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Collisional Penrose process with spinning particles

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ABSTRACT

In this article, we have investigated collisional Penrose process (CPP) using spinning particles in a Kerr spacetime. Recent studies have shown that the collision between two spinning particles can produce a significantly high energy in the center of mass frame. Here, we explicitly compute the energy extraction and efficiency as measured by an observer at infinity. We consider the colliding particles as well as the escaping particles may contain spins. It has been shown that the energy extraction is larger than the non-spinning case and also their possibility to escape to infinity is wider than the geodesics.

1. Introduction

In the late sixties, Penrose and Floyd demonstrated a way to extract the rotational energy from a black hole [1]. This is related to the existence of negative energy in the ergoregion of a Kerr black hole [2]. Recent studies have shown that Penrose process worked out in a collisional framework may result in sufficiently large energy extraction and produce energized particles [3]. But these high energy particles seldom can escape to infinity [4]. This is because, most of them either get absorbed by the black hole or trapped inside the ergoregion [5]. In recent years, several works have been carried out to model an extraction process from a black hole which gives rise to observable ejecta of high energy particles [6–8]. Schnittman has shown that the extraction efficiency may increase with a slight change in collisional mechanism [9]. This idea has been extended by Berti et al. considering the colliding particles are confined in the vicinity of the black hole rather than arriving from infinity [10]. But it has been argued that even if these type of collisions may produce energetic particles, their origination has to be scrutinized carefully [11–13]. Patil and Joshi have shown the problem can be readdressed with Kerr superspinors, where the escaping ratio of the produced particles is larger than the extremal Kerr black hole [14,15]. In the present article, we consider the collision between two spinning particles and compute the energy release in this process. Unlike non-spinning particles, they deviate from the geodesic motion due to the spin-curvature coupling and described by the Mathisson–Papapetrou equations. Obviously this will lead us towards a more involved and practical situation. In reality, the astrophysical bodies are extended objects and likely to have internal moments. So, working with spinning particles rather than geodesics sounds more relevant from astrophysical point of view.

One can expect to have striking departure from the geodesic motion whenever the spin magnitude is close to unity, $S/M \approx \mathcal{O}(1)$ [16], here M is the mass of the black hole. Recently Armaza et al. has proposed that the collision between two spinning particles in the Schwarzschild black hole can produce large energy in the center of mass frame [17]. This idea has been used exclusively in refs. [18,19] and suggested similar situation may arise in case of a rotating black hole. These studies have shown the existence of a divergence radius at, $r_c = M(S/M)^{2/3}$ where the energy in the center of mass frame diverges and the theory reaches a breakdown point. It is, in fact, related to the violation of timelike characteristic of the particle while a similar work to avoid *superluminal* motion can be found in [20]. In this article we carefully extend the general mechanism used to formulate the energy extraction with the geodesics and study the effect of spin. We chiefly concentrate on two particular cases: firstly, the collision of two spinning particles produce two massless photons which follow the geodesic motion and secondly, the produced particles in the collision are massive and contain spins. The later is the most general case. Finally we use this mechanism for trapped spinning particles in the ergoregion and compute the energy extraction in the process. Similar to the geodesics, these particles can produce higher energy ejecta, observable at infinity. In each case, we assume all the particles are confined on the equatorial plane.

The rest of the paper is organized as follows. In section 2 we introduce the evolution equations for a spinning particle and concentrate on some important features of its trajectory. Section 3 is devoted to set up the basic equations for a collisional theory and compute the energy extraction in the process. We discuss two specific scenarios, firstly when the emerging particles are massless

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geodesics and secondly, when they are massive spinning particles. A detail discussion on the efficiency of each process is carried out for different spin parameters. In section 4, we allow bound spinning particles to collide and compute the energy extraction in these processes. Finally we conclude the article with a brief remark.

2. Orbits of a spinning particle

In general relativity, very often one neglect the complexity of a test body by approximating it as a single pole particle. But to address a more realistic astrophysical event, we may need to consider the structure of the test particle as well. This may include a complicated internal structure with multiple moments. Mathisson first described this problem and computed equation of motion of a spinning pole-dipole test particle in a linearized gravitational field [21]. These equations have been extensively used as well as modified by several authors. Full general relativistic equation of motion of a pole-dipole particle was carried out first by Papapetrou [22]. Later Dixon modified these equations for extended bodies in a covariant formalism [23].

The Mathisson–Papapetrou equations for a spinning particle is given by:

$$\dot{P^a} = -\frac{1}{2}R^a{}_{bcd}\mathcal{U}^b S^{cd},$$
$$\dot{S^{ab}} = P^a\mathcal{U}^b - P^b\mathcal{U}^a, \tag{1}$$

where P^a defines the four momentum, S^{ab} is the spin tensor and $\mathcal{U}^a$ is the four velocity. It should be noted that the above equations are not sufficient to determine the complete trajectory of a spinning particle and additional supplementary conditions are essential. For a vanishing mass dipole moment ($S^{i0}=0$) in the rest frame of the object [24], we may employ Tulczyjew–Dixon 'Spin Supplementary Condition (S.S.C)' [23,25],

$$S^{ab}P_b = 0. \tag{2}$$

It is easy to see that physically, the S.S.C conserves the dynamical mass of the spinning body (μ) [26], and we may now define a normalized momentum $\mathcal{V}^a$ as,

$$\mathcal{V}^a = P^a/\mu; \quad \mathcal{V}^a\mathcal{V}_a = -1. \tag{3}$$

This is important to note that under the supplementary condition given by Eq. (2), the four velocity, $\mathcal{U}^a$, in general, is not normalized. But to remain a timelike particle, it has to follow,

$$\mathcal{U}^a\mathcal{U}_a < 0. \tag{4}$$

For a simplicity, we normalize the four velocity with the condition [27],

$$\mathcal{U}^a\mathcal{V}_a = -1. \tag{5}$$

From Eqs. (1)–(5), one can establish a relation between four-velocity and the four-momentum,

$$\mathcal{V}^b - \mathcal{U}^b = \frac{1}{2\mu^2}\mathcal{R}_{aefg}\mathcal{U}^e S^{fg} S^{ab}. \tag{6}$$

For the single pole particle, the underlying symmetries of the Kerr spacetime leads to conserved quantities such as energy and angular momentum. Similarly, for spinning particles, with a given killing vector field $\mathcal{K}^a$, the conserved quantity is given as [28,29]

$$\mathcal{C} = \mathcal{K}^a P_a - \frac{1}{2}S^{ab}\mathcal{K}_{a;b}. \tag{7}$$

It should be reminded carefully that for a spinning particle, neither energy ($-\mathcal{K}^t P_t$) nor angular momentum ($\mathcal{K}^\phi P_\phi$) is conserved. Instead, the conserved quantities are merely a spin dependent deviation from them. So throughout the text, the conserved quantities are defined as, $E=-\mathcal{C}_t$ and $J_z=\mathcal{C}_\phi$, while the energy is denoted as E^∞.

Saijo et al. has successfully derived the equation of motion of a spinning particle on the equatorial plane of a Kerr black hole [27]. These are given by,

$$(\Sigma_s\Lambda_s\mathcal{U}^r)^2 = P_s^2 - \Delta\left(\frac{\Sigma_s^2}{r^2} + \{J_z-(a+S)E\}^2\right),$$
$$(\Sigma_s\Lambda_s\mathcal{U}^t) = a\left(1+\frac{3S^2}{r\Sigma_s}\right)\{J_z-(a+S)E\} + \frac{r^2+a^2}{\Delta}P_s,$$
$$(\Sigma_s\Lambda_s\mathcal{U}^\phi) = \left(1+\frac{3S^2}{r\Sigma_s}\right)\{J_z-(a+S)E\} + \frac{a}{\Delta}, \tag{8}$$

where 'a' and 'M' are angular momentum and mass parameter of the black hole respectively, and P_s, Λ_s and Σ_s are given as,

$$P_s = E\left(r^2+a^2+aS+\frac{aSM}{r}\right) - \left(a+\frac{MS}{r}\right)J_z,$$
$$\Sigma_s = r^2\left(1-MS^2/r^3\right); \quad \Delta = r^2+a^2-2Mr,$$
$$\Lambda_s = 1-\frac{3MS^2r}{\Sigma_s^3}\{J_z-(a+S)E\}^2 \tag{9}$$

The quantity 'S' is defined as the z-component of the spin. We are considering a simplest case when the spin is perpendicular to the orbital plane. When $S>0$, the spin is parallel to the black hole spin, and antiparallel for $S<0$. Using Eqs. (8), (9), one can rewrite the timelike condition in Eq. (4) as,

$$W(S,E,J_z) = r^5\left(1-\frac{MS^2}{r^3}\right)^4 - 3MS^2\left(2+\frac{MS^2}{r^3}\right)\{J_z-(a+S)E\}^2 > 0. \tag{10}$$

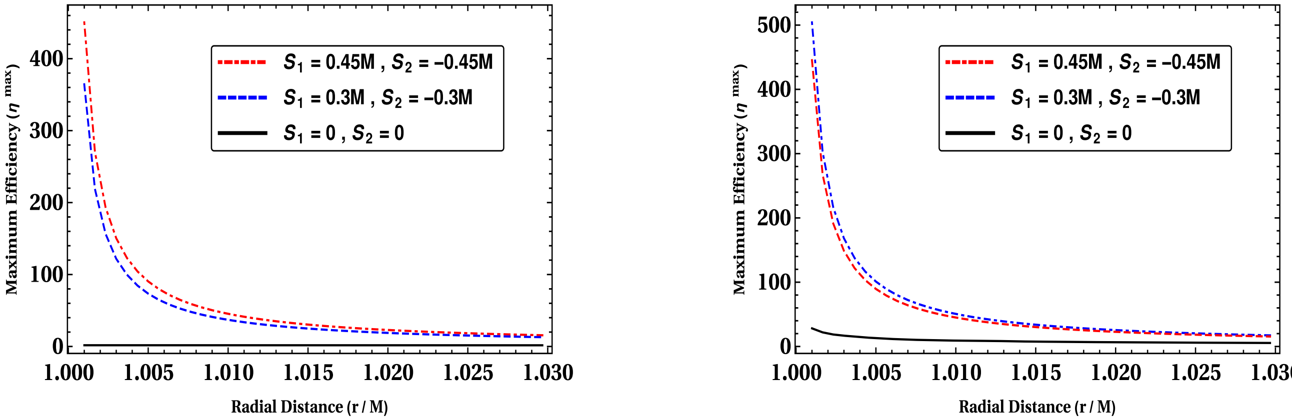
The above equation is important to determine the closest interaction of the spinning particle with the black hole and valid for every discussion throughout the text. We can express $W(r,S,E,J_z)$ in terms of energy, $E^\infty = -P_t$ of the particle:

$$W(S,E^\infty,J_z) = r^5\left(1-\frac{MS^2}{r^3}\right)^4 - \left(\frac{3MS^2}{(r^3+aMS^2)^2}\right)\times \cdot\left(2+\frac{MS^2}{r^3}\right)\{J_z-(a+S)E^\infty\}^2\left(r^3-MS^2\right)^2. \tag{11}$$

It can be seen that for $J_z=(a+S)E$, $W(S,E,J_z)$ is always positive and the additional constraint is trivially satisfied. Furthermore, for this condition E behaves as the energy of the particle and the orbital angular momentum (P_ϕ) become aE. In this case the spinning particle behaves as a geodesic and the acceleration become zero.

3. Collisional Penrose process

Let us consider two spinning particles with four momentum P_1^μ and P_2^μ respectively, collide in the ergoregion of a maximally rotating Kerr black hole and produce two more particles with momentum P_3^μ and P_4^μ respectively. To extract energy, it is necessary



(a) Figure shows energy extraction when the emerging photons are initially outgoing and can be observed at infinity. There is a considerable difference of efficiency for a spinning and a non-spinning particle.

(b) Here the produced massless partilces are intially ingoing and they either bouce back from a turning point or trapped inside the ergoregion. The efficiency is larger than the initially outgoing particles.

Fig. 1. Collisional Penrose process when the ejected particles are null geodesics.

that one of these ejected particles carry negative energy while its counterpart can have more energy than the total initial energies. Considering the collisional process is solely confined within the equatorial plane, we may assume the sum of initial spins is remain unaltered after collision. That is,

$$S_1 + S_2 = S_3 + S_4 \,. \tag{12}$$

The above equation can be regarded as a supplementary condition which is useful to solve the equations consistently. Though it is a simple and unbiased relation, it is not unique and one of many possibilities. One may substitute it with more complicated and physically appealing conditions. According to Eq. (12) the final particles may or may not contain spin, it depends on the sum of initial spins. For simplicity, we further consider $S_3 = S_4$. Such choice is helpful to express our equations analytically, for example if we choose $S_3 = -S_4$ the equations get more complicated. Nevertheless, in principle one can assume such choice and numerically compute the energy efficiency. We plan to come up with a better approach to scan all such possibilities of energy extraction in a future work. In the case of collision between two spinning particles, neither axial angular momentum nor energy is conserved in general. But we still can consider the sum of each component of four momentum remain conserved throughout the process.

$$\begin{aligned} P_1^t + P_2^t &= P_3^t + P_4^t, \\ \epsilon_1^r P_1^r + \epsilon_2^r P_2^r &= \epsilon_3^r P_3^r + \epsilon_4^r P_4^r, \\ P_1^\phi + P_2^\phi &= P_3^\phi + P_4^\phi, \end{aligned} \tag{13}$$

where $\epsilon_i^r = \pm 1$, and i runs from 1 to 4. For a radially ingoing particle $\epsilon_i^r = -1$ and $\epsilon_i^r = 1$ for outgoing trajectories. Expression for momentum can be derived from Eqs. (7)–(8),

$$\begin{aligned} P_r &= \Lambda_s \mathcal{U}_r, \\ P_t &= -\frac{1}{\Sigma_s}\left(E\left\{r^2 + \frac{aMS}{r}\right\} - \frac{J_z MS}{r}\right), \\ P_\phi &= \frac{1}{\Sigma_s}\left(J_z\left\{r^2 - \frac{aMS}{r}\right\} - ES\left\{r^2 - \frac{a^2 M}{r}\right\}\right). \end{aligned} \tag{14}$$

It is easy to find that in the $S \to 0$ limit, the P_t and P_ϕ has usual meaning of energy and angular momentum respectively.

3.1. Production of light-like particles

In this section, we shall consider two equal and oppositely aligned massive spinning particles, P_1 and P_2, collide in the ergoregion and produce two spinless photons P_3 and P_4. One of these photons say P_4, carries negative energy and crosses the horizon, while the other photon P_3 contains more energy than the sum of initial energies. Let us assume P_1 is turned back at the event horizon and collide with the in-falling particle P_2. It should be noted that both P_1 and P_2 has to follow Eq. (4). Now we may employ Eqs. (8)–(13) and easily compute energy, E_3 for P_3 [30],

$$E_3 = \frac{E_T^2 \Sigma_1 - J_T^2 \Sigma_2 - 4arE_T J_T - \Sigma^2 (P_T^r)^2}{4r\alpha(b3 - a) - 2b_3 J_T r^2 + 2E_T(r^4 + a^2 r^2) + C_1 P_T^r} \tag{15}$$

With P_T^r is the total radial momentum before the collision and b_3 is the impact parameter of the third particle, $b_3 = J_3/E_3$. Note J_3 is the conserved momentum of the third particle P_3 and simply written as, $J_3 = (C_\phi)_{P_3}$. It can be seen that E_3 depends on initial parameters such as, $E_T = -(P_1)_t - (P_2)_t$ and $J_T = (P_1)_\phi + (P_2)_\phi$, and also b_3. We define Σ's, C_1 and α as,

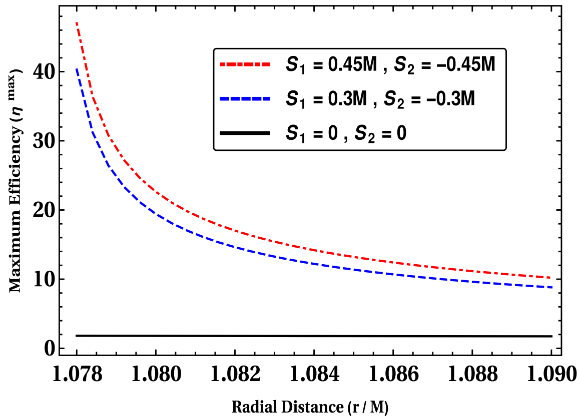
$$\begin{aligned} \Sigma_1 &= r^4 + r^2 a^2 + 2a^2 r, \\ \Sigma_2 &= r^2 - 2r, \\ \alpha &= (J_T - aE_T), \\ C_1 &= 2r^2\sqrt{2(b_3 - a)^2 r + (a^2 - b_3^2) r^2 + r^4}. \end{aligned} \tag{16}$$

Here we calculate the energy as measured by an observer rest at infinity, that is $E_3^\infty = E_3$, as we assume the photon follows a geodesic trajectory. We may choose our initial parameter such as, $(E_1^\infty, E_2^\infty, J_1, J_2, m_1, m_2) = (1, 1, 2M, 0.2M, 1, 1)$ and the black hole is consider to be maximally Kerr ($a = M$).

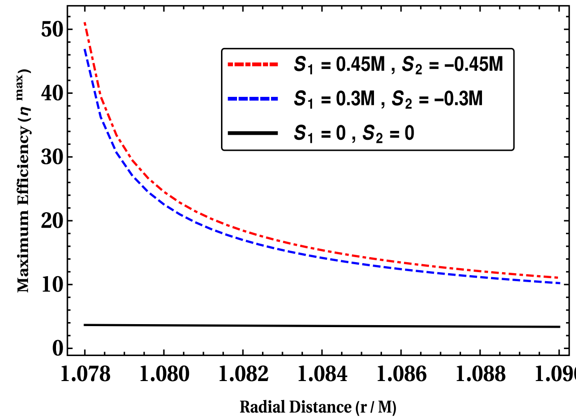
In Fig. 1(a), we have plotted maximum energy efficiency (η_{max}), when the null geodesics are initially outgoing ($\epsilon_i^r = 1$). Maximum efficiency is defined as,

$$\eta^{max} = E_3^{\infty(max)}/(E_1^\infty + E_2^\infty)\,,$$

where E_3 has been maximized w.r.t. the parameter b_3. They could escape to infinity with large energies. The nature of the graph remain same for initially ingoing particles ($\epsilon_i^r = -1$), see Fig. 1(b).

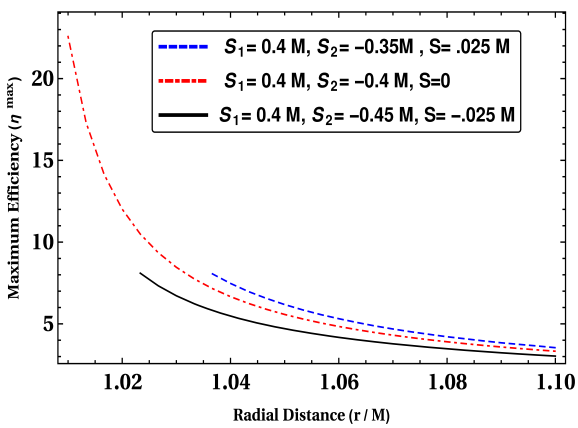


(a) Figure shows energy extraction when the emerging photons are initially outgoing and can be observed at infinity.

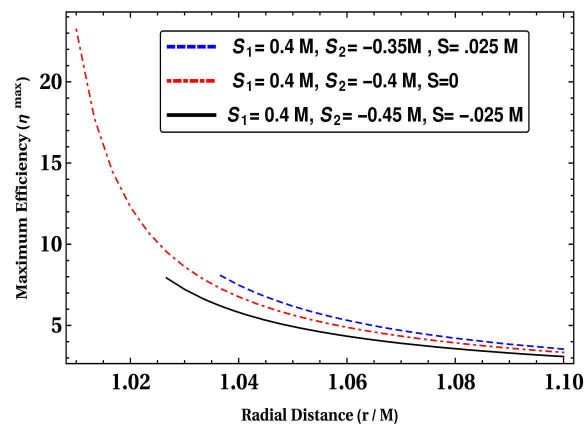


(b) Here the produced massless partilces are intially ingoing.

Fig. 2. The energy extraction with spinning particles for a black hole with, $a = 0.997M$.



(a) The above figure shows the energy efficiency when the emerging spinning particles are initially outgoing. It should be noted that for $S = -0.025M$, the collision has to take place at r $\gtrsim$ 1.023 M, otherwise Eq. (11) will be violated and the process become unphysical. A similar situation arises for $S = 0.025M$.



(b) Here the produced particles in the collision are initially ingoing. The plot is almost inseperable from the initially outgoing case.

Fig. 3. Collisional Penrose process when ejected particles are massive spinning particles. The initial parameters are chosen as, $(E_1^\infty, E_2^\infty, J_1, J_2, m_1, m_2) = (1, 1, 2M, 1M, 1, 1)$ and also we assume, $S_3 = S_4 = S$.

Even if the energy extraction is slightly larger in this case, most of photons are absorbed by the black hole. It can be seen from both the figures, that the energy extraction increases with the increase of S_1 and S_2. As the initial spin parameters for the colliding particles increases, the radial momentum in the center of mass frame become less negative and ejected particle are boosted up with more energy. Interestingly, none of the spin parameters reaches a divergence radius at $r = r_c = M(S/M)^{2/3}$ and the theory is well consistent throughout the considered region.

A similar approach can be done for black holes in the near extremal regime but well inside the *Throne limit* $a \lesssim 0.998M$ [31], see Fig. 2. From an astrophysical standpoint, this is relevant as black holes are unlikely to be extremal or maximally rotating and consistently obey $a < M$. In this case, the maximum efficiency decreases with an approximate factor of 10 while the nature remain same as the previous cases.

3.2. Production of non geodesic massive particles

In this section we are considering the ejected particles are massive and contain spins. From our previous discussion we conclude that the four velocity has to follow Eq. (4) to retain its timelike character. It can be seen from Fig. 3(b) that, no energy extraction occurs when emerging particle has a nonzero spin and colliding point is close to the event horizon. The maximized energy of the escaping particle would depend on initial spins as well as the final spin. Similar to ingoing null geodesics, most of the ingoing massive non geodesics will eventually cross the horizon and absorbed by the black hole. The particles which can have some astrophysical interest and be observed at infinity, are the initially outgoing particles, see Fig. 3(a). Unlike the geodesic case, the difference of energy extraction for initially ingoing and initially outgoing trajectories is very small. The outgoing particles are very useful to have an observable ejecta in a collisional Penrose process. In addition,

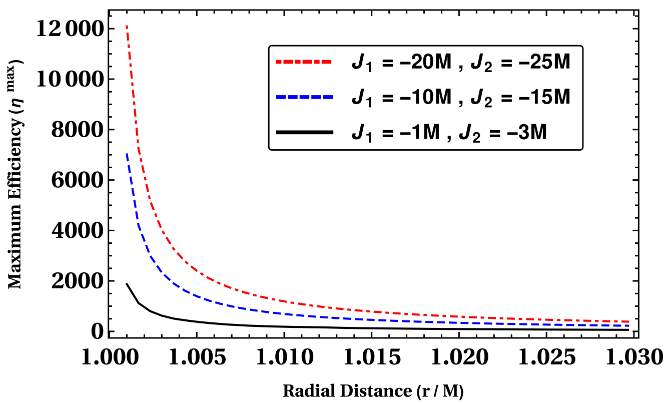


Fig. 4. Collision with the trapped Particles, with $S_1 = 0.01M$ and $S_2 = -0.01M$. All the particles are initially outgoing and can be observed at infinity.

there will be a spin-spin interaction between the ejected particle and the black hole. For parallel spins, the force is repulsive and it is attractive for antiparallel arrangement. So as the emerging particle has a spin parallel to the black hole, it gets repel back and boosted with energy. At the same time, the energy of the escaping particle would decrease if $S_3 < 0$.

4. Collision with trapped particles

In the case of a collision between two trapped particles, the energy extraction can be arbitrarily large, see Fig. 4. This idea was first coined in [9] and explicitly used in [10]. The trapped particles required in the process, can be produced in the ergoregion as a consequence of previous scattering events [32]. The basic structure of this process remain same, as one of the outgoing particle with positive radial momentum collide with an ingoing particle and result in two other particles. We consider the emerging particles after the collision are photons and essentially they follow null geodesics. These photons are initially outgoing and hence, can escape to infinity. This could be astrophysically relevant as we can observe an ejecta of high energy light-like particles.

5. Conclusion

In this article we carried out an investigation of collisional Penrose process in a maximally rotating Kerr black hole, while colliding particles as well as the produced particles may contain spins. We closely follow ref. [27] for the equations of motion of a spinning particle on the equatorial plane of a Kerr black hole. The study of the Penrose process with spinning particles in the inclined orbits would have been a intriguing problem in a theoretical point of view. Though from our previous knowledge, it is expected that energy extraction is maximum when the collision takes place on the equatorial plane, this is due to the symmetry of the spacetime [33]. Here we mainly discuss three events with an assumption on the spin of these particles. Firstly, we consider two equal and oppositely aligned spinning particles collide and produce two non-spinning photons. The energy release is extremely high as compared to the geodesics. In the second case, we consider the produced particles are massive and endowed with spins. Depending on their spin, they may or may not reach close to the black hole and energy release is minimal. In the final case, we let trapped spinning particle to collide and produce massless photons. This produce enormous energy release in the process.

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