Quantum mirrors of log Calabi-Yau surfaces and higher genus curve counting

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Abstract

We present three results, at the intersection of tropical geometry, enumerative geometry, mirror symmetry and non-commutative algebra.

1. A correspondence between Block-Göttsche q-refined tropical curve counting and higher genus log Gromov-Witten theory of toric surfaces.
2. A correspondence between q-refined two-dimensional Kontsevich-Soibelman scattering diagrams and higher genus log Gromov-Witten theory of log Calabi-Yau surfaces.
3. A q-deformation of the Gross-Hacking-Keel mirror construction, producing a deformation quantization with canonical basis for the Gross-Hacking-Keel families of log Calabi-Yau surfaces.

These results are logically dependent: the proof of the third result relies on the second, whose proof itself relies on the first. Nevertheless, each of them is of independent interest.

To my parents.

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Contents

Introduction

In this thesis, we present some contributions at the intersection of tropical geometry, enumerative geometry, mirror symmetry and non-commutative algebra. The text is divided in three chapters.

Chapter 1 is about enumerative geometry, more precisely log Gromov-Witten invariants, of complex toric surfaces, and tropical geometry of the real plane. We solve the all genus log Gromov-Witten theory, with insertion of the top lambda class, of toric surfaces. The answer is formulated in terms of q-refined counts of tropical curves and conversely gives a previously unknown geometric meaning to these q-refined counts.

Chapter 2 is about enumerative geometry, more precisely log Gromov-Witten invariants, of log Calabi-Yau surfaces with maximal boundary, i.e. of pairs (Y, D), where Y is a smooth projective complex surface and D is a singular reduced normal crossing effective anticanonical divisor. The class of log Calabi-Yau surfaces is a natural extension of the class of toric surfaces. In particular, the complement $U = Y - D$ is a non-compact algebraic symplectic surface, generalization of $(\mathbb{C}^*)^2$, and non-compact analogue of K3 surfaces. We solve the all genus log Gromov-Witten theory, with insertion of the top lambda class, of log Calabi-Yau surfaces. The answer is formulated in terms of algebraic and combinatorial objects: q-refined scattering diagrams. The proof is done by reduction to the toric case, for which the main result of Chapter 1 is used.

Chapter 3 is about deformation quantization of log Calabi-Yau surfaces. Using the log Gromov-Witten invariants studied in Chapter 2 as input, we construct non-commutative algebras, deformation quantizations of Poisson algebras of regular functions on the non-compact surfaces U. It seems to be a new way to construct non-commutative algebras.

The genus zero/unrefined/commutative versions of these results were previously known. More precisely, our Chapters 1-2-3 can be viewed as a higher genus/q-refined/non-commutative generalization of the series of papers [Mik05][NS06]-[GPS10]-[GHK15a].

We give below detailed Introductions to each of the three Chapters.

Introduction to Chapter 1

Tropical geometry gives a combinatorial way to approach problems in complex and real algebraic geometry. An early success of this approach is Mikhalkin's correspondence theorem [Mik05], proved differently and generalized by Nishinou and Siebert [NS06], between counts of complex algebraic curves in complex toric surfaces and counts with multiplicity of tropical curves in $\mathbb{R}^2$. The main result of Chapter 1, Theorem 1, is an extension to a correspondence between some generating series of higher genus log Gromov-Witten invariants of toric surfaces and counts with q-multiplicity of tropical curves in $\mathbb{R}^2$.

Counts of tropical curves in $\mathbb{R}^2$ with q-multiplicity were introduced by Block and Göttsche [BG16]. The usual multiplicity of a tropical curve is defined as a product of integer multiplicities attached to the vertices. Block and Göttsche remarked that one can obtain a refinement by replacing the multiplicity m of a vertex by its q-analogue

$$[m]_q \coloneqq \frac{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{-\frac{m-1}{2}} (1 + q + \cdots + q^{m-1}) .$$

The q-multiplicity of a tropical curve is then the product of the q-multiplicities of the vertices. The count with q-multiplicity of tropical curves specializes for $q = 1$ to the ordinary count with multiplicity. This definition is done at the tropical level so is combinatorial in nature and its geometric meaning is *a priori* unclear.

Let Δ be a balanced collection of vectors in $\mathbb{Z}^2$ and let n be a non-negative integer[1]. This determines a complex toric surface X_Δ and a counting problem of virtual dimension zero for complex algebraic curves in X_Δ of some genus $g_{\Delta,n}$, of some class β_Δ, satisfying some tangency conditions with respect to the toric boundary divisor, and passing through n points of X_Δ in general position. Let $N^{\Delta,n} \in \mathbb{N}$ be the solution to this counting problem. According to Mikhalkin's correspondence theorem, $N^{\Delta,n}$ is a count with multiplicity of tropical curves in $\mathbb{R}^2$, and so it has a Block-Göttsche refinement $N^{\Delta,n}(q) \in \mathbb{N}[q^{\pm\frac{1}{2}}]$.

For every $g \geqslant g_{\Delta,n}$, we consider the same counting problem as before—same curve class, same tangency conditions—but for curves of genus g. The virtual dimension is now $g - g_{\Delta,n}$. To obtain a number, we integrate a class of degree $g - g_{\Delta,n}$, the lambda class $\lambda_{g-g_{\Delta,n}}$, over the virtual fundamental class of a corresponding moduli space of stable log maps. For every $g \geqslant g_{\Delta,n}$, we get a log Gromov-Witten invariant $N_g^{\Delta,n} \in \mathbb{Q}$.

Theorem 1. *For every Δ balanced collection of vectors in $\mathbb{Z}^2$, and for every non-negative integer n such that $g_{\Delta,n} \geqslant 0$, we have the equality*

$$\sum_{g \geqslant g_{\Delta,n}} N_g^{\Delta,n} u^{2g-2+|\Delta|} = N^{\Delta,n}(q) \left((-i)(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \right)^{2g_{\Delta,n}-2+|\Delta|}$$

of power series in u with rational coefficients, where

$$q = e^{iu} = \sum_{n \geqslant 0} \frac{(iu)^n}{n!} ,$$

and $|\Delta|$ is the cardinality of Δ.

Remarks

- According to Theorem 1, the knowledge of the Block-Göttsche invariant $N^{\Delta,n}(q)$ is equivalent to the knowledge of the log Gromov-Witten invariants $N_g^{\Delta,n}$ for all $g \geqslant g_{\Delta,n}$. This provides a geometric meaning to Block-Göttsche invariants, independent of any choice of tropical limit, making their deformation invariance manifest.

[1] Precise definitions are given in Section 1.1.

- Given a family $\pi: \mathcal{C} \to B$ of nodal curves, the Hodge bundle $\mathbb{E}$ is the rank g vector bundle over B whose fiber over $b \in B$ is the space $H^0(C_b, \omega_{C_b})$ of sections of the dualizing sheaf ω_{C_b} of the curve $C_b = \pi^{-1}(b)$. The lambda classes are classically [Mum83] the Chern classes of the Hodge bundle:

$$\lambda_j \coloneqq c_j(\mathbb{E}) \,.$$

  The log Gromov-Witten invariants $N_g^{\Delta,n}$ are defined by an insertion of $(-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}}$ to cut down the virtual dimension from $g - g_{\Delta,n}$ to zero.

- One can interpret Theorem 1 as establishing integrality and positivity properties for higher genus log Gromov-Witten invariants of X_Δ with one lambda class inserted.

- The change of variables $q = e^{iu}$ makes the correspondence of Theorem 1 quite non-trivial. In particular, it cannot be reduced to an easy enumerative correspondence. It is essential to have a virtual/non-enumerative count on the Gromov-Witten side: for g large enough, most of the contributions to $N_g^{\Delta,n}$ come from maps with contracted components.

- In Theorem 1.5, we present a generalization of Theorem 1 where some intersection points with the toric boundary divisor can be fixed.

- One could ask for a generalization of Theorem 1 including descendant log Gromov-Witten invariants, i.e. with insertion of psi classes. In the simplest case of a trivalent vertex with insertion of one psi class, we will show in Section 1.9 that it is possible to reproduce the numerator $q^{\frac{m}{2}} + q^{-\frac{m}{2}}$ of the multiplicity introduced by Göttsche and Schroeter [GS16a] in the context of refined broccoli invariants, in a way similar to how we reproduce the numerator $q^{\frac{m}{2}} - q^{-\frac{m}{2}}$ of the Block-Göttsche multiplicity in Theorem 1.

Relation with previous works

q-analogues

It is a general principle in mathematics, going back at least to Heine's introduction of q-hypergeometric series in 1846, that many "classical" notions have a q-analogue, recovering the classical one in the limit $q \to 1$. The Block-Göttsche refinement of the tropical curve counts in $\mathbb{R}^2$ is clearly an example of this principle. In many other examples, it is well known that it is a good idea to write $q = e^{\hbar}$, the limit $q \to 1$ becoming the limit $\hbar \to 0$. From this point of view, the change of variable $q = e^{iu}$ in Theorem 1 is maybe not so surprising.

Göttsche-Shende refinement by Hirzebruch genus

Whereas the specialization of Block-Göttsche invariants at $q = 1$ recovers a count of complex algebraic curves, the specialization $q = -1$ recovers in some cases a count of real algebraic curves in the sense of Welschinger [Wel05]. This strongly suggests a motivic interpretation of the Block-Göttsche invariants and indeed one of the original motivations of Block and

Göttsche was the fact that, under some ampleness assumptions, the refined tropical curve counts should coincide with the refined curve counts on toric surfaces defined by Göttsche and Shende [GS14] in terms of Hirzebruch genera of Hilbert schemes. Using motivic integration, Nicaise, Payne and Schroeter [NPS16] have reduced this conjecture to a conjecture about the motivic measure of a semialgebraic piece of the Hilbert scheme attached to a given tropical curve.

Our approach to the Block-Göttsche refined tropical curve counting is clearly different from the Göttsche-Shende approach: we interpret the refined variable q as coming from the resummation of a genus expansion whereas they interpret it as a formal parameter keeping track of the refinement from some Euler characteristic to some Hirzebruch genus.

The Göttsche-Shende refinement makes sense for surfaces more general than toric ones, as do the higher genus log Gromov-Witten invariants with one lambda class inserted. So one might ask if Theorem 1 can be extended to more general surfaces, as a relation between Göttsche-Shende refined invariants and generating series of higher genus log Gromov-Witten invariants. In Theorem 1.29 and 1.32, we show by combining known results that this is indeed the case for K3 and abelian surfaces. In particular, Theorem 1 is not an isolated fact but part of a family of similar results. The case of a log Calabi-Yau surface obtained as complement of a smooth anticanonical divisor in a del Pezzo surface, and its relation with, in physics terminology, a worldsheet definition of the refined topological string of local del Pezzo 3-folds, will be discussed in a future work.

MNOP

The change of variables $q = e^{iu}$ is reminiscent of the MNOP, [MNOP06a], [MNOP06b], Gromov-Witten/ Donaldson-Thomas (DT) correspondence on 3-folds. The log Gromov-Witten invariants $N_g^{\Delta,n}$ can be rewritten as $\mathbb{C}^*$-equivariant log Gromov-Witten invariants of the 3-fold $X_\Delta \times \mathbb{C}$, where $\mathbb{C}^*$ acts by scaling on $\mathbb{C}$, see Lemma 7 of Maulik-Pandharipande-Thomas [MPT10]. If a log DT theory and a log MNOP correspondence were developed, this would predict that the generating series of $N_g^{\Delta,n}$ is a rational function in $q = e^{iu}$, which is indeed true by Theorem 1. But it would not be enough to imply Theorem 1 because the relation between log DT invariants and Block-Göttsche invariants is *a priori* unclear. In fact, the Göttsche-Shende conjecture and the result of Filippini and Stoppa suggest that Block-Göttsche invariants are refined DT invariants whereas the MNOP correspondence involves unrefined DT invariants. This topic will be discussed in more details elsewhere.

BPS integrality

When the log Gromov-Witten invariants of $X_\Delta \times \mathbb{C}$ coincide with ordinary Gromov-Witten invariants of $X_\Delta \times \mathbb{C}$, which is probably the case if $|v| = 1$ for every $v \in \Delta$ and if the toric boundary divisor of X_Δ is positive enough, then the integrality implied by Theorem 1 coincides with the BPS integrality predicted by Pandharipande [Pan99], and proved via symplectic methods by Zinger [Zin11], for generating series of Gromov-Witten invariants of a 3-fold and of curve class intersecting positively the anticanonical divisor.

Mikhalkin refined real count

Mikhalkin [Mik15] has given an interpretation of some particular Block-Göttsche invariants in terms of counts of real curves. We do not understand the relation with our approach in terms of higher genus log Gromov-Witten invariants. We merely remark that both for us and for Mikhalkin, it is the numerator of the Block-Göttsche multiplicities which appear naturally.

Parker theory of exploded manifolds

This Chapter owes a great intellectual debt towards the paper [Par16] of Brett Parker, where a correspondence theorem between tropical curves in $\mathbb{R}^3$ and some generating series of curve counts in exploded versions of toric 3-folds is proven. Indeed, a conjectural version of Theorem 1 was known to the author around April 2016[2] but it was only after the appearance of [Par16] in August 2016 that it became clear that this result should be provable with existing technology. In particular, the idea to reduce the amount of explicit computations by exploiting the consistency of some gluing formula (see Section 1.7) follows [Par16].

Plan of Chapter 1

In Section 1.1, we fix our notations and we describe precisely the objects involved in the formulation of Theorem 1. In Section 1.2, we review some gluing and vanishing properties of the lambda classes.

The next five Sections form the proof of Theorem 1.

The first step of the proof, described in Section 1.3, is an application of the decomposition formula of Abramovich, Chen, Gross and Siebert [ACGS17a] to the toric degeneration of Nishinou, Siebert [NS06]. This gives a way to write our log Gromov-Witten invariants as a sum of contributions indexed by tropical curves.

In the second step of the proof, described in Sections 1.5 and 1.6, we prove a gluing formula which gives a way to write the contribution of a tropical curve as a product of contributions of its vertices. Here, gluing and vanishing properties of the lambda classes reviewed in Section 1.2, combined with a structure result for non-torically transverse stable log maps proved in Section 1.4, play an essential role. In particular, we only have to glue torically transverse stable log maps and we don't need to worry about the technical issues making the general gluing formula in log Gromov-Witten theory difficult (see Abramovich, Chen, Gross, Siebert [ACGS17b]).

After the decomposition and gluing steps, what remains to do is to compute the contribution to the log Gromov-Witten invariants of a tropical curve with a single trivalent vertex. The third and final step of the proof of Theorem 1, carried out in Section 1.7, is the explicit evaluation of this vertex contribution. Consistency of the gluing formula leads to non-trivial

[2]And was for example presented at the Workshop: Curves on surfaces and 3-folds, EPFL, Lausanne, 21 June 2016.

relations between these vertex contributions, which enable us to reduce the problem to particularly simple vertices. The contribution of these simple vertices is computed explicitly by reduction to Hodge integrals previously computed by Bryan and Pandharipande [BP05] and this ends the proof of Theorem 1.

In Section 1.8, we prove Theorem 1.29 and Theorem 1.32, which are analogues for K3 and abelian surfaces of Theorem 1 for toric surfaces.

In Section 1.9, we make contact in a simple case with refined broccoli invariants.

Introduction to Chapter 2

Statements

We start by giving slightly imprecise versions of the main results of this Chapter. For us, a log Calabi-Yau surface is a pair (Y, D), where Y is a smooth complex projective surface and D is a reduced effective normal crossing anticanonical divisor on Y. A log Calabi-Yau surface (Y, D) has maximal boundary[3] if D is singular.

Theorem 2. *The functions attached to the rays of the q-deformed 2-dimensional Kontsevich-Soibelman scattering diagrams are, after the change of variables $q = e^{i\hbar}$, generating series of higher genus log Gromov-Witten invariants—with maximal tangency condition and insertion of the top lambda class—of log Calabi-Yau surfaces with maximal boundary.*

A precise version of Theorem 2 is given by Theorems 2.6 and 2.7 in Section 2.3.

Theorem 3. *Higher genus log Gromov-Witten invariants–with maximal tangency condition and insertion of the top lambda class–of log Calabi-Yau surfaces with maximal boundary satisfy an Ooguri-Vafa/open BPS integrality property.*

A precise version of Theorem 3 is given by Theorem 2.30 in Section 2.8.

We also formulate a new conjecture.

Conjecture 4. *Higher genus relative Gromov-Witten invariants-with maximal tangency condition and insertion of the top lambda class–of a del Pezzo surface S relatively to a smooth anticanonical divisor are related to refined counts of dimension one stable sheaves on the local Calabi-Yau 3-fold $\mathrm{Tot}K_S$, total space of the canonical line bundle of S.*

A precise version of Conjecture 4 is given by Conjecture 2.41 in Section 2.8.6.

[3] In Chapter 3, following [GHK15a], a log Calabi-Yau surface with maximal boundary is called a Looijenga pair.

SYZ

The Strominger-Yau-Zaslow [SYZ96] picture of mirror symmetry suggests a two steps construction of the mirror of a Calabi-Yau variety admitting a Lagrangian torus fibration: first, construct the "semi-flat" mirror by dualizing the non-singular torus fibers; second, correct the complex structure of the "semi-flat" mirror such that it extends across the locus of singular fibers. It is expected, [SYZ96], [Fuk05], that the corrections involved in the second step are determined by some counts of holomorphic discs in the original variety with boundary on torus fibers.

KS

In dimensional two and with at most nodal singular fibers in the torus fibration, Kontsevich-Soibelman [KS06] had the insight that algebraic self-consistency constraints on the corrections were strong enough to determine these corrections uniquely. More precisely, they reduced the problem to an algebraic computation of commutators in a group of formal families of symplectomorphisms of the dimension two algebraic torus.

This algebraic formalism, graphically encoded under the form of scattering diagrams, was generalized and extended to higher dimensions by Gross-Siebert [GS11] and plays an essential role in the Gross-Siebert algebraic approach to mirror symmetry.

GPS

In [GPS10], Gross-Pandharipande-Siebert made some progress in connecting the original enumerative expectation and the algebraic recipe of scattering diagrams. They showed that the 2-dimensional Kontsevich-Soibelman scattering diagrams indeed have an enumerative meaning: they compute some genus zero log Gromov-Witten invariants of some log Calabi-Yau surfaces with maximal boundary, i.e. complements of a singular normal crossing anticanonical divisor in a smooth projective surface.

This agrees with the original expectation because these geometries admit Lagrangian torus fibrations and these genus zero log Gromov-Witten invariants should be thought as algebraic definitions of some counts of holomorphic discs with boundary on Lagrangian torus fibers[4].

The combination of 2-dimensional scattering diagrams with their enumerative interpretation given by [GPS10] was the main tool in the Gross-Hacking-Keel [GHK15a] construction of mirrors for log Calabi-Yau surfaces with maximal boundary.

[4]For some symplectic approach, relating counts of holomorphic discs in hyperkähler manifolds of real dimension 4 and the Konstevich-Soibelman wall-crossing formula, we refer to the works of Lin [Lin17] and Iacovino [Iac17].

Higher genus GPS = refined KS

At the end of their paper, Section 11.8 of [KS06] (see also [Soi09]), Kontsevich-Soibelman already remarked that the 2-dimensional scattering diagram formalism has a natural q-deformation, with the group of formal families of symplectomorphisms of the 2-dimensional algebraic torus replaced by a group a formal families of automorphisms of the 2-dimensional quantum torus, a natural non-commutative deformation of the 2-dimensional algebraic torus. The enumerative meaning of this q-deformed scattering diagram was *a priori* unclear.

In Section 5.8 of [GPS10], Gross-Pandharipande-Siebert remarked that the genus zero log Gromov-Witten invariants they consider have a natural extension to higher genus, by integration of the top lambda class, and they asked if there is an interpretation of these higher genus invariants in terms of scattering diagrams.

The main result of the present Chapter, Theorem 2, is that the two previous questions, the enumerative meaning of the algebraic q-deformation and the algebraic meaning of the higher genus deformation, are answers to each other.

OV

The higher genus log Gromov-Witten invariants of log Calabi-Yau surfaces that we are considering–with insertion of the top lambda class–should be thought as an algebro-geometric definition of some counts of higher genus Riemann surfaces with boundary on a Lagrangian torus fiber in a Calabi-Yau 3-fold geometry, essentially the product of the log Calabi-Yau surface by a third trivial direction, see Section 2.2.4. For such counts of higher genus open curves in a Calabi-Yau 3-fold geometry, Ooguri-Vafa [OV00] have conjectured an open BPS integrality structure. Theorem 3, which is a consequence of Theorem 2 and of non-trivial algebraic properties of q-deformed scattering diagrams, can be viewed as a check of this BPS integrality structure.

DT

The non-trivial integrality properties of q-deformed scattering diagrams are well-known to be related to integrality properties of refined Donaldson-Thomas (DT) invariants, [KS08]. Indeed, q-deformed scattering diagrams control the wall-crossing behavior of refined DT invariants.

The fact that the integrality structure of DT invariants coincides with the Ooguri-Vafa integrality structure of higher genus open Gromov-Witten invariants of Calabi-Yau 3-folds, essentially involving the quantum dilogarithm in both cases, can be viewed as an early indication that something like Theorem 2 should be true.

As consequence of Theorem 2, we get explicit relations between refined DT invariants of some quivers and higher genus log Gromov-Witten invariants of log Calabi-Yau surfaces, see Section 2.8.5, generalizing the unrefined/genus zero relation of [GP10], [RW13].

CV

In fact, Cecotti-Vafa [CV09] have given a physical derivation of the wall-crossing formula in DT theory going through the higher genus open Gromov-Witten theory of some Calabi-Yau 3-fold. We will explain in Section 2.9 that Theorem 2 and 3 are indeed fully compatible with the Cecotti-Vafa argument. In particular, Theorem 2 can be viewed as a highly non-trivial mathematical check of the connection predicted by Witten [Wit95] between higher genus open A-model and quantum Chern-Simons theory.

del Pezzo

Theorem 2 and 3 are about log Calabi-Yau surfaces with maximal boundary, i.e. with a singular normal crossing anticanonical divisor. Similar questions can be asked for log Calabi-Yau surfaces with respect to a smooth anticanonical divisor. Conjecture 4 gives a non-trivial correspondence in such case, suggested by the similarities between refined DT theory and open higher genus Gromov-Witten invariants discussed above.

Comments on the proof of Theorem 2

The curve counting invariants appearing in Theorem 2 are log Gromov-Witten invariants, as defined by Gross and Siebert [GS13], and Abramovich and Chen [Che14b], [AC14]. The proof of Theorem 2 relies on recently developed general properties of log Gromov-Witten invariants, such as the decomposition formula of [ACGS17a].

The main tool of [GPS10] is a reduction to a tropical setting using the correspondence theorem of Mikhalkin [Mik05] and Nishinou-Siebert [NS06] between counts of curves in complex toric surfaces and counts of tropical curves in $\mathbb{R}^2$. Similarly, the main tool of the present Chapter is a reduction to a tropical setting using the main result, Theorem 1, of Chapter 1.

Given the fact that the relation between q-deformed tropical invariants and q-deformed scattering diagrams has already been worked out by Filippini-Stoppa [FS15], Theorem 2 should really be viewed as a combination of Theorem 1 and [FS15]. The new results required for the proof of Theorem 2 are: the check that the degeneration step used in [GPS10] to go from a log Calabi-Yau setting to a toric setting extends to the higher genus case and the check that the correspondence given by Theorem 1 has exactly the correct form to be used as input in [FS15].

The most technical part is the higher genus version of the degeneration step. As the general version of the degeneration formula in log Gromov-Witten theory is not yet known, we combine the general decomposition formula of [ACGS17a] with some situation specific vanishing statements, which, as in Chapter 1, reduce the gluing operations to some torically transverse locus where they are under control, for example thanks to [KLR18].

Comments on the proof of Theorem 3.

The proof of Theorem 3 is a combination of Theorem 2 and of the non-trivial integrality results about q-deformed scattering diagrams proved by Kontsevich and Soibelman in Section 6 of [KS11]. In fact, to get the most general form of Theorem 3, the results contained in [KS11] do not seem to be enough. We use an induction argument on scattering diagrams, parallel to the one used in Appendix C3 of [GHKK18], to reduce the most general case to a case which can be treated by [KS11].

A small technical point is to keep track of signs, because of the difference between quantum tori and twisted quantum tori, see Section 2.8.3 on the quadratic refinement for details.

Plan of Chapter 2

In Section 2.1, we review the notion of 2-dimensional scattering diagrams, both classical and quantum, with an emphasis on the symplectic/Hamiltonian aspects. In Section 2.2, we introduce a class of log Calabi-Yau surfaces and their log Gromov-Witten invariants.

In Section 2.3, we state our main result, Theorem 2.6, precise version of Theorem 2, relating 2-dimensional quantum scattering diagrams and generating series of higher genus log Gromov-Witten invariants of log Calabi-Yau surfaces. We also state a generalization of Theorem 2.6, Theorem 2.7, phrased in terms of orbifold log Gromov-Witten invariants.

Sections 2.4, 2.5, 2.6, 2.7 are dedicated to the proof of Theorems 2.6 and 2.7. The general structure of the proof is parallel to [GPS10]. In Section 2.4, we introduce higher genus log Gromov-Witten invariants of toric surfaces. In Section 2.5, the most technical part of this Chapter, we prove a degeneration formula relating log Gromov-Witten invariants of log Calabi-Yau surfaces defined in Section 2.2 and appearing in Theorem 2.6, with log Gromov-Witten invariants of toric surfaces defined in Section 2.4.2. In Section 2.6, following Filippini-Stoppa [FS15], we review the connection between quantum scattering diagrams and refined counts of tropical curves. We finish the proof of Theorem 2.6 in Section 2.7, combining the results of Sections 2.5 and 2.6 with Theorem 1. The orbifold Gromov-Witten computation needed to finish the proof of Theorem 2.7 is done in Section 2.7.2.

In Section 2.8.1, we formulate a BPS integrality conjecture for higher genus log Gromov-Witten invariants of log Calabi-Yau surfaces. In Section 2.8.2, we state Theorem 2.30, precise form of Theorem 3. The proof of Theorem 2.30 takes Sections 2.8.3, 2.8.4. In Section 2.8.5, Proposition 2.38 gives an explicit connection with refined DT invariants of quivers. Finally, in Section 2.8.6, we state Conjecture 2.41, precise version of Conjecture 4.

In Section 2.9, we explain how Theorem 2 can be viewed as a mathematical check of the physics work of Cecotti-Vafa [CV09] and how Theorem 3 is compatible with the Ooguri-Vafa integrality conjecture [OV00].

Introduction to Chapter 3

Context and motivations

Mirror symmetry

The Strominger-Yau-Zaslow [SYZ96] picture of mirror symmetry suggests an original way of constructing algebraic varieties: given a Calabi-Yau variety, its mirror geometry should be constructed in terms of the enumerative geometry of holomorphic discs in the original variety. This picture has been developed by Fukaya [Fuk05], Kontsevich-Soibelman [KS06], Gross-Siebert [GS11], Auroux [Aur07] and many others. In particular, Gross and Siebert have developed an algebraic approach in which the enumerative geometry of holomorphic discs is replaced by some genus zero log Gromov-Witten invariants. Given the recent progress in log Gromov-Witten theory, in particular the definition of punctured invariants by Abramovich-Chen-Gross-Siebert [ACGS17b], it is likely that this approach will lead to some general mirror symmetry construction in the algebraic setting, see Gross-Siebert [GS16b] for an announcement.

The work of Gross-Hacking-Keel

An early version of this mirror construction has been used by Gross-Hacking-Keel [GHK15a] to construct mirror families of log Calabi-Yau surfaces, with non-trivial applications to the theory of surface singularities and in particular a proof of the Looijenga's conjecture on smoothing of cusp singularities. More precisely, the construction of [GHK15a] applies to Looijenga pairs, i.e. to pairs (Y, D), where Y is a smooth projective complex surface and D is some reduced effective normal crossing anticanonical divisor on Y. The upshot is in general a formal flat family $\mathcal{X} \to S$ of surfaces over a formal completion, near some point s_0, the "large volume limit of Y", of an algebraic approximation to a compactification of the complexified Kähler cone of Y.

Furthermore, $\mathcal{X}$ is an affine Poisson formal variety with a canonical linear basis of so-called theta functions and the map $\mathcal{X} \to S$ is Poisson if S is equipped with the zero Poisson bracket. Under some positivity assumptions on (Y, D), this family can be in fact extended to an algebraic family over an algebraic base and the generic fiber is then a smooth algebraic symplectic surface.

The first step of the construction involves defining the fiber $\mathcal{X}_{s_0}$, i.e. the "large complex structure limit" of the family $\mathcal{X}$. This step is essentially combinatorial and can be reduced to some toric geometry: $\mathcal{X}_{s_0}$ is a reducible union of toric varieties.

The second step is to construct $\mathcal{X}$ by smoothing of $\mathcal{X}_{s_0}$. This construction is based on the consideration of an algebraic object, a scattering diagram, notion introduced by Kontsevich-Soibelman [KS06] and further developed by Gross-Siebert [GS11], whose definition encodes genus zero log Gromov-Witten invariants[5] of (Y, D). The key non-trivial property to check

[5] In fact, in [GHK15a], an ad hoc definition of genus zero Gromov-Witten invariants is used,which was supposed to coincide with genus zero log Gromov-Witten invariants. This fact follows from the Remark at

is the so-called consistency of the scattering diagram. In [GHK15a], the consistency relies on the work of Gross-Pandharipande-Siebert [GPS10], which itself relies on connection with tropical geometry [Mik05], [NS06]. Once the consistency of the scattering diagram is guaranteed, some combinatorial objects, the broken lines [Gro10], [CPS10], are well-defined and can be used to construct the algebra of functions $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ with its linear basis of theta functions.

Quantization[6]

The variety $\mathcal{X}$ being a Poisson variety over S, it is natural to ask about its quantization, for example in the sense of deformation quantization. As $\mathcal{X}$ and S are affine, the deformation quantization problem takes its simplest form: to construct a structure of non-commutative $H^0(S, \mathcal{O}_S)[[\hbar]]$-algebra on $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})[[\hbar]]$ whose commutator is given at the linear order in $\hbar$ by the Poisson bracket on $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. There are general existence results, [Kon01], [Yek05], for deformation quantizations of smooth affine Poisson varieties. Some useful reference on deformation quantization of algebraic symplectic varieties is [BK04]. In fact, on the smooth locus of $\mathcal{X} \to S$, we have something relative symplectic of relative dimension two and then the existence of a deformation is easy because the obstruction space vanishes for dimension reasons. But they are no known general results which would guarantee a priori the existence of a deformation quantization of $\mathcal{X}$ over S because $\mathcal{X} \to S$ is singular, e.g. over $s_0 \in S$ to start with. Specific examples of deformation quantization of such geometries usually involve some situation-specific representation theory or geometry, e.g. see [Obl04], [EOR07], [EG10], [AK17].

Main results.

The main result of the present Chapter is a construction of a deformation quantization of $\mathcal{X} \to S$. Our construction follows the lines of Gross-Hacking-Keel [GHK15a] except that, rather than to use only genus zero log Gromov-Witten invariants, we use higher genus log Gromov-Witten invariants, the genus parameter playing the role of the quantization parameter $\hbar$ on the mirror side.

We construct a quantum version of a scattering diagram and we prove its consistency using the main result of Chapter 2. Once the consistency of the quantum scattering diagram is guaranteed, some quantum version of the broken lines are well-defined and can be used to construct a deformation quantization of $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. In fact, it follows from Chapter 2 that the dependence on the deformation parameter $\hbar$ is in fact algebraic[7] in $q = e^{i\hbar}$, something which in general cannot be obtained from some general deformation theoretic argument. In other words, the main result of the present Chapter can be phrased in the following slightly

the end of Section 4 of Chapter 1. In the present Chapter, we use log Gromov-Witten theory systematically.

[6] The existence of theta functions is related to the geometric quantization of the real integrable system formed by a Calabi-Yau manifold with a SYZ fibration. We do NOT refer to this quantization story. For us, quantization always means deformation quantization of a holomorphic symplectic/Poisson variety.

[7] Because in general $\mathcal{X}$ is already a formal object, this claim has to be stated more precisely, see Theorem 3.9. It is correct in the most naive sense if (Y, D) is positive enough and $\mathcal{X}$ is then really an algebraic family.

vague terms (see Theorems 3.7, 3.8 and 3.9 for precise statements).

Theorem 5. *The Gross-Hacking-Keel [GHK15a] Poisson family $\mathcal{X} \to S$, mirror of a Looijenga pair (Y, D), admits a deformation quantization, which can be constructed in a synthetic way from the higher genus log Gromov-Witten theory of (Y, D). Furthermore, the dependence on the deformation quantization parameter $\hbar$ is algebraic in $q = e^{i\hbar}$.*

The notion of quantum scattering diagram is already suggested at the end of Section 11.8 of [KS06] and was used by Soibelman [Soi09] to construct non-commutative deformations of non-archimedean K3 surfaces. The connection with quantization, e.g. in the context of cluster varieties [FG09a], [FG09b], was expected, and quantum broken lines have been studied by Mandel [Man15]. The key novelty is the connection between these algebraic/combinatorial q-deformations and the geometric deformation given by higher genus log Gromov-Witten theory.

This connection between higher genus Gromov-Witten theory and quantization is perhaps a little surprising, even if similarly looking statement are known or expected. In Section 3.6, we explain that Theorem 5 should be viewed as an example of higher genus mirror symmetry relation, the deformation quantization being a 2-dimensional reduction of the 3-dimensional higher genus B-model (BCOV theory). We also comment on the relation with some string theoretic expectation, in a way parallel to Section 2.9 of Chapter 2.

In the context of mirror symmetry, there is a well-known symplectic interpretation of some non-commutative deformations on the B-side, involving deformation of the complexified symplectic form which do not preserve the Lagrangian nature of the fibers of the SYZ fibration. An example of this phenomenon has been studied by Auroux-Katzarkov-Orlov [AKO06] in the context of mirror symmetry for del Pezzo surfaces. Further examples should appear in some work of Sheridan and Pascaleff. This approach remains entirely into the traditional realm of genus zero holomorphic curves and so is completely different[8] from our approach using higher genus curves.

It is natural to ask how is the deformation quantization given by Theorem 5 related to previously known examples of quantization. In Section 3.5, we treat a simple example and we recover a well-known description of the A_2 quantum $\mathcal{X}$-cluster variety [FG09a].

For Y a cubic surface in $\mathbb{P}^3$ and D a triangle of lines on Y, the quantum scattering diagram can be explicitly computed and so using techniques similar to those developed in [GHK], one should be able to show that the deformation quantization given by Theorem 5 coincides with the one constructed by Oblomkov [Obl04] using Cherednik algebras (double affine Hecke algebras). We leave this verification, and the general relation to quantum $\mathcal{X}$-cluster varieties, to some future work.

Similarly, if Y is a del Pezzo surface of degree 1, 2 or 3 and D a nodal cubic, it would be interesting to compare Theorem 5 with the construction of Etingof, Oblomkov, Rains [EOR07] using Cherednik algebras. In these cases, the quantum scattering diagrams are extremely complicated and new ideas are probably required.

Finally, we mention that Gross-Hacking-Keel-Siebert [GHKS] have given a mirror construc-

[8]The compatibility of these two approaches can be understood via a chain of string theoretic dualities.

tion for K3 surfaces, producing canonical bases of theta functions for homogeneous coordinate rings. This construction uses scattering diagrams whose initial data are the scattering diagrams considered in [GHK15a] for the log Calabi-Yau surfaces which are irreducible components of the special fiber of a maximal degeneration of K3 surfaces. By using the quantum scattering diagrams leading to the proof of Theorem 5, we expect to be able to construct deformation quantizations with canonical bases for K3 surfaces.

Comments on the proof of Theorem 5

Our proof of Theorem 5 follows closely the structure of [GHK15a]. When an argument in the quantum case is formally parallel to its classical version, we often simply refer to [GHK15a]. The parts that we treat with care are those involving the non-commutative rings, building blocks of the gluing construction, and in particular the computations potentially affected with ordering issues, which have no analogue in the commutative context of [GHK15a].

Plan of Chapter 3

In Section 3.1, we set-up our notations and we give precise versions of the main results. In Section 2.1, we describe the formalism of quantum scattering diagrams and quantum broken lines. In Section 3.3, we explain how to associate to every Looijenga pair (Y, D) a canonical quantum scattering diagram constructed in terms of higher genus log Gromov-Witten invariants of (Y, D). The key result in our construction is Theorem 3.26 establishing the consistency of the canonical quantum scattering diagram. The proof of Theorem 3.26 follows the reduction steps used by Gross-Hacking-Keel [GHK15a] in the genus zero case. In the final step, we use the main result of Chapter 2 in place of the main result of [GPS10]. In Section 3.4, we finish the proofs of the main theorems. In Section 3.5, we work out some explicit example. Finally, in Section 3.6, we discuss the relation of our main result, Theorem 5, with higher genus mirror symmetry and some string theoretic arguments.

1

Tropical refined curve counting from higher genera

1.1 Precise statement of the main result

1.1.1 Toric geometry

Let Δ be a balanced collection of vectors in $\mathbb{Z}^2$, i.e. a finite collection of vectors in $\mathbb{Z}^2 - \{0\}$ summing to zero[1]. Let $|\Delta|$ be the cardinality of Δ. For $v \in \mathbb{Z}^2 - \{0\}$, let $|v|$ the divisibility of v in $\mathbb{Z}^2$, i.e. the largest positive integer k such that we can write $v = kv'$ with $v' \in \mathbb{Z}^2$. Then the balanced collection Δ defines the following data by standard toric geometry.

- A projective[2] toric surface X_Δ over $\mathbb{C}$, whose fan has rays $\mathbb{R}_{\geqslant 0} v$ generated by the vectors $v \in \mathbb{Z}^2 - \{0\}$ contained in Δ. We denote ∂X_Δ the toric boundary divisor of X_Δ.
- A curve class β_Δ on X_Δ, whose polytope is dual to Δ. If ρ is a ray in the fan of X_Δ, we write D_ρ for the prime toric divsisor of X_Δ dual to ρ and Δ_ρ the set of elements $v \in \Delta$ such that $\mathbb{R}_{\geqslant 0} v = \rho$. Then we have

$$\beta_\Delta . D_\rho = \sum_{v \in \Delta_\rho} |v| \,,$$

and these intersection numbers uniquely determine β_Δ. The total intersection number

[1] A given element of $\mathbb{Z}^2 - \{0\}$ can appear several times in Δ. Here we follow the notation used by Itenberg and Mikhalkin in [IM13].

[2] This is true only if the elements in Δ are not all collinear. If they are, we replace X_Δ by a toric compactification whose choice will be irrelevant for our purposes.

of β_Δ with the toric boundary divisor ∂X_Δ is given by

$$\beta_\Delta.(-K_{X_\Delta}) = \sum_{v \in \Delta} |v| \,.$$

- Tangency conditions for curves of class β_Δ with respect to the toric boundary divisor of X_Δ. We say that a curve C is of type Δ if it is of class β_Δ and if for every ray ρ in the fan of X_Δ, the curve C intersects D_ρ in $|\Delta_\rho|$ points with multiplicities $|v|$, $v \in \Delta_\rho$. Similarly, we have a notion of stable log map of type Δ.
- An asymptotic form for a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ in $\mathbb{R}^2$. We say that a parametrized tropical curve in $\mathbb{R}^2$ is of type Δ if it has $|\Delta|$ unbounded edges, with directions v and with weights $|v|$, $v \in \Delta$.

1.1.2 Log Gromov-Witten invariants

The moduli space of n-pointed genus g stable maps to X_Δ of class β_Δ intersecting properly the toric boundary divisor ∂X_Δ with tangency conditions prescribed by Δ is not proper: a limit of curves intersecting ∂X_Δ properly does not necessarily intersect ∂X_Δ properly. A nice compactification of this space is obtained by considering stable log maps. The idea is to allow maps intersecting ∂X_Δ non-properly, but to remember some additional information under the form of log structures, which give a way to make sense of tangency conditions even for non-proper intersections. The theory of stable log maps has been developed by Gross and Siebert [GS13], and Abramovich and Chen [Che14b], [AC14]. By stable log maps, we always mean basic stable log maps in the sense of [GS13]. We refer to Kato [Kat89] for elementary notions of log geometry.

We consider the toric divisorial log structure on X_Δ and use it to view X_Δ as a log scheme. Let $\overline{M}_{g,n,\Delta}$ be the moduli space of n-pointed genus g stable log maps to X_Δ of type Δ. By n-pointed, we mean that the source curves are equipped with n marked points *in addition* to the marked points keeping track of the tangency conditions with respect to the toric boundary divisor. We consider that the latter are notationally already included in Δ.

By the work of Gross, Siebert [GS13] and Abramovich, Chen [Che14b], [AC14], $\overline{M}_{g,n,\Delta}$ is a proper Deligne-Mumford stack[3] of virtual dimension

$$\operatorname{vdim} \overline{M}_{g,n,\Delta} = g - 1 + n + \beta_\Delta.(-K_{X_\Delta}) - \sum_{v \in \Delta} (|v| - 1) = g - 1 + n + |\Delta| \,,$$

and it admits a virtual fundamental class

$$[\overline{M}_{g,n,\Delta}]^{\text{virt}} \in A_{\operatorname{vdim} \overline{M}_{g,n,\Delta}}(\overline{M}_{g,n,\Delta}, \mathbb{Q}) \,.$$

The problem of counting n-pointed genus g curves passing though n fixed points has virtual

[3]Moduli spaces of stable log maps have a natural structure of log stack. The structure of log stack is particularly important to treat correctly evaluation morphisms in log Gromov-Witten theory in general, see [ACGM10]. We will always consider these moduli spaces as stacks over the category of schemes, not as log stacks, and we will always work with naive evaluation morphisms between stacks, not log stacks. This will be enough for us. See the remark at the end of Section 1.3.2 for some justification.

dimension zero if

$$\mathrm{vdim}\,\overline{M}_{g,n,\Delta} = 2n\,,$$

i.e. if the genus g is equal to

$$g_{\Delta,n} \coloneqq n + 1 - |\Delta|\,.$$

In this case, the corresponding count of curves is given by

$$N^{\Delta,n} \coloneqq \langle \tau_0(\mathrm{pt})^n \rangle_{g_{\Delta,n},n,\Delta} \coloneqq \int_{[\overline{M}_{g_{\Delta,n},n,\Delta}]^{\mathrm{virt}}} \prod_{j=1}^{n} \mathrm{ev}_j^*(\mathrm{pt})\,,$$

where $\mathrm{pt} \in A^2(X_\Delta)$ is the class of a point and ev_j is the evaluation map at the j-th marked points.

According to Mandel and Ruddat [MR16], Mikhalkin's correspondence theorem can be reformulated in terms of these log Gromov-Witten invariants. Our refinement of the correspondence theorem will involve curves of genus $g \geqslant g_{\Delta,n}$.

For $g > g_{\Delta,n}$, inserting n points is no longer enough to cut down the virtual dimension to zero. The idea is to consider the Hodge bundle $\mathbb{E}$ over $\overline{M}_{g,n,\Delta}$. If $\pi\colon \mathcal{C} \to \overline{M}_{g,n,\Delta}$ is the universal curve, of relative dualizing[4] sheaf ω_π, then

$$\mathbb{E} \coloneqq \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{M}_{g,n,\Delta}$. The Chern classes of the Hodge bundles are classically [Mum83] called the lambda classes and denoted as

$$\lambda_j \coloneqq c_j(\mathbb{E})\,,$$

for $j = 0, \dots, g$. Because the virtual dimension of $\overline{M}_{g,n,\Delta}$ is given by

$$\mathrm{vdim}\,\overline{M}_{g,n,\Delta} = g - g_{\Delta,n} + 2n\,,$$

inserting the lambda class $\lambda_{g-g_{\Delta,n}}$ and n points will cut down the virtual dimension to zero, so it is natural to consider the log Gromov-Witten invariants with one lambda class inserted

$$N_g^{\Delta,n} \coloneqq \langle (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} \tau_0(\mathrm{pt})^n \rangle_{g,n,\Delta}$$

$$\coloneqq \int_{[\overline{M}_{g,n,\Delta}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} \prod_{j=1}^{n} \mathrm{ev}_j^*(\mathrm{pt})\,.$$

Our refined correspondence result, Theorem 1.4, gives an interpretation of the generating series of these invariants in terms of refined tropical curve counting.

[4]The dualizing line bundle of a nodal curve coincides with the log cotangent bundle up to some twist by marked points and so is a completely natural object from the point of view of log geometry.

1.1.3 Tropical curves

We refer to Mikhalkin [Mik05], Nishinou, Siebert [NS06], Mandel, Ruddat [MR16], and Abramovich, Chen, Gross, Siebert [ACGS17a] for basics on tropical curves. Each of these references uses a slightly different notion of parametrized tropical curve. We will use a variant of [ACGS17a], Definition 2.5.3, because it is the one which is the most directly related to log geometry. It is easy to go from one to the other.

For us, a graph Γ has a finite set $V(\Gamma)$ of vertices, a finite set $E_f(\Gamma)$ of bounded edges connecting pairs of vertices and a finite set $E_\infty(\Gamma)$ of legs attached to vertices that we view as unbounded edges. By edge, we refer to a bounded or unbounded edge. We will always consider connected graphs.

A parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is the following data:

- A non-negative integer $g(V)$ for each vertex V, called the genus of V.
- A bijection of the set $E_\infty(\Gamma)$ of unbounded edges with

$$\{1, \dots, |E_\infty(\Gamma)|\} \,,$$

  where $|E_\infty(\Gamma)|$ is the cardinality of $E_\infty(\Gamma)$.
- A vector $v_{V,E} \in \mathbb{Z}^2$ for every vertex V and E an edge adjacent to V. If $v_{V,E}$ is not zero, the divisibility $|v_{V,E}|$ of $v_{V,E}$ in $\mathbb{Z}^2$ is called the weight of E and is denoted $w(E)$. We require that $v_{V,E} \neq 0$ if E is unbounded and that for every vertex V, the following balancing condition is satisfied:

$$\sum_E v_{V,E} = 0 \,,$$

  where the sum is over the edges E adjacent to V. In particular, the collection Δ_V of non-zero vectors $v_{\Delta,E}$ for E adjacent to V is a balanced collection as in Section 1.1.1.
- A non-negative real number $\ell(E)$ for every bounded edge of E, called the length of E.
- A proper map $h\colon \Gamma \to \mathbb{R}^2$ such that
  - If E is a bounded edge connecting the vertices V_1 and V_2, then h maps E affine linearly on the line segment connecting $h(V_1)$ and $h(V_2)$, and $h(V_2) - h(V_1) = \ell(E) v_{V_1,E}$.
  - If E is an unbounded edge of vertex V, then h maps E affine linearly to the ray $h(V) + \mathbb{R}_{\geqslant 0} v_{V,E}$.

The genus g_h of a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is defined by

$$g_h \coloneqq g_\Gamma + \sum_{V \in V(\Gamma)} g(V) \,,$$

where g_Γ is the genus of the graph Γ.

We fix Δ a balanced collection of vectors in $\mathbb{Z}^2$, as in Section 1.1.1, and we fix a bijection of Δ with $\{1, \dots, |\Delta|\}$. We say that a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ is of type Δ if there exists a bijection between Δ and $\{v_{V,E}\}_{E\in E_\infty(\Gamma)}$ compatible with the fixed bijections to

$$\{1,\dots,|\Delta|\} = \{1,\dots,|E_\infty(\Gamma)|\}\,.$$

Remark that

$$\sum_{E\in E_\infty(\Gamma)} v_{V,E} = 0$$

by the balancing condition.

We say that a parametrized tropical curve $h:\Gamma \to \mathbb{R}^2$ is n-pointed if we have chosen a distribution of the labels $1,\dots,n$ over the vertices of Γ, a vertex having the possibility to have several labels. Vertices without any label are said to be unpointed whereas those with labels are said to be pointed. For $j = 1,\dots,n$, let V_j be the pointed vertex having the label j. Let $p = (p_1,\dots,p_n)$ be a configuration of n points in $\mathbb{R}^2$. We say that a n-pointed parametrized tropical curve $h:\Gamma \to \mathbb{R}^2$ passes through p if $h(V_j) = p_j$ for every $j = 1,\dots,n$. We say that a n-pointed parametrized tropical curve $h:\Gamma \to \mathbb{R}^2$ passing through p is rigid if it is not contained in a non-trivial family of n-pointed parametrized tropical curves passing through p of the same combinatorial type.

Proposition 1.1. *For every balanced collection Δ of vectors in $\mathbb{Z}^2$, and n a non-negative integer such that $g_{\Delta,n} \geqslant 0$, there exists an open dense subset $U_{\Delta,n}$ of $(\mathbb{R}^2)^n$ such that if $p = (p_1,\dots,p_n) \in U_{\Delta,n}$ then $p_j \neq p_k$ for $j \neq k$ and if $h:\Gamma \to \mathbb{R}^2$ is a rigid[5] n-pointed parametrized tropical curve of genus $g \leq g_{\Delta,n}$ and of type Δ passing through p, then*

- *$g = g_{\Delta,n}$.*
- *We have $g(V) = 0$ for every vertex V of Γ. In particular, the graph Γ has genus $g_{\Delta,n}$.*
- *Images by h of distinct vertices are distinct.*
- *No edge is contracted to a point.*
- *Images by h of two distinct edges intersect in at most one point.*
- *Unpointed vertices are trivalent.*
- *Pointed vertices are bivalent.*

Proof. This is essentially Proposition 4.11 of Mikhalkin [Mik05], which itself is essentially some counting of dimensions. In [Mik05], there is no genus attached to the vertices but if we have a parametrized tropical curve of genus $g \leq g_{\Delta,n}$ with some vertices of non-zero genus, the underlying graph has genus strictly less than g and so strictly less than $g_{\Delta,n}$, which is impossible by Proposition 4.11 of [Mik05] for p general enough. ☐

Proposition 1.2. *If $p \in U_{\Delta,n}$, then the set $T_{\Delta,p}$ of rigid n-pointed genus $g_{\Delta,n}$ parametrized tropical curves $h:\Gamma \to \mathbb{R}^2$ of type Δ passing through p is finite.*

[5] Here, the rigidity assumption is only necessary to forbid contracted edges. It happens to be the natural assumption in the general form of the decomposition formula of [ACGS17a], as explained and used in Section 1.3.3.

Proof. This is Proposition 4.13 if Mikhalkin [Mik05]: there are finitely many possible combinatorial types for a parametrized tropical curve as in Proposition 1.1, and for a fixed combinatorial type, the set of such tropical curves passing trough p is a zero dimensional intersection of a linear subspace with an open convex polyhedron, so is a point. ☐

Lemma 1.3. *Let $h: \Gamma \to \mathbb{R}^2$ be a parametrized tropical curve in $T_{\Delta,p}$. Then Γ has*

$$2g_{\Delta,n} - 2 + |\Delta|$$

trivalent vertices.

Proof. By definition of $T_{\Delta,p}$, the graph Γ is of genus $g_{\Delta,n}$ and its vertices are either trivalent or bivalent. Replacing the two edges adjacent to each bivalent vertex by a unique edge, we obtain a trivalent graph $\hat{\Gamma}$ with the same genus and the same number of unbounded edges as Γ. Let $|V(\hat{\Gamma})|$ be the number of vertices of $\hat{\Gamma}$ and let $|E_f(\hat{\Gamma})|$ be the number of bounded edges of $\hat{\Gamma}$. A count of half-edges using that $\hat{\Gamma}$ is trivalent gives

$$3|V(\hat{\Gamma})| = 2|E_f(\hat{\Gamma})| + |\Delta|\,.$$

By definition of the genus, we have

$$1 - g_{\Delta,n} = |V(\hat{\Gamma})| - |E_f(\hat{\Gamma})|\,.$$

Eliminating $|E_f(\hat{\Gamma})|$ from the two previous equalities gives the desired formula and so finishes the proof of Lemma 1.3. ☐

For $h: \Gamma \to \mathbb{R}^2$ a parametrized tropical curve in $\mathbb{R}^2$ and V a trivalent vertex of adjacent edges E_1, E_2 and E_3, the multiplicity of V is the integer defined by

$$m(V) \coloneqq |\det(v_{V,E_1}, v_{V,E_2})|\,.$$

Thanks to the balancing condition

$$v_{V,E_1} + v_{V,E_2} + v_{V,E_3} = 0\,,$$

we also have

$$m(V) = |\det(v_{V,E_2}, v_{V,E_3})| = |\det(v_{V,E_3}, v_{V,E_1})|\,.$$

For $(h: \Gamma \to \mathbb{R}^2) \in T_{\Delta,p}$, the multiplicity of h is defined by

$$m_h \coloneqq \prod_{V \in V^{(3)}(\Gamma)} m(V)\,,$$

where the product is over the trivalent, i.e. unpointed, vertices of Γ.

Let $N_{\text{trop}}^{\Delta,p}$ be the count with multiplicity of n-pointed genus $g_{\Delta,n}$ parametrized tropical curves of type Δ passing through p, i.e.

$$N_{\mathrm{trop}}^{\Delta,p} \coloneqq \sum_{h \in T_{\Delta,p}} m_h \,.$$

This tropical count with multiplicity has a natural refinement, first suggested by Block and Göttsche [BG16]. We can replace the integer valued multiplicity m_h of a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ by the $\mathbb{N}[q^{\pm\frac{1}{2}}]$-valued multiplicity

$$m_h(q) \coloneqq \prod_{V \in V^{(3)}(\Gamma)} \frac{q^{\frac{m(V)}{2}} - q^{-\frac{m(V)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = \prod_{V \in V^{(3)}(\Gamma)} \left(\sum_{j=0}^{m(V)-1} q^{-\frac{m(V)-1}{2}+j} \right),$$

where the product is taken over the trivalent vertices of Γ. The specialization $q = 1$ recovers the usual multiplicity:

$$m_h(1) = m_h \,.$$

Counting the parametrized tropical curves in $T_{\Delta,p}$ as above but with q-multiplicities, we obtain a refined tropical count

$$N_{\mathrm{trop}}^{\Delta,p}(q) \coloneqq \sum_{h \in T_{\Delta,p}} m_h(q) \in \mathbb{N}[q^{\pm\frac{1}{2}}] \,,$$

which specializes to the tropical count $N_{\mathrm{trop}}^{\Delta,p}$ at $q = 1$:

$$N_{\mathrm{trop}}^{\Delta,p}(1) = N_{\mathrm{trop}}^{\Delta,p} \,.$$

1.1.4 Unrefined correspondence theorem

Let Δ be a balanced collection of vectors in $\mathbb{Z}^2$, as in Section 1.1.1, and let n be a non-negative integer and $p \in U_{\Delta,n}$. Then we have some log Gromov-Witten count $N^{\Delta,n}$ of n-pointed genus $g_{\Delta,n}$ curves of type Δ passing through n points in the toric surface X_Δ (see Section 1.1.2), and we have some count with multiplicity $N_{\mathrm{trop}}^{\Delta,n}$ of n-pointed genus $g_{\Delta,n}$ tropical curves of type Δ passing through n points $p = (p_1, \dots, p_n)$ in $\mathbb{R}^2$ (see Section 1.1.3). The (unrefined) correspondence theorem then takes the simple form

$$N^{\Delta,n} = N_{\mathrm{trop}}^{\Delta,p}.$$

The result proved by Mikhalkin [Mik05] and generalized by Nishinou, Siebert [NS06] is an equality between the tropical count $N_{\mathrm{trop}}^{\Delta,n}$ and an enumerative count of algebraic curves. The fact that this enumerative count coincides with the log Gromov-Witten count $N^{\Delta,n}$ is proved by Mandel and Ruddat in [MR16].

1.1.5 Refined correspondence theorem

The Block-Göttsche refinement from $N^{\Delta,p}$ to $N^{\Delta,p}(q)$, reviewed in Section 1.1.3, is done at the tropical level so is combinatorial in nature and its geometric meaning is a priori unclear.

The main result of the present Chapter is a new non-tropical interpretation of Block-Göttsche invariants in terms of the higher genus log Gromov-Witten invariants with one lambda class inserted $N_{\Delta,n}^g$ that we introduced in Section 1.1.2. In particular, this geometric interpretation is independent of any tropical limit and makes the tropical deformation invariance of Block-Göttsche invariants manifest.

More precisely, we prove a refined correspondence theorem, already stated as Theorem 1 in the Introduction.

Theorem 1.4. *For every Δ balanced collection of vectors in $\mathbb{Z}^2$, for every non-negative integer n such that $g_{\Delta,n} \geqslant 0$, and for every $p \in U_{\Delta,n}$, we have the equality*

$$\sum_{g \geqslant g_{\Delta,n}} N_g^{\Delta,n} u^{2g-2+|\Delta|} = N_{\text{trop}}^{\Delta,p}(q) \left((-i)(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\right)^{2g_{\Delta,n}-2+|\Delta|}$$

of power series in u with rational coefficients, where

$$q = e^{iu} = \sum_{n \geqslant 0} \frac{(iu)^n}{n!} .$$

Remarks

- The change of variables $q = e^{iu}$ makes the above correspondence quite non-trivial. In particular, in contrast to its unrefined version, it cannot be reduced to a finite to one enumerative correspondence. It is essential to have a virtual/non-enumerative count on the Gromov-Witten side: for g large enough, most of the contributions to $N_g^{\Delta,n}$ come from maps with contracted components.

- The refined tropical count has the symmetry $N_{\text{trop}}^{\Delta,n}(q) = N_{\text{trop}}^{\Delta,n}(q^{-1})$ and so, after the change of variables $q = e^{iu}$, is a even power series in u. In particular, as

  $$(-i)(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \in u\mathbb{Q}[[u^2]] ,$$

  the tropical side of Theorem 1.4 lies in

  $$u^{2g_{\Delta,n}-2+|\Delta|}\mathbb{Q}[[u^2]] ,$$

  as does the Gromov-Witten side. Taking the leading order terms on both sides in the limit $u \to 0$, $q \to 1$, we recover the unrefined correspondence theorem $N^{\Delta,n} = N_{\text{trop}}^{\Delta,p}$.

- By Lemma 1.3, we know that $2g_{\Delta,n} - 2 + |\Delta|$ is the number of trivalent vertices of a parametrized tropical curve in $T_{\Delta,p}$. In particular, the tropical side of Theorem 1.4 can be obtained directly by considering only the numerators of the Block-Göttsche multiplicities, i.e. Theorem 1.4 can be rewritten

  $$\sum_{g \geqslant g_{\Delta,n}} N_g^{\Delta,n} u^{2g-2+|\Delta|} = \sum_{h \in T_{\Delta,p}} \prod_V (-i) \left(q^{\frac{m(V)}{2}} - q^{-\frac{m(V)}{2}}\right) ,$$

  where $q = e^{iu}$.

1.1.6 Fixing points on the toric boundary

It is possible to generalize Theorem 1.4 by fixing the position of some of the intersection points with the toric boundary divisor. Let Δ^F be a subset of Δ and let

$$\mathrm{ev}_{\Delta^F} \colon \overline{M}_{g,n,\Delta} \to (\partial X_\Delta)^{|\Delta^F|}$$

be the evaluation map at the intersection points with the toric boundary divisor ∂X_Δ indexed by the elements of Δ^F.

The problem of counting n-pointed genus g curves of type Δ passing through n given points of X_Δ and with fixed position of the intersection points with ∂X_Δ indexed by Δ^F, has virtual dimension zero if the genus is equal to

$$g_{\Delta,n}^{\Delta^F} \coloneqq n + 1 - |\Delta| + |\Delta^F| \,.$$

For every $g \geqslant g_{\Delta,n}^{\Delta^F}$, we define the invariants

$$N_{g,\Delta^F}^{\Delta,n} \coloneqq \int_{[\overline{M}_{g,n,\Delta}]^{virt}} (-1)^{g-g_{\Delta,n}^{\Delta^F}} \lambda_{g-g_{\Delta,n}^{\Delta^F}} \mathrm{ev}_{\Delta^F}^* (r^{|\Delta^F|}) \prod_{j=1}^n \mathrm{ev}_j^*(\mathrm{pt}) \,,$$

where $r \in A^1(\partial X_\Delta)$ is the class of a point on ∂X_Δ.

We can consider the corresponding tropical problem. Fix a generic configuration $x = (x_v)_{v \in \Delta^F}$ of points in $\mathbb{R}^2$ and say that a tropical curve of type Δ is of type (Δ, Δ^F) if the unbounded edges in correspondence with Δ^F asymptotically coincide with the half-lines $x_v + \mathbb{R}_{\geqslant 0} v$, $v \in \Delta^F$.

We define a refined tropical count

$$N_{\mathrm{trop},\Delta^F}^{\Delta,p,x}(q) \in \mathbb{N}[q^{\pm\frac{1}{2}}] \,,$$

by counting with q-multiplicity the tropical curves of genus $g_{\Delta,n}^{\Delta^F}$ and of type (Δ, Δ^F) passing through a generic configuration $p = (p_1, \dots, p_n)$ of n points in $\mathbb{R}^2$.

The following result is the generalization of Theorem 1.4 to the case of non-empty Δ^F.

Theorem 1.5. *For every Δ balanced collection of vectors in $\mathbb{Z}^2$, for every Δ^F subset of Δ and for every n non-negative integer such that $g_{\Delta,n}^{\Delta^F} \geqslant 0$, we have the equality*

$$\sum_{g \geqslant g_{\Delta,n}^{\Delta^F}} N_{g,\Delta^F}^{\Delta,n} u^{2g-2+|\Delta|} = \left(\prod_{v \in \Delta^F} \frac{1}{|v|} \right) N_{\mathrm{trop}}^{\Delta,p,x}(q) \left((-i)(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \right)^{2g_{\Delta,n}^{\Delta^F} - 2 + |\Delta|}$$

of power series in u with rational coefficients, where $q = e^{iu}$.

The proof of Theorem 1.5 is entirely parallel to the proof of Theorem 1.4 (Theorem 1 of the Introduction). The required modifications are discussed at the end of Section 1.7.4.

1.1.7 An explicit example

In the present Section, we check by a direct computation one of the consequences of Theorem 1. Let us consider the problem of counting rational cubic curves in $\mathbb{P}^2$ passing through 8 points in general position. To match the notations of the Introduction, we choose Δ containing three times the vector $(1,0)$, three times the vector $(0,1)$ and three times the vector $(-1,-1)$.

The toric surface X_Δ is then $\mathbb{P}^2$ and the curve class β_Δ is the class of a cubic curve in $\mathbb{P}^2$. We have $|\Delta| = 9$, $n = 8$, $g_{\Delta,n} = 0$. Let us write N_g for $N_g^{\Delta,n}$. We have $N_0 = 12$ and the corresponding Block-Göttsche invariant is $q + 10 + q^{-1}$ (see Example 1.3 of [NPS16] for pictures of tropical curves). From the point of view of Göttsche-Shende [GS14], the relevant relative Hilbert scheme to consider happens to be the pencil of cubics passing trough the 8 given points, i.e. $\mathbb{P}^2$ blown-up in 9 points, whose Hirzebruch genus is indeed $1 + 10q + q^2$.

According to Theorem 1.4, we have

$$\sum_{g\geq 0} N_g u^{2g-2+9} = i(q+10+q^{-1})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^7$$

$$= i(q^{\frac{9}{2}} + 3q^{\frac{7}{2}} - 48q^{\frac{5}{2}} + 168q^{\frac{3}{2}} - 294q^{\frac{1}{2}} + 294q^{-\frac{1}{2}} - 168q^{-\frac{3}{2}} + 48q^{-\frac{5}{2}} - 3q^{-\frac{7}{2}} - q^{-\frac{9}{2}})$$

$$= 12u^7 - \frac{9}{2}u^9 + \frac{137}{160}u^{11} - \frac{1253}{11520}u^{13} + \dots$$

We will check directly that $N_1 = -\frac{9}{2}$. Remark that a Block-Göttsche invariant equal to 12 rather than to $q + 10 + q^{-1}$ would lead to $N_1 = -\frac{7}{2}$. In particular, the value of N_1 is already sensitive to the choice of the correct refinement.

We have [6]

$$N_1 = \int_{[\overline{M}_{1,8}(\mathbb{P}^2,3)]^{\text{virt}}} (-1)^1 \lambda_1 \prod_{j=1}^{8} \mathrm{ev}_j^*(\mathrm{pt}),$$

where $\mathrm{pt} \in A^2(\mathbb{P}^2)$ is the class of a point. Introducing an extra marked point and using the divisor equation, one can write

$$N_1 = \frac{1}{3} \int_{[\overline{M}_{1,8+1}(\mathbb{P}^2,3)]^{\text{virt}}} (-1)^1 \lambda_1 \left(\prod_{j=1}^{8} \mathrm{ev}_j^*(\mathrm{pt}) \right) \mathrm{ev}_9^*(h),$$

where $h \in A^1(\mathbb{P}^2)$ is the class of a line. On $\overline{M}_{1,1}$, we have

$$\lambda_1 = \frac{1}{12}\delta_0,$$

where δ_0 is the class of a point. Taking for representative of δ_0 the point corresponding to the nodal genus one curve, with j-invariant $i\infty$, and resolving the node, we can write

$$N_1 = -\frac{1}{12} \cdot \frac{1}{2} \cdot \frac{1}{3} \int_{[\overline{M}_{0,8+1+2}(\mathbb{P}^2,3)]^{\text{virt}}} \left(\prod_{j=1}^{8} \mathrm{ev}_j^*(\mathrm{pt}) \right) \mathrm{ev}_9^*(h)(\mathrm{ev}_{10}^* \times \mathrm{ev}_{11}^*)(D),$$

[6] A general choice of representative for λ_1 cuts out a locus in the moduli space made entirely of torically transverse stable maps. In particular, we do not have to worry about the difference between log and usual stable maps. A general form of this argument is used in the proof of the gluing formula in Section 1.6.

where the factor $\frac{1}{2}$ comes from the two ways of labeling the two points resolving the node, and D is the class of the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$. We have

$$D = 1 \times \mathrm{pt} + \mathrm{pt} \times 1 + h \times h\,.$$

The first two terms do not contribute to N_1 for dimension reasons so

$$N_1 = -\frac{1}{12} \cdot \frac{1}{2} \cdot \frac{1}{3} \int_{[\overline{M}_{0,8+1+2}(\mathbb{P}^2,3)]^{\mathrm{virt}}} \left(\prod_{j=1}^{8} \mathrm{ev}_j^*(\mathrm{pt}) \right) \mathrm{ev}_9^*(h) \mathrm{ev}_{10}^*(h) \mathrm{ev}_{11}^*(h)\,.$$

Using the divisor equation, we obtain

$$N_1 = -\frac{1}{12} \cdot \frac{1}{2} \cdot 3 \cdot 3 \int_{[\overline{M}_{0,8}(\mathbb{P}^2,3)]^{\mathrm{virt}}} \left(\prod_{j=1}^{8} \mathrm{ev}_j^*(\mathrm{pt}) \right) = -\frac{9}{24} N_0 = -\frac{9}{2}\,,$$

as expected.

1.2 Gluing and vanishing properties of lambda classes

In this Section, we review some well-known facts: a gluing result for lambda classes, Lemma 1.6, and then a vanishing result, Lemma 1.7.

Lemma 1.6. *Let B be a scheme over $\mathbb{C}$. Let Γ be a graph, of genus g_Γ, and let $\pi_V : \mathcal{C}_V \to B$ be prestable curves over B indexed by the vertices V of Γ. For every edge E of Γ, connecting vertices V_1 and V_2, let $s_{E,1}$ and $s_{E,2}$ be smooth sections of π_{V_1} and π_{V_2} respectively. Let $\pi : \mathcal{C} \to B$ be the prestable curve over B obtained by gluing together the sections $s_{V_1,E}$ and $s_{V_2,E}$ corresponding to a same edge E of Γ. Then, we have an exact sequence*

$$0 \to \bigoplus_{V \in V(\Gamma)} (\pi_V)_* \omega_{\pi_V} \to \pi_* \omega_\pi \to \mathcal{O}^{\oplus g_\Gamma} \to 0\,,$$

where ω_{π_V} and ω_π are the relative dualizing line bundles.

Proof. Let $s_E : B \to \mathcal{C}$ be the gluing sections. Then we have an exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}} \to \bigoplus_{V \in V(\Gamma)} \mathcal{O}_{\mathcal{C}_V} \to \bigoplus_{E \in E(\Gamma)} \mathcal{O}_{s_E(B)} \to 0\,.$$

Applying $R\pi_*$, we obtain an exact sequence

$$0 \to \pi_* \mathcal{O}_{\mathcal{C}} \to \bigoplus_{V \in V(\Gamma)} \pi_* \mathcal{O}_{\mathcal{C}_V} \to \bigoplus_{E \in E(\Gamma)} \pi_* \mathcal{O}_{s_E(B)}$$

$$\to R^1 \pi_* \mathcal{O}_{\mathcal{C}} \to \bigoplus_{V \in V(\Gamma)} R^1 \pi_* \mathcal{O}_{\mathcal{C}_V} \to 0\,.$$

The kernel of

$$R^1 \pi_* \mathcal{O}_{\mathcal{C}} \to \bigoplus_{V \in V(\Gamma)} R^1 \pi_* \mathcal{O}_{\mathcal{C}_V}$$

is a free sheaf of rank $|E(\Gamma)| - |V(\Gamma)| + 1 = g_\Gamma$. We obtain the desired exact sequence by Serre duality.

Equivalently, if we choose g_Γ edges of Γ whose complement is a tree, we can understand the morphism

$$\pi_* \omega_\pi \to \mathcal{O}^{\oplus g_\Gamma}$$

as taking the residues at the corresponding g_Γ sections. □

Lemma 1.7. *Let B be a scheme over $\mathbb{C}$. Let $\pi: \mathcal{C} \to B$ be a prestable curve of arithmetic genus g over B. For every integer g' such that $0 \le g' \le g$, let $B_{g'}$ be the closed subset of B of points b such that the dual graph of the curve $\pi^{-1}(b)$ is of genus $\geqslant g'$. Then the lambda classes $\lambda_j \in H^{2j}(B, \mathbb{Q})$, defined by $\lambda_j = c_j(\pi_* \omega_\pi)$, satisfy*

$$\lambda_j|_{B_{g'}} = 0$$

in $H^{2j}(B_{g'}, \mathbb{Q})$ for all $j > g - g'$.

Proof. Let $\tilde{B}_{g'}$ be the finite cover of $B_{g'}$ given by the possible choices of g' fully separating nodes, i.e. of nodes whose complement is of genus 0. Separating these g' fully separating nodes gives a way to write the pullback of $\mathcal{C}$ to $\tilde{B}_{g'}$ as the gluing of curves according to a dual graph Γ of genus g'. According to Lemma 1.6, the Hodge bundle of this family of curves has a trivial rank g' quotient. As $\tilde{B}_{g'}$ is finite over B'_g, it is enough to guarantee the desired vanishing in rational cohomology.

□

1.3 Toric degeneration and decomposition formula

In Section 1.3.1, we review the natural link between log geometry and tropical geometry given by tropicalization. In Section 1.3.2, we start the proof of Theorem 1 by considering the Nishinou-Siebert toric degeneration. In Section 1.3.3, we apply the decomposition formula of Abramovich, Chen, Gross, Siebert [ACGS17a] to this toric degeneration to write the log Gromov-Witten invariants $N_g^{\Delta,n}$ in terms of log Gromov-Witten invariants $N_g^{\Delta,h}$ indexed by parametrized tropical curves $h: \Gamma \to \mathbb{R}^2$. We use the vanishing result of Section 1.2 to restrict the tropical curves appearing.

1.3.1 Tropicalization

Log geometry is naturally related to tropical geometry. Every log scheme X admits a tropicalization $\Sigma(X)$.

Recall that a log scheme is a scheme X endowed with a sheaf of monoids $\mathcal{M}_X$ and a morphism of sheaves of monoids[7]

$$\alpha_X: \mathcal{M}_X \to \mathcal{O}_X \,,$$

[7] All the monoids considered will be commutative and with an identity element.

where $\mathcal{O}_X$ is seen as a sheaf of multiplicative monoids, such that the restriction of α_X to $\alpha_X^{-1}(\mathcal{O}_X^*)$ is an isomorphism.

The ghost sheaf of a log scheme X is the sheaf of monoids

$$\overline{\mathcal{M}}_X \coloneqq \mathcal{M}_X / \alpha^{-1}(\mathcal{O}_X^*) .$$

For the kind of log schemes that we are considering, fine and saturated, the ghost sheaf is of combinatorial nature. In this case, one can think of the log geometry of X as a combination of the geometry of the underlying scheme X and of the combinatorics of the ghost sheaf $\overline{\mathcal{M}}_X$. Non-trivial interactions between these two aspects of log geometry are encoded in the sequence

$$\mathcal{O}_X^* \to \mathcal{M}_X \to \overline{\mathcal{M}}_X .$$

A cone complex is an abstract gluing of convex rational cones along their faces. If X is a log scheme, the tropicalization $\Sigma(X)$ of X is the cone complex defined by gluing together the convex rational cones $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geqslant 0})$ for all $x \in X$ according to the natural specialization maps. Tropicalization is a functorial construction. For more details on tropicalization of log schemes, we refer to Appendix B of [GS13] and Section 2 of [ACGS17a]. Tropicalization gives a pictorial way to describe the combinatorial part of log geometry contained in the ghost sheaf.

Examples

- Let X be a toric variety. We can view X as a log scheme for the toric divisorial log structure, i.e. the divisorial log stucture with respect to the toric boundary divisor ∂X. The sheaf $\mathcal{M}_X$ is the sheaf of functions non-vanishing outside ∂X and α_X is the natural inclusion of $\mathcal{M}_X$ in $\mathcal{O}_X$. The tropicalization $\Sigma(X)$ of X is naturally isomorphic as cone complex to the fan of X.

- Let $\overline{\mathcal{M}}$ be a monoid whose only invertible element is 0. Let X be the log scheme of underlying scheme the point $\mathrm{pt} = \mathrm{Spec}\ \mathbb{C}$, with $\mathcal{M}_X = \overline{\mathcal{M}} \oplus \mathbb{C}^*$ and

$$\alpha_X : \overline{\mathcal{M}} \oplus \mathbb{C}^* \to \mathbb{C}$$

$$(m, a) \mapsto a \delta_{m,0} .$$

  We denote this log scheme as $\mathrm{pt}_{\overline{\mathcal{M}}}$ and such a log scheme is called a log point. By construction, we have $\overline{\mathcal{M}}_{\mathrm{pt}_{\overline{\mathcal{M}}}} = \overline{\mathcal{M}}$ and so the tropicalization $\Sigma(\mathrm{pt}_{\overline{\mathcal{M}}})$ is the cone $\mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$, i.e. the fan of the affine toric variety $\mathrm{Spec}\ \mathbb{C}[\overline{\mathcal{M}}]$.

- The log point $\mathrm{pt}_{\mathbb{N}}$ obtained for $\overline{\mathcal{M}} = \mathbb{N}$ is called the standard log point. Its tropicalization is simply $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{R}_{\geqslant 0}$, the fan of the affine line $\mathbb{A}^1$.

- The log point pt_0 obtained for $\overline{\mathcal{M}} = 0$ is called the trivial log point. Its tropicalization $\Sigma(\mathrm{pt}_0)$ is reduced to a point.

- A stable log map to some relative log scheme $X \to S$ determines a commutative diagram in the category of log schemes,

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \longrightarrow & S\,, \end{array}$$

where $\mathrm{pt}_{\overline{\mathcal{M}}}$ is a log point and π is a log smooth proper integral curve. In particular, the scheme underlying C is a projective nodal curve with a natural set of smooth marked points. We can take the tropicalization of this diagram to obtain a commutative diagram of cone complexes

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X) \\ \downarrow{\scriptstyle \Sigma(\pi)} & & \downarrow \\ \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}}) & \longrightarrow & \Sigma(S)\,. \end{array}$$

$\Sigma(C)$ is a family of graphs over the cone $\Sigma(\mathrm{pt}_{\overline{M}}) = \mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$: the fiber of $\Sigma(\pi)$ over a point in the interior of the cone is the dual graph of C. Fibers over faces of the cone are contractions of the dual graph. In particular, the fiber over the origin of the cone is obtained by fully contracting the dual graph of C to a graph with a unique vertex. If X is a toric variety with the toric divisorial log structure and S is the trivial log point, then $\Sigma(f)$ is a family of parametrized tropical curves in the fan of X. We refer to Section 2.5 of [ACGS17a] for more details.

1.3.2 Toric degeneration

Let Δ be a balanced configuration of vectors, as in Section 1.1.1, and let n be a non-negative integer such that $g_{\Delta,n} \geqslant 0$. We fix $p = (p_1, \dots, p_n)$ a configuration of n points in $\mathbb{R}^2$ belonging to the open dense subset $U_{\Delta,n}$ of $(\mathbb{R}^2)^n$ given by Proposition 1.1. Let $T_{\Delta,p}$ be the set of n-pointed genus $g_{\Delta,n}$ parametrized tropical curves in $\mathbb{R}^2$ of type Δ passing through p. The set $T_{\Delta,p}$ is finite by Proposition 1.2. Proposition 1.1 shows that the elements of $T_{\Delta,p}$ are particularly nice parametrized tropical curves.

We can slightly modify p such that $p \in (\mathbb{Q}^2)^n \cap U_{\Delta,n}$ without changing the combinatorial type of the elements of $T_{\Delta,p}$ and so without changing the tropical counts $N_{\mathrm{trop}}^{\Delta,p}$ and $N_{\mathrm{trop}}^{\Delta,p}(q)$. In that case, for every parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$ and for every vertex V of Γ, we have $h(V) \in \mathbb{Q}^2$ and for every edge E of Γ, we have $\ell(E) \in \mathbb{Q}$. Indeed, the positions $h(V)$ of vertices in $\mathbb{R}^2$ and the lengths $\ell(E)$ of edges are natural parameters on the moduli space of genus $g_{\Delta,n}$ parametrized tropical curves of type Δ and this moduli space is a rational polyhedron in the space of these parameters. The set $T_{\Delta,p}$ is obtained as zero dimensional intersection of this rational polyhedron with the rational (because $p \in (\mathbb{Q}^2)^n$) linear space imposing to pass through p. It follows that the parameters $h(V)$ and $\ell(E)$ are rational for elements of $T_{\Delta,p}$.

We follow the toric degeneration approach introduced by Nishinou and Siebert [NS06] (see also Mandel and Ruddat [MR16]). According to [NS06] Proposition 3.9 and [MR16] Lemma 3.1, there exists a rational polyhedral decomposition $\mathcal{P}_{\Delta,p}$ of $\mathbb{R}^2$ such that

- The asymptotic fan of $\mathcal{P}_{\Delta,p}$ is the fan of X_Δ.

- For every parametrized tropical curve $h: \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$, the images $h(V)$ of vertices V of Γ are vertices of $\mathcal{P}_{\Delta,p}$ and the images $h(E)$ of edges E of Γ are contained in union of edges of $\mathcal{P}_{\Delta,p}$

Remark that the points p_j in $\mathbb{R}^2$ are image of vertices of parametrized tropical curves in $T_{\Delta,p}$ and so are vertices of $\mathcal{P}_{\Delta,p}$.

Given a parametrized tropical curve $h: \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$, we construct a new parametrized tropical curve $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}^2$ by simply adding a genus zero bivalent unpointed vertex to Γ at each point $h^{-1}(V)$ for V a vertex of $\mathcal{P}_{\Delta,p}$ which is not the image by h of a vertex of Γ. The image $\tilde{h}(E)$ of each edge E of $\tilde{\Gamma}$ is now exactly an edge of $\mathcal{P}_{\Delta,p}$. The graph $\tilde{\Gamma}$ has three types of vertices:

- Trivalent unpointed vertices, coming from Γ.
- Bivalent pointed vertices, coming from Γ.
- Bivalent unpointed vertices, not coming from Γ.

Doing a global rescaling of $\mathbb{R}^2$ if necessary, we can assume that $\mathcal{P}_{\Delta,p}$ is an integral polyhedral decomposition, i.e. that all the vertices of $\mathcal{P}_{\Delta,p}$ are in $\mathbb{Z}^2$, and that all the lengths $\ell(E)$ of edges E of parametrized tropical curves $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}^2$, coming from $h: \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$, are integral.

Taking the cone over $\mathcal{P}_{\Delta,p} \times \{1\}$ in $\mathbb{R}^2 \times \mathbb{R}$, we obtain the fan of a three dimensional toric variety $X_{\mathcal{P}_{\Delta,p}}$ equipped with a morphism

$$\nu: X_{\mathcal{P}_{\Delta,p}} \to \mathbb{A}^1$$

coming from the projection $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ on the third $\mathbb{R}$ factor. We have $\nu^{-1}(t) \simeq X_\Delta$ for every $t \in \mathbb{A}^1 - \{0\}$. The special fiber $X_0 \coloneqq \nu^{-1}(0)$ is a reducible surface whose irreducible components X_V are toric surfaces in one to one correspondence with the vertices V of $\mathcal{P}_{\Delta,p}$,

$$X_0 = \bigcup_V X_V \,.$$

In other words, $\nu: X_{\mathcal{P}_{\Delta,p}} \to \mathbb{A}^1$ is a toric degeneration of X_Δ.

We consider the toric varieties $\mathbb{A}^1$, $X_{\mathcal{P}_{\Delta,p}}$, X_Δ and X_V as log schemes with respect to the toric divisorial log structure. In particular, the toric morphism ν induces a log smooth morphism

$$\nu: X_{\mathcal{P}_{\Delta,n}} \to \mathbb{A}^1 \,.$$

Restricting to the special fiber gives a structure of log scheme on X_0 and a log smooth morphism to the standard log point

$$\nu_0: X_0 \to \mathrm{pt}_{\mathbb{N}}.$$

From now on, we will denote $\underline{X}_0$ the scheme underlying the log scheme X_0. Beware that the toric divisorial log structure that we consider on X_V is not the restriction of the log structure that we consider on X_0.

For every $j = 1, \dots, n$, the ray $\mathbb{R}_{\geqslant 0}(p_j, 1)$ in $\mathbb{R}^2 \times \mathbb{R}$ defines a one-parameter subgroup $\mathbb{C}^*_{p_j}$ of $(\mathbb{C}^*)^3 \subset X_{\mathcal{P}_{\Delta,n}}$. We choose a point $P_j \in (\mathbb{C}^*)^2$ and we write Z_{P_j} the affine line in $X_{\mathcal{P}_{\Delta,n}}$ defined as the closure of the orbit of $(P_j, 1)$ under the action of $\mathbb{C}^*_{p_j}$. We have

$$Z_{P_j} \cap \nu^{-1}(1) = Z_{P_j} \cap X_\Delta = P_j \,,$$

and

$$P_j^0 \coloneqq Z_{P_j} \cap \nu^{-1}(0)$$

is a point in the dense torus $(\mathbb{C}^*)^2$ contained in the toric component of X_0 corresponding to the vertex p_j of $\mathcal{P}_{\Delta,p}$. In other words, Z_{P_j} is a section of ν degenerating $P_j \in X_\Delta$ to some $P_j^0 \in X_0$.

Recall from Section 1.1.2 that the log Gromov-Witten invariants $N_g^{\Delta,n}$ are defined using stable log maps of target X_Δ,

$$N_g^{\Delta,n} \coloneqq \int_{[\overline{M}_{g,n,\Delta}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} \prod_{j=1}^{n} \mathrm{ev}_j^*(\mathrm{pt}) \,,$$

where $\overline{M}_{g,n,\Delta}$ is the moduli space of n-pointed stable log maps to X_Δ of genus g and of type Δ.

Let $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ be the moduli space of n-pointed stable log maps to $\pi_0 \colon X_0 \to \mathrm{pt}_{\mathbb{N}}$ of genus g and of type Δ. It is a proper Deligne-Mumford stack of virtual dimension

$$\mathrm{vdim}\, \overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}) = \mathrm{vdim}\, \overline{M}_{g,n,\Delta} = g - g_{\Delta,n} + 2n$$

and it admits a virtual fundamental class

$$[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})]^{\mathrm{virt}} \in A_{g-g_{\Delta,n}+2n}(\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}), \mathbb{Q}) \,.$$

Considering the evaluation morphism

$$\mathrm{ev} \colon \overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}) \to \underline{X}_0^n$$

and the inclusion

$$\iota_{P^0} \colon (P^0 \coloneqq (P_1^0, \dots, P_n^0)) \hookrightarrow \underline{X}_0^n \,,$$

we can define the moduli space[8]

$$\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0) \coloneqq \overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}) \times_{\underline{X}_0^n} P^0 \,,$$

of stable log maps passing through P^0, and by the Gysin refined homomorphism (see

[8]As already mentioned in Section 1.1.2, we consider moduli spaces of stable log maps as stacks, not log stacks. In particular, the morphisms ev, ι_{P^0} and the fiber product defining $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)$ are defined in the category of stacks, not log stacks.

Section 6.2 of [Ful98]), a virtual fundamental class

$$[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}} \coloneqq \iota^!_{P^0}[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})]^{\mathrm{virt}}$$
$$\in A_{g-g_{\Delta,n}}(\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0), \mathbb{Q}) .$$

Remark that this definition is compatible with [ACGS17a] because each P^0_j, seen as a log morphism $P^0_j\colon \mathrm{pt}_{\mathbb{N}} \to X_0$, is strict. This follows from the fact that we have chosen P^0_j in the dense torus $(\mathbb{C}^*)^2$ contained in the toric component of X_0 dual to the vertex p_j of $\mathcal{P}_{\Delta,p}$. If it were not the case[9], then, following Section 6.3.2 of [ACGS17a], the definition of $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)$ should have been replaced by a fiber product in the category of fs log stacks and $[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}}$ should have been defined by some perfect obstruction theory directly on $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)$.

By deformation invariance of the virtual fundamental class on moduli spaces of stable log maps in log smooth families, we have

$$N_g^{\Delta,n} = \int_{[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} .$$

1.3.3 Decomposition formula

As the toric degeneration breaks the toric surface X_Δ into many pieces, irreducible components of the special fiber X_0, one can similarly expect that it breaks the moduli space $\overline{M}_{g,n,\Delta}$ of stable log maps to X_Δ into many pieces, irreducible components of the moduli space $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ of stable log maps to X_0. Tropicalization gives a way to understand the combinatorics of this breaking into pieces.

As we recalled in Section 1.3.1, a n-pointed stable log maps to $X_0/\mathrm{pt}_{\mathbb{N}}$ of type Δ gives a commutative diagram of log schemes

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \nu_0} \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}} , \end{array}$$

which can be tropicalized in a commutative diagram of cone complexes

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X_0) \\ \downarrow{\scriptstyle \Sigma(\pi)} & & \downarrow{\scriptstyle \Sigma(\nu_0)} \\ \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}}) & \xrightarrow{\Sigma(g)} & \Sigma(\mathrm{pt}_{\mathbb{N}}) . \end{array}$$

We have $\Sigma(\mathrm{pt}_{\mathbb{N}}) \simeq \mathbb{R}_{\geqslant 0}$ and the fiber $\Sigma(\nu_0)^{-1}(1)$ is naturally identified with $\mathbb{R}^2$ equipped with the polyhedral decomposition $\mathcal{P}_{\Delta,p}$, whose asymptotic fan is the fan of X_Δ. So the above

[9] In Section 6.3.2 of [ACGS17a], sections defining point constraints have to interact non-trivially with the log structure of the special fiber to produce something interesting because the degeneration considered there is a trivial product, whereas we are considering a non-trivial degeneration

diagram gives a family over the polyhedron $\Sigma(g)^{-1}(1)$ of n-pointed parametrized tropical curves in $\mathbb{R}^2$ of type Δ

The moduli space $\overline{M}_{g,n,\Delta}^{\mathrm{trop}}$ of n-pointed genus g parametrized tropical curves in $\mathbb{R}^2$ of type Δ is a rational polyhedral complex. If $\overline{M}_{g,n,\Delta}^{\mathrm{trop}}$ were the tropicalization of $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ (seen as a log stack over $\mathrm{pt}_{\mathbb{N}}$), then $\overline{M}_{g,n,\Delta}^{\mathrm{trop}}$ would be the dual intersection complex of $\overline{M}_{g,n,\Delta}^{\mathrm{trop}}$. In particular, irreducible components of $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ would be in one to one correspondence with the 0-dimensional faces of $\overline{M}_{g,n,\Delta}^{\mathrm{trop}}$. As the polyhedral decomposition of $\overline{M}_{g,n,\Delta}^{\mathrm{trop}}$ is induced by the combinatorial type of tropical curves, the 0-dimensional faces of $\overline{M}_{g,n,\Delta}^{\mathrm{trop}}$ correspond to the rigid parametrized tropical curves, see Definition 4.3.1 of [ACGS17a], i.e. to parametrized tropical curves which are not contained in a non-trivial family of parametrized tropical curves of the same combinatorial type.

According to the decomposition formula of Abramovich, Chen, Gross and Siebert [ACGS17a], this heuristic description of the pieces of $\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}})$ is correct at the virtual level: one can express $[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}}$ as a sum of contributions indexed by rigid tropical curves.

Let $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}^2$ be a n-pointed genus g rigid parametrized tropical curve to $\mathbb{R}^2$ of type Δ passing through p. For every V vertex of $\tilde{\Gamma}$, let Δ_V be the balanced collection of vectors $v_{V,E}$ for all edges E adjacent to V. Using the notations of Section 1.1.1 that we used all along for Δ but now for Δ_V, the toric surface X_{Δ_V} is the irreducible component of X_0 corresponding to the vertex $h(V)$ of the polyhedral decomposition $\mathcal{P}_{\Delta,p}$.

A n-pointed genus g stable log map to X^0 of type Δ passing through P^0 and marked by $\tilde{h}$ is the following data, see [ACGS17a], Definition 4.4.1[10],

- A n-pointed genus g stable log map $f: C/\mathrm{pt}_{\overline{\mathcal{M}}} \to X_0/\mathrm{pt}_{\mathbb{N}}$ of type Δ passing through P^0.
- For every vertex V of $\tilde{\Gamma}$, an ordinary stable map $f_V: C_V \to X_{\Delta_V}$ of class β_{Δ_V} with marked points x_v for every $v \in \Delta_V$, such that $f_V(x_v) \in D_v$, where D_v is the prime toric divisor of X_{Δ_V} dual to the ray $\mathbb{R}_{\geqslant 0} v$.

These data must satisfy the following compatibility conditions: the gluing of the curves C_V along the points corresponding to the edges of $\tilde{\Gamma}$ is isomorphic to the curve underlying the log curve C, and the corresponding gluing of the maps f_V is the map underlying the log map f.

According to [ACGS17a], the moduli space $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}$ of n-pointed genus g stable log maps of type Δ passing through P^0 and marked by $\tilde{h}$ is a proper Deligne-Mumford stack, equipped with a natural virtual fundamental class $[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}$. Forgetting the marking by $\tilde{h}$ gives a morphism

$$i_{\tilde{h}}: \overline{M}_{g,n,\Delta}^{\tilde{h},P^0} \to \overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)\,.$$

[10]In [ACGS17a], the marking includes also a choice of curve classes for the stable maps f_V. In our case, the curve classes are uniquely determined because a curve class in a toric variety is uniquely determined by its intersection numbers with the components of the toric boundary divisor.

According to the decomposition formula, [ACGS17a] Theorem 6.3.9, we have

$$[\overline{M}_{g,n,\Delta}(X_0/\mathrm{pt}_{\mathbb{N}}, P^0)]^{\mathrm{virt}} = \sum_{\tilde{h}} \frac{n_{\tilde{h}}}{|\mathrm{Aut}(\tilde{h})|} (i_{\tilde{h}})_* [\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}} ,$$

where the sum is over the n-pointed genus g rigid parametrized tropical curves to $(\mathbb{R}^2, \mathcal{P}_{\Delta,p})$ of type Δ passing through p, $n_{\tilde{h}}$ is the smallest positive integer such that the scaling of $\tilde{h}$ by $n_{\tilde{h}}$ has integral vertices and integral lengths, and $|\mathrm{Aut}(\tilde{h})|$ is the order of the automorphism group of $\tilde{h}$.

Recall from Proposition 1.1 that a parametrized tropical curve $h\colon \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$ has a source graph Γ of genus $g_{\Delta,n}$ and that all vertices V of Γ are of genus zero: $g(V) = 0$. In Section 1.3.2, we explained that the polyhedral decomposition $\mathcal{P}_{\Delta,p}$ defines a new parametrized tropical $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$, for each $h\colon \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}$, by addition of unmarked genus zero bivalent vertices. Given such parametrized tropical curve $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$, one can construct genus g parametrized tropical curves by changing only the genus of vertices $g(V)$ so that

$$\sum_{V \in V(\Gamma)} g(V) = g - g_{\Delta,n} .$$

We denote $T_{\Delta,p}^g$ the set of genus g parametrized tropical curves obtained in this way.

Lemma 1.8. *Parametrized tropical curves $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ in $T_{\Delta,p}^g$ are rigid. Furthermore, for such $\tilde{h}$, we have $n_{\tilde{h}} = 1$ and $|\mathrm{Aut}(\tilde{h})| = 1$.*

Proof. The rigidity of parametrized tropical curves in $T_{\Delta,p}^g$ follows from the rigidity of parametrized tropical curves in $T_{\Delta,p}$ because the genera attached to the vertices cannot change under a deformation preserving the combinatorial type, and added bivalent vertices to go from Γ to $\tilde{\Gamma}$ are mapped to vertices of $\mathcal{P}_{\Delta,p}$ and so cannot move without changing the combinatorial type.

We have $n_{\tilde{h}} = 1$ because in Section 1.3.2, we have chosen the polyhedral decomposition $\mathcal{P}_{\Delta,p}$ to be integral: vertices of $\tilde{h}$ map to integral points of $\mathbb{R}^2$ and edges E of $\tilde{\Gamma}$ have integral lengths $\ell(E)$. We have $|\mathrm{Aut}(\tilde{h})| = 1$ because $\tilde{h}$ is an immersion. The genus of vertices never enters in the above arguments. □

For every $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ parametrized tropical curve in $T_{\Delta,p}^g$, we define

$$N_{g,\tilde{h}}^{\Delta,n} \coloneqq \int_{[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} .$$

Proposition 1.9. *For every Δ, n and $g \geqslant g_{\Delta,n}$, we have*

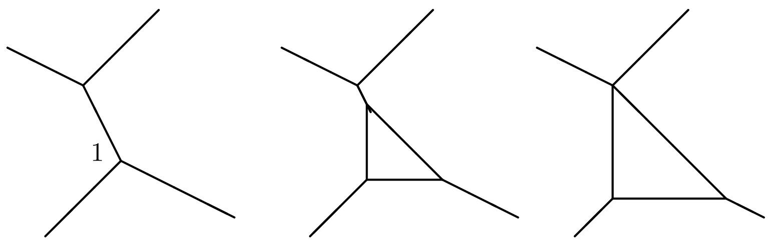
$$N_g^{\Delta,n} = \sum_{\tilde{h} \in T_{\Delta,p}^g} N_{g,\tilde{h}}^{\Delta,n} .$$

Proof. This follows from the decomposition formula and from the vanishing property of lambda classes.

If $\tilde{h}$ is a rigid parametrized tropical curve of genus $g > g_{\Delta,n}$, then every point in $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}$ is a stable log map whose tropicalization has genus $g > g_{\Delta,n}$. In particular, the dual intersection complex of the source curve has genus $g > g_{\Delta,n}$. By Lemma 1.7, $\lambda_{g-g_{\Delta,n}}$ is zero on restriction to such family of curves. □

Example. The generic way to deform a parametrized tropical curve in $T^g_{\Delta,p}$ is to open $g(V)$ small cycles in place of a vertex of genus $g(V)$. When the cycles coming from various vertices grow and meet, we can obtain curves with vertices of valence strictly greater than three which can be rigid. Proposition 1.9 guarantees that such rigid curves do not contribute in the decomposition formula after integration of the lambda class.

Below is an illustration of a genus one vertex opening in one cycle and growing until forming a 4-valent vertex.



1.4 Non-torically transverse stable log maps in X_Δ

Let Δ be a balanced collections of vectors in $\mathbb{Z}^2$, as in Section 1.1.1. We consider the toric surface X_Δ with the toric divisorial log structure. In this Section, we prove some general properties of stable log maps of type Δ in X_Δ, using as tool the tropicalization procedure reviewed in Section 1.3.1.

We say that a stable log map $(f\colon C/\mathrm{pt}_{\overline{\mathcal{M}}} \to X_\Delta)$ to X_Δ is torically transverse[11] if its image does not contain any of the torus fixed points of X_Δ, i.e. if its image does not pass through the "corners" of the toric boundary divisor ∂X_Δ. The difficulty of log Gromov-Witten theory, with respect to relative Gromov-Witten theory for example, comes from the log stable maps which are not torically transverse: the "corners" of ∂X_Δ are the points where ∂X_Δ is not smooth and so are exactly the points where the log structure of X_Δ is locally more complicated that the divisorial log structure along a smooth divisor.

The following result is a structure result for log stable maps of type Δ which are not torically transverse. Combined with vanishing properties of lambda classes reviewed in Section 1.2, this will give us in Section 1.6 a way to completely discard log stable maps which are not torically transverse.

[11]We allow a torically transverse stable log map to have components contracted to points of ∂X_Δ which are not torus fixed points. In particular, we use a notion of torically transverse map which is slightly different from the one used by Nishinou and Siebert in [NS06].

Proposition 1.10. *Let $f: C/\mathrm{pt}_{\overline{\mathcal{M}}} \to X_\Delta$ be a stable log map to X_Δ of type Δ. Let*

$$\Sigma(f): \Sigma(C)/\Sigma(\mathrm{pt}_{\mathbb{N}}) \to \Sigma(X_\Delta)$$

be the family of tropical curves obtained as tropicalization of f. Assume that f is not torically transverse and that the unbounded edges of the fibers of $\Sigma(f)$ are mapped to rays of the fan of X_Δ. Then the dual graph of C has positive genus, i.e. C contains at least one non-separating node.

Proof. Recall that $\Sigma(f)$ is a family over the cone $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$ of parametrized tropical curves in $\mathbb{R}^2$. We assume that the unbounded edges of these parametrized tropical curves are mapped to rays of the fan of X_Δ.

We fix a point in the interior of the cone $\mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$ and we consider the corresponding parametrized tropical curve $h: \Gamma \to \mathbb{R}^2$ in $\mathbb{R}^2$. Combinatorially, Γ is the dual graph of C.

Lemma 1.11. *There exists a vertex V of Γ mapping away from the origin in $\mathbb{R}^2$ and a non-contracted edge E adjacent to V such that $h(E)$ is not included in a ray of the fan of X_Δ.*

Proof. We are assuming that f is not torically transverse. This means that at least one component of C maps dominantly to a component of the toric boundary divisor ∂X_Δ or that at least one component of C is contracted to a torus fixed point of X_Δ.

If one component of C is contracted to a torus fixed point of X_Δ, then we are done because the corresponding vertex V of Γ is mapped away from the origin and from the rays of the fan of X_Δ, and any non-contracted edge of Γ adjacent to V is not mapped to a ray of the fan of X_Δ. Remark that there exists such non-contracted edge because if not, as Γ is connected, all the vertices of Γ would be mapped to $h(V)$ and so the curve C would be entirely contracted to a torus fixed point, contradicting $\beta_\Delta \neq 0$.

So we can assume that no component of C is contracted to a torus fixed point, i.e. that all the vertices of Γ are mapped either to the origin or to a point on a ray of the fan of X_Δ, and that at least one component of C maps dominantly to a component of ∂X_Δ. We argue by contradiction by assuming further that every edge of Γ is either contracted to a point or mapped inside a ray of the fan of X_Δ.

Let Γ_0 be the subgraph of Γ formed by vertices mapping to the origin and edges between them. For every ray ρ of the fan of X_Δ, let Δ_ρ be the set of $v \in \Delta$ such that $\mathbb{R}_{\geqslant 0} v = \rho$, and let Γ_ρ be the subgraph of Γ formed by vertices of Γ mapping to the ray ρ away from the origin and the edges between them.

By our assumption, there is no edge in Γ connecting Γ_ρ and $\Gamma_{\rho'}$ for two different rays ρ and ρ'. For every ray ρ, let $E(\Gamma_0, \Gamma_\rho)$ the set of edges of Γ connecting a vertex $V_0(E)$ of Γ_0 and a vertex $V_\rho(E)$ of Γ_ρ. It follows from the balancing condition that, for every ray ρ, we have

$$\sum_{E \in E(\Gamma_0, \Gamma_\rho)} v_{V_0(E), E} = \sum_{v \in \Delta_\rho} v \,.$$

Let C_0 be the curve obtained by taking the components of C intersecting properly the toric boundary divisor ∂X_Δ. The dual graph of C_0 is Γ_0 and the total intersection number of C_0 with the toric divisor D_ρ is

$$\sum_{E \in E(\Gamma_0, \Gamma_\rho)} |v_{V_0(E),E}|,$$

where $|v_{V_0(E),E}|$ is the divisibility of $v_{V_0(E),E}$ in $\mathbb{Z}^2$, i.e. the multiplicity of the corresponding intersection point of C_0 and D_ρ.

From the previous equality, we obtain that the intersection numbers of C_0 with the components of ∂X_Δ are equal to the intersection numbers of C with the components of ∂X_Δ so $[f(C_0)] = \beta_\Delta$. It follows that all the components of C not in C_0 are contracted, which contradicts the fact that at least one component of C maps dominantly to a component of ∂X_Δ. ☐

We continue the proof of Proposition 1.10. By Lemma 1.11, there exists a vertex V of Γ mapping away from the origin in $\mathbb{R}^2$ and a non-contracted edge E adjacent to V such that $h(E)$ is not included in a ray of the fan of X_Δ. We will use (V, E) as initial data for a recursive construction of a non-trivial cycle in Γ.

There exists a unique two-dimensional cone of the fan of X_Δ, containing $h(V) \in \mathbb{R}^2 - \{0\}$ and delimited by rays ρ_1 and ρ_2, such that the rays ρ_1, $\mathbb{R}_{\geqslant 0} h(V)$ and ρ_2 are ordered in the clockwise way and such that $h(V) \in \rho_1$ if $h(V)$ is on a ray. Let v_1 and v_2 be vectors in $\mathbb{R}^2 - \{0\}$ such that $\rho_1 = \mathbb{R}_{\geqslant 0} v_1$ and $\rho_2 = \mathbb{R}_{\geqslant 0} v_2$. The vectors v_1 and v_2 form a basis of $\mathbb{R}^2$ and for every $v \in \mathbb{R}^2$, we write (v, v_1) and (v, v_2) for the coordinates of v in this basis, i.e. the real numbers such that

$$v = (v, v_1) v_1 + (v, v_2) v_2 \,.$$

By construction, we have $(h(V), v_1) > 0$ and $(h(V), v_2) \geqslant 0$. As $v_{V,E} \neq 0$, we have $(v_{V,E}, v_1) \neq 0$ or $(v_{V,E}, v_2) \neq 0$.

If $(v_{V,F}, v_2) = 0$ for every edge F adjacent to V, then $(v_{V,E}, v_1) \neq 0$ and $(h(V), v_2) > 0$. In particular, E is not an unbounded edge. By the balancing condition, up to replacing E by another edge adjacent to V, one can assume that $(v_{V,E}, v_1) > 0$. Then, the edge E is adjacent to another vertex V' with $(h(V'), v_1) > (h(V), v_1)$ and $(h(V'), v_2) = (h(V), v_2)$. By the balancing condition, there exists an edge E' adjacent to V' such that $(v_{V',E'}, v_1) > 0$. If $(v_{V,F'}, v_2) = 0$ for every edge F' adjacent to V', then in particular we have $(v_{V,E'}, v_2) = 0$ and so E' is adjacent to another vertex V'' with $(h(V''), v_1) > (h(V'), v_1)$ and $(h(V''), v_2) = (h(V'), v_2)$, and we can iterate the argument. Because Γ has finitely many vertices, this process has to stop: there exists a vertex $\tilde{V}$ in the cone generated by ρ_1 and ρ_2 and an edge $\tilde{E}$ adjacent to $\tilde{V}$ such that $(v_{\tilde{V},\tilde{E}}, v_2) \neq 0$.

The upshot of the previous paragraph is that, up to changing V and E, one can assume that $(v_{V,E}, v_2) \neq 0$. By the balancing condition, up to replacing E by another edge adjacent to V, one can assume that $(v_{V,E}, v_2) > 0$. The edge E is adjacent to another vertex V' with $(h(V'), v_2) > (h(V), v_2)$. By the balancing condition, one can find an edge E' adjacent to V' such that $(v_{V',E'}, v_2) > 0$. If $h(V')$ is in the interior of the cone generated by ρ_1 and ρ_2, then E' is not an unbounded edge and so is adjacent to another vertex V'' with

$(h(V''), v_2) > (h(V'), v_2)$. Repeating this construction, we obtain a sequence of vertices of image in the cone generated by ρ_1 and ρ_2. Because Γ has finitely many vertices, this process has to terminate: there exists a vertex $\tilde{V}$ of Γ such that $h(\tilde{V}) \in \rho_2$ and connected to V by a path of edges mapping to the interior of the cone delimited by ρ_1 and ρ_2.

Repeating the argument starting from $\tilde{V}$, and so on, we construct a path of edges in Γ whose projection in $\mathbb{R}^2$ intersects successive rays in the clockwise order. Because the combinatorial type of Γ is finite, this path has to close eventually and so Γ contains a non-trivial closed cycle, i.e. Γ has positive genus. □

Remark: It follows from Proposition 1.10 that the ad hoc genus zero Gromov-Witten invariants defined in terms of relative Gromov-Witten invariants of some open geometry used by Gross, Pandharipande, Siebert in [GPS10](Section 4.4), and Gross, Hacking, Keel in [GHK15a] (Section 3.1), coincide with log Gromov-Witten invariants[12]. In fact, our proof of Proposition 1.10 can be seen as a tropical analogue of the main properness argument of [GPS10] (Proposition 4.2) which guarantees that the ad hoc invariants are well-defined.

1.5 Statement of the gluing formula

We continue the proof of Theorem 1 started in Section 1.3. In Section 1.5, we state a gluing formula, Corollary 1.15, expressing the invariants $N_{g,\tilde{h}}^{\Delta,n}$ attached to a parametrized tropical curve $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}^2$ in terms of invariants $N_{g,V}^{1,2}$ attached to the vertices V of Γ. This gluing formula is proved in Section 1.6, using the structure result of Section 1.4 and the vanishing result of Section 1.2 to reduce the argument to the locus of torically transverse stable log maps.

1.5.1 Preliminaries

We fix $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}^2$ a parametrized tropical curve in $T_{\Delta,p}^{g}$. The purpose of the gluing formula is to write the log Gromov-Witten invariant

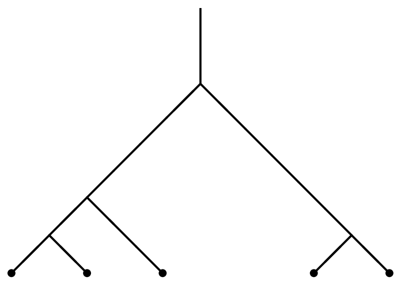
$$N_{g,\tilde{h}}^{\Delta,n} = \int_{[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\text{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} ,$$

introduced in Section 1.3.3, in terms of log Gromov-Witten invariants of the toric surfaces X_{Δ_V} attached to the vertices V of $\tilde{\Gamma}$. Recall from Section 1.3.2 that $\tilde{\Gamma}$ has three types of vertices:

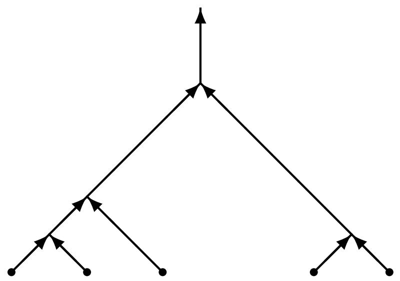
- Trivalent unpointed vertices, coming from Γ.
- Bivalent pointed vertices, coming from Γ.
- Bivalent unpointed vertices, not coming from Γ.

[12]This result was expected: see Remark 3.4 of [GHK15a] but it seems that no proof was published until now.

According to Lemma 4.20 of Mikhalkin [Mik05], the connected components of the complement of the bivalent pointed vertices of $\tilde{\Gamma}$ are trees with exactly one unbounded edge.



In particular, we can fix an orientation of edges of $\tilde{\Gamma}$ consistently from the bivalent pointed vertices to the unbounded edges. Every trivalent vertex of $\tilde{\Gamma}$ has two ingoing and one outgoing edges with respect to this orientation. Every bivalent pointed vertex has two outgoing edges with respect to this orientation. Every bivalent unpointed vertex has one ingoing and one outgoing edges with respect to this orientation.



1.5.2 Contribution of trivalent vertices

Let V be a trivalent vertex of $\tilde{\Gamma}$. Let $\overline{M}_{g,\Delta_V}$ be the moduli space of stable log maps to X_{Δ_V} of genus g and of type Δ_V. It has virtual dimension

$$\mathrm{vdim}\, \overline{M}_{g,\Delta_V} = g + 2\,,$$

and admits a virtual fundamental class

$$[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}} \in A_{g+2}(\overline{M}_{g,\Delta_V}, \mathbb{Q}).$$

Let $E_V^{\mathrm{in},1}$ and $E_V^{\mathrm{in},2}$ be the two ingoing edges adjacent to V, and let E_V^{out} be the outgoing edge adjacent to V. Let $D_{E_V^{\mathrm{in},1}}$, $D_{E_V^{\mathrm{in},2}}$ and $D_{E_V^{\mathrm{out}}}$ be the corresponding toric divisors of

X_{Δ_V}. We have evaluation morphisms

$$(\mathrm{ev}_V^{E_V^{\mathrm{in},1}}, \mathrm{ev}_V^{E_V^{\mathrm{in},2}}, \mathrm{ev}_V^{E_V^{\mathrm{out}}}) \colon \overline{M}_{g,\Delta_V} \to D_{E_V^{\mathrm{in},1}} \times D_{E_V^{\mathrm{in},2}} \times D_{E_V^{\mathrm{out}}} .$$

We define

$$N_{g,V}^{1,2} \coloneqq \int_{[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g (\mathrm{ev}_V^{E_V^{\mathrm{in},1}})^* (\mathrm{pt}_{E_V^{\mathrm{in},1}}) (\mathrm{ev}_V^{E_V^{\mathrm{in},2}})^* (\mathrm{pt}_{E_V^{\mathrm{in},2}}) ,$$

where $\mathrm{pt}_{E_V^{\mathrm{in},1}} \in A^1(D_{E_V^{\mathrm{in},1}})$, $\mathrm{pt}_{E_V^{\mathrm{in},2}} \in A^1(D_{E_V^{\mathrm{in},2}})$ are classes of a point on $D_{E_V^{\mathrm{in},1}}$, $D_{E_V^{\mathrm{in},2}}$ respectively.

1.5.3 Contribution of bivalent pointed vertices

Let V is a bivalent pointed vertex of $\tilde{\Gamma}$. Let $\overline{M}_{g,\Delta_V}$ be the moduli space of 1-pointed[13] stable log maps to X_{Δ_V} of genus g and of type Δ_V. It has virtual dimension

$$\mathrm{vdim}\, \overline{M}_{g,\Delta_V} = g + 2 ,$$

and admits a virtual fundamental class

$$[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}} \in A_{g+2}(\overline{M}_{g,\Delta_V}, \mathbb{Q}).$$

We have the evaluation morphism at the extra marked point,

$$\mathrm{ev} \colon \overline{M}_{g,\Delta_V} \to X_{\Delta_V} ,$$

and we define

$$N_{g,V}^{1,2} \coloneqq \int_{[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g \mathrm{ev}^* (\mathrm{pt}) ,$$

where $\mathrm{pt} \in A^2(X_{\Delta_V})$ is the class of a point on X_{Δ_V}.

1.5.4 Contribution of bivalent unpointed vertices

Let V is a bivalent unpointed vertex of $\tilde{\Gamma}$. Let $\overline{M}_{g,\Delta_V}$ be the moduli space of stable log maps to X_{Δ_V} of genus g and of type Δ_V. It has virtual dimension

$$\mathrm{vdim}\, \overline{M}_{g,\Delta_V} = g + 1 ,$$

and admits a virtual fundamental class

$$[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}} \in A_{g+1}(\overline{M}_{g,\Delta_V}, \mathbb{Q}).$$

[13] As in Section 1.1.2, 1-pointed means that the source curves are equipped with one marked point in addition to the marked points keeping track of the tangency conditions.

Let E_V^{in} be the ingoing edge adjacent to V and E_V^{out} the outgoing edge adjacent to V. Let $D_{E_V^{\mathrm{in}}}$ and $D_{E_V^{\mathrm{out}}}$ the corresponding toric divisors ofX_{Δ_V}. We have evaluation morphisms

$$(\mathrm{ev}_V^{E_V^{\mathrm{in}}}, \mathrm{ev}_V^{E_V^{\mathrm{out}}}) \colon \overline{M}_{g,\Delta_V} \to D_{E_V^{\mathrm{in}}} \times D_{E_V^{\mathrm{out}}} .$$

We define

$$N_{g,V}^{1,2} \coloneqq \int_{[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g (\mathrm{ev}_V^{E_V^{\mathrm{in}}})^* (\mathrm{pt}_{E_V^{\mathrm{in}}}) ,$$

where $\mathrm{pt}_{E_V^{\mathrm{in}}} \in A^1(D_{E_V^{\mathrm{in},1}})$ is the class of a point on $D_{E_V^{\mathrm{in}}}$.

1.5.5 Statement of the gluing formula

The following gluing formula expresses the log Gromov-Witten invariant $N_{g,\tilde{h}}^{\Delta,n}$ attached to a parametrized tropical curve $\tilde{h} \colon \tilde{\Gamma} \to \mathbb{R}^2$ in terms of the log Gromov-Witten invariants $N_{g,V}^{1,2}$ attached to the vertices V of $\tilde{\Gamma}$ and of the weights $w(E)$ of the edges of $\tilde{\Gamma}$.

Proposition 1.12. *For every $\tilde{h} \colon \tilde{\Gamma} \to \mathbb{R}^2$ parametrized tropical curve in $T_{\Delta,p}^g$, we have*

$$N_{g,\tilde{h}}^{\Delta,n} = \left(\prod_{V \in V(\tilde{\Gamma})} N_{g(V),V}^{1,2} \right) \left(\prod_{E \in E_f(\tilde{\Gamma})} w(E) \right) ,$$

where the first product is over the vertices of $\tilde{\Gamma}$ and the second product is over the bounded edges of $\tilde{\Gamma}$.

The proof of Proposition 1.12 is given in Section 1.6.

In the following Lemmas, we compute the contributions $N_{g(V),V}^{1,2}$ of the bivalent vertices.

Lemma 1.13. *Let V be a bivalent pointed vertex of $\tilde{\Gamma}$. Then we have*

$$N_{g,V}^{1,2} = 0$$

for every $g > 0$, and

$$N_{0,V}^{1,2} = 1$$

for $g = 0$.

Proof. Let w be the weight of the two edges of $\tilde{\Gamma}$ adjacent to V. We can take $X_{\Delta_V} = \mathbb{P}^1 \times \mathbb{P}^1$ and $\beta_{\Delta_V} = w([\mathbb{P}^1] \times [\mathrm{pt}])$. We have the evaluation map at the extra marked point

$$\mathrm{ev} \colon \overline{M}_{g,\Delta_V} \to \mathbb{P}^1 \times \mathbb{P}^1 .$$

We fix a point $p = (p_1, p_2) \in \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^1 \times \mathbb{P}^1$ and we denote $\iota_p \colon p \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and $\iota_{p_1} \colon p \hookrightarrow \mathbb{P}^1 \times \{p_2\} \simeq \mathbb{P}^1$ the inclusion morphism.

Let $\overline{M}_{g,1}(\mathbb{P}^1 / \{0\} \cup \{\infty\}, w; w, w)$ be the moduli space of genus g 1-pointed stable maps to $\mathbb{P}^1$, of degree w, relative to the divisor $\{0\} \cup \{\infty\}$, with intersection multiplicities w both along $\{0\}$ and $\{\infty\}$. We have an evaluation morphism at the extra marked point

$$\mathrm{ev}_1 \colon \overline{M}_{g,1}(\mathbb{P}^1/\{0\} \cup \{\infty\}, w; w, w) \to \mathbb{P}^1 \,,$$

Because an element $(f \colon C \to \mathbb{P}^1 \times \mathbb{P}^1)$ of $\mathrm{ev}^{-1}(p)$ factors through $\mathbb{P}^1 \times \{p_2\} \simeq \mathbb{P}^1$, we have a natural identification of moduli spaces $\mathrm{ev}^{-1}(p) = \mathrm{ev}_1^{-1}(p)$, but the natural virtual fundamental classes are different. The class $\iota_p^! [\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}$, defined by the refined Gysin homomorphism (see Section 6.2 of [Ful98]), has degree g whereas the class

$$\iota_{p_1}^! [\overline{M}_{g,1}(\mathbb{P}^1/\{0\} \cup \{\infty\}, w; w, w)]^{\mathrm{virt}}$$

is of degree

$$2g - 2 + 2w - (w-1) - (w-1) + (1-1) = 2g \,.$$

The two obstruction theories differ by the bundle whose fiber at

$$f \colon C \to \mathbb{P}^1$$

is $H^1(C, f^* N_{f(C)|\mathbb{P}^1 \times \mathbb{P}^1})$. Because $\beta_{\Delta_V}^2 = 0$, the normal bundle $N_{f(C)|\mathbb{P}^1 \times \mathbb{P}^1}$ is trivial of rank one, so the pullback $f^* N_{f(C)|\mathbb{P}^1 \times \mathbb{P}^1}$ is trivial of rank one and the two obstruction theories differ by the dual of the Hodge bundle. Therefore, we have

$$\iota_p^! [\overline{M}_{g,\Delta_V}]^{\mathrm{virt}} = c_g(\mathbb{E}^*) \cap \iota_{p_1}^! [\overline{M}_{g,1}(\mathbb{P}^1/\{0\} \cup \{\infty\}, w; w, w)]^{\mathrm{virt}} \,,$$

and so

$$N_{g,V}^1 = \int_{\iota_p^! [\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g = \int_{\iota_{p_1}^! [\overline{M}_{g,1}(\mathbb{P}^1/\{0\} \cup \{\infty\}, w; w, w)]^{\mathrm{virt}}} \lambda_g^2 \,.$$

But $\lambda_g^2 = 0$ for $g > 0$, as follows from Mumford's relation [Mum83]

$$c(\mathbb{E}) c(\mathbb{E}^*) = 1 \,,$$

and so $N_{g,V}^1 = 0$ if $g > 0$.

If $g = 0$, we have $\lambda_0^2 = 1$, the moduli space is a point, given by the degree w map $\mathbb{P}^1 \to \mathbb{P}^1$ fully ramified over 0 and ∞, with trivial automorphism group (there is no non-trivial automorphism of $\mathbb{P}^1$ fixing 0, ∞ and the extra marked point), and so

$$N_{0,V}^{1,2} = 1 \,.$$

□

Lemma 1.14. *Let V be a bivalent unpointed vertex of $\tilde{\Gamma}$ and $w(E_V)$ the common weight of the two edges adjacent to V. Then we have*

$$N_{g,V}^{1,2} = 0$$

for every $g > 0$, and

$$N_{0,V}^{1,2} = \frac{1}{w(E_V)}$$

for $g = 0$.

Proof. The argument is parallel to the one used to prove Lemma 1.13. The only difference is that the vertex is no longer pointed and the invariant $N_{g,V}^{1,2}$ is defined using the evaluation map at one of the tangency point. The vanishing for $g > 0$ still follows from $\lambda_g^2 = 0$. For $g = 0$, the moduli space is a point, given by the degree $w(E_V)$ map $\mathbb{P}^1 \to \mathbb{P}^1$ fully ramified over 0 and ∞, but now with an automorphism group $\mathbb{Z}/w(E_V)$ (the extra marked point in Lemma 1.13 is no longer there to kill all non-trivial automorphisms). It follows that $N_{0,V}^{1,2} = \frac{1}{w(E_V)}$.

□

Corollary 1.15. *Let* $\tilde{h}: \tilde{\Gamma} \to \mathbb{R}^2$ *be a parametrized tropical curve in* $T_{\Delta,p}^g$.

- *If there exists one bivalent vertex* V *of* $\tilde{\Gamma}$ *with* $g(V) \neq 0$*, then*

$$N_{g,\tilde{h}}^{\Delta,n} = 0\,.$$

- *If* $g(V) = 0$ *for all the bivalent vertices* V *of* $\tilde{\Gamma}$*, then*

$$N_{g,\tilde{h}}^{\Delta,n} = \left(\prod_{V \in V^{(3)}(\tilde{\Gamma})} N_{g(V),V}^{1,2} \right) \left(\prod_{E \in E_f(\Gamma)} w(E) \right),$$

where the first product is over the trivalent vertices of Γ *(or* $\tilde{\Gamma}$*), and the second product is over the bounded edges of* Γ *(not* $\tilde{\Gamma}$*).*

Proof. If $\tilde{\Gamma}$ has a bivalent vertex V with $g(V) > 0$, then, according to Lemmas 1.13 and 1.14, we have $N_{g(V),V}^{1,2} = 0$ and so $N_{g,\tilde{h}}^{\Delta,n} = 0$ by Proposition 1.12.

If $g(V) = 0$ for all the bivalent vertices V of $\tilde{\Gamma}$, then, according to Lemma 1.13, we have $N_{g(V),V}^{1,2} = 1$ for all the bivalent pointed vertices V of $\tilde{\Gamma}$ and according to Lemma 1.14, we have $N_{g(V),V}^{1,2} = \frac{1}{w(E_V)}$ for all the bivalent unpointed vertices V of $\tilde{\Gamma}$. It follows that Proposition 1.12 can be rewritten

$$N_{g,\tilde{h}}^{\Delta,n} = \left(\prod_{V \in V^{(3)}(\tilde{\Gamma})} N_{g(V),V}^{1,2} \right) \left(\prod_{V \in V^{(2up)}(\tilde{\Gamma})} \frac{1}{w(E_V)} \right) \left(\prod_{E \in E_f(\tilde{\Gamma})} w(E) \right),$$

where the first product is over the trivalent vertices of $\tilde{\Gamma}$ (which can be naturally identified with the trivalent vertices of Γ) and the second product is over the bivalent unpointed vertices of $\tilde{\Gamma}$. Recalling from Section 1.3.2 that the edges of $\tilde{\Gamma}$ are obtained as subdivision of the edges of Γ by adding the bivalent unpointed vertices, we have

$$\left(\prod_{V \in V^{(2up)}(\tilde{\Gamma})} \frac{1}{w(E_V)} \right) \left(\prod_{E \in E_f(\tilde{\Gamma})} w(E) \right) = \prod_{E \in E_f(\Gamma)} w(E)\,.$$

□

1.6 Proof of the gluing formula

This Section is devoted to the proof of Proposition 1.12. Part of it is inspired the proof by Chen [Che14a] of the degeneration formula for expanded stable log maps, and the proof by Kim, Lho and Ruddat [KLR18] of the degeneration formula for stable log maps in degenerations along a smooth divisor. In Section 1.6.1, we define a cut morphism. Restricted to some open substack of torically transverse stable maps, we show in Section 1.6.2 that the cut morphism is étale, and in Section 1.6.3, that the cut morphism is compatible with the natural obstruction theories of the pieces. Using in addition Proposition 1.10 and the results of Section 1.2, we prove a gluing formula in Section 1.6.4. To finish the proof of Proposition 1.12, we explain in Section 1.6.5 how to organize the glued pieces.

1.6.1 Cutting

Let $\tilde{h}\colon \tilde{\Gamma} \to \mathbb{R}^2$ be a parametrized tropical curve in $T^g_{\Delta,p}$. We denote $V^{(2p)}(\tilde{\Gamma})$ the set of bivalent pointed vertices of $\tilde{\Gamma}$ and $V^{(2up)}(\tilde{\Gamma})$ the set of bivalent unpointed vertices of $\tilde{\Gamma}$.

Evaluations $\mathrm{ev}_V^E\colon \overline{M}_{g(V),\Delta_V} \to D_E$ at the tangency points dual to the bounded edges of $\tilde{\Gamma}$ give a morphism

$$\mathrm{ev}^{(e)}\colon \prod_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \to \prod_{E\in E_f(\tilde{\Gamma})} (D_E)^2\,,$$

where D_E is the divisor of X_0 dual to an edge E of $\tilde{\Gamma}$.

Evaluations $\mathrm{ev}_V^{(p)}\colon \overline{M}_{g(V),\Delta_V} \to X_{\Delta_V}$ at the extra marked points corresponding to the bivalent pointed vertices give a morphism

$$\mathrm{ev}^{(p)}\colon \prod_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \to \prod_{V\in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V}\,.$$

Let

$$\delta\colon \prod_{E\in E_f(\tilde{\Gamma})} D_E \to \prod_{E\in E_f(\tilde{\Gamma})} (D_E)^2$$

be the diagonal morphism. Let

$$\iota_{P^0}\colon \left(P^0 = (P_V^0)_{V\in V^{(2p)}(\tilde{\Gamma})}\right) \hookrightarrow \prod_{V\in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V}\,,$$

be the inclusion morphism of P^0.

Using the fiber product diagram in the category of stacks

$$\begin{array}{ccc}
\underset{V\in V(\tilde{\Gamma})}{\times} \overline{M}_{g(V),\Delta_V} & \xrightarrow{(\delta\times\iota_{P^0})_M} & \prod_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \\
\downarrow & & \downarrow {\scriptstyle \mathrm{ev}^{(e)}\times \mathrm{ev}^{(p)}} \\
\left(\prod_{E\in E_f(\tilde{\Gamma})} D_E\right)\times P^0 & \xrightarrow{\delta\times\iota_{P^0}} & \prod_{E\in E_f(\tilde{\Gamma})} (D_E)^2 \times \prod_{V\in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V}\,,
\end{array}$$

we define the substack $\times_{V\in V(\tilde\Gamma)} \overline{M}_{g(V),\Delta_V}$ of $\prod_{V\in V(\tilde\Gamma)} \overline{M}_{g(V),\Delta_V}$ consisting of curves whose marked points keeping track of the tangency conditions match over the divisors D_E and whose extra marked points associated to the bivalent pointed vertices map to P^0.

Lemma 1.16. *Let*

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \nu_0} \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}}\,, \end{array}$$

be a n-pointed genus g stable log map of type Δ passing through P^0 and marked by $\tilde h:\tilde\Gamma \to \mathbb{R}^2$, i.e. a point of $\overline{M}_{g,n,\Delta}^{\tilde h,P^0}$. Let

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X_0) \\ \downarrow{\scriptstyle \Sigma(\pi)} & & \downarrow{\scriptstyle \Sigma(\nu_0)} \\ \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}}) & \xrightarrow{\Sigma(g)} & \Sigma(\mathrm{pt}_{\mathbb{N}})\,. \end{array}$$

be its tropicalization. For every $b \in \Sigma(g)^{-1}(1)$, let

$$\Sigma(f)_b : \Sigma(C)_b \to \Sigma(\nu_0)^{-1}(1) \simeq \mathbb{R}^2$$

be the fiber of $\Sigma(f)$ over b. Let E be an edge of Γ and let $E_{f,b}$ be the edge of $\Sigma(C)_b$ marked by E. Then $\Sigma(f)_b(E_{f,b}) \subset \tilde h(E)$.

Proof. We recalled in Section 1.5 that the connected components of the complement of the bivalent pointed vertices of $\tilde\Gamma$ are trees with exactly one unbounded edge. We prove mboxLemma 1.16 by induction, starting with the edges connected to the bivalent pointed vertices and then we go through each tree following the orientation introduced in Section 1.5.

Let E be an edge of $\tilde\Gamma$ adjacent to a bivalent pointed vertex V of $\tilde\Gamma$. Let $P_V^0 \in X_{\Delta_V}$ be the corresponding marked point. As f is marked by $\tilde h$, we have an ordinary stable map $f_V : C_V \to X_{\Delta_V}$, a marked point x_E in C_V such that $f(x_E) \in D_E$ and $f_V(C_V)$ contains P_V^0. We can assume that $X_{\Delta_V} = \mathbb{P}^1 \times \mathbb{P}^1$, $D_E = \{0\} \times \mathbb{P}^1$, $\beta_{\Delta_V} = w(E)([\mathbb{P}^1] \times [\mathrm{pt}])$, and $P_V^0 = (P_{V,1}^0, P_{V,2}^0) \in \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{P}^1 \times \mathbb{P}^1$. Then f_V factors through $\mathbb{P}^1 \times \{P_{V,2}^0\}$ and $x_E = (0, P_{V,2}^0)$. It follows that $\Sigma(f)_b(E_{f,b}) \subset \tilde h(E)$.

Let E be the outgoing edge of a trivalent vertex of $\tilde\Gamma$, of ingoing edges E^1 and E^2. By the induction hypothesis, we know that $\Sigma(f)_b(E_{f,b}^1) \subset \tilde h(E^1)$ and $\Sigma(f)_b(E_{f,b}^2) \subset \tilde h(E^2)$. We conclude that $\Sigma(f)_b(E_{f,b}) \subset \tilde h(E)$ by an application of the balancing condition, as in Proposition 30 (tropical Menelaus theorem) of Mikhalkin [Mik15]. □

For a stable log map

$$\begin{array}{ccc} C & \xrightarrow{f} & X_0 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \nu_0} \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}} \end{array}$$

marked by $\tilde{h}$, we have nodes of C in correspondence with the bounded edges of $\tilde{\Gamma}$. Cutting C along these nodes, we obtain a morphism

$$\mathrm{cut}\colon \overline{M}_{g,n,\Delta}^{\tilde{h},P^0} \to \bigtimes_{V \in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} .$$

Let us give a precise definition of the cut morphism[14]. By definition of the marking, for every vertex V of $\tilde{\Gamma}$, we have an ordinary stable map $f_V\colon C_V \to X_{\Delta_V}$, such that the underlying stable map to f is obtained by gluing together the maps f_V along nodes corresponding to the edges of $\tilde{\Gamma}$.

We have to give C_V the structure of a log curve, and enhance f_V to a log morphism. In particular, we need to construct a monoid $\overline{\mathcal{M}}_V$.

We fix a point b in the interior of $\Sigma(g)^{-1}(1)$. Let $\Sigma(f)_b\colon \Sigma(C)_b \to \mathbb{R}^2$ be the corresponding parametrized tropical curve. Let $\Sigma(C)_{V,b}$ be the subgraph of $\Sigma(C)_b$ obtained by taking the vertices of $\Sigma(C)_b$ dual to irreducible components of C_V, the edges between them, and considering the edges to other vertices of $\Sigma(C)_b$ as unbounded edges. Let $\Sigma(f)_{V,b}$ be the restriction of $\Sigma(f)_b$ to $\Sigma(C)_{V,b}$. It follows from Lemma 1.16 that one can view $\Sigma(f)_{V,b}$ as a parametrized tropical curve of type Δ_V to the fan of X_{Δ_V}.

We define $\overline{\mathcal{M}}_V$ as being the monoid whose dual is the monoid of integral points of the moduli space of deformations of $\Sigma(f)_{V,b}$ preserving its combinatorial type[15]. Let $i_{C_V}\colon C_V \to C$ and $i_{X_{\Delta_V}}\colon X_{\Delta_V} \to X_0$ be the inclusion morphisms of ordinary (not log) schemes. The parametrized tropical curves $\Sigma(f)_V$ encode a sheaf of monoids $\overline{\mathcal{M}}_{C_V}$ and a map $f_V^{-1}\overline{\mathcal{M}}_{X_{\Delta_V}} \to \overline{\mathcal{M}}_{C_V}$. We define a log structure on C_V by

$$\mathcal{M}_{C_V} = \overline{\mathcal{M}}_{C_V} \times_{i_{C_V}^{-1}\overline{\mathcal{M}}_C} i_{C_V}^{-1}\mathcal{M}_C .$$

The natural diagram

$$\begin{array}{ccc} f_V^{-1}\mathcal{M}_{X_{\Delta_V}} & & \mathcal{M}_{C_V} \\ \downarrow & & \downarrow \\ f_V^{-1} i_{X_{\Delta_V}}^{-1}\mathcal{M}_{X_0} & \longrightarrow & i_{C_V}^{-1}\mathcal{M}_C \end{array}$$

can be uniquely completed, by restriction, with a map

$$f_V^{-1}\mathcal{M}_{X_{\Delta_V}} \to \mathcal{M}_{C_V}$$

[14] We are considering a stable log map over a point. It is a notational exercise to extend the argument to a stable log map over a general base, which is required to really define a morphism between moduli spaces

[15] The base monoid of a basic stable log map has always such description in terms of deformations of tropical curves. See Remark 1.18 and Remark 1.21 of [GS13] for more details

compatible with $f_V^{-1}\overline{\mathcal{M}}_{X_{\Delta_V}} \to \overline{\mathcal{M}}_{C_V}$. This defines a log enhancement of f_V and finishes the construction of the cut morphism.

Remark: If one considers a general log smooth degeneration and if one applies the decomposition formula, it is in general impossible to write the contribution of a tropical curves in terms of log Gromov-Witten invariants attached to the vertices. This is already clear at the tropical level. The theory of punctured invariants developed by Abramovich, Chen, Gross, Siebert in [ACGS17b] is the correct extension of log Gromov-Witten theory which is needed in order to be able to write down a general gluing formula. In our present case, the Nishinou-Siebert toric degeneration is extremely special because it has been constructed knowing a priori the relevant tropical curves. It follows from Lemma 1.16 that we always cut edges contained in an edge of the polyhedral decomposition, and so we don't have to consider punctured invariants.

1.6.2 Counting log structures

We say that a map to X_0 is torically transverse if its image does not contain any of the torus fixed points of the toric components X_{Δ_V}. In other words, its corestriction to each toric surface X_{Δ_V} is torically transverse in the sense of Section 1.4.

Let $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ}$ be the open locus of $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}$ formed by the torically transverse stable log maps to X_0, and for every vertex V of $\tilde{\Gamma}$, let $\overline{M}_{g(V),\Delta_V}^{\circ}$ be the open locus of $\overline{M}_{g(V),\Delta_V}$ formed by the torically transverse stable log maps to X_{Δ_V}. The morphism cut restricts to a morphism

$$\mathrm{cut}^\circ \colon \overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ} \to \bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ}\,.$$

Proposition 1.17. *The morphism*

$$\mathrm{cut}^\circ \colon \overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ} \to \bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ}$$

is étale of degree

$$\prod_{E\in E_f(\tilde{\Gamma})} w(E)\,,$$

where the product is over the bounded edges of $\tilde{\Gamma}$.

Proof. Let $(f_V \colon C_V \to X_{\Delta_V})_V \in \bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ}$. We have to glue the stable log maps f_V together. Because we are assuming that the maps f_V are torically transverse, the image in X_0 by f_V of the curves C_V is away from the torus fixed points of the components X_{Δ_V}. The gluing operation corresponding to the bounded edge E of $\tilde{\Gamma}$ happens entirely along the torus $\mathbb{C}^*$ contained in the divisor D_E.

It follows that it is enough to study the following local model. Denote $\ell \coloneqq \ell(E)w(E)$, where $\ell(E)$ is the length of E and $w(E)$ the weight of E. Let X_E be the toric variety Spec $\mathbb{C}[x,y,u^{\pm},t]/(xy = t^{\ell})$, equipped with a morphism $\nu_E \colon X_E \to \mathbb{C}$ given by the coordinate t. Using the natural toric divisorial log structures on X_E and $\mathbb{C}$, we define by restriction a log structure on the special fiber $X_{0,E} \coloneqq \nu_E^{-1}(0)$ and a log smooth morphism to the standard log point $\nu_{0,E}\colon X_{0,E} \to \mathrm{pt}_{\mathbb{N}}$. The scheme underlying $X_{0,E}$ has two irreducible components, $X_{1,E} \coloneqq \mathbb{C}_x \times \mathbb{C}_u^*$ and $X_{2,E} \coloneqq \mathbb{C}_y \times \mathbb{C}_u^*$, glued along the smooth divisor $D_E^\circ \coloneqq \mathbb{C}_u^*$. We endow $X_{1,E}$ and $X_{2,E}$ with their toric divisorial log structures.

Let $f_1\colon C_1/\mathrm{pt}_{\overline{\mathcal{M}}_1} \to X_{1,E}$ be the restriction to $X_{1,E}$ of a torically transverse stable log map to some toric compactification of $X_{1,E}$, with one point p_1 of tangency order $w(E)$ along D_E, and let $f_2\colon C_2/\mathrm{pt}_{\overline{\mathcal{M}}_2} \to X_{2,E}$ be the restriction to $X_{2,E}$ of a torically transverse stable log map to some toric compactification of $X_{2,E}$, with one point p_2 of tangency order $w(E)$ along D_E. We assume that $f(p_1) = f(p_2)$ and so we can glue the underlying maps $\underline{f}_1\colon \underline{C}_1 \to \underline{X}_{1,E}$ and $\underline{f}_2\colon \underline{C}_2 \to \underline{X}_{2,E}$ to obtain a map $\underline{f}\colon \underline{C} \to \underline{X}_{0,E}$ where $\underline{C}$ is the curve obtained from $\underline{C}_1$ and $\underline{C}_2$ by identification of p_1 and p_2. We denote p the corresponding node of $\underline{C}$. We have to show that there are $w(E)$ ways to lift this map to a log map in a way compatible with the log maps f_1 and f_2 and with the basic condition. If C_1 and C_2 had no component contracted to $f(p) \in D_E^\circ$, this would follow from Proposition 7.1 of Nishinou, Siebert [NS06]. But we allow contracted components, so we have to present a variant of the proof of Proposition 7.1 of [NS06].

We first give a tropical description of the relevant objects. The tropicalization of $X_{0,E}$ is the cone $\Sigma(X_{0,E}) = \mathrm{Hom}(\overline{\mathcal{M}}_{X_{0,E},f(p)}, \mathbb{R}_{\geqslant 0})$. It is the fan of X_E, a two-dimensional cone generated by rays ρ_1 and ρ_2 dual to the divisors $X_{1,E}$ and $X_{2,E}$. The toric description $X_E = \mathrm{Spec}\,\mathbb{C}[x,y,u^\pm,t]/(xy = t^\ell)$ defines a natural chart for the log structure of $X_{0,E}$. Denote s_x, s_y, s_t the corresponding elements of $\mathcal{M}_{X_{0,E},f(p)}$ and $\overline{s}_x, \overline{s}_y, \overline{s}_t$ their projections in $\overline{\mathcal{M}}_{X_{0,E},f(p)}$. We have $s_x s_y = s_t^\ell$. Seeing elements of $\overline{\mathcal{M}}_{X_{0,E},f(p)}$ as functions on $\Sigma(X_{0,E})$, we have $\rho_1 = \overline{s}_y^{-1}(0)$, $\rho_2 = \overline{s}_x^{-1}(0)$ and $\overline{s}_t\colon \Sigma(X_{0,E}) \to \mathbb{R}_{\geqslant 0}$ is the tropicalization of the projection $X_{0,E} \to \mathrm{pt}_{\mathbb{N}}$. Level sets $\overline{s}_t^{-1}(c)$ are line segments $[P_1, P_2]$ in $\Sigma(X_{0,E})$, connecting a point P_1 of ρ_1 to a point P_2 of ρ_2, of length lc.

Denote $\underline{C}_{1,E}$ and $\underline{C}_{2,E}$ the irreducible components of $\underline{C}_1$ and $\underline{C}_2$ containing p_1 and p_2 respectively. We can see them as the two irreducible components of $\underline{C}$ meeting at the node p. Fix $j = 1$ or $j = 2$. The tropicalization of $C_j/\mathrm{pt}_{\overline{\mathcal{M}}_j}$ is a family $\Sigma(C_j)$ of tropical curves $\Sigma(C_j)_b$ parametrized by $b \in \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}_j}) = \mathrm{Hom}(\overline{\mathcal{M}}_j, \mathbb{R}_{\geqslant 0})$. Let $V_{j,E}$ be the vertex of these tropical curves dual to the irreducible component $\underline{C}_{j,E}$. The image $\Sigma(f_j)(V_{j,E})$ of $V_{j,E}$ by the tropicalization $\Sigma(f_j)$ of f_j is a point in the tropicalization $\Sigma(X_{j,E}) = \mathbb{R}_{\geqslant 0}$. This induces a map $\mathrm{Hom}(\overline{\mathcal{M}}_j, \mathbb{R}_{\geqslant 0}) \to \mathbb{R}_{\geqslant 0}$ defined by an element $v_j \in \overline{\mathcal{M}}_j$. The component $\underline{C}_{j,E}$ is contracted by f_j onto $f_j(p_j)$ if and only if $v_j \neq 0$. In other words, v_j is the measure according to the log structures of "how" $\underline{C}_{j,E}$ is contracted by f_j. The marked point p_j on $C_{j,E}$ defines an unbounded edge E_j, of weight $w(E)$, whose image by $\Sigma(f_j)$ is the unbounded interval $[\Sigma(f_j)(V_{j,E}), +\infty) \subset \Sigma(X_{j,E}) = \mathbb{R}_{\geqslant 0}$.

We explain now the gluing at the tropical level. Let $j = 1$ or $j = 2$. Let $[0, l_j] \subset \Sigma(X_{j,E}) = \mathbb{R}_{\geqslant 0}$ be an interval. If c is a large enough positive real number, we denote $\varphi_c^j\colon [0, l_j] \hookrightarrow \overline{s}_t^{-1}(c) = [P_1, P_2]$ the linear inclusion such that $\varphi_c^j(0) = P_j$ and $\varphi_c^j([0, l_j])$ is a subinterval of $[P_1, P_2]$ of length l_j. Let $b_j \in \Sigma(\mathrm{pt}_{\overline{\mathcal{M}}_j})$. There exists l_j large enough such that all images by $\Sigma(f_j)$ of vertices of $\Sigma(f_j)_{b_j}$ are contained in $[0, l_j] \subset \Sigma(X_{j,E}) = \mathbb{R}_{\geqslant 0}$.

For c large enough, the line segments $\varphi_c^1([0,l_1])$ and $\varphi_c^2([0,l_2])$ are disjoint. We have

$$[P_1, P_2]$$

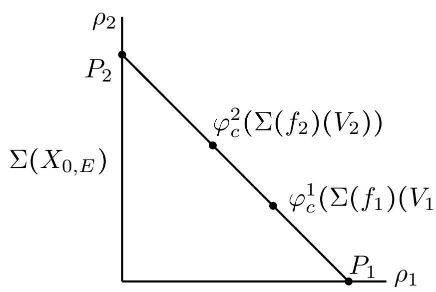
$$= [P_1, \varphi_c^1(\Sigma(f_1)(V_1))) \cup [\varphi_c^1(\Sigma(f_1)(V_1)), \varphi_c^2(\Sigma(f_2)(V_2))] \cup (\varphi_c^2(\Sigma(f_2)(V_2)), P_2]\,.$$

We construct a new tropical curve $\Sigma_{b_1,b_2,c}$ by removing the unbounded edges E_1 and E_2 of $\Sigma(f_1)_{b_1}$ and $\Sigma(f_2)_{b_2}$, and gluing the remaining curves by an edge F connecting $V_{1,E}$ and $V_{2,E}$, of weight $w(E)$, and length $\frac{1}{w(E)}$ times the length of the line segment $[\varphi_c^1(\Sigma(f_1)(V_1)), \varphi_c^2(\Sigma(f_2)(V_2))]$. We construct a tropical map $\Sigma_{b_1,b_2,c} \to \Sigma(X_{0,E})$ using $\Sigma(f_1)_{b_1}$, $\Sigma(f_2)_{b_2}$ and mapping the edge F to $[\varphi_c^1(\Sigma(f_1)(V_1)), \varphi_c^2(\Sigma(f_2)(V_2))]$. We define $\overline{\mathcal{M}}$ as being the monoid whose dual is the monoid of integral points of the moduli space of deformations of these tropical maps.

We have $\overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}$. The element $(0,0,1) \in \overline{\mathcal{M}}$ defines the function on the moduli space of tropical curves $\Sigma(\mathrm{pt}_{\overline{\mathcal{M}}}) = \mathrm{Hom}(\overline{\mathcal{M}}, \mathbb{R}_{\geqslant 0})$ given by the length of the gluing edge F. The function given by $\frac{1}{l}$ times the length of the line segment $[P_1, P_2]$ defines an element $\overline{s}_t^{\overline{\mathcal{M}}} \in \overline{\mathcal{M}}$. The morphism of monoids $\mathbb{N} \to \overline{\mathcal{M}}$, $1 \mapsto \overline{s}_t^{\overline{\mathcal{M}}}$, induces a map $g\colon \mathrm{pt}_{\overline{\mathcal{M}}} \to \mathrm{pt}_{\mathbb{N}}$. The decomposition of $[P_1, P_2]$ into the three intervals $[P_1, \varphi_c^1(\Sigma(f_1)(V_1)))$, $[\varphi_c^1(\Sigma(f_1)(V_1)), \varphi_c^2(\Sigma(f_2)(V_2))]$ and $(\varphi_c^2(\Sigma(f_2)(V_2)), P_2]$, implies the relation

$$l\, \overline{s}_t^{\overline{\mathcal{M}}} = (v_1, 0, 0) + (0, 0, w(E)) + (0, v_2, 0)$$

in $\overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}$.



From the tropical description of the gluing and from the fact that we want to obtain a basic log map, we find that there is a unique structure of log smooth curve $C/\mathrm{pt}_{\overline{\mathcal{M}}}$ compatible with the structures of log smooth curves on C_1 and C_2. As p is a node of C, we have for the ghost sheaf of C at p: $\overline{\mathcal{M}}_{C,p} = \overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2$, with $\mathbb{N} \to \mathbb{N}^2$, $1 \mapsto (1,1)$, and $\mathbb{N} \to \overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}$, $1 \mapsto \rho_p = (0,0,1)$.

It remains to lift $\underline{f}\colon \underline{C} \to \underline{X}_{0,E}$ to a log map $f\colon C \to X_{0,E}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X_{0,E} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \nu_{0,E}} \\ \mathrm{pt}_{\overline{\mathcal{M}}} & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}} \end{array}$$

commutes. The restriction of f to $C_j/\mathrm{pt}_{\overline{\mathcal{M}}_j}$ has to coincide with f_j, for $j = 1$ and $j = 2$. It follows from the explicit description of $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_C$ that such f exists and is unique away from the node p.

It follows from the tropical description of the gluing that at the ghost sheaves level, f at p is given by

$$\overline{f}^\flat \colon \overline{\mathcal{M}}_{X_{0,E},f(p)} \to \overline{\mathcal{M}}_{C,p} = \overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2 = (\overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}) \oplus_{\mathbb{N}} \mathbb{N}^2$$

$$\begin{aligned} \overline{s}_x &\mapsto ((v_1, 0, 0), (w(E), 0)) \\ \overline{s}_y &\mapsto ((0, v_2, 0), (0, w(E))) \\ \overline{s}_t &\mapsto \overline{\pi}^\flat(\overline{s}_t^{\overline{\mathcal{M}}}) = (\overline{s}_t^{\overline{\mathcal{M}}}, (0,0)) \,. \end{aligned}$$

The relation $l\,\overline{s}_t^{\overline{\mathcal{M}}} = (v_1, v_2, w(E))$ in $\overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2 \oplus \mathbb{N}$ implies that

$$\begin{aligned} \overline{f}^\flat(\overline{s}_x) + \overline{f}^\flat(\overline{s}_y) &= ((v_1, v_2, 0), (w(E), w(E))) = ((v_1, v_2, w(E)), (0,0)) \\ &= \overline{f}^\flat(l\overline{s}_t^{\overline{\mathcal{M}}}) \,, \end{aligned}$$

and so that this map is indeed well-defined.

The log maps $f_1 \colon C_1/\mathrm{pt}_{\overline{\mathcal{M}}_1} \to X_{1,E}$ and $f_2 \colon C_2/\mathrm{pt}_{\overline{\mathcal{M}}_2} \to X_{2,E}$ define morphisms

$$f_1^\flat \colon \mathcal{M}_{X_{1,E},f(p_1)} \to \mathcal{M}_{C_1,p_1} \,,$$

and

$$f_2^\flat \colon \mathcal{M}_{X_{2,E},f(p_2)} \to \mathcal{M}_{C_2,p_2} \,.$$

For $j = 1$ or $j = 2$, let $\overline{\mathcal{M}}_j \oplus \mathbb{N} \to \mathcal{O}_{C_j,p_j}$ be a chart of the log structure of C_j at p_j. This realizes $\mathcal{M}_{C_j,p_j}$ as a quotient of $(\overline{\mathcal{M}}_j \oplus \mathbb{N}) \oplus \mathcal{O}_{C,p}^*$. Denote $s_{j,m} \in \mathcal{M}_{C_j,p_j}$ the image of $(m,1)$ for $m \in \overline{\mathcal{M}}_j \oplus \mathbb{N}$.

We fix a coordinate u on C_1 near p_1 such that

$$f_1^\flat(s_x) = s_{1,(v_1,0)} u^{w(E)}$$

and a coordinate v on C_2 near p_2 such that

$$f_2^\flat(s_y) = s_{2,(v_2,0)} v^{w(E)} \,.$$

We are trying to define some $f^\flat \colon \mathcal{M}_{X_{0,E},f(p)} \to \mathcal{M}_{C,p}$, lift of $\overline{f}^\flat$, compatible with $f_1^\flat$ and $f_2^\flat$. For every ζ a $w(E)$-th root of unity, the map

$$\overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2 \to \mathcal{O}_{C,p}$$

$$(m, (a,b)) \mapsto \begin{cases} \zeta^a u^a v^b & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \,. \end{cases}$$

defines a chart for the log structure of C at p. This realizes $\mathcal{M}_{C,p}$ as a quotient of $(\overline{\mathcal{M}} \oplus_{\mathbb{N}}$

$\mathbb{N}^2) \oplus \mathcal{O}_{C,p}^*$. Denote $s_m^\zeta \in \mathcal{M}_{C,p}$ the image of $(m,1)$ for $m \in \overline{\mathcal{M}} \oplus_{\mathbb{N}} \mathbb{N}^2$. Remark that $s^\zeta_{((v_1,0,0),(0,0))}$, $s^\zeta_{((0,v_2,0),(0,0))}$ and $s^\zeta_{((0,0,0),(1,1))}$ are independent of ζ and we denote them simply as $s_{((v_1,0,0),(0,0))}$, $s_{((0,v_2,0),(0,0))}$ and $s_{((0,0,0),(1,1))}$.

Then

$$\begin{aligned} f^{\flat,\zeta} \colon \mathcal{M}_{X_{0,E},f(p)} &\to \mathcal{M}_{C,p} \\ s_x &\mapsto s^\zeta_{((v_1,0,0),(w(E),0))} \\ s_y &\mapsto s^\zeta_{((0,v_2,0),(0,w(E)))} \\ s_t &\mapsto \pi^\flat((\bar{s}_t^{\overline{\mathcal{M}}},1)) \end{aligned}$$

is a lift of $\overline{f}^\flat$, compatible with $f_1^\flat$ and $f_2^\flat$.

Assume that $f^{\flat,\zeta} \simeq f^{\flat,\zeta'}$ for ζ and ζ' two $w(E)$-th roots of unity. It follows from the compatibility with $f_1^\flat$ and $f_2^\flat$ that there exists $\varphi_1 \in \mathcal{O}_{C,p}^*$ and $\varphi_2 \in \mathcal{O}_{C,p}^*$ such that $s^{\zeta'}_{((0,0,0),(1,0))} = \varphi_1 s^\zeta_{((0,0,0),(0,1))}$ and $s^{\zeta'}_{((0,0,0),(0,1))} = \varphi_2 s^\zeta_{((0,0,0),(0,1))}$. It follows from the definition of the charts that $\varphi_1 = \zeta'\zeta^{-1}$ in $\mathcal{O}_{C_1,p_1}$ and $\varphi_2 = 1$ in $\mathcal{O}_{C_2,p_2}$. Compatibility with $\mathrm{pt}_{\overline{\mathcal{M}}} \to \mathrm{pt}_{\mathbb{N}}$ implies that $\varphi_1\varphi_2 = 1$. This implies that $\varphi_1 = \varphi_2 = 1$ and $\zeta = \zeta'$.

It remains to show that any $f^\flat$, lift of $\overline{f}^\flat$ compatible with $f_1^\flat$ and $f_2^\flat$, is of the form $f^{\flat,\zeta}$ for some ζ a $w(E)$-th root of unity. For such $f^\flat$, there exists unique $s'_{(1,0)} \in \mathcal{M}_{C,p}$ and $s'_{(0,1)} \in \mathcal{M}_{C,p}$ such that $\alpha_C(s'_{(1,0)}) = u$, $\alpha_C(s'_{(0,1)}) = v$, and $f^\flat(s_x) = s_{((v_1,0,0),(0,0))}(s'_{(1,0)})^{w(E)}$ and $f^\flat(s_y) = s_{((0,v_2,0),(0,0))}(s'_{(0,1)})^{w(E)}$. From $s_x s_y = s_t^\ell$, we get $(s'_{(1,0)}s'_{(0,1)})^{w(E)} = s^{w(E)}_{((0,0,0),(1,1))}$ and so $s'_{(1,0)}s'_{(0,1)} = \zeta^{-1}s_{((0,0,0),(1,1))}$ for some ζ a $w(E)$-th root of unity. It is now easy to check that $s'_{(1,0)} = \zeta^{-1}s^\zeta_{((0,0,0),(1,0))}$, $s'_{(0,1)} = s^\zeta_{((0,0,0),(0,1))}$ and $f^\flat = f^{\flat,\zeta}$.

□

Remarks:

- When $v_1 = v_2 = 0$, i.e. when the components $C_{1,E}$ and $C_{2,E}$ are not contracted, the above proof reduces to the proof of Proposition 7.1 of [NS06] (see also the proof of Proposition 4.23 of [Gro11]). In general, log geometry remembers enough information about the contracted components, such as v_1 and v_2, to make possible a parallel argument.
- The gluing of stable log maps along a smooth divisor is discussed in Section 6 of [KLR18], proving the degeneration formula along a smooth divisor. In the above proof, we only have to glue along one edge connecting two vertices. In Section 6 of [KLR18], further work is required to deal with pair of vertices connected by several edges.

1.6.3 Comparing obstruction theories

As in the previous Section 1.6.2, let $\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}$ be the open locus of $\overline{M}^{\tilde{h},P^0}_{g,n,\Delta}$ formed by the torically transverse stable log maps to X_0, and for every vertex V of $\tilde{\Gamma}$, let $\overline{M}^\circ_{g(V),\Delta_V}$ be the open locus of $\overline{M}_{g(V),\Delta_V}$ formed by the torically transverse stable log maps to X_{Δ_V}. The morphism cut restricts to a morphism

$$\mathrm{cut}^\circ : \overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta} \to \bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}^\circ_{g(V),\Delta_V} .$$

The goal of the present Section is to use the morphism cut° to compare the virtual classes $[\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}]^{\mathrm{virt}}$ and $[\overline{M}^\circ_{g(V),\Delta_V}]^{\mathrm{virt}}$, which are obtained by restricting the virtual classes $[\overline{M}^{\tilde{h},P^0}_{g,n,\Delta}]^{\mathrm{virt}}$ and $[\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}$ to the open loci of torically transverse stable log maps.

Recall that $X_0 = \nu^{-1}(0)$, where $\nu: X_{\mathcal{P}_{\Delta,n}} \to \mathbb{A}^1$. Following Section 4.1 of [ACGS17a], we define $\mathcal{X}_0 := \mathcal{A}_X \times_{\mathcal{A}_{\mathbb{A}^1}} \{0\}$, where $\mathcal{A}_X$ and $\mathcal{A}_{\mathbb{A}^1}$ are Artin fans, see Section 2.2 of [ACGS17a]. It is an algebraic log stack over $\mathrm{pt}_{\mathbb{N}}$. There is a natural morphism $X_0 \to \mathcal{X}_0$.

Following Section 4.5 of [ACGS17a], let $\mathfrak{M}^{\tilde{h}}_{g,n,\Delta}$ be the stack of n-pointed genus g prestable basic log maps to $\mathcal{X}_0/\mathrm{pt}_{\mathbb{N}}$ marked by $\tilde{h}$ and of type Δ. There is a natural morphism of stacks $\overline{M}^{\tilde{h},P^0}_{g,n,\Delta} \to \mathfrak{M}^{\tilde{h}}_{g,n,\Delta}$. Let $\pi:\mathcal{C} \to \overline{M}^{\tilde{h},P^0}_{g,n,\Delta}$ be the universal curve and let $f:\mathcal{C} \to X_0/\mathrm{pt}_{\mathbb{N}}$ be the universal stable log map. According to Proposition 4.7.1 and Section 6.3.2 of [ACGS17a], the virtual fundamental class $[\overline{M}^{\tilde{h},P^0}_{g,n,\Delta}]^{\mathrm{virt}}$ is defined by $\mathbf{E}$, the cone of the morphism $(\mathrm{ev}^{(p)})^* L_{\iota_{P^0}}[-1] \to (R\pi_* f^* T_{X_0|\mathcal{X}_0})^\vee$, seen as a perfect obstruction theory relative to $\mathfrak{M}^{\tilde{h}}_{g,n,\Delta}$. Here, $T_{X_0|\mathcal{X}_0}$ is the relative log tangent bundle, and $L_{\iota_{P^0}} = \oplus_{V\in V^{(2p)}(\tilde{\Gamma})} (T_{X_{\Delta_V}}|_{P^0_V})^\vee[1]$ is the cotangent complex of ι_{P^0}. As $\mathcal{X}_0$ is log étale over $\mathrm{pt}_{\mathbb{N}}$, we have $T_{X_0|\mathcal{X}_0} = T_{X_0|\mathrm{pt}_{\mathbb{N}}}$. We denote $\mathbf{E}^\circ$ the restriction of $\mathbf{E}$ to the open locus $\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}$ of torically transverse stable log maps.

For every vertex V of $\tilde{\Gamma}$, let $\pi_V: \mathcal{C}_V \to \overline{M}_{g(V),\Delta_V}$ be the universal curve and let $f_V:\mathcal{C}_V \to X_{\Delta_V}$ be the universal stable log map. Let $\mathcal{A}_{X_{\Delta_V}}$ be the Artin fan of X_{Δ_V} and let $\mathfrak{M}_{g(V),\Delta_V}$ be the stack of prestable basic log maps to $\mathcal{A}_{X_{\Delta_V}}$, of genus $g(V)$ and of type Δ_V. There is a natural morphism of stacks $\overline{M}_{g(V),\Delta_V} \to \mathfrak{M}_{g(V),\Delta_V}$. According to Section 6.1 of [AW13], the virtual fundamental class $[\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}$ is defined by $(R(\pi_V)_* f_V^* T_{X_{\Delta_V}})^\vee$, seen as a perfect obstruction theory relative to $\mathfrak{M}_{g(V),\Delta_V}$. Here, $T_{X_{\Delta_V}}$ is the log tangent bundle.

Recall that $\bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}$ is defined by the fiber product diagram

$$\begin{array}{ccc}
\bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} & \xrightarrow{(\delta\times\iota_{P^0})_M} & \prod_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \\
\big\downarrow \mathrm{ev}^{(e)}\times\mathrm{ev}^{(p)} & & \big\downarrow \mathrm{ev}^{(e)}\times\mathrm{ev}^{(p)} \\
\left(\prod_{E\in E_f(\tilde{\Gamma})} D_E\right) \times P^0 & \xrightarrow{\delta\times\iota_{P^0}} & \prod_{E\in E_f(\tilde{\Gamma})} (D_E)^2 \times \prod_{V\in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V} ,
\end{array}$$

We compare the deformation theory of the individual stable log maps f_V and the deformation theory of the stable log maps f_V constrained to match at the gluing nodes. Let $\mathbf{F}$ be the cone of the natural morphism

$$(\mathrm{ev}^{(e)} \times \mathrm{ev}^{(p)})^* L_{\delta\times\iota_{P^0}}[-1] \to (\delta\times\iota_{P^0})_M^* \left(\boxtimes_{V\in V(\tilde{\Gamma})} (R(\pi_V)_* f_V^* T_{X_{\Delta_V}})^\vee \right),$$

where $L_{\delta\times\iota_{P^0}}$ is the cotangent complex of the morphism $\delta\times\iota_{P^0}$. It defines a perfect obstruction theory on $\bigtimes_{V\in V(\tilde\Gamma)} \overline{M}_{g(V),\Delta_V}$ relative to $\prod_{V\in V(\tilde\Gamma)} \mathfrak{M}_{g(V),\Delta_V}$, whose corresponding virtual fundamental class is, using Proposition 5.10 of [BF97],

$$(\delta\times\iota_{P^0})^! \prod_{V\in V(\tilde\Gamma)} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}},$$

where $(\delta\times\iota_{P^0})^!$ is the refined Gysin homomorphism (see Section 6.2 of [Ful98]). We denote $\mathbf{F}^\circ$ the restriction of $\mathbf{F}$ to the open locus $\bigtimes_{V\in V(\tilde\Gamma)} \overline{M}^\circ_{g(V),\Delta_V}$ of torically transverse stable log maps.

The cut operation naturally extends to prestable log maps to $\mathcal{X}_0/\mathrm{pt}_{\mathbb{N}}$ marked by $\tilde h$, and so we have a commutative diagram

$$\begin{array}{ccc}
\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta} & \xrightarrow{\mathrm{cut}^\circ} & \bigtimes_{V\in V(\tilde\Gamma)} \overline{M}^\circ_{g(V),\Delta_V} \\
\downarrow{\scriptstyle\mu} & & \downarrow \\
\mathfrak{M}^{\tilde h}_{g,n,\Delta} & \xrightarrow{\mathrm{cut}_C} & \prod_{V\in V(\tilde\Gamma)} \mathfrak{M}_{g(V),\Delta_V}\,.
\end{array}$$

By Proposition 1.17, the morphism cut° is étale and so $(\mathrm{cut}^\circ)^*\mathbf{F}^\circ$ defines a perfect obstruction theory on $\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta}$ relative to $\prod_{V\in V(\tilde\Gamma)} \mathfrak{M}_{g(V),\Delta_V}$.

The maps $\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta} \xrightarrow{\mu} \mathfrak{M}^{\tilde h}_{g,n,\Delta}(\mathcal{X}_0/\mathrm{pt}_{\mathbb{N}}) \xrightarrow{\mathrm{cut}_C} \prod_{V\in V(\tilde\Gamma)} \mathfrak{M}_{g(V),\Delta_V}$ define an exact triangle of cotangent complexes

$$L_{\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta}|\prod_{V\in V(\tilde\Gamma)}\mathfrak{M}_{g(V),\Delta_V}} \to L_{\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta}|\mathfrak{M}^{\tilde h}_{g,n,\Delta}} \to \mu^* L_{\mathfrak{M}^{\tilde h}_{g,n,\Delta}|\prod_{V\in V(\tilde\Gamma)}\mathfrak{M}_{g(V),\Delta_V}}[1] \xrightarrow{[1]}\,.$$

Adding the perfect obstruction theories $(\mathrm{cut}^\circ)^*\mathbf{F}^\circ$ and $\mathbf{E}^\circ$, we get a diagram

$$\begin{array}{ccccc}
(\mathrm{cut}^\circ)^*\mathbf{F}^\circ & & \mathbf{E}^\circ & & \\
\downarrow & & \downarrow & & \\
L_{\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta}|\prod_{V\in V(\tilde\Gamma)}\mathfrak{M}_{g(V),\Delta_V}} & \to & L_{\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta}|\mathfrak{M}^{\tilde h}_{g,n,\Delta}} & \to & \mu^* L_{\mathfrak{M}^{\tilde h}_{g,n,\Delta}|\prod_{V\in V(\tilde\Gamma)}\mathfrak{M}_{g(V),\Delta_V}}[1] \xrightarrow{[1]}\,.
\end{array}$$

Proposition 1.18. *The above diagram can be completed into a morphism of exact triangles*

$$\begin{array}{ccccc}
(\mathrm{cut}^\circ)^*\mathbf{F}^\circ & \longrightarrow & \mathbf{E}^\circ & \longrightarrow & \mu^* L_{\mathfrak{M}^{\tilde h}_{g,n,\Delta}|\prod_{V\in V(\tilde\Gamma)}\mathfrak{M}_{g(V),\Delta_V}}[1] \xrightarrow{[1]}\,. \\
\downarrow & & \downarrow & & \| \\
L_{\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta}|\prod_{V\in V(\tilde\Gamma)}\mathfrak{M}_{g(V),\Delta_V}} & \to & L_{\overline{M}^{\tilde h,P^0,\circ}_{g,n,\Delta}|\mathfrak{M}^{\tilde h}_{g,n,\Delta}} & \to & \mu^* L_{\mathfrak{M}^{\tilde h}_{g,n,\Delta}|\prod_{V\in V(\tilde\Gamma)}\mathfrak{M}_{g(V),\Delta_V}}[1] \xrightarrow{[1]}\,.
\end{array}$$

Proof. Denote X_0°, $X^\circ_{\Delta_V}$, D°_E the objects obtained from X_0, X_{Δ_V}, D_E by removing the torus fixed points of the toric surfaces X_{Δ_V}. Denote $\iota_{X^\circ_{\Delta_V}}$ the inclusion morphism of $X^\circ_{\Delta_V}$ in X°_0.

If E is a bounded edge of $\tilde{\Gamma}$, we denote V_E^1 and V_E^2 the two vertices of E. Let $\mathcal{F}$ be the sheaf on the universal curve $\mathcal{C}|_{\overline{M}^{\tilde{h},P^0,\circ}_{g,n,\Delta}}$ defined as the kernel of

$$\bigoplus_{V\in V(\tilde{\Gamma})} f^*(\iota_{X^\circ_{\Delta_V}})_* T_{X^\circ_{\Delta_V}} \to \bigoplus_{E\in E_f(\tilde{\Gamma})} (\iota_E)_* (\mathrm{ev}^E)^* T_{D^\circ_E}$$

$$(s_V)_V \mapsto (s_{V_E^1}|_{D^\circ_E} - s_{V_E^2}|_{D^\circ_E})_E\,,$$

where ev^E is the evaluation at the node p_E dual to E, and ι_E the section of $\mathcal{C}$ given by p_E. It follows from the exact triangle obtained by applying $R\pi_*$ to the short exact sequence defining $\mathcal{F}$ and from $L_\delta = \oplus_{E\in E_f(\tilde{\Gamma})} T^\vee_{D_E}[1]$ that $(\mathrm{cut}^\circ)^*\mathbf{F}^\circ$ is given by the cone of the morphism $(\mathrm{ev}^{(p)})^* L_{\iota_{P^0}}[-1] \to (R\pi_*\mathcal{F})^\vee$. So in order to compare $\mathbf{E}^\circ$ and $(\mathrm{cut}^\circ)^*\mathbf{F}^\circ$, we have to compare $f^* T_{X^\circ_0|\mathrm{pt}_\mathbb{N}}$ and $\mathcal{F}$. The sheaf $f^* T_{X^\circ_0|\mathrm{pt}_\mathbb{N}}$ can be written as the kernel of

$$f^* \bigoplus_{V\in V(\tilde{\Gamma})} (\iota_{X^\circ_{\Delta_V}})_*(\iota_{X^\circ_{\Delta_V}})^* T_{X^\circ_0|\mathrm{pt}_\mathbb{N}} \to \bigoplus_{E\in E_f(\tilde{\Gamma})} (\iota_E)_*(\mathrm{ev}^E)^* T_{X^\circ_0|\mathrm{pt}_\mathbb{N}}\,.$$

$$(s_V)_V \mapsto (s_{V_E^1}|_{D^\circ_E} - s_{V_E^2}|_{D^\circ_E})_E\,.$$

Remark that because X_0 is the special fiber of a toric degeneration, all the log tangent bundles T_{X_0}, $T_{X_{\Delta_V}}$, T_{D_E} are free sheaves (see e.g. Section 7 of [NS06]). In particular, the restrictions $(\iota_{X^\circ_{\Delta_V}})^* T_{X^\circ_0|\mathrm{pt}_\mathbb{N}} \to T_{X^\circ_{\Delta_V}}$ are isomorphisms, the restriction

$$\bigoplus_{E\in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* T_{X^\circ_0|\mathrm{pt}_\mathbb{N}} \to \bigoplus_{E\in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* T_{D^\circ_E}$$

has kernel $\oplus_{E\in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* \mathcal{O}_{D^\circ_E}$ and so there is an induced exact sequence

$$0 \to f^* T_{X^\circ_0|\mathrm{pt}_\mathbb{N}} \to \mathcal{F} \to \bigoplus_{E\in E_f(\tilde{\Gamma})} (\iota_E)_*(\mathrm{ev}^E)^* \mathcal{O}_{D^\circ_E} \to 0\,,$$

which induces an exact triangle on $\overline{M}^{\tilde{h},P^0\circ}_{g,n,\Delta}$:

$$(\mathrm{cut}^\circ)^*\mathbf{F}^\circ \to \mathbf{E}^\circ \to \bigoplus_{E\in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* \mathcal{O}_{D^\circ_E}[1] \xrightarrow{[1]}\,.$$

It remains to check the compatibility of this exact triangle with the exact triangle of cotangent complexes. We have

$$\mu^* L_{\mathfrak{M}^{\tilde{h}}_{g,n,\Delta}|\prod_{V\in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}} = \bigoplus_{E\in E_f(\tilde{\Gamma})} (\iota_E)^* \mathcal{O}_{p_E}\,.$$

Indeed, restricted to the locus of torically transverse stable log maps, cut_C is smooth, and, given a torically transverse stable log map to $\mathcal{X}_0/\mathrm{pt}_\mathbb{N}$, a basis of first order infinitesimal deformations fixing its image by cut_C in $\prod_{V\in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}$ is indexed by the cutting nodes.

The dual of the natural map

$$\bigoplus_{E\in E_f(\tilde{\Gamma})} (\mathrm{ev}^E)^* \mathcal{O}_{D_E^\circ} \to \mu^* L_{\mathfrak{M}_{g,n,\Delta}^{\tilde{h}}|_{\prod_{V\in V(\tilde{\Gamma})} \mathfrak{M}_{g(V),\Delta_V}}} = \bigoplus_{E\in E_f(\tilde{\Gamma})} (\iota_E)^* \mathcal{O}_{p_E}$$

sends the canonical first order infinitesimal deformation indexed by the cutting node p_E to the canonical summand $\mathcal{O}_{D_E^\circ}$ in the normal bundle to the diagonal $\prod_{E\in E_f(\tilde{\Gamma})} D_E^\circ$ in $\prod_{E\in E_f(\tilde{\Gamma})} (D_E^\circ)^2$, and so is an isomorphism. This guarantees the compatibility with the exact triangle of cotangent complexes. □

Remark: Restricted to the open locus of torically transverse stable maps, the discussion is essentially reduced to a collection of gluings along the smooth divisors D_E°. A comparison of the obstruction theories in the context of the degeneration formula along a smooth divisor is given with full details in Section 7 of [KLR18].

Proposition 1.19. *We have*

$$(\mathrm{cut}^\circ)_* \left([\overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ}]^{\mathrm{virt}} \right)$$

$$= \left(\prod_{E\in E_f(\tilde{\Gamma})} w(E) \right) \left((\delta \times \iota_{P^0})_M^! \prod_{V\in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}^\circ]^{\mathrm{virt}} \right).$$

Proof. It follows from Proposition 1.18 and from Theorem 4.8 of [Man12] that the relative obstruction theories $\mathbf{E}^\circ$ and $(\mathrm{cut}^\circ)^* \mathbf{F}^\circ$ define the same virtual fundamental class on $\overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ}$. By Proposition 1.17, cut° is étale, and so, by Proposition 7.2 of [BF97], the virtual fundamental class defined by $(\mathrm{cut}^\circ)^* \mathbf{F}^\circ$ is the image by $(\mathrm{cut}^\circ)^*$ of the virtual fundamental class defined by $\mathbf{F}^\circ$. It follows that

$$[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0,\circ}]^{\mathrm{virt}} = (\mathrm{cut}^\circ)^* (\delta \times \iota_{P^0})_M^! \prod_{V\in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}^\circ]^{\mathrm{virt}}.$$

According to Proposition 1.17, the morphism cut° is étale of degree $\prod_{E\in E_f(\tilde{\Gamma})} w(E)$, and so the result follows from the projection formula. □

1.6.4 Gluing

Recall that we have the morphism

$$(\delta \times \iota_{P^0})_M \colon \bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} \to \prod_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}.$$

For every $V \in V(\tilde{\Gamma})$, we have a projection morphism

$$\mathrm{pr}_V \colon \prod_{V'\in V(\tilde{\Gamma})} \overline{M}_{g(V'),\Delta_{V'}} \to \overline{M}_{g(V),\Delta_V}.$$

On each moduli space $\overline{M}_{g(V),\Delta_V}$, we have the top lambda class $(-1)^{g(V)} \lambda_{g(V)}$.

Proposition 1.20. *We have*

$$N_{g,\tilde{h}}^{\Delta,n} = \int_{(\delta\times\iota_{P^0})^! \prod_{V\in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta\times\iota_{P^0})_M^* \prod_{V\in V(\tilde{\Gamma})} \mathrm{pr}_V^*\left((-1)^{g(V)}\lambda_{g(V)}\right) .$$

Proof. By definition (see Section 1.3.3), we have

$$N_{g,\tilde{h}}^{\Delta,n} = \int_{[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}}\lambda_{g-g_{\Delta,n}} .$$

Using the gluing properties of lambda classes given by Lemma 1.6, we obtain that

$$(-1)^{g-g_{\Delta,n}}\lambda_{g-g_{\Delta,n}} = (\mathrm{cut})^*(\delta\times\iota_{P^0})_M^* \prod_{V\in V(\tilde{\Gamma})} \mathrm{pr}_V^*\left((-1)^{g(V)}\lambda_{g(V)}\right) .$$

It follows from the projection formula that

$$N_{g,\tilde{h}}^{\Delta,n} = \int_{(\mathrm{cut})_*[\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}} (\delta\times\iota_{P^0})_M^* \prod_{V\in V(\tilde{\Gamma})} \mathrm{pr}_V^*\left((-1)^{g(V)}\lambda_{g(V)}\right) .$$

According to Proposition 1.19, the cycles

$$(\mathrm{cut})_*\left([\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}\right)$$

and

$$\left(\prod_{E\in E_f(\tilde{\Gamma})} w(E)\right)\left((\delta\times\iota_{P^0})^! \prod_{V\in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}\right)$$

have the same restriction to the open substack

$$\bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{\circ}$$

of

$$\bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V} .$$

It follows, by Proposition 1.8 of [Ful98], that their difference is rationally equivalent to a cycle supported on the closed substack

$$Z \coloneqq \left(\bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}\right) - \left(\bigtimes_{V\in V(\tilde{\Gamma})} \overline{M}_{g(V),\Delta_V}^{0}\right) .$$

If we have

$$(f_V\colon C_V\to X_{\Delta_V})_{V\in V(\tilde{\Gamma})}\in Z ,$$

then at least one stable log map $f_V\colon C_V\to X_{\Delta_V}$ is not torically transverse. By Lemma 1.16, the unbounded edges of the tropicalization of f_V are contained in the rays of the fan of X_{Δ_V}. It follows that we can apply Proposition 1.10 to obtain that at least one of the source curves

C_V contains a non-trivial cycle of components. By the vanishing result of Lemma 1.7, this implies that

$$\int_Z (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right) = 0 \,.$$

It follows that

$$\int_{(\mathrm{cut})_* [\overline{M}_{g,n,\Delta}^{\tilde{h},P^0}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right)$$

$$= \int_{(\delta \times \iota_{P^0})^! \prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right) .$$

This finishes the proof of Proposition 1.20.

□

1.6.5 Identifying the pieces

Proposition 1.21. *We have*

$$\int_{(\delta \times \iota_{P^0})^! \prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right) = \prod_{V \in V(\tilde{\Gamma})} N_{g(V),V}^{1,2} \,.$$

Proof. Using the definitions of δ and ι_{P^0}, we have

$$\int_{(\delta \times \iota_{P^0})^! \prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\delta \times \iota_{P^0})_M^* \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right)$$

$$= \int_{\prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}} (\mathrm{ev}^{(p)})^* ([P^0]) (\mathrm{ev}^{(e)})^* ([\delta]) \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)}\right) ,$$

where

$$[P^0] = \prod_{V \in V^{(2p)}(\tilde{\Gamma})} P_V^0 \in A^* \left(\prod_{V \in V^{(2p)}(\tilde{\Gamma})} X_{\Delta_V} \right)$$

is the class of P^0 and

$$[\delta] \in A^* \left(\prod_{E \in E_f(\tilde{\Gamma})} (D_E)^2 \right)$$

is the class of the diagonal $\prod_{E \in E_f(\tilde{\Gamma})} D_E$. As each D_E is a projective line, we have

$$[\delta] = \prod_{E \in E_f(\tilde{\Gamma})} (\mathrm{pt}_E \times 1 + 1 \times \mathrm{pt}_E) \,,$$

where $\mathrm{pt}_E \in A^1(D_E)$ is the class of a point.

We fix an orientation of edges of $\tilde{\Gamma}$ as described in Section 1.5. In particular, every trivalent vertex has two ingoing and one outgoing adjacent edges, every bivalent pointed vertex has two outgoing adjacent edges, every bivalent unpointed vertex has one ingoing and one outgoing edges. For every bounded edge E of $\tilde{\Gamma}$, we denote V_E^s the source vertex of E and V_E^t the target vertex of E, as defined by the orientation. Furthermore, the connected components of the complement of the bivalent pointed vertices of $\tilde{\Gamma}$ are trees with exactly one unbounded edge.

We argue that the effect of the insertion $(\mathrm{ev}^{(p)})^*([P^0])(\mathrm{ev}^{(e)})^*([\delta])$ can be computed in terms of the combinatorics of ingoing and outgoing edges of $\tilde{\Gamma}$[16]. More precisely, we claim that the only term in

$$(\mathrm{ev}^{(e)})^*([\delta]) = \prod_{E \in E_f(\tilde{\Gamma})} \left((\mathrm{ev}^E_{V^s_E})^*(\mathrm{pt}_E) + (\mathrm{ev}^E_{V^t_E})^*(\mathrm{pt}_E) \right) ,$$

giving a non-zero contribution after multiplication by

$$\left(\prod_{V \in V^{(2p)}(\tilde{\Gamma})} (\mathrm{ev}_V^{(p)})^*(P_V^0) \right) \prod_{V \in V(\tilde{\Gamma})} \mathrm{pr}_V^* \left((-1)^{g(V)} \lambda_{g(V)} \right)$$

and integration over $\prod_{V \in V(\tilde{\Gamma})} [\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}$ is $\prod_{E \in E_f(\tilde{\Gamma})} (\mathrm{ev}^E_{V^t_E})^*(\mathrm{pt}_E)$.

We prove this claim by induction, starting at the bivalent pointed vertices, where things are constrained by the marked points P^0, and propagating these constraints following the flow on $\tilde{\Gamma}$ defined by the orientation of edges.

Let V be a bivalent pointed vertex, E an edge adjacent to V and V' the other vertex of E. The edge E is outgoing for V and ingoing for V', so $V' = V^t_E$. We have in $(\mathrm{ev}^{(p)})^*([P^0])(\mathrm{ev}^{(e)})^*([\delta])$ a corresponding factor

$$(\mathrm{ev}_V^{(p)})^*(P_V^0) \left((\mathrm{ev}_V^E)^*(\mathrm{pt}_E) + (\mathrm{ev}_{V'}^E)^*(\mathrm{pt}_E) \right) .$$

But

$$(\mathrm{ev}_V^{(p)})^*(P_V^0)(\mathrm{ev}_V^E)^*(\mathrm{pt}_E)(-1)^{g(V)} \lambda_{g(V)} = 0$$

for dimension reasons (its insertion over $\overline{M}_{g(V),\Delta_V}$ defines an enumerative problem of virtual dimension -1) and so only the factor

$$(\mathrm{ev}_V^{(p)})^*(P_V^0)(\mathrm{ev}_{V'}^E)^*(\mathrm{pt}_E)$$

survives, which proves the initial step of the induction.

Let E be an outgoing edge of a trivalent vertex V, of ingoing edges E^1 and E^2. Let V^t_E be the target vertex of E. By the induction hypothesis, every possibly non-vanishing term contains the insertion of $(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})(\mathrm{ev}_V^{E^2})^*(\mathrm{pt}_{E^2})$. But

$$(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})(\mathrm{ev}_V^{E^2})^*(\mathrm{pt}_{E^2})(\mathrm{ev}_V^E)^*(\mathrm{pt}_E)(-1)^{g(V)} \lambda_{g(V)} = 0$$

for dimension reasons (its insertion over $\overline{M}_{g(V),\Delta_V}$ defines an enumerative problem of virtual

[16] It is essentially a cohomological reformulation and generalization of the way the gluing is organized in Mikhalkin's proof of the tropical correspondence theorem, [Mik05].

dimension −1) and so only the factor

$$(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_E^1)(\mathrm{ev}_V^{E^2})^*(\mathrm{pt}_E^2)(\mathrm{ev}_{V_E^t}^E)^*(\mathrm{pt}_E)$$

survives.

Let E be an outgoing edge of a bivalent unpointed vertex V, of ingoing edges E^1. Let V_E^t the target vertex of E. By the induction hypothesis, every possibly non-vanishing term contains the insertion of $(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})$. But

$$(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})(\mathrm{ev}_V^{E})^*(\mathrm{pt}_E)(-1)^{g(V)}\lambda_{g(V)} = 0$$

for dimension reasons (its insertion over $\overline{M}_{g(V),\Delta_V}$ defines an enumerative problem of virtual dimension −1) and so only the factor

$$(\mathrm{ev}_V^{E^1})^*(\mathrm{pt}_{E^1})(\mathrm{ev}_{V_E^t}^E)^*(\mathrm{pt}_E)$$

survives. This finishes the proof by induction of the claim.

Using the notations introduced in Section 1.5, we can rewrite

$$\prod_{E\in E_f(\tilde{\Gamma})}(\mathrm{ev}_{V_E^t}^E)^*(\mathrm{pt}_E)$$

as

$$\left(\prod_{V\in V^{(3)}(\tilde{\Gamma})}(\mathrm{ev}_V^{E_V^{\mathrm{in},1}})^*(\mathrm{pt}_{E_V^{\mathrm{in},1}})(\mathrm{ev}_V^{E_V^{\mathrm{in},2}})^*(\mathrm{pt}_{E_V^{\mathrm{in},2}})\right)\left(\prod_{V\in V^{(2up)}(\tilde{\Gamma})}(\mathrm{ev}_V^{E_V^{\mathrm{in}}})^*(\mathrm{pt}_{E_V^{\mathrm{in}}})\right),$$

and so we proved

$$\int_{(\delta\times\iota_{P^0})^! \prod_{V\in V(\tilde{\Gamma})}[\overline{M}_{g(V),\Delta_V}]^{\mathrm{virt}}}(\delta\times\iota_{P^0})_M^*\prod_{V\in V(\tilde{\Gamma})}\mathrm{pr}_V^*\left((-1)^{g(V)}\lambda_{g(V)}\right)$$

$$=\left(\prod_{V\in V^{(3)}(\tilde{\Gamma})}N_{g(V),V}^{1,2}\right)\left(\prod_{V\in V^{(2p)}(\tilde{\Gamma})}N_{g(V),V}^{1,2}\right)\left(\prod_{V\in V^{(2up)}(\tilde{\Gamma})}N_{g(V),V}^{1,2}\right).$$

This finishes the proof of Proposition 1.21. □

1.6.6 End of the proof of the gluing formula

The gluing identity given by Proposition 1.12 follows from the combination of Proposition 1.20 and Proposition 1.21.

1.7 Vertex contribution

In this Section, we evaluate the invariants $N_{g,V}^{1,2}$ attached to the vertices V of Γ and appearing in the gluing formula of Corollary 1.15. The first step, carried out in Section 1.7.1 is to rewrite these invariants in terms of more symmetric invariants $N_{g,V}$ depending only on the multiplicity of the vertex V. In Section 1.7.2, we use the consistency of the gluing formula to deduce non-trivial relations between these invariants and to reduce the question to the computation of the invariants attached to vertices of multiplicity one and two. Invariants attached to vertices of multiplicity one and two are explicitly computed in Section 1.7.3 and this finishes the proof of Theorem 1. Modifications needed to prove Theorem 1.5 are discussed at the end of Section 1.7.4.

1.7.1 Reduction to a function of the multiplicity

The gluing formula of the previous Section, Corollary 1.15, expresses the log Gromov-Witten invariant $N_{g,h}^{\Delta,n}$ attached to a parametrized tropical curve $h: \Gamma \to \mathbb{R}^2$ as a product of log Gromov-Witten $N_{g(V),V}^{1,2}$ attached to the trivalent vertices V of Γ, and of the weights $w(E)$ of the edges E of Γ. The definition of $N_{g(V),V}^{1,2}$ given in Section 1.5 depends on a specific choice of orientation on the edges of Γ. In particular, the definition of $N_{g(V),V}^{1,2}$ does not treat the three edges adjacent to V in a symmetric way.

Let $E_V^{\mathrm{in},1}$ and $E_V^{\mathrm{in},2}$ be the two ingoing edges adjacent to V, and let E_V^{out} be the outgoing edge adjacent to V. Let $D_{E_V^{\mathrm{in},1}}$, $D_{E_V^{\mathrm{in},2}}$ and $D_{E_V^{\mathrm{out}}}$ be the corresponding toric divisors of X_{Δ_V}. We have evaluation morphisms

$$\mathrm{ev} = (\mathrm{ev}_1, \mathrm{ev}_2, \mathrm{ev}_{\mathrm{out}}): \overline{M}_{g,\Delta_V} \to D_{E_V^{\mathrm{in},1}} \times D_{E_V^{\mathrm{in},2}} \times D_{E_V^{\mathrm{out}}} .$$

In Section 1.5, we defined

$$N_{g,V}^{1,2} = \int_{[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g \mathrm{ev}_1^*(\mathrm{pt}_1) \mathrm{ev}_2^*(\mathrm{pt}_2) ,$$

where $\mathrm{pt}_1 \in A^1(D_{E_V^{\mathrm{in},1}})$ and $\mathrm{pt}_2 \in A^1(D_{E_V^{\mathrm{in},2}})$ are classes of a point on $D_{E_V^{\mathrm{in},1}}$ and $D_{E_V^{\mathrm{in},2}}$ respectively.

But one could similarly define

$$N_{g,V}^{2,\mathrm{out}} \coloneqq \int_{[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g \mathrm{ev}_1^*(\mathrm{pt}_1) \mathrm{ev}_{\mathrm{out}}^*(\mathrm{pt}_{\mathrm{out}}) ,$$

and

$$N_{g,V}^{\mathrm{out},1} \coloneqq \int_{[\overline{M}_{g,\Delta_V}]^{\mathrm{virt}}} (-1)^g \lambda_g \mathrm{ev}_{\mathrm{out}}^*(\mathrm{pt}_{\mathrm{out}}) \mathrm{ev}_1^*(\mathrm{pt}_1) ,$$

where $\mathrm{pt}_{\mathrm{out}} \in A^*(D_{E_V^{\mathrm{out}}})$ is the class of a point on E_V^{out}. The following Lemma gives a relation between these various invariants.

Lemma 1.22. *We have*

$$N_{g,V}^{1,2} w(E_V^{\mathrm{in},1}) w(E_V^{\mathrm{in},2}) = N_{g,V}^{2,\mathrm{out}} w(E_V^{\mathrm{in},2}) w(E_V^{\mathrm{out}}) = N_{g,V}^{\mathrm{out},1} w(E_V^{\mathrm{out}}) w(E_V^{\mathrm{in},1})$$

and we denote by $N_{g,V}$ this number.

Proof. Let Γ_V be the trivalent tropical curve given by V and its three edges $E_V^{\mathrm{in},1}$, $E_V^{\mathrm{in},2}$ and

E_V^{out}. Let $\Gamma_{V'}$ be the trivalent tropical curve with a unique vertex V' and edges $E_{V'}^{\text{in},1}$, $E_{V'}^{\text{in},2}$ and $E_{V'}^{\text{out}}$, such that

$$w(E_V^{\text{in},1}) = w(E_{V'}^{\text{in},1})\,, w(E_V^{\text{in},2}) = w(E_{V'}^{\text{in},2})\,, w(E_V^{\text{out}}) = w(E_{V'}^{\text{out}})\,,$$

and

$$v_{V,E_V^{\text{in},1}} = -v_{V',E_{V'}^{\text{in},1}}\,, v_{V,E_V^{\text{in},2}} = -v_{V',E_{V'}^{\text{in},2}}\,, v_{V,E_V^{\text{out}}} = -v_{V',E_{V'}^{\text{out}}}\,.$$

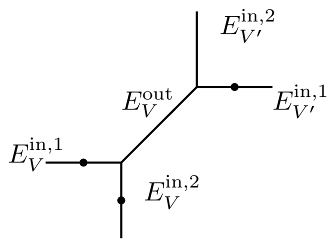
Let $\Gamma_{V,V'}$ be the tropical curve obtained by gluing E_V^{out} and $E_{V'}^{\text{out}}$ together.

Taking

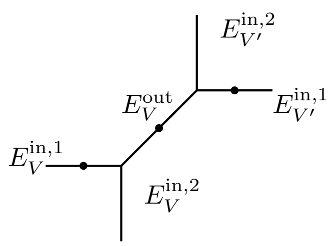
$$\Delta = \{v_{V,E_V^{\text{in},1}}, -v_{V',E_{V'}^{\text{in},1}}, v_{V,E_V^{\text{in},2}}, -v_{V',E_{V'}^{\text{in},2}}\}$$

and $n = 3$, we have $g_{\Delta,n} = 0$ and $T_{\Delta,p}$ consists of a unique tropical curve $\Gamma_{V,V'}^p$, obtained from $\Gamma_{V,V'}$ by adding three bivalent vertices corresponding to the three point p_1, p_2 and p_3 in $\mathbb{R}^2$.

Choosing differently $p = (p_1, p_2, p_3)$, the tropical curve $\Gamma_{V,V'}^p$ can look like



or like



But the log Gromov-Witten invariants $N_g^{\Delta,3}$ are independent of the choice of p and so can be computed for any choice of p. For each of the two above choice of p, the gluing formula of Corollary 1.15 gives an expression for $N_g^{\Delta,3}$. These two expressions have to be equal. Writing

$$F(u) = \sum_{g \geqslant 0} N_g u^{2g+1}$$

we obtain[17]

$$\begin{aligned}&F_V^{1,2}(u)F_{V'}^{1,\mathrm{out}}(u)w(E_V^{\mathrm{in},1})w(E_V^{\mathrm{in},2})w(E_V^{\mathrm{out}})w(E_{V'}^{\mathrm{in},1})\\ =\;&F_V^{1,\mathrm{out}}(u)F_{V'}^{1,\mathrm{out}}(u)w(E_V^{\mathrm{in},1})w(E_V^{\mathrm{out}})w(E_V^{\mathrm{out}})w(E_{V'}^{\mathrm{in},1})\,,\end{aligned}$$

and so after simplification

$$F_V^{1,2}(u)F_{V'}^{1,\mathrm{out}}(u)w(E_V^{\mathrm{in},2}) = F_V^{1,\mathrm{out}}(u)F_{V'}^{1,\mathrm{out}}(u)w(E_V^{\mathrm{out}})\,.$$

By $GL_2(\mathbb{Z})$ invariance, we have $F_V^{1,2}(u) = F_{V'}^{1,2}(u)$ and $F_V^{1,\mathrm{out}}(u) = F_{V'}^{1,\mathrm{out}}(u)$. By the unrefined correspondence theorem, we know that $F_V^{1,\mathrm{out}}(u) \neq 0$, so we obtain

$$F_V^{1,2}(u)w(E_V^{\mathrm{in},2}) = F_V^{1,\mathrm{out}}(u)w(E_V^{\mathrm{out}})\,,$$

which finishes the proof of Lemma 1.22. ☐

We define the contribution $F_V(u) \in \mathbb{Q}[[u]]$ of a trivalent vertex V of Γ as being the power series

$$F_V(u) = \sum_{g\geqslant 0} N_{g,V}u^{2g+1}.$$

Proposition 1.23. *For every Δ and n such that $g_{\Delta,n} \geqslant 0$, and for every $p \in U_{\Delta,n}$, we have*

$$\sum_{g\geqslant g_{\Delta,n}} N_g^{\Delta,n}u^{2g-2+|\Delta|} = \sum_{(h:\Gamma\to\mathbb{R}^2)\in T_{\Delta,p}} \prod_{V\in V^{(3)}(\Gamma)} F_V(u)$$

where the product is over the trivalent vertices of Γ.

Proof. This follows from the decomposition formula, Proposition 1.9, from the gluing formula, Corollary 1.15, and from Lemma 1.22. Indeed, every bounded edge of Γ is an ingoing edge for exactly one trivalent vertex of Γ and every trivalent vertex of Γ has exactly two ingoing edges. Combining the invariant $N_{g(V),V}^{1,2}$ of a trivalent vertex V with the weights of its two ingoing edges, one can rewrite the double product of Corollary 1.15 as a single product in terms of the invariants defined by Lemma 1.22. ☐

Proposition 1.24. *The contribution $F_V(u)$ of a vertex V only depends on the multiplicity $m(V)$ of V.*

In particular, for every m positive integer, one can define the contribution $F_m(u) \in \mathbb{Q}[[u]]$ as the contribution $F_V(u)$ of a vertex V of multiplicity m.

Proof. We follow closely Brett Parker, [Par16] (Section 3).

For $v_1, v_2 \in \mathbb{Z}^2 - \{0\}$, let us denote by $F_{v_1,v_2}(u)$ the contribution $F_V(u)$ of a vertex V of adjacent edges E_1, E_2 and E_3 such that $v_{V,E_1} = v_1$ and $v_{V,E_2} = v_2$. The contribution $F_{v_1,v_2}(u)$ depends on (v_1, v_2) only up to linear action of $GL_2(\mathbb{Z})$ on $\mathbb{Z}^2$. In particular, we can change the sign of v_1 and/or v_2 without changing $F_{v_1,v_2}(u)$.

[17] Recall that we are considering marked points as bivalent vertices and that this affects the notion of bounded edge. According to the gluing formula of Corollary 1.15, we need to include one weight factor for each bounded edge.

By the balancing condition, we have $v_{V,E_3} = -v_{V,E_1} - v_{V,E_2}$ and so

$$F_{v_1,v_2}(u) = F_{-v_1,v_2}(u) = F_{v_1-v_2,v_2}(u)\,.$$

By $GL_2(\mathbb{Z})$ invariance, we can assume $v_1 = (|v_1|, 0)$ and $v_2 = (v_{2x}, *)$ with $v_{2x} \geqslant 0$. If $|v_1|$ divides v_{2x}, $v_{2x} = a|v_1|$, then replacing v_2 by $v_2 - av_1$, which does not change F_{v_1,v_2}, we can assume that $v_1 = (|v_1|, 0)$ and $v_2 = (0, *)$. If not, we do the Euclidean division of v_{2x} by $|v_1|$, $v_{2x} = a|v_1| + b$, $0 \leq b < |v_1|$, and we replace v_2 by $v_2 - av_1$ to obtain $v_2 = (b, *)$. Exchanging the roles of v_1 and v_2, we can assume by $GL_2(\mathbb{Z})$ invariance that $v_1 = (|v_1|, 0)$, for some $|v_1| \leq b$ and $v_2 = (v_{2x}, *)$ for some $v_{2x} \geqslant 0$, and we repeat the above procedure. By the Euclidean algorithm, this process terminates and at the end we have $v_1 = (|v_1|, 0)$ and $v_2 = (0, |v_2|)$. In particular, for every $v_1, v_2 \in \mathbb{Z}^2 - \{0\}$, the contribution F_{v_1,v_2} only depends on $\gcd(|v_1|, |v_2|)$ and on the multiplicity $|\det(v_1, v_2)|$.

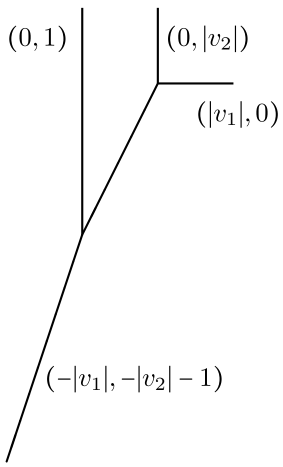
By the previous paragraph, we can assume that $v_1 = (|v_1|, 0)$ and $v_2 = (0, |v_2|)$.

Taking

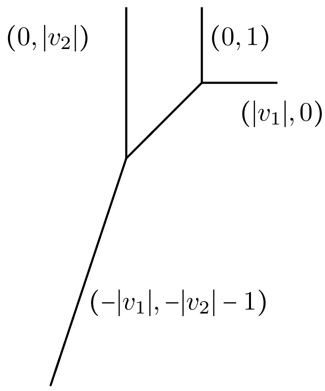
$$\Delta = \{(|v_1|, 0), (0, |v_2|), (0, 1), (-|v_1|, -|v_2| - 1)\}\,,$$

and $n = 3$, we have $g_{\Delta,n} = 0$ and $T_{\Delta,p}$ contains a unique tropical curve Γ^p.

Choosing differently $p = (p_1, p_2, p_3)$, the tropical curve Γ^p can look like



or like



But the log Gromov-Witten invariants $N_g^{\Delta,3}$ are independent of the choice of p and so can be computed for any choice of p. For each of the two above choices of p, the gluing formula of Proposition 1.23 gives an expression for $N_g^{\Delta,3}$. These two expressions have to be equal and we obtain

$$F_{(|v_1|,0),(0,|v_2|)}(u)F_{(0,1),(-|v_1|,-|v_2|-1)}(u) = F_{(|v_1|,0),(0,1)}(u)F_{(0,|v_2|),(-|v_1|,-|v_2|-1)}(u)\,.$$

For both pairs of vectors $(|v_1|,0),(0,1)$ and $(0,1),(-|v_1|,-|v_2|-1)$, the gcd of the divisibilities is equal to one and the absolute value of the determinant is equal to $|v_1|$, so we have

$$F_{(0,1),(-|v_1|,-|v_2|-1)}(u) = F_{(|v_1|,0),(0,1)}(u)\,.$$

As this quantity is non-zero by the unrefined correspondence theorem, we can simplify it from the previous equality to obtain

$$F_{(|v_1|,0),(0,|v_2|)}(u) = F_{(0,|v_2|),(-|v_1|,-|v_2|-1)}(u)\,.$$

As

$$\gcd(|(0,|v_2|)|,|(-|v_1|,-|v_2|-1)|) = 1\,,$$

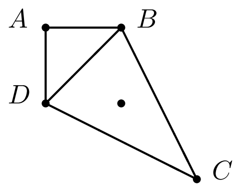
we obtain the desired result. ☐

1.7.2 Reduction to vertices of multiplicity 1 and 2

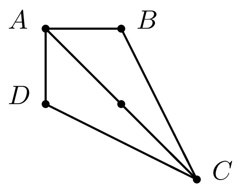
We start reviewing the key step in the argument of Itenberg and Mikhalkin [IM13] proving the tropical deformation invariance of Block-Göttsche invariants. We consider a tropical curve with a 4-valent vertex V. Let Q be the quadrilateral dual to V. We assume that Q has no pair of parallel sides. In that case, there exists a unique parallelogram P having two sides in common with Q and being contained in Q. Let A,B,C and D denote the four vertices of Q, such that A,B and D are vertices of P. Let E be the fourth vertex of P, contained in the interior of Q. There are three combinatorially distinct ways to deform this tropical curve into a simple one, corresponding to the three ways to decompose Q into triangles or parallelograms:

1. We can decompose Q into the triangles ABD and BCD.
2. We can decompose Q into the triangles ABC and ACD.
3. We can decompose Q into the triangles BCE, DEC and the parallelogram P.

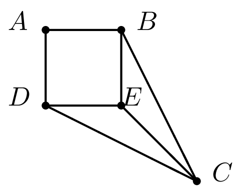
Case (1):



Case (2):



Case (3):



The deformation invariance result then follows from the identity

$$(q^{|ACD|} - q^{-|ACD|})(q^{|ABC|} - q^{-|ABC|})$$

$$= (q^{|BCD|} - q^{-|BCD|})(q^{|ABD|} - q^{-|ABD|}) + (q^{|BCE|} - q^{-|BCE|})(q^{|DEC|} - q^{-|DEC|})$$

where $|-|$ denotes the area. This identity can be proved by elementary geometry considerations.

The following result goes in the opposite direction and shows that the constraints imposed by tropical deformation invariance are quite strong. The generating series of log Gromov-Witten invariants $F_m(u)$ will satisfy these constraints. Indeed, they are defined independently of any tropical limit, so applications of the gluing formula to different degenerations have to give the same result.

Proposition 1.25. *Let $F\colon \mathbb{Z}_{>0} \to R$ be a function of positive integers valued in a commutative ring R, such that, for any quadrilateral Q as above, we have*[18]

$$F(2|BCD|)F(2|ABD|) = F(2|ACD|)F(2|ABC|) + F(2|BCE|)F(2|DEC|).$$

Then for every integer $n \geqslant 2$, we have

$$F(n)^2 = F(2n-1)F(1) + F(n-1)^2$$

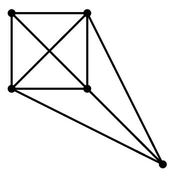
and for every integer $n \geqslant 3$, we have

$$F(n)^2 = F(2n-2)F(2) + F(n-2)^2.$$

In particular, if $F(1)$ and $F(2)$ are invertible in R, then the function F is completely determined by its values $F(1)$ and $F(2)$.

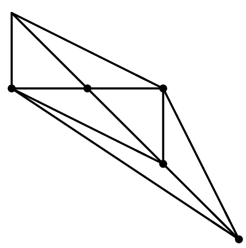
Proof. The first equality is obtained by taking Q to be the quadrilateral of vertices $(-1,0)$, $(-1,1)$, $(0,1)$, $(n-1,-(n-1))$.

Picture of Q for $n = 2$:



The second equality is obtained by taking Q to be the quadrilateral of vertices $(-1,0)$, $(-1,1)$, $(1,0)$, $(n-1,-(n-1))$.

Picture of Q for $n = 3$:



[18] All the relevant areas are half-integers and so their doubles are indeed integers.

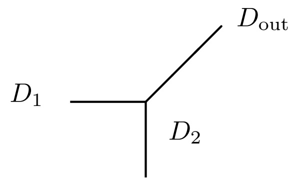
□

1.7.3 Contribution of vertices of multiplicity 1 and 2

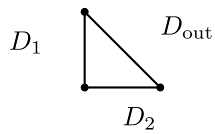
Vertex of multiplicity one

We now evaluate the contribution $F_1(u)$ of a vertex of multiplicity 1 by direct computation.

We consider $\Delta = \{(-1,0),(0,-1),(1,1)\}$. The corresponding toric surface X_Δ is simply $\mathbb{P}^2$, of fan



and of dual polygon



Let D_1, D_2 and D_{out} be the toric boundary divisors of $\mathbb{P}^2$. The class β_Δ is simply the class of a curve of degree one, i.e. of a line, on $\mathbb{P}^2$. Let $\overline{M}_{g,\Delta}$ be the moduli space of genus g stable log maps of type Δ. We have evaluation maps

$$(\mathrm{ev}_1, \mathrm{ev}_2)\colon \overline{M}_{g,\Delta} \to D_1 \times D_2\,,$$

and in Section 1.5, we defined

$$N_{g,\Delta}^{1,2} = \int_{[\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g \mathrm{ev}_1^*(\mathrm{pt}_1)\mathrm{ev}_2^*(\mathrm{pt}_2)\,,$$

where $\mathrm{pt}_1 \in A^*(D_1)$ and $\mathrm{pt}_2 \in A^*(D_2)$ are classes of a point on D_1 and D_2 respectively.

By definition (see Section 1.7.1), we have

$$F_1(u) = \sum_{g \geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}\,.$$

Proposition 1.26. *The contribution of a vertex of multiplicity one is given by*

$$F_1(u) = 2\sin\left(\frac{u}{2}\right) = -i(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$$

where $q = e^{iu}$.

Proof. Let P_1 and P_2 be points on D_1 and D_2 respectively, away from the torus fixed points. Let S be the surface obtained by blowing-up $\mathbb{P}^2$ at P_1 and P_2. Denote by D the strict transform of the class of a line in $\mathbb{P}^2$ and by E_1, E_2 the exceptional divisors. Denote ∂S the strict transform of the toric boundary $\partial\mathbb{P}^2$ of $\mathbb{P}^2$. We endow S with the divisorial log structure with respect to ∂S. Let $\overline{M}_g(S)$ be the moduli space of genus g stable log maps to S of class $D - E_1 - E_2$ with tangency condition to intersect ∂S in one point with multiplicity one. It has virtual dimension g and we define

$$N_g^S \coloneqq \int_{[\overline{M}_g(S)]^{\mathrm{virt}}} (-1)^g \lambda_g \,.$$

The strict transform C of the line L in $\mathbb{P}^2$ passing through P_1 and P_2 is the unique genus zero curve satisfying these conditions and has normal bundle $N_{C|S} = \mathcal{O}_{\mathbb{P}^1}(-1)$ in S. All the higher genus maps factor through C, and as C is away from the preimage of the torus fixed points of $\mathbb{P}^2$, log invariants coincide with relative invariants [AMW14]. More precisely, we can consider the moduli space $\overline{M}_g(\mathbb{P}^1/\infty,1,1)$ genus g stable maps to $\mathbb{P}^1$, of degree one, and relative to a point $\infty \in \mathbb{P}^1$. If $\pi\colon \mathcal{C} \to \overline{M}_g(\mathbb{P}^1/\infty,1,1)$ is the universal curve and $f\colon \mathcal{C} \to \mathbb{P}^1 \simeq C$ is the universal map, the difference in obstruction theories between stable maps to S and stable maps to $\mathbb{P}^1$ comes from $R^1\pi_* f^* N_{C|S} = R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)$.

These integrals have been computed by Bryan and Pandharipande[BP05], (see the proof of the Theorem 5.1), and the result is

$$\sum_{g \geqslant 0} N_g^S u^{2g-1} = \frac{1}{2\sin\left(\frac{u}{2}\right)} \,.$$

So we obtain

$$N_g^S = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,1,1)]^{\mathrm{virt}}} (-1)^g \lambda_g \, e\left(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)\right) ,$$

where $e(-)$ is the Euler class. Rewriting

$$(-1)^g \lambda_g = e(R^1\pi_*\mathcal{O}_{\mathcal{C}}) = e(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}) \,,$$

we get

$$N_g^S = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,1,1)]^{\mathrm{virt}}} e\left(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right) .$$

As in [GPS10], we will work with the non-compact varieties $(\mathbb{P}^2)^\circ$, D_1°, D_2°, S° obtained by removing the torus fixed points of $\mathbb{P}^2$ and their preimages in S.

Denote $\mathbb{P}_1^\circ$ the projectivized normal bundle to D_1° in $(\mathbb{P}^2)^\circ$, coming with two natural sections $(D_1^\circ)_0$ and $(D_1^\circ)_\infty$. Denote $\tilde{\mathbb{P}}_1^\circ$ the blow-up of $\mathbb{P}_1^\circ$ at the point $P_1 \in (D_1^\circ)_\infty$, $\tilde{E}_1$ the corresponding exceptional divisor and C_1 the strict transform of the fiber of $\mathbb{P}_1^\circ$ passing through P_1. In particular, $\tilde{E}_1$ and C_1 are both projective lines with degree -1 normal bundle in $(\tilde{\mathbb{P}}_1)^\circ$. Furthermore, $\tilde{E}_1$ and C_1 intersect in one point. Similarly, denote $\mathbb{P}_2^\circ$ the projectivized normal bundle to D_2° in $(\mathbb{P}^2)^\circ$, coming with two natural sections $(D_2^\circ)_0$ and $(D_2^\circ)_\infty$. Denote $\tilde{\mathbb{P}}_2^\circ$ the blow-up of $\mathbb{P}_2^\circ$ at the point $P_2 \in (D_2^\circ)_\infty$, $\tilde{E}_2$ the corresponding exceptional divisor and C_2 the strict transform of the fiber of $\mathbb{P}_2^\circ$ passing through P_2. In particular, $\tilde{E}_2$ and C_2 are both projective lines with degree -1 normal bundle in $(\tilde{\mathbb{P}}_2)^\circ$. Furthermore, $\tilde{E}_2$ and C_2 intersect in one point.

We degenerate S° as in Section 5.3 of [GPS10]. We first degenerate $(\mathbb{P}^2)^\circ$ to the normal cone of $D_1^\circ \cup D_2^\circ$, i.e. we blow-up $(D_1^\circ \cup D_2^\circ) \times \{0\}$ in $(\mathbb{P}^2)^\circ \times \mathbb{C}$. The fiber over $0 \in \mathbb{C}$ has three irreducible components: $(\mathbb{P}^2)^\circ$, $\mathbb{P}_1^\circ$, $\mathbb{P}_2^\circ$, with $\mathbb{P}_1^\circ$ and $\mathbb{P}_2^\circ$ glued along $(D_1^\circ)_0$ and $(D_2^\circ)_0$ to D_1° and D_2° in $(\mathbb{P}^2)^\circ$. We then blow-up the strict transforms of the sections $P_1 \times \mathbb{C}$ and $P_2 \times \mathbb{C}$. The fiber of the resulting family away from $0 \in \mathbb{C}$ is isomorphic to S°. The fiber over zero has three irreducible components: $(\mathbb{P}^2)^\circ$, $\tilde{\mathbb{P}}_1^\circ$, $\tilde{\mathbb{P}}_2^\circ$.

We would like to apply a degeneration formula to this family in order to compute N_g^S. As discussed above, all the maps in $\overline{M}_g(S)$ factor through C and so N_g^S can be seen as a relative Gromov-Witten invariant of the non-compact surface S°, relatively to the strict transforms of D_1° and D_2°.

The key point is that for homological degree reasons, the degenerating relative stable maps do not leave the non-compact geometries we are considering. More precisely, any limiting relative stable map has to factor through $C_1 \cup L \cup C_2$, with degree one over each of the components C_1, L and C_2. So, even if the target geometry is non-compact, all the relevant moduli spaces of relative stable maps are compact. It follows that we can apply the ordinary degeneration formula in relative Gromov-Witten theory [Li02].

We obtain

$$\sum_{g\geqslant 0} N_g^S u^{2g-1} = \left(\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1} \right) \left(\sum_{g\geqslant 0} N_g^{C_1} u^{2g-1} \right) \left(\sum_{g\geqslant 0} N_g^{C_2} u^{2g-1} \right).$$

The invariants $N_g^{C_1}$ and $N_g^{C_2}$, coming from curves factoring through C_1 and C_2, which are (-1)-curves in $\tilde{\mathbb{P}}_1^\circ$ and $\tilde{\mathbb{P}}_2^\circ$ respectively, can be written as relative invariants of $\mathbb{P}^1$:

$$N_g^{C_1} = N_g^{C_2} = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,1,1)]^{\text{virt}}} e\left(R^1 \pi_* f^* (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right),$$

which is exactly the formula giving N_g^S, and so

$$\sum_{g\geqslant 0} N_g^{C_1} u^{2g-1} = \sum_{g\geqslant 0} N_g^{C_2} u^{2g-1} = \frac{1}{2\sin\left(\frac{u}{2}\right)}.$$

Remark that this equality is a higher genus version of Proposition 5.2 of [GPS10]. Combining the previous equalities, we obtain

$$\frac{1}{2\sin\left(\frac{u}{2}\right)} = \left(\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1} \right) \left(\frac{1}{2\sin\left(\frac{u}{2}\right)} \right)^2,$$

and so

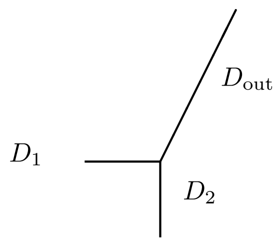
$$\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1} = 2\sin\left(\frac{u}{2}\right).$$

□

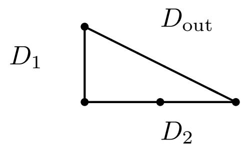
Vertex of multiplicity 2

We now evaluate the contribution $F_2(u)$ of a vertex of multiplicity 2 by direct computation.

We consider $\Delta = \{(-1,0),(0,-2),(1,2)\}$. The corresponding toric surface X_Δ is simply the weighted projective plane $\mathbb{P}^{1,1,2}$, of fan



and of dual polygon



Let D_1, D_2 and D_{out} be the toric boundary divisors of $\mathbb{P}^{1,1,2}$. We have the following numerical properties:

$$2D_1 = D_2 = 2D_{\text{out}}\,,$$

$$D_1.D_2 = 1,\, D_1.D_{\text{out}} = \frac{1}{2},\, D_2.D_{\text{out}} = 1\,,$$

$$D_1^2 = \frac{1}{2}, D_2^2 = 2, D_{\text{out}}^2 = \frac{1}{2}\,.$$

The class β_Δ satisfies $\beta_\Delta.D_1 = 1$, $\beta_\Delta.D_2 = 2$, $\beta_\Delta.D_{\text{out}} = 1$ and so

$$\beta_\Delta = 2D_1 = D_2 = 2D_{\text{out}}\,.$$

Let $\overline{M}_{g,\Delta}$ be the moduli space of genus g stable log maps of type Δ. We have evaluation maps

$$(\text{ev}_1,\text{ev}_2)\colon \overline{M}_{g,\Delta} \to D_1 \times D_2\,,$$

and in Section 1.5, we defined

$$N_{g,\Delta}^{1,2} = \int_{[\overline{M}_{g,\Delta_V}]^{\text{virt}}} (-1)^g \lambda_g \text{ev}_1^*(\text{pt}_1)\text{ev}_2^*(\text{pt}_2)\,,$$

where $\text{pt}_1 \in A^*(D_1)$ and $\text{pt}_2 \in A^*(D_2)$ are classes of a point on D_1 and D_2 respectively.

By definition (see Section 1.7.1), we have

$$F_1(u) = 2\left(\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}\right)\,.$$

Proposition 1.27. *The contribution of a vertex of multiplicity two is given by*

$$F_2(u) = 2\sin(u) = (-i)(q - q^{-1})$$

where $q = e^{iu}$.

Proof. We have to prove that

$$\sum_{g\geqslant 0} N^{1,2}_{g,\Delta} u^{2g+1} = \sin(u)\,.$$

Let P_2 be a point on D_2 away from the torus fixed points. Let S be the surface obtained by blowing-up $\mathbb{P}^{1,1,2}$ at P_2. Still denote β_Δ the strict transform of the class β_Δ and by E_2 the exceptional divisor. Denote ∂S the strict transform of the toric boundary $\partial\mathbb{P}^{1,1,2}$ of $\mathbb{P}^{1,1,2}$. We endow S with the divisorial log structure with respect to ∂S. Let $\overline{M}_g(S)$ be the moduli space of genus g stable log maps to S of class $\beta_\Delta - 2E_2$ with tangency condition to intersect D_1 in one point with multiplicity one and D_{out} in one point with multiplicity one. It has virtual dimension g and we have an evaluation map

$$\mathrm{ev}_1 \colon \overline{M}_{g,S} \to D_1$$

We define

$$N_g^S \coloneqq \int_{[\overline{M}_g(S)]^{\text{virt}}} (-1)^g \lambda_g \mathrm{ev}_1^*(\mathrm{pt}_1)\,,$$

where $\mathrm{pt}_1 \in A^1(D_1)$ is the class of a point on D_1.

In fact, because a curve in the linear system $\beta_\Delta - 2E_2$ is of arithmetic genus g_a given by

$$\begin{aligned}
2g_a - 2 &= (\beta_\Delta - 2E_2)\cdot(\beta_\Delta - 2E_2 + K_S) \\
&= (2D_1 - 2E_2)\cdot(2D_1 - 4D_1 - E_2) \\
&= -4D_1^2 + 2E_2^2 \\
&= -4\,,
\end{aligned}$$

i.e. $g_a = -1 < 0$, all the moduli spaces $\overline{M}_g(S)$ are empty and so

$$\sum_{g\geqslant 0} N_g^S u^{2g-1} = 0\,.$$

We write $\tilde{\Delta} = \{(-1,0),(0,-1),(0,-1),(1,2)\}$ and $\overline{M}_{g,\tilde{\Delta}}$ the moduli space of genus g stable log maps of type $\tilde{\Delta}$. We have evaluation maps

$$(\mathrm{ev}_1, \mathrm{ev}_2, \mathrm{ev}_{2'}) \colon \overline{M}_{g,\tilde{\Delta}} \to D_1 \times D_2 \times D_2\,,$$

and we define

$$N^{1,2,2'}_{g,\tilde{\Delta}} \coloneqq \int_{[\overline{M}_{g,\tilde{\Delta}}]^{\text{virt}}} (-1)^g \lambda_g \mathrm{ev}_1^*(\mathrm{pt}_1)\mathrm{ev}_2^*(\mathrm{pt}_2)\mathrm{ev}_{2'}^*(\mathrm{pt}_2)\,,$$

where $\mathrm{pt}_1 \in A^*(D_1)$ and $\mathrm{pt}_2 \in A^*(D_2)$ are classes of a point on D_1 and D_2 respectively.

As in [GPS10], we will work with the non-compact varieties $(\mathbb{P}^{1,1,2})^\circ$, D_1°, D_2°, S° obtained by removing the torus fixed points of $\mathbb{P}^{1,1,2}$ and their preimages in S. Denote $\mathbb{P}_2^\circ$ the projectivized normal bundle to D_2° in $(\mathbb{P}^2)^\circ$, coming with two natural sections $(D_2^\circ)_0$ and $(D_2^\circ)_\infty$. Denote $\tilde{\mathbb{P}}_2^\circ$ the blow-up of $\mathbb{P}_2^\circ$ at the point $P_2 \in (D_2^\circ)_\infty$, $\tilde{E}_2$ the corresponding exceptional divisor and C_2 the strict transform of the fiber of $\mathbb{P}_2^\circ$ passing through P_2. In particular, $\tilde{E}_2$ and C_2 are both projective lines with degree -1 normal bundle in $(\tilde{\mathbb{P}}_2)^\circ$. Furthermore, $\tilde{E}_2$ and C_2 intersect in one point.

We degenerate S° as in Section 5.3 of [GPS10]. We first degenerate $(\mathbb{P}^{1,1,2})^\circ$ to the normal cone of D_2°, i.e. we blow-up $D_2^\circ \times \{0\}$ in $(\mathbb{P}^{1,1,2})^\circ \times \mathbb{C}$. The fiber over $0 \in \mathbb{C}$ has two components: $(\mathbb{P}^{1,1,2})^\circ$ and $\mathbb{P}_2^\circ$, with $\mathbb{P}_2^\circ$ glued along $(D_2^\circ)_0$ to D_2° in $(\mathbb{P}^{1,1,2})^\circ$. We then blow-up the strict transform of the section $P_2 \times \mathbb{C}$. The fiber of the resulting family away from $0 \in \mathbb{C}$ is isomorphic to S°. The fiber over zero has two components: $(\mathbb{P}^{1,1,2})^\circ$ and $\tilde{\mathbb{P}}_2^\circ$.

We would like to apply a degeneration formula to this family in order to compute N_g^S. The key point is that for homological degree reasons, the relevant degenerating relative stable maps do not leave the non-compact geometries we are considering. More precisely, after fixing a point $P_1 \in D_1^\circ$, realizing the insertion $\mathrm{ev}_1^*(\mathrm{pt}_1)$, any limiting relative stable map has to factor through $L \cup C_2$, with degree one over L and degree two over C_2, where L is the unique curve in $\mathbb{P}^{1,1,2}$, of class β_Δ, passing through P_1 and through P_2 with tangency order two along D_2°. So, even if the target geometry is non-compact, all the relevant moduli spaces of relative stable maps are compact. It follows that we can apply the ordinary degeneration formula in relative Gromov-Witten theory [Li02].

The application of the degeneration formula gives two terms, corresponding to the two partitions $2 = 1 + 1$ and $2 = 2$ of the intersection number

$$(\beta_\Delta - 2E_2).E_2 = 2\,.$$

For the first term, the invariants on the side of $\mathbb{P}^{1,1,2}$ are $N_{g,\tilde{\Delta}}^{1,2,2'}$, whereas on the side of $\tilde{\mathbb{P}}_2$, we have disconnected invariants, corresponding to two degree one maps to C_2. As in the proof of Proposition 1.26, the relevant connected degree one invariants of C_2 are given by

$$N_g^{C_2} = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,1,1)]^{\mathrm{virt}}} e\left(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right)\,,$$

satisfying

$$\sum_{g\geqslant 0} N_g^{C_2} u^{2g-1} = \frac{1}{2\sin\left(\frac{u}{2}\right)}\,.$$

For the second term, the invariants on the side of $\mathbb{P}^{1,1,2}$ are $N_{g,\Delta}^{1,2}$, whereas on the side of $\tilde{\mathbb{P}}_2$, we have connected invariants, corresponding to one degree two map to C_2. More precisely, the relevant connected degree two invariants of C_2 are given by

$$N_g^{2C_2} = \int_{[\overline{M}_g(\mathbb{P}^1/\infty,2,2)]^{\mathrm{virt}}} e\left(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right)\,,$$

where $\overline{M}_g(\mathbb{P}^1/\infty, 2, 2)$ is the moduli space of genus g stable maps to $\mathbb{P}^1$, of degree two, and relative to a point $\infty \in \mathbb{P}^1$ with maximal tangency order two. According to [BP05] (see the proof of Theorem 5.1), we have

$$\sum_{g\geqslant 0} N_g^{2C_2} u^{2g-1} = -\frac{1}{2}\frac{1}{2\sin(u)} .$$

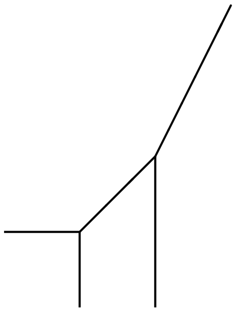
It follows that the degeneration formula takes the form

$$\sum_{g\geqslant 0} N_g^S u^{2g-1}$$

$$= \frac{1}{2}\left(\sum_{g\geqslant 0} N_{g,\tilde{\Delta}}^{1,2,2'} u^{2g+2}\right)\left(\frac{1}{2\sin\left(\frac{u}{2}\right)}\right)^2 + 2\left(\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}\right)\frac{(-1)}{2}\frac{1}{2\sin(u)} .$$

The factor $\frac{1}{2}$ in front of the fist term is a symmetry factor and the factor 2 in front of the second term is a multiplicity.

There exists a unique tropical curve of type $\tilde{\Delta}$, which looks like



This tropical curve has two vertices of multiplicity one, so using the gluing formula of Proposition 1.23 and Proposition 1.26, we find

$$\sum_{g\geqslant 0} N_{g,\tilde{\Delta}}^{1,2,2'} u^{2g+2} = (F_1(u))^2 = \left(2\sin\left(\frac{u}{2}\right)\right)^2 .$$

Combining the previous results, we obtain

$$0 = \frac{1}{2} - \frac{1}{2\sin(u)}\left(\sum_{g\geqslant 0} N_{g,\Delta}^{1,2} u^{2g+1}\right) ,$$

and so the desired formula. □

Remark: The proofs of Propositions 1.26 and 1.27 rely on the fact that the involved curves have low degree. More precisely, in each case, the key point is that the dual polygon does not contain any interior integral point, i.e. a generic curve in the corresponding linear system on the surface has genus zero. This implies that, after imposing tangency constraints, all the higher genus stable maps factor through some rigid genus zero curve in the surface. This guarantees the compactness result needed to work as we did with relative Gromov-Witten theory of non-compact geometries. The higher genus generalization of the most general case of the degeneration argument of Section 5.3 of [GPS10] cannot be dealt with in the same way. This generalization is one of the main topics of Chapter 2.

1.7.4 Contribution of a general vertex

Proposition 1.28. *The contribution of a vertex of multiplicity m is given by*

$$F_m(u) = (-i)(q^{\frac{m}{2}} - q^{-\frac{m}{2}}).$$

Proof. By Proposition 1.26, the result is true for $m = 1$ and by Proposition 1.27, the result is true for $m = 2$. By consistency of the gluing formula of Proposition 1.23, the function $F(m) \coloneqq F_m$ valued in the ring $R \coloneqq \mathbb{Q}[[u]]$ satisfies the hypotheses of Proposition 1.25. The result follows by induction on m using Proposition 1.25. □

The proof of Theorem 1 (Theorem 1.4 in Section 1.1.5) follows from the combination of Proposition 1.23, Proposition 1.24 and Proposition 1.28.

To prove Theorem 1.5, generalizing Theorem 1 by allowing to fix the positions of some of the intersection points with the toric boundary, we only have to organize the gluing procedure slightly differently. The connected components of the complement of the bivalent vertices of Γ, as at the beginning of Section 1.5, are trees with one unfixed unbounded edge and possibly several fixed unbounded edges. We fix an orientation of the edges such that edges adjacent to bivalent pointed vertices go out of the bivalent pointed vertices, such that the fixed unbounded edges are ingoing and such that the unfixed unbounded edge is outgoing. With respect to this orientation, every trivalent vertex has two ingoing and one outgoing edges, and so, without any modification, we obtain the analogue of the gluing formula of Corollary 1.15:

$$N_{g,h}^{\Delta,n} = \left(\prod_{V \in V^{(3)}(\Gamma)} N_{g(V),V}^{1,2} \right) \left(\prod_{E \in E_f(\Gamma)} w(E) \right).$$

In Lemma 1.22, we defined $N_{g,V} \coloneqq N_{g(V),V}^{1,2} w(E_V^{\mathrm{in},1}) w(E_V^{\mathrm{in},2})$, where $E_V^{\mathrm{in},1}$ and $E_V^{\mathrm{in},1}$ are the ingoing edges adjacent to V. Every bounded edge is an ingoing edge to some vertex but some ingoing edges are fixed unbounded edges and so

$$N_{g,h}^{\Delta,n} = \left(\prod_{E_\infty^F \in E_\infty^F(\Gamma)} \frac{1}{w(E_\infty^F)} \right) \left(\prod_{V \in V^{(3)}(\Gamma)} N_{g(V),V} \right),$$

where the first product is over the fixed unbounded edges of Γ. Theorem 1.5 then follows from Proposition 1.28.

1.8 Comparison with known results for K3 and abelian surfaces

In this Section, we prove two results, Theorem 1.29 and Theorem 1.32, which are analogues for K3 and abelian surfaces of Theorem 1 for toric surfaces. We treat both cases in completely parallel ways.

1.8.1 K3 surfaces

Some of the remarks below were already made by Göttsche and Shende in [GS14] (see Theorem 71 and Conjecture 72) and merely interpreted as coincidences. The goal of this section is to formulate these remarks in a way that makes clear their compatibility with our work. More precisely, Theorem 1.29 is an analogue for K3 surfaces of Theorem 1 for toric surfaces.

Let S be a smooth projective K3 surface over $\mathbb{C}$ and let $\beta \in H_2(S,\mathbb{Z})$ be a non-zero effective curve class. The moduli space $\overline{M}_g(S,\beta)$ of genus g stable maps to S of class β admits a reduced virtual class $[\overline{M}_g(S,\beta)]^{\mathrm{red}}$ of degree g (see [MPT10] and references there).

Let us consider the problem of counting genus g_0 curves of class β passing through g_0 given points. A Gromov-Witten definition of this counting problem is given by

$$\langle \tau_0(\mathrm{pt})^{g_0}\rangle_{g_0,\beta} \coloneqq \int_{[\overline{M}_{g_0,g_0}(S,\beta)]^{\mathrm{red}}} \prod_{j=1}^{g_0} \mathrm{ev}_j^*(\mathrm{pt})\,,$$

where $\mathrm{pt} \in A^2(S)$ is the class of a point. We assume for now that β is primitive.

We consider the same problem for curves of genus g, i.e. curves of genus g of class β passing through g_0 points, and we cut down the virtual dimension from $g-g_0$ to zero by introducing a $(-1)^{g-g_0}\lambda_{g-g_0}$. In other words, we consider

$$\langle (-1)^{g-g_0}\lambda_{g-g_0}\tau_0(\mathrm{pt})^{g_0}\rangle_{g,\beta} \coloneqq \int_{[\overline{M}_{g,g_0}(S,\beta)]^{\mathrm{red}}} (-1)^{g-g_0}\lambda_{g-g_0} \prod_{j=1}^{g_0} \mathrm{ev}_j^*(\mathrm{pt})\,.$$

Because we are assuming β is primitive, $\langle \tau_0(\mathrm{pt})^{g_0}\rangle_{g_0,\beta}$ coincides with the Severi degree considered by Göttsche and Shende in [GS14] and so has a natural refinement, defined by replacing Euler characteristics by Hirzebruch genera in a description in terms of Hilbert schemes,

$$\langle \tau_0(\mathrm{pt})^{g_0}\rangle_{g_0,\beta}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]\,.$$

Comparing explicit formulas for $\langle (-1)^{g-g_0}\lambda_{g-g_0}\tau_0(\mathrm{pt})^{g_0}\rangle_{g,\beta}$ obtained in [MPT10] with explicit formulas for $\langle \tau_0(\mathrm{pt})^{g_0}\rangle_{g_0,\beta}(q)$ obtained in [GS14], we obtain:

Theorem 1.29. *If β is primitive, then*

$$\sum_{g\geq g_0} \langle (-1)^{g-g_0}\lambda_{g-g_0}\tau_0(\mathrm{pt})^{g_0}\rangle_{g,\beta} u^{2g-2} = (-1)^{g_0+1}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2g_0-2}\langle \tau_0(\mathrm{pt})^{g_0}\rangle_{g_0,\beta}(q)\,,$$

where $q=e^{iu}$.

Proof. We introduce the notations[19]

$$\Delta(q,z) \coloneqq z\prod_{n\geq 1}(1-z^n)^{20}(1-qz^n)^2(1-q^{-1}z^n)^2$$

[19] Beware the change of notations: we use q for what is y in [GS14], and z for what is q in [GS14] and [MPT10].

and

$$DG_2(q,z) \coloneqq \sum_{m\geq 1} m z^m \sum_{d|m} \frac{1}{d} \left(\frac{q^{\frac{d}{2}} - q^{-\frac{d}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right)^2 .$$

Both sides of the equality in Theorem 1.29 only depends on $\beta^2 = 2h-2$ and we write $\langle \dots \rangle_{g,h}$ for $\langle \dots \rangle_{g,\beta}$.

According to [MPT10] (Theorem 3), we have, after some easy rewriting:

$$\sum_{h\geq 0} \sum_{g\geq g_0} \langle (-1)^{g-g_0} \lambda_{g-g_0} \tau_0(\mathrm{pt})^{g_0} \rangle_{g,h} u^{2g-2} z^{h-1}$$

$$= (-1)^{g_0+1} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g_0-2} \frac{(DG_2(q,z))^{g_0}}{\Delta(q,z)} ,$$

where $q = e^{iu}$. According to [GS14] (Conjecture 68, proven in [GS15]), we have

$$\sum_{h\geq 0} \langle \tau_0(\mathrm{pt})^{g_0} \rangle_{g_0,h}(q) z^{h-1} = \frac{(DG_2(q,z))^{g_0}}{\Delta(q,z)} .$$

Comparing the two previous formulas, we obtain the desired identity. □

Theorem 1.29 is a perfect analogue of Theorem 1. In particular, the prefactors in $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ are remarkably similar.

If β is not primitive, one should extract multicover contributions to formulate the analogue of Theorem 1.29. In [OP16] (Conjecture C2), a general conjecture is formulated for the multicovering structure of Gromov-Witten invariants of K3 surfaces with descendant insertions. For the invariants we are considering, this conjecture takes the following form:

Conjecture 1.30. *We have*

$$\langle (-1)^{g-g_0} \lambda_{g-g_0} \tau_0(\mathrm{pt})^{g_0} \rangle_{g,\beta} = \sum_{\beta = k\beta'} k^{2(g+g_0)-3} \langle (-1)^{g-g_0} \lambda_{g-g_0} \tau_0(\mathrm{pt})^{g_0} \rangle_{g_0,\tilde{\beta}'} ,$$

where $\tilde{\beta}'$ is a primitive class such that $(\tilde{\beta}')^2 = (\beta')^2$.

Combining Theorem 1.29 with this conjecture, we obtain:

Conjecture 1.31. *For general β, we have*

$$\sum_{g\geq g_0} \langle (-1)^{g-g_0} \lambda_{g-g_0} \tau_0(\mathrm{pt})^{g_0} \rangle_{g,\beta} u^{2g-2}$$

$$= (-1)^{g_0+1} \sum_{\beta = k\beta'} k^{2g_0-1} (q^{\frac{k}{2}} - q^{-\frac{k}{2}})^{2g_0-2} \langle \tau_0(\mathrm{pt})^{g_0} \rangle_{g_0,\tilde{\beta}'}(q^k) ,$$

where $\tilde{\beta}'$ is a primitive class such that $(\tilde{\beta}')^2 = (\beta')^2$, and $q = e^{iu}$.

1.8.2 Abelian surfaces

Let A be a smooth projective abelian surface over $\mathbb{C}$ and let $\beta \in H_2(A,\mathbb{Z})$ be a non-zero effective curve class. The moduli space $\overline{M}_g(A,\beta)$ of genus g stable maps to A of class β admits a reduced virtual class $[\overline{M}_g(A,\beta)]^{\mathrm{red}}$ of degree $g-2$ (see [BOPY15] and references there).

Let us consider the problem of counting genus $g_0 \geq 2$ curves of class β passing through g_0-2 points. A Gromov-Witten definition of this counting problem is given by

$$\langle \tau_0(\mathrm{pt})^{g_0-2}\rangle_{g_0,\beta} \coloneqq \int_{[\overline{M}_{g_0,g_0-2}(A,\beta)]^{\mathrm{red}}} \prod_{j=1}^{g_0-2} \mathrm{ev}_j^*(\mathrm{pt})\,,$$

where $\mathrm{pt} \in A^2(A)$ is the class of a point. Let us assume that β is primitive.

We consider the same problem for curves of genus g, i.e. curves of genus g of class β passing through g_0-2 points, and we cut down the virtual dimension from $g-g_0$ to zero by introducing a $(-1)^{g-g_0}\lambda_{g-g_0}$. In other words, we consider

$$\langle (-1)^{g-g_0}\lambda_{g-g_0}\tau_0(\mathrm{pt})^{g_0-2}\rangle_{g,\beta} \coloneqq \int_{[\overline{M}_{g,g_0}(S,\beta)]^{\mathrm{red}}} (-1)^{g-g_0}\lambda_{g-g_0} \prod_{j=1}^{g_0-2} \mathrm{ev}_j^*(\mathrm{pt})\,.$$

Because we are assuming β is primitive, $\langle \tau_0(p)^{g_0-2}\rangle_{g_0,\beta}$ coincides with the Severi degree considered by Göttsche and Shende in [GS14] and so has a natural refinement, defined by replacing Euler characteristics by Hirzebruch genera in a description in terms of Hilbert schemes,

$$\langle \tau_0(\mathrm{pt})^{g_0-2}\rangle_{g_0,\beta}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]\,.$$

Theorem 1.32. *If β is primitive, then*

$$\sum_{g\geq g_0} \langle (-1)^{g-g_0}\lambda_{g-g_0}\tau_0(\mathrm{pt})^{g_0-2}\rangle_{g,\beta} u^{2g-2}$$

$$= (-1)^{g_0+1}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{2g_0-2}\langle \tau_0(\mathrm{pt})^{g_0-2}\rangle_{g_0,\beta}(q)\,,$$

where $q=e^{iu}$.

Proof. We introduce the notation[20]

$$DG_2(q,z) \coloneqq \sum_{m\geq 1} m z^m \sum_{d|m} \frac{1}{d}\left(\frac{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\right)^2 .$$

According to [BOPY15] (Theorem 2), we have, after some easy rewriting (one has to remark that the function S of [BOPY15] is equal to $-(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2 DG_2$):

$$\sum_{h\geq 0}\sum_{g\geq g_0} \langle (-1)^{g-g_0}\lambda_{g-g_0}\tau_0(\mathrm{pt})^{g_0-2}\rangle_{g,h} u^{2g-2} z^h$$

[20] Beware the change of notations: we use q for what is y in [GS14] and p in [BOPY15], and we use z for what is q in [GS14] and [BOPY15].

$$= (-1)^{g_0+1}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g_0-2}(DG_2(q,z))^{g_0-2}\left(z\frac{d}{dz}\right)DG_2(q,z)\,,$$

where $q = e^{iu}$. According to [GS14] (statement before Proposition 74, proven in [GS15]), we have

$$\sum_{h\geq 0}\langle \tau_0(\mathrm{pt})^{g_0}\rangle_{g_0,h}(q)z^h = (DG_2(q,z))^{g_0-2}\left(z\frac{d}{dz}\right)DG_2(q,z)\,.$$

Comparing the two previous formulas, we obtain the desired identity. ☐

Theorem 1.32 is a perfect analogue of Theorem 1. In particular, the prefactors in $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ are remarkably similar.

1.9 Descendants and refined broccoli invariants

In [MR16], Mandel and Ruddat have extended the unrefined correspondence theorem to include descendant log Gromov-Witten invariants, i.e. log Gromov-Witten invariants with insertion of psi classes, in the case of genus zero curves[21]. On the tropical side, one needs to introduce extra markings corresponding to the various insertions of psi classes.

The simplest local model is a parametrized tropical curve $h\colon\Gamma\to\mathbb{R}^2$, of some type Δ, where Γ has a unique vertex, three unbounded edges, and l markings corresponding to insertions of psi classes $\psi_1^{k_1}$, …, $\psi_l^{k_l}$. In addition to the usual multiplicity, this tropical curve has to be counted with an extra factor

$$\binom{l}{k_1,\dots,k_l} = \frac{l!}{k_1!\dots k_l!},$$

corresponding to the fact that

$$\int_{\overline{M}_{0,l+3}}\prod_{i=1}^{l}\psi_i^{k_i} = \binom{l}{k_1,\dots,k_l},$$

where $\overline{M}_{0,l+3}$ is the moduli space of $(l+3)$-pointed genus zero stable curves.

To include descendants in Theorem 1, one should study generating series of descendant log Gromov-Witten invariants with a further insertion of one lambda class.

In this Appendix, we study the simplest possible case of a trivalent vertex with insertion of only one psi class and we recover the numerator $q^{\frac{m}{2}} + q^{-\frac{m}{2}}$ of the multiplicity introduced by Göttsche and Schroeter [GS16a] in the context of refined broccoli invariants.

Let $h\colon\Gamma\to\mathbb{R}^2$ be a parametrized tropical curve, of some type Δ, where Γ has a unique vertex, three unbounded edges, and one extra marking corresponding to the insertion of one psi class ψ_1. For the corresponding log Gromov-Witten invariants with one psi class and one lambda class inserted, one can argue as in Sections 1.5, 1.6 and 1.7 to prove a gluing

[21] In positive genus with insertion of psi classes, superabundant tropical curves generically arise and so the result of [MR16] cannot be applied.

formula and use its consistency to reduce the problem to a generating series

$$F_m^{\psi}(u) = \sum_{g\geq 0} N_{g,V}^{\psi} u^{2g+1}$$

depending only on the multiplicity of the vertex.

Recall that we denoted $F_m(u)$ the analogue generating series without psi class and that by Proposition 1.28, we have $F_m(u) = (-i)(q^{\frac{m}{2}} - q^{-\frac{m}{2}})$, i.e. essentially the numerator of the Block-Göttsche multiplicity.

In the following proposition, we show that $F_m^{\psi}(u)$ is essentially the numerator of the Göttsche-Schroeter multiplicity.

Proposition 1.33. *For every nonnegative integer m, we have*

$$F_m^{\psi}(u) = u\cos\left(\frac{mu}{2}\right)$$

where $q = e^{iu}$.

Proof. In Section 1.7.2, we use steps in the proof by Itenberg and Mikhalkin of the tropical deformation invariance of the Block-Göttsche invariants to obtain identities which have to be satisfied by the generating series $F_m(u)$ by consistency of the gluing formula.

Similarly, looking at the proof of Theorem 4.1 of [GS16a], we obtain that, by consistency of the gluing formula, we have

$$F_m(u)F_m^{\psi}(u) = F_{2m-1}(u)F_1^{\psi}(u) - F_{m-1}(u)F_{m-1}^{\psi}(u)\,.$$

Using that $F_m(u) = (-i)(q^{\frac{m}{2}} - q^{-\frac{m}{2}})$, it is enough to compute $F_1^{\psi}(u)$ to determine all $F_m^{\psi}(u)$ by induction.

Thus, we have to show that $F_1(u) = u\cos\left(\frac{u}{2}\right)$. We follow the argument used in the proof of Proposition 1.26 to compute $F_1(u)$. We consider $\Delta = \{(-1,0),(0,-1),(1,1)\}$. The corresponding toric surface X_Δ is simply $\mathbb{P}^2$. Let D_1, D_2 and D_{out} be the toric boundary divisors of $\mathbb{P}^2$. The class β_Δ is simply the class of a curve of degree one, i.e. of a line, on $\mathbb{P}^2$. Let $\overline{M}_{g,\Delta}^{\psi}$ be the moduli space of genus g stable log maps of type Δ with an extra marking x_3. We denote ψ_3 the insertion of one psi class at this extra marking. We have the usual evaluation maps

$$(\text{ev}_1, \text{ev}_2)\colon \overline{M}_{g,\Delta} \to D_1 \times D_2\,.$$

We consider

$$N_{g,\Delta}^{1,2,\psi} = \int_{[\overline{M}_{g,\Delta_V}^{\psi}]^{\text{virt}}} (-1)^g \lambda_g \text{ev}_1^*(\text{pt}_1)\text{ev}_2^*(\text{pt}_2)\psi_3\,,$$

where $\text{pt}_1 \in A^*(D_1)$ and $\text{pt}_2 \in A^*(D_2)$ are classes of a point on D_1 and D_2 respectively.

By definition, we have

$$F_1^{\psi}(u) = \sum_{g\geq 0} N_{g,\Delta}^{1,2,\psi} u^{2g+1}\,.$$

Let P_1 and P_2 be points on D_1 and D_2 respectively, away from the torus fixed points. Let

S be the surface obtained by blowing-up $\mathbb{P}^2$ at P_1 and P_2. Denote by D the strict transform of the class of a line in $\mathbb{P}^2$ and by E_1, E_2 the exceptional divisors. Denote ∂S the strict transform of the toric boundary $\partial \mathbb{P}^2$ of $\mathbb{P}^2$. We endow S with the divisorial log structure with respect to ∂S. Let $\overline{M}_g(S)$ be the moduli space of genus g stable log maps to S of class $D - E_1 - E_2$ with tangency condition to intersect ∂S in one point with multiplicity one. Let $\overline{M}_g^{\psi}(S)$ be the moduli space of the same stable log maps but with an extra marked point. We define

$$N_g^S \coloneqq \int_{[\overline{M}_g(S)]^{\mathrm{virt}}} (-1)^g \lambda_g \,,$$

and

$$N_g^{S,\psi} \coloneqq \int_{[\overline{M}_g^{\psi}(S)]^{\mathrm{virt}}} (-1)^g \lambda_g \psi \,,$$

where ψ is a psi class inserted at the extra marked point.

The strict transform C of the line in $\mathbb{P}^2$ passing through P_1 and P_2 is the unique genus 0 curves satisfying these conditions and has normal bundle $N_{C|S} = \mathcal{O}_{\mathbb{P}^1}(-1)$ in S. All the higher genus maps factor through C, and as C is away from the preimage of the torus fixed points of $\mathbb{P}^2$, log invariants coincide with relative invariants [AMW14].

By the proof of Proposition 1.26, we know that

$$\sum_{g \geq 0} N_g^S u^{2g-1} = \frac{1}{2\sin\left(\frac{u}{2}\right)} \,.$$

We consider the moduli space $\overline{M}_g^{\psi}(\mathbb{P}^1/\infty, 1, 1)$ of degree 1 genus g stable maps to $\mathbb{P}^1$, relative to a point $\infty \in \mathbb{P}^1$, and with an extra marking. Let $\pi \colon \mathcal{C} \to \overline{M}_g(\mathbb{P}^1/\infty, 1, 1)$ be the universal curve and let $f \colon \mathcal{C} \to \mathbb{P}^1 \simeq C$ be the universal map. We need to compute

$$N_{g,\mathbb{P}^1}^{\psi} \coloneqq \int_{[\overline{M}_g^{\psi}(\mathbb{P}^1/\infty,1,1)]^{\mathrm{virt}}} c_g\left(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))\right)\psi \,,$$

where ψ is a psi class inserted at the extra marked point. Following the proof by Bryan and Pandharipande of Theorem 5.1 in [BP05], and inserting the psi class, we find

$$N_{g,\mathbb{P}^1}^{\psi} = \int_{\overline{M}_{g,2}} \psi_1^{2g-2} \psi_2 \lambda_g \,.$$

Applying formula (6) of [GP98], we obtain

$$N_{g,\mathbb{P}^1}^{\psi} = (2g-1) \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g \,,$$

and so

$$\sum_{g \geq 0} N_{g,\mathbb{P}^1}^{\psi} u^{2g-2} = \frac{d}{du}\left(\sum_{g\geq 0}\left(\int_{\overline{M}_{g,1}} \psi_1^{2g-2}\lambda_g\right) u^{2g-1}\right)$$

$$= \frac{d}{du}\left(\frac{1}{2\sin\left(\frac{u}{2}\right)}\right) = -\frac{\cos\left(\frac{u}{2}\right)}{\left(2\sin\left(\frac{u}{2}\right)\right)^2} \,.$$

We degenerate S as in Section 5.3 of [GPS10], and we apply the degeneration formula in relative Gromov-Witten theory [Li02]. There are three ways to distribute the psi class. Using the previous results, we obtain

$$-\frac{u\cos\left(\frac{u}{2}\right)}{\left(2\sin\left(\frac{u}{2}\right)\right)^2} = \left(\sum_{g\geq 0} N_{g,\Delta}^{1,2,\psi} u^{2g+1}\right)\left(\frac{1}{2\sin\left(\frac{u}{2}\right)}\right)^2 - 2\frac{u\cos\left(\frac{u}{2}\right)}{\left(2\sin\left(\frac{u}{2}\right)\right)^2},$$

and so the desired identity. □

2

The quantum tropical vertex

2.1 Scattering

In this Section, we first fix our notations for the basic objects considered in this Chapter: tori, quantum tori and automorphisms of formal families of them. We then introduce scattering diagrams, both classical and quantum, following [KS06], [GS11], [GPS10] and [FS15].

2.1.1 Torus

We fix $T = (\mathbb{C}^*)^2$ a 2-dimensional complex algebraic torus. Let $M \coloneqq \mathrm{Hom}(T, \mathbb{C}^*)$ be the 2-dimensional lattice of characters of T. Characters form a linear basis of the algebra of functions on T,

$$\Gamma(\mathcal{O}_T) = \bigoplus_{m \in M} \mathbb{C} z^m \,,$$

with the product given by $z^m \cdot z^{m'} = z^{m+m'}$. In other words, the algebra of functions on T is the algebra of the lattice M: $\Gamma(\mathcal{O}_T) = \mathbb{C}[M]$.

We fix

$$\langle -,- \rangle \colon \bigwedge^2 M \xrightarrow{\sim} \mathbb{Z}$$

an orientation of M, i.e. an integral unimodular skew-symmetric bilinear form on M. This defines a Poisson bracket on $\Gamma(\mathcal{O}_T)$, given by

$$\{z^m, z^{m'}\} = \langle m, m' \rangle z^{m+m'} \,,$$

and a corresponding algebraic symplectic form Ω on T.

If we choose a basis (m_1, m_2) of M such that $\langle m_1, m_2 \rangle = 1$, then, denoting $z_1 \coloneqq z^{m_1}$ and $z_2 \coloneqq z^{m_2}$, we have identifications $T = (\mathbb{C}^*)^2$, $M = \mathbb{Z}^2$, $\Gamma(\mathcal{O}_T) = \mathbb{C}[z_1^{\pm}, z_2^{\pm}]$ and $\Omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$.

2.1.2 Quantum torus

Given the symplectic torus (T, Ω), or equivalently the Poisson algebra $(\Gamma(\mathcal{O}_T), \{-,-\})$, it is natural to look for a "quantization". The quantum torus $\hat{T}^q$ is the non-commutative "space" whose algebra of functions is the non-commutative $\mathbb{C}[q^{\pm\frac{1}{2}}]$-algebra $\Gamma(\mathcal{O}_{\hat{T}^q})$, with linear basis indexed by the lattice M,

$$\Gamma(\mathcal{O}_{\hat{T}^q}) = \bigoplus_{m \in M} \mathbb{C}[q^{\pm\frac{1}{2}}]\hat{z}^m ,$$

and with product defined by

$$\hat{z}^m \cdot \hat{z}^{m'} = q^{\frac{1}{2}\langle m,m'\rangle} \hat{z}^{m+m'} .$$

The quantum torus $\hat{T}^q$ is a quantization of the torus T in the sense that writing $q = e^{i\hbar}$ and taking the limit $\hbar \to 0$, $q \to 1$, the linear term in $\hbar$ of the commutator $[\hat{z}^m, \hat{z}^{m'}]$ is determined by the Poisson bracket $\{z^m, z^{m'}\}$:

$$[\hat{z}^m, \hat{z}^{m'}] = (q^{\frac{1}{2}\langle m,m'\rangle} - q^{-\frac{1}{2}\langle m,m'\rangle})\hat{z}^{m+m'} = \langle m, m'\rangle i\hbar \hat{z}^{m+m'} + \mathcal{O}(\hbar^2) .$$

We denote $\hat{T}^\hbar$ the non-commutative "space" whose algebra of functions is the $\mathbb{C}((\hbar))$-algebra $\Gamma(\mathcal{O}_{\hat{T}^\hbar}) \coloneqq \Gamma(\mathcal{O}_{\hat{T}^q}) \otimes_{\mathbb{C}[q^{\pm\frac{1}{2}}]} \mathbb{C}((\hbar))$.

2.1.3 Automorphisms of formal families of tori

Let R be a complete local $\mathbb{C}$-algebra and let $\mathfrak{m}_R$ be the maximal ideal of R. By definition of completeness, we have

$$R = \varprojlim_{\ell} R/\mathfrak{m}_R^\ell .$$

We denote $S \coloneqq \mathrm{Spf}\ R$ the corresponding formal scheme and s_0 the closed point of S defined by $\mathfrak{m}_R$. Let T_S be the trivial family of 2-dimensional complex algebraic tori parametrized by S, i.e. $T_S \coloneqq S \times T$. The corresponding algebra of functions is given by

$$\Gamma(\mathcal{O}_{T_S}) = \varprojlim_{\ell} (R/\mathfrak{m}_R^\ell \otimes \Gamma(\mathcal{O}_T)) = \varprojlim_{\ell} (R/\mathfrak{m}_R^\ell \otimes \mathbb{C}[M]) .$$

Let $\hat{T}_S^\hbar$ be the trivial family of non-commutative 2-dimensional tori parametrized by S, i.e. $\hat{T}_S^\hbar \coloneqq S \times \hat{T}^\hbar$. The corresponding algebra of functions is simply given by

$$\Gamma(\mathcal{O}_{\hat{T}_S^\hbar}) = \varprojlim_{\ell} (R/\mathfrak{m}_R^\ell \otimes \Gamma(\mathcal{O}_{\hat{T}^\hbar})) .$$

The family T_S of tori has a natural Poisson structure, whose symplectic leaves are the torus fibers, and whose Poisson center is R. Explicitly, we have

$$\{H_m z^m, H_{m'} z^{m'}\} = H_m H_{m'} \{z^m, z^{m'}\} ,$$

for every $H_m, H_{m'} \in R$ and $m, m' \in M$. The family $\hat{T}_S^\hbar$ of non-commutative tori is a quantization of the Poisson variety T_S.

Let

$$H = \sum_{m \in M} H_m z^m$$

be a function on T_S whose restriction to the fiber over the closed point $s_0 \in S$ vanishes, i.e. such that $H = 0 \mod \mathfrak{m}_R$. Then $\{H, -\}$ defines a derivation of the algebra of functions on T_S and so a vector field on T_S, the Hamiltonian vector field defined by H, whose restriction to the fiber over the closed point $s_0 \in S$ vanishes.

The time one flow of this vector field defines an automorphism

$$\Phi_H \coloneqq \exp\left(\{H, -\}\right)$$

of T_S, whose restriction to the fiber over the closed point $s_0 \in S$ is the identity. Remark that Φ_H is well-defined because of the assumptions that $H = 0 \mod \mathfrak{m}_R$ and R is a complete local algebra, i.e. exp makes sense formally.

Let $\mathbb{V}_R$ be the subset of automorphisms of T_S which are of the form Φ_H for H as above. By the Baker-Campbell-Hausdorff formula, $\mathbb{V}_R$ is a subgroup of the group of automorphisms of T_S. In [GPS10], $\mathbb{V}_R$ is called the tropical vertex group.

Let

$$\hat{H} = \sum_{m \in M} \hat{H}_m \hat{z}^m$$

be a function on $\hat{T}_S^\hbar$ whose restriction to the fiber over the closed point $s_0 \in S$ vanishes, i.e. such that $\hat{H} = 0 \mod \mathfrak{m}_R$. Conjugation by $\exp\left(\hat{H}\right)$ defines an automorphism

$$\hat{\Phi}_{\hat{H}} \coloneqq \mathrm{Ad}_{\exp(\hat{H})} = \exp\left(\hat{H}\right)(-)\exp\left(-\hat{H}\right)$$

of $\hat{T}_S^\hbar$ whose restriction to the fiber over the closed point $s_0 \in S$ is the identity. Remark that $\hat{\Phi}_{\hat{H}}$ is well-defined because of the assumption that $\hat{H} = 0 \mod \mathfrak{m}_R$ and R is a compete local algebra, i.e. everything makes sense formally. Let $\hat{\mathbb{V}}_R^\hbar$ be the subset of automorphisms of $\hat{T}_S^\hbar$ which are of the form $\hat{\Phi}_{\hat{H}}$ for $\hat{H}$ as above. By the Baker-Campbell-Hausdorff formula, $\hat{\mathbb{V}}_R^\hbar$ is a subgroup from the group of automorphisms of $\hat{T}_S^\hbar$. We call $\hat{\mathbb{V}}_R^\hbar$ the quantum tropical vertex group[1].

If the limit

$$H \coloneqq \lim_{\hbar \to 0} \left(i\hbar \hat{H}\right)$$

exists, then, replacing $\hat{z}^m$ by z^m, H can be naturally viewed as a function on T_S and is the classical limit of $\hat{H}$. It is easy to check that Φ_H is the classical limit of $\hat{\Phi}_{\hat{H}}$.

[1] This group is much bigger that the "quantum tropical vertex group" of [KS11]. We will meet the group of [KS11] in Section 2.8, under the name "BPS quantum tropical vertex group".

2.1.4 Scattering diagrams

In this section, we work in the 2-dimensional real plane $M_{\mathbb{R}} \coloneqq M \otimes_{\mathbb{Z}} \mathbb{R}$. We call ray $\mathfrak{d}$ a half-line of rational slope in $M_{\mathbb{R}}$, and we denote $m_{\mathfrak{d}} \in M - \{0\}$ its primitive integral direction, pointing away from the origin.

Definition 2.1. *A* scattering diagram $\mathfrak{D}$ *over R is a set of rays $\mathfrak{d}$ in $M_{\mathbb{R}}$, equipped with functions $H_{\mathfrak{d}}$ such that either*

$$H_{\mathfrak{d}} \in \varprojlim_{\ell} (R/\mathfrak{m}_R^{\ell} \otimes \mathbb{C}[z^{m_{\mathfrak{d}}}]) ,$$

or

$$H_{\mathfrak{d}} \in \varprojlim_{\ell} (R/\mathfrak{m}_R^{\ell} \otimes \mathbb{C}[z^{-m_{\mathfrak{d}}}]) ,$$

and such that $H_{\mathfrak{d}} = 0 \mod \mathfrak{m}_R$, and for every $\ell \geqslant 1$, only finitely many rays $\mathfrak{d}$ have $H_{\mathfrak{d}} \neq 0 \mod \mathfrak{m}_R^{\ell}$.

A ray $(\mathfrak{d}, H_{\mathfrak{d}})$ such that

$$H_{\mathfrak{d}} \in \varprojlim_{\ell} (R/\mathfrak{m}_R^{\ell} \otimes \mathbb{C}[z^{m_{\mathfrak{d}}}]) ,$$

is called outgoing *and a ray $(\mathfrak{d}, H_{\mathfrak{d}})$ such that*

$$H_{\mathfrak{d}} \in \varprojlim_{\ell} (R/\mathfrak{m}_R^{\ell} \otimes \mathbb{C}[z^{-m_{\mathfrak{d}}}]) ,$$

is called ingoing.

Given a ray $(\mathfrak{d}, H_{\mathfrak{d}})$, we denote $m(H_{\mathfrak{d}}) \coloneqq m_{\mathfrak{d}}$ if $(\mathfrak{d}, H_{\mathfrak{d}})$ is outgoing, and $m(H_{\mathfrak{d}}) \coloneqq -m_{\mathfrak{d}}$ if $(\mathfrak{d}, H_{\mathfrak{d}})$ is ingoing. In both cases, we have

$$H_{\mathfrak{d}} \in \varprojlim_{\ell} (R/\mathfrak{m}_R^{\ell} \otimes \mathbb{C}[z^{m(H_{\mathfrak{d}})}]) ,$$

We will always consider scattering diagrams up to the following simplifying operations:

- A ray $(\mathfrak{d}, H_{\mathfrak{d}})$ with $H_{\mathfrak{d}} = 0$ is considered as trivial and can be safely removed from the scattering diagram.
- If two rays $(\mathfrak{d}_1, H_{\mathfrak{d}_1})$ and $(\mathfrak{d}_2, H_{\mathfrak{d}_2})$ are such that $\mathfrak{d}_1 = \mathfrak{d}_2$ and are both ingoing or outgoing, then they can be replaced by a single ray $(\mathfrak{d}, H_{\mathfrak{d}})$, where $\mathfrak{d} = \mathfrak{d}_1 = \mathfrak{d}_2$ and $H_{\mathfrak{d}} = H_{\mathfrak{d}_1} + H_{\mathfrak{d}_2}$. Remark that, because $\{H_{\mathfrak{d}_1}, H_{\mathfrak{d}_2}\} = 0$, we have $\Phi_{H_{\mathfrak{d}}} = \Phi_{H_{\mathfrak{d}_1}} \Phi_{H_{\mathfrak{d}_2}} = \Phi_{H_{\mathfrak{d}_2}} \Phi_{H_{\mathfrak{d}_1}}$.

Let $\mathfrak{D}$ be a scattering diagram. We call singular locus of $\mathfrak{D}$ the union of the set of initial points of rays and of the set of non-trivial intersection points of rays. Let $\gamma \colon [0,1] \to M_{\mathbb{R}}$ be a smooth path. We say that γ is admissible if γ does not intersect the singular locus of $\mathfrak{D}$, if the endpoints of γ are not on rays of $\mathfrak{D}$, and if γ intersects transversely all the rays of $\mathfrak{D}$.

Let γ be an admissible smooth path in $M_{\mathbb{R}}$. Let $\ell \geqslant 1$ be a positive integer. By definition, $\mathfrak{D}$ contains only finitely many rays $(\mathfrak{d}, H_{\mathfrak{d}})$ with $H_{\mathfrak{d}} \neq 0 \mod \mathfrak{m}_R^{\ell}$. So there exist finitely many

$0 < t_1 \leqslant \dots \leqslant t_s < 1$, the times of intersection of γ with rays $(\mathfrak{d}_1, H_{\mathfrak{d}_1}), \dots, (\mathfrak{d}_s, H_{\mathfrak{d}_s})$ of $\mathfrak{D}$ such that $H_{\mathfrak{d}_r} \neq 0 \mod \mathfrak{m}_l$. For $r = 1, \dots, s$, we define $\epsilon_r \in \{\pm 1\}$ to be the sign of $\langle m(H_{\mathfrak{d}_i}), \gamma'(t_r)\rangle$. We then define

$$\theta_{\gamma,\mathfrak{D},\ell} \coloneqq \Phi^{\epsilon_s}_{H_{\mathfrak{d}_s}} \dots \Phi^{\epsilon_1}_{H_{\mathfrak{d}_1}} .$$

Taking the limit $\ell \to +\infty$, we define

$$\theta_{\gamma,\mathfrak{D}} \coloneqq \lim_{\ell \to +\infty} \theta_{\gamma,\mathfrak{D},\ell} .$$

Definition 2.2. *A scattering diagram $\mathfrak{D}$ over R is* consistent *if, for every closed admissible smooth path $\gamma \colon [0,1] \to M_{\mathbb{R}}$, we have $\theta_{\gamma,\mathfrak{D}} = \mathrm{id}$.*

The following result is due to Kontsevich-Soibelman [KS06], Theorem 6 (see also Theorem 1.4 of [GPS10]).

Proposition 2.3. *Any scattering diagram $\mathfrak{D}$ can be canonically completed by adding only outgoing rays to form a consistent scattering diagram $S(\mathfrak{D})$.*

Proof. It is enough to show that for every non-negative integer ℓ, it is possible to add outgoing rays to $\mathfrak{D}$ to get a scattering diagram $\mathfrak{D}_\ell$ consistent at the order ℓ, i.e. such that $\theta_{\gamma,\mathfrak{D}_\ell} = \mathrm{id} \mod \mathfrak{m}_R^{\ell+1}$. The construction is done by induction on l, starting with $\mathfrak{D}_0 = \mathfrak{D}$. Let us assume we have constructed $\mathfrak{D}_{\ell-1}$, consistent at the order $\ell - 1$. Let p be a point in the singular locus of $\mathfrak{D}_{\ell-1}$ and let γ be a small anticlockwise closed loop around p. As $\mathfrak{D}_{\ell-1}$ is consistent at the order $\ell - 1$, we can write $\theta_{\gamma,\mathfrak{D}_{\ell-1}} = \Phi_H$ for some H with $H = 0 \mod \mathfrak{m}_R^\ell$. There are finitely many $m_j \in M - \{0\}$ primitive such that we can write

$$H = \sum_j H_j \mod \mathfrak{m}_R^{\ell+1}$$

with $H_j \in \mathfrak{m}_R^\ell R \otimes \mathbb{C}[z_j^m]$. We construct $\mathfrak{D}_\ell$ by adding to $\mathfrak{D}_{\ell-1}$ the outgoing rays $(p + \mathbb{R}_{\geqslant 0} m_j, \Phi_{-H_j})$. □

Adding hats everywhere, we get the definition of a quantum scattering diagram $\hat{\mathfrak{D}}$, with functions

$$\hat{H}_{\mathfrak{d}} \in \varprojlim_\ell (R/\mathfrak{m}_R^\ell \otimes \mathbb{C}((\hbar))[\hat{z}^{m_{\mathfrak{d}}}]) ,$$

for outgoing rays and

$$\hat{H}_{\mathfrak{d}} \in \varprojlim_\ell (R/\mathfrak{m}_R^\ell \otimes \mathbb{C}((\hbar))[\hat{z}^{-m_{\mathfrak{d}}}]) ,$$

for ingoing rays, the notion of consistent quantum scattering diagram, and the fact that every quantum scattering diagram $\hat{\mathfrak{D}}$ can be canonically completed by adding only outgoing rays to form a consistent quantum scattering diagram $S(\hat{\mathfrak{D}})$.

We will often call $\hat{H}_{\mathfrak{d}}$ the Hamiltonian attached to the ray $\mathfrak{d}$.

Remark: A general notion of scattering diagram, as in Section 2 of [KS13], takes as input a lattice M and a M-graded Lie algebra $\mathfrak{g}$. What we call a (classical) scattering diagram is the special case where M is the lattice of characters of a 2-dimensional symplectic torus T

and where $\mathfrak{g} = (\Gamma(\mathcal{O}_{T_S}), \{-,-\})$. What we call a quantum scattering diagram is the special case where M is the lattice of characters of a 2-dimensional symplectic torus T and where $\mathfrak{g} = (\Gamma(\mathcal{O}_{\hat{T}_S^\hbar}), [-,-])$.

Remark: In our definition of a scattering diagram, we attach to each ray $\mathfrak{d}$ a function

$$H_{\mathfrak{d}} = \sum_{\ell \geqslant 0} H_l z^{\ell m(H_{\mathfrak{d}})} ,$$

such that $H_{\mathfrak{d}} = 0 \mod \mathfrak{m}_R$, which can be interpreted as Hamiltonian generating an automorphism

$$\Phi_{H_{\mathfrak{d}}} = \exp\left(\{H_{\mathfrak{d}}, -\}\right) .$$

In [GPS10], [GS11] or [FS15], the terminology is slightly different. To a ray $\mathfrak{d}$, they attach a function

$$f_{\mathfrak{d}} = \sum_{\ell \geqslant 0} c_\ell z^{\ell m(H_{\mathfrak{d}})} ,$$

such that $f_{\mathfrak{d}} = 1 \mod \mathfrak{m}_R$, and, to a path $\gamma(t)$ intersecting transversely $\mathfrak{d}$ at time t_0, an automorphism

$$\theta_{f_{\mathfrak{d}},\gamma} \colon z^m \mapsto z^m f_{\mathfrak{d}}^{\langle n_{\mathfrak{d}}, m\rangle} ,$$

where $n_{\mathfrak{d}}$ is the primitive generator of $\mathfrak{d}$ such that $\langle n_{\mathfrak{d}}, \gamma'(t_0)\rangle > 0$. These two choices are equivalent. Indeed, if ϵ is the sign of $\langle m(H_{\mathfrak{d}}), \gamma'(t_0)\rangle$, we have

$$\Phi_{H_{\mathfrak{d}}}^{\epsilon} = \theta_{f_{\mathfrak{d}},\gamma}$$

if $H_{\mathfrak{d}}$ and $f_{\mathfrak{d}}$ are related by

$$\log f_{\mathfrak{d}} = \sum_{\ell \geqslant 0} \ell H_\ell z^{\ell m(H_{\mathfrak{d}})} .$$

The formalism of [GS11] is more general because it treats the Calabi-Yau case and not just a holomorphic symplectic case. For our purposes, focused on a holomorphic symplectic situation, using the Hamiltonians $H_{\mathfrak{d}}$ rather than the functions $f_{\mathfrak{d}}$ makes the quantization step transparent. The quantum version of the functions $f_{\mathfrak{d}}$ will be studied and used in Chapter 3.

2.2 Gromov-Witten theory of log Calabi-Yau surfaces

Our main result, Theorem 2.6, is an enumerative interpretation of a class of quantum scattering diagrams, as introduced in the previous Section 2.1, in terms of higher genus log Gromov-Witten invariants of a class of log Calabi-Yau surfaces. In Section 2.2.1 we review the definition of these log Calabi-Yau surfaces, following [GPS10]. We define the relevant higher genus log Gromov-Witten invariants in Sections 2.2.2 and 2.2.3. We give a 3-dimensional interpretation of these invariants in Section 2.2.4. Finally, we give a generalization of these invariants to some orbifold context in Section 2.2.5.

2.2.1 Log Calabi-Yau surfaces

We fix $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors of $M = \mathbb{Z}^2$. The fan in $\mathbb{R}^2$ with rays $-\mathbb{R}_{\geqslant 0} m_1, \dots, -\mathbb{R}_{\geqslant 0} m_n$ defines a toric surface $\overline{Y}_{\mathfrak{m}}$. Let $D_{m_1}, \dots, D_{m_n}$ be the corresponding toric divisors. If $m_1, \dots, m_n$ do not span M, i.e. if $\overline{Y}_{\mathfrak{m}}$ is non-compact, we add some extra rays to the fan to make it span M and we still denote $\overline{Y}_{\mathfrak{m}}$ the corresponding compact toric surfaces. The choice of the added rays will be irrelevant for us (because of the log birational invariance result in log Gromov-Witten theory proved in [AW13]).

For every $j = 1, \dots, n$, we blow-up a point x_j in general position on the toric divisor D_{m_j}[2]. Remark that it is possible to have $\mathbb{R}_{\geqslant 0} m_j = \mathbb{R}_{\geqslant 0} m_{j'}$, and so $D_{m_j} = D_{m_{j'}}$, for $j \neq j'$, and that in this case we blow-up several distinct points on the same toric divisor. We denote $Y_{\mathfrak{m}}$ the resulting projective surface and $\nu : Y_{\mathfrak{m}} \to \overline{Y}_{\mathfrak{m}}$ the blow-up morphism. Let $E_j \coloneqq \nu^{-1}(x_j)$ be the exceptional divisor over x_j. We denote $\partial Y_{\mathfrak{m}}$ the strict transform of the toric boundary divisor. The divisor $\partial Y_{\mathfrak{m}}$ is an anticanonical cycle of rational curves and so the pair $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$ is an example of log Calabi-Yau surface with maximal boundary.

2.2.2 Curve classes

We want to consider curves in $Y_{\mathfrak{m}}$ meeting $\partial Y_{\mathfrak{m}}$ in a unique point. We first explain how to parametrize the relevant curve classes in terms of their intersection numbers p_j with the exceptional divisors E_j.

Let $p \coloneqq (p_1, \dots, p_n) \in P \coloneqq \mathbb{N}^n$. We assume that $\sum_{j=1}^n p_j m_j \neq 0$ and so we can uniquely write

$$\sum_{j=1}^n p_j m_j = \ell_p m_p \,,$$

with $m_p \in M$ primitive and $\ell_p \in \mathbb{N}$.

We explain now how to define a curve class $\beta_p \in H_2(Y_{\mathfrak{m}}, \mathbb{Z})$. In short, β_p is the class of a curve in $Y_{\mathfrak{m}}$ having for every $j = 1, \dots, n$, intersection number p_j with the exceptional divisor E_j, and exactly one intersection point with the anticanonical cycle $\partial Y_{\mathfrak{m}}$.

More precisely, the vector $m_p \in M$ belongs to some cone of the fan of $\overline{Y}_{\mathfrak{m}}$ and we write the corresponding decomposition

$$m_p = a_p^L m_p^L + a_p^R m_p^R \,,$$

where m_p^L, $m_p^R \in M$ are primitive generators of rays of the fan of $\overline{Y}_{\mathfrak{m}}$ and where a_p^L, $a_p^R \in \mathbb{N}$. Remark that there is only one term in this decomposition if the ray $\mathbb{R}_{\geqslant 0} m_p$ coincides with one of the rays of the fan of $\overline{Y}_{\mathfrak{m}}$. Let D_p^L and D_p^R be the toric divisors corresponding to the rays $\mathbb{R}_{\geqslant 0} m_p^L$ and $\mathbb{R}_{\geqslant 0} m_p^R$. Let $\beta \in H_2(\overline{Y}_{\mathfrak{m}}, \mathbb{Z})$ be determined by the following intersection numbers with the toric divisors:

[2] By deformation invariance of log Gromov-Witten invariants, the precise choice of x_j will be irrelevant for us.

- Intersection number with D_{m_j}, $1 \leqslant j \leqslant n$, distinct from D_p^L and D_p^R:

$$\beta.D_{m_j} = \sum_{j', D_{m_{j'}} = D_{m_j}} p_{j'} \,.$$

- Intersection number with D_p^L:

$$\beta.D_p^L = \ell_p a_p^L + \sum_{j, D_{m_j} = D_p^L} p_j \,.$$

- Intersection number with D_p^R:

$$\beta.D_p^R = \ell_p a_p^R + \sum_{j, D_{m_j} = D_p^R} p_j \,.$$

- Intersection number with every toric divisor D different from the D_{m_j}, $j = 1, \dots, n$, and from D_p^L and D_p^R: $\beta.D = 0$.

Such class $\beta \in H_2(\overline{Y}_{\mathfrak{m}}, \mathbb{Z})$ exists by standard toric geometry because of the relation

$$\sum_{j=1}^{n} p_j m_j = \ell_p m_p \,.$$

Finally, we define

$$\beta_p \coloneqq \nu^* \beta - \sum_{j=1}^{n} p_j E_j \in H_2(Y_{\mathfrak{m}}, \mathbb{Z}) \,.$$

By construction, we have

$$\beta_p.E_j = p_j \,,$$

for $j = 1, \dots, n$,

$$\beta_p.D_p^L = \ell_p a_p^L \,,$$

$$\beta_p.D_p^R = \ell_p a_p^R \,,$$

and

$$\beta_p.D = 0 \,,$$

for every component D of $\partial Y_{\mathfrak{m}}$ distinct from D_p^L and D_p^R.

2.2.3 Log Gromov-Witten invariants

For every $p = (p_1, \dots, p_n) \in P = \mathbb{N}^n$, we defined in the previous Section 2.2.2 positive integers ℓ_p, a_p^L, a_p^R, some components D_p^L and D_p^R of the divisor $\partial Y_{\mathfrak{m}}$ and a curve class $\beta_p \in H_2(Y_{\mathfrak{m}}, \mathbb{Z})$. We would like to consider genus g stable maps $f \colon C \to Y_{\mathfrak{m}}$ of class β_p, intersecting properly the components of $\partial Y_{\mathfrak{m}}$, and meeting ∂Y_m in a unique point. At this point, such a map necessarily has an intersection number $\ell_p a_p^L$ with D_p^L and $\ell_p a_p^R$ with D_p^R.

The space of such stable maps is not proper in general: a limit of curves intersecting properly $\partial Y_{\mathfrak{m}}$ does not necessarily intersect $\partial Y_{\mathfrak{m}}$ properly. A nice compactification of this space is obtained by considering stable log maps. The idea is to allow maps intersecting $\partial Y_{\mathfrak{m}}$ non-properly, but to remember some additional information under the form of log structures, which give a way to make sense of tangency conditions even for non-proper intersections. The theory of stable log maps[3] has been developed by Gross and Siebert [GS13], and Abramovich and Chen [Che14b], [AC14]. We refer to Kato [Kat89] for elementary notions of log geometry. We consider the divisorial log structure on $Y_{\mathfrak{m}}$ defined by the divisor $\partial Y_{\mathfrak{m}}$ and use it to see $Y_{\mathfrak{m}}$ as a smooth log scheme.

Let $\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})$ be the moduli space of genus g stable log maps to $Y_{\mathfrak{m}}$, of class β_p, with contact order along $\partial Y_{\mathfrak{m}}$ given by $\ell_p m_p$. It is a proper Deligne-Mumford stack of virtual dimension g and it admits a virtual fundamental class

$$[\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})]^{\mathrm{virt}} \in A_g(\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}), \mathbb{Q})\,.$$

If $\pi\colon \mathcal{C} \to \overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})$ is the universal curve, of relative dualizing sheaf ω_π, then the Hodge bundle

$$\mathbb{E} \coloneqq \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})$. Its Chern classes are classically called the lambda classes, $\lambda_j \coloneqq c_j(\mathbb{E})$ for $j = 0, \dots, g$. Finally, we define the genus g log Gromov-Witten invariants of $Y_{\mathfrak{m}}$ which will be of interest for us by

$$N_{g,p}^{Y_{\mathfrak{m}}} \coloneqq \int_{[\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})]^{\mathrm{virt}}} (-1)^g \lambda_g \in \mathbb{Q}\,.$$

Remark that the top lambda class λ_g has exactly the right degree to cut down the virtual dimension from g to zero, so that $N_{g,p}^{Y_{\mathfrak{m}}}$ is not obviously zero.

The fact that the top lambda class should be the natural insertion to consider for some higher genus version of [GPS10] was already suggested in Section 5.8 of [GPS10]. From our point of view, higher genus invariants with the top lambda class inserted are the correct objects because it is to them that the correspondence tropical theorem of Chapter 1 applies. In Section 2.9, we will explain how our main result Theorem 2.6 fits into an expected story for higher genus open holomorphic curves in Calabi-Yau 3-folds. This is probably the most conceptual understanding of the role of the invariants $N_{g,\beta}^{Y_{\mathfrak{m}}}$: they are really higher genus invariants of the log Calabi-Yau 3-fold $Y_{\mathfrak{m}} \times \mathbb{P}^1$, and the top lambda class is simply a measure of the difference between surface and 3-fold obstruction theories. This will be made precise in the following Section 2.2.4, whose analogue for K3 surfaces is well-known, see Lemma 7 of [MPT10].

2.2.4 3-dimensional interpretation of the invariants $N_{g,p}^{Y_{\mathfrak{m}}}$

In this Section, we rewrite the log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ of the log Calabi-Yau surface $Y_{\mathfrak{m}}$ in terms of 3-dimensional geometries, first $S \times \mathbb{C}$ and then $S \times \mathbb{P}^1$.

We endow the 3-fold $Y_{\mathfrak{m}} \times \mathbb{C}$ with the smooth log structure given by the divisorial log structure

[3] By stable log maps, we always mean basic stable log maps in the sense of [GS13].

along the divisor $\partial Y_{\mathfrak{m}} \times \mathbb{C}$. Let

$$\overline{M}_{g,p}(Y_{\mathfrak{m}} \times \mathbb{C}/\partial Y_{\mathfrak{m}} \times \mathbb{C})$$

be the moduli space of genus g stable log maps to $Y_{\mathfrak{m}}$, of class β_p, with contact order along $\partial Y_{\mathfrak{m}} \times \mathbb{C}$ given by $\ell_p m_p$. It is a Deligne-Mumford stack of virtual dimension 1 and it admits a virtual fundamental class

$$[\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})]^{\mathrm{virt}} \in A_1(\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}), \mathbb{Q})\,.$$

Because $\mathbb{C}$ is not compact, $\overline{M}_{g,p}(Y_{\mathfrak{m}} \times \mathbb{C}/\partial Y_{\mathfrak{m}} \times \mathbb{C})$ is not compact and so one cannot simply integrate over the virtual class. Using the standard action of $\mathbb{C}^*$ on $\mathbb{C}$, fixing $0 \in \mathbb{C}$, we get an action of $\mathbb{C}^*$ on $\overline{M}_{g,p}(Y_{\mathfrak{m}} \times \mathbb{C}/\partial Y_{\mathfrak{m}} \times \mathbb{C})$, with its perfect obstruction theory, whose fixed point locus is the space of stable log maps mapping to $Y_{\mathfrak{m}} \times \{0\}$, i.e. $\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})$, with its natural perfect obstruction theory. Given the virtual localization formula [GP99], it is natural to define equivariant log Gromov-Witten invariants

$$N_{g,p}^{Y_{\mathfrak{m}} \times \mathbb{C}} := \int_{[\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})]^{\mathrm{virt}}} \frac{1}{e(\mathrm{Nor}^{\mathrm{virt}})} \in \mathbb{Q}(t)\,,$$

where $\mathrm{Nor}^{\mathrm{virt}}$ is the equivariant virtual normal bundle of $\overline{M}_{g,p}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})$ in

$$\overline{M}_{g,p}(Y_{\mathfrak{m}} \times \mathbb{C}/\partial Y_{\mathfrak{m}} \times \mathbb{C})\,,$$

$e(\mathrm{Nor}^{\mathrm{virt}})$ is its equivariant Euler class, and t is the generator of the $\mathbb{C}^*$-equivariant cohomology of a point.

Lemma 2.4. *We have*

$$N_{g,p}^{Y_{\mathfrak{m}} \times \mathbb{C}} = \frac{1}{t} N_{g,p}^{Y_{\mathfrak{m}}}\,.$$

Proof. Because the 3-dimensional geometry $Y_{\mathfrak{m}} \times \mathbb{C}$, including the log/tangency conditions, is obtained from the 2-dimensional geometry $Y_{\mathfrak{m}}$ by a trivial product with a trivial factor $\mathbb{C}$, with $\mathbb{C}^*$ scaling this trivial factor, the virtual normal at a stable log map $f\colon C \to Y_{\mathfrak{m}}$ is $H^0(C, f^*\mathcal{O}) - H^1(C, f^*\mathcal{O}) = t - \mathbb{E}^\vee \otimes t$ so

$$\frac{1}{e(\mathrm{Nor}^{\mathrm{virt}})} = \frac{1}{t}\left(\sum_{i=0}^{g} (-1)^i \lambda_i t^{g-i}\right),$$

and

$$N_{g,p}^{Y_{\mathfrak{m}} \times \mathbb{C}} = \int_{[\overline{M}_{g,p}(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})]^{\mathrm{virt}}} \frac{(-1)^g \lambda_g}{t} = \frac{1}{t} N_{g,p}^{Y_{\mathfrak{m}}}\,.$$

□

Remark: The proof of Lemma 2.4 is identical to the proof of Lemma 7 in [MPT10] up to some small point: in [MPT10], counts of expected dimensions work because of the use of a reduced Gromov-Witten theory of K3 surfaces, whereas for us, counts of expected dimensions work because of the use of log Gromov-Witten theory.

We consider now the 3-fold $Z_{\mathfrak{m}} \coloneqq Y_{\mathfrak{m}} \times \mathbb{P}^1$ with the smooth log structure given by the divisorial log structure along the divisor

$$\partial Z_{\mathfrak{m}} \coloneqq (\partial Y_{\mathfrak{m}} \times \mathbb{P}^1) \cup (Y_{\mathfrak{m}} \times \{0\}) \cup (Y_{\mathfrak{m}} \times \{\infty\}) .$$

The divisor $\partial Z_{\mathfrak{m}}$ is anticanonical, containing zero-dimensional strata, and so the pair $(Z_{\mathfrak{m}}, \partial Z_{\mathfrak{m}})$ is an example of log Calabi-Yau 3-fold with maximal boundary.

Let

$$\overline{M}_{g,p}(Z_{\mathfrak{m}}/\partial Z_{\mathfrak{m}})$$

be the moduli space of genus g stable log maps to $Z_{\mathfrak{m}}$, of class β_p, with contact order along $\partial Z_{\mathfrak{m}}$ given by $\ell_p m_p$. It is a proper Deligne-Mumford stack of virtual dimension 1 and it admits a virtual fundamental class

$$[\overline{M}_{g,p}(Z_{\mathfrak{m}}/\partial Z_{\mathfrak{m}})]^{\mathrm{virt}} \in A_1(\overline{M}_{g,p}(Z_{\mathfrak{m}}/\partial Z_{\mathfrak{m}}), \mathbb{Q}) .$$

Composing the natural evaluation map at the contact order point with $\partial Z_{\mathfrak{m}}$ with the projection $\partial Z_{\mathfrak{m}} \to \mathbb{P}^1$, we get a map $\rho\colon \overline{M}_{g,p}(Z_{\mathfrak{m}}/\partial Z_{\mathfrak{m}}) \to \mathbb{P}^1$ and we define log Gromov-Witten invariants

$$N_{g,p}^{Z_{\mathfrak{m}}} \coloneqq \int_{[\overline{M}_{g,p}(Z_{\mathfrak{m}}/\partial Z_{\mathfrak{m}})]^{\mathrm{virt}}} \rho^*(\mathrm{pt}) \in \mathbb{Q} ,$$

where $\mathrm{pt} \in A^1(\mathbb{P}^1)$ is the class of a point.

Lemma 2.5. *We have*

$$N_{g,p}^{Z_{\mathfrak{m}}} = N_{g,p}^{Y_{\mathfrak{m}}} .$$

Proof. We use virtual localization [GP99] with respect to the action of $\mathbb{C}^*$ on the $\mathbb{P}^1$-factor with weight t at 0 and weight $-t$ at ∞. We choose pt_0 as equivariant lift of $\mathrm{pt} \in A^1(\mathbb{P}^1)$. Because of the insertion of $\mathrm{pt}_0 = t$, only the fixed point $0 \in \mathbb{P}^1$, and not $\infty \in \mathbb{P}^1$, contributes to the localization formula, and we get

$$N_{g,p}^{Z_{\mathfrak{m}}} = t N_{g,p}^{Y_{\mathfrak{m}} \times \mathbb{C}} ,$$

hence the result by Lemma 2.4. ☐

2.2.5 Orbifold Gromov-Witten theory

We give an orbifold generalization of Sections 2.2.1, 2.2.2, 2.2.3, which will be necessary to state Theorem 2.7 in Section 2.7.2.

As in Section 2.2.1, we fix $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors of $M = \mathbb{Z}^2$ and this defines a toric surface $\overline{Y}_{\mathfrak{m}}$, with toric divisors D_{m_j}, $1 \leqslant j \leqslant n$. For every $\mathfrak{r} = (r_1, \dots, r_n)$ an n-tuple of positive integers, we define a projective surface $Y_{\mathfrak{m},\mathfrak{r}}$ by blowing-up a subscheme of length r_j in general position on the toric divisor D_{m_j}, for every $1 \leqslant j \leqslant n$. For $\mathfrak{r} = (1, \dots, 1)$, we simply have $Y_{\mathfrak{m},\mathfrak{r}} = Y_{\mathfrak{m}}$ defined in Section 2.2.1.

Let $\nu\colon Y_{\mathfrak{m},\mathfrak{r}} \to \overline{Y}_{\mathfrak{m}}$ be the blow-up morphism. If $r_j \geqslant 2$, then $Y_{\mathfrak{m},\mathfrak{r}}$ has a A_{r_j-1}-singularity on the exceptional divisor $E_j \coloneqq \nu^{-1}(x_j)$. We will consider $Y_{\mathfrak{m},\mathfrak{r}}$ as a Deligne-Mumford stack by taking the natural structure of smooth Deligne-Mumford stack on a A_{r_j-1} singularity. The exceptional divisor E_j is then a stacky projective line $\mathbb{P}^1[r_j,1]$, with a single $\mathbb{Z}/r_j$ stacky point $0 \in \mathbb{P}^1[r_j,1]$. The normal bundle to E_j in $Y_{\mathfrak{m},\mathfrak{r}}$ is the orbifold line bundle $\mathcal{O}_{\mathbb{P}^1[r_j,1]}(-[0]/(\mathbb{Z}/r_j))$ of degree $-1/r_j$, and in particular we have $E_j^2 = -1/r_j$.

Denote $P_{\mathfrak{r}}$ the set of $(p_1,\dots,p_n) \in P = \mathbb{N}^n$ such that r_j divides p_j, for every $1 \leqslant j \leqslant n$. Exactly as in Section 2.2.2, we define for every $p \in P_{\mathfrak{r}}$ a curve class $\beta_p \in H_2(Y_{\mathfrak{m},\mathfrak{r}},\mathbb{Z})$. The only difference is that now we have

$$\beta_p.E_j = \frac{p_j}{r_j}\,.$$

We denote $\partial Y_{\mathfrak{m},\mathfrak{r}}$ the strict transform of the toric boundary divisor $\partial \overline{Y}_{\mathfrak{m}}$ of $\overline{Y}_{\mathfrak{m}}$, and we endow $Y_{\mathfrak{m}}$ with the divisorial log structure define by $\partial Y_{\mathfrak{m}}$. So we see $Y_{\mathfrak{m},\mathfrak{r}}$ as a smooth Deligne-Mumford log stack. Because the non-trivial stacky structure is disjoint from the divisor $\partial Y_{\mathfrak{m},\mathfrak{r}}$ supporting the non-trivial log structure, there is no difficulty in combining orbifold Gromov-Witten theory, [AGV08], [CR02], with log Gromov-Witten theory, [GS13], [Che14b], [AC14], to get a moduli space $\overline{M}_{g,p}(Y_{\mathfrak{m},\mathfrak{r}}/\partial Y_{\mathfrak{m},\mathfrak{r}})$ of genus g stable log maps to $Y_{\mathfrak{m},\mathfrak{r}}$, of class β_p, with contact order along $\partial Y_{\mathfrak{m},\mathfrak{r}}$ given by $\ell_p m_p$. It is a proper Deligne-Mumford stack of virtual dimension g, admitting a virtual fundamental class

$$[\overline{M}_{g,p}(Y_{\mathfrak{m},\mathfrak{r}}/\partial Y_{\mathfrak{m},\mathfrak{r}})]^{\mathrm{virt}} \in A_g(\overline{M}_{g,p}(Y_{\mathfrak{m},\mathfrak{r}}/\partial Y_{\mathfrak{m},\mathfrak{r}}),\mathbb{Q})\,.$$

We finally define genus g orbifold log Gromov-Witten invariants of $Y_{\mathfrak{m},\mathfrak{r}}$ by

$$N_{g,p}^{Y_{\mathfrak{m},\mathfrak{r}}} \coloneqq \int_{[\overline{M}_{g,p}(Y_{\mathfrak{m},\mathfrak{r}}/\partial Y_{\mathfrak{m},\mathfrak{r}})]^{\mathrm{virt}}} (-1)^g \lambda_g \in \mathbb{Q}\,.$$

2.3 Main results

In Section 2.3.1, we state the main result of the present Chapter, Theorem 2.6, precise form of Theorem 2 mentioned in the Introduction. In Section 2.3.2, we give elementary examples illustrating Theorem 2.6. In Section 2.3.3, we state Theorem 2.7, a generalization of Theorem 2.6 including orbifold geometries. Finally, we give in Section 2.3.4 some brief comments about the level of generality of Theorems 2.6 and 2.7.

2.3.1 Statement

Using the notations of Section 2.1, we define a family of consistent quantum scattering diagrams. Our main result, Theorem 2.6, is that the Hamiltonians attached to the rays of these quantum scattering diagrams are generating series of the higher genus log Gromov-Witten invariants defined in Section 2.2.

We fix $\mathfrak{m} = (m_1,\dots,m_n)$ an n-tuple of primitive non-zero vectors of M. We denote $P \coloneqq \mathbb{N}^n$ and we take $R \coloneqq \mathbb{C}[[P]] = \mathbb{C}[[t_1,\dots,t_n]]$ as complete local $\mathbb{C}$-algebra. Let $\hat{\mathfrak{D}}_{\mathfrak{m}}$ be the quantum scattering diagram over R consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j})$, $1 \leqslant j \leqslant n$, where

$$\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0} m_j \,,$$

and

$$\hat{H}_{\mathfrak{d}_j} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} t_j^\ell \hat{z}^{\ell m_j} = \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} t_j^\ell \hat{z}^{\ell m_j} \,,$$

where $q = e^{i\hbar}$.

Let $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ be the corresponding consistent quantum scattering diagram given by Proposition 2.3, obtained by adding outgoing rays to $\hat{\mathfrak{D}}_{\mathfrak{m}}$. We can assume that, for every $m \in M - \{0\}$ primitive, $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ contains a unique outgoing ray of support $\mathbb{R}_{\geqslant 0} m$.

For every $m \in M - \{0\}$ primitive, let P_m be the subset of $p = (p_1, \dots, p_n) \in P = \mathbb{N}^n$ such that $\sum_{j=1}^n p_j m_j$ is positively collinear with m:

$$\sum_{j=1}^n p_j m_j = \ell_p m$$

for some $\ell_p \in \mathbb{N}$.

Recall that in Section 2.2, for every $\mathfrak{m} = (m_1, \dots, m_n)$, we introduced a log Calabi-Yau surface $Y_{\mathfrak{m}}$ and for every $p = (p_1, \dots, p_n) \in P = \mathbb{N}^n$, we defined some genus g log Gromov-Witten $N_{g,p}^{Y_{\mathfrak{m}}}$ of $Y_{\mathfrak{m}}$.

Theorem 2.6. *For every $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in M and for every $m \in M - \{0\}$ primitive, the Hamiltonian $\hat{H}_m$ attached to the outgoing ray $\mathbb{R}_{\geqslant 0} m$ in the consistent quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ is given by*

$$\hat{H}_m = \left(-\frac{i}{\hbar}\right) \sum_{p \in P_m} \left(\sum_{g \geqslant 0} N_{g,p}^{Y_{\mathfrak{m}}} \hbar^{2g} \right) \left(\prod_{j=1}^n t_j^{p_j} \right) \hat{z}^{\ell_p m} \,.$$

Remarks:

- In the classical limit $\hbar \to 0$, Theorem 2.6 reduces to the main result (Theorem 5.4) of [GPS10], expressing the classical scattering diagram $S(\mathfrak{D}_{\mathfrak{m}})$ in terms of the genus zero log Gromov-Witten invariants $N_{0,p}^{Y_{\mathfrak{m}}}$[4].
- The proof of Theorem 2.6 takes Sections 2.4, 2.5 2.6, and 2.7. In Section 2.2.3, we define higher genus log Gromov-Witten invariants $N_{g,w}^{\overline{Y}_{\mathfrak{m}}}$ of toric surfaces $\overline{Y}_{\mathfrak{m}}$. In Section 2.5, we prove a degeneration formula expressing the log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ of the log Calabi-Yau surface $Y_{\mathfrak{m}}$ in terms of log Gromov-Witten invariants $N_{g,w}^{\overline{Y}_{\mathfrak{m}}}$ of the toric surface $\overline{Y}_{\mathfrak{m}}$. In Section 2.6, we review, following [FS15], the relation between quantum scattering diagrams and Block-Göttsche q-deformed tropical curve count. In Section 2.7, we conclude the proof by using Theorem 1.4, the main

[4]In [GPS10], the genus zero invariants are defined as relative Gromov-Witten invariants of some open geometry. The fact that they coincide with genus zero log Gromov-Witten invariants follows from the cycle arguments used in the proofs of Proposition 1.10 and Lemma 2.12.

result of Chapter 1, relating q-deformed tropical curve count and higher genus log Gromov-Witten invariants of toric surfaces.

- The consistency of the quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ translates into the fact that the product, ordered according to the phase of the rays, of the elements $\hat{\Phi}_{\hat{H}_j}$, $j = 1, \ldots, n$, and $\hat{\Phi}_{\hat{H}_{\mathfrak{m}}}$, $\mathfrak{m} \in M - \{0\}$ primitive, of the quantum tropical vertex group $\hat{\mathbb{V}}_R^{\hbar}$, is equal to the identity. So one can paraphrase Theorem 2.6 by saying that the log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ produce relations in the quantum tropical vertex group $\hat{\mathbb{V}}_R^{\hbar}$, or conversely that relations in $\hat{\mathbb{V}}_R^{\hbar}$ give constraints on the log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$.
- The automorphism $\hat{\Phi}_{\hat{H}_j}$ attached to the incoming rays $\mathfrak{d}_j$ of the quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ are conjugation by $e^{\hat{H}_{\mathfrak{d}_j}}$, i.e. by

$$\exp\left(\sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} t_j^{\ell} \hat{z}^{\ell m_j}\right),$$

  which can be written as $\Psi_q(-t_j \hat{z}^{m_j})$ where

$$\Psi_q(x) \coloneqq \exp\left(-\sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{x^{\ell}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}}\right) = \prod_{k \geqslant 0} \frac{1}{1 - q^{k+\frac{1}{2}} x},$$

  is the quantum dilogarithm[5]. We refer for example to [Zag07] for a nice review of the many aspects of the dilogarithm, including its quantum version.

  As the incoming rays of $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ are expressed in terms of quantum dilogarithms, it is natural to ask if the outgoing rays, which by Theorem 2.6 are generating series of higher genus log Gromov-Witten invariants, can be naturally expressed in terms of quantum dilogarithms. This question is related to the multicover/BPS structure of higher genus log Gromov-Witten theory and is fully answered by Theorem 3 in Section 2.8.

2.3.2 Examples

In this Section, we give some elementary examples illustrating Theorem 2.6.

Trivial scattering: propagation of a ray.

We take $n = 1$ and $\mathfrak{m} = (m_1)$ with $m_1 = (1,0) \in M = \mathbb{Z}^2$. In this case, $R = \mathbb{C}[[t_1]]$, and the quantum scattering diagram $\hat{\mathfrak{D}}_{\mathfrak{m}}$ contains a unique incoming ray: $\mathfrak{d}_1 = -\mathbb{R}_{\geqslant 0}(1,0) = \mathbb{R}_{\geqslant 0}(-1,0)$ equipped with

$$\hat{H}_{\mathfrak{d}_1} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2 \sin\left(\frac{\ell \hbar}{2}\right)} t_1^{\ell} \hat{z}^{(\ell,0)}.$$

[5]Warning: various conventions are used for the quantum dilogarithm throughout the literature.

Then the consistent scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ is obtained by simply propagating the incoming ray, i.e. by adding the outgoing ray $\mathbb{R}_{\geqslant 0}(1,0)$ equipped with

$$\hat{H}_{(1,0)} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} t_1^\ell \hat{z}^{(\ell,0)} .$$

We start with a fan consisting of the ray $\mathbb{R}_{\geqslant 0}(-1,0)$. To get a proper toric surface, we add to the fan the rays $\mathbb{R}_{\geqslant 0}(1,0)$, $\mathbb{R}_{\geqslant 0}(0,1)$ and $\mathbb{R}_{\geqslant 0}(0,-1)$. The corresponding toric surface $\overline{Y}_{\mathfrak{m}}$ is simply $\mathbb{P}^1 \times \mathbb{P}^1$. We get $Y_{\mathfrak{m}}$ by blowing-up a point on $\{0\} \times \mathbb{C}^*$, e.g. $\{0\} \times \{1\}$. Denote E the exceptional divisor and F the strict transform of $\mathbb{P}^1 \times \{1\}$. We have $E^2 = F^2 = -1$ and $E.F = 1$. For $\ell \in P = \mathbb{N}$, we have $\beta_\ell = \ell[F]$. So, according to Theorem 2.6, one should have, for every $\ell \geqslant 1$,

$$\sum_{g \geqslant 0} N_{g,\ell}^{Y_{\mathfrak{m}}} \hbar^{2g-1} = \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} .$$

As F is rigid, contributions to $N_{g,\ell}$ only come from ℓ to 1 multicoverings of F and the computation of $N_{g,\ell}$ can be reduced to a computation in relative Gromov-Witten theory of $\mathbb{P}^1$. Using Theorem 5.1 of [BP05], one can check that the above formula is indeed correct. We refer for more details to Lemma 2.20 which plays a crucial role in the proof of Theorem 2.6.

Simple scattering of two rays

We take $n = 2$ and $\mathfrak{m} = (m_1, m_2)$ with $m_1 = (1,0) \in M = \mathbb{Z}^2$ and $m_2 = (0,1) \in M = \mathbb{Z}^2$. In this case, $R = \mathbb{C}[[t_1, t_2]]$, and the quantum scattering diagram $\hat{\mathfrak{D}}_{\mathfrak{m}}$ contains two incoming rays $\mathfrak{d}_1 = \mathbb{R}_{\geqslant 0}(-1,0)$ and $\mathfrak{d}_2 = \mathbb{R}_{\geqslant 0}(0,-1)$, respectively equipped with

$$\hat{H}_{\mathfrak{d}_1} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} t_1^\ell \hat{z}^{(\ell,0)} ,$$

and

$$\hat{H}_{\mathfrak{d}_2} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} t_2^\ell \hat{z}^{(0,\ell)} .$$

Then, because of the Faddeev-Kashaev [FK94] pentagon identity

$$\Psi_q(z^{(1,0)})\Psi_q(z^{(0,1)}) = \Psi_q(z^{(0,1)})\Psi_q(z^{(1,1)})\Psi_q(z^{(1,0)})$$

satisfied by the quantum dilogarithm Ψ_q, the consistent scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ is obtained by propagation of the two incoming rays in outgoing rays, as in 2.3.2, and by addition of a third outgoing ray $\mathbb{R}_{\geqslant 0}(1,1)$ equipped with

$$\hat{H}_{(1,1)} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} t_1^\ell t_2^\ell \hat{z}^{(\ell,\ell)} .$$

We start with the fan consisting of the rays $\mathbb{R}_{\geqslant 0}(-1,0)$ and $\mathbb{R}_{\geqslant 0}(0,-1)$. To get a proper toric surface, we can for example add to the fan the ray $\mathbb{R}_{\geqslant 0}(1,1)$. The corresponding toric surface $\overline{Y}_{\mathfrak{m}}$ is simply $\mathbb{P}^2$, with its toric divisors D_1, D_2, D_3. We get $Y_{\mathfrak{m}}$ by blowing a point p_1 on D_1 and a point p_2 on D_2, both away from the torus fixed points. We denote E_1 and

E_2 the corresponding exceptional divisors and F the strict transform of the unique line in $\mathbb{P}^2$ passing through p_1 and p_2. We have $E_1^2 = E_2^2 = F^2 = -1$ and $E_1.F = E_2.F = 1$. For $\ell \in \mathbb{N}$ and $(\ell, \ell) \in P = \mathbb{N}^2$, we have $\beta_{(\ell,\ell)} = \ell[F]$. So according to Theorem 2.6, one should have, for every $\ell \geqslant 1$,

$$\sum_{g \geqslant 0} N_{g,(\ell,\ell)}^{Y_{\mathfrak{m}}} \hbar^{2g-1} = \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2 \sin\left(\frac{\ell \hbar}{2}\right)} .$$

As F is rigid, contributions to $N_{g,(\ell,\ell)}$ only come from ℓ to 1 multicoverings of F and the computations of $N_{g,(\ell,\ell)}$ reduces to a computation identical to the one used for $N_{g,\ell}$ in the case of trivial scattering.

More complicated scatterings

Already at the classical level of [GPS10], general scattering diagrams can be very complicated. A fortiori, general quantum scattering diagrams are extremely complicated. Direct computation of the higher genus log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ is a difficult problem in general. In particular, unlike what happens in the two previously described examples, linear systems defined by β_p and the tangency condition contain in general curves of positive genus, and so genus $g > 0$ stable log maps appearing in the moduli space defining $N_{g,p}^{Y_{\mathfrak{m}}}$ do not factor through genus zero curves in general. As consistent scattering diagrams can be algorithmically computed, one can view Theorem 2.6 as an answer to the problem of effectively computing the higher genus log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$.

2.3.3 Orbifold generalization

As in Section 5.5 and 5.6 of [GPS10] for the classical case, we can give an enumerative interpretation of quantum scattering diagrams more general than those considered in Theorem 2.6 if we allow ourself to work with orbifold Gromov-Witten invariants.

We fix $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors of $M = \mathbb{Z}^2$ and $\mathfrak{r} = (r_1, \dots, r_n)$ an n-tuple of positive integers. We denote $P \coloneqq \mathbb{N}^n$ and we take $R \coloneqq \mathbb{C}[[P]] \coloneqq \mathbb{C}[[t_1, \dots, t_n]]$ as complete local $\mathbb{C}$-algebra. Let $P_{\mathfrak{r}}$ be the set of $p = (p_1, \dots, p_n) \in P$ such that r_j divides p_j for every $1 \leqslant j \leqslant n$. Let $\hat{\mathfrak{D}}_{\mathfrak{m},\mathfrak{r}}$ be the quantum scattering diagram over R consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j})$, $1 \leqslant j \leqslant n$, where

$$\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0} m_j ,$$

and

$$\hat{H}_{\mathfrak{d}_j} = -i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2 \sin\left(\frac{r_j \ell \hbar}{2}\right)} t_j^{r_j \ell} \hat{z}^{r_j \ell m_j} = \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{r_j \ell}{2}} - q^{-\frac{r_j \ell}{2}}} t_j^{r_j \ell} \hat{z}^{r_j \ell m_j} ,$$

where $q = e^{i\hbar}$. Let $S(\hat{\mathfrak{D}}_{\mathfrak{m},\mathfrak{r}})$ be the corresponding consistent quantum scattering diagram given by Proposition 2.3, obtained by adding outgoing rays to $\hat{\mathfrak{D}}_{\mathfrak{m},\mathfrak{r}}$. For every $m \in M - \{0\}$, let $P_{\mathfrak{r},m}$ be the subset of $p = (p_1, \dots, p_n) \in P_{\mathfrak{r}}$ such that $\sum_{j=1}^n p_j m_j$ is positively collinear with m:

$$\sum_{j=1}^{n} p_j m_j = \ell_p m$$

for some $\ell_p \in \mathbb{N}$.

Recall that in Section 2.2.5, for every $\mathfrak{m} = (m_1, \dots, m_n)$ and $\mathfrak{r} = (r_1, \dots, r_n)$, we introduced an orbifold log Calabi-Yau surface $Y_{\mathfrak{m},\mathfrak{r}}$ and for every $p = (p_1, \dots, p_n) \in P_{\mathfrak{r}}$, we defined some genus g orbifold log Gromov-Witten $N_{g,p}^{Y_{\mathfrak{m},\mathfrak{r}}}$ of $Y_{\mathfrak{m},\mathfrak{r}}$.

Theorem 2.7. *For every $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in M, every $\mathfrak{r} = (r_1, \dots, r_n)$ an n-tuple of positive integers and for every $m \in M - \{0\}$ primitive, the Hamiltonian $\hat{H}_m$ attached to the outgoing ray $\mathbb{R}_{\geqslant 0} m$ in the consistent quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m},\mathfrak{r}})$ is given by*

$$\hat{H}_m = \left(-\frac{i}{\hbar}\right) \sum_{p \in P_{\mathfrak{r},m}} \left(\sum_{g \geqslant 0} N_{g,p}^{Y_{\mathfrak{m},\mathfrak{r}}} \hbar^{2g}\right) \left(\prod_{j=1}^n t_j^{p_j}\right) \hat{z}^{\ell_p m}.$$

Remarks:

- For $\mathfrak{r} = (1, \dots, 1)$, Theorem 2.7 reduces to Theorem 2.6.
- In the classical limit $\hbar \to 0$, Theorem 2.6 reduces to Theorem 5.6 of [GPS10].
- The proof of Theorem 2.7 is entirely parallel to the proof of its special case Theorem 2.6. The key point is that orbifold and logarithmic questions never interact in a non-trivial way. The only major needed modification is an orbifold version of the multicovering formula of Lemma 2.20. This is done in Lemma 2.25, Section 2.7.2.

2.3.4 More general quantum scattering diagrams

We still fix $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive vectors of $M = \mathbb{Z}^2$ and we continue to denote $P = \mathbb{N}^n$, so that $R = \mathbb{C}[[P]] = \mathbb{C}[[t_1, \dots, t_n]]$. One could try to further generalize Theorem 2.7 by starting with a quantum scattering diagram over R consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j})$, $1 \leqslant j \leqslant n$, where $\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0} m_j$, and where

$$\hat{H}_{\mathfrak{d}_j} = \sum_{\ell \geqslant 1} \hat{H}_{\mathfrak{d}_j, \ell} t_j^{\ell} \hat{z}^{\ell m_j},$$

for arbitrary

$$\hat{H}_{\mathfrak{d}_j, \ell} \in \mathbb{C}[[\hbar]].$$

In the classical limit $\hbar \to 0$, Theorem 5.6 of [GPS10], classical limit of our Theorem 2.7, is enough to give an enumerative interpretation of the resulting consistent scattering diagram in such generality. Indeed, the genus zero orbifold Gromov-Witten story takes as input classical Hamiltonians

$$H_r = \sum_{\ell \geqslant 1} \frac{(-1)^{\ell-1}}{r \ell^2} t^{r\ell} z^{r\ell} = \frac{1}{r}(tz)^r + \mathcal{O}((tz)^{r+1}),$$

for all $r \geqslant 0$, which form a basis of $\mathbb{C}[(tz)]$. In particular, at every finite order in $\mathfrak{m}_R$, every classical scattering diagram consisting of n incoming rays meeting at $0 \in \mathbb{R}^2$ coincides with a classical scattering diagram whose consistent completion has an enumerative interpretation in terms of genus zero orbifold Gromov-Witten invariants.

In the quantum story, because of the extra dependence in $\hbar$, things are more complicated. Theorem 2.7 only covers a class of Hamiltonians $\hat{H}_{\mathfrak{d}_j}$ whose form is dictated by the multi-covering structure of higher genus orbifold Gromov-Witten theory.

2.4 Gromov-Witten theory of toric surfaces

For every $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in $M = \mathbb{Z}^2$, we defined in Section 2.2.1 a log Calabi-Yau surface $Y_{\mathfrak{m}}$ obtained as blow-up of some toric surface $\overline{Y}_{\mathfrak{m}}$, and we introduced in Section 2.2.3 a collection of log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ of $Y_{\mathfrak{m}}$. In the present Section, we define analogue log Gromov-Witten invariants $N_{g,w}^{\overline{Y}_{\mathfrak{m}}}$ of the toric surface $\overline{Y}_{\mathfrak{m}}$. In the next Section 2.5, we will compare the invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ of $Y_{\mathfrak{m}}$ and $N_{g,w}^{\overline{Y}_{\mathfrak{m}}}$ of $\overline{Y}_{\mathfrak{m}}$.

2.4.1 Curve classes on toric surfaces

Recall from Section 2.2.1 that $\overline{Y}_{\mathfrak{m}}$ is a proper toric surface whose fan contains the rays $-\mathbb{R}_{\geqslant 0} m_j$ for $j = 1, \dots, n$. We denote $\partial \overline{Y}_{\mathfrak{m}}$ the union of toric divisors of $\overline{Y}_{\mathfrak{m}}$. We want to consider curves in $\overline{Y}_{\mathfrak{m}}$ meeting $\partial \overline{Y}_{\mathfrak{m}}$ in a number of prescribed points with prescribed tangency conditions and at one unprescribed point with prescribed tangency condition. In this Section, we explain how to parametrize the relevant curve classes in terms of the prescribed tangency conditions w_j at the prescribed points.

Let s be a positive integer and let $w = (w_1, \dots, w_s)$ be a s-tuple of non-zero vectors in M such that for every $r = 1, \dots, s$, there exists $1 \leqslant j \leqslant n$ such that $-\mathbb{R}_{\geqslant 0} w_r = -\mathbb{R}_{\geqslant 0} m_j$. In particular, the ray $-\mathbb{R}_{\geqslant 0} w_r$ belongs to the fan of $\overline{Y}_{\mathfrak{m}}$ and we denote D_{w_r} the corresponding toric divisor of $\overline{Y}_{\mathfrak{m}}$. Remark that we can have $D_{w_r} = D_{w_{r'}}$ even if $r \neq r'$. We denote $|w_r| \in \mathbb{N}$ the divisibility of $w_r \in M = \mathbb{Z}^2$, i.e. the largest positive integer k such that one can write $w_r = kv$ with $v \in M$. One should think about w_r as defining a toric divisor D_{w_r} and an intersection number $|w_r|$ with D_{w_r} for a curve in $\overline{Y}_{\mathfrak{m}}$.

We assume that $\sum_{r=1}^s w_r \neq 0$ and so we can uniquely write

$$\sum_{r=1}^s w_r = \ell_w m_w \,,$$

with $m_w \in M$ primitive and $\ell_w \in \mathbb{N}$.

We explain now how to define a curve class $\beta_w \in H_2(\overline{Y}_{\mathfrak{m}}, \mathbb{Z})$. In short, β_w is the class of a curve in $\overline{Y}_{\mathfrak{m}}$ having for every $r = 1, \dots, s$, an intersection point of intersection number $|w_r|$ with D_{w_r}, and exactly one other intersection point with the toric boundary $\partial \overline{Y}_{\mathfrak{m}}$.

More precisely, the vector $m_w \in M$ belongs to some cone of the fan of $\overline{Y}_{\mathfrak{m}}$ and we write the corresponding decomposition

$$m_w = a_w^L m_w^L + a_w^R m_w^R \,,$$

where m_w^L, $m_w^R \in M$ are primitive generators of rays of the fan of $\overline{Y}_{\mathfrak{m}}$ and where a_w^L, $a_w^R \in \mathbb{N}$. Remark that there is only one term in this decomposition if the ray $\mathbb{R}_{\geqslant 0} m_w$ coincides with one of the rays of the fan of $\overline{Y}_{\mathfrak{m}}$. Let D_w^L and D_w^R be the toric divisors of $\overline{Y}_{\mathfrak{m}}$ corresponding to the rays $\mathbb{R}_{\geqslant 0} m_w^L$ and $\mathbb{R}_{\geqslant 0} m_w^R$. Let $\beta_w \in H_2(\overline{Y}_{\mathfrak{m}}, \mathbb{Z})$ be determined by the following intersection numbers with the toric divisors:

- Intersection number with D_{w_r}, $1 \leqslant r \leqslant s$, distinct from D_w^L and D_w^R:

$$\beta_w . D_{w_r} = \sum_{r', D_{w_{r'}} = D_{w_r}} |w_{r'}| \,,$$

- Intersection number with D_w^L:

$$\beta_w . D_w^L = \ell_w a_w^L + \sum_{r, D_{w_r} = D_w^L} |w_r| \,.$$

- Intersection number with D_w^R:

$$\beta_w . D_w^R = \ell_w a_w^R + \sum_{r, D_{w_r} = D_w^R} |w_r| \,.$$

- Intersection number with every toric divisor D different from the D_{w_r}, $1 \leqslant r \leqslant s$, and from D_w^L and D_w^R: $\beta_w . D = 0$.

Such class $\beta_w \in H_2(\overline{Y}_{\mathfrak{m}}, \mathbb{Z})$ exists by standard toric geometry because of the relation

$$\sum_{r=1}^{s} w_r = \ell_w m_w \,.$$

2.4.2 Log Gromov-Witten invariant of toric surfaces

In the previous Section, given $w = (w_1, \dots, w_s)$ a s-tuple of non-zero vectors in M, we defined some positive integers ℓ_w, a_w^L, a_w^R, some toric divisors D_w^L and D_w^R of $\overline{Y}_{\mathfrak{m}}$, and a curve class $\beta_w \in H_2(\overline{Y}_{\mathfrak{m}}, \mathbb{Z})$.

We would like to consider genus g stable map $f\colon C \to \overline{Y}_{\mathfrak{m}}$ of class β_w, intersecting $\partial Y_{\mathfrak{m}}$ in $s+1$ points, s of them being intersection with D_{w_r} at a point of intersection number $|w_r|$ for $r = 1, \dots, s$, and the last one being a point of intersection number $\ell_w a_w^L$ with D_w^L and $\ell_w a_w^R$ with D_w^R. We also would like to fix the position of the s intersection numbers with the divisors D_{w_r}. It is easy to check that the expected dimension of this enumerative problem is g. As in Section 2.2.3, we will cut down the virtual dimension from g to zero by integration of the top lambda cass.

As in Section 2.2.3, to get proper moduli spaces, we work with stable log maps. We consider the divisorial log structure on $\overline{Y}_{\mathfrak{m}}$ defined by the toric divisor $\partial \overline{Y}_{\mathfrak{m}}$ and use it to view $\overline{Y}_{\mathfrak{m}}$ as a smooth log scheme. Let $\overline{M}_{g,w}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}})$ be the moduli space of genus g stable log maps to $\overline{Y}_{\mathfrak{m}}$, of class β_w, with $s+1$ tangency conditions along $\partial \overline{Y}_{\mathfrak{m}}$ defined by the $s+1$ vectors

$-w_1, \dots, -w_s, \ell_w m_w$ in M. It is a proper Deligne-Mumford stack of virtual dimension $g+s$ and it admits a virtual fundamental class

$$[\overline{M}_{g,w}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}})]^{\mathrm{virt}} \in A_{g+s}(\overline{M}_{g,w}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}}), \mathbb{Q}) .$$

For every $r = 1, \dots, s$, we have an evaluation map

$$\mathrm{ev}_r \colon \overline{M}_{g,w}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}}) \to D_{w_r} .$$

If $\pi \colon \mathcal{C} \to \overline{M}_{g,w}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}})$ is the universal curve, of relative dualizing sheaf ω_π, then the Hodge bundle $\mathbb{E} \coloneqq \pi_* \omega_\pi$ is a rank g vector bundle over $\overline{M}_{g,w}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}})$, of top Chern class $\lambda_g \coloneqq c_g(\mathbb{E})$.

We define

$$N_{g,w}^{\overline{Y}_{\mathfrak{m}}} \coloneqq \int_{[\overline{M}_{g,w}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}})]^{\mathrm{virt}}} (-1)^g \lambda_g \prod_{r=1}^{s} \mathrm{ev}_r^*(\mathrm{pt}_r) \in \mathbb{Q} ,$$

where $\mathrm{pt}_r \in A^1(D_{w_i})$ is the class of a point. It is a rigorous definition of the enumerative problem sketched at the beginning of this Section.

2.5 Degeneration from log Calabi-Yau to toric

2.5.1 Degeneration formula: statement

We fix $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in $M = \mathbb{Z}^2$. In Section 2.2.1, we defined a log Calabi-Yau surface $Y_{\mathfrak{m}}$ obtained as blow-up of some toric surface $\overline{Y}_{\mathfrak{m}}$. In Section 2.2.3, we introduced a collection of log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ of $Y_{\mathfrak{m}}$, indexed by n-tuples $p = (p_1, \dots, p_n) \in P = \mathbb{N}^n$. In Section 2.4.2, we defined log Gromov-Witten invariants $N_{g,w}^{\overline{Y}_{\mathfrak{m}}}$ of the toric surface $\overline{Y}_{\mathfrak{m}}$ indexed by s-tuples $w = (w_1, \dots, w_s) \in M^s$. The main result of the present Section, Proposition 2.8, is the statement of an explicit formula expressing the invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ in terms of the invariants $N_{g,p}^{\overline{Y}_{\mathfrak{m}}}$.

We first need to introduce some notations to relate the indices $p = (p_1, \dots, p_n)$ indexing the invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ and the indices $w = (w_1, \dots, w_s)$ indexing the invariants $N_{g,w}^{\overline{Y}_{\mathfrak{m}}}$. The way it goes is imposed by the degeneration formula in Gromov-Witten theory and hopefully will become conceptually clear in Section 2.5.4.

We fix $p = (p_1, \dots, p_n) \in P = \mathbb{N}^n$. We call k a partition of p, and we write $k \vdash p$, if k is an n-tuple $(k_1, \dots, k_n)$, with k_j a partition of p_j, for $1 \leqslant j \leqslant n$. We encode a partition k_j of p_j as a sequence $k_j = (k_{\ell j})_{\ell \geqslant 1}$ of non-negative integers, all zero except finitely many of them, such that

$$\sum_{\ell \geqslant 1} \ell k_{\ell j} = p_j .$$

Given k a partition of p, we denote

$$s(k) \coloneqq \sum_{j=1}^{n} \sum_{\ell \geqslant 1} k_{\ell j} .$$

We now define, given a partition k of p, a $s(k)$-tuple

$$w(k) = (w_1(k), \dots, w_{s(k)}(k))$$

of non-zero vectors in $M = \mathbb{Z}^2$, by the following formula:

$$w_r(k) \coloneqq \ell m_j$$

if

$$1 + \sum_{j'=1}^{j} \sum_{\ell'=1}^{\ell-1} k_{\ell' j'} \leqslant r \leqslant k_{\ell j} + \sum_{j'=1}^{j} \sum_{\ell'=1}^{\ell-1} k_{\ell' j'} \,.$$

In particular, for every $1 \leqslant j \leqslant n$ and $\ell \geqslant 1$, the $s(k)$-tuple $w(k)$ contains $k_{\ell j}$ copies of the vector $\ell m_j \in M$. Remark that because m_j is primitive in M, we have $\ell = |w_r(k)|$, where $|w_r(k)|$ is the divisibility of $w_r(k)$ in M. Remark also that

$$\sum_{r=1}^{s(k)} w_r(k) = \sum_{j=1}^{n} \sum_{\ell \geqslant 1} k_{\ell j} \ell m_j = \sum_{j=1}^{n} p_j m_j = \ell_p m_p \,,$$

and so, comparing notations of Sections 2.2.2 and 2.4.1, $\ell_{w(k)} = \ell_p$ and $m_{w(k)} = m_p$.

Using the above notations, we can now state Proposition 2.8.

Proposition 2.8. *For evey $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in $M = \mathbb{Z}^2$, and for every $p = (p_1, \dots, p_n) \in P = \mathbb{N}^n$, the log Gromov-Witten invariants $N_{g,p}^{Y_\mathfrak{m}}$ of the log Calabi-Yau surface $Y_\mathfrak{m}$ are expressed in terms of the log Gromov-Witten invariants $N_{g,w}^{\overline{Y}_\mathfrak{m}}$ of the toric surface $\overline{Y}_\mathfrak{m}$ by the following formula:*

$$\sum_{g \geqslant 0} N_{g,p}^{Y_\mathfrak{m}} \hbar^{2g-1}$$

$$= \sum_{k \vdash p} \left(\sum_{g \geqslant 0} N_{g,w(k)}^{\overline{Y}_\mathfrak{m}} \hbar^{2g-1+s(k)} \right) \prod_{j=1}^{n} \prod_{\ell \geqslant 1} \frac{1}{k_{\ell j}!} \ell^{k_{\ell j}} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{1}{2 \sin\left(\frac{\ell \hbar}{2}\right)} \right)^{k_{\ell j}} ,$$

where the first sum is over all partitions k of p.

The proof of Proposition 2.8 takes Sections 2.5.2, 2.5.3, 2.5.4, 2.5.5, 2.5.6. We consider the degeneration from $Y_\mathfrak{m}$ to $\overline{Y}_\mathfrak{m}$ introduced in Section 5.3 of [GPS10] and we apply a higher genus version of the argument of [GPS10]. Because the general degeneration formula in log Gromov-Witten theory is not yet available, we give a proof of the needed degeneration formula following the general strategy used in Chapter 1, which uses specific vanishing properties of the top lambda class.

2.5.2 Degeneration set-up

We first review the construction of the degeneration considered in Section 5.3 of [GPS10].

We fix $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in $M = \mathbb{Z}^2$. Recall from Section 2.2.1 that $\overline{Y}_\mathfrak{m}$ is a proper toric surface whose fan contains the rays $-\mathbb{R}_{\geqslant 0} m_j$ for

$j = 1, \dots, n$, and that we denote D_{m_j} the corresponding toric divisors. For every $j = 1, \dots, n$, we also choose a point x_j in general position on the toric divisor D_{m_j}. Let $\overline{Y}_{\mathfrak{m}} \times \mathbb{C} \to \mathbb{C}$ be the trivial family over $\mathbb{C}$ and let $\{x_j\} \times \mathbb{C}$ be the sections determined by the points x_j. Up to doing some toric blow-ups, which do not change the log Gromov-Witten invariants we are considering by [AW13], we can assume that the divisors D_{m_j} are disjoint.

The degeneration of $\overline{Y}_{\mathfrak{m}}$ to the normal cone of $D_{m_1} \cup \dots \cup D_{m_n}$,

$$\epsilon_{\overline{\mathcal{Y}}_{\mathfrak{m}}} \colon \overline{\mathcal{Y}}_{\mathfrak{m}} \to \mathbb{C} \,,$$

is obtained by blowing-up the loci $D_{m_1}, \dots, D_{m_n}$ over $0 \in \mathbb{C}$ in $\overline{Y}_{\mathfrak{m}} \times \mathbb{C}$. The special fiber is given by

$$\epsilon_{\overline{\mathcal{Y}}_{\mathfrak{m}}}^{-1}(0) = \overline{Y}_{\mathfrak{m}} \cup \bigcup_{j=1}^{n} \mathbb{P}_j \,,$$

where $N_{D_{m_j}|\overline{Y}_{\mathfrak{m}}}$ is the normal line bundle to D_{m_j} in $\overline{Y}_{\mathfrak{m}}$, and $\mathbb{P}_j$ is the projective bundle over D_{m_j} obtained by projectivization of the rank two vector bundle $\mathcal{O}_{D_{m_j}} \oplus N_{D_{m_j}|\overline{Y}_{\mathfrak{m}}}$ over D_{m_j}. The embeddings $\mathcal{O}_{D_{m_j}} \hookrightarrow \mathcal{O}_{D_{m_j}} \oplus N_{D_{m_j}|\overline{Y}_{\mathfrak{m}}}$ and $N_{D_{m_j}|\overline{Y}_{\mathfrak{m}}} \hookrightarrow \mathcal{O}_{D_{m_j}} \oplus N_{D_{m_j}|\overline{Y}_{\mathfrak{m}}}$ induce two sections of $\mathbb{P}_j \to D_{m_j}$ that we denote respectively $D_{m_j,\infty}$ and $D_{m_j,0}$. In $\epsilon_{\overline{\mathcal{Y}}_{\mathfrak{m}}}^{-1}(0)$, the divisor D_{m_j} in $\overline{Y}_{\mathfrak{m}}$ is glued to the divisor $D_{m_j,0}$ in $\mathbb{P}_j$. The strict transform of the section $\{x_j\} \times \mathbb{C}$ of $\overline{Y}_{\mathfrak{m}} \times \mathbb{C}$ is a section S_j of $\epsilon_{\overline{\mathcal{Y}}_{\mathfrak{m}}}$, whose intersection with $\epsilon_{\overline{\mathcal{Y}}_{\mathfrak{m}}}^{-1}(0)$ is a point $x_{j,\infty} \in D_{m_j,\infty}$.

We then blow-up the sections S_j, $j = 1, \dots, n$, in $\overline{\mathcal{Y}}_{\mathfrak{m}}$ and we obtain a family

$$\epsilon_{\mathcal{Y}_{\mathfrak{m}}} \colon \mathcal{Y}_{\mathfrak{m}} \to \mathbb{C},.$$

whose fibers away from zero are isomorphic to the surface $Y_{\mathfrak{m}}$, and whose special fiber is given by

$$\mathcal{Y}_{m,0} \coloneqq \epsilon_{\mathcal{Y}_{\mathfrak{m}}}^{-1}(0) = \overline{Y}_{\mathfrak{m}} \cup \bigcup_{j=1}^{n} \tilde{\mathbb{P}}_j \,,$$

where $\tilde{\mathbb{P}}_j$ is the blow-up of $\mathbb{P}_j$ at all the points $x_{j',\infty}$ such that $\mathbb{P}_{j'} = \mathbb{P}_j$. We denote $E_{j'}$ the corresponding exceptional divisor in $\mathbb{P}_j$ and $C_{j'}$ the strict transform in $\tilde{\mathbb{P}}_j$ of the unique $\mathbb{P}^1$-fiber of $\mathbb{P}_j \to D_{m_j}$ containing $x_{j',\infty}$. We have $E_{j'}.C_{j'} = 1$ in $\tilde{\mathbb{P}}_j$.

We would like to get Proposition 2.8 by application of a degeneration formula in log Gromov-Witten theory to the family

$$\epsilon_{\mathcal{Y}_{\mathfrak{m}}} \colon \mathcal{Y}_{\mathfrak{m}} \to \mathbb{C} \,,$$

to relate the invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ of the general fiber $Y_{\mathfrak{m}}$ to the invariants $N_{g,w}^{\overline{Y}_{\mathfrak{m}}}$ of $\overline{Y}_{\mathfrak{m}}$ which appears as component of the special fiber $\mathcal{Y}_{\mathfrak{m},0}$. In [GPS10], Gross-Pandharipande-Siebert work with an ad hoc definition of the genus 0 invariants as relative Gromov-Witten invariants of some open geometry and they only need to apply the usual degeneration formula in relative Gromov-Witten theory. In our present setting, with log Gromov-Witten invariants in arbitrary genus, we cannot follow exactly the same path.

Because the general degeneration formula in log Gromov-Witten theory is not yet available, we follow the strategy used in Chapter 1. We apply the decomposition formula of Abramovich-Chen-Gross-Siebert [ACGS17a], we use the vanishing property of the top lambda class to restrict the terms appearing in this formula and to prove a gluing formula by working only with torically transverse stable log maps. We review the decomposition formula of [ACGS17a] in Section 2.5.3. In Section 2.5.4, we identify the various terms contributing to the decomposition formula. In Section 2.5.5, we prove a gluing formula computing each of these terms. We finish the proof of Proposition 2.8 in Section 2.5.6.

2.5.3 Statement of the decomposition formula

We consider $\mathcal{Y}_{\mathfrak{m}}$ as a smooth log scheme for the divisorial log structure defined by the divisor $\mathcal{Y}_{\mathfrak{m},0}$ union the strict transforms of the divisors $\partial \overline{Y}_{\mathfrak{m}} \times \mathbb{C}$ in $\overline{Y}_{\mathfrak{m}} \times \mathbb{C}$ for $j = 1, \dots, n$. Considering $\mathbb{C}$ as a smooth log scheme for the divisorial log structure defined by the divisor $\{0\}$, we get that $\epsilon_{\mathcal{Y}_{\mathfrak{m}}}$ is a log smooth morphism. Restricting to the special fiber gives a structure of log scheme on $\mathcal{Y}_{\mathfrak{m},0}$ and a log smooth morphism to the standard log point $\mathrm{pt}_{\mathbb{N}}$ (the point $\{0\}$ equipped with the log structure restricted by $\{0\} \hookrightarrow \mathbb{C}$ of the divisorial log structure on $\mathbb{C}$):

$$\epsilon_{\mathcal{Y}_{\mathfrak{m},0}} \colon \mathcal{Y}_{\mathfrak{m},0} \to \mathrm{pt}_{\mathbb{N}} \,.$$

Let $\overline{M}_{g,p}(\mathcal{Y}_{\mathfrak{m},0})$ be the moduli space of genus g stable log maps to $\epsilon_{\mathcal{Y}_{\mathfrak{m},0}} \colon \mathcal{Y}_{\mathfrak{m},0} \to \mathrm{pt}_{\mathbb{N}}$, of class β_p, with a marked point of contact order $\ell_p m_p$. It is a proper Deligne-Mumford stack of virtual dimension g and it admits a virtual fundamental class

$$[\overline{M}_{g,p}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}} \in A_g(\overline{M}_{g,p}(\mathcal{Y}_{\mathfrak{m},0}), \mathbb{Q}) \,.$$

By deformation invariance of the virtual fundamental class on moduli spaces of stable log maps in log smooth families, we have

$$N_{g,p}^{Y_{\mathfrak{m}}} = \int_{[\overline{M}_{g,p}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}}} (-1)^g \lambda_g \,.$$

The decomposition formula of [ACGS17a] gives a decomposition of $[\overline{M}_{g,p}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}}$ indexed by tropical curves mapping to the tropicalization of $\mathcal{Y}_{\mathfrak{m},0}$. These tropical curves encode the intersection patterns of irreducible components of stable log maps mapping to the special fiber of the degeneration. We refer to Appendix B of [GS13] and Section 2 of [ACGS17a] for the general notion of tropicalization of a log scheme. We denote $\Sigma(X)$ the tropicalization of a log scheme X, it is a cone complex, i.e. an abstract gluing of cones.

We start by describing the tropicalization $\Sigma(\mathcal{Y}_{\mathfrak{m},0})$ of $\mathcal{Y}_{\mathfrak{m},0}$. Tropicalizing the log morphism $\epsilon_{\mathcal{Y}_{\mathfrak{m},0}} \colon \mathcal{Y}_{\mathfrak{m},0} \to \mathrm{pt}_{\mathbb{N}}$, we get a morphism of cone complexes $\Sigma(\epsilon_{\mathcal{Y}_{\mathfrak{m},0}}) \colon \Sigma(\mathcal{Y}_{\mathfrak{m},0}) \to \Sigma(\mathrm{pt}_{\mathbb{N}})$. We have $\Sigma(\mathrm{pt}_{\mathbb{N}}) = \mathbb{R}_{\geqslant 0}$ and $\Sigma(\mathcal{Y}_{\mathfrak{m},0})$ is naturally identified with the cone over the fiber $\Sigma(\epsilon_{\mathcal{Y}_{\mathfrak{m},0}})^{-1}(1)$ at $1 \in \mathbb{R}_{\geqslant 0}$. It is thus enough to describe the cone complex $\Sigma(\epsilon_{\mathcal{Y}_{\mathfrak{m},0}})^{-1}(1)$. We denote

$$\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}} \coloneqq \Sigma(\epsilon_{\mathcal{Y}_{\mathfrak{m},0}})^{-1}(1) \,.$$

The cone complex $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ is the tropicalization of

$$\mathcal{Y}_{\mathfrak{m},0} = \overline{Y}_{\mathfrak{m}} \cup \bigcup_{j=1}^{n} \tilde{\mathbb{P}}_j \,,$$

equipped with the divisorial log structure defined by the divisor $\partial \overline{Y}_{\mathfrak{m}} \cup \bigcup_{j=1}^n \partial \tilde{\mathbb{P}}_j$. In particular, it has one vertex v_0 dual to $\overline{Y}_{\mathfrak{m}}$ and vertices v_j dual to $\tilde{\mathbb{P}}_j$, $j = 1, \dots, n$. For every $j = 1, \dots, n$, there is an edge $e_{j,0}$ of integral length 1, connecting v_0 and v_j, dual to $D_{m_j,0}$, and an unbounded edge $e_{j,\infty}$ attached to v_j, dual to $D_{m_j,\infty}$.

The best way to understand $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ is probably to think about it as a modification of the tropicalization of $\overline{Y}_{\mathfrak{m}}$. As $\overline{Y}_{\mathfrak{m}}$ is simply a toric surface, its tropicalization $\Sigma(\overline{Y}_{\mathfrak{m}})$ can be naturally identified with $\mathbb{R}^2$ endowed with the fan decomposition. In particular, $\Sigma(\overline{Y}_{\mathfrak{m}})$ has one vertex $v_0 = 0 \in \mathbb{R}^2$ and unbounded edges $-\mathbb{R}_{\geqslant 0} m_j$, attached to v_0 and dual to the toric boundary divisors D_{m_j}. To go from $\Sigma(\overline{Y}_{\mathfrak{m}})$ to $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$, one adds a vertex v_j on each primitive integral point of $-\mathbb{R}_{\geqslant} m_j$, which has the effect to split $-\mathbb{R}_{\geqslant 0} v_j$ into a bounded edge $e_{j,0}$ and an unbounded edge $e_{j,\infty}$. One still has to cut along $e_{j,\infty}$ and to insert there two two-dimensional cones dual to the two "corners" of $\partial \tilde{\mathbb{P}}_j$ which are on $D_{m_j,\infty}$. In particular, for $j = 1, \dots, n$, the vertex v_j is 4-valent and looks locally as the fan of the Hirzebruch surface $\mathbb{P}_j$. In general, there is no global linear embedding of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ in $\mathbb{R}^2$.

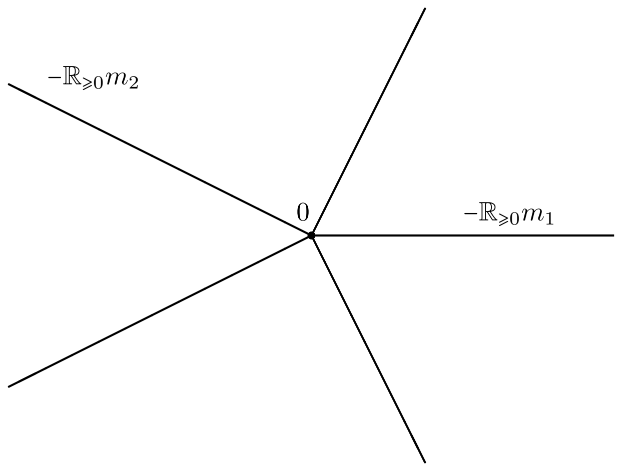


Figure: tropicalization of $\overline{Y}_{\mathfrak{m}}$.

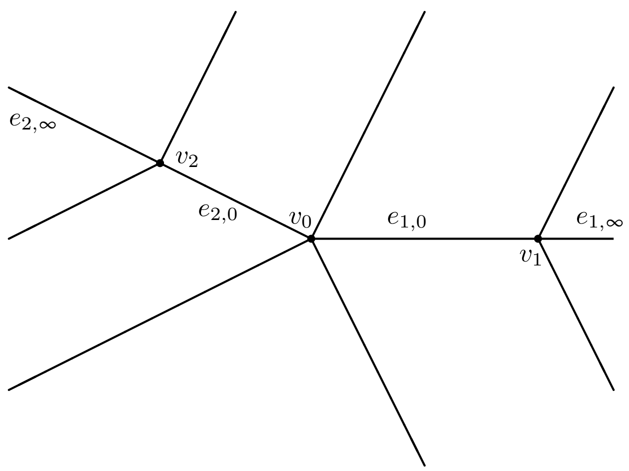


Figure: picture of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$.

We refer to Definition 2.5.3 of [ACGS17a] for the general definition of parametrized tropical curve $h: \Sigma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$. It is a natural generalization of the notion of parametrized tropical curve in $\mathbb{R}^2$ that we will use and review in Section 2.6.1. In particular, Σ is a graph, with bounded and unbounded edges mapped by h to $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ in an affine linear way and vertices V of Σ are decorated by some genus $g(V)$. The total genus g of the parametrized tropical curve is defined by $g_\Gamma + \sum_V g(V)$, where g_Γ is the genus of the graph Γ.

Some distinction between $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ and $\mathbb{R}^2$, related to the fact that the components $\tilde{\mathbb{P}}_j$ of $\mathcal{Y}_{\mathfrak{m},0}$ are non-toric, is that the usual form of the balancing condition for tropical curve in $\mathbb{R}^2$ is not necessarily valid at vertices of Γ mapping to one of the vertices v_j, $j = 1, \dots, n$, of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$. For vertices of Γ mapping away from v_j, $j = 1, \dots, n$, the usual balancing condition applies.

Following Definition 4.2.1 of [ACGS17a], a decorated parametrized tropical curve is a parametrized tropical curve $h: \Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ where each vertex has a further decoration by a curve class in the stratum of $\mathcal{Y}_{\mathfrak{m},0}$ dual to the stratum of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ where this vertex is mapped. In short, a decorated parametrized tropical curve to $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ encodes all the necessary combinatorial information to be a fiber of the tropicalization of a stable log maps to $\mathcal{Y}_{\mathfrak{m},0}$.

The decomposition formula of [ACGS17a] involves decorated parametrized tropical curves which are rigid in their combinatorial type. This is easy to understand intuitively: the decomposition formula is supposed to describe how the moduli space of stable log maps breaks into pieces under degeneration. If the moduli space of tropical curves were the tropicalization, and so the dual intersection complex, of the moduli space of stable log maps, components of the moduli space of stable log maps should be in bijection with the zero dimensional strata of the moduli space of tropical curves, i.e. with exactly rigid tropical curves. The decomposition formula [ACGS17a] expresses that this intuitive picture is correct, at least at the virtual level.

The tropical curves relevant in the study of $\overline{M}_{g,p}(\mathcal{Y}_{\mathfrak{m},0})$ are genus g decorated parametrized tropical curve $\Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ of type p, i.e. with only one unbounded edge, of weight ℓ_p and of direction m_p, and with total curve class β_p.

According to Section 4.4 of [ACGS17a], for every $h: \Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ rigid genus g decorated parametrized topical curve of type p, there exists a notion of stable log map marked by h, and a moduli space $\mathcal{Y}_{\mathfrak{m},0} \to \mathrm{pt}_{\mathbb{N}}$ of stable log maps marked by h, which is a proper Deligne-Mumford stack equipped with a virtual fundamental class $[\overline{M}_{g,p}^{h_k}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}}$. Forgetting the marking by h gives a morphism

$$i_h : \overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathfrak{m},0}) \to \overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathfrak{m},0}) \,.$$

We can finally state the decomposition formula, Theorem 4.8.1 of [ACGS17a]: we have

$$[\overline{M}_{g,p}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}} = \sum_h \frac{n_h}{|\mathrm{Aut}(h)|} (i_h)_* [\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}} \,,$$

where the sum is over rigid genus g decorated parametrized tropical curves $h: \Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ of type p, n_h is the smallest positive integer such that after scaling by n_h, h gets integral vertices and integral lengths, and $|\mathrm{Aut}(h)|$ is the order of the automorphism group of h.

2.5.4 Classification of rigid tropical curves

In order to extract some explicit information from the decomposition formula, the first step is to identify the rigid decorated parametrized tropical curves $h:\Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ of type p. It is in general a difficult question. But because we are only interested in invariants obtained by integration of the top lambda class λ_g, and not in the full virtual class, the situation is much simpler by the following Lemma.

Lemma 2.9. *Let $h:\Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ be a genus g rigid decorated parametrized tropical curve of type p with Γ of positive genus. Then we have*

$$\int_{[\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}}} (-1)^g \lambda_g = 0\,.$$

Proof. If $f:C \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ is a stable log map in $\overline{M}_{g,p}^{h}(\mathcal{Y}_{\mathfrak{m},0})$, then, by definition of the marking by h, the dual intersection complex of C retracts onto Γ and in particular, has genus bigger than the genus of Γ, which is positive by hypothesis. It follows that C contains a cycle of irreducible components. By Lemma 1.7, the class λ_g vanishes on families of curves containing cycles of irreducible components. ☐

By Lemma 2.9, we only have to determine the rigid decorated parametrized tropical curves $h:\Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ of type p with Γ of genus zero.

Recall that we defined in Section 2.5.1 what is a partition of p and that we associated to such partition k of p a positive integer $s(k)$ and a $s(k)$-tuple $(w_1,\dots,w_{s(k)})$ of non-zero vectors in $M=\mathbb{Z}^2$. In particular, each $w_r(k)$ can be naturally written $w_r(k)=\ell m_j$ for some $\ell \geqslant 0$ and some $1 \leqslant j \leqslant n$.

We first explain how to construct a genus g rigid decorated parametrized tropical curve $h_{k,\vec{g}}:\Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ with $\Gamma_{k,\vec{g}}$ of genus zero, for every partition k of p and for every $\vec{g} = (g_0,g_1,\dots,g_{s(k)})$, $(s(k)+1)$-tuple of non-negative integers such that $|\vec{g}| \coloneqq g_0 + \sum_{r=1}^{s(k)} g_r = g$. We refer to Section 2.5 of [ACGS17a] for details on the general notion of decorated parametrized tropical curve.

Let $\Gamma_{k,\vec{g}}$ be the genus zero graph[6] consisting of vertices $V_0,V_1,\dots,V_{s(k)}$, bounded edges E_r, $r=1,\dots,s(k)$, connecting V_0 to V_r, and an unbounded edge E_p attached to V_0.

We define a structure of tropical curve on $\Gamma_{k,\vec{g}}$ by assigning:

- Genera to the vertices. We assign g_0 to V_0, and g_r to V_r, for all $1 \leqslant r \leqslant s(k)$.
- Lengths to the bounded edges. We assign the length

$$\ell(E_r) \coloneqq \frac{1}{|w_r(k)|} = \frac{1}{\ell}$$

[6]We assume for simplicity that m_p does not coincide with any of the $-m_j$. If not, we need to add a 2-valent vertex V_p on E_p and we have $h_{k,\vec{g}}(V_p)=v_j$ for j such that $m_p=-m_j$.

to the bounded edge E_r, for all $1 \leqslant r \leqslant s(k)$.

Finally, we define a decorated parametrized tropical curve

$$h_{k,\vec{g}}\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$$

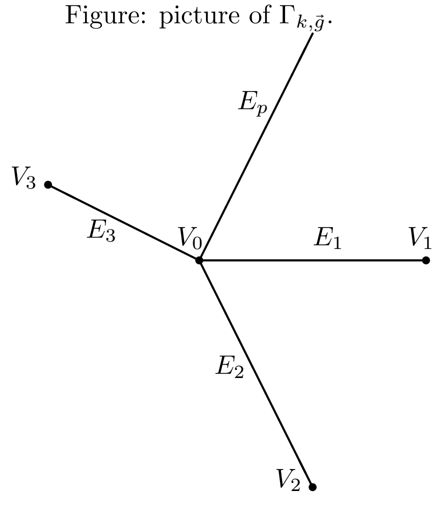
by the following data:

- We define $h_{k,\vec{g}}(V_0) \coloneqq v_0$, and, writing $w_r(k) = lm_j$, $h_{k,\vec{g}}(V_r) \coloneqq v_j$, for all $1 \leqslant r \leqslant s(k)$.
- Edge markings of bounded edges. We define $v_{V_0,E_r} \coloneqq w_r$ for all $1 \leqslant r \leqslant s(k)$. In particular, the bounded edge E_r has weight $|w_r(k)| = \ell$. This is a valid choice because

$$h(V_r) - h(V_0) = m_j = \frac{1}{\ell}\ell m_j = \ell(E_r) v_{V_0,E_r} \,.$$

  This uniquely specifies an affine linear map $h_{k,\vec{g}}|_{E_r}$.
- Edge marking of the unbounded edge. We define $v_{V_0,E_p} \coloneqq \ell_p m_p$. In particular, the unbounded edge E_p has weight ℓ_p. This uniquely specifies an affine linear map $h_{k,\vec{g}}|_{E_p}$.
- Decoration of vertices by curve classes. We decorate V_0 with the curve class $\beta_{w(k)} \in H_2(\overline{Y}_{\mathfrak{m}}, \mathbb{Z})$. Writing $w_r(k) = \ell m_j$, we decorate the vertex V_r with the curve class $\ell[C_j] \in H_2(\tilde{\mathbb{P}}_j, \mathbb{Z})$.

Figure: picture of $\Gamma_{k,\vec{g}}$.



Lemma 2.10. *The genus g decorated parametrized tropical curve*

$$h_{k,\vec{g}}\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$$

is rigid.

Proof. This is obvious because $h_{k,\vec{g}}$ has no contracted edge and all vertices of $h_{k,\vec{g}}$ are mapped to vertices of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$: it is not possible to deform $h_{k,\vec{g}}$ without changing its combinatorial type. □

Proposition 2.11. *Every genus g rigid decorated parametrized tropical curve $h: \Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ of type p, with Γ of genus zero, is of the form $h_{k,\vec{g}}$ for some k partition of p and $\vec{g} = (g_0, g_1, \dots, g_{s(k)})$ some $(s(k)+1)$-tuple of non-negative integers such that $|\vec{g}| = g$.*

Proof. The argument[7] is similar to the one used in the proof of Proposition 1.10, itself a tropical version of the properness argument, Proposition 4.2, of [GPS10]. By iterative application of the balancing condition, we will argue that the source Γ of a rigid decorated parametrized tropical curve $h: \Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ of type p not of the form $h_{k,\vec{g}}$, necessarily contains a closed cycle and so has positive genus.

Let $h: \Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ be a genus g rigid parametrized tropical curve of type p. As h is rigid, there is no edge of Γ contracted by h. The fact that h has type p implies that h has only one unbounded edge and this unbounded edge has weight ℓ_p and direction m_p.

Lemma 2.12. *Assume that there exists a vertex V of Γ such that*

$$h(V) \notin \{v_0, v_1, \dots, v_n\},$$

then Γ has positive genus.

Proof. We first assume that $h(V)$ is contained in the interior of one of the two-dimensional cones $\mathcal{C}$ of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$. Because $h(V)$ is away from the vertices v_j, the situation is locally toric and the balancing condition has to be satisfied in $h(V)$. If $h(V) \notin \mathbb{R}_{\geqslant 0} m_p$, there is no unbounded edge of Γ ending at V, and so by balancing, not all edges attached to $h(V)$ can point towards the vertex of $\mathcal{C}$, i.e. at least one edge of $\mathcal{C}$ points towards a boundary ray of $\mathcal{C}$. If $h(V) \in \mathbb{R}_{\geqslant 0} m_p$, we can get the same conclusion: if all edges passing through $h(V)$ were parallel to $\mathbb{R}_{\geqslant 0} m_p$, this would contradict the rigidity of h because one could move $h(V)$ along $\mathbb{R}_{\geqslant 0} m_p$.

Then, we follow the proof of Proposition 1.10. Fixing a cyclic orientation on the collection of cones and rays of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$, we can assume that this edge points towards the left (from the point of view of the vertex of the cone, looking inside the cone) ray of $\mathcal{C}$. If this edge ends on some vertex still contained in the interior of $\mathcal{C}$, then the balancing condition still applies and so there is still an edge attached to this vertex pointing towards the left ray of $\mathcal{C}$. Because Γ has finitely many vertices, iterating this construction finitely many times, we construct a path starting from $h(V)$ and ending at some vertex $h(V')$ on the left boundary ray of $\mathcal{C}$.

Let $\mathcal{C}'$ be the two-dimensional cone of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ adjacent to $\mathcal{C}$ near $h(V')$. Then we claim that by the balancing condition, there exists an edge attached to $h(V')$ pointing towards the left ray of $\mathcal{C}'$. Indeed, the only case for which the balancing condition is not a priori satisfied is if $h(V') = v_j$ for some j. But at v_j, the non-toric nature of $\tilde{\mathbb{P}}_j$ only modifies the balancing condition in the direction parallel to $e_{j,0}$ and $e_{j,\infty}$: if there is an incoming edge with non-zero transversal direction, then there is still an outgoing edge with non-zero transversal direction ($\tilde{\mathbb{P}}_j$ is obtained from the Hirzebruch surface $\mathbb{P}_j \to D_{m_j}$ by blowing-up points on the divisors

[7]We assume for simplicity that m_p is distinct from all $-m_j$. It is easy to adapt the argument in this special case.

$D_{m_j,\infty}$: this does not affect the fact that the general fibers of $\tilde{\mathbb{P}}_j \to D_{m_j}$ are still linearly equivalent).

Iterating this construction, we get a path in Γ whose image by h in $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ is a path which intersects successive rays in the anticlockwise order. Because Γ has finitely many edges, this path has to close eventually and so Γ contains a non-trivial closed cycle, i.e. Γ has positive genus.

It remains to treat the case where $h(V)$ is in the interior of a one dimensional ray of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$. If all the edges attached to $h(V)$ were parallel to the ray, this would contradict the ridigity of h because one could move $h(V)$ along the ray. So at least one of the edges attached to $h(V)$ is not parallel to the ray and by balancing, we can assume that there is an edge attached to $h(V)$ pointing towards the 2-dimensional cone of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ left to the ray. We can then apply the iterative argument described above. ☐

We continue the proof of Proposition 2.11. Let us assume that Γ has genus zero. By Lemma 2.12, every vertex V of Γ maps to one of the vertices $v_0, v_1, \dots, v_n$ of Γ. If there were an edge connecting a vertex mapped to v_j with a vertex mapped to $v_{j'}$, $1 \leqslant j, j' \leqslant n$ with $j \neq j'$, then we could apply the iterative argument used in the proof of Lemma 2.12, and this would contradict the assumption that Γ has genus zero.

It follows that every edge in Γ adjacent to some vertex mapped to v_j for some $1 \leqslant j \leqslant n$ is also adjacent to some vertex mapped to v_0. As Γ is connected and there is no contracted edges, this implies that there is a unique vertex V_0 of Γ such that $h(V_0) = v_0$. As Γ is of type p, the curve classes of the vertices mapping to v_j, $1 \leqslant j \leqslant n$, naturally define a partition k of p, the genera of the various vertices define some $\vec{g}$ and it is easy to check that $h = h_{k,\vec{g}}$. ☐

Define

$$N_{g,p}^{h_{k,\vec{g}}} \coloneqq \int_{[\overline{M}_{g,p}^{h_{k,\vec{g}}}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}}} (-1)^g \lambda_g \in \mathbb{Q}\,.$$

Proposition 2.13. *We have*

$$N_{g,p}^{Y_{\mathfrak{m}}} = \sum_{k \vdash p} \sum_{\substack{\vec{g} \\ |\vec{g}|=g}} \frac{n_{h_{k,\vec{g}}}}{|\operatorname{Aut}(h_{k,\vec{g}})|} N_{g,p}^{h_{k,\vec{g}}}\,.$$

Proof. This follows from integrating $(-1)^g \lambda_g$ over the decomposition formula of [ACGS17a], reviewed at the end of Section 2.5.3. By Lemma 2.9, rigid tropical curves $h\colon \Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$, with Γ of positive genus, do not contribute, and by Proposition 2.11, all the relevant rigid tropical curves $h\colon \Gamma \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ are of the form $h_{k,\vec{g}}\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ for some k partition of p and some $\vec{g}$ such that $|\vec{g}| = g$. ☐

We fix k a partition of p and $\vec{g}$ such that $|\vec{g}| = g$ and we consider the decorated parametrized tropical curve $h\colon \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$.

Lemma 2.14. *We have*

$$n_{h_{k,\vec{g}}} = \operatorname{lcm}\{|w_r(k)|, 1 \leqslant r \leqslant s(k)\}\,.$$

Proof. Recall that $n_{h_{k,\vec{g}}}$ is the smallest positive integer such that after scaling by $n_{k_{k,\vec{g}}}$, $h_{k,\vec{g}}$ gets integral vertices and integral lengths. By definition of $h_{k,\vec{g}}$, vertices of $h_{k,\vec{g}}$ are already mapped to integral points of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$. On the other hand, bounded edges E_r of $\Gamma_{k,\vec{g}}$ have fractional lengths $1/|w_r(k)|$. It follows that $n_{k,\vec{g}}$ is the least common multiple of the positive integers $|w_r(k)|$, $1 \leqslant r \leqslant s(k)$. □

For $1 \leqslant j \leqslant n$, $\ell \geqslant 1$ and $a \geqslant 0$, denote $k_{\ell j a}$ the number of vertices of $\Gamma_{k,\vec{g}}$ having genus a among the $k_{\ell j}$ ones having curve class decoration $\ell[C_j]$. Remark that we have

$$k_{\ell j} = \sum_{a \geqslant 0} k_{\ell j a} ,$$

and

$$\sum_{r=1}^{s(k)} g_r = \sum_{j=1}^{n} \sum_{\ell \geqslant 1} \sum_{a \geqslant 0} a k_{\ell j a} .$$

Lemma 2.15. *The order of the automorphism group of the decorated parametrized tropical curve $h_{k,\vec{g}}: \Gamma_{k,\vec{g}} \to \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$ is given by*

$$|\operatorname{Aut}(h_{k,\vec{g}})| = \prod_{j=1}^{n} \prod_{\ell \geqslant 1} \prod_{a \geqslant 0} k_{\ell j a}! .$$

Proof. For every $1 \leqslant j \leqslant n$, $\ell \geqslant 1$ and $a \geqslant 0$, there are $k_{\ell j a}$ of the vertices V_r having the same curve class decoration $\ell[C_j]$, the same genus a, and the attached edges have the same weight $ellm_j$, so permutations of these $k_{\ell j a}$ vertices define automorphisms of the decorated tropical curve $h_{k,\vec{g}}$. Any other permutation of the vertices of Σ_k permutes vertices having different curve class decorations and/or different genus, and so cannot be an automorphism of the decorated tropical curve. □

Corollary 2.16. *We have*

$$N_{g,p}^{Y_{\mathfrak{m}}} = \sum_{k \vdash p} \sum_{\substack{\vec{g} \\ |\vec{g}|=g}} (\operatorname{lcm}\{|w_r(k)|, 1 \leqslant r \leqslant s(k)\}) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \prod_{a \geqslant 0} \frac{1}{k_{\ell j a}!} \right) N_{g,p}^{h_{k,\vec{g}}} .$$

Proof. Combination of Proposition 2.13, Lemma 2.14 and Lemma 2.15. □

2.5.5 Gluing formula

The previous Section has reduced the computation of the log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ to the computation of invariants

$$N_{g,p}^{h_{k,\vec{g}}} \coloneqq \int_{[\overline{M}_{g,p}^{h_{k,\vec{g}}}(\mathcal{Y}_{\mathfrak{m},0})]^{\mathrm{virt}}} (-1)^g \lambda_g .$$

where $\overline{M}_{g,p}^{h_{k,\vec{g}}}(\mathcal{Y}_{\mathfrak{m},0})$ is a moduli space of stable log maps to $\mathcal{Y}_{\mathfrak{m},0}$ marked by $h_{k,\vec{g}}$, i.e. whose tropicalization is equipped with a retraction on $h_{k,\vec{g}}$.

Let $f: C \to \mathcal{Y}_{\mathfrak{m},0}$ be a stable log map of tropicalization $h_{k,\bar{g}}$. Then C has irreducible components $C_0, C_1, \dots, C_{s(k)}$, of genus $g_0, g_1, \dots, g_{s(k)}$ and we have

- $f|_{C_0}$ is a genus g_0 stable map to $\overline{Y}_{\mathfrak{m}}$ of class $\beta_{w(k)}$, transverse to $\partial \overline{Y}_{\mathfrak{m}}$, with $s+1$ tangency conditions along $\partial \overline{Y}_{\mathfrak{m}}$ defined by the $s+1$ vectors $-w_1, \dots, -w_s, \ell_p m_p \in M$[8].
- For all $1 \leqslant r \leqslant s(k)$, $w_r(k) = l m_j$, $f|_{C_r}$ is a genus g_r stable map to $\tilde{\mathbb{P}}_j$, of class $\ell[C_j]$, with a full tangency condition of order ℓ along $D_{m_j,0}$.

This suggests to consider the moduli space $\overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial \tilde{\mathbb{P}}_j)$ of genus a stable log maps to $\tilde{\mathbb{P}}_j$, equipped with the divisorial log structure with respect to the divisor $\partial \tilde{\mathbb{P}}_j$, of class $\ell[C_j]$, with a full tangency condition of order ℓ along D_{m_j}. It is a proper Deligne-Mumford stack of virtual dimension a, admitting a virtual fundamental class

$$[\overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial \tilde{\mathbb{P}}_j)]^{\mathrm{virt}} \in A_a(\overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial \tilde{\mathbb{P}}_j), \mathbb{Q}) .$$

We define

$$N_a^{\ell \tilde{\mathbb{P}}_j} \coloneqq \int_{[\overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial \tilde{\mathbb{P}}_j)]^{\mathrm{virt}}} (-1)^a \lambda_a \in \mathbb{Q} .$$

The decomposition of a stable log map $f: C \to \mathcal{Y}_{\mathfrak{m},0}$ into irreducible components suggests that we should be able to express $N_{g,p}^{h_{k,\bar{g}}}$ in terms of $N_{g_0,w(k)}^{\overline{Y}_{\mathfrak{m}}}$ and $N_a^{\ell \tilde{\mathbb{P}}_j}$.

The following Proposition 2.17 gives a gluing formula showing that it is indeed the case.

Proposition 2.17. *We have*

$$N_{g,p}^{h_{k,\bar{g}}} = \frac{N_{g_0,w(k)}^{\overline{Y}_{\mathfrak{m}}}}{\mathrm{lcm}\{|w_r(k)|, 1 \leqslant r \leqslant s(k)\}} \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \ell^{k_{\ell j}} \prod_{a \geqslant 0} (N_a^{\ell \tilde{\mathbb{P}}_j})^{k_{\ell j a}} \right) .$$

Proof. We gave a brief description of stable log maps whose tropicalization is $h_{k,\bar{g}}$ as a motivation for why a gluing formula like Proposition 2.17 should be true. But the moduli space of such stable log maps is not proper. The relevant proper moduli space $\overline{M}_{g,p}^{h_{k,\bar{g}}}(\mathcal{Y}_{\mathfrak{m},0})$ is a moduli space of stable log maps marked by $h_{k,\bar{g}}$, containing stable log maps whose tropicalization only retracts onto $h_{k,\bar{g}}$. These stable log maps interact in a complicated way with the log structure of $\mathcal{Y}_{\mathfrak{m},0}$ and the gluing of such stable log maps has not been worked out yet.

We go around this issue by following the strategy used in Section 1.6. On an open locus of torically transverse stable maps, the above mentioned problems do not arise and the difficulty of the gluing problem is of the same level as the usual degeneration formula in relative Gromov-Witten theory. The log version of this gluing problem has been recently treated in full details by Kim, Lho and Ruddat [KLR18]. On the complement of the nice locus of torically transverse stable log maps, a combinatorial argument of Proposition 1.10 implies that one of the relevant curves will always contain a non-trivial cycle of components.

[8]For simplicity, we are assuming that m_p is distinct from all $-m_j$. It is easy to adapt the argument in this special case. The gluing formula remains unchanged, for the same reason that 2-valent vertices play a trivial role in Chapter 1: see Lemma 1.14.

By standard vanishing properties of the lambda class, it follows that we can ignore this bad locus if we only care about numerical invariants obtained by integration of a top lambda class, which is our case.

We give now an outline of the proof, referring to [KLR18] and Section 1.6 for some of the steps.

We have an evaluation morphism

$$\mathrm{ev}\colon \overline{M}_{g_0,w(k)}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}}) \times \prod_{\substack{1\leqslant j\leqslant n\\ \ell\geqslant 1\\ a\geqslant 0}} \overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial\tilde{\mathbb{P}}_j)^{k_{\ell j a}} \to \prod_{r=1}^{s(k)} (D_{w_r(k)})^2 \,.$$

Let

$$\delta\colon \prod_{r=1}^{s(k)} D_{w_r(k)} \to \prod_{r=1}^{s(k)} (D_{w_r(k)})^2$$

be the diagonal morphism. Using the morphisms ev and δ, we define the fiber product

$$\mathcal{M} \coloneqq \left(\overline{M}_{g_0,w(k)}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}}) \times \prod_{\substack{1\leqslant j\leqslant n\\ \ell\geqslant 1\\ a\geqslant 0}} \overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial\tilde{\mathbb{P}}_j)^{k_{\ell j a}} \right) \times_{\left(\prod_{r=1}^{s(k)} (D_{w_r(k)})^2\right)} \left(\prod_{r=1}^{s(k)} D_{w_r(k)} \right) .$$

We define a cycle class $[\mathcal{M}]^{\mathrm{virt}}$ on $\mathcal{M}$ by

$$[\mathcal{M}]^{\mathrm{virt}} \coloneqq \delta^! \left([\overline{M}_{g_0,w(k)}(\overline{Y}_{\mathfrak{m}}, \partial \overline{Y}_{\mathfrak{m}})]^{\mathrm{virt}} \times \prod_{\substack{1\leqslant j\leqslant n\\ \ell\geqslant 1\\ a\geqslant 0}} [\overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial\tilde{\mathbb{P}}_j)^{k_{\ell j a}}]^{\mathrm{virt}} \right),$$

where $\delta^!$ is the refined Gysin morphism (see Section 6.2 of [Ful98]) defined by δ.

The following Lemma will play for us the same role played by Lemma 1.16 in Section 1.6.

Lemma 2.18. *Let*

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathcal{Y}_{\mathfrak{m},0} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \epsilon_{\mathcal{Y}_{\mathfrak{m},0}}} \\ W & \xrightarrow{g} & \mathrm{pt}_{\mathbb{N}} \,, \end{array}$$

be a point of $\overline{M}_{g,p}^{h_k,\bar{g}}(\mathcal{Y}_{\mathfrak{m},0})$. *Let*

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(\mathcal{Y}_{\mathfrak{m},0}) \\ \downarrow{\scriptstyle \Sigma(\pi)} & & \downarrow{\scriptstyle \Sigma(\epsilon_{\mathcal{Y}_{\mathfrak{m},0}})} \\ \Sigma(W) & \xrightarrow{\Sigma(g)} & \Sigma(\mathrm{pt}_{\mathbb{N}}) \,. \end{array}$$

be its tropicalization. For every $b \in \Sigma(g)^{-1}(1)$, *let*

$$\Sigma(f)_b \colon \Sigma(C)_b \to \Sigma(\epsilon_{\mathcal{Y}_{\mathfrak{m},0}})^{-1}(1) = \mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$$

be the fiber of $\Sigma(f)$ *over* b. *For every* $r = 1, \dots, s(k)$, *let* $E_r^{\Sigma(f)_b}$ *be the edge of* $\Sigma(f)_b$ *marked by the edge* E_r *of* $\Gamma_{k,\bar{g}}$. *Then, we have*

$$h(E_r^{\Sigma(f)_b}) \subset h_{k,\bar{g}}(E_r)\,.$$

Proof. This follows from the fact that for every $1 \leqslant j \leqslant n$, the curve C_j is rigid in $\tilde{\mathbb{P}}_j$, so the vertices of Γ marked by the vertex V_r of $\Gamma_{k,\bar{g}}$ are mapped on $h_{k,\bar{g}}(E_r)$. ☐

Given a stable log map $f\colon C \to \mathcal{Y}_{\mathfrak{m},0}$ marked by $h_{k,\bar{g}}$, we have nodes of C in correspondence with the bounded edges of Γ. Cutting C along these nodes, we obtain a morphism

$$\mathrm{cut}\colon \overline{M}_{g,p}^{h_{k,\bar{g}}}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}) \to \mathcal{M}\,.$$

Because of Lemma 2.18, each cut is locally identical to the corresponding cut in a degeneration along a smooth divisor and so we can refer to Section 1.6 or Section 5 of [KLR18] for a precise definition of the cut morphism, dealing with log structures.

We say that a stable log map $f\colon C \to \overline{Y}_{\mathfrak{m}}$ is torically transverse if its image does not contain any of the torus fixed points of $\overline{Y}_{\mathfrak{m}}$, i.e. if its image does not pass through the "corners" of the toric boundary divisor $\partial\overline{Y}_{\mathfrak{m}}$, i.e. if its tropicalization has no vertex mapping in the interior of one of the two-dimensional cones of the fan of $\overline{Y}_{\mathfrak{m}}$.

Let $\overline{M}^0_{g_0,w(k)}(\overline{Y}_{\mathfrak{m}}, \partial\overline{Y}_{\mathfrak{m}})$ be the open substack of $\overline{M}_{g_0,w(k)}(\overline{Y}_{\mathfrak{m}}, \partial\overline{Y}_{\mathfrak{m}})$ consisting of torically transverse stable log maps. We define

$$\mathcal{M}^0 \coloneqq \left(\overline{M}^0_{g_0,w(k)}(\overline{Y}_{\mathfrak{m}}, \partial\overline{Y}_{\mathfrak{m}}) \times \prod_{\substack{1 \leqslant j \leqslant n \\ \ell \geqslant 1 \\ a \geqslant 0}} \overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial\tilde{\mathbb{P}}_j)^{k_{\ell j a}} \right) \times_{\left(\prod_{r=1}^{s(k)} (D_{w_r(k)})^2\right)} \left(\prod_{r=1}^{s(k)} D_{w_r(k)} \right),$$

$$\overline{M}_{g,p}^{h_{k,\bar{g}},0}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}) \coloneqq \mathrm{cut}^{-1}(\mathcal{M}^0)\,,$$

and we denote

$$\mathrm{cut}^0\colon \overline{M}_{g,p}^{h_{k,\bar{g}},0}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}) \to \mathcal{M}^0$$

the corresponding restriction of the cut morphism.

Lemma 2.19. *The morphism*

$$\mathrm{cut}^0\colon \overline{M}_{g,p}^{h_{k,\bar{g}},0}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}) \to \mathcal{M}^0$$

is étale of degree

$$\frac{\prod_{j=1}^n \prod_{\ell \geqslant 1} \ell^{k_{\ell j}}}{\mathrm{lcm}\{|w_r(k)|, 1 \leqslant r \leqslant s(k)\}}\,.$$

Proof. Because of the restriction to the torically transverse locus, the gluing question is locally isomorphic to the corresponding gluing question in a degeneration along a smooth divisor, and so the result follows from formula (6.13) and Lemma 9.2 of [KLR18][9]. □

Restricted to the torically transverse locus, the comparison of obstruction theories on $\overline{M}_{g,p}^{h_k,\vec{g}}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})$ and $\mathcal{M}$ reduces to the same question studied in Section 9 of[KLR18] for a degeneration along a smooth divisor. In particular, combining Lemma 2.19 with formula 9.14 of [KLR18], we obtain that the cycle classes

$$(\mathrm{cut})_*([\overline{M}_{g,p}^{h_k,\vec{g}}(Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}})]^{\mathrm{virt}})$$

and

$$\frac{\prod_{j=1}^n \prod_{\ell\geqslant 1} \ell^{k_{\ell j}}}{\mathrm{lcm}\{|w_r(k)|, 1\leqslant r\leqslant s(k)\}}[\mathcal{M}]^{\mathrm{virt}}$$

have the same restriction to the open substack $\mathcal{M}^0$ of $\mathcal{M}$. It follows by [Ful98] Proposition 1.8, that their difference is rationally equivalent to a cycle supported on the closed substack

$$Z \coloneqq \mathcal{M} - \mathcal{M}^0 .$$

At a point of Z, the corresponding stable log map $f\colon C\to \overline{Y}_{\mathfrak{m}}$ to $\overline{Y}_{\mathfrak{m}}$ is not torically transverse. Using Lemma 2.18, we can apply Proposition 1.10 to get that C contains a non-trivial cycle of components. Vanishing properties of lambda classes given by Lemma 1.7, combined with gluing properties of lambda classes given by Lemma 1.6, finally imply the gluing formula stated in Proposition 2.17, as in the end of Section 1.6. □

Remark: The most general form of the gluing formula in log Gromov-Witten theory, work in progress of Abramovich-Chen-Gross-Siebert, requires the use of punctured Gromov-Witten invariants, see [ACGS17b]. We do not see punctured invariants in our gluing formula because we only consider rigid tropical curves contained in the polyhedral decomposition of $\mathcal{Y}_{\mathfrak{m},0}^{\mathrm{trop}}$.

2.5.6 End of the proof of the degeneration formula

We now finish the proof of the degeneration formula, Proposition 2.8.

Combining Corollary 2.16 with Proposition 2.17, we get

$$N_{g,p}^{Y_{\mathfrak{m}}} = \sum_{k\vdash p} \sum_{\substack{\vec{g} \\ |\vec{g}|=g}} N_{g_0,w(k)}^{\overline{Y}_{\mathfrak{m}}} \left(\prod_{j\geqslant 1}\prod_{\ell\geqslant 1} \ell^{k_{\ell j}} \prod_{a\geqslant 0} \frac{1}{k_{\ell j a}!} (N_a^{\ell\tilde{\mathbb{P}}_j})^{k_{\ell j a}} \right).$$

Denote

$$F^{\ell\tilde{\mathbb{P}}_j}(\hbar) \coloneqq \sum_{a\geqslant 0} N_a^{\ell\tilde{\mathbb{P}}_j} \hbar^{2a-1} .$$

[9] In the corresponding argument in Section 1.6, the denominator of the formula did not appear because the relevant tropical curves had all edges of integral length.

We have

$$(F^{\ell\tilde{\mathbb{P}}_j}(\hbar))^{k_{\ell j}} = \sum_{k_{\ell j}=\sum_{a\geqslant 0} k_{\ell j a}} \frac{k_{\ell j}!}{\prod_{a\geqslant 0} k_{\ell j a}!} \left(\prod_{a\geqslant 0} (N_a^{\ell\tilde{\mathbb{P}}_j})^{k_{\ell j a}} \right) \hbar^{\sum_{a\geqslant 0}(2a-1)k_{\ell j a}} .$$

Using $k_{\ell j} = \sum_{a\geqslant 0} k_{\ell j a}$, $s(k) = \sum_{j=1}^n \sum_{\ell \geqslant 1} k_{\ell j}$ and $g - g_0 = \sum_{j=1}^n \sum_{\ell\geqslant 0} \sum_{a\geqslant} a k_{\ell j a}$ to count the powers of $\hbar$, we get

$$\sum_{g\geqslant 0} N_{g,p}^{Y_{\mathrm{m}}} \hbar^{2g-1} = \sum_{k\vdash p} \left(\sum_{g\geqslant 0} N_{g,w(k)}^{\overline{Y}_{\mathrm{m}}} \hbar^{2g-1+s(k)} \right) \prod_{j=1}^n \prod_{\ell\geqslant 1} \frac{1}{k_{\ell j}!} \ell^{k_{\ell j}} (F^{\ell\tilde{\mathbb{P}}_j}(\hbar))^{k_{\ell j}} .$$

It follows that the proof of the degeneration formula, Proposition 2.8, is finished by the following Lemma.

Lemma 2.20. *For every $1 \leqslant j \leqslant n$ and $\ell \geqslant 1$, we have*

$$F^{\ell\tilde{\mathbb{P}}_j}(\hbar) = \frac{(-1)^{\ell-1}}{\ell} \frac{1}{2\sin\left(\frac{\ell\hbar}{2}\right)} .$$

Proof. It is a higher genus version of Proposition 5.2 of [GPS10]. As the curve $C_j \simeq \mathbb{P}^1$ is rigid in $\tilde{\mathbb{P}}_j$, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, every stable log map, element of $\overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial\tilde{\mathbb{P}}_j)$, factors through $C_j \simeq \mathbb{P}^1$.

Let $\overline{M}_{a,\ell}(\mathbb{P}^1/\infty)$ be the moduli space of genus a stable log maps to $\mathbb{P}^1$, relative to $\infty \in \mathbb{P}^1$, of degree ℓ and with maximal tangency order ℓ along ∞. It has virtual dimension $2a-1+\ell$.

We have $\overline{M}_{a,\ell}(\tilde{\mathbb{P}}_j, \partial\tilde{\mathbb{P}}_j) = \overline{M}_{a,\ell}(\mathbb{P}^1/\infty)$ as stacks but their natural obstruction theories are different. Denoting $\pi: \mathcal{C} \to \overline{M}_{a,\ell}(\mathbb{P}^1/\infty)$ the universal source log curve and $f: \mathcal{C} \to \mathbb{P}^1$ the universal log map, the two obstruction theories differ by $R^1\pi_* f^* N_{C_j|\tilde{\mathbb{P}}_j} = R^1\pi_* f^* \mathcal{O}_{\mathbb{P}^1}(-1)$. So we obtain

$$N_a^{\ell\tilde{\mathbb{P}}_j} = \int_{[\overline{M}_{a,\ell}(\mathbb{P}^1/\infty)]^{\mathrm{virt}}} e\left(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\right) ,$$

where $e(-)$ is the Euler class. We are now in a setting relative to a smooth divisor so numerical invariants extracted from log Gromov-Witten theory coincide with those extracted from relative Gromov-Witten theory by [AMW12]. These integrals in relative Gromov-Witten theory have been computed by Bryan and Pandharipande ([BP05], see proof of Theorem 5.1) and the result is

$$\sum_{a\geqslant 0} N_a^{\ell\tilde{\mathbb{P}}_j} \hbar^{2a-1} = \frac{(-1)^{\ell-1}}{\ell} \frac{1}{2\sin\left(\frac{\ell\hbar}{2}\right)} .$$

□

2.6 Scattering and tropical curves

In this Section, we review the connection established in [FS15] between quantum scattering diagrams and refined tropical curve counting.

2.6.1 Refined tropical curve counting

In this Section, we review the definition of the refined tropical curve counts used in [FS15]. The relevant tropical curves are identical to those considered in [GPS10]. The only difference is that they are counted with the Block-Göttsche refined multiplicity [BG16], q-deformation of the usual Mikhalkin multiplicity [Mik05].

We first recall the definition of a parametrized tropical curve to $\mathbb{R}^2$ by simply repeating the presentation we gave in Chapter 1.

For us, a graph Γ has a finite set $V(\Gamma)$ of vertices, a finite set $E_f(\Gamma)$ of bounded edges connecting pairs of vertices and a finite set $E_\infty(\Gamma)$ of legs attached to vertices that we view as unbounded edges. By edge, we refer to a bounded or unbounded edge. We will always consider connected graphs.

A parametrized tropical curve $h: \Gamma \to \mathbb{R}^2$ is the following data:

- A nonnegative integer $g(V)$ for each vertex V, called the genus of V.
- A labeling of the elements of the set $E_\infty(\Gamma)$.
- A vector $v_{V,E} \in \mathbb{Z}^2$ for every vertex V and E an edge adjacent to V. If $v_{V,E}$ is not zero, the divisibility $|v_{V,E}|$ of $v_{V,E}$ in $\mathbb{Z}^2$ is called the weight of E and is denoted $w(E)$. We require that $v_{V,E} \neq 0$ if E is unbounded and that for every vertex V, the following balancing condition is satisfied:
$$\sum_E v_{V,E} = 0 \,,$$
where the sum is over the edges E adjacent to V. If E is an unbounded edge, we denote v_E for $v_{V,E}$, where V is the unique vertex to which E is attached.
- A nonnegative real number $\ell(E)$ for every bounded edge of E, called the length of E.
- A proper map $h: \Gamma \to \mathbb{R}^2$ such that
  - If E is a bounded edge connecting the vertices V_1 and V_2, then h maps E affine linearly on the line segment connecting $h(V_1)$ and $h(V_2)$, and $h(V_2) - h(V_1) = \ell(E) v_{V_1,E}$.
  - If E is an unbounded edge of vertex V, then h maps E affine linearly to the ray $h(V) + \mathbb{R}_{\geq 0} v_{V,E}$.

The genus g_h of a parametrized tropical curve $h: \Gamma \to \mathbb{R}^2$ is defined by
$$g_h \coloneqq g_\Gamma + \sum_{V \in V(\Gamma)} g(V) \,,$$
where g_Γ is the genus of the graph Γ.

Let $w = (w_1, \dots, w_s)$ be a s-tuple of non-zero vectors in M. We fix $x = (x_1, \dots, x_s) \in (\mathbb{R}^2)^s$. We say that a parametrized tropical curve $h: \Gamma \to \mathbb{R}^2$ is of type (w, x) if Γ has exactly $s+1$ unbounded edges, labeled $E_0, E_1, \dots, E_s$, such that

- $v_{E_0} = \sum_{r=1}^{s} w_r$,
- $v_{E_r} = -w_r$ for all $r = 1, \dots, s$,
- E_r asymptotically coincides with the half-line $-\mathbb{R}_{\geqslant 0} w_r + x_r$, for all $r = 1, \dots, s$.

Let $T_{w,x}$ be the set of genus zero[10] parametrized tropical curves $h\colon \Gamma \to \mathbb{R}^2$ of type (w, x) without contracted edges. If $x \in (\mathbb{R}^2)^s$ is general enough (in some appropriate open dense subset), then it follows from [Mik05] or [NS06] that $T_{w,x}$ is a finite set, and that if $(h\colon \Gamma \to \mathbb{R}^2)$, then Γ is trivalent and h is an immersion (distinct vertices have distinct images and two distinct edges have at most one point in common in their images).

For $h\colon \Gamma \to \mathbb{R}^2$ a parametrized tropical curve in $\mathbb{R}^2$ and V a trivalent vertex of adjacent edges E_1, E_2 and E_3, the multiplicity of V is the integer defined by

$$m(V) \coloneqq |\det(v_{V,E_1}, v_{V,E_2})| \,.$$

Thanks to the balancing condition

$$v_{V,E_1} + v_{V,E_2} + v_{V,E_3} = 0 \,,$$

this definition is symmetric in E_1, E_2, E_3. The Block-Göttsche [BG16] multiplicity of V is a Laurent polynomial in a formal variable $q^{\frac{1}{2}}$:

$$[m_V]_q \coloneqq \frac{q^{\frac{m(V)}{2}} - q^{-\frac{m(V)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{-\frac{m(V)-1}{2}} (1 + q + \dots + q^{\frac{m(V)-1}{2}}) \in \mathbb{N}[q^{\pm \frac{1}{2}}] \,.$$

For $(h\colon \Gamma \to \mathbb{R}^2)$ a parametrized tropical curve with Γ trivalent, the refined multiplicity of h is defined by

$$m_h(q^{\frac{1}{2}}) \coloneqq \prod_{V \in V(\Gamma)} [m(V)]_q \,,$$

where the product is over the vertices of Γ.

If $x \in (\mathbb{R}^2)^s$ is in general position, we count the elements of $T_{w,x}$ with refined multiplicities and we get a refined count of tropical curves:

$$N_{w,x}^{\mathrm{trop}}(q^{\frac{1}{2}}) \coloneqq \sum_{h\colon \Gamma \to \mathbb{R}^2} m_h(q^{\frac{1}{2}}) \in \mathbb{N}[q^{\pm \frac{1}{2}}] \,.$$

According to Itenberg-Mikhalkin [IM13], $N_{w,x}^{\mathrm{trop}}(q^{\frac{1}{2}})$ does not depend on x if x is general[11], and we simply denote $N_{w}^{\mathrm{trop}}(q^{\frac{1}{2}})$ the corresponding invariant.

2.6.2 Elementary quantum scattering

Let m_1 and m_2 be two non-zero vectors in $M = \mathbb{Z}^2$. Let $\hat{\mathfrak{D}}$ be the quantum scattering diagram over an Artinian ring R consisting of two incoming rays $-\mathbb{R}_{\geqslant 0} m_1$ and $-\mathbb{R}_{\geqslant 0} m_2$ equipped with

[10] In particular, the graph Γ has genus zero and all the vertices have genus zero.

[11] This also follows from Theorem 1.4

the Hamiltonians

$$\hat{H}_1 = \frac{f_1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{z}^{m_1} \,,$$

and

$$\hat{H}_2 = \frac{f_2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{z}^{m_2} \,,$$

where $f_1, f_2 \in R$ satisfy $f_1^2 = f_2^2 = 0$. Let $S(\hat{\mathfrak{D}})$ be the resulting consistent quantum scattering diagram given by Proposition 2.3. The following result is Lemma 4.3 of [FS15].

Lemma 2.21. *The consistent quantum scattering diagram $S(\hat{\mathfrak{D}})$ is obtained from $\hat{\mathfrak{D}}$ by adding three outgoing rays:*

- $(\mathbb{R}_{\geqslant 0} m_1, \hat{H}_1)$
- $(\mathbb{R}_{\geqslant 0} m_2, \hat{H}_2)$
- $(\mathbb{R}_{\geqslant 0}(m_1 + m_2), \hat{H}_{12})$, *where*

$$\hat{H}_{12} \coloneqq [\langle m_1, m_2 \rangle]_q \frac{f_1 f_2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{z}^{m_1+m_2} \,,$$

and

$$[\langle m_1, m_2 \rangle]_q \coloneqq \frac{q^{\frac{\langle m_1, m_2 \rangle}{2}} - q^{-\frac{\langle m_1, m_2 \rangle}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \,.$$

Proof. Using

$$[\hat{z}^{m_1}, \hat{z}^{m_2}] = \left(q^{\frac{\langle m_1, m_2 \rangle}{2}} - q^{-\frac{\langle m_1, m_2 \rangle}{2}} \right) \hat{z}^{m_1+m_2} \,,$$

we compute that

$$[\hat{H}_1, \hat{H}_2] = [\langle m_1, m_2 \rangle]_q \frac{f_1 f_2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{z}^{m_1+m_2} \,.$$

As $f_1^2 = f_2^2 = 0$, it follows that $\hat{H}_1$ and $\hat{H}_2$ commute with $[\hat{H}_1, \hat{H}_2]$. Using an easy case of the Baker-Campbell-Hausdorff formula, according to which $e^a e^b = e^{a+b+\frac{1}{2}[a,b]}$ if a and b commute with $[a,b]$, we obtain

$$e^{\hat{H}_2} e^{-\hat{H}_1} e^{-\hat{H}_2} e^{\hat{H}_1} = e^{[\hat{H}_1, \hat{H}_2]} \,,$$

and so

$$\hat{\Phi}_{\hat{H}_2}^{-1} \hat{\Phi}_{\hat{H}_1} \hat{\Phi}_{\hat{H}_2} \hat{\Phi}_{\hat{H}_1}^{-1} = \hat{\Phi}_{[\hat{H}_1, \hat{H}_2]} \,,$$

hence the result ☐

2.6.3 Quantum scattering from refined tropical curve counting

In this Section, we review the result of Filippini and Stoppa [FS15] expressing the Hamiltonians attached to the rays of the consistent quantum scattering diagram $S(\hat{\mathfrak{D}}_m)$, defined in Section 2.3.1, in tropical terms. We use the notations introduced at the beginning of Section 2.5.1.

Proposition 2.22. *For every $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in M and for every $m \in M - \{0\}$, the Hamiltonian $\hat{H}_m$ attached to the outgoing ray $\mathbb{R}_{\geqslant 0} m$ in the consistent quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ is given by*

$$\hat{H}_m = \sum_{p \in P_m} \sum_{k \vdash p} N^{\text{trop}}_{w(k)}(q^{\frac{1}{2}}) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \frac{1}{k_{\ell j}!} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \right)^{k_{\ell j}} \right) \left(\prod_{j=1}^{n} t_j^{p_j} \right) \frac{\hat{z}^{\ell_p m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \,,$$

where $q = e^{i\hbar}$, and the first sum is over all partitions k of p.

Proof. This follows from the main result, Corollary 4.9, of [FS15], which is a q-deformed version of the proof of Theorem 2.8 of [GPS10]. A higher dimensional generalization of this argument has been given by Mandel in [Man15]. For completeness and because we organize the combinatorics in a slightly different way, we provide a proof.

By definition, $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ is the consistent quantum scattering diagram obtained from the quantum scattering diagram $\hat{\mathfrak{D}}_{\mathfrak{m}}$ consisting of incoming rays $(\mathfrak{d}_i, \hat{H}_{\mathfrak{d}_j})$, $j = 1, \dots, n$, where

$$\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0} m_j \,,$$

and

$$\hat{H}_{\mathfrak{d}_j} = \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} t_j^{\ell} \hat{z}^{\ell m_j} \,.$$

Let us work over the ring $\mathbb{C}[t_1, \dots, t_n]/(t_1^{N+1}, \dots, t_n^{N+1})$. We embed this ring into

$$\mathbb{C}[\{u_{ja} | 1 \leqslant j \leqslant n, 1 \leqslant a \leqslant N\}]/\langle \{u_{ja}^2 | 1 \leqslant j \leqslant n, 1 \leqslant a \leqslant N\} \rangle$$

by

$$t_j = \sum_{a=1}^{N} u_{ja}$$

for all $1 \leqslant j \leqslant n$. We then have

$$t_j^{\ell} = \sum_{\substack{A \subset \{1, \dots, N\} \\ |A| = \ell}} \ell! \prod_{a \in A} u_{ja} \,,$$

and so

$$\hat{H}_{\mathfrak{d}_j} = \sum_{\ell=1}^{N} \sum_{\substack{A \subset \{1, \dots, N\} \\ |A| = \ell}} \left(\frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \right) \ell! \left(\prod_{a \in A} u_{ja} \right) \hat{z}^{\ell m_j} \,.$$

This suggests to consider the quantum scattering diagram $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ consisting of incoming rays $(\mathfrak{d}_{j\ell A}, \hat{H}_{\mathfrak{d}_{j\ell A}})$, $1 \leqslant \ell \leqslant N$, $A \subset \{1, \dots, N\}$, $|A| = \ell$, where

$$\mathfrak{d}_{j\ell A} = -\mathbb{R}_{\geqslant 0} m_j + c_{j\ell A} \,,$$

for $c_{j\ell A} \in \mathbb{R}^2$ in general position, and

$$\hat{H}_{\mathfrak{d}_{j\ell A}} = \left(\frac{1}{\ell} \frac{(-1)^{\ell-1}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \right) \ell! \left(\prod_{a \in A} u_{ja} \right) \hat{z}^{\ell m_j} \,.$$

If we had taken all $c_{j\ell A} = 0$, then $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ would have been equivalent to $\hat{\mathfrak{D}}_{\mathfrak{m}}$. But for $c_{j\ell A} \in \mathbb{R}^2$ in general position, $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ is a perturbation of $\hat{\mathfrak{D}}_{\mathfrak{m}}$: each ray $(\mathfrak{d}_j, \hat{H}_{\partial_j})$ of $\mathfrak{D}_{\mathfrak{m}}$ splits into various rays $(\mathfrak{d}_{j\ell A}, \hat{H}_{\mathfrak{d}_{j\ell A}})$ of $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$.

The key simplifying fact is that the consistent scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$ can be obtained from $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ by a recursive procedure involving only elementary scatterings in the sense of Lemma 2.21. When two rays of $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ intersect, we are in the situation of Lemma 2.21 because $u_{ja}^2 = 0$. The local consistency at this intersection is then guaranteed by emitting a third ray according to Lemma 2.21. Further intersections of the old and newly created rays can similarly be treated by application of Lemma 2.21. Indeed, the assumptions of general position of the $c_{j\ell A}$ guarantees that only double intersections occur.

The asymptotic scattering diagram of $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$ is the scattering diagram obtained by taking all the rays of $S(\hat{\mathfrak{D}})$ and placing their origin at $0 \in \mathbb{R}^2$. By uniqueness of the consistent completion, the asymptotic scattering diagram is precisely $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$. To get the Hamiltonian $\hat{H}_m$ attached to an outgoing ray $\mathbb{R}_{\geqslant 0} m$ in $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$, it is then enough to collect the various contributions to the corresponding asymptotic ray of $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$ coming from the recursive construction of $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$.

Let us study how the recursive construction of $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}})$ can produce a ray $\mathfrak{d}$ asymptotic to $\mathbb{R}_{\geqslant 0} m$ and equipped with a function $\hat{H}_{\mathfrak{d}}$ proportional to $\hat{z}^{\ell_{\mathfrak{d}} m}$, for some $\ell_{\mathfrak{d}} \geqslant 1$. Such a ray is obtained by successive applications of Lemma 2.21 starting from a subset of the initial incoming rays of $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$.

We focus on one particular sequence of such elementary scatterings. Such sequence naturally defines a graph $\bar{\Gamma}$ in $\mathbb{R}^2$. This graph starts with unbounded edges given by the initial rays taking part to the sequence of scatterings. When two of these rays meet, they scatter and produce a third ray given by Lemma 2.21. If this third ray does not contribute to further scatterings ultimately contributing to $\hat{H}_{\mathfrak{d}}$, we do not include it in $\bar{\Gamma}$ and we continue $\bar{\Gamma}$ by propagating the two initial rays. In particular, $\bar{\Gamma}$ contains a 4-valent vertex given by the two initial rays crossing without non-trivial interaction.

If the third ray does contribute to further scatterings ultimately contributing to $\hat{H}_{\mathfrak{d}}$, we include it in $\bar{\Gamma}$ and we do not propagate the two initial rays. In particular, $\bar{\Gamma}$ gets a trivalent vertex given by the two initial rays meeting and producing the third ray. Iterating this construction, we get one trivalent vertex for each elementary scattering ultimately giving a contribution to $\hat{H}_{\mathfrak{d}}$. At the end of this process, the last elementary scattering produces the ray $\mathfrak{d}$ which becomes an unbounded edge of the graph.

The graph $\bar{\Gamma}$ has two kinds of vertices: trivalent vertices where a non-trivial elementary scattering happens and 4-valent vertices where two rays cross without non-trivial interaction. For every 4-valent vertices, we can separate the two rays crossing, and we get a trivalent graph Γ and a map $h: \Gamma \to \bar{\Gamma} \subset \mathbb{R}^2$ which is one to one except over the 4-valent vertices of $\bar{\Gamma}$ where it is two to one. It follows from the iterative construction that the trivalent graph Γ is a tree, i.e. a graph of genus zero.

The function attached to initial ray of $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\text{split}}$ is a monomial in $\hat{z}$, whose power is proportional to the direction of the ray. By Lemma 2.21, this property is preserved under elementary scattering. Each edge E of our Γ is thus equipped with a function proportional to $\hat{z}^{m_E}$ for some $m_E \in M = \mathbb{Z}^2$ proportional to the direction of E. Furthermore, in an elementary scattering of two edges E_1 and E_2 equipped with m_{E_1} and m_{E_2}, the produced edge E_3 is equipped with $m_{E_1} + m_{E_2}$ by Lemma 2.21. In other words, the balancing condition is satisfied at each vertex and so we can view $h\colon \Gamma \to \mathbb{R}^2$ as a parametrized tropical curve to $\mathbb{R}^2$ in the sense of Section 2.6.1.

For every $1 \leqslant j \leqslant n$ and $\ell \geqslant 1$, there is a number $k_{\ell j}$ of subsets A of $\{1,\dots,n\}$, of size ℓ, such that $\mathfrak{d}_{j\ell A}$ is one of the initial ray appearing in Γ. Denote by $\mathcal{A}_{j\ell}^{\Gamma}$ this set of subsets of $\{1,\dots,n\}$. Writing $p_j \coloneqq \sum_{\ell \geq 1} \ell k_{\ell j}$, we have by the balancing condition

$$\sum_{j=1}^{n} p_j = \ell_{\mathfrak{d}} m \,,$$

and in particular $\ell_{\mathfrak{d}} = \ell_p$.

It follows from an iterative application of Lemma 2.21 that the contribution of Γ to $\hat{H}_{\mathfrak{d}}$ is given by

$$m_\Gamma(q^{\frac{1}{2}}) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \right)^{k_{\ell j}} (\ell!)^{k_{\ell j}} \left(\prod_{A \in \mathcal{A}_{j\ell}^{\Gamma}} \prod_{a \in A} u_{ja} \right) \right) \frac{\hat{z}^{\ell_p m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \,,$$

where $m_\Gamma(q^{\frac{1}{2}})$ is the refined multiplicity of the tropical curve Γ.

To get the complete expression for $\hat{H}_{\mathfrak{d}}$, we have to sum over the possible Γ.

If we fix $p = (p_1,\dots,p_n) \in P = \mathbb{N}^n$, k a partition of p and for every $1 \leqslant j \leqslant n$ and $\ell \geqslant 1$, a set $\mathcal{A}_{j\ell}$ of $k_{\ell j}$ disjoint subsets of $\{1,\dots,N\}$ of size ℓ, we can consider the set $T_{j\ell \mathcal{A}_{j\ell}}$ of genus zero tropical curves Γ having one unbounded edge of asymptotic direction $\mathbb{R}_{\geqslant 0} m$ and weight $\ell_p m$, and for every $1 \leqslant j \leqslant n$, $\ell \geqslant 1$, $A \in \mathcal{A}_{j\ell}$, an unbounded edge of weight ℓm_j asymptotically coinciding with $\mathfrak{d}_{j\ell A}$. By Section 2.6.1, this set is finite.

We already saw how a sequence of elementary scatterings contributing to $\hat{H}_{\mathfrak{d}}$ produces an element $\Gamma \in T_{j\ell \mathcal{A}_{j\ell}}$. Conversely, any $\Gamma \in T_{j\ell\mathcal{A}_{j\ell}}$ will define a sequence of elementary scatterings appearing in the construction of $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{split}})$ and contributing to $\hat{H}_{\mathfrak{d}}$.

It follows that, for every $m \in M - \{0\}$, we have

$$\hat{H}_m =$$

$$\sum_{p \in P_m} \sum_{k \vdash p} \sum_{\mathcal{A}_{j\ell}} \left(\sum_{\Gamma \in T_{j\ell\mathcal{A}_{j\ell}}} m_\Gamma(q^{\frac{1}{2}}) \right) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \right)^{k_{\ell j}} (\ell!)^{k_{\ell j}} \left(\prod_{A \in \mathcal{A}_{j\ell}} \prod_{a \in A} u_{ja} \right) \right) \frac{\hat{z}^{\ell_p m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \,,$$

But by Section 2.6.1, we have

$$\sum_{\Gamma \in T_{j\ell\mathcal{A}_{j\ell}}} m_\Gamma(q^{\frac{1}{2}}) = N_{w(k)}^{\mathrm{trop}}(q^{\frac{1}{2}}) \,,$$

which is in particular independent of $\mathcal{A}_{j\ell}$. So we can do the sum over $\mathcal{A}_{j\ell}$. Given an $\mathcal{A}_{j\ell}$, we can form

$$B := \bigcup_{A \in \mathcal{A}_{j\ell}} A\,,$$

a subset of $\{1,\dots,N\}$ of size $\sum_{\ell \geqslant 1} \ell k_{\ell j} = p_j$. Conversely, the number of ways to write a set B of $p_j = \sum_{\ell \geqslant 1} \ell k_{\ell j}$ elements as a disjoint union of subsets, $k_{\ell j}$ of them being of size ℓ, is equal to

$$\frac{p_j!}{\prod_{\ell \geqslant 1} k_{\ell j}!(\ell!)^{k_{\ell j}}}\,.$$

Replacing the sum over $\mathcal{A}_{j\ell}$ by a sum over B, we get

$$\hat{H}_m =$$

$$\sum_{p \in P_m} \sum_{k \vdash p} N^{\mathrm{trop}}_{w(k)}(q^{\frac{1}{2}}) \left(\prod_{j=1}^n \prod_{\ell \geqslant 1} \frac{1}{k_{\ell j}!} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \right)^{k_{\ell j}} \right) \left(\prod_{j=1}^n \sum_{\substack{B \subset \{1,\dots,N\} \\ |B| = p_j}} p_j! \prod_{b \in B} u_{jb} \right) \frac{\hat{z}^{\ell_p m}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,,$$

Finally, using that

$$t_j^{p_j} = \sum_{\substack{B \subset \{1,\dots,N\} \\ |B| = p_j}} p_j! \prod_{b \in B} u_{jb}\,,$$

we obtain the desired formula for $\hat{H}_m$.

□

Corollary 2.23. *We have*

$$\hat{H}_m = \sum_{p \in P_m} \sum_{k \vdash p} N^{\mathrm{trop}}_{w(k)}(q^{\frac{1}{2}}) \left(\prod_{j=1}^n \prod_{\ell \geqslant 1} \frac{1}{k_{\ell j}!} \left(\frac{(-1)^{\ell-1}}{\ell} \frac{1}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \right)^{k_{\ell j}} \right) (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{s(k)-1} \hat{z}^{\ell_p m}\,.$$

Proof. We simply rearrange $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})$ factors in Proposition 2.22 and use that

$$s(k) = \sum_{j=1}^n \sum_{\ell \geqslant 1} k_{\ell j}\,.$$

□

2.7 End of the proof of Theorems 2.6 and 2.7

2.7.1 End of the proof of Theorem 2.6

In this Section, we finish the proof of Theorem 2.6. We have to express the Hamiltonians attached to the rays of the consistent quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ in terms of the log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ of the log Calabi-Yau surface $Y_{\mathfrak{m}}$.

We know already:

- Corollary 2.23, expressing the Hamiltonians attached to the rays of $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ in terms of the refined counts $N_w^{\mathrm{trop}}(q^{\frac{1}{2}})$ of tropical curves in $\mathbb{R}^2$.

- Proposition 2.8, relating the log Gromov-Witten invariants $N_{g,p}^{Y_\mathfrak{m}}$ of the log Calabi-Yau surface $Y_\mathfrak{m}$ to the log Gromov-Witten invariants $N_{g,w}^{\overline{Y}_\mathfrak{m}}$ of the toric surface $\overline{Y}_\mathfrak{m}$.

It remains to connect the refined tropical counts $N_w^{\text{trop}}(q^{\frac{1}{2}})$ to the log Gromov-Witten invariants $N_{g,w}^{\overline{Y}_\mathfrak{m}}$ of the toric surface $\overline{Y}_\mathfrak{m}$. This is given by the following Proposition 2.24, which is a special case of the main result, Theorem 1.4, of Chapter 1.

Proposition 2.24. *For every $\mathfrak{m} = (m_1, \dots, m_n)$ n-tuple of non-zero primitive vectors in $M = \mathbb{Z}^2$, every $p = (p_1, \dots, p_n) \in P = \mathbb{N}^n$, and every k partition of p, we have*

$$\sum_{g \geqslant 0} N_{g,w(k)}^{\overline{Y}_\mathfrak{m}} \hbar^{2g-1+s(k)} = N_{w(k)}^{\text{trop}}(q^{\frac{1}{2}}) \left(\prod_{r=1}^{s(k)} \frac{1}{|w_r|} \right) \left(2 \sin\left(\frac{\hbar}{2}\right)\right)^{s(k)-1}$$
$$= N_{w(k)}^{\text{trop}}(q^{\frac{1}{2}}) \left(\prod_{j=1}^{n} \prod_{\ell \geqslant 1} \frac{1}{\ell^{k_{\ell j}}} \right) \left(2 \sin\left(\frac{\hbar}{2}\right)\right)^{s(k)-1} .$$

Proof. We simply explain the change in notations needed to translate from Theorem 1.4.

In Chapter 1, we fixed a Δ a balanced collection of vectors in $\mathbb{Z}^2$, specifying a toric surface X_Δ and tangency conditions for a curve along the toric divisors. We fixed a subset Δ^F of Δ, for which the corresponding tangency conditions happen at prescribed positions on the toric divisors. Finally, we fixed a non-negative integer n. Theorem 1.4 is a correspondence theorem between log Gromov-Witten invariants of X_Δ, counting curves in X_Δ satisfying the tangency constraints imposed by Δ and Δ^F, and passing through n points in general position, and refined counts of tropical curves in $\mathbb{R}^2$ satisfying the tropical analogue of these constraints.

To get Proposition 2.24, we take $\Delta = (w_1(k), \dots, w_{s(k)}(k), k_w m_w)$[12], $\Delta^F = (w_1(k), \dots, w_{s(k)}(k))$ and $n = 0$. Using the notations of Chapter 1, we have $|\Delta| = s(k)+1$, $|\Delta^F| = s(k)$ and $g_{\Delta,n}^{\Delta^F} = 0$. Using finally that the variable u keeping track of the genus in Chapter 1 is denoted $\hbar$ in the present Chapter, we see that Theorem 1.4 reduces to Proposition 2.24. □

By comparison of the explicit formulas of Corollary 2.23, Proposition 2.24 and Proposition 2.8, and using the relation

$$s(k) = \sum_{j=1}^{n} \sum_{\ell \geqslant 1} k_{\ell j}$$

to collect the powers of i, we find exactly the formula given in Theorem 2.6 for the Hamiltonians of the quantum scattering diagram $S(\hat{\mathfrak{D}}_\mathfrak{m})$ in terms of the log Gromov-Witten invariants $N_{g,p}^{Y_\mathfrak{m}}$ of the log Calabi-Yau surface $Y_\mathfrak{m}$. This ends the proof of Theorem 2.6.

2.7.2 End of the proof of Theorem 2.7

The proof of Theorem 2.7 follows the one of Theorem 2.6, up to minor notational changes. The only needed serious modification is an orbifold version of the multicovering formula of Lemma 2.20. This is provided by Lemma 2.25 below.

[12] We then have $X_\Delta = \overline{Y}_\mathfrak{m}$ up to some toric blow ups, which do not change the relevant log Gromov-Witten invariants by [AW13].

We fix positive integers r and ℓ. Let $\mathbb{P}^1[r,1]$ be the stacky projective line with a single orbifold point of isotropy group $\mathbb{Z}/r$ at 0. Let $\overline{M}_{g,\ell}(\mathbb{P}^1[r,1]/\infty)$ be the moduli space of genus g orbifold stable maps to $\mathbb{P}^1[r,1]$, relative to $\infty \in \mathbb{P}^1[r,\infty]$, of degree $r\ell$, with maximal tangency order $r\ell$ along ∞. It is a proper Deligne-Mumford stack of virtual dimension $2g-1+\ell$, admitting a virtual fundamental class

$$[\overline{M}_{g,\ell}(\mathbb{P}^1[r,1]/\infty)]^{\mathrm{virt}} \in A_{2g-1+\ell}(\overline{M}_{g,\ell}(\mathbb{P}^1[r,1]/\infty),\mathbb{Q})\,.$$

Let $\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))$ be the orbifold line bundle on $\mathbb{P}^1[r,1]$ of degree $-1/r$. Denoting $\pi\colon\mathcal{C}\to\overline{M}_{g,\ell}(\mathbb{P}^1[r,1]/\infty)$ the universal source curve and $f\colon\mathcal{C}\to\mathbb{P}^1[r,1]$ the universal map, we define

$$N_{g,r}^{\ell} \coloneqq \int_{[\overline{M}_{g,\ell}(\mathbb{P}^1[r,1]/\infty)]^{\mathrm{virt}}} (-1)^g\lambda_g\, e\left(R^1\pi_* f^*\left(\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))\right)\right) \in \mathbb{Q}\,,$$

where $e(-)$ is the Euler class.

Lemma 2.25. *For every positive integers r and ℓ, we have*

$$\sum_{g\geqslant 0} N_{g,r}^{\ell}\hbar^{2g-1} = \frac{(-1)^{\ell-1}}{\ell}\frac{1}{2\sin\left(\frac{r\ell\hbar}{2}\right)}\,.$$

Proof. It is an higher genus version of Proposition 5.7 of [GPS10] and an orbifold version of Theorem 5.1 of [BP05]. Very similar localization computations of higher genus orbifold Gromov-Witten invariants can be found in [JPT11]. The main thing we need to explain is the replacement in the orbifold case for the Mumford relation $c(\mathbb{E})c(\mathbb{E}^\vee)=1$ playing a key role in the proof of Theorem 5.1 of [BP05]. We will simply have to twist the usual Hodge theoretic argument of [Mum83] by a local system.

We consider the action of $\mathbb{C}^*$ on $\mathbb{P}^1[r,1]$ with tangent weights $[1/r,-1]$ at the fixed points $[0,\infty]$. We choose the equivariant lifts of

$$\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))$$

and $\mathcal{O}_{\mathbb{P}^1[r,1]}$ having fibers over the fixed points $[0,\infty]$ of weight $[-1/r,0]$ and $[0,0]$ respectively. For such choices, the argument given in the proof of Theorem 5.1 of [BP05] shows that only one graph Γ contributes to the $\mathbb{C}^*$-localization formula computing $N_{g,r}^{\ell}$. The graph Γ consists of a genus g vertex over 0, a unique edge of degree $r\ell$ and a degenerate genus zero vertex over ∞.

The contribution of Γ is computed using the virtual localization formula of [GP99]. The corresponding $\mathbb{C}^*$-fixed locus is[13] the fiber product

$$\overline{M}_{g,1}(B\mathbb{Z}/r)\times_{\overline{I}B\mathbb{Z}/r} B\mathbb{Z}/(rd)\,,$$

where $\overline{M}_{g,1}(B\mathbb{Z}/r)$ is the moduli stack of 1-pointed[14] genus g orbifold stable maps to the classifying stack $B\mathbb{Z}/r$, $\overline{I}B\mathbb{Z}/r$ is the rigidified inertia stack of $B\mathbb{Z}/r$, and the classifying

[13] We are assuming $g>0$. The case $g=0$ is simpler and treated in Proposition 5.7 of [GPS10].
[14] With a trivial stacky structure at the marked point.

stack $B\mathbb{Z}/(rd)$ appears as moduli space of $\mathbb{C}^*$-invariant Galois covers $\mathbb{P}^1 \to \mathbb{P}^1[r,1]$ of degree $r\ell$. This fibered product is a cover of $\overline{M}_{g,1}(B\mathbb{Z}/r)$ of degree $r/(r\ell)$.

We denote $\pi_0 \colon \mathcal{C}_0 \to \overline{M}_{g,1}(B\mathbb{Z}/r)$ the universal source curve over $\overline{M}_{g,1}(B\mathbb{Z}/r)$. The data of an orbifold stable map $f_0 \colon C_0 \to B\mathbb{Z}/r$ is equivalent to the data of an (orbifold) $\mathbb{Z}/r$-local system L on C_0. We denote by t the generator of the $\mathbb{C}^*$-equivariant cohomology of a point.

The computation of the inverse of the equivariant Euler class of the equivariant virtual bundle is done in Section 2.2 [JPT11] and gives

$$e\left(R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0} \otimes L \otimes \frac{t}{r}\right)\right) \frac{(r\ell)^\ell}{t^\ell \ell!} \frac{1}{\frac{t}{r\ell} - \psi} \left(\frac{r}{t}\right)^{\delta_{L,0}} \frac{t}{r}\,,$$

where $\delta_{L,0} = 1$ if L is the trivial $\mathbb{Z}/r$-local system and 0 else. The vector bundle

$$R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0} \otimes L \otimes \frac{t}{r}\right)$$

over $\overline{M}_{g,1}(B\mathbb{Z}/r)$ comes from the equivariant orbifold line bundle $T_{\mathbb{P}^1[r,1]}(-\infty)|_{[0]/(\mathbb{Z}/r)}$ over $B\mathbb{Z}/r$, restriction over $[0]/(\mathbb{Z}/r)$ of the degree $1/r$ orbifold line bundle $T_{\mathbb{P}^1[r,1]}(-\infty)$ over $\mathbb{P}^1[r,1]$.

The contribution of the integrand in the definition of $N_{g,r}^\ell$ is

$$(-1)^g \lambda_g e\left(R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0} \otimes \left(L \otimes \frac{t}{r}\right)^\vee\right)\right)\left(-\frac{t}{r}\right)^{1-\delta_{R,0}} (-1)^{\ell-1}\frac{(\ell-1)!}{(r\ell)^{\ell-1}} t^{\ell-1}\,.$$

The vector bundle $R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0} \otimes \left(L \otimes \frac{t}{r}\right)^\vee\right)$ over $\overline{M}_{g,1}(B\mathbb{Z}/r)$ comes from the equivariant orbifold line bundle $\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))|_{[0]/(\mathbb{Z}/r)}$ over $B\mathbb{Z}/r$, restriction over $[0]/(\mathbb{Z}/r)$ of the degree $-1/r$ orbifold line bundle $\mathcal{O}_{\mathbb{P}^1[r,1]}(-[0]/(\mathbb{Z}/r))$ over $\mathbb{P}^1[r,1]$.

By Serre duality, we have

$$R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0} \otimes \left(L \otimes \frac{t}{r}\right)^\vee\right) = \left((\pi_0)_*\left(\omega_{\pi_0} \otimes L \otimes \frac{t}{r}\right)\right)^\vee\,,$$

and so

$$\begin{aligned} e\left(R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0} \otimes \left(L \otimes \frac{t}{r}\right)^\vee\right)\right) &= (-1)^{\mathrm{rk}} e\left((\pi_0)_*\left(\omega_{\pi_0} \otimes L \otimes \frac{t}{r}\right)\right) \\ &= (-1)^{\mathrm{rk}}\left(\frac{t}{r}\right)^{\mathrm{rk}} \sum_{j=0}^{\mathrm{rk}} \left(\frac{r}{t}\right)^j c_j\left((\pi_0)_*\left(\omega_{\pi_0} \otimes L\right)\right) \\ &= (-1)^{\mathrm{rk}}\left(\frac{t}{r}\right)^{\mathrm{rk}} c_{\frac{r}{t}}\left((\pi_0)_*(\omega_{\pi_0} \otimes L)\right)\,, \end{aligned}$$

where rk is the rank of $(\pi_0)_*(\omega_{\pi_0} \otimes L)$, locally constant function on $\overline{M}_{g,1}(B\mathbb{Z}/r)$, equal to g on the component with L trivial and to $g-1$ on the components with L non-trivial, and where

$$c_x(E) \coloneqq \sum_{j \geqslant 0} x^j c_j(E)$$

is the Chern polynomial of a vector bundle E. Similarly, we have

$$e\left(R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0}\otimes L\otimes\frac{t}{r}\right)\right)=\left(\frac{t}{r}\right)^{\mathrm{rk}}\sum_{j=0}^{\mathrm{rk}}\left(\frac{r}{t}\right)^j c_j\left(R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0}\otimes L\right)\right)$$
$$=\left(\frac{t}{r}\right)^{\mathrm{rk}} c_{\frac{r}{t}}\left(R^1(\pi_0)_*\left(\mathcal{O}_{\mathcal{C}_0}\otimes L\right)\right).$$

We twist now the Hodge theoretic argument of [Mum83] (see formulas (5.4) and (5.5)) (see also Proposition 3.2 of [BGP08]) by the local system L. The complex

$$\omega^\bullet_{\mathcal{C}_0}: 0\to\mathcal{O}_{\mathcal{C}_0}\xrightarrow{d}\omega_{\pi_0}\to 0\,,$$

twisted by L, gives rise to an exact sequence

$$0\to(\pi_0)_*(\omega_{\pi_0}\otimes L)\to R^1(\pi_0)_*(\omega^\bullet_{\mathcal{C}_0}\otimes L)\to R^1(\pi_0)_*(\mathcal{O}_{\mathcal{C}_0}\otimes L)\to 0\,.$$

By Hodge theory, we have the Gauss-Manin connection on the restriction of $R^1(\pi_0)_*(\omega^\bullet_{\mathcal{C}_0}\otimes L)$ to the open dense subset of $\overline{M}_{g,1}(B\mathbb{Z}/r)$ given by smooth curves, with regular singularities and nilpotent residue along the divisor of nodal curves. This is enough to imply

$$c_x\left(R^1(\pi_0)_*(\omega^\bullet_{\mathcal{C}_0}\otimes L)\right)=1\,,$$

and so

$$c_x\left((\pi_0)_*(\omega_{\pi_0}\otimes L)\right)c_x\left(R^1(\pi_0)_*(\mathcal{O}_{\mathcal{C}_0}\otimes L)\right)=1\,.$$

Using this relation to simplify the above expressions, we get

$$N^\ell_{g,r}=\frac{r}{r\ell}\int_{\overline{M}_{g,1}(B\mathbb{Z}/r)}(-1)^{\ell-1}(-1)^{g+\mathrm{rk}+1-\delta_{L,0}}\left(\frac{t}{r}\right)^{2\,\mathrm{rk}-2\delta_{L,0}+1}\frac{\lambda_g}{\frac{t}{r\ell}-\psi}\,.$$

Using that $\mathrm{rk}=g-1+\delta_{R,0}$, this can be rewritten as

$$N^\ell_{g,r}=\int_{\overline{M}_{g,1}(B\mathbb{Z}/r)}\frac{(-1)^{\ell-1}}{\ell}\left(\frac{t}{r}\right)^{2g-1}\frac{\lambda_g}{\frac{t}{r\ell}-\psi}\,.$$

As the dimension of $\overline{M}_{g,1}(B\mathbb{Z}/r)$ is $3g-2$, we have to extract the term proportional to ψ^{2g-2} and we get

$$N^\ell_{g,r}=\int_{\overline{M}_{g,1}(B\mathbb{Z}/r)}\frac{(-1)^{\ell-1}}{\ell}\ell^{2g-1}\lambda_g\psi^{2g-2}\,.$$

The integrand is now pullback from the moduli space $\overline{M}_{g,1}$ of 1-pointed genus g stable maps. As the forgetful map $\overline{M}_{g,1}(B\mathbb{Z}/r)\to\overline{M}_{g,n}$ has degree[15] r^{2g-1}, we have

$$N^\ell_{g,r}=\frac{(-1)^{\ell-1}}{\ell}(r\ell)^{2g-1}\int_{\overline{M}_{g,1}}\lambda_g\psi^{2g-2}\,,$$

and the result then follows, as in the proof of Theorem 5.1 of [BP05], from the Hodge integrals computations of [FP00]. □

[15] There are r^{2g} $\mathbb{Z}/r$-local systems on a smooth genus g curve, each with a $\mathbb{Z}/r$ group of automorphisms.

2.8 Integrality results and conjectures

In Section 2.8.1, we state Conjecture 2.28, a log BPS integrality conjecture. In Section 2.8.2, we state Theorem 2.30, precise version of Theorem 3 of the Introduction, establishing the validity of Conjecture 2.28 for $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$. The proof of Theorem 3 takes Sections 2.8.3 and 2.8.4. In Section 2.8.5, we describe some explicit connection with refined Donaldson-Thomas theory of quivers. Finally, in Section 2.8.6, we discuss del Pezzo surfaces with a smooth anticanonical divisor and we formulate Conjecture 2.41, precise form of Conjecture 4 of the Introduction.

2.8.1 Integrality conjecture

We formulate a higher genus analogue of the log BPS integrality conjecture, Conjecture 6.2, of [GPS10]. We start by formulating a rationality conjecture, Conjecture 2.26, before stating the integrality conjecture, Conjecture 2.28.

Let Y be a smooth projective surface and let $\partial Y \subset Y$ be a reduced normal crossing effective divisor. We endow Y with the divisorial log structure defined by ∂Y and we get a smooth log scheme. Following Section 6.1 of [GPS10], we say that $(Y, \partial Y)$ is log Calabi-Yau with respect to some non-zero class $\beta \in H_2(Y, \mathbb{Z})$ if $\beta.(\partial Y) = \beta.(-K_Y)$.

Two basic examples are:

- For every $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in $M = \mathbb{Z}^2$, the pair $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$[16] defined in Section 2.2.1. Then $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$ is log Calabi-Yau with respect to every class $\beta \in H_2(Y_{\mathfrak{m}}, \mathbb{Z})$ and so in particular with respect to the classes $\beta_p \in H_2(Y_{\mathfrak{m}}, \mathbb{Z})$ defined in Section 2.2.2.
- Y a del Pezzo surface and ∂Y a smooth anticanonical divisor. Then $(Y, \partial Y)$ is log Calabi-Yau with respect to every class $\beta \in H_2(Y, \mathbb{Z})$.

We fix $(Y, \partial Y)$ log Calabi-Yau with respect to some $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta.(\partial Y) \neq 0$. Let $\overline{M}_{g,\beta}(Y/\partial Y)$ be the moduli space of genus g stable log maps to Y of class β and full tangency of order $\beta.(\partial Y)$ at a single unspecified point of D. It is a proper Deligne-Mumford stack of virtual dimension g admitting a virtual fundamental class

$$[\overline{M}_{g,\beta}(Y/\partial Y)]^{\mathrm{virt}} \in A_g(\overline{M}_{g,\beta}(Y/\partial Y), \mathbb{Q}) .$$

We define

$$N_{g,\beta}^{Y/\partial Y} \coloneqq \int_{[\overline{M}_{g,\beta}(Y/\partial Y)]^{\mathrm{virt}}} (-1)^g \lambda_g \in \mathbb{Q} .$$

Remark that if $(Y, \partial Y)$ is of the form $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$ and β is of the form β_p, see Section 2.2.2, then we have $N_{g,\beta}^{Y/\partial Y} = N_{g,p}^{Y_{\mathfrak{m}}}$ where $N_{g,p}^{Y_{\mathfrak{m}}}$ are the invariants defined in Section 2.2.3.

We can now formulate the rationality conjecture.

[16] Strictly speaking, $Y_{\mathfrak{m}}$ is not smooth, but log smooth. We can either make $Y_{\mathfrak{m}}$ smooth by toric blow-ups or allow log smooth objects in the definition of log Calabi-Yau.

Conjecture 2.26. *Let $(Y, \partial Y)$ be a log Calabi-Yau pair with respect to some class $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta.(\partial Y) \neq 0$. Then there exists a rational function*

$$\overline{\Omega}_\beta(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$$

such that we have the equality of power series in $\hbar$,

$$\overline{\Omega}_\beta(q^{\frac{1}{2}}) = (-1)^{\beta.(\partial Y)+1} \left(2\sin\left(\frac{\hbar}{2}\right)\right) \left(\sum_{g \geqslant 0} N_{g,\beta}^{Y/\partial Y} \hbar^{2g-1}\right),$$

after the change of variables $q = e^{i\hbar}$.

Remarks:

- $\overline{\Omega}_\beta(q^{\frac{1}{2}})$ is unique if it exists.
- If the rational function $\overline{\Omega}_\beta(q^{\frac{1}{2}})$ exists, then it is invariant under $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$, because its power series expansion in $\hbar$ after $q = e^{i\hbar}$ has real coefficients.
- Given the 3-dimensional interpretation of the invariants $N_{g,\beta}^{Y,\partial Y}$ given in Section 2.2.4, Conjecture 2.26 should follow from a log version of the MNOP conjectures, [MNOP06a], [MNOP06b], once an appropriate theory of log Donaldson-Thomas invariants is developed. If ∂Y is smooth, then Conjecture 2.26 indeed follows from the relative MNOP conjectures, see Section 3.3 of [MNOP06b].

Let $(Y, \partial Y)$ be a log Calabi-Yau pair with respect to some primitive class $\beta \in H_2(Y, \mathbb{Z})$ such that $\beta.(\partial Y) \neq 0$. Let us assume that Conjecture 2.26 is true for all the classes multiple of β. So, for every $n \geqslant 1$, we have a rational function $\overline{\Omega}_{n\beta}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$. We define a collection of rational functions $\Omega_{n\beta}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$, $n \geqslant 1$, invariant under $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$, by the relations

$$\overline{\Omega}_{n\beta}(q^{\frac{1}{2}}) = \sum_{\ell | n} \frac{1}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \Omega_{\frac{n}{\ell}\beta}(q^{\frac{\ell}{2}}).$$

Lemma 2.27. *These relations have a unique solution, given by*

$$\Omega_{n\beta}(q^{\frac{1}{2}}) = \sum_{\ell | n} \frac{\mu(\ell)}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \overline{\Omega}_{\frac{n}{\ell}\beta}(q^{\frac{\ell}{2}}),$$

where μ is the Möbius function.

Proof. Indeed, we have

$$\sum_{\ell | n} \frac{1}{\ell} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \left(\sum_{\ell' | \frac{n}{\ell}} \frac{\mu(\ell')}{\ell'} \frac{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}}{q^{\frac{\ell\ell'}{2}} - q^{-\frac{\ell\ell'}{2}}} \overline{\Omega}_{\frac{n}{\ell\ell'}\beta}(q^{\frac{\ell\ell'}{2}}) \right)$$

$$= \sum_{\ell | n} \sum_{\ell' | \frac{n}{\ell}} \frac{\mu(\ell')}{\ell\ell'} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell\ell'}{2}} - q^{-\frac{\ell\ell'}{2}}} \overline{\Omega}_{\frac{n}{\ell\ell'}\beta}(q^{\frac{\ell\ell'}{2}}) = \sum_{m | n} \frac{1}{m} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{m}{2}} - q^{-\frac{m}{2}}} \Omega_{\frac{n}{m}\beta}(q^{\frac{m}{2}}) \left(\sum_{\ell' | m} \mu(\ell') \right)$$

$$= \sum_{m|n} \frac{1}{m} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{m}{2}} - q^{-\frac{m}{2}}} \Omega_{\frac{n}{m}\beta}(q^{\frac{m}{2}}) \delta_{m,1} = \overline{\Omega}_{n\beta}(q^{\frac{1}{2}}) \,,$$

where we used the Möbius inversion formula $\sum_{\ell'|m} \mu(\ell') = \delta_{m,1}$. ☐

We can now formulate the integrality conjecture.

Conjecture 2.28. *Let $(Y, \partial Y)$ be a log Calabi-Yau pair with respect to some class $\beta \in H_2(Y, \mathbb{Z})$, such that $\beta.(\partial Y) \neq 0$, and such that the rationality Conjecture 2.26 is true for all multiples of β, so that the rational functions $\Omega_{n\beta}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})$, are defined. Then, in fact, for every $n \geqslant 1$, $\Omega_{n\beta}(q^{\frac{1}{2}})$ is a Laurent polynomial in $q^{\frac{1}{2}}$ with integer coefficients, i.e.*

$$\Omega_{n\beta}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}] \,,$$

invariant under $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$.

Remark:

- In Section 2.9.3, we explain why this integrality conjecture can be interpreted in some cases as a mathematically well-defined example of the general integrality for open Gromov-Witten invariants in Calabi-Yau 3-folds predicted by Ooguri-Vafa [OV00]. In particular, the log BPS invariants $\Omega_\beta(q^{\frac{1}{2}})$ should be thought as examples of Ooguri-Vafa/open BPS invariants.
- In the classical limit $\hbar \to 0$, the integrality of $\Omega_{n\beta} \coloneqq \Omega_{n\beta}(q^{\frac{1}{2}} = 1)$ is equivalent to Conjecture 6.2 of [GPS10].
- If $\beta^2 = -1$, $\beta.(\partial Y) = 1$, and the class β only contains a smooth rational curve, then it follows from Lemma 2.20 that Conjecture 2.28 is true. More precisely, we have $\overline{\Omega}_{n\beta}(q^{\frac{1}{2}}) = \frac{1}{n} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}$ for every $n \geqslant 1$, and so $\Omega_\beta(q^{\frac{1}{2}}) = 1$ and $\Omega_{n\beta}(q^{\frac{1}{2}}) = 0$ for $n > 1$.

2.8.2 Integrality result

Lemma 2.29. *For every $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in $M = \mathbb{Z}^2$ and $p \in P = \mathbb{N}^n$, the rationality Conjecture 2.26 is true for the log Calabi-Yau pair $(Y_\mathfrak{m}, \partial Y_\mathfrak{m})$ with respect to the curve class $\beta_p \in H_2(Y, \mathbb{Z})$.*

Proof. This follows from Theorem 2.6, expressing the generating series of invariants $N_{g,p}^{Y_\mathfrak{m}}$ as a Hamiltonian $\hat{H}_m$ attached to some ray of the quantum scattering diagram $S(\hat{\mathfrak{D}}_\mathfrak{m})$, and from Proposition 2.22, giving a formula for $\hat{H}_\mathfrak{m}$ whose coefficients are manifestly in $\mathbb{Q}[q^{\pm\frac{1}{2}}][(1-q^\ell)^{-1}]_{\ell \geqslant 1}$.

Alternatively, one could argue that, because the initial quantum scattering diagram $\hat{\mathfrak{D}}_\mathfrak{m}$ is defined over $\mathbb{Q}[q^{\pm\frac{1}{2}}][(1-q^\ell)^{-1}]_{\ell \geqslant 1}$, the resulting consistent quantum scattering diagram $S(\hat{\mathfrak{D}}_\mathfrak{m})$ is also defined over $\mathbb{Q}[q^{\pm\frac{1}{2}}][(1-q^\ell)^{-1}]_{\ell \geqslant 1}$ and so Lemma 2.29 follows directly from Theorem 2.6. ☐

By Lemma 2.29, we have rational functions

$$\overline{\Omega}_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}}) \,,$$

such that

$$\overline{\Omega}_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}}) = (-1)^{\ell_p+1}\left(2\sin\left(\frac{\hbar}{2}\right)\right)\left(\sum_{g\geqslant 0} N_{g,p}^{Y_{\mathfrak{m}}}\hbar^{2g-1}\right),$$

as power series in $\hbar$, after the change of variables $q = e^{i\hbar}$. Remark that we used the fact that $\beta_p.(\partial Y_{\mathfrak{m}}) = \ell_p$.

The following result is, after Theorem 2.6, the second main result of this Chapter. It is the precise form of Theorem 3 in the Introduction.

Theorem 2.30. *For every $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in $M = \mathbb{Z}^2$ and $p \in P = \mathbb{N}^n$, the integrality Conjecture 2.28 is true for the log Calabi-Yau pair $(Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$ with respect to the class $\beta_p \in H_2(Y_m, \mathbb{Z})$. In other words, there exists $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ such that*

$$\overline{\Omega}_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}}) = \sum_{p=\ell p'} \frac{1}{\ell}\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}}-q^{-\frac{\ell}{2}}}\Omega_{p'}^{Y_{\mathfrak{m}}}(q^{\frac{\ell}{2}}) \,.$$

The proof of Theorem 2.30 takes the next Sections 2.8.3 and 2.8.4.

2.8.3 Quadratic refinement

According to Theorem 2.6, generating series of the log Gromov-Witten invariants $N_{g,p}^{Y_{\mathfrak{m}}}$ are Hamiltonians attached to the rays of some quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$. Our integrality result, Theorem 2.30, will follow from a general integrality result for scattering diagrams. Our main input, the integrality result of [KS11], is phrased in terms of twisted quantum scattering diagrams, i.e. scattering diagrams valued in automorphisms of twisted quantum tori. The comparison with quiver DT invariants, done in Section 2.8.5, also requires to consider twisted quantum scattering diagrams.

In the present Section, we explain how to compare the quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ with a twisted quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{tw}})$. This comparison requires the notion of quadratic refinement. A short and to the point discussion by Neitzke can be found in [Nei14]. Some related discussion can be found in Appendix A of [Lin17].

We start with $P = \mathbb{N}^n = \oplus_{j=1}^n \mathbb{N}e_j$. For $p = (p_1, \dots, p_n) \in P = \mathbb{N}^n$, we denote $\mathrm{ord}(p) \coloneqq \sum_{j=1}^n p_j$. An n-tuple $\mathfrak{m} = (m_1, \dots, m_n)$ of primitive non-zero vectors in $M = \mathbb{Z}^2$ naturally defines an additive map

$$r \colon P \to M$$
$$e_j \mapsto m_j \,.$$

For every A a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra, we denote $\hat{T}_{P,\mathrm{tw}}^A$ the non-commutative "space" whose algebra of functions is the algebra $\Gamma(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^A})$ given by $A[[P]]$, powers series in $\hat{x}^p$, $p \in P$, with coefficients in A, with the product defined by

$$\hat{x}^p.\hat{x}^{p'} = (-1)^{\langle r(p), r(p')\rangle} q^{\frac{1}{2}\langle r(p), r(p')\rangle} \hat{x}^{p+p'}\,.$$

The main difference with respect to the formalism of Section 2.1 is the twist by the extra sign $(-1)^{\langle r(p), r(p')\rangle}$.

We will use $A = \mathbb{Z}[[q^{\pm\frac{1}{2}}]]$, $\mathbb{Z}((q^{\frac{1}{2}}))$ and $\mathbb{Q}((q^{\frac{1}{2}}))$. We have obviously the inclusions

$$\Gamma\left(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^{\mathbb{Z}[[q^{\pm 1/2}]]}}\right) \subset \Gamma\left(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^{\mathbb{Z}((q^{1/2}))}}\right) \subset \Gamma\left(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^{\mathbb{Q}((q^{1/2}))}}\right).$$

Every

$$\hat{H}^{\mathrm{tw}} = \sum_{p\in P} \hat{H}_p^{\mathrm{tw}} \hat{x}^p \in \Gamma\left(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^{\mathbb{Q}((q^{1/2}))}}\right),$$

such that $\hat{H}^{\mathrm{tw}} = 0 \mod P$, defines via conjugation by $\exp\left(\hat{H}^{\mathrm{tw}}\right)$ an automorphism

$$\hat{\Phi}^{\mathrm{tw}}_{\hat{H}^{\mathrm{tw}}} = \mathrm{Ad}_{\exp\left(\hat{H}^{\mathrm{tw}}\right)} = \exp\left(\hat{H}^{\mathrm{tw}}\right)(-)\exp\left(-\hat{H}^{\mathrm{tw}}\right)$$

of $\Gamma\left(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^{\mathbb{Q}((q^{1/2}))}}\right)$.

Definition 2.31. *A twisted quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{tw}}$ over $(r\colon P \to M)$ is a set of rays $\mathfrak{d}$ in $M_{\mathbb{R}}$, equipped with elements*

$$\hat{H}^{\mathrm{tw}}_{\mathfrak{d}} \in \Gamma\left(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^{\mathbb{Q}((q^{1/2}))}}\right),$$

such that:

- *There exists (a necessarily unique) $p \in P$ primitive such that $\hat{H}^{\mathrm{tw}}_{\mathfrak{d}} \in \hat{x}^p\mathbb{Q}((q^{\frac{1}{2}}))[[\hat{x}^p]]$ and either $r(p) \in -\mathbb{N}_{\geqslant 1} m_{\mathfrak{d}}$ or $r(p) \in \mathbb{N}_{\geqslant 1} m_{\mathfrak{d}}$. We say that the ray $(\mathfrak{d}, \hat{H}^{\mathrm{tw}}_{\mathfrak{d}})$ is ingoing if $r(p) \in -\mathbb{N}_{\geqslant 1} m_{\mathfrak{d}}$ and outgoing if $r(p) \in \mathbb{N}_{\geqslant 1} m_{\mathfrak{d}}$. We call p the P-direction of the ray $(\mathfrak{d}, \hat{H}^{\mathrm{tw}}_{\mathfrak{d}})$.*
- *For every $\ell \geqslant 0$, there are only finitely many rays $\mathfrak{d}$ of P-direction p satisfying $\mathrm{ord}(p) \leqslant \ell$.*

Using the automorphisms $\hat{\Phi}^{\mathrm{tw}}_{\hat{H}^{\mathrm{tw}}}$, we define as in Section 2.1.4 the notion of consistent twisted quantum scattering diagram and one can prove that every twisted quantum scattering diagram $\mathfrak{D}^{\mathrm{tw}}$ can be canonically completed by adding only outgoing rays to form a consistent twisted quantum scattering diagram $S(\mathfrak{D}^{\mathrm{tw}})$.

The following Lemma will give us a way to go back and forth between quantum scattering diagrams and twisted quantum scattering diagrams.

Lemma 2.32. *The map $\sigma_M\colon M \to \{\pm 1\}$, defined by $\sigma_M(0) = 1$ and $\sigma_M(m) = (-1)^{|m|}$ for $m \in M$ non-zero, where $|m|$ is the divisibility of m in M, is a quadratic refinement of*

$$\wedge^2 M \to \{\pm 1\}$$

$$(m_1, m_2) \mapsto (-1)^{\langle m_1, m_2 \rangle} ,$$

i.e. we have

$$\sigma_M(m_1 + m_2) = (-1)^{\langle m_1, m_2\rangle} \sigma_M(m_1)\sigma_M(m_2) ,$$

for every $m_1, m_2 \in M$*. It is the unique quadratic refinement such that* $\sigma_M(m) = -1$ *for every* $m \in M$ *primitive.*

Proof. We fix a basis of M and we denote $m = (m^x, m^y)$ the coordinates of some $m \in M$ in this basis. We define $\sigma'_M : M \to \{\pm 1\}$ by

$$\sigma'_M(m) = (-1)^{m^x m^y + m^x + m^y} .$$

It is easy to check that σ'_M is a quadratic refinement of $(-1)^{\langle -,- \rangle}$: the parity of

$$(m_1^x + m_2^x)(m_1^y + m_2^y) + m_1^x + m_2^x + m_1^y + m_2^y$$

differs from the parity of

$$m_1^x m_1^y + m_1^x + m_1^y + m_2^x m_2^y + m_2^x + m_2^y$$

by $m_1^x m_2^y + m_2^x m_1^y$, which has the parity of $\langle m_1, m_2 \rangle$.

If $m \in M$ is primitive, then (m^x, m^y) is equal to $(1,0)$, $(0,1)$ or $(1,1)$ modulo two, and in all these three cases, we get $\sigma'_M(m) = -1$. Combined with the fact that σ'_M is a quadratic refinement, this implies that, for every $m \in M$, we have $\sigma'_M(m) = (-1)^{|m|}$, i.e. $\sigma'_M = \sigma_M$. In particular, σ_M is a quadratic refinement and σ'_M is independent of the choice of basis.

The uniqueness statement follows from the fact that a quadratic refinement is determined by its value on a basis of M. □

Let $\hat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{tw}}$ be the twisted quantum scattering diagram consisting of incoming rays $(\mathfrak{d}_j, \hat{H}_{\mathfrak{d}_j}^{\mathrm{tw}})$, $1 \leqslant j \leqslant n$, where

$$\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0} m_j ,$$

and

$$\hat{H}_{\mathfrak{d}_j}^{\mathrm{tw}} = -\sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{1}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \hat{x}^{\ell e_j} \in \Gamma\left(\mathcal{O}_{\hat{T}_{P,\mathrm{tw}}^{\mathbb{Q}((q^{1/2}))}}\right) ,$$

where we consider

$$\frac{1}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} = -q^{\frac{\ell}{2}} \sum_{k \geqslant 0} q^{k\ell} \in \mathbb{Q}((q^{\frac{1}{2}})) .$$

Let $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{tw}})$ be the corresponding consistent twisted quantum scattering diagram obtained by adding only outgoing rays.

Define $\sigma_P : P \to \{\pm 1\}$ by $\sigma_P \coloneqq \sigma_M \circ r$. It follows from Lemma 2.32 that σ_P is a quadratic refinement and so

$$\left(\prod_{j=1}^n t_j^{p_j}\right) \hat{z}^{r(p)} \mapsto \sigma_P(p) \hat{x}^p ,$$

is an algebra isomorphism between quantum tori and twisted quantum tori. Using this isomorphism, we can construct a twisted quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})^{\mathrm{tw}}$ from the quantum scattering diagram $\hat{\mathfrak{D}}_{\mathfrak{m}}$.

The incoming rays of $S(\hat{\mathfrak{D}}_{\mathfrak{m}})^{\mathrm{tw}}$ are $(\mathfrak{d}_j, \hat{H}^{\mathrm{tw}}_{\mathfrak{d}_j})$, $1 \leqslant j \leqslant n$, where $\mathfrak{d}_j = -\mathbb{R}_{\geqslant 0} m_j$ and

$$\hat{H}^{\mathrm{tw}}_{\mathfrak{d}_j} = -\sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{1}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \hat{x}^{\ell e_j} \,.$$

The outgoing rays of $S(\hat{\mathfrak{D}}_{\mathfrak{m}})^{\mathrm{tw}}$ are $(\mathbb{R}_{\geqslant 0} m, \hat{H}^{\mathrm{tw}}_m)$ where

$$\hat{H}^{\mathrm{tw}}_m = -\sum_{p \in P_m} \frac{\overline{\Omega}_p(q^{\frac{1}{2}})}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \hat{x}^p = -\sum_{p \in P_m} \sum_{p = \ell p'} \frac{1}{\ell} \frac{1}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \Omega^{Y_{\mathfrak{m}}}_{p'}(q^{\frac{\ell}{2}}) \hat{x}^p \,.$$

Lemma 2.33. *We have* $S(\hat{\mathfrak{D}}^{\mathrm{tw}}_{\mathfrak{m}}) = S(\hat{\mathfrak{D}}_{\mathfrak{m}})^{\mathrm{tw}}$.

Proof. As $(\prod_{j=1}^n t_j^{p_j}) \hat{z}^{r(p)} \mapsto \sigma_P(p) \hat{x}^p$ is an algebra isomorphism, the twisted quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})^{\mathrm{tw}}$ is consistent and so the result follows from the uniqueness of the consistent completion of twisted quantum scattering diagrams. □

2.8.4 Proof of the integrality theorem

We give below the proof of Theorem 2.30. It is a combination of the scattering arguments of Appendix C3 of [GHKK18] with the formalism of quantum admissible series of [KS11]. Because of the structure of the induction argument, we will in fact prove a more general statement than Theorem 2.30. We will prove, Proposition 2.35, that the consistent completion of any (twisted) quantum scattering with incoming rays equipped with Hamiltonians satisfying some BPS integrality condition has outgoing rays equipped Hamiltonians satisfying the BPS integrality condition.

We fix $p \in P$ primitive. Consider

$$\hat{H}^{\mathrm{tw}} = \sum_{\ell \geqslant 1} \hat{H}^{\mathrm{tw}}_\ell (q^{\frac{1}{2}}) \hat{x}^{\ell p} \in \hat{x}^p \mathbb{Q}((q^{\frac{1}{2}}))[[\hat{x}^p]] \,.$$

We define

$$\overline{\Omega}_\ell(q^{\frac{1}{2}}) \coloneqq -(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \hat{H}^{\mathrm{tw}}_\ell(q^{\frac{1}{2}}) \in \mathbb{Q}((q^{\frac{1}{2}})) \,,$$

and

$$\Omega_\ell(q^{\frac{1}{2}}) \coloneqq \sum_{\ell' | \ell} \frac{\mu(\ell')}{\ell'} \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \overline{\Omega}_{\frac{\ell}{\ell'}}(q^{\frac{\ell}{2}}) \in \mathbb{Q}((q^{\frac{1}{2}})) \,.$$

It follows from Lemma 2.27 that we have

$$\hat{H}^{\mathrm{tw}} = -\sum_{n \geqslant 1} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{\Omega_n(q^{\frac{\ell}{2}})}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \hat{x}^{\ell n p} \,.$$

Definition 2.34. *We say that* $\hat{H}^{\mathrm{tw}} \in \hat{x}^p \mathbb{Q}((q^{\frac{1}{2}}))[[\hat{x}^p]]$ *satisfies the BPS integrality condition if the corresponding* $\Omega_l(q^{\frac{1}{2}}) \in \mathbb{Q}((q^{\frac{1}{2}}))$ *are in fact Laurent polynomials with integer coefficients, i.e.* $\Omega_\ell(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\frac{1}{2}}]$.

Remarks:

- $\hat{H}^{\mathrm{tw}}$ satisfies the BPS integrality condition if and only if $\exp\left(\hat{H}^{\mathrm{tw}}\right)$ is admissible in the sense of Section 6 of [KS11].
- It follows from the product form of the quantum dilogarithm, as recalled in Section 2.3.1, that if $\hat{H}^{\mathrm{tw}}$ satisfies the BPS integrality condition, then $\hat{\Phi}^{\mathrm{tw}}_{\hat{H}^{\mathrm{tw}}}$ preserves the subring $\Gamma\left(\mathcal{O}_{\hat{T}^{\mathbb{Z}[[q^{1/2}]]}_{P,\mathrm{tw}}}\right)$ of $\Gamma\left(\mathcal{O}_{\hat{T}^{\mathbb{Q}((q^{1/2}))}_{P,\mathrm{tw}}}\right)$. We call BPS quantum tropical vertex group[17] the subgroup of automorphisms of $\Gamma\left(\mathcal{O}_{\hat{T}^{\mathbb{Z}[[q^{1/2}]]}_{P,\mathrm{tw}}}\right)$ generated by automorphisms of the form $\hat{\Phi}^{\mathrm{tw}}_{\hat{H}^{\mathrm{tw}}}$ with $\hat{H}^{\mathrm{tw}}$ satisfying the BPS integrality condition.

We fix a choice of twisted quantum scattering diagram in each equivalence class by considering as distinct rays with different P-directions and by merging rays with coinciding supports and with the same P-direction.

Proposition 2.35. *Let n_I be a positive integer and $(p^1,\dots,p^{n_I})$ be an n_I-tuple of primitive vectors in P. Let $\hat{\mathfrak{D}}^{\mathrm{tw}}$ be a twisted quantum scattering diagram over $(r\colon P\to M)$, consisting of incoming rays $(\mathfrak{d}_j,\hat{H}^{\mathrm{tw}}_{\mathfrak{d}_j})$, $1\leqslant j\leqslant n_I$, with $\mathfrak{d}_j=-\mathbb{R}_{\geqslant 0}r(p^j)$ and $\hat{H}^{\mathrm{tw}}_{\mathfrak{d}_j}\in \hat{x}^{p^j}\mathbb{Q}((q^{\frac{1}{2}}))[[\hat{x}^{p^j}]]$ satisfying the BPS integrality condition. Then the consistent twisted quantum scattering diagram $S(\hat{\mathfrak{D}}^{\mathrm{tw}})$ is such that for every outgoing ray $(\mathfrak{d},\hat{H}^{\mathrm{tw}}_{\mathfrak{d}})$, of P-direction $p\in P$, we have that $\hat{H}^{\mathrm{tw}}_{\mathfrak{d}}\in\hat{x}^p\mathbb{Q}((q^{\frac{1}{2}}))[[\hat{x}^p]]$ satisfies the BPS integrality condition.*

Proof. If $n_I=2$, or if more generally all the initial rays $-\mathbb{R}r(p_j)$ are contained in a common half-plane of $M_{\mathbb{R}}$, then the result follows directly from Proposition 9 of [KS11].

We will reduce the general case to the case $n_I=2$ by using an argument parallel to the one used in Appendix C.3 of [GHKK18] to prove some positivity property of classical scattering diagrams.

For $p=(p_1,\dots,p_n)\in P=\mathbb{N}^n$, we denote $\mathrm{ord}(p)\coloneqq\sum_{j=1}^n p_j$. It is simply the total degree of the monomial in several variables $\prod_{j=1}^n t_j^{p_j}$.

The result we will prove by induction over some positive integer N is:

Proposition 2.36. *Let n be a positive integer and $r\colon P=\mathbb{N}^n\to M$ be an additive map. Let n_I be a positive integer and $(p^1,\dots,p^{n_I})$ be an n_I-tuple of primitive vectors in P. Let $\hat{\mathfrak{D}}^{\mathrm{tw}}$ be a twisted quantum scattering diagram over $(r\colon P\to M)$, consisting of incoming rays $(\mathfrak{d}_j,\hat{H}^{\mathrm{tw}}_{\mathfrak{d}_j})$, $1\leqslant j\leqslant n_I$, with $\mathfrak{d}_j=-\mathbb{R}_{\geqslant 0}r(p^j)$ and $\hat{H}^{\mathrm{tw}}_{\mathfrak{d}_j}\in\hat{x}^{p^j}\mathbb{Q}((q^{\frac{1}{2}}))[[\hat{x}^{p^j}]]$ satisfying the BPS integrality condition. Then every outgoing ray $(\mathfrak{d},\hat{H}^{\mathrm{tw}}_{\mathfrak{d}})$ of the consistent twisted quantum scattering diagram $S(\hat{\mathfrak{D}}^{\mathrm{tw}})$, whose P-direction p satisfies $\mathrm{ord}(p)\leqslant N$, is such that $\hat{H}^{\mathrm{tw}}_{\mathfrak{d}}\in\hat{x}^p\mathbb{Q}((q^{\frac{1}{2}}))[[\hat{x}^p]]$ satisfies the BPS integrality condition.*

Proposition 2.36 is obviously true for $N=1$: the only outgoing rays with P-direction p satisfying $\mathrm{ord}(p)=1$ are obtained by straight propagation of the initial rays and so satisfy the BPS integrality condition if it is the case for the initial rays.

[17] Called the quantum tropical vertex group in [KS11].

Let $N > 1$ be an integer. We assume by induction that Proposition 2.36 is true for all integers strictly less than N and we want to prove it for N. As in Step III of Appendix C3 of [GHKK18], up to applying the perturbation trick, consisting in separating transversally and generically the initial rays with the same support and then looking at the new local scatterings, we can assume that at most two initial rays have order one.

We now use the change of monoid trick, as in Steps I and IV of Appendix C3 of [GHKK18]. Denote $P' = \oplus_{j=1}^{n_I} \mathbb{N} e'_j$ and

$$\begin{aligned} r' \colon P' &\to M \\ e'_j &\mapsto r'(e_j) \coloneqq r(p^j) \,. \end{aligned}$$

Let $\hat{\mathfrak{D}}^{\mathrm{tw}'}$ be the twisted quantum scattering diagram over $(r' \colon P' \to M)$ obtained by replacing $\hat{x}^{p^j}$ by $\hat{x}^{e'_j}$ in $\hat{H}^{\mathrm{tw}}_{\mathfrak{d}_j}$. Denote

$$\begin{aligned} u \colon P' &\to P \\ e'_j &\mapsto p^j \,. \end{aligned}$$

Let $(\mathfrak{d}, \hat{H}^{\mathrm{tw}}_{\mathfrak{d}})$ be an outgoing ray of $S(\hat{\mathfrak{D}}^{\mathrm{tw}})$, whose P-direction p satisfies $\mathrm{ord}(p) = N$. Then $(\mathfrak{d}, \hat{H}^{\mathrm{tw}}_{\mathfrak{d}})$ is the sum of images by u of outgoing rays of $S(\hat{\mathfrak{D}}^{\mathrm{tw}'})$, of P'-direction mapping to p by u. Let $(\mathfrak{d}', \hat{H}^{\mathrm{tw}}_{\mathfrak{d}'})$ be such outgoing ray of $S(\hat{\mathfrak{D}}^{\mathrm{tw}'})$.

Writing $p' = \sum_{j=1}^{n_I} p'_j e'_j$, $(p'_1, \dots, p'_n) \in \mathbb{N}^{n_I}$, we have

$$\mathrm{ord}(p') = \mathrm{ord}\left(\sum_{j=1}^{n_I} p'_j e'_j \right) = \sum_{j=1}^{n_I} p'_j \,,$$

whereas

$$\mathrm{ord}(p) = \mathrm{ord}\left(\sum_{j=1}^{n_I} p'_j p^j \right) = \sum_{j=1}^{n_I} p'_j \, \mathrm{ord}(p^j) \,.$$

If only two p'_j are non-zero, then the ray $(\mathfrak{d}', \hat{H}^{\mathrm{tw}}_{\mathfrak{d}'})$ belongs to a twisted quantum scattering diagram with two incoming rays and so its BPS integrality follows from Proposition 9 of [KS11]. If more than two of the p'_j are non-zero, then, at least one of the p^j with $n_j \neq 0$ satisfies $\mathrm{ord}(p^j) \geqslant 2$ and so $\mathrm{ord}(p') < \mathrm{ord}(p)$. The BPS integrality of the ray $(\mathfrak{d}', \hat{H}^{\mathrm{tw}}_{\mathfrak{d}'})$ then follows by the induction hypothesis. □

We can now finish the proof of Theorem 2.30. By Theorem 2.6 and Lemma 2.33, it is enough to show that the outgoing rays of the twisted quantum scattering diagram $S(\hat{\mathfrak{D}}^{\mathrm{tw}}_{\mathfrak{m}})$ satisfy the BPS integrality condition. As the initial rays of $S(\hat{\mathfrak{D}}^{\mathrm{tw}}_{\mathfrak{m}})$ satisfy the BPS integrality condition, the result follows from Proposition 2.35.

2.8.5 Integrality and quiver DT invariants

We refer to [KS08], [JS12], [Rei10], [Rei11], [MR17] for Donaldson-Thomas (DT) theory of quivers.

For every $\mathfrak{m} = (m_1, \dots, m_n)$ an n-tuple of primitive non-zero vectors in $M = \mathbb{Z}^2$, we define a quiver $Q_{\mathfrak{m}}$, with set of vertices $\{1,2,\dots,n\}$ and, for every $1 \leqslant j,k \leqslant n$, $\langle m_j, m_k\rangle_+ \coloneqq \max(\langle m_j, m_k\rangle, 0)$ arrows from the vertex j to the vertex k. We identify $P = \oplus_{j=1}^n \mathbb{N} e_j$ with the set of dimension vectors for the quiver $Q_{\mathfrak{m}}$.

Lemma 2.37. *The quiver $Q_{\mathfrak{m}}$ is acyclic, i.e. does not contain any oriented cycle, if and only if the n vectors $m_1,\dots,m_n$ are all contained in a closed half-plane of $M_{\mathbb{R}} = \mathbb{R}^2$.*

Proof. The quiver $Q_{\mathfrak{m}}$ contains an arrow from the vertex i to the vertex j if and only if (m_i, m_j) is an oriented basis of $\mathbb{R}^2$. ☐

Let us assume that the quiver $Q_{\mathfrak{m}}$ is acyclic. Every $\theta = (\theta_j)_{1\leqslant j\leqslant n} \in \mathbb{Z}^n$ defines a notion of stability for representations of $Q_{\mathfrak{m}}$. For every $p \in P$, we then have a projective variety $M_p^{\theta-ss}$, moduli space of θ-semistable representations of $Q_{\mathfrak{m}}$ of dimension p, containing the open smooth locus $M_p^{\theta-st}$ of θ-stable representations. Denote $\iota\colon M_p^{\theta-st} \to M_p^{\theta-ss}$ the natural inclusion. The main result of [MR17] is that the Laurent polynomials

$$\Omega_p^{Q_{\mathfrak{m}},\theta}(q^{\frac{1}{2}}) \coloneqq (-1)^{\dim M_p^{\theta-ss}} q^{-\frac{1}{2}\dim M_p^{\theta-ss}} \sum_{j=0}^{\dim M_p^{\theta-st}} \left(\dim H^{2j}(M_p^{\theta-ss}, \iota_{!*}\mathbb{Q})\right) q^j$$

$$\in (-1)^{\dim M_p^{\theta-ss}} q^{-\frac{1}{2}\dim M_p^{\theta-ss}} \mathbb{N}[q]$$

are the refined DT invariants of $Q_{\mathfrak{m}}$ for the stability θ. In the above formula, $\iota_{!*}$ is the intermediate extension functor defined by ι and so $\iota_{!*}\mathbb{Q}$ is a perverse sheaf on $M_p^{\theta-ss}$.

As $Q_{\mathfrak{m}}$ is acyclic, we can assume, up to relabeling $m_1,\dots,m_n$, that $\langle m_j, m_k\rangle \geqslant 0$ if $j \leqslant k$. If $\theta_1 < \theta_2 < \cdots < \theta_n$, then $\Omega_{e_j}^{Q_{\mathfrak{m}},\theta}(q^{\frac{1}{2}}) = 1$, for all $1 \leqslant j \leqslant n$, and $\Omega_p^{Q_{\mathfrak{m}},\theta}(q^{\frac{1}{2}}) = 0$ for $p \in P - \{e_1,\dots,e_n\}$. We call such θ a trivial stability condition.

If $\theta_1 > \theta_2 > \cdots > \theta_n$, we call θ a maximally non-trivial stability condition. We simply denote $\Omega_p^{Q_{\mathfrak{m}}}(q^{\frac{1}{2}})$ for $\Omega_p^{Q_{\mathfrak{m}},\theta}(q^{\frac{1}{2}})$ and θ a maximally non-trivial stability condition.

Proposition 2.38. *For every $\mathfrak{m} = (m_1,\dots,m_n)$ such that the quiver $Q_{\mathfrak{m}}$ is acyclic, we have, for every $p \in P = \mathbb{N}^n$, the equality*

$$\Omega_p^{Q_{\mathfrak{m}}}(q^{\frac{1}{2}}) = \Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$$

between the refined DT invariant $\Omega_p^{Q_{\mathfrak{m}}}(q^{\frac{1}{2}})$ of the quiver $Q_{\mathfrak{m}}$ and the log BPS invariant $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$ of the log Calabi-Yau surface $Y_{\mathfrak{m}}$.

Proof. The twisted quantum scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}}^{\mathrm{tw}})$ controls the wall-crossing of refined DT invariants of $Q_{\mathfrak{m}}$ from the trivial stability condition to the maximally non-trivial stability condition. ☐

Remarks:

- In the limit $q^{\frac{1}{2}} \to 1$, and if $Q_{\mathfrak{m}}$ is complete bipartite, then Proposition 2.38 reduces to the Gromov-Witten/Kronecker correspondence of [GP10], [RW13], [RSW12].

- Proposition 2.38 can be viewed as a concrete example of equality between open BPS invariants and DT invariants of quivers. The expectation for this kind of relation goes back at least to [CV09], as reviewed in Section 2.9. Related recent stories include [KRSS17a], [KRSS17b], where some knot invariants, which via some string theoretic duality should be examples of open BPS invariants, are identified with some quiver DT invariants, and [Zas18], where a precise correspondence between open BPS invariants of some class of Lagrangian submanifolds in $\mathbb{C}^3$ and some DT invariants of quivers is conjectured.

- Proposition 2.38 gives a different proof of Theorem 2.30 when $Q_{\mathfrak{m}}$ is acyclic. When $Q_{\mathfrak{m}}$ is not acyclic, it is unclear a priori how to relate the log BPS invariants $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$ to some DT quiver theory. In the physics language, one should remove the contributions of non-trivial single-centered (pure Higgs) indices (see [MPS13] and follow-ups). It is still an open question to define mathematically the corresponding operation in DT quiver theory. The fact that the integrality given by Theorem 2.30 holds even if $Q_{\mathfrak{m}}$ is not acyclic is probably an additional evidence that it should be possible.

- When $Q_{\mathfrak{m}}$ is acyclic, Proposition 2.38 gives a positivity result for the log BPS invariants $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$. It is unclear how to prove a similar positivity result if $Q_{\mathfrak{m}}$ is not acyclic.

We finish this Section by some remark about signs. The definition of $\Omega_p^{Y_{\mathfrak{m}}}(q^{\frac{1}{2}})$ given in Section 2.8.2 includes a global sign $(-1)^{\ell_p-1} = (-1)^{\beta_p.(\partial Y_{\mathfrak{m}})-1}$, whereas the formula given above for $\Omega_p^{Q_{\mathfrak{m}}}(q^{\frac{1}{2}})$ includes a global sign $(-1)^{\dim M_p^{\theta-ss}}$. Using that $\beta_p.(\partial Y_{\mathfrak{m}})$ and β_p^2 have the same parity by Riemann-Roch on $Y_{\mathfrak{m}}$, the following result gives a direct proof that these two signs are identical.

Lemma 2.39. *For every $p \in P$, we have*

$$\dim M_p^{\theta-ss} = \beta_p^2 + 1\,.$$

Proof. We write $p = \sum_{j=1}^n p_j e_j \in P$. By standard quiver theory, we have

$$\dim M_p^{\theta-ss} = \sum_{j=1}^n \sum_{k=1}^n \langle m_j, m_k\rangle_+ p_j p_k - \sum_{j=1}^n p_j^2 + 1\,.$$

By definition, Section 2.2.2, we have

$$\beta_p = \nu^* \beta - \sum_{j=1}^n p_j E_j\,,$$

where $\nu\colon Y_{\mathfrak{m}} \to \overline{Y}_{\mathfrak{m}}$ is the blow-up morphism and $\beta \in H_2(\overline{Y}_{\mathfrak{m}}, \mathbb{Z})$ is defined by some intersection numbers. It follows that

$$\beta_p^2 = \beta^2 - \sum_{j=1}^n p_j^2\,.$$

From the intersection numbers defining β, we see that the convex polygon dual to β is obtained by successively adding the vectors $p_j m_j$ and $\ell_p m_p$, in the order given by the counterclockwise ordering of the m_j and m_p given by their argument. By standard toric geometry, β^2 is given by twice the area of the dual polygon and so we have

$$\beta^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle m_j, m_k \rangle_+ p_j p_k \,.$$

It follows that

$$\beta_p^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle m_j, m_k \rangle_+ p_j p_k - \sum_{j=1}^{n} p_j^2 = \dim M_p^{\theta-ss} - 1 \,.$$

□

2.8.6 del Pezzo surfaces

In this Section, we study the conjectures of Section 2.8.1 in the case where Y is a del Pezzo surface S and ∂Y is a smooth anticanonical divisor E of Y. In particular, E is a smooth genus one curve. We formulate Conjecture 2.41, precise form of Conjecture 4 of the Introduction.

Lemma 2.40. *Let S be a del Pezzo surface, and E be a smooth anticanonical divisor of S. Then, for every $\beta \in H_2(Y, \mathbb{Z})$, the rationality Conjecture 2.26 is true for the log Calabi-Yau pair (S, E) with respect to the curve class β.*

Proof. As in Section 2.2.4, the invariants $N_{g,\beta}^{S/E}$ can be written as equivariant Gromov-Witten invariants of the 3-fold $S \times \mathbb{C}$ relative to the divisor $E \times \mathbb{C}$. The rationality result then follows from the Gromov-Witten/stable pairs correspondence for the relative 3-fold geometry $S \times \mathbb{C}/E \times \mathbb{C}$.

This case of the Gromov-Witten/stable pairs correspondence can be proved following Section 5.3 of [MPT10]. This involves considering the degeneration of $S \times \mathbb{C}$ to the normal cone of $E \times \mathbb{C}$. Denote N the normal bundle to E in S. The degeneration formula expresses equivariant Gromov-Witten/stable pairs theories of $S \times \mathbb{C}$, without insertions, in terms of the relative equivariant Gromov-Witten/stable pairs theories, without insertions, of $S \times \mathbb{C}/E \times \mathbb{C}$ and $\mathbb{P}(N \oplus \mathcal{O}_E) \times \mathbb{C}$.

As $S \times \mathbb{C}$ is deformation equivalent to a toric 3-fold[18], the Gromov-Witten/stable pairs correspondence, without insertions, for $S \times \mathbb{C}$ follows from Section 5.1 of [MPT10].

The equivariant Gromov-Witten/stable pairs theory of $\mathbb{P}(N \oplus \mathcal{O}_E) \times \mathbb{C}/E \times \mathbb{C}$ coincides with the non-equivariant Gromov-Witten theory of $\mathbb{P}(N \oplus \mathcal{O}_E) \times E/E \times E$. The 3-fold $\mathbb{P}(N \oplus \mathcal{O}_E) \times E$ is a $\mathbb{P}^1$-bundle over $E \times E$ and we are considering curves of degree 0 over the second E factor. As $E \times E$ is holomorphic symplectic, the Gromov-Witten/stable pairs theories vanish unless the curve class has also degree 0 over the first E factor. The Gromov-Witten/stable pairs correspondence for $\mathbb{P}(N \oplus \mathcal{O}_E) \times E/E \times E$, without insertions, thus follows from the Gromov-Witten/stable pairs correspondence, without insertions, for local curves.

It follows from Proposition 6 of [PP13] that the degeneration formula can be inverted to

[18]Indeed, a del Pezzo surface is deformation equivalent to a (non-necessarily del Pezzo) toric surface: if S is a blow-up of $\mathbb{P}^2$ in n points, then S is deformation equivalent to a surface obtained by n successive toric blow-ups of $\mathbb{P}^2$.

imply the Gromov-Witten/stable pairs correspondence, without insertions, for $S \times \mathbb{C}/E \times \mathbb{C}$. $\square$

By Lemma 2.40, we have rational functions

$$\overline{\Omega}_\beta^{S/E}(q^{\frac{1}{2}}) \in \mathbb{Q}(q^{\pm\frac{1}{2}})\,,$$

such that

$$\overline{\Omega}_\beta^{S/E}(q^{\frac{1}{2}}) = (-1)^{\beta.E+1}\left(2\sin\left(\frac{\hbar}{2}\right)\right)\left(\sum_{g\geqslant 0} N_{g,\beta}^{S/E}\hbar^{2g-1}\right),$$

as power series in $\hbar$, after the change of variables $q = e^{i\hbar}$.

We define

$$\Omega_\beta^{S/E}(q^{\frac{1}{2}}) = \sum_{\beta=\ell\beta'} \frac{\mu(\ell)}{\ell}\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{q^{\frac{\ell}{2}}-q^{-\frac{\ell}{2}}}\overline{\Omega}_{\beta'}(q^{\frac{\ell}{2}}) \in \mathbb{Q}(q^{\frac{1}{2}})\,.$$

According to Conjecture 2.28, one should have $\Omega_\beta^{S/E}(q^{\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Let M_β be the moduli space of dimension one stable sheaves on S, of class $\beta \in H_2(S,\mathbb{Z})$, and Euler characteristic 1. It is a smooth projective variety of dimension β^2+1. We denote

$$\chi_q(M_\beta) \coloneqq q^{-\frac{1}{2}(\beta^2+1)} \sum_{j,k=0}^{\beta^2+1} (-1)^{j+k} h^{j,k}(M_\beta) q^j \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$$

the normalized Hirzebruch genus of M_β, where $h^{j,k}$ are the Hodge numbers. It follows from Theorem 2 of [Mar07], following [ESm93] and [Bea95], that $h^{j,k}(M_\beta)=0$ if $j \neq k$. In particular, $\chi_q(M_\beta)$ coincides with the normalized Poincaré polynomial of M_β.

Conjecture 2.41. *We have*

$$\Omega_\beta^{S/E}(q^{\frac{1}{2}}) = (-1)^{\beta^2+1}(\beta.E)\chi_q(M_\beta)\,.$$

Remarks:

- We have $\beta^2 = \beta.E \mod 2$ by Riemann-Roch.
- In the limit $q^{\frac{1}{2}} \to 1$, Conjecture 2.41 reduces to

$$\begin{aligned} N_{0,\beta}^{S/E} &= (-1)^{\beta.E-1}\sum_{\beta=\ell\beta'}(-1)^{(\beta')^2+1}\frac{(\beta'.E)}{\ell^2}e(M_{\beta'}) \\ &= (-1)^{\beta.E-1}(\beta.E)\sum_{\beta=\ell\beta'}\frac{1}{\ell^3}(-1)^{(\beta')^2+1}e(M_{\beta'})\,, \end{aligned}$$

  which is a known result. Indeed, by an application of the degeneration formula originally due to Graber-Hassett and generalized in [vGGR17], we have $N_{0,\beta}^{S/E} = (-1)^{\beta.E+1}(\beta.E)N_{0,\beta}^X$, where X is the local Calabi-Yau 3-fold given by the total space of the canonical line bundle K_S of S, and $N_{0,\beta}^X$ is the genus 0, class β, Gromov-Witten invariant of X. So the above formula is equivalent to

$$N_{0,\beta}^{X} = \sum_{\beta=\ell\beta'} \frac{1}{\ell^3} (-1)^{(\beta')^2+1} e(M_{\beta'})\,,$$

which is exactly the Katz conjecture (Conjecture 2.3 of [Kat08]) for X. As X is deformation equivalent to a toric Calabi-Yau 3-fold, the Katz conjecture for X follows from the combination of the Gromov-Witten/stable pairs correspondence (Section 5.1 of [MPT10]), the integrality result of [Kon06] and Theorem 6.4 of [Tod12].

- The right-hand side $(-1)^{\beta^2+1}\chi_q(M_\beta)$ should be thought as a refined DT invariant of X, counting dimension one sheaves. From this point of view, Conjecture 2.41 is an equality between a log BPS invariant on one side and a refined DT invariant on the other side, in a way completely parallel to Proposition 2.38.
- Further conceptual evidences for Conjecture 2.41 and a further refinement of Conjecture 2.41 will be presented elsewhere.

2.9 Relation with Cecotti-Vafa

In [CV09], Cecotti-Vafa have given a physical derivation of the fact that the refined BPS indices of a $\mathcal{N} = 2$ 4d quantum field theory admitting a Seiberg-Witten curve satisfy the refined Kontsevich-Soibelman wall-crossing formula. To make connection with Theorem 2.6, we focus on only one part of the argument, establishing the relation between open Gromov-Witten invariants and wall-crossing formula via Chern-Simons theory. In particular, we do not discuss the application to the BPS spectrum of $\mathcal{N} = 2$ 4d quantum field theories, which would be related to our Section 2.8.5 on quiver DT invariants.

2.9.1 Summary of the Cecotti-Vafa argument

Let U be a non-compact hyperkähler manifold[19], (I, J, K) be a quaternionic triple of compatible complex structures, $(\omega_I, \omega_J, \omega_K)$ be the corresponding triple of real symplectic forms and $(\Omega_I, \Omega_J, \Omega_K)$ be the corresponding triple of holomorphic symplectic forms.

Let $\Sigma \subset U$ be a I-holomorphic Lagrangian subvariety of U, i.e. a submanifold such that $\Omega_I|_\Sigma = 0$. It is an example of (B, A, A)-brane in U: it is a complex subvariety for the complex structure I and a real Lagrangian for any of the real symplectic forms $(\cos\theta)\omega_J + (\sin\theta)\omega_K$, $\theta \in \mathbb{R}$. There is in fact a twistor sphere J_ζ, $\zeta \in \mathbb{P}^1$, of compatible complex structures, such that $I = J_0$, $J = J_1$ and $K = J_i$.

Let X be the non-compact Calabi-Yau 3-fold, of underlying real manifold $U \times \mathbb{C}^*$ and equipped with a complex structure twisted in a twistorial way, i.e. such that the fiber over $\zeta \in \mathbb{C}^*$ is the complex variety (U, J_ζ). Consider $S^1 \subset \mathbb{C}^*$ and $L \coloneqq \Sigma \times S^1 \subset X$.

We consider the open topological string A-model on (X, L), i.e. the count of holomorphic

[19] In [CV09], Cecotti-Vafa consider $U = \mathbb{C}^2$ but the generalization to an arbitrary hyperkähler surface is clear and is considered for example in [CNV10] (in particular Appendix B).

maps $(C, \partial C) \to (X, L)$ from an open Riemann surface C to X with boundary ∂C mapping to L[20]. We restrict ourselves to open Riemann surfaces with only one boundary component. Given a class $\beta \in H_2(X,L)$, let $N_{g,\beta} \in \mathbb{Q}$ be the "count" of holomorphic maps $\varphi\colon (C,\partial C) \to (X,L)$ with C a genus g Riemann surface with one boundary component and $[\varphi(C,\partial C)] = \beta$. We denote

$$\partial \beta = [\partial C] \in H_1(L)\,,$$

i.e. the image of β by the natural boundary map $H_2(X,L) \to H_1(L)$. A holomorphic map $\varphi\colon (C,\partial C) \to (X,L)$ of class $\beta \in H_2(X,L)$ is a $J_{e^{i\theta}}$-holomorphic map to U, at a constant value $e^{i\theta} \in S^1$, where θ is the argument of $\int_\beta \Omega_I$.

According to Witten [Wit95], in absence of non-constant worldsheet instantons, the effective spacetime theory of the A-model on the A-brane L is Chern-Simons theory of gauge group $U(1)$. The field of this theory is a $U(1)$ gauge field A and its action is

$$I_{CS}(A) \coloneqq \frac{1}{2}\int_L A \wedge dA\,.$$

The non-constant worldsheet instantons deform this result, see Section 4.4 of [Wit95]. The effective spacetime theory on the A-brane L is still a $U(1)$-gauge theory but the Chern-Simons action is deformed by additional terms involving the worldsheet instantons:

$$I(A) = I_{CS}(A) + \sum_\beta \sum_{g \geqslant 0} N_{g,\beta} \hbar^{2g} e^{-\int_\beta \omega} e^{\int_{\partial\beta} A}\,.$$

The partition function of the deformed theory can be written as a correlation function in Chern-Simons theory

$$\begin{aligned} Z &= \int DA\, e^{i\frac{I(A)}{\hbar}} \\ &= \left\langle \exp\left(i \sum_{\beta \in H_2(X,L)} \sum_{g\geqslant 0} N_{g,\beta} \hbar^{2g-1} e^{-\int_\beta \omega} e^{\int_{\partial\beta} A}\right)\right\rangle_{CS}\,. \end{aligned}$$

As $L = \Sigma \times S^1$, we can adopt a Hamiltonian description where S^1 plays the role of the time direction. The classical phase space of $U(1)$ Chern-Simons theory on $L = \Sigma \times S^1$ is the space of $U(1)$ flat connections on Σ. When Σ is a torus, the classical phase space is the dual torus T. For every $m \in H_1(L)$, the holonomy around m defined a function z^m on T, i.e. a classical observable,

$$z^m(A) \coloneqq e^{\int_m A}\,.$$

The algebra structure is given by $z^m z^{m'} = z^{m+m'}$ and the Poisson structure by $\{z^m, z^{m'}\} = \langle m, m'\rangle z^{m+m'}$. The algebra of quantum observables is given by the non-commutative torus, $\hat{z}^m \hat{z}^{m'} = q^{\frac{1}{2}\langle m,m'\rangle} \hat{z}^{m+m'}$, where $q = e^{i\hbar}$. Writing $t^\beta = e^{-\int_\beta \omega}$, we get

$$Z = \mathrm{Tr}_{\mathcal{H}}\left(T \prod_{\beta \in H_2(X,L)} \mathrm{Ad}_{\exp\left(-i\sum_{g\geqslant 0} N_{g,\beta} \hbar^{2g-1} t^\beta \hat{z}^m\right)}\right)\,,$$

[20] Usually, A-branes, i.e. boundary conditions for the A-model, have to be Lagrangian submanifolds. In fact, L is not Lagrangian in X but only totally real. Combined with specific aspects of the twistorial geometry, it is probably enough to have well-defined worldsheet instantons contributions. As suggested in [CV09], it would be interesting to clarify this point.

where $\mathcal{H}$ is the Hilbert space of quantum Chern-Simons theory and where $T\prod_\beta$ is a time ordered product, with ordering according to the phase of $\int_\beta \Omega_I$.

The key physical input used by Cecotti-Vafa [CV09] is the continuity of the partition function Z as function of the position of L in X. It follows that the jump of the invariants $N_{g,\beta}$ under variation of L in X is controlled by the refined Kontsevich-Soibelman wall-crossing formula formulated in terms of products of automorphisms of the quantum torus.

2.9.2 Comparison with Theorem 2.6

Our main result, Theorem 2.6, expresses the log Gromov-Witten theory of a log Calabi-Yau surface $(Y_\mathfrak{m}, \partial Y_\mathfrak{m})$ in terms of the 2-dimensional Kontsevich-Soibelman scattering diagram. The complement $U_\mathfrak{m} \coloneqq Y_\mathfrak{m} - \partial Y_\mathfrak{m}$ is a non-compact holomorphic symplectic surface admitting a SYZ real Lagrangian torus fibration. In some cases, $U_\mathfrak{m}$ admits a hyperkähler metric, such that the original complex structure of $U_\mathfrak{m}$ is the compatible complex structure J, and such that the SYZ fibration becomes I-holomorphic Lagrangian. Typical examples include 2-dimensional Hitchin moduli spaces, see [Boa12] for a nice review. In such cases, we can apply the Cecotti-Vafa story summarized above to $U \coloneqq U_\mathfrak{m}$, with Σ a torus fiber of the SYZ fibration.

The log Gromov-Witten invariants with insertion of a top lambda class $N_{g,\beta}$, introduced in Section 2.2, should be viewed as a rigorous definition of the open Gromov-Witten invariants in the twistorial geometry X, with boundary on a torus fiber Σ "near infinity"[21]. This is in part justified by the 3-dimensional interpretation of the invariants $N_{g,\beta}^{Y_\mathfrak{m}}$ given in Section 2.2.4 and in particular by Lemma 2.5.

Automorphisms of the quantum torus appearing in Section 2.9.1 coincide with the automorphisms of the quantum torus appearing in Theorem 2.6. It follows that Theorem 2.6 can be viewed as a mathematically rigorous check of the physical argument given by Cecotti-Vafa [CV09], based on the continuity of Chern-Simons correlation functions and on the connection predicted by Witten [Wit95] between A-model topological string and quantum Chern-Simons theory.

2.9.3 Ooguri-Vafa integrality

Using the review of the Cecotti-Vafa paper [OV00] given in 2.9.1, we can explain the relation between Conjecture 2.28 and Theorem 2.30 of Section 2.8 and the integrality conjecture of Ooguri-Vafa [OV00].

If $(Y, \partial Y)$ is a log Calabi-Yau surface, the complement $U \coloneqq Y - \partial Y$ is a non-compact holomorphic symplectic surface. Assuming that U admits a hyperkähler metric such that the original complex structure of U is the compatible complex structure, and an I-holomorphic Lagrangian torus fibrations, we can apply Section 2.9.1, taking for Σ a fiber of the torus

[21] An early reference for the interpretation of some open Gromov-Witten invariants in terms of relative stable maps is [LS06]. The intuitive picture to have in mind is that an open Riemann surface with a boundary on a torus fiber very close to the divisor at infinity can be capped off by a holomorphic disc meeting the divisor at infinity in one point.

fibration. As in Section 2.9.2, the log Gromov-Witten invariants with insertion of a top lambda class defined in Section 2.8 should be viewed as a rigorous definition of the open Gromov-Witten invariants in the twistorial geometry X, with boundary on a torus fiber Σ "near infinity", i.e. near the anticanonical divisor ∂Y of Y. Ooguri-Vafa have given a physical derivation of an integrality result for these open Gromov-Witten invariants, analogue to the Gopakumar-Vafa [GV98a] [GV98b] integrality for closed Gromov-Witten invariants.

The open topological string A-model has a natural embedding in physical string theory. More precisely, in type IIA string theory on $\mathbb{R}^4 \times X$, it computes F-terms in the $\mathcal{N} = (2,2)$ 2d quantum field theory on the non-compact worldvolume of a D4-brane on $\mathbb{R}^2 \times L$. In the strong coupling limit of type IIA string theory, we get M-theory on $\mathbb{R}^5 \times X$, with some M5-brane on $\mathbb{R}^3 \times L$, and fundamental strings become M2-branes. Let $\Omega_{r,\beta} \in \mathbb{Z}$ be the BPS index given by counting M2-branes with boundary on L, of class $\beta \in H_2(X,L)$, and defining BPS states of spin $r \in \frac{1}{2}\mathbb{Z}$ in $\mathbb{R}^3$. Comparing the type IIA string description and the M-theory description, Ooguri-Vafa [OV00] obtained the relation

$$\sum_{g \geqslant 0} N_{g,\beta} \hbar^{2g-1} = \sum_{\beta = \ell \beta'} \frac{(-1)^{\ell-1}}{\ell} \frac{1}{2 \sin\left(\frac{\ell \hbar}{2}\right)} \left(\sum_{r \in \frac{1}{2}\mathbb{Z}} \Omega_{r,\beta'} e^{i \ell r \hbar} \right).$$

The corresponding integrality obviously coincides with the integrality of Conjecture 2.28 and Theorem 2.30.

3

Deformation quantization of log Calabi-Yau surfaces

3.1 Basics and main results

3.1.1 Looijenga pairs

Let (Y, D) be a Looijenga pair[1]: Y is a smooth projective complex surface and D is a singular reduced normal crossings anticanonical effective divisor on Y. Writing the irreducible components

$$D = D_1 + \cdots + D_r ,$$

D is a cycle of r irreducible smooth rational curves D_j if $r \geqslant 2$, or an irreducible nodal rational curve if $r = 1$. The complement $U \coloneqq Y - D$ is a non-compact Calabi-Yau surface, equipped with a holomorphic symplectic form Ω_U, defined up to non-zero scaling and having first order poles along D. We refer to [Loo81], [Fri15], [GHK15a], [GHK15b], for more background on Looijenga pairs.

There are two basic operations on Looijenga pairs:

- Corner blow-up. If (Y, D) is a Looijenga pair, then the blow-up $\tilde{Y}$ of Y at one of the corners of D, equipped with the preimage $\tilde{D}$ of D, is a Looijenga pair.
- Boundary blow-up. If (Y, D) is a Looijenga pair, then the blow-up $\tilde{Y}$ of Y at a smooth point of D, equipped with the strict transform $\tilde{D}$ of D, is a Looijenga pair.

A corner blow-up does not change the interior U of a Looijenga pair (Y, D). An interior blow-up changes the interior of a Looijenga pair: if $(\tilde{Y}, \tilde{D})$ is an interior blow-up of (Y, D),

[1] We follow the terminology of Gross-Hacking-Keel [GHK15a]

then, for example, we have

$$e(\tilde{U}) = e(U) + 1\,,$$

where U is the interior of (Y, D), $\tilde{U}$ is the interior of $(\tilde{Y}, \tilde{D})$, and $e(-)$ denotes the topological Euler characteristic.

If $\bar{Y}$ is a smooth toric variety and $\bar{D}$ is its toric boundary divisor, then $(\bar{Y}, \bar{D})$ is a Looijenga pair, of interior $U = (\mathbb{C}^*)^2$. In particular, we have $e(U) = e((\mathbb{C}^*)^2) = 0$. Such Looijenga pairs are called toric. A Looijenga pair (Y, D) is toric if and only if its interior $U = Y - D$ has a vanishing Euler topological characteristic: $e(U) = 0$.

A toric model of a Looijenga pair (Y, D) is a toric Looijenga pair $(\bar{Y}, \bar{D})$ such that (Y, D) is obtained from $(\bar{Y}, \bar{D})$ by applying successively a finite number of boundary blow-ups.

If (Y, D) is a Looijenga pair, then, by Proposition 1.3 of [GHK15a], there exists a Looijenga pair $(\tilde{Y}, \tilde{D})$, obtained from (Y, D) by applying successively a finite number of corner blow-ups, which admits a toric model. In particular, we have $e(U) \geqslant 0$, where U is the interior of (Y, D).

Let $(\bar{Y}, \bar{D})$ be a toric model of a Looijenga pair (Y, D) of interior U. Let $\bar{\omega}$ be a torus invariant real symplectic form on $(\mathbb{C}^*)^2 = \bar{Y} - \bar{D}$. Then the corresponding moment map for the torus action gives to $\bar{Y}$ the structure of toric fibration, whose restriction to U is a smooth fibration in Lagrangian tori. By definition of a toric model, we have a map $p \colon (Y, D) \to (\bar{Y}, \bar{D})$, composition of successive boundary blow-ups. Let E_j denote the exceptional divisors, $j = 1, \dots, e(U)$. Then for ϵ_j small enough positive real numbers, there exists a symplectic form ω in the class

$$p^*[\bar{\omega}] - \sum_{j=1}^{e(U)} \epsilon_j E_j$$

with respect to which Y admits an almost toric fibration, whose restriction to U is a fibration in Lagrangian tori with $e(U)$ nodal fibers, [AAK16].

Toric models of a given Looijenga pair are very far from being unique but are always related by sequences of corner blow-ups/blow-downs and boundary blow-ups/blow-downs. The corresponding almost toric fibrations are related by nodal trades, [Sym03].

Following Section 6.3 of [GHK15a], we say that (Y, D) is positive if one of the following equivalent conditions is satisfied:

- There exists positive integers $a_1, \dots, a_r$ such that, for all $1 \leqslant k \leqslant r$, we have

$$\left(\sum_{j=1}^{r} a_j D_j \right) . D_k > 0\,.$$

- U is deformation equivalent to an affine surface.
- U is the minimal resolution of Spec $H^0(U, \mathcal{O}_U)$, which is an affine surface with at worst Du Val singularities.

3.1.2 Tropicalization of Looijenga pairs

We refer to Sections 1.2 and 2.1 of [GHK15a] and to Section 1 of [GHKS16] for details. Let (Y,D) be a Looijenga pair. Let $D_1,\dots,D_r$ be the component of D, ordered in a cyclic order, the index j of D_j being considered modulo r. For every j modulo r, we consider an integral affine cone $\sigma_{j,j+1}=(\mathbb{R}_{\geqslant 0})^2$, of edges ρ_j and ρ_{j+1}. We abstractly glue together the cones $\sigma_{j-1,j}$ and $\sigma_{j,j+1}$ along the edge ρ_j. We obtain a topological space B, homeomorphic to $\mathbb{R}^2$, equipped with a cone decomposition Σ in two-dimensional cones $\sigma_{j,j+1}$, all meeting at a point that we call $0\in B$, and pairwise meeting along one-dimensional cones ρ_j. The pair (B,Σ) is the dual intersection complex of (Y,D). We define an integral linear structure on $B_0=B-\{0\}$ by the charts

$$\psi_j : U_j \to \mathbb{R}^2\,,$$

where $U_j \coloneqq \mathrm{Int}(\sigma_{j-1,j}\cup\sigma_{j,j+1})$ and ψ_j is defined on the closure of U_j by

$$\psi_j(v_{j-1})=(1,0)\,,\psi_j(v_j)=(0,1)\,,\psi_j(v_{j+1})=(-1,-D_j^2)\,,$$

where v_j is a primitive generator of ρ_j and ψ_j is defined linearly on the two-dimensional cones. Let Λ be the sheaf of integral tangent vectors of B_0. It is a locally constant sheaf on B_0 of fiber $\mathbb{Z}^2$.

The integral linear structure on B_0 extends to B through 0 if and only if (Y,D) is toric. In this case, B can be identified with $\mathbb{R}^2$ as integral linear manifold and Σ is simply the fan of the toric variety Y. In general, the integral linear structure is singular at 0, with a non-trivial monodromy along a loop going around 0.

As B_0 is an integral linear manifold, its set $B_0(\mathbb{Z})$ of integral points is well-defined. We denote $B(\mathbb{Z})\coloneqq B_0(\mathbb{Z})\cup\{0\}$. If (Y,D) is toric, with $Y-D=(\mathbb{C}^*)^2$, then $B(\mathbb{Z})$ is the lattice of cocharacters of $(\mathbb{C}^*)^2$, i.e. the lattice of one-parameter subgroups $\mathbb{C}^*\to(\mathbb{C}^*)^2$. Thus, intuitively, a point of $B_0(\mathbb{Z})$ is a way to go to infinity in $(\mathbb{C}^*)^2$. This intuition remains true in the non-toric case: a point in $B_0(\mathbb{Z})$ is a way to go to infinity in the interior U of the pair (Y,D).

More precisely, if we equip (Y,D) with its divisorial log structure, then $p\in B(\mathbb{Z})$ defines a tangency condition along D for a marked point x on a stable log curve $f: C\to(Y,D)$. If $p=0$, then $f(x)\notin D$. If $p=m_jv_j$, $m_j\in\mathbb{N}$, then $f(x)\in D_j$ with tangency order m_j along D_j and tangency order zero along D_{j-1} and D_{j+1}. If $p=m_jv_{j+1}+m_{j+1}v_{j+1}$, $m_j,m_{j+1}\in\mathbb{N}$, then $f(x)\in D_j\cap D_{j+1}$ with tangency order m_j along D_j and tangency order m_{j+1} along D_{j+1}[2].

Let P be a toric monoid and P^{gp} be its group completion, a finitely generated abelian group. Denote $P^{\mathrm{gp}}_{\mathbb{R}}\coloneqq P^{\mathrm{gp}}\otimes_{\mathbb{Z}}\mathbb{R}$, a finite dimensional $\mathbb{R}$-vector space. Let φ be a convex $P^{\mathrm{gp}}_{\mathbb{R}}$-valued multivalued Σ-piecewise linear function on B_0. Let Λ_j be the fiber of the sheaf Λ of integral tangent vectors over the chart U_j. Let $n_{j-1,j},n_{j,j+1}\in\Lambda_i^\vee\otimes P^{\mathrm{gp}}$ be the slopes of $\varphi|_{\sigma_{j-1,j}}$ and $\varphi|_{\sigma_{j,j+1}}$. Let Λ_{ρ_j} be the fiber of the sheaf of integral tangent vectors to the ray ρ_j. Let $\delta_j:\Lambda_j\to\Lambda_j/\Lambda_{\rho_j}\simeq\mathbb{Z}$ be the quotient map. We fix signs by requiring that δ_j is nonnegative on tangent vectors pointing from ρ_j to $\sigma_{j,j+1}$. Then $(n_{j,j+1}-n_{j-1,j})(\Lambda_{\rho_j})=0$ and hence

[2] This makes sense precisely because we are using log geometry.

there exists $\kappa_{\rho_j,\varphi} \in P$ with

$$n_{j,j+1} - n_{j-1,j} = \delta_j \kappa_{\rho_j,\varphi} \,,$$

called the kink of φ along ρ_j.

Let $\mathbb{B}_{0,\varphi}$ be the $P_{\mathbb{R}}^{\mathrm{gp}}$-torsor, which is set-theoretically $B_0 \times P_{\mathbb{R}}^{\mathrm{gp}}$ but with an integral affine structure twisted by φ: for each chart $\psi_j : U_j \to \mathbb{R}^2$ of B_0, we define a chart on $\mathbb{B}_{0,\varphi}$ by

$$(x,p) \mapsto \begin{cases} (\psi_j(x), p) & \text{if } x \in \sigma_{j-1,j} \\ (\psi_j(x), p + \tilde{\delta}_j(x)\kappa_{\rho_j,\varphi}) & \text{if } x \in \sigma_{j,j+1} \,, \end{cases}$$

where $\tilde{\delta}_j : \sigma_{j,j+1} \to \mathbb{R}_{\geqslant 0}$ is the integral affine map of differential δ_j. By definition, φ can be viewed as a section of the projection $\pi : \mathbb{B}_{0,\varphi} \to B_0$. Then $\mathcal{P} \coloneqq \varphi^* \Lambda_{\mathbb{B}_{0,\varphi}}$ is a locally constant sheaf on $\mathbb{B}_{0,\varphi}$, of fiber $\mathbb{Z}^2 \oplus P^{\mathrm{gp}}$, and the projection $\pi : \mathbb{B}_{0,\varphi} \to B_0$ induces a short exact sequence

$$0 \to \underline{P}^{\mathrm{gp}} \to \mathcal{P} \xrightarrow{\mathfrak{r}} \Lambda \to 0$$

of locally constant sheaves on B_0, where $\underline{P}^{\mathrm{gp}}$ is the constant sheaf on B_0 of fiber P^{gp}, and where $\mathfrak{r}$ is the derivative of π.

The sheaf Λ is naturally a sheaf of symplectic lattices: we have a skew-symmetric non-degenerate form

$$\langle -,- \rangle : \Lambda \otimes \Lambda \to \mathbb{Z} \,.$$

We extend $\langle -,- \rangle$ to a skew-symmetric form on $\mathcal{P}$ of kernel $\underline{P}^{\mathrm{gp}}$.

3.1.3 Algebras and quantum algebras

When we write "A is an R-algebra", we mean that A is an associative algebra with unit over a commutative ring with unit R. In particular, R is naturally contained in the center of A.

We fix $\Bbbk$ an algebraically closed field of characteristic zero and $i \in \Bbbk$ a square root of -1.

For every monoid[3] M equipped with a skew-symmetric bilinear form

$$\langle -,- \rangle : M \times M \to \mathbb{Z} \,,$$

we denote $\Bbbk[M]$ the monoid algebra of M, consisting of monomials z^m, $m \in M$, such that $z^m . z^{m'} = z^{m+m'}$. It is a Poisson algebra, of Poisson bracket determined by

$$\{z^m, z^{m'}\} = \langle m, m' \rangle z^{m+m'} \,.$$

We denote $\Bbbk_q \coloneqq \Bbbk[q^{\pm \frac{1}{2}}]$ and $\Bbbk_q[M]$ the possibly non-commutative $\Bbbk_q$-algebra structure on $\Bbbk[M] \otimes_{\Bbbk} \Bbbk_q$ such that

$$\hat{z}^m . \hat{z}^{m'} = q^{\frac{1}{2}\langle m, m' \rangle} \hat{z}^{m+m'} \,.$$

[3]All the monoids considered will be commutative and with an identity element.

We denote $\Bbbk_\hbar \coloneqq \Bbbk[[\hbar]]$. We view $\Bbbk_\hbar$ as a complete topological ring for the $\hbar$-adic topology and in particular, we will use the operation of completed tensor product $\hat{\otimes}$ with $\Bbbk_\hbar$:

$$(-)\hat{\otimes}_\Bbbk \Bbbk_\hbar \coloneqq \varprojlim_j (-) \otimes_\Bbbk (\Bbbk[\hbar]/\hbar^j)\,.$$

We view $\Bbbk_\hbar$ as a $\Bbbk_q$-module by the change of variables

$$q = e^{i\hbar} = \sum_{k \geqslant 0} \frac{(i\hbar)^k}{k!}\,.$$

We denote $\Bbbk_\hbar[M] \coloneqq \Bbbk_q[M]\hat{\otimes}_{\Bbbk_q}\Bbbk_\hbar$. The possibly non-commutative algebra $\Bbbk_\hbar[M]$ is a deformation quantization of the Poisson algebra $\Bbbk[M]$ in the sense that $\Bbbk_\hbar[M]$ is flat as $\Bbbk_\hbar$-module, and taking the limit $\hbar \to 0$, $q \to 1$, the linear term in $\hbar$ of the commutator $[\hat{z}^m, \hat{z}^{m'}]$ in $\Bbbk_\hbar[M]$ is determined by the Poisson bracket $\{\hat{z}^m, \hat{z}^{m'}\}$ in $\Bbbk[M]$:

$$[\hat{z}^m, \hat{z}^{m'}] = (q^{\frac{1}{2}\langle m,m'\rangle} - q^{-\frac{1}{2}\langle m,m'\rangle})\hat{z}^{m+m'} = \langle m, m'\rangle i\hbar \hat{z}^{m+m'} + \mathcal{O}(\hbar^2)\,.$$

We will often apply the constructions $\Bbbk[M]$ and $\Bbbk_\hbar[M]$ to M a fiber of the locally constant sheaves Λ or $\mathcal{P}$.

In particular, considering the toric monoid P with the zero skew-symmetric form, we denote

$$R \coloneqq \Bbbk[P]$$

and

$$R^\hbar \coloneqq \Bbbk_\hbar[P] = R\hat{\otimes}_\Bbbk \Bbbk_\hbar\,.$$

For every monomial ideal I of R, we denote

$$R_I \coloneqq R/I$$

$$R_I^q \coloneqq R/I \otimes_\Bbbk \Bbbk_q = R_I[q^{\pm\frac{1}{2}}]$$

and

$$R_I^\hbar \coloneqq R^\hbar/I = R_I\hat{\otimes}_\Bbbk \Bbbk_\hbar = R_I[[\hbar]]\,.$$

Remark that the algebras $R^\hbar$, R_I^q and $R_I^\hbar$ are commutative.

3.1.4 Ore localization

As should be clear from the previous Section, we will be dealing with non-commutative rings. Unlike what happens for commutative rings, it is not possible in general to localize with respect to an arbitrary multiplicative subset of a non-commutative ring, because of left-right issues. These left-right issues are absent by definition if the multiplicative subset satisfies the so-called Ore conditions.

We refer for example to Section 2.1 of [Kap98] and Section 1.3 of [Gin98] for short presentations of these elementary notions of non-commutative algebra. A multiplicative subset

$S \subset A - \{0\}$ of an associative ring A is said to satisfy the Ore conditions if

- For all $a \in A$ and $s \in S$, there exists $b \in A$ and $t \in S$ such that[4] $ta = bs$.
- For all $a \in A$, if there exists $s \in S$ such that $as = 0$, then there exists $t \in S$ such that $ta = 0$.
- For all $b \in A$ and $t \in S$, there exists $a \in A$ and $s \in S$ such that[5] $ta = bs$.
- For all $a \in A$, if there exists $s \in S$ such that $sa = 0$, then there exists $t \in S$ such that $at = 0$.

If S is a multiplicative subset of an associative ring A and if S satisfies the Ore conditions, then there is a well-defined localized ring $A[S^{-1}]$.

Let R be a commutative ring. Denote $R^\hbar \coloneqq R[[\hbar]]$.

Lemma 3.1. *Let A be a $R^\hbar$-algebra such that $A_0 \coloneqq A/\hbar A$ is a commutative R-algebra. Assume that A is $\hbar$-nilpotent, i.e. that there exists j such that $\hbar^j A = 0$. Denote $\pi\colon A \to A_0$ the natural projection. Let $\overline{S} \subset A_0 - \{0\}$ be a multiplicative subset. Then the multiplicative subset $S \coloneqq \pi^{-1}(\overline{S})$ of A satisfies the Ore conditions.*

Proof. See the proof of Proposition 2.1.5 in [Kap98]. □

Definition 3.2. *Let A be a $R^\hbar$-algebra such that $A_0 \coloneqq A/\hbar A$ is a commutative R-algebra. Assume that A is $\hbar$-complete, i.e. that $A = \varprojlim_j A/\hbar^j A$. By Lemma 3.1, each $A/\hbar^j A$ defines a sheaf of algebras on $X_0 \coloneqq \mathrm{Spec}\, A_0$, that we denote $\mathcal{O}^\hbar_{X_0}/\hbar^j$. We define*

$$\mathcal{O}^\hbar_{X_0} \coloneqq \varprojlim_j \mathcal{O}^\hbar_{X_0}/\hbar^j \,.$$

It is a sheaf in $R^\hbar$-algebras over X_0, such that $\mathcal{O}^\hbar_{X_0}/\hbar = \mathcal{O}_{X_0}$.

Remark: Definition 3.2 gives us a systematic way to turn certain non-commutative algebras into sheaves of non-commutatives algebras.

3.1.5 The Gross-Hacking-Keel mirror family

We fix (Y,D) a Looijenga pair. Let $NE(Y)_\mathbb{R} \subset A_1(Y,\mathbb{R})$ be the cone generated by curve classes and let $NE(Y)$ be the monoid $NE(Y)_\mathbb{R} \cap A_1(Y,\mathbb{Z})$.

Let $\sigma_P \subset A_1(Y,\mathbb{R})$ be a strictly convex polyhedral cone containing $NE(Y)_\mathbb{R}$. Let $P \coloneqq \sigma_P \cap A_1(Y,\mathbb{Z})$ be the associated toric monoid and let $R \coloneqq \Bbbk[P]$ be the corresponding $\Bbbk$-algebra. We denote t^β the monomial in R defined by $\beta \in P$. Let $\mathfrak{m}_R$ be the maximal

[4] Informally, $as^{-1} = t^{-1}b$, i.e. every fraction with a denominator on the right can be rewritten as a fraction with a denominator on the left.

[5] Informally, $t^{-1}b = as^{-1}$, i.e. every fraction with a denominator on the left can be rewritten as a fraction with a denominator on the right.

monomial ideal of R. For every monomial ideal I of R with radical $\mathfrak{m}_R$, we denote $R_I \coloneqq R/I$ and $S_I \coloneqq \mathrm{Spec}\ R_I$.

Let $T^D \coloneqq \mathbb{G}_m^r$ be the torus whose character group has a basis e_{D_j} indexed by the irreducible components D_j of D. The map

$$\beta \mapsto \sum_{j=1}^{r} (\beta.D_j) e_{D_j}$$

induces an action of T^D on S_I.

Theorem 0.1 of [GHK15a] gives the existence of a flat T^D-equivariant morphism

$$X_I \to S_I \,,$$

with X_I affine. The algebra of functions of X_I is given as R_I-module by

$$H^0(X_I, \mathcal{O}_{X_I}) = A_I \coloneqq \bigoplus_{p \in B(\mathbb{Z})} R_I \vartheta_p \,,$$

The algebra structure on $H^0(X_I, \mathcal{O}_{X_I})$ is determined by genus zero log Gromov-Witten invariants of (Y, D).

By Theorem 0.2. of [GHK15a], there exists a unique smallest radical monomial ideal $J_{\min} \subset R$ such that,

- For every monomial ideal I of R of radical containing $J_{\min}$, there is a finitely generated R_I-algebra structure on A_I compatible with the $R_{I+\mathfrak{m}^N}$-algebra structure on $A_{I+\mathfrak{m}^N}$ given by Theorem 0.1 of [GHK15a] for all $N > 0$.
- The zero locus $V(J_{\min}) \subset \mathrm{Spec}\ R$ contains the union of the closed toric strata corresponding to faces F of σ_P such that there exists $1 \leqslant j \leqslant r$ such that $[D_j] \notin F$. If (Y, D) is positive, then $J_{\min} = 0$ and $V(J_{\min}) = \mathrm{Spec}\ R$.
- Let $\hat{R}^{J_{\min}}$ denote the $J_{\min}$-adic completion of R. The algebras A_I determine a T^D-equivariant formal flat family of affine surfaces $\mathcal{X}^{J_{\min}} \to \mathrm{Spf}\ \hat{R}^{J_{\min}}$. The theta functions ϑ_p determine a canonical embedding $\mathcal{X}^{J_{\min}} \subset \mathbb{A}^{\max(r,3)} \times \mathrm{Spf}\ \hat{R}^{J_{\min}}$. In particular, if (Y, D) is positive, then we get an algebraic family $\mathcal{X} \to \mathrm{Spec}\ R$ and the theta functions ϑ_p determine a canonical embedding $\mathcal{X} \subset \mathbb{A}^{\max(r,3)} \times \mathrm{Spec}\ R$.

3.1.6 Deformation quantization

We discuss below the notion of deformation quantization. There are two technical aspects to keep in mind: first, we work relatively to a non-trivial base, second, we work in general with formal schemes. We refer to [Kon01], [Yek05], [BK04], for general facts about deformation quantization in algebraic geometry.

Definition 3.3. *A Poisson scheme over a scheme S is a scheme $\pi\colon X \to S$ over S, equipped with a $\pi^{-1}\mathcal{O}_S$-bilinear Poisson bracket, i.e. a $\pi^{-1}\mathcal{O}_S$-bilinear skew-symmetric map of sheaves*

$$\{-,-\}\colon \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X \,,$$

which is a biderivation

$$\{a, bc\} = \{a, b\}c + \{a, c\}b\,,$$

and a Lie bracket

$$\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0\,.$$

The two definitions below give two notions of deformation quantization of a Poisson scheme.

Definition 3.4. *Let $\pi\colon (X, \{-,-\}) \to S$ be a Poisson scheme over a scheme S. A* deformation quantization *of $(X, \{-,-\})$ over S is a sheaf $\mathcal{O}_X^\hbar$ of associative flat $\pi^{-1}\mathcal{O}_S \hat{\otimes} \mathbb{k}_\hbar$-algebras on X, complete in the $\hbar$-adic topology, equipped with an isomorphism $\mathcal{O}_X^\hbar / \hbar \mathcal{O}_X^\hbar \simeq \mathcal{O}_X$, such that for every f and g in $\mathcal{O}_X$, and $\tilde{f}$ and $\tilde{g}$ lifts of f and g in $\mathcal{O}_X^\hbar$, we have*

$$[\tilde{f}, \tilde{g}] = i\hbar\{f, g\} \mod \hbar^2\,,$$

where $[\tilde{f}, \tilde{g}] \coloneqq \tilde{f}\tilde{g} - \tilde{g}\tilde{f}$ is the commutator in $\mathcal{O}_X^\hbar$.

Definition 3.5. *Let $\pi\colon (X, \{-,-\}) \to S$ be a Poisson scheme over a scheme S. Assume that both X and S are affine. A* deformation quantization *of $(X, \{-,-\})$ over S is a flat $H^0(S, \mathcal{O}_S) \hat{\otimes} \mathbb{k}_\hbar$-algebra A, complete in the $\hbar$-adic topology, equipped with an isomorphism $A/\hbar A \simeq H^0(X, \mathcal{O}_X)$, such that for every f and g in $H^0(X, \mathcal{O}_X)$, and $\tilde{f}$ and $\tilde{g}$ lifts of f and g in A, we have*

$$[\tilde{f}, \tilde{g}] = i\hbar\{f, g\} \mod \hbar^2\,,$$

where $[\tilde{f}, \tilde{g}] \coloneqq \tilde{f}\tilde{g} - \tilde{g}\tilde{f}$ is the commutator in A.

The compatibility of these two definitions is guaranteed by the following lemma.

Lemma 3.6. *When both X and S are affine, the notions of deformation quantization given by Definitions 3.5 and 3.4 are equivalent.*

Proof. One goes from a sheaf quantization to an algebra quantization by taking global sections. One goes from an algebra quantization to a sheaf quantization by Ore localization, see Section 3.1.4. □

Definitions 3.4 and 3.5 and Lemma 3.6 have obvious analogues if one replaces schemes by formal schemes[6].

3.1.7 Main results

We fix (Y, D) a Looijenga pair and we use notations introduced in Section 3.1.5. The Gross-Hacking-Keel mirror family $X_I \to S_I$ has a natural structure of Poisson scheme over S_I. Indeed, the Gross-Hacking-Keel construction glues together simple pieces having natural Poisson structures by Poisson preserving gluing maps. Our main result, Theorem 3.7, is the construction of a deformation quantization of the Gross-Hacking-Keel mirror family by a higher genus deformation of the Gross-Hacking-Keel construction.

[6]or, in fact, any locally ringed space.

Theorem 3.7. *Let I be a monomial ideal of R with radical $\mathfrak{m}_R$. Then there exists a flat T^D-equivariant $R_I^\hbar$-algebra $A_I^\hbar$, such that $A_I^\hbar$ is a deformation quantization over S_I of the Gross-Hacking-Keel mirror family $X_I \to S_I$, and $A_I^\hbar$ is given as $R_I^\hbar$-module by*

$$A_I^\hbar = \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p \,,$$

where the algebra structure is determined by higher genus log Gromov-Witten invariants of (Y, D), with genus expansion parameter identified with the quantization parameter $\hbar$.

Remark: Taking the limit over all monomial ideals I of R with radical $\mathfrak{m}_R$, we get a deformation quantization of the formal family

$$\varinjlim_I X_I \to \varinjlim_I S_I \,.$$

The following Theorem is a quantum version of Theorem 0.2 of [GHK15a].

Theorem 3.8. *There is a unique smallest radical monomial $J_{\min}^\hbar \subset R$ such that,*

- *For every monomial ideal I of R of radical containing $J_{\min}^\hbar$, there is a finitely generated $R_I^\hbar$-algebra structure on*
$$A_I^\hbar = \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p \,,$$
compatible with the $R^\hbar_{I+\mathfrak{m}_R^k}$-algebra structure on $A^\hbar_{I+\mathfrak{m}_R^k}$ given by Theorem 3.7 for all $k > 0$.
- *The zero locus $V(J_{\min}) \subset R$ contains the union of the closed toric strata corresponding to faces F of σ_P such that there exists $1 \leqslant j \leqslant r$ such that $[D_j] \notin F$. If (Y, D) is positive, then $J_{\min}^\hbar = 0$, i.e. $V(J_{\min}^\hbar) = \operatorname{Spec} R$ and $A_0^\hbar$ is a deformation quantization of the mirror family $\mathcal{X} \to \operatorname{Spec} R$.*

The following result, Theorem 3.9, controls the dependence in $\hbar$ of the deformation quantization given by Theorem 3.8: this dependence is algebraic in $q = e^{i\hbar}$.

Theorem 3.9. *Let I be a monomial ideal of R with radical containing $J_{\min}^\hbar$. Then there exists a flat R_I^q-algebra A_I^q such that*

$$A_I^\hbar = A_I^q \hat{\otimes}_{\mathbb{k}_q} \mathbb{k}_\hbar \,,$$

where $\mathbb{k}_\hbar$ is viewed as a $\mathbb{k}_q$-module via $q = e^{i\hbar}$.

The proof of Theorems 3.7, 3.8, and 3.9 takes Sections 3.2, 3.3, 3.4. In Section 3.2, we explain how a consistent quantum scattering diagram can be used as input to a construction of quantum modified Mumford degeneration, giving a deformation quantization of the modified Mumford degeneration of [GHK15a], [GHKS16], constructed from a classical scattering diagram. In Section 3.3, we explain how to construct a quantum scattering diagram from higher genus log Gromov-Witten theory of a Looijenga pair and we prove its consistency using Theorem 2.6, the main result of Chapter 2. We finish the proof of Theorems 3.7, 3.8 and 3.9 in Section 3.4.

3.2 Quantum modified Mumford degenerations

In this Section, we explain how to construct a quantization of the mirror family of a given Looijenga pair (Y, D) starting from its tropicalization (B, Σ) and a consistent quantum scattering diagram.

In Section 3.2.1, we describe the rings $R^{\hbar}_{\sigma,I}$ and $R^{\hbar}_{\sigma,I}$ involved in the construction of the quantum version of modified Mumford degenerations. In Section 3.2.2, we review the notion of quantum scattering diagrams. In Section 3.2.4, we explain how a consistent quantum scattering diagram gives a way to glue together the rings $R^{\hbar}_{\sigma,I}$ and $R^{\hbar}_{\sigma,I}$ to produce a quantum modified Mumford degeneration. In Section 3.2.6, we review the notions of quantum broken lines and theta functions and we use them in Section 3.2.7 to prove that the quantum modified Mumford degeneration is indeed a deformation quantization of the modified Mumford degeneration of [GHK15a]. In Section 3.2.8, we express the structure constants of the quantum algebra of global sections in terms of quantum broken lines.

3.2.1 Building blocks

The goal of this Section is to define non-commutative deformations $R^{\hbar}_{\sigma,I}$ and $R^{\hbar}_{\rho,I}$ of the rings $R_{\sigma,I}$ and $R_{\rho,I}$ defined in Sections 2.1 and 2.2 of [GHK15a]. The way to go from $R_{\sigma,I}$ to $R^{\hbar}_{\sigma,I}$ is fairly obvious. The deformation $R^{\hbar}_{\rho,I}$ of $R_{\rho,I}$ is maybe not so obvious.

We fix (Y, D) a Looijenga pair, (B, Σ) its tropicalization, P a toric monoid, J a radical monomial ideal of P, and φ a $P^{\mathrm{gp}}_{\mathbb{R}}$-valued multivalued convex Σ-piecewise linear function on B.

For any locally constant sheaf $\mathcal{F}$ on B_0 and any simply connected subset τ of B_0, we write $\mathcal{F}_\tau$ for the stalk of this local system at any point of τ. We will constantly use this notation for τ a cone of Σ.

If τ is a cone of Σ, we define the localized fan $\tau^{-1}\Sigma$ as being the fan in $\Lambda_{\mathbb{R},\tau}$ defined as follows:

- If τ is two-dimensional, then $\tau^{-1}\Sigma$ consists just of the entire space $\Lambda_{\mathbb{R},\tau}$.
- If τ is one-dimensional, then $\tau^{-1}\Sigma$ consists of the tangent line of τ in $\Lambda_{\mathbb{R},\tau}$ along with the two half-planes with boundary this tangent line.

For each τ cone of Σ, the Σ-piecewise P-convex function $\varphi \colon B_0 \to \mathbb{B}_{0,\varphi}$ determines a $\tau^{-1}\Sigma$-piecewise linear P-convex function $\varphi_\tau \colon \Lambda_{\mathbb{R},\tau} \to \mathcal{P}_{\mathbb{R},\tau}$. We define the toric monoid $P_{\varphi_\tau} \subset \mathcal{P}_\tau$ by

$$P_{\varphi_\tau} \coloneqq \{ s \in \mathcal{P}_\tau \mid s = p + \varphi_\tau(m) \text{ for some } p \in P,\ m \in \Lambda_\tau \}.$$

If ρ is a one-dimensional cone of Σ, bounding the two-dimensional cones σ_+ and σ_- of Σ, we have $P_{\varphi_\rho} \subset P_{\varphi_{\sigma_+}}$, $P_{\varphi_\rho} \subset P_{\varphi_{\sigma_-}}$, and

$$P_{\varphi_{\sigma_+}} \cap P_{\varphi_{\sigma_-}} = P_{\varphi_\rho}\,.$$

Figure: $P_{\varphi_{\sigma_+}}$

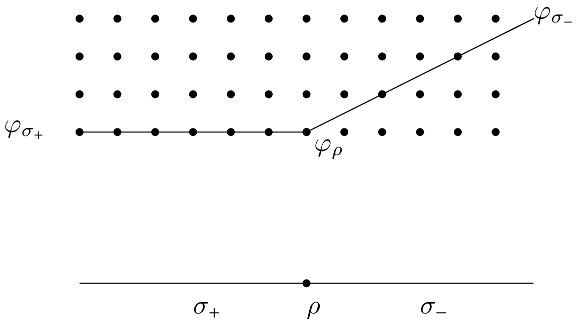


Figure: $P_{\varphi_{\sigma_-}}$

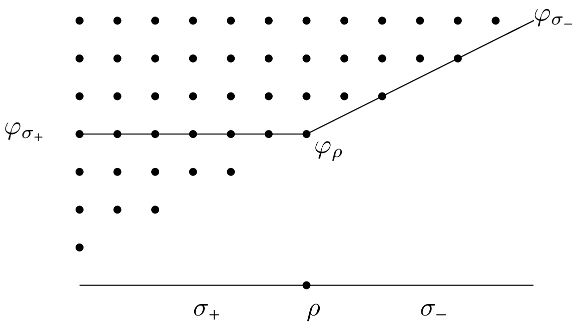
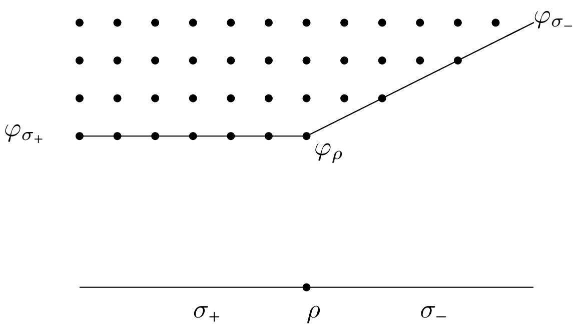


Figure: P_{φ_ρ}



For every σ two-dimensional cone of Σ, we define $R^{\hbar}_{\sigma,I} \coloneqq \mathbb{k}_{\hbar}[P_{\varphi_\sigma}]/I$, deformation quantization of $R_{\sigma,I} \coloneqq \mathbb{k}[P_{\varphi_\sigma}]/I$. We have a natural trivialization $P_{\varphi_\sigma} = P \oplus \Lambda_\sigma$ and so $R^{\hbar}_{\sigma,I}$ is simply the algebra of functions on a trivial family of two-dimensional quantum tori parametrized by Spec R_I.

Let ρ be a one-dimensional cone of Σ. Let $\kappa_{\rho,\varphi} \in P$ be the kink of φ across ρ, so that $z^{\kappa_{\rho,\varphi}} \in R_I$. Let X be an invertible formal variable. We fix elements $\hat{f}_{\rho^{\text{out}}} \in R^{\hbar}_I[X^{-1}]$ and $\hat{f}_{\rho^{\text{in}}} \in R^{\hbar}_I[X]$.

Let $R^{\hbar}_{\rho,I}$ be the $R^{\hbar}_I$-algebra generated by formal variables X_+, X_- and X, with X invertible, and with relations

$$XX_+ = qX_+X\,,$$
$$XX_- = q^{-1}X_-X\,,$$
$$X_+X_- = q^{\frac{1}{2}D^2_\rho}\hat{z}^{\kappa_{\rho,\varphi}}\hat{f}_{\rho^{\text{out}}}(q^{-1}X)\hat{f}_{\rho^{\text{in}}}(X)X^{-D^2_\rho}\,,$$
$$X_-X_+ = q^{-\frac{1}{2}D^2_\rho}\hat{z}^{\kappa_{\rho,\varphi}}\hat{f}_{\rho^{\text{out}}}(X)\hat{f}_{\rho^{\text{in}}}(qX)X^{-D^2_\rho}\,,$$

where $q = e^{i\hbar}$. The $R^{\hbar}_I$-algebra $R^{\hbar}_{\rho,I}$ is flat as $R^{\hbar}_I$-module and so is a deformation quantization of

$$R_{\rho,I} \coloneqq R_I[X_+,X_-,X^{\pm}]/(X_+X_- - z^{\kappa_{\rho,\varphi}}X^{-D^2_\rho}f_{\rho^{\text{out}}}(X)f_{\rho^{\text{in}}}(X))\,.$$

Let σ_+ and σ_- be the two-dimensional cones of Σ bounding ρ, and let ρ_+ and ρ_- be the other boundary rays of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order.

The precise form of $R^{\hbar}_{\rho,I}$ is justified by the following Proposition.

Proposition 3.10. *The map of $R^{\hbar}_I$-algebras*

$$\tilde{\psi}_{\rho,-}\colon R^{\hbar}_I\langle X_+,X_-,X^{\pm}\rangle \to R^{\hbar}_{\sigma_-,I}$$

defined by

$$\begin{aligned}\tilde{\psi}_{\rho,-}(X) &= \hat{z}^{\varphi_\rho(m_\rho)}\,,\\ \tilde{\psi}_{\rho,-}(X_-) &= \hat{z}^{\varphi_\rho(m_{\rho_-})}\,,\\ \tilde{\psi}_{\rho,-}(X_+) &= \hat{f}_{\rho^{\text{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}_{\rho^{\text{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\\ &= \hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}_{\rho^{\text{in}}}(q\hat{z}^{\varphi_\rho(m_\rho)})\hat{f}_{\rho^{\text{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\,,\end{aligned}$$

induces a map of $R^{\hbar}_I$-algebras

$$\hat{\psi}_{\rho,-}\colon R^{\hbar}_{\rho,I} \to R^{\hbar}_{\sigma_-,I}\,.$$

The map of $R^{\hbar}_I$-algebras

$$\tilde{\psi}_{\rho,+}\colon R^{\hbar}_I\langle X_+,X_-,X^{\pm}\rangle \to R^{\hbar}_{\sigma_+,I}\,,$$

defined by

$$\begin{aligned}
\tilde{\psi}_{\rho,+}(X) &= \hat{z}^{\varphi_\rho(m_\rho)}\,,\\
\tilde{\psi}_{\rho,+}(X_-) &= \hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_-})}\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\\
&= \hat{z}^{\varphi_\rho(m_{\rho_-})}\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}\hat{z}^{\varphi_\rho(m_\rho)})\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\,,\\
\tilde{\psi}_{\rho,+}(X_+) &= \hat{z}^{\varphi_\rho(m_{\rho_+})}\,,
\end{aligned}$$

induces a map of $R_I^\hbar$-algebras

$$\hat{\psi}_{\rho,+}\colon R^\hbar_{\rho,I} \to R^\hbar_{\sigma_+,I}\,.$$

Proof. We have to check that $\tilde{\psi}_{\rho,-}$ and $\tilde{\psi}_{\rho,+}$ map the relations defining $R^\hbar_{\rho,I}$ to zero. We have $\langle m_\rho, m_{\rho_+}\rangle = 1$ and $\langle m_\rho, m_{\rho_-}\rangle = -1$. It follows that

$$\tilde{\psi}_{\rho,-}(XX_+ - qX_+X) = 0\,,$$

$$\tilde{\psi}_{\rho,+}(XX_+ - qX_+X) = 0\,,$$

and

$$\tilde{\psi}_{\rho,-}(XX_- - q^{-1}X_-X) = 0\,,$$

$$\tilde{\psi}_{\rho,+}(XX_- - q^{-1}X_-X) = 0\,.$$

Furthermore, we have

$$m_{\rho_-} + D_\rho^2 m_\rho + m_{\rho_+} = 0$$

so

$$\langle m_{\rho_+}, m_{\rho_-}\rangle = D_\rho^2$$

and

$$\varphi_\rho(m_{\rho_-}) + \varphi_\rho(m_{\rho_+}) = \kappa_{\rho,\varphi} - D_\rho^2\varphi_\rho(m_\rho)\,.$$

It follows that

$$\begin{aligned}
\tilde{\psi}_{\rho,-}(X_+X_-) &= \hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_-})}\\
&= q^{\frac{1}{2}D_\rho^2}\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\kappa_{\rho,\varphi}-D_\rho^2\varphi_\rho(m_\rho)}\\
&= q^{\frac{1}{2}D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}}\tilde{\psi}_{\rho,-}\left(\hat{f}_{\rho^{\mathrm{in}}}(X)\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)X^{-D_\rho^2}\right)\,,
\end{aligned}$$

$$\begin{aligned}
\tilde{\psi}_{\rho,+}(X_+X_-) &= \hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_-})}\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\\
&= q^{\frac{1}{2}D_\rho^2}\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\kappa_{\rho,\varphi}-D_\rho^2\varphi_\rho(m_\rho)}\\
&= q^{\frac{1}{2}D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}}\tilde{\psi}_{\rho,+}\left(\hat{f}_{\rho^{\mathrm{in}}}(X)\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)X^{-D_\rho^2}\right)\,,
\end{aligned}$$

and

$$\begin{aligned}\tilde{\psi}_{\rho,-}(X_-X_+) &= \hat{z}^{\varphi_\rho(m_{\rho_-})} f_{\rho^{\text{in}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \hat{z}^{\varphi_\rho(m_{\rho_+})} f_{\rho^{\text{out}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \\ &= q^{-\frac{1}{2}D_\rho^2} \hat{z}^{\kappa_{\rho,\varphi} - D_\rho^2 \varphi_\rho(m_\rho)} f_{\rho^{\text{in}}}(\hat{z}^{q\varphi_\rho(m_\rho)}) f_{\rho^{\text{out}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \\ &= q^{-\frac{1}{2}D_\rho^2} \hat{z}^{\kappa_{\rho,\varphi}} \tilde{\psi}_{\rho,-}\left(\hat{f}_{\rho^{\text{in}}}(qX)\hat{f}_{\rho^{\text{out}}}(X) X^{-D_\rho^2}\right),\end{aligned}$$

$$\begin{aligned}\tilde{\psi}_{\rho,+}(X_-X_+) &= \hat{f}_{\rho^{\text{out}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \hat{z}^{\varphi_\rho(m_{\rho_-})} f_{\rho^{\text{in}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \hat{z}^{\varphi_\rho(m_{\rho_+})} \\ &= q^{-\frac{1}{2}D_\rho^2} \hat{f}_\rho(\hat{z}^{\varphi_\rho(m_\rho)}) f_{\rho^{\text{in}}}(q\hat{z}^{\varphi_\rho(m_\rho)}) \hat{z}^{\kappa_{\rho,\varphi} - D_\rho^2 \varphi_\rho(m_\rho)} \\ &= q^{-\frac{1}{2}D_\rho^2} \hat{z}^{\kappa_{\rho,\varphi}} \tilde{\psi}_{\rho,+}\left(\hat{f}_{\rho^{\text{in}}}(qX)\hat{f}_{\rho^{\text{out}}}(X) X^{-D_\rho^2}\right).\end{aligned}$$

□

Remark:

- In the special case where $D_\rho^2 = 0$ and $\hat{f}_{\rho^{\text{in}}} = 1$, our description of $R^\hbar_{\rho,I}$ by generators and relations coincides with the description given by Soibelman in Section 7.5 of [Soi09] of a local model for deformation quantization of a neighborhood of a focus-focus singularity.
- $R^\hbar_{\sigma,I}$ is a deformation quantization of $R_{\sigma,I}$, and $R^\hbar_{\rho,I}$ is a deformation quantization of $R_{\rho,I}$. The maps $\hat{\psi}_{\rho,+}$ and $\hat{\psi}_{\rho,-}$ are quantizations of the maps $\psi_{\rho,-}$ and $\psi_{\rho,+}$ defined by formula (2.8) of [GHK15a]. Following [GHK15a], we denote $U_{\sigma,I} \coloneqq \operatorname{Spec} R_{\sigma,I}$ and $U_{\rho,I} \coloneqq \operatorname{Spec} R_{\rho,I}$. If ρ is a one-dimensional cone of Σ, and σ_+ and σ_- are the two-dimensional cones of Σ bounding ρ, then the maps $\psi_{\rho,-}$ and $\psi_{\rho,+}$ induce open immersions
$$U_{\sigma-,I} \hookrightarrow U_{\rho,I}$$
and
$$U_{\sigma+,I} \hookrightarrow U_{\rho,I}\,.$$

Using Ore localization (see Definition 3.2), we can produce from $R^\hbar_{\sigma,I}$ and $R^\hbar_{\rho,I}$ some sheaves of flat $\Bbbk_\hbar$-algebras $\mathcal{O}^\hbar_{U_{\sigma,I}}$ and $\mathcal{O}^\hbar_{U_{\rho,I}}$ on $U_{\sigma,I}$ and $U_{\rho,I}$, such that

$$\mathcal{O}^\hbar_{U_{\sigma,I}}/\hbar\mathcal{O}^\hbar_{U_{\sigma,I}} \simeq \mathcal{O}_{U_{\sigma,I}}$$

and

$$\mathcal{O}^\hbar_{U_{\rho,I}}/\hbar\mathcal{O}^\hbar_{U_{\rho,I}} \simeq \mathcal{O}_{U_{\rho,I}}$$

respectively.

3.2.2 Quantum scattering diagrams

Quantum scattering diagrams have been studied by Filippini-Stoppa [FS15] in dimension two and by Mandel [Man15] in higher dimensions. We have already seen this kind of quantum scattering diagram in Chapter 2. Mandel [Man15] also studied quantum broken lines and quantum theta functions. The quantum scattering diagrams studied in [FS15], [Man15] or in our Chapter 2 live on a smooth integral affine manifold. We need to make some changes to include the case we care about, where the integral affine manifold is the tropicalization B of a Looijenga pair and has a singularity at the origin with a non-trivial monodromy around it.

As in the previous Section, we fix (Y,D) a Looijenga pair, (B,Σ) its tropicalization, P a toric monoid, J a radical monomial ideal of P, and φ a $P_{\mathbb{R}}^{\mathrm{gp}}$-valued multivalued convex Σ-piecewise linear function on B. Recall from Section 3.1.2 that we then have an exact sequence

$$0 \to \underline{P}^{\mathrm{gp}} \to \mathcal{P} \xrightarrow{\mathfrak{r}} \Lambda \to 0$$

of locally constant sheaves on B_0.

We explained in Section 3.2.1 how to define for every cone τ of Σ a toric monoid P_{φ_τ}. We denote by

$$\Bbbk_\hbar \widehat{[P_{\varphi_\tau}]}$$

the J-adic completion of the $\Bbbk_\hbar$-algebra $\Bbbk_\hbar[P_{\varphi_\tau}]$. The map $r: \mathcal{P} \to \Lambda$ induces a morphism of monoids $r: P_{\varphi_\tau} \to \Lambda_\tau$.

Definition 3.11. *A* quantum scattering diagram $\hat{\mathfrak{D}}$ *for the data* (B,Σ)*,* P*,* J *and* φ *is a set*

$$\hat{\mathfrak{D}} = \{(\mathfrak{d}, \hat{H}_{\mathfrak{d}})\}$$

where

- $\mathfrak{d} \subset B$ *is a ray of rational slope in* B *with endpoint the origin* $0 \in B$.
- *Let* $\tau_{\mathfrak{d}}$ *be the smallest cone of* Σ *containing* $\mathfrak{d}$ *and let* $m_{\mathfrak{d}} \in \Lambda_{\tau_{\mathfrak{d}}}$ *be the primitive generator of* $\mathfrak{d}$ *pointing away from the origin. Then we have either*

$$\hat{H}_{\mathfrak{d}} = \sum_{\substack{p \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ \mathfrak{r}(p) \in \mathbb{Z}_{<0} m_{\mathfrak{d}}}} H_p \hat{z}^p \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]},$$

  or

$$\hat{H}_{\mathfrak{d}} = \sum_{\substack{p \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ \mathfrak{r}(p) \in \mathbb{Z}_{>0} m_{\mathfrak{d}}}} H_p \hat{z}^p \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]}.$$

  In the first case, we say that the ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ *is* outgoing*, and in the second case, we say that the ray* $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ *is* ingoing.
- *Let* $\tau_{\mathfrak{d}}$ *be the smallest cone of* Σ *containing* $\mathfrak{d}$*. If* $\dim \tau_{\mathfrak{d}} = 2$*, or if* $\dim \tau_{\mathfrak{d}} = 1$ *and* $\kappa_{\tau_{\mathfrak{d}},\varphi} \notin J$*, then* $\hat{H}_{\mathfrak{d}} = 0 \mod J$.
- *For any ideal* $I \subset P$ *of radical* J*, there are only finitely many rays* $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ *such that* $\hat{H}_{\mathfrak{d}} \neq 0 \mod I$.

Remark: Given a ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ of a quantum scattering diagram, we call $\hat{H}_{\mathfrak{d}}$ the Hamiltonian attached to ρ. This terminology is justified by the following Section 3.2.3, where we attach to $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ the automorphism $\hat{\Phi}_{\hat{H}_{\mathfrak{d}}}$ given by the time one evolution according to the quantum Hamiltonian $\hat{H}_{\mathfrak{d}}$.

3.2.3 Quantum automorphisms

Let $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ is a ray of a quantum scattering diagram $\hat{\mathfrak{D}}$ for the data (B, Σ), P, J and φ. Let $\tau_{\mathfrak{d}}$ be the smallest cone of Σ containing $\mathfrak{d}$ and let $m_{\mathfrak{d}} \in \Lambda_{\tau_{\mathfrak{d}}}$ be the primitive generator of $\mathfrak{d}$ pointing away from the origin. Denote $m(\hat{H}_{\mathfrak{d}}) = m_{\mathfrak{d}}$ if $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ is outgoing and $m(\hat{H}_{\mathfrak{d}}) = -m_{\mathfrak{d}}$ if $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ is ingoing. Writing

$$\hat{H}_{\mathfrak{d}} = \sum_{p \in P_{\varphi_{\tau_{\mathfrak{d}}}}} H_p \hat{z}^p \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]},$$

we denote

$$\hat{f}_{\mathfrak{d}} \coloneqq \exp\left(\sum_{\substack{p \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ \mathfrak{r}(p) = \ell m(\hat{H}_{\mathfrak{d}})}} (q^\ell - 1) H_p \hat{z}^p \right) \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]},$$

where $q = e^{i\hbar}$. Remark that, by our definition of $m(\hat{H}_{\mathfrak{d}})$, we have $\ell \leqslant 0$ when writing $r(p) = \ell m(\hat{H}_{\mathfrak{d}})$.

We write

$$\hat{f}_{\mathfrak{d}} = \sum_{p \in P_{\varphi_{\tau_{\mathfrak{d}}}}} f_p \hat{z}^p .$$

For every $j \in \mathbb{Z}$, we define

$$\hat{f}_{\mathfrak{d}}(q^j \hat{z}) \coloneqq \sum_{\substack{p \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ \mathfrak{r}(p) = \ell m(\hat{H}_{\mathfrak{d}})}} q^{\ell j} f_p \hat{z}^p \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]},$$

where $q = e^{i\hbar}$.

Lemma 3.12. *The automorphism $\hat{\Phi}_{\hat{H}_{\mathfrak{d}}}$ of $\Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]}$ given by conjugation by $\exp\left(\hat{H}_{\mathfrak{d}}\right)$,*

$$\hat{z}^p \mapsto \exp\left(\hat{H}_{\mathfrak{d}}\right) \hat{z}^p \exp\left(-\hat{H}_{\mathfrak{d}}\right),$$

is equal to

$$\hat{z}^p \mapsto \begin{cases} \hat{z}^p \prod_{j=0}^{\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle - 1} \hat{f}_{\mathfrak{d}}(q^j \hat{z}) & \text{if } \langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle \geqslant 0 \\ \hat{z}^p \prod_{j=0}^{|\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle| - 1} \hat{f}_{\mathfrak{d}}(q^{-j-1} \hat{z})^{-1} & \text{if } \langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle < 0 . \end{cases}$$

Proof. Using that $\hat{z}^{p'} \hat{z}^p = q^{\langle r(p'), \mathfrak{r}(p)\rangle} \hat{z}^p \hat{z}^{p'}$, we get

$$\exp\left(\hat{H}_{\mathfrak{d}}\right) \hat{z}^p \exp\left(-\hat{H}_{\mathfrak{d}}\right) = \hat{z}^p \exp\left(\sum_{\substack{p' \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ r(p') = \ell m(\hat{H}_{\mathfrak{d}})}} (q^{\ell \langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle} - 1) H_{p'} \hat{z}^{p'} \right).$$

If $\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle \geqslant 0$, this can be written

$$\hat{z}^p \exp\left(\sum_{\substack{p' \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ r(p') = \ell m(\hat{H}_{\mathfrak{d}})}} \frac{1 - q^{\ell \langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle}}{1 - q^\ell} (q^\ell - 1) H_{p'} \hat{z}^{p'} \right)$$

$$= \hat{z}^p \exp\left(\sum_{\substack{p' \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ r(p') = \ell m(\hat{H}_{\mathfrak{d}})}} \sum_{j=0}^{\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle - 1} q^{\ell j} (q^\ell - 1) H_{p'} \hat{z}^{p'} \right)$$

$$= \hat{z}^p \prod_{j=0}^{\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle - 1} \hat{f}_{\mathfrak{d}}(q^j \hat{z}) \,.$$

If $\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle < 0$, this can be written

$$\hat{z}^p \exp\left(- \sum_{\substack{p' \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ r(p') = \ell m(\hat{H}_{\mathfrak{d}})}} \frac{1 - q^{-\ell |\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle|}}{1 - q^{-\ell}} q^{-\ell} (q^\ell - 1) H_{p'} \hat{z}^{p'} \right)$$

$$= \hat{z}^p \exp\left(- \sum_{\substack{p' \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ r(p') = \ell m(\hat{H}_{\mathfrak{d}})}} \sum_{j=0}^{|\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle| - 1} (q^{-j-1})^\ell (q^\ell - 1) H_{p'} \hat{z}^{p'} \right)$$

$$= \hat{z}^p \prod_{j=0}^{|\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle| - 1} \hat{f}_{\mathfrak{d}}(q^{-j-1} \hat{z})^{-1} \,.$$

□

Remark: One can equivalently write $\hat{\Phi}_{\hat{H}_{\mathfrak{d}}}$ as

$$\hat{z}^p \mapsto \begin{cases} \left(\prod_{j=0}^{\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle - 1} \hat{f}_\rho(q^{-j-1} z) \right) \hat{z}^p & \text{if } \langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle \geqslant 0 \\ \left(\prod_{j=0}^{|\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle| - 1} \hat{f}_\rho(q^j z)^{-1} \right) \hat{z}^p & \text{if } \langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p)\rangle < 0 \,. \end{cases}$$

A direct application of the definition of $\hat{f}_{\mathfrak{d}}$ gives the following Lemma.

Lemma 3.13. *If*

$$\hat{H} = i \sum_{\ell \geqslant 1} \frac{(-1)^{\ell-1}}{\ell} \frac{\hat{z}^{-\ell \varphi(m_{\mathfrak{d}})}}{2 \sin\left(\frac{\ell \hbar}{2}\right)} = - \sum_{\ell \geqslant 1} \frac{(-1)^{\ell-1}}{\ell} \frac{\hat{z}^{-\ell \varphi(m_{\mathfrak{d}})}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \,,$$

where $q = e^{i\hbar}$, we have $m(\hat{H}) = m_{\mathfrak{d}}$ and

$$\hat{f} = \exp\left(- \sum_{\ell \geqslant 1} \frac{(-1)^{\ell-1}}{\ell} \frac{q^{-\ell}-1}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \hat{z}^{-\ell\varphi(m_{\mathfrak{d}})} \right) = 1 + q^{-\frac{1}{2}} \hat{z}^{-\varphi(m_{\mathfrak{d}})} \,.$$

If

$$\hat{H} = i \sum_{\ell \geqslant 1} \frac{(-1)^{\ell-1}}{\ell} \frac{\hat{z}^{\ell\varphi(m_{\mathfrak{d}})}}{2\sin\left(\frac{\ell\hbar}{2}\right)} = - \sum_{\ell \geqslant 1} \frac{(-1)^{\ell-1}}{\ell} \frac{\hat{z}^{\ell\varphi(m_{\mathfrak{d}})}}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \,,$$

where $q = e^{i\hbar}$, we have $m(\hat{H}) = -m_{\mathfrak{d}}$ and

$$\hat{f} = \exp\left(- \sum_{\ell \geqslant 1} \frac{(-1)^{\ell-1}}{\ell} \frac{q^{-\ell}-1}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} \hat{z}^{\ell\varphi(m_{\mathfrak{d}})} \right) = 1 + q^{-\frac{1}{2}} \hat{z}^{\varphi(m_{\mathfrak{d}})} \,.$$

3.2.4 Gluing

We fix a quantum scattering diagram $\hat{\mathfrak{D}}$ for the data (B, Σ), P, J and φ, and an ideal I of radical J.

Let ρ be a one-dimensional cone of Σ, bounding the two-dimensional cones σ_+ and σ_-, such that σ_-, ρ, σ_+ are in an anticlockwise order. Identifying X with $\hat{z}^{\varphi_\rho(m_\rho)}$, we define $\hat{f}_{\rho^{\text{out}}} \in R_I^\hbar[X^{-1}]$ by

$$\hat{f}_{\rho^{\text{out}}} \coloneqq \prod_{\substack{\mathfrak{d} \in \hat{\mathfrak{D}}, \mathfrak{d} = \rho \\ \text{outgoing}}} \hat{f}_{\mathfrak{d}} \mod I \,,$$

where the product is over the outgoing rays of $\hat{\mathfrak{D}}$ of support ρ, and we define $\hat{f}_{\rho^{\text{in}}} \in R_I^\hbar[X]$ by

$$\hat{f}_{\rho^{\text{in}}} \coloneqq \prod_{\substack{\mathfrak{d} \in \hat{\mathfrak{D}}, \mathfrak{d} = \rho \\ \text{ingoing}}} \hat{f}_{\mathfrak{d}} \mod I \,,$$

where the product is over the ingoing rays of $\hat{\mathfrak{D}}$ of support ρ.

By Section 3.2.1, we then have $R_I^\hbar$-algebras $R_{\sigma_+,I}^\hbar$, $R_{\sigma_-,I}^\hbar$, $R_{\rho,I}^\hbar$.

Let $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ be a ray of $\hat{\mathfrak{D}}$ such that $\tau_{\mathfrak{d}} = \sigma$ is a two-dimensional cone of Σ. Let $m_{\mathfrak{d}} \in \Lambda_{\tau_{\mathfrak{d}}}$ be the primitive generator of $\mathfrak{d}$ pointing away from the origin. Let γ be a path in B_0 which crosses $\mathfrak{d}$ transversally at time t_0. We define

$$\hat{\theta}_{\gamma,\mathfrak{d}} \colon R_{\sigma,I}^\hbar \to R_{\sigma,I}^\hbar \,,$$

$$\hat{z}^p \mapsto \hat{\Phi}_{\hat{H}_{\mathfrak{d}}}^\epsilon (\hat{z}^p) \,,$$

where $\epsilon \in \{\pm 1\}$ is the sign of $-\langle m(\hat{H}_{\mathfrak{d}}), \gamma'(t_0) \rangle$.

Let $\hat{\mathfrak{D}}_I \subset \hat{\mathfrak{D}}$ be the finite set of rays $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ with $\hat{H}_{\mathfrak{d}} \neq 0 \mod I$, i.e. $\hat{f}_{\mathfrak{d}} \neq 1 \mod I$. If γ is a path in B_0 entirely contained in the interior of a two-dimensional cone σ of Σ, and crossing elements of $\mathfrak{D}_I$ transversally, we define

$$\hat{\theta}_{\gamma,\hat{\mathfrak{D}}_I} \coloneqq \hat{\theta}_{\gamma,\mathfrak{d}_n} \circ \cdots \circ \hat{\theta}_{\gamma,\mathfrak{d}_1} \,,$$

where γ crosses the elements $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ of $\hat{\mathfrak{D}}_I$ in the given order.

For every σ two-dimensional cone of Σ, bounded by rays ρ_+ and ρ_-, such that ρ_-, σ, ρ_+ are in anticlockwise order, we choose $\gamma_\sigma\colon [0,1] \to B_0$ a path whose image is entirely contained in the interior of σ, with $\gamma(0)$ close to ρ_- and $\gamma(1)$ close to ρ_+, such that γ_σ crosses every ray of $\hat{\mathfrak{D}}_I$ contained in σ transversally exactly once. Let

$$\hat{\theta}_{\gamma_\sigma, \hat{\mathfrak{D}}_I}\colon R^{\hbar}_{\sigma,I} \to R^{\hbar}_{\sigma,I}$$

be the corresponding automorphism. In the classical limit, $\hat{\theta}_{\gamma,\hat{\mathfrak{D}}_I}$ induces an automorphism $\theta_{\gamma,\mathfrak{D}_I}$ of $U_{\sigma,I}$. Gluing together the open sets $U_{\sigma,I} \subset U_{\rho_-,I}$ and $U_{\sigma,I} \subset U_{\rho_+,I}$ along these automorphisms, we get the scheme $X^\circ_{I,\mathfrak{D}}$ defined in [GHK15a].

Recall from the end of Section 3.2.1 that by Ore localization the algebras $R^{\hbar}_{\sigma,I}$ and $R^{\hbar}_{\rho,I}$ produce sheaves $\mathcal{O}^{\hbar}_{U_{\sigma,I}}$ and $\mathcal{O}^{\hbar}_{U_{\rho,I}}$ on $U_{\sigma,I}$ and $U_{\rho,I}$ respectively. Using $\hat{\theta}_{\gamma_\sigma,\hat{\mathfrak{D}}_I}$, we can glue together the sheaves $\mathcal{O}^{\hbar}_{U_{\rho,I}}$ to get a sheaf of $R^{\hbar}_I$-algebras $\mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}$ on $X^\circ_{I,\mathfrak{D}}$.

From the fact that the sheaves $\mathcal{O}^{\hbar}_{U_{\rho,I}}$ are deformation quantizations of $U_{\rho,I}$, we deduce that the sheaf $\mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}$ is a deformation quantization of $X^\circ_{I,\mathfrak{D}}$. In particular, we have $\mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}/\hbar\mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}} = \mathcal{O}_{X^\circ_{I,\mathfrak{D}}}$ and $\mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}$ is a sheaf a flat $R^{\hbar}_I$-algebras.

Remark: Let ρ be a one-dimensional cone of Σ. Let σ_+ and σ_- be the two two-dimensional cones of Σ bounding ρ, and let ρ_+ and ρ_- be the other boundary rays of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order. According to Remark 2.6 of [GHK15a], we have, in $U_{\rho,I}$,

$$U_{\rho_-,I} \cap U_{\rho_+,I} \simeq (\mathbb{G}_m)^2 \times \operatorname{Spec}(R_I)_{z^{\kappa_{\rho,\varphi}}} \,,$$

where $(R_I)_{z^{\kappa_{\rho,\varphi}}}$ is the localization of R_I defined by inverting $z^{\kappa_{\rho,\varphi}}$. Similarly, the restriction of $\mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}$ to $U_{\rho_-,I} \cap U_{\rho_+,I}$ is the Ore localization of $\Bbbk_\hbar[M]\hat{\otimes}(R_I)_{z^{\kappa_{\rho,\varphi}}}$, where $M = \mathbb{Z}^2$ is the character lattice of $(\mathbb{G}_m)^2$, equipped with the standard unimodular integral symplectic pairing. We have a natural identification $M = \Lambda_\rho$. Restricted to $\Bbbk_\hbar[M]\hat{\otimes}(R_I)_{z^{\kappa_{\rho,\varphi}}}$, and assuming that $\hat{f}_{\rho^{\text{in}}} = 1 \mod \hat{z}^{\kappa_{\rho,\varphi}}$ and $\hat{f}_{\rho^{\text{out}}} = 1 \mod \hat{z}^{\kappa_{\rho,\varphi}}$, the expression $\hat{\psi}_{\rho_+} \circ \hat{\psi}^{-1}_{\rho_-}$ makes sense[7] and is given by

$$(\hat{\psi}_{\rho_+} \circ \hat{\psi}^{-1}_{\rho_-})(\hat{z}^{\varphi_\rho(m_\rho)}) = \hat{z}^{\varphi_\rho(m_\rho)} \,,$$

$$(\hat{\psi}_{\rho_+} \circ \hat{\psi}^{-1}_{\rho_-})(\hat{z}^{\varphi_\rho(m_{\rho_-})}) = \hat{f}_{\rho^{\text{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_-})}\hat{f}_{\rho^{\text{in}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \,,$$

$$(\hat{\psi}_{\rho_+} \circ \hat{\psi}^{-1}_{\rho_-})(\hat{z}^{\varphi_\rho(m_{\rho_+})}) = \hat{f}^{-1}_{\rho^{\text{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}^{-1}_{\rho^{\text{out}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \,.$$

As $\langle m_\rho, m_{\rho_-}\rangle = -1$ and $\langle m_\rho, m_{\rho_+}\rangle = 1$, this implies that $\hat{\psi}_{\rho_+} \circ \hat{\psi}^{-1}_{\rho_-}$ coincides with the transformation

$$\hat{\theta}_{\gamma,\rho} = \prod_{\mathfrak{d}\in\mathfrak{D}, \mathfrak{d}=\rho} \hat{\theta}_{\gamma,\mathfrak{d}}, \,,$$

where $\hat{\theta}_{\gamma,\mathfrak{d}}$ is defined by the same formulas as above and with γ a path intersecting ρ in a single point and going from σ_- to σ_+.

[7] Without restriction, $\hat{\psi}_{\rho_-}$ is not invertible and so $\hat{\psi}^{-1}_{\rho_-}$ does not make sense.

3.2.5 Result of the gluing for $I = J$.

Assume $r \geqslant 3$ and $\kappa_{\rho,\varphi} \in J$ for every ρ one-dimensional cone of Σ. The following Lemma 3.14 gives an explicit description of $\mathcal{O}^{\hbar}_{X^{\circ}_{I,\mathfrak{D}}}$ for $I = J$.

Denote $\Bbbk[\Sigma]$ the $\Bbbk$-algebra with a $\Bbbk$-basis $\{z^m \,|m \in B(\mathbb{Z})\}$ with multiplication given by

$$z^m \cdot z^{m'} = \begin{cases} z^{m+m'} & \text{if } m \text{ and } m' \text{ lie in a common cone of } \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Let 0 be the closed point of Spec $\Bbbk[\Sigma]$ whose ideal is generated by $\{z^m \,|m \neq 0\}$. Denote $R_J[\Sigma] \coloneqq R_J \otimes_{\Bbbk} \Bbbk[\Sigma]$. According to Lemma 2.12 of [GHK15a], we have

$$X_J^{\circ} \simeq (\mathrm{Spec}\, R_J[\Sigma]) - ((\mathrm{Spec}\, R_J) \times \{0\}) .$$

Denote $\Bbbk_{\hbar}[\Sigma]$ the $\Bbbk_{\hbar}$-algebra with a $\Bbbk_{\hbar}$-basis $\{\hat{z}^m \,|m \in B(\mathbb{Z})\}$ with multiplication given by

$$\hat{z}^m \cdot \hat{z}^{m'} = \begin{cases} q^{\frac{1}{2}\langle m,m'\rangle} \hat{z}^{m+m'} & \text{if } m \text{ and } m' \text{ lie in a common cone of } \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Denote $R_J^{\hbar}[\Sigma] \coloneqq R_J \hat{\otimes}_{\Bbbk} \Bbbk_{\hbar}[\Sigma]$.

Lemma 3.14. *Assume $r \geqslant 3$ and $\kappa_{\rho,\varphi} \in J$ for every ρ one-dimensional cone of Σ. Then $\Gamma(X^{\circ}_{J,\mathfrak{D}}, \mathcal{O}^{\hbar}_{X^{\circ}_{J,\mathfrak{D}}}) = R^{\hbar}_J[\Sigma]$, and the sheaf $\mathcal{O}^{\hbar}_{X^{\circ}_{J,\mathfrak{D}}}$ is the restriction to X°_J of the Ore localization, see Section 3.1.4, of $R^{\hbar}_J[\Sigma]$ over* Spec $R_J[\Sigma]$.

Proof. By definition of a quantum scattering diagram, if $\mathfrak{d}$ is contained in the interior of a two-dimensional cone of Σ, we have $\hat{H}_{\mathfrak{d}} = 0 \mod J$ and so the corresponding automorphism $\hat{\Phi}_{\hat{H}_{\mathfrak{d}}}$ is the identity. As we are assuming $\kappa_{\rho,\varphi} \in J$, $R^{\hbar}_{\rho,J}$ is the $R^{\hbar}_J$-algebra generated by formal variables X_+, X_- and X, with X invertible, and with relations

$$XX_+ = qXX_+ \,,$$

$$XX_- = q^{-1} X_- X \,,$$

$$X_+ X_- = X_- X_+ = 0 \,,$$

where $q = e^{i\hbar}$. Let σ_+ and σ_- be the two two-dimensional cones of Σ bounding ρ, and let ρ_+ and ρ_- be the other boundary rays of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order.

From $\varphi_\rho(m_{\rho_-}) + \varphi_\rho(m_{\rho_+}) = \kappa_{\rho,\varphi} - D_\rho^2 \varphi_\rho(m_\rho)$ and $\kappa_{\rho,\varphi} \in J$, we deduce that $\hat{z}^{\varphi_\rho(m_{\rho_-})} \hat{z}^{\varphi_\rho(m_{\rho_+})} = 0$ in $R^{\hbar}_{\rho,I}$, $R^{\hbar}_{\sigma_- I}$ and $R^{\hbar}_{\sigma_+,I}$. As $\hat{z}^{\varphi_\rho(m_{\rho_-})}$ is invertible in $R^{\hbar}_{\sigma_- I}$, we have $\hat{z}^{\varphi_\rho(m_{\rho_+})} = 0$ in $R^{\hbar}_{\sigma_- I}$. Similarly, as $\hat{z}^{\varphi_\rho(m_{\rho_+})}$ is invertible in $R^{\hbar}_{\sigma_+ I}$, we have $\hat{z}^{\varphi_\rho(m_{\rho_-})} = 0$ in $R^{\hbar}_{\sigma_+ I}$.

So the map $\hat{\psi}_{\rho,-} \colon R^{\hbar}_{\rho,J} \to R^{\hbar}_{\sigma_-,J}$ is given by $\hat{\psi}_{\rho,-}(X) = \hat{z}^{\varphi_\rho(m_\rho)}$, $\hat{\psi}_{\rho,-}(X_-) = \hat{z}^{\varphi_\rho(m_{\rho_-})}$, $\hat{\psi}_{\rho,-}(X_+) = 0$. Similarly, the map $\hat{\psi}_{\rho,+} \colon R^{\hbar}_{\rho,J} \to R^{\hbar}_{\sigma_+,J}$ is given by $\hat{\psi}_{\rho,+}(X) = \hat{z}^{\varphi_\rho(m_\rho)}$, $\hat{\psi}_{\rho,+}(X_-) = 0$, $\hat{\psi}_{\rho,+}(X_+) = \hat{z}^{\varphi_\rho(m_{\rho_+})}$. The result follows. ☐

We fix (Y, D) a Looijenga pair, (B, Σ) its tropicalization, P a toric monoid, J a radical monomial ideal of P, φ a $P_{\mathbb{R}}^{\mathrm{gp}}$-valued multivalued convex Σ-piecewise linear function on B, and $\hat{\mathfrak{D}}$ a quantum scattering diagram for the data (B, Σ), P, J and φ.

Quantum broken lines and quantum theta functions have been studied by Mandel [Man15], for smooth integral affine manifolds. We make below the easy combination of the notion of quantum broken lines and theta functions used by [Man15] with the notion of classical broken lines and theta functions used in Section 2.3 of [GHK15a] for the tropicalization B of a Looijenga pair.

Definition 3.15. *A* quantum broken line *of charge $p \in B_0(\mathbb{Z})$ with endpoint Q in B_0 is a proper continuous piecewise integral affine map*

$$\gamma \colon (-\infty, 0] \to B_0$$

with only finitely many domains of linearity, together with, for each $L \subset (-\infty, 0]$ a maximal connected domain of linearity of γ, a choice of monomial $m_L = c_L \hat{z}^{p_L}$ where $c_L \in \mathbb{k}_{\hbar}^*$ and $p_L \in \Gamma(L, \gamma^{-1}(\mathcal{P})|_L)$, such that

- *For each L and $t \in L$, we have $-r(p_L) = \gamma'(t)$, i.e. the direction of the line is determined by the monomial attached to it.*
- *We have $\gamma(0) = Q \in B_0$.*
- *For the unique unbounded domain of linearity L, $\gamma|_L$ goes off for $t \to -\infty$ to infinity in the cone σ of Σ containing p and $m_L = \hat{z}^{\varphi_\sigma(p)}$, i.e. the charge p is the asymptotic direction of the broken line.*
- *Let $t \in (-\infty, 0)$ be a point at which γ is not linear, passing from the domain of linearity L to the domain of linearity L'. Let τ be the cone of Σ containing $\gamma(t)$. Let $(\mathfrak{d}_1, \hat{H}_{\mathfrak{d}_1}), \dots, (\mathfrak{d}_N, \hat{H}_{\mathfrak{d}_N})$ be the rays of $\hat{\mathfrak{D}}$ that contain $\gamma(t)$. Then γ passes from one side of these rays to the other side at time t.*

  Expand the product of

  $$\prod_{\substack{1 \leqslant k \leqslant N \\ \langle m(H_{\mathfrak{d}_k}), \mathfrak{r}(p_L)\rangle > 0}} \prod_{j=0}^{\langle m(H_k), \mathfrak{r}(p_L)\rangle - 1} \hat{f}_{\mathfrak{d}_k}(q^j \hat{z})$$

  and

  $$\prod_{\substack{1 \leqslant k' \leqslant N \\ \langle m(H_{\mathfrak{d}_{k'}}), \mathfrak{r}(p_L)\rangle < 0}} \prod_{j'=0}^{|\langle m(H_{k'}), \mathfrak{r}(p_L)\rangle| - 1} \hat{f}_{\mathfrak{d}_{k'}}(q^{-j'-1} \hat{z}),$$

  as a formal power series in $\mathbb{k}_{\hbar}\widehat{[P_{\varphi_\tau}]}$. Then there is a term $c\hat{z}^s$ in this sum with

  $$m_{L'} = m_L . (c\hat{z}^s).$$

Let $Q \in B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ be in the interior of a two-dimensional cone σ of Σ. Let γ be a quantum broken line with endpoint Q. We denote $\mathrm{Mono}(\gamma) \in \mathbb{k}_\hbar[P_{\varphi_\sigma}]$ the monomial attached to the last domain of linearity of γ.

The following finiteness result is formally identical to Lemma 2.25 of [GHK15a].

Lemma 3.16. *Let $Q \in B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ be in the interior of a two-dimensional cone σ of Σ. Fix $p \in B_0(\mathbb{Z})$. Let I be an ideal of radical J. Assume that $\kappa_{\rho,\varphi} \in J$ for at least one ray ρ of Σ. Then*

- *The collection of quantum broken lines γ of charge p with endpoint Q and such that $\mathrm{Mono}(\gamma) \notin I\mathbb{k}_\hbar[P_{\varphi_\sigma}]$ is finite.*
- *If one boundary ray of the connected component of $B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ containing Q is a ray ρ of Σ, then for every quantum broken line γ of charge p with endpoint Q, we have $\mathrm{Mono}(\gamma) \in \mathbb{k}_\hbar[P_{\varphi_\rho}]$.*

Proof. Identical to the proof of Lemma 2.25 of [GHK15a]. □

Let $Q \in B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ be in the interior of a two-dimensional cone σ of Σ. Fix $p \in B_0(\mathbb{Z})$. Let I be an ideal of radical J. We define

$$\mathrm{Lift}_Q(p) \coloneqq \sum_\gamma \mathrm{Mono}(\gamma) \in \mathbb{k}_\hbar[P_{\varphi_\sigma}]/I\,,$$

where the sum is over all the quantum broken lines γ of charge p with endpoint Q. According to Lemma 3.16, there are only finitely many such γ with $\mathrm{Mono}(\gamma) \notin I\mathbb{k}_\hbar[P_{\varphi_\sigma}]$ and so $\mathrm{Lift}_Q(p)$ is well-defined.

The following definition is formally identical to Definition 2.26 of [GHK15a].

Definition 3.17. *Assume that $\kappa_{\rho,\varphi} \in J$ for at least one one-dimensional cone ρ of Σ. We say that a quantum scattering diagram $\hat{\mathfrak{D}}$ for the data (B,Σ), P, J and φ is* consistent *if for every ideal I of P of radical J and for all $p \in B_0(\mathbb{Z})$, the following holds. Let $Q \in B_0$ be chosen so that the line joining the origin and Q has irrational slope, and $Q' \in B_0$ similarly. Then*

- *If Q and Q' are contained in a common two-dimensional cone σ of Σ, then we have*

$$\mathrm{Lift}_{Q'}(p) = \hat{\theta}_{\gamma,\hat{\mathfrak{D}}_I}(\mathrm{Lift}_Q(p))$$

in $R^\hbar_{\sigma,I}$, for every γ path in the interior of σ connecting Q and Q', and intersecting transversely the rays of $\hat{\mathfrak{D}}$.

- *If Q_- is contained in a two-dimensional cone σ_- of Σ, and Q_+ is contained in a two-dimensional cone σ_+ of Σ, such that σ_+ and σ_- intersect along a one-dimensional cone ρ of Σ, and furthermore Q_- and Q_+ are contained in connected components of $B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ whose closures contain ρ, then $\mathrm{Lift}_{Q_+}(p) \in R^\hbar_{\sigma_+,I}$ and $\mathrm{Lift}_{Q_-}(p) \in R^\hbar_{\sigma_-,I}$ are both images under $\hat{\psi}_{\rho,+}$ and $\hat{\psi}_{\rho,-}$ respectively of a single element $\mathrm{Lift}_\rho(p) \in R^\hbar_{\rho,I}$.*

The following construction is formally identical to Construction 2.27 of [GHK15a]. Suppose that D has $r \geqslant 3$ irreducible components, and that $\hat{\mathfrak{D}}$ is a consistent quantum scattering diagram for the data (B,Σ), P, J and φ. Assume that $\kappa_{\rho,\varphi} \in J$ for all one-dimensional cones ρ of Σ. Let I be an ideal of P of radical J. We construct below an element

$$\hat{\vartheta}_p \in \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}})$$

for each $p \in B(\mathbb{Z}) = B_0(\mathbb{Z}) \cup \{0\}$.

We first define $\hat{\vartheta}_0 \coloneqq 1$. Let $p \in B_0(\mathbb{Z})$. Recall that $X^\circ_{I,\mathfrak{D}}$ is defined by gluing together schemes $U_{\rho,I}$, indexed by ρ rays of Σ, and that $\mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}$ is defined by gluing together sheaves $\mathcal{O}^{\hbar}_{U_{\rho,I}}$ on $U_{\rho,I}$, such that $\Gamma(U_{\rho,I}, \mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}) = R^{\hbar}_{\rho,I}$. So, to define $\hat{\vartheta}_p$, it is enough to define elements of $R^{\hbar}_{\rho,I}$ compatible with the gluing functions. But, by definition, the consistency of $\hat{\mathfrak{D}}$ gives us such elements $\mathrm{Lift}_\rho(p) \in R^{\hbar}_{\rho,I}$.

The quantum theta functions $\hat{\vartheta}_p \in \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}})$ reduce in the classical limit to the theta functions $\vartheta_p \in \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$ defined in [GHK15a].

3.2.7 Deformation quantization of the mirror family

Suppose D has $r \geqslant 3$ irreducible components, and let φ be a $P^{\mathrm{gp}}_{\mathbb{R}}$-valued convex Σ-piecewise linear function on B such that $\kappa_{\rho,\varphi} \in J$ for all one-dimensional cones ρ of Σ. Let $\hat{\mathfrak{D}}$ be a consistent quantum scattering diagram for the data (B,Σ), P, J and φ. Let I be an ideal of P of radical J.

Denote

$$X_{I,\mathfrak{D}} \coloneqq \mathrm{Spec}\, \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$$

the affinization of $X^\circ_{I,\mathfrak{D}}$ and $j\colon X^\circ_{I,\mathfrak{D}} \to X_{I,\mathfrak{D}}$ the affinization morphism. It is proved in [GHK15a], Theorem 2.28, that j is an open immersion, that $j_\star \mathcal{O}_{X^\circ_{I,\mathfrak{D}}} = \mathcal{O}_{X_{I,\mathfrak{D}}}$, and that X_I is flat over R_I. More precisely, the R_I-algebra

$$A_I \coloneqq \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}}) = \Gamma(X_{I,\mathfrak{D}}, \mathcal{O}_{X_{I,\mathfrak{D}}})$$

is free as R_I-module and the set of theta functions ϑ_p, $p \in B(\mathbb{Z})$ is a R_I-module basis of A_I.

Theorem 3.18. *Suppose D has $r \geqslant 3$ irreducible components, and let φ be a $P^{\mathrm{gp}}_{\mathbb{R}}$-valued convex Σ-piecewise linear function on B such that $\kappa_{\rho,\varphi} \in J$ for all one-dimensional cones ρ of Σ. Let $\hat{\mathfrak{D}}$ be a consistent quantum scattering diagram for the data (B,Σ), P, J and φ. Let I be an ideal of P of radical J. Then*

- *The sheaf $\mathcal{O}^{\hbar}_{X_{I,\mathfrak{D}}} \coloneqq j_\star \mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}$ of $R^{\hbar}_I$-algebras is a deformation quantization of $X_{I,\mathfrak{D}}$ over R_I in the sense of Definition 3.4.*
- *The $R^{\hbar}_I$-algebra*
$$A^{\hbar}_I \coloneqq \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}}}) = \Gamma(X_{I,\mathfrak{D}}, \mathcal{O}^{\hbar}_{X_{I,\mathfrak{D}}})$$
is a deformation quantization of $X_{I,\mathfrak{D}}$ over R_I in the sense of Definition 3.5.

- *The $R_I^\hbar$-algebra $A_I^\hbar$ is free as $R_I^\hbar$-module.*
- *The set of quantum theta functions*

$$\{\vartheta_p^\hbar | p \in B(\mathbb{Z})\}$$

is a $R_I^\hbar$-module basis for $A_I^\hbar$.

Proof. We follow the structure of the proof of Theorem 2.28 of [GHK15a].

We first prove the result for $I = J$. As $r \geqslant 3$ and $\kappa_{\rho,\varphi} \in J$ for all one-dimensional cones ρ of Σ, the only broken line contributing to $\mathrm{Lift}_Q(p)$, for every Q in B_0 and $p \in B_0(\mathbb{Z})$, is the straight line of endpoint Q and direction p, and this provides a non-zero contribution only if Q and p lie in the same two-dimensional cone of Σ. Combined with Lemma 3.14, this implies that the map

$$\bigoplus_{p \in B(\mathbb{Z})} R_J^\hbar \hat{\vartheta}_p \to A_J^\hbar \coloneqq \Gamma(X_{J,\mathfrak{D}}^\circ, \mathcal{O}_{X_{J,\mathfrak{D}}^\circ}^\hbar) = R_J^\hbar[\Sigma]$$

is given by

$$\hat{\vartheta}_p \mapsto \hat{z}^p$$

and so is an isomorphism.

We now treat the case of a general ideal I of P of radical J. By construction, $\mathcal{O}_{X_{I,\mathfrak{D}}^\circ}^\hbar$ is a deformation quantization of $X_{I,\mathfrak{D}}^\circ$ over R_I. In particular, $\mathcal{O}_{X_{I,\mathfrak{D}}^\circ}^\hbar$ is a sheaf in flat $R_I^\hbar$-algebras. As used in [GHK15a], the fibers of $X_{J,\mathfrak{D}} \to \mathrm{Spec}\, R_J$ satisfy Serre's condition S_2 by [Ale02]. We have $\mathcal{O}_{X_{J,\mathfrak{D}}}^\hbar \simeq \mathcal{O}_{X_{J,\mathfrak{D}}} \hat{\otimes} \Bbbk_\hbar$ as $\Bbbk_\hbar$-module and so it follows that $j_* j^* \mathcal{O}_{X_{J,\mathfrak{D}}}^\hbar = \mathcal{O}_{X_{J,\mathfrak{D}}}^\hbar$. The existence of quantum theta functions $\hat{\vartheta}_p$ guarantees that the natural map

$$\mathcal{O}_{X_{I,\mathfrak{D}}}^\hbar \coloneqq j_* \mathcal{O}_{X_{I,\mathfrak{D}}}^\hbar \to j_* j^* \mathcal{O}_{X_{J,\mathfrak{D}}}^\hbar = \mathcal{O}_{X_{J,\mathfrak{D}}}^\hbar$$

is surjective. So the result follows from the following Lemma, analogue of Lemma 2.29 of [GHK15a].

Lemma 3.19. *Let X_0/S_0 be a flat family of surfaces such whose fibers satisfy Serre's condition S_2. Let $j: X_0^\circ \subset X_0$ be the inclusion of an open subset such that the complement has finite fiber. Let $S_0 \subset S$ be an infinitesimal thickening of S_0, and X/S a flat deformation of X_0/S_0, inducing a flat deformation X°/S of X_0°/S_0. Let $\mathcal{O}_{X_0}^\hbar$ be a deformation quantization of X_0/S_0 such that $\mathcal{O}_{X_0}^\hbar \simeq \mathcal{O}_{X_0} \hat{\otimes} \Bbbk_\hbar$ as $\mathcal{O}_{S_0} \hat{\otimes} \Bbbk_\hbar$-module, and so $j_* j^* \mathcal{O}_{X_0}^\hbar = \mathcal{O}_{X_0}^\hbar$ by the relative S_2 condition satisfied by X_0/S_0. Let $\mathcal{O}_{X^\circ}^\hbar$ be a deformation quantization of X°/S, restricting to $j^* \mathcal{O}_{X_0}^\hbar$ over X_0°. If the natural map*

$$\mathcal{O}_X^\hbar \coloneqq j_* \mathcal{O}_{X^\circ}^\hbar \to j_* j^* \mathcal{O}_{X_0}^\hbar = \mathcal{O}_{X_0}^\hbar$$

is surjective, then $\mathcal{O}_X^\hbar$ is a deformation quantization of X/S.

Proof. We have to prove that $\mathcal{O}_X^\hbar$ is flat over $\mathcal{O}_S \hat{\otimes} \Bbbk_\hbar$.

Let $\mathcal{I} \subset \mathcal{O}_S$ be the nilpotent ideal defining $S_0 \subset S$. Let X_n, X_n°, S_n be the nth order infinitesimal thickening of X_0, X_0°, S_0 in S, i.e. $\mathcal{O}_{X_n} = \mathcal{O}_X/\mathcal{I}^{n+1}$, $\mathcal{O}_{X_n^\circ} = \mathcal{O}_{X^\circ}/\mathcal{I}^{n+1}$ and $\mathcal{O}_{S_n} = \mathcal{O}_S/\mathcal{I}^{n+1}$.

We define $\mathcal{O}_{X_n}^\hbar \coloneqq j_* \mathcal{O}_{X_n^\circ}^\hbar$. We show by induction on n that $\mathcal{O}_{X_n}^\hbar$ is flat over $\mathcal{O}_{S_n} \hat{\otimes} \mathbb{k}_\hbar$.

For $n = 0$, we have $j_* \mathcal{O}_{X_0^\circ}^\hbar = j_* j^* \mathcal{O}_{X_0}^\hbar = \mathcal{O}_{X_0}^\hbar$, which is flat over $\mathcal{O}_{S_0} \hat{\otimes} \mathbb{k}_\hbar$ by assumption.

Assume that the induction hypothesis is true for $n-1$. Since $\mathcal{O}_{X_n^\circ}^\hbar$ is flat over $\mathcal{O}_{S_n} \hat{\otimes} \mathbb{k}_\hbar$, we have an exact sequence

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}_{X_0^\circ}^\hbar \to \mathcal{O}_{X_n^\circ}^\hbar \to \mathcal{O}_{X_{n-1}^\circ}^\hbar \to 0\,.$$

Applying j_*, we get an exact sequence

$$0 \to j_*(\mathcal{I}^n/\mathcal{I}^{n+1} \otimes j^* \mathcal{O}_{X_0}^\hbar) \to \mathcal{O}_{X_n}^\hbar \to \mathcal{O}_{X_{n-1}}^\hbar\,.$$

We have $j_*(\mathcal{I}^n/\mathcal{I}^{n+1} \otimes j^* \mathcal{O}_{X_0}^\hbar) = \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}_{X_0}^\hbar$.

By assumption, the natural map $\mathcal{O}_X^\hbar \to j_* j^* \mathcal{O}_{X_0}^\hbar = \mathcal{O}_{X_0}^\hbar$ is surjective. By the induction hypothesis, we have $\mathcal{O}_{X_{n-1}}^\hbar/\mathcal{I} = \mathcal{O}_{X_0}^\hbar$. As $\mathcal{I}$ is nilpotent, it follows that the map $\mathcal{O}_{X_n}^\hbar \to \mathcal{O}_{X_{n-1}}^\hbar$ is surjective. So we have an exact sequence

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}_{X_0}^\hbar \to \mathcal{O}_{X_n}^\hbar \to \mathcal{O}_{X_{n-1}}^\hbar \to 0\,,$$

implying that $\mathcal{O}_{X_n}^\hbar$ is flat over $\mathcal{O}_{S_n} \hat{\otimes} \mathbb{k}_\hbar$. □

□

3.2.8 The algebra structure

This Section is a q-deformed version of Section 2.4 of [GHK15a].

We saw in the previous Section that the $R_I^\hbar$-algebra

$$A_I^\hbar \coloneqq \Gamma(X_{I,\mathfrak{D}}^\circ, \mathcal{O}_{X_{I,\mathfrak{D}}^\circ}^\hbar)$$

is free as $R_I^\hbar$-module, admitting a basis of quantum theta functions $\hat{\vartheta}_p$, $p \in B(\mathbb{Z})$. Theorem 3.20 below gives a combinatorial expression for the structure constants of the algebra $A_I^\hbar$ in the basis of quantum theta functions.

If γ is a quantum broken line of endpoint Q in a cone τ of Σ, we can write the monomial $\mathrm{Mono}(\gamma)$ attached to the segment ending at Q as

$$\mathrm{Mono}(\gamma) = c(\gamma) \hat{z}^{\varphi_\tau(s(\gamma))}$$

with $c(\gamma) \in \mathbb{k}_\hbar[P_{\varphi_\tau}]$ and $s(\gamma) \in \Lambda_\tau$.

Theorem 3.20. *Let $p \in B(\mathbb{Z})$ and let $z \in B - \mathrm{Supp}_I(\hat{\mathfrak{D}}^{\mathrm{can}})$ be very close to p. For every p_1,*

$p_2 \in B(\mathbb{Z})$*, the structure constants* $C^p_{p_1,p_2} \in R^\hbar_I$ *in the product expansion*

$$\hat{\vartheta}_{p_1}\hat{\vartheta}_{p_2} = \sum_{p \in B(\mathbb{Z})} C^p_{p_1,p_2}\hat{\vartheta}_p$$

are given by

$$C^p_{p_1,p_2} = \sum_{\gamma_1,\gamma_2} c(\gamma_1)c(\gamma_2)q^{\frac{1}{2}\langle s(\gamma_1),s(\gamma_2)\rangle}\,,$$

where the sum is over all broken lines γ_1 *and* γ_2*, of asymptotic charges* p_1 *and* p_2*, satisfying* $s(\gamma_1)+s(\gamma_2)=p$*, and both ending at the point* $z \in B_0$*.*

Proof. Let τ be the smallest cone of Σ containing p. Working in the algebra $\Bbbk_\hbar[P_{\varphi_\tau}]/I$, we have

$$\mathrm{Lift}_z(p_1)\,\mathrm{Lift}_z(p_2) = \sum_{p\in B(\mathbb{Z})} C^p_{p_1,p_2}\,\mathrm{Lift}_z(p)\,.$$

By definition, we have

$$\mathrm{Lift}_z(p_1) = \sum_{\gamma_1} c(\gamma_1)\hat{z}^{\varphi_\tau(s(\gamma_1))}\,,$$

and

$$\mathrm{Lift}_z(p_2) = \sum_{\gamma_2} c(\gamma_2)\hat{z}^{\varphi_\tau(s(\gamma_2))}\,.$$

As p and z belongs to the cone τ, the only quantum broken line of charge p ending at z is the straight line $z+\mathbb{R}_{\geqslant 0}$ equipped with the monomial $\hat{z}^{\varphi_\tau(p)}$, and so we have

$$\mathrm{Lift}_z(p) = \hat{z}^{\varphi_\tau(p)}\,.$$

The result then follows from the multiplication rule

$$\hat{z}^{\varphi_\tau(s(\gamma_1))}\hat{z}^{\varphi_\tau(s(\gamma_2))} = q^{\frac{1}{2}\langle s(\gamma_1),s(\gamma_2)\rangle}\hat{z}^{\varphi_\tau(p)}\,.$$

□

Remark: In the formula given by the previous theorem, the non-commutativity of the product of the quantum theta functions comes from the twist by the power of q,

$$q^{\frac{1}{2}\langle s(\gamma_1),s(\gamma_2)\rangle}\,,$$

which is obviously not symmetric in γ_1 and γ_2 as $\langle -,-\rangle$ is skew-symmetric.

Taking the classical limit $\hbar \to 0$, we get an explicit formula for the Poisson bracket of classical theta functions, which could have been written and proved in [GHK15a].

Corollary 3.21. *Let* $p \in B(\mathbb{Z})$ *and let* $z \in B - \mathrm{Supp}_I(\mathfrak{D}^{\mathrm{can}})$ *be very close to* p*. For every* $p_1, p_2 \in B(\mathbb{Z})$*, the Poisson bracket of the classical theta functions* ϑ_{p_1} *and* ϑ_{p_2} *is given by*

$$\{\vartheta_{p_1},\vartheta_{p_2}\} = \sum_{p\in B(\mathbb{Z})} P^p_{p_1,p_2}\vartheta_p\,,$$

where

$$P^p_{p_1,p_2} \coloneqq \sum_{\gamma_1,\gamma_2} \langle s(\gamma_1), s(\gamma_2)\rangle c(\gamma_1)c(\gamma_2)\,,$$

where the sum is over all broken lines γ_1 and γ_2 pf asymptotic charges p_1 and p_2, satisfying $s(\gamma_1)+s(\gamma_2)=p$, and both ending at the point $z\in B_0$.

3.3 The canonical quantum scattering diagram

In this Section, we construct a quantum deformation of the canonical scattering diagram constructed in Section 3 of [GHK15a] and we prove its consistency. In Section 3.3.1, we give the definition of a family of higher genus log Gromov-Witten invariants of a Looijenga pair. In Section 3.3.2, we use these invariants to construct the quantum canonical scattering diagram of a Looijenga pair and we state its consistency, Theorem 3.26. The proof of Theorem 3.26 takes Sections 3.3.4, 3.3.5, 3.3.6, 3.3.7, and 3.3.8, and follows the general structure of the proof given in the classical case by [GHK15a], the use of [GPS10] being replaced by the use of Theorem 2.6.

3.3.1 Log Gromov-Witten invariants

We fix (Y,D) a Looijenga pair, (B,Σ) its tropicalization, P a toric monoid and $\eta\colon NE(Y)\to P$ a morphism of monoids. Let φ be the unique[8] (up to addition of a linear function) $P_{\mathbb{R}}^{\mathrm{gp}}$-valued multivalued convex Σ-piecewise linear function on B such that $\kappa_{\rho,\varphi}=\eta([D_\rho])$ for every ρ one-dimensional cone of Σ, where $[D_\rho]\in NE(Y)$ is the class of the divisor D_ρ dual to ρ.

Let $\mathfrak{d}\subset B$ be a ray with endpoint the origin and with rational slope. Let $\tau_{\mathfrak{d}}\in\Sigma$ be the smallest cone containing $\mathfrak{d}$ and let $m_{\mathfrak{d}}\in\Lambda_{\tau_{\mathfrak{d}}}$ be the primitive generator of $\mathfrak{d}$ pointing away from the origin.

Let us first assume that $\tau=\sigma$ is a two-dimensional cone of Σ. The ray $\mathfrak{d}$ is then contained in the interior of σ. Let ρ_+ and ρ_- be the two rays of Σ bounding σ. Let $m_{\rho_\pm}\in\Lambda_\sigma$ be primitive generators of $\rho_\pm$ pointing away from the origin. As σ is isomorphic as integral affine manifold to the standard positive quadrant $(\mathbb{R}_{\geqslant 0})^2$ of $\mathbb{R}^2$, there exists a unique decomposition

$$m_{\mathfrak{d}} = n_+ m_{\rho_+} + n_- m_{\rho_-}$$

with n_+ and n_- positive integers. Let $NE(Y)_{\mathfrak{d}}$ be the set of classes $\beta\in NE(Y)$ such that there exists a positive integer ℓ_β such that

$$\beta.D_{\rho_+} = \ell_\beta n_+\,,$$

$$\beta.D_{\rho_-} = \ell_\beta n_-\,,$$

and

$$\beta.D_\rho = 0\,,$$

[8]See Lemma 1.13 of [GHK15a].

for every one-dimensional cone ρ of Σ distinct of ρ_+ and ρ_-.

If $\tau = \rho$ is a one-dimensional cone of Σ, we define $NE(Y)_{\mathfrak{d}}$ as being the set of classes $\beta \in NE(Y)$ such that there exists a positive integer ℓ_β such that

$$\beta.D_\rho = \ell_\beta \,,$$

and

$$\beta.D_{\rho'} = 0 \,,$$

for every one-dimensional cone ρ' of Σ distinct of ρ.

The upshot of the preceding discussion is that, for any ray $\mathfrak{d}$ with endpoint the origin and of rational slope, we have defined a subset $NE(Y)_{\mathfrak{d}}$ of $NE(Y)$.

We equip Y with the divisorial log structure defined by the normal crossing divisor D. The resulting log scheme is log smooth. As reviewed in Section 3.1.2, integral points $p \in B(\mathbb{Z})$ of the tropicalization naturally define tangency conditions for stable log maps to Y.

For every $\beta \in NE(Y)_{\mathfrak{d}}$, let $\overline{M}_g(Y/D,\beta)$ be the moduli space of genus g stable log maps to (Y,D), of class β, and satisfying the tangency condition $\ell_\beta m_{\mathfrak{d}} \in B(\mathbb{Z})$. By the work of Gross, Siebert [GS13] and Abramovich, Chen [Che14b], [AC14], $\overline{M}_g(Y/D,\beta)$ is a proper Deligne-Mumford stack of virtual dimension g and it admits a virtual fundamental class

$$[\overline{M}_g(Y/D,\beta)]^{\text{virt}} \in A_g(\overline{M}_g(Y/D,\beta),\mathbb{Q}) \,.$$

If $\pi\colon \mathcal{C} \to \overline{M}_g(Y/D,\beta)$ is the universal curve, of relative dualizing sheaf ω_π, then the Hodge bundle

$$\mathbb{E} \coloneqq \pi_* \omega_\pi$$

is a rank g vector bundle over $\overline{M}_g(Y/D,\beta)$. Its Chern classes are classically called the lambda classes,

$$\lambda_j \coloneqq c_j(\mathbb{E}) \,,$$

for $j = 0,\dots,g$. We define genus g log Gromov-Witten invariants of (Y,D) by

$$N_{g,\beta}^{Y/D} \coloneqq \int_{[\overline{M}_g(Y/D,\beta)]^{\text{virt}}} (-1)^g \lambda_g \in \mathbb{Q} \,.$$

3.3.2 Definition

Using the higher genus log Gromov-Witten invariants defined in the previous Section, we can define a natural deformation of the canonical scattering diagram defined in Section 3.1 of [GHK15a].

Definition 3.22. *We define $\hat{\mathfrak{D}}^{\text{can}}$ as being the set of pairs $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$, where $\mathfrak{d}$ is a ray of rational slope in B with endpoint the origin, and, denoting $\tau_{\mathfrak{d}}$ the smallest cone of Σ containing*

$\mathfrak{d}$, *and* $m_{\mathfrak{d}} \in \Lambda_{\tau_{\mathfrak{d}}}$ *the primitive generator of* $\mathfrak{d}$ *pointing away from the origin,* $\hat{H}_{\mathfrak{d}}$ *is given by*

$$\hat{H}_{\mathfrak{d}} \coloneqq \left(\frac{i}{\hbar}\right) \sum_{\beta \in NE(Y)_{\mathfrak{d}}} \left(\sum_{g \geqslant 0} N_{g,\beta}^{Y/D} \hbar^{2g}\right) \hat{z}^{\eta(\beta) - \varphi_{\tau_{\mathfrak{d}}}(\ell_\beta m_{\mathfrak{d}})} \in \mathbb{k}_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]} \,.$$

The following Lemma is almost formally identical to Lemma 3.5 of [GHK15a].

Lemma 3.23. *Let* J *be a radical ideal of* P*. Suppose that the map* $\eta \colon NE(Y) \to P$ *satisfies the following conditions*

- *If* $\mathfrak{d}$ *is contained in the interior of a two-dimensional cone of* Σ*, then* $\eta(\beta) \in J$ *for every* $\beta \in NE(Y)_{\mathfrak{d}}$ *such that* $N_{g,\beta} \neq 0$ *for some* g*.*
- *If* $\mathfrak{d}$ *is a ray* ρ *of* Σ *and* $\kappa_{\rho,\varphi} \notin J$*, then* $\eta(\beta) \in J$ *for every* $\beta \in NE(Y)_{\mathfrak{d}}$ *such that* $N_{g,\beta} \neq 0$ *for some* g*.*
- *For any ideal* I *in* P *of radical* J*, there are only finitely may classes* $\beta \in NE(Y)$ *such that* $N_{g,\beta} \neq 0$ *for some* g *and such that* $\eta(\beta) \notin I$*.*

Then $\hat{\mathfrak{D}}^{\mathrm{can}}$ *is a quantum scattering diagram for the data* (B, Σ)*,* P*,* J*, and* φ*. Furthermore, the quantum scattering diagram* $\hat{\mathfrak{D}}^{\mathrm{can}}$ *has only outgoing rays.*

Proof. The assumptions guarantee the finiteness requirements in the definition of a quantum scattering diagram, see Section 3.2.2. The ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ is outgoing because

$$r(\eta(\beta) - \varphi_{\tau_{\mathfrak{d}}}(\ell_\beta m_{\mathfrak{d}})) = -\ell_\beta m_{\mathfrak{d}} \in \mathbb{Z}_{<0} m_{\mathfrak{d}} \,.$$

□

Lemma 3.24. *The canonical quantum scattering diagram* $\hat{\mathfrak{D}}^{\mathrm{can}}$ *is invariant under flat deformation of* (Y, D)*.*

Proof. This follows from deformation invariance of the log Gromov-Witten invariants $N_{g,\beta}^{Y/D}$. □

Lemma 3.25. *The classical limit of the canonical quantum scattering diagram* $\hat{\mathfrak{D}}^{\mathrm{can}}$ *is the canonical scattering diagram defined in Section 3.1 of [GHK15a].*

Proof. It follows from the cycle argument used in the proofs of Proposition 1.10 and 2.12, and from the log birational invariance of log Gromov-Witten invariants [AW13], that the relative genus zero Gromov-Witten invariants of non-compact surfaces used in [GHK15a] coincide with the genus zero log Gromov-Witten invariants $N_{0,\beta}^{Y/D}$. □

3.3.3 Consistency

The following result states that the quantum scattering diagram $\mathfrak{D}^{\mathrm{can}}$, defined in Section 3.3.2, is consistent in the sense of Section 3.2.6.

Theorem 3.26. *Suppose that*

- *For any class $\beta \in NE(Y)$ such that $N_{g,\beta} \neq 0$ for some g, we have $\eta(\beta) \in J$.*
- *For any ideal I of P of radical J, there are only finitely many classes $\beta \in NE(Y)$ such that $N_{g,\beta} \neq 0$ for some g and $\eta(\beta) \notin I$.*
- *$\eta([D_\rho]) \in J$ for at least one boundary component $D_\rho \subset D$.*

Then the canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ is consistent.

Let us review the various steps taken by [GHK15a] to prove the consistency of the canonical scattering diagram in the classical case.

- Step I. We can replace (Y, D) by a corner blow-up of (Y, D).
- Step II. Changing the monoid P.
- Step III. Reduction to the Gross-Siebert locus.
- Step IV. Pushing the singularities at infinity.
- Step V. $\bar{\mathfrak{D}}$ satisfies the required compatibility condition.

Step I, see Proposition 3.10 of [GHK15a], is easy in the classical case. The quantum case is similar: the scattering diagram changes only in a trivial way under corner blow-up and we will not say more.

Step II, see Proposition 3.12 of [GHK15a], is more subtle and involves some regrouping of monomials in the comparison of the broken lines for two different monoids. Exactly the same regrouping operation deals with the quantum case too.

Step III in [GHK15a] requires an understanding of genus zero multicover contributions of exceptional divisors of a toric model. We explain below, Section 3.3.4, how the quantum analogue is obtained from the knowledge of higher genus multicover contributions.

Step IV in [GHK15a] is the reduction of the consistency of $\mathfrak{D}^{\mathrm{can}}$ to the consistency of a scattering diagram $\nu(\mathfrak{D}^{\mathrm{can}})$ on an integral affine manifold without singularities. We explain in Sections 3.3.5, 3.3.7, 3.3.8, how the consistency of the quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ can be reduced to the consistency of a quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ on an integral affine manifold without singularities.

Step V in [GHK15a] is the proof of consistency of $\nu(\mathfrak{D}^{\mathrm{can}})$ and ultimately relies on the main result of [GPS10]. We explain in Section 3.3.6 how its q-analogue, i.e. the consistency of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$, ultimately relies on Theorem 2.6.

3.3.4 Reduction to the Gross-Siebert locus.

We start recalling some notations from Chapter 2.

Let $\mathfrak{m} = (m_1, \dots, m_n)$ be an n-tuple of primitive non-zero vectors of $M = \mathbb{Z}^2$. The fan in $\mathbb{R}^2$ with rays $-\mathbb{R}_{\geqslant 0} m_1, \dots, -\mathbb{R}_{\geqslant 0} m_n$ defines a toric surface $\bar{Y}_{\mathfrak{m}}$. Denote $\partial \bar{Y}_{\mathfrak{m}}$ the anticanonical toric divisor of $\bar{Y}_{\mathfrak{m}}$, and let $D_{m_1}, \dots, D_{m_n}$ be the irreducible components of $\partial \bar{Y}_{\mathfrak{m}}$ dual to the rays $-\mathbb{R}_{\geqslant 0} m_1, \dots, -\mathbb{R}_{\geqslant 0} m_n$.

For every $j = 1, \dots, n$, we blow-up a point x_j in general position on the toric divisor D_{m_j}. Remark that it is possible to have $\mathbb{R}_{\geqslant 0} m_j = \mathbb{R}_{\geqslant 0} m_{j'}$, and so $D_{m_j} = D_{m_{j'}}$, for $j \neq j'$, and that in this case we blow-up several distinct points on the same toric divisor. We denote $Y_{\mathfrak{m}}$ the resulting projective surface and $\pi \colon Y_{\mathfrak{m}} \to \bar{Y}_{\mathfrak{m}}$ the blow-up morphism. Let $E_j \coloneqq \pi^{-1}(x_j)$ be the exceptional divisor over x_j. We denote $\partial Y_{\mathfrak{m}}$ the strict transform of $\partial \bar{Y}_{\mathfrak{m}}$.

Using Steps I and II and the deformation invariance property of $\hat{\mathfrak{D}}^{\mathrm{can}}$, we can make the following assumptions (see Assumptions 3.13 of [GHK15a]):

- There exists $\mathfrak{m} = (m_1, \dots, m_n)$ a n-tuple of primitive non-zero vectors of $M = \mathbb{Z}^2$ such that $(Y, D) = (Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$.
- The map $\eta \colon NE(Y) \to P$ is an inclusion and $P^\times = \{0\}$.
- There is an ample divisor H on $\overline{Y}$ such that there is a face of P whose intersection with $NE(Y)$ is the face $NE(Y) \cap (p^* H)^\perp$ generated by the classes $[E_j]$ of exceptional divisors. Let G be the prime monomial ideal of R generated by the complement of this face.
- $J = P - \{0\}$.

Following Definition 3.14 of [GHK15a], we call Gross-Siebert locus the open torus orbit T^{gs} of the toric face Spec $\Bbbk[P]/G$ of Spec $\Bbbk[P]$.

Proposition 3.27. *For each ray ρ of Σ,with primitive generator $m_\rho \in \Lambda_\rho$ pointing away from the origin, the Hamiltonian $\hat{H}_\rho$ attached to ρ in the scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ satisfies*

$$\hat{H}_\rho = i \sum_{j, D_{m_j} = D_\rho} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell - 1}}{2 \sin\left(\frac{\ell \hbar}{2}\right)} \hat{z}^{\ell \eta([E_j]) - \ell \varphi_\rho(m_\rho)} \mod G\,.$$

Proof. The only contributions to $\hat{H}_\rho \mod G$ come from the multiple covers of the exceptional divisors E_j. The result then follows from Lemma 2.20. □

Proposition 3.28. *The canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ is a scattering diagram for the data (B, Σ), P, G and φ. Concretely, for every ideal I of P of radical G, there are only finitely many rays such $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ such that $\hat{H}_{\mathfrak{d}} \neq 0 \mod I$.*

Proof. This follows from the argument given in the proof of Corollary 3.16 in [GHK15a]. It is a geometric argument about curve classes and the genus of the curves plays no role. □

Corollary 3.29. *If $\hat{\mathfrak{D}}^{\mathrm{can}}$ is consistent as a quantum scattering diagram for the data (B, Σ), P, G and φ, then $\hat{\mathfrak{D}}^{\mathrm{can}}$ is consistent as a quantum scattering diagram for the data (B, Σ), P, J and φ.*

Following Remark 3.18 of [GHK15a], we denote $E \subset P^{\mathrm{gp}}$ the sublattice generated by the face $P \setminus G$. We have naturally $T^{\mathrm{gs}} = \mathrm{Spec}\ \Bbbk[E] \subset \mathrm{Spec}\ \Bbbk[P]$. Denote $\mathfrak{m}_{P+E} = (P+E) \setminus E$.

The following Lemma is formally identical to Lemma 3.19 of [GHK15a].

Lemma 3.30. *If $\hat{\mathfrak{D}}^{\mathrm{can}}$, viewed as a quantum scattering diagram for the data (B, Σ), $P+E$, φ and $\mathfrak{m}_{P+E}$, is consistent, then $\hat{\mathfrak{D}}^{\mathrm{can}}$, viewed as a quantum scattering diagram for the data (B, Σ), P, φ and G, is consistent.*

Proof. Identical to the proof of Lemma 3.19 of [GHK15a]. □

It follows that we can replace P by $P+E$, and so from now on, we assume that $P^* = E$ and $G = P \setminus E$. Concretely, this means that it is enough to check the consistency of $\hat{\mathfrak{D}}^{\mathrm{can}}$ by working in rings in which the monomials $\hat{z}^{\eta([E_j] - \varphi_\rho(m_\rho))}$ are invertible.

3.3.5 Pushing the singularities at infinity

We first recall the notations introduced at the beginning of Step IV of [GHK15a].

We denote $M = \mathbb{Z}^2$ the lattice of cocharacters of the torus acting on the toric surface $(\bar{Y}, \partial \bar{Y}_{\mathfrak{m}})$. Let $(\bar{B}, \bar{\Sigma})$ be the tropicalization of $(\bar{Y}_m, \partial \bar{Y}_{\mathfrak{m}})$. The affine manifold $\bar{B}$ has no singularity at the origin and so is naturally isomorphic to $M_{\mathbb{R}} = \mathbb{R}^2$. The cone decomposition $\bar{\Sigma}$ of $M_{\mathbb{R}} = \mathbb{R}^2$ is simply the fan of $\bar{Y}$. Let $\bar{\varphi}$ be the single-valued $P_{\mathbb{R}}^{\mathrm{gp}}$-valued on $\bar{B}$ such that

$$\kappa_{\bar{\rho}, \bar{\varphi}} = \pi^* [\bar{D}_{\bar{\rho}}] \,,$$

for every $\bar{\rho}$ one-dimensional cone of $\bar{\Sigma}$ and where $\bar{D}_{\bar{\rho}}$ is the toric divisor dual to $\bar{\rho}$. Since $\bar{\varphi}$ is single-valued and $\bar{B}$ has no singularities, the sheaf $\bar{\mathcal{P}}$, as defined in Section 3.1.2 is constant with fiber $P^{\mathrm{gp}} \oplus M$.

There is a canonical piecewise linear map $\nu \colon B \to \bar{B}$ which restricts to an integral affine isomorphism $\nu|_\sigma \colon \sigma \to \bar{\sigma}$ from each two-dimensional cone σ of Σ to the corresponding two-dimensional cone $\bar{\sigma}$ of $\bar{\Sigma}$. This map naturally identifies $B(\mathbb{Z})$ with $\bar{B}(\mathbb{Z})$. Restricted to each two-dimensional cone σ of Σ, the derivative ν_* of ν induces a identification $\Lambda_{B,\sigma} \simeq \Lambda_{\bar{B},\bar{\sigma}}$, an isomorphism of monoids

$$\tilde{\nu}_\sigma \colon P_{\varphi_\sigma} \to P_{\bar{\varphi}_{\bar{\sigma}}}$$

$$p + \varphi_\sigma(m) \mapsto p + \bar{\varphi}_{\bar{\sigma}}(\nu_*(m)) \,,$$

for $p \in P$ and $m \in \Lambda_\sigma$, and so an identification of algebras of $\Bbbk_\hbar[P_{\varphi_\sigma}]$ and $\Bbbk_\hbar[P_{\bar{\varphi}_{\bar{\sigma}}}]$.

If ρ is a one-dimensional cone of Σ, then ν_* is only defined on the tangent space to ρ (not on the full Λ_ρ because ν is only piecewise linear) and so give an identification

$$\tilde{\nu}_\rho \colon \{p + \varphi_\rho(m) |\ m \text{ tangent to } \rho,\ p \in P\} \to \{p + \bar{\varphi}_{\bar{\rho}}(m) |\ m \text{ tangent to } \bar{\rho},\ p \in P\}$$

$$p + \varphi_\rho(m) \mapsto p + \bar{\varphi}_{\bar{\rho}}(\nu_*(m)) \,.$$

We define below a quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ for the data $(\bar{B}, \bar{\Sigma})$, P, $\bar{\varphi}$ and G.

- For every ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ of $\hat{\mathfrak{D}}^{\mathrm{can}}$ contained in the interior of a two-dimensional cone of Σ, the quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ contains the ray

$$(\nu(\mathfrak{d}), \tilde{\nu}_{\tau_\sigma}(\hat{H}_{\mathfrak{d}}))\,,$$

which is outgoing.

- For every ray $(\rho, \hat{H}_\rho)$, with ρ a one-dimensional cone of Σ, and so by Proposition 3.27,

$$\hat{H}_\rho = \hat{G}_\rho + i \sum_{j, D_{m_j} = D_\rho} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} \hat{z}^{\ell[E_j] - \ell\varphi_\rho(m_\rho)}\,,$$

with $\hat{G}_\rho = 0 \mod G$, the quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ contains two rays:

$$(\bar{\rho}, \tilde{\nu}_{\tau_{\mathfrak{d}}}(\hat{G}_\rho))\,,$$

which is outgoing, and

$$\left(\bar{\rho},\, i \sum_{j, D_{m_j} = D_\rho} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} \hat{z}^{\ell\bar{\varphi}(m_\rho) - \ell[E_j]}\right),$$

which is ingoing.

Remark: In going from $\hat{\mathfrak{D}}^{\mathrm{can}}$ to $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$, we invert $\hat{z}^{\ell[E_j] - \ell\varphi_{\bar{\rho}}(m_\rho)}$, which becomes $\hat{z}^{\ell\varphi_{\bar{\rho}}(m_\rho) - \ell[E_j]}$. This makes sense because we are assuming $P^* = E$.

3.3.6 Consistency of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$

Let $\hat{\mathfrak{D}}_{\mathfrak{m}}$ be the quantum scattering diagram for the data $(\bar{B}, \bar{\Sigma})$, P, φ and G, having, for each $\bar{\rho}$ one-dimensional cone of $\bar{\Sigma}$, a ray $(\bar{\rho}, \hat{H}_{\bar{\rho}})$ where

$$\hat{H}_{\bar{\rho}} \coloneqq i \sum_{j, D_{m_j} = D_\rho} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} \hat{z}^{\ell\bar{\varphi}(m_\rho) - \ell[E_j]}\,.$$

Writing $\ell\bar{\varphi}(m_\rho) - \ell[E_j] = (\ell m_\rho, \varphi(\ell m_\rho) - \ell[E_j])$, it is clear that $\hat{H}_{\bar{\rho}} \in \Bbbk_\hbar\widehat{[P_{\overline{\varphi}}]}$, where the monoid

$$P_{\overline{\varphi}} = \{(m, \bar{\varphi}(m) + p) | m \in M, p \in P\}$$

is independent of $\bar{\rho}$.

For such quantum scattering diagram $\hat{\mathfrak{D}}$, with all Hamiltonians valued in the same ring, it makes sense to define an automorphism $\hat{\theta}_{\gamma, \hat{\mathfrak{D}}}$ of this ring, as in Section 3.2.4, but for γ an arbitrary path in $\bar{B}_0$ transverse to the rays of the diagram. By [KS06], Theorem 6, there exists another scattering diagram $S(\hat{\mathfrak{D}})$ containing $\hat{\mathfrak{D}}$, such that $S(\hat{\mathfrak{D}}) - \hat{\mathfrak{D}}$ consists only of outgoing rays and $\hat{\theta}_{\gamma, S(\hat{\mathfrak{D}})}$ is the identity for γ a loop in $\bar{B}_0$ going around the origin. We can assume that there is at most one ray of $S(\hat{\mathfrak{D}}) - \hat{\mathfrak{D}}$ in each possible outgoing direction.

The scattering diagram $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ was the main object of study of Chapter 2[9].

For every $m \in M - \{0\}$, let P_m be the subset of $p = (p_1, \dots, p_n) \in \mathbb{N}^n$ such that $\sum_{i=1}^n p_i m_i$ is positively collinear with m:

$$\sum_{i=1}^{n} p_i m_i = \ell_p m$$

for some $\ell_p \in \mathbb{N}$. Given $p \in P_m$, we defined in Section 2.2.2 a curve class $\beta_p \in A_1(Y_{\mathfrak{m}}, \mathbb{Z})$.

Recall that if $\mathfrak{d} \subset \bar{B}$ is a ray with endpoint the origin and with rational slope, we denote $m_{\mathfrak{d}} \in M$ the primitive generator of $\mathfrak{d}$ pointing away from the origin.

The following Proposition expresses $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ in terms of the log Gromov-Witten invariants $N_{g,\beta}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}}$ defined in Section 3.3.1 and entering in the definition of $\hat{\mathfrak{D}}^{\mathrm{can}}$.

Proposition 3.31. *The Hamiltonian $\hat{H}_{\mathfrak{d}}$ attached to an outgoing ray $\mathfrak{d}$ of $S(\hat{\mathfrak{D}}_{\mathfrak{m}}) - \hat{\mathfrak{D}}_{\mathfrak{m}}$ is given by*

$$\hat{H}_{\mathfrak{d}} = \left(\frac{i}{\hbar}\right) \sum_{p \in P_{m_{\mathfrak{d}}}} \left(\sum_{g \geqslant 0} N_{g,\beta_p}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}} \hbar^{2g} \right) \hat{z}^{(-\ell_\beta m_{\mathfrak{d}}, \beta_p - \bar{\varphi}(\ell_\beta m_{\mathfrak{d}}))} \,,$$

where $(-\ell_\beta m_{\mathfrak{d}}, \beta_p - \bar{\varphi}(\ell_\beta m_{\mathfrak{d}})) \in P_{\bar{\varphi}}$.

Proof. This is Theorem 2.6. □

Proposition 3.32. *We have $S(\hat{\mathfrak{D}}_{\mathfrak{m}}) = \nu(\hat{\mathfrak{D}}^{\mathrm{can}})$.*

Proof. We compare the explicit description of $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ given by Proposition 3.31 with the explicit description of $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ obtained from its definition in Section 3.3.5 and from the definition of $\hat{\mathfrak{D}}^{\mathrm{can}}$ in Section 3.3.2.

The ingoing rays obviously coincide.

Let $\mathfrak{d}$ be an outgoing ray. The corresponding Hamiltonian in $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ involves the log Gromov-Witten invariants $N_{g,\beta}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}}$, for

$$\beta \in NE(Y)_{\mathfrak{d}} \cap G \,,$$

whereas the corresponding Hamiltonian in $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ involves the log Gromov-Witten invariants $N_{g,\beta_p}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}}$ for $p \in P_{m_{\mathfrak{d}}}$. The only thing to show is that $N_{g,\beta}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}} = 0$ if $\beta \in NE(Y)_{\mathfrak{d}} \cap G$ is not of the form β_p for some $p \in P_{m_{\mathfrak{d}}}$.

Recall that we have the blow-up morphism $\pi \colon Y_{\mathfrak{m}} \to \bar{Y}_{\mathfrak{m}}$. Let $\beta \in NE(Y)_{\mathfrak{d}} \cap G$. We can uniquely write $\beta = \pi^* \pi_* \beta - \sum_{j=1}^n p_j E_j$ for some $p_j \in \mathbb{Z}$, $j = 1, \dots, n$. If $p_j \geqslant 0$ for every $j = 1, \dots, n$, then $p = (p_1, \dots, p_n) \in \mathbb{N}^n$ and $\beta = \beta_p$.

Assume that there exists $1 \leqslant j \leqslant n$ such that $p_j < 0$. Then $\beta . E_j = p_j < 0$ and so every stable log map $f \colon C \to Y_{\mathfrak{m}}$ of class β has a component dominating E_j. If $\mathfrak{d} \neq -\mathbb{R}_{\geqslant 0} m_j$, then we can do an analogue of the cycle argument of Proposition 1.10 and Lemma 2.12. Knowing the asymptotic behavior of the tropical map to the tropicalization B of $Y_{\mathfrak{m}}$, imposed by

[9]Comparing the conventions of the present Chapter and of Chapter 2, the notions of outgoing and ingoing rays are exchanged. This implies that a global sign must be included in comparing Hamiltonians of the present Chapter and those of Chapter 2.

the tangency condition $\ell_\beta m_{\mathfrak{d}}$, and using repetitively the balancing condition, we get that C needs to contain a cycle of components mapping surjectively to $\partial Y_{\mathfrak{m}}$. Vanishing properties of the lambda class given by Lemma 1.7 then imply that $N_{g,\beta}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}} = 0$. If $\mathfrak{d} = -\mathbb{R}_{\geqslant 0} m_j$ for some j, then the same argument implies the vanishing of $N_{g,\beta}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}}$, except if β is a multiple of some E_j, which is not the case by the assumption $\beta \in G$.

□

The following Proposition is the quantum version of Theorem 3.30 of [GHK15a].

Proposition 3.33. *Let I be an ideal of P of radical G. If Q and Q' are two points in general position in $M_{\mathbb{R}} - \mathrm{Supp}(S(\hat{\mathfrak{D}}_{\mathfrak{m}}))_I$, and γ is a path connecting Q and Q' for which $\hat{\theta}_{\gamma, S(\hat{\mathfrak{D}}_{\mathfrak{m}})_I}$ is defined, then*

$$\mathrm{Lift}_{Q'}(p) = \hat{\theta}_{\gamma, S(\hat{\mathfrak{D}}_{\mathfrak{m}})_I}(\mathrm{Lift}_Q(p))$$

as elements of $\Bbbk_\hbar[P_{\bar{\varphi}}]/I$.

Proof. The key input is that, by construction, $\hat{\theta}_{\gamma, S(\hat{\mathfrak{D}}_{\mathfrak{m}})}$ is the identity for γ a loop in $\bar{B}_0$ going around the origin. Proofs of the classical statement can be found in [CPS10], Section 5.4 of [Gro11] and Section 3.2 of the first arxiv version of [GHK15a]. Putting hats everywhere, the same argument proves the quantum version, without extra complication. □

3.3.7 Comparing $\hat{\mathfrak{D}}^{\mathrm{can}}$ and $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$

In order to obtain the consistency of $\hat{\mathfrak{D}}^{\mathrm{can}}$ from some properties of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$, we need to compare the rings $R^\hbar_{\sigma,I}$, $R^\hbar_{\rho,I}$ coming from (B,Σ), φ, and the corresponding rings $\bar{R}^\hbar_{\sigma,I}$, $\bar{R}^\hbar_{\rho,I}$ coming from $(\bar{B},\bar{\Sigma})$, $\bar{\varphi}$. Such comparison is done in the following Lemma.

Lemma 3.34. *There are isomorphisms $p_\rho \colon R^\hbar_{\rho,I} \to \bar{R}^\hbar_{\bar{\rho},I}$ and $p_\sigma \colon R^\hbar_{\sigma,I} \to \bar{R}^\hbar_{\bar{\sigma},I}$, intertwining*

- *the maps $\hat{\psi}_{\rho,-} \colon R^\hbar_{\rho,I} \to R^\hbar_{\sigma_-,I}$ and $\hat{\psi}_{\bar{\rho},-} \colon \bar{R}^\hbar_{\bar{\rho},I} \to \bar{R}^\hbar_{\bar{\sigma}_-,I}$,*
- *the maps $\hat{\psi}_{\rho,+} \colon R^\hbar_{\rho,I} \to R^\hbar_{\sigma_+,I}$ and $\hat{\psi}_{\bar{\rho},+} \colon \bar{R}^\hbar_{\bar{\rho},I} \to \bar{R}^\hbar_{\bar{\sigma}_+,I}$,*
- *the automorphisms $\hat{\theta}_{\gamma,\hat{\mathfrak{D}}^{\mathrm{can}}} \colon R^\hbar_{\sigma,I} \to R^\hbar_{\sigma,I}$ and $\hat{\theta}_{\bar{\gamma},\nu(\hat{\mathfrak{D}}^{\mathrm{can}})} \colon R^\hbar_{\bar{\sigma},I} \to R^\hbar_{\bar{\sigma},I}$, where γ is a path in σ for which $\hat{\theta}_{\gamma,\mathfrak{D}^{\mathrm{can}}}$ is defined and $\bar{\gamma} = \nu \circ \gamma$.*

Proof. It is a quantum version of Lemma 3.27 of [GHK15a]. The isomorphism p_σ simply comes from the isomorphism of monoids $\tilde{\nu}_\sigma \colon P_{\varphi_\sigma} \to P_{\bar{\varphi}_{\bar{\sigma}}}$.

Recall from Section 3.2.1 that the rings $R^\hbar_{\rho,I}$ and $\bar{R}^\hbar_{\bar{\rho},I}$ are generated by variables X_+, X_-, X and $\bar{X}_+$,$\bar{X}_-$, $\bar{X}$ respectively and we define p_ρ as the morphism of $R^\hbar_I$-algebras such that $p_\rho(X_+) = \bar{X}_+$, $p_\rho(X_-) = \bar{X}_-$, $p_\rho(X) = \bar{X}$. We have to check that p_ρ is compatible with the relations defining $R^\hbar_{\rho,I}$ and $\bar{R}^\hbar_{\bar{\rho},I}$.

We have $\hat{f}_{\rho^{\text{in}}} = 1$. Using Proposition 3.27 and Lemma 3.13, we can write

$$\hat{f}_{\rho^{\text{out}}}(X) = \hat{g}_\rho(X) \prod_{j, D_{m_j} = D_\rho} (1 + q^{-\frac{1}{2}} \hat{z}^{E_j} X^{-1}),$$

for some $\hat{g}_\rho(X) = 1 \mod G$. Using the definition of $\nu(\hat{\mathfrak{D}}^{\text{can}})$ given in Section 3.3.5, and Lemma 3.13, we have

$$\hat{f}_{\bar{\rho}^{\text{in}}}(\bar{X}) = \prod_{j, D_{m_j} = D_\rho} (1 + q^{-\frac{1}{2}} \hat{z}^{-E_j} \bar{X}),$$

and

$$\hat{f}_{\bar{\rho}^{\text{out}}}(\bar{X}) = \hat{g}_\rho(\bar{X}).$$

We need to check that

$$p_\rho \left(q^{\frac{1}{2} D_\rho^2} \hat{z}^{D_\rho} \hat{f}_{\rho^{\text{in}}}(X) \hat{f}_{\rho^{\text{out}}}(q^{-1}X) X^{-D_\rho^2} \right) = q^{\frac{1}{2} D_{\bar{\rho}}^2} \hat{z}^{D_{\bar{\rho}}} \hat{f}_{\bar{\rho}^{\text{in}}}(\bar{X}) \hat{f}_{\bar{\rho}^{\text{out}}}(q^{-1}\bar{X}) \bar{X}^{-D_{\bar{\rho}}^2}.$$

We have $D_\rho^2 = D_{\bar{\rho}}^2 - l_\rho$ and $D_\rho = D_{\bar{\rho}} - \sum_{j, D_{m_j} = D_\rho} E_j$ and so the desired identity follows from

$$(1 + q^{-\frac{1}{2}} \hat{z}^{E_j} (q^{-1}X)^{-1}) = (1 + q^{\frac{1}{2}} \hat{z}^{E_j} X^{-1}) = q^{\frac{1}{2}} \hat{z}^{E_j} X^{-1} (1 + q^{-\frac{1}{2}} \hat{z}^{-E_j} X).$$

Similarly, the relation

$$p_\rho (q^{-\frac{1}{2} D_\rho^2} \hat{z}^{D_\rho} \hat{f}_{\rho^{\text{out}}}(X) \hat{f}_{\rho^{\text{in}}}(qX) X^{-D_\rho^2}) = q^{-\frac{1}{2} D_{\bar{\rho}}^2} \hat{z}^{D_{\bar{\rho}}} \hat{f}_{\bar{\rho}^{\text{out}}}(X) \hat{f}_{\bar{\rho}^{\text{in}}}(qX) X^{-D_{\bar{\rho}}^2}$$

follows from

$$(1 + q^{-\frac{1}{2}} \hat{z}^{E_j} X^{-1}) = q^{-\frac{1}{2}} \hat{z}^{E_j} X^{-1} (1 + q^{\frac{1}{2}} \hat{z}^{-E_j} X) = q^{-\frac{1}{2}} \hat{z}^{E_j} X^{-1} (1 + q^{-\frac{1}{2}} \hat{z}^{-E_j} (qX)).$$

□

Lemma 3.35. *The piecewise linear map $\nu: B \to \bar{B}$ induces a bijection between broken lines of $\hat{\mathfrak{D}}^{\text{can}}$ and broken lines of $\nu(\hat{\mathfrak{D}}^{\text{can}})$.*

Proof. It is a quantum version of Lemma 3.27 of [GHK15a].

It is enough to compare bending and attached monomials of broken lines near a one-dimensional cone ρ of Σ. Indeed, away from such ρ, ν is linear and so the claim is obvious.

Let ρ be a one-dimensional cone of Σ. Let σ_+ and σ_- be the two-dimensional cones of Σ bounding ρ, and let ρ_+, ρ_- be the other boundary one-dimensional cones of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order. Let m_ρ be the primitive generator of ρ pointing away from the origin. We continue to use the notations introduced in the proof of Lemma 3.34.

Let γ be a quantum broken line in B_0, passing from σ_- to σ_+ across ρ. Let $c\hat{z}^s$, $s \in P_{\varphi_{\sigma_-}}$, be the monomial attached to the domain of linearity of γ preceding the crossing with ρ. Without loss of generality, we can assume $s = \varphi_{\sigma_-}(m_{\rho_-})$. Indeed, the pairing $\langle -, - \rangle$ is trivial on P, $r(s)$ is a linear combination of m_ρ and m_{ρ_-}, and $\hat{z}^{\varphi_{\sigma_-}(m_\rho)}$ transforms trivially across

ρ.

By the Definition 3.15 of a quantum broken line, we have to show that

$$p_{\sigma_+}\left(\hat{z}^{\varphi_{\sigma_-}(m_{\rho_-})}\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)\right) = \hat{z}^{\bar{\varphi}_{\sigma_-}(m_{\bar{\rho}_-})}\hat{f}_{\bar{\rho}^{\mathrm{out}}}(q^{-1}\bar{X})\hat{f}_{\bar{\rho}^{\mathrm{in}}}(\bar{X})\,.$$

From the relations

$$\hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{z}^{\varphi_\rho(m_{\rho_-})} = q^{\frac{1}{2}D_\rho^2}\hat{z}^{D_\rho}X^{-D_\rho^2}$$

in $\Bbbk[P_{\varphi_\rho}]$,

$$\hat{z}^{\bar{\varphi}_{\bar{\rho}}(m_{\bar{\rho}_+})}\hat{z}^{\bar{\varphi}_{\bar{\rho}}(m_{\bar{\rho}_-})} = q^{\frac{1}{2}D_{\bar{\rho}}^2}\hat{z}^{D_{\bar{\rho}}}\bar{X}^{-D_{\bar{\rho}}^2}$$

in $\Bbbk[P_{\bar{\varphi}_{\bar{\rho}}}]$, and using $D_\rho^2 = D_{\bar{\rho}}^2 - l_\rho$ and $D_\rho = D_{\bar{\rho}} - \sum_{j, D_{m_j}=D_\rho} E_j$, we get

$$p_{\sigma_+}(\hat{z}^{\varphi_\rho(m_{\rho_-})}) = \hat{z}^{\bar{\varphi}_{\bar{\rho}}(m_{\bar{\rho}_-})}\prod_{j, D_{m_j}=D_\rho}(q^{-\frac{1}{2}}\bar{X}\hat{z}^{-E_j})\,.$$

The result follows from the identity

$$q^{-\frac{1}{2}}\bar{X}\hat{z}^{-E_j}(1+q^{-\frac{1}{2}}\hat{z}^{E_j}(q^{-1}\bar{X})^{-1}) = 1+q^{-\frac{1}{2}}\hat{z}^{-E_j}\overline{X}\,.$$

□

Lemma 3.36. *Let σ be a two-dimensional cone of Σ. For every $Q \in \sigma$ and for every $p \in B_0(\mathbb{Z})$, we have*

$$p_\sigma(\mathrm{Lift}_Q(p)) = \mathrm{Lift}_{\nu(Q)}(\nu(p))\,.$$

Proof. It is a direct consequence of Lemma 3.35. □

3.3.8 End of the proof of Theorem 3.26

This Section is parallel to the proof of Theorem 3.30 of [GHK15a]. We have to show that $\hat{\mathfrak{D}}^{\mathrm{can}}$ satisfies the two conditions entering the Definition 3.17 of consistency of a quantum scattering diagram.

- Let Q and Q' be generic points in B_0 contained in a common two-dimensional cone σ of Σ, and let γ be a path in the interior of σ connecting Q and Q', and intersecting transversely the rays of $\hat{\mathfrak{D}}$. We have to show that

  $$\mathrm{Lift}_{Q'}(p) = \hat{\theta}_{\gamma,\hat{\mathfrak{D}}^{\mathrm{can}}}(\mathrm{Lift}_Q(p))\,.$$

  By Lemma 3.34 and Lemma 3.36, it is enough to show that

  $$\mathrm{Lift}_{\nu(Q')}(\nu(p)) = \hat{\theta}_{\nu(\gamma),\nu(\hat{\mathfrak{D}}^{\mathrm{can}})}(\mathrm{Lift}_{\nu(Q)}(\nu(p)))\,,$$

  which follows from the combination of Proposition 3.32 and Proposition 3.33.

- Let Q_- and Q_+ be two generic points in B_0, contained respectively in two-dimensional cones σ_- and σ_+ of Σ, such that σ_+ and σ_- intersect along a one-dimensional cone ρ

of Σ. Assuming further that Q_- and Q_+ are contained in connected components of $B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ whose closures contain ρ, we have to show that $\mathrm{Lift}_{Q_+}(p) \in R^{\hbar}_{\sigma_+,I}$ and $\mathrm{Lift}_{Q_-}(p) \in R^{\hbar}_{\sigma_-,I}$ are both images under $\hat{\psi}_{\rho,+}$ and $\hat{\psi}_{\rho,-}$ respectively of a single element $\mathrm{Lift}_{\rho}(p) \in R^{\hbar}_{\rho,I}$. By Lemma 3.34 and Lemma 3.36, it is enough to prove the corresponding statement after application of ν. This result follows from the combination of the Remark at the end of Section 3.2.4 and of the second point of Lemma 3.16.

3.4 Extension over boundary strata

3.4.1 Torus equivariance

Recall from Section 3.1.5 that $T^D \coloneqq \mathbb{G}_m^r$ is the torus whose character group $\chi(T^D)$ has a basis e_{D_j} indexed by the irreducible components D_j of D, $1 \leqslant j \leqslant r$. The map

$$\beta \mapsto \sum_{j=1}^{r} (\beta.D_j) e_{D_j}$$

induces an action of T^D on $S_I = \mathrm{Spec}\, R_I$.

Following Section 5 of [GHK15a], we consider

$$w: B \to \chi(T_D) \otimes \mathbb{R}\,,$$

the unique piecewise linear map such that $w(0) = 0$ and $w(m_{\rho_j}) = e_{D_j}$ for all $1 \leqslant j \leqslant r$, where m_{ρ_j} is the primitive generator of the ray ρ_j.

According to Theorem 5.2 of [GHK15a], for every I monomial ideal of R such that $X_{I,\mathfrak{D}^{\mathrm{can}}} \to \mathrm{Spec}\, R_I$ is defined, the T^D-action on $\mathrm{Spec}\, R_I$ has a natural lift to $X_{I,\mathfrak{D}^{\mathrm{can}}}$, such that the decomposition

$$H^0(X_{I,\mathfrak{D}^{\mathrm{can}}}, \mathcal{O}_{X_{I,\mathfrak{D}^{\mathrm{can}}}}) = A_I = \bigoplus_{p \in B(\mathbb{Z})} R_I \vartheta_p$$

as R_I-module is a weight decomposition, T^D acting on ϑ_p with weight $w(p)$.

We extend the action of T^D on R_I by $\mathbb{k}$-algebra automorphisms to an action of T^D on $R_I^{\hbar}$ by $\mathbb{k}_{\hbar}$-automorphism by assigning weight zero to $\hbar$.

Proposition 3.37. *The T^D-action on A_I by $\mathbb{k}$-algebra automorphisms, equivariant for the structure of R_I-algebra, lifts to a T^D-action on $A_I^{\hbar}$ by $\mathbb{k}_{\hbar}$-automorphisms, equivariant for the structure of $R_I^{\hbar}$-algebra. Furthermore, the decomposition*

$$A_I^{\hbar} = \bigoplus_{p \in B(\mathbb{Z})} R_I^{\hbar} \hat{\vartheta}_p$$

as $R_I^{\hbar}$-module is a weight decomposition, T^D acting on $\hat{\vartheta}_p$ with weight $w(p)$.

Proof. It is a quantum deformation of the proof of Theorem 5.2 of [GHK15a]. As $A_I^{\hbar} = \Gamma(X^{\circ}_{I,\mathfrak{D}^{\mathrm{can}}}, \mathcal{O}^{\hbar}_{X^{\circ}_{I,\mathfrak{D}^{\mathrm{can}}}})$, it is enough to define the T^D-action on $\mathcal{O}^{\hbar}_{X^{\circ}_{I,\mathfrak{D}^{\mathrm{can}}}}$.

Remark that for every ray $\mathfrak{d}$ of $\hat{\mathfrak{D}}^{\mathrm{can}}$, the monomials appearing in $\hat{H}_{\mathfrak{d}}$ and so in $\hat{f}_{\mathfrak{d}}$ have weight zero. Indeed they are of the form $\hat{z}^{\beta-\varphi_{\tau_{\mathfrak{d}}}(\ell_\beta m_{\mathfrak{d}})}$ with $\beta \in NE(Y)_{\mathfrak{d}}$, which by definition means that $\beta.D_j = \ell_\beta(m_{\mathfrak{d}}, D_j)$ for all $1 \leqslant j \leqslant r$. In other words, the scattering automorphisms have weight zero.

Let ρ be a one-dimensional cone of Σ. Let σ_+ and σ_- be the two-dimensional cones of Σ bounding ρ, and let ρ_+, ρ_- be the other boundary one-dimensional cones of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order. From the explicit description of $R^{\hbar}_{\rho,I}$ by generator and relations given in Section 3.2.1, and recalling that φ is such that $\kappa_{\rho,\varphi} = [D_\rho]$, we define an action of T^D on $R^{\hbar}_{\rho,I}$, equivariant for the structure of equivariant $R^{\hbar}_I$-algebra by acting on X with the character e_{D_ρ}, on X_+ with the character $e_{D_{\rho_+}}$, and on X_- with the character $e_{D_{\rho_-}}$. The fact that $\hat{f}_{\rho^{\mathrm{in}}}$ and $\hat{f}_{\rho^{\mathrm{out}}}$ have weight zero implies that the relations defining $R^{\hbar}_{\rho,I}$ are equivariant and so this T^D-action is indeed well-defined on $R^{\hbar}_{\rho,I}$.

As the scattering automorphisms have weight zero, these T^D-actions on the various $R^{\hbar}_{\rho,I}$ glue to define a T^D-action on $\mathcal{O}^{\hbar}_{X^\circ_{I,\mathfrak{D}^{\mathrm{can}}}}$.

The check that $\hat{\vartheta}_p$ is an eigenfunction of the T^D-action with weight $w(p)$ is now formally identical to the corresponding classical check given in the proof of Theorem 5.2 of [GHK15a]. As the scattering automorphisms have weight zero, the weights of the monomials on the various domains of linearity of a broken line are identical and so it is enough to consider the unbounded domain of linearity. In this case, the monomial is $\hat{z}^{\varphi_{\tau_p}(p)}$, which has weight $w(p)$. □

3.4.2 End of the proof of Theorem 3.7

We fix (Y,D) a Looijenga pair. Let $\sigma_P \subset A_1(Y,\mathbb{R})$ be a strictly convex polyhedral cone containing $NE(Y)_{\mathbb{R}}$. Let $P \coloneqq \sigma_P \cap A_1(Y,\mathbb{Z})$ be the associated monoid and let $R \coloneqq \Bbbk[P]$ be the corresponding $\Bbbk$-algebra. For $J = \mathfrak{m}_R$ the maximal ideal monomial of R, the assumptions of Theorem 3.26 are satisfied and so the canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ is consistent.

If (Y,D) admits a toric model, then D has $r \geqslant 3$ irreducible components, and so we can apply Theorem 3.18. Combined with Proposition 3.37, this proves Theorem 3.7 in this case.

In general, it is proven in Section 6.2 of [GHK15a] that

$$H^0(X_I, \mathcal{O}_{X_I}) = A_I \coloneqq \bigoplus_{p \in B(\mathbb{Z})} R_I \theta_p \,,$$

with the R_I-algebra structure determined by the classical version of the product formula given in Theorem 3.20. So Theorem 3.7 follows from the following Proposition 3.38.

Proposition 3.38. *For every monomial ideal I of R of radical $\mathfrak{m}_R$, the multiplication rule of Theorem 3.20 defines a structure of $R^{\hbar}_I$-algebra on the $R^{\hbar}_I$-module*

$$A^{\hbar}_I \coloneqq \bigoplus_{p \in B(\mathbb{Z})} R^{\hbar}_I \hat{\vartheta}_p \,.$$

Proof. If (Y, D) admits a toric model, then D has $r \geqslant 3$ components and so the result follows from Theorem 3.20.

In general, there is a corner blow-up (Y', D') of (Y, D) admitting a toric model. The result for (Y', D') implies the result for (Y, D) as in Section 6.2 of [GHK15a].

□

3.4.3 Quantization of $\mathbb{V}_1$ and $\mathbb{V}_2$

By Poposition 3.38, for every monomial ideal I of R of radical $\mathfrak{m}_R$, we have a structure of $R_I^\hbar$-algebra on

$$A_I^\hbar = \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p \,.$$

In this Section, we describe explicitly this algebra for $I = \mathfrak{m}_R$.

In the classical limit $\hbar = 0$, we get a commutative R_I-algebra which, by [GHK15a] is the algebra of functions on the variety $\mathbb{V}_r$, where r is the number of irreducible components of D, and

- If $r \geqslant 3$, $\mathbb{V}_r$ is the r-cycle of coordinates planes in the affine space $\mathbb{A}^r$, $\mathbb{V}_r = \mathbb{A}^2_{x_1,x_2} \cup \mathbb{A}^2_{x_2,x_3} \cup \cdots \cup \mathbb{A}^2_{x_r,x_1} \subset \mathbb{A}^r_{x_1,\dots,x_r}$.
- If $r = 2$, $\mathbb{V}_2$ is a union of two affine planes[10],

$$\mathbb{V}_2 = \mathrm{Spec}\ \Bbbk[x,y,z]/(xyz - z^2)\,,$$

affine cone over the union of the two rational curves $z = 0$ and $xy - z = 0$, intersecting in two points, embedded in the weighted projective plane $\mathbb{P}^{1,1,2}$.
- If $r = 1$, $\mathbb{V}_1 = \mathrm{Spec}\ \Bbbk[x,y,z]/(xyz - x^2 - z^3)$, affine cone over a nodal curve embedded in the weighted projective plane $\mathbb{P}^{(3,1,2)}_{x,y,z}$.

When $r \geqslant 3$, the explicit description of $A^\hbar_{\mathfrak{m}_R}$ follows from the combination of Section 3.2.5 and the beginning of the proof of Theorem 3.18: we have

$$\hat{\vartheta}_m \cdot \hat{\vartheta}_{m'} = \begin{cases} q^{\frac{1}{2}\langle m,m'\rangle} \hat{\vartheta}_{m+m'} & \text{if } m \text{ and } m' \text{ lie in a common cone of } \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

In particular, denoting $v_1, \dots, v_r$ the primitive generators of the one-dimensional cones $\rho_1, \dots, \rho_r$ of Σ, $A^\hbar_{\mathfrak{m}_R}$ is generated as $\Bbbk_\hbar$-algebra by $\hat{\vartheta}_{v_1}, \dots, \hat{\vartheta}_{v_r}$.

For $r = 2$ and $r = 1$, computing $A^\hbar_{\mathfrak{m}_R}$ is slightly more subtle and the answer is given below in Propositions 3.39 and 3.40.

Both $\mathbb{V}_1$ and $\mathbb{V}_2$ are hypersurfaces in $\mathbb{A}^3_{x,y,z}$. Evey hypersurface $F(x,y,z) = 0$ in $\mathbb{A}^3_{x,y,z}$ has

[10] In [GHK15a], the description $\mathbb{V}_2 = \mathrm{Spec}\ \Bbbk[u,v,w]/(w^2 - u^2v^2)$ is given. It is equivalent to our description via the change of variables $x = \sqrt{2}u$, $y = \sqrt{2}v$, $z = w + uv$.

a natural Poisson structure defined by

$$\{x,y\} = \frac{\partial F}{\partial z}\,, \{y,z\} = \frac{\partial F}{\partial x}\,, \{z,x\} = \frac{\partial F}{\partial y}\,,$$

see [EG10] for example.

For $\mathbb{V}_2$ and $F(x,y,z) = z^2 - xyz$, we get

$$\{x,y\} = 2z - xy\,, \{y,z\} = -yz\,, \{z,x\} = -zx\,.$$

It follows from $\{y,z\} = -yz$ and $\{z,x\} = -zx$ that this bracket coincides with the one coming from the standard symplectic form on the two natural copies of $(\mathbb{G}_m)^2$ contained in $\mathbb{V}_1$.

For $\mathbb{V}_1$ and $F(x,y,z) = z^3 + x^2 - xyz$, we get

$$\{x,y\} = 3z^2 - xy\,, \{y,z\} = 2x - yz\,, \{z,x\} = -zx\,.$$

It follows from $\{x,z\} = xz$ that the above Poisson structure is indeed the one induced by the standard symplectic form on the natural copy of $(\mathbb{G}_m)^2$ contained in $\mathbb{V}_2$.

We first explain how to recover the above Poisson brackets from the formula given by Corollary 3.21 in terms of classical broken lines. We then use the formula of Theorem 3.20 in terms of quantum broken lines to compute the q-commutators deforming these Poisson brackets.

For $\mathbb{V}_2$, the tropicalization B contains two two-dimensional cones σ_1, and σ_2, and two one-dimensional cones ρ_1 and ρ_2. Let v_1 and v_2 in $B(\mathbb{Z})$ be the primitive generators of ρ_1 and ρ_2. Cutting B along ρ_1, we can identify B as the upper half-plane in $\mathbb{R}^2$ with an identification of the two boundary horizontal rays. Denote $w = (1,0)$, $w' = (-1,0)$, $v_2 = (0,1)$. We have $x = \vartheta_{v_1} = \vartheta_w = \vartheta_{w'}$, $y = \vartheta_{v_2}$, $z = \vartheta_{w+v_2}$. The broken lines description of the product gives

$$xy = \vartheta_{v_1}\vartheta_{v_2} = \vartheta_{w+v_2} + \vartheta_{w'+v_2}\,,$$

and

$$\vartheta_{w+v_2}\vartheta_{w'+v_2} = 0\,,$$

so $\vartheta_{w'+v_2} = xy - z$ and $(xy-z)z = 0$, which is indeed the equation defining $\mathbb{V}_2$. We have

$$\{x,y\} = \{\vartheta_{v_1},\vartheta_{v_2}\} = \langle(1,0),(0,1)\rangle\vartheta_{w+v_2} + \langle(-1,0),(0,1)\rangle\vartheta_{w'+v_2} = \vartheta_{w+v_2} - \vartheta_{w'+v_2}$$

Using $\vartheta_{w'+v_2} = xy - z$, we get $\{x,y\} = 2z - xy$. We have

$$\{y,z\} = \{\vartheta_{v_2},\vartheta_{w+v_2}\} = \langle(0,1),(1,1)\rangle\vartheta_{w+2v_2} = -\vartheta_{v_2}\vartheta_{w+v_2} = -yz\,.$$

Finally, we have

$$\{z,x\} = \langle(1,1),(1,0)\rangle\vartheta_{2w+v_2} = -\vartheta_w\vartheta_{w+v_2} = -zx\,.$$

Using the formula of Theorem 3.20, we compute the q-commutators deforming the above

Poisson brackets. We have

$$\hat{x}\hat{y} = \hat{\vartheta}_{v_1}\hat{\vartheta}_{v_2} = q^{\frac{1}{2}}\hat{\vartheta}_{w+v_2} + q^{-\frac{1}{2}}\hat{\vartheta}_{w'+v_2}\,,$$

so $\hat{\vartheta}_{w'+v_2} = q^{\frac{1}{2}}\hat{x}\hat{y} - q\hat{z}^2$. On the other hand, we have

$$\hat{y}\hat{x} = \hat{\vartheta}_{v_2}\hat{\vartheta}_{v_1} = q^{-\frac{1}{2}}\hat{\vartheta}_{w'+v_2} + q^{\frac{1}{2}}\hat{\vartheta}_{w'+v_2}\,,$$

$$q^{-\frac{1}{2}}\hat{y}\hat{x} = q^{-1}\hat{\vartheta}_{w'+v_2} + \hat{\vartheta}_{w'+v_2}\,,$$

and so

$$q^{\frac{1}{2}}\hat{x}\hat{y} - q^{-\frac{1}{2}}\hat{y}\hat{x} = (q - q^{-1})\hat{z}^2\,.$$

We have

$$\hat{y}\hat{z} = \hat{\vartheta}_{v_2}\hat{\vartheta}_{w+v_2} = q^{-\frac{1}{2}}\hat{\vartheta}_{w+2v_2}\,,$$

and

$$\hat{z}\hat{y} = \hat{\vartheta}_{w+v_2}\hat{\vartheta}_{v_2} = q^{\frac{1}{2}}\hat{\vartheta}_{w+2v_2}\,,$$

so

$$q^{\frac{1}{2}}\hat{y}\hat{z} - q^{-\frac{1}{2}}\hat{z}\hat{y} = 0\,.$$

We have

$$\hat{z}\hat{x} = \hat{\vartheta}_{w+v_2}\hat{\vartheta}_{w} = q^{-\frac{1}{2}}\hat{\vartheta}_{2w+v_2}$$

and

$$\hat{x}\hat{z} = \hat{\vartheta}_{w}\hat{\vartheta}_{w+v_2} = q^{\frac{1}{2}}\hat{\vartheta}_{2w+v_2}\,,$$

so

$$q^{\frac{1}{2}}\hat{z}\hat{x} - q^{-\frac{1}{2}}\hat{x}\hat{z} = 0\,.$$

Finally, we compute the q-deformation of the cubic relation $F = 0$:

$$\hat{x}\hat{y}\hat{z} = \hat{\vartheta}_{w}q^{-\frac{1}{2}}\hat{\vartheta}_{w+2v_2} = q^{\frac{1}{2}}\hat{z}^2\,.$$

In summary, we proved:

Proposition 3.39. *The deformation quantization of $\mathbb{V}_2$ given by the product formula of Theorem 3.20 is the associative $\Bbbk_\hbar$-algebra generated by variables $\hat{x}, \hat{y}, \hat{z}$ and with relations*

$$q^{\frac{1}{2}}\hat{x}\hat{y} - q^{-\frac{1}{2}}\hat{y}\hat{x} = (q - q^{-1})\hat{z}\,,$$

$$q^{\frac{1}{2}}\hat{y}\hat{z} - q^{-\frac{1}{2}}\hat{z}\hat{y} = 0\,,$$

$$q^{\frac{1}{2}}\hat{z}\hat{x} - q^{-\frac{1}{2}}\hat{x}\hat{z} = 0\,,$$

$$\hat{x}\hat{y}\hat{z} = q^{\frac{1}{2}}\hat{z}^2\,.$$

For $\mathbb{V}_1$, the tropicalization B contains one two-dimensional cone σ, and one one-dimensional cone ρ. Let v in $B(\mathbb{Z})$ be the primitive generator of ρ. Cutting B along ρ, we can identify B as a quadrant in $\mathbb{R}^2$ with an identification of the two boundary rays. Denote $w = (1, 0)$

and $w' = (0,1)$. The description of the product of classical theta functions by broken lines is given in Section 6.2 of [GHK15a]. We have $x = \vartheta_{2w+w'}$, $y = \vartheta_v = \vartheta_w = \vartheta_{w'}$, $z = \vartheta_{w+w'}$. We have

$$\{x,y\} = \{\vartheta_{2w+w'}, \vartheta_v\} = \langle (2,1),(1,0)\rangle \vartheta_{3w+w'} + \langle (2,1),(0,1)\rangle \vartheta_{2w+2w'}$$

$$= -\vartheta_{3w+w'} + 2\vartheta_{2w+2w'}\,.$$

On the other hand, we have $xy = \vartheta_{3w+w'} + \vartheta_{2w+2w'}$ and $z^2 = \vartheta_{2w+2w'}$, and so $\{x,y\} = 3z^2 - xy$. We have

$$\{y,z\} = \{\vartheta_v, \vartheta_{w+w'}\} = \langle (1,0),(1,1)\rangle \vartheta_{2w+w'} + \langle (0,1),(1,1)\rangle \vartheta_{w+2w'}$$

$$= \vartheta_{2w+w'} - \vartheta_{w+2w'}\,.$$

On the other hand, we have $yz = \vartheta_{2w+w'} + \vartheta_{w+2w'}$ and $x = \vartheta_{2w+w'}$, and so $\{y,z\} = 2x - yz$. We have

$$\{z,x\} = \{\vartheta_{w+w'}, \vartheta_{2w+w'}\} = \langle (1,1),(2,1)\rangle \vartheta_{3w+2w'} = -\vartheta_{3w+2w'}\,.$$

On the other hand, we have $zx = \vartheta_{3w+2w'}$ and so $\{z,x\} = -zx$.

Using the formula of Theorem 3.20, we compute the q-commutators deforming the above Poisson brackets. We have

$$\hat{x}\hat{y} = \hat{\vartheta}_{2w+w'}\hat{\vartheta}_v = q^{-\frac{1}{2}}\hat{\vartheta}_{3w+w'} + q\hat{\vartheta}_{2w+2w'}\,,$$

so

$$\hat{\vartheta}_{3w+w'} = q^{\frac{1}{2}}\hat{x}\hat{y} - q^{\frac{3}{2}}\hat{z}^2\,.$$

On the other hand, we have

$$\hat{y}\hat{x} = \hat{\vartheta}_v\hat{\vartheta}_{2w+w'} = q^{\frac{1}{2}}\hat{\vartheta}_{3w+w'} + q^{-1}\hat{\vartheta}_{2w+2w'}\,,$$

$$q^{-\frac{1}{2}}\hat{y}\hat{x} = \hat{\vartheta}_{3w+w'} + q^{-\frac{3}{2}}\hat{z}^2\,,$$

and so

$$q^{\frac{1}{2}}\hat{x}\hat{y} - q^{-\frac{1}{2}}\hat{y}\hat{x} = (q^{\frac{3}{2}} - q^{-\frac{3}{2}})\hat{z}^2\,.$$

We have

$$\hat{y}\hat{z} = \hat{\vartheta}_v\hat{\vartheta}_{w+w'} = q^{\frac{1}{2}}\hat{\vartheta}_{2w+w'} + q^{-\frac{1}{2}}\hat{\vartheta}_{w+2w'}\,,$$

so

$$\hat{\vartheta}_{w+2w'} = q^{\frac{1}{2}}\hat{y}\hat{z} - q\hat{x}\,.$$

On the other hand, we have

$$\hat{z}\hat{y} = \hat{\vartheta}_{w+w'}\hat{\vartheta}_v = q^{-\frac{1}{2}}\hat{\vartheta}_{2w+w'} + q^{\frac{1}{2}}\hat{\vartheta}_{w+2w'}\,,$$

$$q^{-\frac{1}{2}}\hat{z}\hat{y} = q^{-1}\hat{x} + \hat{\vartheta}_{w+2w'}\,,$$

and so

$$q^{\frac{1}{2}}\hat{y}\hat{z} - q^{-\frac{1}{2}}\hat{z}\hat{y} = (q - q^{-1})\hat{x}\,.$$

We have

$$\hat{z}\hat{x} = \hat{\vartheta}_{w+w'}\hat{\vartheta}_{2w+w'} = q^{-\frac{1}{2}}\hat{\vartheta}_{3w+2w'}$$

$$\hat{\vartheta}_{3w+2w'} = q^{\frac{1}{2}}\hat{z}\hat{x}\,.$$

On the other hand, we have

$$\hat{x}\hat{z} = \hat{\vartheta}_{2w+w'}\hat{\vartheta}_{w+w'} = q^{\frac{1}{2}}\hat{\vartheta}_{3w+2w'}\,,$$

and so

$$q^{\frac{1}{2}}\hat{z}\hat{x} - q^{-\frac{1}{2}}\hat{x}\hat{z} = 0\,.$$

Finally, we compute the q-deformation of the cubic relation $F = 0$:

$$\hat{x}\hat{y}\hat{z} = \hat{\vartheta}_{2w+w'}(q^{\frac{1}{2}}\hat{\vartheta}_{2w+w'} + q^{-\frac{1}{2}}\hat{\vartheta}_{w+2w'}) = q^{\frac{1}{2}}\hat{\vartheta}^2_{2w+w'} + q^{-\frac{1}{2}}q^{\frac{3}{2}}\hat{\vartheta}_{3w+3w'}\,,$$

$$\hat{x}\hat{y}\hat{z} = q^{\frac{1}{2}}\hat{x}^2 + q\hat{z}^3\,.$$

In summary, we proved:

Proposition 3.40. *The deformation quantization of $\mathbb{V}_1$ given by the product formula of Theorem 3.20 is the associative $\Bbbk_\hbar$-algebra generated by variables $\hat{x}, \hat{y}, \hat{z}$ and with relations*

$$q^{\frac{1}{2}}\hat{x}\hat{y} - q^{-\frac{1}{2}}\hat{y}\hat{x} = (q^{\frac{3}{2}} - q^{-\frac{3}{2}})\hat{z}^2\,,$$

$$q^{\frac{1}{2}}\hat{y}\hat{z} - q^{-\frac{1}{2}}\hat{z}\hat{y} = (q - q^{-1})\hat{x}\,,$$

$$q^{\frac{1}{2}}\hat{z}\hat{x} - q^{-\frac{1}{2}}\hat{x}\hat{z} = 0\,,$$

$$\hat{x}\hat{y}\hat{z} = q^{\frac{1}{2}}\hat{x}^2 + q\hat{z}^3\,.$$

3.4.4 End of the proof of Theorem 3.8

In this Section, we finish the proof of Theorem 3.8, which is done by combination of Proposition 3.41 and Proposition 3.42. We follow Section 6.1 of [GHK15a].

For every I monomial ideal of P, we define the free $R_I^\hbar$-module

$$A_I^\hbar = \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p\,.$$

According to Proposition 3.38, if I has radical $\mathfrak{m}_R$, then there is a natural $R_I^\hbar$-algebra structure on $A_I^\hbar$.

Let $\Gamma \subset B(\mathbb{Z})$ be a finite collection of integral points such that the corresponding quantum theta functions $\hat{\vartheta}_p$ generate the $\Bbbk_\hbar$-algebra $A_{\mathfrak{m}_R}^\hbar$. Using the notations of Section 3.4.3, we can take $\Gamma = \{v_1, \dots, v_r\}$ if $r \geqslant 3$, $\Gamma = \{v_1, v_2, w + v_2\}$ if $r = 2$, and $\Gamma = \{v, w + w', 2w + w'\}$ if $r = 1$.

Proposition 3.41. *There exists a unique minimal radical monomial ideal $J_{\min}^\hbar$ of P such that, for every I monomial ideal of P of radical containing $J_{\min}^\hbar$,*

- *There exists a $R_I^\hbar$-algebra structure on $A_I^\hbar$ such that, for every $k > 0$, the natural isomorphism of $R_{I+\mathfrak{m}^k}^\hbar$-modules $A_I^\hbar \otimes R_{I+\mathfrak{m}^k}^\hbar = A_{I+\mathfrak{m}^k}^\hbar$ is an isomorphism of $R_{I+\mathfrak{m}^k}^\hbar$-algebras.*
- *The quantum theta functions $\hat{\vartheta}_p$, $p \in \Gamma$, generate $A_I^\hbar$ as an $R_I^\hbar$-algebra.*

Proof. Follows as its classical version in Section 6.1 of [GHK15a]. □

As in Section 6.1 of [GHK15a], the first point of Proposition 3.41 is equivalent to the fact that for every $p_1, p_2 \in B(\mathbb{Z})$, at most finitely many terms $\hat{z}^\beta \hat{\vartheta}_p$ with $\beta \notin I$ appear in the expansion given by Theorem 3.20 for $\hat{\vartheta}_{p_1} \hat{\vartheta}_{p_2}$.

Proposition 3.42. *Suppose that $F \subset \sigma_P$ is a face such that F does not contain the class of every component of D, then $J_{\min}^\hbar \subset P - F$. If (Y, D) is positive, then $J_{\min}^\hbar = 0$.*

Proof. The proof is formally identical to the proof of its classical version, Proposition 6.6 of [GHK15a]. The main input, the T^D-equivariance, is given in our case by Proposition 3.37. □

Remark: Let $J_{\min}$ be the ideal defined by Proposition 6.5 of [GHK15a]. We obviously have $J_{\min} \subset J_{\min}^\hbar$, as the vanishing of all genus Gromov-Witten invariants includes the vanishing of genus zero Gromov-Witten invariants. If (Y, D) is positive then $J_{\min} = J_{\min}^\hbar = 0$. In general, it is unclear if we always have $J_{\min} = J_{\min}^\hbar$ or if there are examples with $J_{\min} \neq J_{\min}^\hbar$. Geometrically, it is the question to know if some vanishing of genus zero Gromov-Witten invariants implies (or not) a vanishing of all higher genus Gromov-Witten invariants.

3.4.5 q-integrality: end of the proof of Theorem 3.9

The $R_I^\hbar$-algebra structure on

$$A_I^\hbar = \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p$$

is given by the product formula of Theorem 3.20,

$$\hat{\vartheta}_{p_1} \hat{\vartheta}_{p_2} = \sum_{p \in B(\mathbb{Z})} C_{p_1,p_2}^p \hat{\vartheta}_p \,.$$

A priori, we have $C_{p_1,p_2}^p \in R_I^\hbar = R_I[[\hbar]]$. Theorem 3.9 follows from the following Proposition.

Proposition 3.43. *For every $p_1, p_2, p_3 \in B(\mathbb{Z})$, we have*

$$C_{p_1,p_2}^p \in R_I^q = R_I[q^{\pm \frac{1}{2}}] \,,$$

where $q = e^{i\hbar}$. More precisely, C_{p_1,p_2}^p is the power series expansion around $\hbar = 0$ of a Laurent polynomial in $q^{\frac{1}{2}}$ after the change of variables $q = e^{i\hbar}$.

Proof. Recall that, if γ is a quantum broken line of endpoint Q in a cone τ of Σ, we write the monomial Mono(γ) attached to the segment ending at Q as

$$\mathrm{Mono}(\gamma) = c(\gamma)\hat{z}^{\varphi_\tau(s(\gamma))}$$

with $c(\gamma) \in \mathbb{k}_\hbar[P_{\varphi_\tau}]$ and $s(\gamma) \in \Lambda_\tau$.

By definition, we have

$$C^p_{p_1,p_2} = \sum_{\gamma_1,\gamma_2} c(\gamma_1)c(\gamma_2)q^{\frac{1}{2}\langle s(\gamma_1),s(\gamma_2)\rangle}\,,$$

where the sum is over all broken lines γ_1 and γ_2, of asymptotic charges p_1 and p_2, satisfying $s(\gamma_1)+s(\gamma_2)=p$, and both ending at the point $z \in B_0$, an where $z \in B - \mathrm{Supp}_I(\mathfrak{D}^{\mathrm{can}})$ very close to p.

So it is enough to show that, for every γ quantum broken line of endpoint Q in a cone τ of Σ, we have $c(\gamma) \in \mathbb{k}_q[P_{\varphi_\tau}]$. We will show more generally that for every quantum broken line γ of $\hat{\mathfrak{D}}^{\mathrm{can}}$, and for every L domain of linearity of γ, the attached monomial $m_L = c_L \hat{z}^{p_L}$ satisfies $c_L \in \mathbb{k}_q$.

It is obviously true if L is the unbounded domain of linearity of γ since then $c_L = 1$. Given the formula in Definition 3.15 specifying the change of monomials when the quantum broken line bends, it is then enough to show that, for every ray $(\mathfrak{d}, \hat{H}_\mathfrak{d})$ of $\hat{\mathfrak{D}}^{\mathrm{can}}$, the corresponding $\hat{f}_\mathfrak{d}$ is in $\mathbb{k}_q\widehat{[P_{\varphi_{\tau_\mathfrak{d}}}]}$.

Given the argument used in Section 6.2 of [GHK15a], we can assume that (Y,D) admits a toric model, and, using the deformation invariance of $\hat{\mathfrak{D}}^{\mathrm{can}}$, see Lemma 3.24, we can assume further that there exists $\mathfrak{m}=(m_1,\dots,m_n)$ such that $(Y,D)=(Y_\mathfrak{m},\partial Y_\mathfrak{m})$, as in Section 3.3.4. In Section 3.3.5, we introduced a quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$. From the definition of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ and the explicit formulas given in the proof of Lemma 3.34 comparing $\hat{\mathfrak{D}}^{\mathrm{can}}$ and $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$, it is enough to prove the result for outgoing rays $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$.

By Proposition 3.32, we have $\nu(\hat{\mathfrak{D}}^{\mathrm{can}}) = S(\hat{\mathfrak{D}}_\mathfrak{m})$. So it remains to show that, for every outgoing ray $(\mathfrak{d},\hat{H}_\mathfrak{d})$ of $S(\hat{\mathfrak{D}}_\mathfrak{m})$, the corresponding $\hat{f}_\mathfrak{d}$ is in $\mathbb{k}_q\widehat{[P_{\bar{\varphi}}]}$.

By Proposition 3.31, the Hamiltonian $\hat{H}_\mathfrak{d}$ attached to an outgoing ray $\mathfrak{d}$ of $S(\hat{\mathfrak{D}}_\mathfrak{m}) - \hat{\mathfrak{D}}_\mathfrak{m}$ is given by

$$\hat{H}_\mathfrak{d} = \left(\frac{i}{\hbar}\right)\sum_{p\in P_{m_\mathfrak{d}}}\left(\sum_{g\geqslant 0} N_{g,\beta_p}^{Y_\mathfrak{m}/\partial Y_\mathfrak{m}}\hbar^{2g}\right)\hat{z}^{\beta_p-\bar{\varphi}(\ell_\beta m_\mathfrak{d})}\,.$$

According to Theorem 2.30, for every $p \in P_{m_\mathfrak{d}}$, there exists

$$\Omega_p^{Y_\mathfrak{m}}(q^{\frac{1}{2}}) = \sum_{j\in\mathbb{Z}}\Omega_{p,j}^{Y_\mathfrak{m}} q^{\frac{j}{2}} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]\,,$$

such that

$$\left(\frac{i}{\hbar}\right)\left(\sum_{g\geqslant 0} N_{g,p}^{Y_\mathfrak{m}}\hbar^{2g-1}\right) = -(-1)^{\beta_p\cdot\partial Y_\mathfrak{m}+1}\sum_{p=\ell p'}\frac{1}{\ell}\frac{1}{q^{\frac{\ell}{2}}-q^{-\frac{\ell}{2}}}\Omega_{p'}^{Y_\mathfrak{m}}(q^{\frac{\ell}{2}})\,,$$

which can be rewritten

$$\left(\frac{i}{\hbar}\right)\left(\sum_{g\geqslant 0} N_{g,p}^{Y_{\mathfrak{m}}} \hbar^{2g-1}\right) = \sum_{j\in\mathbb{Z}} \sum_{p=\ell p'} \frac{1}{\ell} \frac{1}{q^{\frac{\ell}{2}} - q^{-\frac{\ell}{2}}} (-1)^{\ell \beta_{p'} . \partial Y_{\mathfrak{m}}} \Omega_{p',j}^{Y_{\mathfrak{m}}} q^{\frac{j\ell}{2}} ,$$

Using Lemma 3.13, we get that

$$\hat{f}_{\mathfrak{d}} = \prod_{p\in P_{m_{\mathfrak{d}}}} \prod_{j\in\mathbb{Z}} (1 + q^{\frac{j-1}{2}} \hat{z}^{\beta_p - \bar{\varphi}(\ell_\beta m_{\mathfrak{d}})})^{\Omega_{p,j}^{Y_{\mathfrak{m}}}} ,$$

which concludes the proof. □

3.5 Example: degree 5 del Pezzo surfaces

Let Y be a del Pezzo surface of degree 5, i.e. a blow-up of $\mathbb{P}^2$ in four points in general position, and let D be an anticanonical cycle of five (-1)-curves on Y. The Looijenga pair (Y, D) is studied in Example 1.9, Example 3.7 and Example 6.12 of [GHK15a]. Remark that the interior $U = Y - D$ has topological Euler characteristic $e(U) = 2$.

Let j be an index modulo 5. We denote D_j the components of D and ρ_j the corresponding one-dimensional cones in the tropicalization (B, Σ) of (Y, D). Let v_j be the primitive generator of ρ_j and E_j be the unique (-1)-curve in Y which is not contained in D and meets D_j transversally in one point.

The only curve classes contributing to the canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ are multiples of some $[E_j]$, and so $\hat{\mathfrak{D}}^{\mathrm{can}}$ consists of five rays $(\rho_j, \hat{H}_{\rho_j})$. By Lemma 2.20, we have

$$\hat{H}_{\rho_j} = i \sum_{\ell\geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin\left(\frac{\ell\hbar}{2}\right)} \hat{z}^{\ell\eta([E_j]) - \ell\varphi_{\rho_j}(v_j)} .$$

and so, by Lemma 3.13, the corresponding $\hat{f}_{\rho_j}$ are given by

$$\hat{f}_{\rho_j} = 1 + q^{-\frac{1}{2}} \hat{z}^{E_j - \varphi_{\rho_j}(v_j)} .$$

Proposition 3.44. *The $\Bbbk[NE(Y)]$-algebra defined by the product formula of Theorem 3.20 is generated by the quantum theta functions $\hat{\vartheta}_j$, satisfying the relations*

$$\hat{\vartheta}_{v_{j-1}} \hat{\vartheta}_{v_{j+1}} = \hat{z}^{[D_j]} (\hat{z}^{[E_j]} + q^{\frac{1}{2}} \hat{\vartheta}_{v_j}) ,$$

$$\hat{\vartheta}_{v_{j+1}} \hat{\vartheta}_{v_{j-1}} = \hat{z}^{[D_j]} (\hat{z}^{[E_j]} + q^{-\frac{1}{2}} \hat{\vartheta}_{v_j}) .$$

Proof. The description of quantum broken lines is identical to the description of classical broken lines given in Example 3.7 of [GHK15a].

The term $\hat{z}^{[D_j]} \hat{z}^{[E_j]}$ is the coefficient of $\hat{\vartheta}_0 = 1$. The final directions of the broken lines γ_1 and γ_2 satisfy $s(\gamma_1) + s(\gamma_2) = 0$, so $\langle s(\gamma_1), s(\gamma_2) \rangle = 0$ and the quantum result is identical to the classical one.

The term $\hat{z}^{[D_j]} \hat{\vartheta}_{v_j}$ corresponds to two straight broken lines for v_{j-1} and v_{j+1}, with endpoint the point v_j of ρ_j. The corresponding extra power of q in Theorem 3.20 is $q^{\pm\frac{1}{2}\langle v_{j-1}, v_{j+1}\rangle} = q^{\pm\frac{1}{2}}$. □

Remark: Setting $[E_j] = [D_j] = 0$, we recover some well-known description of the A_2 quantum $\mathcal{X}$-cluster algebra, see formula (60) in Section 3.3 of [FG09a].

3.6 Higher genus mirror symmetry and string theory

3.6.1 From higher genus to quantization via Chern-Simons

In Section 2.9, we compared our enumerative interpretation of the q-refined 2-dimensional Kontsevich-Soibelman scattering diagrams in terms of higher genus log Gromov-Witten invariants of log Calabi-Yau surfaces with the physical derivation of the refined wall-crossing formula from topological string given by Cecotti-Vafa [CV09].

A parallel discussion shows that the main result of the present Chapter, the connection between higher genus log Gromov-Witten invariants of log Calabi-Yau surfaces and quantization of the mirror geometry, also fits naturally into this story.

Let (Y, D) be a Looijenga pair. The complement $U \coloneqq Y - D$ is a non-compact holomorphic symplectic surface admitting a real Lagrangian torus fibration. According to the SYZ picture of mirror symmetry, the mirror of U should be obtained by taking the dual Lagrangian torus fibration, corrected by counts of holomorphic discs in U with boundary on the torus fibers.

As in Section 2.9, we assume that U admits a hyperkähler metric, such that the original complex structure of U is the compatible complex structure J, and such that the torus fibration is I-holomorphic Lagrangian. Let X be the non-compact Calabi-Yau 3-fold, of underlying real manifold $U \times \mathbb{C}^*$ and equipped with a complex structure twisted in a twistorial way, i.e. such that the fiber over $\zeta \in \mathbb{C}^*$ is the complex variety (U, J_ζ). Consider $S^1 \subset \mathbb{C}^*$ and $L \coloneqq \Sigma \times S^1 \subset X$. Again following Section 2.9, the log Gromov-Witten invariants with insertion of a top lambda class $N_{g,\beta}$, introduced in Section 3.3, should be viewed as a rigorous definition of the open Gromov-Witten invariants in the twistorial geometry X, with boundary on a torus fiber L "near infinity".

Always following Section 2.9, according to Witten [Wit95], in absence of non-constant worldsheet instantons, the effective spacetime theory of the A-model on the A-brane L is Chern-Simons theory of gauge group $U(1)$. The non-constant worldsheet instantons deform this result. The effective spacetime theory on the A-brane L is still a $U(1)$-gauge theory but the Chern-Simons action is deformed by additional terms involving the worldsheet instantons. The genus zero worldsheet instantons correct the classical action whereas higher genus worldsheet instantons give higher quantum corrections.

We now arrive at the key point, i.e. the relation between the SYZ mirror construction in terms of dual tori and the Chern-Simons story, whose quantization is supposed to be naturally related to higher genus curves. As $L = \Sigma \times S^1$, we can adopt a Hamiltonian description where S^1 plays the role of the time direction. The key point is that the classical phase space of $U(1)$ Chern-Simons theory on $L = \Sigma \times S^1$ is the space of $U(1)$ flat connections on Σ, i.e. it is exactly the dual torus of Σ used in the construction of the SYZ mirror. The genus zero worldsheet instantons corrections to $U(1)$ Chern-Simons theory then translate into the genus zero worldsheet instantons corrections in the SYZ construction of the mirror.

The Poisson structure on the mirror comes from the natural Poisson structure on the classical phase spaces of Chern-Simons theory. It is then natural to think that a quantization of the mirror should be obtained from quantization of Chern-Simons theory. Quantization of the torus of flat connections gives a quantum torus and higher genus worldsheet instantons corrections to quantum Chern-Simons theory imply that these quantum tori should be glued together in a non-trivial way. We recover the main construction of the present Chapter: gluing quantum tori together using higher genus curve counts in the gluing functions. The fact that we have been able to give a rigorous version of this construction should be viewed as a highly non-trivial mathematical check of the above string-theoretic expectations.

3.6.2 Quantization and higher genus mirror symmetry

In the previous Section, we explained how to understand the connection between higher genus log Gromov-Witten invariants and deformation quantization using Chern-Simons theory as an intermediate step. In this explanation, a key role is played by the non-compact Calabi-Yau 3-fold X, partial twistor family of U.

In the present Section, we adopt a slightly different point of view, and we also consider a similar non-compact Calabi-Yau 3-fold on the mirror side: $Y = V \times \mathbb{C}^*$. It is natural to expect that the mirror symmetry relation between U and V lifts to a mirror symmetry relation between the Calabi-Yau 3-folds X and Y.

As explained in the previous Section, the higher genus log Gromov-Witten invariants that we are considering should be viewed as part of an algebraic version of the open higher genus A-model on X. Open higher genus A-model should be mirror to open higher genus B-model on Y. We briefly explain below why the open higher genus B-model on $Y = V \times \mathbb{C}^*$ has something to do with quantization of the holomorphic symplectic variety V.

String field theory of open higher genus B-model for a single B-brane wrapping Y is holomorphic Chern-Simons theory, of field a $(0,1)$-connection A and action

$$S(A) = \int_Y \Omega_Y \wedge A \wedge \bar{\partial} A \,,$$

where Ω_Y is the holomorphic volume form of V. We will be rather interested in a single B-brane wrapping a curve $\mathbb{C}^*_v \coloneqq \{v\} \times \mathbb{C}^* \subset Y$, where v is a point in V. The study of the dimensional reduction of holomorphic Chern-Simons to describe a B-brane wrapping a curve was first done by Aganagic and Vafa [AV00] (Section 4). Writing locally

$$\Omega_Y = dx \wedge dp \wedge \frac{dz}{z} \,,$$

where (x,p) are local holomorphic Darboux coordinates on V near v and z a linear coordinate along $\mathbb{C}^*$, the fields of the reduced theory on $\mathbb{C}^*_v$ are functions $(x(z,\bar{z}), p(z,\bar{z}))$ and the action is

$$S(x,p) = \int_{\mathbb{C}_v^*} \frac{dz}{z} \wedge p \wedge \bar{\partial} x \,.$$

A further dimensional reduction from the cylinder $\mathbb{C}_v^*$ to a real line $\mathbb{R}_t$ leads to a theory of a particle moving on V, of position $(x(t), z(t))$, of action

$$S(x,p) = \int_{\mathbb{R}_t} p(t) dx(t) \,.$$

In particular, $p(t)$ and $x(t)$ are canonically conjugate variables and in the corresponding quantum theory, obtained as dimensional reduction of the higher genus B-model, they should become operator satisfying the canonical commutation relations $[x,p] = \hbar$. We conclude that the higher genus B-model of the B-branes $\mathbb{C}_v^*$ should lead to a quantization of the holomorphic symplectic surface V. The same relation between higher genus B-model and quantization appears in [ADK⁺06] and follow-ups.

We conclude that Theorem 5 should be viewed as an example of higher genus mirror symmetry relation.

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