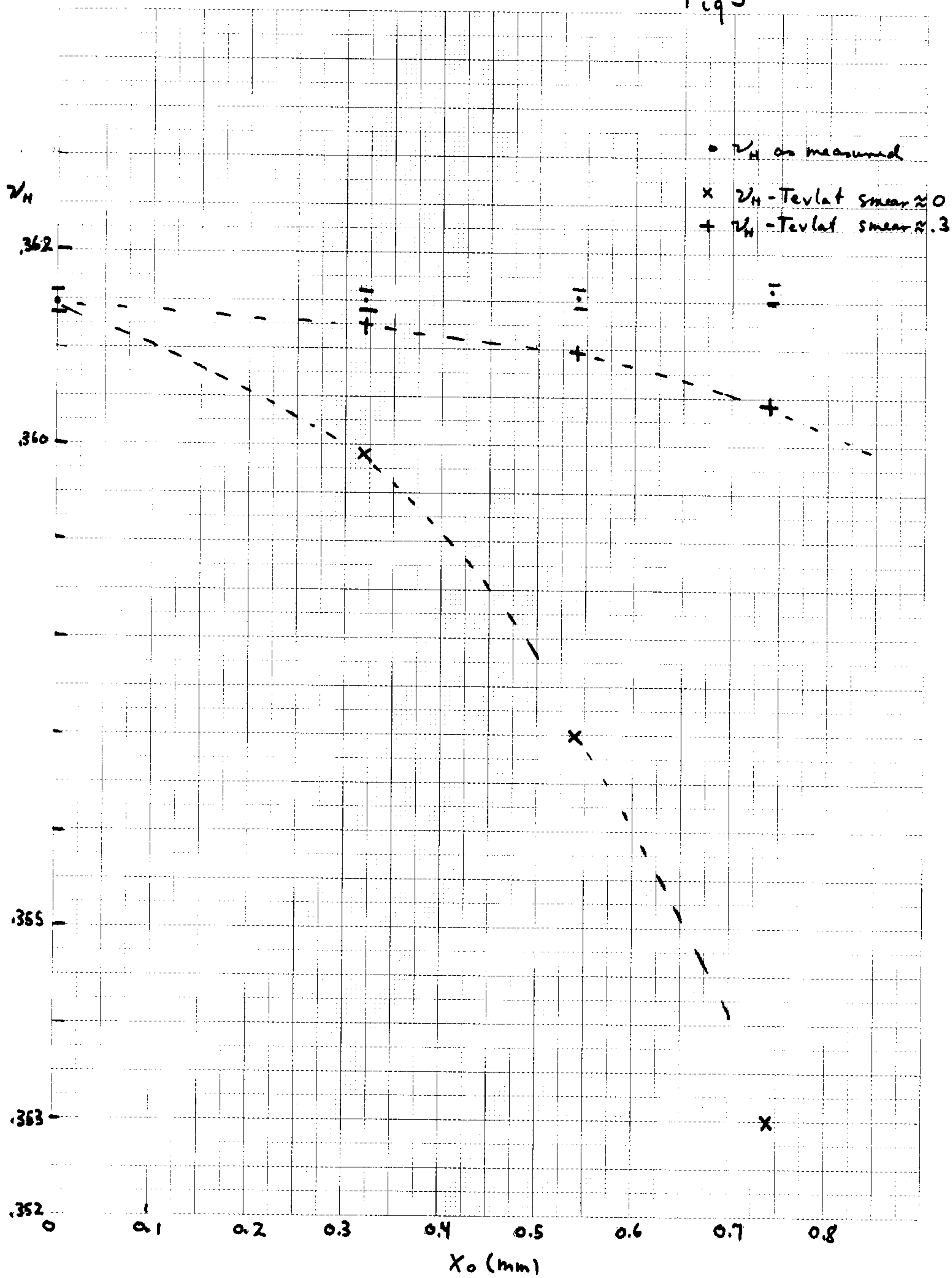

-4
-3
-2
-1
$\Delta\nu_x$ $\times 10^4$
0.1
0.2
SMEAR
-4
-3
-1
$\Delta\nu_x$ $\times 10^4$
0
0
1
2
3
4
R_0

Fig 3



• ν_H as measured
× ν_H - Tevlat smear ≈ 0
+ ν_H - Tevlat smear ≈ .3
ν_H
.362
.360
.355
.353
.352
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
X_0 (mm)

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