BLED WORKSHOPS
IN PHYSICS
VOL. 5, NO. 2

Proceedings to the 7th Workshop
What Comes Beyond ... *(p. 94)*
Bled, Slovenia, July 19-31, 2004

8 Physics Would Be Impossible in Any Dimension But 3+1 — There Could Be Only Empty Universes

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Abstract. Our universe has dimension 3+1: three of space, one of time. And it must — a universe could not exist if the dimension were different. Physics has to be inconsistent — not possible — in any other dimension ([6], chap. 7, p. 122). Why?

8.1 There must be observers

A most fundamental fact of nature is that there are different observers — physical objects. They must all be able to observe, their observations must make sense and must be related. And these are related by geometry. Mathematically all physical laws have to be expressible — in a consistent way — in different coordinate systems.

Thus we can use coordinates x, y or

$$x = x' \cos\theta + y' \sin\theta, \tag{8.1}$$

$$y = y' \cos\theta - x' \sin\theta. \tag{8.2}$$

This — rotation — is merely a change of symbols with no physics involved. If it gave an inconsistent set of laws then there could be no laws, physics would not be possible, thus nor would a universe.

These transformations need not be symmetries (although it is quite provocative that they are symmetries also). They form transformation groups (sets of transformations) ([6], sec. A.2, p. 178; [9], sec. I.7.b, p. 37), but need not be symmetry groups (sets of transformations leaving space or other systems invariant).

Were a direction of space different (simulated by the vertical) it would not matter. Arguments would not be affected. Since that is all these are, transformations not symmetries, requirements on space and nature are very weak thus quite strong. They are weak because very little (actually it seems nothing) is put in, is needed. So they are quite strong as it is (almost?) impossible to avoid them, to have anything else thus avoid what a universe must be like. And that is not only quite strong but quite disturbing.

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8.2 So a homomorphism is required

But why are transformations — just rotations — so restrictive, so difficult? Why is it impossible to have them for physical objects in any space but one, that of dimension 3+1, and that just barely? Coordinates are real numbers thus transformed by orthogonal (rotation) groups. Physical objects are given by statefunctions (a better term than wavefunction since nothing waves). And these are complex numbers, thus transformed by unitary groups.

Unitary and orthogonal transformations have to be related (essentially the same), else there could be no physical objects.

Consider an electron with spin up along z. It is not up along z'. Its statefunction expressed in terms of z' is different from that expressed in terms of z. One gives spin up, the other gives it at an angle. They are different, but related. Knowing that along z and the angle of rotation we know it along z'. The statefunction is transformed — and by a unitary transformation. For each rotation there are corresponding unitary transformations.

Any rotation can be written in an infinite number of ways as a product of rotations. This is also true of unitary transformations. Rotations and unitary transformations both form groups.

A fundamental requirement is that the products coincide (up to a possible sign) — the groups are homomorphic. Consider a rotation and then the inverse, but one written as a product. This can be done in an infinite number of ways going from a state back to the original. The final state is the same as that from which we started. These transformations are purely mathematical, just changes of variables, as above.

Suppose that for each product the direction of spin of the electron were different — even though the orientation of the coordinate system (the observer) is the same. That is observers, originally and finally identical, carrying out these (mathematical) transformations would see different spins even though these observers are the same. Obviously physics would be inconsistent — not possible, nor would a universe.

8.3 Dirac's equation shows this

This can be seen, perhaps more rigorously, using Dirac's equation. Writing equations for two objects and an electromagnetic potential

$$i\gamma_\mu \frac{\partial \psi(x)_j}{\partial x_\mu} - m_j \psi(x)_j + I(\psi(x)_j, A) = 0; \quad j = 1, 2, \tag{8.3}$$

with statefunction $\psi(x)$ giving the probability of finding the object at x with some spin direction, A (written schematically), the electromagnetic potential, for which there is another equation, and I the interaction term (whose form is irrelevant — the argument is very general). We can take the z axis along the spin of one electron or along the spin of the other (or pick any axes) — there is no way we can restrict the direction of arbitrary axes. We must be able to transform (by changing variables as in the equations above) so such equations for the statefunctions have to be properly transformable under (arbitrary) rotations. But coordinates are real and statefunctions complex. Therefore transformations on terms in the equations are different.

Were conditions not fulfilled then a set of (mathematical) rotations returning to the initial coordinates — so nothing is changed — would give different equations. These would depend on how rotations were (mathematically) carried out. Electrons would get very confused, they would not know what equation to obey. Thus there is a set of equations, one for each object (including the electromagnetic potential). However if conditions were not met then how we (mathematically) rotate determines these equations, clearly ridiculous — they would be inconsistent. Each set of rotations, returning to the original orientation, would give different equations. Statefunctions would have to — simultaneously — satisfy all equations obtained by the infinite set of possible rotations. Obviously that could not be.

If these equations were inconsistent — and if the transformations were not properly related that proves that they are — the only solutions are all zero. Every statefunction would be zero, thus would the probability of finding any object be — the universe is then empty.

What determines the dimension then is that equations of physics must be invariant under space transformations, here rotations. But these involve objects that are real (coordinates), and ones that are complex (statefunctions). Hence these must transform the same way, which they do not do — unitary and orthogonal groups are not homomorphic. Fortunately there is one dimension, 3+1, with unitary and orthogonal groups that are. And that thus is the only possible dimension allowing a universe with matter.

8.4 Mathematical analysis

We consider this in more depth.

For mere existence there must be one dimension whose orthogonal group is homomorphic to a unitary group. Given a space it cannot contain matter unless there is a unitary group homomorphic to the (pseudo-)rotation group of that space. The Lie algebras of these groups have to be isomorphic. Is this possible? Seemingly no. The number of generators (parameters) and the number of commuting generators of unitary and orthogonal groups are different. Can there be a space in which they are the same? The analysis is essentially trivial, just counting. For orthogonal groups it requires counting the number of planes, each giving a generator, and the number of planes that do not share an axis, each giving commuting generators. Counting need not be done here since the results are known and the counting is trivial. (Existence is based on trivialities.)

The number of sets of rotations in a space of dimension d is

$$N_d = \frac{(d-1)(d-1+1)}{2} = \frac{d(d-1)}{2}. \tag{8.4}$$

Counting nonintersecting planes gives the number of sets of com- muting rotations,

$$C_c = \frac{d}{2}, \text{ for d even, and } C_c = \frac{(d-1)}{2}, \text{ for d odd.} \tag{8.5}$$

For a unitary group in a space of dimension p there are $p^2 - 1$ transformations, of which $p - 1$ commute.

These quite elementary formulas for the number of rotations and the number of commuting ones determine whether any universe is possible.

A space of dimension d cannot contain matter unless there is a unitary group, in some complex space of dimension p, whose Lie algebra is isomorphic to that of the orthogonal group of the space. Setting the numbers of generators, and the numbers of commuting ones, equal we find that this is possible only for $d = 3$ or 6. Which does the universe choose?

There is more.

Rotations are defined as those transformations (on real numbers) leaving angles and lengths unchanged. Angles of rotations are real numbers but there are also complex parameters for which these are preserved ([6], sec. 7.2, p. 124). Both imaginary and real parts are limited as the formulas show but both can be nonzero. In general such a transformation in a plane relates coordinates by

$$x' = x\alpha + y\beta, \tag{8.6}$$

with α, β complex.

Mathematically if the sets of transformations on real and complex numbers are the same for real parameters they must be the same for complex ones. We write

$$\alpha = a + ib, \tag{8.7}$$

using two real parameters a, b. The equations for the real and imaginary parts are essentially the same. We then get two equivalent sets of transformations, for the real parts of the parameters and for the imaginary parts. However the number of sets of commuting transformations does not depend on whether parameters are real or complex. Transformations in planes sharing an axis do not commute but do commute if they do not share an axis. Whether parameters are complex or only real does not change this.

If transformations on complex numbers (statefunctions) are the same for real parts of parameters of the generalization of rotations they must be the same for imaginary parts. This necessitates another condition — which is fortunate. And fortunately space gives another transformation: inversion.

A statefunction for an object with spin $\frac{1}{2}$ has two parts, giving spin up and spin down. No matter what the direction of its spin the statefunction can always be written as a sum of these two taken along any axis (which does not mean, as sometimes believed, that it can only point up or down — it can be in any direction). Under an inversion the statefunction goes to a different one thus it has four components (the Dirac bispinor). The two two-dimensional spinors behave somewhat differently under all these transformations with complex parameters (the CO(3,1) group) as there are differences in minus signs for the boosts ([2], p. 216), ([12], p. 312), ([13], p. 79).

Thus the bispinor really has 4 components, which is exactly what is needed. The universe could not exist were this not true.

What condition do these, complex parameters and the inversion, give? Consider that the statefunction of an object has p parts (here 4). It transforms under SU(p). Is there any space whose orthogonal algebra is isomorphic to that of SU(p), for some p? We know the number of generators for these algebras. We take the statefunction in such a space to have j (here 2) blocks, each then of size $\frac{p}{j}$.

The number of parameters, now complex, for the transformations are thus twice those of the set of rotations. The rotations are the same so the number of commuting ones does not depend on whether the parameters are real or complex. On this complex statefunction there are transformations going with the rotations plus others mixing the j blocks. The number of these is the sum of those of the two sets. The total number of parameters must be equal to the total number of transformations on this p-dimensional complex space. So

$$(d^2 - d) + j^2 = p^2, \tag{8.8}$$

and for the commuting ones

$$\frac{d}{2} + j = p \quad \text{or} \quad \frac{(d-1)}{2} + j = p, \tag{8.9}$$

for d even and d odd.

It can easily be checked that the only solution is

$$d = 4, \quad p = 4, \quad j = 2. \tag{8.10}$$

Neither 3 nor 6 satisfies. This gives that the dimension is 4. Fortunately j equals 2 (the Dirac bispinor) as it must since it is the inversion — which can only interchange one block with another — that requires blocks. There can thus be only two.

For larger spin, statefunctions transform as reduced products of this fundamental representation — for the same p.

The dimension then must be 3 or 6, but can only be 4. That seems to imply that any universe is impossible. Fortunately the choices are 3 and 4, thus can be both. Uniquely 3 + 3 = 6; the three generators of SO(3) plus the three of SO(3) gives the same number of generators as the six of SO(4). That group is not simple but only semisimple. It is easy to prove that this is the only orthogonal group that is not simple — only for dimension 4 can this argument work. And that is just the dimension the argument gives.

Each of the SO(3) groups has the correct number of transformations to satisfy the condition from real angles, the two together satisfy the condition for complex ones. One group acts on each of the two-dimensional spinors, the other group acts on the pair treating it as a two-dimensional spinor (each of which has two components). The six generators of SO(4) break into two sets, each of the three generators of SO(3). Only for dimension 4 is this splitting possible.

Why does this work? The (complex) statefunction must split into two corresponding parts (since space also allows an inversion), here each themselves of two parts giving four (components) altogether. Rotations act (identically) on each of the two pieces but boosts (changes of speed) do not, differing in minus signs. However as there are two parts an additional set of transformations mixes them. Hence the group of transformations of real space, CO(4), has twelve parameters. But there are an additional three parameters because the inversion gives a pair of spinors and this group mixes them. Coordinate systems obtained by these transformations on space are possible — it must be possible to write physical laws using any. Equations have to be form-invariant under them. Thus space allows a 15-parameter set of transformations, the same number as that of SU(4).

The electron statefunction illustrates this. Rotations mix up and down states. But we have pairs of these. So another set of rotations mixes the pairs (each a pair). Two sets of rotations are thus needed giving two sets of transformations on complex statefunctions. Each of the two sets of (three) rotations has a set of (homomorphic) transformations on complex variables — each acts in a three-dimensional space which is one of the two possible spaces allowed by the first condition. Thus the set (of the two sets together), with six parameters (three complex parameters so six real numbers) plus the set of unitary transformations mixing the blocks (spinors) have a (homomorphic) set of unitary transformations going with it.

In summary what transformations are there for this 4-dimensional complex statefunction? Clearly those of SU(4). However if we rotate (with complex parameters) we induce transformations on each of the two-dimensional spinors. There are two 3-dimensional real spaces whose transformations act independently on these spinors (because SO(4), being uniquely semisimple, splits into two parts). This gives 12 transformations. Also the pair of spinors is a 2-dimensional complex number, giving three more transformations. The total number is thus 15, equal to that of SU(4).

Space must have dimension 4. Yet that is not unique. Is it 4 of space, 3 of space and 1 of time, or 2 of each? The signature is irrelevant to these arguments; only the number of parameters mat- ters. Both SO(4) and SO(2,2) are only semisimple ([1], p. 868; [15], p. 274), which is necessary for this argument ([3], p. 52; [4], p. 340). Each consists of two parts these acting on 3-dimensional spaces. But 3-dimensional spaces do not satisfy as we see. However SO(3,1) fortunately is simple. Physics, a universe, is only possible in a space of dimension 3+1, three of space plus one of time.

These conditions are necessary — only dimension 3+1 is possible — but that does not mean that the groups are homomorphic, only that they cannot be in any other dimension. It is well-known that they are ([1], p. 65), ([14], p. 6), so this need not be discussed here.

The universe is possible, but just barely. The equations must have integer solutions. There is no reason to expect that any have integer solutions, that there is any integer that satisfies any one — and certainly not all. Change only one number in any of the equations even by 1. Then its solution would not be an integer. That mere change would make universes hopeless. And there must be two blocks, which is just what the inversion gives. But the inversion has nothing to do with the counting arguments that give the requirement of two blocks. As we see there are several — independent — arguments; none should be satisfied and definitely not all, all together. Yet strangely there is one dimension and one only — it is unique — that does satisfy all — and all at once.

The dimension emphasizes again, as seen in so many ways [5,6,7,8,9,10], that geometry and physics are deeply intertwined. Geometry limits, perhaps determines, physics. Physics is possible only in a universe with the proper geometry. They almost seem, perhaps are, one subject.

Derivation of the dimension was given previously in a somewhat different manner but with additional analysis ([6], chap. 7, p. 122), and will be elsewhere [11]. That is on a more elementary level with greater detail including reasons that the only possible dimension is just the right dimension — certainly for life. What we learn from that more extensive analysis of nature (going well beyond the dimension) is that the laws of physics really do love us.

Acknowledgements

This could not have existed without the kindness of Norma Mankoc Borstnik.

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