MONASH University

Enumerative geometry, topological recursion, and the semi-infinite wedge

Ellena Moskovsky
BSc (Hons)

A thesis submitted for the degree of Doctor of Philosophy at Monash University in 2022.
School of Mathematics

Copyright notice

Abstract

The field of enumerative geometry has seen an explosion of activity over recent decades, primarily due to emerging connections with other areas, such as representation theory, integrability, and mathematical physics. This thesis showcases new results and conjectures for four different enumerative problems motivated by the aforementioned connections.

First, I prove that a certain enumeration of lattice points in the Deligne–Mumford compactification of the moduli space of curves is governed by local topological recursion. Local topological recursion is a relatively recent generalisation of the topological recursion of Chekhov, Eynard and Orantin, however its benefits remain unclear. This result resolves a question of Do and Norbury, and serves as one of the first demonstrations of local topological recursion governing a natural combinatorial problem.

Second, this thesis contains proofs of three key results for double Hurwitz numbers: I prove that double Hurwitz numbers satisfy a polynomiality structure, that they are governed by topological recursion, and finally, that they can be expressed in terms of intersection numbers on moduli spaces of curves. These results subsume analogous results that have previously appeared in the literature for the simpler settings of single and orbifold Hurwitz numbers. Further, the techniques used to prove these results can be adapted for a variety of other enumerative problems, with the relative Gromov–Witten invariants of the sphere providing a likely fruitful example.

Third, I derive a sequence of Virasoro operators that annihilate the partition function for fully simple maps. This is achieved by combining a known relation between the enumerations of ordinary and fully simple maps with techniques from the semi-infinite wedge formalism. The thesis describes preliminary findings stemming from this result, including work towards a Tutte-like recursion for fully simple maps, and a direct relation between the enumerations of ordinary and fully simple disks.

Finally, I use topological recursion to motivate the definition and study of a new generalisation of the Narayana polynomials that can be considered a deformation of monotone Hurwitz numbers. This family of so-called topological Narayana polynomials continues to satisfy certain recursive and symmetry properties possessed by their original counterparts. I prove these and posit explicit conjectures pertaining to the real-rootedness and interlacing of this new family of polynomials. Thus it appears that one can "topologise" sequences of polynomials via topological recursion while preserving key properties, a new phenomenon that prompts further study.

The novel results for these four enumerative problems have standalone merit, yet they share underlying, unifying themes. The work on the lattice point enumeration, double Hurwitz numbers, and topological Narayana polynomials all prove some form of topological recursion. For double Hurwitz numbers and the enumeration of fully simple maps, the semi-infinite wedge formalism is a vital tool. And, as is typical for such problems in enumerative geometry, these results form part of a rich tapestry of ongoing work that connects various areas of mathematics.

Declaration

This thesis is an original work of my research and contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Ellena Moskovsky
25 August 2022

Acknowledgements

First, this research was supported by an Australian Government Research Training Program (RTP) Scholarship.

I thank my supervisor, Norman Do. Norm has been extremely generous with his time and his ideas, and through his guidance I have learnt much about mathematics (but also about many other things!). His patience appeared to be infinite in size, and his passion for mathematics is infectious. Mostly, I'd like to thank him for being my friend, and for being there for me, every time.

I would certainly not be where I am today without the unwavering support of my parents and their honest belief that I can achieve anything to which I aspire. I am extremely lucky; their confidence in me has propelled me through life and I can't thank them enough.

And finally, I especially thank the two people who mean most to me: my sister and best friend, Neda; and my life-partner, James. I wouldn't have made it without them.

Preface

This thesis is split into two distinct halves: Part I describes background material while Part II presents novel research.

In Part I, Chapter 1 introduces the semi-infinite wedge space, providing foundational knowledge on aspects of the semi-infinite wedge formalism that are required for some results in Part II. Chapter 2 defines the topological recursion of Chekhov, Eynard and Orantin, fixing key notations for the remainder of the thesis. Chapters 3 and 4 provide gentle introductions to maps and Hurwitz numbers respectively. Chapter 3 also defines the combinatorial object that is associated to the enumeration discussed in Chapter 5 and describes the sense in which this combinatorial object enumerates lattice points in the moduli space of curves. Chapter 4 also presents a proof of the previously known polynomiality of single Hurwitz numbers via the semi-infinite wedge.

Chapter 5 signifies the start of Part II and describes joint work with Anupam Chaudhuri and Norman Do proving that the enumeration of lattice points in the Deligne–Mumford compactification of the moduli space of curves is governed by local topological recursion. The results described in this chapter can also be found in the arxiv preprint [24] and answers a long-standing question asked by Do and Norbury in their original work on this lattice point enumeration.

Chapter 6 contains joint work with Gaëtan Borot, Norman Do, Maksim Karev and Danilo Lewański which appears in [8]. This chapter proves that double Hurwitz numbers satisfy a polynomiality structure, are governed by topological recursion, and can be expressed via intersection theory on the moduli spaces of curves. These results resolve a conjecture of Do and Karev.

Chapter 7 deduces a sequence of Virasoro operators that annihilate the partition function for fully simple maps. This chapter, motivated originally by the now-proven conjecture of Borot and Garcia-Failde that fully simple maps are governed by topological recursion, also presents work towards deriving a Tutte-like recursion for fully simple maps, and deducing an explicit relation between ordinary and fully simple maps.

Finally, Chapter 8 uses topological recursion to motivate the definition of a new enumeration called topological Narayana polynomials. This chapter contains proofs that the enumeration is governed by topological recursion, along with two other recursions, and that it satisfies a particular symmetry property. Further, it states two conjectures that topological Narayana polynomials are real-rooted and that they interlace.

Contents

Part I

Background

Chapter 1

Semi-infinite wedge

1.1 Introduction

The semi-infinite wedge, sometimes referred to as the infinite wedge space or the fermionic Fock space, originally emerged from the study of infinite-dimensional Lie algebras [73] and has applications to enumerative geometry [40, 92], integrable systems [29, 83, 100], random partitions [90], and modular forms [5].

Use of the semi-infinite wedge is instrumental for deducing results for double Hurwitz numbers and fully simple maps. In Chapter 6 on double Hurwitz numbers, the semi-infinite wedge formalism was used to prove a polynomiality structure theorem. This result acted as a catalyst for proving topological recursion for double Hurwitz numbers, from which an ELSV-like formula was obtained. In other words, all substantial results for double Hurwitz numbers in this thesis were proved as a consequence of the semi-infinite wedge analysis. In Chapter 7 on fully simple maps, the semi-infinite wedge provides a particularly nice setting for deriving a set of Virasoro operators that annihilate the partition function for the enumeration.

The purpose of this chapter is to provide foundational knowledge on aspects of the semi-infinite wedge formalism that are required for Chapters 6 and 7. Previous work of Okounkov [90] is used as the main reference for definitions and notations, while the survey paper of Ríos-Zertuche [96] contains a number of useful results that are cited here.

The structure of this chapter is as follows. Section 1.2 provides the key definitions and notations (Section 1.2.1) as well as common diagrammatic representations for vectors in the semi-infinite wedge (Section 1.2.2). Section 1.3 defines the operators that will be used throughout this thesis, namely, fermionic and bosonic operators; the $\mathcal{E}$-operators appearing in the work of Okounkov and Pandharipande [92]; vertex operators; and $\mathcal{F}$-operators. Section 1.4 contains results required in later chapters: Section 1.4.1 describes the Murnaghan–Nakayama rule via operators acting on the semi-infinite wedge, and Section 1.4.2 details the so-called boson-fermion correspondence.

1.2 Preliminaries

1.2.1 Definitions and notations

Let V be a $\mathbb{C}$-vector space with basis $\{\underline{s} \mid s \in \mathbb{Z} + \frac{1}{2}\}$.

Definition 1.2.1. The *semi-infinite wedge space*, denoted $\Lambda^{\frac{\infty}{2}} V$, is a $\mathbb{C}$-vector space with a preferred basis containing vectors

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \cdots ,$$

where $S = \{s_1 > s_2 > \cdots\} \subset \mathbb{Z} + \frac{1}{2}$ is such that the sets

$$S_+ = S \setminus (\mathbb{Z}_{\leqslant 0} - \tfrac{1}{2}), \qquad S_- = (\mathbb{Z}_{\leqslant 0} - \tfrac{1}{2}) \setminus S \tag{1.1}$$

are finite.

We equip the semi-infinite wedge space with an inner product $\langle \cdot \, , \, \cdot \rangle$, for which the basis $\{v_S\}$ is orthonormal. That is, define

$$\langle v_S, v_T \rangle = \delta_{S,T}.$$

Example 1.2.2. An example of a basis element in the semi-infinite wedge is

$$\underline{\tfrac{7}{2}} \wedge \underline{\tfrac{3}{2}} \wedge \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{9}{2}} \wedge \underline{-\tfrac{11}{2}} \wedge \underline{-\tfrac{13}{2}} \wedge \underline{-\tfrac{15}{2}} \wedge \cdots .$$

The condition (1.1) that the sets S_+ and S_- are finite implies that there are only finitely many positive half-integers present and only finitely many negative half-integers missing. In this example, there are only three positive half-integers present, $\frac{7}{2}, \frac{3}{2}, \frac{1}{2}$, and three negative half-integers missing, $-\frac{1}{2}, -\frac{5}{2}, -\frac{7}{2}$.

A particular basis vector that arises naturally in many calculations is the ***vacuum vector***, denoted $v_\emptyset$ and given by

$$v_\emptyset := \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \underline{-\tfrac{9}{2}} \wedge \cdots . \tag{1.2}$$

Section 1.2.2 explains the way in which $v_\emptyset$ corresponds to the empty partition, and hence gives context for the notation.

Finally, define the ***vacuum expectation*** of an operator $\mathcal{O}$ on $\Lambda^{\frac{\infty}{2}} V$ to be

$$\langle \mathcal{O} \rangle := \langle \mathcal{O} v_\emptyset, v_\emptyset \rangle. \tag{1.3}$$

1.2.2 Diagrammatics

Basis vectors in the semi-infinite wedge are represented via two common diagram types; the first is a Young diagram and the second is a Maya diagram. Here both are considered, using the basis vector from Example 1.2.2 as an example. In the subsequent sections of this chapter, the Young diagram representation is used exclusively. Maya diagrams are defined here for completeness.

In the Young diagram presentation of the semi-infinite wedge the basis vectors are represented as continuous piecewise linear functions with gradient ± 1 defined up to an additive constant in the following way. For a given basis vector v_S, $s \in S$ corresponds to a gradient of -1 (a ***down-step***) in the interval $[s - \frac{1}{2}, s + \frac{1}{2}]$, whereas $s \notin S$ corresponds to a gradient of $+1$ (an ***up-step***) in the same interval. The fact that the sets S_+ and S_- (defined in equation (1.1) above) are finite dictates that this piecewise linear function has positive gradient for sufficiently positive x-values and negative gradient for sufficiently negative x-values. By drawing the "V-shape" created by these two lines of positive and negative gradients, one encloses a union of squares, each of area 2, creating a Young diagram. The Young diagram corresponding to Example 1.2.2 above can be seen in Figure 1.1.

$$v_S = \underline{\tfrac{7}{2}} \wedge \underline{\tfrac{3}{2}} \wedge \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{9}{2}} \wedge \underline{-\tfrac{11}{2}} \wedge \underline{-\tfrac{13}{2}} \wedge \underline{-\tfrac{15}{2}} \wedge \cdots .$$

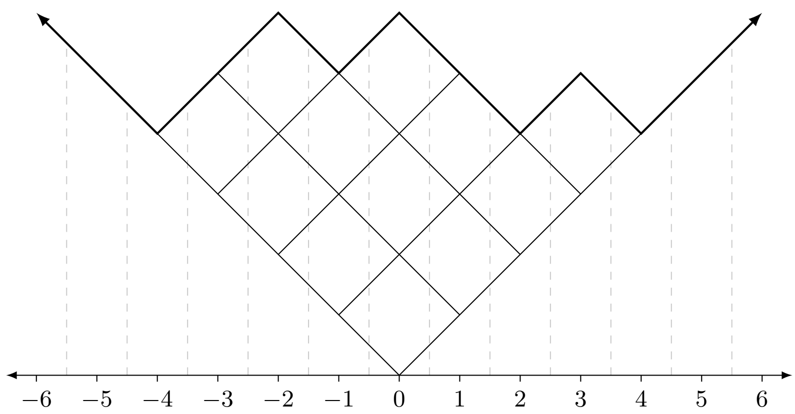

Figure 1.1: Young diagram of v_S in the semi-infinite wedge corresponding to the partition $\lambda = (4, 3, 3, 2)$ and charge $c = 0$.

From the Young diagram of a basis vector one can associate a pair (λ, c). The partition λ corresponds to the Young diagram, where the rows appear along lines of gradient $+1$, and c is an integer that corresponds to the x-value of the vertex of the V-shape and is called the *charge*. The Young diagram associated to Figure 1.1 produces the partition $\lambda = (4, 3, 3, 2)$ (rather than its transpose $(4, 4, 3, 1)$) and the charge $c = 0$.

Charge zero basis elements span a subspace of the semi-infinite wedge, referred to as the *charge zero subspace* and denoted $\Lambda_0^{\frac{\infty}{2}} V$. Further, it is often useful to describe a basis vector by the partition corresponding to its Young diagram, using the notation v_λ. This is done primarily when $v_\lambda \in \Lambda_0^{\frac{\infty}{2}} V$ or if the charge is clear from context. For example, v_S in Figure 1.1 can instead be written as $v_{(4,3,3,2)}$.

The vacuum vector defined in equation (1.2) is the element of the semi-infinite wedge corresponding to the empty partition in the charge zero subspace; see Figure 1.2.

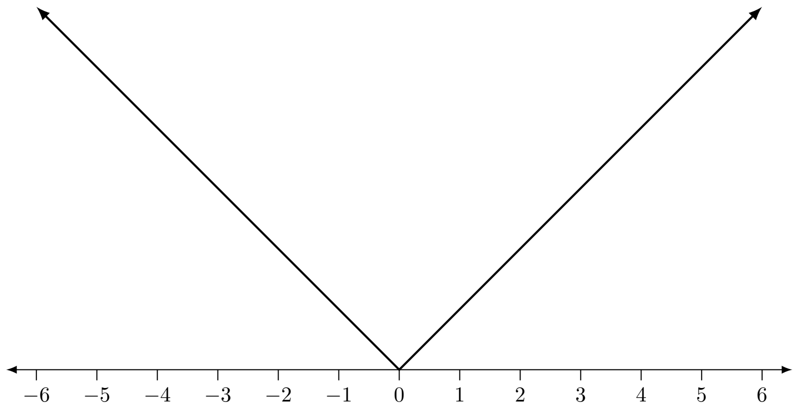

Figure 1.2: The vacuum vector, $v_\emptyset$.

Alternatively, Maya diagrams are given by the placements of black or white beads at each position in $\mathbb{Z} + \frac{1}{2}$; a black bead is used if $s \in S$ and a white bead is used if $s \notin S$. The Maya diagram corresponding to Example 1.2.2 is shown in Figure 1.3.

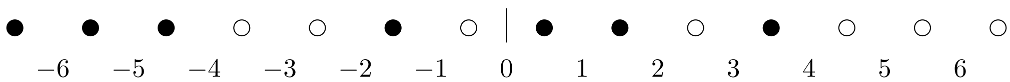

Figure 1.3: Maya diagram corresponding to the vector $v_{(4,3,3,2)}$.

1.3 Operators

1.3.1 Fermionic operators

Definition 1.3.1. For $k \in \mathbb{Z} + \frac{1}{2}$, define the *fermionic operator* $\psi_k : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ by

$$\psi_k v_S = \underline{k} \wedge v_S.$$

Define ψ_k^* to be the adjoint operator of ψ_k; that is, define ψ_k^* to be the operator that satisfies $\langle \psi_k u, v \rangle = \langle u, \psi_k^* v \rangle$ for all $u, v \in \Lambda^{\frac{\infty}{2}} V$.

Using the usual skew-symmetry of the wedge product, $a \wedge b = -b \wedge a$, which in turn implies $a \wedge a = 0$, the actions of ψ_k and ψ_k^* on basis vectors can be expressed as

$$\psi_k v_S = \begin{cases} \pm v_{S \cup \{k\}}, & \text{if } k \notin S, \\ 0, & \text{if } k \in S, \end{cases} \qquad \psi_k^* v_S = \begin{cases} \pm v_{S \setminus \{k\}}, & \text{if } k \in S, \\ 0, & \text{if } k \notin S. \end{cases}$$

In both cases, the sign is given by $(-1)^{|\{s\in S | s>k\}|}$.

For example, consider the element in the charge zero subspace given by the Young diagram $(3,2)$; that is, $v_{(3,2)} = \underline{\frac{5}{2}} \wedge \underline{\frac{1}{2}} \wedge \underline{-\frac{5}{2}} \wedge \underline{-\frac{7}{2}} \wedge \cdots$. Applying the fermionic operator ψ_k with $k = \frac{3}{2}, \frac{1}{2}$ respectively gives

$$\psi_{3/2} v_{(3,2)} = \underline{\tfrac{3}{2}} \wedge \underline{\tfrac{5}{2}} \wedge \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \cdots = -\left(\underline{\tfrac{5}{2}} \wedge \underline{\tfrac{3}{2}} \wedge \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \cdots\right)$$
$$\psi_{1/2} v_{(3,2)} = \underline{\tfrac{1}{2}} \wedge \underline{\tfrac{5}{2}} \wedge \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \cdots = -\left(\underline{\tfrac{5}{2}} \wedge \left(\underline{\tfrac{1}{2}} \wedge \underline{\tfrac{1}{2}}\right) \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \cdots\right) = 0.$$

On the other hand, applying the adjoint fermionic operators $\psi^*_{(3,2)}$ and $\psi^*_{(1,2)}$ to $v_{(3,2)}$ respectively yields

$$\psi^*_{3/2} v_{(3,2)} = 0, \qquad \text{and} \qquad \psi^*_{1/2} v_{(3,2)} = -\left(\underline{\tfrac{5}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \underline{-\tfrac{7}{2}} \wedge \cdots\right).$$

Hence, one thinks of ψ_k as adding k to S if missing (up to sign) and returning zero otherwise, while ψ^*_k removes k from S if present (up to sign) and returns zero otherwise. Figure 1.4 shows the action of $\psi_{3/2}$ on $v_{(3,2)}$ in the Young diagram representation. We can see that $\psi_{3/2}$ has shifted the charge of $v_{(3,2)}$ by $+1$; this is true in general, ψ_k increases the charge by 1 while its adjoint ψ^*_k decreases the charge by 1.

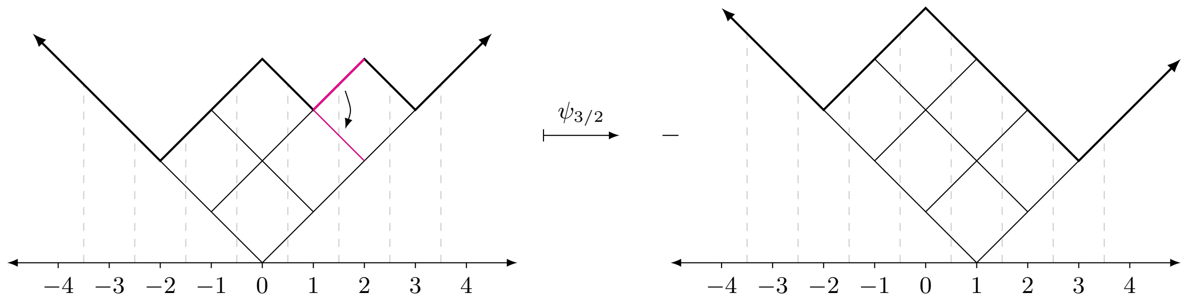

Figure 1.4: Young diagram of $v_{(3,2)}$ and $\psi_{3/2} v_{(3,2)}$, showing the action of ψ_k.

The fermionic operators satisfy canonical anti-commutation relations.

Proposition 1.3.2. *The fermionic operators satisfy the anti-commutation relations*

$$[\psi_i, \psi^*_j]_+ = \psi_i \psi^*_j + \psi^*_j \psi_i = \delta_{ij} \tag{1.4}$$
$$[\psi_i, \psi_j]_+ = [\psi^*_i, \psi^*_j]_+ = 0. \tag{1.5}$$

Proof. The anti-commutation relation $[\psi_i, \psi_j]_+ = 0$ follows immediately from the skew-symmetry of the wedge product, while the relation $[\psi^*_i, \psi^*_j]_+ = 0$ follows from the first by taking adjoints.

For (1.4), if $i \neq j$, then the operator $\psi_i \psi^*_j + \psi^*_j \psi_i$ acting on v_S will only contribute if $i \in S$ but $j \notin S$. In this case, both terms $\psi_i \psi^*_j$ and $\psi^*_j \psi_i$ acting on v_S will yield the same result, up to sign, where the signs are necessarily opposite to each other. Hence, if $i \neq j$, $(\psi_i \psi^*_j + \psi^*_j \psi_i) v_S = 0$ for all basis vectors $v_S \in \Lambda^{\frac{\infty}{2}} V$. If $i = j$, then either $i, j \in S$, in which case $\psi^*_i \psi_i v_S = 0$ and $\psi_i \psi^*_i v_S = v_S$, or $i, j \notin S$, in which case $\psi^*_i \psi_i v_S = v_S$ and $\psi_i \psi^*_i v_S = 0$. Hence, $\psi_i \psi^*_j + \psi^*_j \psi_i = \delta_{ij}$, as required. ∎

Also, define the *normally ordered product* of fermionic operators as follows.

Definition 1.3.3. The *normally ordered product* of fermionic operators is defined to be

$$:\psi_i \psi^*_j: := \begin{cases} \psi_i \psi^*_j, & \text{if } j > 0, \\ -\psi^*_j \psi_i, & \text{if } j < 0. \end{cases} \tag{1.6}$$

The normal ordering is introduced to cater for the possibility of non-convergent infinite sums; consider for example, the expression $\sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_k \psi^*_k$. The operator $\psi_k \psi^*_k$ acts on a basis vector v_λ and returns v_λ if there is a down-step at k in the corresponding Young diagram, and 0 otherwise. Since the number of down-steps in any Young diagram is infinite, the action of $\sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k\psi_k^*$ is not well-defined. On the other hand, $\sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_k\psi_k^*:$ is a well-defined diagonal operator, for which the eigenvalue associated to v_λ is equal to the number of down-steps at positive k minus the number of up-steps at negative k in the corresponding Young diagram. That is, it precisely enumerates the difference in the sizes of the sets S_+ and S_- defined in (1.1), and hence, it detects the charge of a basis vector.

1.3.2 Bosonic operators

Definition 1.3.4. For $n \in \mathbb{Z} \setminus \{0\}$, define the *bosonic operator* $\alpha_n \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ by

$$\alpha_n = \sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_{k-n}\psi_k^*: . \tag{1.7}$$

The adjoint α_n^* is given by

$$\alpha_n^* = \Big(\sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_{k-n}\psi_k^*: \Big)^* = \sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_k\psi_{k-n}^* := \sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_{k+n}\psi_k^* := \alpha_{-n}.$$

Observe that the bosonic operators preserve charge, so they can be considered as operators on the charge zero subspace $\Lambda_0^{\frac{\infty}{2}} V$.

Consider the action of α_{-1} on some v_λ in the charge zero subspace:

$$\alpha_{-1}\, v_\lambda = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k+1}\psi_k^*\, v_\lambda .$$

In the Young diagram representation, α_{-1} corresponds to detecting all down-up sequences, and converting them to up-down sequences. In other words, $\alpha_{-1}v_\lambda$ is the linear combination of basis vectors obtained by adding one box to the Young diagram for λ in every possible way. (In general, each term in this sum picks up a sign based on the height of the ribbon, but in the case of α_{-1} the sign is always positive.)

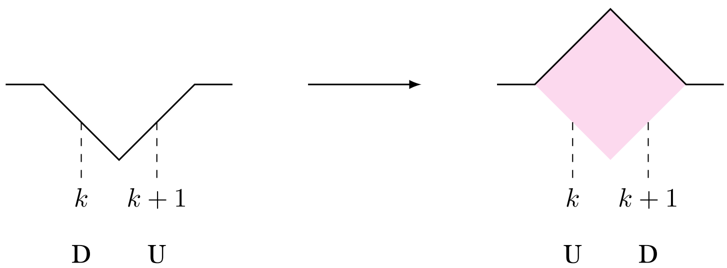

Figure 1.5: The local action of α_{-1}.

For example, consider $v_{(2,1,1)}$ in Figure 1.6: there are three down-up sequences in the Young diagram, hence there are three possible locations where a box can be added (or if you like, three possible locations where a box can land when dropped in from above; think Tetris). Thus

$$\alpha_{-1}\, v_{(2,1,1)} = v_{(3,1,1)} + v_{(2,2,1)} + v_{(2,1,1,1)}.$$

One can extend this idea to all $\alpha_{\pm n}$; to do this, the notion of an n-ribbon is required.

A *skew shape*, denoted $\lambda \setminus \mu$, is a pair of partitions (λ, μ) with $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ and $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$ such that $\ell(\mu) \leqslant \ell(\lambda)$ and $\mu_i \leqslant \lambda_i$ for all $i \in \{1, 2, \ldots, \ell(\mu)\}$. A *skew Young diagram* of shape $\lambda \setminus \mu$ is obtained by removing a Young diagram of shape μ from the Young diagram of shape λ. A Young diagram is connected if, for any two cells, a path via adjacent cells can be drawn between them. An n-*ribbon* is a

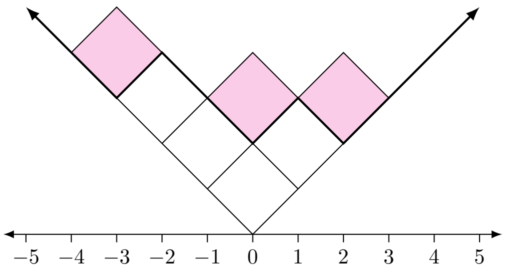

Figure 1.6: Here we see $v_{(2,1,1)}$ and the three possible locations for an added box in pink.

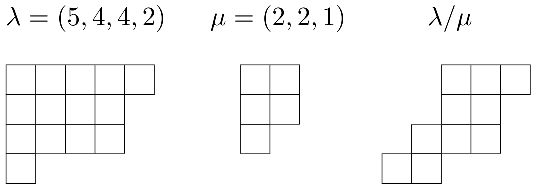

Figure 1.7: Young diagrams of shape $(5,4,4,2)$, $(2,2,1)$ and $(5,4,4,2)/(2,2,1)$.

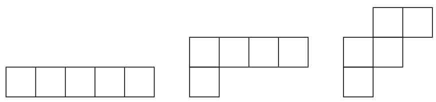

Figure 1.8: Three 5-ribbons of heights 0, 1, and 2, from left-to-right.

connected skew Young diagram comprising n boxes that does not contain any 2×2 blocks. The *height* of a ribbon is one fewer than the number of rows it occupies.

The operator α_{-n} for $n > 0$ acts on basis vectors v_λ by adding an n-ribbon to λ in all possible ways, with each term picking up a sign depending on the parity of the height of the ribbon. Conversely, α_n removes n-ribbons in all possible ways, again with each term picking up the appropriate sign depending on the height of the ribbon removed.

For example, $\alpha_{-3}\, v_{(4,3,1)}$ adds a 3-ribbon to $\lambda = (4,3,1)$ in all possible ways (with sign), while $\alpha_3\, v_{(4,3,1)}$ removes a 3-ribbon in all possible ways. That is,

$$\begin{aligned} \alpha_{-3}\, v_{(4,3,1)} &= v_{(7,3,1)} - v_{(5,5,1)} - v_{(4,3,2,2)} + v_{(4,3,1,1,1,1)} \\ \alpha_3\, v_{(4,3,1)} &= -v_{(2,2,1)}. \end{aligned}$$

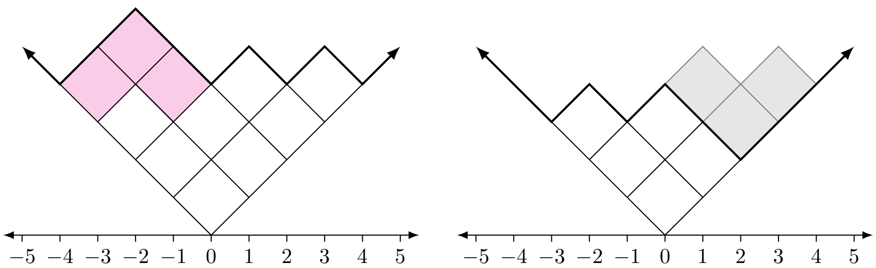

Figure 1.9: Left: $v_{(4,3,1)}$ with an added 3-ribbon, becoming $v_{(4,3,2,2)}$. Right: $v_{(4,3,1)}$ with a 3-ribbon removed, making $v_{(2,2,1)}$.

In contrast to the fermionic operators and their anti-commutation relations, the bosonic operators obey canonical commutation relations.

Proposition 1.3.5. *The bosonic operators α_n satisfy the commutation relation*

$$[\alpha_m, \alpha_n] = m\, \delta_{m,-n}. \tag{1.8}$$

This can be proven by using the definition of the bosonic operators in terms of fermionic operators (1.7), then applying the anti-commutation relations of the latter. A cute alternative proof proceeds as follows.

Proof of Proposition 1.3.5. One can prove that $[\alpha_m, \alpha_n] = 0$ for $m+n \neq 0$ directly via the fermionic definition of α_n; this is left as a straightforward though tedious exercise for the enthusiastic reader.

To prove that $[\alpha_m, \alpha_{-m}] = m$, begin with the case of $m = 1$. For any $v_\lambda \in \Lambda^{\frac{\infty}{2}} V$ with some fixed charge c,

$$[\alpha_1, \alpha_{-1}]\, v_\lambda = \alpha_1 \alpha_{-1} v_\lambda - \alpha_{-1} \alpha_1 v_\lambda.$$

The term $\alpha_1 \alpha_{-1} v_\lambda$ corresponds to adding then removing a box from λ, while $\alpha_{-1}\alpha_1 v_\lambda$ removes then adds a box to λ. If the boxes are being added and removed from different locations the order doesn't matter, hence, in those cases the commutator $[\alpha_1, \alpha_{-1}]$ doesn't contribute. Thus, any contribution to $\alpha_1\alpha_{-1}v_\lambda - \alpha_{-1}\alpha_1 v_\lambda$ arises from the difference between the number of ways to add then remove the same box versus the number of ways to remove then add the same box. This, in turn, is equal to the number of valleys (down-up sequences) minus the number of mountains (up-down sequences). The inherent V-shape underlying the Young diagram corresponding to a basis vector implies that the number of valleys will always be one greater than the number of mountains. Thus conclude that $[\alpha_1, \alpha_{-1}] = 1$.

To prove the commutation relation for general $m > 0$, upgrade the argument above in the following way. The commutation $[\alpha_m, \alpha_{-m}]\, v_\lambda$ corresponds to adding then removing an m-ribbon versus removing then adding an m-ribbon. Adding an m-ribbon to v_λ arises from changing a down-step to an up-step at some position k, and doing the opposite (changing an up-step to a down-step) m places away at position $k + m$.

Take v_λ and label each up- or down-step, in order from left-to-right, with a "colour" from $1, 2, \ldots, m$. That is, from left-to-right, the colour labellings read $1, 2, \ldots, m, 1, 2, \ldots$, and so on. For each $i \in \{1, 2, \ldots, m\}$, define $v_{\lambda^{(i)}}$ to be the basis vector corresponding to the sequence of up- and down-steps labelled with the colour i.

Adding or removing an m-ribbon to v_λ corresponds to adding or removing a single box to $v_{\lambda^{(i)}}$ for some $i \in \{1, 2, \ldots, m\}$. Hence, the difference between the number of ways to add then remove an m-ribbon from v_λ versus removing then adding an m-ribbon to v_λ is equal to the number of ways to add then remove a box minus the number of ways to remove then add a box to some $v_{\lambda^{(i)}}$. As argued above, $[\alpha_1, \alpha_{-1}] = 1$, hence $[\alpha_m, \alpha_{-m}] = m$.

For $m < 0$, $[\alpha_m, \alpha_{-m}] = -[\alpha_{-m}, \alpha_m] = m$. This concludes the proof. ■

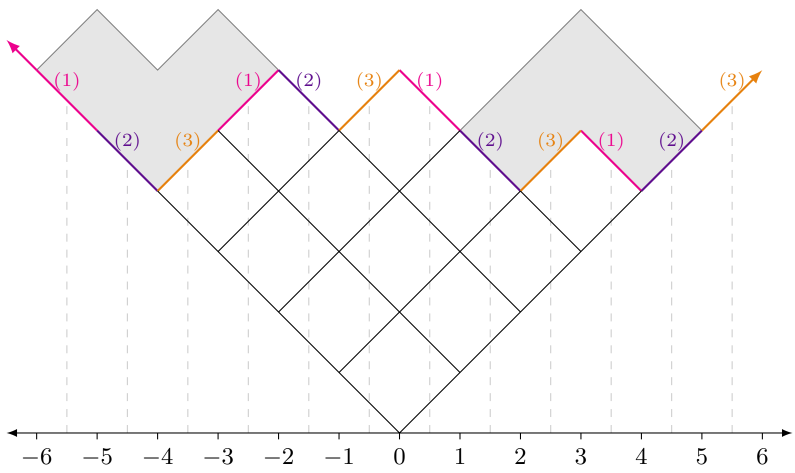

Figure 1.10: The basis element v_λ for $\lambda = (4, 3, 2, 2)$ with every third step labelled with the same colour. Adding a 3-ribbon corresponds to adding a box to one of $v_{\lambda^{(1)}}$, $v_{\lambda^{(2)}}$ or $v_{\lambda^{(3)}}$.

1.3.3 $\mathcal{E}$-operators

First, define $\varsigma(z) := e^{z/2} - e^{-z/2}$.

Definition 1.3.6. For $n \in \mathbb{Z}$, define the operator $\mathcal{E}_n(z) \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ by

$$\mathcal{E}_n(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z\left(k-\frac{n}{2}\right)} : \psi_{k-n}\psi_k^* : + \frac{\delta_{n,0}}{\varsigma(z)}.$$

The constant term, $\delta_{n,0}/\varsigma(z)$, is a regularisation term that adjusts for the normal ordering of $:\psi_k\psi_k^*:$. One can also consider these operators without the regularisation term, denoted $\tilde{\mathcal{E}}$, and defined to be

$$\tilde{\mathcal{E}}_n(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z\left(k-\frac{n}{2}\right)} : \psi_{k-n}\psi_k^* : .$$

The $\mathcal{E}$-operator satisfies the following commutation relation, the statement of which can be found in the work of Okounkov and Pandharipande [92].

Proposition 1.3.7. *For all $a, b \in \mathbb{Z}$, the $\mathcal{E}$-operator satisfies the following commutation relation:*

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma(aw - bz)\, \mathcal{E}_{a+b}(z+w). \tag{1.9}$$

The $\mathcal{E}$-operator specialises to the bosonic operator in the case of $z = 0$; that is, for $n \in \mathbb{Z} \setminus \{0\}$,

$$\mathcal{E}_n(0) = \alpha_n.$$

The $\mathcal{E}$-operator also has a nice expression in terms of the bosonic operators. That is,

$$\mathcal{E}_n(z) = \frac{1}{\varsigma(z)} \sum_{\ell \geqslant 0} \frac{1}{\ell!} \sum_{\substack{i_1+\cdots+i_\ell = n \\ i_k \in \mathbb{Z}\setminus\{0\}}} \frac{\varsigma(i_1 z)\cdots\varsigma(i_\ell z)}{i_1 \cdots i_\ell} : \alpha_{i_1} \cdots \alpha_{i_\ell} : .$$

This expression can be proven using the boson-fermion correspondence; see Lemma 1.4.8 in Section 1.4.2. The normal ordering of bosonic operators denotes that we write bosonic operators with positive subscripts to the right and those with negative subscripts to the left; that is, we remove boxes first before adding boxes. Explicitly, applying the normal ordering to the expression above gives

$$\mathcal{E}_n(z) = \frac{1}{\varsigma(z)} \sum_{\substack{\ell, s \geqslant 0 \\ q \geqslant 0}} \frac{1}{\ell!\, s!} \left[\sum_{\substack{j_1+\cdots+j_\ell = q \\ j_k \geqslant 1}} \prod_{k=1}^{\ell} \frac{\varsigma(j_k z)}{j_k} \alpha_{-j_k} \right] \left[\sum_{\substack{i_1+\cdots+i_s = q+n \\ i_k \geqslant 1}} \prod_{k=1}^{s} \frac{\varsigma(i_k z)}{i_k} \alpha_{i_k} \right].$$

1.3.4 Other operators

For $n \geqslant 0$, define the operator $\mathcal{F}_n \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ by

$$\mathcal{F}_n := \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{k^n}{n!} : \psi_k \psi_k^* : = [z^n]\, \mathcal{E}_0(z). \tag{1.10}$$

Here the notation $[z^n]\, \mathcal{E}_0(z)$ denotes the coefficient of z^n in the series expansion of the operator $\mathcal{E}_0(z)$.

The operator $\mathcal{F}_1$ is called the *energy operator* and it satisfies $\mathcal{F}_1 v_\lambda = |\lambda|\, v_\lambda$. This is true because, for each $k \in \mathbb{Z} + \frac{1}{2}$, $\mathcal{F}_1$ counts down-steps with weight k for $k > 0$ and up-steps with weight $-k$ for $k < 0$. This in turn returns the number of boxes in λ. One can use the operator $\mathcal{F}_1$ to define the energy of an operator; that is, one can say that an operator $\mathcal{O}$ on $\Lambda^{\frac{\infty}{2}} V$ has *energy* $n \in \mathbb{Z}$ if

$$[\mathcal{O}, \mathcal{F}_1] = n\, \mathcal{O}.$$

The operators $\mathcal{F}_n$ all have energy zero, while α_n and $\mathcal{E}_n$ both have energy n.

The operator $\mathcal{F}_2$ plays a particularly vital role in the context of Hurwitz numbers, single and double, appearing in the vacuum expectations of both enumerations. Its appearance functions by enumerating the number of simple branch points for both single and double Hurwitz numbers, and the reason for this is as follows. The operator $\mathcal{F}_2$ is a diagonal operator with eigenvalue $f_2(\lambda)$ for eigenvector v_λ; that is, $\mathcal{F}_2\, v_\lambda = f_2(\lambda)\, v_\lambda$, where

$$f_2(\lambda) = |C_{(2,1,\ldots,1)}| \frac{\chi^\lambda_{(2,1,\ldots,1)}}{\dim \lambda}. \tag{1.11}$$

Here, $|C_{(2,1,\ldots,1)}|$ is the size of the conjugacy class of transpositions, $\chi^\lambda_{(2,1,\ldots,1)}$ is the character of the irreducible representation corresponding to λ evaluated on an element of $C_{(2,1,\ldots,1)}$ (that is, any transposition); and $\dim \lambda = \chi^\lambda_{(1,1,\ldots,1)}$ is the dimension of the irreducible representation corresponding to λ.

As an aside, one can more generally define

$$f_\mu(\lambda) = \binom{|\lambda|}{|\mu|} |C_\mu| \frac{\chi^\lambda_\mu}{\dim \lambda},$$

where, if $|\mu| < |\lambda|$, the character χ^λ_μ is defined via the natural inclusion of symmetric groups $S_{|\mu|} \subset S_{|\lambda|}$, and if $|\mu| > |\lambda|$ the binomial vanishes. These operators arise in the general setting of enumerating Hurwitz numbers with arbitrary ramification; see the work of Okounkov and Pandharipande [92].

An explicit expression for $f_2(\lambda)$ is given by

$$f_2(\lambda) = \frac{1}{2} \sum_{i=1}^{\ell(\lambda)} \Big[\big(\lambda_i - i + \tfrac{1}{2}\big)^2 - \big(-i + \tfrac{1}{2}\big)^2 \Big]. \tag{1.12}$$

Via this last equation, one can show that $f_2(\lambda)$ is equal to the sum of the *contents* of λ, which is defined as follows. For a Young diagram corresponding to a partition λ, the *content* of a box in column j and row i is $j - i$. For example, consider the Young diagram given by $\lambda = (5, 4, 4, 2, 2)$; Figure 1.11 shows the Young diagram with filling given by its contents. In this case, the sum of the contents of λ is -2, and one can verify that $f_2(\lambda) = -2$.

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			
−4	−3			

Figure 1.11: Young diagram with contents for the partition $\lambda = (5, 4, 4, 2, 2)$.

To see that (1.12) is indeed the sum of the contents, first note

$$f_2(\lambda) = \frac{1}{2} \sum_{i=1}^{\ell(\lambda)} \Big[\big(\lambda_i - i + \tfrac{1}{2}\big)^2 - \big(-i + \tfrac{1}{2}\big)^2 \Big] = \frac{1}{2} \sum_{i=1}^{\ell(\lambda)} \lambda_i \big(\lambda_i - 2i + 1\big).$$

The second equality is using the difference of squares. Then, observe that the sum of the contents of row i is equal to the number of boxes in the row, λ_i, multiplied by the average of the fillings; the contents of row i are given by $1 - i, 2 - i, \ldots, \lambda_i - i$, hence the average of the contents of row i is $\frac{1}{2}(\lambda_i - 2i + 1)$. Sum over all rows to conclude.

Definition 1.3.8. Define the *vertex operators* $\Gamma_\pm(\vec{p})$ on the semi-infinite wedge space by

$$\Gamma_\pm(\vec{p}) = \exp \Bigg(\sum_{m \geqslant 1} \frac{p_m}{m} \alpha_{\pm m} \Bigg),$$

where $p_1, p_2, \ldots$ are formal variables.

The vertex operators satisfy the following commutation relation [73]:

$$\Gamma_+(\vec{p})\,\Gamma_-(\vec{q}) = \exp\left(\sum_{m\geqslant 1}\frac{p_m\,q_m}{m}\right)\Gamma_-(\vec{q})\,\Gamma_+(\vec{p}). \tag{1.13}$$

The vertex operators are related to the Schur symmetric polynomials in the following way. Begin with Γ_-, write the exponential of the sum as a product of exponentials, then expand each exponential as a power series to obtain

$$\begin{aligned}\Gamma_-(\vec{p}) &= \exp\left(\sum_{m\geqslant 1}\frac{p_m}{m}\alpha_{-m}\right) = \prod_{m\geqslant 1}\exp\left(\frac{p_m}{m}\alpha_{-m}\right) = \prod_{m\geqslant 1}\left[\sum_{k_m\geqslant 0}\frac{p_m^{k_m}}{m^{k_m}\,k_m!}\alpha_{-m}^{k_m}\right] \\ &= \sum_{k_1,k_2,\ldots\geqslant 0}\frac{p_1^{k_1}\,p_2^{k_2}\cdots}{1^{k_1}\,2^{k_2}\cdots k_1!\,k_2!\cdots}\alpha_{-1}^{k_1}\alpha_{-2}^{k_2}\cdots.\end{aligned} \tag{1.14}$$

Now rewrite this sum over $k_1, k_2, \ldots$ as a sum over partitions $\mu = (1^{k_1}, 2^{k_2}, \ldots, m^{k_m})$. In this case,

$$\vec{p}_\mu = p_1^{k_1}\,p_2^{k_2}\cdots \qquad \prod_{i=1}^{\ell(\mu)}\mu_i = 1^{k_1}\,2^{k_2}\cdots$$

$$|\mathrm{Aut}\;\mu| = k_1!\,k_2!\cdots \qquad \alpha_{-\mu_1}\cdots\alpha_{-\mu_{\ell(\mu)}} = \alpha_{-1}^{k_1}\alpha_{-2}^{k_2}\cdots.$$

Thus,

$$\Gamma_-(\vec{p}) = \sum_{\mu\in\mathcal{P}}\frac{\vec{p}_\mu}{|\mathrm{Aut}\;\mu|\,\prod_{i=1}^{\ell(\mu)}\mu_i}\alpha_{-\mu_1}\cdots\alpha_{-\mu_{\ell(\mu)}} = \sum_{\mu\in\mathcal{P}}\frac{\vec{p}_\mu}{z(\mu)}\alpha_{-\mu_1}\cdots\alpha_{-\mu_{\ell(\mu)}}, \tag{1.15}$$

using the common notation $z(\mu) = |\mathrm{Aut}\;\mu|\,\prod_{i=1}^{\ell(\mu)}\mu_i$.

Now apply the vertex operator Γ_- to the vacuum vector $v_\emptyset$ to obtain

$$\Gamma_-(\vec{p})v_\emptyset = \sum_{\mu\in\mathcal{P}}\frac{\vec{p}_\mu}{z(\mu)}\alpha_{-\mu_1}\cdots\alpha_{-\mu_{\ell(\mu)}}v_\emptyset = \sum_{\mu\in\mathcal{P}}\frac{\vec{p}_\mu}{z(\mu)}\sum_{\lambda\vdash|\mu|}\chi_\mu^\lambda\,v_\lambda = \sum_{\lambda\in\mathcal{P}}\left(\sum_{\mu\in\mathcal{P}}\frac{\chi_\mu^\lambda\,\vec{p}_\mu}{z(\mu)}\right)v_\lambda = \sum_{\lambda\in\mathcal{P}}s_\lambda(\vec{p})\,v_\lambda.$$

Here, $\mathcal{P}$ is the set of all partitions (including the empty partition), the notation $|\mu|$ denotes the sum of the parts of μ: $|\mu| = \mu_1 + \cdots + \mu_{\ell(\mu)}$, the notation $\lambda \vdash |\mu|$ denotes that λ is an integer partition of $|\mu|$, and $s_\lambda(\vec{p})$ is the Schur symmetric polynomial indexed by λ, written in terms of the power sum symmetric polynomials $p_1, p_2, \ldots$. (It is generally to our benefit to secretly think of the formal variables $p_1, p_2, \ldots$ as the power sum symmetric polynomials). The second equality is using the Murnaghan–Nakayama rule; see Theorem 1.4.3 in the following section. The third equality is swapping the order of the summations and using the fact that $\chi_\mu^\lambda = 0$ for λ and μ such that $|\lambda| \neq |\mu|$. The final equality is using the change of basis between the power sum symmetric polynomials and the Schur symmetric polynomials in the ring of symmetric functions [81].

1.4 Results

1.4.1 Murnaghan–Nakayama rule

The Murnaghan–Nakayama rule provides a combinatorial identity for calculating irreducible characters of symmetric groups; the original statement of the theorem can be found in [81]. The Murnaghan–Nakayama rule also has an alternate description in terms of operators acting on the semi-infinite wedge. This result plays a fundamental role in the vacuum expectation derivation for Hurwitz numbers, both single and double, but also appears in Chapter 7 on fully simple maps.

Before I can state the semi-infinite wedge version of the Murnaghan–Nakayama rule I will first state the original theorem, and to do that I will first define a number of necessary objects.

First, recall that the *filling* of a Young diagram is a labelling of each box in the diagram with a positive integer. A *Young tableau* is any filling of a Young diagram, while a *semistandard Young tableau* is a filling that is weakly increasing across each row and strictly increasing down each column. Although not used here, for reference, a *standard Young tableau* is a semistandard Young tableau where the boxes are filled with $1, 2, \ldots, d$, each occurring precisely once.

And finally, a *ribbon tableau of shape* λ *and type* μ is a Young tableau that satisfies the following: the filling is weakly increasing across each row and down each column; and the filling of the tableau is prescribed by μ. That is, the tableau has a μ_1-ribbon of 1s, a μ_2-ribbon of 2s, and so on. Recall that an n-*ribbon* is a connected skew Young diagram comprising n boxes that does not contain any 2×2 blocks, and the *height* of a ribbon is one fewer than the number of rows it occupies.

For example, a Young diagram, Young tableau, semistandard Young tableau and standard Young tableau all of shape $(4, 3, 1, 1)$ are shown in Figure 1.12, while all possible ribbon tableaux of shape $(3, 2)$ and type $(3, 1, 1)$ are given in Figure 1.13.

$$
\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline 5 & 2 & 4 & 4 \\ \hline 2 & 5 & 6 \\ \cline{1-3} 1 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 8 & 9 \\ \cline{1-3} 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}
$$

Figure 1.12: From left-to-right: a Young diagram, Young tableau, semistandard Young tableau and standard Young tableau.

$$
\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 1 & 3 \\ \cline{1-2} \end{array}
\qquad
\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 1 & 2 \\ \cline{1-2} \end{array}
$$

Figure 1.13: All ribbon tableaux of shape $(3, 2)$ and type $(3, 1, 1)$.

Theorem 1.4.1 (Murnaghan–Nakayama rule). *The character evaluated on a permutation in the conjugacy class corresponding to* μ *in the unique irreducible representation of the symmetric group corresponding to* λ, χ^{λ}_{μ}, *satisfies*

$$\chi^{\lambda}_{\mu} = \sum_{T} (-1)^{h(T)}.$$

Here, the summation is over is the set of ribbon tableaux T *of shape* λ *and type* μ, *and* $h(T)$ *is the sum of the heights of the ribbons in* T.

For more on the representation theory of the symmetric group, see [97].

Example 1.4.2. Let us use Theorem 1.4.1 to calculate $\chi^{(4,3,2)}_{(4,3,1,1)}$. Here $\lambda = (4, 3, 2)$ determines the shape of the Young diagram, while $\mu = (4, 3, 1, 1)$ determines the type. There are seven possible tableaux of shape λ and type μ, and these are given below.

$$
\begin{array}{|c|c|c|c|}\hline 1&1&1&1 \\ \hline 2&2&2 \\ \cline{1-3} 3&4 \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 1&1&1&1 \\ \hline 2&2&3 \\ \cline{1-3} 2&4 \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 1&1&1&1 \\ \hline 2&2&4 \\ \cline{1-3} 2&3 \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 1&1&1&3 \\ \hline 1&2&4 \\ \cline{1-3} 2&2 \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 1&1&1&4 \\ \hline 1&2&3 \\ \cline{1-3} 2&2 \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 1&1&2&3 \\ \hline 1&2&2 \\ \cline{1-3} 1&4 \\ \cline{1-2} \end{array}
\quad
\begin{array}{|c|c|c|c|}\hline 1&1&2&4 \\ \hline 1&2&2 \\ \cline{1-3} 1&3 \\ \cline{1-2} \end{array}
$$

Let these be $T_1, \ldots, T_7$ respectively. The heights of the ribbons in T_4 are 1, 1, 0 and 0 for μ_1, μ_2, μ_3 and μ_4 respectively, hence $h(T_1) = 1 + 1 + 0 + 0 = 2$. Following the same process for the remaining six tableaux leads to the following calculation for $\chi^{(4,3,2)}_{(4,3,1,1)}$:

$$
\begin{aligned}
\chi^{(4,3,2)}_{(4,3,1,1)} &= (-1)^0 + (-1)^1 + (-1)^1 + (-1)^2 + (-1)^2 + (-1)^3 + (-1)^3 \\
&= 1 - 1 - 1 + 1 + 1 - 1 - 1 = -1.
\end{aligned}
$$

The Murnaghan–Nakayama rule given via operators acting on the semi-infinite wedge is the following.[1]

[1]Actually, the following result is an immediate consequence of the Murnaghan–Nakayama rule given in Theorem 1.4.1, but is often also simply referred to as the Murnaghan–Nakayama rule.

Theorem 1.4.3 (Murnaghan–Nakayama rule via the semi-infinite wedge). *Let $(\mu_1, \ldots, \mu_n)$ be a tuple of positive integers such that $\mu_1 + \cdots + \mu_n = d$. Then*

$$\alpha_{-\mu_1} \cdots \alpha_{-\mu_n} v_\emptyset = \sum_{\lambda \vdash d} \chi_\mu^\lambda v_\lambda.$$

Here, χ_μ^λ is the character evaluated on a permutation in the conjugacy class corresponding to μ in the unique irreducible representation of the symmetric group corresponding to λ.

Example 1.4.4. Let us use Theorem 1.4.3 to calculate $\chi_{(2,1,1)}^{(3,1)}$. The theorem tells us that

$$\alpha_{-2}\alpha_{-1}^2 v_\emptyset = \sum_{\lambda \vdash 4} \chi_{(2,1,1)}^\lambda v_\lambda.$$

Because α_{-m} commutes with any α_{-n}, it does not matter in which order we apply the $\alpha_{-\mu_i}$. Thus,

$$\begin{aligned}\sum_{\lambda \vdash 4} \chi_{(2,1,1)}^\lambda v_\lambda = \alpha_{-1}^2\alpha_{-2}v_\emptyset &= \alpha_{-1}^2 v_{(2)} - \alpha_{-1}^2 v_{(1,1)} \\ &= \alpha_{-1}v_{(3)} + \alpha_{-1}v_{(2,1)} - \alpha_{-1}v_{(2,1)} - \alpha_{-1}v_{(1,1,1)} \\ &= v_{(4)} + v_{(3,1)} - v_{(2,1,1)} - v_{(1,1,1,1)}.\end{aligned}$$

Hence, $\chi_{(2,1,1)}^{(3,1)} = 1$.

1.4.2 Boson-fermion correspondence

The bosonic operators are, by definition, written in terms of the fermionic operators. At heart, the boson-fermion correspondence states that the reverse is true; that the fermionic operators can, in turn, be written in terms of the bosonic operators. More precisely and more generally, the boson-fermion correspondence describes a vector space isomorphism between the so-called bosonic and fermionic Fock spaces. To describe this isomorphism and the description of fermions in terms of bosons, a number of new definitions and notations will be introduced. The first half of this section will primarily follow Miwa, Jimbo and Date [83], however it is important to note that some conventional choices differ between this thesis and [83].

Begin by defining the bosonic Fock space. For n a positive integer, define operators α_n, α_n^* acting on polynomials $f(\vec{x}) \in \mathbb{C}[x_1, x_2, \ldots]$ by the following rules:

$$(\alpha_n f)(\vec{x}) = x_n f(\vec{x}), \qquad \text{and} \qquad (\alpha_n^* f)(\vec{x}) = n\frac{\partial f}{\partial x_n}(\vec{x}). \tag{1.16}$$

These operators satisfy the canonical commutation relations

$$[\alpha_m, \alpha_n] = 0, \quad [\alpha_m^*, \alpha_n^*] = 0, \quad \text{and} \quad [\alpha_m, \alpha_n^*] = m\,\delta_{n,m}. \tag{1.17}$$

Define $\mathcal{B}$ to be the algebra generated by the abstract symbols $\{\alpha_n, \alpha_n^*\}_{n=1,2,\ldots}$ satisfying the relations (1.17); $\mathcal{B}$ is called the *Heisenberg algebra.*

Define an algebra representation of the Heisenberg algebra $\mathcal{B}$ on the polynomial ring $\mathbb{C}[\vec{x}]$; that is, define $\rho : \mathcal{B} \to \mathrm{End}(\mathbb{C}[\vec{x}])$ by $\alpha_n \mapsto x_n$, and $\alpha_n^* \mapsto n\frac{\partial}{\partial x_n}$. The representation space $\mathbb{C}[\vec{x}]$ is called the *bosonic Fock space.*

The operators α_n, α_n^* have been suggestively labelled, and indeed, the commutation relations (1.17) coincide with the commutation relations for the bosonic operators defined in Definition 1.3.4; thus, one obtains a representation of the Heisenberg algebra $\mathcal{B}$ on the semi-infinite wedge space $\Lambda^{\frac{\infty}{2}} V$ via the bosonic operators $\alpha_{\pm n}$ defined in Definition 1.3.4.

Define $\Lambda_\ell^{\frac{\infty}{2}} V$ to be the charge ℓ subspace of $\Lambda^{\frac{\infty}{2}} V$. There is a natural isomorphism between the charge ℓ subspace, for each $\ell \in \mathbb{Z}$, and the bosonic Fock space $\mathbb{C}[\vec{x}]$. This isomorphism can be described for all charges at once. To do this, define

$$\mathbb{C}[w, w^{-1}; x_1, x_2, \ldots] = \bigoplus_{\ell \in \mathbb{Z}} w^\ell\, \mathbb{C}[x_1, x_2, \ldots].$$

Let $v_\ell \in \Lambda_\ell^{\frac{\infty}{2}} V$ denote the vacuum vector in the charge ℓ subspace; that is,

$$v_\ell := \underline{\ell - \frac{1}{2}} \wedge \underline{\ell - \frac{3}{2}} \wedge \underline{\ell - \frac{5}{2}} \wedge \cdots .$$

And finally, define $\Phi\colon \Lambda^{\frac{\infty}{2}} V \to \mathbb{C}[w^{\pm 1}; \vec{x}]$ by

$$\Phi(v) = \sum_{\ell \in \mathbb{Z}} w^\ell \Big\langle v, \exp\Big(\sum_{m \geqslant 1} \frac{x_m}{m} \alpha_{-m}\Big) v_\ell \Big\rangle.$$

Theorem 1.4.5 (Miwa, Jimbo and Date [83, Theorem 5.1]). *The correspondence*

$$\Phi\colon \Lambda^{\frac{\infty}{2}} V \to \mathbb{C}[w, w^{-1}; x_1, x_2, \ldots]$$

is an isomorphism of vector spaces. Moreover, we have

$$\Phi(\alpha_n v) = \begin{cases} n \frac{\partial}{\partial x_n} \Phi(v), & \text{if } n > 0, \\ x_{-n} \Phi(v), & \text{if } n < 0. \end{cases}$$

Theorem 1.4.5 allows us to identify the fermionic and bosonic Fock spaces, which, in turn, means that one ought to be able to represent the action of the fermionic operators in terms of operators acting on the bosonic Fock space. This is the content of the following theorem, but first I state a number of definitions.

Define the fermionic operator generating functions by

$$\psi(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k z^{k - \frac{1}{2}} \qquad \text{and} \qquad \psi^*(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k^* z^{-k - \frac{1}{2}}.$$

Introduce operators z^C and R acting on the space $\mathbb{C}[w^{\pm 1}; \vec{x}]$ by

$$(z^C f)(w, \vec{x}) = f(zw, \vec{x}) \qquad \text{and} \qquad (Rf)(w, \vec{x}) = w f(w, \vec{x}).$$

Loosely, the operator z^C measures the charge while the operator R shifts the charge by $+1$. Define

$$\xi(\vec{x}, z) = \sum_{n \geqslant 1} z^n x_n.$$

And finally define

$$\begin{aligned} \Psi(z) &= R\, z^C \exp\big(\xi(\vec{x}, z)\big) \exp\big(-\xi(\vec{\partial}, z^{-1})\big) \\ \Psi^*(z) &= R^{-1} z^{-C} \exp\big(-\xi(\vec{x}, z)\big) \exp\big(\xi(\vec{\partial}, z^{-1})\big), \end{aligned} \tag{1.18}$$

where $\vec{\partial}$ denotes

$$\vec{\partial} = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots\Big), \quad \text{and} \quad \xi(\vec{\partial}, z^{-1}) = \sum_{m \geqslant 1} z^{-m} \frac{\partial}{\partial x_m}.$$

Theorem 1.4.6 (Miwa, Jimbo and Date [83, Theorem 5.2]). *The fermionic generating functions $\psi(z)$ and $\psi^*(z)$ are realised in the bosonic Fock space by (1.18). That is, for any $v \in \Lambda^{\frac{\infty}{2}} V$ we have*

$$\Phi(\psi(z)\, v) = \Psi(z)\, \Phi(v) \qquad \textit{and} \qquad \Phi(\psi^*(z)\, v) = \Psi^*(z)\, \Phi(v).$$

Together, Theorems 1.4.5 and 1.4.6 imply that there's a description of fermionic operators ψ_k, ψ_k^* in terms of the bosonic operators $\alpha_{\pm n}$ defined in Definition 1.3.4.[2] There is indeed such an expression; this is the content of the following theorem, which can be found in the work of Kac [73].

First, define the charge C and translation R operators on the semi-infinite wedge space by

$$C = \sum_{k \in \mathbb{Z} + \frac{1}{2}} :\psi_k \psi_k^*:, \qquad \text{and} \qquad R\,(\underline{s_1} \wedge \underline{s_2} \wedge \cdots) = \underline{s_1 + 1} \wedge \underline{s_2 + 1} \wedge \cdots .$$

Note that the operator C featured in the discussion immediately after Definition 1.3.3; it enumerates the difference in the sizes of the sets S_+ and S_- and hence detects the charge of a basis element.

[2] Note that I often confuse the bosonic operators introduced in Definition 1.3.4 and those defined by (1.16), using the term "bosonic operators" to mean either of the operators.

Theorem 1.4.7 (Kac [73]). *The fermionic operator generating functions $\psi(z)$ and $\psi^*(z)$ have the following expressions in terms of the bosonic operators:*

$$\begin{aligned}\psi(z) &= z^C\, R\, \Gamma_-(\vec{z})\, \Gamma_+(-\vec{z}^{\,-1}) \\ \psi^*(z) &= R^{-1}\, z^{-C}\, \Gamma_-(-\vec{z})\, \Gamma_+(\vec{z}^{\,-1}).\end{aligned} \tag{1.19}$$

One can use Theorem 1.4.7 to prove the bosonic operator form of the $\mathcal{E}$-operator that appeared in Section 1.3.3; that is, to prove the following lemma. This bosonic expression of the $\mathcal{E}$-operator is a necessary component of the proof of the main Virasoro result for fully simple maps in Chapter 7.

Lemma 1.4.8. *The $\mathcal{E}$-operator introduced in Definition 1.3.6 can be described in terms of bosonic operators. That is,*

$$\mathcal{E}_n(z) = \frac{1}{\varsigma(z)} \sum_{\ell \geqslant 0} \frac{1}{\ell!} \sum_{i_1+\cdots+i_\ell=n} \frac{\varsigma(i_1 z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} : \alpha_{i_1}\cdots\alpha_{i_\ell} : . \tag{1.20}$$

Proof. Define $\mathcal{Q}_n(z)$ to be operator given on the right side of (1.20) above and rewrite to obtain

$$\begin{aligned}
\mathcal{Q}_n(z) &= \frac{1}{\varsigma(z)} \sum_{\ell \geqslant 0} \frac{1}{\ell!} \sum_{i_1+\cdots+i_\ell=n} \frac{\varsigma(i_1 z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} : \alpha_{i_1}\cdots\alpha_{i_\ell} : \\
&= \frac{1}{\varsigma(z)} \sum_{\substack{\ell, s \geqslant 0 \\ q \geqslant 0}} \frac{1}{\ell!\, s!} \Bigg[\sum_{\substack{j_1+\cdots+j_\ell=q \\ j_k \geqslant 1}} \prod_{k=1}^{\ell} \frac{\varsigma(j_k z)}{j_k} \alpha_{-j_k} \Bigg] \Bigg[\sum_{\substack{i_1+\cdots+i_s=q+n \\ i_k \geqslant 1}} \prod_{k=1}^{s} \frac{\varsigma(i_k z)}{i_k} \alpha_{i_k} \Bigg] \\
&= \frac{1}{\varsigma(z)} \sum_{q \geqslant 0} \Bigg[\sum_{\lambda \vdash q} \frac{1}{z(\lambda)} \prod_{i=1}^{\ell(\lambda)} \varsigma(\lambda_i z) \alpha_{-\lambda_i} \Bigg] \Bigg[\sum_{\mu \vdash q+n} \frac{1}{z(\mu)} \prod_{j=1}^{\ell(\mu)} \varsigma(\mu_j z) \alpha_{\mu_j} \Bigg],
\end{aligned}$$

where $z(\mu) = |\text{Aut}\ \mu| \prod_{i=1}^{\ell(\mu)} \mu_i$. The second equality is using the normal ordering of bosonic operators, while the third equality is rewriting the sum over tuples $(i_1, \ldots, i_s)$ as a sum over partitions $\mu \in \mathcal{P}$, and each tuple $(i_1, \ldots, i_s)$ arises $s!/|\text{Aut}\ \mu|$ times; see (1.14) for this calculation in detail.

Sum the $\mathcal{Q}_n(z)$-operators over all n to give

$$\begin{aligned}
\sum_{n \in \mathbb{Z}} \mathcal{Q}_n(z) &= \frac{1}{\varsigma(z)} \sum_{n \in \mathbb{Z}} \sum_{q \geqslant 0} \Bigg[\sum_{\lambda \vdash q} \frac{1}{z(\lambda)} \prod_{i=1}^{\ell(\lambda)} \varsigma(\lambda_i z) \alpha_{-\lambda_i} \Bigg] \Bigg[\sum_{\mu \vdash q+n} \frac{1}{z(\mu)} \prod_{j=1}^{\ell(\mu)} \varsigma(\mu_j z) \alpha_{\mu_j} \Bigg] \\
&= \frac{1}{\varsigma(z)} \Bigg[\sum_{\lambda \in \mathcal{P}} \frac{1}{z(\lambda)} \prod_{i=1}^{\ell(\lambda)} \varsigma(\lambda_i z) \alpha_{-\lambda_i} \Bigg] \Bigg[\sum_{\mu \in \mathcal{P}} \frac{1}{z(\mu)} \prod_{j=1}^{\ell(\mu)} \varsigma(\mu_j z) \alpha_{\mu_j} \Bigg] \\
&= \frac{1}{\varsigma(z)}\, \Gamma_-(\{\varsigma(k\,z)\})\, \Gamma_+(\{\varsigma(k\,z)\}) \\
&= \frac{1}{\varsigma(z)}\, \Gamma_-(\{-e^{-kz/2}\})\, \Gamma_-(\{e^{kz/2}\})\, \Gamma_+(\{e^{kz/2}\})\, \Gamma_+(\{-e^{-kz/2}\}).
\end{aligned}$$

Here, the notation $\Gamma(\{k\,z\})$ denotes $\Gamma(z, 2z, 3z, \ldots)$. The third equality is using (1.15). The final equality is using the fact that $\varsigma(z) = e^{z/2} - e^{-z/2}$, hence $\Gamma_\pm(\{e^{kz/2} - e^{-kz/2}\}) = \Gamma_\pm(\{e^{kz/2}\})\, \Gamma_\pm(\{-e^{-kz/2}\})$, along with the fact that Γ_--operators commute with each other. Using the commutation relation (1.13) on the inner two vertex operators gives

$$\begin{aligned}
\Gamma_-(\{e^{kz/2}\})\, \Gamma_+(\{e^{kz/2}\}) &= \exp\Bigg(-\sum_{k \geqslant 1} \frac{e^{kz}}{k} \Bigg) \Gamma_+(\{e^{kz/2}\})\, \Gamma_-(\{e^{kz/2}\}) \\
&= (1 - e^z)\, \Gamma_+(\{e^{kz/2}\})\, \Gamma_-(\{e^{kz/2}\}).
\end{aligned}$$

Here, the second equality is using the Taylor series expansion of $\log(1 - x)$; that is,

$$(1 - e^z) = \exp(\log(1 - e^z)) = \exp\Bigg(-\sum_{k \geqslant 1} \frac{e^{kz}}{k} \Bigg).$$

Therefore,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \mathcal{Q}_n(z) &= \frac{(1-e^z)}{\varsigma(z)}\,\Gamma_-(\{-e^{-kz/2}\})\,\Gamma_+(\{e^{kz/2}\})\,\Gamma_-(\{e^{kz/2}\})\,\Gamma_+(\{-e^{-kz/2}\})\\ &= -e^{z/2}\,\Gamma_-(\{-e^{-kz/2}\})\,\Gamma_+(\{e^{kz/2}\})\,\Gamma_-(\{e^{kz/2}\})\,\Gamma_+(\{-e^{-kz/2}\}).\end{aligned}$$

Use Theorem 1.4.7 to rewrite the product of vertex operators in terms of the fermionic operator generating functions as follows:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \mathcal{Q}_n(z) &= -e^{z/2}\,\Gamma_-(\{-e^{-kz/2}\})\,\Gamma_+(\{e^{kz/2}\})\,\Gamma_-(\{e^{kz/2}\})\,\Gamma_+(\{-e^{-kz/2}\})\\ &= -e^{z/2}\left(e^{Cz/2}\,R\,\psi^*(e^{-z/2})\right)\left(R^{-1}e^{-Cz/2}\,\psi(e^{z/2})\right)\\ &= -e^{z/2}\,e^{Cz/2}\sum_{j\in\mathbb{Z}+\frac{1}{2}}\psi^*_{j+1}e^{jz/2}e^{-z/4}\,e^{-Cz/2}\sum_{i\in\mathbb{Z}+\frac{1}{2}}\psi_i e^{iz/2}e^{z/4}\\ &= -\sum_{i,j\in\mathbb{Z}+\frac{1}{2}} e^{(i+j)z/2}\psi^*_j\psi_i.\end{aligned}$$

The second equality is using Theorem 1.4.7. The third equality is using the conjugation of ψ_j by R; that is, $R\psi_j R^{-1} = \psi_{j+1}$ [90]. The fourth equality is relabelling $j \mapsto j-1$ and using the fact that, because ψ^*_k decreases the charge by 1, the charge operator C commutes with ψ^*_k by $\psi^*_k e^C = e^{C+1}\psi^*_k$.

Now split the sum over j into j positive and j negative and use the commutation relation for the fermionic operators (1.4) along with the definition of the normally ordered product of fermions (1.6) to obtain

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \mathcal{Q}_n(z) &= -\sum_{\substack{i,j\in\mathbb{Z}+\frac{1}{2}\\ j<0}} e^{(i+j)z/2}\psi^*_j\psi_i + \sum_{\substack{i,j\in\mathbb{Z}+\frac{1}{2}\\ j>0}} e^{(i+j)z/2}\psi_i\psi^*_j - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} e^{kz}\\ &= \sum_{i,j\in\mathbb{Z}+\frac{1}{2}} e^{(i+j)z/2} :\psi_i\psi^*_j: + \frac{1}{\varsigma(z)}\\ &= \sum_{n\in\mathbb{Z}}\left[\sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z\left(k-\frac{n}{2}\right)} :\psi_{k-n}\psi^*_k: + \frac{\delta_{n,0}}{\varsigma(z)}\right]\\ &= \sum_{n\in\mathbb{Z}} \mathcal{E}_n(z).\end{aligned}$$

The third equality is using the fact that

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} e^{kz} = e^{z/2} + e^{3z/2} + \cdots = e^{z/2}\left(1 + e^z + e^{2z} + \cdots\right) = -\frac{e^{z/2}}{e^z - 1} = -\frac{1}{\varsigma(z)}.$$

Equating operators with equal energy on both sides yields $\mathcal{Q}_n(z) = \mathcal{E}_n(z)$, as required. ■

Chapter 2

Topological recursion

2.1 Introduction

The topological recursion of Chekhov, Eynard and Orantin was borne out of the abstraction of loop equations from the theory of matrix models [25, 52]. Since its inception, it is has been proven or conjectured to govern a widespread collection of problems in mathematics and physics, including: the enumeration of ribbon graphs and hypermaps [34, 36, 41, 88]; simple, monotone, orbifold, spin, and double Hurwitz numbers [8, 14, 17, 31, 32, 33, 39, 49, 84]; intersection theory on moduli spaces of curves [52]; Weil–Petersson volumes of moduli spaces of hyperbolic surfaces [55]; Gromov–Witten invariants of $\mathbb{CP}^1$ [42, 89] and toric Calabi–Yau threefolds [16, 51, 56]; coloured HOMFLY-PT polynomials of torus knots [38]; and the asymptotics of the coloured Jones polynomials of knots [9, 30].

Topological recursion persists as a theme throughout this thesis, playing a key role in results described in Chapters 5, 6 and 8. The present chapter will serve the purpose of introducing the original Chekhov–Eynard–Orantin topological recursion (hereafter referred to as CEO topological recursion) that will form the basis of all topological recursion discussion throughout this thesis.

In Section 2.2.1, I define the necessary components of topological recursion: the input data, base cases, and the recursion itself. Since topological recursion was first introduced, a number of variations and generalisations have been defined; Section 2.2.2 provides a brief description of some that have appeared in the literature. Section 2.3 works through two hallmark examples from the history of topological recursion: the Airy curve (Section 2.3.1), and the ribbon graph spectral curve (Section 2.3.2). Section 2.3 is aimed at the topological recursion novice; while the seasoned topological recurser may be all-too-familiar with these examples, those readers looking for a gentle introduction to topological recursion may enjoy working through this section for valuable hands-on experience.

2.2 Preliminaries

2.2.1 Definitions

At heart, topological recursion is just a recursion; it is a tool that can be used to inductively calculate numerical information, and the values it produces are determined by the input chosen. The "miracle" of topological recursion lies in the variety and breadth of the problems that it has been proven to govern. Depending on the input data used, topological recursion may generate information from such diverse problems as map enumeration, Gromov–Witten invariants of toric Calabi–Yau threefolds, and asymptotics of coloured Jones polynomials of knots (conjecturally); these are just three examples of many. While in this section I provide the full definition of the so-called Chekhov–Eynard–Orantin topological recursion, Section 2.3 is intended to be more illustrative in terms of gaining an understanding of how to calculate with topological recursion.

Topological recursion takes as input a spectral curve and produces a family of meromorphic multidifferentials $\omega_{g,n}$ for all integers $g \geqslant 0$ and $n \geqslant 1$. I will refer to these multidifferentials $\omega_{g,n}$ as *correlation differentials.* More precisely, $\omega_{g,n}$ is a meromorphic section of the line bundle $\pi_1(T^*\mathcal{C}) \otimes \pi_2(T^*\mathcal{C}) \otimes \cdots \otimes \pi_n(T^*\mathcal{C})$ over $\mathcal{C}^n$, where the Riemann surface $\mathcal{C}$ is part of the initial data and $\pi_i \colon \mathcal{C}^n \to \mathcal{C}$ is projection onto the ith factor. One can define topological recursion explicitly as follows.

Input. A *spectral curve* $(\mathcal{C}, x, y, T)$ consists of a compact Riemann surface $\mathcal{C}$, two meromorphic functions $x, y \colon \mathcal{C} \to \mathbb{CP}^1$, and a Torelli marking T on $\mathcal{C}$; that is, a choice of a symplectic basis of the first homology group $H_1(\mathcal{C}; \mathbb{Z})$. It is required that all zeros of $\mathrm{d}x$ are simple and disjoint from the zeros and poles of $\mathrm{d}y$. We refer to these zeros of $\mathrm{d}x$ as *branch points.*

Base cases. The base cases are $\omega_{0,1}(z_1) := y(z_1)\,\mathrm{d}x(z_1)$ and $\omega_{0,2}(z_1, z_2)$, where $z_1, z_2 \in \mathcal{C}$. Here, $\omega_{0,2}(z_1, z_2)$ is the unique bidifferential on $\mathcal{C} \times \mathcal{C}$ that satisfies the following: symmetric in its arguments; has double poles along the diagonal $z_1 = z_2$ and is holomorphic away from the diagonal; and is normalised on the $\mathcal{A}$-cycles of the Torelli marking. That is, it has the form

$$\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2} + [\text{symmetric, holomorphic bidifferential}]$$

and satisfies

$$\oint_{\mathcal{A}_i} \omega_{0,2}(z_1, z_2) = 0,$$

for $i = 1, 2, \ldots, \text{genus}(\mathcal{C})$.

Recursion. For all $(g, n) \neq (0, 1), (0, 2)$, define the multidifferentials $\omega_{g,n}$ recursively by the equation

$$\omega_{g,n}(z_1, \vec{z}_S) = \sum_{\alpha} \operatorname*{Res}_{z=\alpha} K_\alpha(z_1, z) \Bigg[\omega_{g-1,n+1}(z, \sigma_\alpha(z), \vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\circ} \omega_{g_1,|I|+1}(z, \vec{z}_I)\, \omega_{g_2,|J|+1}(\sigma_\alpha(z), \vec{z}_J) \Bigg]. \tag{2.1}$$

Here $S = \{2, 3, \ldots, n\}$, and for $I = \{i_1, \ldots, i_k\}$, the shorthand notation $\vec{z}_I$ denotes $z_{i_1}, \ldots, z_{i_k}$. The outer summation is over the zeros α of $\mathrm{d}x$, while the $\circ$ superscript on the inner summation denotes that we exclude all terms containing $\omega_{0,1}$. The constraint that the zeros of $\mathrm{d}x$ are simple implies that for each zero α of $\mathrm{d}x$ there exists a unique meromorphic function $z \mapsto \sigma_\alpha(z)$ such that $x(\sigma_\alpha(z)) = x(z)$ for all z in a neighbourhood of α but $\sigma_\alpha(z) \neq z$. Finally, for each branch point α, the kernel $K_\alpha(z_1, z)$ is defined by

$$K_\alpha(z_1, z) = \frac{\int_o^z \omega_{0,2}(z_1, \cdot\,)}{[y(z) - y(\sigma_\alpha(z))]\ \mathrm{d}x(z)}.$$

It turns out that the topological recursion is not sensitive to the choice of basepoint o nor the path of integration on the spectral curve [52].

Observe that the equation given by (2.1) is indeed a recursion, on the negative Euler characteristic $2g-2+n$.

I make a number of conventional remarks. First, note that the sign convention used here differs to that of Eynard and Orantin [52]; here I have omitted the negative signs in their definitions of $\omega_{0,1}$ and $K_\alpha(z_1, z)$. This is just convention; over time it has become increasingly popular in the literature to omit these signs.

Second, throughout this chapter, the Riemann surface $\mathcal{C}$ underlying the spectral curve will be taken to be the Riemann sphere $\mathbb{CP}^1$, in which case the Torelli marking is trivial; and indeed there are no non-zero holomorphic differentials on the Riemann sphere and this in turn forces $\omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}$. Hence, for this chapter, the Torelli marking will be omitted from the spectral curve input data and $\omega_{0,2}$ will be taken to be the canonical bidifferential satisfying the conditions given above; that is, $\omega_{0,2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}$.

Third, while $\omega_{g,n}$ is technically defined to be an nth tensor product of forms, the tensor products will be omitted for convenience. That is, $\mathrm{d}z_1 \otimes \cdots \otimes \mathrm{d}z_n$ will instead be written in the shorthand $\mathrm{d}z_1 \cdots \mathrm{d}z_n$.

Topological recursion admits a number of striking features. It is remarkable that, despite the fact that the recursion given in equation (2.1) is inherently asymmetrical with respect to z_1 versus $z_2, \ldots, z_n$, the correlation differentials $\omega_{g,n}(z_1, \ldots, z_n)$ are symmetric in their arguments [52]. Further, $\omega_{g,n}$ possesses a certain pole structure. Namely, that for $2g-2+n > 0$ the correlation differential $\omega_{g,n}$ satisfies the following: for all branch points α, $\omega_{g,n}(z_1, \ldots, z_n) + \omega_{g,n}(\sigma_\alpha(z_1), z_2, \ldots, z_n)$ is holomorphic at $z_1 = \alpha$; and that $\omega_{g,n}$ has a pole of order $6g - 4 + 2n$ at each of the branch points and no poles elsewhere [10, 46, 52]. Another remarkable feature of topological recursion is that the resulting correlation differentials can be expressed via intersection theory on moduli spaces of curves [46].

Note that the definition I give here is somewhat classical, in the sense that the Riemann surface $\mathcal{C}$ given as part of the spectral curve data is assumed to be compact. On the other hand, the main result of Chapter 5 proves that the enumeration of lattice points in $\overline{\mathcal{M}}_{g,n}$ is governed by a more recent generalisation known as *local topological recursion*, which notably does not require $\mathcal{C}$ to be compact. It is also worth noting that the topological recursion for single and double Hurwitz numbers uses x that is not meromorphic due to the appearance of the natural logarithm. However, one still has that $\mathrm{d}x(z)$ is meromorphic, and this weaker assumption is all that is required to apply the topological recursion. See Section 2.2.2 below or Chapter 5 for a more thorough discourse on local topological recursion, and see Chapter 6 for the precise statement of topological recursion in the case of double Hurwitz numbers.

2.2.2 Generalisations and variations

Since its introduction into the literature, topological recursion has been generalised, essentially, to encompass a wider range of problems and hence widen the scope of its applicability. Here I give a brief description of three notable such generalisations: global topological recursion; local topological recursion; and the Kontsevich–Soibelman formulation of topological recursion.

Global topological recursion. The definition of topological recursion given above—which aligns with the original CEO topological recursion—imposes the condition that the zeros of $\mathrm{d}x$ are simple, however it has been proven by Bouchard, Hutchinson, Loliencar, Meiers and Rupert [15] and Bouchard and Eynard [13] that topological recursion can be naturally generalised to include spectral curves that do not satisfy this condition. The work of Bouchard, Hutchinson, Loliencar, Meiers and Rupert [15] defined a generalisation of topological recursion that included spectral curves with higher order branching. Bouchard and Eynard [13] then proved that one can actually consider all branch points of a spectral curve as a branch point of order d, where d is the degree of x. They then used this to give a so-called "global topological recursion", which is defined globally, rather than locally around the branch points. This global topological recursion then allows one to, for example, take the limit as two or more branch points approach each other.

Local topological recursion. It was observed that the definition of CEO topological recursion depends only on local information around each of the branch points of the spectral curve; that is, the CEO topological recursion does not use the information of the global underlying Riemann surface $\mathcal{C}$ of the spectral curve. This led to the definition of "local topological recursion" [42] where, instead of including a compact Riemann surface $\mathcal{C}$ as part of the spectral curve input data, one instead defines local neighbourhoods D_i with canonical coordinates around each of the N branch points and the input data becomes the disjoint union $D_1 \sqcup \cdots \sqcup D_N$. In this case, given that there is no compact Riemann surface underlying the spectral curve, and hence no Torelli marking, it is necessary to include $\omega_{0,2}$ as part of the spectral curve input data. See the work of Dunin-Barkowski, Orantin, Shadrin and Spitz [42] for a detailed and precise definition of local topological recursion.

Kontsevich–Soibelman topological recursion. The reformulation of topological recursion due to Kontsevich and Soibelman [76] takes a more algebraic approach. It treats the partition function as the central object and defines the input data to be a *quantum Airy structure*, namely, a sequence of at-most quadratic differential operators that form a subalgebra of the overarching space of formal differential operators acting on some finite-dimensional $\mathbb{C}$-vector space. This approach emphasizes the role of the Virasoro algebra, as well as other W-algebras, and, by the nature of its definition, extends the existing topological recursion framework to include a broader variety enumerative problems. More explicitly, quantum Airy structures have underlying abstract Lie algebras associated to them. The abstract Lie algebras coming from CEO topological recursion are formed by taking a copy of the Virasoro algebra for each branch point. Quantum Airy structures also capture not only the local and global topological recursions, but their definition allows for the input data to be even more general than this.

2.3 Examples

2.3.1 Airy spectral curve

Input data. Define the *Airy spectral curve* to be $(\mathbb{CP}^1, x, y)$ with

$$x(z) = \frac{1}{2}\, z^2, \qquad \text{and} \qquad y(z) = z. \tag{2.2}$$

Here, $\omega_{0,2}$ is the canonical bilinear differential, $\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}$.

It was proven by Eynard [47] that the correlation differentials resulting from applying topological recursion to the Airy spectral curve (2.2) store ψ-class intersection numbers on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. This is the content of the following theorem, which is a special case of Theorem 1.1 in the work of Eynard [47]. (One can obtain Theorem 2.3.1 below from Theorem 1.1 in [47] by setting $\tilde{t}_k = 0$ for all $k \geqslant 1$.)

Theorem 2.3.1 (Eynard [47]). *For (g, n) satisfying $2g - 2 + n > 0$, the correlation differentials resulting from applying topological recursion to the spectral curve $(\mathbb{CP}^1, x, y)$ defined in equation (2.2) satisfy*

$$\omega_{g,n} = \frac{(-1)^n}{2^{3g-3+n}} \sum_{d_1+\cdots+d_n=3g-3+n} \prod_{i=1}^{n} \frac{(2d_i+1)!}{d_i!} \frac{\mathrm{d}z_i}{z_i^{2d_i+2}} \left[\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \right]. \tag{2.3}$$

Here, $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is the first Chern class of the cotangent bundle to the ith marked point, and the integral denotes taking the cup product $\psi_1^{d_1} \cdots \psi_n^{d_n} \in H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ and pairing it with the fundamental class $[\overline{\mathcal{M}}_{g,n}] \in H_*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$; see the book of Harris and Morrison [68] for a precise definition of such intersection numbers.

To apply topological recursion to this spectral curve, we first calculate: the branch points; that is, the points $z \in \mathbb{CP}^1$ that satisfy $\mathrm{d}x(z) = 0$; the involution σ at each of the branch points; and the kernel $K(z_1, z)$.

Branch points. The branch points of the spectral curve are the points satisfying $\mathrm{d}x(z) = 0$:

$$\mathrm{d}x(z) = 0 \qquad \Rightarrow \qquad z\,\mathrm{d}z = 0 \qquad \Rightarrow \qquad z = 0.$$

Involutions. The involution σ at the branch point $z = 0$ is the unique non-identity meromorphic function such that $x(\sigma(z)) = x(z)$ for all $z \in \mathbb{CP}^1$ in a neighbourhood of $z = 0$. Here, the meromorphic function $\sigma(z) = -z$ satisfies these conditions.

Recursion kernel. The recursion kernel can be taken to be

$$K(z_1, z) = \frac{\int_{\infty}^{z} \omega_{0,2}(z_1, \cdot\,)}{[y(z) - y(\sigma(z))]\,\mathrm{d}x(z)} = \frac{\int_{\infty}^{z} \frac{\mathrm{d}z_1\,\mathrm{d}t}{(z_1-t)^2}}{[z-(-z))]\,z\,\mathrm{d}z} = \frac{1}{2z^2\,(z_1 - z)} \frac{\mathrm{d}z_1}{\mathrm{d}z}.$$

Base cases: $2g - 2 + n = -1$ and $2g - 2 + n = 0$. The base cases $\omega_{0,1}$ and $\omega_{0,2}$ are given by

$$\omega_{0,1}(z_1) = y(z_1)\,\mathrm{d}x(z_1) = z_1^2\,\mathrm{d}z_1, \qquad \text{and} \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}.$$

We are now well-prepared to calculate $\omega_{g,n}$ for the first non-trivial case, when $2g - 2 + n = 1$; namely, $(g, n) = (0, 3)$ and $(g, n) = (1, 1)$.

The case $2g - 2 + n = 1$.

- $(g, n) = (1, 1)$

$$\begin{aligned}
\frac{\omega_{1,1}(z_1)}{\mathrm{d}z_1} &= \operatorname*{Res}_{z=0} \frac{K(z_1, z)}{\mathrm{d}z_1}\,\omega_{0,2}(z, \sigma(z)) = \operatorname*{Res}_{z=0} \frac{1}{2z^2\,(z_1 - z)} \frac{1}{\mathrm{d}z} \frac{\mathrm{d}z\,\mathrm{d}(-z)}{(z - (-z))^2} \\
&= -\operatorname*{Res}_{z=0} \frac{1}{8z^4} \frac{1}{(z_1 - z)}\,\mathrm{d}z = -\operatorname*{Res}_{z=0} \frac{1}{8z^4} \frac{1}{z_1} \frac{1}{1 - \frac{z}{z_1}}\,\mathrm{d}z \\
&= -\operatorname*{Res}_{z=0} \frac{1}{8z^4} \frac{1}{z_1} \left(1 + \frac{z}{z_1} + \frac{z^2}{z_1^2} + \frac{z^3}{z_1^3} + \frac{z^4}{z_1^4} + \cdots \right) \mathrm{d}z \\
&= -\frac{1}{8z_1^4}
\end{aligned}$$

This residue can be calculated by a computer (using, for example, SageMath); while the computer will be the main tool for such calculations subsequently, the above calculation is simple enough that one is able to do it by hand, and doing so can be illustrative.

- $(g, n) = (0, 3)$

$$\begin{aligned}\frac{\omega_{0,3}(z_1, z_2, z_3)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3} &= \operatorname*{Res}_{z=0} \frac{K(z_1, z)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3} \left[\omega_{0,2}(z, z_2)\,\omega_{0,2}(\sigma(z), z_3) + \omega_{0,2}(z, z_3)\,\omega_{0,2}(\sigma(z), z_2)\right] \\ &= \operatorname*{Res}_{z=0} \frac{1}{2z^2(z_1 - z)} \frac{1}{\mathrm{d}z\,\mathrm{d}z_2\,\mathrm{d}z_3} \left[\frac{\mathrm{d}z\,\mathrm{d}z_2}{(z - z_2)^2} \frac{\mathrm{d}(-z)\,\mathrm{d}z_3}{(-z - z_3)^2} + \frac{\mathrm{d}z\,\mathrm{d}z_3}{(z - z_3)^2} \frac{\mathrm{d}(-z)\,\mathrm{d}z_2}{(-z - z_2)^2}\right] \\ &= -\frac{1}{z_1^2\,z_2^2\,z_3^2}\end{aligned}$$

The case $2g - 2 + n = 2$.

- $(g, n) = (1, 2)$

$$\begin{aligned}\frac{\omega_{1,2}(z_1, z_2)}{\mathrm{d}z_1\,\mathrm{d}z_2} &= \operatorname*{Res}_{z=0} \frac{K(z_1, z)}{\mathrm{d}z_1\,\mathrm{d}z_2} \left[\omega_{0,3}(z, \sigma(z), z_2) + \omega_{0,2}(z, z_2)\,\omega_{1,1}(\sigma(z)) + \omega_{1,1}(z)\,\omega_{0,2}(\sigma(z), z_2)\right] \\ &= \operatorname*{Res}_{z=0} \frac{1}{2z^2(z_1 - z)} \frac{1}{\mathrm{d}z\,\mathrm{d}z_2} \left[-\frac{\mathrm{d}z\,\mathrm{d}(-z)\,\mathrm{d}z_2}{z^2\,(-z)^2\,z_2^2} - \frac{\mathrm{d}z\,\mathrm{d}z_2}{(z - z_2)^2} \frac{\mathrm{d}(-z)}{8(-z)^4} - \frac{\mathrm{d}z}{8z^4} \frac{\mathrm{d}(-z)\,\mathrm{d}z_2}{(-z - z_2)^2}\right] \\ &= \frac{5}{8} \frac{1}{z_1^2\,z_2^6} + \frac{3}{8} \frac{1}{z_1^4\,z_2^4} + \frac{5}{8} \frac{1}{z_1^6\,z_2^2}\end{aligned}$$

- $(g, n) = (0, 4)$

$$\begin{aligned}&\frac{\omega_{0,4}(z_1, z_2, z_3, z_4)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4} \\ &\quad = \operatorname*{Res}_{z=0} \frac{K(z_1, z)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4} \sum_{i=2}^{4} \left[\omega_{0,2}(z, z_i)\,\omega_{0,3}(\sigma(z), \vec{z}_{S\setminus\{i\}}) + \omega_{0,3}(z, \vec{z}_{S\setminus\{i\}})\,\omega_{0,2}(\sigma(z), z_i)\right] \\ &\quad = \frac{3}{z_1^2\,z_2^2\,z_3^2\,z_4^2} \left(\frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{1}{z_3^2} + \frac{1}{z_4^2}\right)\end{aligned}$$

Extracting coefficients. One can use the correlation differentials $\omega_{g,n}$ to calculate all intersection numbers of ψ-classes on $\overline{\mathcal{M}}_{g,n}$. In this case, extracting coefficients from $\omega_{g,n}$ when expanded in a power series about $z_i = \infty$ for $i \in \{1, 2, \ldots, n\}$ yields these intersection numbers. Specifically, use (2.3), extract a coefficient by calculating a residue, then rearrange to obtain

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} = 2^{3g-3+n} (-1)^n \prod_{i=1}^{n} \frac{d_i!}{(2d_i + 1)!} \operatorname*{Res}_{z_1=\infty} \cdots \operatorname*{Res}_{z_n=\infty} \omega_{g,n} \prod_{i=1}^{n} z_i^{2d_i+1}.$$

Using this to calculate intersection numbers of ψ-classes leads to the following data.

- $(g, n) = (1, 1)$

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1^{d} = \begin{cases} \frac{1}{24}, & \text{if } d = 1, \\ 0, & \text{otherwise.} \end{cases}$$

- $(g, n) = (0, 3)$

$$\int_{\overline{\mathcal{M}}_{0,3}} \psi_1^{d_1} \psi_2^{d_2} \psi_3^{d_3} = \begin{cases} 1, & \text{if } d_1 = d_2 = d_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

- $(g, n) = (1, 2)$

$$\int_{\overline{\mathcal{M}}_{1,2}} \psi_1^{d_1} \psi_2^{d_2} = \begin{cases} \frac{1}{24}, & \text{if } d_1 = d_2 = 1, \\ \frac{1}{24}, & \text{if } d_1 = 2, d_2 = 0 \text{ or } d_1 = 0, d_2 = 2, \\ 0, & \text{otherwise.} \end{cases}$$

- $(g, n) = (0, 4)$

$$\int_{\overline{\mathcal{M}}_{0,4}} \psi_1^{d_1} \psi_2^{d_2} \psi_3^{d_3} \psi_4^{d_4} = \begin{cases} 1, & \text{if } d_1 + d_2 + d_3 + d_4 = 1, \\ 0, & \text{otherwise.} \end{cases}$$

These values can be (and have been) verified using the base cases $\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$ and $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1^1 = \frac{1}{24}$, and the string and dilaton equations [68].

2.3.2 Ribbon graph spectral curve

Input data. Define the *ribbon graph spectral curve* to be $(\mathbb{CP}^1, x, y)$ with

$$x(z) = z + \frac{1}{z}, \qquad \text{and} \qquad y(z) = z. \tag{2.4}$$

Here, $\omega_{0,2}$ is the canonical bilinear differential, $\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{(z_1 - z_2)^2}$.

It was proven by Eynard and Orantin [54] that the correlation differentials resulting from applying topological recursion to the ribbon graph spectral curve (2.4) store the enumerations of ribbon graphs. This is the content of the following theorem, which is a special case of Theorem 7.3 in the work of Eynard and Orantin [54]. (One can obtain Theorem 2.3.2 below from Theorem 7.3 in [54], this time by setting $t_k = 0$ for all $k \geqslant 1$.)

Theorem 2.3.2 (Eynard and Orantin [54]). *For $(g, n) \neq (0, 2)$, the correlation differentials resulting from applying topological recursion to the spectral curve $(\mathbb{CP}^1, x, y)$ defined in equation (2.4) satisfy*

$$\omega_{g,n} = \mathrm{d}_1 \cdots \mathrm{d}_n \sum_{\mu_1, \ldots, \mu_n \geqslant 1} \mathrm{R}_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} x_i^{-\mu_i}. \tag{2.5}$$

Here, x_i is a shorthand notation for $x(z_i)$, d_i denotes applying the exterior derivative in the ith variable, and $\mathrm{R}_{g,n}(\mu_1, \ldots, \mu_n)$ is the weighted enumeration of ribbon graphs of type (g, n) where the degree of boundary face i is μ_i. See Definition 3.2.10 for a precise definition of $\mathrm{R}_{g,n}$.

While the theorem excludes $(g, n) = (0, 2)$, one can "correct" $\omega_{0,2} = \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{(z_1 - z_2)^2}$ to obtain the generating function for the $(0, 2)$-enumeration in the following way:

$$\omega_{0,2} = \frac{\mathrm{d}x_1 \, \mathrm{d}x_2}{(x_1 - x_2)^2} + \mathrm{d}_1 \, \mathrm{d}_2 \sum_{\mu_1, \mu_2 \geqslant 1} \mathrm{R}_{0,2}(\mu_1, \mu_2) \, x_1^{-\mu_1} \, x_2^{-\mu_2}.$$

To apply topological recursion to this spectral curve, we first calculate: the branch points; that is, the points $z \in \mathbb{CP}^1$ that satisfy $\mathrm{d}x(z) = 0$; the involution σ at each of the branch points; and the kernel $K(z_1, z)$.

Branch points. The branch points of the spectral curve are the points satisfying $\mathrm{d}x(z) = 0$:

$$\mathrm{d}x(z) = 0 \qquad \Rightarrow \qquad \left(1 - \frac{1}{z^2}\right) \mathrm{d}z = 0 \qquad \Rightarrow \qquad z = \pm 1.$$

Involutions. The involutions $\sigma_{\pm 1}$ at the branch points $z = \pm 1$ are the unique non-identity meromorphic functions such that $x(\sigma_{\pm 1}(z)) = x(z)$ for all $z \in \mathbb{CP}^1$ in neighbourhoods of $z = \pm 1$. Here, the meromorphic function $\sigma(z) = \frac{1}{z}$ satisfies these conditions for both branch points; so $\sigma(z) = \sigma_{+1}(z) = \sigma_{-1}(z) = \frac{1}{z}$.

Recursion kernel. The recursion kernel can be taken to be

$$K(z_1, z) = \frac{\int_{\infty}^{z} \omega_{0,2}(z_1, \cdot\,)}{[y(z) - y(\sigma(z))] \, \mathrm{d}x(z)} = \frac{\int_{\infty}^{z} \frac{\mathrm{d}z_1 \, \mathrm{d}t}{(z_1 - t)^2}}{\left[z - \frac{1}{z}\right] \left(1 - \frac{1}{z^2}\right) \mathrm{d}z} = \frac{1}{(z_1 - z)} \frac{\mathrm{d}z_1}{\mathrm{d}z} \frac{z^3}{(1 - z^2)^2}.$$

Base cases: $2g - 2 + n = -1$ **and** $2g - 2 + n = 0$. The base cases $\omega_{0,1}$ and $\omega_{0,2}$ are given by

$$\omega_{0,1}(z_1) = y(z_1) \, \mathrm{d}x(z_1) = \left(z_1 - \frac{1}{z_1}\right) \mathrm{d}z_1, \qquad \text{and} \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

The case $2g-2+n=1$.

- $(g,n)=(1,1)$

$$\frac{\omega_{1,1}(z_1)}{\mathrm{d}z_1} = \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{K(z_1,z)}{\mathrm{d}z_1}\,\omega_{0,2}(z,\sigma(z)) = \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{1}{(z_1-z)}\frac{1}{\mathrm{d}z}\frac{z^3}{(1-z^2)^2}\frac{\mathrm{d}z\,\mathrm{d}\big(\frac{1}{z}\big)}{(z-\frac{1}{z})^2}$$
$$= -\frac{z_1^3}{(1-z_1^2)^4}$$

- $(g,n)=(0,3)$

$$\frac{\omega_{0,3}(z_1,z_2,z_3)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3}$$
$$= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{K(z_1,z)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3}\,[\omega_{0,2}(z,z_2)\,\omega_{0,2}(\sigma(z),z_3)+\omega_{0,2}(z,z_3)\,\omega_{0,2}(\sigma(z),z_2)]$$
$$= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{1}{(z_1-z)}\frac{z^3}{(1-z^2)^2}\frac{1}{\mathrm{d}z\,\mathrm{d}z_2\,\mathrm{d}z_3}\left[\frac{\mathrm{d}z\,\mathrm{d}z_2}{(z-z_2)^2}\frac{\mathrm{d}\big(\frac{1}{z}\big)\,\mathrm{d}z_3}{(\frac{1}{z}-z_3)^2}+\frac{\mathrm{d}z\,\mathrm{d}z_3}{(z-z_3)^2}\frac{\mathrm{d}\big(\frac{1}{z}\big)\,\mathrm{d}z_2}{(\frac{1}{z}-z_2)^2}\right]$$
$$= -\frac{1}{2}\left[\prod_{i=1}^{3}\frac{1}{(1-z_i)^2}-\prod_{i=1}^{3}\frac{1}{(1+z_i)^2}\right]$$

The case $2g-2+n=2$.

- $(g,n)=(1,2)$

$$\frac{\omega_{1,2}(z_1,z_2)}{\mathrm{d}z_1\,\mathrm{d}z_2}$$
$$= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{K(z_1,z)}{\mathrm{d}z_1\,\mathrm{d}z_2}\,[\omega_{0,3}(z,\sigma(z),z_2)+\omega_{0,2}(z,z_2)\,\omega_{1,1}(\sigma(z))+\omega_{1,1}(z)\,\omega_{0,2}(\sigma(z),z_2)]$$
$$= \frac{5}{32\,(1-z_1)^2(1-z_2)^2}\sum_{i=1}^{2}\left(\frac{z_i^2}{(1-z_i)^4}-\frac{z_i}{4(1-z_i)^2}\right)+\frac{3z_1z_2}{32\,(1-z_1)^4(1-z_2)^4}$$
$$+\frac{5}{32\,(1+z_1)^2(1+z_2)^2}\sum_{i=1}^{2}\left(\frac{z_i^2}{(1+z_i)^4}+\frac{z_i}{4(1+z_i)^2}\right)+\frac{3z_1z_2}{32\,(1+z_1)^4(1+z_2)^4}$$
$$+\frac{z_1z_2}{8(1-z_1^2)^2(1-z_2^2)^2}$$

- $(g,n)=(0,4)$

$$\frac{\omega_{0,4}(z_1,z_2,z_3,z_4)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}$$
$$= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{K(z_1,z)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}\left[\sum_{i=2}^{4}\omega_{0,2}(z,z_i)\,\omega_{0,3}(\sigma(z),\vec{z}_{S\setminus\{i\}})+\omega_{0,3}(z,\vec{z}_{S\setminus\{i\}})\,\omega_{0,2}(\sigma(z),z_i)\right]$$
$$= \left[\frac{3}{4}\prod_{i=1}^{3}\frac{1}{(1-z_i)^2}\sum_{j=1}^{3}\frac{z_j}{(1-z_j)^2}-\frac{3}{4}\prod_{i=1}^{3}\frac{1}{(1+z_i)^2}\sum_{j=1}^{3}\frac{z_j}{(1+z_j)^2}\right.$$
$$\left.+\frac{1}{2}\prod_{i=1}^{3}\frac{1}{(1-z_i^2)^2}\sum_{\{i,j,k,\ell\}=\{1,2,3,4\}} z_iz_j(1+z_k^2)(1+z_\ell^2)\right]$$

Extracting coefficients. Again, one can use the correlation differentials $\omega_{g,n}$ to calculate all enumerations of ribbon graphs. In this case, extracting coefficients from $\omega_{g,n}$ when expanded in a power series about $x_i=\infty$ for $i\in\{1,2,\dots,n\}$ yields the numbers $\mathrm{R}_{g,n}(\mu_1,\dots,\mu_n)$. Specifically, use equation (2.5), extract a coefficient by calculating a residue, then rearrange to obtain

$$\mathrm{R}_{g,n}(\mu_1,\ldots,\mu_n) = \operatorname*{Res}_{x_1=\infty} \cdots \operatorname*{Res}_{x_n=\infty} \omega_{g,n} \prod_{i=1}^{n} \frac{x_i^{\mu_i}}{-\mu_i}.$$

Recall that the general theory of topological recursion asserts that the correlation differential $\omega_{g,n}$ satisfies the property that α, $\omega_{g,n}(z_1,\ldots,z_n) + \omega_{g,n}(\sigma_\alpha(z_1), z_2,\ldots,z_n)$ is holomorphic at $z_1 = \alpha$, and only has poles at the branch points, each of degree $6g-4+2n$. One can use this known pole structure to deduce a quasi-polynomiality structure for the ribbon graph enumeration, although we do not pursue this here. This structure is analogous to the polynomiality for Hurwitz numbers.

Calculating the appropriate residues of $\omega_{g,n}$ in the cases above gives the data in the following tables. Note that $\mathrm{R}_{g,n}(\mu_1,\ldots,\mu_n) = 0$ if $|\mu| = \mu_1 + \cdots + \mu_n$ is odd, hence values for these cases have been omitted.

Data.

$n=1$		
(μ)	$\mathrm{R}_{0,1}(\mu)$	$\mathrm{R}_{1,1}(\mu)$
(2)	$\frac{1}{2}$	0
(4)	$\frac{1}{2}$	$\frac{1}{4}$
(6)	$\frac{5}{6}$	$\frac{5}{3}$
(8)	$\frac{7}{4}$	$\frac{35}{4}$
(10)	$\frac{21}{5}$	42
(12)	11	$\frac{385}{2}$
(14)	$\frac{429}{14}$	858
(16)	$\frac{715}{8}$	$\frac{15015}{4}$
(18)	$\frac{2431}{9}$	$\frac{48620}{3}$
(20)	$\frac{4199}{5}$	$\frac{138567}{2}$

$n=2$		
$(\vec{\mu})$	$\mathrm{R}_{0,2}(\vec{\mu})$	$\mathrm{R}_{1,2}(\vec{\mu})$
$(1,1)$	1	0
$(3,1)$	1	0
$(2,2)$	$\frac{1}{2}$	0
$(5,1)$	2	1
$(4,2)$	1	$\frac{1}{2}$
$(3,3)$	$\frac{4}{3}$	$\frac{1}{3}$
$(7,1)$	5	10
$(6,2)$	$\frac{5}{2}$	5
$(5,3)$	3	4
$(4,4)$	$\frac{9}{4}$	$\frac{15}{4}$

$n=3$	
$(\vec{\mu})$	$\mathrm{R}_{0,3}(\vec{\mu})$
$(2,1,1)$	1
$(4,1,1)$	3
$(3,2,1)$	2
$(2,2,2)$	1
$(5,2,1)$	6
$(4,3,1)$	6
$(4,2,2)$	3
$(3,3,2)$	4
$(5,4,1)$	18
$(5,3,2)$	12

$n=4$	
$(\vec{\mu})$	$\mathrm{R}_{0,4}(\vec{\mu})$
$(3,1,1,1)$	2
$(2,2,1,1)$	2
$(5,1,1,1)$	12
$(4,2,1,1)$	9
$(3,3,1,1)$	8
$(3,2,2,1)$	6
$(5,2,2,1)$	24
$(4,3,2,1)$	24
$(5,4,2,1)$	90
$(3,3,3,1)$	24

Chapter 3

Maps, fully simple maps, and ribbon graphs

3.1 Introduction

Loosely speaking, maps are obtained by gluing together polygons to create a surface. Maps and their combinatorics have been studied extensively since the pioneering work of Tutte [104]. The enumeration of fully simple maps, meanwhile, was defined only recently by Borot and Garcia-Failde [11]: a map is said to be fully simple if it satisfies further conditions restricting how boundary faces can interact with each other and themselves. (The overarching subset of maps may henceforth be referred to as ordinary maps to avoid confusion.)

Since the early studies of maps in the literature, a number of significant results have been proven. For example, the enumeration of ordinary maps satisfies a recursion known as the Tutte recursion; this was proven first in the case of planar maps by Tutte [105], then in higher genus by Walsh and Lehman [106]. And, the enumeration of maps is equivalent to a particular enumeration of triples of permutations, so maps can therefore be studied via this so-called permutation model [79].

The enumeration of ordinary maps is also known to be governed by the 1-Hermitian matrix model [18, 79]. The partition function for ordinary maps with no boundary faces Z^M is given by

$$Z^M = \int_{\mathcal{H}_N} \exp\left(N \operatorname{Tr}\left[V(M)\right]\right) \mathrm{d}M$$

where the integral is over the space $\mathcal{H}_N$ of $N \times N$ Hermitian matrices, $V(x) = \sum_{k\geqslant 1} \frac{t_k x^k}{k}$ is called the ***potential***, and $\mathrm{d}M$ is the Gaussian measure. For more on matrix models and maps, see [79] or [108].

The enumeration of ordinary maps was also a significant progenitor of topological recursion. It was the first enumeration shown to be governed by topological recursion, and the theory of topological recursion evolved from the abstraction of loop equations from the theory of matrix models in the particular context of the specific matrix model that governs the enumeration of maps, the 1-Hermitian matrix model mentioned above [25, 48, 52]. The enumeration of fully simple maps has also now been shown to be governed by topological recursion [7, 19]. For more details on maps and topological recursion, see the introduction in Chapter 7 on fully simple maps.

In this chapter, Section 3.2 synthesises relevant literature and provides a guided and thorough introduction to maps, ordinary and fully simple, and their enumerations. Section 3.2.1 gives the formal definition of both ordinary maps and fully simple maps as embeddings of graphs on surfaces satisfying certain conditions, aided by a number of illustrative examples. Section 3.2.2 defines the permutation model for both ordinary and fully simple maps, again illustrated with examples, and uses the permutation model to provide an alternate perspective on the automorphisms of a map. Section 3.2.3 briefly introduces a third viewpoint on maps as certain branched covers of $\mathbb{CP}^1$. Section 3.2.4 defines the enumerations of ordinary and fully simple maps $\mathrm{M}_{g,n}(\mu_1, \ldots, \mu_n)$ and $\mathrm{FS}_{g,n}(\mu_1, \ldots, \mu_n)$ respectively. Finally, Section 3.2.5 gives the well-known Tutte recursion for ordinary maps.

The purpose of Section 3.3 is to complement the results in Chapter 5. In Chapter 5, I describe joint work with Chaudhuri and Do which proves that a certain lattice point enumeration of the Deligne–Mumford compactification of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ is governed by local topological recursion. Section 3.3

describes the correspondence between lattice points of $\overline{\mathcal{M}}_{g,n}$ and a combinatorial object called a stable ribbon graph, establishing the connection with the enumeration defined in Chapter 5. Specifically, in Chapter 5, I define the lattice point enumeration as a count of branched covers of $\mathbb{CP}^1$ satisfying particular conditions; in Section 3.3 I will define the combinatorial object that is associated to this count and I will describe the sense in which this combinatorial object—the stable ribbon graph—enumerates lattice points in $\overline{\mathcal{M}}_{g,n}$.

3.2 Maps and fully simple maps

3.2.1 Maps

One can informally think of a map as a way to glue polygons to form a surface. A formal definition of a map that captures this intuition is as follows.

Definition 3.2.1 (Map). A *map* is a finite graph embedded in a compact oriented surface such that the complement of the graph on the surface is a disjoint union of topological disks, called *faces*.

A *half-edge*, or *oriented edge*, is an edge with a choice of orientation. A half-edge is *adjacent* to the face on its left and *incident* to the vertex to which it points. Choose n faces to be marked and label these $1, 2, \ldots, n$; these are called *boundary faces* while other faces are *internal faces*. The number of half-edges adjacent to a face is the *degree* of that face. If the underlying surface has genus g and n boundary faces, then the map is said to be of *type* (g, n).

Two maps are *isomorphic* if there exists an orientation-preserving homeomorphism of the underlying surface that maps all vertices, half-edges and faces of the first map bijectively to the second, preserving all incidences, adjacencies and labelled boundary faces.

A *ribbon graph* is a map without internal faces.

Example 3.2.2. In Figure 3.1 the diagram (1) is not a map — one component of the graph's complement is not homeomorphic to a disk. Diagrams (2)-(5) display maps of type $(1, 1)$, $(0, 2)$, $(0, 1)$, $(0, 1)$ and $(2, 2)$ respectively. Although diagrams (3)-(5) are drawn on the plane, we consider them as graphs on the sphere by compactifying the plane.

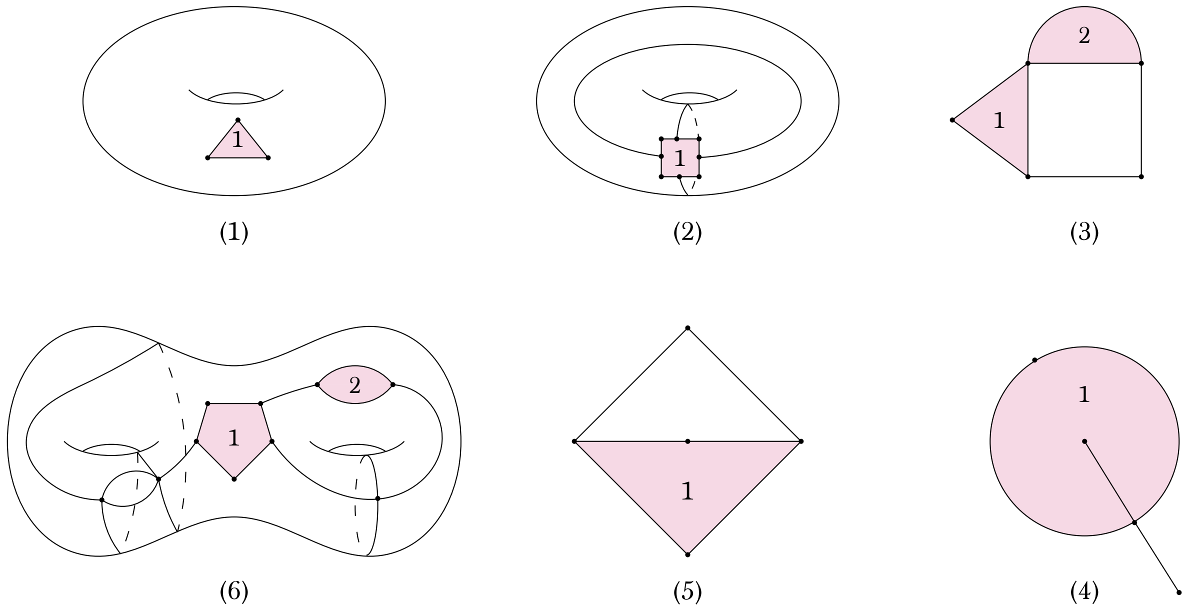

Figure 3.1: Clockwise from top-left the diagrams depict: (1) a graph embedded on a surface that is not a map; and (2-5) maps of type $(1, 1)$, $(0, 2)$, $(0, 1)$, $(0, 1)$ and $(2, 2)$ respectively.

Remark 3.2.3. Note that Definition 3.2.1 includes as a map the graph of a single vertex on the sphere, sometimes referred to as a *degenerate map*. It will be convenient to indeed consider this as a map, although it will need to be treated separately when discussing the permutation model below.

A sidebar on rooted maps. Instead of choosing n boundary faces, one can alternatively choose n half-edges adjacent to distinct faces and dictate that these are boundary faces; in essence, each boundary face has a distinguished adjacent half-edge. The corresponding objects are called *rooted maps*, defined below, and are the objects defined and used in related literature such as the work of Borot, Charbonnier, Do and Garcia-Failde [6] and Garcia-Failde [58]. The enumeration of rooted maps introduces a simple combinatorial factor when compared to the unrooted analogue.

Definition 3.2.4 (Rooted map). A *rooted map* is a map along with a tuple of distinct half-edges, called *roots* and depicted by arrows, such that no two are adjacent to the same face. The faces adjacent to the roots are the boundary faces.

Two rooted maps are isomorphic if there exists an orientation-preserving isomorphism of the underlying maps that preserves the tuple of roots.

Although this thesis will primarily discuss maps rather than rooted maps, it will be useful for both objects to be defined for maximum versatility.

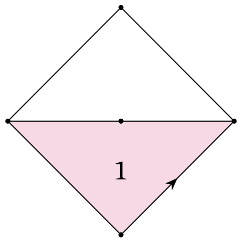

Figure 3.2: A rooted map, where the root is represented by an arrow.

Definition 3.2.1 allows for distinct boundary faces to intersect via edges or vertices, or for a boundary face to intersect with itself. In Figure 3.1, (3) has two distinct boundary faces that intersect at a common vertex, while (4) shows a map in which a boundary face intersects itself along an edge. Informally, a map is fully simple if it does not exhibit these types of behaviour. This is captured in the following definition.

Definition 3.2.5 (Fully simple map). A half-edge of a map is a *boundary edge* if it is adjacent to a boundary face. A map is *fully simple* if each vertex is incident to at most one boundary edge.

In instances of ambiguity, the class of all maps may be referred to as *ordinary maps* to distinguish from the subclass of fully simple maps.

A sidebar on simple maps. There is also a notion of a *simple map* in which boundary faces are not allowed to intersect themselves at vertices or along edges, but distinct boundary faces are allowed to intersect. A simple map can include the behaviour seen in (3) in Figure 3.1, but not the behaviour seen in (4). The notion of a simple map will not arise naturally in this thesis and their definition is merely included here for completeness.

In Figure 3.1, (2), (5) and (6) are examples of fully simple maps, (3) is simple but not fully simple, and (4) is not simple and hence not fully simple.

Thinking about fully simple maps informally as ordinary maps where the boundary faces are not self-adjacent may lead one to the conclusion that the map depicted in Figure 3.3 is not fully simple. However, referring to the definition of fully simple maps, Definition 3.2.5, we see that the two half-edges are boundary edges and the two vertices are incident to precisely one boundary edge each. Thus, the unique map consisting of one degree two boundary face and no internal faces is indeed fully simple.

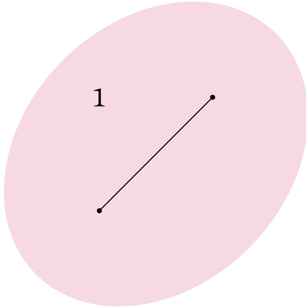

Figure 3.3: The slightly surprising fully simple map with one degree two boundary face and no internal faces.

A map is *connected* if the underlying surface is connected and *disconnected* otherwise. Note that we define the genus for a disconnected surface by taking the Euler characteristic to be additive over disjoint union. In particular, this implies that the genus of a disconnected surface can be negative. Although we ultimately strive to calculate enumerations for connected maps, for our analysis in Chapter 7 on fully simple maps it will be necessary to first consider the possibly disconnected enumeration, and thus here I introduce both the connected and possibly disconnected objects.

3.2.2 Permutation model

What additional information does one need to attach to a graph in order to define a map? Considering a neighbourhood of a vertex in a map, we see that the embedding imposes a cyclic ordering of the half-edges about the vertex, arising from the orientation of the underlying surface. For example, the two diagrams shown in Figure 3.4 depict the same graph but two distinct maps—the left map has two faces of degrees 7 and 3, while the right map has two faces both of degree 5. This notion of the cyclic ordering of edges about a vertex can be captured by a permutation of the half-edges, and this idea leads to the permutation model for maps. Although I aim to give a complete and accessible definition of the permutation model for maps, the interested reader may refer to work of Lando and Zvonkin [79] for an alternate introduction. One should note that the conventions here differ to those used in this reference.

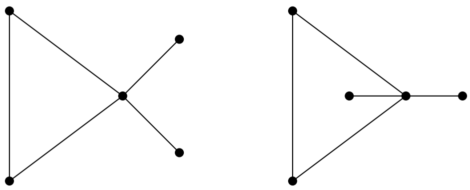

Figure 3.4: The two diagrams depict maps that are not isomorphic, yet their underlying graphs are isomorphic.

In the following, I will represent an edge as a two-way street, as in Figure 3.5. (Because I live in Australia) I will opt for the convention where we "drive on the left side of the road". More precisely, if you view edges from the point of view of a vertex, then the half-edges incident to that vertex are on the right; see Figure 3.6 to observe this perspective.

We then define a permutation on the set of half-edges of a map that rotates each half-edge anticlockwise about the vertex to which it is incident. This encodes the cyclic ordering of the half-edges about the vertices of a map; the choice of anticlockwise rotation is consistent with the conventional orientation of a surface.

This information, however, is not yet sufficient to recover a map. Currently, the cyclic ordering of edges about each vertex gives local information of how the half-edges are organised about each vertex, but no

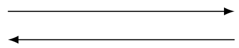

Figure 3.5: The two half-edges comprising a single edge viewed as a "two-way street" where convention dictates that we drive on the left.

global information of how the half-edges are glued to form a map. To define a map, it suffices to further define a permutation that prescribes how the half-edges are glued together to create edges. That is, define a permutation that swaps every pair of half-edges belonging to the same underlying edge. The information of this permutation (swapping half-edges) coupled with the previous (anticlockwise rotation of edges about a vertex) recover a unique map. As we will see, it will be convenient to furthermore define a third permutation that rotates half-edges anticlockwise about the face to which they are adjacent.

We can now introduce the permutation model for ordinary maps. The correspondence described will be between triples of permutations and labelled maps. A ***labelled map*** is a map where the half-edges are labelled with the integers $1, 2, \dots, d$, where d is the number of half-edges.

The information of a labelled ordinary map is encoded in a triple of permutations $(\sigma_0, \sigma_1, \sigma_2) \in S_d$ acting on the set of labelled half-edges in the following way:

- σ_0 rotates half-edges anticlockwise about the vertices to which they are incident;
- σ_1 is a fixed point free involution that swaps half-edges belonging to the same underlying edge; and
- σ_2 rotates half-edges anticlockwise about the faces to which they are adjacent.

It follows that $\sigma_0\sigma_1\sigma_2 = \mathrm{id}$, where I use the convention of multiplying permutations from right to left. Further, to choose the n boundary faces required in Definition 3.2.1 it suffices to choose a tuple B of n distinct cycles of σ_2. See Figure 3.6 for a depiction of the actions of σ_0, σ_1 and σ_2.

From now on, I will write $(\sigma_0, \sigma_1, \sigma_2, B)$; these will be the objects in bijection with labelled maps.

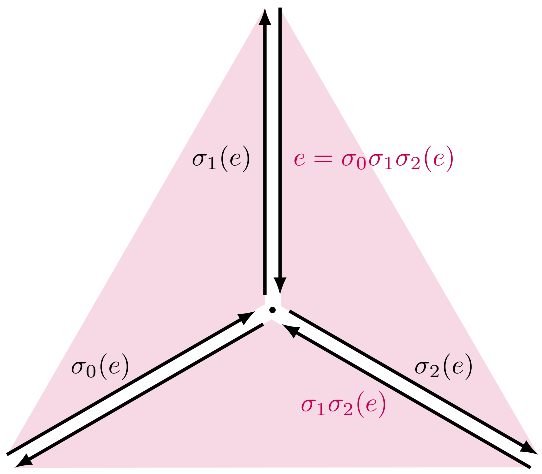

Figure 3.6: A local diagram of a vertex depicting the actions of σ_0, σ_1 and σ_2 on a half-edge e. Edge labels in pink show why one has $\sigma_0\sigma_1\sigma_2 = \mathrm{id}$.

From the way the permutation model was constructed, this automatically leads to the following bijection between the set of labelled maps and the set of tuples $(\sigma_0, \sigma_1, \sigma_2, B)$. And further, there is a correspondence between isomorphism classes of these objects, hence to state the equivalence fully we first need the notion of an isomorphism of triples of permutations. An ***isomorphism*** between $(\sigma_0, \sigma_1, \sigma_2, B)$ and $(\sigma_0', \sigma_1', \sigma_2', B')$ is a permutation $\phi \in S_d$ that satisfies $\sigma_i' = \phi\sigma_i\phi^{-1}$ for $i \in \{0, 1, 2\}$ and sends B to B'.

Proposition 3.2.6. *There is a one-to-one correspondence between labelled ordinary maps (excluding the map consisting of a single vertex on the sphere) and tuples $(\sigma_0, \sigma_1, \sigma_2, B)$, where $\sigma_0, \sigma_1, \sigma_2 \in S_d$, σ_1 is a fixed point free involution, $\sigma_0\sigma_1\sigma_2 = id$, and B is a tuple of distinct cycles in σ_2.*

Further, there is a one-to-one correspondence between isomorphism classes of ordinary maps (excluding the map consisting of a single vertex on the sphere) and isomorphism classes of triples of permutations $\sigma_0, \sigma_1, \sigma_2 \in S_d$ such that σ_1 is a fixed point free involution and $\sigma_0\sigma_1\sigma_2 = id$, along with a tuple B of distinct cycles in σ_2.

The subgroup generated by the triple, $\langle \sigma_0, \sigma_1, \sigma_2 \rangle \leqslant S_d$, acts transitively on $\{1, 2, \ldots, d\}$ if and only if the corresponding labelled ordinary map is connected.

I may refer to a tuple $(\sigma_0, \sigma_1, \sigma_2, B)$ as a *combinatorial map* in contrast to the object from Definition 3.2.1 which I may call a *topological map*.

The automorphism group of a labelled map depends only on the underlying map and not on any choice of labelling. This allows us to define the automorphism of a (topological) map as an automorphism of the associated combinatorial map. Thus, define an automorphism of a map to be an automorphism of a corresponding tuple $(\sigma_0, \sigma_1, \sigma_2, B)$; that is, a permutation $\phi \in S_d$ that satisfies $\sigma_i = \phi\sigma_i\phi^{-1}$ for $i \in \{0, 1, 2\}$ and sends B to B.

Example 3.2.7. Figure 3.7 shows a labelled map, and the corresponding triple of permutations is

$$\begin{aligned}\sigma_0 &= (1863)(2)(45)(7)\\ \sigma_1 &= (15)(23)(46)(78)\\ \sigma_2 &= (1234)(5678).\end{aligned}$$

And one can verify that

$$\sigma_1^{-1}\sigma_0^{-1} = (15)(23)(46)(78) \circ (1368)(45) = (1234)(5678) = \sigma_2,$$

or equivalently that $\sigma_0\sigma_1\sigma_2 = \text{id}$.

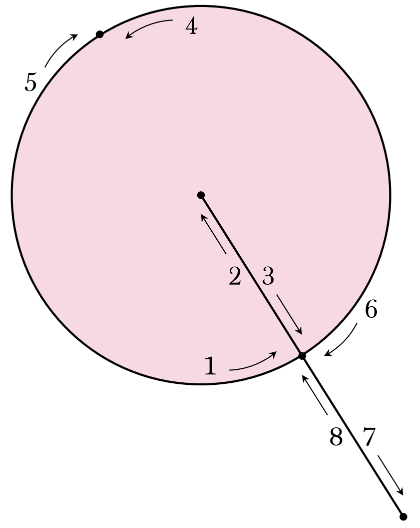

Figure 3.7: One way to represent a labelled map.

As observed above, from the information of the tuple $(\sigma_0, \sigma_1, \sigma_2, B)$, one can construct the labelled topological map. However, if one were to write down all $d!$ labellings of the same underlying topological map, not all of these would be unique combinatorial maps. The orbit-stabiliser theorem tells us that the ratio between the size of the isomorphism class of the tuple $(\sigma_0, \sigma_1, \sigma_2, B)$—denoted $|\text{orb}(\sigma_0, \sigma_1, \sigma_2)|$—and the total number of labellings, $d!$, is precisely the number of automorphisms of the combinatorial map. That is,

$$d! = |\text{Aut}(\sigma_0, \sigma_1, \sigma_2)| \cdot |\text{orb}(\sigma_0, \sigma_1, \sigma_2)|.$$

Example 3.2.8. Figure 3.8 shows all 4! labellings of the ordinary map with one degree four boundary face and no internal faces. Each labelled map has a "partner" which is represented by the same tuple $(\sigma_0, \sigma_1, \sigma_2, B)$, signifying a non-trivial automorphism between them. By the orbit-stabiliser theorem

$$24 = d! = |\mathrm{Aut}(\sigma_0, \sigma_1, \sigma_2)| \cdot |\mathrm{orb}(\sigma_0, \sigma_1, \sigma_2)| = 2 \cdot 12,$$

as expected. ("As expected" because we secretly knew that this map has two automorphisms, though usually one might use the orbit-stabiliser theorem to calculate the number of automorphisms of a topological map.)

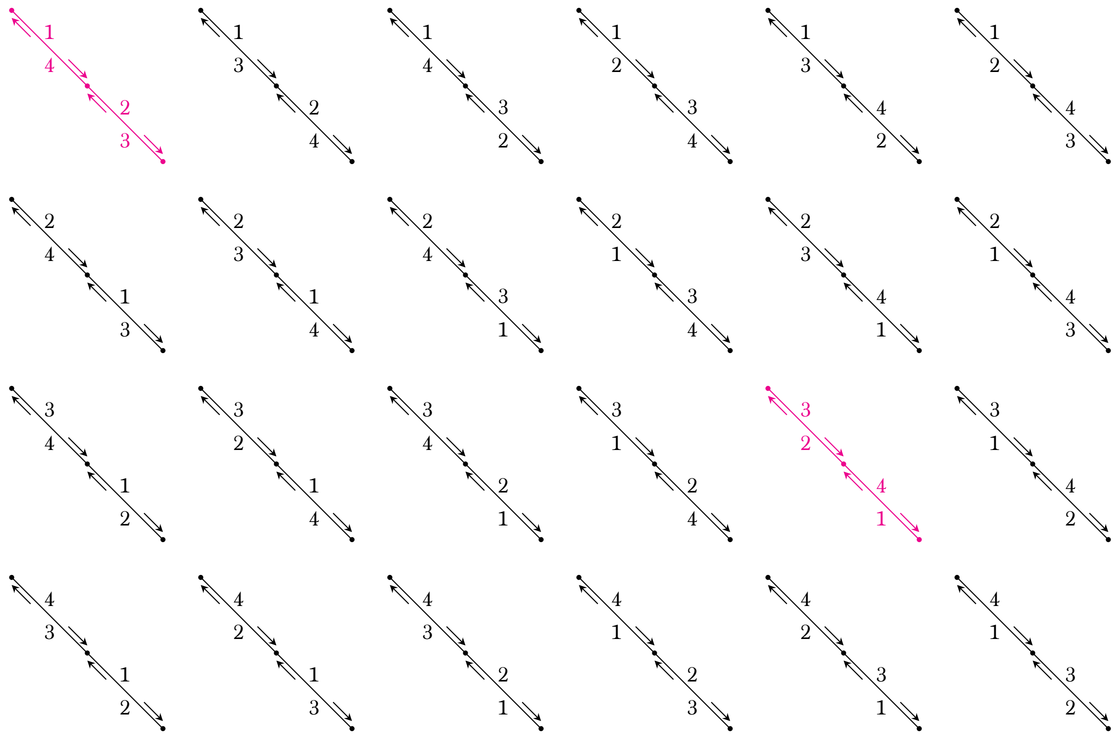

Figure 3.8: All possible relabellings of the ordinary map with one degree four boundary face and no internal faces. The two maps drawn in pink are represented by the same tuple $(\sigma_0, \sigma_1, \sigma_2, B)$, signifying a non-trivial automorphism between them.

The following example demonstrates an isomorphism of combinatorial maps.

Example 3.2.9. Figure 3.9 shows three labellings of the same topological map. The leftmost two diagrams represent the same labelled map $(\sigma_0, \sigma_1, \sigma_2, B)$, demonstrating a non-trivial automorphism $\phi \in S_d$ satisfying $\phi\sigma_i\phi^{-1} = \sigma_i$ for $i \in \{0, 1, 2\}$ and $\phi(B) = B$. The outer two labellings depict isomorphic combinatorial maps. The triples of permutations for the outer two maps are

$$\begin{aligned} \sigma_0 &= (1\,8)(2\,7\,10)(3\,9)(4\,12\,5)(6\,11) & \sigma_0' &= (1\,9)(2\,12\,10)(3\,11)(4\,6\,8)(5\,7) \\ \sigma_1 &= (1\,5)(2\,8)(3\,10)(4\,9)(6\,12)(7\,11) & \sigma_1' &= (1\,12)(2\,3)(4\,9)(5\,10)(6\,11)(7\,8) \\ \sigma_2 &= (1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11\,12) & \sigma_2' &= (1\,4\,7\,10)(2\,5\,8\,11)(3\,6\,9\,12) \\ B &= (5\,6\,7\,8) & B' &= (2\,5\,8\,11). \end{aligned}$$

The isomorphism is given by $\phi = (1\,3\,9)(2\,6\,5)(4\,12\,10)(7\,8\,11)$, and one can again verify that $\phi\sigma_i\phi^{-1} = \sigma_i'$ for $i \in \{0, 1, 2\}$ and $\phi(B) = B'$.

By considering the combinatorics of graph isomorphisms, one can reason that any labelling of this topological map has only two automorphisms, the identity and the one that corresponds to the automorphism between the leftmost two diagrams in Figure 3.9. To see this, consider the half-edge labelled 5 in the leftmost map. It is incident to a 3-valent vertex, and adjacent to the boundary face. Hence this half-edge must either stay where it is or be sent to the edge labelled 7, allowing us to deduce that the number of automorphisms is at most two. Yet, Figure 3.9 shows that both of these choices give rise to valid automorphisms. So the number of automorphisms is exactly two. Because an isomorphism between maps must preserve all adjacencies and incidences, the image of one half-edge determines the images of all half-edges, as long as the maps are connected.

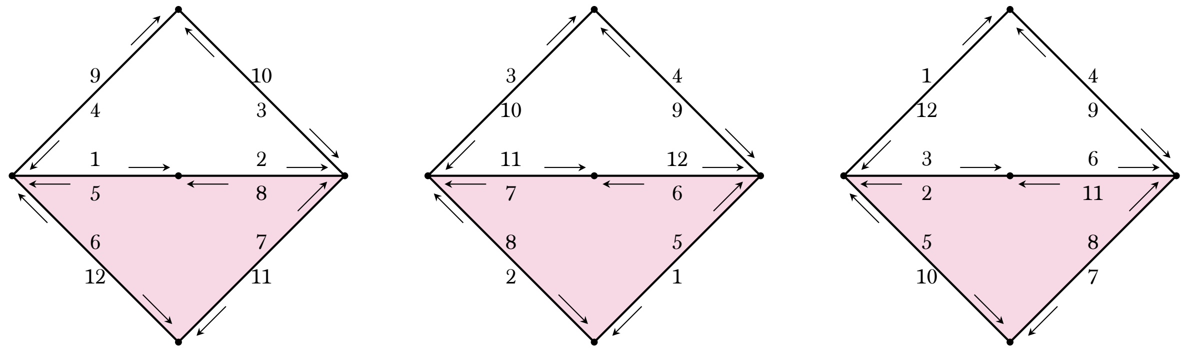

Figure 3.9: Three labellings of the same topological map. The two labellings on the left are the same labelled map, while the outer labellings represent isomorphic maps.

The condition for an ordinary map to be fully simple in the permutation model can be given as follows. For a combinatorial map $(\sigma_0, \sigma_1, \sigma_2, B)$, let $\mathbb{B}$ be the union of half-edges in B. The map is fully simple if and only if the elements of $\mathbb{B}$ lie in different σ_0-orbits.

Finally, I will briefly describe the one-to-one correspondence by outlining how to obtain a triple of permutations from a map and vice versa. (The forward direction was loosely and intuitively described by way of motivation at the start of this section, but I restate it here more precisely.) Begin with a map and label the half-edges. There is no canonical way to do this, but recall that the correspondence is between isomorphism classes of both maps and triples of permutations. Now one can write down the permutations σ_0, σ_1 and σ_2: the cycles of σ_0 are given by the anticlockwise rotation of half-edges about a vertex; σ_1 is a fixed point free involution associating a pair of half-edges to an underlying edge; and the cycles of σ_2 are given by the anticlockwise rotation of half-edges incident to a face. By the construction of the permutation model, it then follows that $\sigma_0\sigma_1\sigma_2 = \text{id}$ and one can indeed check that $\sigma_2 = \sigma_1^{-1}\sigma_0^{-1}$.

Conversely, begin with a triple of permutations $(\sigma_0, \sigma_1, \sigma_2)$ and construct a map as follows. Associate to each cycle of σ_2 of length k a polygon of degree k, where the sides of the polygon are half-edges labelled and oriented anticlockwise according to the cycle. Next glue half-edges together according to σ_1 such that the orientation of two half-edges being glued is always opposite. The result will be a surface where the embedded graph is given by the edges and vertices of the polygons. The cyclic ordering of the vertices will automatically be glued according to $\sigma_0 = \sigma_2^{-1}\sigma_1^{-1}$.

3.2.3 Maps as branched covers of the sphere

In addition to the permutation model, maps are also in natural bijection with certain branched covers of $\mathbb{CP}^1$. The bijection occurs in the following way. For an ordinary map of type (g, n) realised by permutations $(\sigma_0, \sigma_1, \sigma_2)$ one can associate a branched cover $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ from a genus g compact Riemann surface $\mathcal{C}$ with n marked points $p_1, \ldots, p_n$ satisfying the following conditions.

- The degree of f is equal to the sum of the degrees of all faces of the corresponding map, and is unramified over $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.
- The monodromy around 0, 1, and $\infty \in \mathbb{CP}^1$ is given by σ_0, σ_1, and σ_2 respectively.

For a tuple $(p_1, \ldots, p_n)$, the notation $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ denotes a map $f\colon \mathcal{C} \to \mathbb{CP}^1$ that satisfies

$f(p_i) = \infty$ for all $i \in \{1, 2, \ldots, n\}$.

Given a morphism satisfying the above conditions, one obtains the corresponding map by taking the preimage $f^{-1}([0,1])$. The preimages of $0 \in \mathbb{CP}^1$ are the vertices of the map; the preimages of $1 \in \mathbb{CP}^1$ are the midpoints of the edges; and the preimages of $\infty \in \mathbb{CP}^1$ are the centres of the faces of the map. The faces containing the marked points $p_1, \ldots, p_n$ correspond to the boundary faces labelled $1, \ldots, n$ respectively.

Conversely, for any triple of permutations $(\sigma_0, \sigma_1, \sigma_2)$ satisfying $\sigma_0\sigma_1\sigma_2 = \mathrm{id}$ the Riemann existence theorem guarantees that there exists a unique holomorphic map $f\colon \mathcal{C} \to \mathbb{CP}^1$ ramified only at 0, 1, and ∞ that realises $(\sigma_0, \sigma_1, \sigma_2)$ as its monodromy representation. Here, $\mathcal{C}$ is a possibly disconnected compact Riemann surface. A statement and proof of this theorem can be found in [22].

This point of view is particularly useful for the perspective used in Section 3.3.1 to identify so-called stable ribbon graphs as lattice points in the Deligne–Mumford compactification of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$.

3.2.4 Generating functions

The following generating functions for the enumerations of ordinary maps and fully simple maps will be useful in Chapter 7.

Definition 3.2.10. Let $\mu_1, \ldots, \mu_n$ be positive integers, and define $\mathrm{M}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n)$ to be the weighted enumeration of (isomorphism classes of) connected genus g ordinary maps with n boundary faces such that the degree of boundary face i is μ_i. The weight of each map M is given by

$$\frac{s^{e(M)}}{|\mathrm{Aut}\ M|} t_1^{f_1(M)} t_2^{f_2(M)} t_3^{f_3(M)} \cdots,$$

where $f_i(M)$ is the number of internal faces of degree i, $e(M)$ denotes the number of edges of M, and $|\mathrm{Aut}\ M|$ is the number of automorphisms of M. Define $\mathrm{FS}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n)$ to be the analogous enumeration of connected fully simple maps, and let $\mathrm{M}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$ and $\mathrm{FS}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$ be the analogous enumerations for possibly disconnected maps and fully simple maps respectively.

Define $\mathrm{R}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n)$ to be the weighted enumeration of ribbon graphs with n faces such that the degree of boundary face i is μ_i. The weight of each ribbon graph R is given by

$$\frac{s^{|\mu|/2}}{|\mathrm{Aut}\ R|}.$$

Here, $|\mu| = \mu_1 + \cdots + \mu_n$. Let $\mathrm{R}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$ be the analogous enumeration for possibly disconnected ribbon graphs.

Finally, let $\hat{\mathrm{M}}^{\circ}_{g,n}, \hat{\mathrm{FS}}^{\circ}_{g,n}, \hat{\mathrm{R}}^{\circ}_{g,n}$ and and $\hat{\mathrm{M}}^{\bullet}_{g,n}, \hat{\mathrm{FS}}^{\bullet}_{g,n}, \hat{\mathrm{R}}^{\bullet}_{g,n}$ denote the corresponding generating functions for the *rooted* connected and *rooted* disconnected enumerations respectively (see Definition 3.2.4).

The enumeration of ribbon graphs is indeed equal to the enumeration of ordinary maps with no internal faces. That is, $\mathrm{R}_{g,n}(\mu_1, \ldots, \mu_n) = \mathrm{M}_{g,n}(\mu_1, \ldots, \mu_n)\big|_{t_i=0}$.

The inclusion of the formal parameter s in the definitions of the generating functions is not necessary since it can be recovered from the remaining parameters in the generating function. However, it can be useful to make use of the operator $\frac{\partial}{\partial s}$, which appears in the evolution equations for these enumerations and is utilised in Chapter 7 on fully simple maps.

The generating functions $\mathrm{M}_{g,n}$ and $\mathrm{FS}_{g,n}$ are formal power series in the t-variables; that is,

$$\begin{aligned}
&\mathrm{M}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n), \mathrm{FS}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n), \mathrm{M}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n), \mathrm{FS}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n) \in \mathbb{Q}[[t_1, t_2, \ldots]] \\
&\hat{\mathrm{M}}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n), \hat{\mathrm{FS}}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n), \hat{\mathrm{M}}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n), \hat{\mathrm{FS}}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n) \in \mathbb{Q}[[t_1, t_2, \ldots]].
\end{aligned}$$

For maps of type (g, n) with boundary face degrees given by $\mu_1, \ldots, \mu_n$, one can obtain the enumeration of ordinary maps with prescribed internal face degrees from $\mathrm{M}_{g,n}(\mu_1, \ldots, \mu_n)$ by extracting the appropriate coefficient of t-monomials.

Finally, as alluded to in Section 3.2.1, the enumeration of maps and rooted maps are related by a combinatorial factor. In fact, the enumeration of rooted maps are obtained from the unrooted enumeration by multiplying by the product of the boundary face degrees. That is, for $\mu_1, \ldots, \mu_n$ positive integers,

$$\hat{\mathrm{M}}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \mu_1 \cdots \mu_n \mathrm{M}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n),$$

and similarly for $\mathrm{FS}^{\bullet}_{g,n}$ and $\mathrm{R}^{\bullet}_{g,n}$. The reason for this is straightforward: for the boundary face labelled i, there are μ_i choices for the root.

Example 3.2.11. Consider connected ordinary maps on the sphere with one boundary face of degree four that are quadrangulations; that is, all internal faces are of degree four. The corresponding generating function is then given by setting $t_i = 0$ for all $i \neq 4$ which yields

$$\mathrm{M}^{\circ}_{0,1}(4)\Big|_{\substack{t_i=0\\ i\neq 4}} = \frac{1}{2}s^2 + \frac{9}{4}t_4 s^4 + \frac{27}{2}t_4^2 s^6 + \frac{189}{2}t_4^3 s^8 + 729 t_4^4 s^{10} + \cdots . \tag{3.1}$$

Figure 3.10 shows (1) the only ordinary map with one boundary of degree four and no internal faces, as well as (2) the set of ordinary maps with one boundary and one internal face, both of degree four. The sizes of the automorphism groups for these maps are 2, 4, 1 and 1, respectively, and hence one can see that these enumerations correspond to the first two terms in the generating function $\mathrm{M}^{\circ}_{0,1}(4)$ above.

In Figure 3.10, only the second map from the left is fully simple. The corresponding generating function for fully simple maps is

$$\mathrm{FS}^{\circ}_{0,1}(4) = \frac{1}{4}t_4 s^4 + \frac{5}{2}t_4^2 s^6 + \frac{45}{2}t_4^3 s^8 + \frac{405}{2}t_4^4 s^{10} + \cdots .$$

The data for both the ordinary map and fully simple map generating functions have been calculated via SageMath [98], the latter using the results from Chapter 7.

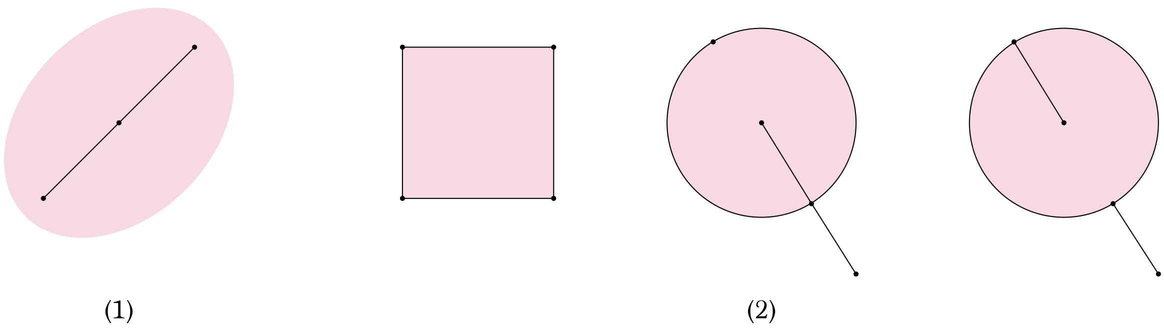

Figure 3.10: Ordinary maps with one boundary face and (1) zero internal faces or (2) one internal face, all of degree four. The sizes of the automorphism groups for these maps are, from left to right, 2, 4, 1 and 1 respectively.

A sidebar on enumerating ordinary and fully simple maps. The data in Example 3.2.11 for ordinary maps has been calculated via SageMath using the following relation to ribbon graphs:

$$\mathrm{M}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\nu \in \mathcal{P}} \mathrm{R}^{\circ}_{g,n}(\vec{\mu}, \vec{\nu}) \frac{\vec{t}_\nu}{|\mathrm{Aut}\ \nu|} s^{\frac{|\nu|}{2}} . \tag{3.2}$$

Here, $\mathcal{P}$ is the set of all partitions, $|\nu| = \nu_1 + \cdots + \nu_{\ell(\nu)}$, $\vec{t}_\nu$ is a shorthand notation for the monomial $t_{\nu_1} \cdots t_{\nu_{\ell(\nu)}}$, and $|\mathrm{Aut}\ \nu|$ denotes the number of automorphisms of the partition ν. Equation (3.2) can be proven in a straightforward combinatorial manner by proving that the coefficient of t_ν on each side is equal.

In SageMath, the sum in (3.2) has been calculated up to partitions ν of size 10 and $\mathrm{R}^{\circ}_{g,n}$ has been calculated using the Tutte recursion for ribbon graphs; see Proposition 3.2.12 in Section 3.2.5 and restrict to ribbon graphs by setting $t_i = 0$ for all positive integers i.

3.2.5 Tutte recursion

Tutte first derived a combinatorial recursion for genus 0 rooted maps with one boundary in 1963 [105]. This was then generalised to higher genus by Walsh and Lehman [106]. The idea behind the recursion for genus 0 rooted ordinary maps with one boundary face is as follows. Begin with an ordinary map on the sphere and remove the rooted edge from the first marked face; that is, the marked face labelled 1. Let the degree of this face be μ. The edge removed separates two faces and either these two faces are the same or they are different. In the former case, the result after removing the chosen edge (and cutting the surface into two and patching each component with disks) is two genus 0 maps, each having one boundary face with lengths α and β where $\alpha + \beta = \mu - 2$. Alternatively, the edge removed separates two distinct faces in which case the other face is an internal face of some degree, say j. Removing the chosen edge results in a new map with boundary length $\mu + j - 2$ and one fewer internal face of degree j. In either case the rooted edges on new boundary faces are chosen canonically by the cyclic ordering of the edges at the vertices. That is, after removing the rooted edge e, in the first case we choose the edges $\sigma_0(e)$ and $\sigma_0\sigma_1(e)$ to be the two new rooted edges, and in the latter case we choose $\sigma_0(e)$.

This leads to the following recursion

$$\frac{1}{s}\hat{\mathrm{M}}^\circ_{0,1}(\mu) = \sum_{\alpha+\beta=\mu-2} \hat{\mathrm{M}}^\circ_{0,1}(\alpha)\hat{\mathrm{M}}^\circ_{0,1}(\beta) + \sum_{t\geqslant 1} t_j \hat{\mathrm{M}}^\circ_{0,1}(\mu + j - 2),$$

valid for all $\mu > 0$, and with the base case given by $\hat{\mathrm{M}}^\circ_{0,1}(0) = 1$ (recall the so-called degenerate map in Remark 3.2.3).

The generalisation of this bijection to all genus g involves a number of new options, however all cases are again mutually exclusive and the combinatorial analysis follows through in a similar manner. For a discussion of the combinatorics behind the generalised Tutte equation, see previous work of Eynard [45].

Proposition 3.2.12 (Tutte recursion). *Let $S = \{2, 3, \ldots, n\}$. For all (g, n) and $\mu_1 + \cdots + \mu_n > 0$, connected (rooted) ordinary maps satisfy the following recursion:*

$$\begin{aligned}\frac{1}{s}\,&\hat{\mathrm{M}}^\circ_{g,n}(\mu_1, \ldots, \mu_n)\\ &= \sum_{i=2}^{n} \mu_i \hat{\mathrm{M}}^\circ_{g,n-1}(\mu_1 + \mu_i - 2, \vec{\mu}_{S\setminus\{i\}}) + \sum_{j\geqslant 1} t_j \hat{\mathrm{M}}^\circ_{g,n}(\mu_1 + j - 2, \vec{\mu}_S)\\ &\quad + \sum_{\alpha+\beta=\mu_1-2} \left[\hat{\mathrm{M}}^\circ_{g-1,n+1}(\alpha, \beta, \vec{\mu}_S) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\ldots,n\}}} \hat{\mathrm{M}}^\circ_{g_1,|I|+1}(\alpha, \vec{\mu}_I)\hat{\mathrm{M}}^\circ_{g_2,|J|+1}(\beta, \vec{\mu}_J) \right].\end{aligned}$$

For $I = \{i_1, \ldots, i_k\}$, the shorthand notation $\vec{\mu}_I$ denotes $\mu_{i_1}, \ldots, \mu_{i_k}$. Along with the base case given by $\mathrm{M}^\circ_{0,1}(0) = 1$, this uniquely determines the value of $\hat{\mathrm{M}}^\circ_{g,n}(\vec{\mu})$ for all g, n and $\vec{\mu}$.

Note the $1/s$ factor on the left side, which arises from the fact that the s-parameter in the definition of $\mathrm{M}_{g,n}(\mu_1, \ldots, \mu_n)$ keeps track of the number of edges. The Tutte recursion acts by removing an edge, hence the number of edges after applying the Tutte recursion has been reduced by one.

3.3 Stable ribbon graphs

3.3.1 Counting lattice points in $\overline{\mathcal{M}}_{g,n}$

The goal of this section is to describe the correspondence between certain branched covers, so-called stable ribbon graphs and lattice points in the Deligne–Mumford compactification of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. In some sense, this is a compactified version of a simpler correspondence between branched covers, ribbon graphs and lattice points in the uncompactified moduli space of curves $\mathcal{M}_{g,n}$. To motivate and provide context for the former correspondence I will first describe the latter.

But first I provide a brief definition of the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. Complete and precise definitions of these spaces are quite subtle and ideally framed in the language of stacks; providing a complete and precise definition here would take me too far afield of the scope of this thesis. For a full introduction to moduli spaces of curves I refer the reader to the book of Harris and Morrison [68], while for more on stacks, see The Stacks Project [71]. Define the moduli space

$$\mathcal{M}_{g,n} = \left\{ (\mathcal{C}; p_1, \ldots, p_n) \;\middle|\; \begin{array}{l} \mathcal{C} \text{ is a smooth algebraic curve of genus} \\ g \text{ with } n \text{ distinct points } p_1, \ldots, p_n \end{array} \right\} \Big/ \sim$$

where $(\mathcal{C}; p_1, \ldots, p_n) \sim (\mathcal{D}; q_1, \ldots, q_n)$ if there exists an isomorphism from $\mathcal{C}$ to $\mathcal{D}$ that sends p_i to q_i for all $i \in \{1, 2, \ldots, n\}$.

The space $\mathcal{M}_{g,n}$ is not compact. While there are many ways to compactify this moduli space, the Deligne–Mumford compactification will be relevant to the work described here. The Deligne–Mumford compactification broadens the definition of the space to allow stable algebraic curves. Hence, define the space

$$\overline{\mathcal{M}}_{g,n} = \left\{ (\mathcal{C}; p_1, \ldots, p_n) \;\middle|\; \begin{array}{l} \mathcal{C} \text{ is a stable algebraic curve of genus } g \\ \text{with } n \text{ distinct smooth points } p_1, \ldots, p_n \end{array} \right\} \Big/ \sim$$

where $(\mathcal{C}; p_1, \ldots, p_n) \sim (\mathcal{D}; q_1, \ldots, q_n)$ if there exists an isomorphism from $\mathcal{C}$ to $\mathcal{D}$ that sends p_i to q_i for all $i \in \{1, 2, \ldots, n\}$. An algebraic curve is ***stable*** if it has at worst nodal singularities and a finite automorphism group. All smooth algebraic curves are stable, hence $\mathcal{M}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$.

This describes the moduli spaces as sets, but they are actually endowed with a lot of geometric structure. In particular, they can be considered as orbifolds, varieties, schemes or stacks. Furthermore, the orbifold structure arises from curves with non-trivial automorphisms and means that we will consider the cohomology with rational (as opposed to integer) coefficients.

As described in Section 3.2.3, maps are in one-to-one correspondence with certain branched covers of $\mathbb{CP}^1$. For the sake of completeness and clarity, I will briefly describe this correspondence explicitly in the context of ribbon graphs. Recall from Definition 3.2.1 that a ribbon graph is a map with no internal faces. More concretely, a ribbon graph of type (g, n) is a finite graph embedded on a compact oriented genus g surface such that the complement of the graph on the surface is a disjoint union of n topological disks which are labelled 1 to n.

In this section, we restrict our attention to ribbon graphs where the degree of every vertex is at least two, motivated by the connection between ribbon graphs and moduli spaces of curves. For this reason, I give a definition for the following enumeration, which is analogous to $\mathrm{R}_{g,n}(\mu_1, \ldots, \mu_n)$ from Definition 3.2.10.

Definition 3.3.1. Let $\mu_1, \ldots, \mu_n$ be positive integers. Define $\mathrm{N}_{g,n}(\mu_1, \ldots, \mu_n)$ to be the weighted enumeration of isomorphism classes of connected genus g ribbon graphs with n boundary components, where the degree of every vertex is at least two and the degree of the boundary component labelled i is μ_i. The weight of each such ribbon graph R is

$$\frac{1}{|\mathrm{Aut}\ R|}.$$

For a ribbon graph of type (g, n) with boundary degrees given by $\mu_1, \ldots, \mu_n$, one can associate a branched cover $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ from a compact genus g Riemann surface $\mathcal{C}$ with n marked points $p_1, \ldots, p_n$ satisfying the following conditions.

- The degree of f is equal to the sum $\mu_1 + \cdots + \mu_n$ and is unramified over $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.
- The ramification profile of $\infty \in \mathbb{CP}^1$ is $(\mu_1, \ldots, \mu_n)$, where $p_k \in \mathcal{C}$ has ramification index μ_k for $k \in \{1, 2, \ldots, n\}$.
- The ramification profile of $1 \in \mathbb{CP}^1$ is $(2, 2, \ldots, 2)$ and each point in the preimage of $0 \in \mathbb{CP}^1$ has ramification index at least 2.

The corresponding ribbon graph can be obtained by considering the preimage of the line segment between $0 \in \mathbb{CP}^1$ and $1 \in \mathbb{CP}^1$; that is, by considering $f^{-1}([0,1]) \subset \mathcal{C}$. The labelling of the points $p_1, \ldots, p_n$ gives rise to the labelling of the faces of the ribbon graph. Let $\mathcal{Z}_{g,n}(\mu_1, \ldots, \mu_n) \subset \mathcal{M}_{g,n}$ be the set of genus g Riemann surfaces $\mathcal{C}$—or equivalently the set of genus g smooth complex algebraic curves—such that there exists a morphism $f\colon \mathcal{C} \to \mathbb{CP}^1$ with the properties described above. Note that if a smooth curve admits such a morphism then that morphism is unique. Furthermore, the automorphisms of the curve and of the morphism agree. See Remark 1.3 in [35]. Hence, the weighted enumeration of such curves equals the enumeration of such morphisms.

The sense in which ribbon graphs enumerate lattice points in the uncompactified moduli space $\mathcal{M}_{g,n}$ is due to a cell decomposition of the *decorated* moduli space of curves, proved independently by Harer [67] using Strebel differentials, and by Penner [94] using hyperbolic geometry. Specifically, to each smooth genus g curve with n marked points, each decorated by a positive real number, one can associate a ***metric ribbon graph*** of type (g, n). A ***metric ribbon graph*** is a ribbon graph whose vertex degrees are at least three and which has a positive real number associated to each edge. This association allows one to decompose the decorated moduli space of curves into cells

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong \bigsqcup_{\Gamma} \mathcal{P}_\Gamma,$$

where the union is over ribbon graphs Γ of type (g, n) whose vertex degrees are at least three, and the cell $\mathcal{P}_\Gamma$ consists of all metric ribbon graphs whose underlying ribbon graph is Γ. Fixing n positive real numbers $(b_1, \ldots, b_n) \in \mathbb{R}_+^n$, we obtain the following decomposition from the previous one

$$\mathcal{M}_{g,n} \cong \bigsqcup_{\Gamma} \mathcal{P}_\Gamma(b_1, \ldots, b_n),$$

where the union is again over ribbon graphs Γ of type (g, n) whose vertex degrees are at least three, and $\mathcal{P}_\Gamma(b_1, \ldots, b_n)$ is the set of metric ribbon graphs with boundary lengths given by $b_1, \ldots, b_n$. Here, the boundary length of a face in a metric ribbon graph is equal to the sum of the positive real numbers associated to the edges adjacent to that face.

Originally proposed by Norbury [87, 88], one can restrict consideration to metric ribbon graphs of type (g, n) with vertex degrees at least three and where the edge lengths are positive integers. Further, these objects are in fact equivalent to ribbon graphs with vertex degrees at least two. To see this correspondence one can simply consider an edge in a metric ribbon graph with an associated integer k as a path of k edges in a ribbon graph. Norbury interprets such ribbon graphs as lattice points in $\mathcal{M}_{g,n}$ and links their enumeration with the geometry of the moduli space.

Kontsevich's proof of Witten's conjecture [75, 107] proceeds by calculating certain volumes of moduli spaces of curves using the cell decomposition described above. Norbury's lattice point count can be seen as a discretisation of this volume calculation.

Therefore, $\mathcal{Z}_{g,n}(\mu_1, \ldots, \mu_n) \subset \mathcal{M}_{g,n}$ is in bijection with ribbon graphs with vertex degrees at least two and can be thought of as capturing the "lattice points" in $\mathcal{M}_{g,n}$. Further the automorphism group of the curve agrees with the automorphism group of the ribbon graph. To enumerate the points in the set $\mathcal{Z}_{g,n}$, one should take into account the natural orbifold nature of $\mathcal{M}_{g,n}$ by counting $\mathcal{C} \in \mathcal{Z}_{g,n}(\mu_1, \ldots, \mu_n)$ with weight equal to one over the size of the automorphism group of $\mathcal{C}$. This enumeration is equal to the orbifold Euler characteristic of $\mathcal{Z}_{g,n}(\mu_1, \ldots, \mu_n)$. This leads to the following result due to Norbury [87, 88],

$$\mathrm{N}_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\mathcal{C} \in \mathcal{Z}_{g,n}(\mu_1, \ldots, \mu_n)} \frac{1}{|\mathrm{Aut}\ \mathcal{C}|} = \chi(\mathcal{Z}_{g,n}(\mu_1, \ldots, \mu_n)),$$

where $\mathrm{N}_{g,n}(\mu_1, \ldots, \mu_n)$ is the enumeration of connected ribbon graphs of type (g, n) with boundary face degrees given by $\mu_1, \ldots, \mu_n$, as defined in Definition 3.3.1.

Do and Norbury [35] generalised this notion to the Deligne–Mumford compactification of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. The generalisation is reasonably natural from the viewpoint of ribbon graphs as branched covers and the set $\mathcal{Z}_{g,n}(\mu_1, \dots, \mu_n)$. That is, instead of considering the set of smooth curves with an associated morphism as described above, one can consider the set of stable curves with an associated morphism satisfying certain conditions and respecting certain stability conditions, motivated by the notion of a stable map in Gromov–Witten theory. This leads to the following definition.

Definition 3.3.2. Let $\mu_1, \dots, \mu_n$ be positive integers, and define $\overline{\mathcal{Z}}_{g,n}(\mu_1, \dots, \mu_n) \subset \overline{\mathcal{M}}_{g,n}$ to be the set of genus g stable curves $\mathcal{C} \in \overline{\mathcal{M}}_{g,n}$ with n marked points $p_1, \dots, p_n$ such that there exists a morphism $f \colon (\mathcal{C}; p_1, \dots, p_n) \to (\mathbb{CP}^1; \infty)$ satisfying the following conditions.

- The degree of f is equal to the sum $\mu_1 + \cdots + \mu_n$ and is unramified over $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$.
- The ramification profile of $\infty \in \mathbb{CP}^1$ is $(\mu_1, \dots, \mu_n)$, where $p_k \in \mathcal{C}$ has ramification index μ_k for $k \in \{1, 2, \dots, n\}$.
- The ramification profile of $1 \in \mathbb{CP}^1$ is $(2, 2, \dots, 2)$ and each point in the preimage of $0 \in \mathbb{CP}^1$ has ramification index at least 2 or is a node.

Unlike in the uncompactified case, the set $\overline{\mathcal{Z}}_{g,n}(\mu_1, \dots, \mu_n)$ is no longer a finite set of points, and indeed also contains components with positive dimension. For example, in the instance that a curve $\mathcal{C} \in \overline{\mathcal{M}}_{g,n}$ contains a *ghost component*—that is, an entire component that is being mapped to 0—then it could be that that ghost component may be continuously deformed to other curves $\overline{\mathcal{Z}}_{g,n}(\mu_1, \dots, \mu_n)$, leading to a positive dimension component. However, as in the uncompactified enumeration, Do and Norbury define the enumeration of the set $\overline{\mathcal{Z}}_{g,n}(\mu_1, \dots, \mu_n)$ via the orbifold Euler characteristic and argue that this enumeration is natural and possesses interesting structure [35].

Definition 3.3.3. For positive integers $\mu_1, \dots, \mu_n$, define

$$\overline{\mathrm{N}}_{g,n}(\mu_1, \dots, \mu_n) = \chi(\overline{\mathcal{Z}}_{g,n}(\mu_1, \dots, \mu_n)).$$

3.3.2 Combinatorial construction

Given that there is a correspondence between lattice points of $\mathcal{M}_{g,n}$ and ribbon graphs via the set of stable curves satisfying certain conditions, $\mathcal{Z}_{g,n}(\mu_1, \dots, \mu_n)$, it is natural to define the corresponding combinatorial object in the case of $\overline{\mathcal{Z}}_{g,n}(\mu_1, \dots, \mu_n)$ and $\overline{\mathcal{M}}_{g,n}$. In this section, I define the notion of a stable ribbon graph. These objects were previously defined by Kontsevich [75] as well as Do and Norbury [35]; for a more thorough introduction I refer the reader to these references. The exposition in this section is largely based on the work of Do and Norbury [35, Section 2].

The idea is as follows. From a map $f \colon (\mathcal{C}; p_1, \dots, p_n) \to (\mathbb{CP}^1; \infty)$ satisfying the conditions of Definition 3.3.2 we would like to associate a combinatorial object—a stable ribbon graph—from which the stable curve $\mathcal{C} \in \overline{\mathcal{M}}_{g,n}$ can be uniquely obtained. Morally, the stable ribbon graph captures the pre-image $f^{-1}([0, 1])$. If a component of $\mathcal{C}$ maps to $\mathbb{CP}^1$ with positive degree, the inverse image of $[0, 1]$ on this component will be a ribbon graph, where the nodes—that must map to 0—are distinguished vertices. If a component, or a connected collection of components, of $\mathcal{C}$ maps to $0 \in \mathbb{CP}^1$ with degree 0, then this component is one of the so-called ghost components and becomes a distinguished vertex in the resulting stable ribbon graph. To be able to recover this entire component from the resulting stable ribbon graph, we require only its genus. Thus we store the genus of these ghost components via a genus function defined on the vertices corresponding to ghost components.

The stable ribbon graph then becomes the information of the ribbon graphs associated to each component of $\mathcal{C}$ that maps under f with positive degree; a set of distinguished vertices containing both the vertices that correspond to the nodes (where two vertices will be identified) along with the vertices that correspond to the ghost components; and a genus function defined on this set of distinguished vertices that records the genus of the ghost components that was collapsed or 0 for a node. This leads to the following definition.

Definition 3.3.4. A *stable ribbon graph* is a possibly disconnected ribbon graph along with the following extra information:

- a subset S of distinguished vertices;
- an equivalence relation $\sim$ on S; and
- a genus function $h\colon S/\sim \to \{0,1,2,\ldots\}$ where for any equivalence class $S_0 \subset S$, if $|S_0| = 1$ then $h(S_0) > 0$.

An *isomorphism* between stable ribbon graphs is an isomorphism of the possibly disconnected ribbon graphs that leaves S invariant and preserves $\sim$ and h.

Definition 3.3.4 has been previously introduced by Kontsevich [75] and separately by Do and Norbury [35].

As we do in the case of ribbon graphs, we would like to enumerate stable ribbon graphs according to their genus and number of boundary components. The genus of a stable ribbon graph is defined to be the genus of the underlying stable curve to which is is associated. The contributions to the genus come from the genera of the original components, the genera of the ghost components (obtained via the genus function), and any extra contributions arising from gluing components together to form "loops".

To calculate the genus of the stable ribbon graph, we will use the notion of the dual graph of a stable ribbon graph Γ. Denote by $\pi_0\Gamma$ the set of connected components of Γ. The genus of a connected component Γ' of Γ is determined by $2 - 2g(\Gamma') = V(\Gamma' \setminus S) - E(\Gamma') + F(\Gamma')$, where the set of distinguished vertices has been removed.

Define the *dual graph* of a stable ribbon graph Γ, denoted $G(\Gamma)$, to be the graph consisting of vertices $V = (S/\sim) \cup \pi_0\Gamma$, edges $E = S \cup F(\Gamma)$ and incidence relations given by inclusion. Extend the definition of the genus function h to $h\colon V(G(\Gamma)) \to \{0,1,2,\ldots\}$ by defining $h(v)$ for some $v \in V(G(\Gamma)) \setminus (S/\sim)$ to be equal to the genus of the corresponding connected component in $\pi_0\Gamma$.

Calculate the genus of a stable ribbon graph Γ by

$$g(\Gamma) = b_1(G(\Gamma)) + \sum_{v \in V(G(\Gamma))} h(v),$$

where $b_1(G(\Gamma))$ is the first Betti number of $G(\Gamma)$.

The term with the first Betti number counts the genus contributed by the "loops" described above, while the terms in the sum corresponding to $v \in S/\sim$ contribute the genera of the ghost components and terms corresponding to the remaining vertices contribute the genera of the connected components in $\pi_0\Gamma$.

Definition 3.3.5. For positive integers $b_1, \ldots, b_n$, define $\mathcal{R}^{\text{stable}}_{g,n}(b_1, \ldots, b_n)$ to be the set of isomorphism classes of genus g stable ribbon graphs, connected after identification of vertices by $\sim$, with n boundary components of lengths $b_1, \ldots, b_n$, and where all valence 1 vertices are contained in S.

The following proposition states that the enumeration of $\overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n)$ given in Definition 3.3.3 matches the enumeration of stable ribbon graphs [35].

Proposition 3.3.6 (Do and Norbury [35]). *For positive integers* $b_1, \ldots, b_n$,

$$\overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n) = \sum_{\Gamma \in \mathcal{R}^{\text{stable}}_{g,n}(b_1,\ldots,b_n)} \frac{1}{|\mathrm{Aut}\ \Gamma|} \prod_{v \in S/\sim} \chi(\overline{\mathcal{M}}_{h(v),n(v)}),$$

where $n(S_0) = |S_0|$ *for an equivalence class* $S_0 \subset S$*, and we define* $\chi(\overline{\mathcal{M}}_{0,2}) := 1$.

The proof of this proposition goes via the set $\overline{\mathcal{Z}}_{g,n}(b_1, \ldots, b_n)$ in the following way. Given a morphism $f\colon \mathcal{C} \to \mathbb{CP}^1$ satisfying the conditions given in Definition 3.3.2, one can construct a stable ribbon graph as follows. First, as described earlier in Section 3.3.1, construct a ribbon graph Γ' by $\Gamma' = f^{-1}([0,1]) \setminus \{\text{nodes, ghost components}\}$, but note that in this case Γ' is possibly disconnected and may have leaves; that is, edges that don't end in a vertex. Define a stable ribbon graph Γ to be the closure of Γ' in the normalisation of $\mathcal{C}$ by adding vertices to the non-compact ends of Γ'. Define the subset of distinguished vertices by $S = \Gamma \setminus \Gamma'$ and dictate that two vertices are in the same equivalence class of S if they coincide in the non-ghost components of $\mathcal{C}$. Define $h\colon S/\!\sim \to \{0, 1, 2, \ldots\}$ by dictating that the genus of an equivalence class $S_0 \subset S$ is equal to the genus of the collapsed ghost component and zero if S_0 corresponds to a node.

This process defines a map $\overline{\mathcal{Z}}_{g,n}(b_1, \ldots, b_n) \to \mathcal{R}^{\text{stable}}_{g,n}(b_1, \ldots, b_n)$. This map is not one-to-one and indeed, each stable ribbon graph describes a component of $\overline{\mathcal{Z}}_{g,n}(b_1, \ldots, b_n)$; some of these components are points and some are positive-dimensional. However, the weight of the orbifold Euler characteristic in Proposition 3.3.6 counts the stable ribbon graphs in precisely the necessary way such that the enumeration is equal to $\overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n)$.

Chapter 4

Hurwitz numbers

4.1 Introduction

In the late nineteenth century, Hurwitz [69] studied enumerations of branched covers of the Riemann sphere with prescribed ramification data. While interest in Hurwitz numbers lulled in the century that followed, the last twenty-five years has seen a boom of activity in the Hurwitz theory literature. One major catalyst for this boom is the rich mathematical structure discovered within single Hurwitz numbers, defined below. In fact, single Hurwitz numbers have since been treated as an archetype for many similar problems and generalisations in the enumerative geometry of curves.

This chapter reviews these major results for single Hurwitz numbers and provides a skeleton or prototype for the generalisations of these results.

Definition 4.1.1. The ***single Hurwitz number*** $H_{g,n}(\mu_1, \ldots, \mu_n)$ is a weighted enumeration of isomorphism classes of connected genus g branched covers of the Riemann sphere $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- the point $p_i \in f^{-1}(\infty)$ has ramification index μ_i for $i \in \{1, 2, \ldots, n\}$; and
- all other branching is simple and occurs at m fixed points of $\mathbb{CP}^1$.

The weight of a cover is given by

$$\frac{1}{m!\,|\mathrm{Aut}\ f|},$$

where $\mathrm{Aut}\ f$ is the group of automorphisms of f. An automorphism of a branched cover $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ is an isomorphism $\phi\colon \mathcal{C} \to \mathcal{C}$ that preserves the marked points $p_1, \ldots, p_n$ and satisfies $f \circ \phi = f$.

Recall, for a tuple $(p_1, \ldots, p_n)$, the notation $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ denotes a map $f\colon \mathcal{C} \to \mathbb{CP}^1$ that satisfies $f(p_i) = \infty$ for all $i \in \{1, 2, \ldots, n\}$.

The factor of $1/m!$ is a normalisation factor which has become increasingly commonplace in the definition of single Hurwitz numbers in the recent literature, primarily to enable cleaner statements for subsequent results (for example, polynomiality and the ELSV formula; see below).

The number of simple branch points is related to the genus of the Riemann surface $\mathcal{C}$ and the tuple $\mu_1, \ldots, \mu_n$ via the Riemann–Hurwitz formula. Recall that the Riemann–Hurwitz formula for a given degree d non-constant holomorphic map $f\colon X \to Y$ between Riemann surfaces is

$$\chi(X) = \chi(Y)\,d - \sum_{x \in X} (k_x - 1),$$

where k_x is the ramification index of f at x and $\chi(X) = 2 - 2g(X)$ relates the Euler characteristic of X to its genus. In the case of single Hurwitz numbers we are considering a genus g branched cover of the Riemann sphere with prescribed ramification profile $(\mu_1, \ldots, \mu_n)$ over ∞ and simple ramification elsewhere.[1] Letting

[1] Recall that *ramification points* live upstairs on $\mathcal{C}$ while *branch points* live downstairs on $\mathbb{CP}^1$. These definitions are infinitely elusive and I personally find I have to re-recall them every time.

m be the number of branch points with simple ramification, the formula reduces to

$$2 - 2g = 2d - \sum_{i=1}^{n} (\mu_i - 1) - m$$
$$m = 2d + 2g - 2 - |\mu| + n,$$

where the notation $|\mu|$ denotes the sum of the parts: $|\mu| = \mu_1 + \cdots + \mu_n$. Here, d is the degree of the map, so $d = |\mu|$, and in this case the Riemann–Hurwitz formula yields

$$m = 2g - 2 + n + |\mu|.$$

It was first observed and conjectured by Goulden, Jackson and Vainshtein [65] that single Hurwitz numbers satisfy the following structural property.

Theorem 4.1.2 (Conjectured by Goulden, Jackson and Vainshtein [65], proven by Ekedahl, Lando, Shapiro and Vainshtein [43]). *For (g, n) satisfying $2g - 2 + n > 0$, there exist symmetric polynomials $P_{g,n}(\mu_1, \ldots, \mu_n)$ of degree $3g - 3 + n$ such that*

$$H_{g,n}(\mu_1, \ldots, \mu_n) = \left[\prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \right] P_{g,n}(\mu_1, \ldots, \mu_n). \tag{4.1}$$

A proof of this polynomiality structure was obtained by Ekedahl, Lando, Shapiro, and Vainshtein as a direct corollary of the celebrated ELSV formula, which relates single Hurwitz numbers to intersection theory on moduli spaces of curves [43]. Specifically, their result is the following.

Theorem 4.1.3 (Ekedahl, Lando, Shapiro and Vainshtein [43]). *For (g, n) satisfying $2g-2+n > 0$, the single Hurwitz number $H_{g,n}(\mu_1, \ldots, \mu_n)$ satisfies*

$$H_{g,n}(\mu_1, \ldots, \mu_n) = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{c(\Lambda^\vee)}{\prod_{i=1}^{n}(1 - \mu_i \psi_i)}. \tag{4.2}$$

Here, $\overline{\mathcal{M}}_{g,n}$ is the Deligne–Mumford compactification of the moduli space of curves, $c(\Lambda^\vee)$ is the total Chern class of the dual Hodge bundle over $\overline{\mathcal{M}}_{g,n}$, and ψ_i is the first Chern class of the cotangent line bundle to the ith marked point. For a thorough introduction to the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ and its characteristic classes, see the book of Harris and Morrison [68].

Further, Bouchard and Mariño [17] conjectured that single Hurwitz numbers are governed by topological recursion. This conjecture was motivated by earlier work of Bouchard, Klemm, Mariño and Pasquetti [16] on the remodelling conjecture, or BKMP conjecture, which states that Gromov–Witten invariants of toric Calabi–Yau threefolds are governed by topological recursion. The conjecture for single Hurwitz numbers arises as a particular limiting case of the BKMP conjecture. Specifically, the Bouchard–Mariño conjecture is as follows.

Theorem 4.1.4 (Conjectured by Bouchard and Mariño [17], proven by Eynard, Mulase and Safnuk [49]). *The correlation differentials resulting from applying topological recursion to the spectral curve $(\mathbb{C}^*, x, y, \omega_{0,2})$ with*

$$x(z) = \ln z - z, \qquad y(z) = z, \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{(z_1 - z_2)^2}$$

satisfy

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\mu_1, \ldots, \mu_n \geqslant 1} H_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \mathrm{d} \exp(\mu_i x(z_i)).$$

Note that for the topological recursion input data here, I have replaced the Torelli marking with $\omega_{0,2}$, utilising local topological recursion instead of CEO. However this is not strictly necessary. Although $x(z)$ is not meromorphic on all of $\mathbb{CP}^1$, the form $\mathrm{d}x(z)$ is, and this is ultimately what is required to apply topological recursion. Thus, one could still treat the single Hurwitz numbers spectral curve as compact.

Since these results were proven, there has been a wealth of subsequent work to prove analogous results—polynomiality, topological recursion, and ELSV-type formulas—for generalisations and variations of Hurwitz numbers. Such variations (as well as combinations thereof) include weakly and strictly monotone, orbifold, and spin; see the introduction of [8] for a more extensive list of such results and references. Chapter 6 provides proofs of analogous results for double Hurwitz numbers, which generalise single and orbifold Hurwitz numbers.

The present chapter provides an introduction to Hurwitz numbers, using as a framework the specific enumeration of single Hurwitz numbers. In particular, I aim to present some well-known results and techniques used in Hurwitz theory in the setting of single Hurwitz numbers. This is the content of Section 4.2, which comprises:

- Section 4.2.1 gives the classical equivalent formulation of single Hurwitz numbers via enumerations of sequences of transpositions in the symmetric group S_d;
- Section 4.2.2 gives the cut-and-join recursion for single Hurwitz numbers, both at the level of the enumeration and at the level of the partition function;
- Section 4.2.3 writes single Hurwitz numbers as a vacuum expectation in the semi-infinite wedge; and
- Section 4.2.4 performs an analysis on the semi-infinite wedge vacuum expectation to prove the polynomiality for single Hurwitz numbers.

Section 4.3 briefly introduces two prominent variations of Hurwitz numbers—monotone and spin—that have been the subject of recent study.

4.2 Single Hurwitz numbers

4.2.1 Hurwitz numbers via permutations

One can use the notion of monodromy to give an equivalent formulation of single Hurwitz numbers as an enumeration of sequences of transpositions in the symmetric group satisfying certain conditions. To motivate this equivalence, consider the following.

Let $\mu_1, \ldots, \mu_n$ be positive integers. Let $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ be a holomorphic map that has ramification profile $(\mu_1, \ldots, \mu_n)$ over ∞, simple ramification at m fixed points, and is unramified elsewhere. Let $B \subset \mathbb{CP}^1$ be the set of branch points of f.

Consider loops $\gamma, \alpha_1, \ldots, \alpha_m \in \pi_1(\mathbb{CP}^1 \setminus B, p)$, where γ is a simple loop based at p that separates ∞ from the other branch points, and each α_i separates the ith simple branch point, as shown in Figure 4.1. And now consider the concatenation $\gamma \cdot \alpha_1 \cdots \alpha_m$; the order has been chosen to match the orientation of the underlying sphere. There exists a homotopy between $\gamma \cdot \alpha_1 \cdots \alpha_m$ and a loop that does not separate any branch points, or equivalently, the constant loop; that is, the identity in $\pi_1(\mathbb{CP}^1 \setminus B; p)$.

Recall the corresponding monodromy representation

$$\varphi\colon \pi_1(\mathbb{CP}^1 \setminus B; p) \to S_d$$

defined by $\alpha \mapsto \sigma_\alpha$, where $\sigma_\alpha\colon f^{-1}(p) \to f^{-1}(p)$ is an element of $\mathrm{Sym}(f^{-1}(p))$, and by labelling the d preimages of p one can consider σ_α as an element in the symmetric group S_d on d elements. Standard results on monodromy assert that the cycle type of the permutation σ_α depends only on the ramification profile of the branch point it encloses. Specifically in our setting, γ has cycle type $(\mu_1, \ldots, \mu_n)$ and α_i has cycle type $(2, 1, \ldots, 1)$ for all $i \in \{1, 2, \ldots, m\}$. And finally, given that the concatenation $\gamma \cdot \alpha_1 \cdots \alpha_m$ is homotopic to the identity in the fundamental group, the composition $\sigma_\gamma \sigma_{\alpha_1} \cdots \sigma_{\alpha_m}$ must equal the identity permutation.

This correspondence between representations and covers is surjective due to the Riemann existence theorem; that is, for a representation $\phi\colon \pi_1(Y \setminus B) \to S_d$, such that the image of $\pi_1(Y \setminus B)$ acts transitively on $\{1, 2, \ldots, d\}$, there is a connected Riemann surface X and a non-constant holomorphic map $f\colon X \to Y$ that realises ϕ as its monodromy representation. Moreover, f and X are unique up to isomorphism. This correspondence between representations and covers leads to the following theorem [22].

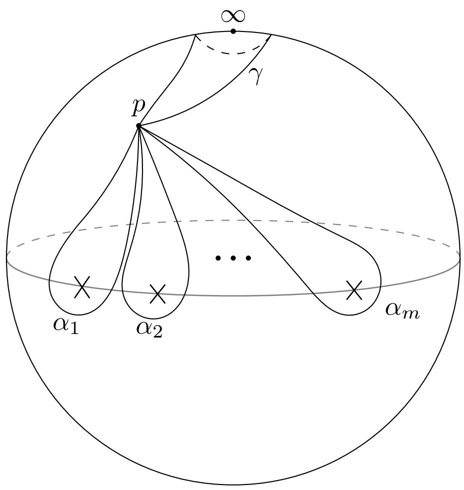

Figure 4.1: Loops $\gamma, \alpha_1, \alpha_2, \ldots, \alpha_m$ on $\mathbb{CP}^1$.

Theorem 4.2.1. *The single Hurwitz number $H_{g,n}(\mu_1, \ldots, \mu_n)$ is equal to $1/(m!\,d!)$ multiplied by the number of tuples $(\tau_1, \ldots, \tau_m)$ of permutations in the symmetric group such that*

- *the cycles of $\tau_1 \cdots \tau_m$ are labelled $1, 2, \ldots, n$ such that cycle i has length μ_i for $i \in \{1, 2, \ldots, n\}$;*
- *τ_i is a transposition for all $i \in \{1, 2, \ldots, m\}$; and*
- *$\langle \tau_1, \ldots, \tau_m \rangle \leqslant S_d$ acts transitively on $\{1, 2, \ldots, d\}$.*

Recall that $d = |\mu|$. The factor of $1/d!$ arises from the fact that the d sheets of the cover are not naturally labelled, but one naturally labels them to produce the monodromy data. As mentioned in Section 4.1, the factor $1/m!$ is primarily aesthetic, and gives rise to cleaner formulas for polynomiality and the ELSV formula.

Example 4.2.2. Calculate $H_{0,2}(2, 1)$ using Theorem 4.2.1. Here, the number of simple branch points, and hence the number of transpositions required, is $m = 2g - 2 + n + |\mu| = 3$. So to calculate $H_{0,2}(2, 1)$, one enumerates the number of tuples $(\tau_1, \tau_2, \tau_3) \in S_3^3$ such that τ_1, τ_2, τ_3 are transpositions and $\tau_1\tau_2\tau_3$ has cycle type $(2, 1)$. Below I list all possible calculations of $\tau_1\tau_2\tau_3$ for τ_1, τ_2, τ_3 transpositions in S_3, where I multiply permutations from right to left.

$(12)(12)(12) = (12)$	$(12)(12)(13) = (13)$	$(12)(12)(23) = (23)$
$(12)(13)(12) = (23)$	$(12)(13)(13) = (12)$	$(12)(13)(23) = (13)$
$(12)(23)(12) = (13)$	$(12)(23)(13) = (23)$	$(12)(23)(23) = (12)$
$(13)(12)(12) = (13)$	$(13)(12)(13) = (23)$	$(13)(12)(23) = (12)$
$(13)(13)(12) = (12)$	$(13)(13)(13) = (13)$	$(13)(13)(23) = (23)$
$(13)(23)(12) = (23)$	$(13)(23)(13) = (12)$	$(13)(23)(23) = (13)$
$(23)(12)(12) = (23)$	$(23)(12)(13) = (12)$	$(23)(12)(23) = (13)$
$(23)(13)(12) = (13)$	$(23)(13)(13) = (23)$	$(23)(13)(23) = (12)$
$(23)(23)(12) = (12)$	$(23)(23)(13) = (13)$	$(23)(23)(23) = (23)$

The only tuples that do not contribute to $H_{0,2}(2,1)$ are the three corresponding to the calculations above highlighted in red text; these three triples of transpositions do not satisfy the transitivity condition. Note that here the labelling of the cycles in $\tau_1\tau_2\tau_3$ does not affect the resulting Hurwitz number. Therefore,

$$H_{0,1}(3) = \frac{1}{m!\,d!} \cdot 24 = \frac{1}{3!\,3!} \cdot 24 = \frac{2}{3}.$$

The transitivity condition in Theorem 4.2.1 ensures the connectedness of the domain Riemann surface of the branched cover. Relaxing this condition leads to an analogous enumerative problem where the source surface may not be connected. The enumeration of such branched covers are called *disconnected Hurwitz numbers* and are denoted $H^{\bullet}_{g,n}(\mu_1,\ldots,\mu_n)$. The enumeration of the connected enumeration may henceforth be denoted $H^{\circ}_{g,n}(\mu_1,\ldots,\mu_n)$ to distinguish it from the disconnected enumeration. It is always possible to calculate the connected numbers from the disconnected enumeration, and vice versa, via inclusion-exclusion; see Section 6.3.1 in Chapter 6 on double Hurwitz numbers for an explicit inclusion–exclusion formula.

Note that a disconnected surface can lead to negative genus. This stems from the fact that the Euler characteristic is a topological invariant that is naturally additive under disjoint union, whereas the genus is not. That is, $\chi(X \sqcup Y) = \chi(X) + \chi(Y)$. For example, consider the genus of the disjoint union of two copies of the Riemann sphere: $X = \mathbb{CP}^1 \sqcup \mathbb{CP}^1$. The Euler characteristic of the Riemann sphere is

$$\chi(\mathbb{CP}^1) = 2 - 2g(\mathbb{CP}^1) = 2,$$

therefore

$$\chi(X) = \chi(\mathbb{CP}^1) + \chi(\mathbb{CP}^1) = 2 + 2 = 4.$$

Using this to calculate the genus of X via $\chi(X) = 2 - 2g(X)$, one finds $g(X) = -1$.

4.2.2 The cut-and-join recursion

Single Hurwitz numbers (and actually, many variations of Hurwitz numbers) satisfy a recursion known as the cut-and-join recursion. If one had a desire to calculate single Hurwitz numbers (by hand or by computer), the cut-and-join recursion would be one of the more efficient ways from a not-so-small number of possible ways to do so.

A sidebar on calculating single Hurwitz numbers. Ways to calculate single Hurwitz numbers include, in approximate order from least to most computationally efficient: by hand(!); via permutations (Theorem 4.2.1); using the character formula (Proposition 4.2.6); via the semi-infinite wedge (Proposition 4.2.7 or equation (4.7)); and by the cut-and-join (Proposition 4.2.3).

The cut-and-join recursion was first formulated in genus 0 by Goulden and Jackson [64], but has since been generalised to all genus, again by Goulden and Jackson [63].

Proposition 4.2.3 (Cut-and-join recursion, Goulden and Jackson [63]). *The single Hurwitz numbers satisfy the equation*

$$\begin{aligned} mH^{\circ}_{g,n}(\mu_1,\ldots,\mu_n) &= \sum_{i<j} (\mu_i+\mu_j) H^{\circ}_{g,n-1}(\vec{\mu}_{S\setminus\{i,j\}}, \mu_i+\mu_j) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \sum_{\alpha+\beta=\mu_i} \alpha\beta \left[H^{\circ}_{g-1,n+1}(\alpha,\beta,\vec{\mu}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}} H^{\circ}_{g_1,|I|+1}(\alpha,\vec{\mu}_I)\, H^{\circ}_{g_2,|J|+1}(\beta,\vec{\mu}_J) \right]. \end{aligned} \tag{4.3}$$

Here $S = \{1,2,\ldots,n\}$, and for $I = \{i_1,\ldots,i_k\}$ the shorthand notation $\vec{\mu}_I$ denotes $\mu_{i_1},\ldots,\mu_{i_k}$. Further, the recursion uniquely determines all Hurwitz numbers from the base case $H_{0,1}(1) = 1$ corresponding to the unique connected branched cover with no ramification, namely, the identity map $\mathbb{CP}^1 \to \mathbb{CP}^1$).

This can be proven combinatorially via the permutation interpretation for single Hurwitz numbers, by analysing the result of multiplying both sides of the equation $\tau_1 \ldots \tau_m = \sigma^{-1}$ on the right by the transposition

τ_m. Alternatively, this can also be proven at the level of branched covers by considering what happens as one of the simple branch points approaches infinity. In particular, note that both of these processes are not sensitive to the ramification over 0 (or elsewhere), and hence variations of Hurwitz numbers (including orbifold and double Hurwitz numbers) also satisfy essentially the same recursion, but necessarily with different base cases.

The cut-and-join recursion can also be expressed at the level of the partition function as a differential equation. First, define the partition function for single Hurwitz numbers as follows:

$$Z(p_1, p_2, \ldots; \hbar; s) = \exp\left[\sum_{g\geqslant 0}\sum_{n\geqslant 1}\sum_{\mu_1,\ldots,\mu_n\geqslant 1} H^\circ_{g,n}(\mu_1, \ldots, \mu_n)\, s^{2g-2+n+|\mu|}\, \frac{\hbar^{2g-2+n}}{n!} p_{\mu_1}\cdots p_{\mu_n}\right].$$

Then we have the following reformulation of the cut-and-join recursion, obtained using the usual generating function tricks Goulden and Jackson [63].

Proposition 4.2.4 (Cut-and-join recursion, Goulden and Jackson [63]). *The partition function for single Hurwitz numbers satisfies the following differential equation*

$$\frac{\partial}{\partial s} Z(p_1, p_2, \ldots; \hbar; s) = \frac{1}{2}\left[\sum_{i,j\geqslant 1} ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} + (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}}\right] Z(p_1, p_2, \ldots; \hbar; s),$$

with initial condition $Z(p_1, p_2, \ldots; \hbar; s)|_{s=0} = \exp(p_1)$.

Obtaining the spectral curve from the cut-and-join recursion

As may be commonly known to those studying topological recursion, the spectral curve often directly stores the $(0, 1)$ data for an enumerative problem. Define the *free energies*

$$F_{g,n}(x_1, \ldots, x_n) = \sum_{\mu_1,\ldots,\mu_n\geqslant 1} H^\circ_{g,n}(\mu_1, \ldots, \mu_n) \exp(\mu_1 x_1)\cdots \exp(\mu_n x_n).$$

Then, the Bouchard–Mariño conjecture, Theorem 4.1.4, gives

$$\omega_{g,n} = \mathrm{d}_1 \cdots \mathrm{d}_n\, F_{g,n}(x_1, \ldots, x_n),$$

where d_i denotes applying the exterior derivative to the ith variable. In particular,

$$y\, \mathrm{d}x = \omega_{0,1} = \mathrm{d}F_{0,1}(x) = \sum_{\mu\geqslant 1} H^\circ_{0,1}(\mu)\, \mu \exp(\mu x)\, \mathrm{d}x.$$

One can often solve the $(0, 1)$ part of the enumerative problem by taking the recursion in the case of $(0, 1)$ and expressing this as an equation in terms of x and y, leading to an expression for the spectral curve. Let us see how this works in the case of single Hurwitz numbers. First, recall the spectral curve for single Hurwitz numbers: $x(z) = \ln z - z$, $y(z) = z$, or in unparametrised form, $e^x = ye^{-y}$.

For $(g, n) = (0, 1)$, the cut-and-join recursion (4.3) reduces to

$$(\mu - 1)H^\circ_{0,1}(\mu) = \frac{1}{2}\sum_{\alpha+\beta=\mu} \alpha\beta\, H^\circ_{0,1}(\alpha)\, H^\circ_{0,1}(\beta).$$

Multiply both sides by $\exp(\mu x)$ and sum over all $\mu \geqslant 1$. This yields

$$\sum_{\mu\geqslant 1}(\mu - 1)H^\circ_{0,1}(\mu) \exp(\mu x) = \frac{1}{2}\sum_{\mu\geqslant 1}\left[\sum_{\alpha+\beta=\mu} \alpha\beta\, H^\circ_{0,1}(\alpha)\, H^\circ_{0,1}(\beta)\right] \exp(\mu x),$$

or in equivalent generating function form,

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} - 1\right] F_{0,1}(x) = \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}x} F_{0,1}(x)\right]^2.$$

One can either solve this differential equation, or in this case where the spectral curve is already known, check that the spectral curve satisfies this differential equation. That is, check that $x(z) = \ln z - z$, $y(z) = z$ satisfies this differential equation where $\mathrm{d}F_{0,1}(x) = y\,\mathrm{d}x$. To do this, first apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{0,1}(x) - \frac{\mathrm{d}}{\mathrm{d}x}F_{0,1}(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_{0,1}(x) \cdot \frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{0,1}(x).$$

From $\mathrm{d}F_{0,1}(x) = y\,\mathrm{d}x$, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{0,1}(x) = \frac{\mathrm{d}}{\mathrm{d}x}y = \frac{1}{\frac{1}{z}-1}\frac{\mathrm{d}}{\mathrm{d}z}y = \frac{z}{1-z}.$$

Substituting both of these into the left side yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{0,1}(x) - \frac{\mathrm{d}}{\mathrm{d}x}F_{0,1}(x) = \frac{z}{1-z} - z = \frac{z - z(1-z)}{1-z} = \frac{z^2}{1-z},$$

while substituting them into the right side gives

$$\frac{\mathrm{d}}{\mathrm{d}x}F_{0,1}(x) \cdot \frac{\mathrm{d}^2}{\mathrm{d}x^2}F_{0,1}(x) = \frac{z^2}{1-z},$$

as required.

4.2.3 Hurwitz numbers via the semi-infinite wedge

Single Hurwitz numbers can be packaged via the semi-infinite wedge as an appropriate vacuum expectation. This section will follow standard procedure to prove this result.

Begin by deriving a character formula for single Hurwitz numbers. A general formula for enumerating sequences of elements in specified conjugacy classes that multiply to give the identity in a finite group G is originally attributable to Frobenius [79, Appendix A]. However, rather than providing the result and proof in full generality, I will instead derive the character formula in the specific setting of symmetric groups.

To enumerate Hurwitz numbers in the context of permutations in the symmetric group (via monodromy; see Section 4.2.1), we wish to count the number of tuples of permutations $(\sigma_1, \ldots, \sigma_k)$ such that σ_i has cycle type λ_i, and the cycles of $\sigma_1 \cdots \sigma_k = \mathrm{id}$. Let $C_\lambda \in Z\mathbb{C}[S_d]$ be the conjugacy class considered as an element of the centre of the symmetric group algebra corresponding to the partition λ of d. That is,

$$C_\lambda = \sum_{\sigma} \sigma,$$

where the sum is over all $\sigma \in S_d$ such that σ has cycle type λ.[2] For example,

$$C_{(3,1)} = (123) + (132) + (124) + (142) + (234) + (243) \in Z\mathbb{C}[S_4].$$

The number of such tuples of permutations $(\sigma_1, \ldots, \sigma_k)$ that satisfy these two conditions is equal to the coefficient of $C_{(1)} = (1)$ in the product $C_{\lambda_1} \cdots C_{\lambda_k} \in Z\mathbb{C}[S_d]$.[3] And indeed, one can calculate possibly disconnected Hurwitz numbers this way and recover the connected enumeration thereafter through inclusion-exclusion.

Example 4.2.5. Calculate the single Hurwitz number $H^{\bullet}_{0,1}(3)$. The number of simple branch points is $m = 2g-2+n+|\mu| = 2$. Note that enumerating the number of tuples $(\tau_1, \ldots, \tau_m)$ of transpositions satisfying the conditions given in Theorem 4.2.1 is equivalent to enumerating tuples $(\sigma, \tau_1, \ldots, \tau_m)$ of permutations in

[2] Note that I may use a slight abuse of notation throughout this thesis, where C_λ can refer to either the conjugacy class (as a set) corresponding to λ in the symmetric group, or the formal sum of all permutations with cycle type λ in the group algebra. I would also like to note that this abuse of notation is common(!) in the literature, in my experience often without explanation. It's a wild world we live in.

[3] Here $C_{(1)}$ is another slight abuse of notation, denoting $C_{(1^d)}$; it is common in this thesis to write (1) in place of (1^d).

the symmetric group where σ has cycle type μ, τ_i is a transposition for $i \in \{1, 2, \ldots, m\}$, and $\sigma\tau_1 \cdots \tau_m = \mathrm{id}$, and counting each tuple with weight $|\mathrm{Aut}\ \mu|$.

Thus one can calculate $H^\bullet_{0,1}(3)$ using the following.

$$\begin{aligned}[C_{(1)}]C_{(3)}C^2_{(2,1)} &= [C_{(1)}]((123) + (132))((12) + (13) + (23))^2 \\ &= 3\,[C_{(1)}]((123) + (132))((1) + (123) + (132)) \\ &= 6\,[C_{(1)}]((1) + (123) + (132)) = 6\,[C_{(1)}](C_{(1)} + C_{(3)}) = 6\end{aligned}$$

Here I am multiplying permutations from right to left. And so we find that the single Hurwitz number is

$$H^\bullet_{0,1}(3) = \frac{|\mathrm{Aut}\ \mu|}{m!\,d!}[C_{(1)}]C_{(3)}C^2_{(2,1)} = \frac{6}{2!\,3!} = \frac{1}{2}.$$

Note that the connected and possibly disconnected enumerations coincide for $n = 1$.

Recall that I aim to obtain a character formula for single Hurwitz numbers. To do this, one can rewrite the product $C_{\lambda_1} \cdots C_{\lambda_k}$ as a product of orthogonal idempotents $E_{\rho_1} \cdots E_{\rho_k}$ using the fact that that both the conjugacy classes $\{C_\lambda\}$ and the orthogonal idempotents $\{E_\rho\}$ form a linear basis for the centre of the group algebra.

First I set a number notations. Denote by χ^ρ_λ the character of the irreducible representation corresponding to ρ evaluated on an element of C_λ; $\dim\rho = \chi^\rho_{(1)}$ is the dimension of the irreducible representation corresponding to ρ, and $\rho \vdash d$ denotes that ρ is a partition of the integer d. For an introduction on the representation theory of the symmetric group, see the book of Sagan [97].

Then following the steps outlined above yields

$$\begin{aligned}[C_{(1)}]C_{\lambda_1} \cdots C_{\lambda_k} &= [C_{(1)}]\prod_{i=1}^{k}\left[|C_{\lambda_i}|\sum_{\rho_i}\frac{\chi^{\rho_i}_{\lambda_i}}{\dim\rho_i}E_{\rho_i}\right] \\ &= [C_{(1)}]\,|C_{\lambda_1}| \cdots |C_{\lambda_k}|\sum_{\rho\vdash d}\frac{\chi^\rho_{\lambda_1}\cdots\chi^\rho_{\lambda_k}}{(\dim\rho)^k}E_\rho \\ &= |C_{\lambda_1}| \cdots |C_{\lambda_k}|\sum_{\rho\vdash d}\frac{\chi^\rho_{\lambda_1}\cdots\chi^\rho_{\lambda_k}}{(\dim\rho)^k}[C_{(1)}]\left[\frac{\dim\rho}{d!}\sum_{\lambda\vdash d}\chi^\rho_\lambda C_\lambda\right] \\ &= |C_{\lambda_1}| \cdots |C_{\lambda_k}|\sum_{\rho\vdash d}\frac{\chi^\rho_{\lambda_1}\cdots\chi^\rho_{\lambda_k}}{(\dim\rho)^k}\left[\frac{\dim\rho}{d!}\chi^\rho_{(1)}\right] \\ &= \frac{|C_{\lambda_1}| \cdots |C_{\lambda_k}|}{d!}\sum_{\rho\vdash d}\frac{\chi^\rho_{\lambda_1}\cdots\chi^\rho_{\lambda_k}}{(\dim\rho)^{k-2}}.\end{aligned}$$

The first and third equalities are using the change of basis relation between the two bases for the centre of the group algebra; that is, C_λ and the orthogonal idempotents E_ρ, which are given by

$$C_\lambda = |C_\lambda|\sum_\rho\frac{\chi^\rho_\lambda}{\dim\rho}E_\rho \qquad \text{and} \qquad E_\rho = \frac{\dim\rho}{d!}\sum_{\lambda\vdash d}\chi^\rho_\lambda C_\lambda,$$

respectively. The second equality is using the fundamental relation for orthogonal idempotents $E_\rho E_\mu = \delta_{\rho,\mu}E_\rho$, while the fifth and final equality uses that $\chi^\rho_{(1)} = \dim\rho$. Thus, the number of tuples of permutations $(\sigma_1, \ldots, \sigma_k)$ such that σ_i has cycle type λ_i, and $\sigma_1 \cdots \sigma_k = (1)$ is equal to

$$\frac{|C_{\lambda_1}| \cdots |C_{\lambda_k}|}{d!}\sum_{\rho\vdash d}\frac{\chi^\rho_{\lambda_1}\cdots\chi^\rho_{\lambda_k}}{(\dim\rho)^{k-2}}.$$

In the case of single Hurwitz numbers, by Theorem 4.2.1, the single Hurwitz number $H_{g,n}(\mu_1, \ldots, \mu_n)$ is equal to $\frac{|\mathrm{Aut}\ \mu|}{m!\,d!}$ multiplied by the number of tuples of permutations $(\sigma, \tau_1, \ldots, \tau_m)$ such that

- $\sigma\tau_1\cdots\tau_m = (1)$;
- σ has cycle type $(\mu_1,\ldots,\mu_n)$ while τ_i is a transposition for all $i \in \{1,\ldots,m\}$; and
- $\langle\tau_1,\ldots,\tau_m\rangle \leqslant S_d$ acts transitively on $\{1,\ldots,d\}$.

If one drops the transitivity condition (since our calculation via the group algebra above imposes no such condition), then using the calculation above one can conclude that the possibly disconnected single Hurwitz number can be calculated by a specific version of the character formula. The exact statement is given in the proposition below, and has used that the size of the conjugacy class corresponding to μ is

$$|C_\mu| = \frac{d!}{|\mathrm{Aut}\ \mu| \prod_{i=1}^n \mu_i}.$$

Proposition 4.2.6. *The disconnected single Hurwitz number $H^\bullet_{g,n}(\mu_1,\ldots,\mu_n)$ satisfies*

$$H^\bullet_{g,n}(\mu_1,\ldots,\mu_n) = \frac{1}{m!\,d!\,\prod_{i=1}^n \mu_i} \sum_{\rho\vdash d} \frac{\left[|C_{(2,1,\ldots,1)}\,\chi^\rho_{(2,1,\ldots,1)}\right]^m}{(\dim\rho)^{m-1}}\,\chi^\rho_\mu. \tag{4.4}$$

The character formula calculates possibly disconnected Hurwitz numbers because, while all other conditions given in Theorem 4.2.1 are satisfied, the character formula does not enforce that $\langle\tau_1,\ldots,\tau_m\rangle \leqslant S_d$ acts transitively on $\{1,2,\ldots,d\}$.

We can now use the character formula of Proposition 4.2.6 to derive a vacuum expectation for single Hurwitz numbers using the semi-infinite wedge. First, define the following generating function for disconnected single Hurwitz numbers

$$h^\bullet_{\vec{\mu}}(s) = \sum_{g\in\mathbb{Z}} H^\bullet_{g,n}(\mu_1,\ldots,\mu_n)s^{2g-2+n}. \tag{4.5}$$

Proposition 4.2.7. *The generating function for disconnected single Hurwitz numbers $h^\bullet_{\vec{\mu}}(s)$ is equal to the following vacuum expectation in the semi-infinite wedge.*

$$h^\bullet_{\vec{\mu}}(s) = \left\langle \exp\left(\frac{\alpha_1}{s}\right)\exp(s\mathcal{F}_2)\prod_{i=1}^n \frac{\alpha_{-\mu_i}}{\mu_i}\right\rangle \tag{4.6}$$

Here, the bosonic operators, $\alpha_{\pm m}$, and the diagonal operator $\mathcal{F}_2$ are defined in Definition 1.3.4 and equation (1.10) *respectively.*

Proof. Consider the action of the product of operators $e^{\alpha_1/s}e^{s\mathcal{F}_2}\prod_{i=1}^n \frac{\alpha_{-\mu_i}}{\mu_i}$ on the vacuum vector.

$$\begin{aligned}
\exp\left(\frac{\alpha_1}{s}\right)\exp(s\mathcal{F}_2)\prod_{i=1}^n \frac{\alpha_{-\mu_i}}{\mu_i}v_\emptyset &= \exp\left(\frac{\alpha_1}{s}\right)\exp(s\mathcal{F}_2)\sum_{\lambda\vdash|\mu|}\frac{\chi^\lambda_\mu}{\prod_{i=1}^n \mu_i}v_\lambda \\
&= \sum_{\lambda\vdash|\mu|}\frac{\chi^\lambda_\mu}{\prod_{i=1}^n \mu_i}\exp\left(\frac{\alpha_1}{s}\right)\sum_{m=0}^\infty \frac{s^m}{m!}\mathcal{F}_2^m v_\lambda \\
&= \sum_{m\geqslant 0}\sum_{\lambda\vdash|\mu|}\frac{\chi^\lambda_\mu}{\prod_{i=1}^n \mu_i}\frac{s^m}{m!}\exp\left(\frac{\alpha_1}{s}\right)f_2(\lambda)^m v_\lambda \\
&= \sum_{m\geqslant 0}\sum_{\lambda\vdash|\mu|}\frac{\chi^\lambda_\mu}{\prod_{i=1}^n \mu_i}\frac{s^m\,f_2(\lambda)^m}{m!}\sum_{k\geqslant 0}\frac{\alpha_1^k}{s^k\,k!}v_\lambda
\end{aligned}$$

The first equality is using the Murnaghan–Nakayama rule, Theorem 1.4.3. The second and fourth equalities are expanding $\exp(s\mathcal{F}_2)$ and $\exp(\alpha_1/s)$ as power series respectively. The third equality is using the fact that $\mathcal{F}_2$ is diagonal with eigenvalue $f_2(\lambda)$ for the eigenvector v_λ, and $f_2(\lambda)$ in turn is given by the sum of the contents of λ as a Young diagram. For a Young diagram corresponding to partition λ, the content of a box in column j and row i is $j - i$.

Recall that we are calculating the inner product

$$\Big\langle \exp\Big(\frac{\alpha_1}{s}\Big)\exp(s\mathcal{F}_2)\prod_{i=1}^{n}\frac{\alpha_{-\mu_i}}{\mu_i}\Big\rangle,$$

hence, only the summand in the sum over k with $k=|\lambda|$ will provide a non-zero contribution. In this case

$$\begin{aligned}\Big\langle \exp\Big(\frac{\alpha_1}{s}\Big)\exp(s\mathcal{F}_2)\prod_{i=1}^{n}\frac{\alpha_{-\mu_i}}{\mu_i}\Big\rangle &= \sum_{m\geqslant 0}\sum_{\lambda\vdash|\mu|}\frac{\chi^\lambda_\mu}{\prod_{i=1}^n\mu_i}\frac{s^m\, f_2(\lambda)^m}{m!}\Big\langle\sum_{k\geqslant 0}\frac{\alpha_1^k}{s^k\,k!}v_\lambda, v_\emptyset\Big\rangle\\ &= \sum_{m\geqslant 0}\sum_{\lambda\vdash|\mu|}\frac{\chi^\lambda_\mu}{\prod_{i=1}^n\mu_i}\frac{s^m\, f_2(\lambda)^m}{m!}\Big\langle\frac{\alpha_1^{|\lambda|}}{s^{|\lambda|}\,|\lambda|!}v_\lambda, v_\emptyset\Big\rangle\\ &= \sum_{m\geqslant 0}\sum_{\lambda\vdash|\mu|}\frac{\chi^\lambda_\mu}{\prod_{i=1}^n\mu_i}\frac{s^m\, f_2(\lambda)^m}{m!}\frac{\dim\lambda}{s^{|\lambda|}\,|\lambda|!}.\end{aligned}$$

The third equality uses that $\langle\alpha_1^{|\lambda|}v_\lambda, v_\emptyset\rangle$ is equal to the number of standard Young tableaux of size λ, and that this in turn is equal to the dimension of the irreducible representation of the symmetric group labelled by λ. That is, $\langle\alpha_1^{|\lambda|}v_\lambda, v_\emptyset\rangle=\dim\lambda$. Now use that, as defined in equation (1.11),

$$f_2(\lambda)=\frac{|C_{(2,1,\dots,2)}|\chi^\lambda_{(2,1,\dots,2)}}{\dim\lambda}.$$

This allows us to rewrite the vacuum expectation

$$\begin{aligned}\Big\langle \exp\Big(\frac{\alpha_1}{s}\Big)\exp(s\mathcal{F}_2)\prod_{i=1}^{n}\frac{\alpha_{-\mu_i}}{\mu_i}\Big\rangle &= \sum_{m\geqslant 0}\sum_{\lambda\vdash|\mu|}\frac{1}{m!\,d!\,\prod_{i=1}^n\mu_i}\frac{\Big[|C_{(2,1,\dots,2)}|\chi^\lambda_{(2,1,\dots,2)}\Big]^m}{(\dim\lambda)^{m-1}}\chi^\lambda_\mu\, s^{m-|\lambda|}\\ &= \sum_{m\geqslant 0}H^\bullet_{g,n}(\mu_1,\dots,\mu_n)s^{m-|\mu|}=h^\bullet_{\vec{\mu}}(s),\end{aligned}$$

as required. The first equality is using the character formula in Proposition 4.2.6 while the final equality is using the definition of the generating function $h^\bullet_{\vec{\mu}}(s)$. ■

Thus, single Hurwitz numbers can be packaged as a vacuum expectation in the semi-infinite wedge, and as we will see in the subsequent section, the semi-infinite wedge provides a suitable environment for proving the polynomiality structure of single Hurwitz numbers. To prove this polynomiality structure, we will rewrite the vacuum expectation for single Hurwitz numbers using the $\mathcal{E}$-operators defined in Definition 1.3.6. This is what is being done in the next lemma.

Lemma 4.2.8. *The generating function for disconnected single Hurwitz numbers $h^\bullet_{\vec{\mu}}(s)$ is equal to the following vacuum expectation in the semi-infinite wedge:*

$$h^\bullet_{\vec{\mu}}(s)=\frac{1}{\prod_{i=1}^n\mu_i}\sum_{k_1+\cdots+k_n=0}\Bigg[\prod_{i=1}^{n}\frac{\mu_i^{\mu_i-k_i}\mathcal{S}(\mu_i s)^{\mu_i-k_i}}{(\mu_i-k_i)!}\Bigg]\Big\langle\mathcal{E}_{-k_1}(\mu_1 s)\cdots\mathcal{E}_{-k_n}(\mu_n s)\Big\rangle, \tag{4.7}$$

where $\mathcal{S}(z)=\frac{\varsigma(z)}{z}=\frac{e^{z/2}-e^{-z/2}}{z}$ and $\mathcal{E}_a(z)$ is as defined in Definition 1.3.6. Or, alternatively,

$$h^\bullet_{\vec{\mu}}(s)=\langle\mathcal{C}(\mu_1,s)\cdots\mathcal{C}(\mu_n,s)\rangle,$$

where

$$\mathcal{C}(\mu,s):=\frac{1}{\mu}\sum_{k\leqslant\mu}\frac{\mu^{\mu-k}\mathcal{S}(\mu s)^{\mu-k}}{(\mu-k)!}\mathcal{E}_{-k}(\mu s).$$

Proof. Begin with the vacuum expectation for the generating function $h^\bullet_{\vec{\mu}}(s)$ given in Proposition 4.2.7,

$$h^\bullet_{\vec{\mu}}(s)=\Big\langle\exp(\alpha_1/s)\exp(s\mathcal{F}_2)\prod_{i=1}^{n}\frac{\alpha_{-\mu_i}}{\mu_i}\Big\rangle.$$

Observe that $\exp(-s\mathcal{F}_2)\exp(-\alpha_1/s)$ fix the vacuum vector. This allows us to write

$$h^\bullet_{\vec{\mu}}(s) = \frac{1}{\prod_{i=1}^n \mu_i}\Big\langle \prod_{i=1}^n \exp(\alpha_1/s)\exp(s\mathcal{F}_2)\alpha_{-\mu_i}\exp(-s\mathcal{F}_2)\exp(-\alpha_1/s)\Big\rangle.$$

One can compute the inner conjugation by observing that $\alpha_{-\mu} = \mathcal{E}_{-\mu}(0)$ and $\mathcal{F}_2 = [z^2]\mathcal{E}_0(z)$, and using Hadamard's lemma, given below as equation (4.8), coupled with the commutation relation (1.9) for the $\mathcal{E}$-operators, $[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma(aw - bz)\mathcal{E}_{a+b}(z+w)$. Or, alternatively, observe that the operator $\mathcal{F}_2$ is diagonal with eigenvalue $f_2(\lambda)$ corresponding to the eigenvector v_λ. As mentioned previously, the function $f_2(\lambda)$ is equal to the sum of the contents of the Young diagram given by the partition λ. In this case, for any partition λ,

$$\begin{aligned} e^{s\mathcal{F}_2}\alpha_{-\mu}e^{-s\mathcal{F}_2}v_\lambda &= e^{-sf_2(\lambda)}e^{s\mathcal{F}_2}\alpha_{-\mu}v_\lambda = e^{-sf_2(\lambda)}e^{s\mathcal{F}_2}\sum_{\lambda^{+\mu}} v_{\lambda^{+\mu}} \\ &= \sum_{\lambda^{+\mu}} e^{s\left(f_2(\lambda^{+\mu}) - f_2(\lambda)\right)}v_{\lambda^{+\mu}} = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{s\mu\left(k+\frac{\mu}{2}\right)} :\psi_{k+\mu}\psi^*_k: v_\lambda \\ &= \mathcal{E}_{-\mu}(\mu s), \end{aligned}$$

where the sum over $\lambda^{+\mu}$ is over all Young diagrams that can be obtained from adding a μ-ribbon to λ.[4] Hence,

$$h^\bullet_{\vec{\mu}}(s) = \frac{1}{\prod_{i=1}^n \mu_i}\Big\langle \prod_{i=1}^n \exp(\alpha_1/s)\,\mathcal{E}_{-\mu_i}(\mu_i s)\exp(-\alpha_1/s)\Big\rangle.$$

Compute the outer conjugation using Hadamard's lemma,

$$e^A B e^{-A} = B + \sum_{k\geqslant 1}\frac{1}{k!}[A,[A,\ldots,[A,B]\cdots]], \tag{4.8}$$

where there are k commutators in the kth summand. Doing so gives

$$\exp(\alpha_1/s)\mathcal{E}_{-\mu}(\mu s)\exp(-\alpha_1/s) = \sum_{m\geqslant 0}\frac{\varsigma(\mu s)^m}{m!\,s^m}\mathcal{E}_{-\mu+m}(\mu s).$$

Therefore,

$$\begin{aligned} h^\bullet_{\vec{\mu}}(s) &= \frac{1}{\prod_{i=1}^n \mu_i}\Big\langle \prod_{i=1}^n \exp(\alpha_1/s)\,\mathcal{E}_{-\mu_i}(\mu_i s)\exp(-\alpha_1/s)\Big\rangle \\ &= \frac{1}{\prod_{i=1}^n \mu_i}\Big\langle \prod_{i=1}^n \sum_{m_i\geqslant 0}\frac{\varsigma(\mu_i s)^{m_i}}{m_i!\,s^{m_i}}\mathcal{E}_{-\mu_i+m_i}(\mu_i s)\Big\rangle \\ &= \frac{1}{\prod_{i=1}^n \mu_i}\sum_{k_1+\cdots+k_n=0}\left[\prod_{i=1}^n \frac{\mu_i^{\mu_i-k_i}\mathcal{S}(\mu_i s)^{\mu_i-k_i}}{(\mu_i-k_i)!}\right]\Big\langle \mathcal{E}_{-k_1}(\mu_1 s)\cdots\mathcal{E}_{-k_n}(\mu_n s)\Big\rangle \\ &= \langle \mathcal{C}(\mu_1,s)\cdots\mathcal{C}(\mu_n,s)\rangle \end{aligned}$$

The third equality above has relabelled $m_i \mapsto \mu_i - k_i$ for $i\in\{1,2,\ldots,n\}$ and used the fact that the energies of the $\mathcal{E}$-operators must sum to zero for the vacuum expectation to provide a contribution. The fact that $k_i \leqslant \mu_i$ stems from $m_i \geqslant 0$. ■

4.2.4 Polynomiality of Hurwitz numbers

This section is dedicated to proving the polynomiality structure of single Hurwitz numbers that was indicated at the start of this chapter. It may be interesting to note that the polynomiality of single Hurwitz

[4]This latter proof that the conjugation of $\alpha_{-\mu}$ by $e^{s\mathcal{F}_2}$ leads to the $\mathcal{E}$-operator is somewhat more enlightening than the route via Hadamard's lemma; and in fact, it gives some sort of justification for the definition of the $\mathcal{E}$-operator itself.

numbers was not originally proved using this method, but was proven as a direct corollary of the ELSV formula, Theorem 4.1.3. Of course in the setting of single Hurwitz numbers, where the ELSV formula was proven using algebro-geometric methods and polynomiality is a direct consequence, it is not necessary to resort to proving the polynomiality structure via the semi-infinite wedge. However, in analogous or generalised settings, obtaining an ELSV-like formula for Hurwitz-type enumerations is a daydream, and hence it is desirable to prove polynomiality another way.

So far, the primary way of doing such a task has been via the semi-infinite wedge; this has been done for orbifold Hurwitz numbers [40], and double Hurwitz numbers (this is the content of Chapter 6). However, a complete and accurate proof of the polynomiality of single Hurwitz numbers using the semi-infinite wedge is lacking. For this reason, I lay out the method here; I do this not only for a sense of completeness, but as a demonstrative tool. My approach is necessarily a specialisation of the results mentioned above for orbifold Hurwitz numbers and double Hurwitz numbers.

One might question why polynomiality-like structures for these types of enumerations is so highly sought-after. The most pertinent reason in recent times is that polynomiality, or a polynomiality-like structure, is a necessary ingredient to prove that an enumeration is governed by topological recursion, which can then lead to deeper geometric information on the original enumeration.

Theorem 4.1.2 (Conjectured by Goulden, Jackson and Vainshtein [65], proven by Ekedahl, Lando, Shapiro and Vainshtein [43]). For $2g-2+n>0$, there exist symmetric polynomials $P_{g,n}(\mu_1,\ldots,\mu_n)$ such that

$$H^\circ_{g,n}(\mu_1,\ldots,\mu_n) = \left[\prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!}\right] P_{g,n}(\mu_1,\ldots,\mu_n).$$

Proof. First, fix $\mu_2,\ldots,\mu_n$ to be positive integers. The aim of the proof is to consider the dependence of $h^\bullet_{\vec{\mu}}(s)$ on μ_1, then to use the symmetry of $H^\circ_{g,n}(\mu_1,\ldots,\mu_n)$ to deduce polynomiality in $\mu_2,\ldots,\mu_n$.

The main steps of the proof are as follows.

1. Use the vacuum expectation from Lemma 4.2.8 to show that, aside from the combinatorial factor present in (4.1), $h^\bullet_{\vec{\mu}}(s)$ is a rational function in μ_1 with at most simple poles at $\mu_1=-j$ for all $j\in\{1,2,\ldots,\mu_2+\cdots+\mu_n\}$ and a double pole at $\mu_1=0$.
2. Study the residues of $h^\bullet_{\vec{\mu}}(s)$ at $\mu_1=-j$ for $j\in\{1,2,\ldots,\mu_2+\cdots+\mu_n\}$ and $j=0$ (Lemmas 4.2.9 and 4.2.10 respectively) to conclude that $H^\circ_{g,n}(\mu_1,\ldots,\mu_n)$ is indeed polynomial in μ_1.
3. Conclude the polynomiality structure of single Hurwitz numbers by invoking the symmetry of the Hurwitz number in $\mu_1,\ldots,\mu_n$.

Begin with the vacuum expectation from Lemma 4.2.8,

$$h^\bullet(s) = \frac{1}{\prod_{i=1}^n \mu_i} \sum_{k_1+\cdots+k_n=0} \left[\prod_{i=1}^n \frac{\mu_i^{\mu_i-k_i}\mathcal{S}(\mu_i s)^{\mu_i-k_i}}{(\mu_i-k_i)!}\right] \Big\langle \mathcal{E}_{-k_1}(\mu_1 s)\cdots\mathcal{E}_{-k_n}(\mu_n s)\Big\rangle,$$

and focus on the dependence on μ_1.

The fact that the leftmost $\mathcal{E}$-operator acts on the covacuum dictates that $k_1\leqslant 0$ for the vacuum expectation to be non-zero. Further, from the proof of Lemma 4.2.8 we have that $-k_i=-\mu_i+m_i$ for m_i a non-negative integer, hence it follows that $-k_i\geqslant -\mu_i$ for all $i\in\{2,\ldots,n\}$. Therefore, using $k_1=-k_2-\cdots-k_n$ gives us the following bounds on k_1:

$$-\mu_2-\cdots-\mu_n\leqslant k_1\leqslant 0.$$

Relabelling $-k_1\mapsto k$, one can write

$$\begin{aligned} h^\bullet_{\vec{\mu}}(s) &= \sum_{k=0}^{\mu_2+\cdots+\mu_n} \frac{\mu_1^{\mu_1+k-1}\mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+k)!}\Big\langle \mathcal{E}_k(\mu_1 s)\,\mathcal{C}(\mu_2,s)\cdots\mathcal{C}(\mu_n,s)\Big\rangle \\ &= \frac{\mu_1^{\mu_1}}{\mu_1!}\Big\langle \mathcal{B}(\mu_1,s)\,\mathcal{C}(\mu_2,s)\cdots\mathcal{C}(\mu_n,s)\Big\rangle, \end{aligned} \tag{4.9}$$

where

$$B(\mu_1, s) := \sum_{k=0}^{\mu_2+\cdots+\mu_n} \frac{\mu_1^{k-1}\mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+1)\cdots(\mu_1+k)}\mathcal{E}_k(\mu_1 s).$$

Observe that the combinatorial factor $\frac{\mu_1^{\mu_1}}{\mu_1!}$ is present, and that $B(\mu_1, s)$ is a finite linear combination of $\mathcal{E}$-operators whose coefficients are power series in s. Hence, for each fixed power of s, its coefficient in $B(\mu_1, s)$ is a rational function in μ_1 with, as stated above, at worst simple poles at negative integers, and a double pole at zero.

Hence, to deduce the polynomiality of the connected single Hurwitz number $H^{\circ}_{g,n}(\mu_1, \ldots, \mu_n)$ in μ_1, it remains to prove that, after inclusion-exclusion has been applied to obtain the connected contribution and the coefficient of s has been extracted, the resultant single Hurwitz number is no longer a rational function in μ_1 but a polynomial in μ_1 (with the combinatorial coefficient). This can be shown by considering an appropriate residue of $\langle \mathcal{B}(\mu_1, s)\,\mathcal{C}(\mu_2, s)\cdots\mathcal{C}(\mu_n, s)\rangle$ at $\mu_1 = -j$ for $j \in \{0, 1, \ldots, \mu_2 + \cdots + \mu_n\}$.

I will now prove two mini-lemmas concerning the poles. Lemmas 6.3.7 and 6.3.8 in Chapter 6 are the analogous results pertaining to the residues in the more general setting of double Hurwitz numbers.

Lemma 4.2.9. *Fix $\mu_2, \ldots, \mu_n$ to be positive integers, and fix $n \geqslant 2$. Then for all $j \in \{1, 2, \ldots, \mu_2 + \cdots + \mu_n\}$,*

$$\operatorname*{Res}_{\mu_1=-j} \Big\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle \, d\mu_1 = F(j) \Big\langle \exp(\alpha_1/s) \exp(s\mathcal{F}_2)\alpha_j \exp(-s\mathcal{F}_2) \exp(-\alpha_1/s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle,$$

where $F(j) = \frac{j^j}{j!}$.

Proof. Given that $h^{\bullet}_{\vec{\mu}}(s)$ has at most simple poles at $\mu_1 = -j$ for $j \in \{1, 2, \ldots, \mu_2 + \cdots + \mu_n\}$, one can calculate the residue of $\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s)\rangle$ at $\mu_1 = -j$ by multiplying by $(\mu_1 + j)$ then taking the limit $\mu_1 \to -j$. That is,

$$\operatorname*{Res}_{\mu_1=-j} \Big\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle d\mu_1 = \lim_{\mu_1=-j} (\mu_1 + j) \sum_{k=0}^{\mu_2+\cdots+\mu_n} \frac{\mu_1^{k-1}\mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+1)\cdots(\mu_1+k)} \Big\langle \mathcal{E}_k(\mu_1 s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle.$$

This is non-zero only when $j \in \{1, 2, \ldots, \mu_2 + \cdots + \mu_n\}$. For any given summand, if $j > k$ then that summand will not contribute to the residue. For this reason, I can rewrite the above as

$$\begin{aligned}
&\operatorname*{Res}_{\mu_1=-j} \Big\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle d\mu_1 \\
&= \lim_{\mu_1=-j} \sum_{k=j}^{\mu_2+\cdots+\mu_n} \frac{\mu_1^{k-1}\mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+1)\cdots(\mu_1+j-1)(\mu_1+j+1)\cdots(\mu_1+k)} \Big\langle \mathcal{E}_k(\mu_1 s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle \\
&= \sum_{k=j}^{\mu_2+\cdots+\mu_n} \frac{1}{(-j+1)\cdots(-1)} \frac{(-j)^{k-1}\mathcal{S}(-js)^{-j+k}}{1\cdots(-j+k)} \Big\langle \mathcal{E}_k(-js) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle \\
&= \frac{(-j)^{j-1}}{(-j+1)\cdots(-1)} \sum_{k=j}^{\mu_2+\cdots+\mu_n} \frac{(-j)^{-j+k}\mathcal{S}(-js)^{-j+k}}{1\cdots(-j+k)} \Big\langle \mathcal{E}_k(-js) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle.
\end{aligned}$$

On the other hand, apply the techniques from the proof of Lemma 4.2.8 to rewrite

$$\begin{aligned}
\exp(\alpha_1/s) \exp(s\mathcal{F}_2)\alpha_j \exp(-s\mathcal{F}_2) \exp(-\alpha_1/s) &= \exp(\alpha_1/s)\mathcal{E}_j(-js) \exp(-\alpha_1/s) \\
&= \sum_{m \geqslant 0} \frac{\varsigma(-js)^m}{m!\, s^m} \mathcal{E}_{j+m}(-js) \\
&= \sum_{k \geqslant j} \frac{(-j)^{-j+k}\,\mathcal{S}(-js)^{-j+k}}{(-j+k)!} \mathcal{E}_k(-js).
\end{aligned}$$

An upper bound on k, namely $k \leqslant \mu_2 + \cdots + \mu_n$ can be deduced when considering the overall vacuum expectation $\langle \mathcal{E}_k(-js) \prod_{i=2}^n \mathcal{C}(\mu_i, s) \rangle$ by making the same observations as at the start of this proof. Defining

$$F(j) := \frac{(-j)^{j-1}}{(-j+1)\cdots(-1)} = \frac{(-j)^j}{(-j)(-j+1)\cdots(-1)} = \frac{j^j}{j!}$$

concludes the proof of the lemma. ■

To state the next lemma, I require the notion of a connected correlator. Define the *connected correlator* of a tuple of operators $(\mathcal{O}_1, \ldots, \mathcal{O}_n)$, denoted $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle^\circ$, to be what one obtains from applying inclusion-exclusion to the disconnected correlator. That is,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle^\circ = \sum_{M \vdash \{1,\ldots,n\}} (-1)^{|M|-1} (|M|-1)! \prod_{i=1}^{|M|} \langle \vec{\mathcal{O}}_{M_i} \rangle, \tag{4.10}$$

where $\vec{\mathcal{O}}_{M_i} = \prod_{j \in M_i} \mathcal{O}_j$.

Lemma 4.2.10. *Fix $\mu_2, \ldots, \mu_n$ to be positive integers, and fix $n \geqslant 2$. Then,*

$$\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle \, \mathrm{d}\mu_1 = \operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}(\mu_1, s) \Big\rangle \Big\langle \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle \, \mathrm{d}\mu_1,$$

and hence, applying the inclusion-exclusion formula to obtain the connected contribution yields

$$\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle^\circ \mathrm{d}\mu_1 = 0.$$

Proof. A pole at $\mu_1 = 0$ of

$$\Big\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle = \sum_{k=0}^{\mu_2+\cdots+\mu_n} \frac{\mu_1^{k-1} \mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+1)\cdots(\mu_1+k)} \Big\langle \mathcal{E}_k(\mu_1 s) \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle$$

can only occur in the summand corresponding to $k = 0$, for which the contribution of $\mathcal{B}(\mu_1, s)$ is given by $\frac{1}{\mu_1} \mathcal{S}(\mu_1 s)^{\mu_1} \mathcal{E}_0(\mu_1 s)$. Given that $\mathcal{E}_0(\mu_1, s)$ is acting on the covacuum, $\langle \mathcal{E}_0(\mu_1 s) \mathcal{O} \rangle = \frac{1}{\varsigma(\mu_1 s)} \langle \mathcal{O} \rangle$, and since

$$\frac{1}{\varsigma(\mu_1 s)} = \frac{1}{\mu_1 s} - \frac{\mu_1 s}{24} + \frac{7\mu_1^3 s^3}{5760} - \cdots,$$

the $k = 0$ summand will contribute a double pole at zero, with zero residue. We can compute the coefficient of the double pole by calculating the residue of $\mu_1 \langle \mathcal{B}(\mu_1, s) \prod_{i=2}^n \mathcal{C}(\mu_i, s) \rangle$ as follows.

$$\begin{aligned}
\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle \mathrm{d}\mu_1 &= \operatorname*{Res}_{\mu_1=0} \mu_1 \frac{1}{\mu_1} \mathcal{S}(\mu_1 s)^{\mu_1} \Big\langle \mathcal{E}_0(\mu_1 s) \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle \, \mathrm{d}\mu_1 \\
&= \operatorname*{Res}_{\mu_1=0} \mu_1 \frac{1}{\mu_1} \mathcal{S}(\mu_1 s)^{\mu_1} \frac{1}{\varsigma(\mu_1 s)} \Big\langle \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle \, \mathrm{d}\mu_1 \\
&= \operatorname*{Res}_{\mu_1=0} \mu_1 \frac{1}{\mu_1} \mathcal{S}(\mu_1 s)^{\mu_1} \Big\langle \mathcal{E}_0(\mu_1 s) \Big\rangle \Big\langle \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle \, \mathrm{d}\mu_1 \\
&= \operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}(\mu_1, s) \Big\rangle \Big\langle \prod_{i=2}^n \mathcal{C}(\mu_i, s) \Big\rangle \, \mathrm{d}\mu_1
\end{aligned} \tag{4.11}$$

The final equality is using

$$\Big\langle \mathcal{B}(\mu_1, s) \Big\rangle = \sum_{k=0}^{\mu_2+\cdots+\mu_n} \frac{\mu_1^{k-1} \mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+1)\cdots(\mu_1+k)} \langle \mathcal{E}_k(\mu_1 s) \rangle = \frac{1}{\mu_1} \mathcal{S}(\mu_1 s)^{\mu_1} \langle \mathcal{E}_0(\mu_1 s) \rangle.$$

Consider the connected correlator $\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s)\rangle^\circ$ by applying inclusion-exclusion to the disconnected correlator (using Equation (4.10)) then take the residue to obtain

$$\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}(\mu_1 s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle^\circ \mathrm{d}\mu_1$$
$$= \operatorname*{Res}_{\mu_1=0} \mu_1 \sum_{\substack{N \subseteq \{2,\ldots,n\} \\ M \vdash \{2,\ldots,n\} \setminus N}} (-1)^{|M|} |M|! \, \langle \mathcal{B}(\mu_1 s) \, \mathcal{C}(\vec{\mu}_N, s) \rangle \prod_{i=1}^{|M|} \langle \mathcal{C}(\vec{\mu}_{M_i}, s) \rangle \, \mathrm{d}\mu_1,$$

where, for $M_i = \{i_1, \ldots, i_k\}$, $\mathcal{C}(\vec{\mu}_{M_i}, s)$ is a convenient shorthand notation that denotes $\prod_{j=1}^{k} \mathcal{C}(\mu_{i_j}, s)$. Use the result (4.11) above, which is true for any $n \geqslant 2$, to obtain

$$\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}(\mu_1 s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \Big\rangle^\circ \mathrm{d}\mu_1$$
$$= \operatorname*{Res}_{\mu_1=0} \mu_1 \sum_{\substack{N \subseteq \{2,\ldots,n\} \\ M \vdash \{2,\ldots,n\} \setminus N}} (-1)^{|M|} |M|! \langle \mathcal{B}(\mu_1 s) \rangle \langle \mathcal{C}(\vec{\mu}_N, s) \rangle \prod_{i=1}^{|M|} \langle \mathcal{C}(\vec{\mu}_{M_i}, s) \rangle \, \mathrm{d}\mu_1.$$

Each term in this sum arises twice: once for $N = \emptyset$, which occurs with coefficient $(-1)^{|M|} \, |M|!$, and $|M|$ times when $N = M_i$ for all $i \in \{1, \ldots, |M|\}$ and each of these arise with coefficient $(-1)^{|M|-1}(|M| - 1)!$. Thus, all terms cancel whenever $n \geqslant 2$. ■

It now remains to conclude that $H^\circ_{g,n}(\mu_1, \ldots, \mu_n)$ is polynomial in μ_1. To do this, I will combine the above results, extract the appropriate coefficient of s from $h^\bullet_{\vec{\mu}}(s)$, and apply inclusion-exclusion. As with the case of double Hurwitz numbers in Chapter 6, it will be necessary to consider the cases $n = 1$, $n = 2$, and $n \geqslant 3$ separately.

Case $n = 1$. When $n = 1$, the connected and disconnected enumerations coincide $H^\bullet_{g,1}(\mu) = H^\circ_{g,1}(\mu) = H_{g,1}(\mu)$. Also, the case $(g, n) = (0, 1)$ constitutes one of the so-called ***unstable terms***, and is omitted from the statement of polynomiality (along with $(g, n) = (0, 2)$, which will be explicitly considered below) for the obvious reason — it is not polynomial. And in fact it is known [69] that

$$H_{0,1}(\mu) = \frac{\mu^\mu}{\mu!} \frac{1}{\mu^2} = \frac{\mu^{\mu-3}}{(\mu - 1)!}.$$

I will show that

$$h^\circ_\mu(s) = \frac{\mu^\mu}{\mu!} \left[\frac{1}{\mu^2 s} + \sum_{g \geqslant 1} P_{g,1}(\mu) s^{2g-1} \right],$$

where $P_{g,1}(\mu)$ is a polynomial in μ for all $g \geqslant 1$, which implies that $H_{0,1}(\mu) = \frac{\mu^\mu}{\mu!} \frac{1}{\mu^2}$ and $H_{g,1}(\mu) = \frac{\mu^\mu}{\mu!} P_{g,1}(\mu)$.

By equation (4.7) from Lemma 4.2.8,

$$\begin{aligned} h^\circ_\mu(s) &= \frac{1}{\mu} \frac{\mu^\mu}{\mu!} \mathcal{S}(\mu s)^\mu \langle \mathcal{E}_0(\mu s) \rangle = \frac{1}{\mu} \frac{\mu^\mu}{\mu!} \mathcal{S}(\mu s)^\mu \frac{1}{\varsigma(\mu s)} = \frac{1}{\mu^2 s} \frac{\mu^\mu}{\mu!} \mathcal{S}(\mu s)^{\mu-1} \\ &= \frac{1}{\mu^2 s} \frac{\mu^\mu}{\mu!} \left(1 + \frac{\mu^2 s^2}{24} + \frac{\mu^4 s^4}{1920} + \cdots \right)^{\mu-1} \\ &= \frac{\mu^\mu}{\mu!} \left[\frac{1}{\mu^2 s} + (\mu - 1) \frac{s}{24} + \left(\frac{(\mu-1)}{1920} + \frac{(\mu-1)(\mu-2)}{2 \cdot 24^2} \right) \mu^2 s^3 + \cdots \right]. \end{aligned}$$

Hence,

$$H_{0,1}(\mu) = [s^{-1}] \, h_\mu(s) = \frac{\mu^\mu}{\mu!} \frac{1}{\mu^2} = \frac{\mu^{\mu-3}}{(\mu - 1)!},$$

as required. For general genus when $n = 1$ we have $2g - 2 + n = 2g - 1$, so

$$H_{g,1}(\mu) = [s^{2g-1}]\, h_\mu(s) = [s^{2g}] \frac{1}{\mu^2} \frac{\mu^\mu}{\mu!} \left(1 + \frac{\mu^2 s^2}{24} + \frac{\mu^4 s^4}{1920} + \cdots\right)^{\mu-1}.$$

Letting $c_{2k} = [z^{2k}]\mathcal{S}(z)$, I can write

$$H_{g,1}(\mu) = \frac{\mu^\mu}{\mu!} \mu^{2g-2} \sum_{a_1+\cdots+a_j=g} c_{2a_1} \cdots c_{2a_j} = \frac{\mu^\mu}{\mu!} P_{g,1}(\mu),$$

where $P_{g,1}(\mu)$ is a polynomial in μ for $g \geqslant 1$. This concludes the proof of polynomiality for the case $n = 1$.

Case $n = 2$. Begin with equation (4.9) specialised to $n = 2$,

$$h^\bullet_{\vec{\mu}}(s) = \frac{\mu_1^{\mu_1}}{\mu_1!} \Big\langle \mathcal{B}(\mu_1, s)\, \mathcal{C}(\mu_2, s) \Big\rangle = \sum_{k=0}^{\mu_2} \frac{\mu_1^{\mu_1+k-1} \mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+k)!} \Big\langle \mathcal{E}_k(\mu_1 s)\, \mathcal{C}(\mu_2, s) \Big\rangle,$$

and now consider the possible structure of the poles. I have already deduced that the only possible poles in μ_1 are at $0, -1, \ldots, -\mu_2$, but Lemma 4.2.10 eliminates the possibility that the connected single Hurwitz number has a pole at $\mu_1 = 0$. Considering the poles at negative integers, Lemma 4.2.9 gives

$$\begin{aligned} \operatorname*{Res}_{\mu_1=-j} \langle \mathcal{B}(\mu_1, s)\, \mathcal{C}(\mu_2, s) \rangle \, \mathrm{d}\mu_1 &= F(j) \Big\langle \exp(\alpha_1/s) \exp(s\mathcal{F}_2) \alpha_j \exp(-s\mathcal{F}_2) \exp(-\alpha_1/s) \mathcal{C}(\mu_2, s) \Big\rangle \\ &= F(j) \Big\langle \exp(\alpha_1/s) \exp(s\mathcal{F}_2) \alpha_j \frac{\alpha_{-\mu_2}}{\mu_2} \Big\rangle \\ &= F(j)\, \delta_{j,\mu_2} \end{aligned} \tag{4.12}$$

where the last equality is using the commutation relation $[\alpha_j, \alpha_{-\mu_2}] = \mu_2 \delta_{j,\mu_2}$, along with the facts that α_j annihilates the vacuum, and $\exp(\alpha_1/s)$ and $\exp(s\mathcal{F}_2)$ both fix the vacuum.

Therefore, the only residue contributing to $H_{g,2}(\mu_1, \mu_2)$ arises when $\mu_1 = -\mu_2$. Look again at equation (4.7), in the case of $n = 2$,

$$\begin{aligned} h^\bullet_{\vec{\mu}}(s) &= \frac{1}{\mu_1\, \mu_2} \sum_{k_1+k_2=0} \left[\frac{\mu_1^{\mu_1-k_1} \mathcal{S}(\mu_1 s)^{\mu_1-k_1}}{(\mu_1-k_1)!} \frac{\mu_2^{\mu_2-k_2} \mathcal{S}(\mu_2 s)^{\mu_2-k_2}}{(\mu_2-k_2)!} \right] \Big\langle \mathcal{E}_{-k_1}(\mu_1 s)\, \mathcal{E}_{-k_2}(\mu_2 s) \Big\rangle \\ &= \frac{1}{\mu_1\, \mu_2} \sum_{k=0}^{\mu_2} \left[\frac{\mu_1^{\mu_1+k} \mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+k)!} \frac{\mu_2^{\mu_2-k} \mathcal{S}(\mu_2 s)^{\mu_2-k}}{(\mu_2-k)!} \right] \Big\langle \mathcal{E}_k(\mu_1 s)\, \mathcal{E}_{-k}(\mu_2 s) \Big\rangle. \end{aligned}$$

Here, the second equality is using the fact that $k = -k_1 = k_2$ is bounded by $0 \leqslant k \leqslant \mu_2$. The inclusion-exclusion formula in Equation (4.10) in the case of $n = 2$ reads

$$\langle \mathcal{E}_k(z_1)\, \mathcal{E}_{-k}(z_2) \rangle^\circ = \langle \mathcal{E}_k(z_1)\, \mathcal{E}_{-k}(z_2) \rangle - \langle \mathcal{E}_k(z_1) \rangle \langle \mathcal{E}_{-k}(z_2) \rangle.$$

Thus, because $\langle \mathcal{E}_k(z) \rangle$ vanishes unless $k = 0$, to pass to the connected generating function it suffices to remove the $k = 0$ term from the summation. That is,

$$h^\circ_{\vec{\mu}}(s) = \frac{1}{\mu_1\, \mu_2} \sum_{k=1}^{\mu_2} \left[\frac{\mu_1^{\mu_1+k} \mathcal{S}(\mu_1 s)^{\mu_1+k}}{(\mu_1+k)!} \frac{\mu_2^{\mu_2-k} \mathcal{S}(\mu_2 s)^{\mu_2-k}}{(\mu_2-k)!} \right] \Big\langle \mathcal{E}_k(\mu_1 s)\, \mathcal{E}_{-k}(\mu_2 s) \Big\rangle^\circ.$$

A residue at $\mu_1 = -\mu_2$ can only occur when $k = \mu_2$, so to analyse the pole it is sufficient to consider only the $k = \mu_2$ term in the sum. That is, consider

$$\begin{aligned} &\frac{1}{\mu_1\, \mu_2} \frac{\mu_1^{\mu_1+\mu_2}}{(\mu_1+\mu_2)!} \mathcal{S}(\mu_1 s)^{\mu_1+\mu_2} \langle \mathcal{E}_{\mu_2}(\mu_1 s) \mathcal{E}_{-\mu_2}(\mu_2 s) \rangle^\circ \\ &= \frac{1}{\mu_1\, \mu_2} \frac{\mu_1^{\mu_1+\mu_2}}{(\mu_1+\mu_2)!} \mathcal{S}(\mu_1 s)^{\mu_1+\mu_2} \frac{\varsigma((\mu_1+\mu_2)\mu_2 s)}{\varsigma((\mu_1+\mu_2)s)} \\ &= \frac{1}{\mu_1\, \mu_2} \frac{\mu_1^{\mu_1+\mu_2}}{(\mu_1+\mu_2)!} \left[\left(1 + \frac{\mu_1^2 s^2}{24} + O(\mu_1^4 s^4)\right)^{\mu_1+\mu_2} \left(1 + \frac{\mu_2^2 - 1}{24}(\mu_1+\mu_2)^2 s^2 + O((\mu_1+\mu_2)^4 s^4)\right) \right]. \end{aligned}$$

The second equality is using the commutation relation for the $\mathcal{E}$-operators given in equation (1.9) in Chapter 1, $[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma(aw - bz)\mathcal{E}_{a+b}(z+w)$, along with the fact that $\langle \mathcal{E}_0(z)\rangle = \frac{1}{\varsigma(z)}$. Observe that the coefficient of s^{2k} for any $k \geqslant 1$ will necessarily include a factor of $\mu_1 + \mu_2$. Combine these facts to conclude that, for $g \geqslant 1$,

$$H_{g,2}(\mu_1, \mu_2) = \frac{\mu_1^{\mu_1}}{\mu_1!} P_{g,2}^{\mu_2}(\mu_1)$$

where $P_{g,2}^{\mu_2}(\mu_1)$ is a polynomial in μ_1.

Although it isn't central to this proof, I also deduce the unstable term, $H_{0,2}^{\circ}(\mu_1, \mu_2)$, for posterity. First, extract the coefficient of s^0 from $h_{\vec{\mu}}^{\circ}(s)$. Using that $\mathcal{E}_k(0) = \alpha_k$ I have

$$[s^0]\, \mathcal{S}(\mu_1 s)^{\mu_1+k} \mathcal{S}(\mu_2 s)^{\mu_2-k} \langle \mathcal{E}_k(\mu_1 s)\, \mathcal{E}_{-k}(\mu_2 s)\rangle^{\circ} = \mathcal{S}(0)^{\mu_1+k}\mathcal{S}(0)^{\mu_2-k}\langle \alpha_k\, \alpha_{-k}\rangle^{\circ} = k,$$

therefore

$$H_{0,2}^{\circ}(\mu_1, \mu_2) = [s^0]\, h_{\vec{\mu}}^{\circ}(s) = \frac{1}{\mu_1\, \mu_2} \sum_{k=1}^{\mu_2} \frac{\mu_1^{\mu_1+k}}{(\mu_1+k)!} \frac{\mu_2^{\mu_2-k}}{(\mu_2-k)!}\, k.$$

Next I calculate the residues for $\mu_1 = -j$ where $j \in \{1, 2, \ldots, \mu_2\}$. By (4.12), for $j \neq \mu_2$,

$$\operatorname*{Res}_{\mu_1=-j} H_{0,2}^{\circ}(\mu_1, \mu_2) = 0,$$

while

$$\operatorname*{Res}_{\mu_1=-\mu_2} \Big\langle \mathcal{B}(\mu_1, s)\, \mathcal{C}(\mu_2, s)\Big\rangle\, \mathrm{d}\mu_1 = F(\mu_2) = \frac{\mu_2^{\mu_2}}{\mu_2!}$$

Hence, I can conclude that

$$H_{0,2}^{\circ}(\mu_1, \mu_2) = \frac{\mu_1^{\mu_1}}{\mu_1!} \frac{\mu_2^{\mu_2}}{\mu_2!} \frac{1}{\mu_1+\mu_2} + P(\mu_1) = \frac{\mu_1^{\mu_1}}{\mu_1!} \frac{\mu_2^{\mu_2}}{\mu_2!} \frac{1}{\mu_1+\mu_2},$$

where $P(\mu_1)$ some polynomial in μ_1. The final equality is using the fact that $H_{0,2}^{\circ}(\mu_1, \mu_2)$ is symmetric in μ_1 and μ_2 to conclude that $P(\mu_1) = 0$. The final expression aligns with the formula deduced by Goulden and Jackson [64].

Case $n \geqslant 3$. Again, begin with equation (4.9),

$$h_{\vec{\mu}}^{\bullet}(s) = \frac{\mu_1^{\mu_1}}{\mu_1!} \Big\langle \mathcal{B}(\mu_1, s)\, \mathcal{C}(\mu_2, s) \cdots \mathcal{C}(\mu_n, s)\Big\rangle. \tag{4.13}$$

Lemma 4.2.10 gives that any poles of $h_{\vec{\mu}}^{\circ}(s)$ at $\mu_1 = 0$ are cancelled via the inclusion-exclusion process. Use Lemma 4.2.9 to consider the residue at negative integers. That is,

$$\begin{aligned}\operatorname*{Res}_{\mu_1=-j} \Big\langle \mathcal{B}(\mu_1, s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s)\Big\rangle\, \mathrm{d}\mu_1 &= F(j)\Big\langle \exp(\alpha_1/s)\exp(s\mathcal{F}_2)\alpha_j \exp(-s\mathcal{F}_2)\exp(-\alpha_1/s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s)\Big\rangle \\ &= F(j)\Big\langle \exp(\alpha_1/s)\exp(s\mathcal{F}_2)\alpha_j \frac{\alpha_{-\mu_2}}{\mu_2} \cdots \frac{\alpha_{-\mu_n}}{\mu_n}\Big\rangle.\end{aligned}$$

Commute α_j to the right. The bosonic commutation relation, $[\alpha_j, \alpha_{-\mu_k}] = j\, \delta_{j,\mu_k}$ implies that the residue vanishes except if $j = \mu_k$ for some $k \in \{2, 3, \ldots, n\}$. In this case, the residue becomes

$$\delta_{j,\mu_k} F(j)\Big\langle \exp(\alpha_1/s)\exp(s\mathcal{F}_2)\frac{\alpha_{-\mu_2}}{\mu_2} \cdots \widehat{\alpha}_{\mu_k} \cdots \frac{\alpha_{-\mu_n}}{\mu_n}\Big\rangle = \delta_{j,\mu_k} F(j)\Big\langle \prod_{\substack{i=2\\ i\neq k}}^{n} \mathcal{C}(\mu_i, s)\Big\rangle.$$

On the right side, the notation $\widehat{\alpha}_{\mu_k}$ denotes that we exclude α_{μ_k} from the product. This simple pole at $\mu_1 = -\mu_k$ cancels via inclusion-exclusion with the simple pole arising from the term

$$\Big\langle \mathcal{B}(\mu_1, s)\, \mathcal{C}(\mu_k, s)\Big\rangle \Big\langle \prod_{\substack{i=2\\ i\neq k}}^{n} \mathcal{C}(\mu_i, s)\Big\rangle.$$

Indeed, by Lemma 4.2.9,

$$\operatorname*{Res}_{\mu_1=-\mu_k} \Big\langle \mathcal{B}(\mu_1,s)\,\mathcal{C}(\mu_k,s)\Big\rangle \Big\langle \prod_{\substack{i=2\\ i\neq k}}^{n} \mathcal{C}(\mu_i,s)\Big\rangle \mathrm{d}\mu_1 = f(\mu_k)\Big\langle \exp(\alpha_1/s)\exp(s\mathcal{F}_2)\alpha_{\mu_k}\frac{\alpha_{-\mu_k}}{\mu_k}\Big\rangle \Big\langle \prod_{\substack{i=2\\ i\neq k}}^{n} \mathcal{C}(\mu_i,s)\Big\rangle$$

$$= f(\mu_k)\Big\langle \prod_{\substack{i=2\\ i\neq k}}^{n} \mathcal{C}(\mu_i,s)\Big\rangle.$$

Thus, for $n \geqslant 3$,

$$H^{\circ}_{g,n}(\mu_1,\ldots,\mu_n) = \frac{\mu_1^{\mu_1}}{\mu_1!}P_{g,n}(\mu_1)$$

where $P_{g,n}(\mu_1)$ is a polynomial in μ_1.

Conclude the polynomiality structure for all $\mu_2,\ldots,\mu_n$ by invoking the symmetry of $H^{\circ}_{g,n}(\mu_1,\ldots,\mu_n)$ in $\mu_1,\ldots,\mu_n$; see Theorem 6.3.11 for a generalised version of this statement. ■

Note that I do not prove the condition that $P_{g,n}(\mu_1,\ldots,\mu_n)$ is degree $3g-3+n$.

4.2.5 Data

4.3 Generalisations and variations

As discussed in the introduction of this chapter, since the profound results regarding single Hurwitz numbers were discovered, many Hurwitz number generalisations and variations have been defined and similarly studied. In particular, mathematicians have aimed to deduce a polynomiality structure, topological recursion, and an ELSV-like formula for many such problems. Some generalisations and variations that have been prominent in the recent literature are double Hurwitz numbers, monotone Hurwitz numbers, and spin Hurwitz numbers.

4.3.1 Monotone Hurwitz numbers

A particularly interesting variation of single Hurwitz numbers is given by the *monotone single Hurwitz numbers*. Whereas single Hurwitz numbers can be described as the number of ways to factorise a permutation with a given cycle type into transpositions, (weakly) monotone Hurwitz numbers impose an extra condition where the factorisation by transpositions $\tau_1\cdots\tau_m$ is monotone; that is, if $\tau_i = (a_i, b_i)$ is written conventionally with $a_i < b_i$, then $b_1 \leqslant \cdots \leqslant b_m$.

Definition 4.3.1. The *weakly monotone Hurwitz number* $H^{\leqslant}_{g,n}(\mu_1,\ldots,\mu_n)$ is equal to $1/d!$ multiplied by the number of tuples $(\tau_1,\ldots,\tau_m)$ of permutations in the symmetric group S_d such that such that

- $\tau_1\cdots\tau_m = (1)$;
- the cycles of $\tau_1\cdots\tau_m$are labelled $1,2,\ldots,n$ such that cycle i has length μ_i for $i \in \{1,2,\ldots,n\}$;
- $\langle\tau_1,\ldots,\tau_m\rangle \leqslant S_d$ acts transitively on $\{1,2,\ldots,d\}$; and
- if $\tau_i = (a_i\, b_i)$ with $a_i < b_i$, then $b_1 \leqslant \cdots \leqslant b_m$.

The monotone Hurwitz numbers first appeared in the literature in a series of papers by Goulden, Guay-Paquet and Novak, where monotone Hurwitz numbers featured as coefficients in the large N expansion of the Harish–Chandra-Itzukson–Zuber (HCIZ) matrix integral [60, 61, 62]. Monotonicity is also natural from the viewpoint of Jucys–Murphy elements in the symmetric group algebra $\mathbb{C}[S_d]$ and the representation theory of the symmetric group S_d [72, 85, 93].

Since their debut in the literature nearly a decade ago, monotone Hurwitz numbers have made recurring cameos, in particular on two occasions relevant to this thesis. First, monotone Hurwitz numbers were proved by Do, Dyer and Mathews [31] to be governed by topological recursion applied to the spectral curve $(\mathbb{CP}^1, x, y)$ with

$$x(z) = \frac{z-1}{z^2}, \qquad y(z) = -z, \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

Second, Borot, Charbonnier, Do and Garcia-Failde [6] and Borot and Garcia-Failde [11] used monotone Hurwitz numbers to prove a relation between ordinary maps and fully simple maps; this relation forms the basis for the research presented in Chapter 7 on fully simple maps.

4.3.2 Spin Hurwitz numbers

One other variation of Hurwitz numbers that has attracted attention recently is the *r-spin Hurwitz number*. The r-spin Hurwitz number enumerates ramified genus g covers of $\mathbb{CP}^1$ with prescribed ramification over ∞ and all other ramification is given by *$(r+1)$-completed cycles*.

First note that here I am referring to the spin Hurwitz numbers introduced by Shadrin, Spitz, and Zvonkine [102], not the enumeration that was introduced by Eskin, Okounkov and Pandharipande [44] and has also been referred to as "spin Hurwitz numbers" by Giacchetto, Kramer and Lewański [59]. Second, I do not define a completed cycle, to do so would take me too far afield of the scope of this chapter; for a description of completed cycles, see previous work of Okounkov and Pandharipande [92].

Definition 4.3.2. The *r-spin Hurwitz number* $H^r_{g,n}(\mu_1, \ldots, \mu_n)$ is a weighted enumeration of isomorphism classes of connected genus g branched covers of the Riemann sphere $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- the point $p_i \in f^{-1}(\infty)$ has ramification index μ_i for $i \in \{1, 2, \ldots, n\}$; and
- all other branching is given by $(r+1)$-completed cycles, and occurs at m fixed points of $\mathbb{CP}^1$.

The weight of a cover is given by

$$\frac{1}{(r!)^m \, |\mathrm{Aut}\ f|},$$

where $\mathrm{Aut}\ f$ is the group of automorphisms of f.

By the Gromov–Witten/Hurwitz correspondence, r-spin Hurwitz numbers are relative Gromov–Witten invariants of $\mathbb{CP}^1$ [92].

4.3.3 Deformed Hurwitz numbers

I finish off this chapter with a new variation of Hurwitz numbers, arising from joint work in progress with Norman Do, which we call *deformed Hurwitz numbers*. First, fix d to be a positive integer.

The motivation for this enumeration was to create an analogue of an enumeration with arbitrary internal faces as in the setting of maps and ribbon graphs.

Definition 4.3.3. The *deformed Hurwitz number* $\overline{H}_{g,n}(\mu_1, \ldots, \mu_n)$ is a weighted enumeration of isomorphism classes of connected genus g branched covers of the Riemann sphere $f\colon (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- the point $p_i \in f^{-1}(\infty)$ has ramification index μ_i for $i \in \{1, 2, \ldots, n\}$;
- every other preimage of ∞ has ramification order at most d; and
- all other branching is simple and occurs at m fixed points of $\mathbb{CP}^1$.

We say that the points $p_1, \ldots, p_n$ are the *marked* preimages of ∞, while all other preimages of ∞ are unmarked. The weight of a cover is given by

$$\frac{1}{|\mathrm{Aut}\ f|} \frac{s^m}{m!} t_1^{r_1(f)} \cdots t_d^{r_d(f)},$$

where $\mathrm{Aut}\ f$ is the group of automorphisms of f, m is the number of simple branch points, and, for $i \in \{1, 2, \ldots, d\}$, $r_i(f)$ denotes the number of unmarked preimages of ∞ with ramification order i.

The deformed Hurwitz number $\overline{H}_{g,n}(\mu_1, \ldots, \mu_n)$ is a formal power series in the ring $\mathbb{Q}[[s; t_1, \ldots, t_d]]$.

One can ask whether this enumeration satisfies similar properties to those satisfied by many other Hurwitz enumerations. That is, does it satisfy a polynomiality structure; is the enumeration governed by topological recursion; can the enumeration expressed via intersection theory on moduli spaces of curves.

Part II

Results

Chapter 5

Local topological recursion governs the enumeration of lattice points in $\overline{\mathcal{M}}_{g,n}$

5.1 Introduction

In this chapter, I prove that a certain enumeration of lattice points in the Deligne–Mumford compactification of the moduli space of curves is governed by local topological recursion. Local topological recursion, first introduced by Dunin-Barkowski, Orantin, Shadrin and Spitz [42], is a generalisation of the topological recursion of Chekhov, Eynard and Orantin (CEO) [25, 52]. In the last 15 years, CEO topological recursion has been revealed to govern a vast array of problems and has thus garnered significant attention. Its contemporary generalisation, local topological recursion, has also sparked some interest but remains largely uninvestigated and its benefits over CEO topological recursion are yet unclear. In this chapter, I provide one of the first instances in which local topological recursion is related to a natural combinatorial problem, and in particular, one which captures some of the geometry of the moduli space $\overline{\mathcal{M}}_{g,n}$.

The moduli space $\overline{\mathcal{M}}_{g,n}$ has a rich structure and has been widely studied due to its ties to many areas of mathematics and physics. It was shown by Norbury [88] that a certain enumeration of lattice points in the uncompactified moduli space of curves $\mathcal{M}_{g,n}$ is governed by CEO topological recursion. Yet through its aforementioned applications to other fields—algebraic and hyperbolic geometry and mathematical physics are some examples—the Deligne–Mumford compactification proves to be the inherently more natural object to study. In this case, one might ask whether an analogous count in $\overline{\mathcal{M}}_{g,n}$ also obeys the recursion of Chekhov, Eynard and Orantin. In fact, Do and Norbury [35] defined an analogous count of lattice points in the compactified space $\overline{\mathcal{M}}_{g,n}$ but were unable to prove that the enumeration satisfied CEO topological recursion, stating:

> *"It would be interesting to know whether the compactified lattice point polynomials can be used to define multidifferentials which also satisfy a topological recursion."*

However, it has not been investigated whether the lattice point enumeration in $\overline{\mathcal{M}}_{g,n}$ is governed by local topological recursion. Thus, the primary motivation of this chapter is to prove that the enumeration $\overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n)$ defined by Do and Norbury [35], when stored as coefficients in a multidifferential power series, is governed by local topological recursion. The key result is the following theorem.

Theorem 5.1.1. *For* (g, n) *satisfying* $2g-2+n > 0$*, the correlation differentials resulting from applying topological recursion to the spectral curve* $(\mathbb{C}^*, x, y, \omega_{0,2})$ *with*

$$x(z) = z + \frac{1}{z}, \qquad y(z) = z, \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2} + \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{z_1\,z_2}, \tag{5.1}$$

satisfy

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{b_1, \ldots, b_n \geqslant 0} \overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n) \prod_{i=1}^{n} [b_i] z_i^{b_i - 1}\,\mathrm{d}z_i.$$

Here, we use the notation $[b] = b$ *for* b *positive and* $[0] = 1$.

The spectral curve in Theorem 5.1.1 is local in the sense that the input data cannot be extended to the compact Riemann surface $\mathbb{CP}^1$ such that $\omega_{0,2}$ satisfies the conditions necessary for CEO topological recursion. Dunin-Barkowski [37] writes *"... local topological recursion (to the moment) lacks interesting applications or profound meaning separate from what originates from ordinary (global) topological recursion."* Yet the key result of this chapter demonstrates that local topological recursion governs a natural enumerative problem which is not governed by CEO topological recursion. Moreover, this phenomenon has been discovered elsewhere, such as in the work of Andersen, Borot, Charbonnier, Delecroix, Giacchetto, Lewański and Wheeler [3], in which they relate Masur–Veech volumes to local topological recursion using the Airy spectral curve with a specific choice for $\omega_{0,2}$.

Relevant background for this chapter, specifically, the definition of the enumeration of Do and Norbury $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$, is given in Section 3.3.1 of Chapter 3 (in particular, see Definitions 3.3.2 and 3.3.3). This background section describes the way in which $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ enumerates lattice points in the compactified moduli space of curves $\overline{\mathcal{M}}_{g,n}$. Further, it shows that $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ can be equivalently defined as a weighted enumeration of stable ribbon graphs.

In this chapter, Section 5.2 presents the local topological recursion as well as relevant calculations specific to the spectral curve (5.1) in Theorem 5.1.1. Section 5.3 then contains the proof of Theorem 5.1.1.

The work in this chapter is a product of collaboration with Anupam Chaudhuri and Norman Do [24].

5.2 Local topological recursion

5.2.1 Definitions and background

Recall from Chapter 2 that CEO topological recursion has as input a spectral curve $(\mathcal{C},x,y,T)$ consisting of a compact Riemann surface $\mathcal{C}$, two meromorphic functions x and y defined on $\mathcal{C}$, and a Torelli marking T on $\mathcal{C}$ [25, 52]. The CEO topological recursion then recursively outputs correlation differentials $\omega_{g,n}$ for $g \geqslant 0$ and $n \geqslant 1$. In particular, $\omega_{0,2}(z_1,z_2)$ is defined implicitly by the fact that it has double poles without residue along the diagonal $z_1 = z_2$, is holomorphic away from the diagonal, and is normalised on the $\mathcal{A}$-cycles of the Torelli marking. The assumption that $\mathcal{C}$ is compact implies that $\omega_{0,2}(z_1,z_2)$ is uniquely defined by the spectral curve data.

In contrast to this, local topological recursion does not require a compact Riemann surface as part of the initial data [42]. It was observed that CEO topological recursion only utilises local information around each of the branch points (where $\mathrm{d}x$ vanishes) and hence one can instead define local neighbourhoods D_i with canonical coordinates around each of the N branch points and the underlying Riemann surface becomes $D_1 \sqcup D_2 \sqcup \cdots \sqcup D_N$, without mention of a way to glue these neighbourhoods together to form a compact Riemann surface. In this case, $\omega_{0,2}(z_1,z_2)$ is no longer uniquely defined and must be included as part of the initial data.

Here, I introduce a definition of local topological recursion which is specifically suitable for use in the context of the lattice point enumeration; one can find a more general formulation in the work of Dunin-Barkowski, Orantin, Shadrin and Spitz [42].

Define a *local spectral curve* $(\mathcal{C},x,y,\omega_{0,2})$ to consist of a Riemann surface $\mathcal{C}$, which may be non-compact and disconnected; two meromorphic functions $x,y\colon \mathcal{C}\to\mathbb{CP}^1$; and a bidifferential $\omega_{0,2}$ which has a double pole at $z_1 = z_2$ and is holomorphic away from the diagonal. And, as in the definition of CEO topological recursion, we require that the zeros of $\mathrm{d}x$ are simple. (Note that due to the work of Bouchard and Eynard [13], we now know that this last condition is not strictly necessary, however I impose it here for simplicity.)

Define the base cases of the recursion to be $\omega_{0,1}(z) = -y(z)\,\mathrm{d}x(z)$ and $\omega_{0,2}$ where the latter is defined in the initial data. And finally, define the recursion to be as in Chapter 2 for CEO topological recursion: for

(g,n) satisfying $2g-2+n>0$, the correlation differentials $\omega_{g,n}$ are defined recursively to be

$$\omega_{g,n}(z_1,\vec{z}_S)=\sum_{\alpha}\operatorname*{Res}_{z=\alpha}K_\alpha(z_1,z)\left[\omega_{g-1,n+1}(z,\sigma_\alpha(z),\vec{z}_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}^{\circ}\omega_{g_1,|I|+1}(z,\vec{z}_I)\,\omega_{g_2,|J|+1}(\sigma_\alpha(z),\vec{z}_J)\right],\tag{5.2}$$

where $S=\{2,3,\ldots,n\}$ and $\vec{z}_I=(z_{i_1},\ldots,z_{i_k})$ for $I=\{i_1,\ldots,i_k\}$. The outer summation is over all zeros α of $\mathrm{d}x$, while $\circ$ over the inner summation denotes that we exclude all terms including $\omega_{0,1}$. The involutions σ_α and the kernel K_α are defined as in the definition for CEO topological recursion. That is, for each branch point α, σ_α is defined to be the non-trivial holomorphic involution defined in a neighbourhood of $z=\alpha$ such that $x(\sigma_\alpha(z))=x(z)$, and K_α is

$$K_\alpha(z_1,z)=-\frac{\int_o^z\omega_{0,2}(z_1,\,\cdot\,)}{[y(z)-y(\sigma_\alpha(z))]\,\mathrm{d}x(z)},$$

where o is any arbitrary point on the spectral curve.

5.2.2 The spectral curve for the lattice point enumeration

This section details calculations using the spectral curve (5.1) that will be necessary in the proof of Theorem 5.1.1 in Section 5.3. Namely, I calculate the kernel appearing in the recursion, $K(z_1,z)$, as well as the correlation differentials $\omega_{0,3}$ and $\omega_{1,1}$ produced by topological recursion. Recall that the spectral curve utilised in Theorem 5.1.1 is defined to be $(\mathbb{C}^*,x,y,\omega_{0,2})$ with

$$x(z)=z+\frac{1}{z},\qquad y(z)=z,\qquad \omega_{0,2}(z_1,z_2)=\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}+\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{z_1\,z_2}.$$

First, calculate the recursion kernel, $K(z_1,z)$. The branch points of the spectral curve are given by $\mathrm{d}x(z)=0$, hence $z=\pm1$. The local involution σ is given by $z\mapsto\frac{1}{z}$ at both branch points. Hence,

$$\begin{aligned}K(z_1,z)&=-\frac{\int_o^z\omega_{0,2}(z_1,\,\cdot\,)}{[y(z)-y(\sigma(z))]\,\mathrm{d}x(z)}\\&=-\frac{\int_o^z\frac{\mathrm{d}z_1\,\mathrm{d}t}{(z_1-t)^2}+\frac{\mathrm{d}z_1\,\mathrm{d}t}{z_1\,t}}{\left[z-\frac{1}{z}\right]\left(1-\frac{1}{z^2}\right)\mathrm{d}z}\\&=-\frac{z^3}{(1-z^2)^2}\frac{\mathrm{d}z_1}{\mathrm{d}z}\left[\int_\infty^z\frac{\mathrm{d}t}{(z_1-t)^2}+\int_1^z\frac{\mathrm{d}t}{z_1\,t}\right]\\&=-\frac{z^3}{(1-z^2)^2}\frac{\mathrm{d}z_1}{\mathrm{d}z}\left[\frac{1}{z_1-z}+\frac{\log(z)}{z_1}\right].\end{aligned}\tag{5.3}$$

The third equality is using the fact that the topological recursion is not sensitive to the choice of base point in the integration — any constants that arise from the integral in the recursion kernel do not contribute to the residue calculations and as such do not affect the resultant multidifferentials produced by topological recursion. For this reason, we can split the integral and choose separate (and convenient) base points for each term.

Hence topological recursion gives the following expression for $\omega_{g,n}$, where $S=\{2,3,\ldots,n\}$:

$$\begin{aligned}\omega_{g,n}(z_1,\vec{z}_S)=\sum_{\alpha\in\{\pm1\}}\operatorname*{Res}_{z=\alpha}\frac{-z^3}{(1-z^2)^2}\frac{\mathrm{d}z_1}{\mathrm{d}z}\left[\frac{1}{z_1-z}+\frac{\log(z)}{z_1}\right]\Bigg[&\omega_{g-1,n+1}(z,\tfrac{1}{z},\vec{z}_S)\\&+\sum_{\substack{g=g_1+g_2\\ I\sqcup J=S}}^{\circ}\omega_{g_1,|I|+1}(z,\vec{z}_I)\,\omega_{g_2,|J|+1}(\tfrac{1}{z},\vec{z}_J)\Bigg].\end{aligned}\tag{5.4}$$

I now calculate the first correlation differentials produced by the topological recursion; that is, $\omega_{g,n}$ for which $2g-2+n=1$, namely $\omega_{0,3}$ and $\omega_{1,1}$. These will be necessary for the proof of the main theorem in Section 5.3. Applying the topological recursion to calculate $\omega_{0,3}$ yields

$$\begin{aligned}
&\omega_{0,3}(z_1,z_2,z_3)\\
&\quad= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} K(z_1,z)\,[\omega_{0,2}(z,z_2)\,\omega_{0,2}(\sigma(z),z_3)+\omega_{0,2}(z,z_3)\,\omega_{0,2}(\sigma(z),z_2)]\\
&\quad= -\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{z^3}{(1-z^2)^2}\frac{\mathrm{d}z_1}{\mathrm{d}z}\left[\frac{1}{z_1-z}+\frac{\log(z)}{z_1}\right]\left[\left(\frac{\mathrm{d}z\,\mathrm{d}z_2}{(z-z_2)^2}+\frac{\mathrm{d}z\,\mathrm{d}z_2}{z\,z_2}\right)\left(\frac{\mathrm{d}(\frac{1}{z})\,\mathrm{d}z_3}{(\frac{1}{z}-z_3)^2}+\frac{\mathrm{d}(\frac{1}{z})\,\mathrm{d}z_3}{\frac{1}{z}\,z_3}\right)\right.\\
&\qquad\qquad\left.+\left(\frac{\mathrm{d}z\,\mathrm{d}z_3}{(z-z_3)^2}+\frac{\mathrm{d}z\,\mathrm{d}z_3}{z\,z_3}\right)\left(\frac{\mathrm{d}(\frac{1}{z})\,\mathrm{d}z_2}{(\frac{1}{z}-z_2)^2}+\frac{\mathrm{d}(\frac{1}{z})\,\mathrm{d}z_2}{\frac{1}{z}\,z_2}\right)\right]\\
&\quad= \frac{1}{2}\left[\prod_{i=1}^{3}\frac{z_i^2-z_i+1}{z_i(z_i-1)^2}+\prod_{i=1}^{3}\frac{z_i^2+z_i+1}{z_i(z_i+1)^2}\right]\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3. \qquad (5.5)
\end{aligned}$$

Here, the last line was calculated by a computer.

Calculating $\omega_{1,1}$ I find

$$\begin{aligned}
\omega_{1,1}(z_1) &= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} K(z_1,z)\,\omega_{0,2}(z,\sigma(z))\\
&= -\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{z^3}{(1-z^2)^2}\frac{\mathrm{d}z_1}{\mathrm{d}z}\left[\frac{1}{z_1-z}+\frac{\log(z)}{z_1}\right]\left(\frac{\mathrm{d}z\,\mathrm{d}(\frac{1}{z})}{(z-\frac{1}{z})^2}+\frac{\mathrm{d}z\,\mathrm{d}(\frac{1}{z})}{\frac{1}{z}z}\right)\\
&= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{z^3}{(1-z^2)^2}\left[\frac{1}{z_1-z}+\frac{\log(z)}{z_1}\right]\left(\frac{1}{(z^2-1)^2}+\frac{1}{z^2}\right)\mathrm{d}z\,\mathrm{d}z_1\\
&= \frac{1}{12}\frac{5z_1^8-8z_1^6+18z_1^4-8z_1^2+5}{z_1(z_1^2-1)^4}\,\mathrm{d}z_1. \qquad (5.6)
\end{aligned}$$

Again the final equality here was calculated with a computer.

5.3 Proof of the main theorem

To prove Theorem 5.1.1 I adopt a general strategy that has previously been used to prove that an enumeration satisfies topological recursion. This strategy has been used for lattice points in uncompactified moduli space of curves [87] and various kinds of Hurwitz numbers, including simple [49], orbifold [14, 33], and monotone Hurwitz numbers [31]. The overarching steps in this strategy are typically thus: begin with the combinatorial recursion for the underlying enumerative problem and write it in terms of multidifferentials; use structural properties of the enumeration to deduce equivalent properties for the multidifferentials, then use these to acquire an asymmetric version of the equation obtained; use the fact that a rational differential form is equal to the sum of its principal parts, then finally match the resulting expression with the topological recursion.

First, define the following formal multidifferentials.

$$\Omega_{g,n}(z_1,\ldots,z_n) = \sum_{b_1,\ldots,b_n\geqslant 0} \overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)\prod_{i=1}^{n}[b_i]\,z_i^{b_i-1}\,\mathrm{d}z_i \qquad (5.7)$$

Theorem 5.1.1 is essentially the statement that the multidifferentials $\omega_{g,n}(z_1,\ldots,z_n)$ arising from applying the topological recursion to the spectral curve in equation (5.1) are equal to the multidifferentials $\Omega_{g,n}(z_1,\ldots,z_n)$ defined above; that is,

$$\Omega_{g,n}(z_1,\ldots,z_n) = \omega_{g,n}(z_1,\ldots,z_n),$$

for (g,n) satisfying $2g-2+n>0$.

Adapting the strategy given above as required in our case results in the following four steps.

1. Use the known quasi-polynomiality of $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ given by Theorem 5.3.1 to deduce analytic and symmetry properties of $\Omega_{g,n}(z_1,\ldots,z_n)$ (Lemma 5.3.5).
2. Rewrite the combinatorial recursion of Theorem 5.3.1 in terms of the multidifferentials $\Omega_{g,n}(z_1,\ldots,z_n)$ (Proposition 5.3.7).
3. Apply the operator $F(z) \mapsto F(z) - \frac{1}{z^2}F(\frac{1}{z})$ and use the symmetry properties of $\Omega_{g,n}$ to asymmetrise the equation obtained in the preceding step (Proposition 5.3.8).
4. Use the fact that a rational multidifferential is equal to the sum of its principal parts, where the principal part of a differential form $\Omega(z_1)$ at $z_1 = \alpha$ can be expressed as

$$\operatorname*{Res}_{z=\alpha} \frac{\mathrm{d}z_1}{z_1 - z}\,\Omega(z).$$

   Finally, compare the resulting recursion for $\Omega_{g,n}(z_1,\ldots,z_n)$ from the above process with the topological recursion for the correlation differentials $\omega_{g,n}(z_1,\ldots,z_n)$ and use induction to prove that $\Omega_{g,n} = \omega_{g,n}$ for all (g,n) satisfying $2g-2+n>0$.

The above four steps are carried out in the following four subsections respectively.

5.3.1 Structure of the enumeration

First begin by recalling the following properties of $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ proven by Do and Norbury [35].

Theorem 5.3.1 (Do and Norbury [35]).

1. Quasi-polynomiality. *For (g,n) satisfying $2g-2+n>0$, $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ is a symmetric quasi-polynomial in $b_1^2,\ldots,b_n^2$ of degree $3g-3+n$. We use the term* quasi-polynomial *to refer to a function on $\mathbb{Z}_+^n$ that is polynomial on each fixed parity class. Observe that $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)=0$ whenever $b_1+\cdots+b_n$ is odd and that quasi-polynomiality allows us to extend $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ to evaluation at $b_i=0$ for all $i\in\{1,2,\ldots,n\}$.*
2. Combinatorial recursion. *For (g,n) satisfying $2g-2+n\geqslant 2$, the compactified lattice point count $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ satisfies the following recursion:*

$$\left(\sum_{i=1}^n b_i\right)\overline{\mathrm{N}}_{g,n}(\vec{b}_S) = \sum_{i<j}\sum_{\substack{p+q=b_i+b_j\\ q \text{ even}}} [p]q\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})$$
$$+\frac{1}{2}\sum_i \sum_{\substack{p+q+r=b_i\\ r\text{ even}}}[p][q]r\left[N_{g-1,n+1}(p,q,\vec{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} \overline{\mathrm{N}}_{g_1,|I|+1}(p,\vec{b}_I)\,\overline{\mathrm{N}}_{g_2,|J|+1}(q,\vec{b}_J)\right],$$

   Here $S=\{1,2,\ldots,n\}$ and for an index set $I=\{i_1,\ldots,i_n\}$, let $\vec{b}_I=(b_{i_1},\ldots,b_{i_n})$. In the summations, p, q and r vary over all non-negative integers, and we use the notation $[p]=p$ for p positive and $[0]=1$. The word stable *over the final summation denotes that we exclude all terms with $\overline{\mathrm{N}}_{0,1}$ and $\overline{\mathrm{N}}_{0,2}$.*
3. Top degree coefficients. *For non-negative integers $\alpha_1+\cdots+\alpha_n=3g-3+n$, the coefficient of $b_1^{2\alpha_1}\cdots b_n^{2\alpha_n}$ in any non-zero polynomial underlying $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ is equal to the psi-class intersection number*

$$\frac{1}{2^{5g-6+2n}\alpha_1!\cdots\alpha_n!}\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n}.$$

4. Orbifold Euler characteristics. *Let $\chi_{g,n}$ denote the orbifold Euler characteristic of $\overline{\mathcal{M}}_{g,n}$. The quasi-polynomial $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ satisfies $\overline{\mathrm{N}}_{g,n}(0,0,\ldots,0)=\chi_{g,n}$. Further, these orbifold Euler characteristics $\chi_{g,n}$ satisfy the following recursion for $2g-2+n>0$, with the convention $\chi_{0,1}=0$ and $\chi_{0,2}=1$.*

$$\chi_{g,n+1} = (2-2g-n)\chi_{g,n} + \frac{1}{2}\chi_{g-1,n+2} + \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ i+j=n}}\binom{n}{i}\chi_{g_1,i+1}\,\chi_{g_2,j+1}$$

Next, define the following vector space, which captures the structure of the family of multidifferentials $\Omega_{g,n}(z_1, \ldots, z_n)$.

Definition 5.3.2. Define the complex vector space of differential forms

$$V(z) = \left\{ \sum_{b=0}^{\infty} [b]\, Q(b)\, z^{b-1}\, \mathrm{d}z \;\middle|\; Q(b) \text{ is a quasi-polynomial in } b^2 \right\}.$$

The quasi-polynomiality of $\overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n)$ given by Theorem 5.3.1 amounts to the fact that, for (g, n) satisfying $2g - 2 + n > 0$,

$$\Omega_{g,n}(z_1, \ldots, z_n) \in V(z_1) \otimes V(z_2) \otimes \cdots \otimes V(z_n).$$

For the purposes of proving Theorem 5.1.1, it will be useful to deduce some analytic and symmetry properties for the forms $\Omega(z) \in V(z)$. I begin by deriving a vector space basis for $V(z)$.

Lemma 5.3.3. *The vector space* $V(z)$ *has basis* $\{\xi_k^{\text{even}}(z), \xi_k^{\text{odd}}(z) \mid k \geqslant 0\}$*, where*

$$\xi_k^{\text{even}}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z^2}{1-z^2}\,\mathrm{d}z + \frac{\delta_{k,0}}{z}\,\mathrm{d}z \qquad \textit{and} \qquad \xi_k^{\text{odd}}(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z}{1-z^2}\,\mathrm{d}z.$$

Proof. First observe that a quasi-polynomial is a unique linear combination of monomials, acting on either even or odd arguments. So the following is a basis for $V(z)$, with k varying over all non-negative integers.

$$\begin{aligned}
\xi_k^{\text{even}}(z) &= \sum_{\substack{b\geqslant 0\\ b \text{ even}}} [b]\cdot b^{2k} z^{b-1}\,\mathrm{d}z \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \sum_{\substack{b>0\\ b \text{ even}}} z^b\,\mathrm{d}z + \frac{\delta_{k,0}}{z}\,\mathrm{d}z \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z^2}{1-z^2}\,\mathrm{d}z + \frac{\delta_{k,0}}{z}\,\mathrm{d}z
\end{aligned}
\qquad
\begin{aligned}
\xi_k^{\text{odd}}(z) &= \sum_{\substack{b\geqslant 0\\ b \text{ odd}}} [b]\cdot b^{2k} z^{b-1}\,\mathrm{d}z \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \sum_{\substack{b>0\\ b \text{ odd}}} z^b\,\mathrm{d}z \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z}{1-z^2}\,\mathrm{d}z
\end{aligned}
\qquad \blacksquare$$

Example 5.3.4. It has been previously shown [35] that

$$\overline{\mathrm{N}}_{0,3}(b_1,b_2,b_3) = \begin{cases} 1, & b_1+b_2+b_3 \text{ even}, \\ 0, & b_1+b_2+b_3 \text{ odd}, \end{cases} \qquad \text{and} \qquad \overline{\mathrm{N}}_{1,1}(b_1) = \begin{cases} \frac{1}{48}(b_1^2+20), & b_1 \text{ even}, \\ 0, & b_1 \text{ odd}. \end{cases}$$

We can calculate the corresponding differentials $\Omega_{0,3}$ and $\Omega_{1,1}$ in terms of the basis elements as follows. First in the case of $\Omega_{0,3}$ we have

$$\Omega_{0,3}(z_1,z_2,z_3) = \sum_{\substack{b_1,b_2,b_3\geqslant 0\\ b_1+b_2+b_3=\text{even}}} [b_1]\,[b_2]\,[b_3]\, z_1^{b_1-1} z_2^{b_2-1} z_1^{b_3-1}\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3$$

Thus $\Omega_{0,3}$ is a linear combination of products of ξ_0^{even} and ξ_0^{odd} satisfying a parity condition resulting from the constraint that $b_1 + b_2 + b_3$ is even. To satisfy this, either b_1, b_2, b_3 are all even or exactly two are odd. Hence

$$\begin{aligned}
\Omega_{0,3}(z_1,z_2,z_3) &= \xi_0^{\text{even}}(z_1)\,\xi_0^{\text{even}}(z_2)\,\xi_0^{\text{even}}(z_3) + \xi_0^{\text{even}}(z_1)\,\xi_0^{\text{odd}}(z_2)\,\xi_0^{\text{odd}}(z_3) \\
&\quad + \xi_0^{\text{odd}}(z_1)\,\xi_0^{\text{even}}(z_2)\,\xi_0^{\text{odd}}(z_3) + \xi_0^{\text{odd}}(z_1)\,\xi_0^{\text{odd}}(z_2)\,\xi_0^{\text{even}}(z_3) \\
&= \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3 \left[\prod_{i=1}^{3} \left(\frac{2z_i}{(1-z_i^2)^2} + \frac{1}{z_i} \right) + \left(\frac{2z_1}{(1-z_1^2)^2} + \frac{1}{z_1} \right) \frac{1+z_2^2}{(1-z_2^2)^2} \frac{1+z_3^2}{(1-z_3^2)^2} \right. \\
&\quad \left. + \left(\frac{2z_2}{(1-z_2^2)^2} + \frac{1}{z_2} \right) \frac{1+z_1^2}{(1-z_1^2)^2} \frac{1+z_3^2}{(1-z_3^2)^2} + \left(\frac{2z_3}{(1-z_3^2)^2} + \frac{1}{z_3} \right) \frac{1+z_1^2}{(1-z_1^2)^2} \frac{1+z_2^2}{(1-z_2^2)^2} \right] \\
&= \frac{1}{2} \left[\prod_{i=1}^{3} \frac{z_i^2 - z_i + 1}{z_i(z_i-1)^2} + \prod_{i=1}^{3} \frac{z_i^2 + z_i + 1}{z_i(z_i+1)^2} \right] \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3.
\end{aligned}$$

This agrees with the calculation of $\omega_{0,3}$ obtained in equation (5.5) by topological recursion.

Similarly, calculate $\Omega_{1,1}(z_1)$ as a linear combination of $\xi_1^{\text{even}}(z_1)$ and $\xi_0^{\text{even}}(z_1)$ in the following way.

$$\begin{aligned}\Omega_{1,1}(z_1) &= \frac{1}{48}\left(\xi_1^{\text{even}}(z_1) + 20\,\xi_0^{\text{even}}(z_1)\right)\\ &= \frac{1}{48}\left[\frac{\mathrm{d}}{\mathrm{d}z_1}\left(z_1\frac{\mathrm{d}}{\mathrm{d}z_1}\right)^2\frac{z_1^2}{1-z_1^2} + 20\left(\frac{\mathrm{d}}{\mathrm{d}z_1}\frac{z_1^2}{1-z_1^2} + \frac{1}{z_1}\right)\right]\mathrm{d}z_1\\ &= \frac{1}{48}\left[\frac{8z_1^5+32z_1^3+8z_1}{(1-z_1^2)^4} + \frac{20(z_1^4+1)}{z_1(1-z_1^2)^2}\right]\mathrm{d}z_1\\ &= \frac{1}{12}\frac{5z_1^8 - 8z_1^6 + 18z_1^4 - 8z_1^2 + 5}{z_1(z_1^2-1)^4}\,\mathrm{d}z_1\end{aligned}$$

Again, this agrees with the calculation of $\omega_{1,1}$ obtained in equation (5.6) by topological recursion.

The next lemma asserts certain pole structure and symmetry properties for $\Omega(z) \in V(z)$ that will be necessary for the proof of Theorem 5.1.1.

Lemma 5.3.5. *For all* $\Omega(z) \in V(z)$,

1. $\Omega(z)$ *can have poles only at* $z = 1, z = -1$ *and* $z = 0$*, with at worst a simple pole occurring at* $z = 0$*; and*
2. $\Omega(z) + \Omega(1/z) = 0$.

Proof. It is sufficient to prove these two statements for the basis elements $\xi_k^{\text{even}}(z), \xi_k^{\text{odd}}(z)$, and the results for general $\Omega(z) \in V(z)$ can then be deduced by linearity. The first statement is a direct consequence of Lemma 5.3.3: the operators $\frac{\mathrm{d}}{\mathrm{d}z}$ and $\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}$ do not introduce poles and therefore it follows that the basis elements $\xi_k^{\text{even}}(z), \xi_k^{\text{odd}}(z)$ can have poles only at $z = 1, z = -1$ and $z = 0$, with at worst a simple pole at $z = 0$. For the second statement, first observe

$$\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}(1/z)} = -z\frac{\mathrm{d}}{\mathrm{d}z}.$$

Expressing $\xi_k^{\text{even}}(z)$ as

$$\xi_k^{\text{even}}(z) = \mathrm{d}\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{z^2}{1-z^2} + \delta_{k,0}\log(z)\right]$$

allows us to write

$$\begin{aligned}\xi_k^{\text{even}}(z) + \xi_k^{\text{even}}(1/z) &= \mathrm{d}\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{z^2}{1-z^2} + \delta_{k,0}\log(z)\right] + \mathrm{d}\left[\left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{(1/z)^2}{1-(1/z)^2} + \delta_{k,0}\log(1/z)\right]\\ &= \mathrm{d}\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{z^2-1}{1-z^2} + \delta_{k,0}\left(\log(z)+\log(1/z)\right)\right]\\ &= 0.\end{aligned}$$

Similarly in the case of $\xi_k^{\text{odd}}(z)$,

$$\xi_k^{\text{odd}}(z) + \xi_k^{\text{odd}}(1/z) = \mathrm{d}\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{z}{1-z^2}\right] + \mathrm{d}\left[\left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k}\frac{(1/z)}{1-(1/z)^2}\right] = 0. \qquad \blacksquare$$

Next I state a further and final lemma regarding the pole structure of the forms $\Omega(z)$. This lemma is necessary for Theorem 5.1.1 to assimilate the logarithmic terms that arise from the extra term in $\omega_{0,2}$.

Lemma 5.3.6. *For all* $\Omega(z) \in V(z)$,

$$\sum_{\alpha\in\{\pm1\}}\operatorname*{Res}_{z=\alpha}\Omega(z)\log(z) = \operatorname*{Res}_{z=0}\Omega(z). \tag{5.8}$$

Proof. As in the proof of Lemma 5.3.5, it is only necessary to prove the statement for the basis elements $\xi_k^{\text{even}}(z)$, $\xi_k^{\text{odd}}(z)$ for all $k \geqslant 0$, and the general result follows by linearity. First consider the case of $\xi_k^{\text{even}}(z)$, and here I will deal with $k = 0$ and $k \geqslant 1$ separately. Observe that the right side is zero for $k \geqslant 1$:

$$\operatorname*{Res}_{z=0} \xi_k^{\text{even}}(z) = \operatorname*{Res}_{z=0} \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z^2}{1-z^2}\,\mathrm{d}z = 0.$$

In this case, the left side is

$$\begin{aligned}\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \xi_k^{\text{even}}(z)\log(z) &= -\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \left[\int \xi_k^{\text{even}}(z)\right] \mathrm{d}\log(z) \\ &= -\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z^2}{1-z^2}\right] \frac{\mathrm{d}z}{z} \\ &= -\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k-1} \left(-1+\frac{1}{1-z^2}\right)\mathrm{d}z \\ &= 0.\end{aligned}$$

The first equality uses the fact that a function $F(z)$ meromorphic at $z=\alpha$ satisfies $\operatorname*{Res}_{z=\alpha} \mathrm{d}F = 0$. It follows by taking $F = fg$ that $\operatorname*{Res}_{z=\alpha} f\,\mathrm{d}g = -\operatorname*{Res}_{z=\alpha} g\,\mathrm{d}f$ for any two functions $f(z)$, $g(z)$ that are meromorphic at $z=\alpha$. The last equality uses the fact that the sum of the residues of a meromorphic form equals zero: given that the operators $\frac{\mathrm{d}}{\mathrm{d}z}$ and $z\frac{\mathrm{d}}{\mathrm{d}z}$ do not introduce any poles, the expression has the same poles as $-1+\frac{1}{(1-z)^2}$, which are $+1$ and -1, and both of which are being summed over.

When $k=0$, the right side is

$$\operatorname*{Res}_{z=0} \xi_0^{\text{even}}(z) = \operatorname*{Res}_{z=0}\left[\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2}{1-z^2}\mathrm{d}z + \frac{1}{z}\,\mathrm{d}z\right] = \operatorname*{Res}_{z=0}\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(-1+\frac{1}{1-z^2}\right)\mathrm{d}z + \frac{1}{z}\,\mathrm{d}z\right] = 1,$$

while the left side can be rewritten as

$$\begin{aligned}\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \xi_0^{\text{even}}(z)\log(z) &= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2}{1-z^2}\mathrm{d}z + \frac{1}{z}\,\mathrm{d}z\right]\log(z) \\ &= \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2}{1-z^2}\,\mathrm{d}z\right]\log(z) \\ &= -\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha}\left[\frac{z^2}{1-z^2}\right]\mathrm{d}\log(z) \\ &= -\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{z}{1-z^2}\mathrm{d}z.\end{aligned}$$

Here the second equality uses the fact that $\log(z)/z$ is holomorphic at $z=\pm1$, while the third equality uses again that $\operatorname*{Res}_{z=\alpha} f\,\mathrm{d}g = -\operatorname*{Res}_{z=\alpha} g\,\mathrm{d}f$ for any two functions $f(z)$, $g(z)$ that are meromorphic at $z=\alpha$. Since $\frac{z}{1-z^2}$ has only simple poles at $z=1$ and $z=-1$, the residue at these points can be calculated by multiplying the expression by $(z-1)$ and $(z+1)$ and substituting $z=1$ and $z=-1$ respectively. Therefore the expression equates to

$$\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \xi_0^{\text{even}}(z)\log(z) = -\sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{z}{1-z^2}\,\mathrm{d}z = -\left.\frac{z(z-1)}{1-z^2}\right|_{z=1} - \left.\frac{z(z+1)}{1-z^2}\right|_{z=-1} = 1,$$

as required. It remains to prove the statement for $\xi_k^{\text{odd}}(z)$ with $k \geqslant 0$. In this case note again that the right side is zero:

$$\operatorname*{Res}_{z=0} \xi_k^{\text{odd}}(z) = \operatorname*{Res}_{z=0} \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z}{1-z^2}\,\mathrm{d}z = 0.$$

Using a similar argument to the case of $\xi_k^{\text{even}}(z)$ for $k \geqslant 0$, the left side is

$$\sum_{\alpha\in\{\pm 1\}} \operatorname*{Res}_{z=\alpha} \xi_k^{\text{odd}}(z)\, \log(z) = -\sum_{\alpha\in\{\pm 1\}} \operatorname*{Res}_{z=\alpha} \left[\int \xi_k^{\text{odd}}(z)\right] \mathrm{d}\log(z)$$
$$= -\sum_{\alpha\in\{\pm 1\}} \operatorname*{Res}_{z=\alpha} \left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^{2k} \frac{z}{1-z^2}\right] \frac{1}{z}\,\mathrm{d}z = 0.$$

The first equality is using the fact that $\operatorname*{Res}_{z=\alpha} f\,\mathrm{d}g = -\operatorname*{Res}_{z=\alpha} g\,\mathrm{d}f$ for any two functions $f(z)$, $g(z)$ that are meromorphic at $z=\alpha$, and the last equality is using the fact that the residues of a meromorphic form sum to zero. ∎

5.3.2 Combinatorial recursion

The aim of this section is to rewrite the combinatorial recursion of Theorem 5.3.1 in terms of natural generating functions. First define the following generating functions, which will be used rather than the multidifferentials $\Omega_{g,n}(z_1,\ldots,z_n)$ defined at the start of the section. That is, define

$$W_{g,n}(z_1,\ldots,z_n) = \frac{\Omega_{g,n}(z_1,\ldots,z_n)}{\mathrm{d}z_1\,\mathrm{d}z_2\,\cdots\,\mathrm{d}z_n} = \sum_{b_1,\ldots,b_n\geqslant 0} \overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n) \prod_{i=1}^n [b_i] z_i^{b_i-1}.$$

Proposition 5.3.7. *For (g,n) satisfying $2g-2+n \geqslant 2$, we have the following equation, where $S=\{1,2,\ldots,n\}$ and $\vec{z}_I = (z_{i_1},\ldots,z_{i_k})$ for $I=\{i_1,\ldots,i_k\}$.*

$$\sum_{i=1}^n \frac{\partial}{\partial z_i} z_i W_{g,n}(z_1,\vec{z}_S) = \sum_{i<j} \left(\frac{\partial}{\partial z_i}\left[\frac{2}{z_j}\frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}(\vec{z}_{S\setminus\{j\}})\right] + \frac{\partial}{\partial z_j}\left[\frac{2}{z_i}\frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\vec{z}_{S\setminus\{i\}})\right]\right.$$
$$\left. + 2\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\left[\frac{z_j}{z_i-z_j}\frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}(\vec{z}_{S\setminus\{j\}}) - \frac{z_i}{z_i-z_j}\frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\vec{z}_{S\setminus\{i\}})\right]\right)$$
$$+ \sum_{i=1}^n \frac{\partial}{\partial z_i}\frac{z_i^4}{(1-z_i^2)^2}\left[W_{g-1,n+1}(z_i,z_i,\vec{z}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} W_{g_1,|I|+1}(z_i,\vec{z}_I)\, W_{g_2,|J|+1}(z_i,\vec{z}_J)\right] \tag{5.9}$$

The term stable *over the final summation denotes that we exclude all terms with $W_{0,1}$ or $W_{0,2}$.*

Proof. First recall from Theorem 5.3.1 the following combinatorial recursion for $\overline{\mathrm{N}}_{g,n}(b_1,\ldots,b_n)$ where (g,n) satisfies $2g-2+n \geqslant 2$ and $b_1,\ldots,b_n \geqslant 0$.

$$\left(\sum_{i=1}^n b_i\right) \overline{\mathrm{N}}_{g,n}(\vec{b}_S) = \sum_{i<j} \sum_{\substack{p+q=b_i+b_j \\ q \text{ even}}} [p]q\, \overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})$$
$$+ \frac{1}{2}\sum_i \sum_{\substack{p+q+r=b_i \\ r \text{ even}}} [p][q]r \left[\overline{\mathrm{N}}_{g-1,n+1}(p,q,\vec{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{i\}}}^{\text{stable}} \overline{\mathrm{N}}_{g_1,|I|+1}(p,\vec{b}_I)\, \overline{\mathrm{N}}_{g_2,|J|+1}(q,\vec{b}_J)\right] \tag{5.10}$$

Define the operators

$$\mathcal{O} = \sum_{b_1,\ldots,b_n\geqslant 0} [\,\cdot\,] \prod_{i=1}^n [b_i]\, z_i^{b_i-1} \qquad \text{and} \qquad \mathcal{O}_J = \sum_{b_i\geqslant 0: i\notin J} [\,\cdot\,] \prod_{i\notin J} [b_i]\, z_i^{b_i-1},$$

for all $J \subseteq \{1,2,\ldots,n\}$.

To obtain the result, apply the operator $\mathcal{O}$ to both sides of the combinatorial recursion. The left side becomes

$$\sum_{b_1,\ldots,b_n\geqslant 0}\left(\sum_{i=1}^n b_i\right)\overline{\mathrm{N}}_{g,n}(\vec{b}_S)\prod_{i=1}^n[b_i]\,z_i^{b_i-1}=\sum_{b_1,\ldots,b_n\geqslant 0}\left(\sum_{i=1}^n\frac{\partial}{\partial z_i}z_i\right)\overline{\mathrm{N}}_{g,n}(\vec{b}_S)\prod_{i=1}^n[b_i]\,z_i^{b_i-1}$$
$$=\sum_{i=1}^n\frac{\partial}{\partial z_i}z_i\,W_{g,n}(\vec{z}_S).$$

Applying $\mathcal{O}$ to the (i,j)th summand in the first term on the right side yields

$$\begin{aligned}
&\sum_{b_1,\ldots,b_n\geqslant 0}\sum_{\substack{p+q=b_i+b_j\\ q\text{ even}}}[p]q\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})\prod_{i=1}^n[b_i]\,z_i^{b_i-1}\\
&=\mathcal{O}_{i,j}\sum_{b_i,b_j\geqslant 0}\sum_{\substack{p+q=b_i+b_j\\ q\text{ even}}}[p]q\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})\,[b_i]\,[b_j]\,z_i^{b_i-1}z_j^{b_j-1}\\
&=\mathcal{O}_{i,j}\sum_{\substack{p,q\geqslant 0\\ q\text{ even}}}[p]q\,\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})\left(\sum_{k=0}^{p+q}[p+q-k]\,[k]\,z_i^{p+q-k-1}z_j^{k-1}\right)\\
&=\mathcal{O}_{i,j}\sum_{\substack{p,q\geqslant 0\\ q\text{ even}}}[p]q\,\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})\left[\frac{\partial}{\partial z_i}z_i^{p+q}z_j^{-1}+\frac{\partial}{\partial z_j}z_i^{-1}z_j^{p+q}\right]\\
&\quad+\mathcal{O}_{i,j}\sum_{\substack{p,q\geqslant 0\\ q\text{ even}}}[p]q\,\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\left(z_i^{p+q-1}z_j^1+z_i^{p+q-2}z_j^2+\cdots+z_i^1z_j^{p+q-1}\right). \qquad (\triangle)
\end{aligned}$$

The third equality has exchanged the sum over b_i and b_j with the sum over p and q. Considering the entire term in the first line of the final equality above,

$$\begin{aligned}
&\mathcal{O}_{i,j}\sum_{\substack{p,q\geqslant 0\\ q\text{ even}}}[p]q\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})\left[\frac{\partial}{\partial z_i}z_i^{p+q}z_j^{-1}+\frac{\partial}{\partial z_j}z_i^{-1}z_j^{p+q}\right]\\
&=\mathcal{O}_{i,j}\frac{\partial}{\partial z_i}z_j^{-1}z_i\sum_{\substack{q\geqslant 0\\ q\text{ even}}}qz_i^q\sum_{p\geqslant 0}[p]\,\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})z_i^{p-1}\\
&\quad+\mathcal{O}_{i,j}\frac{\partial}{\partial z_j}z_i^{-1}z_j\sum_{\substack{q\geqslant 0\\ q\text{ even}}}qz_j^q\sum_{p\geqslant 0}[p]\,\overline{\mathrm{N}}_{g,n-1}(p,\vec{b}_{S\setminus\{i,j\}})z_j^{p-1}\\
&=\frac{\partial}{\partial z_i}\left[\frac{2}{z_j}\frac{z_i^3}{(1-z_i^2)^2}W_{g,n-1}\left(\vec{z}_{S\setminus\{j\}}\right)\right]+\frac{\partial}{\partial z_j}\left[\frac{2}{z_i}\frac{z_j^3}{(1-z_j^2)^2}W_{g,n-1}\left(\vec{z}_{S\setminus\{i\}}\right)\right]. \qquad (*)
\end{aligned}$$

The final equality has used the following:

$$\sum_{\substack{q\geqslant 0\\ q\text{ even}}}qz_i^q=z_i\sum_{\substack{q\geqslant 0\\ q\text{ even}}}qz_i^{q-1}=z_i\frac{\partial}{\partial z_i}\frac{1}{1-z_i^2}=\frac{2z_i^2}{(1-z_i^2)^2}. \qquad (5.11)$$

The sum in brackets in the second line of the final equality in the expression above $(\triangle)$ is a geometric series with ratio z_j/z_i and $p+q$ terms, hence it is equal to

$$z_i^{p+q-1}z_j^1+z_i^{p+q-2}z_j^2+\cdots+z_i^1z_j^{p+q-1}=\frac{z_i^{p+q}z_j-z_i\,z_j^{p+q}}{z_i-z_j}.$$

Therefore,

$$\begin{aligned}
&\mathcal{O}_{i,j} \sum_{\substack{p,q\geqslant 0\\ q \text{ even}}} [p]q\, \overline{\mathrm{N}}_{g,n-1}(p, \vec{b}_{S\setminus\{i,j\}}) \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \left(z_i^{p+q-1} z_j^1 + z_i^{p+q-2} z_j^2 + \cdots + z_i^1 z_j^{p+q-1}\right) \\
&\qquad = \mathcal{O}_{i,j} \sum_{\substack{p,q\geqslant 0\\ q \text{ even}}} [p]q\, \overline{\mathrm{N}}_{g,n-1}(p, \vec{b}_{S\setminus\{i,j\}}) \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \frac{z_i^{p+q} z_j - z_i\, z_j^{p+q}}{z_i - z_j} \\
&\qquad = \mathcal{O}_{i,j} \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \left[\frac{1}{z_i - z_j} \sum_{\substack{p,q\geqslant 0\\ q \text{ even}}} [p]q\, \overline{\mathrm{N}}_{g,n-1}(p, \vec{b}_{S\setminus\{i,j\}}) \left(z_i^{p+q} z_j - z_i\, z_j^{p+q}\right) \right] \\
&\qquad = 2 \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} \left[\frac{z_j}{z_i - z_j} \frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}\left(\vec{z}_{S\setminus\{j\}}\right) - \frac{z_i}{z_i - z_j} \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}\left(\vec{z}_{S\setminus\{i\}}\right) \right]. \qquad (*)
\end{aligned}$$

The ith summand of the third term in (5.10) under the operator is

$$\begin{aligned}
&\sum_{\substack{b_1,\ldots,b_n\geqslant 0\\ p+q+r=b_i\\ r \text{ even}}} \frac{1}{2}[p][q]r \left[\overline{\mathrm{N}}_{g-1,n+1}(p, q, \vec{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J = S\setminus\{i\}}}^{\text{stable}} \overline{\mathrm{N}}_{g_1,|I|+1}(p, \vec{b}_I)\, \overline{\mathrm{N}}_{g_2,|J|+1}(q, \vec{b}_J) \right] \prod_{i=1}^{n} [b_i]\, z_i^{b_i-1} \\
&= \mathcal{O}_i \sum_{\substack{b_i\geqslant 0\\ p+q+r=b_i\\ r \text{ even}}} \frac{1}{2}[p][q]r \left[\overline{\mathrm{N}}_{g-1,n+1}(p, q, \vec{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J = S\setminus\{i\}}}^{\text{stable}} \overline{\mathrm{N}}_{g_1,|I|+1}(p, \vec{b}_I)\, \overline{\mathrm{N}}_{g_2,|J|+1}(q, \vec{b}_J) \right] [b_i]\, z_i^{b_i-1} \\
&= \mathcal{O}_i \frac{\partial}{\partial z_i} z_i \sum_{\substack{p,q,r\geqslant 0\\ r \text{ even}}} \frac{1}{2}[p][q]r \left[\overline{\mathrm{N}}_{g-1,n+1}(p, q, \vec{b}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J = S\setminus\{i\}}}^{\text{stable}} \overline{\mathrm{N}}_{g_1,|I|+1}(p, \vec{b}_I)\, \overline{\mathrm{N}}_{g_2,|J|+1}(q, \vec{b}_J) \right] z_i^{p+q+r-1} \\
&= \frac{\partial}{\partial z_i} \frac{z_i^4}{(1-z_i^2)^2} \left[W_{g-1,n+1}(z_i, z_i, \vec{z}_{S\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J = S\setminus\{i\}}}^{\text{stable}} W_{g_1,|I|+1}(z_i, z_I)\, W_{g_2,|J|+1}(z_i, z_J) \right]. \qquad (*)
\end{aligned}$$

In the final line, the sums over p, q and factors of z_i^{p-1}, z_i^{q-1} have been absorbed into the generating functions $W_{g,n}$, while for the sum over r I have used the same idea as in equation (5.11). In other words,

$$\frac{z_i^3}{2} \sum_{\substack{r\geqslant 0\\ r \text{ even}}} r z_i^{r-1} = \frac{z_i^3}{2} \frac{\partial}{\partial z_i} \frac{1}{1-z_i^2} = \frac{z_i^4}{(1-z_i^2)^2}.$$

Combine the contributions from the expressions marked by $(*)$ to obtain the desired result. ■

5.3.3 Breaking the symmetry

Thus far we have acquired a symmetric expression for the combinatorial recursion of $\overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n)$ in terms of the generating functions $W_{g,n}(z_1, \ldots, z_n)$; however, the topological recursion given in Section 5.2 is inherently asymmetric with the variable z_1 playing a special role. In this section, I apply the operator

$$F(z_1, \ldots, z_n) \mapsto F(z_1, \ldots, z_n) - \frac{1}{z_1^2} F(\tfrac{1}{z_1}, z_2, \ldots, z_n), \qquad (5.12)$$

and use the properties of multidifferentials $\Omega(z) \in V(z)$ in Lemma 5.3.5 to break the symmetry in (5.9). This yields an appropriate asymmetric recursion that is ultimately compared with the topological recursion. By the second result of Lemma 5.3.5,

$$\Omega_{g,n}(z_1, \ldots, z_n) + \Omega_{g,n}(1/z_1, z_2, \ldots, z_n) = 0,$$

and at the level of generating functions this property becomes

$$W_{g,n}(z_1, \ldots, z_n) - \frac{1}{z_1^2} W_{g,n}(\tfrac{1}{z_1}, z_2, \ldots, z_n) = 0. \qquad (5.13)$$

This motivates the definition of the operator (5.12).

Proposition 5.3.8. *For (g,n) satisfying $2g-2+n \geqslant 2$, we have the following equation, where $S = \{2,3,\ldots,n\}$ and $\vec{z}_I = (z_{i_1},\ldots,z_{i_k})$ for $I = \{i_1,\ldots,i_k\}$.*

$$\begin{aligned} W_{g,n}(z_1,\vec{z}_S) - \operatorname*{Res}_{p=0} \frac{W_{g,n}(p,\vec{z}_S)\,\mathrm{d}p}{z_1} &= \sum_{j=2}^{n} \left(\frac{2}{z_1 z_j} + \frac{1}{(z_1-z_j)^2} + \frac{1}{(1-z_1z_j)^2} \right) \frac{z_1^3}{(1-z_1^2)^2} W_{g,n-1}(z_1,\vec{z}_{S\setminus\{j\}}) \\ &\quad - \sum_{j=2}^{n} \frac{\partial}{\partial z_j} \left[\left(\frac{1}{z_1-z_j} + \frac{z_j}{1-z_1z_j} \right) \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\vec{z}_S) \right] \\ &\quad + \frac{z_1^3}{(1-z_1^2)^2} \left[W_{g-1,n+1}(z_1,z_1,\vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S}}^{\text{stable}} W_{g_1,|I|+1}(z_1,\vec{z}_I)\, W_{g_2,|J|+1}(z_1,\vec{z}_J) \right] \end{aligned} \tag{5.14}$$

Proof. Recall equation (5.9) of Proposition 5.3.7, but note that I have used the notation $S' = \{1,2,\ldots,n\}$ (while $S = \{2,3,\ldots,n\}$ as in the statement of the proposition),

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial z_i} z_i W_{g,n}(\vec{z}_{S'}) &= \sum_{i<j} \Bigg(\frac{\partial}{\partial z_i} \left[\frac{2}{z_j} \frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}(\vec{z}_{S'\setminus\{j\}}) \right] + \frac{\partial}{\partial z_j} \left[\frac{2}{z_i} \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\vec{z}_{S'\setminus\{i\}}) \right] \\ &\quad + 2 \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \left[\frac{z_j}{z_i-z_j} \frac{z_i^3}{(1-z_i^2)^2} W_{g,n-1}(\vec{z}_{S'\setminus\{j\}}) - \frac{z_i}{z_i-z_j} \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\vec{z}_{S'\setminus\{i\}}) \right] \Bigg) \\ &\quad + \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \frac{z_i^4}{(1-z_i^2)^2} \left[W_{g-1,n+1}(z_i,z_i,\vec{z}_{S'\setminus\{i\}}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S'\setminus\{i\}}}^{\text{stable}} W_{g_1,|I|+1}(z_i,\vec{z}_I)\, W_{g_2,|J|+1}(z_i,\vec{z}_J) \right]. \end{aligned}$$

The result can be obtained by applying the operator (5.12) to all terms. The left side becomes

$$\begin{aligned} &\frac{\partial}{\partial z_1} z_1 W_{g,n}(z_1,\vec{z}_S) - \frac{1}{z_1^2} \frac{\partial}{\partial(\frac{1}{z_1})} \frac{1}{z_1} W_{g,n}(\tfrac{1}{z_1},\vec{z}_S) + \sum_{i=2}^{n} \left[\frac{\partial}{\partial z_i} z_i W_{g,n}(z_1,\vec{z}_S) - \frac{1}{z_1^2} \frac{\partial}{\partial z_i} z_i W_{g,n}(\tfrac{1}{z_1},\vec{z}_S) \right] \\ &= \frac{\partial}{\partial z_1} z_1 W_{g,n}(z_1,\vec{z}_S) + \frac{1}{z_1^2} z_1^2 \frac{\partial}{\partial z_1} \frac{1}{z_1} W_{g,n}(\tfrac{1}{z_1},\vec{z}_S) + \sum_{i=2}^{n} \frac{\partial}{\partial z_i} z_i \left[W_{g,n}(z_1,\vec{z}_S) - \frac{1}{z_1^2} W_{g,n}(\tfrac{1}{z_1},\vec{z}_S) \right] \\ &= 2 \frac{\partial}{\partial z_1} z_1 W_{g,n}(z_1,\vec{z}_S). \end{aligned}$$

Here the final equality has used (5.13) to write $W_{g,n}(\frac{1}{z_1},\vec{z}_S)$ in terms of $W_{g,n}(z_1,\vec{z}_S)$, and further, to deduce that each summand in the sum over i is equal to zero.

For the first two lines on the right side the property (5.13) will ensure that the only terms to contribute will be when $i = 1$ and $j \in \{2,3,\ldots,n\}$. In this case, applying the operator (5.12) to the jth summand in the first term yields

$$\begin{aligned} &\frac{\partial}{\partial z_1} \left[\frac{2}{z_j} \frac{z_1^3}{(1-z_1^2)^2} W_{g,n-1}(z_1,\vec{z}_{S\setminus\{j\}}) \right] - \frac{1}{z_1^2} \frac{\partial}{\partial \frac{1}{z_1}} \left[\frac{2}{z_j} \frac{\frac{1}{z_1^3}}{(1-\frac{1}{z_1^2})^2} W_{g,n-1}(\tfrac{1}{z_1},\vec{z}_{S\setminus\{j\}}) \right] \\ &\qquad = 2 \frac{\partial}{\partial z_1} \left[\frac{2}{z_j} \frac{z_1^3}{(1-z_1^2)^2} W_{g,n-1}(z_1,\vec{z}_{S\setminus\{j\}}) \right]. \end{aligned}$$

Doing so for the second term gives

$$\frac{\partial}{\partial z_j} \left[\frac{2}{z_1} \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\vec{z}_S) \right] - \frac{1}{z_1^2} \frac{\partial}{\partial z_j} \left[2z_1 \frac{z_j^3}{(1-z_j^2)^2} W_{g,n-1}(\vec{z}_S) \right] = 0.$$

The second line becomes

$$
\begin{aligned}
&2\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_j}\left[\frac{z_j}{z_1-z_j}\frac{z_1^3}{(1-z_1^2)^2}W_{g,n-1}(z_1,\vec{z}_{S\setminus\{j\}})\right]-\frac{2}{z_1^2}\frac{\partial}{\partial\frac{1}{z_1}}\frac{\partial}{\partial z_j}\left[\frac{z_j}{\frac{1}{z_1}-z_j}\frac{\frac{1}{z_1^3}}{(1-\frac{1}{z_1^2})^2}W_{g,n-1}(\tfrac{1}{z_1},\vec{z}_{S\setminus\{j\}})\right]\\
&\qquad-2\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_j}\left[\frac{z_1}{z_1-z_j}\frac{z_j^3}{(1-z_j^2)^2}W_{g,n-1}(\vec{z}_S)\right]+2\frac{2}{z_1^2}\frac{\partial}{\partial\frac{1}{z_1}}\frac{\partial}{\partial z_j}\left[\frac{\frac{1}{z_1}}{\frac{1}{z_1}-z_j}\frac{z_j^3}{(1-z_j^2)^2}W_{g,n-1}(\vec{z}_S)\right]\\
&\quad=2\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_j}\left[\left(\frac{z_j}{z_1-z_j}+\frac{z_1z_j}{1-z_1z_j}\right)\frac{z_1^3}{(1-z_1^2)^2}W_{g,n-1}(z_1,\vec{z}_{S\setminus\{j\}})\right.\\
&\qquad\qquad\left.-\left(\frac{z_1}{z_1-z_j}+\frac{1}{1-z_1z_j}\right)\frac{z_j^3}{(1-z_j^2)^2}W_{g,n-1}(\vec{z}_S)\right]
\end{aligned}
$$

Finally, and similarly to previously, after application of the operator (5.12) only terms corresponding to $i=1$ in the third and final line will contribute. In this case, we have

$$
\begin{aligned}
&\frac{\partial}{\partial z_1}\frac{z_1^4}{(1-z_1^2)^2}\left[W_{g-1,n+1}(z_1,z_1,\vec{z}_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}^{\text{stable}}W_{g_1,|I|+1}(z_1,\vec{z}_I)\,W_{g_2,|J|+1}(z_1,\vec{z}_J)\right]\\
&\quad-\frac{1}{z_1^2}\frac{\partial}{\partial\frac{1}{z_1}}\frac{\frac{1}{z_1^4}}{(1-\frac{1}{z_1^2})^2}\left[W_{g-1,n+1}(\tfrac{1}{z_1},\tfrac{1}{z_1},\vec{z}_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}^{\text{stable}}W_{g_1,|I|+1}(\tfrac{1}{z_1},\vec{z}_I)\,W_{g_2,|J|+1}(\tfrac{1}{z_1},\vec{z}_J)\right]\\
&\qquad=2\frac{\partial}{\partial z_1}\frac{z_1^4}{(1-z_1^2)^2}\left[W_{g-1,n+1}(z_1,z_1,\vec{z}_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}^{\text{stable}}W_{g_1,|I|+1}(z_1,\vec{z}_I)\,W_{g_2,|J|+1}(z_1,\vec{z}_J)\right]
\end{aligned}
$$

Collecting all contributions obtained thus far—and dividing throughout by 2—gives

$$
\begin{aligned}
\frac{\partial}{\partial z_1}z_1W_{g,n}(z_1,\vec{z}_S)=&\sum_{j=2}^{n}\frac{\partial}{\partial z_1}\left[\frac{2}{z_j}\frac{z_1^3}{(1-z_1^2)^2}W_{g,n-1}(z_1,\vec{z}_{S\setminus\{j\}})\right]\\
&+\sum_{j=2}^{n}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_j}\left[\left(\frac{z_j}{z_1-z_j}+\frac{z_1z_j}{1-z_1z_j}\right)\frac{z_1^3}{(1-z_1^2)^2}W_{g,n-1}(z_1,\vec{z}_{S\setminus\{j\}})\right.\\
&\left.-\left(\frac{z_1}{z_1-z_j}+\frac{1}{1-z_1z_j}\right)\frac{z_j^3}{(1-z_j^2)^2}W_{g,n-1}(\vec{z}_S)\right]\\
&+\frac{\partial}{\partial z_1}\frac{z_1^4}{(1-z_1^2)^2}\left[W_{g-1,n+1}(z_1,z_1,\vec{z}_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}^{\text{stable}}W_{g_1,|I|+1}(z_1,\vec{z}_I)\,W_{g_2,|J|+1}(z_1,\vec{z}_J)\right].
\end{aligned}
$$

Computing the differentiation

$$\frac{\partial}{\partial z_j}\left(\frac{z_j}{z_1-z_j}+\frac{z_1z_j}{1-z_1z_j}\right)=\frac{z_1}{(z_1-z_j)^2}+\frac{z_1}{(1-z_1z_j)^2},$$

using the fact that

$$\frac{\partial}{\partial z_j}\frac{1}{1-z_1z_j}=\frac{\partial}{\partial z_j}\left(1+\frac{z_1z_j}{1-z_1z_j}\right)=\frac{\partial}{\partial z_j}\frac{z_1z_j}{1-z_1z_j},$$

and grouping the first two terms on the right side allows me to write

$$
\begin{aligned}
\frac{\partial}{\partial z_1}z_1W_{g,n}(\vec{z}_S)=&\sum_{j=2}^{n}\frac{\partial}{\partial z_1}\left[\left(\frac{2}{z_j}+\frac{z_1}{(z_1-z_j)^2}+\frac{z_1}{(1-z_1z_j)^2}\right)\frac{z_1^3}{(1-z_1^2)^2}W_{g,n-1}(z_1,\vec{z}_{S\setminus\{j\}})\right]\\
&-\sum_{j=2}^{n}\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_j}\left[\left(\frac{z_1}{z_1-z_j}+\frac{z_1z_j}{1-z_1z_j}\right)\frac{z_j^3}{(1-z_j^2)^2}W_{g,n-1}(\vec{z}_S)\right]\\
&+\frac{\partial}{\partial z_1}\frac{z_1^4}{(1-z_1^2)^2}\left[W_{g-1,n+1}(z_1,z_1,\vec{z}_S)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=S}}^{\text{stable}}W_{g_1,|I|+1}(z_1,\vec{z}_I)W_{g_2,|J|+1}(z_1,\vec{z}_J)\right].
\end{aligned}
$$

Integrate both sides with respect to z_1 and divide throughout by z_1 to obtain

$$\begin{aligned}W_{g,n}(z_1, \vec{z}_S) + \frac{c}{z_1} &= \sum_{j=2}^{n} \left[\left(\frac{2}{z_1 z_j} + \frac{1}{(z_1 - z_j)^2} + \frac{1}{(1 - z_1 z_j)^2} \right) \frac{z_1^3}{(1 - z_1^2)^2} W_{g,n-1}(z_1, \vec{z}_{S\setminus\{j\}}) \right] \\ &\quad - \sum_{j=2}^{n} \frac{\partial}{\partial z_j} \left[\left(\frac{1}{z_1 - z_j} + \frac{z_j}{1 - z_1 z_j} \right) \frac{z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\vec{z}_S) \right] \\ &\quad + \frac{z_1^3}{(1 - z_1^2)^2} \left[W_{g-1,n+1}(z_1, z_1, \vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} W_{g_1,|I|+1}(z_1, \vec{z}_I) W_{g_2,|J|+1}(z_1, \vec{z}_J) \right],\end{aligned}$$

where c is constant with respect to z_1. Recalling that $W_{g,n}(z_1, \ldots, z_n)$ has at worst a simple pole at $z_1 = 0$ we can infer that the right side has no pole at $z_1 = 0$: the factors of z_1^3 in the first and third lines on the right side will cancel with any possible pole at $z_1 = 0$. Therefore,

$$c = -\operatorname*{Res}_{p=0} W_{g,n}(p, \vec{z}_S)\, \mathrm{d}p,$$

and this concludes the proof. ■

5.3.4 Proof of topological recursion

I am now equipped to prove the main result.

Proof of Theorem 5.1.1. I use induction to prove

$$\Omega_{g,n}(z_1, \ldots, z_n) = \omega_{g,n}(z_1, \ldots, z_n)$$

for all (g, n) such that $2g - 2 + n > 0$.

The base cases for the induction are given by (g, n) satisfying $2g - 2 + n = 1$; that is, $(g, n) = (0, 3)$ and $(1, 1)$. First, on the topological recursion side $\omega_{0,3}$ and $\omega_{1,1}$ were calculated in equations (5.5) and (5.6) to be

$$\omega_{0,3}(z_1, z_2, z_3) = \frac{1}{2} \left[\prod_{i=1}^{3} \frac{z_i^2 - z_i + 1}{z_i(z_i - 1)^2} + \prod_{i=1}^{3} \frac{z_i^2 + z_i + 1}{z_i(z_i + 1)^2} \right] \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3,$$

and

$$\omega_{1,1}(z_1) = \frac{1}{12} \frac{5z_1^8 - 8z_1^6 + 18z_1^4 - 8z_1^2 + 5}{z_1(z_1^2 - 1)^4}\, \mathrm{d}z_1.$$

Comparing these expressions with $\Omega_{0,3}(z_1, z_2, z_3)$ and $\Omega_{1,1}(z_1)$ of Example 5.3.4 we have $\Omega_{g,n} = \omega_{g,n}$ for (g, n) satisfying $2g - 2 + n = 1$. Now fix (g, n) and assume

$$\Omega_{g',n'}(z_1, \ldots, z_n) = \omega_{g',n'}(z_1, \ldots, z_n)$$

for all (g', n') such that $2g - 2 + n > 2g' - 2 + n' > 0$.

Begin with the recursion proven in Proposition 5.3.8 and multiply both sides by $\mathrm{d}z_1 \cdots \mathrm{d}z_n$ to obtain a recursion in terms of $\Omega_{g,n}(z_1, \ldots, z_n)$. This gives

$$\begin{aligned}&\Omega_{g,n}(z_1, \vec{z}_S) - \frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \vec{z}_S) \\ &= \sum_{j=2}^{n} \left(\frac{2}{z_1 z_j} + \frac{1}{(z_1 - z_j)^2} + \frac{1}{(1 - z_1 z_j)^2} \right) \frac{z_1^3}{(1 - z_1^2)^2} \Omega_{g,n-1}(z_1, \vec{z}_{S\setminus\{j\}})\, \mathrm{d}z_j \\ &\quad - \sum_{j=2}^{n} \frac{\partial}{\partial z_j} \left[\left(\frac{1}{z_1 - z_j} + \frac{z_j}{1 - z_1 z_j} \right) \frac{z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\vec{z}_S) \right] \mathrm{d}z_1 \ldots \mathrm{d}z_n \\ &\quad + \frac{z_1^3}{(1 - z_1^2)^2} \frac{1}{\mathrm{d}z_1} \left[\Omega_{g-1,n+1}(z_1, z_1, \vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} \Omega_{g_1,|I|+1}(z_1, \vec{z}_I)\, \Omega_{g_2,|J|+1}(z_1, \vec{z}_J) \right].\end{aligned} \tag{5.15}$$

Use the fact that a rational differential is equal to the sum of its principal parts, where the principal part of a rational differential $\Omega(z_1)$ at $z_1 = \alpha$ can be written

$$\operatorname*{Res}_{z=\alpha} \frac{\mathrm{d}z_1}{z_1 - z}\, \Omega(z).$$

By Lemma 5.3.5, $\Omega(z_1)$ is indeed a rational differential and has at worst a simple pole at $z_1 = 0$ and poles at $z_1 = \pm 1$. Therefore, given that the left side of (5.15) above has poles only at $z_1 = \pm 1$, we can write

$$\begin{aligned}
&\Omega_{g,n}(z_1, \vec{z}_S) - \frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \vec{z}_S) \\
&= \sum_{\alpha \in \{\pm 1\}} \operatorname*{Res}_{z=\alpha} \frac{\mathrm{d}z_1}{z_1 - z} \left[\Omega_{g,n}(z, \vec{z}_S) - \frac{\mathrm{d}z}{z} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \vec{z}_S) \right] \\
&= \sum_{\alpha \in \{\pm 1\}} \operatorname*{Res}_{z=\alpha} \frac{1}{z_1 - z} \frac{z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \left[\sum_{j=2}^{n} \mathrm{d}z\, \mathrm{d}z_j \left(\frac{2}{z z_j} + \frac{1}{(z - z_j)^2} + \frac{1}{(1 - z z_j)^2} \right) \Omega_{g,n-1}(z, \vec{z}_{S \setminus \{j\}}) \right. \\
&\qquad \left. + \; \Omega_{g-1,n+1}(z, z, \vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} \Omega_{g_1,|I|+1}(z, \vec{z}_I)\, \Omega_{g_2,|J|+1}(z, \vec{z}_J) \right] \\
&= \sum_{\alpha \in \{\pm 1\}} \operatorname*{Res}_{z=\alpha} \frac{1}{z_1 - z} \frac{-z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \left[\sum_{j=2}^{n} \omega_{0,2}(z, z_j)\, \Omega_{g,n-1}(\tfrac{1}{z}, \vec{z}_{S \setminus \{j\}}) + \omega_{0,2}(\tfrac{1}{z}, z_j)\, \Omega_{g,n-1}(z, \vec{z}_{S \setminus \{j\}}) \right. \\
&\qquad \left. + \; \Omega_{g-1,n+1}(z, \tfrac{1}{z}, \vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\text{stable}} \Omega_{g_1,|I|+1}(z, \vec{z}_I)\, \Omega_{g_2,|J|+1}(\tfrac{1}{z}, \vec{z}_J) \right] \\
&= \sum_{\alpha \in \{\pm 1\}} \operatorname*{Res}_{z=\alpha} \frac{1}{z_1 - z} \frac{-z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \left[\omega_{g-1,n+1}(z, \tfrac{1}{z}, \vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}}^{\circ} \omega_{g_1,|I|+1}(z, \vec{z}_I)\, \omega_{g_2,|J|+1}(\tfrac{1}{z}, \vec{z}_J) \right].
\end{aligned} \tag{5.16}$$

The second equality has substituted in the right side of equation (5.15) and used the fact that, since the entire second line of the right side of (5.15) is analytic at $z_1 = \pm 1$, it will not contribute. The third equality has used the definition of $\omega_{0,2}$,

$$\omega_{0,2}(z, z_j) = \frac{\mathrm{d}z\, \mathrm{d}z_j}{(z - z_j)^2} + \frac{\mathrm{d}z\, \mathrm{d}z_j}{z\, z_j},$$

which implies that

$$\omega_{0,2}(z, z_j) - \omega_{0,2}(\tfrac{1}{z}, z_j) = \frac{2\, \mathrm{d}z\, \mathrm{d}z_j}{z z_j} + \frac{\mathrm{d}z\, \mathrm{d}z_j}{(z - z_j)^2} + \frac{\mathrm{d}z\, \mathrm{d}z_j}{(1 - z z_j)^2}.$$

The third equality is also using the fact that $\Omega_{g,n}(z_1, \ldots, z_n) = -\Omega_{g,n}(\frac{1}{z_1}, z_2, \ldots, z_n)$ for all (g, n). To obtain the final equality, I have absorbed the terms including $\omega_{0,2}$ into the sum over $g_1 + g_2 = g$ and $I \sqcup J = S$ and used the inductive hypothesis. The symbol $\circ$ over the sum indicates that we exclude all terms that involve $\omega_{0,1}$.

Lemma 5.3.6 tells us that

$$\frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \vec{z}_S) = \frac{\mathrm{d}z_1}{z_1} \sum_{\alpha \in \{\pm 1\}} \operatorname*{Res}_{z=\alpha} \Omega_{g,n}(z, \vec{z}_S)\, \log(z).$$

Substitute the expression for $\Omega_{g,n}$ given by equation (5.15) into the right side of the equation immediately above and observe that the terms

$$\frac{\mathrm{d}z}{z} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \vec{z}_S)$$

and

$$\sum_{j=2}^{n} \frac{\partial}{\partial z_j} \left[\left(\frac{1}{z - z_j} + \frac{z_j}{1 - z z_j} \right) \frac{z_j^3}{(1 - z_j^2)^2} W_{g,n-1}(\vec{z}_S) \right] \mathrm{d}z\, \mathrm{d}z_2 \cdots \mathrm{d}z_n$$

are analytic at $z = \pm 1$. Doing so, then following the same steps of manipulation as was done above to derive (5.16), I obtain

$$\begin{aligned}
&\frac{\mathrm{d}z_1}{z_1} \operatorname*{Res}_{p=0} \Omega_{g,n}(p, \vec{z}_S) \\
&\quad = \frac{\mathrm{d}z_1}{z_1} \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \Omega_{g,n}(z, \vec{z}_S) \log(z) \\
&\quad = \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{\log(z)}{z_1} \frac{z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \Bigg[\sum_{j=2}^{n} \mathrm{d}z\, \mathrm{d}z_j \left(\frac{2}{zz_j} + \frac{1}{(z-z_j)^2} + \frac{1}{(1-zz_j)^2} \right) \Omega_{g,n-1}(z, \vec{z}_{S\setminus\{j\}}) \\
&\qquad + \Omega_{g-1,n+1}(z, z, \vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S}}^{\text{stable}} \Omega_{g_1,|I|+1}(z, \vec{z}_I)\, \Omega_{g_2,|J|+1}(z, \vec{z}_J) \Bigg] \\
&\quad = \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \frac{\log(z)}{z_1} \frac{-z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \Bigg[\omega_{g-1,n+1}(z, \tfrac{1}{z}, \vec{z}_S) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=S}}^{\circ} \omega_{g_1,|I|+1}(z, \vec{z}_I)\, \omega_{g_2,|J|+1}(\tfrac{1}{z}, \vec{z}_J) \Bigg].
\end{aligned} \tag{5.17}$$

Substituting (5.17) into (5.16) yields

$$\begin{aligned}
\Omega_{g,n}(z_1, \vec{z}_S) = \sum_{\alpha\in\{\pm1\}} \operatorname*{Res}_{z=\alpha} \left[\frac{1}{z_1 - z} + \frac{\log(z)}{z_1} \right] \frac{-z^3}{(1-z^2)^2} \frac{\mathrm{d}z_1}{\mathrm{d}z} \Bigg[&\omega_{g-1,n+1}(z, \tfrac{1}{z}, \vec{z}_{S\setminus\{1\}}) \\
&+ \sum_{\substack{g=g_1+g_2 \\ I\sqcup J=S}}^{\circ} \omega_{g_1,|I|+1}(z, \vec{z}_I)\, \omega_{g_2,|J|+1}(\tfrac{1}{z}, \vec{z}_J) \Bigg].
\end{aligned}$$

The right side coincides with (5.4) and hence I deduce that, by induction, $\Omega_{g,n} = \omega_{g,n}$ for all (g, n) satisfying $2g - 2 + n > 0$, and this concludes the proof. ■

5.4 Remarks

I conclude this chapter with a number of brief remarks.

The ribbon graph spectral curve and its stable analogue differ only in $\omega_{0,2}$. While the preceding sections prove that this difference does indeed capture the stable part of the enumeration, one might still seek an explanation for why this works. Loosely speaking, gluing information, and hence nodal information, is stored in $\omega_{0,2}$. Hence $\omega_{0,2}$ stores the information of $\overline{\mathrm{N}}_{0,2}(b_1, b_2)$, and, by virtue of the geometric setup, nodes are captured by evaluating at zero. Morally, one wants to incorporate nodes by defining $\overline{\mathrm{N}}_{0,2}(0, 0) = 1$. This would then, by equation (5.7), introduce an additional term of $\frac{\mathrm{d}z_1\,\mathrm{d}z_2}{z_1\, z_2}$ to $\omega_{0,2}$. This term then propagates via the topological recursion machinery to account for the stable contributions in all topologies. Of course, in the uncompactified lattice point enumeration, there is no $\mathrm{N}_{0,2}(0, 0)$ contribution.

The enumeration of ribbon graphs in general and the restricted enumeration in which vertices have degree at least two are both captured by topological recursion on the same spectral curve (2.4). The former is obtained by expanding the correlation differentials at $x_i = \infty$, while the latter is obtained by expanding at $z_i = 0$. This relation should extend to the stable case; that is, expanding the correlation differential obtained from applying topological recursion to the spectral curve (5.1) at $x_i = \infty$ yields an analogous enumeration to $\mathrm{N}_{g,n}(b_1, \ldots, b_n)$ in which the condition on the ramification order being at least 2 over $0 \in \mathbb{CP}^1$ is removed.

One would expect that using our amended $\omega_{0,2}$ for the well-studied ordinary map spectral curve (7.1) would produce a naturally defined stable analogue of the ordinary map enumeration, which, to the best of my knowledge, has not yet been written down.

Looking at the data in Section 5.5 it should be immediately clear that the coefficients of the quasi-polynomials are non-negative. This suggests the following conjecture.

Conjecture 5.4.1. *The polynomials underlying the quasi-polynomial* $\overline{\mathrm{N}}_{g,n}$ *have non-negative coefficients.*

This positivity conjecture is not immediate from the definition of the enumeration nor the perspective of the combinatorial recursion of Theorem 5.3.1, nor topological recursion. A potential approach to prove non-negativity is to derive a geometric interpretation for the coefficients. The general theory of topological recursion should allow one to relate the enumeration of lattice points in $\overline{\mathcal{M}}_{g,n}$ with intersection theory on $\overline{\mathcal{M}}_{g,n}$, via the formula of Eynard [46], or the connection to cohomological field theories [42]. Work along these lines for the uncompactified lattice point enumeration was carried out by Andersen, Chekhov, Norbury and Penner [4]. It would be interesting to have an analogous result for the compactified case, which may potentially shed light on the positivity conjecture.

5.5 Data

By the result of Do and Norbury [35] given in Theorem 5.3.1, $\overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n)$ is a quasi-polynomial in $b_1^2, \ldots, b_n^2$ that is fixed on each parity class of $(b_1, \ldots, b_n)$ and zero if $b_1 + \cdots + b_n$ is odd. Hence the enumeration $\overline{\mathrm{N}}_{g,n}(b_1, \ldots, b_n)$ can be described by polynomials $N_{g,n}^{(k)}(b_1, \ldots, b_n)$ for k an even non-negative integer where $b_1, \ldots, b_k$ are odd and $b_{k+1}, \ldots, b_n$ are even.

The following data has been sourced from the literature [35] and shows these polynomials for low (g, n).

g	n	k	$\mathrm{N}_{g,n}^{(k)}(b_1, \ldots, b_n)$
0	3	0	1
0	3	2	1
1	1	0	$\frac{1}{48}(b_1^2 + 20)$
0	4	0	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
0	4	2	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 2)$
0	4	4	$\frac{1}{4}(b_1^2 + b_2^2 + b_3^2 + b_4^2 + 8)$
1	2	0	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2 b_2^2 + 36b_1^2 + 36b_2^2 + 192)$
1	2	2	$\frac{1}{384}(b_1^4 + b_2^4 + 2b_1^2 b_2^2 + 36b_1^2 + 36b_2^2 + 84)$
0	5	0	$\frac{1}{32}\sum b_i^4 + \frac{1}{8}\sum b_i^2 b_j^2 + \frac{7}{8}\sum b_i^2 + 7$
0	5	2	$\frac{1}{32}\sum b_i^4 + \frac{1}{8}\sum b_i^2 b_j^2 + \frac{5}{16}(b_1^2 + b_2^2) + \frac{1}{8}(b_3^2 + b_4^2 + b_5^2) + \frac{19}{16}$
0	5	4	$\frac{1}{32}\sum b_i^4 + \frac{1}{8}\sum b_i^2 b_j^2 + \frac{5}{16}(b_1^2 + b_2^2 + b_3^2 + b_4^2) + \frac{7}{8}b_5^2 + \frac{7}{8}$
1	3	0	$\frac{1}{4608}\sum b_i^6 + \frac{1}{768}\sum b_i^4 b_j^2 + \frac{1}{384}b_1^2 b_2^2 b_3^2 + \frac{13}{1152}\sum b_i^4 + \frac{1}{24}\sum b_i^2 b_j^2 + \frac{29}{144}\sum b_i^2 + \frac{17}{12}$
1	3	2	$\frac{1}{4608}\sum b_i^6 + \frac{1}{768}\sum b_i^4 b_j^2 + \frac{1}{384}b_1^2 b_2^2 b_3^2 + \frac{43}{4608}\sum b_i^4 + \frac{1}{24}\sum b_i^2 b_j^2 + \frac{277}{4608}\sum b_i^2 + \frac{1}{512}b_3^4 + \frac{1}{1536}b_3^2 + \frac{81}{256}$
2	1	0	$\frac{1}{1769472}b_1^8 + \frac{3}{40960}b_1^6 + \frac{133}{61440}b_1^4 + \frac{1087}{34560}b_1^2 + \frac{247}{1440}$
0	6	0	$\frac{1}{384}\sum b_i^6 + \frac{3}{28}\sum b_i^4 b_j^2 + \frac{3}{32}\sum b_i^2 b_j^2 b_k^2 + \frac{1}{6}\sum b_i^4 + \frac{9}{6}\sum b_i^2 b_j^2 + \frac{109}{24}b_i^2 + 34$

Chapter 6

Polynomiality, topological recursion, and an ELSV-like formula for double Hurwitz numbers

6.1 Introduction

Hurwitz numbers enumerate branched covers of the Riemann sphere with prescribed ramification data. First studied by Hurwitz in the late nineteenth century [69], recent decades have seen a resurgence of interest in Hurwitz numbers due to their connections to algebraic geometry and mathematical physics. The resulting study into these connections has catalysed significant expansion of Hurwitz theory.

Single Hurwitz numbers provide a particular enumeration of branched covers of the Riemann sphere. The structural results proven for single Hurwitz numbers—which were found to be quite surprising at the time—have instigated the study of further Hurwitz numbers. A particular generalisation of these single Hurwitz numbers, Double Hurwitz numbers have been shown to satisfy a wealth of structure, including piecewise polynomiality and wall-crossing [21, 66, 101], and arise as coefficients in a τ-function for the Toda integrable hierarchy [90]. The results given in this chapter, first presented in joint work with Borot, Do, Karev, and Lewański [8], are motivated by results proven for single Hurwitz numbers (described in Chapter 4), and significantly generalise them.

This chapter provides results for polynomiality, topological recursion, and an ELSV-like formula for the enumeration of double Hurwitz numbers. These results now resolve the conjectures given by Do and Karev [32], which originally posited that double Hurwitz numbers are governed by topological recursion, and, relatedly, that they satisfy a polynomiality structure analogous to the polynomiality of single Hurwitz numbers.

Double Hurwitz numbers, defined below, are a generalisation of single Hurwitz numbers where the ramification profiles of two points—chosen to be ∞ and 0 by convention—is prescribed, rather than just one point in the case of single Hurwitz numbers. In this case, the results given here for double Hurwitz numbers subsume not only the results described above for single Hurwitz numbers, but also analogous results in the literature, such as those pertaining to orbifold Hurwitz numbers [33, 40].

Before defining the double Hurwitz number, henceforth fix d to be a positive integer, s a complex parameter, and let $q_1, \ldots, q_d \in \mathbb{C}$ be a set of weights.

Definition 6.1.1. The *double Hurwitz number* $DH_{g,n}(\mu_1, \ldots, \mu_n)$ is the weighted enumeration of connected genus g branched covers $f : (\mathcal{C}; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- the point $p_i \in f^{-1}(\infty)$ has ramification index μ_i for $i \in \{1, 2, \ldots, n\}$;
- each preimage of $0 \in \mathbb{CP}^1$ has ramification order at most d; and
- all other branch points have simple ramification and occur at m fixed points of $\mathbb{CP}^1$.

The weight of such a branched cover is

$$\frac{s^{2g-2+n}}{m!} \cdot \frac{q_{\lambda_1} \cdots q_{\lambda_\ell}}{|\mathrm{Aut}(f)|},$$

where the ramification profile over $0 \in \mathbb{CP}^1$ is given by the partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $m = 2g-2+n+\ell$ is the number of simple branch points, as determined by the Riemann–Hurwitz formula.

Double Hurwitz numbers have not previously been defined this way. Historically, double Hurwitz numbers have been defined to depend on two partitions, the ramification profiles of both 0 and ∞. This approach, however, defines them to depend on one partition, the ramification profile of ∞ only, and we allow the ramification profile of 0 to vary, introducing parameters to record this ramification.

The following theorems state that double Hurwitz numbers satisfy a polynomiality-like structure (which we simply call polynomiality), are governed by topological recursion, and can be expressed as intersection numbers on moduli spaces of curves.

Theorem 6.1.2 (Polynomiality). *For $2g-2+n > 0$, there exist $C_{g,n}\binom{j_1,\ldots,j_n}{m_1,\ldots,m_n} \in \mathbb{C}(q_1, \ldots, q_d, s)$, which vanish for all but finitely many values of $j_1, \ldots, j_n \in \{1, \ldots, d\}$ and non-negative integers $m_1, \ldots, m_n$, such that*

$$DH_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\substack{1 \leqslant j_1, \ldots, j_n \leqslant d \\ m_1, \ldots, m_n \geqslant 0}} C_{g,n}\binom{j_1,\ldots,j_n}{m_1,\ldots,m_n} \prod_{i=1}^{n} A_{m_i}^{j_i}(\mu_i)$$

where

$$A_m^j(\mu) = j \sum_{\lambda \vdash \mu - j} \frac{\mu^{\ell(\lambda)+m}}{|\mathrm{Aut}(\lambda)|} q_{\lambda_1} \cdots q_{\lambda_{\ell(\lambda)}}.$$

Here, λ represents a partition with $\ell(\lambda)$ parts and $\mathrm{Aut}(\lambda)$ is the set of permutations of the tuple $(\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ that leave it invariant.

Theorem 6.1.3 (Topological recursion). *Let $Q(z) = q_1 z + \cdots + q_d z^d$. The correlation differentials resulting from applying topological recursion to the spectral curve $(\mathbb{C}^*, x, y, \omega_{0,2})$ with*

$$x(z) = \ln z - Q(z), \qquad y(z) = \frac{1}{s} Q(z), \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \, \mathrm{d}z_2}{(z_1 - z_2)^2}$$

satisfy

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\mu_1, \ldots, \mu_n \geqslant 1} DH_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \mathrm{d}(\exp(\mu_i x(z_i))).$$

The following ELSV-like formula involves certain tautological classes $\Omega^{[d]}_{g;a_1,\ldots,a_n} \in H^*(\overline{\mathcal{M}}_{g,n+\ell}; \mathbb{Q})$ indexed by $a_1, \ldots, a_n \in \{0, 1, \ldots, d-1\}$, which come from the moduli space of d-spin curves. This class is defined in Section 6.4.1, but for a more thorough introduction see [26] or [80]. For an introduction to the moduli spaces of curves and its characteristic classes, see the book of Harris and Morrison [68].

Theorem 6.1.4 (ELSV-like formula). *For (g, n) satisfying $2g-2+n > 0$ and $d \geqslant 2$, the double Hurwitz number $DH_{g,n}(\mu_1, \ldots, \mu_n)$ may be expressed in terms of the following intersection numbers over the moduli spaces of curves $\overline{\mathcal{M}}_{g,n+\ell}$ for $\ell > 0$.*

$$DH_{g,n}(\mu_1, \ldots, \mu_n) = (d/s)^{2g-2+n} \prod_{i=1}^{n} \frac{(\mu_i/d)^{\lfloor \mu_i/d \rfloor}}{\lfloor \mu_i/d \rfloor !}$$
$$\times \sum_{\substack{\ell \geqslant 0 \\ p_1, \ldots, p_\ell \geqslant 0 \\ 1 \leqslant k_1, \ldots, k_\ell \leqslant d-1}} \frac{(d/s)^\ell}{\ell!} \frac{d^{\sum(\mu_i/d + k_i/d)} \, q_d^{\sum \mu_i/d - \sum(d-k_i)/d}}{(sd)^{\sum k_i}} \left(\int_{g,n+\ell} \frac{\Omega^{[d]}_{g;-\overline{\mu},d-k}}{\prod_{i=1}^{n} (1 - \frac{\mu_i}{d} \psi_i)} \right) \prod_{i=1}^{\ell} \frac{Q^{(p_i)}_{d-k_i}}{(p_i+1)!}.$$

Here, $Q^{(p)}_{d-k} \in \mathbb{C}(q_1, \ldots, q_d)$ are defined by equation (6.21), $\Omega^{[d]}_{g;-\overline{\mu},d-k} \in H^*(\overline{\mathcal{M}}_{g,n+\ell}; \mathbb{Q})$ is a Chiodo class, where $-\overline{\mu}$ is a shorthand for $-\overline{\mu} = (-\overline{\mu}_1, \ldots, -\overline{\mu}_n)$, and $-\overline{\mu}_i \in \{0, 1, \ldots, d-1\}$ is the unique residue of $-\mu_i \mod d$.

Note that this formula is explicit. One can calculate $Q^{(p)}_{d-k} \in \mathbb{C}(q_1, \ldots, q_d)$ by (6.21) then use known double Hurwitz numbers along with the above formula to calculate the Chiodo classes $\Omega^{[d]}_{g;-\overline{\mu},d-k} \in H^*(\overline{\mathcal{M}}_{g,n+\ell};\mathbb{Q})$ (or vice versa).

As mentioned above, these results generalise previously known results, specifically the analogous results for single Hurwitz numbers [17, 43, 49] and orbifold Hurwitz numbers [33, 40]. These can be obtained from the results given above by taking the specialisations $q_1 = 1, q_i = 0$ for $i > 1$, and $q_k = 1, q_i = 0$ for $i \neq k$ respectively.

It is interesting to note that, while in the case of double Hurwitz numbers the structure theorem was proved first and all other results followed, this is not always so. For single Hurwitz numbers, the famed ELSV formula was originally used to prove the polynomiality as a direct consequence, and the polynomiality then became an ingredient in the proof of topological recursion. However, no analogue of the ELSV formula is known in the more general setting of double Hurwitz numbers; the approach in this chapter is to prove the polynomiality structure first. To date, without access to an analogue of the ELSV formula, such polynomiality results have exclusively been proven using the semi-infinite wedge formalism (another strong advertisement for the power of the semi-infinite wedge).

The theorems above are obtained in the following logical order. First, using previous work of Do and Karev [32], we reduce the proof of Theorem 6.1.3 (topological recursion) to Theorem 6.1.2 (polynomiality). The proof of the polynomiality structure comprises the greatest proportion of the work in this chapter and therein lies the main difficulty in proving these results. The method here goes via the semi-infinite wedge space and is one that has successfully been used previously for other enumerative problems in Hurwitz theory [40, 77]. In the case of double Hurwitz numbers this process is somewhat more intricate and involved, but follows through nonetheless. The correlation differentials produced by topological recursion can be expressed as intersection numbers on moduli spaces of "coloured" stable curves—where the components of the curve are coloured by the branch points of the spectral curve—via a process outlined by Eynard [46]. For the double Hurwitz number spectral curve above, we do not follow this procedure, which appears to be difficult and does not immediately lead to classes on $\overline{\mathcal{M}}_{g,n}$ that are geometrically natural. For this reason, to obtain an ELSV-like formula for double Hurwitz numbers, we have taken an alternate approach which hinges on a variational result of Eynard and Orantin [52]. This approach's starting point is the known analogue of the ELSV formula for orbifold Hurwitz numbers due to Johnson, Pandharipande, and Tseng [70, 80]. Encoding the spectral curves for orbifold and double Hurwitz numbers within a one-parameter family of spectral curves, we then use the variational result to flow from the Johnson–Pandharipande–Tseng formula to a formula that expresses the double Hurwitz numbers as a linear combination of intersection numbers on moduli spaces of curves.

In some sense, the ELSV-like formula obtained in Theorem 6.1.4 is not in an ideal form. For instance, one would like an ELSV formula to directly imply the polynomiality structure as in the case of the ELSV formula (4.2) for single Hurwitz numbers. This is not immediately obvious from the ELSV-like formula for double Hurwitz numbers given in this chapter, and this is in part due to the fact that the formula involves intersection numbers over $\overline{\mathcal{M}}_{g,n+\ell}$ for non-negative integers ℓ. One would like to obtain an expression involving the intersection theory of $\overline{\mathcal{M}}_{g,n}$ alone, which could potentially be obtained via pushforward. However, the pushforward of Chiodo classes by forgetting marked points is not sufficiently well understood at present.

This chapter is structured as follows.

- Section 6.2 relies on previous work of Do and Karev [32] to reduce the proof of topological recursion to the polynomiality structure. This is done by studying the vector space of rational functions which satisfy the linear loop equations, and detailing the structure they exhibit; necessarily implying that the double Hurwitz numbers must satisfy this structure in order to be governed by topological recursion. In this section I also deal with the technical issue where the ramification points of $x(z)$ may have higher order branching.

- Section 6.3 is devoted to the proof of polynomiality via a detailed analysis using the semi-infinite wedge formalism.
- In Section 6.4 we derive the ELSV-like formula for double Hurwitz numbers by using the known ELSV formula for orbifold Hurwitz numbers [70, 80] and a variational result of Eynard and Orantin [52].

This chapter contains joint work with Gaëtan Borot, Norman Do, Maksim Karev, and Danilo Lewański that appears in [8].

6.2 Topological recursion

In this section, I follow the previous work of Do and Karev [32] to reduce the proof of Theorem 6.1.3 to the polynomiality structure of Theorem 6.1.2. First, however, I deal with a technicality; that is, the definition of topological recursion provided in Chapter 2 requires that the zeroes of $\mathrm{d}x$ are simple, which is not necessarily the case for the spectral curve defined in Theorem 6.1.3.

6.2.1 Higher order zeroes

The topological recursion of Chekhov, Eynard and Orantin requires the assumption that the zeroes of $\mathrm{d}x$ are simple, yet this is not necessarily the case for the spectral curve stated in Theorem 6.1.3, because $\mathrm{d}x = \left(\frac{1}{z} - Q'(z)\right)\mathrm{d}z$ may have higher order zeroes. This section addresses this issue by appealing to a result of Bouchard and Eynard [13]. Specifically, Bouchard and Eynard introduce a so-called global topological recursion in which the recursion is defined globally, not locally around the ramification points, and hence only depends on the degree of the branched cover, not the multiplicity of the ramification points. Thus, for a spectral curve with non-simple ramification points, one can use the global topological recursion of Bouchard and Eynard to calculate the corresponding correlation differentials. Bouchard and Eynard then prove that the result of using the global topological recursion is equivalent to using CEO topological recursion then taking the limit as two or more branch points approach each other. We can use their work in our setting with the following lemma.

Lemma 6.2.1. *If Theorem 6.1.3 holds in the case that $1 - zQ'(z)$ has simple roots, then it also holds in the case that $1 - zQ'(z)$ has roots of arbitrary order.*

Proof. In the case that the zeroes of $1 - zQ'(z)$ are simple, the proof of Theorem 6.1.3 provided in the subsequent section tells us that

$$DH_{g,n}(\mu_1, \ldots, \mu_n) = \operatorname*{Res}_{z_n=0} \cdots \operatorname*{Res}_{z_1=0} \omega_{g,n}(z_1, \ldots, z_n) \prod_{i=1}^{n} \frac{\exp(-\mu_i x(z_i))}{\mu_i}.$$

We can then invoke the result of Bouchard and Eynard [13, Section 3.5]. Bouchard and Eynard prove that $\omega_{g,n}$ is continuous in neighbourhoods of the ramification points, and therefore one can take the limit of the right-side above as two branch points approach each other. They prove that obtaining the double Hurwitz number in this way is equivalent to applying their so-called global topological recursion to the spectral curve defined in Theorem 6.1.3, where this recursion does not require the assumption that the roots of $\mathrm{d}x$ are simple. ■

Thus we now can, and will, proceed with the assumption that the spectral curve has simple branch points.

6.2.2 Structure of the enumeration

The correlation differentials $\omega_{g,n}$ satisfy the so-called linear loop equations [10, 12], hence, we are motivated to investigate the vector space spanned by such functions. To this end, introduce the following vector space. Let $A = \{a_1, \ldots, a_d\}$ be the set of ramification points of the meromorphic function $x(z)$; that is, the points that satisfy $\mathrm{d}x(z) = 0$.

Definition 6.2.2. Let $V(z)$ be the $\mathbb{C}(q_1, \ldots, q_d, s)$-vector space consisting of rational functions $f(z)$ such that

- $f(z)$ has poles only at the ramification points $a_1, \ldots, a_d$; and
- $f(z) + f(\sigma(z))$ is analytic at $z = a_i$ for all $i \in \{1, 2, \ldots, d\}$.

Here, σ is the local involution that satisfies $x(\sigma(z)) = x(z)$, as defined in Section 2.2.

We now define a basis for the vector space $V(z)$. For each $j \in \{1, 2, \ldots, d\}$, let $\phi^j_{-1}(z) = z^j$ and define $\phi^j_m(z)$ inductively by

$$\phi^j_{m+1}(z) = \frac{\partial}{\partial x}\phi^j_m(z) = \frac{z}{1 - zQ'(z)}\frac{\partial}{\partial z}\phi^j_m(z).$$

Lemma 6.2.3. *The family* $\{\phi^j_m(z)\}$ *for* $j \in \{1, 2, \ldots, d\}$ *and non-negative integers* m *provide a vector space basis for* $V(z)$.

Proof. For each ramification point a, define a local coordinate w_a to be such that $x = w_a^2 + x(a)$, and hence the local involution is given by $w_a \mapsto -w_a$. Observe that for each j, $\phi^j_{-1}(z) = z^j$ only has a pole at ∞, while $\phi^j_0(z) = \frac{jz^j}{1-zQ'(z)}$ does not hence $\phi^j_0(z) = \frac{jz^j}{1-zQ'(z)} \in V(z)$. Given that the $\phi^j_m(z)$ are defined iteratively by applying $\frac{\partial}{\partial x}$, showing that $\phi^j_m(z) \in V(z)$ for $m \geqslant 0$ and $j \in \{1, 2, \ldots, d\}$ reduces to showing that $\frac{\partial}{\partial x}$ preserves $V(z)$.

Note that $\frac{\partial}{\partial x} = \frac{z}{1-zQ'(z)}\frac{\partial}{\partial z}$ introduces no new poles outside of $\{a_1, \ldots, a_d\}$, and consider the action of $\frac{\partial}{\partial x}$ on a function in $V(z)$ locally around a single ramification point a. In the local coordinate w_a, the principal part of any $\phi(z) \in V(z)$ is an odd polynomial in w_a^{-1}, and

$$\frac{\partial}{\partial x} = \frac{1}{2w_a}\frac{\partial}{\partial w_a}$$

preserves the parity of the power of any monomial in w_a. Further, $\frac{\partial}{\partial x}\mathbb{C}[[w_a]] \subseteq \mathbb{C}w_a^{-2} \oplus \mathbb{C}[[w_{a_i}]]$. Hence, $\frac{\partial}{\partial x}V(z) \subseteq V(z)$, so $\phi^j_m(z) \in V(z)$ for all $j \in \{1, 2, \ldots, d\}$ and non-negative integers m.

It remains to show that the $\phi^j_m(z)$ span $V(z)$, and for this, it is useful to consider principal parts. Iterating over possible m for each j means that there exists a $\phi^j_m(z)$ with every possible odd pole order, and hence the principal parts of $\phi^j_m(z)$ form a basis for $\bigoplus_{\alpha \in A} w_\alpha^{-1}\mathbb{C}(q_1, \ldots, q_d)[w_\alpha^{-2}]$. The fact that the principal parts of a meromorphic function determines the whole function up to a constant—given by Liouville's theorem—means that one can uniquely find $\phi^j_m(z)$ from its principal part only. And therefore, $\{\phi^j_m(z)\}$ form a basis for $V(z)$. ■

Thus, given that the $\omega_{g,n}$ satisfy the linear loop equations, it follows that they can be written as a linear combination of the basis elements $\phi^j_m(z)$. Precisely, for (g, n) satisfying $2g - 2 + n > 0$, there exist $C_{g,n}\binom{j_1,\ldots,j_n}{m_1,\ldots,m_n} \in \mathbb{C}(q_1, \ldots, q_d, s)$ which vanish for all but finitely many $j_1, \ldots, j_n \in \{1, 2, \ldots, d\}$ and non-negative integers $m_1, \ldots, m_n$, such that

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\substack{1 \leqslant j_1,\ldots,j_n \leqslant d \\ m_1,\ldots,m_n \geqslant 0}} C_{g,n}\binom{j_1,\ldots,j_n}{m_1,\ldots,m_n} \mathrm{d}\phi^{j_1}_{m_1}(z_1) \otimes \cdots \otimes \mathrm{d}\phi^{j_n}_{m_n}(z_n). \tag{6.1}$$

The double Hurwitz numbers $DH_{g,n}(\mu_1, \ldots, \mu_n)$ are stored in the correlation differentials as coefficients of $X(z_i) = \exp(x(z_i))$ in the expansion of the correlation differentials at $X = 0$ (or equivalently $z = 0$). For this reason, consider the expansion of the basis $\phi^j_m(z)$ around $X = 0$. That is, write

$$\phi^j_m(z) = \sum_{\mu \geqslant 1} A^j_m(\mu) X(z)^\mu, \tag{6.2}$$

and note that the expansion of this series has no constant term because, by definition $\phi(0) = 0$, and $X(z) = z\exp(-Q(z)) = z + O(z^2)$. By definition

$$\phi^j_m(z) = \left(\frac{\partial}{\partial x}\right)^{m+1}\phi^j_{-1}(z) = \left(X\frac{\partial}{\partial X}\right)^{m+1} z^j,$$

hence, observing that applications of $X\frac{\partial}{\partial X}$ to $\phi_m^j(z)$ in (6.2) only introduces factors of μ in the expansion of $\phi_m^j(z)$ at $X=0$, it is sufficient to consider (6.2) for $m=-1$. That is, consider

$$\phi_{-1}^j(z) = z^j = \sum_{\mu\geqslant 1} A_{-1}^j(\mu) X^\mu.$$

Compute the coefficients $A_{-1}^j(\mu)$ by

$$\begin{aligned} A_{-1}^j(\mu) &= \operatorname*{Res}_{z=0} \frac{z^j}{X(z)^{\mu+1}}\,\mathrm{d}X(z) = \frac{j}{\mu}\operatorname*{Res}_{z=0}\frac{z^{j-1}}{X(z)^{\mu}}\,\mathrm{d}z \\ &= \frac{j}{\mu}\operatorname*{Res}_{z=0} z^{j-\mu-1}\exp(\mu Q(z))\,\mathrm{d}z = \frac{j}{\mu}[z^{\mu-j}]\exp(\mu Q(z)) \\ &= \frac{j}{\mu}[z^{\mu-j}]\prod_{i=1}^{d}\sum_{k_i=0}^{\infty}\frac{\mu^{k_i}q_i^{k_i}z^{ik_i}}{k_i!} = \frac{j}{\mu}[z^{\mu-j}]\sum_{\lambda=(1^{k_1},\ldots,d^{k_d})}\frac{\vec{q}_\lambda \mu^{\ell(\lambda)}z^{|\lambda|}}{|\mathrm{Aut}\ \lambda|} \\ &= j\sum_{\lambda\vdash\mu-j}\frac{\vec{q}_\lambda\mu^{\ell(\lambda)-1}}{|\mathrm{Aut}\ \lambda|}. \end{aligned}$$

The second equality uses the fact that $\operatorname*{Res}_{z=\alpha} f\,\mathrm{d}g = -\operatorname*{Res}_{z=\alpha} g\,\mathrm{d}f$ for any two functions $f(z)$ and $g(z)$ that are meromorphic at $z=\alpha$. (Note: This trick will be used again throughout this chapter and shall henceforth be referred to simply as integration by parts.) Also, because residues are coordinate-invariant this allows us to write $A_{-1}^j(\mu)$ as the residue over $z=0$ rather than $X=0$.

The upshot of this calculation is that, because $A_m^j(\mu)$ is obtained from $A_{-1}^j(\mu)$ by iteratively applying $\frac{\partial}{\partial X}$, we can deduce a formula for $A_m^j(\mu)$ for all $m\geqslant 0$ and $j\in\{1,2,\ldots,d\}$.

It will be useful to define a vector space which is the span of these coefficients. Note that here I use a slight abuse of notation and write $V(\mu)$ for the space spanned by the coefficients of the $X=0$ expansions of elements in $V(z)$.

Definition 6.2.4. Define the vector space $V(\mu)$ to be the $\mathbb{C}(q_1,\ldots,q_d,s)$-span of $A_m^j(\mu)$ defined by

$$A_m^j(\mu) = j\sum_{\lambda\vdash\mu-j}\frac{\vec{q}_\lambda\mu^{\ell(\lambda)+m}}{|\mathrm{Aut}\ \lambda|}$$

for $j\in\{1,2,\ldots,d\}$ and non-negative integers m.

6.2.3 Equivalence between topological recursion and polynomiality

We are now equipped to prove the equivalence of the polynomiality result, Theorem 6.1.2, and topological recursion, Theorem 6.1.3.

Theorem 6.2.5. *Theorem 6.1.2 (polynomiality) and Theorem 6.1.3 (topological recursion) are equivalent.*

Proof. The fact that the correlation differentials $\omega_{g,n}$ satisfy the linear loop equations, and Lemma 6.2.3 giving a basis for the vector space of functions $V(z)$ that satisfy the linear loop equations, both imply the following. For all (g,n) satisfying $2g-2+n>0$, there exist $C_{g,n}\binom{j_1,\ldots,j_n}{m_1,\ldots,m_n}\in\mathbb{C}(q_1,\ldots,q_d,s)$ which vanish for all but finitely many $j_1,\ldots,j_n\in\{1,2,\ldots,d\}$ and non-negative integers $m_1,\ldots,m_n$, such that

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{\substack{1\leqslant j_1,\ldots,j_n\leqslant d\\ m_1,\ldots,m_n\geqslant 0}} C_{g,n}\binom{j_1,\ldots,j_n}{m_1,\ldots,m_n}\,\mathrm{d}\phi_{m_1}^{j_1}(z_1)\otimes\cdots\otimes\mathrm{d}\phi_{m_n}^{j_n}(z_n).$$

Further, $\phi(z)$ written as an expansion in $X(z)=\exp(x(z))$ around $z=0$ is given by

$$\phi(z) = \sum_{\mu\geqslant 1} A_m^j(\mu)\,X(z)^\mu = \sum_{\mu\geqslant 1} j\sum_{\lambda\vdash\mu-j}\frac{\vec{q}_\lambda\mu^{\ell(\lambda)+m}}{|\mathrm{Aut}\ \lambda|}X(z)^\mu.$$

Therefore, one can write

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{\substack{1\leqslant j_1,\ldots,j_n\leqslant d\\ m_1,\ldots,m_n\geqslant 0}} C_{g,n}\binom{j_1,\ldots,j_n}{m_1,\ldots,m_n} \prod_{i=1}^{n} \sum_{\mu_i\geqslant 1} A^{j_i}_{m_i}(\mu_i)\, \mathrm{d}X(z_1)^{\mu_1}\otimes\cdots\otimes \mathrm{d}X(z_n)^{\mu_n},$$

and this gives us that Theorem 6.1.3 (topological recursion) implies Theorem 6.1.2 (polynomiality). The converse was proved in previous work by Do and Karev [32, Theorem 26]. ■

The converse statement follows a similar approach to the one taken in Chapter 5 to prove that the lattice point enumeration is governed by topological recursion.

6.3 Polynomiality

The aim of this section is to prove Theorem 6.1.2. This is achieved by proving that $DH_{g,n}(\mu_1,\ldots,\mu_n)$ satisfies the polynomiality structure for μ_1 when $\mu_2,\ldots,\mu_n$ are fixed, then using the fact that double Hurwitz numbers are symmetric in $\mu_1,\ldots,\mu_n$ to deduce the result. Specifically, I first aim to prove that for fixed positive integers $\mu_2,\ldots,\mu_n$ and for all (g,n) satisfying $2g-2+n>0$, the connected double Hurwitz generating function $\mathbb{H}^\circ(\mu_1,\ldots,\mu_n;s)$ can be written

$$\mathbb{H}^\circ(\mu_1,\ldots,\mu_n;s) = \sum_{r=1}^{d}\left[\sum_{\lambda\vdash\mu_1-r}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\right]\sum_{g\geqslant 0} P^r_{g,n}(\mu_1,\ldots,\mu_n;\vec{q})s^{2g-2+n},$$

where $P^r_{g,n}(\mu_1,\ldots,\mu_n;\vec{q})$ is a polynomial in μ_1 and $\vec{q}$.

The proof of Theorem 6.1.2 is divided into the following five parts.

1. Begin with a previously known vacuum expectation for double Hurwitz numbers (Theorem 6.3.2) and rewrite it in a form that will be convenient for the proof of the polynomiality structure (Proposition 6.3.3).
2. Use the vacuum expectation derived in Proposition 6.3.3 for $\mathbb{H}^\bullet(\mu_1,\ldots,\mu_n;s)$ and consider the dependence on μ_1. This yields an expression including a sum over partitions of size μ_1+a for a positive integer a. Use a "peeling lemma", Lemma 6.3.4, to reduce this to a sum over partitions of size μ_1-r for $r\in\{1,2,\ldots,d\}$ (Lemma 6.3.6).
3. Use of Lemma 6.3.4 in step 2 results in a rational expression with distinct factors of the form $\frac{\mu_1}{\mu_1+k}$ for k a non-negative integer. Therefore, in the aim toward showing that $DH_{g,n}(\mu_1,\ldots,\mu_n)$ satisfies a polynomiality structure, I prove two useful lemmas regarding the residue of $\mathbb{H}^\bullet(\mu_1,\ldots,\mu_n;s)$ at $\mu_1=-a$ for all non-negative integers a (Lemmas 6.3.7 and 6.3.8).
4. Use the results in step 3 to deduce that

$$\mathbb{H}^\circ(\mu_1,\ldots,\mu_n;s) = \sum_{r=1}^{d}\sum_{\lambda\vdash\mu_1-r}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s)\sum_{k\geqslant 0} Q^r_{k,n}(\mu_1,\ldots,\mu_n;\vec{q})s^k,$$

   where $Q^r_{k,n}(\mu_1,\ldots,\mu_n;\vec{q})$ is a polynomial in μ_1 and $\vec{q}$, then extract the coefficient of s^{2g-2+n} to yield the desired structural result about the double Hurwitz numbers; that is, $DH_{g,n}(\mu_1,\ldots,\mu_n)\in V(\mu_1)$ (Theorem 6.3.10).
5. Use the symmetry of $DH_{g,n}(\mu_1,\ldots,\mu_n)$ in $\mu_1,\ldots,\mu_n$ to conclude that $DH_{g,n}(\mu_1,\ldots,\mu_n)$ satisfies the polynomiality structure for $\mu_1,\ldots,\mu_n$ for all (g,n) satisfying $2g-2+n>0$ (Theorem 6.3.11).

These five steps are carried out in Sections 6.3.2 to 6.3.6 respectively. Section 6.3.1 below introduces some preliminary notations, including the distinction between connected and disconnected double Hurwitz numbers, and the relation between these enumerations.

6.3.1 Preliminaries

Before proceeding with the proof of polynomiality, I first introduce some useful notions. First, define the *possibly disconnected double Hurwitz number*, denoted $DH^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$, to be as in Definition 6.1.1, but where the source surface Σ in the branched cover $f\colon (\Sigma; p_1, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ may be disconnected. Hereafter for clarity, I will write $DH^{\circ}_{g,n}(\mu_1, \ldots, \mu_n)$ when referring to the connected double Hurwitz number. Next, for positive integers $\mu_1, \ldots, \mu_n$, define the generating functions for connected and disconnected double Hurwitz numbers to be

$$\mathbb{H}^{\circ}(\mu_1, \ldots, \mu_n; s) = \sum_{g \geqslant 0} DH^{\circ}_{g,n}(\mu_1, \ldots, \mu_n)\, s^{2g-2+n},$$

and

$$\mathbb{H}^{\bullet}(\mu_1, \ldots, \mu_n; s) = \sum_{g \in \mathbb{Z}} DH^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n)\, s^{2g-2+n},$$

respectively. The genus of a disconnected surface is defined by its Euler characteristic which is naturally additive under disjoint union; this results in a disconnected surface possibly having negative genus and thus the sum over the genus above runs over all integers. (However, observe that for fixed $\mu_1, \ldots, \mu_n$, the genus is bounded and the genus cannot be arbitrarily large in magnitude.)

To relate disconnected double Hurwitz numbers to the connected counts consider the possible ways in which the source surface can be disconnected. We are necessarily forced to have at least one of the preimages $f^{-1}(\infty) = \{p_1, \ldots, p_n\}$ in each component, which predicates that the number of components is limited to at most n. Given that these preimages are labelled and the fact that the genus of the disconnected surface is dictated by its Euler characteristic leads to the following sum over set partitions of $\{\mu_1, \ldots, \mu_n\}$. That is,

$$DH^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{M \vdash \{1,\ldots,n\}} \sum_{g_1 + \cdots + g_{|M|} = g-1+|M|} \prod_{i=1}^{|M|} DH^{\circ}_{g_i,|M_i|}(\vec{\mu}_{M_i}),$$

where if $M_i = \{i_1, \ldots, i_j\}$, then $\vec{\mu}_{M_i} = (\mu_{i_1}, \ldots, \mu_{i_j})$. At the level of the generating function, the sum over the possible genera $g_1, \ldots, g_{|M|}$ is taken care of by the parameter s and this gives the relation

$$\mathbb{H}^{\bullet}(\vec{\mu}; s) = \sum_{M \vdash \{1,\ldots,n\}} \prod_{i=1}^{|M|} \mathbb{H}^{\circ}(\vec{\mu}_{M_i}; s). \tag{6.3}$$

This relation is invertible via the inclusion-exclusion formula, which is given by the following lemma.

Lemma 6.3.1. *The generating function for connected double Hurwitz numbers can be written in terms of the disconnected enumeration via*

$$\mathbb{H}^{\circ}(\mu_1, \ldots, \mu_n; s) = \sum_{M \vdash \{1,\ldots,n\}} (-1)^{|M|-1}\, (|M|-1)! \prod_{i=1}^{|M|} \mathbb{H}^{\bullet}(\vec{\mu}_{M_i}; s).$$

Here, M is a set partition of $\{1, \ldots, n\}$ and if $M_i = \{i_1, \ldots, i_k\}$ then $\vec{\mu}_{M_i} = (\mu_{i_1}, \ldots, \mu_{i_k})$.

Proof. Begin with the right side of the statement above and substitute using (6.3). This gives

$$\sum_{M \vdash \{1,\ldots,n\}} (-1)^{|M|-1}\, (|M|-1)! \prod_{i=1}^{|M|} \mathbb{H}^{\bullet}(\vec{\mu}_{M_i}; s)$$
$$= \sum_{M \vdash \{1,\ldots,n\}} (-1)^{|M|-1}\, (|M|-1)! \prod_{i=1}^{|M|} \sum_{N \vdash M_i} \prod_{j=1}^{|N|} \mathbb{H}^{\circ}(\vec{\mu}_{N_j}; s).$$

Consider the coefficient of each term on the right. An arbitrary term $\mathbb{H}^{\circ}(\vec{\mu}_{N_1}; s) \cdots \mathbb{H}^{\circ}(\vec{\mu}_{N_{|N|}}; s)$ can arise from any partition $\{M_1, \ldots, M_{|M|}\}$ where $|M|$ is within the range $1 \leqslant |M| \leqslant |N|$. For each distinct $|M|$, the coefficient of the term $\mathbb{H}^\circ(\vec{\mu}_{N_1}; s) \cdots \mathbb{H}^\circ(\vec{\mu}_{N_{|N|}}; s)$ is given by $(-1)^{|M|-1}(|M|-1)!$ multiplied by the number of times it can occur in this way, which is the number of ways one can group $|N|$ objects into $|M|$ groups—also known as Stirling numbers of the second kind—and this is given by the formula

$$\sum_{j=1}^{|M|} \frac{(-1)^{|M|-j} j^{|N|}}{j!\,(|M|-j)!}.$$

Therefore, summing over possible sizes $|M|$, the coefficient of $\mathbb{H}^\circ(\vec{\mu}_{N_1}; s) \cdots \mathbb{H}^\circ(\vec{\mu}_{N_{|N|}}; s)$ is given by

$$\begin{aligned}
\sum_{|M|=1}^{|N|} (-1)^{|M|-1}(|M|-1)! \sum_{j=1}^{|M|} \frac{(-1)^{|M|-j} j^{|N|}}{j!\,(|M|-j)!} &= \sum_{j=1}^{|N|} \sum_{|M|=j}^{|N|} (-1)^{|M|-1}(|M|-1)! \frac{(-1)^{|M|-j} j^{|N|}}{j!\,(|M|-j)!} \\
&= \sum_{j=1}^{|N|} (-1)^{j+1} j^{|N|-1} \sum_{|M|=j}^{|N|} \binom{|M|-1}{j-1} \\
&= \sum_{j=1}^{|N|} (-1)^{j+1} j^{|N|-1} \binom{|N|}{j} \\
&= \begin{cases} 1, & \text{if } |N| = 1, \\ 0, & \text{if } |N| > 1. \end{cases}
\end{aligned}$$

The first equality has switched the two sums, while the third equality is true by the hockey-stick identity. The fourth equality for $|N| = 1$ is clear by computation. For $|N| > 1$, consider

$$\begin{aligned}
\left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^{|N|-1} \left(-(1-x)^{|N|}\right) &= \left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^{|N|-1} \sum_{j=0}^{|N|} (-1)^{j+1} x^j \binom{|N|}{j} \\
&= \sum_{j=1}^{|N|} (-1)^{j+1} j^{|N|-1} x^j \binom{|N|}{j}.
\end{aligned}$$

The expression $(1-x)^{|N|}$ has a root of order $|N|$ at $x = 1$, and given that applications of $\mathrm{d}/\mathrm{d}x$ reduce the order of the root by 1 while applying x does not affect the root, it follows that $(x\,\mathrm{d}/\mathrm{d}x)^{|N|-1}(1-x)^{|N|}$ has a simple root at $x = 1$. Substituting $x = 1$ into the expression above yields

$$\sum_{j=1}^{|N|} (-1)^{j+1} j^{|N|-1} \binom{|N|}{j} = 0$$

when $|N| > 1$, as claimed. Hence

$$\sum_{M \vdash \{1,\ldots,n\}} (-1)^{|M|-1} (|M|-1)! \prod_{i=1}^{|M|} \mathbb{H}^\bullet(\vec{\mu}_{M_i}; s) = \mathbb{H}^\circ(\mu_1, \ldots, \mu_n; s),$$

as required. ■

Define the *connected correlator* of a tuple of operators $(\mathcal{O}_1, \ldots, \mathcal{O}_n)$, denoted $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle^\circ$, to be what one obtains from applying inclusion-exclusion to the disconnected correlator. That is,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle^\circ = \sum_{M \vdash \{1,\ldots,n\}} (-1)^{|M|-1} (|M|-1)! \prod_{i=1}^{|M|} \langle \vec{\mathcal{O}}_{M_i} \rangle, \tag{6.4}$$

where $\vec{\mathcal{O}}_{M_i} = \prod_{j \in M_i} \mathcal{O}_j$.

6.3.2 Double Hurwitz numbers via the semi-infinite wedge

Begin with the previously derived vacuum expectation for the double Hurwitz generating function given below [91, 103].

Theorem 6.3.2. *The generating function for disconnected double Hurwitz numbers satisfies the following equation.*

$$\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s) = \Big\langle \exp\Big(\sum_{j=1}^{\infty} \frac{q_j}{js}\alpha_j\Big) \exp(s\mathcal{F}_2) \prod_{i=1}^{n} \frac{\alpha_{-\mu_i}}{\mu_i} \Big\rangle \tag{6.5}$$

Here α_m and $\mathcal{F}_2$ are operators defined in Definition 1.3.4 and equation (1.10) respectively.

The following theorem rewrites the above vacuum expectation in a way that will be convenient for the proof of the polynomiality structure.

Proposition 6.3.3. *The generating function for disconnected double Hurwitz numbers is given by the following vacuum expectation in the semi-infinite wedge space:*

$$\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s) = \big\langle \mathcal{C}(\mu_1, s)\, \mathcal{C}(\mu_2, s) \cdots \mathcal{C}(\mu_n, s) \big\rangle, \tag{6.6}$$

where

$$\mathcal{C}(\mu, s) = \frac{1}{\mu} \sum_{i \in \mathbb{Z}} \left[\sum_{\lambda \vdash \mu - i} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu\lambda_k s) \right] \mathcal{E}_{-i}(\mu s),$$

and $\vec{q}_\lambda = q_{\lambda_1} q_{\lambda_2} \cdots q_{\lambda_{\ell(\lambda)}}$. The function $\mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{e^{z/2} - e^{-z/2}}{z}$ and operator $\mathcal{E}_a(z)$ is defined in Definition 1.3.6.

Proof. Begin with the vacuum expectation (6.5) given in Theorem 6.3.2 and, observing that $\exp(-s\mathcal{F}_2)$ and $\exp\big(-\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js}\big)$ fix the vacuum vector, rewrite it as

$$\mathbb{H}^\bullet(\vec{\mu}; s) = \frac{1}{\prod_{i=1}^{n} \mu_i} \Big\langle \prod_{i=1}^{n} \exp\Big(\sum_{j=1}^{\infty} \frac{q_j}{js}\alpha_j\Big) \exp(s\mathcal{F}_2)\, \alpha_{-\mu_i} \exp(-s\mathcal{F}_2) \exp\Big(-\sum_{j=1}^{\infty} \frac{q_j}{js}\alpha_j\Big) \Big\rangle.$$

Consider first the inner conjugation $e^{s\mathcal{F}_2}\alpha_{-\mu}e^{-s\mathcal{F}_2}$. One can compute this conjugation by noting that $\alpha_{-\mu} = \mathcal{E}_{-\mu}(0)$, $\mathcal{F}_2 = [z^2]\mathcal{E}_0(z)$, and using Hadamard's lemma (6.7) coupled with the commutation relation for the $\mathcal{E}$-operators (1.9). Or, alternatively, observe that the operator $\mathcal{F}_2$ is diagonal, with eigenvalue $f_2(\lambda)$ corresponding to the eigenvector v_λ. The function $f_2(\lambda)$ returns the sum of the contents of the Young diagram given by the partition λ, where the content of the box in column j and row i is $j - i$. In this case, for any partition λ,

$$\begin{aligned} e^{s\mathcal{F}_2}\alpha_{-\mu}e^{-s\mathcal{F}_2} v_\lambda &= e^{-sf_2(\lambda)} e^{s\mathcal{F}_2} \alpha_{-\mu} v_\lambda = e^{-sf_2(\lambda)} e^{s\mathcal{F}_2} \sum_{\lambda^{+\mu}} v_{\lambda^{+\mu}} \\ &= \sum_{\lambda^{+\mu}} e^{s\left(f_2(\lambda^{+\mu}) - f_2(\lambda)\right)} v_{\lambda^{+\mu}} = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{s\mu\left(k+\frac{\mu}{2}\right)} : \psi_{k+\mu}\psi_k^* : v_\lambda \\ &= \mathcal{E}_{-\mu}(\mu s), \end{aligned}$$

where the sums over $\lambda^{+\mu}$ are over all Young diagrams that can be obtained from adding a μ-ribbon to λ.

Compute subsequent conjugations by iteratively applying the Hadamard Lemma,

$$e^A B e^{-A} = B + \sum_{k \geqslant 1} \frac{1}{k!} [A, [A, \ldots, [A, B] \cdots]], \tag{6.7}$$

where there are k commutators in the summand. The first iteration gives

$$\begin{aligned} \exp\Big(\frac{\alpha_d q_d}{ds}\Big) \mathcal{E}_{-\mu}(\mu s) \exp\Big(-\frac{\alpha_d q_d}{ds}\Big) &= \sum_{k_d \geqslant 0} \frac{\varsigma(\mu d s)^{k_d} q_d^{k_d}}{k_d!(ds)^{k_d}} \mathcal{E}_{-\mu+dk_d}(\mu s) \\ &= \sum_{k_d \geqslant 0} \frac{(\mu q_d)^{k_d} \mathcal{S}(\mu d s)^{k_d}}{k_d!} \mathcal{E}_{-\mu+dk_d}(\mu s). \end{aligned}$$

Further iterations therefore result in

$$\begin{aligned}
&\exp\Big(\sum_{j=1}^{\infty}\frac{q_j}{js}\alpha_j\Big)\,\mathcal{E}_{-\mu}(\mu s)\,\exp\Big(-\sum_{j=1}^{\infty}\frac{q_j}{js}\alpha_j\Big)\\
&=\sum_{k_1\geqslant 0}\frac{(\mu q_1)^{k_1}\mathcal{S}(\mu s)^{k_1}}{k_1!}\sum_{k_2\geqslant 0}\frac{(\mu q_2)^{k_2}\mathcal{S}(2\mu s)^{k_2}}{k_2!}\cdots\sum_{k_d\geqslant 0}\frac{(\mu q_d)^{k_d}\mathcal{S}(d\mu s)^{k_d}}{k_d!}\,\mathcal{E}_{-\mu+k_1+\cdots+dk_d}(\mu s)\\
&=\sum_{\lambda=(1^{k_1},\ldots,d^{k_d})}\frac{\vec{q}_\lambda\mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu\lambda_k s)\,\mathcal{E}_{-\mu+|\lambda|}(\mu s).
\end{aligned}$$

Here λ is a partition, hence the number of permutations that fix λ is given by the multiplicities; that is, $|\mathrm{Aut}\ \lambda| = k_1!\,k_2!\cdots k_d!$. Letting $-i = |\lambda| - \mu$ yields the result. ■

6.3.3 Dependence on μ_1

With the aim of proving that the double Hurwitz number $DH^{\circ}_{g,n}(\mu_1,\ldots,\mu_n)$ satisfies the polynomiality structure, first consider the dependence of $\mathbb{H}^{\bullet}(\mu_1,\ldots,\mu_n;s)$ on μ_1. Now fix $\mu_2,\ldots,\mu_n$. As detailed in the outline at the start of this section, the dependence on μ_1 yields an expression with a sum over partitions of size $\mu_1 + a$ and it is desirable to reduce this to a sum over partitions of size $\mu_1 - r$. For this, we require the "peeling lemma", Lemma 6.3.4. First, allowing a small abuse of notation, define $\hat{A}^k(x)$ by

$$\hat{A}^k(x) = \sum_{\lambda\vdash k}\frac{\vec{q}_\lambda x^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|},$$

where x is a formal variable and $j \in \mathbb{Z}$. Recalling the definition of $A^j_m(\mu)$ in Definition 6.2.4, observe that

$$j\hat{A}^{\mu-j}(x)\big|_{x=\mu} = A^j_0(\mu) \in V(\mu)$$

for $j \in \{1,2,\ldots,d\}$. Defining the expression $\hat{A}^{\mu-j}(x)$ that distinguishes between μ in the sum and $x^{\ell(\lambda)}$ in the numerator will be useful in proving Lemma 6.3.4 and, later in the section, Lemma 6.3.9.

Lemma 6.3.4 (Peeling lemma). *For a positive integer μ and any integer a,*

$$\sum_{\lambda\vdash\mu+a}\frac{\vec{q}_\lambda\mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu\lambda_k s) = \frac{\mu}{\mu+a}\sum_{r=1}^{d} rq_r\mathcal{S}(\mu rs)\sum_{\lambda\vdash\mu+a-r}\frac{\vec{q}_\lambda\mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu\lambda_k s).$$

Proof. I first aim to show

$$\hat{A}^{\mu+a}(x) = \frac{x}{\mu+a}\sum_{r=1}^{d} rq_r\hat{A}^{\mu+a-r}(x). \tag{6.8}$$

To do this, consider the coefficient of $\vec{q}_\nu x^{\ell(\nu)}$ of both sides. The left side gives

$$[\vec{q}_\nu x^{\ell(\nu)}]\sum_{\lambda\vdash\mu+a}\frac{\vec{q}_\lambda x^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} = \frac{1}{|\mathrm{Aut}\ \nu|},$$

where $|\nu| = |\lambda| = \mu + a$. For the right side, note that the sum will be over all possible partitions λ acquired by removing one part of ν, and this will result in a sum over distinct parts in ν. Thus, if $\nu = (1^{i_1},\ldots,d^{i_d})$,

$$[\vec{q}_\nu x^{\ell(\nu)}]\frac{x}{\mu+a}\sum_{r=1}^{d} rq_r\sum_{\lambda\vdash\mu+a-r}\frac{\vec{q}_\lambda x^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} = \sum_{k=1}^{d}\frac{k}{|\nu|\,|\mathrm{Aut}\ \nu\setminus k|},$$

where I have used that $\ell(\lambda) = \ell(\nu) - 1$. Observing that

$$\sum_{k=1}^{d}\frac{k\,|\mathrm{Aut}\ \nu|}{|\mathrm{Aut}\ \nu\setminus k|} = |\nu|,$$

we have

$$[\vec{q}_\nu x^{\ell(\nu)}]\frac{x}{\mu+a}\sum_{r=1}^{d} rq_r \sum_{\lambda\vdash\mu+a-r} \frac{\vec{q}_\lambda x^{\ell(\lambda)}}{|\text{Aut }\lambda|} = \frac{1}{|\text{Aut }\nu|},$$

proving equation (6.8). Letting $x=\mu$ and applying the rescaling $q_k \mapsto q_k\mathcal{S}(\mu ks)$ yields the result. ■

The following definition of admissible sequences is also required, to be utilised in Lemma 6.3.6.

Definition 6.3.5. Define $\mathcal{P}_d$ to be the set of (possibly empty) sequences $P = (p_1, \ldots, p_{\ell(P)})$ that satisfy $1 \leqslant p_i \leqslant d$ for all $i \in \{1, 2, \ldots, \ell(P)\}$. Fix a positive integer $r \in \{1, 2, \ldots, d\}$, and define $\mathcal{P}_{d,r} \subset \mathcal{P}_d$ to be the set of all non-empty sequences P such that $p_{\ell(P)} \geqslant r$. Denote by $e(P)$ the sum of the terms of P; that is, $e(P) = p_1 + \cdots + p_d$. Further, define $\mathcal{P}_d(e) \subset \mathcal{P}_d$ to be the set of all possible sequences where $e = e(P)$ is fixed.

Lemma 6.3.6. *For fixed positive integers $\mu_2, \ldots, \mu_n$, the double Hurwitz generating function $\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s)$ can be written*

$$\mathbb{H}^\bullet(\vec{\mu}; s) = \sum_{r=1}^{d}\sum_{\lambda\vdash\mu_1-r}\left[\frac{\vec{q}_\lambda \mu_1^{\ell(\lambda)}}{|\text{Aut }\lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s)\right]\Big\langle \mathcal{B}_r(\mu_1,s)\,\mathcal{C}(\mu_2,s)\cdots\mathcal{C}(\mu_n,s)\Big\rangle, \tag{6.9}$$

where

$$\mathcal{B}_r(\mu_1,s) = \frac{1}{\mu_1}\sum_{P\in\mathcal{P}_{r,d}}\left[\prod_{i=1}^{\ell(P)}\frac{\mu_1}{\mu_1 - r + \sum_{j=i}^{\ell(P)} p_i}\, p_i q_{p_i}\mathcal{S}(\mu_1 p_i s)\right]\mathcal{E}_{-r+e(P)}(\mu_1 s).$$

Proof. Begin with the vacuum expectation (6.6) for the double Hurwitz generating function as in Proposition 6.3.3,

$$\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s) = \big\langle \mathcal{C}(\mu_1,s)\,\mathcal{C}(\mu_2,s)\,\cdots\,\mathcal{C}(\mu_n,s)\big\rangle,$$

where

$$\mathcal{C}(\mu,s) = \frac{1}{\mu}\sum_{i\in\mathbb{Z}}\left[\sum_{\lambda\vdash\mu-i}\frac{\vec{q}_\lambda\mu^{\ell(\lambda)}}{|\text{Aut }\lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu\lambda_k s)\right]\mathcal{E}_{-i}(\mu s).$$

Observe that, for the vacuum expectation to return a non-zero value, the energies of the $\mathcal{E}$ operators must sum to zero; that is, $i_1 + i_2 + \cdots + i_n = 0$. Hence, $i_1 = -i_2 - i_3 - \cdots - i_n$. Further, given that $\mathcal{E}_{-i_1}(\mu_1 s)$ is acting on the covacuum, i_1 has an upper bound $i_1 \leqslant 0$. Finally, for each $j \in \{1, 2, \ldots, n\}$, the relation $-i_j = -\mu_j + |\lambda^j|$ gives a bound $-i_j \geqslant -\mu_j$ for all i_j, and thus $i_1 \geqslant -\mu_2 - \mu_3 - \cdots - \mu_n$. Given that $\mu_2, \mu_3, \ldots, \mu_n$ are fixed positive integers, this establishes a lower bound for i_1. Hence,

$$\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s) = \frac{1}{\mu_1}\sum_{a=0}^{\mu_2+\cdots+\mu_n}\left[\sum_{\lambda\vdash\mu_1+a}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\text{Aut }\lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s)\right]\left\langle \mathcal{E}_a(\mu_1 s)\prod_{j=2}^{n}\mathcal{C}(\mu_j,s)\right\rangle.$$

Apply Lemma 6.3.4 repeatedly to decrease $\mu_1 + a$ to $\mu_1 - r$ for some $r \in \{1, 2, \ldots, d\}$. The first application gives

$$\frac{1}{\mu_1}\sum_{a=0}^{\mu_2+\cdots+\mu_n}\sum_{p_1=1}^{d}\frac{\mu_1}{\mu_1+a}p_1 q_{p_1}\mathcal{S}(\mu_1 p_1 s)\sum_{\lambda\vdash\mu_1+a-p_1}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\text{Aut }\lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s).$$

Apply Lemma 6.3.4 again, this time with the shift $a \mapsto a - p_1$ and leaving summands corresponding to negative $a - p_1$ unchanged. Thus the second application will include terms of the form

$$\frac{1}{\mu_1}\frac{\mu_1}{\mu_1+a}\frac{\mu_1}{\mu_1+a-p_1}p_1 q_{p_1}\mathcal{S}(\mu_1 p_1 s)\,p_2 q_{p_2}\mathcal{S}(\mu_1 p_2 s)\sum_{\lambda\vdash\mu_1+a-p_1-p_2}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\text{Aut }\lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s).$$

Repeat this process until $a - p_1 - p_2 - \cdots - p_{\ell(P)}$ is negative for all terms in the summation. The set of all possible sequences $(p_1, p_2, \ldots, p_{\ell(P)})$ that can be constructed thus for all possible a gives rise to the set

$\mathcal{P}_{d,r}$ in Definition 6.3.5 (and summing over this set $\mathcal{P}_{d,r}$ absorbs the sum over a). The condition that each p_i satisfies $1 \leqslant p_i \leqslant d$ ensures that the peeling process will terminate after a finite number of iterations. The condition $p_{\ell(P)} \geqslant r$ is implied by the stopping condition of the algorithm, which is given by the two inequalities

$$a - p_1 - p_2 - \cdots - p_{\ell(P)-1} \geqslant 0,$$
$$a - p_1 - p_2 - \cdots - p_{\ell(P)-1} + p_{\ell(P)} < 0.$$

This concludes the proof of the lemma. ■

6.3.4 Calculating the residue

Observe that in the expression for $\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s)$ in Lemma 6.3.6, the constraints on the set $\mathcal{P}_{d,r}$ dictate that the operator $\mathcal{B}_r(\mu_1, s)$ is a finite linear combination of $\mathcal{E}$-operators whose coefficients are power series in s. Further, for each fixed power of s, its coefficient is a rational function in μ_1 (tensored with parameters $q_1, \ldots, q_d$). Therefore, speaking of the poles of $\mathcal{B}_r(\mu_1, s)$ is well-defined. First note that the factors in the term

$$\prod_{i=1}^{\ell(P)} \frac{\mu_1}{\mu_1 - r + \sum_{j=i}^{\ell(P)} p_i}$$

contribute at most simple poles at negative integers. To ultimately show that $DH^\circ_{g,n}(\mu_1, \ldots, \mu_n)$ satisfies a polynomiality structure, I consider the residue of $\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s)$ at $\mu = -a$ for positive integers a; this is the content of Lemma 6.3.7.

A pole at zero can only occur in the summand corresponding to the sequence $P = (p_1 = r)$; in this case the factor $\frac{1}{\mu_1}$ in $\mathcal{B}_r(\mu_1, s)$ introduces a pole at zero, and further, as this term must involve an $\mathcal{E}$-operator with zero energy, evaluation of $\mathcal{E}_0(\mu_1 s)$ on the covacuum makes it a second-order pole. However, it can be shown that for $n \geqslant 2$, terms of this form correspond precisely to a disconnected contribution; this is shown in Lemma 6.3.8.

Lemma 6.3.7. *Fix $\mu_2, \ldots, \mu_n$ to be positive integers and fix $n \geqslant 2$. Then, for all positive integers a and for $r \in \{1, 2, \ldots, d\}$,*

$$\operatorname*{Res}_{\mu_1=-a} \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1$$
$$= M_r(a; s) \Big\langle \exp\Big(\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js}\Big) \exp(s\mathcal{F}_2)\, \alpha_a \exp(-s\mathcal{F}_2) \exp\Big(-\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js}\Big) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle,$$

where

$$M_r(a; s) = \sum_{P' \in \mathcal{P}_{d,r}} (-a)^{\ell(P')-1} \frac{\prod_{i=1}^{\ell(P')} p'_i q_{p'_i} \mathcal{S}(a p'_i s)}{\prod_{i=2}^{\ell(P')} (-a - r + \sum_{j=i}^{\ell(P')} p'_j)}.$$

Proof. As observed at the start of this section, $\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s)$ has at most simple poles at $\mu_1 = -a$ for positive integers a. Hence, calculate the residue at $-a$ by multiplying $\big\langle \mathcal{B}_r(\mu_1, s) \prod_{i=2}^{n} \mathcal{C}(\mu_i, s) \big\rangle$ by $(\mu_1 + a)$ then taking the limit $\mu_1 \to -a$. That is,

$$\operatorname*{Res}_{\mu_1=-a} \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1$$
$$= \lim_{\mu_1=-a} \left[\frac{\mu_1 + a}{\mu_1} \sum_{P \in \mathcal{P}_{d,r}} \left[\prod_{i=1}^{\ell(P)} \frac{\mu_1}{\mu_1 - r + \sum_{j=i}^{\ell(P)} p_j} p_i q_{p_i} \mathcal{S}(\mu_1 p_i s) \right] \Big\langle \mathcal{E}_{-r+e(P)}(\mu_1 s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \right].$$

This is only non-zero when $a = -r + p_k + p_{k+1} + \cdots + p_{\ell(P)}$ for some $k \in \{1, 2, \ldots, \ell(P)\}$. Cancelling this factor with $(\mu_1 + a)$, substituting in $\mu_1 = -a$ and noting that $\mathcal{S}$ is an even function gives

$$\operatorname*{Res}_{\mu_1=-a} \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1 = \sum_{\substack{P \in \mathcal{P}_{d,r} \\ a+r=\sum_{j=k}^{\ell(P)} p_j}} (-a)^{\ell(P)-1} \prod_{i=1}^{k-1} \frac{p_i q_{p_i} \mathcal{S}(a p_i s)}{-a - r + \sum_{j=i}^{\ell(P)} p_j} \cdot p_k q_{p_k} \mathcal{S}(a p_k s)$$
$$\times \prod_{i=k+1}^{\ell(P)} \frac{p_i q_{p_i} \mathcal{S}(a p_i s)}{-a - r + \sum_{j=i}^{\ell(P)} p_j} \Big\langle \mathcal{E}_{-r+e(P)}(-as) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle.$$

Rewrite the denominators of the fractions in the first product using the condition on a; that is, rewrite using $a = -r + p_k + p_{k+1} + \cdots + p_{\ell(P)}$. Split $P = (p_1, p_2, \ldots, p_{\ell(P)})$ into two sequences

$$P' := (p_k, p_{k+1}, \ldots, p_{\ell(P)}) = (p'_1, p'_2, \ldots, p'_{\ell(P')}) \quad \text{and} \quad P'' := (p_{k-1}, p_{k-2}, \ldots, p_1) = (p''_1, p''_2, \ldots, p''_{\ell(P'')})$$

so that $P' \sqcup P'' = P$ as sets. Observe that $-r + e(P) = a + e(P'')$. Therefore,

$$\operatorname*{Res}_{\mu_1=-a} \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1$$
$$= \left[\sum_{P' \in \mathcal{P}_{d,r}} (-a)^{\ell(P')-1} \frac{\prod_{i=1}^{\ell(P')} p'_i q_{p'_i} \mathcal{S}(a p'_i s)}{\prod_{i=2}^{\ell(P')} -a - r + \sum_{j=i}^{\ell(P')} p'_j} \right]$$
$$\times \sum_{P'' \in \mathcal{P}_d} (-a)^{\ell(P'')} \prod_{i=1}^{\ell(P'')} \frac{p''_i q_{p''_i} \mathcal{S}(a p''_i s)}{\sum_{j=1}^{i} p''_j} \Big\langle \mathcal{E}_{a+e(P'')}(-as) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle$$

On the other hand, use the same technique as in Proposition 6.3.3 for $a \geqslant 1$ to calculate

$$\exp\Big(\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) e^{s\mathcal{F}_2} \alpha_a e^{-s\mathcal{F}_2} \exp\Big(-\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) = \sum_{\lambda} \frac{\vec{q}_\lambda (-a)^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{j=1}^{\ell(\lambda)} \mathcal{S}(a\lambda_j s) \mathcal{E}_{a+|\lambda|}(-as)$$
$$= \sum_{i \geqslant a} \sum_{\lambda \vdash -a+i} \frac{\vec{q}_\lambda (-a)^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{j=1}^{\ell(\lambda)} \mathcal{S}(a\lambda_j s) \mathcal{E}_i(-as).$$

Now apply Lemma 6.3.4 iteratively to reduce $-a + i$ to 0. The first application gives

$$\sum_{i > a} \sum_{\lambda \vdash -a+i} \frac{q_\lambda (-a)^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{j=1}^{\ell(\lambda)} \mathcal{S}(a\lambda_j s) \mathcal{E}_i(-as)$$
$$= \sum_{i > a} \sum_{p_1=1}^{d} \frac{-a}{-a+i} p_1 q_{p_1} \mathcal{S}(a p_1 s) \sum_{\lambda \vdash -a+i-p_1} \frac{q_\lambda (-a)^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{j=1}^{\ell(\lambda)} \mathcal{S}(a\lambda_j s) \mathcal{E}_i(-as).$$

Repeating the process yields a sum over (possibly empty) sequences $P = (p_1, p_2, \ldots, p_{\ell(P)})$ such that $p_1 + p_2 + \cdots + p_{\ell(P)} = -a + i$ for all possible $i \geqslant a$. We include the empty sequence to take into account the term $i = a$. As per Definition 6.3.5, all possible such sequences comprise the set $\mathcal{P}_d(-a + i)$. Thus, applying the process above iteratively gives

$$\sum_{i \geqslant a} \sum_{\lambda \vdash -a+i} \frac{q_\lambda (-a)^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{j=1}^{\ell(\lambda)} \mathcal{S}(a\lambda_j s) \mathcal{E}_i(-as)$$
$$= \sum_{P \in \mathcal{P}_d(-a+i)} \frac{-a}{-a+i} \prod_{k=1}^{\ell(P)-1} \frac{-a}{-a+i-\sum_{j=1}^{k} p_j} \prod_{j=1}^{\ell(P)} p_j q_{p_j} \mathcal{S}(a p_j s) \left[\sum_{\lambda=\emptyset} \frac{\vec{q}_\lambda (-a)^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{j=1}^{\ell(\lambda)} \mathcal{S}(a\lambda_j s) \right] \mathcal{E}_i(-as)$$
$$= \sum_{P \in \mathcal{P}_d(-a+i)} (-a)^{\ell(P)} \prod_{j=1}^{\ell(P)} \frac{p_j q_{p_j} \mathcal{S}(a p_j s)}{\sum_{b=1}^{i} p_b} \mathcal{E}_{a+e(P)}(-as),$$

where the last line has applied the relabelling $(p_1, p_2, \ldots, p_{\ell(P)}) \mapsto (p_{\ell(P)}, p_{\ell(P)-1}, \ldots, p_2, p_1)$. ■

It remains to consider the double pole at zero, which is treated by the following lemma.

Lemma 6.3.8. *Fix $\mu_2, \ldots, \mu_n$ to be positive integers and fix $n \geqslant 2$. Then for all $r \in \{1, 2, \ldots, d\}$,*

$$\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1 = \operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}_r(\mu_1, s) \Big\rangle \Big\langle \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1,$$

and hence, applying the inclusion-exclusion formula,

$$\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle^{\circ} \mathrm{d}\mu_1 = 0.$$

Proof. As stated at the start of this section, a pole at zero can only occur in the summand corresponding to $P = (p_1 = r)$, and in this case this term in $\mathcal{B}_r(\mu_1, s)$ is

$$s r q_r \mathcal{S}(\mu_1 r s) \mathcal{E}_0(\mu_1 s).$$

The factor of μ_1^{-1} in $\mathcal{B}_r(\mu_1, s)$ makes it a second-order pole. Observe that the expression has no simple pole at zero; given that $\mathcal{S}$ is an even function, the residue is indeed 0. Compute the double pole as

$$\begin{aligned}
\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1 &= \operatorname*{Res}_{\mu_1=0} \mu_1 \frac{s r q_r \mathcal{S}(\mu_1 r s)}{\mu_1} \Big\langle \mathcal{E}_0(\mu_1 s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1 \\
&= \operatorname*{Res}_{\mu_1=0} s r q_r \mathcal{S}(\mu_1 r s) \Big\langle \mathcal{E}_0(\mu_1 s) \Big\rangle \Big\langle \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1 \qquad (6.10) \\
&= \operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}_r(\mu_1, s) \Big\rangle \Big\langle \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1,
\end{aligned}$$

where the second equality is using the fact that for any operator $\mathcal{O}$, $\langle \mathcal{E}_0(z) \mathcal{O} \rangle = \varsigma(z)^{-1} \langle \mathcal{O} \rangle = \langle \mathcal{E}_0(z) \rangle \langle \mathcal{O} \rangle$. Apply the residue to the inclusion-exclusion formula (6.4),

$$\begin{aligned}
&\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle^{\circ} \mathrm{d}\mu_1 \\
&\qquad = \operatorname*{Res}_{\mu_1=0} \mu_1 \sum_{\substack{N \subseteq \{2, \ldots, n\} \\ M \vdash \{2, \ldots, n\} \setminus N}} (-1)^{|M|} |M|! \langle \mathcal{B}_r(\mu_1, s) \mathcal{C}(\vec{\mu}_N, s) \rangle \prod_{i=1}^{|M|} \langle \mathcal{C}(\vec{\mu}_{M_i}, s) \rangle \mathrm{d}\mu_1,
\end{aligned}$$

where, for $M_i = \{i_1, \ldots, i_k\}$, $\mathcal{C}(\vec{\mu}_{M_i}, s)$ is a convenient shorthand notation that denotes $\prod_{j=1}^{k} \mathcal{C}(\mu_{i_j}, s)$. Use the result (6.10), which is true for any $n \geqslant 2$, to obtain

$$\begin{aligned}
&\operatorname*{Res}_{\mu_1=0} \mu_1 \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle^{\circ} \mathrm{d}\mu_1 \\
&\qquad = \operatorname*{Res}_{\mu_1=0} \mu_1 \sum_{\substack{N \subseteq \{2, \ldots, n\} \\ M \vdash \{2, \ldots, n\} \setminus N}} (-1)^{|M|} |M|! \langle \mathcal{B}_r(\mu_1, s) \rangle \langle \mathcal{C}(\vec{\mu}_N, s) \rangle \prod_{i=1}^{|M|} \langle \mathcal{C}(\vec{\mu}_{M_i}, s) \rangle \, \mathrm{d}\mu_1,
\end{aligned}$$

Each term in this sum arises twice — either when $N = \emptyset$, which occurs with coefficient $(-1)^{|M|} |M|!$ or if $N = M_i$ for some $i \in \{1, 2, \ldots, |M|\}$, and this occurs with a factor $|M| \cdot (-1)^{|M|-1} (|M| - 1)!$. Thus, each term cancels whenever $n \geqslant 2$. ■

6.3.5 Polynomiality in μ_1

Before giving the main result of this section, I first prove a lemma about extracting the coefficient of s from $\prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s)$.

Lemma 6.3.9. *For all non-negative integers m,*

$$[s^{2m}] \sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\text{Aut } \lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s) = \sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\text{Aut } \lambda|} f(\lambda),$$

where $f(\lambda)$ is a symmetric function in $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$. Further, for any symmetric function $f(\lambda)$,

$$\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\text{Aut } \lambda|} f(\lambda) \in V(\mu).$$

Proof. First, that extracting the coefficient of s from the product $\prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s)$ yields a symmetric function of λ is evident by expanding the functions $\mathcal{S}$ as power series in s. This can be shown through direct computation; for example, in the case $m = 2$,

$$\begin{aligned}
&[s^4] \mathcal{S}(\mu_1 \lambda_1 s) \mathcal{S}(\mu_2 \lambda_2 s) \cdots \mathcal{S}(\mu_1 \lambda_{\ell(\lambda)} s) \\
&= [s^4] \left(1 + \frac{\mu_1^2 \lambda_1^2 s^2}{24} + \frac{\mu_1^4 \lambda_1^4 s^4}{1920} + \cdots \right) \left(1 + \frac{\mu_1^2 \lambda_2^2 s^2}{24} + \frac{\mu_1^4 \lambda_2^4 s^4}{1920} + \cdots \right) \\
&\qquad \cdots \left(1 + \frac{\mu_1^2 \lambda_{\ell(\lambda)}^2 s^2}{24} + \frac{\mu_1^4 \lambda_{\ell(\lambda)}^4 s^4}{1920} + \cdots \right) \\
&= \mu_1^4 \left(\sum_{i=1}^{\ell(\lambda)} \frac{\lambda_i^4}{1920} + \sum_{i,j=1}^{\ell(\lambda)} \frac{\lambda_i^2 \lambda_j^2}{24^2} \right).
\end{aligned}$$

It remains to show that

$$\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\text{Aut } \lambda|} f(\lambda) \in V(\mu).$$

By linearity, it suffices to show the above statement in the case where $g(\lambda)$ is a product of power sum symmetric polynomials $p_{a_1}(\lambda), p_{a_2}(\lambda), \ldots, p_{a_m}(\lambda)$, where $p_a(\lambda) = \lambda_1^a + \lambda_2^a + \cdots + \lambda_{\ell(\lambda)}^a$, since the power sum symmetric polynomials provide a basis for the ring of symmetric functions. To do this, define the operator $\mathcal{Q}_a$ to be

$$\mathcal{Q}_a = \sum_{k=1}^{d} k^a q_k \frac{\partial}{\partial q_k},$$

for a a positive integer. The action of $\mathcal{Q}_a$ on $A_0^j(\mu)$ for $j \in \{1, 2, \ldots, d\}$ is

$$\mathcal{Q}_a A_0^j(\mu) = j \sum_{\lambda \vdash \mu - j} \frac{\mu^{\ell(\lambda)}}{|\text{Aut } \lambda|} \sum_{k=1}^{d} k^a q_k \frac{\partial}{\partial q_k} \vec{q}_\lambda = j \sum_{\lambda \vdash \mu - j} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\text{Aut } \lambda|} p_a(\lambda),$$

hence, applying a product of operators $\mathcal{Q}_{a_1} \cdots \mathcal{Q}_{a_m}$ yields

$$\mathcal{Q}_{a_1} \cdots \mathcal{Q}_{a_m} A_0^j(\mu) = j \sum_{\lambda \vdash \mu - j} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\text{Aut } \lambda|} p_{a_1}(\lambda) \cdots p_{a_m}(\lambda).$$

It remains to show that the right side lives in the vector space $V(\mu)$. We first claim that

$$\sum_{\lambda \vdash \mu - j} \frac{\vec{q}_\lambda x^{\ell(\lambda)}}{|\text{Aut } \lambda|} p_a(\lambda) = x \sum_{k=1}^{d} k^a q_k \sum_{\lambda \vdash \mu - j - k} \frac{\vec{q}_\lambda x^{\ell(\lambda)}}{|\text{Aut } \lambda|}, \tag{6.11}$$

which we can prove by considering the coefficient of $\vec{q}_\nu x^{\ell(\nu)}$ of both sides. On the left side

$$[\vec{q}_\nu x^{\ell(\nu)}] \sum_{\lambda \vdash \mu - j} \frac{\vec{q}_\lambda x^{\ell(\lambda)}}{|\text{Aut } \lambda|} p_a(\lambda) = \frac{p_a(\nu)}{|\text{Aut } \nu|},$$

with $|\nu| = \mu - j$. Similar to the argument in the proof of Lemma 6.3.4, the right side will be a sum over partitions λ acquired by removing one part of ν which will result in a sum over distinct parts in ν, and again $\ell(\nu) = \ell(\lambda) - 1$. Observing that

$$\sum_{\substack{k\in\nu \\ \text{distinct}}} \frac{k^a}{|\mathrm{Aut}\ \nu \setminus k|} = \frac{p_a(\nu)}{|\mathrm{Aut}\ \nu|},$$

the right side yields

$$[\vec{q}_\nu x^{\ell(\nu)}]\ x \sum_{k=1}^{d} k^a q_k \sum_{\lambda\vdash\mu-j-k} \frac{\vec{q}_\lambda x^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} = \sum_{\substack{k\in\nu \\ \text{distinct}}} \frac{k^a}{|\mathrm{Aut}\ \nu \setminus k|} = \frac{p_a(\nu)}{|\mathrm{Aut}\ \nu|}.$$

Substituting $x = \mu$ into equation (6.11), we therefore have

$$\begin{aligned}\mathcal{Q}_a A_0^j(\mu) &= \sum_{\lambda\vdash\mu-j} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} p_a(\lambda) = \mu \sum_{k=1}^{d} k^a q_k \sum_{\lambda\vdash\mu-j-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \\ &= \sum_{k=1}^{d-j} k^a q_k \sum_{\lambda\vdash\mu-j-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)+1}}{|\mathrm{Aut}\ \lambda|} + \mu \sum_{k=d-j+1}^{d} k^a q_k \sum_{\lambda\vdash\mu-j-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}.\end{aligned}$$

Hence, proving the second statement of the lemma has been reduced to showing that $\mathcal{Q}_a$ preserves the vector space $V(\mu)$. The first term in the expression above is a linear combination of $A_{j+k,1}(\mu)$ where $j + k \in \{2, \ldots, d\}$, hence

$$\sum_{k=1}^{d-j} k^a q_k \sum_{\lambda\vdash\mu-j-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)+1}}{|\mathrm{Aut}\ \lambda|} \in V(\mu).$$

To prove that the second term also lives in $V(\mu)$, it suffices to show $\mu A_0^{d+i}(\mu) \in V(\mu)$ for $i \in \{1, 2, \ldots, d\}$. Proceed by induction on i. When $i = 1$,

$$\mu A_0^{d+1}(\mu) = \mu \sum_{\lambda\vdash\mu-d-1} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}.$$

From the proof Lemma 6.3.4, we have

$$\sum_{\lambda\vdash\mu+a} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} = \frac{\mu}{\mu + a} \sum_{k=1}^{d} k q_k \sum_{\lambda\vdash\mu+a-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}, \tag{6.12}$$

for all integers a. Using $a = -1$ and rearranging, we find

$$\mu \sum_{\lambda\vdash\mu-1-d} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} = \frac{\mu - 1}{dq_d} \sum_{\lambda\vdash\mu-1} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} - \frac{1}{dq_d} \sum_{k=1}^{d-1} k q_k \sum_{\lambda\vdash\mu-1-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)+1}}{|\mathrm{Aut}\ \lambda|},$$

and hence $\mu A_0^{d+1}(\mu) \in V(\mu)$. Now assume $\mu A_0^{d+1}(\mu), \ldots, \mu A_0^{d+m-1}(\mu) \in V(\mu)$ and consider

$$\mu A_0^{d+m}(\mu) = \mu \sum_{\lambda\vdash\mu-m-d} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}.$$

Using (6.12) with $a = -m$ and rearranging gives

$$\begin{aligned}&\mu \sum_{\lambda\vdash\mu-m-d} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \\ &= \frac{\mu - m}{dq_d} \sum_{\lambda\vdash\mu-m} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} - \frac{\mu}{dq_d} \sum_{k=1}^{d-1} k q_k \sum_{\lambda\vdash\mu-m-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \\ &= \frac{\mu - m}{dq_d} \sum_{\lambda\vdash\mu-m} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} - \frac{1}{dq_d} \sum_{k=1}^{d-m} k q_k \sum_{\lambda\vdash\mu-m-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)+1}}{|\mathrm{Aut}\ \lambda|} - \frac{\mu}{dq_d} \sum_{k=d-m+1}^{d-1} k q_k \sum_{\lambda\vdash\mu-m-k} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}.\end{aligned}$$

The first two terms live in $V(\mu)$ because they are linear combinations of the basis elements $A^j_m(\mu)$, while the last term lives in $V(\mu)$ by the induction hypothesis. Therefore $\mu A_0^{d+i}(\mu) \in V(\mu)$ for $i \in \{1, 2, \ldots, d\}$, and we can conclude that $\mathcal{Q}_a A^j_0(\mu) \in V(\mu)$.

Given that the vector spaces $V(z)$ and $V(\mu)$ are isomorphic, the fact that $\mathcal{Q}_a$ (acting on elements in $V(z)$ via the isomorphism) and $\frac{\partial}{\partial x}$ commute, and that $\frac{\partial}{\partial x}$ acts on $A^j_m(\mu)$ by a shift $m \mapsto m+1$, it follows that $\mathcal{Q}_a$ preserves the vector space $V(\mu)$, and this concludes the proof of the result. ■

We are now equipped to prove that $DH^\circ_{g,n}(\mu_1, \ldots, \mu_n)$ satisfies the polynomiality structure. Specifically, we prove the following theorem.

Theorem 6.3.10. *Fix $\mu_2, \ldots, \mu_n$ to be positive integers. Then, for all (g,n) satisfying $2g-2+n>0$, the connected double Hurwitz generating function $\mathbb{H}^\circ(\mu_1, \ldots, \mu_n; s)$ can be written*

$$\mathbb{H}^\circ(\mu_1, \ldots, \mu_n; s) = \sum_{r=1}^{d} \left(\sum_{\lambda \vdash \mu_1 - r} \frac{\vec{q}_\lambda \mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \right) \sum_{g \geqslant 0} P^r_{g,n}(\mu_1, \mu_2, \ldots, \mu_n; \vec{q}) s^{2g-2+n},$$

where $P^r_{g,n}(\mu_1, \mu_2, \ldots, \mu_n; \vec{q})$ is a polynomial in μ_1 and $\vec{q}$.

Proof. It will be convenient to consider the cases $n=1$, $n=2$ and $n \geqslant 3$ separately.

Case $n=1$. In this case, I aim to prove that the generating function for double Hurwitz numbers can be written as

$$\mathbb{H}^\circ(\mu; s) = \sum_{r=1}^{d} \left(\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \right) \left(\frac{r q_r}{\mu^2 s} + \sum_{g=1}^{\infty} P^r_{g,1}(\mu; \vec{q}) s^{2g-1} \right),$$

where $P^r_{g,1}$ is a polynomial in μ and $\vec{q}$ for all $g \geqslant 1$.

When $n=1$, the connected and disconnected double Hurwitz numbers coincide, in which case, by Proposition 6.3.3, we have

$$\mathbb{H}^\bullet(\mu; s) = \mathbb{H}^\circ(\mu; s) = \frac{1}{\mu} \sum_{i \in \mathbb{Z}} \left(\sum_{\lambda \vdash \mu - i} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s) \right) \langle \mathcal{E}_{-i}(\mu s) \rangle.$$

Only the summand corresponding to $i=0$ contributes non-trivially, and in this case $\langle \mathcal{E}_0(x) \rangle = \frac{1}{\varsigma(x)}$. Apply Lemma 6.3.4 once yields a sum over all sequences of the form $P = (p_1 = r)$ which gives

$$\begin{aligned}
\mathbb{H}^\circ(\mu; s) &= \frac{1}{\mu} \sum_{r=1}^{d} \left(\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s) \right) r q_r \mathcal{S}(\mu r s) \frac{1}{\varsigma(\mu s)} \\
&= \sum_{r=1}^{d} \left(\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s) \right) \frac{r q_r \mathcal{S}(\mu r s)}{\mu^2 s\, \mathcal{S}(\mu s)}.
\end{aligned}$$

Extracting the coefficient of s^{2g-1} yields

$$\begin{aligned}
DH^\circ_{g,1}(\mu) &= [s^{2g-1}] \sum_{r=1}^{d} \left(\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s) \right) \frac{r q_r \mathcal{S}(\mu r s)}{\mu^2 s\, \mathcal{S}(\mu s)} \\
&= \sum_{\substack{a+b=2g \\ a,b \text{ even}}} \sum_{r=1}^{d} \left(\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} [s^a] \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s) \right) \frac{r q_r}{\mu^2} [s^b] \frac{\mathcal{S}(\mu r s)}{\mathcal{S}(\mu s)}.
\end{aligned}$$

The case $g=0$ corresponds to choosing the constant from each $\mathcal{S}$ term and this gives $P^r_{0,1}(\mu; \vec{q}) = \frac{r q_r}{\mu^2}$, as expected. When $g > 0$,

$$\sum_{\substack{a+b=2g\\ a,b \text{ even}}} \sum_{r=1}^{d} \left(\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} [s^a] \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu \lambda_k s) \right) \frac{r q_r}{\mu^2} [s^b] \frac{\mathcal{S}(\mu r s)}{\mathcal{S}(\mu s)}$$

$$= \sum_{\substack{a+b=2g\\ a,b \text{ even}}} \sum_{r=1}^{d} \left(\sum_{\lambda \vdash \mu - r} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \mu^a f(\lambda) \right) \frac{r q_r}{\mu^2} \mu^b c_b(r),$$

where $c_b(r)$ is a degree b polynomial in r. Applying the second statement of Lemma 6.3.9 yields the desired result, concluding the proof of the case $n = 1$.

Case $n = 2$. The expression given in Lemma 6.3.6 in the case of $n = 2$ is

$$\mathbb{H}^\bullet(\mu_1, \mu_2; s) = \sum_{r=1}^{d} \sum_{\lambda \vdash \mu_1 - r} \left(\frac{\vec{q}_\lambda \mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu_1 \lambda_k s) \right) \Big\langle \mathcal{B}_r(\mu_1, s)\, \mathcal{C}(\mu_2, s) \Big\rangle.$$

By Lemma 6.3.8 any poles at zero are cancelled when passing to the connected count via inclusion-exclusion. Hence, it remains to treat the poles at negative integers; here Lemma 6.3.7 gives

$$\begin{aligned}
&\operatorname*{Res}_{\mu_1 = -a} \Big\langle \mathcal{B}_r(\mu_1, s) \mathcal{C}(\mu_2, s) \Big\rangle \mathrm{d}\mu_1 \\
&\quad = M_r(a; s) \Big\langle \exp\Big(\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) \exp(s\mathcal{F}_2) \alpha_a \exp(-s\mathcal{F}_2) \exp\Big(-\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) \mathcal{C}(\mu_2, s) \Big\rangle \\
&\quad = M_r(a; s) \Big\langle \exp\Big(\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) \exp(s\mathcal{F}_2) \alpha_a \frac{\alpha_{-\mu_2}}{\mu_2} \Big\rangle \\
&\quad = M_r(a; s)\, \delta_{a,\mu_2}.
\end{aligned}$$

The last line uses the commutation relation for the bosonic operators $\alpha_{\pm m}$ (1.8), and the fact that the vacuum expectation $\langle e^{\sum_j \frac{\alpha_j q_j}{js} + s\mathcal{F}_2} \rangle$ contributes 1. This shows that the residue is only non-zero when $\mu_1 = -\mu_2$. Consider the expression for double Hurwitz numbers as given by Proposition 6.3.3 and note that in the case $n = 2$ the constraint on the energies is $i_1 = -i_2 = a$ with $a \in \{0, 1, \ldots, \mu_2\}$. Thus,

$$\begin{aligned}
&\mathbb{H}^\bullet(\mu_1, \mu_2; s) \\
&\quad = \frac{1}{\mu_1\, \mu_2} \sum_{a=0}^{\mu_2} \left(\sum_{\lambda^1 \vdash \mu_1 + a} \frac{\vec{q}_{\lambda^1} \mu_1^{\ell(\lambda^1)}}{|\mathrm{Aut}\ \lambda^1|} \prod_{k=1}^{\ell(\lambda^1)} \mathcal{S}(\mu_1 \lambda^1_k s) \right) \left(\sum_{\lambda^2 \vdash \mu_2 - a} \frac{\vec{q}_{\lambda^2} \mu_2^{\ell(\lambda^2)}}{|\mathrm{Aut}\ \lambda^2|} \prod_{k=1}^{\ell(\lambda^2)} \mathcal{S}(\mu_2 \lambda^2_k s) \right) \\
&\qquad \Big\langle \mathcal{E}_a(\mu_1 s) \mathcal{E}_{-a}(\mu_2 s) \Big\rangle.
\end{aligned}$$

The inclusion-exclusion formula in the case of $n = 2$ is

$$\langle \mathcal{E}_a(z_1) \mathcal{E}_{-a}(z_2) \rangle^\circ = \langle \mathcal{E}_a(z_1) \mathcal{E}_{-a}(z_2) \rangle - \langle \mathcal{E}_a(z_1) \rangle \, \langle \mathcal{E}_{-a}(z_1) \rangle,$$

hence, because $\langle \mathcal{E}_a(z) \rangle$ vanishes unless $a = 0$, one can pass to the connected generating function by excluding the $a = 0$ term from the summand. In the aim of showing $DH^\circ_{g,2}(\mu_1, \mu_2) \in V(\mu_1)$ for $g > 0$, one would apply Lemma 6.3.4 to reduce the sum over $\lambda^1 \vdash \mu_1 + a$ to one over $\mu_1 - r$ for some $r = 1, 2, \ldots, d$. Recall that applying Lemma 6.3.4 to an expression that is being summed over $\lambda \vdash \mu_1 + a$ introduces a factor of $\frac{\mu_1}{\mu_1 + a}$. With this in mind, a pole at $\mu_1 = -\mu_2$ can only arise when applying Lemma 6.3.4 to an expression including a sum over $\lambda \vdash \mu_1 + \mu_2$ (and this would yield a factor of $\frac{\mu_1}{\mu_1 + \mu_2}$). Observe that a sum of this form corresponds to the $a = \mu_2$ term in the sum over a. Hence, by the argument above that the residue is only non-zero when $\mu_1 = -\mu_2$, it suffices to consider the $a = \mu_2$ summand in the sum — and in this case $\lambda^2 = \emptyset$.

That is, to consider the poles of $\mathbb{H}^\bullet(\mu_1,\mu_2;s)$, it suffices to consider only the following expression:

$$\sum_{\lambda\vdash\mu_1+\mu_2}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s)\Big\langle\mathcal{E}_{\mu_2}(\mu_1 s)\mathcal{E}_{-\mu_2}(\mu_2 s)\Big\rangle^\circ$$

$$=\sum_{\lambda\vdash\mu_1+\mu_2}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s)\frac{\mathcal{S}(\mu_2(\mu_1+\mu_2)s)}{\mathcal{S}((\mu_1+\mu_2)s)}.$$

The expansion of the ratio of the $\mathcal{S}$ functions begins

$$\frac{\mathcal{S}(\mu_2(\mu_1+\mu_2)s)}{\mathcal{S}((\mu_1+\mu_2)s)}=1+\frac{(\mu_2^2-1)}{24}(\mu_1+\mu_2)^2s^2+O((\mu_1+\mu_2)^2s^4),$$

and, extracting the coefficient of any positive power of s from this term would include a factor of $(\mu_1+\mu_2)^k$ for k even and $k\geqslant 2$, which annihilates any simple pole at $\mu_1=-\mu_2$. Therefore, it remains to study the coefficient of s^{2g} in the expression

$$f(\vec{\mu};s;\vec{q}):=\sum_{\lambda\vdash\mu_1+\mu_2}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s)$$

for $g>0$. I achieve this by showing that the coefficient of s^{2g} for $g>0$ can in fact be written as a linear combination of terms of the form

$$\sum_{b=1}^{g}\sum_{\lambda'\vdash\mu_1+\mu_2-b}\frac{\vec{q}_{\lambda'}\mu_1^{\ell(\lambda')}}{|\mathrm{Aut}\ \lambda'|}.$$

That is, for $g>0$, extracting the coefficient of s from $f(\vec{\mu};s;\vec{q})$ reduces the sum over $\lambda\vdash\mu_1+\mu_2-b$ for b strictly positive. Hence, applying Lemma 6.3.4 to $f(\vec{\mu};s;\vec{q})$ after extracting a coefficient of s will only introduce factors of the form $\frac{\mu_1}{\mu_1+\mu_2-b}$ for b strictly positive, and therefore, a pole at $\mu_1=-\mu_2$ cannot be introduced. Combining this with the argument above that a pole can only arise at $\mu_1=-\mu_2$ would conclude the proof in this case.

In the case $g=0$, extracting the coefficient of s^0 corresponds to choosing the constant from each $\mathcal{S}(\mu_1\lambda_k s)$ term. Hence,

$$[s^0]\sum_{\lambda\vdash\mu_1+\mu_2}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s)=\sum_{\lambda\vdash\mu_1+\mu_2}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|},$$

which, when applying Lemma 6.3.4, as described above, will pick up a factor of $\frac{1}{\mu_1+\mu_2}$, as expected.

To prove that $[s^{2g}]f(\vec{\mu};s\vec{q})$ can be written as a linear combination of terms of the appropriate form, first rewrite

$$f(\vec{\mu};s;\vec{q})=\sum_{\lambda\vdash\mu_1+\mu_2}\frac{\vec{q}_\lambda\mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|}\prod_{k=1}^{\ell(\lambda)}\mathcal{S}(\mu_1\lambda_k s)=\sum_{\lambda\vdash\mu_1+\mu_2}\prod_{j=1}^{d}\frac{q_j^{k_j}\mu_1^{k_j}}{k_j!}\mathcal{S}(\mu_1 js)^{k_j},$$

recalling that $\lambda=(1^{k_1},2^{k_2},\ldots,d^{k_d})$. Now consider the product $\mathcal{S}(\mu_1 js)^{k_j}$. If $\mathcal{S}(x)=\sum_k c_{2k}x^{2k}$, then

$$\begin{aligned}\mathcal{S}(\mu_1 js)^{k_j}&=\Big(1+\frac{1}{24}s^2\mu_1^2j^2+\frac{1}{1920}s^4\mu_1^4j^4+\frac{1}{5760}s^6\mu_1^6j^6+\cdots\Big)^{k_j}\\
&=1+\binom{k_j}{1}\frac{1}{24}\mu_1^2s^2j^2+\mu_1^4s^4j^4\left(\frac{1}{1920}\binom{k_j}{1}+\frac{1}{24^2}\binom{k_j}{2}\right)\\
&\qquad+\mu_1^4s^6j^6\left(\frac{1}{5760}\binom{k_j}{1}+\frac{1}{1920\cdot 24}\binom{k_j}{1,1}+\frac{1}{24^3}\binom{k_j}{3}\right)+\cdots\\
&=1+\binom{k_j}{1}c_2\,c_0^{k_j-1}\mu_1^2s^2j^2+\mu_1^4s^4j^4\left(c_4\,c_0^{k_j-1}\binom{k_j}{1}+c_2^2\,c_0^{k_j-2}\binom{k_j}{2}\right)\\
&\qquad+\mu_1^4s^6j^6\left(c_6\,c_0^{k_j-1}\binom{k_j}{1}+c_4\,c_2\,c_0^{k_j-2}\binom{k_j}{1,1}+c_2^3\,c_0^{k_j-3}\binom{k_j}{3}\right)+\cdots\\
&=\sum_{i\geqslant 0}\mu_1^{2i}s^{2i}j^{2i}\sum_{\substack{|\rho|=2i\\ \ell(\rho)=k_j}}\frac{\vec{c}_\rho k_j!}{m_0!\,m_2!\cdots m_{2i}!},\end{aligned}$$

where $\rho = (0^{m_0}, 2^{m_2}, \ldots, 2i^{m_{2i}})$. Extracting the coefficient of s^{2g}, for $g > 0$,

$$
\begin{aligned}
[s^{2g}]f(\vec{\mu}; s; \vec{q}) &= \sum_{\substack{\lambda \vdash \mu_1+\mu_2 \\ a_1+\cdots+a_d=g}} \prod_{j=1}^{d} \frac{q_j^{k_j}\,\mu_1^{k_j}}{k_j!} [s^{2a_j}]\mathcal{S}(\mu_1 j s)^{k_j} \\
&= \sum_{\substack{\lambda \vdash \mu_1+\mu_2 \\ a_1+\cdots+a_d=g}} \prod_{j=1}^{d} \frac{q_j^{k_j}\,\mu_1^{k_j}}{k_j!} \sum_{\substack{|\rho|=2a_j \\ \ell(\rho)=k_j}} \frac{\vec{c}_\rho\,\mu_1^{2a_j}\,j^{2a_j}\,k_j!}{m_0!\,m_2!\cdots m_{2a_j}!} \\
&= \sum_{\lambda \vdash \mu_1+\mu_2} \sum_{\substack{\rho^1,\rho^2,\ldots,\rho^d \\ \sum_{j=1}^d |\rho^j|=2g \\ \ell(\rho^j)=k_j}} \prod_{j=1}^{d} \frac{\mu_1^{m_0}\,q_j^{m_0}}{m_0!} \frac{\vec{c}_\rho\,(\mu_1 q_j)^{k_j-m_0}(\mu_1 j)^{2a_j}}{m_2!\,m_4!\cdots m_{2a_j}!} \\
&= \sum_{\lambda \vdash \mu_1+\mu_2} \left[\prod_{j=1}^{d} \sum_{m_0^j=0}^{k_j} \frac{\mu_1^{m_0^j}\,q_j^{m_0^j}}{m_0^j!}\right] \sum_{\substack{\overline{\rho}^1,\overline{\rho}^2,\ldots,\overline{\rho}^d \\ \sum_{j=1}^d |\overline{\rho}^j|=2g}} \prod_{j=1}^{d} \frac{\vec{c}_{\overline{\rho}^j}\,(\mu_1 q_j)^{\ell(\overline{\rho}^j)}(\mu_1 j)^{2a_j}}{\overline{m}_2^j!\,\overline{m}_4^j!\cdots \overline{m}_{2a_j}^j!},
\end{aligned}
$$

where $\overline{\rho}_i$ is equal to ρ_i but with all zero-parts removed; that is, $\overline{\rho}_i$ is now a partition of $2a_i$ with only even parts.

The key step in the above working lies in the observation that extracting the coefficient of s^{2a_j} from $\mathcal{S}^{k_j}$ for each $j \in \{1, 2, \ldots, d\}$ yields a factor of $k_j!$ which cancels with the same factor in the denominator, and allows us to rewrite the sum over $\lambda \vdash \mu_1 + \mu_2$ with $\lambda = (1^{k_1}, 2^{k_2}, \ldots, d^{k_d})$ as a sum over $\lambda' = (1^{m_0^1}, 2^{m_0^2}, \ldots, d^{m_0^d})$ with $\lambda' \vdash \mu_1 + \mu_2 - b$ for all possible b.

The contribution for

$$b = \sum_{j=1}^{d} j\,\ell(\overline{\rho}_j)$$

is given by

$$\sum_{\substack{\overline{\rho}^1,\overline{\rho}^2,\ldots,\overline{\rho}^d \\ \sum_{i=1}^d |\overline{\rho}^i|=2g}} \prod_{j=1}^{d} \frac{\vec{c}_{\overline{\rho}^j}\,(\mu_1 q_j)^{k_j-\overline{m}_0^j}(\mu_1 j)^{2a_j}}{\overline{m}_2^j!\,\overline{m}_4^j!\cdots \overline{m}_{2a_j}^j!}.$$

Hence,

$$
\begin{aligned}
&[s^{2g}] \sum_{\lambda \vdash \mu_1+\mu_2} \frac{\vec{q}_\lambda \mu_1^{\ell(\lambda)}}{|\text{Aut }\lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu_1 \lambda_k s) \\
&\qquad = \sum_{b=1}^{g} \sum_{\lambda' \vdash \mu_1+\mu_2-b} \frac{\vec{q}_{\lambda'} \mu_1^{\ell(\lambda')}}{|\text{Aut }\lambda'|} \sum_{\substack{\overline{\rho}^1,\overline{\rho}^2,\ldots,\overline{\rho}^d \\ \sum_{i=1}^d |\overline{\rho}^i|=2g}} \prod_{j=1}^{d} \frac{\vec{c}_{\overline{\rho}^j}\,(\mu_1 q_j)^{k_j-\overline{m}_0^j}(\mu_1 j)^{2a_j}}{\overline{m}_2^j!\,\overline{m}_4^j!\cdots \overline{m}_{2a_j}^j!},
\end{aligned}
$$

where the sum over b starts at 1 because $g > 0$ and concludes with $b = g$ because $\sum_{j=1}^{d} |\overline{\rho}^j| = 2g$. Applying Lemma 6.3.4 now will only introduce rational terms with denominators $\mu_1 + \mu_2 - b$ for $b > 0$; hence this process cannot introduce a pole at $\mu_1 = -\mu_2$. Therefore, we can conclude that, in the case $n = 2$,

$$\mathbb{H}^\circ(\mu_1, \mu_2; s) = \sum_{r=1}^{d} \left(\sum_{\lambda' \vdash \mu_1 - r} \frac{\vec{q}_{\lambda'} \mu_1^{\ell(\lambda')}}{|\text{Aut }\lambda'|} \right) \sum_{g \geqslant 0} P_{g,2}^{r}(\mu_1, \mu_2; \vec{q}) s^{2g}$$

where $P_{g,2}^{r}(\mu_1, \mu_2; \vec{q})$ is a polynomial in μ_1 and $\vec{q}$ for all $g > 0$. This concludes the case $n = 2$.

Case $n \geqslant 3$. Begin with the expression given in Lemma 6.3.6,

$$\mathbb{H}^\bullet(\mu_1, \ldots, \mu_n; s) = \sum_{r=1}^{d} \sum_{\lambda \vdash \mu_1 - r} \left(\frac{\vec{q}_\lambda \mu_1^{\ell(\lambda)}}{|\text{Aut }\lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu_1 \lambda_k s) \right) \Big\langle \mathcal{B}_r(\mu_1, s)\, \mathcal{C}(\mu_2, s)\, \mathcal{C}(\mu_3, s) \cdots \mathcal{C}(\mu_n, s) \Big\rangle.$$

As in the $n = 2$ case, Lemma 6.3.8 gives that any poles at zero are cancelled via the inclusion-exclusion process. Use Lemma 6.3.7 to calculate the residue at negative integers. That is,

$$\begin{aligned}
&\operatorname*{Res}_{\mu_1=-a} \Big\langle \mathcal{B}_r(\mu_1, s) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \mathrm{d}\mu_1 \\
&\quad = M_r(a;s) \Big\langle \exp\Big(\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) \exp(s\mathcal{F}_2) \alpha_a \exp(-s\mathcal{F}_2) \exp\Big(-\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) \prod_{j=2}^{n} \mathcal{C}(\mu_j, s) \Big\rangle \\
&\quad = M_r(a;s) \Big\langle \exp\Big(\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) \exp(s\mathcal{F}_2)\, \alpha_a \frac{\alpha_{-\mu_2}}{\mu_2} \frac{\alpha_{-\mu_3}}{\mu_3} \cdots \frac{\alpha_{-\mu_n}}{\mu_n} \Big\rangle.
\end{aligned}$$

Commute α_a to the right. The commutation relation $[\alpha_a, \alpha_{-\mu_k}] = a\, \delta_{a,\mu_k}$ implies that the residue vanishes except possibly if a is equal to one of the fixed positive integers μ_k for some $k \in \{2, 3, \ldots, n\}$. In this case, the residue reads

$$\delta_{a,\mu_k} M_r(\mu_k; s) \Big\langle \exp\Big(\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) \exp(s\mathcal{F}_2) \frac{\alpha_{-\mu_2}}{\mu_2} \cdots \widehat{\alpha}_{-\mu_k} \cdots \frac{\alpha_{-\mu_n}}{\mu_n} \Big\rangle = \delta_{a,\mu_k} M_r(\mu_k; s) \Big\langle \prod_{\substack{i=2 \\ i \neq k}}^{n} \mathcal{C}(\mu_i, s) \Big\rangle.$$

This simple pole at $\mu_1 = -\mu_k$ simplifies via the inclusion-exclusion formula against the simple pole arising from the term

$$\Big\langle \mathcal{B}_r(\mu_1, s)\, \mathcal{C}(\mu_k, s) \Big\rangle \Big\langle \prod_{\substack{i=2 \\ i \neq k}}^{n} \mathcal{C}(\mu_i, s) \Big\rangle.$$

Indeed, by Lemma 6.3.7,

$$\begin{aligned}
&\operatorname*{Res}_{\mu_1=-\mu_k} \Big\langle \mathcal{B}_r(\mu_1, s)\, \mathcal{C}(\mu_k, s) \Big\rangle \Big\langle \prod_{\substack{i=2 \\ i \neq k}}^{n} \mathcal{C}(\mu_i, s) \Big\rangle \mathrm{d}\mu_1 \\
&\qquad = M_r(\mu_k; s) \Big\langle \exp\Big(\sum_{j=1}^{\infty} \frac{\alpha_j q_j}{js} \Big) \exp(s\mathcal{F}_2)\, \alpha_a \frac{\alpha_{-\mu_2}}{\mu_2} \Big\rangle \Big\langle \prod_{\substack{i=2 \\ i \neq k}}^{n} \mathcal{C}(\mu_i, s) \Big\rangle \\
&\qquad = M_r(\mu_k; s) \Big\langle \prod_{\substack{i=2 \\ i \neq k}}^{n} \mathcal{C}(\mu_i, s) \Big\rangle.
\end{aligned}$$

Applying inclusion-exclusion and induction on n as in the case of the double pole at zero (see Lemma 6.3.8) proves that for $n \geqslant 3$ the connected generating function $\mathbb{H}^\circ(\mu_1, \ldots, \mu_n; s)$ can be written

$$\mathbb{H}^\circ(\mu_1, \ldots, \mu_n; s) = \sum_{r=1}^{d} \Bigg(\sum_{\lambda \vdash \mu_1 - r} \frac{\vec{q}_\lambda \mu_1^{\ell(\lambda)}}{|\mathrm{Aut}\ \lambda|} \prod_{k=1}^{\ell(\lambda)} \mathcal{S}(\mu_1 \lambda_k s) \Bigg) \sum_{h \geqslant 0} Q^r_{h,n}(\mu_1, \mu_2, \ldots, \mu_n; \vec{q}) s^{2h-2+n}.$$

Now apply Lemma 6.3.9 to conclude the result. ■

6.3.6 Invoking symmetry

To conclude the proof of Theorem 6.1.2, it remains to use the symmetry of $DH^\circ_{g,n}(\mu_1, \ldots, \mu_n)$ in $\mu_1, \ldots, \mu_n$ to show that $DH^\circ_{g,n}(\mu_1, \ldots, \mu_n) \in V(\mu_1) \otimes V(\mu_2) \otimes \cdots \otimes V(\mu_n)$.

Theorem 6.3.11. *Suppose that $F(\mu_1, \ldots, \mu_n)$ is a symmetric function of positive integers $\mu_1, \ldots, \mu_n$. If, for fixed positive integers $\mu_2, \ldots, \mu_n$ we have $F(\mu_1, \ldots, \mu_n) \in V(\mu_1)$, then*

$$F(\mu_1, \ldots, \mu_n) \in V(\mu_1) \otimes V(\mu_2) \otimes \cdots \otimes V(\mu_n).$$

Proof. By assumption $F(\mu_1, \ldots, \mu_n) \in V(\mu_1)$, and considering the definition of $V(\mu)$ as given in Definition 6.2.4, we can write $F(\mu_1, \ldots, \mu_n)$ as

$$F(\mu_1, \ldots, \mu_n) = \sum_{k=0}^{m-1} A_k(\mu_1) b_k(\mu_2, \mu_3, \ldots, \mu_n)$$

where $m-1$ is a finite integer. Here, we have taken an index set X to be the sensible relabelling of $\{j, m\}$ where $j \in \{1, 2, \ldots, d\}$ and m is a non-negative integer, and we write $A_k(\mu_1) = A^j_m(\mu_1)$ for $k \in X$. Prove $F(\mu_1, \ldots, \mu_n) \in V(\mu_2)$ by showing that $b_k(\mu_2, \mu_3, \ldots, \mu_n) \in V(\mu_2)$ for all $k \in \{0, 1, \ldots, m-1\}$.

Begin by computing the values of F at $\mu_1 \in \{1, \ldots, m\}$, resulting in the following system of equations.

$$\begin{aligned}
F(1, \mu_2, \ldots, \mu_n) &= \sum_{k=0}^{m-1} A_k(1) b_k(\mu_2, \ldots, \mu_n) \\
F(2, \mu_2, \ldots, \mu_n) &= \sum_{k=0}^{m-1} A_k(2) b_k(\mu_2, \ldots, \mu_n) \\
&\vdots \\
F(m, \mu_2, \ldots, \mu_n) &= \sum_{k=0}^{m-1} A_k(m) b_k(\mu_2, \ldots, \mu_n)
\end{aligned}$$

Now invoke the symmetry of $F(\mu_1, \ldots, \mu_n)$ in $\mu_1, \ldots, \mu_n$ to write

$$F(\mu_1, \ldots, \mu_n) = F(\mu_2, \ldots, \mu_n, \mu_1),$$

and find the following equivalent system of linear equations.

$$\begin{aligned}
F(\mu_2, \ldots, \mu_n, 1) &= \sum_{k=0}^{m-1} A_k(\mu_2) b_k(\mu_3, \ldots, \mu_n, 1) \\
F(\mu_2, \ldots, \mu_n, 2) &= \sum_{k=0}^{m-1} A_k(\mu_2) b_k(\mu_3, \ldots, \mu_n, 2) \\
&\vdots \\
F(\mu_2, \ldots, \mu_n, m) &= \sum_{k=0}^{m-1} A_k(\mu_2) b_k(\mu_3, \ldots, \mu_n, m)
\end{aligned}$$

Equating the right hand sides of the two systems above gives

$$\begin{aligned}
\sum_{k=0}^{m-1} A_k(1) b_k(\mu_2, \ldots, \mu_n) &= \sum_{k=0}^{m-1} A_k(\mu_2) b_k(\mu_3, \ldots, \mu_n, 1) \\
\sum_{k=0}^{m-1} A_k(2) b_k(\mu_2, \ldots, \mu_n) &= \sum_{k=0}^{m-1} A_k(\mu_2) b_k(\mu_3, \ldots, \mu_n, 2) \\
&\vdots \\
\sum_{k=0}^{m-1} A_k(m) b_k(\mu_2, \ldots, \mu_n) &= \sum_{k=0}^{m-1} A_k(\mu_2) b_k(\mu_3, \ldots, \mu_n, m).
\end{aligned}$$

As outlined above, we ultimately wish to show that $b_k(\mu_2, \mu_3, \ldots, \mu_n) \in V(\mu_2)$ for all $k \in \{0, 1, \ldots, m-1\}$. Note that $A_k(\mu_2) \in V(\mu_2)$ for all $k \in \{0, 1, \ldots, m-1\}$ by definition, and $b_k(\mu_3, \ldots, \mu_n, c)$ is constant with respect to μ_2. Hence, as the right hand sides of the equations above are linear combinations of terms $A_k(\mu_2)\, b_k(\mu_3, \ldots, \mu_n, c)$ for $k \in \{0, 1, \ldots, m-1\}$ and $c \in \{1, 2, \ldots, m\}$, it follows that both sides of the equations above live in $V(\mu_2)$. If the system of equations is solvable for $b_0, \ldots, b_m$, then it will follow that each of these functions must live in the vector space $V(\mu_2)$. Thus, it remains to show that the system of linear equations above is solvable, or equivalently, that the matrix

$$\begin{bmatrix}
A_0(1) & A_1(1) & A_2(1) & \cdots & A_{m-1}(1) \\
A_0(2) & A_1(2) & A_2(2) & \cdots & A_{m-1}(2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_0(m-1) & A_1(m-1) & A_2(m-1) & \cdots & A_{m-1}(m-1)
\end{bmatrix}$$

is invertible. But

$$A_k(\mu) = A^j_m(\mu) = \sum_{\lambda \vdash \mu - j} \frac{\vec{q}_\lambda \mu^{\ell(\lambda)+m}}{|\mathrm{Aut}\ \lambda|}$$

are linearly independent if and only if the $\phi^j_m(z)$ are linearly independent. To show this, first observe that for $j \in \{1, 2, \ldots, d\}$, the $\phi^j_{-1}(z) = z^j$ are linearly independent, and for distinct m, $\phi^j_m(z) = (\partial/\partial x)^{m+1}\phi^j_{-1}(z)$ have distinct pole orders, hence any set of $\phi^j_m(z)$ with distinct m are linearly independent. It remains then to show that $\{\phi^1_m(z), \ldots, \phi^d_m(z)\}$ are linearly independent for all m. If $\{\phi^1_{m-1}(z), \ldots, \phi^d_{m-1}(z)\}$ are linearly independent, then it follows that $\{\frac{\partial}{\partial x}\phi^1_{m-1}, \ldots, \frac{\partial}{\partial x}\phi^d_{m-1}\} = \{\phi^1_m(z), \ldots, \phi^d_m(z)\}$ are also linearly independent. Since $\{\phi^1_{-1}(z), \ldots, \phi^d_{-1}(z)\}$ are linearly independent, it follows that $\{\phi^1_m(z), \ldots, \phi^d_m(z)\}$ are for all non-negative integers m.

Therefore, $b_k(\mu_2, \mu_3, \ldots, \mu_n) \in V(\mu_2)$ for all $k \in \{0, 1, \ldots, m-1\}$ and hence $F(\mu_1, \ldots, \mu_n) \in V(\mu_2)$. Iterating this argument for $\mu_3, \ldots, \mu_n$, it follows that $F(\mu_1, \ldots, \mu_n) \in V(\mu_1) \otimes \cdots \otimes V(\mu_n)$, as required. ■

6.4 An ELSV-like formula for double Hurwitz numbers

This section works toward obtaining an ELSV-like formula for double Hurwitz numbers. This is done by considering the spectral curve for double Hurwitz numbers within a 1-parameter deformation of the spectral curve of a similar enumeration, orbifold Hurwitz numbers. One can then use a variational result of Eynard and Orantin [52, Theorem 5.1] to relate the free energies of the enumeration.

First, we introduce orbifold Hurwitz numbers and recall the relevant results for this enumeration.

6.4.1 Orbifold Hurwitz numbers

The ***orbifold Hurwitz number*** $H^{[d]}_{g,n}(\mu_1, \ldots, \mu_n)$ is the weighted enumeration of connected genus g branched covers $f\colon (\Sigma; p_1, p_2, \ldots, p_n) \to (\mathbb{CP}^1; \infty)$ such that

- the point $p_i \in f^{-1}(\infty)$ has ramification index μ_i for $i \in \{1, 2, \ldots, n\}$;
- the ramification profile of $0 \in \mathbb{CP}^1$ is $(d, d, \ldots, d)$; and
- all other branching is simple and occurs over prescribed points of $\mathbb{CP}^1$.

The weight of a branched cover is given by

$$\frac{1}{m!\,|\mathrm{Aut}\ f|},$$

where $m = 2g - 2 + n + |\mu|/d$ is the number of simple branch points, as given by the Riemann–Hurwitz formula.

Note that orbifold Hurwitz numbers are a specialisation of double Hurwitz numbers, and their enumeration can be obtained from Definition 6.1.1 by setting $q_d = 1$ and $q_k = 0$ for all $k \neq d$.

Orbifold Hurwitz numbers have previously been expressed via intersection theory on moduli spaces of curves in a number of ways, due to Johnson, Pandharipande and Tseng [70], and, separately by Lewański, Popolitov, Shadrin and Zvonkine [80]. A third formula, and the starting point for the work in this chapter, can be obtained by either of the ELSV formulas found in these references by pushforward to $\overline{\mathcal{M}}_{g,n}$, which gives

$$H^{[d]}_{g,n}(\mu_1, \ldots, \mu_n) = d^{2g-2+n+\sum \mu_i/d} \prod_{i=1}^{n} \frac{(\mu_i/d)^{\lfloor \mu_i/d \rfloor}}{\lfloor \mu_i/d \rfloor !} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega^{[d]}_{g;-\overline{\mu}}}{\prod_{i=1}^{n}(1 - \frac{\mu_i}{d}\psi_i)}. \tag{6.13}$$

Here,

$$\Omega^{[d]}_{g;a_1,\ldots,a_n} := \tilde{\epsilon}_* C^{d,d}_{g;a_1,\ldots,a_n} = \epsilon_* \left(\sum_{k=0}^{g} (-1)^k \lambda^U_k \right) \in H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}),$$

where $C^{r,s}_{g;a_1,\ldots,a_n}$ is the *Chiodo class* first defined and explicitly calculated by Chiodo [26].

Given that orbifold Hurwitz numbers are a specialisation of double Hurwitz numbers for specific values for the q-weights, Theorem 6.1.3 implies that orbifold Hurwitz numbers too are governed by topological recursion. In fact, this was previously known in the literature, having been proven by Bouchard, Hernández Serrano, Liu and Mulase [14], and separately by Do, Leigh and Norbury [33]. For orbifold Hurwitz numbers, we define the spectral curve $\mathcal{S}^{[d]}$ by

$$x^{[d]}(z) = \ln z - z^d, \quad y^{[d]}(z) = z^d, \quad \omega_{0,2}^{[d]}(z_1,z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2}.$$

Also define $X^{[d]}(z) = \exp(x^{[d]}(z)) = z\exp(-z^d)$. Denote by $\omega_{g,n}^{[d]}(z_1,\ldots,z_n)$ the correlation differentials produced by topological recursion applied to $\mathcal{S}^{[d]}$. Define the free energies $F_{g,n}^{[d]}(z_1,\ldots,z_n)$ for orbifold Hurwitz numbers

$$F_{g,n}^{[d]}(z_1,\ldots,z_n) = \int_0^{z_n}\cdots\int_0^{z_1}\omega_{g,n}^{[d]}.$$

Theorem 6.4.1 (Bouchard, Hernández Serrano, Liu and Mulase [14] and Do, Leigh and Norbury [33]). *For* $(g,n)\neq(0,2)$*, we have the expansion*

$$F_{g,n}^{[d]}(z_1,\ldots,z_n) = \sum_{\mu_1,\ldots,\mu_n\geqslant 1} H_{g,n}^{[d]}(\mu_1,\ldots,\mu_n)\prod_{i=1}^n X^{[d]}(z_i)^{\mu_i}.$$

We can write $F_{g,n}^{[d]}$ in a basis of natural meromorphic functions. For $a\in\{1,2,\ldots,d\}$ let $\phi_{j,-1}^{[d]}(z) = \frac{z^j}{j}$ and for $m\geqslant -1$ iteratively define

$$\phi_{m+1}^{j,[d]}(z) = \frac{\partial}{\partial x^{[d]}}\,\phi_{j,m}^{[d]}(z) = X^{[d]}\frac{\partial}{\partial X^{[d]}}\,\phi_{j,m}^{[d]}(z) = \frac{z}{1-dz^d}\frac{\mathrm{d}}{\mathrm{d}z}\,\phi_{j,m}^{[d]}(z). \tag{6.14}$$

Expanding in the variable $X^{[d]}(z)$ and using Lagrange inversion to determine the coefficients of $X^{[d]}(z)^\mu$, we can write

$$\phi_{j,-1}^{[d]}(z) = \sum_{k\geqslant 0}\frac{(kd+j)^{k-1}}{k!}X^{[d]}(z)^{kd+j},$$

hence

$$\phi_{j,m}^{[d]}(z) = \left(X^{[d]}\frac{\partial}{\partial X^{[d]}}\right)^{m+1}\phi_{j,-1}^{[d]}(z) = \sum_{k\geqslant 0}\frac{(kd+j)^{k+m}}{k!}X^{[d]}(z)^{kd+j}, \tag{6.15}$$

Now rewrite the ELSV formula for orbifold Hurwitz numbers in terms of the free energies by substituting equation (6.13) into Theorem 6.4.1. That is,

$$\begin{aligned}F_{g,n}^{[d]}(z_1,\ldots,z_n) &= \sum_{\mu_1,\ldots,\mu_n\geqslant 1} H_{g,n}^{[d]}(\mu_1,\ldots,\mu_n)\prod_{i=1}^n X^{[d]}(z_i)^{\mu_i}\\ &= \sum_{\mu_1,\ldots,\mu_n\geqslant 1}\left[d^{2g-2+n+\sum\mu_i/d}\prod_{i=1}^n\frac{(\mu_i/d)^{\lfloor\mu_i/d\rfloor}}{\lfloor\mu_i/d\rfloor!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Omega_{g;-\overline{\mu}}^{[d]}}{\prod_{i=1}^n(1-\frac{\mu_i}{d}\psi_i)}\right]\prod_{i=1}^n X^{[d]}(z_i)^{\mu_i}.\end{aligned}$$

Using equation (6.15) then yields

$$F_{g,n}^{[d]}(z_1,\ldots,z_n) = d^{2g-2+n}\sum_{\substack{1\leqslant j_1,\ldots,j_n\leqslant d\\ m_1,\ldots,m_n\geqslant 0}} d^{\sum(j_i/d-m_i)}\left(\int_{\overline{\mathcal{M}}_{g,n}}\Omega_{g;d-j}^{[d]}\prod_{i=1}^n\psi_i^{m_i}\right)\prod_{i=1}^n\phi_{j_i,m_i}^{[d]}(z_i), \tag{6.16}$$

where $d-j$ is a shorthand notation $d-j = (d-j_1,\ldots,d-j_n)$. Also note that, due to considerations of the degree of relevant moduli spaces, the sum is only over tuples $j_1,\ldots,j_n$ whose sum is a multiple of d, and second, because of cohomological degree considerations the sum is only over $m_1,\ldots,m_n$ whose sum is at most $3g-3+n$. Equation (6.16) will be the starting point for the ELSV-like formula given in Theorem 6.1.4.

6.4.2 Deformation to double Hurwitz numbers

Let $\hat{Q}(z) = q_{d-1}z^{d-1} + \cdots + q_1 z$, and define a 1-parameter family of spectral curves $\mathcal{S}^t$ by

$$x^t(z) = \ln z - (q_d z^d + t\hat{Q}(z)), \quad y^t(z) = \frac{1}{s}\left(q_d z^d + t\hat{Q}(z)\right), \quad \omega_{0,2}^t(z_1, z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2}.$$

Denote by $\omega_{g,n}^t$ the correlation differentials corresponding to applying topological recursion to the above spectral curve $\mathcal{S}^t$ and define the free energies $F_{g,n}^t$ by

$$F_{g,n}^t(z_1, \ldots, z_n) = \int_0^{z_n} \cdots \int_0^{z_1} \omega_{g,n}^t.$$

Substituting $t = 0$ and $q_d = 1$ into $\mathcal{S}^t$ recovers the orbifold spectral curve $\mathcal{S}^{t=0} = \mathcal{S}^{[d]}$, while $t = 1$ gives the double Hurwitz spectral curve $\mathcal{S}^{t=1} = \mathcal{S}$ and Theorem 6.1.3 confirms that $\omega_{g,n} = \omega_{g,n}^{t=1}$ is a generating function for double Hurwitz numbers $DH_{g,n}(\mu_1, \ldots, \mu_n)$.

Let $\tilde{z} = q_d^{1/d} z$, in which case

$$x^0(z) = x^{[d]}(\tilde{z}) - \frac{1}{d}\ln q_d, \quad y^0(z) = s y^{[d]}(\tilde{z}), \quad \omega_{0,2}^0(z_1, z_2) = \omega_{0,2}^{[d]}(\tilde{z}_1, \tilde{z}_2).$$

The homogeneity property of topological recursion under rescaling gives the relation

$$\omega_{g,n}^0(z_1, \ldots, z_n) = s^{2-2g-n}\omega_{g,n}^{[d]}(\tilde{z}_1, \ldots, \tilde{z}_n),$$

hence

$$F_{g,n}^0(z_1, \ldots, z_n) = s^{2-2g-n}F_{g,n}^{[d]}(\tilde{z}_1, \ldots, \tilde{z}_n). \tag{6.17}$$

The spectral curve $\mathcal{S}^t$ is smooth in t in a neighbourhood $\mathcal{U}$ of $[0, 1]$ provided that s is chosen small enough relative to $q_1, \ldots, q_d$ ensuring that the branch points of the deformed curve remain simple. In this case $F_{g,n}^t$ is an analytic function of $t \in \mathcal{U}$ and we compute its value at $t = 1$ via the Taylor series

$$F_{g,n}(z_1, \ldots, z_n) = F_{g,n}^{t=1}(z_1, \ldots, z_n) = \sum_{i\geqslant 0} \frac{1}{i!}\frac{\partial^i}{\partial t^i}F_{g,n}^t(z_1, \ldots, z_n)\Big|_{t=0},$$

where the t-derivatives are computed while keeping $x^t(z_i) = x^1(z_i)$ fixed. Now apply the result of Eynard and Orantin [52, Theorem 5.1] to compute these derivatives with respect to t. First, find $f(w)$ satisfying

$$\Big(\frac{\partial}{\partial t}y^t(z)\Big)\mathrm{d}x^t(z) - \Big(\frac{\partial}{\partial t}x^t(z)\Big)\mathrm{d}y^t(z) = -\operatorname*{Res}_{w=\infty} \omega_{0,2}^t(z, w)f(w).$$

In this case, choosing

$$f(w) = \frac{1}{s}\sum_{i=1}^{d-1}\frac{q_i}{i}w^i$$

satisfies the equation above. The result of Eynard and Orantin then implies

$$\frac{\partial}{\partial t}\omega_{g,n}^t(z_1, \ldots, z_n) = -\operatorname*{Res}_{w=\infty} \omega_{g,n+1}^t(z_1, \ldots, z_n, w)f(w), \tag{6.18}$$

where the derivative is taken for fixed $x_t(z_i)$. Now apply the operator $\partial/\partial t$ iteratively but note that, because w is a function of t, one must apply the product rule to the right side. That is, applying the derivative again yields

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\omega_{g,n}^t(z_1, \ldots, z_n) &= -f(w_1)\operatorname*{Res}_{w_1=\infty}\frac{\partial}{\partial t}\omega_{g,n+1}^t(z_1, \ldots, z_n, w_1) - \operatorname*{Res}_{w_1=\infty}\omega_{g,n+1}^t(z_1, \ldots, z_n, w_1)\frac{\partial}{\partial t}f(w_1)\\ &= \operatorname*{Res}_{w_1=\infty}\omega_{g,n+2}^t(z_1, \ldots, z_n, w_1, w_2)f(w_1)f(w_2) - \operatorname*{Res}_{w_1=\infty}\omega_{g,n+1}^t(z_1, \ldots, z_n, w_1)\frac{\partial}{\partial t}f(w_1).\end{aligned}$$

Applying this process i times and dividing both sides by $i!$ yields the following sum over tuples:

$$\frac{1}{i!}\frac{\partial^i}{\partial t^i}\omega^t_{g,n}(z_1,\dots,z_n) = \sum_{\ell=1}^{i}\frac{(-1)^\ell}{\ell!}\operatorname*{Res}_{w_\ell=\infty}\cdots\operatorname*{Res}_{w_1=\infty}\omega^t_{g,n+\ell}(z_1,\dots,z_n,w_1,\dots,w_\ell)$$
$$\sum_{\substack{p_1,\dots,p_\ell\geqslant 0\\ \sum_{j=1}^{\ell}p_j=i-\ell}}\frac{1}{(p_1+1)!}\frac{\partial^{p_1}}{\partial t^{p_1}}f(w_1)\cdots\frac{1}{(p_\ell+1)!}\frac{\partial^{p_\ell}}{\partial t^{p_\ell}}f(w_\ell). \tag{6.19}$$

Now, take equation (6.19) and compute the antiderivatives with respect to $z_1,\dots,z_n$ on both sides. The left side is

$$\int_0^{z_n}\cdots\int_0^{z_1}\frac{1}{i!}\frac{\partial^i}{\partial t^i}\omega^t_{g,n}(z'_1,\dots,z'_n)\Big|_{t=0} = \frac{1}{i!}\frac{\partial^i}{\partial t^i}F^t_{g,n}(z_1,\dots,z_n)\Big|_{t=0}.$$

Letting

$$f^{(k)}(z) := \left[\frac{\partial^k}{\partial t^k}f(z)\right]_{t=0},$$

the right side gives

$$\int_0^{z_n}\cdots\int_0^{z_1}\sum_{\ell=1}^{i}\frac{(-1)^\ell}{\ell!}\operatorname*{Res}_{w_\ell=\infty}\cdots\operatorname*{Res}_{w_1=\infty}\omega^0_{g,n+\ell}(z'_1,\dots,z'_n,w_1,\dots,w_\ell)\sum_{\substack{p_1,\dots,p_\ell\geqslant 0\\ \sum_{j=1}^{\ell}p_j=i-\ell}}\prod_{j=1}^{\ell}\frac{f^{(p_j)}(w_j)}{(p_j+1)!}$$
$$= \int_0^{w_\ell}\cdots\int_0^{w_1}\int_0^{z_n}\cdots\int_0^{z_1}\sum_{\ell=1}^{i}\frac{1}{\ell!}\operatorname*{Res}_{w_\ell=\infty}\cdots\operatorname*{Res}_{w_1=\infty}\omega^0_{g,n+\ell}(z'_1,\dots,z'_n,w'_1,\dots,w'_\ell)$$
$$\times\sum_{\substack{p_1,\dots,p_\ell\geqslant 0\\ \sum_{j=1}^{\ell}p_j=i-\ell}}\prod_{j=1}^{\ell}\mathrm{d}\Big(\frac{f^{(p_j)}(w_j)}{(p_j+1)!}\Big)$$
$$= \sum_{\ell=1}^{i}\sum_{\substack{p_1,\dots,p_\ell\geqslant 0\\ \sum_{j=1}^{\ell}p_j=i-\ell}}\frac{1}{\ell!}\operatorname*{Res}_{w_\ell=\infty}\cdots\operatorname*{Res}_{w_1=\infty}F^0_{g,n+\ell}(z_1,\dots,z_n,w_1,\dots,w_\ell)\prod_{j=1}^{\ell}\mathrm{d}\Big(\frac{f^{(p_j)}(w_j)}{(p_j+1)!}\Big)$$
$$= \sum_{\ell=1}^{i}\sum_{\substack{p_1,\dots,p_\ell\geqslant 0\\ \sum_{j=1}^{\ell}p_j=i-\ell}}\frac{s^{2-2g-n-\ell}}{\ell!}\operatorname*{Res}_{w_\ell=\infty}\cdots\operatorname*{Res}_{w_1=\infty}F^{[d]}_{g,n+\ell}(\tilde z_1,\dots,\tilde z_n,\tilde w_1,\dots,\tilde w_\ell)\prod_{j=1}^{\ell}\mathrm{d}\Big(\frac{f^{(p_j)}(w_j)}{(p_j+1)!}\Big),$$

where $\tilde z = q_d^{1/d}z$ and we define $\tilde w = q_d^{1/d}w$ similarly. The second equality is using integration by parts, while the fourth equality is applying (6.17). Thus, at this point we have

$$F_{g,n}(z_1,\dots,z_n) = \sum_{i\geqslant 0}\frac{1}{i!}\frac{\partial^i}{\partial t^i}F^t_{g,n}(z_1,\dots,z_n)\Big|_{t=0}$$
$$= \sum_{i\geqslant 0}\sum_{\ell=1}^{i}\sum_{\substack{p_1,\dots,p_\ell\geqslant 0\\ \sum_{j=1}^{\ell}p_j=i-\ell}}\frac{s^{2-2g-n-\ell}}{\ell!}\operatorname*{Res}_{w_\ell=\infty}\cdots\operatorname*{Res}_{w_1=\infty}F^{[d]}_{g,n+\ell}(\tilde z_1,\dots,\tilde z_n,\tilde w_1,\dots,\tilde w_\ell)\prod_{j=1}^{\ell}\mathrm{d}\Big(\frac{f^{(p_j)}(w_j)}{(p_j+1)!}\Big). \tag{6.20}$$

The free energies $F^{[d]}_{g,n+\ell}$ can be written as a linear combination of the basis elements $\phi^{[d]}_{j,m}(\tilde z)$, in which case to calculate the expression above it suffices to compute

$$R^{(p)}_{j,m} := \operatorname*{Res}_{z=\infty}\phi^{[d]}_{j,m}(\tilde z)\,\mathrm{d}(f^{(p)}(z)).$$

We can write $f^{(p)}(z)$ in the form

$$f^{(p)}(z) = \frac{1}{s}\sum_{i=1}^{d-1}\frac{Q^{(p)}_i}{i}z^i,$$

where $Q_j^{(p)} \in \mathbb{C}(q_1, \dots, q_d)$, and hence

$$R_{j,m}^{(p)} = \operatorname*{Res}_{w=0} \phi_{j,m}^{[d]}(q_d^{1/d} w^{-1}) \, \frac{1}{s} \sum_{i=1}^{d-1} Q_i^{(p)} \frac{1}{w^{i-1}} \mathrm{d}(w^{-1}) = -\frac{1}{s} \operatorname*{Res}_{w=0} \sum_{i=1}^{d-1} \frac{Q_i^{(p)}}{w^{i+1}} \phi_{j,m}^{[d]}(q_d^{1/d} w^{-1}) \, \mathrm{d}w$$
$$= -\frac{1}{s} \sum_{i=1}^{d-1} Q_i^{(p)} [z^{-i}] \phi_{j,m}^{[d]}(\tilde{z}),$$

where $[z^{-i}]f(z)$ denotes the coefficient of z^{-i} in the series expansion of $f(z)$ at $z = \infty$. By induction on m, one can show that

$$\phi_{j,m}^{[d]}(\tilde{z}) = \frac{z^j \, p_{j,m}(\tilde{z}^d)}{(1 - dz^d)^{2j+1}}, \tag{6.21}$$

for some polynomial $p_{j,m}$, which has degree m if $j \in \{1, 2, \dots, d-1\}$ or degree $m-1$ if $j = d$. This implies that $R_{j,m}^{(p)} = 0$ for $m \geqslant 1$. Now, calculate $R_{j,0}^{(p)}$ using (6.14) to obtain

$$\begin{aligned} R_{j,0}^{(p)} &= -\frac{1}{s} \sum_{i=1}^{d-1} Q_i^{(p)} [z^{-i}] \phi_{j,0}^{[d]}(\tilde{z}) = -\frac{1}{s} \sum_{i=1}^{d-1} Q_i^{(p)} [z^{-i}] \frac{\tilde{z}^j}{1 - q_d \tilde{z}^d} \\ &= -\frac{1}{s} \sum_{i=1}^{d-1} Q_i^{(p)} [z^{-i}] \frac{q_d^{j/d} z^j}{1 - d q_d z^d} \\ &= \begin{cases} \dfrac{Q_{d-j}^{(p)}}{s d q_d^{(d-j)/d}}, & \text{if } j \in \{1, 2, \dots, d-1\} \\ 0, & \text{if } j = d. \end{cases} \end{aligned}$$

Substituting equation (6.16) into equation (6.20) we find

$$F_{g,n}(z_1, \dots, z_n) = (d/s)^{2g-2+n} \sum_{\ell \geqslant 0} \sum_{p_1, \dots, p_\ell \geqslant 0} \frac{(d/s)^\ell}{\ell!} \sum_{\substack{1 \leqslant j_1, \dots, j_n \leqslant d \\ 1 \leqslant k_1, \dots, k_\ell \leqslant d \\ m_1, \dots, m_{n+\ell} \geqslant 0}} d^{\sum (j_i/d + k_i/d - m_i)}$$
$$\left(\int_{\overline{\mathcal{M}}_{g,n+\ell}} \Omega_{g;d-j,d-k}^{[d]} \prod_{i=1}^{n+\ell} \psi_i^{m_i} \right) \prod_{i=1}^{n} \phi_{j_i,m_i}^{[d]}(\tilde{z}_i) \prod_{i=1}^{\ell} \operatorname*{Res}_{w_i=\infty} \phi_{k_i,m_{n+i}}^{[d]}(\tilde{w}_i) \, \mathrm{d}\Big(\frac{f^{(p_i)}(w_i)}{(p_i+1)!} \Big).$$

Using the expression for $R_{k,m}^{(p)}$ calculated above gives

$$F_{g,n}(z_1, \dots, z_n) = (d/s)^{2g-2+n} \sum_{\ell \geqslant 0} \sum_{p_1, \dots, p_\ell \geqslant 0} \frac{(d/s)^\ell}{\ell!} \sum_{\substack{1 \leqslant j_1, \dots, j_n \leqslant d \\ 1 \leqslant k_1, \dots, k_\ell \leqslant d-1 \\ m_1, \dots, m_n \geqslant 0}} \frac{d^{\sum (j_i/d + k_i/d - m_i)}}{(sd)^{\sum k_i} q_d^{\sum (d-k_i)/d}}$$
$$\left(\int_{\overline{\mathcal{M}}_{g,n+\ell}} \Omega_{g;d-j,d-k}^{[d]} \prod_{i=1}^{n} \psi_i^{m_i} \right) \prod_{i=1}^{n} \phi_{j_i,m_i}^{[d]}(\tilde{z}_i) \prod_{i=1}^{\ell} \frac{Q_{d-k_i}^{(p_i)}}{(p_i+1)!}.$$

Double Hurwitz numbers are stored in the expansion of $F_{g,n}(z_1, \dots, z_n)$ at $X(z_i) = 0$. Using the series expansion for $\phi_{j_i,m_i}^{[d]}(z_i)$ at $X^{[d]}(z_i) = 0$ given in (6.15) and the fact that $X(z_i) = q_d^{-1/d} X^{[d]}(\tilde{z}_i)$ gives

$$DH_{g,n}(\mu_1, \dots, \mu_n) = (d/s)^{2g-2+n} \prod_{i=1}^{n} \frac{(\mu_i/d)^{\lfloor \mu_i/d \rfloor}}{\lfloor \mu_i/d \rfloor!}$$
$$\times \sum_{\substack{\ell \geqslant 0 \\ p_1, \dots, p_\ell \geqslant 0 \\ 1 \leqslant k_1, \dots, k_\ell \leqslant d-1}} \frac{(d/s)^\ell}{\ell!} \frac{d^{\sum (\mu_i/d + k_i/d)} \, q_d^{\sum \mu_i/d - \sum (d-k_i)/d}}{(sd)^{\sum k_i}} \left(\int_{\overline{\mathcal{M}}_{g,n+\ell}} \frac{\Omega_{g;-\overline{\mu},d-k}^{[d]}}{\prod_{i=1}^{n} (1 - \frac{\mu_i}{d} \psi_i)} \right) \prod_{i=1}^{\ell} \frac{Q_{d-k_i}^{(p_i)}}{(p_i+1)!},$$

and this concludes the proof of Theorem 6.1.4.

6.5 Remarks

Theorem 6.1.4 connects double Hurwitz numbers to intersection theory on moduli spaces of curves, however I refer to this as an ELSV-like formula because it is not exactly what one would obtain by using the approach of Eynard [46] or Dunin-Barkowski, Orantin, Shadrin and Spitz [42]. In particular, the double Hurwitz number $DH_{g,n}(\mu_1, \ldots, \mu_n)$ is related to intersection numbers on $\overline{\mathcal{M}}_{g,n+\ell}$ for arbitrary $\ell \geqslant 0$. Further, the polynomiality structure of double Hurwitz numbers is not immediate from the ELSV-like formula.

It is possible to obtain a bona fide ELSV formula using, as stated above, the approach of Eynard [46] or Dunin-Barkowski, Orantin, Shadrin and Spitz [42]. One might then wonder whether this matches what one would obtain by pushing forward the ELSV-like formula of Theorem 6.1.4 to $\overline{\mathcal{M}}_{g,n}$. At present, the behaviour of Chiodo classes under pushforward is not well-enough understood to carry this out.

The idea of using deformation of spectral curves to obtain ELSV-like formulas is largely unexplored but may have applications to other settings. For example, from the viewpoint of considering double Hurwitz numbers with simple branching as relative Gromov–Witten invariants with τ_1-insertions, it is natural to generalise to arbitrary insertions. Utilising a new family of parameters by associating a w-weight to each τ-insertion, this leads to the following conjecture.

Conjecture 6.5.1 (Topological recursion). *Let* $Q(z) = q_1 z + \cdots + q_d z^d$ *and* $W(z) = w_1 z + \cdots + w_k z^k$. *The correlation differentials resulting from applying topological recursion to the spectral curve* $(\mathbb{C}^*, x, y, \omega_{0,2})$ *with*

$$x(z) = \ln z - W(Q(z)), \qquad y(z) = Q(z), \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1\, \mathrm{d}z_2}{(z_1 - z_2)^2}$$

satisfy

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\mu_1, \ldots, \mu_n = 1}^{\infty} GW_{g,n}(\mu_1, \ldots, \mu_n) \prod_{i=1}^{n} \mathrm{d}(\exp(\mu_i x(z_i))).$$

Here, the coefficient $GW_{g,n}(\mu_1, \ldots, \mu_n) \in \mathbb{Q}(q_1, \ldots, q_d; w_1, \ldots, w_k)$ *stores the genus* g *stationary Gromov–Witten invariants of* $\mathbb{CP}^1$ *relative to* 0 *and* ∞ *via*

$$GW_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\substack{\nu \vdash d \\ \alpha \vdash 2g-2+\ell(\mu)+\ell(\nu)}} \langle\, \nu \,|\, \tau_{\alpha_1} \cdots \tau_{\alpha_{\ell(\alpha)}} \,|\, \mu \,\rangle_g \, \frac{\vec{q}_\nu\, \vec{w}_\alpha}{|\mathrm{Aut}\ \nu|\, |\mathrm{Aut}\ \alpha|}.$$

One ought to be able to push through the techniques of this chapter to prove this conjecture. Further, one could implement the approach of spectral curve variation to obtain an ELSV-like formula for these Gromov–Witten invariants by deforming the known spectral curve of Dunin-Barkowski, Kramer, Popolitov and Shadrin [39] for orbifold spin Hurwitz numbers.

6.6 Data

The following data was calculated in Maple [82] using the semi-infinite wedge vacuum expectation for double Hurwitz numbers given in Proposition 6.3.3.

g	$(\mu_1, \ldots, \mu_n)$	$DH_{g,n}(\mu_1, \ldots, \mu_n)$ evaluated at $s = 1$
0	(1)	q_1
0	(2)	$\frac{1}{2}q_1^2 + \frac{1}{2}q_2$
0	(3)	$\frac{1}{2}q_1^3 + q_1 q_2 + \frac{1}{3}q_3$
0	(4)	$\frac{2}{3}q_1^4 + 2q_1^2 q_2 + \frac{1}{2}q_2^2 + q_1 q_3 + \frac{1}{4}q_4$
0	(5)	$\frac{25}{24}q_1^5 + \frac{25}{6}q_1^3 q_2 + \frac{5}{2}q_1 q_2^2 + \frac{5}{2}q_1^2 q_3 + q_2 q_3 + q_1 q_4 + \frac{1}{5}q_5$
0	(6)	$\frac{9}{5}q_1^6 + 9q_1^4 q_2 + 9q_1^2 q_2^2 + q_2^3 + 6q_1^3 q_3 + 6q_1 q_2 q_3 + \frac{1}{2}q_3^2 + 3q_1^2 q_4 + q_2 q_4 + q_1 q_5 + \frac{1}{6}q_6$

g	$(\mu_1, \ldots, \mu_n)$	$DH_{g,n}(\mu_1, \ldots, \mu_n)$ evaluated at $s = 1$
0	$(1,1)$	$\frac{1}{2}q_1^2 + q_2$
0	$(2,1)$	$\frac{2}{3}q_1^3 + 2q_1q_2 + q_3$
0	$(3,1)$	$\frac{9}{8}q_1^4 + \frac{9}{2}q_1^2q_2 + \frac{3}{2}q_2^2 + 3q_1q_3 + q_4$
0	$(2,2)$	$q_1^4 + 4q_1^2q_2 + q_2^2 + 3q_1q_3 + q_4$
0	$(4,1)$	$\frac{32}{15}q_1^5 + \frac{32}{3}q_1^3q_2 + 8q_1q_2^2 + 8q_1^2q_3 + 4q_2q_3 + 4q_1q_4 + q_5$
0	$(3,2)$	$\frac{9}{5}q_1^5 + 9q_1^3q_2 + 6q_1q_2^2 + \frac{15}{2}q_1^2q_3 + 3q_2q_3 + 4q_1q_4 + q_5$
0	$(1,1,1)$	$q_1^3 + 4q_1q_2 + 3q_3$
0	$(2,1,1)$	$2q_1^4 + 10q_1^2q_2 + 4q_2^2 + 9q_1q_3 + 4q_4$
0	$(3,1,1)$	$\frac{9}{2}q_1^5 + 27q_1^3q_2 + 24q_1q_2^2 + \frac{51}{2}q_1^2q_3 + 15q_2q_3 + 16q_1q_4 + 5q_5$
0	$(2,2,1)$	$4q_1^5 + 24q_1^3q_2 + 20q_1q_2^2 + 24q_1^2q_3 + 12q_2q_3 + 16q_1q_4 + 5q_5$
1	(2)	$\frac{1}{12}q_1^2 + \frac{1}{4}q_2$
1	(3)	$\frac{3}{8}q_1^3 + \frac{3}{2}q_1q_2 + q_3$
1	(4)	$\frac{4}{3}q_1^4 + \frac{20}{3}q_1^2q_2 + \frac{7}{3}q_2^2 + 6q_1q_3 + \frac{5}{2}q_4$
1	(5)	$\frac{625}{144}q_1^5 + \frac{625}{24}q_1^3q_2 + \frac{125}{6}q_1q_2^2 + \frac{625}{24}q_1^2q_3 + \frac{25}{2}q_2q_3 + \frac{50}{3}q_1q_4 + 5q_5$
1	$(1,1)$	$\frac{1}{24}q_1^2 + \frac{1}{6}q_2$
1	$(2,1)$	$\frac{1}{3}q_1^3 + \frac{5}{3}q_1q_2 + \frac{3}{2}q_3$
1	$(3,1)$	$\frac{27}{16}q_1^4 + \frac{81}{8}q_1^2q_2 + \frac{9}{2}q_2^2 + \frac{45}{4}q_1q_3 + 6q_4$
1	$(2,2)$	$\frac{4}{2}q_1^4 + 8q_1^2q_2 + \frac{10}{3}q_2^2 + 9q_1q_3 + \frac{14}{3}q_4$
1	$(4,1)$	$\frac{64}{9}q_1^5 + \frac{448}{9}q_1^3q_2 + 48q_1q_2^2 + \frac{176}{3}q_1^2q_3 + \frac{104}{3}q_2q_3 + \frac{136}{3}q_1q_4 + \frac{50}{3}q_5$
1	$(3,2)$	$\frac{21}{4}q_1^5 + \frac{147}{4}q_1^3q_2 + 34q_1q_2^2 + \frac{355}{8}q_1^2q_3 + 24q_2q_3 + \frac{104}{3}q_1q_4 + \frac{25}{2}q_5$
2	(2)	$\frac{1}{240}q_1^2 + \frac{1}{48}q_2$
2	(3)	$\frac{9}{80}q_1^3 + \frac{27}{40}q_1q_2 + \frac{3}{4}q_3$
2	(4)	$\frac{52}{45}q_1^4 + \frac{364}{45}q_1^2q_2 + \frac{61}{15}q_2^2 + \frac{54}{5}q_1q_3 + \frac{41}{6}q_4$
2	(5)	$\frac{3125}{384}q_1^5 + \frac{3125}{48}q_1^3q_2 + \frac{625}{9}q_1q_2^2 + \frac{4375}{48}q_1^2q_3 + \frac{1375}{24}q_2q_3 + \frac{250}{3}q_1q_4 + \frac{425}{12}q_5$
2	$(1,1)$	$\frac{1}{720}q_1^2 + \frac{1}{120}q_2$
2	$(2,1)$	$\frac{13}{180}q_1^3 + \frac{91}{180}q_1q_2 + \frac{27}{40}q_3$
2	$(3,1)$	$\frac{729}{640}q_1^4 + \frac{729}{80}q_1^2q_2 + \frac{27}{5}q_2^2 + \frac{567}{40}q_1q_3 + \frac{54}{5}q_4$
2	$(2,2)$	$\frac{13}{15}q_1^4 + \frac{104}{15}q_1^2q_2 + \frac{182}{45}q_2^2 + \frac{54}{5}q_1q_3 + \frac{122}{15}q_4$
2	$(4,1)$	$\frac{1472}{135}q_1^5 + \frac{1472}{15}q_1^3q_2 + \frac{1808}{15}q_1q_2^2 + \frac{7024}{45}q_1^2q_3 + \frac{1736}{15}q_2q_3 + \frac{7448}{45}q_1q_4 + \frac{250}{3}q_5$
2	$(3,2)$	$\frac{303}{40}q_1^5 + \frac{2727}{40}q_1^3q_2 + \frac{412}{5}q_1q_2^2 + \frac{1747}{16}q_1^2q_3 + \frac{1561}{20}q_2q_3 + \frac{1736}{15}q_1q_4 + \frac{1375}{24}q_5$

Chapter 7

Virasoro constraints for fully simple maps

7.1 Introduction

Maps, which informally correspond to ways to glue polygons together to create surfaces, are a cornerstone in the field of topological recursion. The enumeration of maps is the prototypical example of a problem that is governed by topological recursion. Indeed, the theory of topological recursion evolved from the abstraction of loop equations from the theory of matrix models, with one of the most fundamental examples given by the 1-Hermitian matrix model. Specifically, the following theorem was developed in a series of works, most notably by Chekhov, Eynard and Orantin [25, 48, 52].

Theorem 7.1.1 (Chekhov, Eynard and Orantin [25, 48, 52]). *The correlation differentials resulting from applying topological recursion to the spectral curve* $(\mathbb{CP}^1, x, y)$ *with*

$$x(z) = \alpha + \gamma\Big(z + \frac{1}{z}\Big), \qquad \textit{and} \qquad y(z) = -\sum_{j=1}^{d-1} u_j z^{-j} \tag{7.1}$$

satisfy

$$\omega_{g,n}(z_1, \ldots, z_n) = \mathrm{d}_1 \cdots \mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n \geqslant 1} \mathrm{M}_{g,n}(\mu_1, \ldots, \mu_n)\, x_1^{-\mu_1} \cdots x_n^{-\mu_n}.$$

Here, u_j *for* $j \in \{0, \ldots, d-1\}$ *are polynomials in* α *and* γ *that satisfy*

$$V'(x(z)) = \sum_{j=0}^{d-1} u_j (z^j - z^{-j})$$

for $V'(x) = x - \sum_{i\geqslant 1} t_i x^{i-1}$. *The parameters* $\alpha, \gamma \in \mathbb{Q}[t_1, \ldots, t_d][[t]]$ *are the unique solutions of* $u_0 = 0$ *and* $u_1 = \frac{t}{\gamma}$ *that behave like* $\alpha = O(t)$ *and* $\gamma = \sqrt{t}(1 + O(t))$.

For a formal definition of maps see Definition 3.2.1 in Chapter 3, and see Definition 3.2.10 in the same chapter for the full definition of the enumeration $\mathrm{M}_{g,n}(\mu_1, \ldots, \mu_n)$.

In the statement of Theorem 7.1.1 above, the parameter t is typically used in the generating function of the enumeration to track the number of vertices of a map, while the positive integer d is fixed at the outset; d is used to provide an upper bound on the possible degree of an internal face. Topological recursion inherently requires analyticity of the generating functions involved and hence the role of d is partly to allow objects that could otherwise only be considered as formal power series to be viewed as analytic. In this chapter I do not need to track the number of edges, and separately, I do not utilise any analytic arguments, hence the parameter t and the integer d are not required.

Relatively recently, the closely related enumeration of fully simple maps was defined by Borot and Garcia-Failde [11]. As discussed in the background Chapter 3 on maps, the enumeration of fully simple maps is a subset of the enumeration of ordinary maps; that is, a fully simple map is an ordinary map that satisfies extra conditions. The definition of fully simple maps arose from the study of random matrix models and a discussion of this viewpoint can be found in previous work of Borot and Garcia-Failde [11].

One of the remarkable features of the enumeration of fully simple maps is the conjecture posited by Borot and Garcia-Failde [11] that the enumeration of fully simple maps is governed by topological recursion but where the spectral curve for the fully simple enumeration is obtained by taking the rational spectral curve for ordinary maps and switching the meromorphic functions x and y that comprise the spectral curve data. This transformation of switching x and y is an example of a symplectic transformation, which is any transformation of x and y that preserves $|\mathrm{d}x \wedge \mathrm{d}y|$. In the setting of topological recursion, symplectic transformations are related to the somewhat mysterious property of symplectic invariance. Symplectic invariance, first proposed by Eynard and Orantin [50, 53] and still an open question today, states that the free energies $F_{g,0}$ are invariant under symplectic transformations of the spectral curve. For a definition of the quantities $F_{g,0}$ and a discussion of symplectic transformations and symplectic invariance, see previous work of Eynard and Orantin [50, 53].

The conjecture of Borot and Garcia-Failde [11] has since been proved simultaneously by Borot, Charbonnier and Garcia-Failde [7] and Bychkov, Dunin-Barkowski, Kazarian and Shadrin [19]. The statement of the theorem is the following.

Theorem 7.1.2 (Borot, Charbonnier and Garcia-Failde [7] and Bychkov, Dunin-Barkowski, Kazarian and Shadrin [19]). *The correlation differentials resulting from applying the topological recursion to the spectral curve* $(\mathbb{CP}^1, x, y)$ *with*

$$x(z) = -\sum_{j=1}^{d-1} u_j z^{-j}, \qquad \text{and} \qquad y(z) = \alpha + \gamma\Big(z + \frac{1}{z}\Big) \tag{7.2}$$

satisfy

$$\omega_{g,n}(z_1, \ldots, z_n) = \mathrm{d}_1 \cdots \mathrm{d}_n \sum_{\mu_1, \ldots, \mu_n \geqslant 1} \mathrm{FS}_{g,n}(\mu_1, \ldots, \mu_n)\, x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

Here, u_j *for* $j \in \{0, \ldots, d-1\}$ *are polynomials in* α *and* γ *that satisfy*

$$V'(y(z)) = \sum_{j=0}^{d-1} u_j (z^j - z^{-j})$$

for $V'(y) = y - \sum_{i \geqslant 1} t_i y^{i-1}$. *The parameters* $\alpha, \gamma \in \mathbb{Q}[t_1, \ldots, t_d][[t]]$ *are the unique solutions of* $u_0 = 0$ *and* $u_1 = \frac{t}{\gamma}$ *that behave like* $\alpha = O(t)$ *and* $\gamma = \sqrt{t}(1 + O(t))$.

Again, for a formal definition of fully simple maps see Definition 3.2.1 in Chapter 3, and see Definition 3.2.10 in the same chapter for the full definition of the enumeration $\mathrm{FS}_{g,n}(\mu_1, \ldots, \mu_n)$.

Beyond topological recursion, a relation between the enumerations of ordinary and fully simple maps via monotone Hurwitz numbers was proven by Borot, Charbonnier, Do, and Garcia-Failde [6, 11]; see Theorem 7.2.3 in Section 7.2.1. One can give an equivalent expression for monotone Hurwitz numbers as a character formula, and hence as a vacuum expectation in the semi-infinite wedge; see Chapter 4 for a demonstration of this process in the case of single Hurwitz numbers. Separately, the semi-infinite wedge offers a particularly nice setting for deriving differential operators that act on, and annihilate, partition functions.

The aim of this chapter is to deduce a sequence of Virasoro operators V_n^F for $n \geqslant -1$ that annihilate the partition function for fully simple maps Z^F. Use of the semi-infinite wedge is instrumental for deducing this result. The steps taken to obtain this result are as follows.

1. Begin by deriving a vacuum expectation for the partition function of ribbon graphs (Lemma 7.3.2), then use a combinatorial argument to pass to a vacuum expectation for the partition function for ordinary maps (Proposition 7.3.3).
2. Use the relation between ordinary maps and fully simple maps, Theorem 7.2.3, along with the known character formula for weakly monotone Hurwitz numbers (7.4) to deduce a vacuum expectation for the partition function for fully simple maps (Theorem 7.3.4).
3. Use the known Virasoro constraints for ordinary maps to derive the Virasoro constraints for fully simple maps as a conjugation of the Virasoro operators for ordinary maps (Theorem 7.4.8).

The structure of this chapter is as follows. Section 7.2 contains useful preliminary definitions and results regarding ordinary maps and fully simple maps (Section 7.2.1) as well as key results pertaining to operators in the semi-infinite wedge (Section 7.2.2). Section 7.3 develops vacuum expectations for the partition functions of ordinary maps (Section 7.3.1) and fully simple maps (Section 7.3.2). Section 7.4 derives the Virasoro constraints of fully simple maps: Section 7.4.1 reproduces the known derivation of the Virasoro operators for ordinary maps from the Tutte recursion; Section 7.4.2 outlines a number of useful preliminary conjugation results that are needed in the following section; and Section 7.4.3 contains the main result of this chapter, the derivation of the Virasoro constraints for the enumeration of fully simple maps.

Section 7.5 describes some work in progress towards further desirable results. Namely, Section 7.5.1 outlines work towards a Tutte-like recursion for fully simple maps; Section 7.5.2 provides a relation between ordinary and fully simple maps in the case of $(g, n) = (0, 1)$; and Section 7.5.3 details an enlightening calculation that recovers the spectral curve for fully simple maps from the $(0, 1)$-Tutte-like recursion.

All work in this chapter is joint work with Norman Do.

7.2 Preliminaries

7.2.1 Maps and fully simple maps

Begin by defining particular generating functions for ordinary and fully simple maps that will be useful throughout this chapter.

Definition 7.2.1. Let $\mu_1, \ldots, \mu_n$ be positive integers, and define $\mathrm{Map}(\mu_1, \ldots, \mu_n)$ to be the weighted enumeration of (isomorphism classes of possibly disconnected) ordinary maps with n boundary faces such that the degree of boundary face i is μ_i. The weight of each map M is given by

$$\frac{\hbar^{2g-2+n}\, s^{e(M)}}{|\mathrm{Aut}\ M|} t_1^{f_1(M)} t_2^{f_2(M)} t_3^{f_3(M)} \cdots ,$$

where g is the genus of the underlying surface, $e(M)$ denotes the number of edges, $f_i(M)$ is the number of internal faces of degree i, and $|\mathrm{Aut}\ M|$ is the number of automorphisms of M.

Define $\mathrm{FSMap}(\mu_1, \ldots, \mu_n)$ to be the analogous enumeration of fully simple maps.

Note that this definition packages possibly disconnected ordinary maps and fully simple maps, and indeed throughout this chapter ordinary maps and fully simple maps will be possibly disconnected unless explicitly stated otherwise.

The generating function $\mathrm{Map}(\mu_1, \ldots, \mu_n)$ is equal to the sum of the enumerations $\mathrm{M}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$ ranging over all genera; that is,

$$\mathrm{Map}(\mu_1, \ldots, \mu_n) = \sum_{g \in \mathbb{Z}} \mathrm{M}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n) \hbar^{2g-2+n},$$

and similarly for FSMap and $\mathrm{FS}^{\bullet}_{g,n}$. Note that the sum over genus above ranges over all integers, positive and negative. This is because Map enumerates disconnected objects and the genus of a disconnected surface may be negative. This particular generating function $\mathrm{Map}(\mu_1, \ldots, \mu_n)$ is useful when packaging the ordinary map enumerative data in the semi-infinite wedge. That is, the generating function $\mathrm{Map}(\mu_1, \ldots, \mu_n)$ can be conveniently written as a vacuum expectation in the semi-infinite wedge. The analogous generating functions were also employed in Chapters 4 and 6 to write single and double Hurwitz numbers via the semi-infinite wedge.

Next, recall the relation between the enumerations of ordinary maps and fully simple maps via monotone Hurwitz numbers, first proven by Borot and Garcia-Failde [11], and separately also proven by Borot, Charbonnier, Do and Garcia-Failde [6]. (Note that although monotone Hurwitz numbers are already defined in Definition 4.3.1 (Section 4.3.1), the way that they are packaged here is sufficiently different that it will be most convenient to simply include the necessary definition here.) A sequence of transpositions $\tau_1, \ldots, \tau_k$

in the symmetric group S_d is called *strictly monotone* if, when writing the transposition $\tau_i = (a_i\, b_i)$ conventionally with $a_i < b_i$, then $b_1 < \cdots < b_n$. Similarly, a sequence of transpositions $\tau_1, \ldots, \tau_k$ is called *weakly monotone* if $b_1 \leqslant \cdots \leqslant b_n$.

Definition 7.2.2. Let λ and μ be partitions of a positive integer d, and let k be a non-negative integer. The *strictly monotone Hurwitz number* $H_k^<(\lambda; \mu)$ is $\frac{1}{d!}$ multiplied by the number of tuples $(\rho_\lambda, \tau_1, \ldots, \tau_k, \rho_\mu)$ of permutations in the symmetric group S_d such that

- ρ_λ and ρ_μ have cycle type λ and μ respectively;
- $\tau_1, \ldots, \tau_k$ is a strictly monotone sequence of transpositions; and
- $\rho_\lambda \tau_1 \cdots \tau_k, \rho_\mu = \text{id}$.

The *weakly monotone Hurwitz number* $H_k^\leqslant(\lambda; \mu)$ is defined analogously where $\tau_1, \ldots, \tau_k$ is a weakly monotone sequence of transpositions.

Also define the following generating functions for these monotone Hurwitz numbers

$$H^<(\lambda; \mu) = \sum_{k \geqslant 0} H_k^<(\lambda; \mu)\, \hbar^k \qquad \text{and} \qquad H^\leqslant(\lambda; \mu) = \sum_{k \geqslant 0} H_k^\leqslant(\lambda; \mu)\, \hbar^k.$$

I also record the following character formulas for strictly and weakly monotone Hurwitz numbers [6], which will be needed to write fully simple maps as a vacuum expectation in the semi-infinite wedge:

$$H^<(\lambda; \mu) = \sum_{\rho \vdash d} \frac{\chi_\lambda^\rho \chi_\mu^\rho}{z(\lambda) z(\mu)} \prod_{\square \in \rho} (1 + c(\square)\hbar) \tag{7.3}$$

$$H^\leqslant(\lambda; \mu) = \sum_{\rho \vdash d} \frac{\chi_\lambda^\rho \chi_\mu^\rho}{z(\lambda) z(\mu)} \prod_{\square \in \rho} \frac{1}{1 - c(\square)\hbar}. \tag{7.4}$$

Here, the notation χ_λ^ρ refers to the character of the symmetric group indexed by the partition ρ evaluated on a permutation of cycle type λ, the notation $\rho \vdash d$ denotes that ρ is a partition of the integer d, and the notation $z(\lambda)$ for a partition λ denotes

$$z(\lambda) = \prod_{i=1}^{\ell(\lambda)} \lambda_i \prod_{j \geqslant 1} m_j(\lambda)!, \tag{7.5}$$

where $m_j(\lambda)$ is the number of occurrences of the positive integer j in the partition λ.

Theorem 7.2.3 (Borot, Charbonnier, Do and Garcia-Failde [6] and Borot and Garcia-Failde [11]). *For any partitions λ and μ,*

$$\lambda_1 \cdots \lambda_\ell \, \text{Map}(\lambda_1, \ldots, \lambda_\ell) = z(\lambda) \sum_{\mu \vdash |\lambda|} H^<(\lambda; \mu)\, \mu_1 \cdots \mu_n \, \text{FSMap}(\mu_1, \ldots, \mu_n) \tag{7.6}$$

$$\mu_1 \cdots \mu_n \, \text{FSMap}(\mu_1, \ldots, \mu_n) = z(\mu) \sum_{\lambda \vdash |\mu|} H^\leqslant(\mu; \lambda)\big|_{\hbar \mapsto -\hbar} \, \lambda_1 \cdots \lambda_\ell \, \text{Map}(\lambda_1, \ldots, \lambda_\ell). \tag{7.7}$$

Here $|\mu| = \mu_1 + \cdots + \mu_n$. Note that this theorem as stated is superficially different to its presentation in the work of Borot, Charbonnier, Do and Garcia-Failde [6]; this is because the generating functions I have defined, Map and FSMap, enumerate unrooted disconnected ordinary maps and fully simple maps, while the objects of study in this reference are their rooted counterparts. Defining $\hat{\text{Map}}$ and $\hat{\text{FSMap}}$ to be the corresponding generating functions for rooted ordinary and fully simple maps respectively, one can easily switch between these enumerations using the relation

$$\hat{\text{Map}}(\mu_1, \ldots, \mu_n) = \mu_1 \cdots \mu_n \text{Map}(\mu_1, \ldots, \mu_n),$$

and similarly for FSMap. As described in Section 3.2.4, this is because, to pass from the unrooted enumeration to the rooted enumeration one needs to choose a root (an edge on the boundary face) for each boundary face and for the boundary face labelled i there are μ_i choices for the root.

Further, the corresponding generating functions defined by Borot, Charbonnier, Do and Garcia-Failde [6] do not include the s-parameter that I am utilising here (that tracks the number of edges). However, following either of the combinatorial bijections described by the authors of this paper one observes that from an ordinary map one obtains a fully simple map with the same number of edges, hence the above formula has been (trivially) amended to include the s-parameter.

7.2.2 Semi-infinite wedge

Lemma 7.2.4. *Let m be a positive integer. Then, the following is true:*

$$\alpha_m \exp\left(\sum_{m\geqslant 1} \frac{p_m}{m}\alpha_{-m}\right)\Big\rangle = p_m \exp\left(\sum_{m=1}^{\infty} \frac{p_m}{m}\alpha_{-m}\right)\Big\rangle$$
$$\alpha_{-m} \exp\left(\sum_{m=1}^{\infty} \frac{p_m}{m}\alpha_{-m}\right)\Big\rangle = m\frac{\partial}{\partial p_m} \exp\left(\sum_{m=1}^{\infty} \frac{p_m}{m}\alpha_{-m}\right)\Big\rangle.$$

Proof. The second equation follows immediately by applying the differential operator $m\frac{\partial}{\partial p_m}$ to the exponential.

To prove the first equation, use the following application of the Baker–Campbell–Hausdorff formula. If $[X, Y]$ is central, then

$$e^{sX} Y e^{-sX} = Y + s[X, Y].$$

Take $X = \frac{p_k}{k}\alpha_{-k}$ and $Y = \alpha_k$. As $[\alpha_{-k}, \alpha_k] = -k$ is indeed central, applying the formula yields

$$\begin{aligned}\exp\Big(\frac{p_k}{k}\alpha_{-k}\Big)\alpha_k \exp\Big(-\frac{p_k}{k}\alpha_{-k}\Big) &= \alpha_k + \frac{p_k}{k}[\alpha_{-k}, \alpha_k] \\ &= \alpha_k - p_k \\ \exp\Big(\frac{p_k}{k}\alpha_{-k}\Big)\alpha_k &= \alpha_k \exp\Big(\frac{p_k}{k}\alpha_{-k}\Big) - p_k \exp\Big(\frac{p_k}{k}\alpha_{-k}\Big).\end{aligned}$$

Apply both sides to the vacuum vector. The left side vanishes and I obtain

$$\alpha_k \exp\Big(\frac{p_k}{k}\alpha_{-k}\Big)\Big\rangle = p_k \exp\Big(\frac{p_k}{k}\alpha_{-k}\Big)\Big\rangle.$$

Apply the operator

$$\exp\left(\sum_{\substack{m=1\\ m\neq k}}^{\infty} \frac{p_m}{m}\alpha_{-m}\right)$$

to both sides to obtain the desired result. ■

Finally, I also define a number of operators that will be needed throughout this chapter. First, define the following operators on the semi-infinite wedge.

Definition 7.2.5. For all $n \in \mathbb{Z}$, define the two sequences of operators

$$\mathcal{M}_n = \frac{1}{6}\sum_{i+j+k=n} :\alpha_i\alpha_j\alpha_k:, \tag{7.8}$$

and

$$\mathcal{K}_n = \frac{1}{2}\sum_{i+j=n} :\alpha_i\alpha_j:, \tag{7.9}$$

where I adopt the convention that $\alpha_0 := 0$. As defined in Definition 1.3.3 the colons surrounding these sets of operators denotes that we use the normal ordering of the product.

Next recall the definition of the $\mathcal{E}$-operator from Definition 1.3.6 in Chapter 1,

$$\mathcal{E}_n(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z\left(k-\frac{n}{2}\right)} : \psi_{k-n}\psi_k^* : + \frac{\delta_{n,0}}{\varsigma(z)},$$

where $\varsigma(z) = e^{z/2} - e^{-z/2}$. Also recall the fact that $\mathcal{E}_n(0) = \alpha_n$. By Lemma 1.4.8 the $\mathcal{E}$-operator can alternatively be written in terms of the bosonic operators as

$$\mathcal{E}_n(z) = \frac{1}{\varsigma(z)} \sum_{\ell\geqslant 0} \frac{1}{\ell!} \sum_{i_1+\cdots+i_\ell=n} \frac{\varsigma(i_1 z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} : \alpha_{i_1}\cdots\alpha_{i_\ell} : .$$

These $\mathcal{E}$-operators satisfy the commutation relation, equation (1.9):

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma(aw - bz)\, \mathcal{E}_{a+b}(z+w).$$

To prove the main result of this chapter, it will be useful to observe that the operator $\mathcal{K}_n$ is related to the z-coefficient of the $\mathcal{E}_n(z)$ operator. That is, using (1.20) for the $\mathcal{E}$-operator, I have that

$$\begin{aligned}
[z]\mathcal{E}_n(z) &= [z]\frac{1}{\varsigma(z)} \sum_{\ell\geqslant 0} \frac{1}{\ell!} \sum_{i_1+\cdots+i_\ell=n} \frac{\varsigma(i_1 z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} : \alpha_{i_1}\cdots\alpha_{i_\ell} : \\
&= \left([z^{-1}]\frac{1}{\varsigma(z)}\right)[z^2] \sum_{\ell\geqslant 0} \frac{1}{\ell!} \sum_{i_1+\cdots+i_\ell=n} \frac{\varsigma(i_1 z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} : \alpha_{i_1}\cdots\alpha_{i_\ell} : \\
&\qquad + \left([z^1]\frac{1}{\varsigma(z)}\right)[z^0] \sum_{\ell\geqslant 0} \frac{1}{\ell!} \sum_{i_1+\cdots+i_\ell=n} \frac{\varsigma(i_1 z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} : \alpha_{i_1}\cdots\alpha_{i_\ell} : \\
&= \frac{1}{2} \sum_{i_1+i_2=n} : \alpha_{i_1}\alpha_{i_2} : - \frac{\delta_{n,0}}{24} = \mathcal{K}_n - \frac{\delta_{n,0}}{24}.
\end{aligned}$$

Here I'm using the following expansions:

$$\begin{aligned}
\frac{1}{\varsigma(z)} &= \frac{1}{z} - \frac{z}{24} + \frac{7z^3}{5760} + O(z^5) \\
\varsigma(z) &= z + \frac{z^3}{24} + \frac{z^5}{1920} + O(z^7).
\end{aligned}$$

Hence,

$$\mathcal{K}_n = [z]\,\mathcal{E}_n(z) + \frac{\delta_{n,0}}{24}. \tag{7.10}$$

The following commutation relations, while not strictly necessary for the results in this chapter, are proven here for the sake of interest.

Lemma 7.2.6. *For all non-zero integers m and all integers n, we have the following commutation relations:*

$$[\mathcal{M}_n, \alpha_m] = -m\,\mathcal{K}_{n+m}, \tag{7.11}$$

and

$$[\mathcal{K}_n, \alpha_m] = \begin{cases} -m\,\alpha_{n+m}, & \textit{if } n+m=0, \\ 0, & \textit{otherwise.} \end{cases} \tag{7.12}$$

Proof. Begin with equation (7.12). Use equation (7.10) and the fact that $\alpha_m = \mathcal{E}_m(0)$: then we can utilise the commutation relation for the $\mathcal{E}$-operator (1.3.7). This gives

$$\begin{aligned}
[\mathcal{K}_n,\alpha_m] &= [z][\mathcal{E}_n(z),\mathcal{E}_m(0)] + [z]\Big[\frac{\delta_{n,0}}{24},\mathcal{E}_m(0)\Big] = [z]\varsigma(-mz)\mathcal{E}_{n+m}(z)\\
&= [z]\varsigma(-mz)\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{i_1+\cdots+i_\ell=n+m}\frac{\varsigma(i_1z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} :\alpha_{i_1}\cdots\alpha_{i_\ell}:\\
&= \Big([z^0]\varsigma(-mz)\frac{1}{\varsigma(z)}\Big)[z^1]\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{i_1+\cdots+i_\ell=n+m}\frac{\varsigma(i_1z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} :\alpha_{i_1}\cdots\alpha_{i_\ell}:\\
&= \begin{cases} -m\,\alpha_{n+m}, & \text{if } n+m\neq 0,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

The third equality is using the fact that $\varsigma(-mz)\frac{1}{\varsigma(z)}$ is even, while the fourth equality is using that, when $m+n=0$,

$$[z^1]\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{i_1+\cdots+i_\ell=0}\frac{\varsigma(i_1z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell} = 0.$$

This is because, given that $i_1+\cdots+i_\ell = 0$ and $i_1,\ldots,i_\ell$ are non-zero, then ℓ must be even and thus $\varsigma(i_1z)\cdots\varsigma(i_\ell z)$ must too be even in z. Hence conclude that $[z^1]\varsigma(i_1z)\cdots\varsigma(i_\ell z) = 0$.

Equation (7.11) uses the fact that $\mathcal{M}_n = [z^2]\mathcal{E}_n(z) - \frac{n^2-1}{24}\alpha_n$ and a similar argument to the one above. ■

Finally, define the following operator that will be central to the work in this chapter.

Definition 7.2.7. Define the operator

$$\begin{aligned}
\mathcal{G} &= \hbar\mathcal{M}_2 + \mathcal{K}_2 + \frac{1}{2\hbar}\alpha_2\\
&= \frac{\hbar}{2}\sum_{i,j\geqslant 1}(\alpha_{2-i-j}\alpha_i\alpha_j + \alpha_{-i}\alpha_{-j}\alpha_{i+j+2}) + \sum_{i\geqslant 1}\alpha_{-i}\alpha_{2+i} + \frac{1}{2}\alpha_1^2 + \frac{1}{2\hbar}\alpha_2.
\end{aligned} \tag{7.13}$$

A side bar on the $\mathcal{G}$-operator. One might find it interesting to observe that the operator $\mathcal{G}$ can be written succinctly as

$$\mathcal{G} = \frac{\hbar}{6}\sum_{i+j+k=2} :\alpha_i\alpha_j\alpha_k:,$$

by defining the convention that $\alpha_0 = \frac{1}{\hbar}$. This might lead one to the definition of the operator

$$\mathcal{G}_n = \frac{\hbar}{6}\sum_{i+j+k=n} :\alpha_i\alpha_j\alpha_k: = \hbar\mathcal{M}_n + \mathcal{K}_n + \frac{1}{2\hbar}\alpha_n,$$

still using the same convention $\alpha_0 = \frac{1}{\hbar}$. It is natural to wonder if there is there a context in which these operators are useful.

I'm also going to define the following differential operators. First, define

$$M_2 = \frac{\hbar}{2}\sum_{i,j\geqslant 1}\Big((i+j-2)p_ip_j\frac{\partial}{\partial p_{i+j-2}} + ijp_{i+j+2}\frac{\partial^2}{\partial p_i\partial p_j}\Big), \tag{7.14}$$

and

$$K_2 = \sum_{i\geqslant 1} ip_{i+2}\frac{\partial}{\partial p_i} + \frac{1}{2}p_1^2. \tag{7.15}$$

And finally, define the operator

$$\begin{aligned}
G &= \hbar M_2 + K_2 + \frac{1}{2\hbar}p_2\\
&= \frac{\hbar}{2}\sum_{i,j\geqslant 1}\Big((i+j-2)p_ip_j\frac{\partial}{\partial p_{i+j-2}} + ijp_{i+j+2}\frac{\partial^2}{\partial p_i\partial p_j}\Big) + \sum_{i\geqslant 1} ip_{i+2}\frac{\partial}{\partial p_i} + \frac{1}{2}p_1^2 + \frac{1}{2\hbar}p_2.
\end{aligned} \tag{7.16}$$

These three operators have been very suggestively labelled: they are the indeed each related to their calligraphic counterparts via Lemma 7.2.4. This can be seen by quick and straightforward calculation.

7.3 Partition functions via the semi-infinite wedge

7.3.1 Partition functions for ribbon graphs and maps

First define the partition function for ribbon graphs:

$$Z^R(\vec{p};\hbar) = \exp\left(\sum_{n\geqslant 1}\sum_{g\geqslant 0}\sum_{\mu_1,\ldots,\mu_n\geqslant 1} \mathrm{R}^{\circ}_{g,n}(\mu_1,\ldots,\mu_n)\,\frac{\hbar^{2g-2+n}}{n!}\,\vec{p}_\mu\right). \tag{7.17}$$

Recall that the definition of $\mathrm{R}_{g,n}(\mu_1,\ldots,\mu_n)$ includes the s-parameter which tracks the number of edges.

Begin with the so-called evolution equation for ribbon graphs (alternatively called the *master equation*) which is a reformulation of the Tutte recursion at the level of the partition function [74], given in the following theorem below.

Theorem 7.3.1. *The partition function for ribbon graphs Z^R satisfies the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial s}&Z^R(\vec{p};\hbar)\\ &= GZ^R(\vec{p};\hbar) = \big(\hbar M_2 + K_2 + \frac{1}{2\hbar}p_2\big)Z^R(\vec{p};\hbar)\\ &= \left(\frac{\hbar}{2}\sum_{i,j\geqslant 1}(i+j-2)p_ip_j\frac{\partial}{\partial p_{i+j-2}} + ijp_{i+j+2}\frac{\partial^2}{\partial p_i\partial p_j} + \sum_{i\geqslant 1} ip_{i+2}\frac{\partial}{\partial p_i} + \frac{1}{2}p_1^2 + \frac{1}{2\hbar}p_2\right)Z^R(\vec{p};\hbar).\end{aligned}$$

One can prove this differential equation, as done by Kazarian and Zograf [74], by proving that the Tutte recursion for ribbon graphs (equation 12 in the same reference [74], or, originally proven by Walsh and Lehman [106]) is equivalent to the Virasoro constraints for ribbon graphs [74, Theorem 3 (i)], then multiplying the nth Virasoro operator L_n by p_{n+2} and summing over all n. The fact that the Virasoro operators annihilate the partition function for ribbon graphs guarantees that the resulting evolution equation will as well.

One can use the evolution equation along with Lemma 7.2.4 to derive a vacuum expectation for the ribbon graph partition function; this is what is being done in the following lemma.

Lemma 7.3.2. *The partition function for ribbon graphs is given by the following vacuum expectation in the semi-infinite wedge:*

$$Z^R(\vec{p};\hbar) = \left\langle \exp(s\mathcal{G})\exp\left(\sum_{m\geqslant 1}\frac{p_m}{m}\alpha_{-m}\right)\right\rangle. \tag{7.18}$$

Here, the operator $\mathcal{G}$ is defined in Definition 7.2.7, while the bosonic operators, $\alpha_{\pm m}$, are defined in Definition 1.3.4.

Proof. Begin with the vacuum expectation on the right side of (7.18) and apply the operator $\frac{\partial}{\partial s}$. Define the result to be $Z^?$ (which I ultimately aim to show is equal to Z^R). Thus

$$\begin{aligned}Z^? &:= \frac{\partial}{\partial s}\left\langle \exp(s\mathcal{G})\exp\left(\sum_{m\geqslant 1}\frac{p_m}{m}\alpha_{-m}\right)\right\rangle\\ &= \left\langle \exp(s\mathcal{G})\,\mathcal{G}\,\exp\left(\sum_{m\geqslant 1}\frac{p_m}{m}\alpha_{-m}\right)\right\rangle\\ &= \left\langle \exp(s\mathcal{G})\left(\frac{\hbar}{6}\sum_{i+j+k=2} :\alpha_i\alpha_j\alpha_k: + \sum_{i\geqslant 1}\alpha_{-i}\alpha_{2+i} + \frac{1}{2}\alpha_1^2 + \frac{1}{2\hbar}\alpha_2\right)\exp\left(\sum_{m\geqslant 1}\frac{p_m}{m}\alpha_{-m}\right)\right\rangle\\ &= \left(\frac{\hbar}{2}\sum_{i,j\geqslant 1}(i+j-2)p_ip_j\frac{\partial}{\partial p_{i+j-2}} + ijp_{i+j+2}\frac{\partial^2}{\partial p_i\partial p_j} + \sum_{i\geqslant 1} ip_{i+2}\frac{\partial}{\partial p_i} + \frac{1}{2}p_1^2 + \frac{1}{2\hbar}p_2\right)Z^?\\ &= GZ^?,\end{aligned}$$

where the penultimate equality has applied Lemma 7.2.4. Hence, $Z^?$ satisfies $\frac{\partial}{\partial s} Z^? = GZ^?$. Given that Z^R also satisfies $\frac{\partial}{\partial s} Z^R = GZ^R$, uniqueness of solutions guarantees that, up to multiplicative constant (in s), they are both equal to $\exp(sG) \cdot 1$.

The s^0 coefficient of Z^R is 1 by definition, therefore,

$$Z^R(\vec{p};\hbar) = \frac{Z^?}{Z^?|_{s=0}} = \frac{\Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m\geqslant 1} \frac{p_m}{m} \alpha_{-m}\Big)\Big\rangle}{\Big\langle \exp\Big(\sum_{m\geqslant 1} \frac{p_m}{m} \alpha_{-m}\Big)\Big\rangle} = \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m\geqslant 1} \frac{p_m}{m} \alpha_{-m}\Big)\Big\rangle,$$

as required. ■

Next, use the vacuum expectation for ribbon graphs to derive a vacuum expectation for the partition function for ordinary maps. First, define the partition function for ordinary maps to be

$$Z^M(\vec{p};\hbar) = \exp\Bigg(\sum_{n\geqslant 1}\sum_{g\geqslant 0}\sum_{\mu_1,\ldots,\mu_n\geqslant 1} \mathrm{M}^{\circ}_{g,n}(\mu_1,\ldots,\mu_n)\,\frac{\hbar^{2g-2+n}}{n!}\,\vec{p}_\mu\Bigg). \tag{7.19}$$

And again recall that the definition of $\mathrm{M}_{g,n}(\mu_1,\ldots,\mu_n)$ includes the s-parameter which tracks the number of edges.

Proposition 7.3.3. *The partition function for ordinary maps is given by the following vacuum expectation in the semi-infinite wedge:*

$$Z^M(\vec{p};\hbar) = \Bigg\langle \exp(s\mathcal{G}) \exp\Bigg(\sum_{m\geqslant 1} \frac{t_m}{m\,\hbar} \alpha_{-m}\Bigg) \exp\Bigg(\sum_{m\geqslant 1} \frac{p_m}{m} \alpha_{-m}\Bigg)\Bigg\rangle.$$

Here, the operator $\mathcal{G}$ is defined in Definition 7.2.7, while the bosonic operators, $\alpha_{\pm m}$, are defined in Definition 1.3.4.

Proof. Let $\mathrm{Map}(\emptyset)$ be the enumeration of ordinary maps from Definition 7.2.1 but with no boundary faces. Observe that

$$\mathrm{Map}(\emptyset) = Z^R\big(\tfrac{\vec{t}}{\hbar};\hbar\big) = Z^R\Big(\frac{t_1}{\hbar},\frac{t_2}{\hbar},\frac{t_3}{\hbar},\ldots;\hbar\Big).$$

To prove this, I will compare the two generating functions and show that they are equal. Intuitively, the mechanism behind this equality is the fact that an ordinary map with no boundary faces can equivalently be thought of as a ribbon graph, where one considers the internal faces of the ordinary map to be the boundary faces of the ribbon graph. And, in this case, the above equality is also stating that the two generating functions $\mathrm{Map}(\emptyset)$ and $Z^R\big(\frac{\vec{t}}{\hbar};\hbar\big)$ are enumerating these same objects in the same way.

Let $\mathcal{M}(\mu_1,\ldots,\mu_n)$ be the set of (isomorphism classes of) possibly disconnected ordinary maps with n boundary faces, where the degree of boundary face i is μ_i. Then,

$$\mathrm{Map}(\emptyset) = \sum_{M\in\mathcal{M}(\emptyset)} \frac{\hbar^{2g-2}\, s^{e(M)}}{|\mathrm{Aut}\ M|} t_1^{f_1(M)}\, t_2^{f_2(M)} t_3^{f_3(M)} \cdots,$$

where g is the genus of the underlying surface, $f_i(M)$ is the number of internal faces of degree i, and $|\mathrm{Aut}\ M|$ is the number of automorphisms of M. As reasoned above, this enumerates ribbon graphs, where the sum is varying over all genus, number of (boundary) faces, and the degrees of faces. I can instead write this as a sum over these variables instead. Doing this yields

$$\mathrm{Map}(\emptyset) = \sum_{\mu\in\mathcal{P}}\sum_{g\in\mathbb{Z}} \mathrm{R}^{\bullet}_{g,n}(\mu_1,\ldots,\mu_n)\frac{\hbar^{2g-2}}{|\mathrm{Aut}\ \mu|}\, t_{\mu_1}\cdots t_{\mu_n},$$

where $\mathcal{P}$ is the set of all partitions, including the empty partition. Recall, from Definition 3.2.10 that $\mathrm{R}^{\bullet}_{g,n}(\mu_1,\ldots,\mu_n)$ is the weighted enumeration of possibly disconnected ribbon graphs with n faces such that the degree of boundary face i is μ_i, and the weight of a ribbon graph R is $\frac{s^{|\mu|/2}}{|\text{Aut } R|}$. It remains to observe that for an ordinary map with no boundary faces M corresponding to a ribbon graph R,

$$|\text{Aut } M| = |\text{Aut } R| \cdot |\text{Aut } \mu|.$$

This is because, for an ordinary map with no boundary faces and n internal faces with degrees $\mu_1, \ldots, \mu_n$, there are $|\text{Aut } \mu|$ unique ways to label the n faces $1, \ldots, n$ to yield the corresponding ribbon graph with n boundary faces. (Recall that internal faces are not labelled, while boundary faces are.)

And hence,

$$\begin{aligned}
\text{Map}(\emptyset) &= 1 + \sum_{n\geqslant 1}\sum_{g\in\mathbb{Z}}\sum_{\mu_1,\ldots,\mu_n\geqslant 1} \text{R}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n)\frac{\hbar^{2g-2}}{n!}\, t_{\mu_1}\cdots t_{\mu_n} \\
&= 1 + \sum_{n\geqslant 1}\sum_{g\in\mathbb{Z}}\sum_{\mu_1,\ldots,\mu_n\geqslant 1} \text{R}^{\bullet}_{g,n}(\mu_1, \ldots, \mu_n)\,\frac{\hbar^{2g-2+n}}{n!}\,\frac{t_{\mu_1}}{\hbar}\cdots\frac{t_{\mu_n}}{\hbar} \\
&= Z^R\big(\tfrac{\vec{t}}{\hbar}; \hbar\big).
\end{aligned}$$

In the first equality, the leading 1 outside the summations corresponds to the empty map, and the factor of $1/n!$ arises because the sum is over $\mu_1, \ldots, \mu_n$ as a tuple rather than $\mu \in \mathcal{P}$ as a partition and each tuple arises $n!/|\text{Aut } \mu|$ times. Next observe that

$$\text{Map}(\mu_1, \ldots, \mu_n) = \left[\prod_{i=1}^{n} \hbar\frac{\partial}{\partial t_{\mu_i}}\right] \text{Map}(\emptyset).$$

This is a combinatorial statement which can be reasoned thus. From an ordinary map with no boundary faces, one can obtain an ordinary map with n boundary faces by choosing n of the internal faces and labelling them $1, 2, \ldots, n$. Thus, in the expression above, we can obtain the generating function for all maps with n boundary faces such that the degree the boundary face labelled i has degree μ_i from the generating function for ordinary maps with no boundary faces by converting n internal faces with degrees $\mu_1, \ldots, \mu_n$ into boundary faces — this is being done by the product of differential operators $\prod_{i=1}^{n} \frac{\partial}{\partial \mu_i}$ — and labelling them accordingly, noting that the labelling is given by $\mu_1, \ldots, \mu_n$.

To prove this statement, it is required to prove that the resultant map obtained from this procedure carries the precise weight with which it would appear in the generating function $\text{Map}(\mu_1, \ldots, \mu_n)$. Or equivalently, we are proving that the coefficients of the two power series are equal.

To do this, begin with a map $M \in \mathcal{M}(\emptyset)$ with weight

$$\frac{\hbar^{2g-2}\, s^{e(M)}}{|\text{Aut } M|} t_1^{f_1(M)}\, t_2^{f_2(M)} t_3^{f_3(M)} \cdots,$$

and call the resulting map after converting n internal faces to n boundary faces $\overline{M}$; the weight of $\overline{M}$ in $\text{Map}(\mu_1, \ldots, \mu_n)$ is

$$\frac{\hbar^{2g'-2+n}\, s^{e(\overline{M})}}{|\text{Aut } \overline{M}|} t_1^{f_1(\overline{M})}\, t_2^{f_2(\overline{M})} t_3^{f_3(\overline{M})} \cdots,$$

Observe that this combinatorial procedure preserves the genus as well as the number of edges of the map, hence $g' = g(\overline{M}) = g(M) = g$, and $e(\overline{M}) = e(M)$. Further, the operator $\prod_{i=1}^{n} \hbar\frac{\partial}{\partial t_{\mu_i}}$ increases the power of $\hbar$ for each boundary face introduced. The number and degrees of the internal faces of $\overline{M}$ is precisely given by the resulting t-monomial after applying $\prod_{i=1}^{n} \hbar\frac{\partial}{\partial t_{\mu_i}}$ to $t_1^{f_1(M)}\, t_2^{f_2(M)} t_3^{f_3(M)} \cdots$, while the weight from applying this differential operator combines with the number of automorphisms of M to give the number of automorphisms of $\overline{M}$. That is, if we write $\mu = (\mu_1, \ldots, \mu_n)$ alternatively as $\mu = (1^{k_1}, 2^{k_2}, 3^{k_3}, \ldots)$ where

k_i is a non-negative integer for all i, then

$$\left[\prod_{i=1}^{n} \frac{\partial}{\partial t_{\mu_i}}\right] t_1^{f_1(M)}\, t_2^{f_2(M)} t_3^{f_3(M)} \cdots$$

$$\begin{aligned} &= \left[\frac{\partial^{k_1}}{\partial t_1^{k_1}} \frac{\partial^{k_2}}{\partial t_2^{k_2}} \frac{\partial^{k_3}}{\partial t_3^{k_3}} \cdots\right] t_1^{f_1(M)}\, t_2^{f_2(M)} t_3^{f_3(M)} \cdots \\ &= \frac{f_1(M)!}{(f_1(M)-k_1)!} t_1^{f_1(M)-k_1} \frac{f_2(M)!}{(f_2(M)-k_2)!} t_2^{f_2(M)-k_2} \frac{f_3(M)!}{(f_3(M)-k_3)!} t_3^{f_3(M)-k_3} \cdots \\ &= \frac{f_1(M)!}{(f_1(M)-k_1)!} t_1^{f_1(\overline{M})} \frac{f_2(M)!}{(f_2(M)-k_2)!} t_2^{f_2(\overline{M})} \frac{f_3(M)!}{(f_3(M)-k_3)!} t_3^{f_3(\overline{M})} \cdots . \end{aligned}$$

It remains to observe that

$$\frac{1}{|\text{Aut } M|} \frac{f_1(M)!}{(f_1(M)-k_1)!} \frac{f_2(M)!}{(f_2(M)-k_2)!} \frac{f_3(M)!}{(f_3(M)-k_3)!} \cdots = \frac{1}{|\text{Aut } \overline{M}|}.$$

This is because $f_i(M)!/(f_i(M)-k_i)!$ is precisely the number of ways to choose k_i faces of degree i to convert from internal faces to boundary faces where the ordering of the resulting boundary faces matters because different labellings of these k_i boundary faces gives rise to different ordinary maps $\overline{M}$. Loosely, if M has $f_i(M)$ internal faces of degree i then $|\text{Aut } M|$ will include a factor of $f_i(M)!$, while $\overline{M}$ now has $(f_i(M)-k_i)$ internal faces of degree i and hence $|\text{Aut } \overline{M}|$ ought to include a factor of $(f_i(M)-k_i)!$.

Combining these observations, I can conclude

$$\left[\prod_{i=1}^{n} \hbar \frac{\partial}{\partial t_{\mu_i}}\right] \frac{\hbar^{2g-2}\, s^{e(M)/2}}{|\text{Aut } M|} t_1^{f_1(M)}\, t_2^{f_2(M)} t_3^{f_3(M)} \cdots = \frac{\hbar^{2g'-2+n}\, s^{e(\overline{M})/2}}{|\text{Aut } \overline{M}|} t_1^{f_1(\overline{M})}\, t_2^{f_2(\overline{M})} t_3^{f_3(\overline{M})} \cdots ,$$

as required.

Therefore,

$$\begin{aligned} \text{Map}(\mu_1, \ldots, \mu_n) &= \left[\prod_{i=1}^{n} \hbar \frac{\partial}{\partial t_{\mu_i}}\right] \text{Map}(\emptyset) = \left[\prod_{i=1}^{n} \hbar \frac{\partial}{\partial t_{\mu_i}}\right] Z^R(\vec{t}/\hbar) \\ &= \left[\prod_{i=1}^{n} \hbar \frac{\partial}{\partial t_{\mu_i}}\right] \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m\geqslant 1} \frac{p_m}{m} \alpha_{-m}\Big)\Big\rangle \\ &= \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m\geqslant 1} \frac{t_m}{m\,\hbar} \alpha_{-m}\Big) \frac{\alpha_{-\mu_1}}{\mu_1} \cdots \frac{\alpha_{-\mu_n}}{\mu_n}\Big\rangle. \end{aligned} \tag{7.20}$$

On the other hand, by the definition of the partition function for ordinary maps (7.19), we have

$$\begin{aligned} Z^M(\vec{p};\hbar) &= 1 + \sum_{n\geqslant 1} \sum_{g\in\mathbb{Z}} \sum_{\mu_1,\ldots,\mu_n\geqslant 1} \frac{1}{n!} \mathrm{M}^\bullet_{g,n}(\mu_1, \ldots, \mu_n)\, \hbar^{2g-2+n}\, \vec{p}_\mu \\ &= 1 + \sum_{n\geqslant 1} \sum_{\mu_1,\ldots,\mu_n\geqslant 1} \frac{1}{n!} \text{Map}(\mu_1, \ldots, \mu_n)\, \vec{p}_\mu \\ &= 1 + \sum_{n\geqslant 1} \sum_{\mu_1,\ldots,\mu_n\geqslant 1} \frac{1}{n!} \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m\geqslant 1} \frac{t_m}{m\,\hbar} \alpha_{-m}\Big) \frac{\alpha_{-\mu_1}}{\mu_1} \cdots \frac{\alpha_{-\mu_n}}{\mu_n}\Big\rangle \vec{p}_\mu \\ &= \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m\geqslant 1} \frac{t_m}{m\,\hbar} \alpha_{-m}\Big) \exp\Big(\sum_{m\geqslant 1} \frac{p_m}{m} \alpha_{-m}\Big)\Big\rangle. \end{aligned}$$

The third equality is using (7.20) above, while the final equality is using the following:

$$\begin{aligned} \exp\Big(\sum_{m\geqslant 1} \frac{p_m}{m} \alpha_{-m}\Big) &= \prod_{m\geqslant 1} \sum_{k_m\geqslant 0} \frac{p_m^{k_m}}{m^{k_m}\, k_m!} \alpha_{-m}^{k_m} = \sum_{\lambda=(1^{k_1},2^{k_2},\ldots)} \frac{p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}}}{\prod_{m=1}^{\ell(\lambda)} \lambda_m\, |\text{Aut } \lambda|} \prod_{m=1}^{\ell(\lambda)} \alpha_{-\lambda_m} \\ &= 1 + \sum_{n\geqslant 1} \sum_{\mu_1,\ldots,\mu_n\geqslant 1} \frac{1}{n!} \frac{\alpha_{-\mu_1}}{\mu_1} \cdots \frac{\alpha_{-\mu_n}}{\mu_n}\, \vec{p}_\mu. \end{aligned} \tag{7.21}$$

The first equality includes in the sum the empty partition $\lambda = \emptyset$, and this corresponds to the leading 1 in the final line (denoting the identity operator). Further, in the final expression, the $1/n!$ arises because the inner sum is over $\mu_1, \ldots, \mu_n$ as a tuple (rather than a partition), and each tuple arises $n!/|\mathrm{Aut}\ \lambda|$ times. ■

7.3.2 Partition function for fully simple maps

Define the partition function for fully simple maps to be

$$Z^F(\vec{p};\hbar) = \exp\Big(\sum_{n\geqslant 1}\sum_{g\geqslant 0}\sum_{\mu_1,\ldots,\mu_n\geqslant 1} \mathrm{FS}^{\circ}_{g,n}(\mu_1,\ldots,\mu_n)\,\frac{\hbar^{2g-2+n}}{n!}\,\vec{p}_\mu\Big). \tag{7.22}$$

Also, define a diagonal operator, denoted $\mathcal{H}$, acting on the semi-infinite wedge by the following action on basis elements:

$$\mathcal{H}v_\lambda = \prod_{\square\in\lambda}\frac{1}{1+c(\square)\hbar}\,v_\lambda. \tag{7.23}$$

The product is over all boxes in the Young diagram corresponding to λ while $c(\square)$ is the content of the box. Recall that, when considering a Young diagram corresponding to a partition λ, the content of a box in column j and row i is $j-i$.

Theorem 7.3.4. *The partition function for fully simple maps Z^F is given by the following vacuum expectation in the semi-infinite wedge space:*

$$Z^F(\vec{p};\hbar) = \Big\langle \exp(s\mathcal{G})\exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big)\mathcal{H}\exp\Big(\sum_{m\geqslant 1}\frac{p_m}{m}\alpha_{-m}\Big)\Big\rangle.$$

The bosonic operators $\alpha_{\pm m}$ are defined in Definition 1.3.4 while the definition of $\mathcal{H}$ is given in equation (7.23) above.

Proof. First, for the ease of the reader, let me use the following shorthand notations: $\alpha_\lambda = \alpha_{\lambda_1}\cdots\alpha_{\lambda_\ell}$, and

$$\Gamma_-(\vec{t}/\hbar) = \exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big),$$

where the latter uses the $\Gamma_\pm$ notation of the vertex operator defined in Chapter 1.

Begin with the partition function for fully simple maps (7.22), rewrite it as a sum over partitions of disconnected fully simple maps, and apply the relation between maps and fully simple maps (Theorem 7.2.3), specifically, apply equation (7.7) that writes the generating function for fully simple maps in terms of the generating functions for ordinary maps and weakly monotone Hurwitz numbers. Doing so gives

$$\begin{aligned}
Z^F(\vec{p};\hbar) &= 1 + \sum_{n\geqslant 1}\sum_{g\in\mathbb{Z}}\sum_{\mu_1,\ldots,\mu_n\geqslant 1}\frac{1}{n!}\,\mathrm{FS}^{\bullet}_{g,n}(\mu_1,\ldots,\mu_n)\,\hbar^{2g-2+n}\vec{p}_\mu \\
&= 1 + \sum_{n\geqslant 1}\sum_{\mu_1,\ldots,\mu_n\geqslant 1}\frac{1}{n!}\,\mathrm{FSMap}(\mu_1,\ldots,\mu_n)\,\vec{p}_\mu \\
&= \sum_{\mu\in\mathcal{P}}\frac{1}{z(\mu)}\,\mu_1\cdots\mu_n\mathrm{FSMap}(\mu_1,\ldots,\mu_n)\,\vec{p}_\mu \\
&= \sum_{\mu\in\mathcal{P}}\Big[\sum_{\lambda\vdash|\mu|} H^{\leqslant}(\mu;\lambda)\big|_{\hbar\mapsto-\hbar}\,\lambda_1\cdots\lambda_\ell\mathrm{Map}(\lambda_1,\ldots,\lambda_\ell)\Big]\;\vec{p}_\mu \\
&= \sum_{\mu\in\mathcal{P}}\Big[\sum_{\lambda\vdash|\mu|}\sum_{\rho\vdash|\mu|}\frac{\chi^\rho_\mu\,\chi^\rho_\lambda}{z(\mu)z(\lambda)}\prod_{\square\in\rho}\frac{1}{1+c(\square)\hbar}\Big\langle\exp(s\mathcal{G})\Gamma_-(\vec{t}/\hbar)\alpha_{-\lambda}\Big\rangle\Big]\;\vec{p}_\mu.
\end{aligned}$$

Here, $\mathcal{P}$ denotes the set of all partitions, including the empty partition. The last equation has used the character formula for weakly monotone Hurwitz numbers (7.4) as well as the vacuum expectation for $\mathrm{Map}(\lambda_1,\ldots,\lambda_\ell)$ given in equation (7.20).

Next use the Murnaghan–Nakayama rule (Theorem 1.4.3):

$$\alpha_\lambda v_\emptyset = \sum_{\nu \vdash |\lambda|} \chi^\nu_\lambda v_\nu,$$

as well as the orthogonality of characters, which in the context of symmetric groups is

$$\delta_{\rho,\nu} = \frac{1}{|S_d|} \sum_{\sigma \in S_d} \chi^\rho_\sigma \chi^\nu_\sigma = \frac{1}{|S_d|} \sum_{\lambda \vdash d} \chi^\rho_\lambda \chi^\nu_\lambda \frac{|S_d|}{z(\lambda)} = \sum_{\lambda \vdash d} \frac{\chi^\rho_\lambda \chi^\nu_\lambda}{z(\lambda)}.$$

The second equality here rewrites the sum over permutations in S_d as a sum over partitions of d and, to each term, adds the weight of the number of elements in the conjugacy class corresponding to that partition. That is, for each $\lambda \vdash d$, the size of the conjugacy class corresponding to λ is

$$|C_\lambda| = \frac{d!}{|\mathrm{Aut}\ \lambda \prod_{i=1}^{\ell(\lambda)} \lambda_i} = \frac{|S_d|}{z(\lambda)},$$

where $z(\lambda)$ is defined in (7.5).

Applying these results to $Z^F(\vec{p};\hbar)$ yields

$$\begin{aligned} Z^F(\vec{p};\hbar) &= \sum_{\mu \in \mathcal{P}} \left[\sum_{\lambda \vdash |\mu|} \sum_{\rho \vdash |\mu|} \frac{\chi^\rho_\lambda \chi^\rho_\mu}{z(\lambda) z(\mu)} \prod_{\square \in \rho} \frac{1}{1 + c(\square)\hbar} \Big\langle \exp(s\mathcal{G})\, \Gamma_-(\vec{t}/\hbar) \sum_{\nu \vdash |\lambda|} \chi^\nu_\lambda v_\nu, v_\emptyset \Big\rangle \right] \vec{p}_\mu \\ &= \sum_{\mu \in \mathcal{P}} \left[\sum_{\rho \vdash |\mu|} \frac{\chi^\rho_\mu}{z(\mu)} \prod_{\square \in \rho} \frac{1}{1 + c(\square)\hbar} \Big\langle \exp(s\mathcal{G})\, \Gamma_-(\vec{t}/\hbar) \sum_{\nu \vdash |\lambda|} \Big(\sum_{\lambda \vdash |\mu|} \frac{\chi^\rho_\lambda \chi^\nu_\lambda}{z(\lambda)} \Big) v_\nu, v_\emptyset \Big\rangle \right] \vec{p}_\mu \\ &= \sum_{\mu \in \mathcal{P}} \left[\sum_{\rho \vdash |\mu|} \frac{\chi^\rho_\mu}{z(\mu)} \Big\langle \exp(s\mathcal{G})\, \Gamma_-(\vec{t}/\hbar) \mathcal{H} \sum_{\nu \vdash |\lambda|} \delta_{\rho,\nu} v_\nu, v_\emptyset \Big\rangle \right] \vec{p}_\mu. \end{aligned}$$

The last equality has also used the definition of $\mathcal{H}$. Use the Murnaghan–Nakayama rule once more, and rewrite the sum over partitions μ as a sum over tuples to obtain

$$\begin{aligned} Z^F(\vec{p};\hbar) &= \sum_{\rho \in \mathcal{P}} \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m \geqslant 1} \frac{t_m}{m\,\hbar} \alpha_{-m} \Big) \mathcal{H} \sum_{\mu \vdash |\rho|} \frac{\chi^\rho_\mu}{z(\mu)} v_\rho, v_\emptyset \Big\rangle \vec{p}_\mu \\ &= \sum_{\mu \in \mathcal{P}} \frac{1}{z(\mu)} \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m \geqslant 1} \frac{t_m}{m\,\hbar} \alpha_{-m} \Big) \mathcal{H} \sum_{\rho \vdash |\mu|} \chi^\rho_\mu v_\rho, v_\emptyset \Big\rangle \vec{p}_\mu \\ &= 1 + \sum_{n \geqslant 1} \sum_{\mu_1, \ldots, \mu_n} \frac{1}{n!} \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m \geqslant 1} \frac{t_m}{m\,\hbar} \alpha_{-m} \Big) \mathcal{H} \frac{\alpha_{-\mu_1}}{\mu_1} \cdots \frac{\alpha_{-\mu_n}}{\mu_n} \Big\rangle \vec{p}_\mu \\ &= \Big\langle \exp(s\mathcal{G}) \exp\Big(\sum_{m \geqslant 1} \frac{t_m}{m\,\hbar} \alpha_{-m} \Big) \mathcal{H} \exp\Big(\sum_{m \geqslant 1} \frac{p_m}{m} \alpha_{-m} \Big) \Big\rangle, \end{aligned}$$

as announced. The penultimate equality is using the fact that the inner sum is over $\mu_1, \ldots, \mu_n$ as a tuple and each tuple arises $n!/|\mathrm{Aut}\ \mu|$ times. The final equality is using (7.21). ■

7.4 Virasoro constraints

7.4.1 Virasoro constraints for ordinary maps

My starting point for deriving the Virasoro constraints for fully simple maps is the Virasoro constraints for ordinary maps (Theorem 7.4.2 below, first proven by Eynard [45]). Out of interest, I will also include a proof of the derivation of the Virasoro constraints for ordinary maps, which are in turn derived from the Tutte recursion (Theorem 7.4.1 below, also first proven by Eynard [45]). The fact that the Virasoro operators for ordinary maps V^M satisfy the commutation relation $[V^M_n, V^M_m] = (n-m)V^M_{n+m}$ I leave as an exercise.

Theorem 7.4.1 (Eynard [45, Equation (1.3.2)]). *For all $g \geqslant 0, n \geqslant 1$ and $\mu_1, \ldots, \mu_n$ such that $\mu_1 + \cdots + \mu_n > 2$, connected ordinary maps satisfy the following recursion:*

$$\frac{\mu_1}{s} \mathrm{M}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{j=2}^{n} (\mu_1 + \mu_j - 2) \mathrm{M}^{\circ}_{g,n-1}(\mu_1 + \mu_j - 2, \vec{\mu}_{S \setminus \{j\}}) + \sum_{j \geqslant 1} t_j (\mu_1 + j - 2) \mathrm{M}^{\circ}_{g,n}(\mu_1 + j - 2, \vec{\mu}_S)$$
$$+ 2(\mu_1 - 2) \mathrm{M}^{\circ}_{g,n}(\mu_1 - 2, \vec{\mu}_S) + \sum_{i+j=\mu_1-2} ij \left[\mathrm{M}^{\circ}_{g-1,n+1}(i, j, \vec{\mu}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \mathrm{M}^{\circ}_{g_1,|I|+1}(i, \vec{\mu}_I) \, \mathrm{M}^{\circ}_{g_2,|J|+1}(j, \vec{\mu}_J) \right]. \tag{7.24}$$

Here, $S = \{2, \ldots, n\}$ and the notation $\vec{\mu}_I$ denotes $\mu_{i_1}, \ldots, \mu_{i_k}$ for $I = \{i_1, \ldots, i_k\}$.

Theorem 7.4.2 (Eynard [45, Proposition 2.6.2]). *For all $n \geqslant -1$, the differential operator*

$$V_n^M = \sum_{i \geqslant 1} (i+n)(\hbar p_i + t_i) \frac{\partial}{\partial p_{i+n}} + \hbar \sum_{i+j=n} ij \frac{\partial^2}{\partial p_i \partial p_j} + 2n \frac{\partial}{\partial p_n} + \delta_{n,-1} \Big(p_1 + \frac{t_1}{\hbar} \Big) + \frac{1}{\hbar} \delta_{n,0} - \frac{1}{s}(n+2) \frac{\partial}{\partial p_{n+2}}$$

annihilates the partition function for ribbon graphs; that is,

$$V_n^M Z^M(\vec{p}; \hbar) = 0.$$

Proposition 7.4.3 (Eynard [45, Proposition 2.6.2]). *For all $n \geqslant -1$, the differential operator*

$$V_n^M = \sum_{i \geqslant 1} (i+n)(\hbar p_i + t_i) \frac{\partial}{\partial p_{i+n}} + \hbar \sum_{i+j=n} ij \frac{\partial^2}{\partial p_i \partial p_j} + 2n \frac{\partial}{\partial p_n} + \delta_{n,-1} \Big(p_1 + \frac{t_1}{\hbar} \Big) + \frac{1}{\hbar} \delta_{n,0} - \frac{1}{s}(n+2) \frac{\partial}{\partial p_{n+2}}$$

satisfy the commutation relation $[V_n^M, V_m^M] = (n-m) V_{n+m}^M$.

I include a quick proof of Theorem 7.4.2.

Proof of Theorem 7.4.2. The Virasoro constraints are a reformulation of the Tutte recursion for ordinary maps. To derive the former from the latter, we multiply each term in the Tutte recursion by

$$\frac{p_{\mu_2} \cdots p_{\mu_n} \, \hbar^{2g-2+n}}{(n-1)!} \tag{$*$}$$

then sum over all possible values of $n \geqslant 1$, $g \geqslant 0$ and $\mu_2, \ldots, \mu_n \geqslant 1$. Denote by $\mathcal{O}$ the application of this operator—that is, multiplying by $(*)$ and summing over all $\mu_2, \ldots, \mu_n \geqslant 1$. Note that I am summing over all $n \geqslant 1$ and $\mu_1 + \cdots + \mu_n \geqslant 1$ but the recursion in Theorem 7.4.1 is only valid for $\mu_1 + \cdots + \mu_n > 2$. This can be fixed by manually adding a finite number of correction terms to the recursion for the base cases $\vec{\mu} \in \{(1), (2), (1,1)\}$. The correction terms in these three cases are t_1, 1, and 1 respectively, therefore the recursion becomes

$$\frac{\mu_1}{s} \mathrm{M}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{j=2}^{n} (\mu_1 + \mu_j - 2) \mathrm{M}^{\circ}_{g,n-1}(\mu_1 + \mu_j - 2, \vec{\mu}_{S \setminus \{j\}}) + \sum_{j \geqslant 1} t_j (\mu_1 + j - 2) \mathrm{M}^{\circ}_{g,n}(\mu_1 + j - 2, \vec{\mu}_S)$$
$$+ 2(\mu_1 - 2) \mathrm{M}^{\circ}_{g,n}(\mu_1 - 2, \vec{\mu}_S) + \sum_{i+j=\mu_1-2} ij \left[\mathrm{M}^{\circ}_{g-1,n+1}(i, j, \vec{\mu}_S) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \mathrm{M}_{g_1,|I|+1}(i, \vec{\mu}_I) \, \mathrm{M}_{g_2,|J|+1}(j, \vec{\mu}_J) \right]$$
$$+ \delta_{g,0} \delta_{n,1} \delta_{\mu_1,1} t_1 + \delta_{g,0} \delta_{n,1} \delta_{\mu_1,2} + \delta_{g,0} \delta_{n,2} \delta_{\mu_1,1} \delta_{\mu_2,1}.$$

Now fix μ_1 and apply the operator $\mathcal{O}$ described above. The term on the left side becomes

$$\sum_{n \geqslant 1} \sum_{g \geqslant 0} \sum_{\mu_2, \ldots, \mu_n \geqslant 1} \frac{\mu_1}{s} \mathrm{M}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n) \frac{p_{\mu_2} \cdots p_{\mu_n} \, \hbar^{2g-2+n}}{(n-1)!}$$
$$= \frac{\mu_1}{s} \frac{\partial}{\partial p_{\mu_1}} \sum_{n \geqslant 1} \sum_{g \geqslant 0} \sum_{\mu_1', \ldots, \mu_n \geqslant 1} \mathrm{M}^{\circ}_{g,n}(\mu_1', \ldots, \mu_n) \frac{p_{\mu_1'} p_{\mu_2} \cdots p_{\mu_n} \, \hbar^{2g-2+n}}{n!}$$
$$= \frac{\mu_1}{s} \frac{\partial F}{\partial p_{\mu_1}}, \tag{$\triangle$}$$

where $F(\vec{p};\hbar)$ is the following generating function of connected objects, typically called the *free energy*:

$$F(\vec{p};\hbar) = \sum_{n\geqslant 1}\sum_{g\geqslant 0}\sum_{\mu_1',\ldots,\mu_n\geqslant 1} \mathrm{M}^{\circ}_{g,n}(\mu_1',\ldots,\mu_n)\frac{p_{\mu_1'}p_{\mu_2}\cdots p_{\mu_n}\,\hbar^{2g-2+n}}{n!}.$$

Also, recall that exponentiating the connected enumeration yields the disconnected one. That is, $Z^M = \exp F$.

Applying $\mathcal{O}$ to the first term on the right side yields

$$\begin{aligned}
&\sum_{n\geqslant 2}\sum_{g\geqslant 0}\sum_{\mu_2,\ldots,\mu_n\geqslant 1}\sum_{j=2}^{n}(\mu_1+\mu_j-2)\mathrm{M}^{\circ}_{g,n-1}(\mu_1+\mu_j-2,\vec{\mu}_{S\setminus\{j\}})\frac{p_{\mu_2}\cdots p_{\mu_n}\,\hbar^{2g-2+n}}{(n-1)!}\\
&\quad=\sum_{i\geqslant 1}\sum_{n\geqslant 2}\sum_{g\geqslant 0}\sum_{\mu_2,\ldots,\hat{\mu}_j,\ldots,\mu_n\geqslant 1} p_i(\mu_1+i-2)\mathrm{M}^{\circ}_{g,n-1}(\mu_1+i-2,\vec{\mu}_{S\setminus\{j\}})\frac{p_{\mu_2}\cdots\hat{p}_{\mu_j}\cdots p_{\mu_n}\hbar^{2g-2+n}}{(n-2)!}\\
&\quad=\hbar\sum_{i\geqslant 1}p_i(\mu_1+i-2)\frac{\partial}{\partial p_{\mu_1+i-2}}\sum_{n\geqslant 2}\sum_{g\geqslant 0}\sum_{\mu_1',\ldots,\mu_{n-1}'\geqslant 1}\mathrm{M}^{\circ}_{g,n-1}(\mu_1',\ldots,\mu_{n-1}')\frac{p_{\mu_1'}\cdots p_{\mu_{n-1}'}\hbar^{2g-2+(n-1)}}{(n-2)!}\\
&\quad=\hbar\sum_{i\geqslant 1}p_i(\mu_1+i-2)\frac{\partial F}{\partial p_{\mu_1+i-2}}. \qquad (\triangle)
\end{aligned}$$

Similarly, the second term on the right becomes

$$\begin{aligned}
&\sum_{n\geqslant 2}\sum_{g\geqslant 0}\sum_{\mu_2,\ldots,\mu_n\geqslant 1}\sum_{j\geqslant 1}t_j(\mu_1+j-2)\mathrm{M}^{\circ}_{g,n-1}(\mu_1+j-2,\vec{\mu}_S)\frac{p_{\mu_2}\cdots p_{\mu_n}\,\hbar^{2g-2+n}}{(n-1)!}\\
&\qquad\qquad=\sum_{i\geqslant 1}t_j(\mu_1+j-2)\frac{\partial F}{\partial p_{\mu_1+j-2}}. \qquad (\triangle)
\end{aligned}$$

Applying the operator to the term $2(\mu_1-2)\mathrm{M}^{\circ}_{g,n}(\mu_1-2,\vec{\mu}_S)$ gives

$$\sum_{n\geqslant 2}\sum_{g\geqslant 0}\sum_{\mu_2,\ldots,\mu_n\geqslant 1}2(\mu_1-2)\mathrm{M}^{\circ}_{g,n}(\mu_1-2,\vec{\mu}_S)\frac{p_{\mu_2}\cdots p_{\mu_n}\,\hbar^{2g-2+n}}{(n-1)!} = 2(\mu_1-2)\frac{\partial F}{\partial p_{\mu_1-2}}, \qquad (\triangle)$$

while the subsequent two terms become

$$\hbar\sum_{i+j=\mu_1-2} ij\frac{\partial^2 F}{\partial p_i\partial p_j}, \qquad \text{and} \qquad \hbar\sum_{i+j=\mu_1-2} ij\frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j}, \qquad (\triangle)$$

respectively. And the final three terms corresponding to the base cases become

$$\frac{t_1}{\hbar}\delta_{\mu_1,1},\quad \frac{1}{\hbar}\delta_{\mu_1,2},\quad p_1\delta_{\mu_1,1} \qquad (\triangle)$$

respectively. Collect all terms labelled $\triangle$, relabel $\mu_1\mapsto n+2$ and multiply throughout by $\exp F$. Note that

$$\frac{\partial F}{\partial x}\exp F = \frac{\partial}{\partial x}(\exp F) = \frac{\partial Z^M}{\partial x},$$

in which case, we obtain

$$\begin{aligned}
\frac{n+2}{s}\frac{\partial Z^M}{\partial p_{n+2}} &= \sum_{i\geqslant 1}(\hbar p_i+t_i)(n+i)\frac{\partial Z^M}{\partial p_{n+i}} + 2n\frac{\partial Z^M}{\partial p_n}\\
&\qquad + \hbar\sum_{i+j=n} ij\frac{\partial^2 Z^M}{\partial p_i\partial p_j} + \Big(p_1+\frac{t_1}{\hbar}\Big)\delta_{n,-1}Z^M + \frac{1}{\hbar}\delta_{n,0}Z^M.
\end{aligned}$$

The first term on the second line is using the fact that

$$\frac{\partial^2}{\partial p_i\partial p_j}(\exp F) = \frac{\partial}{\partial p_i}\Big(\frac{\partial F}{\partial p_i}\exp F\Big) = \left(\frac{\partial^2 F}{\partial p_i\partial p_j} + \frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j}\right)\exp F.$$

Defining

$$V_n^M := \sum_{i\geqslant 1}(i+n)(\hbar p_i + t_i)\frac{\partial}{\partial p_{i+n}} + \hbar\sum_{i+j=n} ij\frac{\partial^2}{\partial p_i \partial p_j} + 2n\frac{\partial}{\partial p_n} + \delta_{n,-1}\left(p_1 + \frac{t_1}{\hbar}\right) + \frac{1}{\hbar}\delta_{n,0} - \frac{1}{s}(n+2)\frac{\partial}{\partial p_{n+2}}$$

for $n \geqslant -1$, it follows that $V_n^M Z^M = 0$. ∎

7.4.2 Conjugation results

I will now prove a key conjugation result that will be needed to derive the Virasoro constraints for fully simple maps. For context, my overarching approach to deriving the Virasoro operator V_n^F for fully simple is by conjugating the Virasoro operator for ordinary maps by the operator $\mathcal{H}$. This will be more explicit in the next section, but nevertheless, my intermediate goal is to derive conjugations of the operators α_n and $\mathcal{K}_n$ by $\mathcal{H}$. First, I obtain a result writing these conjugations in terms of the fermionic operators ψ, ψ^*, then use results stated by Kramer, Lewański, and Shadrin [77] to translate the conjugated operators into bosonic form.

First, begin with the following lemma.

Lemma 7.4.4. *For all positive integers n,*

$$\mathcal{H}^{-1}\alpha_n\mathcal{H} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\prod_{i=1}^{n}\frac{1}{1+(k-i+\frac{1}{2})\hbar} :\psi_{k-n}\psi_k^*:, \tag{7.25}$$

$$\mathcal{H}^{-1}\mathcal{K}_n\mathcal{H} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\left(k+\tfrac{n}{2}\right)\prod_{i=1}^{n}\frac{1}{1+(k-i+\frac{1}{2})\hbar} :\psi_{k-n}\psi_k^*: . \tag{7.26}$$

and

$$\mathcal{H}^{-1}\alpha_{-n}\mathcal{H} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\prod_{i=1}^{n}(1+(k+i-\tfrac{1}{2})\hbar) :\psi_{k+n}\psi_k^*:, \tag{7.27}$$

$$\mathcal{H}^{-1}\mathcal{K}_{-n}\mathcal{H} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\left(k-\tfrac{n}{2}\right)\prod_{i=1}^{n}(1+(k+i-\tfrac{1}{2})\hbar) :\psi_{k+n}\psi_k^*: . \tag{7.28}$$

The bosonic operators $\alpha_{\pm k}$ are defined in Definition 1.3.4 the operator $\mathcal{K}_n$ is defined in Definition 7.2.5, equation (7.9), while the fermionic operators ψ_i, ψ_j^* are defined in Definition 1.3.1.

Proof. Begin with $\mathcal{H}^{-1}\alpha_n\mathcal{H}$ (7.25), and first consider the case where n is a positive integer. Looking at the action of this operator on a basis vector v_λ leads to the following argument:

$$\begin{aligned}\mathcal{H}^{-1}\alpha_n\mathcal{H}\,v_\lambda &= \prod_{\square\in\lambda}\frac{1}{1+c(\square)\hbar}\mathcal{H}^{-1}\alpha_n v_\lambda = \prod_{\square\in\lambda}\frac{1}{1+c(\square)\hbar}\mathcal{H}^{-1}\sum_{\rho=\lambda_-}\operatorname{sgn}(\lambda\setminus\lambda_-)\,v_\rho \\ &= \prod_{\square\in\lambda}\frac{1}{1+c(\square)\hbar}\sum_{\rho=\lambda_-}\operatorname{sgn}(\lambda\setminus\lambda_-)\prod_{\square\in\rho}(1+c(\square)\hbar)v_\rho = \sum_{\rho=\lambda_-}\operatorname{sgn}(\lambda\setminus\lambda_-)\prod_{\square\in\lambda\setminus\rho}\frac{1}{1+c(\square)\hbar}v_\rho.\end{aligned}$$

The sums over $\rho=\lambda_-$ are over all Young diagrams ρ that can be obtained from λ by removing an n-ribbon, and $\operatorname{sgn}(\lambda\setminus\lambda_-)$ is equal to the parity of the height of the n-ribbon that was removed less one (the usual sign introduced by the action of the α-operators). Thus, for n positive, $\mathcal{H}^{-1}\alpha_n\mathcal{H}$ acts by removing all possible n-ribbons and for each contribution, dividing by $(1+c(\square)\hbar)$ for each box in the removed ribbon. Writing this action in terms of the fermionic operators yields the desired expression (7.25).

For n negative this works analogously; that is, the operator acts by summing over all the ways to add an n-ribbon and in each case multiplying by the product of $(1+c(\square)\hbar)$ for each box in the ribbon added.

For $\mathcal{H}^{-1}\mathcal{K}_n\mathcal{H}$, use (7.10) with $n\neq 0$,

$$\mathcal{K}_n = [z]\,\mathcal{E}_n(z),$$

and consider

$$\mathcal{H}^{-1}\mathcal{K}_n\mathcal{H} = [z]\mathcal{H}^{-1}\mathcal{E}_n(z)\mathcal{H} = [z]\mathcal{H}^{-1}\sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{n}{2})} : \psi_{k-n}\psi_k^* : \mathcal{H}$$
$$= \mathcal{H}^{-1}\sum_{k\in\mathbb{Z}+\frac{1}{2}} \left(k-\tfrac{n}{2}\right) : \psi_{k-n}\psi_k^* : \mathcal{H}.$$

Use the above reasoning for $\mathcal{H}^{-1}\alpha_n\mathcal{H}$ to conclude. ∎

The goal now is to write these conjugated operators in bosonic form. For this we use the following two results that appear in Kramer, Lewański, and Shadrin [77].

Lemma 7.4.5 (Kramer, Lewański, and Shadrin [77, Lemma 3.4]). *If the variables in a symmetric polynomial are all offset by the same amount, they can be re-expressed as a linear combination of non-offset symmetric polynomials as follows:*

$$\begin{aligned} h_m(x_1+A,\dots,x_n+A) &= \sum_{i=0}^{m}\binom{m+n-1}{i}h_{m-i}(x_1,\dots,x_n)A^i \\ e_m(x_1+A,\dots,x_n+A) &= \sum_{i=0}^{m}\binom{n+i-m}{i}e_{m-i}(x_1,\dots,x_n)A^i. \end{aligned} \tag{7.29}$$

Here, h and e are the elementary and homogeneous symmetric polynomials respectively.

The elementary and homogeneous symmetric polynomials when evaluated at integers have nice descriptions in terms of the first and second kind of Stirling numbers respectively. Recall that the *(unsigned) Stirling number of the first kind* $\left[{n \atop k}\right]$ is the number of permutations of $\{1, 2, \dots, n\}$ which have exactly k cycles, while the *Stirling number of the second kind* $\left\{{n \atop k}\right\}$ is the number of set partitions of $\{1, 2, \dots, n\}$ into k parts. Then, the elementary and symmetric polynomials evaluated at integers have the following descriptions:

$$\begin{aligned} e_v(1,2,\dots,t-1) &= \left[{t \atop t-v}\right] \\ h_v(1,2,\dots,t) &= \left\{{t+v \atop t}\right\}. \end{aligned} \tag{7.30}$$

For a thorough discourse on symmetric functions, see the book of Macdonald [81]. The second result from Kramer, Lewański, and Shadrin [77] that I require is the following.

Lemma 7.4.6 (Kramer, Lewański, and Shadrin [77, Lemma 3.6]). *We have*

$$\left[{j \atop t}\right] = [y^{j-t}]\frac{(j-1)!}{(t-1)!}\mathcal{S}(y)^{-j}e^{yj/2}, \quad \textit{and} \quad \left\{{j \atop t}\right\} = [y^{j-t}]\frac{j!}{t!}\mathcal{S}(y)^{t}e^{yt/2}. \tag{7.31}$$

Here, $\mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{e^{z/2}-e^{-z/2}}{z}$.

I can now write the conjugated operators $\mathcal{H}^{-1}\alpha_n\mathcal{H}$ and $\mathcal{H}^{-1}\mathcal{K}_n\mathcal{H}$ in terms of bosonic operators, or, more precisely, in terms of the operator $\mathcal{E}$.

Lemma 7.4.7. *For all positive integers n,*

$$\mathcal{H}^{-1}\alpha_{-n}\mathcal{H} = \sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}[z^m]\mathcal{S}(z)^{-n-1}\mathcal{E}_{-n}(z)$$
$$\mathcal{H}^{-1}\mathcal{K}_{-n}\mathcal{H} = \sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}[z^m]\mathcal{S}(z)^{-n-1}\frac{\partial}{\partial z}\mathcal{E}_{-n}(z),$$

and

$$\mathcal{H}^{-1}\alpha_n\mathcal{H} = \sum_{m=0}^{\infty}\frac{(n+m-1)!\,(-\hbar)^m}{(n-m)!}[z^m]\mathcal{S}(z)^{n-1}\mathcal{E}_n(z)$$
$$\mathcal{H}^{-1}\mathcal{K}_n\mathcal{H} = \sum_{m=0}^{\infty}\frac{(n+m-1)!\,(-\hbar)^m}{(n-m)!}[z^m]\mathcal{S}(z)^{n-1}\frac{\partial}{\partial z}\mathcal{E}_n(z).$$

Here, the bosonic operators $\alpha_{\pm k}$ are defined in Definition 1.3.4 the operators $\mathcal{K}_n$ and $\mathcal{E}_n(z)$ are defined in Definition 7.2.5 and Definition 1.3.6 respectively, and $\mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{e^{z/2}-e^{-z/2}}{z}$.

Proof. Begin with (7.27) from Lemma 7.4.4 for $\mathcal{H}^{-1}\alpha_{-n}\mathcal{H}$ and rewrite it in terms of elementary symmetric polynomials. Doing so gives

$$\begin{aligned}\mathcal{H}^{-1}\alpha_{-n}\mathcal{H} &= \sum_{k\in\mathbb{Z}+\frac12}\prod_{i=1}^{n}(1+(k+i-\tfrac12)\hbar) : \psi_{k+n}\psi_k^* : \\ &= \sum_{k\in\mathbb{Z}+\frac12}\sum_{m=0}^{n} e_m(k+\tfrac12, k+\tfrac32,\dots,k+n-\tfrac12)\hbar^m : \psi_{k+n}\psi_k^* : .\end{aligned}$$

Apply the result of Lemma 7.4.5 for elementary symmetric polynomials, followed by the expression for $e_m(x_1,\dots,x_n)$ in terms of Stirling numbers of the first kind (7.30), then the result in Lemma 7.4.6. We obtain

$$\begin{aligned}e_m(k+\tfrac12,k+\tfrac32,\dots,k+n-\tfrac12) &= \sum_{i=1}^{m}\binom{n+i-m}{i}e_{m-i}(1,\dots,n)\Big(k-\frac12\Big)^i \\ &= \sum_{i=1}^{m}\binom{n+i-m}{i}\begin{bmatrix}n+1\\ n+1-m+i\end{bmatrix}\Big(k-\frac12\Big)^i \\ &= \sum_{i=1}^{m}\binom{n+i-m}{i}[y^{m-i}]\frac{n!}{(n-m+i)!}\mathcal{S}(y)^{-n-1}e^{y(n+1)/2}\Big(k-\frac12\Big)^i.\end{aligned}$$

Substitute this result back into the expression for $\mathcal{H}^{-1}\alpha_{-n}\mathcal{H}$ and rewrite in the following way to obtain

$$\begin{aligned}&\mathcal{H}^{-1}\alpha_{-n}\mathcal{H} \\ &= \sum_{k\in\mathbb{Z}+\frac12}\sum_{m=0}^{n}\sum_{i=1}^{m}\binom{n+i-m}{i}[y^{m-i}]\frac{n!}{(n-m+i)!}\mathcal{S}(y)^{-n-1}e^{y(n+1)/2}\Big(k-\frac12\Big)^i\hbar^m : \psi_{k+n}\psi_k^* : \\ &= \sum_{k\in\mathbb{Z}+\frac12}\sum_{m=0}^{n}\sum_{i=1}^{m}\frac{(n+i-m)!}{(n-m)!\,i!}[y^{m-i}]\frac{n!}{(n-m+i)!}\mathcal{S}(y)^{-n-1}e^{y(n+1)/2}[z^i]i!e^{z(k-\frac12)}\hbar^m : \psi_{k+n}\psi_k^* : \\ &= \sum_{k\in\mathbb{Z}+\frac12}\sum_{m=0}^{n}\sum_{i=1}^{m}\frac{n!\,\hbar^m}{(n-m)!}[y^{m-i}]\mathcal{S}(y)^{-n-1}e^{y(n+1)/2}[z^i]e^{z(k-\frac12)} : \psi_{k+n}\psi_k^* : \\ &= \sum_{k\in\mathbb{Z}+\frac12}\sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}[z^m]\mathcal{S}(z)^{-n-1}e^{z(k+\frac{n}{2})} : \psi_{k+n}\psi_k^* : \\ &= \sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}[z^m]\mathcal{S}(z)^{-n-1}\mathcal{E}_{-n}(z),\end{aligned}$$

as announced. The second equality is using the fact that $e^z = \sum_{i\geqslant 0}\frac{z^i}{i!}$, the third equality is merely simplifying, and the fourth equality is using $[z^m]f(z)g(z) = \sum_{i=0}^{m}[z^{m-i}]f(z)\,[z^i]g(z)$.

Beginning with (7.28) and following an identical calculation for $\mathcal{H}^{-1}\mathcal{K}_{-n}\mathcal{H}$ yields

$$\begin{aligned}\mathcal{H}^{-1}\mathcal{K}_{-n}\mathcal{H} &= \sum_{m=0}^{n} \frac{n!\,\hbar^m}{(n-m)!}\,[z^m]\mathcal{S}(z)^{-n-1} \sum_{k\in\mathbb{Z}+\frac{1}{2}} \left(k+\frac{n}{2}\right) e^{z\left(k+\frac{n}{2}\right)} :\psi_{k+n}\psi_k^*: \\ &= \sum_{m=0}^{n} \frac{n!\,\hbar^m}{(n-m)!}\,[z^m]\mathcal{S}(z)^{-n-1}\frac{\partial}{\partial z}\mathcal{E}_{-n}(z),\end{aligned}$$

as required. For $\mathcal{H}^{-1}\alpha_n\mathcal{H}$, we follow a very similar process as with $\mathcal{H}^{-1}\alpha_{-n}\mathcal{H}$. Expressing equation (7.25) in terms of the homogeneous symmetric functions gives

$$\begin{aligned}\mathcal{H}^{-1}\alpha_n\mathcal{H} &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \prod_{i=1}^{n} \frac{1}{1+(k-i+\frac{1}{2})\hbar} :\psi_{k-n}\psi_k^*: \\ &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{m\geqslant 0} h_m(k-\tfrac{1}{2}, k-\tfrac{3}{2},\dots,k-n+\tfrac{1}{2})(-\hbar)^m :\psi_{k-n}\psi_k^*: \\ &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{m\geqslant 0} h_m(k+\tfrac{1}{2}, k+\tfrac{3}{2},\dots,k+n-\tfrac{1}{2})(-\hbar)^m :\psi_k\psi_{k+n}^*:,\end{aligned}$$

where the last line has applied the relabelling $k\mapsto k+n$. Again, apply the result of Lemma 7.4.5 along with equation (7.30), this time in the case of homogeneous symmetric functions. This gives

$$\begin{aligned}h_m(k+\tfrac{1}{2}, k+\tfrac{3}{2},\dots,k+n-\tfrac{1}{2}) &= \sum_{i=0}^{m} \binom{m+n-1}{i} h_{m-i}(0,1,\dots,n-1)\left(k+\frac{1}{2}\right)^i \\ &= \sum_{i=0}^{m} \binom{m+n-1}{i} \left\{ {n-1+m-i \atop n-1} \right\} \left(k+\frac{1}{2}\right)^i \\ &= \sum_{i=0}^{m} \frac{(m+n-1)!}{i!\,(n-1)!}\,[y^{m-i}]\mathcal{S}(y)^{n-1}e^{y(n-1)/2}\left(k+\frac{1}{2}\right)^i.\end{aligned}$$

Reinserting this expression into $\mathcal{H}^{-1}\alpha_n\mathcal{H}$ and utilising the same tricks as in the case of $\mathcal{H}^{-1}\alpha_{-n}\mathcal{H}$ above yields

$$\begin{aligned}&\mathcal{H}^{-1}\alpha_n\mathcal{H} \\ &\quad= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{m\geqslant 0} \sum_{i=0}^{m} \frac{(m+n-1)!}{i!\,(n-1)!}\,[y^{m-i}]\mathcal{S}(y)^{n-1}e^{y(n-1)/2}\left(k+\frac{1}{2}\right)^i (-\hbar)^m :\psi_k\psi_{k+n}^*: \\ &\quad= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{m\geqslant 0} \sum_{i=0}^{m} \frac{(m+n-1)!}{(n-1)!}\,[y^{m-i}]\mathcal{S}(y)^{n-1}e^{y(n-1)/2}\,[z^i]e^{z\left(k+\frac{1}{2}\right)}\,(-\hbar)^m :\psi_k\psi_{k+n}^*: \\ &\quad= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{m\geqslant 0} \frac{(m+n-1)!\,(-\hbar)^m}{(n-1)!}\,[z^m]\mathcal{S}(z)^{n-1}e^{z\left(k+\frac{n}{2}\right)} :\psi_k\psi_{k+n}^*: \\ &\quad= \sum_{m\geqslant 0} \frac{(m+n-1)!\,(-\hbar)^m}{(n-1)!}\,[z^m]\mathcal{S}(z)^{n-1}\mathcal{E}_n(z).\end{aligned}$$

The result for $\mathcal{H}^{-1}\mathcal{K}_n\mathcal{H}$ is now immediate. ∎

7.4.3 Virasoro constraints for fully simple maps

I am now equipped to derive a sequence of Virasoro operators for fully simple maps. The overarching idea to obtain said operators is by conjugating the Virasoro operators for ordinary maps by the operator $\mathcal{H}$. That is, the Virasoro operators for fully simple maps will be obtained by conjugating V_n^M by $\mathcal{H}$. First, define

$$\hat{\mathcal{E}}_n(z) := \frac{1}{\varsigma(z)} \sum_{\substack{\ell,s\geqslant 0\\ q\geqslant 0}} \frac{1}{\ell!\,s!}\left[\sum_{\substack{i_1+\cdots+i_s=q\\ i_k\geqslant 1}} \prod_{k=1}^{s} \frac{\varsigma(i_k z)}{i_k} p_{i_k}\right]\left[\sum_{\substack{j_1+\cdots+j_\ell=q+n\\ j_k\geqslant 1}} \prod_{k=1}^{\ell} \varsigma(j_k z)\frac{\partial}{\partial p_{j_k}}\right]. \tag{7.32}$$

This operator is the counterpart to $\mathcal{E}_n(z)$ via Lemma 7.2.4. Note that, while the differential counterparts to calligraphic operators acting on the semi-infinite wedge are usually denoted by straight letters throughout this chapter, I adopt an alternate notation here to forsake any possible ambiguity.

Theorem 7.4.8. *The differential operators*

$$V_n^F = \hbar \sum_{m=0}^{n} \frac{n!\,\hbar^m}{(n-m)!} [z^m]\mathcal{S}(z)^{-n-1} \frac{\partial}{\partial z}\hat{\mathcal{E}}_{-n}(z) + \sum_{i\geqslant 1} t_i \sum_{m=0}^{n+i} \frac{(n+i)!\,\hbar^m}{(n+i-m)!} [z^m]\mathcal{S}(z)^{-n-i-1}\hat{\mathcal{E}}_{-n-i}(z)$$
$$+ \frac{\hbar}{2} \sum_{i+j=n} \left(\sum_{m=0}^{i} \frac{i!\,\hbar^m}{(i-m)!} [z^m]\mathcal{S}(z)^{-i-1}\hat{\mathcal{E}}_{-i}(z) \right) \left(\sum_{m=0}^{j} \frac{j!\,\hbar^m}{(j-m)!} [z^m]\mathcal{S}(z)^{-j-1}\hat{\mathcal{E}}_{-j}(z) \right)$$
$$+ 2 \sum_{m=0}^{n} \frac{n!\,\hbar^m}{(n-m)!} [z^m]\mathcal{S}(z)^{-n-1}\hat{\mathcal{E}}_{-n}(z) - \frac{1}{s} \sum_{m=0}^{n+2} \frac{(n+2)!\,\hbar^m}{(n+2-m)!} [z^m]\mathcal{S}(z)^{-n-3}\hat{\mathcal{E}}_{-n-2}(z) \quad (7.33)$$

for $n \geqslant 1$ *and*

$$V_{-1}^F = \hbar \sum_{m\geqslant 0} \frac{m!\,(-\hbar)^m}{(1-m)!} [z^m] \frac{\partial}{\partial z}\hat{\mathcal{E}}_1(z) + \sum_{i\geqslant 2} t_i \sum_{m=0}^{i-1} \frac{(i-1)!\,\hbar^m}{(i-1-m)!} [z^m]\mathcal{S}(z)^{-i}\hat{\mathcal{E}}_{1-i}(z)$$
$$+ \sum_{m\geqslant 0} \frac{m!\,(-\hbar)^m}{(1-m)!} [z^m]\hat{\mathcal{E}}_1(z) + \frac{t_1}{\hbar} - \frac{1}{s} \sum_{m=0}^{1} \frac{\hbar^m}{(1-m)!} [z^m]\mathcal{S}(z)^{-2}\hat{\mathcal{E}}_{-1}(z) \quad (7.34)$$
$$V_0^F = \hbar\,[z]\hat{\mathcal{E}}_0(z) + \frac{\hbar}{24} + \sum_{i\geqslant 1} t_i \sum_{m=0}^{i} \frac{i!\,\hbar^m}{(i-m)!} [z^m]\mathcal{S}(z)^{-i-1}\hat{\mathcal{E}}_{-i}(z) + \frac{1}{\hbar} - \frac{1}{s} \sum_{m=0}^{2} \frac{2!\,\hbar^m}{(2-m)!} [z^m]\mathcal{S}(z)^{-3}\hat{\mathcal{E}}_{-2}(z) \quad (7.35)$$

annihilate the partition function for fully simple maps and satisfy the Virasoro constraints; that is, $V_n^F Z^F = 0$ *for all* $n \geqslant -1$*, and* $[V_n^F, V_m^F] = (n-m)V_{n+m}^F$*. Here,* $\hat{\mathcal{E}}$ *is defined in equation (7.32), while* $\mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{e^{z/2}-e^{-z/2}}{z}$.

Proof. The ultimate aim will be to derive the Virasoro operators for fully simple maps, V_n^F, by a conjugation for the Virasoro operators for ordinary maps, V_n^M, using the semi-infinite wedge as the medium. That is, I will define $\mathcal{V}_n^F := \mathcal{H}^{-1}\mathcal{V}_n^M\mathcal{H}$ via the semi-infinite wedge and prove that the corresponding differential operator V_n^F, obtained by applying Lemma 7.2.4 to $\mathcal{V}_n^F$, annihilates the partition function.

As an aside, it is possible to follow this process entirely without the semi-infinite wedge formalism, but in the world of differential operators acting on the partition function for fully simple maps. The semi-infinite wedge formalism simply provides a particularly nice language for deriving the conjugation $\mathcal{H}^{-1}\mathcal{V}_n^M\mathcal{H}$.

Begin with the Virasoro operators for ordinary maps, which I restate here for convenience:

$$V_n^M = \sum_{i\geqslant 1}(i+n)(\hbar p_i + t_i)\frac{\partial}{\partial p_{i+n}} + \hbar \sum_{i+j=n} ij \frac{\partial^2}{\partial p_i \partial p_j} + 2n\frac{\partial}{\partial p_n} + \delta_{n,-1}\left(p_1 + \frac{t_1}{\hbar}\right) + \frac{1}{\hbar}\delta_{n,0} - \frac{1}{s}(n+2)\frac{\partial}{\partial p_{n+2}}.$$

Use the fact that $V_n^M Z^M = 0$ for all $n \geqslant -1$ as well as Lemma 7.2.4 to write the following:

$$0 = HV_n^M Z^M = HV_n^M \left\langle \exp(s\mathcal{G}) \exp\left(\sum_{m\geqslant 1} \frac{t_m}{m\,\hbar}\alpha_{-m}\right) \exp\left(\sum_{m\geqslant 1} \frac{p_m}{m}\alpha_{-m}\right) \right\rangle$$
$$= H\left\langle \exp(s\mathcal{G}) \exp\left(\sum_{m\geqslant 1} \frac{t_m}{m\,\hbar}\alpha_{-m}\right) \mathcal{V}_n^M \exp\left(\sum_{m\geqslant 1} \frac{p_m}{m}\alpha_{-m}\right) \right\rangle$$
$$= \left\langle \exp(s\mathcal{G}) \exp\left(\sum_{m\geqslant 1} \frac{t_m}{m\,\hbar}\alpha_{-m}\right) \mathcal{V}_n^M \mathcal{H} \exp\left(\sum_{m\geqslant 1} \frac{p_m}{m}\alpha_{-m}\right) \right\rangle$$
$$= \left\langle \exp(s\mathcal{G}) \exp\left(\sum_{m\geqslant 1} \frac{t_m}{m\,\hbar}\alpha_{-m}\right) \mathcal{H}\mathcal{B} \exp\left(\sum_{m\geqslant 1} \frac{p_m}{m}\alpha_{-m}\right) \right\rangle,$$

where $\mathcal{B} = \mathcal{H}^{-1}\mathcal{V}_n^M\mathcal{H}$. Here, H is the differential operator that is the counterpart to $\mathcal{H}$ via Lemma 7.2.4. Such a differential operator H exists because it can be written in terms of bosonic operators [2]. More precisely, Alexandrov, Lewański, and Shadrin [2] show that $\mathcal{H}$ can also be written as

$$\mathcal{H} = \exp\left(\left[\frac{\tilde{\mathcal{E}}_0\left(-\hbar^2\frac{\mathrm{d}}{\mathrm{d}\hbar}\right)}{\varsigma\left(-\hbar^2\frac{\mathrm{d}}{\mathrm{d}\hbar}\right)} - \mathcal{F}_1\right]\log\hbar\right),$$

where $\varsigma(z) = e^{z/2} - e^{-z/2}$ as usual, $\mathcal{F}_1$ is the diagonal operator $\mathcal{F}_1 = [z]\mathcal{E}_0(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} k : \psi_k\psi_k^* :$, and $\tilde{\mathcal{E}}_0(z)$ is the $\mathcal{E}$-operator in energy zero without the usual correction term; that is,

$$\tilde{\mathcal{E}}_0(z) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} e^{zk} : \psi_k\psi_k^* : .$$

Equation (1.20) writes $\mathcal{E}$ in terms of bosonic operators, hence it is also possible to do so for $\tilde{\mathcal{E}}_0$ and $\mathcal{F}_1$ (the latter because $\mathcal{F}_1$ is the z-coefficient of $\mathcal{E}_0(z)$).

Thus it follows that one can apply Lemma 7.2.4 to $\mathcal{H}$ and obtain a corresponding differential operator H. (It is not problematic that such an H will inevitably be unwieldy, for the purposes of this chapter, it only matters that such an operator exists.)

Thus, define $\mathcal{V}_n^F := \mathcal{H}^{-1}\mathcal{V}_n^M\mathcal{H}$. It follows that

$$\left\langle \exp(s\mathcal{G})\exp\left(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\right)\mathcal{H}\mathcal{V}_n^F\exp\left(\sum_{m\geqslant 1}\frac{p_m}{m}\alpha_{-m}\right)\right\rangle = 0$$

for all $n \geqslant -1$. This in turn yields that $V_n^F Z^F = 0$ where V_n^F is the differential operator given by applying Lemma 7.2.4 to $\mathcal{V}_n^F$. That V_n^F satisfies the Virasoro constraints is given immediately by the fact that $\mathcal{V}_n^F$ is defined to be the conjugation of an operator that satisfies the Virasoro constraints, and hence the same is true of V_n^F; this is easily verified as follows. Begin with the Virasoro constraints for ordinary maps,

$$(n-m)V_{n+m}^M = [V_n^M, V_m^M] = V_n^M V_m^M - V_m^M V_n^M,$$

and apply H on the left and H^{-1} on the right. This gives

$$\begin{aligned}(n-m)V_{n+m}^F &= (n-m)HV_{n+m}^M H^{-1} = HV_n^M V_m^M H^{-1} - HV_m^M V_n^M H^{-1}\\ &= \left(HV_n^M H^{-1}\right)\left(HV_m^M H^{-1}\right) - \left(HV_m^M H^{-1}\right)\left(HV_n^M H^{-1}\right) = V_n^F V_m^F - V_m^F V_n^F,\end{aligned}$$

hence, $(n-m)V_{n+m}^F = [V_n^F, V_m^F]$ as required.

It remains to conjugate $\mathcal{V}_n^M$ by $\mathcal{H}$ and hence derive expression for $\mathcal{V}_n^F$ and V_n^F for all $n \geqslant -1$.

The operator $\mathcal{V}_n^M$ obtained from applying Lemma 7.2.4 to V_n^M is given by the following expression in terms of bosonic operators:

$$\mathcal{V}_n^M = \sum_{i\geqslant 1}(\hbar\alpha_{-i-n}\alpha_i + t_i\alpha_{-i-n}) + \hbar\sum_{i+j=n}\alpha_{-i}\alpha_{-j} + 2\alpha_{-n} + \delta_{n,-1}\left(\alpha_1 + \frac{t_1}{\hbar}\right) + \frac{1}{\hbar}\delta_{n,0} - \frac{1}{s}\alpha_{-n-2},$$

using the convention that $\alpha_0 = 0$. Note the slight abuse of notation: by Lemma 7.2.4, $m\frac{\partial}{\partial p_m}$ only corresponds to α_{-n} for n positive, hence the term $2\alpha_{-n}$ in $\mathcal{V}_n^M$ is only present for n positive.

Using the fact that

$$\mathcal{K}_{-n} = \frac{1}{2}\sum_{i+j=-n} :\alpha_i\alpha_j: = \sum_{i\geqslant 1}\alpha_{-n-i}\alpha_i + \frac{1}{2}\sum_{i+j=n}\alpha_{-i}\alpha_{-j},$$

we can rewrite $\mathcal{V}_n^M$ as

$$\mathcal{V}_n^M = \hbar\mathcal{K}_{-n} + \frac{\hbar}{2}\sum_{i+j=n}\alpha_{-i}\alpha_{-j} + \sum_{i\geqslant 1}t_i\alpha_{-i-n} + 2\alpha_{-n} + \delta_{n,-1}\left(\alpha_1 + \frac{t_1}{\hbar}\right) + \frac{1}{\hbar}\delta_{n,0} - \frac{1}{s}\alpha_{-n-2}.$$

I will now conjugate $\mathcal{V}_n^M$ by $\mathcal{H}$. To do this, I consider the cases of $n=-1$, $n=0$ and $n\geqslant 1$ separately. First in the case of $n=-1$ I obtain

$$\begin{aligned}\mathcal{V}_{-1}^F &:= \mathcal{H}^{-1}\mathcal{V}_{-1}^M\mathcal{H} = \hbar\,\mathcal{H}^{-1}\mathcal{K}_1\mathcal{H} + \sum_{i\geqslant 2} t_i\mathcal{H}^{-1}\alpha_{-i+1}\mathcal{H} + \mathcal{H}^{-1}\alpha_1\mathcal{H} + \frac{t_1}{\hbar} - \frac{1}{s}\mathcal{H}^{-1}\alpha_{-1}\mathcal{H}\\ &= \hbar\sum_{m\geqslant 0}\frac{m!\,(-\hbar)^m}{(1-m)!}[z^m]\frac{\partial}{\partial z}\mathcal{E}_1(z) + \sum_{i\geqslant 2} t_i\sum_{m=0}^{i-1}\frac{(i-1)!\,\hbar^m}{(i-1-m)!}[z^m]\mathcal{S}(z)^{-i}\mathcal{E}_{1-i}(z)\\ &\quad+\sum_{m\geqslant 0}\frac{m!\,(-\hbar)^m}{(1-m)!}[z^m]\mathcal{E}_1(z) + \frac{t_1}{\hbar} - \frac{1}{s}\sum_{m=0}^{1}\frac{\hbar^m}{(1-m)!}[z^m]\mathcal{S}(z)^{-2}\mathcal{E}_{-1}(z).\end{aligned}$$

The second equality is using the fact that the operator $\mathcal{H}$ commutes with constants. In the case of $n=0$ we have

$$\begin{aligned}\mathcal{V}_0^F &:= \mathcal{H}^{-1}\mathcal{V}_0^M\mathcal{H} = \hbar\,\mathcal{H}^{-1}\mathcal{K}_0\mathcal{H} + \sum_{i\geqslant 1} t_i\mathcal{H}^{-1}\alpha_{-i}\mathcal{H} + \frac{1}{\hbar} - \frac{1}{s}\mathcal{H}^{-1}\alpha_{-2}\mathcal{H}\\ &= \hbar\,[z]\mathcal{E}_0(z) + \frac{\hbar}{24} + \sum_{i\geqslant 1} t_i\sum_{m=0}^{i}\frac{i!\,\hbar^m}{(i-m)!}[z^m]\mathcal{S}(z)^{-i-1}\mathcal{E}_{-i}(z) + \frac{1}{\hbar} - \frac{1}{s}\sum_{m=0}^{2}\frac{2!\,\hbar^m}{(2-m)!}[z^m]\mathcal{S}(z)^{-3}\mathcal{E}_{-2}(z).\end{aligned}$$

Here, the last equality is using the fact that $\mathcal{K}_0$ is diagonal and the operator $\mathcal{H}$ commutes with diagonal operators (as well as constants), along with equation (7.10),

$$\mathcal{K}_0 = [z]\,\mathcal{E}_0(z) + \frac{1}{24}.$$

The operator $\mathcal{H}$ commutes with diagonal operators because $\mathcal{H}$ itself is diagonal. Finally, for $n\geqslant 1$, we define $\mathcal{V}_n^F$ to be

$$\begin{aligned}\mathcal{V}_n^F &:= \mathcal{H}^{-1}\mathcal{V}_n^M\mathcal{H}\\ &= \hbar\,\mathcal{H}^{-1}\mathcal{K}_{-n}\mathcal{H} + \frac{\hbar}{2}\sum_{i+j=n}\mathcal{H}^{-1}\alpha_{-i}\alpha_{-j}\mathcal{H} + \sum_{i\geqslant 1} t_i\mathcal{H}^{-1}\alpha_{-i-n}\mathcal{H} + 2\mathcal{H}^{-1}\alpha_{-n}\mathcal{H} - \frac{1}{s}\mathcal{H}^{-1}\alpha_{-n-2}\mathcal{H}\\ &= \hbar\sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}[z^m]\mathcal{S}(z)^{-n-1}\frac{\partial}{\partial z}\mathcal{E}_{-n}(z) + \sum_{i\geqslant 1} t_i\sum_{m=0}^{n+i}\frac{(n+i)!\,\hbar^m}{(n+i-m)!}[z^m]\mathcal{S}(z)^{-n-i-1}\mathcal{E}_{-n-i}(z)\\ &\quad+\frac{\hbar}{2}\sum_{i+j=n}\left(\sum_{m=0}^{i}\frac{i!\,\hbar^m}{(i-m)!}[z^m]\mathcal{S}(z)^{-i-1}\mathcal{E}_{-i}(z)\right)\left(\sum_{m=0}^{j}\frac{j!\,\hbar^m}{(j-m)!}[z^m]\mathcal{S}(z)^{-j-1}\mathcal{E}_{-j}(z)\right)\\ &\quad+2\sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}[z^m]\mathcal{S}(z)^{-n-1}\mathcal{E}_{-n}(z) - \frac{1}{s}\sum_{m=0}^{n+2}\frac{(n+2)!\,\hbar^m}{(n+2-m)!}[z^m]\mathcal{S}(z)^{-n-3}\mathcal{E}_{-n-2}(z).\end{aligned}$$

The third equality is using the fact that $\mathcal{H}^{-1}\alpha_{-i}\alpha_{-j}\mathcal{H} = (\mathcal{H}^{-1}\alpha_{-i}\mathcal{H})(\mathcal{H}^{-1}\alpha_{-j}\mathcal{H})$. Recall the bosonic operator form of the $\mathcal{E}$-operator:

$$\begin{aligned}\mathcal{E}_n(z) &= \frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{i_1+\cdots+i_\ell=n}\frac{\varsigma(i_1z)\cdots\varsigma(i_\ell z)}{i_1\cdots i_\ell}:\alpha_{i_1}\cdots\alpha_{i_\ell}:\\ &= \frac{1}{\varsigma(z)}\sum_{\substack{\ell,s\geqslant 0\\ q\geqslant 0}}\frac{1}{\ell!\,s!}\left[\sum_{\substack{j_1+\cdots+j_\ell=q\\ j_k\geqslant 1}}\prod_{k=1}^{\ell}\frac{\varsigma(j_kz)}{j_k}\alpha_{-j_k}\right]\left[\sum_{\substack{i_1+\cdots+i_s=q+n\\ i_k\geqslant 1}}\prod_{k=1}^{s}\frac{\varsigma(i_kz)}{i_k}\alpha_{i_k}\right].\end{aligned}$$

Here the second equality has implemented the normal ordering. Recalling the definition for the $\hat{\mathcal{E}}$-operator given by equation (7.32), the statement of the theorem now follows. ■

7.5 Recursions for fully simple maps

7.5.1 A Tutte-like recursion for fully simple maps

One can use the Virasoro operators for fully simple maps to derive a Tutte-like recursion; this is done by following the reverse process to what was done in the proof of Theorem 7.4.2 to derive the Tutte recursion for ordinary maps. Explicitly, to derive a Tutte-like recursion for (connected) fully simple maps in full generality for all (g, n), one applies $\frac{1}{Z^F} p_{n+2} V_n$ to Z^F and extracts the coefficient of $p_{\mu_1} \cdots p_{\mu_n} \hbar^{2g-2+n}$. The role of dividing by Z^F after applying V_n is a trick that ensures the resulting recursion is in terms of the connected enumeration. To see this, define the *free energy* for fully simple maps, a generating function for the connected enumeration,

$$F(\vec{p}; \hbar) := \sum_{n \geqslant 1} \sum_{g \geqslant 0} \sum_{\mu_1, \ldots, \mu_n \geqslant 1} \mathrm{FS}^{\circ}_{g,n}(\mu_1, \ldots, \mu_n) \frac{p_{\mu_1} p_{\mu_2} \cdots p_{\mu_n} \hbar^{2g-2+n}}{n!}.$$

Then observe

$$\frac{1}{Z^F} \frac{\partial}{\partial x} Z^F = \frac{1}{Z^F} \frac{\partial}{\partial x} \exp F = \frac{1}{Z^F} \frac{\partial F}{\partial x} \exp F = \frac{\partial F}{\partial x},$$

hence, the result on the right-most side is in terms of connected fully simple maps only.

In this section, I derive a Tutte-like recursion for fully simple maps for some special cases of low (g, n); namely, $(g, n) = (0, 1)$, $(0, 2)$, and $(1, 1)$.

Ideally one would like to be able to do this process in full generality. However, the Virasoro operator V_n^F for fully simple maps involves extracting coefficients from the $\mathcal{S}(z)$ and $\varsigma(z)$ functions: $[z^a] \mathcal{S}(z)^{-n-1}$ and $[z^b] \frac{1}{\varsigma(z)} \prod_{k=1}^{m} \varsigma(j_k z)$. For low (g, n) this process is highly constrained (as we will see) but in full generality this is not the case and the contributions of these terms become intractable.

The disk case: $(g, n) = (0, 1)$

My first aim is to find a Tutte-like recursion for fully simple maps in the case of $(g, n) = (0, 1)$. This is done by applying $\frac{1}{Z^F} p_{n+2} V_n$ to Z^F and extracting the coefficient of $p_{n+2} \hbar^{-1}$. In this case (and actually for all $n = 1$),

$$[p_{n+2} \hbar^{-1}] \frac{1}{Z^F} p_{n+2} V_n Z^F = [\hbar^{-1}] \frac{1}{Z^F} V_n Z^F$$

and hence it follows that we would like to extract terms from $\frac{1}{Z^F} V_n Z^F$ that are constant in the p-variables. This is equivalent to disregarding any terms of V_n that include an application of p, or applying $\frac{1}{Z^F} V_n Z^F$ then setting $\vec{p} = \vec{0}$.

This leads to the following Tutte-like recursion for fully simple maps in the disk case; that is, when $(g, n) = (0, 1)$. This recursion has been used to reproduce the data calculated by Garcia-Failde [58, Section 2.2.2].

Proposition 7.5.1. *The enumeration of fully simple maps in the disk case satisfies the following recursion for all $n \geqslant 1$:*

$$2\hat{A}_{0,1}(n) + 2A_{0,1}(n) = \frac{1}{s} A_{0,1}(n+2) - \sum_{i \geqslant 1} t_i \, A_{0,1}(n+i), \tag{7.36}$$

where

$$A_{0,1}(N) = \sum_{m=0}^{N} \frac{N!}{(N-m)!(m+1)!} \sum_{j_1 + \cdots + j_{m+1} = N} \prod_{k=1}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)$$

$$\hat{A}_{0,1}(N) = \sum_{m=0}^{N} \frac{N! \, (m+1)}{(N-m)!(m+2)!} \sum_{j_1 + \cdots + j_{m+2} = N} \prod_{k=1}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k).$$

Proof. First recall V_n^F for convenience.

$$\begin{aligned} V_n^F = \hbar \sum_{m=0}^{n} \frac{n!\,\hbar^m}{(n-m)!}\,[z^m]\mathcal{S}(z)^{-n-1}\frac{\partial}{\partial z}\hat{\mathcal{E}}_{-n}(z) + \sum_{i\geqslant 1} t_i \sum_{m=0}^{n+i}\frac{(n+i)!\,\hbar^m}{(n+i-m)!}\,[z^m]\mathcal{S}(z)^{-n-i-1}\hat{\mathcal{E}}_{-n-i}(z) \\ + \frac{\hbar}{2}\sum_{i+j=n}\left(\sum_{m=0}^{i}\frac{i!\,\hbar^m}{(i-m)!}\,[z^m]\mathcal{S}(z)^{-i-1}\hat{\mathcal{E}}_{-i}(z)\right)\left(\sum_{m=0}^{j}\frac{j!\,\hbar^m}{(j-m)!}\,[z^m]\mathcal{S}(z)^{-j-1}\hat{\mathcal{E}}_{-j}(z)\right) \\ + 2\sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}\,[z^m]\mathcal{S}(z)^{-n-1}\hat{\mathcal{E}}_{-n}(z) - \frac{1}{s}\sum_{m=0}^{n+2}\frac{(n+2)!\,\hbar^m}{(n+2-m)!}\,[z^m]\mathcal{S}(z)^{-n-3}\hat{\mathcal{E}}_{-n-2}(z) \end{aligned} \tag{7.37}$$

Let us begin with the first term on the third line. That is, let me consider extracting the coefficient of $\hbar^{-1}$ from the application of this term applied to Z^F:

$$[\hbar^{-1}]\frac{1}{Z^F}\sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}\,[z^m]\mathcal{S}(z)^{-n-1}\hat{\mathcal{E}}_{-n}(z)Z^F.$$

Let me also restate the definition of $\hat{\mathcal{E}}_{-n}(z)$:

$$\hat{\mathcal{E}}_n(z) := \frac{1}{\varsigma(z)}\sum_{\substack{\ell,s\geqslant 0\\ q\geqslant 0}}\frac{1}{\ell!\,s!}\left[\sum_{\substack{i_1+\cdots+i_s=q\\ i_k\geqslant 1}}\prod_{k=1}^{s}\frac{\varsigma(i_k z)}{i_k}p_{i_k}\right]\left[\sum_{\substack{j_1+\cdots+j_\ell=q+n\\ j_k\geqslant 1}}\prod_{k=1}^{\ell}\varsigma(j_k z)\frac{\partial}{\partial p_{j_k}}\right].$$

Recalling that because $n=1$, after the application of the operator to Z^F we are forced to extract the terms that are constant in the p-variables. Therefore, we can consider only the terms in $\hat{\mathcal{E}}_n(z)$ where $s=0$. That is,

$$\begin{aligned} &[\hbar^{-1}]\frac{1}{Z^F}\sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}\,[z^m]\mathcal{S}(z)^{-n-1}\hat{\mathcal{E}}_{-n}(z)Z^F \\ &= \sum_{m=0}^{n}\frac{n!}{(n-m)!}\sum_{a+b=m}[z^a]\mathcal{S}(z)^{-n-1}[z^b\,\hbar^{-m-1}]\frac{1}{Z^F}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\left[\sum_{\substack{j_1+\cdots+j_\ell=n\\ j_k\geqslant 1}}\prod_{k=1}^{\ell}\varsigma(j_k z)\frac{\partial}{\partial p_{j_k}}\right]Z^F. \end{aligned} \tag{$*$}$$

From here it follows that the contribution of this term is

$$\sum_{m=0}^{n}\frac{n!}{(n-m)!\,(m+1)!}\sum_{j_1+\cdots+j_{m+1}=n}\prod_{k=1}^{m+1}j_k\mathrm{FS}^\circ_{0,1}(j_k). \tag{$\triangle$}$$

This is not a trivial statement, and one can reason this in the following way. We are extracting a negative power of $\hbar$ from $\frac{1}{Z^F}\hat{\mathcal{E}}_n(z)Z^F$ that is constant in the p-variables, and the only terms that carry a negative power of $\hbar$ are $(0,1)$-terms where $\hbar^{2g-2+n}=\hbar^{-1}$. Hence, in the mth summand of the m-sum, we need to apply at least $m+1$ derivatives for the term to contribute:

$$[\hbar^{-m-1}]\frac{1}{Z^F}\prod_{k=1}^{m+1}\frac{\partial}{\partial p_{j_k}}Z^F = [\hbar^{-m-1}]\frac{1}{Z^F}\prod_{k=1}^{m+1}\mathrm{FS}^\circ_{0,1}(j_k)\hbar^{-1}Z^F = \prod_{k=1}^{m+1}\mathrm{FS}^\circ_{0,1}(j_k).$$

Thus, $\ell\geqslant m+1$. Yet it is also true that we can apply at most $m+1$ derivatives. This is because of the ς-functions. The expansions of ς and $1/\varsigma$ begin

$$\frac{1}{\varsigma(z)} = \frac{1}{z} - \frac{z}{24} + O(z^3) \qquad \text{and} \qquad \varsigma(z) = z + \frac{z^3}{24} + O(z^5),$$

hence each copy of ς in the inner product over k, $\prod_{k=1}^{\ell}\varsigma(j_k z)$, contributes at least one power of z, while from the $\frac{1}{\varsigma(z)}$ in front of the ℓ-sum we can collect either one negative power of z or odd positive powers of z. From this latter half of the expression,

$$[z^b\,\hbar^{-m-1}]\frac{1}{Z^F}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\left[\sum_{\substack{j_1+\cdots+j_\ell=q\\ j_k\geqslant 1}}\prod_{k=1}^{\ell}\varsigma(j_k z)\frac{\partial}{\partial p_{j_k}}\right]Z^F,$$

I am extracting the coefficient of z^b for $0 \leqslant b \leqslant m$ and this therefore forces $m \geqslant b \geqslant \ell - 1$, or equivalently, $\ell \leqslant m+1$. Recalling that I also had the constraint $\ell \geqslant m+1$, one can conclude that $\ell = m+1$, $b = m$ and $a = 0$. Applying these constraints to the original expression $(*)$ results in the contribution given by $\triangle$, as announced.

Returning to the Virasoro operator for fully simple maps (7.37), one can use an identical argument for the second terms on the first and third lines to deduce the following contributions respectively:

$$\sum_{i \geqslant 1} t_i \sum_{m=0}^{n+i} \frac{(n+i)!}{(n+i-m)!\,(m+1)!} \sum_{j_1+\cdots+j_{m+1}=n+i} \prod_{k=1}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k), \qquad (\triangle)$$

$$-\frac{1}{s} \sum_{m=0}^{n+2} \frac{(n+2)!}{(n+2-m)!\,(m+1)!} \sum_{j_1+\cdots+j_{m+1}=n+2} \prod_{k=1}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k). \qquad (\triangle)$$

For the first term on the first line—the $\frac{\partial}{\partial z}\hat{\mathcal{E}}_n(z)$ term—I apply a similar argument but also utilise the trick that, as operators, $[z^a]\frac{\partial}{\partial z} = (a+1)[z^{a+1}]$. This gives

$$\begin{aligned}
&[\hbar^{-1}] \frac{1}{Z^F} \sum_{m=0}^{n} \frac{n!\,\hbar^{m+1}}{(n-m)!} [z^m] \mathcal{S}(z)^{-n-1} \frac{\partial}{\partial z} \hat{\mathcal{E}}_{-n}(z) Z^F \\
&= \sum_{m=0}^{n} \frac{n!}{(n-m)!} \sum_{a+b=m} [z^a]\mathcal{S}(z)^{-n-1} [z^b\,\hbar^{-m-2}] \frac{1}{Z^F} \frac{\partial}{\partial z} \frac{1}{\varsigma(z)} \sum_{\ell \geqslant 0} \frac{1}{\ell!} \Bigg[\sum_{\substack{j_1+\cdots+j_\ell=n \\ j_k \geqslant 1}} \prod_{k=1}^{\ell} \varsigma(j_k z) \frac{\partial}{\partial p_{j_k}} \Bigg] Z^F \\
&= \sum_{m=0}^{n} \frac{n!}{(n-m)!} [\hbar^{-m-2}] \frac{1}{Z^F} (m+1) [z^{m+1}] \frac{1}{\varsigma(z)} \sum_{\ell \geqslant 0} \frac{1}{\ell!} \Bigg[\sum_{\substack{j_1+\cdots+j_\ell=n \\ j_k \geqslant 1}} \prod_{k=1}^{\ell} \varsigma(j_k z) \frac{\partial}{\partial p_{j_k}} \Bigg] Z^F \\
&= \sum_{m=0}^{n} \frac{n!\,(m+1)}{(n-m)!(m+2)!} \sum_{j_1+\cdots+j_{m+2}=n} \prod_{k=1}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k). \qquad (\triangle)
\end{aligned}$$

The second equality is using the fact that, by similar arguments to the case above, $\ell = m+2$.

It remains to treat the term in V_n^F given by the product of $\hat{\mathcal{E}}$-operators:

$$\frac{\hbar}{2} \sum_{i+j=n} \left(\sum_{m=0}^{i} \frac{i!\,\hbar^m}{(i-m)!} [z^m] \mathcal{S}(z)^{-i-1} \hat{\mathcal{E}}_{-i}(z) \right) \left(\sum_{m=0}^{j} \frac{j!\,\hbar^m}{(j-m)!} [z^m] \mathcal{S}(z)^{-j-1} \hat{\mathcal{E}}_{-j}(z) \right). \qquad (7.38)$$

Apply this operator to the partition function for fully simple maps, divide by Z^F, and extract the coefficient of $\hbar^{-1}$. Again utilise the fact that $n=1$ to discard any term of the $\hat{\mathcal{E}}$-operator that multiplies by some p_i and deduce that the contribution is equal to

$$\begin{aligned}
[\hbar^{-1}] \frac{1}{Z^F} \frac{\hbar}{2} \sum_{i+j=n} &\Bigg[\sum_{m_1=0}^{i} \frac{i!\,\hbar^{m_1}}{(i-m_1)!} [z_1^{m_1}] \mathcal{S}(z_1)^{-i-1} \frac{1}{\varsigma(z_1)} \sum_{\ell_1 \geqslant 0} \frac{1}{\ell_1!} \sum_{i_1+\cdots+i_{\ell_1}=i} \prod_{k=1}^{\ell_1} \varsigma(i_k z_1) \frac{\partial}{\partial p_{i_k}} \Bigg] \\
&\Bigg[\sum_{m_2=0}^{j} \frac{j!\,\hbar^{m_2}}{(i-m_2)!} [z_2^{m_2}] \mathcal{S}(z_2)^{-j-1} \frac{1}{\varsigma(z_2)} \sum_{\ell_2 \geqslant 0} \frac{1}{\ell_2!} \sum_{j_1+\cdots+j_{\ell_2}=j} \prod_{k=1}^{\ell_2} \varsigma(j_k z_2) \frac{\partial}{\partial p_{j_k}} \Bigg] Z^F.
\end{aligned}$$

Note that in contrast to the preceding expression above I have utilised explicitly distinct indexing variables m_1 and m_2 and formal variables z_1 and z_2 for clarity. Again, as previously, for the m_1th summand of the m_1-sum and m_2th summand of the m_2-sum, the overall number of derivatives applied must be at least $m_1 + m_2 + 2$ to yield a term resulting in an overall factor of $\hbar^{-1}$ (after being multiplied by $\hbar^{m_1+m_2+1}$). However, due to the fact that we are extracting the coefficient of $z_1^{m_1}$ from the first bracket and $z_2^{m_2}$ from the second, the maximum number of derivatives that can be applied is $m_1 + 1 + m_2 + 1 = m_1 + m_2 + 2$. Therefore $\ell_1 + \ell_2 = m_1 + m_2 + 2$. Further, because of the series expansions for $\frac{1}{\varsigma(z)}$ and $\varsigma(z)$, it follows that

$m_1 \geqslant \ell_1 - 1$ and $m_2 \geqslant \ell_2 - 1$. Conclude that $\ell_i = m_i + 1$ for $i \in \{1, 2\}$. This collapses the above expression to

$$\frac{1}{2} \sum_{i+j=n} \left[\sum_{m_1=0}^{i} \frac{i!}{(i-m_1)!} \frac{1}{(m_1+1)!} \sum_{i_1+\cdots+i_{m_1+1}=i} \prod_{k=1}^{m_1+1} i_k \mathrm{FS}^{\circ}_{0,1}(i_k) \right] \left[\sum_{m_2=0}^{j} \frac{j!}{(j-m_2)!} \frac{1}{(m_2+1)!} \sum_{j_1+\cdots+j_{m_2+1}=j} \prod_{k=1}^{m_2+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \right].$$

In fact, this expression is equal to the contribution from the $\frac{\partial}{\partial z}\hat{\mathcal{E}}_n(z)$ term; that is,

$$\begin{aligned} \frac{1}{2} \sum_{i+j=n} \left[\sum_{m_1=0}^{i} \frac{i!}{(i-m_1)!} \frac{1}{(m_1+1)!} \sum_{i_1+\cdots+i_{m_1+1}=i} \prod_{k=1}^{m_1+1} i_k \mathrm{FS}^{\circ}_{0,1}(i_k) \right] & \left[\sum_{m_2=0}^{j} \frac{j!}{(j-m_2)!} \frac{1}{(m_2+1)!} \sum_{j_1+\cdots+j_{m_2+1}=j} \prod_{k=1}^{m_2+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \right] \\ &= \sum_{m=0}^{n} \frac{n!\,(m+1)}{(n-m)!\,(m+2)!} \sum_{j_1+\cdots+j_{m+2}=n} \prod_{k=1}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k). \end{aligned} \tag{7.39}$$

This can be argued in the following way. What I'm calling the $\frac{\partial}{\partial z}\hat{\mathcal{E}}_n(z)$ term—the first term on the first line of (7.37)—stems from, in the setting of operators in the semi-infinite wedge, conjugating the $\mathcal{K}_{-n}$ operator; that is,

$$\hbar \sum_{m=0}^{n} \frac{n!\,\hbar^m}{(n-m)!} [z^m] \mathcal{S}(z)^{-n-1} \frac{\partial}{\partial z} \hat{\mathcal{E}}_{-n}(z) = \hbar \mathcal{H}^{-1} \mathcal{K}_{-n} \mathcal{H} = \hbar \mathcal{H}^{-1} \Big(\sum_{i\geqslant 1} \alpha_{-n-i}\alpha_i + \frac{1}{2} \sum_{i+j=n} \alpha_{-i}\alpha_{-j} \Big) \mathcal{H}.$$

In the setting of differential operators acting on a partition function the $\mathcal{K}_{-n}$ operator corresponds to

$$\hbar \sum_{i\geqslant 1} (i+n) p_i \frac{\partial}{\partial p_{i+n}} + \frac{\hbar}{2} \sum_{i+j=n} ij \frac{\partial^2}{\partial p_i \partial p_j}.$$

On the other hand, the product term (7.38) stems from the conjugating the operator $\frac{\hbar}{2}\sum_{i+j=n} \alpha_{-i}\alpha_{-j}$, which in turn also corresponds to $\frac{\hbar}{2}\sum_{i+j=n} ij \frac{\partial^2}{\partial p_i \partial p_j}$ in terms of differential operators. In the case of $n = 1$ and any genus, the term that corresponds to conjugating $\hbar \sum_{i\geqslant 1}(i+n)p_i \frac{\partial}{\partial p_{i+n}}$ is not going to contribute because this term multiplies the resulting expression by p_i; recall that in the case of $n = 1$ we extract terms that are constant in the p-variables. It remains to observe that $\mathcal{H}$ is a diagonal operator and hence the application of multiplying by p_i is unchanged by the conjugation by $\mathcal{H}$. Therefore, when $n = 1$,

$$\begin{aligned} [\hbar^{2g-1}] \frac{1}{Z^F} \frac{\hbar}{2} \sum_{i+j=n} \left(\sum_{m=0}^{i} \frac{i!\,\hbar^m}{(i-m)!} [z^m] \mathcal{S}(z)^{-i-1} \hat{\mathcal{E}}_{-i}(z) \right) & \left(\sum_{m=0}^{j} \frac{j!\,\hbar^m}{(j-m)!} [z^m] \mathcal{S}(z)^{-j-1} \hat{\mathcal{E}}_{-j}(z) \right) Z^F \\ &= [\hbar^{2g-1}] \frac{1}{Z^F} \sum_{m=0}^{n} \frac{n!\,\hbar^{m+1}}{(n-m)!} [z^m] \mathcal{S}(z)^{-n-1} \frac{\partial}{\partial z} \hat{\mathcal{E}}_{-n}(z) Z^F. \end{aligned} \tag{7.40}$$

Collect all terms labelled with a triangle ($\triangle$) to conclude. ■

One might be able to prove the identity in equation (7.39) directly; via a combinatorial bijection or perhaps using generating functions. At the time of writing, such a proof has eluded me.

The cylinder case: $(g, n) = (0, 2)$

Find a Tutte-like recursion for fully simple maps in the case of $(g, n) = (0, 2)$ by applying $\frac{1}{Z^F} p_{n+2} V_n$ and extracting the coefficient of $p_{\mu_1} p_{\mu_2} \hbar^0$. This is equivalent to applying $\frac{1}{Z^F} V_n$ to Z^F and extracting the coefficient of $p_\mu \hbar^0$.

This leads to the following Tutte-like recursion for fully simple maps in the cylinder case; that is, when $(g,n)=(0,2)$. Again, this recursion has been used to reproduce the data calculated by Garcia-Failde [58].

Proposition 7.5.2. *Fix μ to be a positive integer. The enumeration of fully simple maps in the cylinder case satisfies the following recursion for all $n \geqslant 1$:*

$$\hat{A}_{0,2}(n,\mu) + \check{A}_{0,2}(n,\mu) + 2A_{0,2}(n,\mu) = \frac{1}{s} A_{0,2}(n+2,\mu) - \sum_{i\geqslant 1} t_i\, A_{0,2}(n+i,\mu), \tag{7.41}$$

where

$$\begin{aligned} A_{0,2}(N,\mu) = \sum_{m=0}^{n} \frac{N!}{(N-m)!} \Bigg[& \frac{1}{m!} \sum_{\substack{j_1+\cdots+j_m=\mu+N \\ j_k\geqslant 1}} \prod_{k=1}^{m} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \\ & + \frac{1}{(m+1)!} \sum_{\substack{j_1+\cdots+j_{m+1}=N \\ j_k\geqslant 1}} \sum_{q=1}^{m+1} j_q \mathrm{FS}^{\circ}_{0,2}(\mu,j_q) \prod_{\substack{k=1\\k\neq q}}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \Bigg] \\ \hat{A}_{0,2}(N,\mu) = \sum_{m=0}^{N} \frac{N!\,(m+1)}{(N-m)!} \Bigg[& \frac{1}{(m+1)!} \sum_{\substack{j_1+\cdots+j_{m+1}=\mu+N \\ j_k\geqslant 1}} \prod_{k=1}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \\ & + \frac{1}{(m+2)!} \sum_{\substack{j_1+\cdots+j_{m+2}=N \\ j_k\geqslant 1}} \sum_{q=1}^{m+2} j_q \mathrm{FS}^{\circ}_{0,2}(\mu,j_q) \prod_{\substack{k=1\\k\neq q}}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \Bigg], \end{aligned}$$

and

$$\begin{aligned} \check{A}_{0,2}(N,\mu) = \sum_{i+j=N} \Bigg[& \left(\sum_{m_1=0}^{i} \frac{i!}{(i-m_1)!} \frac{1}{m_1!} \sum_{i_1+\cdots+i_{m_1}=i+\mu} \prod_{k=1}^{m_1} i_k \mathrm{FS}^{\circ}_{0,1}(i_k) \right) \\ & \times \left(\sum_{m_2=0}^{j} \frac{j!}{(j-m_2)!} \frac{1}{(m_2+1)!} \sum_{j_1+\cdots+j_{m_2+1}=j} \prod_{k=1}^{m_2+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \right) \\ + & \left(\sum_{m_1=0}^{i} \frac{i!}{(i-m_1)!} \frac{1}{(m_1+1)!} \sum_{i_1+\cdots+i_{m_1+1}=i} \sum_{q=1}^{m_1+1} j_q \mathrm{FS}_{0,2}(\mu,j_q) \prod_{\substack{k=1\\k\neq q}}^{m_1+1} i_k \mathrm{FS}^{\circ}_{0,1}(i_k) \right) \\ & \times \left(\sum_{m_2=0}^{j} \frac{j!}{(j-m_2)!} \frac{1}{(m_2+1)!} \sum_{j_1+\cdots+j_{m_2+1}=j} \prod_{k=1}^{m_2+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \right) \Bigg]. \end{aligned}$$

Proof. Begin with

$$[p_\mu \hbar^0] \frac{1}{Z^F} \sum_{m=0}^{n} \frac{n!\,\hbar^m}{(n-m)!} [z^m] \mathcal{S}(z)^{-n-1} \hat{\mathcal{E}}_{-n}(z) Z^F. \tag{$*$}$$

Again recall the definition of the $\hat{\mathcal{E}}_{-n}(z)$ operator:

$$\hat{\mathcal{E}}_n(z) := \frac{1}{\varsigma(z)} \sum_{\substack{\ell,s\geqslant 0\\ q\geqslant 0}} \frac{1}{\ell!\,s!} \Bigg[\sum_{\substack{i_1+\cdots+i_s=q\\ i_k\geqslant 1}} \prod_{k=1}^{s} \frac{\varsigma(i_k z)}{i_k} p_{i_k} \Bigg] \Bigg[\sum_{\substack{j_1+\cdots+j_\ell=q+n\\ j_k\geqslant 1}} \prod_{k=1}^{\ell} \varsigma(j_k z) \frac{\partial}{\partial p_{j_k}} \Bigg].$$

The p_μ can either arise from an $s=1$ term from the $\hat{\mathcal{E}}_{-n}(z)$ operator, or from an application of a differential operator applied to the partition function $\frac{\partial}{\partial p_j} Z^F$. Therefore $s=0$ or $s=1$; I'll treat these separately. First consider the case when $s=1$. Then $q=\mu$ and we have

$$\sum_{m=0}^{n} \frac{n!}{(n-m)!} [z^m\,\hbar^{-m}] \frac{1}{Z^F} \mathcal{S}(z)^{-n-1} \frac{1}{\varsigma(z)} \sum_{\ell\geqslant 0} \frac{1}{\ell!} \frac{\varsigma(\mu z)}{\mu} \Bigg[\sum_{\substack{j_1+\cdots+j_\ell=\mu+n\\ j_k\geqslant 1}} \prod_{k=1}^{\ell} \varsigma(j_k z) \frac{\partial}{\partial p_{j_k}} \Bigg] Z^F.$$

Using a similar argument as in the case of $(g,n)=(0,1)$, because we are ultimately exacting a coefficient of $\hbar^{-m}$ we must apply at least m derivatives to the partition function; hence $\ell \geqslant m$. However, in the process of extracting the coefficient of z^m we obtain the constraint that $m \geqslant \ell$. This is because, minimally, one can take the constant term from $\mathcal{S}(z)^{-n-1}$, and the leading terms from $\frac{1}{\varsigma(z)}$, $\varsigma(\mu z)$ and each of the $\varsigma(j_k z)$ terms. Therefore, $\ell = m$ and this leads to the following contribution:

$$\sum_{m=0}^{n} \frac{n!}{(n-m)!\,m!} \sum_{\substack{j_1+\cdots+j_m=\mu+n \\ j_k \geqslant 1}} \prod_{k=1}^{m} j_k \mathrm{FS}^{\circ}_{0,1}(j_k).$$

When $s=0$, the contribution becomes

$$\sum_{m=0}^{n} \frac{n!}{(n-m)!} [p_\mu\, z^m\, \hbar^{-m}] \frac{1}{Z^F} \mathcal{S}(z)^{-n-1} \frac{1}{\varsigma(z)} \sum_{\ell \geqslant 0} \frac{1}{\ell!} \Bigg[\sum_{\substack{j_1+\cdots+j_\ell=n \\ j_k \geqslant 1}} \prod_{k=1}^{\ell} \varsigma(j_k z) \frac{\partial}{\partial p_{j_k}} \Bigg] Z^F.$$

Using an analogous reasoning as just above, we have that $\ell \geqslant m$ and $m \geqslant \ell - 1$; that is, $m \leqslant \ell \leqslant m+1$. When $\ell = m$ the expression reduces to

$$\sum_{m=0}^{n} \frac{n!}{(n-m)!\,m!} [p_\mu\, z^m\, \hbar^{-m}] \frac{1}{Z^F} \mathcal{S}(z)^{-n-1} \frac{1}{\varsigma(z)} \Bigg[\sum_{\substack{j_1+\cdots+j_m=n \\ j_k \geqslant 1}} \prod_{k=1}^{m} \varsigma(j_k z) \frac{\partial}{\partial p_{j_k}} \Bigg] Z^F.$$

This term cannot contribute. Because the functions $\varsigma(z)$ and $\mathcal{S}(z)^{-n-1}$ are even in z while the function $\frac{1}{\varsigma(z)}$ is odd, overall the expression is odd in z. The minimal power of z we can take is $m-1$, thus there is no z^m term. When $\ell = m+1$ the term is equal to

$$\begin{aligned} &\sum_{m=0}^{n} \frac{n!}{(n-m)!(m+1)!} [p_\mu\, z^m\, \hbar^{-m}] \frac{1}{Z^F} \mathcal{S}(z)^{-n-1} \frac{1}{\varsigma(z)} \Bigg[\sum_{\substack{j_1+\cdots+j_{m+1}=n \\ j_k \geqslant 1}} \prod_{k=1}^{m+1} \varsigma(j_k z) \frac{\partial}{\partial p_{j_k}} \Bigg] Z^F \\ &\qquad = \sum_{m=0}^{n} \frac{n!}{(n-m)!(m+1)!} \sum_{\substack{j_1+\cdots+j_{m+1}=n \\ j_k \geqslant 1}} \sum_{q=1}^{m+1} j_q \mathrm{FS}^{\circ}_{0,2}(\mu, j_q) \prod_{\substack{k=1 \\ k \neq q}}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k). \end{aligned}$$

Therefore, the overall contribution of the term labelled $(*)$ is

$$\begin{aligned} \sum_{m=0}^{n} \frac{n!}{(n-m)!} \Bigg[&\frac{1}{m!} \sum_{\substack{j_1+\cdots+j_m=\mu+n \\ j_k \geqslant 1}} \prod_{k=1}^{m} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \\ &+ \frac{1}{(m+1)!} \sum_{\substack{j_1+\cdots+j_{m+1}=n \\ j_k \geqslant 1}} \sum_{q=1}^{m+1} j_q \mathrm{FS}^{\circ}_{0,2}(\mu, j_q) \prod_{\substack{k=1 \\ k \neq q}}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) \Bigg]. \end{aligned} \qquad (\triangle)$$

Next consider both the terms

$$[p_\mu \hbar^0] \frac{1}{Z^F} \hbar \sum_{m=0}^{n} \frac{n!\,\hbar^m}{(n-m)!} [z^m] \mathcal{S}(z)^{-n-1} \frac{\partial}{\partial z} \hat{\mathcal{E}}_{-n}(z) Z^F, \qquad (**)$$

$$[p_\mu \hbar^0] \frac{1}{Z^F} \frac{\hbar}{2} \sum_{i+j=n} \left(\sum_{m=0}^{i} \frac{i!\,\hbar^m}{(i-m)!} [z^m] \mathcal{S}(z)^{-i-1} \hat{\mathcal{E}}_{-i}(z) \right) \left(\sum_{m=0}^{j} \frac{j!\,\hbar^m}{(j-m)!} [z^m] \mathcal{S}(z)^{-j-1} \hat{\mathcal{E}}_{-j}(z) \right) Z^F. \qquad (***)$$

For these contributions I apply very similar arguments but for the first term, as in the disk case, I utilise the trick that $[z^a]\frac{\partial}{\partial z} = (a+1)[z^{a+1}]$. I will omit the details; the necessary ideas have been demonstrated for the first term $(\triangle)$ above as well as in the disk case, the arguments are analogous. Very briefly, it suffices to consider, for each instance of the $\hat{\mathcal{E}}$-operator, the cases $s=0$ and $s=1$ separately, and, within each of those cases, the possible values of ℓ that can arise. Conclude that the term $(**)$ contributes

$$\sum_{m=0}^{n} \frac{n!\,(m+1)}{(n-m)!}\Bigg[\frac{1}{(m+1)!}\sum_{\substack{j_1+\cdots+j_{m+1}=\mu+n\\ j_k\geqslant 1}}\prod_{k=1}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) + \frac{1}{(m+2)!}\sum_{\substack{j_1+\cdots+j_{m+2}=n\\ j_k\geqslant 1}}\sum_{q=1}^{m+2} j_q \mathrm{FS}^{\circ}_{0,2}(\mu,j_q)\prod_{\substack{k=1\\ k\neq q}}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\Bigg]. \qquad (\triangle)$$

The product term $(***)$ contributes

$$\sum_{i+j=n}\Bigg[\left(\sum_{m_1=0}^{i}\frac{i!}{(i-m_1)!}\frac{1}{m_1!}\sum_{i_1+\cdots+i_{m_1}=i+\mu}\prod_{k=1}^{m_1} i_k \mathrm{FS}^{\circ}_{0,1}(i_k)\right) \times\left(\sum_{m_2=0}^{j}\frac{j!}{(j-m_2)!}\frac{1}{(m_2+1)!}\sum_{j_1+\cdots+j_{m_2+1}=j}\prod_{k=1}^{m_2+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\right)$$
$$+\left(\sum_{m_1=0}^{i}\frac{i!}{(i-m_1)!}\frac{1}{(m_1+1)!}\sum_{i_1+\cdots+i_{m_1+1}=i}\sum_{q=1}^{m_1+1} j_q \mathrm{FS}_{0,2}(\mu,j_q)\prod_{\substack{k=1\\ k\neq q}}^{m_1+1} i_k \mathrm{FS}^{\circ}_{0,1}(i_k)\right) \times\left(\sum_{m_2=0}^{j}\frac{j!}{(j-m_2)!}\frac{1}{(m_2+1)!}\sum_{j_1+\cdots+j_{m_2+1}=j}\prod_{k=1}^{m_2+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\right)\Bigg]. \qquad (\triangle)$$

Collect all terms given by $\triangle$ to conclude the result. ■

In the case of $(g,n)=(1,1)$

Finding a Tutte-like recursion in the case of $(g,n)=(1,1)$ is similar to the case of $(g,n)=(0,1)$, in that this is obtained by applying $\frac{1}{Z^F}\, p_{n+2}V_n$ to Z^F but in this case we must extract the coefficient of $p_\mu\hbar$.

This leads to the following Tutte-like recursion for fully simple maps in the punctured torus case; that is, when $(g,n)=(1,1)$. Again, this recursion has been used to reproduce the data calculated by Garcia-Failde [58].

Proposition 7.5.3. *The enumeration of fully simple maps in the punctured torus case* $(g,n)=(1,1)$ *satisfies the following recursion for all* $n\geqslant 1$*:*

$$2\hat{A}_{1,1}(n)+2A_{1,1}(n)=\frac{1}{s}A_{1,1}(n+2)-\sum_{i\geqslant 1} t_i\, A_{1,1}(n+i), \qquad (7.42)$$

where

$$A_{1,1}(N)=\sum_{m=0}^{N}\frac{N!}{(N-m)!}\Bigg[\frac{1}{(m-1)!}\left(\sum_{\substack{j_1+\cdots+j_{m-1}=N\\ j_k\geqslant 1}}\frac{1}{24}\left(j_1^2+\cdots+j_{m-1}^2-N-2\right)\prod_{k=1}^{m-1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\right)$$
$$+\frac{1}{(m+1)!}\sum_{\substack{j_1+\cdots+j_{m+1}=N\\ j_k\geqslant 1}}\left(\sum_{1\leqslant p<q\leqslant m+1} j_p j_q \mathrm{FS}^{\circ}_{0,2}(j_p,j_q)\prod_{\substack{k=1\\ k\neq p,q}}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) + \sum_{q=1}^{m+1} j_q \mathrm{FS}^{\circ}_{1,1}(j_q)\prod_{\substack{k=1\\ k\neq q}}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\right)\Bigg],$$

and

$$\hat{A}_{1,1}(N) = \sum_{m=0}^{n} \frac{N!}{(N-m)!}\Bigg[\frac{m+1}{24\,m!}\Bigg(\sum_{\substack{j_1+\cdots+j_m=N\\ j_k\geqslant 1}} (j_1^2+\cdots+j_m^2-1)\prod_{k=1}^{m} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\Bigg)$$

$$-\frac{(m-1)(N+1)}{24\,m!}\sum_{\substack{j_1+\cdots+j_m=N\\ j_k\geqslant 1}}\prod_{k=1}^{m} j_k\mathrm{FS}^{\circ}_{0,1}(j_k)$$

$$+\frac{m+1}{(m+2)!}\sum_{\substack{j_1+\cdots+j_{m+2}=N\\ j_k\geqslant 1}}\Bigg(\sum_{1\leqslant p<q\leqslant m+2} j_p j_q \mathrm{FS}^{\circ}_{0,2}(j_p,j_q)\prod_{\substack{k=1\\ k\neq p,q}}^{m+2} j_k\mathrm{FS}^{\circ}_{0,1}(j_k)$$

$$+\sum_{q=1}^{m+2} j_q\mathrm{FS}^{\circ}_{1,1}(j_q)\prod_{\substack{k=1\\ k\neq q}}^{m+2} j_k\mathrm{FS}^{\circ}_{0,1}(j_k)\Bigg)\Bigg].$$

Proof. Again, I'll begin by considering the following term from the V_n^F operator

$$[\hbar]\frac{1}{Z^F}\sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}\,[z^m]\mathcal{S}(z)^{-n-1}\hat{\mathcal{E}}_{-n}(z)Z^F.$$

Using a similar argument in the case of $(g,n)=(0,1)$, the $\hat{\mathcal{E}}$-operator cannot contain any terms that multiply by any p_j, thus the above expression reduces to

$$[\hbar]\frac{1}{Z^F}\sum_{m=0}^{n}\frac{n!\,\hbar^m}{(n-m)!}\,[z^m]\mathcal{S}(z)^{-n-1}\hat{\mathcal{E}}_{-n}(z)Z^F$$

$$=\sum_{m=0}^{n}\frac{n!}{(n-m)!}[z^m\,\hbar^{-m+1}]\frac{1}{Z^F}\mathcal{S}(z)^{-n-1}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\Bigg[\sum_{\substack{j_1+\cdots+j_\ell=n\\ j_k\geqslant 1}}\prod_{k=1}^{\ell}\varsigma(j_k z)\frac{\partial}{\partial p_{j_k}}\Bigg]Z^F. \qquad (*)$$

Again using a similar argument as in the disk case, given that we are extracting a coefficient of $\hbar^{-m+1}$ we must apply at least $m-1$ derivatives, thus $\ell\geqslant m-1$. On the other hand, as always $m\geqslant \ell-1$: this is because, minimally one can take the constant term from $\mathcal{S}(z)^{-n-1}$, the z^{-1} term from $\frac{1}{\varsigma(z)}$ and the linear terms from all ℓ of the $\varsigma(j_k z)$ terms. Conclude that ℓ is constrained by $m-1\leqslant \ell\leqslant m+1$. I will consider each of these in turn in decreasing order. That is, first consider the case where $\ell=m+1$. This case yields

$$\sum_{m=0}^{n}\frac{n!}{(n-m)!\,(m+1)!}[\hbar^{-m+1}]\frac{1}{Z^F}\Bigg[\sum_{\substack{j_1+\cdots+j_{m+1}=n\\ j_k\geqslant 1}}\prod_{k=1}^{m+1} j_k\frac{\partial}{\partial p_{j_k}}\Bigg]Z^F$$

$$=\sum_{m=0}^{n}\frac{n!}{(n-m)!\,(m+1)!}\sum_{\substack{j_1+\cdots+j_{m+1}=n\\ j_k\geqslant 1}}\Bigg[\sum_{1\leqslant p<q\leqslant m+1} j_p j_q\mathrm{FS}^{\circ}_{0,2}(j_p,j_q)\prod_{\substack{k=1\\ k\neq p,q}}^{m+1} j_k\mathrm{FS}^{\circ}_{0,1}(j_k)$$

$$+\sum_{q=1}^{m+1} j_q\mathrm{FS}^{\circ}_{1,1}(j_q)\prod_{\substack{k=1\\ k\neq q}}^{m+1} j_k\mathrm{FS}^{\circ}_{0,1}(j_k)\Bigg].$$

The two distinct types of terms arise in the following way. For the term containing $\mathrm{FS}^{\circ}_{0,2}$, $m-1$ of the derivatives act on $(0,1)$-terms contributing $\hbar^{-m+1}$, while the last two derivatives both act on the same $\mathrm{FS}^{\circ}_{0,2}$ contributing $\hbar^0$. For the term containing $\mathrm{FS}^{\circ}_{1,1}$, m of the derivatives act on $(0,1)$-terms contributing $\hbar^{-m}$, while the last derivative acts on a $\mathrm{FS}_{1,1}$ term contributing $\hbar^1$. These two options are the only possible contributions; while this may not be immediately clear, a few minutes consideration should be enough for one to convince themselves. For $\ell=m$ we have

$$\sum_{m=0}^{n}\frac{n!}{(n-m)!}[z^m\,\hbar^{-m+1}]\frac{1}{Z^F}\mathcal{S}(z)^{-n-1}\frac{1}{\varsigma(z)}\frac{1}{m!}\Bigg[\sum_{\substack{j_1+\cdots+j_m=n\\ j_k\geqslant 1}}\prod_{k=1}^{m}\varsigma(j_k z)\frac{\partial}{\partial p_{j_k}}\Bigg]Z^F.$$

Considering extracting the coefficient of z^m from the product of ς- and $\mathcal{S}$-functions allows us to conclude that this term cannot contribute using the same parity argument as in the case of $(g,n) = (0,2)$.

Finally consider $\ell = m-1$:

$$\sum_{m=0}^{n} \frac{n!}{(n-m)!}[z^m\,\hbar^{-m+1}]\frac{1}{Z^F}\mathcal{S}(z)^{-n-1}\frac{1}{\varsigma(z)}\frac{1}{(m-1)!}\left[\sum_{\substack{j_1+\cdots+j_{m-1}=n\\ j_k\geqslant 1}}\prod_{k=1}^{m-1}\varsigma(j_kz)\frac{\partial}{\partial p_{j_k}}\right]Z^F$$

Because we are applying $m-1$ derivatives and extracting the coefficient of $\hbar^{-m+1}$, each derivative must act on a $(0,1)$-term. Extracting the coefficient of z^m can be rewritten as follows:

$$[z^m]\mathcal{S}(z)^{-n-1}\frac{1}{\varsigma(z)}\prod_{k=1}^{m-1}\varsigma(j_kz) = \sum_{a+b+c=m}[z^a]\mathcal{S}(z)^{-n-1}[z^b]\frac{1}{\varsigma(z)}[z^c]\prod_{k=1}^{m-1}\varsigma(j_kz).$$

There are three possibilities for a, b and c, summarised in the following table.

a	b	c	$[z^a]\mathcal{S}(z)^{-n-1}$	$[z^b]\frac{1}{\varsigma(z)}$	$[z^c]\prod_{k=1}^{m-1}\varsigma(j_kz)$
2	-1	$m-1$	$-\frac{n+1}{24}$	1	$\prod_{k=1}^{m-1} j_k$
0	1	$m-1$	1	$-\frac{1}{24}$	$\prod_{k=1}^{m-1} j_k$
0	-1	$m+1$	1	1	$\frac{1}{24}(j_1^2+\cdots+j_{m-1}^2)\prod_{k=1}^{m-1} j_k$

The contribution when $\ell = m-1$ is

$$\sum_{m=0}^{n}\frac{n!}{(n-m)!\,(m-1)!}\left[\sum_{\substack{j_1+\cdots+j_{m-1}=n\\ j_k\geqslant 1}}\frac{1}{24}\big(j_1^2+\cdots+j_{m-1}^2-n-2\big)\prod_{k=1}^{m-1}j_k\mathrm{FS}^{\circ}_{0,1}(j_k)\right].$$

Therefore, the overall contribution for the term labelled $(*)$ is

$$\begin{aligned}\sum_{m=0}^{n}\frac{n!}{(n-m)!}\Bigg[&\frac{1}{(m-1)!}\Bigg(\sum_{\substack{j_1+\cdots+j_{m-1}=n\\ j_k\geqslant 1}}\frac{1}{24}\big(j_1^2+\cdots+j_{m-1}^2-n-2\big)\prod_{k=1}^{m-1}j_k\mathrm{FS}^{\circ}_{0,1}(j_k)\Bigg)\\ &+\frac{1}{(m+1)!}\sum_{\substack{j_1+\cdots+j_{m+1}=n\\ j_k\geqslant 1}}\Bigg(\sum_{1\leqslant p<q\leqslant m+1}j_pj_q\mathrm{FS}^{\circ}_{0,2}(j_p,j_q)\prod_{\substack{k=1\\ k\neq p,q}}^{m+1}j_k\mathrm{FS}^{\circ}_{0,1}(j_k)\\ &\qquad+\sum_{q=1}^{m+1}j_q\mathrm{FS}^{\circ}_{1,1}(j_q)\prod_{\substack{k=1\\ k\neq q}}^{m+1}j_k\mathrm{FS}^{\circ}_{0,1}(j_k)\Bigg)\Bigg].\end{aligned}\qquad(\triangle)$$

Next I consider the $\frac{\partial}{\partial z}\hat{\mathcal{E}}_{-n}(z)$ term and again use the trick $[z^a]\frac{\partial}{\partial z} = (a+1)[z^{a+1}]$. This gives

$$\begin{aligned}&[\hbar]\frac{1}{Z^F}\sum_{m=0}^{n}\frac{n!\,\hbar^{m+1}}{(n-m)!}[z^m]\mathcal{S}(z)^{-n-1}\frac{\partial}{\partial z}\hat{\mathcal{E}}_{-n}(z)Z^F\\ &=\sum_{m=0}^{n}\frac{n!}{(n-m)!}\sum_{a+b=m}[z^a]\mathcal{S}(z)^{-n-1}[z^b\,\hbar^{-m}]\frac{1}{Z^F}\frac{\partial}{\partial z}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\left[\sum_{\substack{j_1+\cdots+j_\ell=n\\ j_k\geqslant 1}}\prod_{k=1}^{\ell}\varsigma(j_kz)\frac{\partial}{\partial p_{j_k}}\right]Z^F\\ &=\sum_{m=0}^{n}\frac{n!}{(n-m)!}[\hbar^{-m}]\sum_{a+b=m}[z^a]\mathcal{S}(z)^{-n-1}(b+1)[z^{b+1}]\frac{1}{Z^F}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\left[\sum_{\substack{j_1+\cdots+j_\ell=n\\ j_k\geqslant 1}}\prod_{k=1}^{\ell}\varsigma(j_kz)\frac{\partial}{\partial p_{j_k}}\right]Z^F.\end{aligned}$$

Now $\ell \geqslant m$ and $m+1 \geqslant \ell - 1$, thus $m \leqslant \ell \leqslant m+2$. Following a completely analogous argument as in the case for the term $(*)$ above, we conclude that this term yields the following contribution:

$$\sum_{m=0}^{n} \frac{n!}{(n-m)!}\Bigg[\frac{m+1}{24\,m!}\Bigg(\sum_{\substack{j_1+\cdots+j_m=n\\ j_k\geqslant 1}} (j_1^2+\cdots+j_m^2-1)\prod_{k=1}^{m} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\Bigg)$$

$$-\frac{(m-1)(n+1)}{24\,m!}\sum_{\substack{j_1+\cdots+j_m=n\\ j_k\geqslant 1}}\prod_{k=1}^{m} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)$$

$$+\frac{m+1}{(m+2)!}\sum_{\substack{j_1+\cdots+j_{m+2}=n\\ j_k\geqslant 1}}\Bigg(\sum_{1\leqslant p<q\leqslant m+2} j_p j_q \mathrm{FS}^{\circ}_{0,2}(j_p,j_q)\prod_{\substack{k=1\\ k\neq p,q}}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)$$

$$+\sum_{q=1}^{m+2} j_q \mathrm{FS}^{\circ}_{1,1}(j_q)\prod_{\substack{k=1\\ k\neq q}}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\Bigg)\Bigg]. \qquad (\triangle)$$

Finally, for the product term, recall the argument given in the case of $(g,n)=(0,1)$; that is, whenever $n=1$,

$$[h^{2g-1}]\frac{1}{Z^F}\frac{\hbar}{2}\sum_{i+j=n}\Bigg(\sum_{m=0}^{i}\frac{i!\,\hbar^m}{(i-m)!}[z^m]\mathcal{S}(z)^{-i-1}\hat{\mathcal{E}}_{-i}(z)\Bigg)\Bigg(\sum_{m=0}^{j}\frac{j!\,\hbar^m}{(j-m)!}[z^m]\mathcal{S}(z)^{-j-1}\hat{\mathcal{E}}_{-j}(z)\Bigg)Z^F$$

$$=[h^{2g-1}]\frac{1}{Z^F}\sum_{m=0}^{n}\frac{n!\,\hbar^{m+1}}{(n-m)!}[z^m]\mathcal{S}(z)^{-n-1}\frac{\partial}{\partial z}\hat{\mathcal{E}}_{-n}(z)Z^F.$$

Thus, the product term yields the same contribution as the $\frac{\partial}{\partial z}\hat{\mathcal{E}}_{-n}(z)$ term. Collecting the terms labelled by $(\triangle)$ yields the statement of proposition (7.42). ∎

7.5.2 A relation between ordinary and fully simple map enumerations

In this section, I deduce a relation between ordinary maps and fully simple maps via the semi-infinite wedge in the case of $(g,n)=(0,1)$. Again, one would ideally like to be able to go this for $n=1$ and all genera, or even more generally for all (g,n).

In the disk case: $(g,n)=(0,1)$

Begin with the vacuum expectation for the enumeration $\mathrm{Map}(\mu_1,\ldots,\mu_n)$ for ordinary maps given in equation (7.20) specialised in the case of $n=1$, and use the fact that $\mathcal{H}\rangle = 1$ to rewrite it in the following way:

$$\mathrm{Map}(\mu) = \Big\langle \exp(s\mathcal{G})\exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big)\frac{\alpha_\mu}{\mu}\Big\rangle$$

$$= \Big\langle \exp(s\mathcal{G})\exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big)\mathcal{H}\mathcal{H}^{-1}\frac{\alpha_\mu}{\mu}\mathcal{H}\Big\rangle$$

$$= \frac{1}{\mu}\Big\langle \exp(s\mathcal{G})\exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big)\mathcal{H}\sum_{m=0}^{\mu}\frac{\mu!\,\hbar^m}{(\mu-m)!}[z^m]\mathcal{S}(z)^{-\mu-1}\mathcal{E}_{-\mu}(z)\Big\rangle$$

$$= \frac{1}{\mu}\Big\langle \exp(s\mathcal{G})\exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big)\mathcal{H}\sum_{m=0}^{\mu}\frac{\mu!\,\hbar^m}{(\mu-m)!}[z^m]\mathcal{S}(z)^{-\mu-1}$$

$$\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{j_1+\cdots+j_\ell=-\mu}\Bigg[\prod_{k=1}^{\ell}\frac{\varsigma(j_k z)}{j_k}\Bigg] :\alpha_{j_1}\cdots\alpha_{j_\ell}:\Big\rangle.$$

Thus conclude

$$\mathrm{Map}(\mu) = \frac{1}{\mu}\sum_{m=0}^{\mu}\frac{\mu!\,\hbar^m}{(\mu-m)!}[z^m]\mathcal{S}(z)^{-\mu-1}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{j_1+\cdots+j_\ell=-\mu}\left[\prod_{k=1}^{\ell}\frac{\varsigma(j_kz)}{j_k}\right]$$
$$\Big\langle \exp(s\mathcal{G})\exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big)\mathcal{H} : \alpha_{j_1}\cdots\alpha_{j_\ell} : \Big\rangle.$$

The third equality is using Lemma 7.4.7, while the fourth equality is using the bosonic form of the $\mathcal{E}$-operator (1.20). The normal ordering in this case will push all bosonic operators with positive subscripts to the right and negative subscripts to the left. Given that α_m for a positive integer m annihilates the vacuum, it follows that the sum is zero unless $i_k < 0$ for all $k \in \{1, 2, \ldots, \ell\}$. Thus,

$$\mathrm{Map}(\mu) = \frac{1}{\mu}\sum_{m=0}^{\mu}\frac{\mu!\,\hbar^m}{(\mu-m)!}[z^m]\mathcal{S}(z)^{-\mu-1}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{\substack{j_1+\cdots+j_\ell=\mu\\ j_k>0}}\left[\prod_{k=1}^{\ell}\frac{\varsigma(j_kz)}{j_k}\right]$$
$$\Big\langle \exp(s\mathcal{G})\exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big)\mathcal{H}\alpha_{-j_1}\cdots\alpha_{-j_\ell}\Big\rangle,$$

where I have relabelled $j_k \mapsto -j_k$ and noted that $\frac{\varsigma(j_kz)}{j_k}$ is even in j_k. Therefore,

$$\mu\mathrm{Map}(\mu) = \sum_{m=0}^{\mu}\frac{\mu!\,\hbar^m}{(\mu-m)!}[z^m]\mathcal{S}(z)^{-\mu-1}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{\substack{j_1+\cdots+j_\ell=\mu\\ j_k>0}}\left[\prod_{k=1}^{\ell}\varsigma(j_kz)\right]\mathrm{FSMap}(j_1,\ldots,j_\ell), \tag{7.43}$$

where I have used the following vacuum expectation for FSMap:

$$\mathrm{FSMap}(j_1,\ldots,j_\ell) = \Big\langle \exp(s\mathcal{G})\exp\Big(\sum_{m\geqslant 1}\frac{t_m}{m\,\hbar}\alpha_{-m}\Big)\mathcal{H}\frac{\alpha_{-j_1}}{j_1}\cdots\frac{\alpha_{-j_\ell}}{j_\ell}\Big\rangle.$$

One could read equation (7.43) as a reformulation of (7.6) from Theorem 7.2.3 in the case of $\mathrm{Map}(\mu)$.

To obtain a specific relation in genus g, extract a coefficient of $\hbar^{2g-1}$ from both sides, recalling that

$$\mathrm{Map}(\mu_1,\ldots,\mu_n) = \sum_{g\in\mathbb{Z}}\mathrm{M}^{\bullet}_{g,n}(\mu_1,\ldots,\mu_n)\,\hbar^{2g-2+n},$$

and that one can use inclusion-exclusion to pass from the disconnected enumeration to the connected and vice versa:

$$\mathrm{Map}(\mu_1,\ldots,\mu_n) = \sum_{M\vdash\{1,\ldots,n\}}\prod_{i=1}^{|M|}\mathrm{Map}^{\circ}(\vec{\mu}_{M_i}). \tag{7.44}$$

In the specific case of $g=0$, extract the coefficient of $\hbar^{-1}$ from both sides and note that all maps (ordinary and fully simple) are connected when $n=1$. The left yields $\mu\mathrm{M}^{\circ}_{0,1}(\mu)$ while the right side gives

$$[\hbar^{-1}]\sum_{m=0}^{\mu}\frac{\mu!\,\hbar^m}{(\mu-m)!}[z^m]\mathcal{S}(z)^{-\mu-1}\frac{1}{\varsigma(z)}\sum_{\ell\geqslant 0}\frac{1}{\ell!}\sum_{\substack{j_1+\cdots+j_\ell=\mu\\ j_k>0}}\left[\prod_{k=1}^{\ell}\varsigma(j_kz)\right]\mathrm{FSMap}(j_1,\ldots,j_\ell)$$
$$= \sum_{m=0}^{\mu}\frac{\mu!}{(\mu-m)!\,(m+1)!}\sum_{j_1+\cdots+j_{m+1}=\mu}[\hbar^{-m-1}]\mathrm{FSMap}(j_1,\ldots,j_{m+1})$$
$$= \sum_{m=0}^{\mu}\frac{\mu!}{(\mu-m)!\,(m+1)!}\sum_{j_1+\cdots+j_{m+1}=\mu}\prod_{k=1}^{m+1}j_k\mathrm{FS}^{\circ}_{0,1}(j_k).$$

This statement is nontrivial. From the fact we are extracting the coefficient of $\hbar^{-1}$ and the expression contains $\hbar^m$ it follows that we require $[\hbar^{-m-1}]\mathrm{FSMap}(j_1,\ldots,j_\ell)$. The smallest power of $\hbar$ that can arise in $\mathrm{FSMap}(j_1,\dots,j_\ell)$ is $-\ell$, which occurs in the term $\hbar^{-\ell}F_{0,1}(j_1)\cdots F_{0,1}(j_\ell)$. Thus $-m-1 \geqslant -\ell$ or equivalently $m+1 \leqslant \ell$. On the other hand the power of z in the expression must be at least $\ell-1$: minimally, we can take the constant term from $\mathcal{S}(z)^{-\mu-1}$, the z term from each of the $\varsigma(j_k z)$ series and the $1/z$ term from $1/\varsigma$. Hence $m \geqslant \ell-1$ (since we are extracting the coefficient of z^m) or equivalently $m+1 \geqslant \ell$. Combining these two inequalities gives that $\ell = m+1$ and this is what is being applied in the first equality above. The second equality is using (7.44) in the case of fully simple maps.

Therefore, conclude with the following relation between the enumerations of ordinary and fully simple disks.

Proposition 7.5.4. *For μ a positive integer, we have the following relation between ordinary and fully simple maps in the disk case:*

$$\mu \mathrm{M}^{\circ}_{0,1}(\mu) = \sum_{m=0}^{\mu} \frac{\mu!}{(\mu-m)!\,(m+1)!} \sum_{j_1+\cdots+j_{m+1}=\mu} \prod_{k=1}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k).$$

Given the form of the result above, it is quite hopeful that there is a direct combinatorial interpretation for the relation.

7.5.3 Recovering the spectral curve for topological recursion

In the setting of topological recursion, it is common knowledge that the spectral curve usually contains the data of the $(0,1)$-enumeration, or, conversely, that from the $(0,1)$ information one can deduce the spectral curve. See Section 4.2.2 in Chapter 4 for how this process can be done in the case of single Hurwitz numbers.

As discussed in the introduction, it has been proven that the enumeration of fully simple maps is governed by topological recursion with the spectral curve in Theorem 7.1.2, and, perhaps more importantly, that this is precisely the spectral curve obtained by taking the spectral curve for ordinary maps in Theorem 7.1.1 and switching the roles of x and y. In this section, I use the Tutte-like recursion in the case of $(g,n)=(0,1)$ obtained in Proposition 7.5.1 and recover the spectral curve for fully simple maps. The form of the spectral I will use here will be different to its presentation in Theorem 7.1.1, namely, I will use its global form which can be written as

$$x^2 = V'(y)\,x - P_{0,1}(y), \tag{7.45}$$

where

$$V'(y) = y - \sum_{i\geqslant 1} t_i y^{i-1}, \qquad \text{and} \qquad P_{0,1}(y) = 1 + A_{0,1}(1)y^{-1} - \sum_{i\geqslant 1} t_i \sum_{j=0}^{i-1} y^{i-j-2} A_{0,1}(j).$$

This form of the spectral curve for fully simple maps aligns with the spectral curve for ordinary maps given by Eynard [45, Equation (3.1.2)] and switching x and y. To avoid confusion, I will use x_{ord} and y_{ord} for the meromorphic functions of the ordinary maps spectral curve, while x and y will continue to be those for the fully simple maps spectral curve. Thus $x = y_{\mathrm{ord}}$ and $y = x_{\mathrm{ord}}$.

Begin with the Tutte-like recursion for fully simple maps in the disk case $(g,n)=(0,1)$ given in Proposition 7.5.1. Throughout this section the s-parameter will not be required, hence set $s=1$. The Tutte-like recursion reads, for all $n \geqslant 1$,

$$2\hat{A}_{0,1}(n) + 2A_{0,1}(n) = A_{0,1}(n+2) - \sum_{i\geqslant 1} t_i\, A_{0,1}(n+i), \tag{$\triangle$}$$

where

$$A_{0,1}(N) = \sum_{m=0}^{N} \frac{N!}{(N-m)!(m+1)!} \sum_{j_1+\cdots+j_{m+1}=N} \prod_{k=1}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)$$

$$\hat{A}_{0,1}(N) = \sum_{m=0}^{N} \frac{N!\,(m+1)}{(N-m)!(m+2)!} \sum_{j_1+\cdots+j_{m+2}=N} \prod_{k=1}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k).$$

To pass to generating functions multiply by $\frac{1}{y^{n+1}}$ and sum over all $n \geqslant 0$. This recursion is only valid for $n \geqslant 1$, thus I need to derive the corresponding recursion when $n = 0$. This is done by following the process outlined at the start of Section 7.5.1; in this case one must apply $\frac{1}{Z^F} p_2 V_0^F$ to Z^F and extract the coefficient of $p_2 \hbar^{-1}$. Recall the Virasoro operator for $n = 0$:

$$V_0^F = \hbar\,[z]\hat{\mathcal{E}}_0(z) + \frac{\hbar}{24} + \sum_{i\geqslant 1} t_i \sum_{m=0}^{i} \frac{i!\,\hbar^m}{(i-m)!}[z^m]\mathcal{S}(z)^{-i-1}\hat{\mathcal{E}}_{-i}(z) + \frac{1}{\hbar} - \sum_{m=0}^{2} \frac{2!\,\hbar^m}{(2-m)!}[z^m]\mathcal{S}(z)^{-3}\hat{\mathcal{E}}_{-2}(z).$$

Following the outlined process yields the recursion

$$1 = A_{0,1}(2) - \sum_{i\geqslant 1} t_i A_{0,1}(i). \tag{$\triangle\triangle$}$$

Multiply the equation $\triangle$ throughout by $\frac{1}{y^{n+1}}$ and sum over all $n \geqslant 1$, then add to it the equation ($\triangle\triangle$) multiplied throughout by $\frac{1}{y}$. Finally, multiply throughout again by $\frac{1}{y}$. This leads to

$$2\sum_{n\geqslant 1} \hat{A}_{0,1}(n)\frac{1}{y^{n+2}} + 2\sum_{n\geqslant 1} A_{0,1}(n)\frac{1}{y^{n+2}} + \frac{1}{y^2} = \sum_{n\geqslant 0} A_{0,1}(n+2)\frac{1}{y^{n+2}} - \sum_{n\geqslant 0}\sum_{i\geqslant 1} t_i A_{0,1}(n+i)\frac{1}{y^{n+2}}. \tag{7.46}$$

Begin with the right side and rewrite it in the following way. The aim of this manipulation is to introduce the potential $V'(y)$.

$$\begin{aligned}
&\sum_{n\geqslant 0} A_{0,1}(n+2)\frac{1}{y^{n+2}} - \sum_{n\geqslant 0}\sum_{i\geqslant 1} t_i A_{0,1}(n+i)\frac{1}{y^{n+2}} \\
&\quad = \sum_{n\geqslant 2} A_{0,1}(n)\frac{1}{y^{n}} - \sum_{i\geqslant 1} t_i y^{i-1} \sum_{n\geqslant i} A_{0,1}(n)\frac{1}{y^{n+1}} \\
&\quad = y\sum_{n\geqslant 1} A_{0,1}(n)\frac{1}{y^{n+1}} - A_{0,1}(1)\frac{1}{y} - \sum_{i\geqslant 1} t_i y^{i-1} \sum_{n\geqslant i} A_{0,1}(n)\frac{1}{y^{n+1}} + \sum_{i\geqslant 1} t_i \sum_{j=1}^{i-1} y^{i-j-2} A_{0,1}(j) \\
&\quad = \Big[y - \sum_{i\geqslant 1} t_i y^{i-1}\Big] \sum_{n\geqslant 1} A_{0,1}(n)\frac{1}{y^{n+1}} - A_{0,1}(1)\frac{1}{y} + \sum_{i\geqslant 1} t_i \sum_{j=1}^{i-1} y^{i-j-2} A_{0,1}(j)
\end{aligned}$$

The first equality is applying the relabelling $n \mapsto n-2$ to the first term and $n \mapsto n-i$ to the second.

Recall the relation derived in Proposition 7.5.4 and observe that the right side of this relation is $A_{0,1}(\mu)$ as defined in Proposition 7.5.1,

$$\mu \mathrm{M}^{\circ}_{0,1}(\mu) = \sum_{m=0}^{\mu} \frac{\mu!}{(\mu-m)!\,(m+1)!} \sum_{j_1+\cdots+j_{m+1}=\mu} \prod_{k=1}^{m+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k) = A_{0,1}(\mu).$$

As defined by Eynard [45],

$$y_{\mathrm{ord}} = \frac{\omega^{\mathrm{ord}}_{0,1}}{\mathrm{d}x_{\mathrm{ord}}} = \frac{1}{x_{\mathrm{ord}}} + \sum_{\mu\geqslant 1} \mu \mathrm{M}_{0,1}(\mu)\frac{1}{x_{\mathrm{ord}}^{\mu+1}},$$

hence

$$x = \frac{1}{y} + \sum_{\mu\geqslant 1} \mu \mathrm{M}_{0,1}(\mu)\frac{1}{y^{\mu+1}} = \frac{1}{y} + \sum_{\mu\geqslant 1} A_{0,1}(\mu)\frac{1}{y^{\mu+1}}. \tag{7.47}$$

Returning to the right side and rewriting further gives

$$\begin{aligned}
&\Big[y - \sum_{i\geqslant 1} t_i y^{i-1}\Big] \sum_{n\geqslant 1} A_{0,1}(n)\frac{1}{y^{n+1}} - A_{0,1}(1)\frac{1}{y} + \sum_{i\geqslant 1} t_i \sum_{j=1}^{i-1} y^{i-j-2} A_{0,1}(j) \\
&= \Big[y - \sum_{i\geqslant 1} t_i y^{i-1}\Big] \Big[\frac{1}{y} + \sum_{n\geqslant 1} A_{0,1}(n)\frac{1}{y^{n+1}}\Big] - A_{0,1}(1)\frac{1}{y} - 1 + \sum_{i\geqslant 1} t_i y^{i-2} + \sum_{i\geqslant 1} t_i \sum_{j=1}^{i-1} y^{i-j-2} A_{0,1}(j) \\
&= V'(y)\,x - P_{0,1}(y),
\end{aligned}$$

where I have absorbed the first sum over $i \geqslant 1$ into the second and, for convenience, defined $A_{0,1}(0) := 1$. Next, take equation (7.47), let $\mu = n$ and square both sides. Doing this gives

$$\begin{aligned} x^2 &= \left[\frac{1}{y} + \sum_{n \geqslant 1} A_{0,1}(n) \frac{1}{y^{n+1}}\right]^2 \\ &= \frac{1}{y^2} + 2 \sum_{n \geqslant 1} A_{0,1}(n) \frac{1}{y^{n+2}} + \sum_{n \geqslant 2} \left[\sum_{i+j=n} A_{0,1}(i) A_{0,1}(j)\right] \frac{1}{y^{n+2}} \\ &= \frac{1}{y^2} + 2 \sum_{n \geqslant 1} A_{0,1}(n) \frac{1}{y^{n+2}} + 2 \sum_{n \geqslant 2} \hat{A}_{0,1}(n) \frac{1}{y^{n+2}}. \end{aligned}$$

The final equality has used the identity for $\hat{A}_{0,1}(n)$ derived in Section 7.5.1, equation (7.39),

$$\begin{aligned} \frac{1}{2} \sum_{i+j=n} &\left[\sum_{m_1=0}^{i} \frac{i!}{(i-m_1)!} \frac{1}{(m_1+1)!} \sum_{i_1+\cdots+i_{m_1+1}=i} \prod_{k=1}^{m_1+1} i_k \mathrm{FS}^{\circ}_{0,1}(i_k)\right] \\ &\left[\sum_{m_2=0}^{j} \frac{j!}{(j-m_2)!} \frac{1}{(m_2+1)!} \sum_{j_1+\cdots+j_{m_2+1}=j} \prod_{k=1}^{m_2+1} j_k \mathrm{FS}^{\circ}_{0,1}(j_k)\right] \\ &= \sum_{m=0}^{n} \frac{n!\,(m+1)}{(n-m)!\,(m+2)!} \sum_{j_1+\cdots+j_{m+2}=n} \prod_{k=1}^{m+2} j_k \mathrm{FS}^{\circ}_{0,1}(j_k). \end{aligned}$$

Hence, the left side of (7.46) is equal to x^2 and I can conclude that

$$x^2 = V'(y)\,x - P_{0,1}(y),$$

as required.

Chapter 8

Topological Narayana polynomials

8.1 Introduction

One relatively unexplored feature of topological recursion is its ability to generalise existing combinatorial problems. A spectral curve, input to topological recursion, stores the $(0,1)$-enumeration of a problem and thus can motivate generalisations of a combinatorial enumeration to higher topologies. For instance, Narayana polynomials arise in a number of combinatorial settings [86, 95] and provide a prime example of an enumeration ripe for such a generalisation. In this chapter, I describe joint work in progress with Xavier Coulter and Norman Do, which introduces a particular generalisation of Narayana polynomials that is motivated by topological recursion.

Narayana polynomials are known to satisfy a number of notable properties, including a linear recursion [57], a quadratic recursion [95], symmetry of coefficients, real-rootedness [57], and interlacing [57]. It is natural to ask if our generalisation preserves these properties; indeed, in this ongoing research, we prove that some of these properties generalise and conjecture that the remaining ones do as well.

For $1 \leqslant k \leqslant n$, define the ***Narayana number*** $N(n,k)$ to be $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Define the ***Narayana polynomial*** $N(n)$ to be the generating function

$$N(n) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} t^k, \tag{8.1}$$

for $n \geqslant 1$ and define $N(0) = 1$. The Narayana numbers and polynomials are a refinement of the Catalan numbers in the sense that

$$\sum_{k=1}^{n} N(n,k) = N(n)\big|_{t=1} = \mathrm{Cat}_n.$$

Narayana numbers and polynomials can be realised as enumerations of a number of combinatorial objects; for example, $N(n,k)$ is equal to:

- the number of words containing n pairs of parentheses, which are correctly matched and contain k distinct nestings [78];
- the number of Dyck paths of length $2n$ with k peaks; that is, a lattice path from $(0,0)$ to (n,n) consisting of n horizontal steps, n vertical steps, contains k peaks (a peak is a vertical step followed immediately by a horizontal step), and where all points (i,j) on the path satisfy $i \leqslant j$ [95]; and
- the number of unlabelled rooted plane trees with n non-rooted vertices and k left-pointing leaves [95].

The Narayana polynomials satisfy the following linear recursion for $n \geqslant 2$ [57]:

$$N(n) = \frac{1}{n+1}\left[(2n-1)(t+1)N(n-1) - (n-2)(t-1)^2 N(n-2)\right]. \tag{8.2}$$

Furthermore, the Narayana polynomials satisfy the following quadratic recursion for $n \geqslant 1$ [95]:

$$N(n) = \sum_{\alpha+\beta=n-1} N(\alpha)\, N(\beta) + (t-1)N(n-1). \tag{8.3}$$

Note that the difference between the definition here and in [95] is given by $N(n) = tC_n(t)$. The choice not to carry n as a subscript here is to avoid possible confusion with the generalised polynomials.

Each Narayana polynomial is symmetric in the sense that it is a polynomial in t whose coefficients form a palindromic sequence. More precisely, the Narayana polynomial $N(n)$ has degree n, and the coefficient of t^k is equal to the coefficient of t^{n+1-k} for each k. This is immediate from the definition (8.1) and leads to interesting consequences once the Narayana number is interpreted as an enumeration of combinatorial objects. For example, the symmetry suggests that there is a natural bijection between the sets of Dyck paths of length $2n$ with k peaks and the sets of Dyck paths of length $2n$ with $n+1-k$ peaks .The bijectively-minded reader might find this an enjoyable exercise.

Narayana polynomials also satisfy two deeper properties that are interrelated; namely, real-rootedness and interlacing. Precisely, the Narayana polynomial $N(n)$ has only real roots, and the Narayana polynomial $N(n)$ interlaces with $N(n+1)$ [57]. Interlacing can be deduced from the linear recursion (8.2) and one obtains real-rootedness as a direct consequence. For a definition of interlacing, see Section 8.2.5.

In this chapter, topological recursion is used to motivate a generalisation of the Narayana polynomials via the following approach. The generating function for Narayana polynomials was used to define a spectral curve—which necessarily carries the polynomial variable t as a parameter—namely, the spectral curve (8.4) of Theorem 8.2.2. Applying topological recursion to this spectral curve produces correlation differentials $\omega_{g,n}$ whose coefficients are polynomials in t, thereby producing a desired generalisation. The definition of the polynomials given below was then devised to match the output of topological recursion.[1]

Given that Narayana polynomials are a refinement of Catalan numbers, one can use the latter as a framework for this process. Note that the Catalan numbers are stored in both the $(0,1)$-enumeration of ribbon graphs as well as the $(0,1)$-enumeration of monotone Hurwitz numbers, thereby providing two natural spectral curves to generalise. Here, we have modelled the generalisation in the framework of monotone Hurwitz numbers, and a combinatorial definition of the enumeration produced by topological recursion can be seen as a t-deformation of monotone Hurwitz numbers. While we have chosen to use the terminology *topological Narayana polynomials*, the enumeration could equally be named *t-deformed monotone Hurwitz numbers.* This leads to the following definition.

Definition 8.1.1. The *topological Narayana polynomial* $H^t_{g,n}(\mu_1,\ldots,\mu_n)$ is $\frac{1}{|\mu|!}$ times the weighted enumeration of tuples $(\tau_1,\ldots,\tau_m)$ of transpositions in the symmetric group $S_{|\mu|}$ such that

- $m = 2g-2+n+|\mu|$;
- the cycles of $\tau_1\cdots\tau_m$ are labelled $1,2,\ldots,n$ such that cycle i has length μ_i for $i \in \{1,2,\ldots,n\}$;
- $\langle\tau_1,\ldots,\tau_m\rangle$ is a transitive subgroup of $S_{|\mu|}$; and
- if $\tau_i = (a_i\, b_i)$ with $a_i < b_i$, then $b_1 \leqslant \cdots \leqslant b_m$.

The weight of such a tuple $((a_1\, b_1),\ldots,(a_m\, b_m))$ is t^w where w is the number of distinct integers in the sequence $b_1,\ldots,b_m$. The weight w is referred to as the ***hive number*** of the tuple.[2]

If one relaxes the transitivity condition, then one obtains the *disconnected topological Narayana polynomial* $H^{t\bullet}_{g,n}(\mu_1,\ldots,\mu_n)$.

Substituting $t=1$ into the definition above recovers the monotone Hurwitz number $H^{\leqslant}_{g,n}(\mu_1,\ldots,\mu_n)$ defined in Definition 4.3.1. Definition 8.1.1 is indeed a generalisation of Narayana polynomials, because the $(0,1)$-enumeration $H^t_{0,1}(\mu)$ recovers the Narayana polynomials. Explicitly,

$$\mu H^t_{0,1}(\mu) = N(\mu-1).$$

The presence of μ on the left is due to the choice of normalisation for $H^t_{g,n}(\mu_1,\ldots,\mu_n)$, while the shift in μ has occurred to ensure that the topological Narayana polynomials are equal to the monotone Hurwitz numbers when $t=1$. A table of topological Narayana polynomials appears in Section 8.4.

[1] While for the organisation of this chapter, it is logical to begin with the enumeration then prove that it is governed by topological recursion, one of the interesting features of this work is that our approach obtained these in the reverse order.

[2] (Because it denotes the number of bs).

This definition compels us to ask: do topological Narayana polynomials satisfy combinatorial properties analogous to those satisfied by Narayana polynomials? We answer this question in the affirmative in the case of the linear recursion, the quadratic recursion, and the symmetry of coefficients. Further, we posit explicit conjectures that generalise real-rootedness and interlacing to topological Narayana polynomials, which are backed by strong computational evidence. Thus, it could be argued that topological recursion has, in a sense, preserved the significant properties of Narayana polynomials.

This chapter is organised as follows. Section 8.2 discusses the following properties of topological Narayana polynomials:

- a representation theoretic expression for topological Narayana polynomials that leads to a proof that they are governed by topological recursion (Section 8.2.1);
- a proof via topological recursion that topological Narayana polynomials are symmetric (Section 8.2.2);
- the generalisation of the linear recursion (8.2) to a one-point recursion for topological Narayana polynomials (Section 8.2.3);
- the generalisation of the quadratic recursion (8.3) to a cut-and-join recursion for topological Narayana polynomials (Section 8.2.4); and
- explicit conjectures on the real-rootedness and interlacing properties of topological Narayana polynomials (Section 8.2.5).

Section 8.3 concludes the chapter with brief remarks concerning the more general phenomenon of "topologising" combinatorial enumerations and potential connections to matrix integrals.

8.2 Properties

8.2.1 Topological recursion

First, the Jucys–Murphy elements in the symmetric group algebra $J_k \in \mathbb{C}[S_d]$ are defined by

$$J_k = (1\,k) + (2\,k) + \cdots + (k-1\,k),$$

for $k \in \{2, 3, \ldots, d\}$. The Jucys–Murphy elements satisfy remarkable properties [72, 85, 93] and are connected to topological Narayana polynomials via the following result.

Proposition 8.2.1. *The disconnected topological Narayana polynomial $H^{t\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$ is given by the formula*

$$H^{t\bullet}_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{\prod_{i=1}^{n} \mu_i} [C_\mu][x^m] \prod_{k=2}^{|\mu|} \frac{1 - xJ_k + txJ_k}{1 - xJ_k}.$$

Here, $J_2, J_3, \ldots \in \mathbb{C}[S_{|\mu|}]$ are the Jucys–Murphy elements, $m = 2g-2+n+|\mu|$ is the number of transpositions in the corresponding monotone sequence, $[C_\mu]$ signifies that we are extracting the coefficient of C_μ from the resulting expression in the centre $Z\mathbb{C}[S_{|\mu|}]$ of the symmetric group algebra, while $[x^m]$ signifies that we are extracting the coefficient of x^m in the resulting power series.

Proof. Recall that the disconnected topological Narayana polynomial $H^{t\bullet}_{g,n}(\mu_1, \ldots, \mu_n)$ is $\frac{1}{|\mu|!}$ times the weighted enumeration of monotone tuples $(\tau_1, \ldots, \tau_m)$ of transpositions in $S_{|\mu|}$ that compose to give a permutation whose cycles are labelled $1, 2, \ldots, n$ such that cycle i has length μ_i for $i \in \{1, 2, \ldots, n\}$, where a tuple carries the weight t to the power of its hive number. On the other hand, consider evaluating the expression

$$[C_\mu][x^m] \prod_{k=2}^{|\mu|} \left(1 + txJ_k + tx^2 J_k^2 + \cdots\right)$$

by expanding the product and expressing each Jucys–Murphy as a sum of transpositions. The result equals $\frac{1}{|C_\mu|}$ times the weighted enumeration of monotone tuples $(\tau_1, \ldots, \tau_m)$ of transpositions in $S_{|\mu|}$ that compose to give a permutation of cycle type μ, where each tuple carries the weight t to the power of its hive number.

Therefore,

$$H_{g,n}^{t\bullet}(\mu_1,\ldots,\mu_n) = \frac{|\mathrm{Aut}\ \mu|}{|\mu|!}\,|C_\mu|\,[C_\mu][x^m]\prod_{k=2}^{|\mu|}\left(1+txJ_k+tx^2J_k^2+\cdots\right)$$
$$= \frac{1}{\prod_{i=1}^n \mu_i}[C_\mu][x^m]\prod_{k=2}^{|\mu|}\left[1+\frac{txJ_k}{1-xJ_k}\right].$$

The factor $|\mathrm{Aut}\ \mu|$ arises to take into account the fact that the definition of the topological Narayana polynomial requires the cycles of μ to be labelled. ■

The form of the expression given for $H_{g,n}^t(\mu_1,\ldots,\mu_n)$ in the proposition above allows us to utilise the work of Alexandrov, Chapuy, Eynard and Harnad [1] to deduce that topological Narayana polynomials are governed by topological recursion. That is, the proposition above gives us that topological Narayana polynomials are an example of so-called weighted Hurwitz numbers discussed in [1], where the weight is given by $G(z) = \frac{1-z+tz}{1-z}$; this weight matches precisely the form of the expression in the product formula above and indeed, the product formula is what motivates the weight function. That is, the following theorem is an application of Theorem 1.1 in [1] with $G(z) = \frac{1-z+tz}{1-z}$, $S(z) = z$ and $\gamma = 1$. I note that, while the work of Alexandrov, Chapuy, Eynard and Harnad [1] has the assumption that $G(z)$ is a polynomial (whereas in our case it is a rational function), the results in this work were later extended by Bychkov, Dunin-Barkowski, Kazarian and Shadrin [20] to include the case where G is a power series. Thus, we deduce the following result.

Theorem 8.2.2. *The correlation differentials resulting from applying topological recursion to the rational spectral curve* $(\mathbb{CP}^1, x, y, \omega_{0,2})$ *with*

$$x(z) = \frac{z(1-z)}{1-z+tz}, \quad y(z) = \frac{1-z+tz}{1-z}, \quad \textit{and} \quad \omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1-z_2)^2} \tag{8.4}$$

satisfy

$$\omega_{g,n}(z_1,\ldots,z_n) = \delta_{g,0}\,\delta_{n,2}\frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(x_1-x_2)^2} + \mathrm{d}_1\cdots\mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n\geqslant 1} H_{g,n}^t(\mu_1,\ldots,\mu_n)\,x_1^{\mu_1}\cdots x_n^{\mu_n}.$$

Setting $t = 1$ into the spectral curve (8.4) recovers the monotone Hurwitz number spectral curve given by Do, Dyer and Mathews [31].

The above spectral curve is a rational parametrisation of the global spectral curve given by

$$xy^2 + (t-1)xy - y + 1 = 0.$$

8.2.2 Symmetry

The topological Narayana polynomial $H_{g,n}^t(\mu_1,\ldots,\mu_n)$ is a polynomial in t of degree $|\mu|-1$, vanishing at $t=0$; these facts are reasonably clear from the definition of $H_{g,n}^t(\mu_1,\ldots,\mu_n)$. In addition, the topological Narayana polynomial $H_{g,n}^t(\mu_1,\ldots,\mu_n)$ is symmetric, which is the content of the next proposition.

Proposition 8.2.3. *The topological Narayana polynomial* $H_{g,n}^t(\mu_1,\ldots,\mu_n)$ *is symmetric in the sense that its coefficients form a palindromic sequence. That is,*

$$H_{g,n}^t(\mu_1,\ldots,\mu_n)\big|_{t=t^{-1}} = t^{-|\mu|}H_{g,n}^t(\mu_1,\ldots,\mu_n).$$

Proof. I will prove this using the topological recursion for topological Narayana polynomials, Theorem 8.2.2. Take the spectral curve $\mathcal{S} = (\mathbb{CP}^1, x, y, \omega_{0,2})$ defined in (8.4) and define $\tilde{\mathcal{S}} = \mathcal{S}|_{t\mapsto t^{-1}}$. Applying topological recursion to $\tilde{\mathcal{S}}$ yields correlation differentials that satisfy

$$\tilde{\omega}_{g,n}(z_1,\ldots,z_n) = \mathrm{d}_1\cdots\mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n\geqslant 1} H_{g,n}^t(\mu_1,\ldots,\mu_n)\big|_{t\mapsto t^{-1}}\,x_1^{\mu_1}\cdots x_n^{\mu_n}. \tag{$*$}$$

for $(g,n) \neq (0,2)$.

On the other hand, define $\hat{\mathcal{S}} = (\mathbb{CP}^1, \hat{x}, \hat{y}, \omega_{0,2})$ with $\hat{x}(z) = t^{-1}x(z)$ and $\hat{y}(z) = ty(z) + 1 - t$. Applying topological recursion to $\hat{\mathcal{S}}$ yields correlation differentials that satisfy

$$\hat{\omega}_{g,n}(z_1, \ldots, z_n) = \mathrm{d}_1 \cdots \mathrm{d}_n \sum_{\mu_1,\ldots,\mu_n \geqslant 1} H^t_{g,n}(\mu_1, \ldots, \mu_n)\, \hat{x}_1^{\mu_1} \cdots \hat{x}_n^{\mu_n}. \tag{$*$}$$

This is a nontrivial statement. First, it is known that rescaling x or y by a constant c rescales $\omega_{g,n}$ by c^{2-2g-n}; this is a direct application of the homogeneity result of Eynard and Orantin [54, Section 4.1]. Thus given that $\hat{\mathcal{S}}$ has rescaled x by t^{-1} and y by t, the resulting factors in $\hat{\omega}_{g,n}$ cancel. Further, the constant term $1-t$ in $\hat{y}$ does not contribute to $\hat{\omega}_{g,n}$. The only role $\hat{y}$ plays in the definition of the recursion is in the occurrence of $\hat{y}$ in denominator of the recursion kernel $K_\alpha(z_1, z)$, but in this case the constant terms cancel:

$$\hat{y}(z) - \hat{y}(\sigma_\alpha(z)) = ty(z) + 1 - t - (ty(\sigma_\alpha(z)) + 1 - t) = t\,[y(z) - y(\sigma_\alpha(z))]\,.$$

One can verify that both $\tilde{\mathcal{S}}$ and $\hat{\mathcal{S}}$ are parametrisations of the same algebraic curve, satisfying

$$xy^2 + \left(\tfrac{1}{t} - 1\right)xy - y + 1 = 0, \qquad \text{and} \qquad \hat{x}\hat{y}^2 + \left(\tfrac{1}{t} - 1\right)\hat{x}\hat{y} - \hat{y} + 1 = 0.$$

Therefore, $\tilde{\omega}_{g,n} = \hat{\omega}_{g,n}$ under the transformation $x \mapsto \hat{x}$ and $y \mapsto \hat{y}$, and equating coefficients in the asterisked equations yields

$$H^t_{g,n}(\mu_1, \ldots, \mu_n)\big|_{t \mapsto t^{-1}} = t^{-|\mu|} H^t_{g,n}(\mu_1, \ldots, \mu_n)$$

for $(g,n) \neq (0,2)$. In the case of $(g,n) = (0,2)$, $\tilde{\omega}_{0,2} = \hat{\omega}_{0,2} = \omega_{0,2}$, hence,

$$\tilde{\omega}_{0,2} - \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(x_1 - x_2)^2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2} - \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(x_1 - x_2)^2} = \mathrm{d}_1\mathrm{d}_2 \sum_{\mu_1,\mu_2 \geqslant 1} H^t_{0,2}(\mu_1, \mu_2)\big|_{t \mapsto t^{-1}}\, x_1^{\mu_1} x_2^{\mu_2},$$

while

$$\hat{\omega}_{0,2} - \frac{\mathrm{d}\hat{x}_1\,\mathrm{d}\hat{x}_2}{(\hat{x}_1 - \hat{x}_2)^2} = \frac{\mathrm{d}z_1\,\mathrm{d}z_2}{(z_1 - z_2)^2} - \frac{\mathrm{d}x_1\,\mathrm{d}x_2}{(x_1 - x_2)^2} = \mathrm{d}_1\mathrm{d}_2 \sum_{\mu_1,\mu_2 \geqslant 1} H^t_{0,2}(\mu_1, \mu_2)\, \hat{x}_1^{\mu_1}\, \hat{x}_2^{\mu_2}.$$

Thus $H^t_{0,2}(\mu_1, \mu_2)\big|_{t \mapsto t^{-1}} = t^{-|\mu|} H^t_{0,2}(\mu_1, \mu_2)$, as required. ■

The use of the topological recursion in the proof the symmetry result above reveals that this type of symmetry is detectable from the spectral curve. Hence one can seek this type of symmetry in spectral curves more generally, and this might lead to sequences of polynomials that are suitable for generalising via topological recursion.

8.2.3 One-point recursion

The topological Narayana polynomials also satisfy a one-point recursion, which generalises the linear recursion for Narayana polynomials of equation (8.2). This one-point recursion can be proven using a computer, using the Gfun package in Maple [99] and following the techniques of Chaudhuri and Do [23].

Proposition 8.2.4. *For $\mu \geqslant 3$, the topological Narayana polynomials satisfy the following recursion:*

$$\mu^2 H^t_{g,1}(\mu) = (t+1)(2\mu - 3)(\mu - 1)H^t_{g,1}(\mu - 1) - (t-1)^2(\mu - 3)(\mu - 2)H^t_{g,1}(\mu - 2) + \mu^2(\mu - 1)^2 H^t_{g-1,1}(\mu). \tag{8.5}$$

Setting $t = 1$ into the recursion (8.5) above recovers the one-point recursion for monotone Hurwitz numbers of Chaudhuri and Do [23], while setting $g = 0$ recovers the recursion (8.2) for Narayana polynomials, using the fact that $\mu H^t_{0,1}(\mu) = N(\mu - 1)$.

8.2.4 Cut-and-join recursion

The following cut-and-join recursion is a higher topology analogue of the quadratic recursion for Narayana polynomials of equation (8.3), in the sense that restricting it to the case of $(g, n) = (0, 1)$ recovers (8.3). Moreover, setting $t = 1$ recovers the known cut-and-join recursion for monotone Hurwitz numbers of Goulden, Guay-Paquet and Novak [61], and the proof below generalises their argument.

Proposition 8.2.5. *Let $S = \{2, 3, \ldots, n\}$. For all (g, n) and $\mu_1 + \cdots + \mu_n > 1$, the topological Narayana polynomials satisfy the following recursion:*

$$\begin{aligned}\mu_1 H^t_{g,n}(\mu_1, \vec{\mu}_S) = \sum_{i=2}^{n} (\mu_1 + \mu_i) H^t_{g,n-1}(\mu_1 + \mu_i, \vec{\mu}_{S\setminus\{i\}}) + \sum_{\alpha+\beta=\mu_1} \alpha\beta H^t_{g-1,n+1}(\alpha, \beta, \vec{\mu}_S) \\ + \sum_{\alpha+\beta=\mu_1} \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} \alpha\beta H^t_{g_1,|I|+1}(\alpha, \vec{\mu}_I)\, H^t_{g_2,|J|+1}(\beta, \vec{\mu}_J) + (t-1)(\mu_1 - 1) H^t_{g,n}(\mu_1 - 1, \vec{\mu}_S). \quad (8.6)\end{aligned}$$

For $I = \{i_1, \ldots, i_k\}$, the shorthand notation $\vec{\mu}_I$ denotes $\mu_{i_1}, \ldots, \mu_{i_k}$. Along with the base case $H^t_{0,1}(1) = 1$, this uniquely defines all topological Narayana polynomials.

Proof. Fix a permutation $\sigma \in S_{|\mu|}$ with cycle type $(\mu_1, \ldots, \mu_n)$ where the element d is in a cycle of length μ_1. For simplicity, I will say that d is in the μ_1-cycle of σ. While this is not strictly correct because the cycles are not labelled at this point, the analysis and resulting recursion will be the same. If, for example, after we label the cycles the element d happens to be in the cycle labelled 2 (corresponding to μ_2), we would have obtained the same recursion but with a relabelling of μ_1 and μ_2.

Let $(\tau_1, \ldots, \tau_m)$ be a tuple of transpositions satisfying the first, third and fourth conditions given in Definition 8.1.1 and such that

$$\tau_1 \cdots \tau_m = \sigma. \quad (8.7)$$

Let $M^t_{g,n}(\mu_1, \ldots, \mu_n; \sigma)$ be the number of such tuples of transpositions.

The transitivity condition forces $d = |\mu|$ to appear at least once in $\tau_1, \ldots, \tau_m$ while the monotonicity insists that $\tau_m = (a\, d)$ for some $a \in \{1, 2, \ldots, d-1\}$.

Apply τ_m to both sides of (8.7) to obtain

$$\tau_1 \cdots \tau_{m-1} = \sigma\, (a\, d).$$

Applying $(a\, d)$ to σ forces the cycle type of σ to change in one of the following two ways. Either a is not in the same cycle as d and applying $(a\, d)$ joins some μ_i cycle to μ_1, for $i \in \{2, 3, \ldots, n\}$, or a is in the same cycle of σ as d, in which case applying $(a\, d)$ cuts the μ_1-cycle into two.

In the former case, for each cycle μ_i for $i \in \{2, 3, \ldots, n\}$, there are μ_i choices for a. Hence, for each i, there are μ_i such transpositions $(a\, d)$ that give rise to this scenario. In this case, the subgroup $\langle \tau_1, \ldots, \tau_{m-1} \rangle$ must be transitive on $\{1, 2, \ldots, d\}$, therefore, the t-weight for $\tau_1, \ldots, \tau_{m-1}$ is the same as for $\tau_1, \ldots, \tau_m$. All such tuples of transpositions are therefore enumerated by

$$\sum_{i=2}^{n} \mu_i M^t_{g,n-1}(\mu_1 + \mu_i, \vec{\mu}_{S\setminus\{i\}}; \sigma).$$

In the latter case where a is in the μ_1-cycle of σ, one can split the μ_1-cycle into two cycles of lengths α and β for all $\alpha + \beta = \mu_1$. For each such α, β, there is only one transposition $(a\, d)$ that cuts the μ_1-cycle of the fixed permutation σ into two cycles with the correct lengths. If $\langle \tau_1, \ldots, \tau_{m-1} \rangle$ is transitive on $\{1, 2, \ldots, d\}$, then the sequence of transpositions $\tau_1, \ldots, \tau_{m-1}$ has again the same t-weight as $\tau_1, \ldots, \tau_m$ and contributes

$$\sum_{\alpha+\beta=\mu_1} M^t_{g-1,n+1}(\alpha, \beta, \vec{\mu}_S; \sigma).$$

In the case that $\langle \tau_1, \ldots, \tau_{m-1} \rangle$ is not transitive on $\{1, 2, \ldots, d\}$, the only time $\tau_1, \ldots, \tau_{m-1}$ has a different t-weight to $\tau_1, \ldots, \tau_m$ is when $\tau_1, \ldots, \tau_{m-1}$ does not contain d. In this case the weight of the tuple of transpositions $\tau_1, \ldots, \tau_{m-1}$ is precisely one fewer than that of $\tau_1, \ldots, \tau_m$. This scenario contributes to $M^{t\sigma}_{0,1}(1)\, M^{t\sigma}_{g,n}(\mu_1 - 1, \vec{\mu}_S)$. Therefore, the case where τ_m cuts μ_1 and $\tau_1, \ldots, \tau_{m-1}$ is not transitive on $\{1, 2, \ldots, d\}$ contributes

$$\sum_{\alpha+\beta=\mu_1} \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} M^t_{g_1,|I|+1}(\alpha, \vec{\mu}_I; \sigma)\, M^t_{g_2,|J|+1}(\beta, \vec{\mu}_J; \sigma) + (t-1) M^t_{g,n}(\mu_1 - 1, \vec{\mu}_S; \sigma).$$

Therefore,

$$\begin{aligned} M^t_{g,n}(\mu_1, \vec{\mu}_S; \sigma) &= \sum_{i=2}^{n} \mu_i M^t_{g,n-1}(\mu_1 + \mu_i, \vec{\mu}_{S \setminus \{i\}}; \sigma) + \sum_{\alpha+\beta=\mu_1} M^t_{g-1,n+1}(\alpha, \beta, \vec{\mu}_S; \sigma) \\ &+ \sum_{\alpha+\beta=\mu_1} \sum_{\substack{g_1+g_2=g \\ I \sqcup J = S}} M^t_{g_1,|I|+1}(\alpha, \vec{\mu}_I; \sigma)\, M^t_{g_2,|J|+1}(\beta, \vec{\mu}_J; \sigma) + (t-1) M^t_{g,n}(\mu_1 - 1, \vec{\mu}_S; \sigma). \end{aligned} \tag{8.8}$$

Note that the $M^t_{g,n}(\mu_1, \ldots, \mu_n; \sigma)$ enumerates tuples of transpositions that compose to give a fixed permutation $\sigma \in S_{|\mu|}$ of cycle type μ, while the topological Narayana polynomial $H^t_{g,n}(\mu_1, \ldots, \mu_n)$ enumerates tuples of transpositions that give any permutation of cycle type μ, but where the cycles of the permutation are labelled and each tuple is weighted by $\frac{1}{|\mu|!}$. Therefore, pass from the tuples I have enumerated to those enumerated by $H^t_{g,n}(\mu_1, \ldots, \mu_n)$ and carrying the correct weight by

$$H^t_{g,n}(\mu_1, \ldots, \mu_n) = \frac{1}{|\mu|!} |C_\mu|\, |\mathrm{Aut}\ \mu| M^t_{g,n}(\mu_1, \ldots, \mu_n; \sigma) = \frac{1}{\prod_{i=1}^{n} \mu_i} M^t_{g,n}(\mu_1, \ldots, \mu_n; \sigma).$$

The second equality is using that the size of the conjugacy class is $|C_\mu| = \frac{|\mu|!}{|\mathrm{Aut}\ \mu| \prod_{i=1}^{n} \mu_i}$. Applying this substitution to (8.8) and dividing throughout by $\mu_2 \cdots \mu_n$ yields the cut-and-join (8.6). ■

8.2.5 Real-rootedness and interlacing conjectures

Computation of topological Narayana polynomials, some of which appear in Section 8.4, gives strong evidence for the conjectures that topological Narayana polynomials have only real roots and that they interlace in the precise sense described below.

Conjecture 8.2.6. *The topological Narayana polynomial $H^t_{g,n}(\mu_1, \ldots, \mu_n)$ has only real roots.*

To state the interlacing conjecture, I first give a definition of interlacing.

Definition 8.2.7. We say that a polynomial $q(x)$ *interlaces* with a polynomial $p(x)$ if

- $q(x)$ has degree n and $p(x)$ has degree $n+1$, for a positive integer n;
- $q(x)$ has n real roots $b_1 \leqslant \cdots \leqslant b_n$ while $p(x)$ has $n+1$ real roots $a_1 \leqslant \cdots \leqslant a_{n+1}$, counted with multiplicity; and
- $a_1 \leqslant b_1 \leqslant \cdots \leqslant a_n \leqslant b_n \leqslant a_{n+1}$.

The sequence of Narayana polynomials interlace; that is, $N(n)$ interlaces with $N(n+1)$ for all positive integers n. On the other hand, the family of topological Narayana polynomials $H^t_{g,n}(\mu_1, \ldots, \mu_n)$ do not form a single sequence, so one can ask what the analogous notion of interlacing is in this generalised setting. The following conjecture posits that the topological Narayana polynomial $H^t_{g,n}(\mu_1, \ldots, \mu_n)$ interlaces with each topological Narayana polynomial in which one of the arguments has been increased by 1.

Conjecture 8.2.8. *The topological Narayana polynomial $H^t_{g,n}(\mu_1, \ldots, \mu_n)$ interlaces with the topological Narayana polynomials*

$$H^t_{g,n}(\mu_1 + 1, \mu_2, \ldots, \mu_n), H^t_{g,n}(\mu_1, \mu_2 + 1, \ldots, \mu_n), \ldots, H^t_{g,n}(\mu_1, \mu_2, \ldots, \mu_n + 1).$$

Interlacing for the usual Narayana polynomials can be proved using what I call the linear recursion (8.2):

$$N(n) = \frac{1}{n+1}\left[(2n-1)(t+1)N(n-1) - (n-2)(t-1)^2N(n-2)\right].$$

The idea of the proof proceeds by induction on n: assume that $N(n-2)$ interlaces with $N(n-1)$, substitute the roots of $N(n-1)$ into the recursion above, and use a sign argument along with the intermediate value theorem to deduce that $N(n-1)$ interlaces with $N(n)$.

Generalising this argument for topological Narayana polynomials is met with a number of obstructions. First, neither the one-point nor the cut-and-join recursion for topological Narayana polynomials has the three-term structure exhibited in the linear recursion, hence the sign argument doesn't follow through in the same way. To use a sign argument in conjunction with the one-point recursion for topological Narayana polynomials (8.5), one comes up against the issue that the term $\mu^2(\mu-1)^2H^t_{g-1,1}(\mu)$ prevents a straightforward inductive argument from carrying through. Furthermore, this would only prove that $H_{g,1}(\mu)$ interlaces with $H_{g,1}(\mu-1)$, and one would require a new idea to extend this to general $n > 1$. On the other hand, to use a sign argument in conjunction with the cut-and-join recursion (8.6) one would require a much stronger inductive assumption on the locations of the roots of the topological Narayana polynomials that feature on the right side of the recursion.

The data given in Section 8.4 affirms both the real-rootedness and interlacing conjectures.

8.3 Remarks

While the idea of using topological recursion to generalise natural and pre-existing combinatorial problems is relatively unexplored, one can wonder whether the apparent phenomenon exhibited here is more general. That is, to what extent does generalising enumerative problems via topological recursion preserve properties of the original enumerations? In particular, it would be interesting to find other examples of the conjectural real-rootedness and interlacing phenomena described above. A natural place to start such an exploration is with sequences of polynomials that are known to satisfy such properties, for example, orthogonal polynomials. On the other hand, one can detect the symmetry property of topological Narayana polynomials from the spectral curve (8.4); this idea could also lead to other natural candidates to study.

In the process of generalising Narayana polynomials via topological recursion, a choice was made in the way that the Narayana polynomials are stored in the spectral curve. As mentioned in the introduction, as a refinement of the Catalan numbers, Narayana numbers are stored in the $(0,1)$-enumeration of both ribbon graphs and monotone Hurwitz numbers; the way in which each of these enumerations is stored in their respective spectral curves differs. Thus, we could have instead deformed the ribbon graph spectral curve to generalise Narayana polynomials and one might wonder what polynomials are generated in this case. Explicitly, the enumerations of ribbon graphs and monotone Hurwitz numbers in the case of $(g,n) = (0,1)$ (that is, Catalan numbers) are stored in the $(0,1)$ correlation differentials in the following different ways:

$$\begin{aligned}
\omega^{\mathrm{MH}}_{0,1} &= \mathrm{d}\sum_{\mu\geqslant 1} H^{\leqslant}_{0,1}(\mu)\, x^{\mu} = \sum_{\mu\geqslant 1} \mu\, H^{\leqslant}_{0,1}(\mu)\, x^{\mu-1}\,\mathrm{d}x = \sum_{\mu\geqslant 1} \mathrm{Cat}_{\mu-1}\, x^{\mu-1}\,\mathrm{d}x\\
\omega^{\mathrm{RG}}_{0,1} &= \mathrm{d}\sum_{\mu\geqslant 1} \mathrm{R}_{0,1}(\mu)\, x^{-\mu} = \sum_{\mu\geqslant 1} -\mu\,\mathrm{R}_{0,1}(\mu)\, x^{-\mu-1}\,\mathrm{d}x = \sum_{\mu\geqslant 1} -\mathrm{Cat}_{\mu}\, x^{-2\mu-1}\,\mathrm{d}x.
\end{aligned}$$

The final equalities in each line are using that $\mu H^{\leqslant}_{0,1}(\mu) = \mathrm{Cat}_{\mu-1}$ and $2\mu\,\mathrm{R}_{0,1}(2\mu) = \mathrm{Cat}_{\mu}$ respectively. If one uses the ribbon graph enumeration as the framework for generalising Narayana polynomials, one obtains the following global spectral curve:

$$xy^2 + ((t-1) - x^2)y + x = 0.$$

This is a genus 1 curve; it is well-known that dealing with positive genus spectral curves is necessarily more difficult than genus 0. The analysis may still be possible though and is deferred to future work.

Monotone Hurwitz numbers first appeared in the literature in a series of papers by Goulden, Guay-Paquet and Novak [60, 61, 62] where they featured as coefficients of the large N expansion of the Harish-Chandra–Itzukson–Zuber (HCIZ) matrix integral. Separately, though through a similar mechanism, monotone Hurwitz numbers make an appearance as the coefficients of the large N expansion of the cumulants of the inverse Laguerre unitary ensemble [28]. It would be interesting to know if either of these matrix models can be deformed to introduce a t-parameter and recover the topological Narayana polynomials. In particular, the representation theoretic interpretation, Proposition 8.2.1, lends itself well to potentially being studied via the Weingarten calculus to reverse engineer a matrix integral that stores the topological Narayana polynomials as coefficients of its large N expansion [27].

8.4 Data

The following data was calculated in SageMath [98] using the cut-and-join recursion in Proposition 8.2.5.

g	$(\mu_1, \ldots, \mu_n)$	$\mu_1 \cdots \mu_n H^t_{g,n}(\mu_1, \ldots, \mu_n)$
0	(1)	1
0	(2)	t
0	(3)	$t^2 + t$
0	(4)	$t^3 + 3t^2 + t$
0	(5)	$t^4 + 6t^3 + 6t^2 + t$
0	(6)	$t^5 + 10t^4 + 20t^3 + 10t^2 + t$
0	(7)	$t^6 + 15t^5 + 50t^4 + 50t^3 + 15t^2 + t$
0	(8)	$t^7 + 21t^6 + 105t^5 + 175t^4 + 105t^3 + 21t^2 + t$
0	$(1,1)$	t
0	$(2,1)$	$2t^2 + 2t$
0	$(3,1)$	$3t^3 + 9t^2 + 3t$
0	$(2,2)$	$4t^3 + 10t^2 + 4t$
0	$(4,1)$	$4t^4 + 24t^3 + 24t^2 + 4t$
0	$(3,2)$	$6t^4 + 30t^3 + 30t^2 + 6t$
0	$(5,1)$	$5t^5 + 50t^4 + 100t^3 + 50t^2 + 5t$
0	$(4,2)$	$8t^5 + 68t^4 + 128t^3 + 68t^2 + 8t$
0	$(3,3)$	$9t^5 + 72t^4 + 138t^3 + 72t^2 + 9t$
0	$(1,1,1)$	$4t^2 + 4t$
0	$(2,1,1)$	$10t^3 + 28t^2 + 10t$
0	$(3,1,1)$	$18t^4 + 102t^3 + 102t^2 + 18t$
0	$(2,2,1)$	$24t^4 + 120t^3 + 120t^2 + 24t$
0	$(4,1,1)$	$28t^5 + 268t^4 + 528t^3 + 268t^2 + 28t$
0	$(3,2,1)$	$42t^5 + 348t^4 + 660t^3 + 348t^2 + 42t$
0	$(2,2,2)$	$56t^5 + 424t^4 + 768t^3 + 424t^2 + 56t$

g	$(\mu_1, \ldots, \mu_n)$	$\mu_1 \cdots \mu_n H^t_{g,n}(\mu_1, \ldots, \mu_n)$
1	(2)	t
1	(3)	$5t^2 + 5t$
1	(4)	$15t^3 + 40t^2 + 15t$
1	(5)	$35t^4 + 175t^3 + 175t^2 + 35t$
1	(6)	$70t^5 + 560t^4 + 1050t^3 + 560t^2 + 70t$
1	(7)	$210t^7 + 3360t^6 + 14700t^5 + 23520t^4 + 14700t^3 + 3360t^2 + 210t$
1	(8)	$330t^8 + 6930t^7 + 41580t^6 + 97020t^5 + 97020t^4 + 41580t^3 + 6930t^2 + 330t$
1	(1, 1)	t
1	(2, 1)	$10t^2 + 10t$
1	(3, 1)	$45t^3 + 120t^2 + 45t$
1	(2, 2)	$50t^3 + 128t^2 + 50t$
1	(4, 1)	$140t^4 + 700t^3 + 700t^2 + 140t$
1	(3, 2)	$168t^4 + 792t^3 + 792t^2 + 168t$
1	(5, 1)	$350t^5 + 2800t^4 + 5250t^3 + 2800t^2 + 350t$
1	(4, 2)	$448t^5 + 3348t^4 + 6128t^3 + 3348t^2 + 448t$
1	(3, 3)	$462t^5 + 3432t^4 + 6312t^3 + 3432t^2 + 462t$
1	(1, 1, 1)	$20t^2 + 20t$
1	(2, 1, 1)	$140t^3 + 368t^2 + 140t$
1	(3, 1, 1)	$588t^4 + 2892t^3 + 2892t^2 + 588t$
1	(2, 2, 1)	$672t^4 + 3168t^3 + 3168t^2 + 672t$
1	(4, 1, 1)	$1848t^5 + 14548t^4 + 27128t^3 + 14548t^2 + 1848t$
1	(3, 2, 1)	$672t^4 + 3168t^3 + 3168t^2 + 672t$
1	(2, 2, 2)	$2688t^5 + 19128t^4 + 34416t^3 + 19128t^2 + 2688t$
2	(2)	t
2	(3)	$21t^2 + 21t$
2	(4)	$161t^3 + 413t^2 + 161t$
2	(5)	$777t^4 + 3612t^3 + 3612t^2 + 777t$
2	(6)	$2835t^5 + 20538t^4 + 37338t^3 + 20538t^2 + 2835t$
2	(7)	$8547t^6 + 88473t^5 + 251328t^4 + 251328t^3 + 88473t^2 + 8547t$
2	(8)	$22407t^7 + 313005t^6 + 1273041t^5 + 1990296t^4 + 1273041t^3 + 313005t^2 + 22407t$
2	(1, 1)	t
2	(2, 1)	$42t^2 + 42t$
2	(3, 1)	$483t^3 + 1239t^2 + 483t$
2	(2, 2)	$504t^3 + 1278t^2 + 504t$
2	(4, 1)	$3108t^4 + 14448t^3 + 14448t^2 + 3108t$
2	(3, 2)	$3402t^4 + 15450t^3 + 15450t^2 + 3402t$
2	(5, 1)	$14175t^5 + 102690t^4 + 186690t^3 + 102690t^2 + 14175t$
2	(4, 2)	$16296t^5 + 114256t^4 + 205376t^3 + 114256t^2 + 16296t$
2	(3, 3)	$16443t^5 + 115344t^4 + 207666t^3 + 115344t^2 + 16443t$
2	(1, 1, 1)	$84t^2 + 84t$
2	(2, 1, 1)	$1470t^3 + 3756t^2 + 1470t$
2	(3, 1, 1)	$12726t^4 + 58794t^3 + 58794t^2 + 12726t$
2	(2, 2, 1)	$13608t^4 + 61800t^3 + 61800t^2 + 13608t$
2	(4, 1, 1)	$72996t^5 + 525016t^4 + 952136t^3 + 525016t^2 + 72996t$
2	(3, 2, 1)	$81774t^5 + 573456t^4 + 1031460t^3 + 573456t^2 + 81774t$
2	(2, 2, 2)	$90552t^5 + 619728t^4 + 1104624t^3 + 619728t^2 + 90552t$

Bibliography

[1] A. Alexandrov, G. Chapuy, B. Eynard and J. Harnad. 'Weighted Hurwitz numbers and topological recursion'. *Comm. Math. Phys.* 375.1 (2020). math-ph/1806.09738, pp. 237–305.

[2] A. Alexandrov, D. Lewański and S. Shadrin. 'Ramifications of Hurwitz theory, KP integrability and quantum curves'. *J. High Energy Phys.* 5 (2016). math-ph/1512.07026, 124, front matter+30.

[3] J. E. Andersen, G. Borot, S. Charbonnier, V. Delecroix, A. Giacchetto, D. Lewański and C. Wheeler. 'Topological recursion for Masur–Veech volumes'. *math.GT/1905.10352* (2019).

[4] J. E. Andersen, L. O. Chekhov, P. Norbury and R. C. Penner. 'Models of discretized moduli spaces, cohomological field theories, and Gaussian means'. *J. Geom. Phys.* 98 (2015), pp. 312–339.

[5] S. Bloch and A. Okounkov. 'The character of the infinite wedge representation'. *Adv. Math.* 149.1 (2000). math.AG/9712009, pp. 1–60.

[6] G. Borot, S. Charbonnier, N. Do and E. Garcia-Failde. 'Relating ordinary and fully simple maps via monotone Hurwitz numbers'. *Electron. J. Combin.* 26.3 (2019). math.CO/1904.02267, Paper No. 3.43, 24.

[7] G. Borot, S. Charbonnier and E. Garcia-Failde. 'Topological recursion for fully simple maps from ciliated maps'. *arXiv preprint arXiv:2106.09002* (2021). math.CO/2106.09002.

[8] G. Borot, N. Do, M. Karev, D. Lewański and E. Moskovsky. 'Double Hurwitz numbers: polynomiality, topological recursion and intersection theory'. *to appear in Math. Ann.* (2022). math-ph/2002.00900.

[9] G. Borot and B. Eynard. 'All order asymptotics of hyperbolic knot invariants from non-perturbative topological recursion of A-polynomials'. *Quantum Topol.* 6.1 (2015). math-ph/1205.2261, pp. 39–138.

[10] G. Borot, B. Eynard and N. Orantin. 'Abstract loop equations, topological recursion and new applications'. *Commun. Number Theory Phys.* 9.1 (2015). math-ph/1303.5808, pp. 51–187.

[11] G. Borot and E. Garcia-Failde. 'Simple maps, Hurwitz numbers, and topological recursion'. *Comm. Math. Phys.* 380.2 (2020). math-ph/1710.07851, pp. 581–654.

[12] G. Borot and S. Shadrin. 'Blobbed topological recursion: properties and applications'. *Math. Proc. Cambridge Philos. Soc.* 162.1 (2017). math-ph/1502.00981, pp. 39–87.

[13] V. Bouchard and B. Eynard. 'Think globally, compute locally'. *J. High Energy Phys.* 2 (2013). math-ph/1211.2302, 143, front matter + 34.

[14] V. Bouchard, D. Hernández Serrano, X. Liu and M. Mulase. 'Mirror symmetry for orbifold Hurwitz numbers'. *J. Differential Geom.* 98.3 (2014). math.AG/1301.4871, pp. 375–423.

[15] V. Bouchard, J. Hutchinson, P. Loliencar, M. Meiers and M. Rupert. 'A generalized topological recursion for arbitrary ramification'. *Ann. Henri Poincaré* 15.1 (2014), pp. 143–169.

[16] V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti. 'Remodeling the B-model'. *Comm. Math. Phys.* 287.1 (2009). hep-th/1301.4871, pp. 117–178.

[17] V. Bouchard and M. Mariño. 'Hurwitz numbers, matrix models and enumerative geometry'. *From Hodge theory to integrability and TQFT tt*-geometry.* Vol. 78. Proc. Sympos. Pure Math. math.AG/0709.1458. Amer. Math. Soc., Providence, RI, 2008, pp. 263–283.

[18] E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber. 'Planar diagrams'. *Comm. Math. Phys.* 59.1 (1978), pp. 35–51.

[19] B. Bychkov, P. Dunin-Barkowski, M. Kazarian and S. Shadrin. 'Generalised ordinary vs fully simple duality for n-point functions and a proof of the Borot–Garcia-Failde conjecture'. *arXiv preprint arXiv:2106.08368* (2021). math-ph/2106.08368.

[20] B. Bychkov, P. Dunin-Barkowski, M. Kazarian and S. Shadrin. 'Topological recursion for Kadomtsev-Petviashvili tau functions of hypergeometric type'. *arXiv preprint arXiv:2106.08368* (2020). math-ph/2012.14723.

[21] R. Cavalieri, P. Johnson and H. Markwig. 'Tropical Hurwitz numbers'. *J. Algebraic Combin.* 32.2 (2010). math.AG/0804.0579, pp. 241–265.

[22] R. Cavalieri and E. Miles. *Riemann surfaces and algebraic curves.* Vol. 87. London Mathematical Society Student Texts. A first course in Hurwitz theory. Cambridge University Press, Cambridge, 2016, pp. xii+183.

[23] A. Chaudhuri and N. Do. 'Generalisations of the Harer-Zagier recursion for 1-point functions'. *J. Algebraic Combin.* 53.2 (2021). math-ph/1812.11885, pp. 469–503.

[24] A. Chaudhuri, N. Do and E. Moskovsky. 'Local topological recursion governs the enumeration of lattice points in $\overline{\mathcal{M}}_{g,n}$'. *arXiv preprint arXiv:1906.06964* (2019).

[25] L. Chekhov and B. Eynard. 'Hermitian matrix model free energy: Feynman graph technique for all genera'. *J. High Energy Phys.* 3 (2006). hep-th/0504116, pp. 014, 18.

[26] A. Chiodo. 'Towards an enumerative geometry of the moduli space of twisted curves and rth roots'. *Compos. Math.* 144.6 (2008). math.AG/0607324, pp. 1461–1496.

[27] B. Collins. 'Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability'. *Int. Math. Res. Not.* 17 (2003). math-ph/0205010, pp. 953–982.

[28] F. D. Cunden, A. Dahlqvist and N. O'Connell. 'Integer moments of complex Wishart matrices and Hurwitz numbers'. *Ann. Inst. Henri Poincaré D* 8.2 (2021). math-ph/1809.10033, pp. 243–268.

[29] E. Date, M. Jimbo, M. Kashiwara and T. Miwa. 'Transformation groups for soliton equations. III. Operator approach to the Kadomtsev-Petviashvili equation'. *J. Phys. Soc. Japan* 50.11 (1981), pp. 3806–3812.

[30] R. Dijkgraaf, H. Fuji and M. Manabe. 'The volume conjecture, perturbative knot invariants, and recursion relations for topological strings'. *Nuclear Phys. B* 849.1 (2011). hep-th/1010.4542, pp. 166–211.

[31] N. Do, A. Dyer and D. V. Mathews. 'Topological recursion and a quantum curve for monotone Hurwitz numbers'. *J. Geom. Phys.* 120 (2017). math.GT/1408.3992, pp. 19–36.

[32] N. Do and M. Karev. 'Towards the topological recursion for double Hurwitz numbers'. *Topological recursion and its influence in analysis, geometry, and topology.* Vol. 100. Proc. Sympos. Pure Math. math.GT/1811.05107. Amer. Math. Soc., Providence, RI, 2018, pp. 151–178.

[33] N. Do, O. Leigh and P. Norbury. 'Orbifold Hurwitz numbers and Eynard-Orantin invariants'. *Math. Res. Lett.* 23.5 (2016). math.AG/1212.6850, pp. 1281–1327.

[34] N. Do and D. Manescu. 'Quantum curves for the enumeration of ribbon graphs and hypermaps'. *Commun. Number Theory Phys.* 8.4 (2014). math.GT/1312.6869, pp. 677–701.

[35] N. Do and P. Norbury. 'Counting lattice points in compactified moduli spaces of curves'. *Geom. Topol.* 15.4 (2011). math.GT/1012.5923, pp. 2321–2350.

[36] O. Dumitrescu, M. Mulase, B. Safnuk and A. Sorkin. 'The spectral curve of the Eynard-Orantin recursion via the Laplace transform'. *Algebraic and geometric aspects of integrable systems and random matrices.* Vol. 593. Contemp. Math. math.AG/1202.1159. Amer. Math. Soc., Providence, RI, 2013, pp. 263–315.

[37] P. Dunin-Barkowski. 'Topological recursion and Givental's formalism: spectral curves for Gromov-Witten theories'. *Topological recursion and its influence in analysis, geometry, and topology.* Vol. 100. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2018, pp. 231–295.

[38] P. Dunin-Barkowski, M. Kazarian, A. Popolitov, S. Shadrin and A. Sleptsov. 'Topological recursion for the extended Ooguri–Vafa partition function of colored HOMFLY-PT polynomials of torus knots'. *arXiv preprint arXiv:2010.11021* (2010). math-ph/2010.11021.

[39] P. Dunin-Barkowski, R. Kramer, A. Popolitov and S. Shadrin. 'Loop equations and a proof of Zvonkine's qr-ELSV formula' (2019). math.AG/1905.04524.

[40] P. Dunin-Barkowski, D. Lewański, A. Popolitov and S. Shadrin. 'Polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson-Pandharipande-Tseng formula'. *J. Lond. Math. Soc. (2)* 92.3 (2015). math-ph/1504.07440, pp. 547–565.

[41] P. Dunin-Barkowski, N. Orantin, A. Popolitov and S. Shadrin. 'Combinatorics of loop equations for branched covers of sphere'. *Int. Math. Res. Not. IMRN* 18 (2018). math-ph/1412.1698, pp. 5638–5662.

[42] P. Dunin-Barkowski, N. Orantin, S. Shadrin and L. Spitz. 'Identification of the Givental formula with the spectral curve topological recursion procedure'. *Comm. Math. Phys.* 328.2 (2014). math-ph/1211.4021, pp. 669–700.

[43] T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein. 'Hurwitz numbers and intersections on moduli spaces of curves'. *Invent. Math.* 146.2 (2001). math.AG/0004096, pp. 297–327.

[44] A. Eskin, A. Okounkov and R. Pandharipande. 'The theta characteristic of a branched covering'. *Adv. Math.* 217.3 (2008). math.AG/0312186, pp. 873–888.

[45] B. Eynard. *Counting surfaces.* Vol. 70. Progress in Mathematical Physics. CRM Aisenstadt chair lectures. Birkhäuser/Springer, [Cham], 2016, pp. xvii+414.

[46] B. Eynard. 'Invariants of spectral curves and intersection theory of moduli spaces of complex curves'. *Commun. Number Theory Phys.* 8.3 (2014). math-ph/1110.2949, pp. 541–588.

[47] B. Eynard. 'Recursion between Mumford volumes of moduli spaces'. *Ann. Henri Poincaré* 12.8 (2011). math.AG/0706.4403, pp. 1431–1447.

[48] B. Eynard. 'Topological expansion for the 1-Hermitian matrix model correlation functions'. *J. High Energy Phys.* 11 (2004). hep-th/0407261, 031, 35 pp. (2005).

[49] B. Eynard, M. Mulase and B. Safnuk. 'The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers'. *Publ. Res. Inst. Math. Sci.* 47.2 (2011). math.AG/0907.5224, pp. 629–670.

[50] B. Eynard and N. Orantin. 'About the x-y symmetry of the F_g algebraic invariants'. *arXiv preprint arXiv:1311.4993* (2013). math-ph/1311.4993.

[51] B. Eynard and N. Orantin. 'Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture'. *Comm. Math. Phys.* 337.2 (2015). math-ph/1205.1103, pp. 483–567.

[52] B. Eynard and N. Orantin. 'Invariants of algebraic curves and topological expansion'. *Commun. Number Theory Phys.* 1.2 (2007). math-ph/0702045, pp. 347–452.

[53] B. Eynard and N. Orantin. 'Topological expansion of mixed correlations in the Hermitian 2-matrix model and x-y symmetry of the F_g algebraic invariants'. *J. Phys. A* 41.1 (2008). math-ph/0705.0958, pp. 015203, 28.

[54] B. Eynard and N. Orantin. 'Topological recursion in enumerative geometry and random matrices'. *J. Phys. A* 42.29 (2009). math-ph/0811.3531, pp. 293001, 117.

[55] B. Eynard and N. Orantin. 'Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models'. *arXiv preprint arXiv:0705.3600* (2007). math-ph/0705.3600.

[56] B. Fang, C.-C. M. Liu and Z. Zong. 'All-genus open-closed mirror symmetry for affine toric Calabi-Yau 3-orbifolds'. *Algebr. Geom.* 7.2 (2020). math.AG/1310.4818, pp. 192–239.

[57] S. Fisk. 'Polynomials, roots, and interlacing'. *arXiv preprint math/0612833* (2006). math.CA/0612833.

[58] E. Garcia-Failde. 'On discrete surfaces: Enumerative geometry, matrix models and universality classes via topological recursion'. *arXiv preprint arXiv:2002.00316* (2020). math-ph/2002.00316.

[59] A. Giacchetto, R. Kramer and D. Lewański. 'A new spin on Hurwitz theory and ELSV via theta characteristics'. *arXiv preprint arXiv:2104.05697* (2021). math-ph/2104.05697.

[60] I. P. Goulden, M. Guay-Paquet and J. Novak. 'Monotone Hurwitz numbers and the HCIZ integral'. *Ann. Math. Blaise Pascal* 21.1 (2014). math.CO/1210.3415, pp. 71–89.

[61] I. P. Goulden, M. Guay-Paquet and J. Novak. 'Monotone Hurwitz numbers in genus zero'. *Canad. J. Math.* 65.5 (2013). math.CO/1204.2618, pp. 1020–1042.

[62] I. P. Goulden, M. Guay-Paquet and J. Novak. 'Polynomiality of monotone Hurwitz numbers in higher genera'. *Adv. Math.* 238 (2013). math.CO/1210.3415, pp. 1–23.

[63] I. P. Goulden and D. M. Jackson. 'The number of ramified coverings of the sphere by the double torus, and a general form for higher genera'. *J. Combin. Theory Ser. A* 88.2 (1999). math.AG/9902011, pp. 259–275.

[64] I. P. Goulden and D. M. Jackson. 'Transitive factorisations into transpositions and holomorphic mappings on the sphere'. *Proc. Amer. Math. Soc.* 125.1 (1997), pp. 51–60.

[65] I. P. Goulden, D. M. Jackson and A. Vainshtein. 'The number of ramified coverings of the sphere by the torus and surfaces of higher genera'. *Ann. Comb.* 4.1 (2000). math.AG/9902125, pp. 27–46.

[66] I. P. Goulden, D. M. Jackson and R. Vakil. 'Towards the geometry of double Hurwitz numbers'. *Adv. Math.* 198.1 (2005). math.AG/0309440, pp. 43–92.

[67] J. L. Harer. 'The virtual cohomological dimension of the mapping class group of an orientable surface'. *Invent. Math.* 84.1 (1986), pp. 157–176.

[68] J. Harris and I. Morrison. *Moduli of curves*. Vol. 187. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xiv+366.

[69] A. Hurwitz. 'Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten'. *Math. Ann.* 39.1 (1891), pp. 1–60.

[70] P. Johnson, R. Pandharipande and H.-H. Tseng. 'Abelian Hurwitz-Hodge integrals'. *Michigan Math. J.* 60.1 (2011). math.AG/0803.0499, pp. 171–198.

[71] A. J. de Jong et al. *The stacks project*. stacks.math.columbia.edu. 2013.

[72] A.-A. A. Jucys. 'Symmetric polynomials and the center of the symmetric group ring'. *Rep. Mathematical Phys.* 5.1 (1974), pp. 107–112.

[73] V. G. Kac. *Infinite-dimensional Lie algebras*. Third. Cambridge University Press, Cambridge, 1990, pp. xxii+400.

[74] M. Kazarian and P. Zograf. 'Virasoro constraints and topological recursion for Grothendieck's dessin counting'. *Lett. Math. Phys.* 105.8 (2015). math.CO/1406.5976v3, pp. 1057–1084.

[75] M. Kontsevich. 'Intersection theory on the moduli space of curves and the matrix Airy function'. *Comm. Math. Phys.* 147.1 (1992), pp. 1–23.

[76] M. Kontsevich and Y. Soibelman. 'Airy structures and symplectic geometry of topological recursion'. *Topological recursion and its influence in analysis, geometry, and topology*. Vol. 100. Proc. Sympos. Pure Math. math.AG/1701.09137. Amer. Math. Soc., Providence, RI, 2018, pp. 433–489.

[77] R. Kramer, D. Lewański and S. Shadrin. 'Quasi-polynomiality of monotone orbifold Hurwitz numbers and Grothendieck's dessins d'enfants'. *Doc. Math.* 24 (2019). math.CO/1610.08376, pp. 857–898.

[78] G. Kreweras and P. Moszkowski. 'A new enumerative property of the Narayana numbers'. *J. Statist. Plann. Inference* 14.1 (1986), pp. 63–67.

[79] S. K. Lando and A. K. Zvonkin. *Graphs on surfaces and their applications*. Vol. 141. Encyclopaedia of Mathematical Sciences. With an appendix by Don B. Zagier, Low-Dimensional Topology, II. Springer-Verlag, Berlin, 2004, pp. xvi+455.

[80] D. Lewański, A. Popolitov, S. Shadrin and D. Zvonkine. 'Chiodo formulas for the r-th roots and topological recursion'. *Lett. Math. Phys.* 107.5 (2017). math-ph/1504.07439, pp. 901–919.

[81] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Second. Oxford Mathematical Monographs. With contributions by A. Zelevinsky, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995, pp. x+475.

[82] Maplesoft, a division of Waterloo Maple Inc.. *Maple.* Version 2016. Waterloo, Ontario, 2019.

[83] T. Miwa, M. Jimbo and E. Date. *Solitons.* Vol. 135. Cambridge Tracts in Mathematics. Differential equations, symmetries and infinite-dimensional algebras, Translated from the 1993 Japanese original by Miles Reid. Cambridge University Press, Cambridge, 2000, pp. x+108.

[84] M. Mulase, S. Shadrin and L. Spitz. 'The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures'. *Commun. Number Theory Phys.* 7.1 (2013). math.AG/1301.5580, pp. 125–143.

[85] G. E. Murphy. 'A new construction of Young's seminormal representation of the symmetric groups'. *J. Algebra* 69.2 (1981), pp. 287–297.

[86] T. V. Narayana. 'Sur les treillis formés par les partitions d'un entier et leurs applications à la théorie des probabilités'. *C. R. Acad. Sci. Paris* 240 (1955), pp. 1188–1189.

[87] P. Norbury. 'Counting lattice points in the moduli space of curves'. *Math. Res. Lett.* 17.3 (2010). math.AG/0801.4590, pp. 467–481.

[88] P. Norbury. 'String and dilaton equations for counting lattice points in the moduli space of curves'. *Trans. Amer. Math. Soc.* 365.4 (2013). math.AG/0905.4141, pp. 1687–1709.

[89] P. Norbury and N. Scott. 'Gromov-Witten invariants of $\mathbb{P}^1$ and Eynard-Orantin invariants'. *Geom. Topol.* 18.4 (2014). math.AG/1106.1337, pp. 1865–1910.

[90] A. Okounkov. 'Infinite wedge and random partitions'. *Selecta Math. (N.S.)* 7.1 (2001). math.RT/9907127, pp. 57–81.

[91] A. Okounkov. 'Toda equations for Hurwitz numbers'. *Math. Res. Lett.* 7.4 (2000). math.AG/0004128, pp. 447–453.

[92] A. Okounkov and R. Pandharipande. 'Gromov-Witten theory, Hurwitz theory, and completed cycles'. *Ann. of Math. (2)* 163.2 (2006). math.AG/0204305, pp. 517–560.

[93] A. Okounkov and A. Vershik. 'A new approach to representation theory of symmetric groups'. *Selecta Math. (N.S.)* 2.4 (1996), pp. 581–605.

[94] R. C. Penner. 'The decorated Teichmüller space of punctured surfaces'. *Comm. Math. Phys.* 113.2 (1987), pp. 299–339.

[95] T. K. Petersen. 'Eulerian numbers'. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. With a foreword by Richard Stanley. Birkhäuser/Springer, New York, 2015, pp. xviii+456.

[96] R. Ríos-Zertuche. 'An introduction to the half-infinite wedge'. *Mexican mathematicians abroad: recent contributions.* Vol. 657. Contemp. Math. math.RT/1309.0054. Amer. Math. Soc., Providence, RI, 2016, pp. 197–237.

[97] B. E. Sagan. *The symmetric group.* Second. Vol. 203. Graduate Texts in Mathematics. Representations, combinatorial algorithms, and symmetric functions. Springer-Verlag, New York, 2001, pp. xvi+238.

[98] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.6).* https://www.sagemath.org. 2022.

[99] B. Salvy and P. Zimmerman. 'Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable'. *ACM Trans. Math. Softw.* 20.2 (June 1994), pp. 163–177.

[100] M. Sato and Y. Sato. 'Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold'. *Nonlinear partial differential equations in applied science (Tokyo, 1982).* Vol. 81. North-Holland Math. Stud. North-Holland, Amsterdam, 1983, pp. 259–271.

[101] S. Shadrin, M. Shapiro and A. Vainshtein. 'Chamber behavior of double Hurwitz numbers in genus 0'. *Adv. Math.* 217.1 (2008). math.AG/0611442, pp. 79–96.

[102] S. Shadrin, L. Spitz and D. Zvonkine. 'Equivalence of ELSV and Bouchard-Mariño conjectures for r-spin Hurwitz numbers'. *Math. Ann.* 361.3-4 (2015). math.AG/1306.6226, pp. 611–645.

[103] S. Shadrin, L. Spitz and D. Zvonkine. 'On double Hurwitz numbers with completed cycles'. *J. Lond. Math. Soc. (2)* 86.2 (2012). math.CO/1103.3120, pp. 407–432.

[104] W. T. Tutte. 'A census of Hamiltonian polygons'. *Canadian J. Math.* 14 (1962), pp. 402–417.

[105] W. T. Tutte. 'A census of planar maps'. *Canadian J. Math.* 15 (1963), pp. 249–271.

[106] T. R. S. Walsh and A. B. Lehman. 'Counting rooted maps by genus. I'. *J. Combinatorial Theory Ser. B* 13 (1972), pp. 192–218.

[107] E. Witten. 'Two-dimensional gravity and intersection theory on moduli space'. *Surveys in differential geometry (Cambridge, MA, 1990).* Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.

[108] A. Zvonkin. 'Matrix integrals and map enumeration: an accessible introduction'. Vol. 26. 8-10. Combinatorics and physics (Marseilles, 1995). 1997, pp. 281–304.