Abstract of "Aspects of Scattering Amplitudes: Symmetry and Duality" by Dung Nguyen, Ph.D., Brown University, May 2011.

In recent years there have been significant progresses in the understanding of scattering amplitudes at both strong and weak coupling. There are new dualities discovered, new symmetries demystified and new hidden structures unearthed. And we will discuss these aspects of planar scattering amplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory in this dissertation.

Firstly, we review the discovery and development of a new symmetry: the dual superconformal symmetry. Inspired by this we present our study of dual conformally invariant off-shell four-point Feynman diagrams. We classify all such diagrams through four loops and evaluate 10 new off-shell integrals in terms of Mellin-Barnes representations, also finding explicit expression for their infrared singularities.

Secondly, we discuss about the recent progress on the calculation of Wilson loops at both strong and weak coupling. A remarkable feature of light-like polygon Wilson loops is their conjectured duality to MHV scattering amplitudes, which apparently holds at all orders in perturbation as well as non-perturbation theories. We demonstrate, by explicit calculation, the completely unanticipated fact that the duality continues to hold at two loops through order epsilon in dimensional regularization for both the four-particle amplitude and the (parity-even part of the) five-particle amplitude.

Finally, we discuss about the structure of scattering amplitudes in $\mathcal{N} = 8$ Super Gravity theory, which shares many features with that of $\mathcal{N} = 4$ Super Yang-Mills. We present and prove a new formula for the MHV scattering amplitude of n gravitons at tree level. Some of the more interesting features of this formula, which set it apart as being significantly different from many more familiar formulas, include the absence of any vestigial reference to a cyclic ordering of the gravitons and the fact that it simultaneously manifests both S_{n-2} symmetry as well as large-z behavior. The formula is seemingly related to others by an enormous simplification provided by $O(n^n)$ iterated Schouten identities, but our proof relies on a complex analysis argument rather than such a brute force manipulation. We find that the formula has a very simple link representation in twistor space, where cancellations that are non-obvious in physical space become manifest.

Aspects of Scattering Amplitudes: Symmetry and Duality

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for the Degree of Doctor of Philosophy in the
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Acknowledgments

I would like to thank my advisor, Professor Marcus Spradlin, for his advice, collaboration, and support through my years at Brown. His enthusiasm for physics and encouragement have always lightened my life and research. I've learned a lots from him.

I am grateful to Professor Antal Jevicki and Professor Chung-I Tan, from whom I have received many supports and learned many things in physics.

I also would like to thank many other professors in the physics department at Brown: Professors Gerald Guralnik for his nice teaching, Anastasia Volovich for her support and collaboration, David Cutts for his counseling, J. Michael Kosterlitz for his teaching and simulating, interesting discussions, Dmitri Feldman for his challenges in class, Greg Landsberg for his support at Fermilab, David Lowe for setting up an useful video course, Robert Pelcovits for his advice, James Valles Jr. for his support, and Meenakshi Narain for her support.

I am certainly indebted to many other people at Brown: Dean Hudek for his help and humorous sense, Ken Silva for his help with TA work, Mary Ann Rotondo for her help with paperwork, Barbara Dailey for her help with student matters, Professor/Dean Jabbar Bennett and Professor/Associate Provost Valerie Petit Wilson for setting up the Brown Executive Scholar Training (BEST) program from which I've learned many useful and important things, and Drew Murphy for his help in a project.

I would like to thank my friends in the High Energy Theory group and my other Vietnamese friends at Brown and in Providence who have supported and shared good as well as bad times with me through all the years.

And last but not least, I am grateful to my family (my parents, my wife, my two sisters, and my aunt), who have always loved, supported, and stayed by my side for all my life. My achievement couldn't be completed without them, to whom it is specially dedicated.

This thesis is dedicated to my late father and my mother.

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Chapter 1

Introduction

1.1 Overview

In this dissertation, we will discuss about recent progress in the understanding of the structure of scattering amplitudes in gauge theories, especially in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. One can wonder about the need to study amplitudes in a more "realistic" theory than $\mathcal{N} = 4$ SYM. We will justify this and give a brief review of the field of scattering amplitudes as below.

The $\mathcal{N} = 4$ SYM is a remarkable model of mathematical physics. It is the gauge theory with maximal supersymmetry and is superconformally invariant at the classical and quantum level with a coupling constant free of renormalization. In its planar limit all observables of the $SU(N)$ model depend on a single tunable parameter, the 't Hooft coupling λ. Moreover the theory is dual to type IIB superstring theory on $AdS_5 \times S^5$ via the AdS/CFT correspondence [6]. The duality implies that the full quantum anomalous dimensions of various series of gauge-invariant composite operators are equal to energies of different gravity modes or configurations of strings in anti-de Sitter space. In recent years we have seen tremendous progress in our understanding of this most symmetric AdS/CFT system due to its rich symmetries, hidden structures and integrability.

The scattering amplitudes in $\mathcal{N} = 4$ SYM have a number of remarkable properties both at weak and at strong coupling. Defined as matrix elements of the $S-$matrix between asymptotic on-shell states, they inherit the symmetries of the underlying gauge theory. In addition, trying to understand the properties of the scattering amplitudes, one can discover new dynamical symmetries of the $\mathcal{N} = 4$ theory.

Although of fundamental interest in jet physics, perturbative QCD amplitudes are notoriously difficult to calculate. The situation is alleviated in maximal $\mathcal{N} = 4$ supersymmetry, but it had been, for a long time during the 1990's, quite a formidable effort to calculate first few orders of gluon scattering amplitudes although a unitarity method was developed [64, 65]. Because $\mathcal{N} = 4$ SYM shares many features with QCD, especially at tree level, an understanding of structures in $\mathcal{N} = 4$ SYM will help to understand QCD, as well as supergravity.

Many developments have been made in the study of on-shell scattering amplitudes in $\mathcal{N} = 4$ SYM since the beginning of 2000's. Back from 1988, maximally helicity violating (MHV) n-gluon scattering amplitudes at tree-level have been given a very simple form when expressed in the spinor helicity formalism conjectured by Park-Taylor and proved by Berends-Giele [25, 26]. This indicates there should be a hidden structure of amplitudes to be discovered. From the pioneer work of Witten in 2003 [218] at tree level there is an interesting perturbative duality to strings in twistor space, that makes computation of Yang-Mills amplitudes feasible and easy. The theory, which known as the twistor string theory, leads to a connected prescription, in which an elegant and simple formula for all tree-level Yang-Mills scattering amplitudes is presented by Roiban, Spradlin and Volovich [21]. Twistor string theory also inspired the development of recursive techniques to construct tree-level gauge theory amplitudes known as the BCFW recursion relations in 2004 [196, 209], and the development of the CSW method to construct higher order amplitudes from MHV diagrams [12].

At the level of loop corrections Bern, Dixon and Smirnov (BDS) in 2005 [39] conjectured an all-loop form of the MHV amplitudes in $\mathcal{N} = 4$ gauge theory, based on an iterative structure [133] found at lower loop levels employing on-shell techniques [64, 65]. The form of the proposed iterative structure of multi-loop planar SYM is based on the understanding of how soft and collinear infrared singularities factorize and exponentiate in gauge theory. Specifically, this form was dictated by the tree-level and one-loop structure and the cusp anomalous dimension, a quantity which is simultaneously the leading ultraviolet singularity of Wilson loops with light-like cusps [108] and the scaling dimension of a particular class of local operators in the high spin limit. Due to this latter property it may be obtained from the above mentioned Bethe equations. In fact the ABDK/BDS conjecture is now known to fail at two loops and six points [60, 88, 121] but the deviations from it are constrained by a novel symmetry, the dual conformal symmetry.

This symmetry was discovered by Drummond, Henn, Smirnov and Sokatchev [87] in 2006. Since then, dual conformal symmetry has been studied extensively to uncover its role in the structure of scattering amplitudes in $\mathcal{N} = 4$ SYM. Thanks to many efforts that we now have a good understanding of this symmetry, though the origin of it at weak coupling is still unknown. Dual conformal symmetry plays an important role not only in the computation of four- and five-loop amplitudes by generalized unitarity method [5, 50], but also in the derivation of a formula for all tree-level scattering amplitudes [225], in the establishment of Yangian symmetry of $\mathcal{N} = 4$ SYM at least at tree level [232], in the discovery of a new duality between amplitudes and Wilson loops [102, 131?] and importantly in the conjecture of a Grassmannian integral that gives all leading singularities of all amplitudes at all loop orders [204, 205, 233].

The dual string theory prescription for computing scattering amplitudes was proposed by Alday and Maldacena in 2007 [59]. There, the problem of computing certain gluon scattering amplitudes at strong coupling was mapped via a T-duality transformation to that of computing Wilson loops with light-like segments. The strong coupling calculation of [59] is insensitive to the helicity structure of the amplitude being calculated. It was a year later then in 2008, Berkovits and Maldacena [268] unveils the fermionic T-duality to explain the nature of dual conformal symmetry and the duality between amplitudes and Wilson loops. This transformation maps the dual superconformal symmetry of the original theory to the ordinary superconformal symmetry of the dual model.

At weak coupling, it was also found that light-like Wilson loops are dual to MHV amplitudes, as was demonstrated for $n = 4, 5, 6$ gluons scattering up to two-loop order [102, 131]. A direct implication of this Wilson loop/MHV amplitude duality is the existence of the dual conformal symmetry of the amplitudes. This symmetry has its interpretation as the ordinary conformal symmetry of the Wilson loops [131, 132].

Indeed the two-loop, six-point calculations of [88, 121] showed that the amplitude and the Wilson loop both differ from the BDS conjecture by the same function of dual conformal invariants. This symmetry extends naturally to dual superconformal symmetry when one considers writing all amplitudes (MHV and non-MHV) in on-shell superspace [127]. In particular, tree-level amplitudes are covariant under dual superconformal symmetry. And when we combine the conventional and the dual superconformal symmetry, what we get is an infinite dimensional symmetry, the Yangian symmetry for scattering amplitudes at, at least, tree level.

Now let us discuss on a different approach to understand the full structure of scattering amplitudes by Mason-Skinner and Arkani-Hamed, Cachazo and collaborators. In 1967, Roger Penrose proposed twistors as a fundamental tool for the physics of spacetime. He argued that they should be relevant for quantum gravity. But for decades, this claim remained nothing else than a wishful thinking. The pioneer work of Witten in 2003 on twistor string theory makes this idea alive again. However this connected prescription works only at tree level and does not apply to loops. In the mean while, the leading singularity method has been "sharpened" by Cachazo [78] to give useful results on loop amplitudes. In 2009, in an effort to make the amplitude manifest dual conformal symmetry, Arkani-Hamed, Cachazo and collaborators [205, 233], together with a parallel work by Mason and Skinner [204], have made an important breakthrough with the discovery of an object, a Grassmannian integral, that is conjectured to capture all leading singularities of all types of scattering amplitudes at all loops level. Later work revealed that this Grassmannian integral is a special case of the amplitude formula in connected prescription at tree level [237]. It was then shown that the form of this object is fixed uniquely by Yangian symmetry

[232]. And an expression for a BCFW all-loop recursion relation for the integrand was proposed [233], giving a way to compute loop amplitudes in momentum twistor space.

So finally with recent developments and so much more to come, the full understand of scattering amplitudes is being realized.

1.2 Scattering Amplitudes: Foundation

In this section we will review the foundations of scattering amplitudes [236] that will be crucial for later work.

On-shell scattering amplitudes are among the basic quantities in any quantum field theory that the standard textbooks relate them through the LSZ reduction to Green's functions, which will be computed in terms of Feynman diagrams. The number of diagrams grows exponentially at loop level. The evaluation of individual diagram is generally more complicated than that of the complete amplitude. This fact implies that Feynman diagrams do not exhibit or take advantage of the symmetries of the theory, neither local nor global and that there are hidden structures yet to be discovered.

Let consider the L-loop $SU(N)$ gauge theory n-point scattering amplitudes, which may be written in general as

$$A^{(L)} = N^L \sum_{\rho \in S_n/\mathbb{Z}_n} \mathrm{Tr}[T^{a_{\rho(1)}} \dots T^{a_{\rho(n)}}] A^{(L)}(k_{\rho(1)} \dots k_{\rho(n)}, N) + \text{multi–traces} \quad (1.2.1)$$

where the sum extends over all non-cyclic permutations ρ of $(1 \dots n)$.

The subleading terms in the $1/N$ expansion and the multi-trace terms are called non-planar amplitudes and do not contribute to our main subject , amplitudes in the large N (planar) limit. In this limit, a generic $n-$particle scattering amplitude in the $\mathcal{N}=4$ SYM theory with an $SU(N)$ gauge group has the form

$$\mathcal{A}_n(\{p_i, h_i, a_i\}) = (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^{n} p_i\Big) \sum_{\sigma \in S_n/Z_n} 2^{n/2} g^{n-2} \mathrm{Tr}[t^{a_{\sigma(1)}} \dots t^{a_{\sigma(n)}}] A_n\left(\sigma(1^{h_1}, \dots, n^{h_n})\right) , \tag{1.2.2}$$

where each scattered particle (scalar, gluino with helicity $\pm 1/2$ or gluon with helicity ± 1) is characterized by its on-shell momentum p_i^μ ($p_i^2 = 0$), helicity h_i and color index a_i. Here the sum runs over all possible non-cyclic permutations σ of the set $\{1, \dots, n\}$ and the color trace involves the generators t^a of $SU(N)$ in the fundamental representation normalized as $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. All particles are treated as incoming, so that the momentum conservation takes the form $\sum_{i=1}^{n} p_i = 0$.

The color-ordered partial amplitudes $A_n\left(\sigma(1^{h_1}, \dots, n^{h_n})\right)$ only depend on the momenta and helicities of the particles and admit a perturbative expansion in powers of ‘t Hooft coupling $a = g^2 N/(8\pi^2)$. The best studied so far are the gluon scattering amplitudes. In some cases, amplitudes with external particles other than gluons can be obtained from them with the help of supersymmetry.

1.2.1 General Properties

Color-ordered scattering amplitudes in Yang-Mills theory satisfy some general properties, follow from their construction in terms of Feynman diagrams, as below [236]:

1. Cyclicity: this is a consequence of the cyclic symmetry of traces

$$A(1,2,\ldots,n) \;=\; A(2,3,\ldots,n,1) \tag{1.2.3}$$

2. Reflection:

$$A(1,2,\ldots,n) \;=\; (-1)^n A(n,n-1,\ldots,1) \tag{1.2.4}$$

3. Parity Invariance: In $(2,2)$ signature, amplitude is invariant if all helicities are reversed and simultaneously all spinors are interchanged $\lambda \leftrightarrow \tilde{\lambda}$ (In Minskowski signature both sides are related by complex conjugation):

$$A(\lambda_i,\tilde{\lambda}_i,\eta_{iA}) = \int d^{4n}\psi\;\exp\left[i\sum_{i=1}^{n}\eta_{iA}\psi_i^A\right]A(\tilde{\lambda}_i,\lambda_i,\psi_i^A). \tag{1.2.5}$$

4. Dual Ward Identity or Photon Decoupling: At tree-level, Ward identity expresses the decoupling of $U(1)$ degree of freedom: fixing one of the external legs (supposed n as below) and summing over cyclic permutations $C(1,\ldots,n-1)$ of the remaining $(n-1)$ legs leads to a vanishing result:

$$\sum_{C(1,\ldots,n-1)} A(1,2,3,\ldots,n) \;=\; 0. \tag{1.2.6}$$

   At loop level, this identity is modified and relates planar and non-planar partial amplitudes.

A generalization of the above identity for Yang-Mills amplitudes is given as follows:

$$\sum_{Perm(i,j)} A(i_1,\ldots,i_m,j_1,\ldots,j_k,n{+}1) \;=\; 0, \qquad 1\le m\le n{-}1, \qquad m{+}k=n, \tag{1.2.7}$$

where the sum is taken over permutations of the set $(i_1,\ldots,i_m,j_1,\ldots,j_k)$ which preserve the order of the $(i_1,\ldots,i_m)$ and $(j_1,\ldots,j_k)$ separately.

5. Soft (momentum) limit: in the limit in which one momentum becomes soft ($p_1\to 0$ as below) the amplitude universally factorizes as

$$A^{\rm tree}(1^+,2,\ldots,n) \longrightarrow \frac{\langle n\,2\rangle}{\langle n\,1\rangle\langle 1\,2\rangle}A^{\rm tree}(2,\ldots,n) \tag{1.2.8}$$

If particle 1 has negative helicity then the expression is conjugated.

6. Collinear limit: in the limit in which two adjacent momenta become collinear $k_{n-1}\cdot k_n\to 0$ a L-loop amplitude factorizes as

$$A_n^{(L)}(1,\ldots,(n{-}1)^{h_{n-1}},n^{h_n}) \mapsto \sum_{l=0}^{L}\sum_h A_{n-1}^{(L-l)}(1,\ldots,k^h){\rm Split}_{-h}^{(l)}((n{-}1)^{h_{n-1}},n^{h_n})\ , \tag{1.2.9}$$

where $k = k_{n-1}+k_n$ and h_i denotes the helicity of the i-th gluon. For a given gauge theory, the l-loop splitting amplitudes ${\rm Split}_{-h}^{(l)}((n{-}1)^{h_{n-1}},n^{h_n})$ are universal functions [22] of the helicities of the collinear particles, the helicity of the external leg of the resulting amplitude and of the momentum fraction z defined as

$$z=\frac{\xi\cdot k_{n-1}}{\xi\cdot(k_{n-1}+k_n)}\quad . \tag{1.2.10}$$

In the strict collinear limit one may also use $k_{n-1} \to zk$ and $k_n \to (1-z)k$ with $k^2 = 0$. For example, the tree-level splitting amplitudes are:

$$\text{Split}_{-}^{(0)}(1^+, 2^+) = \frac{1}{\sqrt{z(1-z)}} \frac{1}{\langle 1\, 2 \rangle} \tag{1.2.11}$$
$$\text{Split}_{-}^{(0)}(1^+, 2^-) = \frac{z^2}{\sqrt{z(1-z)}} \frac{1}{[1\, 2]} \qquad \text{Split}_{+}^{(0)}(1^+, 2^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}} \frac{1}{\langle 1\, 2 \rangle}$$

7. Multi-particle factorization: color-ordered amplitudes can only have poles where the square of the sum of some cyclically adjacent momenta vanishes. At tree-level this pole corresponds to some propagator going on-shell. At higher loops, the amplitude decomposes into a completely factorized part given by the sum of products of lower loop amplitudes and a non-factorized part, given in terms of additional universal functions. At one-loop level and in the limit $k_{1,m}^2 \equiv (k_1 + \ldots k_m)^2 \to 0$ we have [23]

$$\begin{aligned} A_n^{1\,\text{loop}}(1, \ldots, n) \;\longrightarrow& \\ \sum_{h_p = \pm} \Big[& A_{m+1}^{\text{tree}}(1, \ldots, m, k^{h_k}) \frac{i}{k_{1,m}^2} A_{n-m+1}^{1\,\text{loop}}((-k)^{-h_k}, m+1, \ldots, n) \\ &+ A_{m+1}^{1\,\text{loop}}(1, \ldots, m, k^{h_k}) \frac{i}{k_{1,m}^2} A_{n-m+1}^{\text{tree}}((-k)^{-h_k}, m+1, \ldots, n) \\ &+ A_{m+1}^{\text{tree}}(1, \ldots, m, k^{h_k}) \frac{i\mathcal{F}(1 \ldots n)}{k_{1,m}^2} A_{n-m+1}^{\text{tree}}((-k)^{-h_k}, m+1, \ldots, n) \Big] \end{aligned} \tag{1.2.12}$$

1.2.2 Regularization and Factorization

In massless gauge theories, infrared singularities of scattering amplitudes, which is a general feature in four dimensions that can not be renormalized away like the ultraviolet divergences, come from two different sources: the small energy region of some virtual particle (soft limit) and the region in which some virtual particle is collinear with some external one (collinear limit). These divergences can occur simultaneously so in general at L-loops we can have up to $1/\epsilon^{2L}$ infrared singularities. They should cancel once gluon scattering amplitudes are combined to compute physical infrared-safe quantities.

The structure of soft and collinear singularities in massless gauge theories in four dimensions plays an important role in the structure of scattering amplitudes, as the amplitudes can be factorized universally as follow [41, 42, 43]:

$$\mathcal{M}_n = \left[\prod_{i=1}^{n} J_i(\frac{Q}{\mu}, \alpha_s(\mu), \epsilon)\right] \times S(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon) \times h_n(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon) \ , \qquad (1.2.13)$$

where Q is factorization scale that separating soft and collinear momenta, μ is the renormalization scale, $\alpha_s(\mu) = \frac{g(\mu)^2}{4\pi}$ is the running coupling at scale μ, $J_i(\frac{Q}{\mu}, \alpha_s(\mu), \epsilon)$ are the "jet" functions that contain information on collinear dynamics of virtual particles, $S(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon)$ is the soft function responsible for the purely infrared poles, $h_n(k, \frac{Q}{\mu}, \alpha_s(\mu), \epsilon)$ contains the effects of highly virtual fields.

In the planar limit, the above scattering amplitudes can be simplified as

$$\mathcal{M}_n = \left[\prod_{i=1}^{n} \mathcal{M}^{gg\to 1}(\frac{s_{i,i+1}}{\mu}, \lambda(\mu), \epsilon)\right]^{1/2} h_n \ , \qquad (1.2.14)$$

where $\lambda(\mu) = g(\mu)^2 N$ is the 't Hooft coupling.

Because the rescaled amplitude is independent of the factorization scale Q, we will have an evolution equation for the soft function, in addition to a renormalization group equation for the factors $\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2}, \lambda(\mu), \epsilon\right)$:

$$\frac{d}{d\ln Q^2}\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2}, \lambda, \epsilon\right) = \frac{1}{2}\left[K(\epsilon, \lambda) + G\left(\frac{Q^2}{\mu^2}, \lambda, \epsilon\right)\right]\mathcal{M}^{[gg\to 1]}\left(\frac{Q^2}{\mu^2}, \lambda, \epsilon\right) , \tag{1.2.15}$$

where the function K contains only poles and no scale dependence. The functions K and G themselves obey renormalization group equations [28, 29, 30, 44, 45]

$$\left(\frac{d}{d\ln\mu} + \beta(\lambda)\frac{d}{dg}\right)(K+G) = 0 \qquad \left(\frac{d}{d\ln\mu} + \beta(\lambda)\frac{d}{dg}\right)K(\epsilon, \lambda) = -\gamma_K(\lambda) \ . \tag{1.2.16}$$

In $\mathcal{N} = 4$ SYM in the planar limit with $\beta(\lambda) = -2\epsilon\lambda$, we can solve these above equations exactly and explicitly in terms of the expansion coefficients of the cusp anomalous dimension

$$f(\lambda) \equiv \gamma_K(\lambda) = \sum_l a^l \gamma_K^{(l)} \tag{1.2.17}$$

and another set of coefficients defining the expansion of G:

$$G\left(\frac{Q^2}{\mu^2}, \lambda, \epsilon\right) = \sum_l \mathcal{G}_0^{(l)} a^l \left(\frac{Q^2}{\mu^2}\right)^{l\epsilon} \tag{1.2.18}$$

where $a = \frac{\lambda}{8\pi^2}(4\pi e^{-\gamma})^\epsilon$ is the coupling constant.

This then leads to the following result for the amplitude [39]:

$$\begin{aligned} \mathcal{M}_n &= \exp\left[-\frac{1}{8}\sum_{l=1}^{\infty} a^l \left(\frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2\mathcal{G}_0^{(l)}}{l\epsilon}\right)\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon}\right] \times h_n \\ &= \exp\left[\sum_{l=1}^{\infty} a^l \left(\frac{1}{4}\gamma_K^{(l)} + \epsilon\,\frac{l}{2}\mathcal{G}_0^{(l)}\right)\hat{I}_n^{(1)}(l\epsilon)\right] \times h_n \ , \end{aligned} \tag{1.2.19}$$

where

$$\hat{I}_n^{(1)} = -\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} . \tag{1.2.20}$$

At weak coupling, we have [39, 46, 47, 48, 49, 50, 51, 52]:

$$\begin{aligned} f(\lambda) &= \frac{\lambda}{2\pi^2}\left(1 - \frac{\lambda}{48} + \frac{11\,\lambda^2}{11520} - \left(\frac{73}{1290240} + \frac{\zeta_3^2}{512\pi^6}\right)\lambda^3 + \cdots\right) , && (1.2.21) \\ G(\lambda) &= -\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2 + \left(6\zeta_5 + 5\zeta_2\zeta_3\right)\left(\frac{\lambda}{8\pi^2}\right)^3 - 2(77.56 \pm 0.02)\left(\frac{\lambda}{8\pi^2}\right)^4 + \cdots , && (1.2.22) \end{aligned}$$

whereas at strong coupling [56, 57, 58, 59]:

$$\begin{aligned} f(\lambda) &= \frac{\sqrt{\lambda}}{\pi}\left(1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{\mathrm{K}}{\lambda} + \cdots\right) , \qquad \lambda \to \infty , && (1.2.23) \\ G(\lambda) &= (1 - \ln 2)\frac{\sqrt{\lambda}}{8\pi} + \cdots , \qquad \lambda \to \infty , && (1.2.24) \end{aligned}$$

with $\mathrm{K} = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2} \simeq 0.9159656\ldots$ is the Catalan constant.

Because the scattering amplitudes have infrared singularities, we will need to regularize them in order to get meaningful results. There are several regularization tools: by dimensional regularization, by dimension reduction or by introducing certain Higgs masses for amplitudes on the Coulomb branch [].

1.2.3 Generalized-Unitarity-based Method

The generalized unitarity method is at present one of the most powerful general tools for obtaining loop-level scattering amplitudes, that could be used in any massless non-supersymmetric or supersymmetric theory for both planar and non-planar contributions. The key feature of the unitarity method is that it constructs loop amplitudes directly from on-shell tree amplitudes, making it possible to carry over any newly identified property or symmetry of tree-level amplitudes to loop level. This may be contrasted with Feynman diagrammatic methods, whose diagrams are inherently gauge dependent and off-shell in intermediate states.

The idea of unitarity method is using unitarity cuts in such a way that if given the discontinuity of an amplitude in some channel (or a cut), we could reconstruct the complete amplitude by a dispersion integral. Let us recap briefly how this idea works. Consider the unitarity condition for the scattering matrix $S = 1 + iT$, we obtain

$$i(T^\dagger - T) = 2\,\Im T = T^\dagger T \ . \tag{1.2.25}$$

The right hand side is the product of lower loop on-shell amplitudes, which could be interpreted as a higher loop amplitude with some number of Feynman propagators replaced by on-shell (or cut) propagators

$$\frac{1}{l^2 + i\epsilon} \mapsto -2\pi i\theta(l^0)\delta(l^2) \ . \tag{1.2.26}$$

As a consequence of the $i\epsilon$ prescription, the difference on the left hand side of equation (1.2.25) is interpreted as the discontinuity in the multi-particle invariant obtained by squaring the sum of the momenta of the cut propagators.

Hence, this discontinuity at L-loops is determined in terms of products of lower-loop amplitudes. There are two types of cuts: singlet (only one type of field crosses the cut) and non-singlet (several types of particles cross the cut).

Then the real part of the amplitude is constructed from a dispersion integral as follows

$$\Re f(s) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dw \frac{\Im f(w)}{w - s} - C_{\infty} \quad , \tag{1.2.27}$$

where s is the momentum invariant flowing across the cut.

Now let us review the the procedure of generalized-unitarity based method in $\mathcal{N} = 4$ super-Yang-Mills theory [238].

In general, the complete amplitude is determined from a spanning set of cuts. Such sets are found by considering all potential independent contributions to the integrand that can enter an amplitude (and which do not integrate to zero), based on power counting or other constraints. One can often construct an ansatz for the entire amplitude using various conjectured properties. Once one has an ansatz, by confirming it over the spanning set, either numerically or analytically, we have a proof of the correctness of the ansatz.

One simple spanning set is obtained from the set of standard unitarity cuts, where a given amplitude is split into two lower-loop amplitudes, each with four or more external legs. At L loops, this is given by all cuts starting from the two-particle cut to the $(L + 1)$-particle cut in all channels. We can convert this to a spanning set involving only tree amplitudes by iterating this process until no loops remain. If the amplitudes are color ordered then we need to maintain a fixed ordering of legs, depending on which planar or non-planar contribution is under consideration. On the other hand, if they are dressed with color then the distinct permutations of legs enter each cut automatically.

Another spanning set that is especially useful in $\mathcal{N} = 4$ super-Yang-Mills theory is obtained by starting from "maximal cuts" (also referred to as "leading singularities" [78]), where the maximum numbers of internal propagators are placed on shell. These maximal cuts decompose the amplitudes into products of three-point tree amplitudes, summed over the states crossing the cuts. To construct a spanning set we systematically release cut conditions one by one, first considering cases with one internal line off shell, then two internal lines off shell and so forth. Each time a cut condition is released, potential contact terms which would not be visible at earlier steps are captured. The process terminates when the only remaining potential contact terms exceed power counting requirements of the theory (or integrate to zero in dimensional regularization).

Given a spanning set of unitarity cuts, the task is to then find an expression for the integrand of the amplitude with the correct cuts in all channels. This can be done either in a forward or reverse direction. In the forward direction the different cuts are merged into integrands with no cut conditions. In the reverse direction we first construct an ansatz for the amplitude containing unknown parameters which are then determined by taking generalized cuts of the ansatz and comparing to the cuts of the amplitude. The reverse direction is usually preferred because we can expose desired properties, simply by imposing them on the ansatz and then checking if its unitarity cuts are correct.

We can also construct four-dimensional unitarity cuts in superspace. The sum over states crossing the unitarity cuts can be expressed simply as an integration over the fermionic variables η^A of the cut legs. The generalized $\mathcal{N}=4$ supercut is then given by

$$\mathcal{C} = \int \Big[\prod_{i=1}^{k} d^4\eta_i\Big] \mathcal{A}^{\rm tree}_{(1)} \mathcal{A}^{\rm tree}_{(2)} \mathcal{A}^{\rm tree}_{(3)} \cdots \mathcal{A}^{\rm tree}_{(m)} \,, \tag{1.2.28}$$

where $\mathcal{A}^{\rm tree}_{(j)}$ are the tree superamplitudes connected by k on-shell cut legs. These cuts then constrain the amplitude, which are now functions in the on-shell superspace.

1.2.4 MHV Tree and Loop Gluon Scattering Amplitudes

With the knowledge of a powerful tool described in the previous section, the generalized-unitarity based method, we will review how it would be applied to obtain the results for MHV tree and loop gluon scattering amplitudes in $\mathcal{N}=4$ Super Yang-Mills theory in the planar limit in this section [236].

It has been possible to derive a set of extremely simple formulae at tree level for "maximally helicity-violating" (MHV) amplitudes with an arbitrary number of external gluons. Parke and Taylor [25] formulated conjectures for these amplitudes in part by using an analysis of collinear limits and they were later proven by Berends and Giele [26] using recursion relations.

$$A_n^{MHV}(1^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) \;=\; i\,(2\pi)^4\;\delta^{(4)}\left(\sum_{i=1}^{n} p_i\right)\frac{\langle j\,k\rangle^4}{\langle 1\,2\rangle\;\langle 2\,3\rangle\cdots\langle n\,1\rangle}\;. \tag{1.2.29}$$

The nonvanishing Parke-Taylor formulae are for amplitudes where two gluons have a given helicity, and the remaining gluons all have the opposite helicity, in the convention where all external particles are treated as outgoing. These amplitudes are called "maximally helicity-violating" because amplitudes with all helicities identical, or all but one identical, vanish at tree-level as a consequence of supersymmetry Ward identities [**?**].

$$A^{\rm tree}(g^+ \dots g^+) = 0 \qquad\qquad A^{\rm tree}(g^- g^+ \dots g^+) = 0 \quad . \tag{1.2.30}$$

However, for $n = 3$ one can construct amplitudes e.g. $A_3(1^-, 2^+, 3^+) \neq 0$ provided the on-shell momenta are complex [218].

A remarkable property in planar SYM is that the finite terms in the MHV scattering amplitudes can be organized into the same exponentiated form as the divergent terms. All perturbative corrections can be factorized into a universal scalar factor

$$A_n^{\rm MHV}(1^+ ... \, j^- ... \, k^- ... \, n^+) = A_{n;0}^{\rm MHV} \, M_n^{\rm MHV} \, . \tag{1.2.31}$$

Here $M_n^{\rm MHV} = M_n^{\rm MHV}(\{s_{ij}\}, a)$ is a function of the Mandelstam kinematical invariants and of the ‘t Hooft coupling but independent of j and k.

We can construct the superamplitude by using the on-shell superspace with fermionic variables η^A, which were first introduced by Ferber to extend twistors, representations of four-dimensional conformal group, to supertwistors. Nair [223] applied these variables to represent MHV superamplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory as

$$\mathcal{A}_n^{\rm MHV}(1, 2, \cdots, n) = \frac{i}{\prod_{j=1}^n \langle j(j+1)\rangle} \, \delta^{(8)}\left(\sum_{j=1}^n \lambda_j^\alpha \eta_j^a\right) \, , \tag{1.2.32}$$

where leg $n + 1$ is to be identified with leg 1, and

$$\delta^{(8)}\left(\sum_{j=1}^n \lambda_j^\alpha \eta_j^a\right) = \prod_{a=1}^4 \sum_{i<j}^n \langle ij\rangle \eta_i^a \eta_j^a \, . \tag{1.2.33}$$

As a simple example to expect that superamplitudes for theories with fewer supersymmetries are simply given by subamplitudes of the maximal theory, the MHV tree amplitudes for the minimal gauge multiplets of $\mathcal{N} < 4$ super-Yang-Mills theory are given by [**?**]

$$\mathcal{A}_n^{\text{MHV}}(1,2,\ldots,n) = \frac{\prod_{a=1}^{\mathcal{N}} \delta^{(2)}(Q^a)}{\prod_{j=1}^{n} \langle j\ (j+1)\rangle} \left(\sum_{i<j}^{n} \langle i\, j\rangle^{4-\mathcal{N}} \prod_{a=\mathcal{N}+1}^{4} \eta_i^a \eta_j^a \right), \tag{1.2.34}$$

with $\mathcal{N}$ counting the number of supersymmetries, $Q^a = \sum_{i=1}^{n} \lambda_i \eta_i^a$, and $n \geq 3$.

Starting from the three-point MHV and $\overline{\text{MHV}}$ amplitude, one can obtain the higher-point amplitudes in superspace via the MHV vertex expansion constructed by Cachazo, Svrček and Witten (CSW) [12] or the on-shell recursion relations of Britto, Cachazo, Feng, and Witten (BCFW) [196, 209].

The expression for the N^mMHV superamplitude constructed via the CSW construction is given by

$$\mathcal{A}_n^{\text{N}^m\text{MHV}} = i^m \sum_{\text{CSW graphs}} \int \left[\prod_{j=1}^{m} \frac{d^4\eta_j}{P_j^2} \right] \mathcal{A}_{(1)}^{\text{MHV}} \mathcal{A}_{(2)}^{\text{MHV}} \cdots \mathcal{A}_{(m)}^{\text{MHV}} \mathcal{A}_{(m+1)}^{\text{MHV}}, \tag{1.2.35}$$

where the integral is over the $4m$ internal Grassmann parameters ($d^4\eta_j \equiv \prod_{a=1}^{4} d\eta_j^a$) associated with the internal legs, and each P_j is the momentum of the j'th internal leg of the graph.

At loop level there are two important consistency constraints on amplitudes, collinear behavior and unitarity, which can be used as a guide in constructing ansätze for amplitudes.

Now let consider the loop contributions of the scattering amplitude. It is convenient to scale out a factor of the tree amplitude, and work with the quantities $M_n^{(L)}$ defined by

$$M_n^{(L)}(\rho;\epsilon) = A_n^{(L)}(\rho)/A_n^{(0)}(\rho)\,. \qquad (1.2.36)$$

Here ρ indicates the dependence on the external momenta, $\rho \equiv \{s_{12}, s_{23}, \ldots\}$, where $s_{i(i+1)} = (k_i + k_{i+1})^2$ are invariants built from color-adjacent momenta.

At one loop, all contributions to the amplitudes of massless supersymmetric theories are determined completely by their four-dimensional cuts. Unfortunately, no such theorem has been proven at higher loops. There is, however, substantial evidence that it holds for four-point amplitudes in this theory through six loops [50]. It is not expected to continue for higher-point amplitudes. Indeed, we know that for two-loop six-point amplitudes, terms which vanish in $D = 4$ do contribute in dimensional regularization [50].

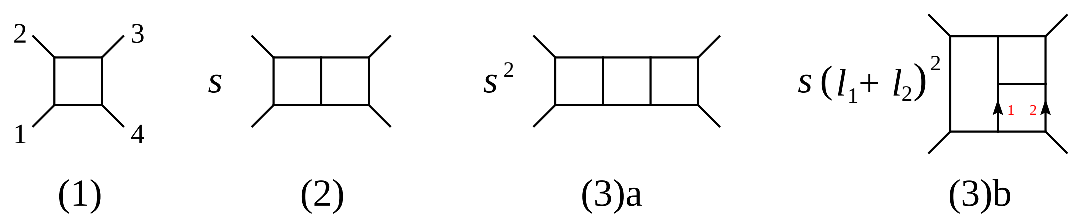


Figure 1.1: Diagrams contribute to the leading-color MHV gluon scattering amplitude: The box (1) at one loop, the planar double box (2) at two loops, and the three-loop ladder ($3a$) and tennis-court ($3b$) at three loops.

The result for the one-loop four-point MHV amplitude is [183]

$$M_4^{(1)}(\epsilon) = -\frac{1}{2}\,\mathcal{I}^{(1)}(s,t)\,, \qquad (1.2.37)$$

where the Mandelstam variables are $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$ and

$$\mathcal{I}^{(1)}(s,t) \equiv st\, I_4^{(1)}(s,t)\,, \tag{1.2.38}$$

The factor of 1/2 in 1.2.37 follows from the normalization convention for $A_n^{(L)}$.

The two-loop four-point MHV amplitude is given by [184]

$$M_4^{(2)}(\epsilon) = \frac{1}{4}\left[\mathcal{I}^{(2)}(s,t) + \mathcal{I}^{(2)}(t,s)\right], \tag{1.2.39}$$

with the two-loop scalar double-box integral

$$\mathcal{I}^{(2)}(s,t) \equiv s^2 t\, I_4^{(2)}(s,t)\,. \tag{1.2.40}$$

The three-loop four-point MHV amplitude is given by [39, 184]

$$M_4^{(3)}(\epsilon) = -\frac{1}{8}\left[\mathcal{I}^{(3)\mathrm{a}}(s,t) + 2\,\mathcal{I}^{(3)\mathrm{b}}(t,s) + \mathcal{I}^{(3)\mathrm{a}}(t,s) + 2\,\mathcal{I}^{(3)\mathrm{b}}(s,t)\right], \tag{1.2.41}$$

where the scalar triple-ladder and non-scalar "tennis-court" integrals are

$$\begin{aligned} \mathcal{I}^{(3)\mathrm{a}} &\equiv s^3 t\, I_4^{(3)\mathrm{a}}(s,t)\,, \\ \mathcal{I}^{(3)\mathrm{b}} &\equiv s t^2\, I_4^{(3)\mathrm{b}}(s,t)\,. \end{aligned} \tag{1.2.42}$$

The explicit results of each $\mathcal{I}(s,t)$ integral through three loops are given in [39].

Using generalized cuts, Bern, Czakon, Dixon, Kosower and Smirnov [50] found that

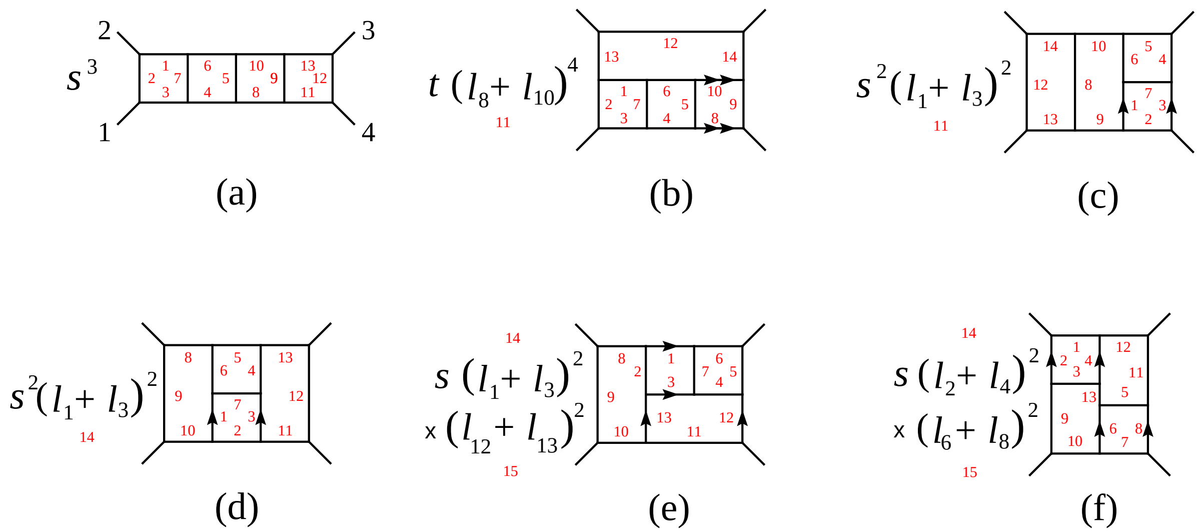


Figure 1.2: Rung-rule contributions to the leading-color four loop MHV gluon scattering amplitude. An overall factor of st is suppressed.

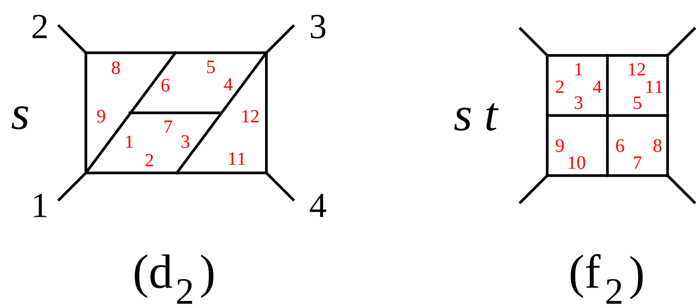


Figure 1.3: Non-rung-rule contributions to the leading-color four loop MHV gluon scattering amplitude. An overall factor of st is suppressed.

the four-loop planar amplitude receives contributions not only from diagrams controlled by the rung rule as in Fig. 1.2, but also from non-rung-rule diagrams in Fig. 1.3. It is given by

$$\begin{aligned} M_4^{(4)}(\epsilon) \;=\; & \frac{1}{16}\Big[\mathcal{I}^{(\mathrm{a})}(s,t)+\mathcal{I}^{(\mathrm{a})}(t,s)+2\,\mathcal{I}^{(\mathrm{b})}(s,t)+2\,\mathcal{I}^{(\mathrm{b})}(t,s)+2\,\mathcal{I}^{(\mathrm{c})}(s,t)+2\,\mathcal{I}^{(\mathrm{c})}(t,s) \\ & +\mathcal{I}_4^{(\mathrm{d})}(s,t)+\mathcal{I}^{(\mathrm{d})}(t,s)+4\,\mathcal{I}^{(\mathrm{e})}(s,t)+4\,\mathcal{I}^{(\mathrm{e})}(t,s)+2\,\mathcal{I}^{(\mathrm{f})}(s,t)+2\,\mathcal{I}^{(\mathrm{f})}(t,s) \\ & -2\,\mathcal{I}^{(\mathrm{d_2})}(s,t)-2\,\mathcal{I}^{(\mathrm{d_2})}(t,s)-\mathcal{I}^{(\mathrm{f_2})}(s,t)\Big]\,, \end{aligned} \tag{1.2.43}$$

with

$$\mathcal{I}^{(x)}(s,t) \equiv (-ie^{\epsilon\gamma}\pi^{-d/2})^4 \int \partial^d p\,\partial^d q\,\partial^d u\,\partial^d v\;\frac{st\,\mathcal{N}^{(x)}}{\prod_j p_j^2}\,, \tag{1.2.44}$$

The explicit result for each $\mathcal{I}^{(x)}(s,t)$ integral is given in [50]. And the total four-loop planar amplitude, $M_4^{(4)}$, has the expansion,

$$\begin{aligned} M_4^{(4)}(s,t) \;=\; & (-t)^{-4\epsilon}\Big\{\frac{2}{3\,\epsilon^8}+\frac{4}{3\,\epsilon^7}L+\frac{1}{\epsilon^6}\Big[L^2-\frac{13}{18}\pi^2\Big] \\ & +\frac{4}{3\,\epsilon^5}\Big[H_{0,0,1}(x)-L\,H_{0,1}(x)+\frac{1}{2}(L^2+\pi^2)H_1(x)+\frac{L^3}{4}-\frac{5}{6}\pi^2\,L-\frac{59}{12}\zeta_3\Big] \\ & +\frac{4}{3\,\epsilon^4}\big[-H_{0,0,0,1}(x)-H_{0,0,1,1}(x)-H_{0,1,0,1}(x)-H_{1,0,0,1}(x) \\ & \qquad +L\Big(\frac{5}{2}\,H_{0,0,1}(x)+H_{0,1,1}(x)+H_{1,0,1}(x)\Big)-\frac{L^2}{2}(4\,H_{0,1}(x)+H_{1,1}(x)) \\ & \qquad -\frac{\pi^2}{2}\Big(H_{1,1}(x)-\frac{3}{2}L\,H_1(x)+\frac{15}{16}L^2\Big) \\ & \qquad +\frac{11}{12}L^3\,H_1(x)+\zeta_3\,H_1(x)+\frac{L^4}{32}-\frac{28}{3}\zeta_3\,L+\frac{637}{17280}\pi^4\Big] \\ & +\,\mathcal{O}(\epsilon^{-3})\Big\}\,. \end{aligned} \tag{1.2.45}$$

Calculation of higher loop scattering amplitudes requires tremendous efforts if one tries to continually use generalized cuts: it is much more complicated with the need to use D-dimensional cuts beside the usual 4-dimensional cuts. However, using a newly discovered symmetry, dual conformal symmetry - which we will discuss in details in chapter 2, Bern, Carrasco, Johansson and Kosower [5] were able to determine the contributing diagrams as given in Fig. 1.4 and Fig. 1.5, and proposed the complete five-loop four-point planar gluon scattering amplitude as below,

$$\begin{aligned} M_4^{(5)}(1,2,3,4) &= -\frac{1}{32}\left[(I_1 + 2I_2 + 2I_3 + 2I_4 + I_5 + I_6 + 2I_7 + 4I_8 + 2I_9 + 4I_{10}\right. \\ &\quad +2I_{11} + 4I_{12} + 4I_{13} + 4I_{14} + 4I_{15} + 2I_{16} + 4I_{17} + 4I_{18} + 4I_{19} + 4I_{20} \\ &\quad +2I_{21} + 2I_{23} + 4I_{24} + 4I_{25} + 4I_{26} + 2I_{27} + 4I_{28} + 4I_{29} + 4I_{30} \\ &\quad \left. +2I_{31} + I_{32} + 4I_{33} + 2I_{34} + \{s \leftrightarrow t\}) + I_{22}\right] . \end{aligned} \qquad (1.2.46)$$

Finally using the supercut described in the previous section, we have the supercut of the L-loop n-particle scattering superamplitude is given by [127]

$$\begin{aligned} \mathcal{A}_n^L\Big|_{\text{cut}} &= \int \prod_{\{ij\}} d^4\eta_{\{ij\}} \times \mathcal{A}_{(1)}^{tree}\mathcal{A}_{(2)}^{tree}\mathcal{A}_{(3)}^{tree}\ldots\mathcal{A}_{(m)}^{tree} \\ &= \delta^4(P)\int \prod_{\{ij\}} d^4\eta_{\{ij\}} \times \prod_{\alpha}\delta^8\left(Q_\alpha\right) f_\alpha \\ &= \delta^4(P)\delta^8(Q)\int \left(\prod_{l_k} d^8\theta_{l_k}\right)\prod_{\{ij\}}\delta^4(\theta_{ij}\lambda_{\{ij\}})\prod_{\alpha} f_\alpha\,, \end{aligned} \qquad (1.2.47)$$

where the product over internal loop label l_k runs over the internal dual point labels and $\mathcal{A}_n^{tree} = \delta^4(P)\,\delta^8(Q) f_n$.

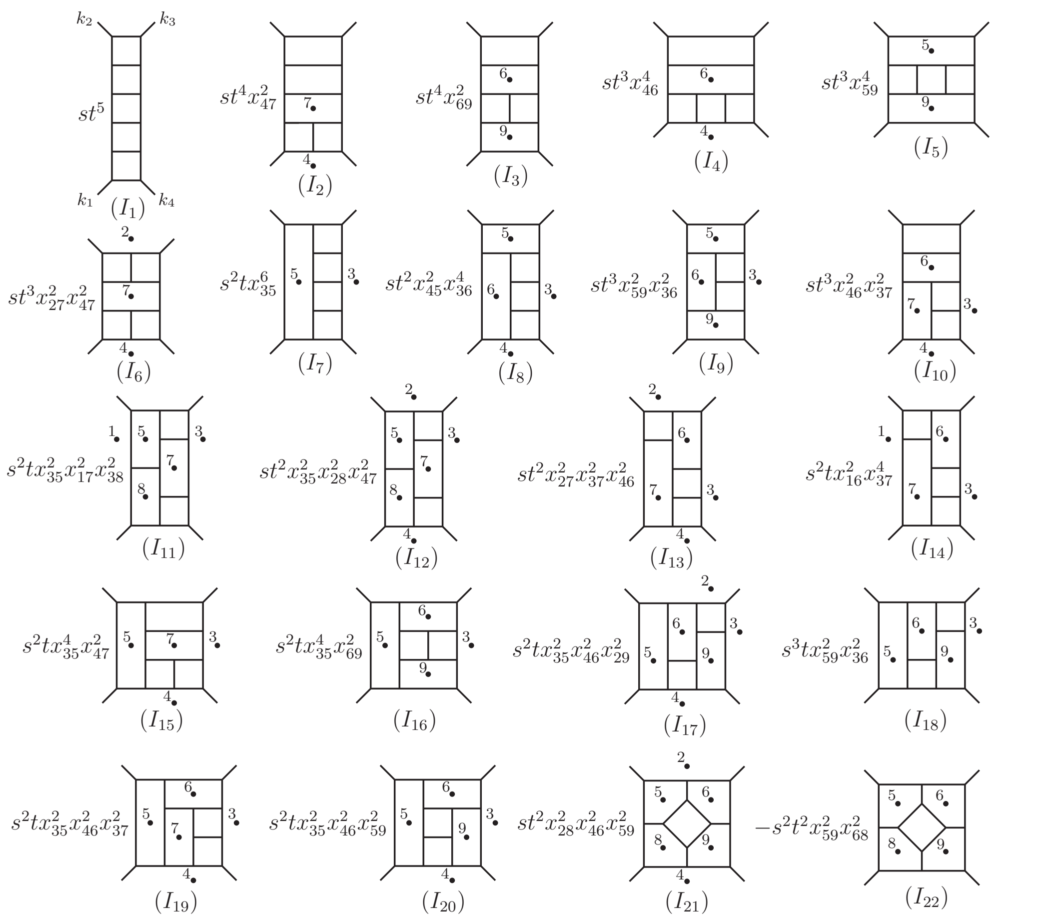


Figure 1.4: Diagrams with only cubic vertices that contribute to the five-loop four-point gluon scattering amplitude.

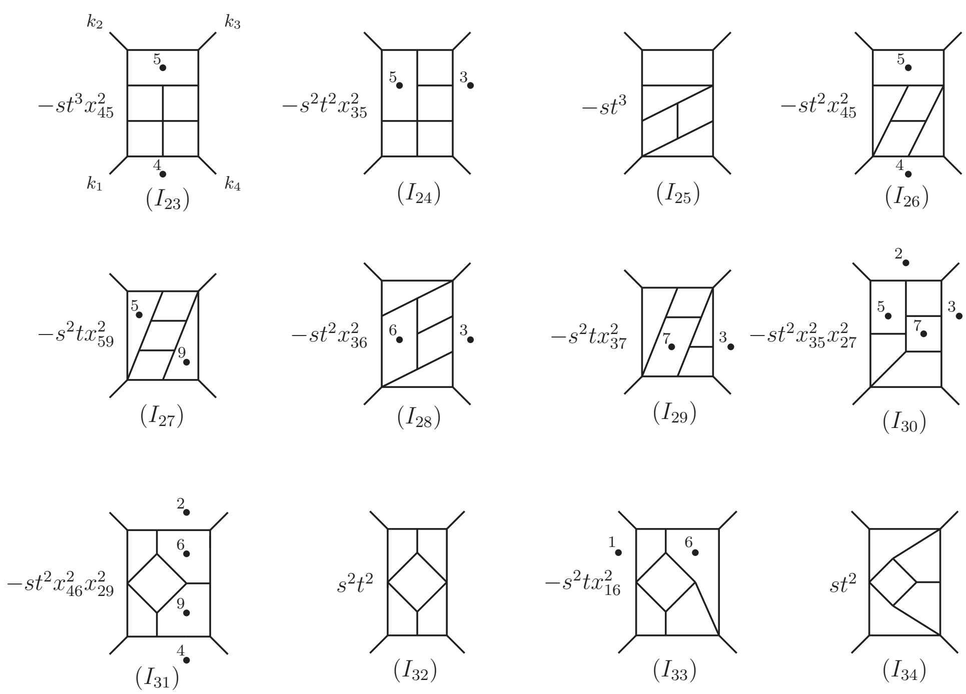


Figure 1.5: Diagrams with both cubic and quartic vertices that contribute to the five-loop four-point gluon scattering amplitude.

1.2.5 ABDK/BDS Ansatz

We will now review an important progress to the understanding of the structure of scattering amplitudes: the ABDK/BDS ansatz for the all loop order n-point MHV amplitude [39].

While evaluating the two-loop four-gluon amplitude in planar $\mathcal{N} = 4$ super Yang-Mills theory, Anastasiou-Bern-Dixon-Kosower (ABDK) [133] discovered a surprising relation between one-loop amplitude and two-loop amplitude as below:

$$M_4^{(2)}(\rho;\epsilon) = \frac{1}{2}\Big[M_4^{(1)}(\rho;\epsilon)\Big]^2 + f^{(2)}(\epsilon)\, M_4^{(1)}(\rho;2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)\,, \tag{1.2.48}$$

where

$$f^{(2)}(\epsilon) = -(\zeta_2 + \zeta_3\epsilon + \zeta_4\epsilon^2)\,, \tag{1.2.49}$$

and

$$C^{(2)} = -\frac{1}{2}\zeta_2^2\,. \tag{1.2.50}$$

The same expression holds for the two-loop splitting amplitude. This relation gives rise to an iterative structure that is hidden in scattering amplitudes. Will this structure hold at higher loop level? Answering this question in a later work, Bern-Dixon-Smirnov (BDS) [39] calculated the three-loop four-gluon amplitude and found the same structure as before:

$$\begin{aligned} M_4^{(3)}(\rho;\epsilon) \;&=\; -\frac{1}{3}\Big[M_4^{(1)}(\rho;\epsilon)\Big]^3 + M_4^{(1)}(\rho;\epsilon)\, M_4^{(2)}(\rho;\epsilon) + f^{(3)}(\epsilon)\, M_4^{(1)}(\rho;3\,\epsilon) \\ &\quad + C^{(3)} + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{1.2.51}$$

where

$$f^{(3)}(\epsilon) = \frac{11}{2}\,\zeta_4 + \epsilon(6\zeta_5 + 5\zeta_2\zeta_3) + \epsilon^2(c_1\zeta_6 + c_2\zeta_3^2)\,, \tag{1.2.52}$$

and

$$C^{(3)} = \left(\frac{341}{216} + \frac{2}{9}c_1\right)\zeta_6 + \left(-\frac{17}{9} + \frac{2}{9}c_2\right)\zeta_3^2\,. \tag{1.2.53}$$

Then it was suggested that the four-loop iteration relation would have the following form,

$$\begin{aligned} M_4^{(4)}(\rho;\epsilon) \;=\; & \frac{1}{4}\Big[M_4^{(1)}(\rho;\epsilon)\Big]^4 - \Big[M_4^{(1)}(\rho;\epsilon)\Big]^2 M_4^{(2)}(\rho;\epsilon) + M_4^{(1)}(\rho;\epsilon)M_4^{(3)}(\rho;\epsilon) \\ & + \frac{1}{2}\Big[M_4^{(2)}(\rho;\epsilon)\Big]^2 + f^{(4)}(\epsilon)\,M_4^{(1)}(\rho;4\,\epsilon) + C^{(4)} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{1.2.54}$$

This leads Bern-Dixon-Smirnov to propose that the all-loop-order n-gluon MHV amplitude is given by (which is then known as the ABDK/BDS ansatz),

$$\mathcal{M}_n(\rho) \equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\rho;\epsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(\rho;l\epsilon) + C^{(l)} + E_n^{(l)}(\rho;\epsilon)\Big)\right]\,. \tag{1.2.55}$$

where

$$a \equiv \frac{N_c\alpha_s}{2\pi}(4\pi e^{-\gamma})^{\epsilon}\,, \tag{1.2.56}$$

and

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}\,. \tag{1.2.57}$$

The objects $f_k^{(l)}$, $k = 0, 1, 2$, and $C^{(l)}$ are pure constants, independent of the external kinematics ρ, and also independent of the number of legs n. $f_0^{(l)}$ are the Taylor coefficients of the cusp anomalous dimension or universal scaling function (1.2.17)

$$f(\lambda) = 4\sum_{l=0}^{\infty} a^l f_0^{(l)} \,. \tag{1.2.58}$$

Similarly, we have

$$g(\lambda) = 2\sum_{l=2}^{\infty} \frac{a^l}{l} f_1^{(l)} \equiv 2\int \frac{d\lambda}{\lambda} G(\lambda) \quad , \qquad k(\lambda) = -\frac{1}{2}\sum_{l=2}^{\infty} \frac{a^l}{l^2} f_2^{(l)} \,, \tag{1.2.59}$$

which can be identified with quantities appearing in the resummed Sudakov form factor (1.2.18).

Their one-loop values are defined to be,

$$f^{(1)}(\epsilon) = 1 \,, \qquad C^{(1)} = 0 \,, \qquad E_n^{(1)}(\rho;\epsilon) = 0 \,. \tag{1.2.60}$$

Let us consider the general L-loop n-particle amplitude

$$M_n^{(L)}(\rho;\epsilon) = X_n^{(L)}\Big[M_n^{(l)}(\rho;\epsilon)\Big] + f^{(L)}(\epsilon) M_n^{(1)}(\rho; L\epsilon) + C^{(L)} + E_n^{(L)}(\rho;\epsilon) \,, \tag{1.2.61}$$

where the quantities $X_n^{(L)} = X_n^{(L)}[M_n^{(l)}]$ only depend on the lower-loop amplitudes $M_n^{(l)}(\rho;\epsilon)$ with $l < L$. The $X_n^{(L)}$ can be computed simply by performing the following Taylor expansion,

$$X_n^{(L)}\Big[M_n^{(l)}\Big] = M_n^{(L)} - \ln\left(1 + \sum_{l=1}^{\infty} a^l M_n^{(l)}\right)\Big|_{a^L \text{ term}} \,. \tag{1.2.62}$$

At lower loops, we have

$$X_n^{(2)}\left[M_n^{(l)}\right] = \frac{1}{2}\left[M_n^{(1)}\right]^2, \qquad (1.2.63)$$

$$X_n^{(3)}\left[M_n^{(l)}\right] = -\frac{1}{3}\left[M_n^{(1)}\right]^3 + M_n^{(1)}M_n^{(2)}, \qquad (1.2.64)$$

$$X_n^{(4)}\left[M_n^{(l)}\right] = \frac{1}{4}\left[M_n^{(1)}\right]^4 - \left[M_n^{(1)}\right]^2 M_n^{(2)} + M_n^{(1)}M_n^{(3)} + \frac{1}{2}\left[M_n^{(2)}\right]^2. \qquad (1.2.65)$$

Now if we define

$$F_n^{(L)}(\rho;\epsilon) = M_n^{(L)} - \sum_{l=0}^{L-1} \hat{I}_n^{(L-l)}\, M_n^{(l)}, \qquad (1.2.66)$$

with $M_n^{(0)} \equiv 1$, then

$$F_n^{(L)}(\rho;0) \equiv X_n^{(L)}\left[F_n^{(l)}(\rho;0)\right] + f^{(L)}(\rho;0)F_n^{(1)}(\rho;0) + C^{(L)}. \qquad (1.2.67)$$

and we obtain

$$\begin{aligned}\mathcal{F}_n(\rho;0) \equiv 1 + \sum_{L=1}^{\infty} \hat{a}^L F_n^{(L)}(\rho;0) &= \exp\left[\sum_{l=1}^{\infty} \hat{a}^l \left(f_0^{(l)} F_n^{(1)}(\rho;0) + C^{(l)}\right)\right] \\ &\equiv \exp\left[\frac{1}{4}\gamma_K(\hat{a})\; F_n^{(1)}(\rho;0) + C(\hat{a})\right], \qquad (1.2.68)\end{aligned}$$

where $C(\hat{a}) = \sum_{l=1}^{\infty} C^{(l)}\hat{a}^l$.

Now let us investigate the divergent structure of the ABDK/BDS ansatz. The infrared poles can be isolated as follow:

$$\mathrm{Div}_n = -\sum_{i=1}^{n}\left[\frac{1}{8\epsilon^2} f^{(-2)}\left(\frac{\lambda\mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^\epsilon}\right) + \frac{1}{4\epsilon} g^{(-1)}\left(\frac{\lambda\mu_{IR}^{2\epsilon}}{(-s_{i,i+1})^\epsilon}\right)\right], \qquad (1.2.69)$$

where the invariants $s_{i,i+1}$ are assumed to be negative.

Then we can rewrite the amplitude as

$$\ln \mathcal{M}_n = \mathrm{Div}_n + \frac{f(\lambda)}{4} F_n^{(1)}(0) + nk(\lambda) + C(\lambda) \tag{1.2.70}$$

with $C(\lambda) = \sum_{l=1}^{\infty} C^{(l)} a^l$.

The finite remainder function $F_n^{(1)}(0)$ is given as below:

For $n = 4$:

$$F_4^{(1)}(0) = \frac{1}{2} \left(\ln \frac{s_{12}}{s_{23}} \right)^2 + 4\zeta_2 \ . \tag{1.2.71}$$

For $n > 4$:

$$F_n^{(1)}(0) = \frac{1}{2} \sum_{i=1}^{n} g_{n,i} \, , \tag{1.2.72}$$

where

$$g_{n,i} = - \sum_{r=2}^{\lfloor n/2 \rfloor - 1} \ln \left(\frac{-t_i^{[r]}}{-t_i^{[r+1]}} \right) \ln \left(\frac{-t_{i+1}^{[r]}}{-t_i^{[r+1]}} \right) + D_{n,i} + L_{n,i} + \frac{3}{2} \zeta_2 \, , \tag{1.2.73}$$

in which $\lfloor x \rfloor$ is the greatest integer less than or equal to x and $t_i^{[r]} = (k_i + \cdots + k_{i+r-1})^2$ are momentum invariants. (All indices are understood to be mod n)

The form of $D_{n,i}$ and $L_{n,i}$ depends upon whether n is odd or even. For the even case ($n = 2m$) these quantities are given by

$$\begin{aligned} D_{2m,i} &= - \sum_{r=2}^{m-2} \mathrm{Li}_2 \left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}} \right) - \frac{1}{2} \mathrm{Li}_2 \left(1 - \frac{t_i^{[m-1]} t_{i-1}^{[m+1]}}{t_i^{[m]} t_{i-1}^{[m]}} \right) \, , \\ L_{2m,i} &= \frac{1}{4} \ln^2 \left(\frac{-t_i^{[m]}}{-t_{i+1}^{[m]}} \right) \, . \end{aligned} \tag{1.2.74}$$

In the odd case ($n = 2m+1$), we have

$$\begin{aligned} D_{2m+1,i} &= -\sum_{r=2}^{m-1} \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i-1}^{[r+2]}}{t_i^{[r+1]} t_{i-1}^{[r+1]}}\right) , \\ L_{2m+1,i} &= -\frac{1}{2} \ln\left(\frac{-t_i^{[m-1]}}{-t_i^{[m+1]}}\right) \ln\left(\frac{-t_{i+1}^{[m]}}{-t_{i-1}^{[m+1]}}\right) . \end{aligned} \tag{1.2.75}$$

By construction, the ABDK/BDS ansatz has the correct infrared singularities as well as the correct behavior under collinear limits. Although this ansatz has a beautiful structure, calculations of scattering amplitudes at strong coupling [] and at weak coupling [] have shown that the finite part of this ansatz is not correct when the number of external legs $n \geq 6$. The finite remainder function needs to include a function of all dual conformal invariant cross-ratios, which are constructed by kinematics. For $n = 4, 5$ case, there are no cross-ratios so there is no correction. It is still an on-going effort to determine the exact form of this remainder function. We will discuss more about this subject in chapter 3.

1.2.6 Recursion Relations

We will briefly review the derivation of the BCF recursion relations for tree-level amplitudes [14, 15]. The important point of these recursion relations is factorization on multi-particle poles. Let consider a particular deformation of an amplitude which shifts two spinors, labelled here as i and j, of n massless external particles as

$$\tilde{\lambda}_i \ \to\ \hat{\tilde{\lambda}}_i \ :=\ \tilde{\lambda}_i + z\tilde{\lambda}_j \ , \qquad \lambda_j \ \to\ \hat{\lambda}_j \ :=\ \lambda_j - z\lambda_i \ , \tag{1.2.76}$$

where z is the complex parameter characterizing the deformation. The spinors λ_i and $\tilde{\lambda}_j$ are left unshifted. The deformations (1.2.76) are chosen in such a way that the corresponding shifted momenta

$$\hat{p}_i(z) := \lambda_i \hat{\tilde{\lambda}}_i \;=\; p_i + z\lambda_i\tilde{\lambda}_j \;, \quad \hat{p}_j(z) := \hat{\lambda}_j\tilde{\lambda}_j \;=\; p_j - z\lambda_i\tilde{\lambda}_j \;, \tag{1.2.77}$$

are on-shell for all complex z. Furthermore, $p_i(z)+p_j(z) = p_i+p_j$. So the amplitude $\mathcal{A}(p_1,\ldots,p_i(z),\ldots,p_j(z),\ldots,p_n)$ is a well-defined one complex parameter function.

Then let's consider the following contour integral

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} dz \frac{\mathcal{A}(z)}{z} \;, \tag{1.2.78}$$

where the contour $\mathcal{C}$ is the circle at infinity in the complex z-plane. The integral in (1.2.78) vanishes if $\mathcal{A}(z) \to 0$ as $z \to \infty$. It then follows from Cauchy's theorem that we can write the amplitude we wish to calculate, $\mathcal{A}(0)$, as a sum of residues of $\mathcal{A}(z)/z$,

$$\mathcal{A}(0) \;=\; -\sum_{\substack{\text{poles of } \mathcal{A}(z)/z \\ \text{excluding } z=0}} \text{Res}\left[\frac{\mathcal{A}(z)}{z}\right] \;. \tag{1.2.79}$$

At tree level, $\mathcal{A}(z)$ has only simple poles in z. A pole at $z = z_P$ is associated with a shifted momentum $\hat{P} := P(z_P)$ flowing through an internal propagator becoming null. The residue at this pole is then obtained by factorizing the shifted amplitude on this pole. The result is that

$$\mathcal{A} \;=\; \sum_P \sum_h \mathcal{A}_L^h(z_P)\frac{i}{P^2}\mathcal{A}_R^{-h}(z_P) \;, \tag{1.2.80}$$

where the sum is over the possible assignments of the helicity h of the intermediate state, and over all possible P such that precisely one of the shifted momenta, say $\hat{p}_i$, is contained in P.

The left and right hand amplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$ are well-defined amplitudes only for $z = z_P$, when $P(z)$ becomes null. We call $\lambda_{\hat{P}}$ and $\tilde{\lambda}_{\hat{P}}$ the spinors associated to the internal, on-shell momentum $\hat{P}$, so that $\hat{P} := \lambda_{\hat{P}}\tilde{\lambda}_{\hat{P}}$. Notice that the intermediate propagator is evaluated with unshifted kinematics.

Since a momentum invariant involving both (or neither) of the shifted legs i and j does not give rise to a pole in z, the shifted legs i and j must always appear on opposite sides of the factorization channel. In order to limit the number of recursive diagrams, it is very convenient to shift adjacent legs. In this case, the sum over P in (1.2.80) is just a single sum. In the following we will do this, so that the shifted legs will always be i and $j = i + 1$. We will denote the shift in (1.2.76) with the standard notation $[i\, i + 1\rangle$.

Now for the supersymmetric version of the BCF recursion relation. Firstly, we notice that it is very easy to describe the shifts (1.2.76), (1.2.77) using dual (or region) momenta. One simply defines

$$\hat{p}_i \ := \ x_i - \hat{x}_{i+1} \ , \qquad \hat{p}_{i+1} \ := \ \hat{x}_{i+1} - x_{i+2} \ , \tag{1.2.81}$$

where we have introduced a shifted region momentum

$$\hat{x}_{i+1} \ := \ x_{i+1} - z\, \lambda_i \tilde{\lambda}_{i+1} \ . \tag{1.2.82}$$

Notice that this is the only region momentum that is affected by the shifts[1]. Therefore in the supersymmetric case we expect that θ_{i+1} is shifted but all other θ's remain unshifted. This implies that

$$\theta_i - \theta_{i+2} = \eta_i \lambda_i + \eta_{i+1}\lambda_{i+1} \ , \tag{1.2.83}$$

should remain unshifted. This is in complete similarity to the fact that the sum of the shifted momenta is unshifted, $\hat{p}_i + \hat{p}_{i+1} = p_i + p_{i+1}$. Now, in the case of the $[i\, i+1\rangle$ shift employed here, we have shifted λ_{i+1} according to (1.2.76) and so we can achieve this by shifting η_i to

$$\hat{\eta}_i = \eta_i + z\, \eta_{i+1} \ , \tag{1.2.84}$$

and leaving η_{i+1} unshifted. This then gives the shifted θ_{i+1}

$$\hat{\theta}_{i+1} \ := \ \theta_{i+1} - z\, \eta_{i+1}\lambda_i \ . \tag{1.2.85}$$

The recursion relation builds up tree-level amplitudes recursively from lower point amplitudes.

The supersymmetric recursion relation follows from arguments similar to those which led to (1.2.80). We have

$$\mathcal{A} \ = \ \sum_P \int d^4\eta_{\hat{P}} \ \mathcal{A}_L(z_P) \frac{i}{P^2} \mathcal{A}_R(z_P) \ , \tag{1.2.86}$$

[1]This is true only if adjacent legs are shifted. If i and j are not adjacent, then region momenta $x_{i+1} \ldots x_j$ are all shifted by $-z\, \lambda_i \tilde{\lambda}_j$.

where $\eta_{\hat{P}}$ is the anticommuting variable associated to the internal, on-shell leg with momentum $\hat{P}$.

In the recursion relation (1.2.86) we have an important constraint on $\mathcal{A}_L$ and $\mathcal{A}_R$, namely the total helicity of $\mathcal{A}_L$ plus the total helicity of $\mathcal{A}_R$ must equal the total helicity of the full amplitude $\mathcal{A}$. This condition replaces the sum over internal helicities in the standard BCF recursion (1.2.80).

1.3 Twistor String Theory and The Connected Prescription

Gluon scattering amplitudes in Yang-Mills theory have some remarkable mathematical properties which are completely obscured in the Feynman diagram expansion by which they are traditionally computed. One of the examples is the tree-level MHV scattering amplitudes can be expressed in terms of a simple holomorphic or antiholomorphic function, which was conjectured by Park and Taylor [25] and proved later by Berends and Giele [26]. Motivated by the desire to find an underlying explanation for this structure, Witten suggested [218] that these mathematical properties hint at the existence of a description of Yang-Mills theory in terms of twistor string theory.

What will happen when the usual momentum space scattering amplitudes are Fourier transformed to Penrose's twistor space? Witten [218] proposed that the perturbative expansion of $\mathcal{N} = 4$ super Yang-Mills theory with $U(N)$ gauge group is equivalent to the D-instanton expansion of the topological B model string theory, whose target space is the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$. This proposal gives a beautiful relation between perturbative gauge theory and string theory, as well as a powerful calculation method to evaluate scattering amplitudes in gauge theory.

We will now give a brief review of the setup. The standard textbook calculation of color-stripped tree-level gluon scattering amplitude in Yang-Mills theory describes the amplitude A of n gluons as a function of n momenta and n polarizations $(p_i^\mu, \epsilon_i^\mu)$, which is highly redundant. A more efficient and sufficient choice of variables is given by the spinor helicity notation $(\lambda_i, \tilde{\lambda}_i, \textit{gluon helicity}\ \pm 1)$, where we have written the momenta as

$$p_i^{a\dot{a}} = \lambda_i^a \tilde{\lambda}_i^{\dot{a}}. \tag{1.3.1}$$

There is still redundancy that the spinor λ and $\tilde{\lambda}$ are not unique, but are determined only modulo the scaling

$$\lambda \to t\ \lambda, \qquad \tilde{\lambda} \to t^{-1}\ \tilde{\lambda}. \tag{1.3.2}$$

In Minkowski signature $(-+++)$, $\tilde{\lambda}$ would in fact be the complex conjugate of λ. It is simpler to work with signature $(--++)$ because λ and $\tilde{\lambda}$ are then independent variables. The split signature $(--++)$ allows for a rather simplified treatment of the transformation to twistor space. We now make a choice of performing a "1/2-Fourier transform" in which only the $\tilde{\lambda}_i^{\dot{a}}$ variables are transformed

$$\tilde{\lambda}_{\dot{a}} \to i\frac{\partial}{\partial \mu^{\dot{a}}}, \qquad -i\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \to \mu_{\dot{a}}, \tag{1.3.3}$$

so that the momentum and special conformal operators become first order, the dilatation operator becomes homogeneous while the Lorentz generators are unchanged.

This choice breaks the symmetry between left and right.

An amplitude $A(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}})$ can then be expressed in twistor space as

$$\tilde{A}(\lambda_i^a, \mu_i^{\dot{a}}) = \int d^{2n}\tilde{\lambda}\ \exp i \sum_{j=1}^{n} \mu_i^{\dot{a}} \tilde{\lambda}_{i\dot{a}}\ A(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}). \tag{1.3.4}$$

Twistor string theory is naturally related not to pure Yang-Mills theory but to the $\mathcal{N} = 4$ supersymmetric version thereof, so we need to include a fermionic variable η_i^A, $(A = 1, 2, 3, 4)$ with the same transformation as in the bosonic case:

$$\eta_A \to i\frac{\partial}{\partial \psi^A}, \qquad\qquad -i\frac{\partial}{\partial \eta_A} \to \psi^A. \tag{1.3.5}$$

So the supersymmetric extension of (nonprojective) twistor space with $\mathcal{N} = 4$ supersymmetry is $\hat{\mathbb{T}} = \mathbb{C}^{4|4}$ with four bosonic coordinates $Z^I = (\lambda^a, \mu^{\dot{a}})$ and four fermionic coordinates ψ^A. Then we make (Z^I, ψ^A) as homogeneous coordinates by using the symmetry group of the projectivized super-twistor space $\widehat{\mathbb{PT}}$. A version of that space in $(2, 2)$ signature where Z^I is real and we only consider functions of (Z^I, ψ^A), is called $\mathbb{RP}^{3|4}$. If Z^I is complex then $\widehat{\mathbb{PT}}$ is a copy of $\mathbb{CP}^{3|4}$ Hence the supersymmetric amplitude $\tilde{A}(\lambda_i^a, \mu_i^{\dot{a}}, \eta_i^A)$ a function on the supermanifold $\mathbb{P}^{3|4}$.

Witten conjectured and checked in several cases that an n-gluon scattering amplitude with q negative helicity gluons and $n - q$ positive helicity gluons is supported on curves in $\mathbb{P}^{3|4}$ of degree $d = q - 1$. This implies that $\tilde{A}$ can be expressed as an integral over the moduli space of degree d curves in $\mathbb{P}^{3|4}$.

Witten's formulation of twistor string theory includes a collection of N D5-branes spanning the the bosonic dimensions of $\mathbb{P}^{3|4}$ and D1-branes. Quantizing the open strings on D5-branes gives rise to the gluons of $\mathcal{N} = 4$ super Yang-Mills theory. The D1-branes can wrap any holomorphic curve (topologically a $\mathbb{P}^1$) inside $\mathbb{P}^{3|4}$. D1-brane instantons give rise to new degrees of freedom, the open strings stretching between the D1-brane and D5-branes. So the n-gluon scattering amplitude takes the form as follows:

$$A_n \sim \int d\mathcal{M}_d \int d^n z \, \langle J(z_1) \dots J(z_n) \rangle \prod_{i=1}^{n} \phi_i(Z(z_i)), \tag{1.3.6}$$

here z_i are n points on $\mathbb{P}^1$ where the open string vertex operator J is inserted, $Z(z)$ is an embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{3|4}$ of degree d describing the curve wrapped by the D1-brane, $\phi_i(Z)$ are the wave-functions for the external gluons, and $d\mathcal{M}_d$ is the measure on the space of degree d curves in $\mathbb{P}^{3|4}$. We proceed by calculating the open string correlator, which gives

$$\langle J(z_1) \dots J(z_n) \rangle = \frac{1}{(z_1 - z_2) \dots (z_n - z_1)}, \tag{1.3.7}$$

writing out the moduli space of curves as,

$$d\mathcal{M}_d = \frac{\prod_{k=0}^{d} d^4\alpha \; d^4\beta}{GL(2, \mathbb{C})}, \tag{1.3.8}$$

with α, β are the moduli of the curve, and finally expressing the wave-function to be,

$$\phi_i(\lambda^a, \mu^{\dot{a}}, \psi^A) = \int \frac{d\xi_i}{\xi_i} \delta^2(\lambda_i^a - \xi_i \lambda^a) \exp\left(i\xi_i [\mu, \tilde{\lambda}_i] \right) \; \exp\left(i\xi_i \psi^A \eta_{iA} \right). \tag{1.3.9}$$

Putting all together and integrating out half of the bosonic and all of the fermionic moduli, we obtain a formula for n-gluon scattering amplitude

$$A(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}, \eta_i^A) = \int \frac{d^{2d+2}\alpha}{GL(2,\mathbb{C})} \frac{d^n z}{(z_1 - z_2)\dots(z_n - z_1)} \frac{d^m \xi}{\xi_1 \dots \xi_m} \prod_{i=1}^{n} \delta^2(\lambda_i^a - \xi_i Z^a(z_i)) \times \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \xi_i z_i^k \tilde{\lambda}_i^{\dot{a}}\right) \delta^4\left(\sum_{i=1}^{4} \xi_i z_i^k \eta_i^A\right). \quad (1.3.10)$$

Witten's proposal leads to the so-called "connected prescription", by only considering connected instantons, for the tree-level S-matrix of Yang-Mills theory. Cachazo, Svrcek and Witten [] used disconnected instantons to also reproduce the gauge theory amplitudes while Bena, Bern and Kosower [] showed that the partially connected instantons lead to the same result. To evaluate formula 1.3.10 for a scattering amplitude of $d+1$ negative helicity gluons and $n-d-1$ positive helicity gluons, we first find all solutions to the $2n+2d+2$ polynomial equations

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \xi_i z_i^k \tilde{\lambda}_i^{\dot{a}}, \qquad (\dot{a} = 1,2;\ k = 0,\dots,d) \\ 0 &= \lambda_i^a - \xi_i \sum_{k=0}^{d} z_i^k \alpha_k^a, \qquad (a = 1,2; i = 1,\dots,n), \end{aligned} \quad (1.3.11)$$

in terms of the $2n+2d+2$ variables (z_i, ξ_i, α_k^a). These equations are guaranteed to have solutions because of momentum conservation.

Then we sum a certain Jacobian, obtained by using

$$\int d^n x \prod_{i=1}^{n} \delta(f_i(x_1,\dots,x_n)) = \frac{1}{det\left(\frac{\partial f_i}{\partial x_j}\right)}, \quad (1.3.12)$$

over the collection of roots from 1.3.10. This remarks an important feature of 1.3.10 is that it is completely localized, not really an integral - that would be complicated to evaluate - but just a simple computation.

The formula 1.3.10 can be generalized to describe any tree-level n-particle scattering amplitudes in super Yang-Mills. The conjectured generalization has similar formula and has been checked [] to satisfy almost all properties of amplitudes: it has cyclic, reflective and parity symmetries, obeys the (generalized) dual Ward identity, has the correct soft and collinear limit. But it has not been proven to have the correct multi-particle factorization, which will then prove the conjecture is true - something that most people believe.

The connected prescription formula can be written again another form, using $\mathcal{Z}_i = (\lambda_i^\alpha, \mu_i^{\dot\alpha}, \eta_i^A)$ the 4|4 component homogeneous coordinates for the i-th particle in $\mathbb{P}^{3|4}$ as follows []:

$$\mathcal{A}(\mathcal{Z}) = \int \frac{d^{4k|4k}\mathcal{A}\ d^n\sigma\ d^m\xi}{vol\ GL(2)} \prod_{i=1}^{n} \frac{\delta^{4|4}(\mathcal{Z}_i - \xi_i \mathcal{P}(\sigma_i))}{\xi_i(\sigma_i - \sigma_{i+1})}, \tag{1.3.13}$$

where $\mathcal{P}$ is the degree $k-1$ polynomial given in terms of its k $\mathbb{C}^{4|4}$-valued superco-efficients $\mathcal{A}_d$ by

$$\mathcal{P}(\sigma) = \sum_{d=0}^{k-1} \mathcal{A}_d \sigma^d. \tag{1.3.14}$$

This formula is a contour integral in a multidimensional complex space. It has a $GL(2)$ invariance in the integrand and the measure, which needs to be gauged. The delta functions here specify the contour of integration, indicating which poles to include in the sum over residues.

Chapter 2

Dual Conformal Symmetry

2.1 Magic Identities: the emerging of a new symmetry

We will review the discovery, formalism and recent developments of the dual superconformal symmetry in this chapter [127]. Four-point correlators in the $\mathcal{N} = 4$ super-Yang-Mills conformal field theory have attracted considerable attention since the formulation of the AdS/CFT conjecture [6]. They can provide non-trivial dynamical information about the CFT side of the correspondence, which can then be compared to its AdS dual. And although it may seem that the two problems, that of the correlators of gauge-invariant composites and that of gluon scattering amplitudes, are unrelated, it is quite significant that in both studies one deals with the same conformal integrals. We will show later that the understanding of these subjects has contributed greatly to the full understanding of scattering amplitudes in recent years.

The scattering amplitudes are typically described in terms of scalar loop integrals. And a new symmetry emerged in 2006 when Drummond, Henn, Smirnov and Sokatchev [87] found some 'magic identities' by studying the four-point off-shell scattering amplitudes. They examined the structure of the loop corrections of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory in the planar limit and deduced an iterative procedure for constructing classes of off-shell four-point conformal integrals ,which are identical, up to 4 loops as shown in Fig. 2.1 [87]. It was proven in there that the three-loop ladder diagram and the three-loop tennis court diagram are precisely equal to each other in four dimensions when taken off-shell. There are also other relations between different integrals at higher loop levels. The origin of why these identities exist wasn't known at the time because there is no trace of these relations when working on-shell. However this procedure does not generate all the contributing diagrams to the scattering amplitudes, which will be shown later in this section.

In addition to the new identities, it has also been found in [87] that all of the contributing integrals obey a new conformal symmetry, which is called 'dual conformal symmetry'. This symmetry, which is unrelated to the conventional conformal symmetry of $\mathcal{N} = 4$ Yang-Mills, is dynamical and does not manifest in the Lagrangian of the theory. It acts as conformal transformations on the variables $x_i \equiv k_i - k_{i+1}$, where k_i are the cyclically ordered momenta of the particles participating in a scattering process. This dual conformal symmetry is formal because the integrals are in fact infrared divergent and the introduction of dimensional regularisation breaks the symmetry. Nevertheless, even broken, the dual conformal symmetry still imposes constraints on the on-shell scattering amplitudes. It is important to emphasize that

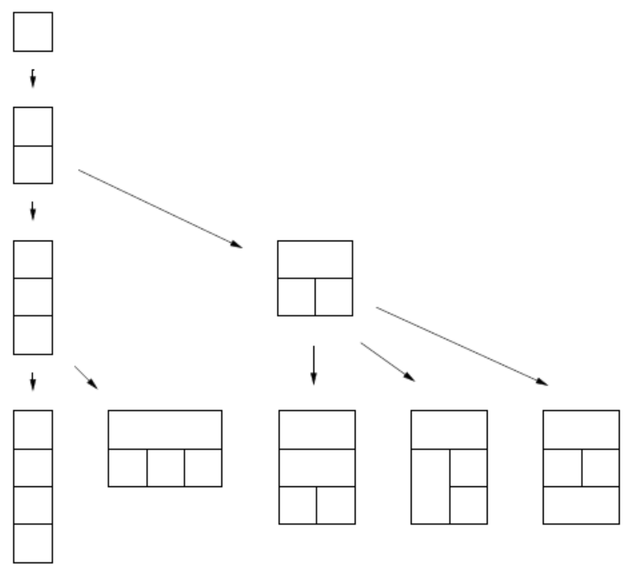


Figure 2.1: Some integral topologies up to four loops. The integrals generated from topologies in a given row are all equivalent. Each topology in the next row is generated by integrating the slingshot in all possible orientations.

dual conformal invariance is a property of planar amplitudes only in super Yang-Mills and not a property of supergravity theory. After its discovery and though its origin is still under study, dual conformal symmetry was used as a guideline to find the contributing diagrams to the scattering amplitudes at four loops [50] and five loops [5].

Although somewhat mysterious at weak coupling, dual conformal symmetry is well understood in the Alday-Maldacena prescription [59] which provides a geometrical interpretation, making it manifest at strong coupling. One of the steps in this construction involves T-dualizing along the four directions of AdS_5 parallel to the boundary, and dual conformal symmetry is the just isometry of this T-dualized AdS_5. We will discuss more about this in the next section.

2.1.1 Properties of Dual Conformal Integrals

We will review here the definition and diagrammatic properties of dual conformal integrals following [102, 132]. By way of illustration we consider first the one-loop diagram shown in Fig. 2.2. Black lines depict the underlying scalar Feynman diagram, with each internal line associated to a $1/p^2$ propagator as usual. Each dotted red line indicates a numerator factor $(p_1 + p_2 + \cdots + p_n)^2$ where the p_i are the momenta flowing through the black lines that it crosses. Of course momentum conservation at each vertex guarantees that a numerator factor only depends on where the dotted red line begins and ends, not on the particular path that it traverses through the diagram. We adopt a standard convention (see for example [255]) that each four-

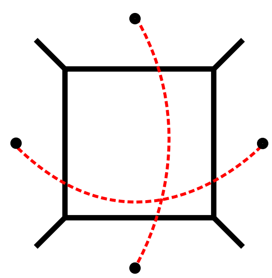


Figure 2.2: The one-loop scalar box diagram with conformal numerator factors indicated by the dotted red lines.

dimensional loop momentum integral comes with a normalization factor of $1/i\pi^2$. Thus the diagram shown in Fig.2.2 corresponds to the integral

$$\mathcal{I}^{(1)}(k_1, k_2, k_3, k_4) = \int \frac{d^4 p_1}{i\pi^2} \frac{(k_1 + k_2)^2 (k_2 + k_3)^2}{p_1^2 (p_1 - k_1)^2 (p_1 - k_1 - k_2)^2 (p_1 + k_4)^2}. \tag{2.1.1}$$

We regulate this infrared divergent integral by taking the external legs off-shell, choosing for simplicity all of the 'masses' $k_i^2 = \mu^2$ to be the same. A different possible infrared regulator that one might consider would be to replace the $1/p^2$ propagators by massive propagators $1/(p^2 - m^2)$, but we keep all internal lines strictly massless.

Following [102, 132] we then pass to dual coordinates x_i by taking

$$k_1 = x_{12}, \qquad k_2 = x_{23}, \qquad k_3 = x_{34}, \qquad k_4 = x_{41}, \qquad p_1 = x_{15}, \tag{2.1.2}$$

where $x_{ij} \equiv x_i - x_j$, so that [**?**] becomes

$$\mathcal{I}^{(1)}(x_1, x_2, x_3, x_4) = \int \frac{d^4 x_5}{i\pi^2} \frac{x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}. \tag{2.1.3}$$

This expression is now easily seen to be invariant under arbitrary conformal transformations on the x_i. This invariance is referred to as 'dual conformal' symmetry in [102] because it should not be confused with the familiar conformal symmetry of $\mathcal{N} = 4$ Yang-Mills. The coordinates x_i here are momentum variables and are not simply related to the position space variables on which the usual conformal symmetry acts.

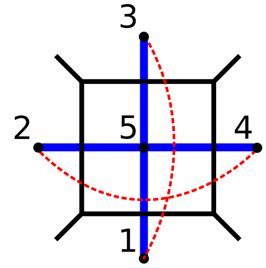


Figure 2.3: The one-loop scalar box with dotted red lines indicating numerator factors and thick blue lines showing the dual diagram.

To analyze the dual conformal invariance of a diagram it is convenient to consider its dual diagram [1]. In Fig.2.3 we have labeled the vertices of the dual diagram to Fig.2.2 in accord with the expression 2.1.3. The numerator factors correspond to dotted red lines as before, while denominator factors in the dual expression 2.1.3 correspond to the thick blue lines in the dual diagram. Neighboring external faces are not connected by thick blue lines because the associated propagator is absent.

With this notation set up it is easy to formulate a diagrammatic rule for determining whether a diagram can correspond to a dual conformal integral. We associate to each face in the diagram (i.e., each vertex in the dual diagram) a weight which is equal to the number of thick blue lines attached to that face minus the number of dotted red lines. Then a diagram is dual conformal if the weight of each of the four external faces is zero and the weight of each internal face is four (to cancel the weight of the corresponding loop integration $\int d^4x$). Consequently, no tadpoles, bubbles or triangles are allowed in the Feynman diagram, each square must be associated with no numerator factors, each pentagon must be associated with one numerator factor, etc.

We distinguish slightly between dual conformal *diagrams* and dual conformal *integrals*. The latter are all diagrams satisfying the diagrammatic rule given above. However in [132] it was pointed out that not all dual conformal diagrams give rise to integrals that are finite off-shell in four dimensions. Those that are not finite can only be defined with a regulator (such as dimensional regularization) that breaks the dual conformal symmetry and hence cannot be considered dual conformal integrals.

[1] The fact that nonplanar graphs do not have duals is consistent with the observation that dual conformal symmetry is a property only of the planar limit.

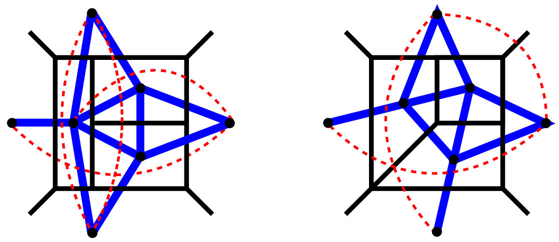


Figure 2.4: Two examples of three-loop dual conformal diagrams.

In Fig. 2.4 we show two diagrams that are easily seen to be dual conformal according to the above rules. The diagram on the left is the well-known three-loop tennis court $\mathcal{I}^{(3)b}$. The diagram on the right demonstrates a new feature that is possible only when the external lines are taken off-shell. The dotted red line connecting the top external face to the external face on the right crosses only one external line and is therefore associated with a numerator factor of $k_i^2 = \mu^2$. Such an integral is absent when we work on-shell. Of course *absent* does not necessarily mean that an integral *vanishes* if we first calculate it for finite μ^2 and then take $\mu^2 \to 0$. Indeed we will see below that $\mathcal{I}^{(3)c} \sim \ln^3(\mu^2)$ in the infrared limit.

An important and well-known feature of four-point dual conformal integrals is that they are constrained by the symmetry to be a function only of the conformally invariant cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{2.1.4}$$

Since we have chosen to take all external momenta to have the same value of $k_i^2 = \mu^2$, we see that u and v are are actually both equal to

$$x \equiv \frac{\mu^4}{st}, \tag{2.1.5}$$

where $s = (p_1+p_2)^2$ and $t = (p_2+p_3)^2$ are the usual Mandelstam invariants.Therefore we can express any dual conformal integral as a function of the single variable x. One immediate consequence of this observation is that any dual conformal integral is invariant under rotations and reflections of the corresponding diagram, since x itself is invariant under such permutations. A second consequence is that any degenerate integral (by which we mean one where two or more of the external momenta enter the diagram at the same vertex) must evaluate to a constant, independent of x.

2.2 Dual Conformal Symmetry at Weak Coupling

We will review here the dual conformal symmetry of gluon scattering amplitudes following [127]. First we recall the conventional conformal symmetry of the scattering amplitudes, then describe how dual conformal symmetry emerges. $\mathcal{N} = 4$ SYM theory involves bosons (gluons and scalars), as well as fermions (gluinos). Each of these particles is characterized by its on-shell momentum p_i^μ $(i = 1 \dots n)$ and helicity $h_i = \pm 1$ (gluons), $\pm 1/2$ (gluinos), 0 (scalars). All particles are treated as incoming, so that the total momentum conservation condition is

$$\sum_{i=1}^{n} (p_i)^{\dot{\alpha}\alpha} = 0\,. \tag{2.2.1}$$

To solve the on-shell condition for p_i, it is convenient to introduce commuting spinor variables

$$(p_i)^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}}\,\lambda_i^{\alpha}\,, \tag{2.2.2}$$

where λ_i^{α} $(\alpha = 1, 2)$ and $\tilde{\lambda}_i^{\dot{\alpha}}$ $(\dot{\alpha} = \dot{1}, \dot{2})$ are two-component Weyl spinors. The standard convention is that the spinors λ and $\tilde{\lambda}$ carry helicities $-1/2$ and $+1/2$, respectively. Thus, the momentum $(p_i)^{\dot{\alpha}\alpha}$ has vanishing helicity. We will refer to the space with coordinates $(\lambda_i, \tilde{\lambda}_i)$ as the 'on-shell' space. We will let the momenta complex so that there will be solutions in further applications. The spinors λ_i^{α} and $\tilde{\lambda}_i^{\dot{\alpha}}$ could be determined in terms of the momentum up to a scale by the little group transformation:

$$\lambda_i^{\alpha} \;\rightarrow\; t_i \lambda_i^{\alpha}\,, \qquad \tilde{\lambda}_i^{\dot{\alpha}} \;\rightarrow\; t_i^{-1}\tilde{\lambda}_i^{\dot{\alpha}}\,. \tag{2.2.3}$$

The advantage of using the spinor variables λ and $\tilde{\lambda}$ is that we can make a bridge between Lorentz and massless Poincaré representations.

Let us introduce the spinor description of gluon states. On-shell gluons are massless Poincaré states of helicity ± 1 described, correspondingly, by $G^{\pm}(p)$. Their Lorentz covariant description makes use of the self-dual and anti-self-dual parts of the gluon field strength tensor, $G_{\alpha\beta}(p) = \lambda_\alpha \lambda_\beta\; G^{+}(p)$ and $\bar{G}_{\dot{\alpha}\dot{\beta}}(p) = \tilde{\lambda}_{\dot{\alpha}} \tilde{\lambda}_{\dot{\beta}}\; G^{-}(p)$ respectively, satisfying the equations of motion $p^{\dot{\alpha}\alpha} G_{\alpha\beta}(p) = \bar{G}_{\dot{\alpha}\dot{\beta}}(p) p^{\dot{\beta}\alpha} = 0$.

Then, as mentioned in chapter 1, the MHV tree-level gluon amplitude with the two negative-helicity gluons occupying sites i and j is given by

$$\mathcal{A}_{n;0}\left(1^{+} \ldots i^{-} \ldots j^{-} \ldots n^{+}\right) = i(2\pi)^4 \delta^{(4)}\left(\sum_{k=1}^{n} p_k\right) \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \ldots \langle n\, 1\rangle} \tag{2.2.4}$$

where

$$\langle i\, j\rangle = -\langle j\, i\rangle = \lambda_i^{\alpha}\, \epsilon_{\alpha\beta}\, \lambda_j^{\beta} = \lambda_i^{\alpha} \lambda_{j\,\alpha}\,. \tag{2.2.5}$$

The above amplitude has a conformal symmetry [218], which is the ordinary conformal symmetry $SO(2,4)$ of the $\mathcal{N}=4$ SYM Lagrangian, but realized on the particle momenta (or, equivalently, on the spinor variables λ and $\tilde{\lambda}$). And the conformal boost generator takes the form of a second-order differential operator

$$k_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda_i^{\alpha}\partial\tilde{\lambda}_i^{\dot{\alpha}}}\,, \tag{2.2.6}$$

leading to $k_{\alpha\dot{\alpha}}\mathcal{A}_{n;0} = 0$.

Now let us introduce new 'dual' coordinates x_i (with $i = 1, \ldots, n$):

$$(p_i)^{\dot{\alpha}\alpha} = (x_i)^{\dot{\alpha}\alpha} - (x_{i+1})^{\dot{\alpha}\alpha}\,, \tag{2.2.7}$$

satisfying the cyclicity condition

$$x_{n+1} \equiv x_1\,. \tag{2.2.8}$$

The new variables x_i have nothing to do with the coordinates in the original configuration space (which are the Fourier conjugates of the particle momenta p_i).

With the scattering amplitudes having the following general form

$$\mathcal{A}_n = i(2\pi)^4\delta^{(4)}(\sum_{i=1}^{n} p_i)A_n(p_1, \ldots, p_n)\,, \tag{2.2.9}$$

we can think of the relation $(x_i - x_{i+1})^{\dot{\alpha}\alpha} = \tilde{\lambda}_i^{\dot{\alpha}}\,\lambda_i^{\alpha}\,, x_{n+1} \equiv x_1$ as defining a surface in the full space and then we can interpret the function $A_n(p_i) = A_n(\lambda_i, \tilde{\lambda}_i, x_i)$ appearing in the amplitude (2.2.9) as a function defined on this surface. This leads to a dual translation invariance

$$P_{\alpha\dot{\alpha}}\, A_n(\lambda_i, \tilde{\lambda}_i, x_i) = 0 \tag{2.2.10}$$

with $P_{\alpha\dot{\alpha}}$ being the generator of translations

$$P_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}\,. \tag{2.2.11}$$

We could rewrite the constraint to eliminate one variable from the Full space $(x, \lambda, \tilde{\lambda})$ to go to the On-shell space $(\lambda, \tilde{\lambda})$ or the Dual space (x, λ). Here we will work in the dual space to exhibit a new conformal symmetry of gluon scattering amplitudes. We shall assume that the dual coordinates $x^{\dot{\alpha}\alpha}$ transform in the standard way under the conformal group $SO(2,4)$, and then deduce the transformation properties of λ^{α} by requiring that the defining relation should remain covariant.

It is well known that the conformal group $SO(2,4)$ can be obtained from the *Poincaré* group by adding the discrete operation of conformal inversion,

$$I[x_{\alpha\dot{\beta}}] = \frac{x_{\beta\dot{\alpha}}}{x^2} \equiv (x^{-1})_{\beta\dot{\alpha}}\,. \tag{2.2.12}$$

We can then use the following transformations

$$I[x_{ij}^2] = \frac{x_{ij}^2}{x_i^2 x_j^2}\,. \tag{2.2.13}$$

$$I\Big[\langle i\ i+1\rangle\Big] = (x_i^2)^{-1}\ \langle i\ i+1\rangle\,, \qquad (2.2.14)$$

$$I\Big[[i\ i+1]\Big] = (x_{i+2}^2)^{-1}\ [i\ i+1]\,.$$

to deduce the conformal weight of the tree-level MHV split-helicity amplitude to be (+1) at all points:

$$I\left[\mathcal{A}_{n;0}^{\rm MHV}\left(1^-2^-3^+\dots n^+\right)\right] = \left(x_1^2x_2^2\dots x_n^2\right)\ \mathcal{A}_{n;0}^{\rm MHV}\left(1^-2^-3^+\dots n^+\right)\,, \qquad (2.2.15)$$

So all tree-level MHV split-helicity amplitudes $\mathcal{A}_{n;0}^{\rm MHV}\left(\dots G_i^- G_{i+1}^- \dots\right)$ are dual conformal covariant. The same property holds for *all split-helicity tree-level non-MHV amplitudes* $\mathcal{A}_{n;0}\left(1^-\dots q^-(q+1)^+\dots n^+\right)$, i.e. those amplitudes in which the negative-helicity gluons appear contiguously.

$$I\left[\mathcal{A}_{n;0}\left(1^-\dots q^-(q+1)^+\dots n^+\right)\right] = \left(x_1^2x_2^2\dots x_n^2\right)\ \mathcal{A}_{n;0}\left(1^-\dots q^-(q+1)^+\dots n^+\right)\,. \qquad (2.2.16)$$

However, the tree-level non-split-helicity MHV amplitudes $\mathcal{A}_{n;0}^{\rm MHV}\left(\dots i^-\dots j^-\dots\right)$ are not covariant for $|i-j|>1$. The full understanding of the role of dual conformal symmetry is achieved when the gluon amplitudes are combined together with the amplitudes involving other particles (gluinos, scalars) into a bigger and unifying object, the complete $\mathcal{N}=4$ superamplitude, which has well defined conformal properties.

With the defined inversion operator, we can choose the generators of the dual special conformal transformation $K = IPI$ in the full space to be

$$K^{\dot{\alpha}\alpha} = \sum_{i=1}^{n} \left[x_i^{\dot{\alpha}\beta} x_i^{\dot{\beta}\alpha} \frac{\partial}{\partial x_i^{\dot{\beta}\beta}} + x_i^{\dot{\alpha}\beta} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\dot{\beta}\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} \right]. \tag{2.2.17}$$

Now we will summarize here the dual conformal properties of the complete MHV amplitude, which is naturally factorized into a tree-level factor and a loop-correction factor as

$$\mathcal{A}^{\text{MHV}} = \mathcal{A}_{n;0}^{\text{MHV}} \; M_n^{MHV}(x_i) \tag{2.2.18}$$

where

$$M_n^{MHV}(x_i) = 1 + aM_n^{(1)}(x_i) + a^2 M_n^{(2)}(x_i) + \dots \tag{2.2.19}$$

and the coupling $a = g^2 N / 8\pi^2$.

The $M_n^{(j)}(x_i)$ is a combination of infrared divergent loop momentum integrals, which needs to be regulated. Here we will use dimensional regularization ($D = 4 - 2\epsilon$ with $\epsilon > 0$) and the amplitudes will therefore depend on the regulator ϵ and some associated scale μ.

One can split $M_n(x_i)$ into two parts:

$$\begin{aligned} \ln M_n^{MHV} &= \ln Z_n^{MHV} + \ln \mathcal{F}_n^{MHV} + O(\epsilon) \\ &= \sum_{l=1}^{\infty} a^l \left[\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma_{\text{col}}^{(l)}}{l\epsilon} \right] \sum_i \left(\frac{\mu_{IR}^2}{-s_{i,i+1}} \right)^{l\epsilon} + F_n^{\text{MHV}}(p_1, \dots, p_n; a) + O(\epsilon). \end{aligned} \tag{2.2.20}$$

The first part is the infrared divergent part, $\ln Z_n^{MHV}$, that includes only simple and double poles, which is controlled by the cusp anomalous dimension $\Gamma_{\text{cusp}}(a) =$

$\sum a^l \Gamma^{(l)}_{\rm cusp}$. And the second part is the finite part, $\ln \mathcal{F}_n^{MHV}$, that is controlled under the anomalous conformal Ward identity as below

$$K^\mu \ln \mathcal{F}_n^{MHV} = \sum_{i=1}^{n} (2x_i^\nu x_i \cdot \partial_i - x_i^2 \partial_i^\nu) \ln \mathcal{F}_n^{MHV} = \frac{1}{2} \Gamma_{\rm cusp}(a) \sum_{i=1}^{n} \ln \frac{x^2_{i,i+2}}{x^2_{i-1,i+1}} x^\nu_{i,i+1} \; . \tag{2.2.21}$$

The general solution of this identity allows some freedom in the form of an arbitrary function of the conformally invariant cross-ratios

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{il}^2 x_{jk}^2} \, , \tag{2.2.22}$$

if $n \geq 6$ (for $n = 4, 5$ there exist no cross-ratios, due to the light-like separation of adjacent points).

2.3 Super Amplitudes

Beside the gluons, the $\mathcal{N} = 4$ SYM theory also have eight fermion states (gluinos) Γ_A and $\bar{\Gamma}^A$ with helicities $1/2$ and $-1/2$, respectively, and six scalars (helicity zero states) $S_{AB} = -S_{BA}$. Here $A, B, C, D = 1, \ldots, 4$ are indices of the (anti)fundamental representation of the R symmetry group $SU(4)$.

We will briefly recall the $\mathcal{N} = 4$ gluon supermultiplet [127]. One can build the massless representation with a choice of Lorentz frame in which $p^\mu = (p, 0, 0, p)$. Then the supersymmetry algebra becomes the Clifford algebra

$$\{q_1^A, \bar{q}_{B\,\dot{1}}\} = 2\delta^A_B \; p \tag{2.3.1}$$

with all the other anticommutators vanishing. In this frame the states (massless Poincaré representations) are labeled by their helicity, the eigenvalue of the Lorentz generator J_{12}. For chiral spinors it is $J_{12} = \frac{1}{2}(\sigma_{12})_\alpha{}^\beta$, and for antichiral spinors $J_{12} = \frac{1}{2}(\tilde{\sigma}_{12})_{\dot\alpha}{}^{\dot\beta}$. The helicity of q_1^A is 1/2 and of $\bar{q}_{A\,\dot{1}}$ is $-1/2$.

Next, we define a vacuum state of helicity h by the condition that it be annihilated by all those supersymmetry generators which anticommute among themselves (annihilation operators):

$$q_1^A|h\rangle = q_2^A|h\rangle = \bar{q}_{A\,\dot{2}}|h\rangle = (J_{12} - h)\,|h\rangle = 0\,. \tag{2.3.2}$$

Then the massless supermultiplet of states is obtained by applying the four creation operators $\bar{q}_{A\,\dot{1}}$ to the vacuum:

State	Helicity	Gluon SuperMultiplet	Multiplicity
$\lvert h\rangle$	h	1	1
$\bar{q}_{A\,\dot{1}}\lvert h\rangle$	$h - 1/2$	1/2	4
$\bar{q}_{A\,\dot{1}}\bar{q}_{B\,\dot{1}}\lvert h\rangle$	$h - 1$	0	6
$\epsilon^{ABCD}\bar{q}_{A\,\dot{1}}\bar{q}_{B\,\dot{1}}\bar{q}_{C\,\dot{1}}\lvert h\rangle$	$h - 3/2$	$-1/2$	4
$\epsilon^{ABCD}\bar{q}_{A\,\dot{1}}\bar{q}_{B\,\dot{1}}\bar{q}_{C\,\dot{1}}\bar{q}_{D\,\dot{1}}\lvert h\rangle$	$h - 2$	-1	1

(2.3.3)

In a physical theory the helicity should be $|h| \leq 2$, so in the case $\mathcal{N} = 4$ the allowed values are $h = 1, 3/2, 2$. We see that the multiplet obtained by choosing $h = 1$ is self-conjugate under PCT, since it contains all the helicities ranging from $+1$ to -1. This is the so-called $\mathcal{N} = 4$ gluon supermultiplet, describing massless particles of helicities ± 1 (gluons), $\pm 1/2$ (gluinos) and 0 (scalars).

However, this construction requires the choice of a special frame, thus manifestly breaking Lorentz invariance. We can reproduce the supermultiplet 2.3.3 in a manifestly Lorentz covariant way. Let us rewrite the supersymmetry algebra using the representation of the on-shell momentum:

$$\{q^A_\alpha, \bar{q}_{B\,\dot{\alpha}}\} = \delta^A_B \; \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \,. \tag{2.3.4}$$

The two-component spinor q^A_α has two Lorentz covariant projections, one 'parallel' to λ_α, $q^A_{||\,\alpha} = \lambda_\alpha q^A$ (with $\lambda^\alpha q^A_{||\,\alpha} = 0$), the other 'orthogonal', $q^A_\perp = \lambda^\alpha q^A_\alpha$. The same applies to $\bar{q}_{A\,\dot{\alpha}}$. Then we obtain the covariant analog of the Clifford algebra 2.3.1 by setting the orthogonal projections to zero:

$$\{q^A, \bar{q}_B\} = \delta^A_B \,. \tag{2.3.5}$$

Such algebras are most naturally realized in terms of anticommuting (Grassmann) variables η^A, which transform in the fundamental representation of $su(4)$ and have helicity 1/2:

$$q^A_\alpha = \lambda_\alpha \eta^A \,, \qquad \bar{q}_{A\,\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \eta^A} \,, \qquad \{\eta^A, \eta^B\} = 0 \,. \tag{2.3.6}$$

We can now use the generators 2.3.6 to reproduce the content of the multiplet 2.3.3 in the convenient and compact form of a super-wave function

$$\Phi(p,\eta) = G^+ + \eta^A \Gamma_A + \tfrac{1}{2!}\eta^A\eta^B S_{AB} + \tfrac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD} \overline{\Gamma}^D + \tfrac{1}{4!}\eta^A\eta^B\eta^C\eta^D \epsilon_{ABCD} G^- . \tag{2.3.7}$$

Here $G^+, \Gamma_A, S_{AB} = \frac{1}{2}\epsilon_{ABCD}\overline{S}^{CD}, \overline{\Gamma}^A, G^-$ are the positive helicity gluon, gluino, scalar, anti-gluino and negative helicity gluon states respectively.

The helicity generator, the central charge of the superconformal algebra $su(2,2|4)$,

$$h = -\tfrac{1}{2}\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} + \tfrac{1}{2}\tilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial\tilde{\lambda}^{\dot{\alpha}}} + \tfrac{1}{2}\eta^A \frac{\partial}{\partial\eta^A}, \tag{2.3.8}$$

gives the helicity condition

$$h\Phi = \Phi \ , \tag{2.3.9}$$

$$h_i\mathcal{A}(\Phi_1, \ldots, \Phi_n) = \mathcal{A}(\Phi_1, \ldots, \Phi_n), \qquad i = 1, \ldots, n. \tag{2.3.10}$$

The superwave function Φ has helicity 1 because we have chosen a holomorphic approach, we could have the same description with a helicity (-1) superwave function $\bar{\Phi}(p, \bar{\eta})$, which is PCT self-conjugate and related to Φ by

$$\bar{\Phi}(p, \bar{\eta}) = \int d^4\eta \, \epsilon^{\bar{\eta}_A \eta^A} \, \Phi(p, \eta) \, . \tag{2.3.11}$$

In a theory with $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supersymmetry the gluon multiplet is not self-conjugate, therefore we would need both a holomorphic *and* an anti-holomorphic super-wave functions for the full theory.

The all-loop superamplitude in $\mathcal{N} = 4$ super Yang-Mills theory is conjectured to be written as follows [],

$$\begin{aligned} \mathcal{A}_n &\equiv \mathcal{A}_n^{\text{MHV}} + \mathcal{A}_n^{\text{NMHV}} + \mathcal{A}_n^{\text{N}^2\text{MHV}} + \ldots + \mathcal{A}_n^{\text{N}^{n-4}\text{MHV}} \\ &= \mathcal{A}_n^{\text{MHV}} \left[R_n(\eta_i, \lambda_i, \tilde{\lambda}_i) + O(\epsilon) \right] \, . \end{aligned} \tag{2.3.12}$$

Here $\mathcal{A}_n^{\mathrm{N^{n-4}MHV}} = \mathcal{A}_n^{\overline{\mathrm{MHV}}}$ is the googly, or anti-MHV amplitude. The ratio function

$$R_n(\eta_i, \lambda_i, \tilde{\lambda}_i) = 1 + R_n^{\mathrm{NMHV}} + R_n^{\mathrm{N^2MHV}} + \ldots + R_n^{\mathrm{N^{n-4}MHV}} \tag{2.3.13}$$

is finite as $\epsilon \to 0$ and satisfies the dual conformal Ward identities

$$K^\mu R_n(\eta_i, \lambda_i, \tilde{\lambda}_i) = D\, R_n(\eta_i, \lambda_i, \tilde{\lambda}_i) = 0\,, \tag{2.3.14}$$

with the dilatation D and conformal boost K^μ operators defined in the on-shell superspace $(\eta_i, \lambda_i, \tilde{\lambda}_i)$.

Specifically, the tree-level superamplitude can be written as

$$\begin{aligned} \mathcal{A}(\Phi_1, \ldots, \Phi_n) \equiv \mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) \quad &= i(2\pi)^4\ \frac{\delta^{(4)}\left(\sum_{j=1}^n p_j\right)\ \delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle \ldots \langle n1\rangle} \mathcal{P}_n(\lambda_i, \tilde{\lambda}_i, \eta_i) \\ &\equiv \mathcal{A}_{n;0}^{\mathrm{MHV}} \mathcal{P}_{n;0}, \end{aligned} \tag{2.3.15}$$

where $\mathcal{P}_{n;0}$ can be expanded as,

$$\mathcal{P}_{n;0} = 1 + \mathcal{P}_{n;0}^{\mathrm{NMHV}} + \mathcal{P}_{n;0}^{\mathrm{NNMHV}} + \ \ldots\ + \mathcal{P}_{n;0}^{\overline{\mathrm{MHV}}}. \tag{2.3.16}$$

and we have used Nair's description of the n-particle MHV tree-level superamplitude

$$\mathcal{A}_{n;0}^{\mathrm{MHV}} = \frac{i(2\pi)^4 \delta^{(4)}(\sum_{i=1}^n\ \lambda_i^\alpha\, \tilde{\lambda}_i^{\dot{\alpha}})\ \delta^{(8)}(\sum_{i=1}^n\ \lambda_i^\alpha\, \eta_i^A)}{\langle 1\, 2\rangle \langle 2\, 3\rangle \ldots \langle n\, 1\rangle}\,. \tag{2.3.17}$$

The explicit form of the function $\mathcal{P}_{n;0}$ which encodes all tree-level amplitudes was found in [225] by solving a supersymmetrised version [173, 229**?**] of the BCFW recursion relations [196, 209].

The all-loop MHV superamplitude is given as

$$\mathcal{A}_n^{\mathrm{MHV}}(p_1,\eta_1;\ldots;p_n,\eta_n) = i(2\pi)^4 \frac{\delta^{(4)}\left(\sum_{j=1}^n p_j\right)\,\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\, M_n^{\mathrm{MHV}}\,, \quad (2.3.18)$$

where $\delta^{(8)}\left(\sum_{i=1}^n \lambda_{i\,\alpha}\eta_i^A\right) = \prod_{\alpha=1,2}\prod_{A=1}^4 \lambda_{i\,\alpha}\eta_i^A$ is a Grassmann delta function.

For NMHV case, $\mathcal{A}_n^{\mathrm{NMHV}}$ transforms covariantly under the dual superconformal transformations and has the same conformal weights as the MHV superamplitude $\mathcal{A}_n^{\mathrm{MHV}}$. As a result, the 'ratio' of the two superamplitudes is given by a linear combination of *superinvariants* of the form (to one-loop order)

$$\mathcal{A}_n^{\mathrm{NMHV}} = \mathcal{A}_n^{\mathrm{MHV}} \times \left(\frac{1}{n}\sum_{p,q,r=1}^{n} c_{pqr}\,\delta^{(4)}\,(\Xi_{pqr})\,[1 + aV_{pqr} + O(\epsilon)] + O(a^2)\right)\,. \quad (2.3.19)$$

Here $\delta^{(4)}\,(\Xi_{pqr}) \equiv \prod_{A=1}^4 \Xi_{pqr}^A$ are Grassmann delta functions. The integers $p \neq q \neq r$ label three points in the dual superspace $(x_i, \lambda_i^\alpha, \theta_i^{A\,\alpha})$.

In [**?**] it was argued that this conjecture holds for NMHV amplitudes at one loop, based on explicit calculations up to nine points using supersymmetric generalised unitarity. Subsequently [**?**] it has been argued to hold for all one-loop amplitudes by analysing the dual conformal anomaly arising from infrared divergent two-particle cuts.

At tree level, we can obtain the explicit form in the NMHV case for the amplitude as in 2.7.1 and 2.3.16 as follows [225**? ?**]:

$$\mathcal{P}_{n;0}^{\mathrm{NMHV}} = \sum_{a,b} R_{n,ab} \quad (2.3.20)$$

where the sum runs over the range $2 \leq a < b \leq n-1$ (with a and b separated by at least two) and

$$R_{n,ab} = \frac{\langle a\, a-1\rangle\langle b\, b-1\rangle\delta^4\Big(\langle n|x_{na}x_{ab}|\theta_{bn}\rangle + \langle n|x_{nb}x_{ba}|\theta_{an}\rangle\Big)}{x_{ab}^2\langle n|x_{na}x_{ab}|b\rangle\langle n|x_{na}x_{ab}|b-1\rangle\langle n|x_{nb}x_{ba}|a\rangle\langle n|x_{nb}x_{ba}|a-1\rangle}, \qquad (2.3.21)$$

is by itself a dual superconformal invariant. This formula was originally constructed in [?] by supersymmetrising the three-mass coefficients of NMHV gluon scattering amplitudes at one loop in [?]. It was then derived from supersymmetric generalised unitarity [?] as the general form of the one-loop three-mass box coefficient.

A simple one-loop expression for the $n = 6$ NMHV superamplitude is given in [] as follows,

$$\mathcal{A}_6^{\rm NMHV} = \mathcal{A}_6^{\rm MHV}\left[\frac{1}{2}c_{146}\,\delta^{(4)}\left(\Xi_{146}\right)\left(1 + aV_{146}\right) + (\text{cyclic}) + O(a^2)\right], \qquad (2.3.22)$$

where

$$\Xi_{146} = \langle 61\rangle\langle 45\rangle\Big(\eta_4[56] + \eta_5[64] + \eta_6[45]\Big), \qquad (2.3.23)$$

$$c_{146} = -\frac{\langle 34\rangle\langle 56\rangle}{x_{46}^2[1\ ]x_{16}x_{63}|3][1\ ]x_{16}x_{64}|4][1\ ]x_{14}x_{45}|5][1\ ]x_{14}x_{46}|6]}, \qquad (2.3.24)$$

and

$$V_{146} = -\ln u_1 \ln u_2 + \frac{1}{2}\sum_{k=1}^{3}\Big[\ln u_k \ln u_{k+1} + \mathrm{Li}_2(1-u_k)\Big], \qquad (2.3.25)$$

with the periodicity condition $u_{i+3} = u_i$ is implied and

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}. \qquad (2.3.26)$$

2.4 Dual SuperConformal Symmetry at Weak Coupling

In the previous section, we have extended the bosonic coordinates $x, \lambda, \tilde{\lambda}$ to include the fermionic coordinates θ, η and by doing so, promoted the dual conformal symmetry $SO(2,4)$ to the dual superconformal symmetry $SU(2,2|4)$. We have also defined our superamplitudes on the on-shell superspace and seen how their structures are controlled by the dual superconformal symmetry. Now we will discuss about the formalism of this symmetry as below, following [127].

Let us introduce dual $x_i^{\dot{\alpha}\alpha}$ coordinates to solve the momentum conservation constraint and chiral dual $\theta_i^{A\,\alpha}$ coordinates to solve the supercharge conservation constraint,

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} = 0 \implies x_i^{\dot{\alpha}\alpha} - x_{i+1}^{\dot{\alpha}\alpha} = \tilde{\lambda}_{ß}^{\dot{\alpha}} \lambda_i^{\alpha}\,, \\ \sum_{i=1}^{n} \lambda_i^{\alpha} \eta_i^{A} = 0 \implies \theta_i^{A\,\alpha} - \theta_{i+1}^{A\,a} = \lambda_i^{\alpha} \eta_i^{A}\,, \end{aligned} \tag{2.4.1}$$

and we impose the cyclicity conditions

$$x_{n+1} \equiv x_1\,, \qquad \theta_{n+1} \equiv \theta_1\,. \tag{2.4.2}$$

Similar to the dual conformal case, the space with coordinates $(\lambda_i, \tilde{\lambda}_i, x_i, \eta_i, \theta_i)$ is called the Full Superspace. And we can use the constraint 2.4.1 to transform the Full Superspace into the On-shell Superspace of $(\lambda_i, \tilde{\lambda}_i, \eta_i)$ or into the Dual Superspace, which is chiral, of $(x_i, \lambda_i, \theta_i)$.

The dual (super)translation invariance is given as,

$$P_{\alpha\dot{\alpha}}\,\mathcal{P}_n = 0\,, \qquad\qquad Q_{A\,\alpha}\mathcal{P}_n = 0 \tag{2.4.3}$$

with

$$P_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}\,, \qquad\qquad Q_{A\,\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_i^{A\,\alpha}}\,, \tag{2.4.4}$$

where they are not related to the usual translation generator $p_{\alpha\dot{\alpha}}$ or supercharge q^A_α. As before, $P_{\alpha\dot{\alpha}}$ generates shifts of dual $x-$variables, while $Q_{A\alpha}$ generates shifts of θ's

$$\delta_Q \theta_i^{A\,\alpha} = \epsilon^{A\,\alpha}\,, \tag{2.4.5}$$

with $\epsilon^{A\,a}$ being a constant odd parameter (a chiral Weyl spinor).

Dual Poincaré supersymmetry In the dual superspace with coordinates $(x_i, \lambda_i, \theta_i)$ we introduce the generators

$$Q_{A\,\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_i^{A\,\alpha}}\,, \qquad \bar{Q}^A_{\dot{\alpha}} = \sum_{i=1}^{n} \theta_i^{A\,\alpha} \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}\,, \qquad P_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}\,, \tag{2.4.6}$$

satisfying the $\mathcal{N} = 4$ Poincaré supersymmetry algebra

$$\{Q_{A\,\alpha}, \bar{Q}^B_{\dot{\alpha}}\} = \delta_A^B\; P_{\alpha\dot{\alpha}}\,. \tag{2.4.7}$$

The generator $\bar{Q}^A_{\dot{\alpha}}$ has an induced action on the on-shell superspace variables η,

$$\bar{Q}^A_{\dot{\alpha}} = \sum_{i=1}^{n} \eta_i^A \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}}\,. \tag{2.4.8}$$

The choice of the chiral dual superspace is determined by the holomorphic description of the on-shell gluon multiplet. We could make the equivalent choice of an antichiral dual superspace, corresponding to the antiholomorphic description of the gluon multiplet in terms of $\bar{\eta}$. The important point is that the specific nature of the $\mathcal{N} = 4$ gluon multiplet allows us to use either the one or the other description, and does not oblige us to mix them.

For Dual superspace, we have:

$$I\left[\theta_i^{A\,\alpha}\right] = (x_i^{-1})^{\dot{\alpha}\beta}\theta_{i\,\beta}^{A}\,, \qquad\qquad I\left[\theta_{i\,\alpha}^{A}\right] = \theta_i^{A\,\beta}\,(x_i^{-1})_{\beta\dot{\alpha}}\,. \tag{2.4.9}$$

The combination of the dual supersymmetry transformation 2.4.5 with inversion implies another continuous symmetry with generator $\bar{S}^{\dot{\alpha}}_A = IQ_{A\,\alpha}I$, in close analogy with the conformal boosts $K^\mu = IP^\mu I$.

$$\bar{S}^{\dot{\alpha}}_A = \sum_{i=1}^{n} \, x_i^{\dot{\alpha}\alpha}\frac{\partial}{\partial\theta_i^{A\,\alpha}}\,. \tag{2.4.10}$$

$$[\bar{S}^{\dot{\alpha}}_A, P_{\beta\dot{\beta}}] = \delta^{\dot{\alpha}}_{\dot{\beta}}Q_{A\,\beta}\,. \tag{2.4.11}$$

$$[Q_{\alpha A}, K_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta}\bar{S}_{A\,\dot{\beta}}\,. \tag{2.4.12}$$

The reason why we have $SU(2,2|4)$ and not $PSU(2,2|4)$ is that the algebra involves a central charge, which is suggested to be identified with helicity.

For On-shell superspace we have:

$$I[\eta_i^A] = \frac{x_i^2}{x_{i+1}^2}\,\left(\eta_i^A - \theta_i^A x_i^{-1}\tilde{\lambda}_i\right)\,, \tag{2.4.13}$$

which is not homogeneous as in the transformation of θ.

$$I[\lambda_i^{\alpha}] = \frac{(\lambda_i x_i)^{\dot{\alpha}}}{x_i^2}, \qquad I[\tilde{\lambda}_i^{\dot{\alpha}}] = \frac{(\tilde{\lambda}_i x_i)^{\alpha}}{x_{i+1}^2}. \tag{2.4.14}$$

And the generator S has the following form

$$\bar{S}_A^{\dot{\alpha}} = \sum_{i=1}^{n} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^A} \tag{2.4.15}$$

The full set of generators of conventional and dual superconformal symmetry is given in Appendix B.

The generator of dual conformal transformation of all variables $x_i, \theta_i, \lambda_i, \tilde{\lambda}_i, \eta_i$ is given as

$$K^{\dot{\alpha}\alpha} = \sum_{i=1}^{n} \Big[x_i^{\dot{\beta}\alpha} x_i^{\dot{\alpha}\beta} \frac{\partial}{\partial x_i^{\dot{\beta}\beta}} + x_i^{\dot{\alpha}\beta} \theta_i^{B\,\alpha} \frac{\partial}{\partial \theta_i^{\beta B}} + x_i^{\dot{\alpha}\beta} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\dot{\beta}\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} + \tilde{\lambda}_i^{\dot{\alpha}} \theta_{i+1}^{B\,\alpha} \frac{\partial}{\partial \eta_i^B}], \tag{2.4.16}$$

For any generator of the superconformal algebra $psu(2,2|4)$,

$$j_a \in \{p^{\alpha\dot{\alpha}}, q^{\alpha A}, \bar{q}_A^{\dot{\alpha}}, m_{\alpha\beta}, \bar{m}_{\dot{\alpha}\dot{\beta}}, r^A{}_B, d, s_A^{\alpha}, \bar{s}_{\dot{\alpha}}^A, k_{\alpha\dot{\alpha}}\}, \tag{2.4.17}$$

at tree-level we have, (up to contact terms which vanish for generic configurations of the external momenta) [**? ? ?**]:

$$j_a \mathcal{A}_n = 0. \tag{2.4.18}$$

Because there are no infrared divergences.

And we can show that the tree-level MHV superamplitude transforms covariantly under inversion and has equal conformal weights +1 at each point

$$I[\mathcal{A}_{n;0}^{\mathrm{MHV}}] = \left(x_1^2 x_2^2 \ldots x_n^2\right) \, \mathcal{A}_{n;0}^{\mathrm{MHV}} \, . \tag{2.4.19}$$

In [**?**] it was conjectured that the full tree-level superamplitude $\mathcal{A}_n^{\mathrm{tree}}$ is covariant under dual superconformal symmetry:

$$K^{\alpha\dot{\alpha}}\mathcal{A}_n = -\sum_i x_i^{\alpha\dot{\alpha}}\mathcal{A}_n, \tag{2.4.20}$$

$$S^{\alpha A}\mathcal{A}_n = -\sum_i \theta_i^{\alpha A}\mathcal{A}_n, \tag{2.4.21}$$

together with $D\mathcal{A}_n = n\mathcal{A}_n$ and $C\mathcal{A}_n = n\mathcal{A}_n$. The remaining generators of the dual superconformal algebra annihilate the amplitudes.

2.5 Off-shell Amplitudes and Classification

Recent work on $\mathcal{N} = 4$ super Yang-Mills theory has unlocked rich hidden structure in planar scattering amplitudes which indicates the exciting possibility of obtaining exact formulas for certain amplitudes. At weak coupling it has been observed at two and three loops[133] that the planar four-particle amplitude satisfies certain iterative relations which, if true to all loops, suggest that the full planar amplitude $\mathcal{A}$ sums to the simple form

$$\log(\mathcal{A}/\mathcal{A}_{\mathrm{tree}}) = (\text{IR divergent terms}) + \frac{f(\lambda)}{8}\log^2(t/s) + c(\lambda) + \cdots \, , \tag{2.5.1}$$

where $f(\lambda)$ is the cusp anomalous dimension, s and t are the usual Mandelstam invariants, and the dots indicate terms which vanish as the infrared regulator is removed. Evidence for similar structure in the five-particle amplitude has been found at two loops . At strong coupling, Alday and Maldacena[59] have recently given a prescription for calculating gluon scattering amplitudes via AdS/CFT and demonstrated that the structure 2.5.1 holds for large λ as well. An important role in this story is evidently played by dual conformal symmetry [132].

The generalized unitarity methods which are used to construct the dimensionally-regulated multiloop four-particle amplitude in $\mathcal{N} = 4$ Yang-Mills theory express the ratio $\mathcal{A}/\mathcal{A}_{\text{tree}}$ as a sum of certain scalar Feynman integrals—the same kinds of integrals that would appear in ϕ^n theory, but with additional scalar factors in the numerator. However dimensional regularization explicitly breaks dual conformal symmetry, so the authors of [132] used an alternative regularization which consists of letting the four external momenta in these scalar integrals satisfy $k_i^2 = \mu^2$ instead of zero. Then they observed that the particular Feynman integrals which contribute to the dimensionally-regulated amplitude are precisely those integrals which, if taken off-shell, are finite and dual conformally invariant in four dimensions (at least through five loops, which is as far as the contributing integrals are currently known).

Off-shell dual conformally invariant integrals have a number of properties which make them vastly simpler to study than their on-shell cousins. First of all they are finite in four dimensions, whereas an L-loop on-shell dimensionally regulated integral has a complicated set of infrared poles starting at order ϵ^{-2L}. Moreover in our experience it has always proven possible to obtain a one-term Mellin-Barnes representation for any off-shell integral, several examples of which are shown explicitly in section 4. In contrast, on-shell integrals typically can only be written as a sum of many (at four loops, thousands or even tens of thousands of) separate terms near $\epsilon = 0$. It would be inconceivable to include a full expression for any such integral in a paper.

Secondly, the relative simplicity of off-shell integrals is such that a simple analytic expression for the off-shell L-loop ladder diagram was already obtained several years ago and generalized to arbitrary dimensions. For the on-shell ladder diagram an analytic expression is well-known at one-loop, but it is again difficult to imagine that a simple analytic formula for any L might even be possible.

Finally, various off-shell integrals have been observed to satisfy apparently highly nontrivial relations, the 'magic identities' in [102], as mentioned above. Moreover a simple diagrammatic argument was given which allows one to relate various classes of integrals to each other at any number of loops. No trace of this structure is evident when the same integrals are taken on-shell in $4 - 2\epsilon$ dimensions.

Hopefully these last few paragraphs serve to explain our enthusiasm for off-shell integrals. Compared to recent experience with on-shell integrals, which required significant supercomputer time to evaluate, we find that the off-shell integrals we study here are essentially trivial to evaluate.

Unfortunately however there is a very significant drawback to working off-shell, which is that although we know (through five loops) which scalar Feynman integrals contribute to the dimensionally-regulated on-shell amplitude, we do not know which integrals contribute to the off-shell amplitude. In fact it is not even clear that one can in general provide a meaningful definition of the 'off-shell amplitude.' Taking $k_i^2 = \mu^2$ in a scalar integral seems to be a relatively innocuous step but we must remember that although they are expressed in terms of scalar integrals, the amplitudes we are interested in are really those of non-abelian gauge bosons. In this light relaxing the on-shell condition $k_i^2 = 0$ does not seem so innocent. If it is possible to consistently define a general off-shell amplitude in $\mathcal{N} = 4$ Yang-Mills then we would expect to see as $\mu^2 \to 0$ the universal leading infrared singularity:

$$\log\left(\mathcal{A}/\mathcal{A}_{\text{tree}}\right) = -\frac{f(\lambda)}{8}\log^2(\mu^4/st) + \text{less singular terms}, \tag{2.5.2}$$

where $f(\lambda)$ is the same cusp anomalous dimension appearing in [**?**]. If an equation of the form 2.5.2 could be made to work with off-shell integrals, it would provide a method for computing the cusp anomalous dimension that is vastly simpler than that of reading it off from on-shell integrals.

2.5.1 Classification of Dual Conformal Diagrams: Algorithm

Let us now explain a systematic algorithm to enumerate all possible off-shell dual conformal diagrams [3]. We use the graph generating program qgraf [243] to generate all scalar 1PI [2] four-point topologies with no tadpoles or bubbles, and throw away any that are non-planar or contain triangles since these cannot be made dual conformal. After these cuts there remain $(1, 1, 4, 25)$ distinct topologies at $(1, 2, 3, 4)$

[2] We do not know of a general proof that no one-particle reducible diagram can be dual conformal but it is easy to check through four loops that there are no such examples.

loops respectively respectively. We adopt the naming conventions which displays 24 out of the 25 four-loop topologies, omitting the one we call h in Fig. 2.10 since it vanishes on-shell in dimensional regularization.

The next step is to try adding numerator factors to render each diagram dual conformal. Through three loops this is possible in a unique way for each topology, but at four loops there are three topologies (shown in Fig. 2.5) that cannot be made dual conformal at all while six topologies (b_1, c, d, e, e_2 and f, shown below) each admit two distinct choices of numerator factors making the diagram dual conformal.

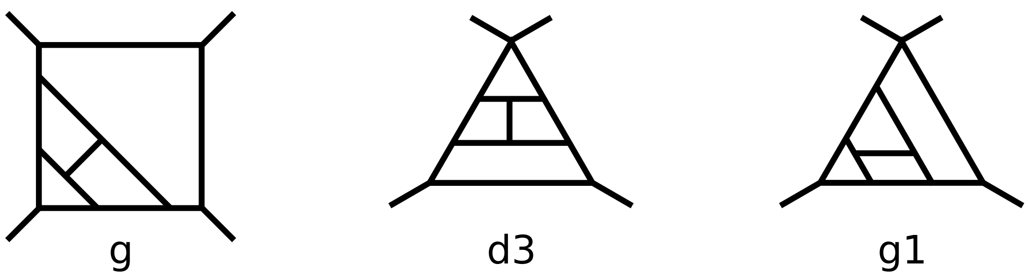


Figure 2.5: These are the three planar four-point 1PI four-loop tadpole-, bubble- and triangle-free topologies that cannot be made into dual conformal diagrams by the addition of any numerator factors. In each case the obstruction is that there is a single pentagon whose excess weight cannot be cancelled by any numerator factor because the pentagon borders on all of the external faces. (There are no examples of this below four loops.)

We remark that we exclude from our classification certain 'trivial' diagrams that can be related to others by rearranging numerator factors connected only to external faces. For example, consider the two diagrams shown in Fig. 2.6. Clearly both are dual conformal, but they differ from each other only by an overall factor of $x = \mu^4/st$. In cases such as this we include in our classification only the diagram with the fewest number of μ^2 powers in the numerator, thereby choosing the integral that is most singular in the $\mu^2 \to 0$ limit. In the example of Fig. 2.6 we therefore exclude the diagram on the right, which actually vanishes in the $\mu^2 \to 0$ limit, in favor of the diagram on the left, which behaves like $\ln^2(\mu^2)$.

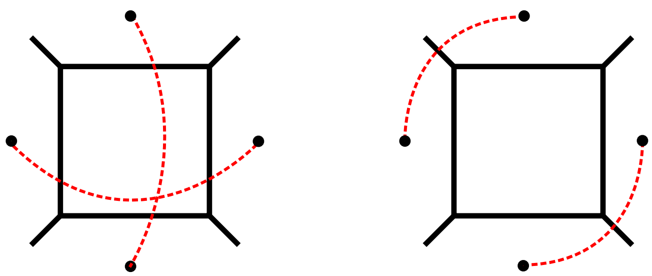


Figure 2.6: Two dual conformal diagrams that differ only by an overall factor. As explained in the text we resolve such ambiguities by choosing the integral that is most singular in the $\mu^2 \to 0$ limit, in this example eliminating the diagram on the right.

Another possible algorithm, which we have used to double check our results, is to first use the results of [253] to generate all planar 1PI vacuum graphs and then enumerate all possible ways of attaching four external legs so that no triangles or bubbles remain.

2.5.2 Classification of Dual Conformal Diagrams: Results

Applying the algorithm just described, we find a total of $(1, 1, 4, 28)$ distinct dual conformal diagrams respectively at $(1, 2, 3, 4)$ loops. While $(1, 1, 2, 10)$ of these diagrams have appeared previously in the literature on dual conformal integrals[102, 265, 267], the remaining $(0, 0, 2, 18)$ that only exist off-shell are new to this thesis. We classify all of these diagrams into four groups, according to whether or not they are finite in four dimensions, and according to whether or not the numerator contains any explicit factors of μ^2 [3]. Hence we define:

Type I diagrams are finite in four dimensions and have no μ^2 factors.

Type II diagrams are divergent in four dimensions and have no μ^2 factors.

Type III diagrams are finite in four dimensions and have μ^2 factors.

Type IV diagrams are divergent in four dimensions and have μ^2 factors.

Type I and II diagrams through five loops have been classified, and some of their properties studied, in[102, 132, 265, 267]. In particular, it has been observed in these references that it is precisely the type I integrals that contribute to the dimensionally regulated on-shell four-particle amplitude (at least through five loops). We display these diagrams through four loops in Figs.2.7 and 2.8. The new type III and IV diagrams that only exist off-shell are shown respectively in Figs.2.9 and 2.10. Each diagram is given a name of the form $\mathcal{I}^{(L)i}$ where L denotes the number of loops and i is a label. The one- and two-loop diagrams are unique and do not require a label. Below we will also use $\mathcal{I}^{(L)}$ to refer to the L-loop ladder diagram (specifically, $\mathcal{I}^{(1)}$, $\mathcal{I}^{(2)}$, $\mathcal{I}^{(3)a}$ and $\mathcal{I}^{(4)a}$ for $L = 1, 2, 3, 4$).

We summarize the results of our classification in the following table showing the number of dual conformal diagrams of each type at each loop order:

L	I	II	III	IV
1	1	0	0	0
2	1	0	0	0
3	2	0	2	0
4	8	2	9	9

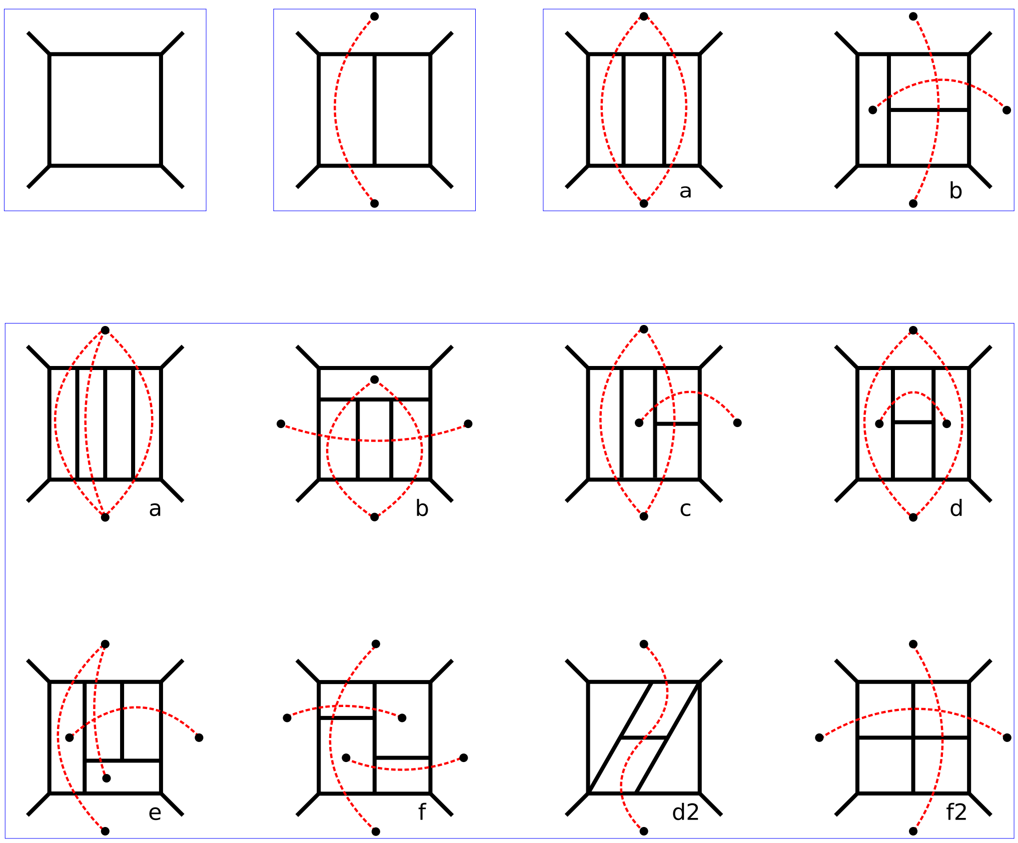


Figure 2.7: Type I: Here we show all dual conformal diagrams through four loops that are finite off-shell in four dimensions and have no explicit numerator factors of μ^2. These are precisely the integrals which contribute to the dimensionally-regulated on-shell four-particle amplitude[242, 249, 258, 265]. For clarity we suppress an overall factor of st in each diagram.

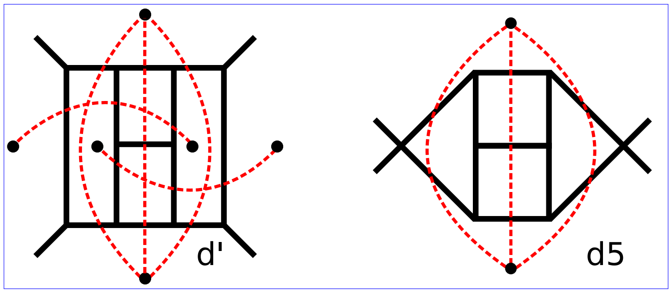


Figure 2.8: Type II: These two diagrams have no explicit factors of μ^2 in the numerator and satisfy the diagrammatic criteria of dual conformality, but the corresponding off-shell integrals diverge in four dimensions [132]. (There are no examples of this type below four loops).

2.5.3 Evaluation of Dual Conformal Integrals: Previously known integrals

In this section we describe the evaluation of dual conformal integrals [3]. Even though dual conformal integrals are finite in four dimensions we evaluate them by first dimensionally regulating the integral to $D = 4-2\epsilon$ dimensions, and then analytically continuing ϵ to zero using for example algorithms described in[251, 252, 260]. For a true dual conformal invariant integral the result of this analytic continuation will be an integral that is finite in four dimensions, so that we can then freely set $\epsilon = 0$. However the type II and type IV diagrams shown in Figs.2.7 and2.9 turn out to not be finite in four dimensions (as can be verified either by direct calculation, or by applying the argument used in [132] to identify divergences). This leaves (1,1,4,17) integrals to be evaluated at (1,2,3,4) loops respectively.

Now we will briefly review the (1,1,2,5) off-shell dual conformal integrals that have already been evaluated in the literature.

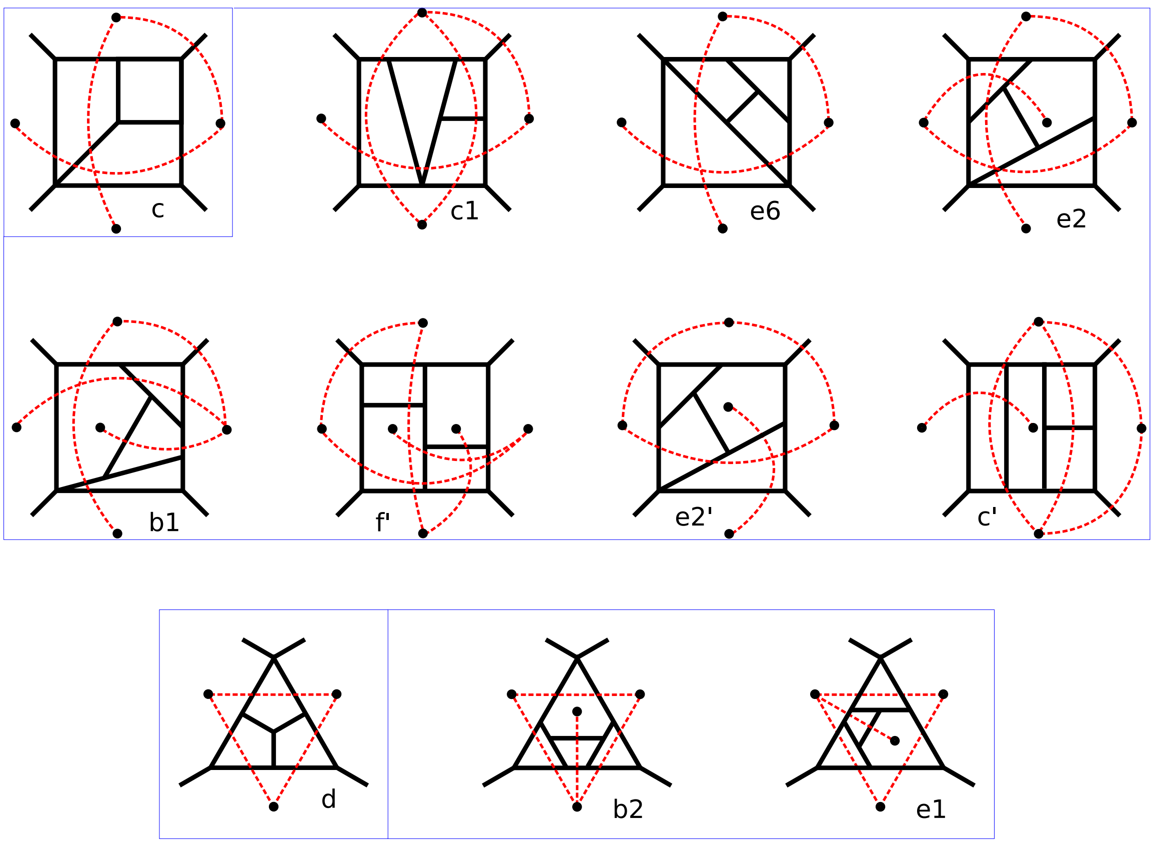


Figure 2.9: Type III: Here we show all diagrams through four loops that correspond to dual conformal integrals in four dimensions with explicit numerator factors of μ^2. (There are no examples of this type below three loops). In the bottom row we have isolated three degenerate diagrams which are constrained by dual conformal invariance to be equal to pure numbers (independent of s, t and μ^2).

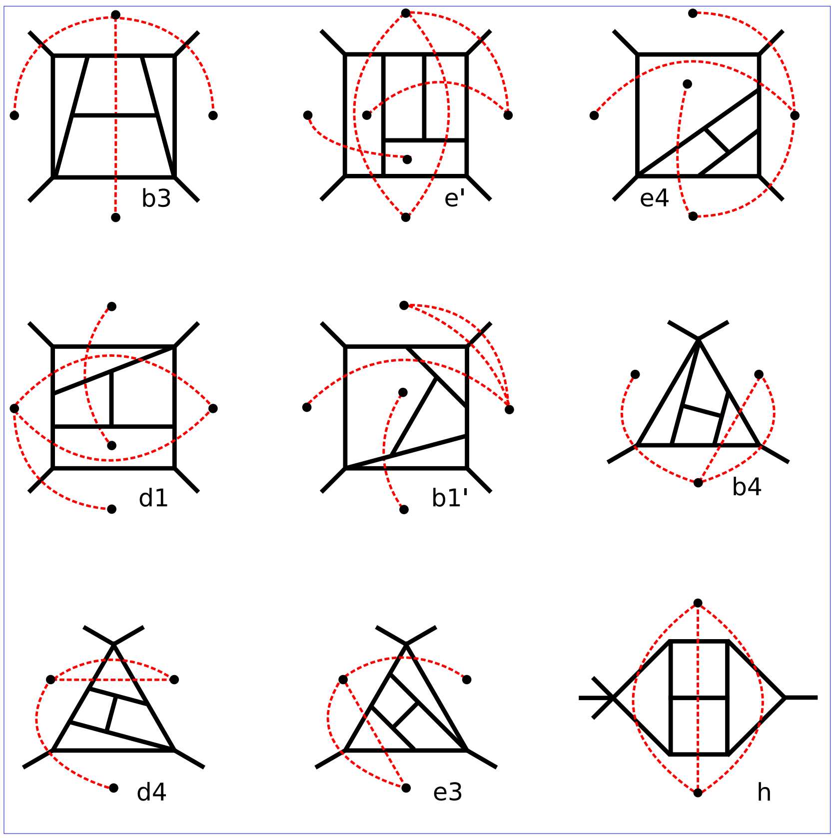


Figure 2.10: Type IV: All remaining dual conformal diagrams. All of the corresponding off-shell integrals diverge in four dimensions.

The first class of integrals that have been evaluated off-shell are those corresponding to the ladder diagrams $\mathcal{I}^{(1)}$, $\mathcal{I}^{(2)}$, $\mathcal{I}^{(3)a} \equiv \mathcal{I}^{(3)}$ and $\mathcal{I}^{(4)a} \equiv \mathcal{I}^{(4)}$. In fact an explicit formula for the off-shell L-loop ladder diagram was given in the remarkable paper [245]. The function $\Phi^{(L)}(x,x)$ in that paper corresponds precisely to our conventions in defining the ladder diagrams $\mathcal{I}^{(L)}$ (including the appropriate conformal numerator factors), so we copy here their result:

$$\mathcal{I}^{(L)}(x) = \frac{2}{\sqrt{1-4x}} \left[\frac{(2L)!}{L!^2} \operatorname{Li}_{2L}(-y) + \sum_{\substack{k,l=0 \\ k+l \text{ even}}}^{L} \frac{(k+l)!(1-2^{1-k-l})}{k!l!(L-k)!(L-l)!} \zeta(k+l) \log^{2L-l-k} y \right], \tag{2.5.3}$$

where

$$y = \frac{2x}{1-2x+\sqrt{1-4x}}. \tag{2.5.4}$$

The second class of integrals that have been evaluated off-shell are those that can be proven equal to $\mathcal{I}^{(L)}$ using the 'magic identities' of [102] as mentioned above. There it was shown that:

$$\mathcal{I}^{(3)a} = \mathcal{I}^{(3)b} = \mathcal{I}^{(3)}, \tag{2.5.5}$$

and

$$\mathcal{I}^{(4)a} = \mathcal{I}^{(4)b} = \mathcal{I}^{(4)c} = \mathcal{I}^{(4)d} = \mathcal{I}^{(4)e} = \mathcal{I}^{(4)}. \tag{2.5.6}$$

These identities appear to be highly nontrivial and are only valid for the off-shell integrals in four dimensions; certainly no hint of any relation between these integrals is apparent when they are taken on-shell and evaluated in $4-2\epsilon$ dimensions as in[258, 265, 273]. Moreover in [102] the relations 2.5.5 and 2.5.6 were given a simple diagrammatic interpretation which can be utilized to systematically identify equalities between certain integrals at any number of loops.

2.5.4 Evaluation of Dual Conformal Integrals: New integrals

As indicated above we evaluate these integrals by starting with Mellin-Barnes representations in $4-2\epsilon$ dimensions and then analytically continuing ϵ to zero. A very useful Mathematica code which automates this process has been written by Czakon [260]. We found however that this implementation was too slow to handle some of the new integrals in a reasonable amount of time so we implemented the algorithm in a C program instead. The most difficult off-shell integral we have evaluated, $\mathcal{I}^{(4)f2}$, starts off in $4-2\epsilon$ dimensions as a 24-fold Mellin-Barnes representation (far more complicated than any of the on-shell four-loop integrals considered in[265, 266], which require at most 14-fold representations), yet the analytic continuation to 4 dimensions takes only a fraction of a second in C. In what follows we display Mellin-Barnes representations for the various integrals in 4 dimensions, after the analytic continuation has been performed and ϵ has been set to zero [3].

The most surprising aspect of the formulas given below is that we are able to write each integral in terms of just a single Mellin-Barnes integral in four dimensions. This stands in stark contrast to dimensionally regulated on-shell integrals, for which the analytic continuation towards $\epsilon = 0$ can generate (at four loops, for example) thousands or even tens of thousands of terms. For the off-shell integrals studied here something rather amazing happens: the analytic continuation still produces thousands of terms (or more), but for each off-shell integral it turns out that only one of the resulting terms is non-vanishing at $\epsilon = 0$, leaving in each case only a single Mellin-Barnes integral in four dimensions.

This surprising result is not automatic but depends on a number of factors, including the choice of initial Mellin-Barnes representation, the choice of integration contour for the Mellin-Barnes variables z_i, and some details of the how the analytic continuation is carried out. All of these steps involve highly non-unique choices, and by making different choices it is easy to end up with more than one term that is finite in four dimensions. However in such cases it is always possible to 'reassemble' the finite terms into the one-term representations shown here by shifting the contours of the remaining integration variables.

For the new 3-loop integrals, both shown in Fig.2.9, we find the Mellin-Barnes representations[3]

$$\begin{aligned} \mathcal{I}^{(3)c} \;=\; & -\int \frac{d^5 z}{(2\pi i)^5}\, x^{z_2}\, \Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-z_2+z_3+1)\Gamma(z_1+z_3-z_4+1) \\ & \times\Gamma(-z_4)^2\Gamma(z_4-z_3)\Gamma(-z_2-z_5)^2\Gamma(z_1-z_4-z_5+1) \\ & \times\Gamma(z_1-z_2+z_3-z_4-z_5+1)\Gamma(-z_1+z_4+z_5)\Gamma(z_2+z_4+z_5+1) \\ & \times\Gamma(-z_1+z_2+z_4+z_5)\Gamma(-z_1-z_3+z_4+z_5-1)/(\Gamma(1-z_4) \\ & \times\Gamma(-z_2-z_5+1)\Gamma(z_1-z_2+z_3-z_4-z_5+2)\Gamma(-z_1-z_2+z_4+z_5) \\ & \times\Gamma(-z_1+z_2+z_4+z_5+1)) \end{aligned} \qquad (2.5.7)$$

[3] All formulas in this section are valid when the integration contours for the z_i are chosen to be straight lines parallel to the imaginary axis and such that the arguments of all Γ functions in the numerator of the integrand have positive real part.

and

$$\begin{aligned}\mathcal{I}^{(3)d} &= \int \frac{d^4 z}{(2\pi i)^4}\,\Gamma(-z_1)\Gamma(z_1+1)\Gamma(-z_2)\Gamma(z_1+z_2-z_3+1)\Gamma(-z_3)^2\Gamma(z_3-z_1)\\ &\quad\times\Gamma(z_2-z_3-z_4+1)\Gamma(z_1+z_2-z_3-z_4+1)\Gamma(-z_4)^2\Gamma(z_3+z_4+1)\\ &\quad\times\Gamma(-z_2+z_3+z_4)\Gamma(-z_1-z_2+z_3+z_4)/(\Gamma(1-z_1)\Gamma(1-z_3)\Gamma(1-z_4)\\ &\quad\times\Gamma(z_1+z_2-z_3-z_4+2)\Gamma(-z_2+z_3+z_4+1)). \qquad (2.5.8)\end{aligned}$$

As expected, $\mathcal{I}^{(3)d}$ is x-independent because the corresponding diagram is degenerate. Upon evaluating 2.5.8 numerically (using CUBA [256]) we find

$$\mathcal{I}^{(3)d} \approx 20.73855510 \qquad (2.5.9)$$

with a reported estimated numerical uncertainty smaller than the last digit shown.

Finally we have 12 off-shell four-loop integrals left to evaluate, corresponding to the 9 diagrams shown in Fig.2.9, along with three of the diagrams ($\mathcal{I}^{(4)f}$, $\mathcal{I}^{(4)d2}$ and $\mathcal{I}^{(4)f2}$) from Fig.2.7. It turns out that 4 of these 12 integrals ($\mathcal{I}^{(4)f}$, $\mathcal{I}^{(4)f'}$, $\mathcal{I}^{(4)e2'}$ and $\mathcal{I}^{(4)c'}$) are significantly more difficult than the rest because they apparently require analytic continuation not only in ϵ but also in a second parameter ν parameterizing the power of the numerator factors. (That is, the integrals initially converge only for $\nu < 1$ and must be analytically continued to $\nu = 1$.) We postpone the study of these more complicated integrals to future work.

In analyzing the remaining 8 off-shell four-loop integrals we have found two new 'magic identities',

$$\mathcal{I}^{(4)e2} = \mathcal{I}^{(4)b1}, \qquad \mathcal{I}^{(4)c1} = -\mathcal{I}^{(4)d2}. \tag{2.5.10}$$

We established these results directly by deriving Mellin-Barnes representations for these integrals and showing that they can be related to each other under a suitable change of integration variables. It would certainly be interesting to understand the origin of the relations [140] and to see whether the insight gained thereby can be used to relate various dual conformal integrals at higher loops to each other.

Mellin-Barnes representations for the 8 off-shell four-loop integrals are:

$$\begin{aligned}
\mathcal{I}^{(4)c1} \quad = \quad & -\mathcal{I}^{(4)d2} = -\int \frac{d^7 z}{(2\pi i)^7}\, x^{z_2}\, \Gamma(-z_1-1)\Gamma(z_1+2)\Gamma(-z_2)\Gamma(z_2+1) \\
& \times\Gamma(-z_1-z_3-1)\Gamma(-z_3)\Gamma(z_3-z_2)\Gamma(z_1-z_2+z_3+1)\Gamma(-z_4) \\
& \times\Gamma(z_1-z_2+z_3+z_5+2)\Gamma(-z_6)^2\Gamma(z_4+z_5-z_7+1) \\
& \times\Gamma(-z_1+z_2+z_4-z_6-z_7)\Gamma(z_4+z_5-z_6-z_7+1)\Gamma(-z_7)^2\Gamma(z_7-z_5) \\
& \times\Gamma(z_6+z_7+1)\Gamma(-z_4+z_6+z_7)\Gamma(-z_3-z_4-z_5+z_6+z_7-1)/ \\
& \times(\Gamma(-z_1-z_2-1)\Gamma(1-z_3)\Gamma(z_1-z_2+z_3+2)\Gamma(1-z_6)\Gamma(1-z_7) \\
& \times\Gamma(z_4+z_5-z_6-z_7+2)\Gamma(-z_4+z_6+z_7+1))
\end{aligned} \tag{2.5.11}$$

$$
\begin{aligned}
\mathcal{I}^{(4)f2} \;=\; & \int \frac{d^{10}z}{(2\pi i)^{10}}\, x^{z_1}\, \Gamma(-z_1-z_{10})^2\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_1+z_{10}+z_3+1)^2\Gamma(-z_4) \\
& \times\Gamma(-z_5)\Gamma(-z_6)\Gamma(z_{10}+z_2+z_6)\Gamma(-z_7)^2\Gamma(z_4+z_7+1)^2 \\
& \times\Gamma(-z_1-z_4-z_5-z_8-2)(-z_1-z_{10}-z_2-z_4-z_6-z_7-z_8-2) \\
& \times\Gamma(-z_8)\Gamma(z_1+z_8+2)(z_2+z_4+z_5+z_7+z_8+2) \\
& \times\Gamma(-z_2-z_3-z_4-z_5-z_9-2)\Gamma(-z_{10}-z_2-z_3-z_6-z_9)\Gamma(-z_9) \\
& \times\Gamma(z_1+z_{10}+z_2+z_3+z_6+z_9+2) \\
& \times\Gamma(z_{10}+z_2+z_3+z_4+z_5+z_6+z_9+2)\Gamma(-z_1+z_3-z_8+z_9) \\
& \times\Gamma(z_2+z_4+z_5+z_6+z_8+z_9+2)/(\Gamma(z_1+z_{10}+z_3+2)\Gamma(-z_4-z_5) \\
& \times\Gamma(-z_1-z_{10}-z_7)\Gamma(z_4+z_7+2)\Gamma(-z_1-z_{10}-z_2-z_6-z_8) \\
& \times\Gamma(-z_3-z_9)\Gamma(-z_1+z_{10}+z_2+z_3+z_6-z_8+z_9) \\
& \times\Gamma(z_1+z_{10}+z_2+z_3+z_4+z_5+z_6+z_8+z_9+4)) \qquad (2.5.12)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}^{(4)e6} \;=\; & -\int \frac{d^5z}{(2\pi i)^5}\, x^{z_2}\, \Gamma(-z_1)\Gamma(-z_2)^4\Gamma(z_2+1)^2\Gamma(-z_3)\Gamma(z_3+1) \\
& \times\Gamma(z_1+z_3-z_4+1)\Gamma(-z_4)^2\Gamma(z_4-z_3)\Gamma(z_1-z_4-z_5+1) \\
& \times\Gamma(z_1+z_3-z_4-z_5+1)\Gamma(-z_5)^2\Gamma(z_4+z_5+1)\Gamma(-z_1+z_4+z_5) \\
& \times\Gamma(-z_1-z_3+z_4+z_5)/(\Gamma(-2z_2)\Gamma(1-z_3)\Gamma(1-z_4)\Gamma(1-z_5) \\
& \times\Gamma(z_1+z_3-z_4-z_5+2)\Gamma(-z_1+z_4+z_5+1)) \qquad (2.5.13)
\end{aligned}
$$

$$\begin{aligned}
\mathcal{I}^{(4)e2} \;\; = \;\; \mathcal{I}^{(4)b1} &= -\int \frac{d^7 z}{(2\pi i)^7}\, x^{z_3}\, \Gamma(-z_1)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(z_3-z_2)\Gamma(-z_4)^2 \\
&\times\Gamma(z_2-z_3+z_4+1)^2\Gamma(-z_5)^2\Gamma(z_1+z_2-z_5-z_6+2)\Gamma(-z_6)^2 \\
&\times\Gamma(z_5+z_6+1)\Gamma(-z_1-z_2+z_5+z_6-1)\Gamma(-z_1-z_2+z_3+z_5+z_6-1) \\
&\times\Gamma(-z_4+z_5-z_7)\Gamma(-z_1-z_2-z_4+z_5+z_6-z_7-1)\Gamma(-z_7) \\
&\times\Gamma(-z_3+z_4+z_7)\Gamma(z_1+z_2-z_3+z_4-z_5+z_7+2) \\
&\times\Gamma(z_1-z_5-z_6+z_7+1)/(\Gamma(z_2-z_3+z_4-z_5+2)\Gamma(-z_4-z_6+1) \\
&\times\Gamma(-z_1-z_2-z_3+z_5+z_6-1)\Gamma(-z_1-z_2+z_3+z_5+z_6) \\
&\times\Gamma(-z_4-z_7+1)\Gamma(z_1+z_2-z_3+z_4-z_5-z_6+z_7+2)) \qquad (2.5.14)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^{(4)b2} \;\; = \;\; \mathcal{I}^{(4)e1} &= \int \frac{d^6 z}{(2\pi i)^6}\, \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_1-z_2-z_4-2) \\
&\times\Gamma(-z_1-z_2-z_3-z_4-2)\Gamma(-z_4) \\
&\times\Gamma(z_1+z_2+z_4+3)\Gamma(z_1+z_2+z_3+z_4+3)\Gamma(z_1+z_3-z_5+1)\Gamma(-z_5) \\
&\times\Gamma(z_5-z_3)\Gamma(z_2+z_4+z_5+2)\Gamma(-z_4-z_5-z_6-1)\Gamma(-z_6)^2 \\
&\times\Gamma(z_4+z_6+1)^2\Gamma(z_2+z_4+z_5+z_6+2)/(\Gamma(1-z_3) \\
&\times\Gamma(-z_1-z_2-z_4-1)\Gamma(z_1+z_2+z_3+z_4+4)\Gamma(1-z_5) \\
&\times\Gamma(z_2+z_4+z_5+3)\Gamma(1-z_6)\Gamma(z_4+z_6+2)) \qquad (2.5.15)
\end{aligned}$$

We do not consider the equality of the two degenerate integrals $\mathcal{I}^{(4)b2} = \mathcal{I}^{(4)e1}$ to be a ‘magic’ identity since it is easily seen to be a trivial consequence of dual conformal invariance. Evaluating them numerically we find

$$\mathcal{I}^{(4)b2} = \mathcal{I}^{(4)e1} = 70.59, \tag{2.5.16}$$

again with a reported estimated numerical uncertainty smaller than the last digit shown.

It would certainly be interesting to obtain fully explicit analytic results for these new integrals. Although this might seem to be a formidable challenge, the fact that it has been possible for the ladder diagrams 2.5.3 suggests that there is hope.

2.5.5 Evaluation of Dual Conformal Integrals: Infrared singularity structure

Finally, it is clearly of interest to isolate the infrared singularities of the various integrals [3]. For the previously known integrals reviewed in subsection 3.1 we expand 2.5.3 for small x, finding

$$\begin{aligned}
\mathcal{I}^{(1)} &= \log^2 x + \mathcal{O}(1), \\
\mathcal{I}^{(2)} &= \frac{1}{4}\log^4 x + \frac{\pi^2}{2}\log^2 x + \mathcal{O}(1), \\
\mathcal{I}^{(3)} &= \frac{1}{36}\log^6 x + \frac{5\pi^2}{36}\log^4 x + \frac{7\pi^4}{36}\log^2 x + \mathcal{O}(1), \\
\mathcal{I}^{(4)} &= \frac{1}{576}\log^8 x + \frac{7\pi^2}{432}\log^6 x + \frac{49\pi^4}{864}\log^4 x + \frac{31\pi^6}{432}\log^2 x + \mathcal{O}(1).
\end{aligned} \tag{2.5.17}$$

For the new integrals evaluated in this paper we obtain the small x expansion directly from the Mellin-Barnes representations given in section 3.2 by writing each one in the form

$$\int \frac{dy}{2\pi i} x^y F(y), \tag{2.5.18}$$

shifting the y contour of integration to the left until it sits directly on the imaginary axis (picking up terms along the way from any poles crossed), expanding the resulting integrand around $y = 0$ and then using the fact that the coefficient of the $1/y^k$ singularity at $y = 0$ corresponds in x space to the coefficient of the $\frac{(-1)^k}{k!} \log^k x$ singularity at $x = 0$. In this manner we find

$$\mathcal{I}^{(3)c} = \frac{\zeta(3)}{3} \log^3 x - \frac{\pi^4}{30} \log^2 x + 14.32388625 \log x + \mathcal{O}(1) \tag{2.5.19}$$

at three loops and

$$\begin{aligned}
\mathcal{I}^{(4)d2} &= -\mathcal{I}^{(4)c1} = -\frac{\zeta(3)}{12} \log^5 x + \frac{7\pi^4}{720} \log^4 x - 6.75193310 \log^3 x \\
&\qquad +15.45727322 \log^2 x - 41.26913 \log x + \mathcal{O}(1), \\
\mathcal{I}^{(4)f2} &= \frac{1}{144} \log^8 x + \frac{7\pi^2}{108} \log^6 x + \frac{149\pi^4}{1080} \log^4 x + 64.34694867 \log^2 x + \mathcal{O}(1), \\
\mathcal{I}^{(4)e6} &= -20.73855510 \log^2 x + \mathcal{O}(1), \\
\mathcal{I}^{(4)e2} &= \mathcal{I}^{(4)b1} = -\frac{\pi^4}{720} \log^4 x + 1.72821293 \log^3 x \\
&\qquad -12.84395616 \log^2 x + 52.34900 \log x + \mathcal{O}(1)
\end{aligned} \tag{2.5.20}$$

at four loops, where some coefficients have only been evaluated numerically with an estimated uncertainly smaller than the last digit shown. Interestingly, the coefficient of $\log^2 x$ in $\mathcal{I}^{(4)e6}$ appears to be precisely (minus) the value of $\mathcal{I}^{(3)d}$ shown in 2.5.9.

Perhaps this can be traced to the diagrammatic relation that is evident in Fig.2.9: $\mathcal{J}^{(3)d}$ appears in the 'upper diagonal' of $\mathcal{J}^{(4)e6}$.

2.6 Dual SuperConformal Symmetry at Strong Coupling

In the previous sections, we have seen many establishment of the dual superconformal symmetry at weak coupling, but its origin and fundamental understanding are still mysterious questions. In this section, we discuss about how one can understand this dual superconformal symmetry using a T-duality symmetry of the full superstring theory on $AdS_5 \times S^5$, and will address the development of this symmetry into a Yangian symmetry in the next section. Berkovits and Maldacena [268] found a new symmetry, called fermionic T-duality, which maps a supersymmetric background to another supersymmetric background with different RR fields and a different dilaton. The T-duality here is a generalization of the Buscher version of T-duality in theories with fermionic worldsheet scalars. The authors study this fermionic T-duality transformation, in the pure spinor sigma model, of the Type II supergravity background fields. This is the most convenient method method because in the pure spinor formalism, the BRST invariance determines the choice of torsion constraints and allows a straightforward identification of the background fields.

Then they show that a certain combination of bosonic and fermionic T-dualities maps the full superstring theory on $AdS_5 \times S^5$ back to itself in such a way that gluon scattering amplitudes in the original theory map to Wilson loops in the dual theory. This duality also maps the dual superconformal symmetry of the original theory to the ordinary superconformal symmetry of the dual model, which explains the dual superconformal invariance of planar scattering amplitudes of $N = 4$ super Yang Mills and also gives the foundation of the connection between amplitudes and Wilson loops, which is the subject of the next chapter.

2.7 Yangian Symmetry

Having both the conventional and dual superconformal symmetries at work, one could come up with a natural question arising from these two symmetry algebras: what mathematical structure arises, when one commutes generators of the conventional superconformal and dual superconformal algebras with each other? One would indeed expect to generate an infinite-dimensional symmetry algebra as a manifestation of the integrability of the theory. We shall show in this section that a Yangian symmetry of scattering amplitudes appears, at least at tree level, following the work of [232]. We also demonstrate how this Yangian symmetry fixes uniquely an important structure of scattering amplitudes in twistor space.

First, let us recall the form of the tree-level superamplitude, mentioned earlier in this chapter:

$$\mathcal{A}(\Phi_1, \ldots, \Phi_n) = \mathcal{A}_n = \frac{\delta^4(p)\delta^8(q)}{\langle 12 \rangle \ldots \langle n1 \rangle} \mathcal{P}_n(\lambda_i, \tilde{\lambda}_i, \eta_i) = \mathcal{A}_n^{\text{MHV}} \mathcal{P}_n \ , \tag{2.7.1}$$

with the helicity condition

$$h_i \mathcal{A}(\Phi_1, \ldots, \Phi_n) = \mathcal{A}(\Phi_1, \ldots, \Phi_n), \qquad i = 1, \ldots, n \tag{2.7.2}$$

and $\mathcal{P}_n$ is a function with no helicity,

$$h_i \mathcal{P}_n = 0, \qquad i = 1, \ldots, n. \tag{2.7.3}$$

where the helicity operator is

$$h_i = -\tfrac{1}{2}\lambda_i^\alpha \frac{\partial}{\partial \lambda_i^\alpha} + \tfrac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} + \tfrac{1}{2}\eta_i^A \frac{\partial}{\partial \eta_i^A}. \tag{2.7.4}$$

The superconformal algebra is generically $SU(2,2|4)$ with central charge $c = \sum_i (1 - h_i)$, but imposing the homogeneity condition (2.7.2) the algebra becomes $PSU(2,2|4)$.

At tree-level, because there are no infrared divergences, amplitudes are annihilated by the generators of the standard (conventional) superconformal symmetry, up to contact terms:

$$j_a \mathcal{A}_n = 0, \tag{2.7.5}$$

where j_a is any generator of $PSU(2,2|4)$,

$$j_a \in \{p^{\alpha\dot{\alpha}}, q^{\alpha A}, \bar{q}^{\dot{\alpha}}_A, m_{\alpha\beta}, \bar{m}_{\dot{\alpha}\dot{\beta}}, r^A{}_B, d, s^\alpha_A, \bar{s}^A_{\dot{\alpha}}, k_{\alpha\dot{\alpha}}\}. \tag{2.7.6}$$

Now, taking into account the dual superconformal symmetry [127] with the dual superspace coordinates,

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}, \qquad \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} = \lambda_i^{\alpha}\eta_i^{A} \ , \tag{2.7.7}$$

we can express the amplitudes in the dual variables as

$$\mathcal{A}_n = \frac{\delta^4(x_1 - x_{n+1})\delta^8(\theta_1 - \theta_{n+1})}{\langle 12\rangle \dots \langle n1\rangle}\mathcal{P}_n(x_i, \theta_i), \tag{2.7.8}$$

which are covariant under certain generators (K, S, D and C) of the dual superconformal algebra. We will denote J_a as a generator of the dual superconformal algebra,

$$J_a \in \{P_{\alpha\dot{\alpha}}, Q_{\alpha A}, \bar{Q}^A_{\dot{\alpha}}, M_{\alpha\beta}, \overline{M}_{\dot{\alpha}\dot{\beta}}, R^A{}_B, D, C, S^A_{\alpha}, \overline{S}^{\dot{\alpha}}_A, K^{\alpha\dot{\alpha}}\}. \tag{2.7.9}$$

By making a redefinition as follows [232],

$$K'^{\alpha\dot{\alpha}} = K^{\alpha\dot{\alpha}} + \sum_i x_i^{\alpha\dot{\alpha}}, \tag{2.7.10}$$

$$S'^{\alpha A} = S^{\alpha A} + \sum_i \theta_i^{\alpha A}, \tag{2.7.11}$$

$$D' = D - n \ , \tag{2.7.12}$$

we can get a vanishing central charge, $C' = 0$ while these generators still satisfy the commutation relations of the superconformal algebra. More importantly, dual superconformal symmetry becomes simply

$$J'_a\mathcal{A}_n = 0 \tag{2.7.13}$$

with

$$J'_a \in \{P_{\alpha\dot{\alpha}}, Q_{\alpha A}, \bar{Q}^A_{\dot{\alpha}}, M_{\alpha\beta}, \overline{M}_{\dot{\alpha}\dot{\beta}}, R^A{}_B, D', C' = 0, S'^A_\alpha, \overline{S}^{\dot{\alpha}}_A, K'^{\alpha\dot{\alpha}}\}. \tag{2.7.14}$$

It was shown in [232] that the generators j_a together with S' (or K') generate the Yangian of the superconformal algebra, $Y(PSU(2,2|4))$.

The generators j_a form the level-zero $PSU(2,2|4)$ subalgebra, with $[O_2, O_1] = (-1)^{1+|O_1||O_2|}[O_1, O_2]$,

$$[j_a, j_b] = f_{ab}{}^c j_c. \tag{2.7.15}$$

The level-one generators $j_a^{(1)}$ are defined by

$$[j_a, j_b^{(1)}] = f_{ab}{}^c j_c^{(1)} \tag{2.7.16}$$

and represented by the bilocal formula,

$$j_a^{(1)} = f_a{}^{cb} \sum_{k<k'} j_{kb} j_{k'c}. \tag{2.7.17}$$

The full symmetry of the tree-level amplitudes can be therefore written as

$$j\mathcal{A}_n = j^{(1)}\mathcal{A}_n = 0. \tag{2.7.18}$$

We could also express the level-zero and level-one generators of the Yangian symmetry, in terms of the twistor space variables $\mathcal{Z}^{\mathcal{A}} = (\tilde{\mu}^{\alpha}, \tilde{\lambda}^{\dot{\alpha}}, \eta^{A})$,

$$j^{\mathcal{A}}{}_{\mathcal{B}} = \sum_i \mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}}, \tag{2.7.19}$$

$$j^{(1)\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j} (-1)^{\mathcal{C}} \Big[\mathcal{Z}_i^{\mathcal{A}} \frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{C}}} \mathcal{Z}_j^{\mathcal{C}} \frac{\partial}{\partial \mathcal{Z}_j^{\mathcal{B}}} - (i,j) \Big]. \tag{2.7.20}$$

We have suppressed the supertrace term $(-1)^{\mathcal{A}} \delta^{\mathcal{A}}_{\mathcal{B}}$. In this representation the generators of superconformal symmetry are first-order operators while the level-one Yangian generators are second order.

We can also have an alternative T-dual representation, demonstrated in [232], that it is the dual superconformal symmetries J_a which give the level-zero generators, while the standard conformal symmetry together with the dual superconformal symmetry provides the level-one generators.

$$J_a \mathcal{P}_n = J_a^{(1)} \mathcal{P}_n = 0 \tag{2.7.21}$$

where

$$J_a^{(1)} = f_a{}^{cb} \sum_{i<j} J_{ib} J_{jc}. \tag{2.7.22}$$

The fact that Yangian symmetry appears at least at tree-level in $\mathcal{N} = 4$ SYM scattering amplitudes in twistor space was suggested many years ago in Witten's original twistor string theory work [218]. Let us see now how this symmetry fixes uniquely an important structure of amplitudes.

Recently a remarkable formula has been proposed which computes leading singularities of scattering amplitudes in the $\mathcal{N}=4$ super Yang-Mills theory [233], which takes the form of an integral over the Grassmannian $G(k,n)$, the space of complex k-planes in $\mathbb{C}^n$:

$$\mathcal{L} = \int_C K. \tag{2.7.23}$$

We can write it in several equivalent forms as follows: In twistor space

$$\mathcal{L}_{\text{ACCK}}(\mathcal{Z}) = \int \frac{D^{k(n-k)}c}{\mathcal{M}_1 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{4|4}\Big(\sum_{i=1}^{n} c_{ai}\mathcal{Z}_i\Big), \tag{2.7.24}$$

where the c_{ai} are complex parameters which are integrated choosing a specific contour. The denominator is the cyclic product of consecutive $(k \times k)$ minors $\mathcal{M}_p$ made from the columns $p, \ldots, p+k-1$ of the $(k \times n)$ matrix of the c_{ai}

$$\mathcal{M}_p \equiv (p\; p+1\; p+2 \ldots p+k-1). \tag{2.7.25}$$

or in momentum space [233],

$$\mathcal{L}_{\text{ACCK}}(\lambda, \tilde{\lambda}, \eta) = \int \frac{D^{k(n-k)}c \prod_a d^2\rho_a}{\mathcal{M}_1 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^2\Big(\sum_{i=1}^{n} c_{ai}\tilde{\lambda}_i\Big) \delta^4\Big(\sum_{i-1}^{n} c_{ai}\eta_i\Big) \prod_{i=1}^{n} \delta^2\Big(\lambda_i - \sum_{a=1}^{k} \rho_a c_{ai}\Big), \tag{2.7.26}$$

or in momentum twistor space [?],

$$\mathcal{L}_{\text{MS}} = \int \frac{D^{k(n-k)}t}{\mathcal{M}_1 \ldots \mathcal{M}_n} \prod_{a=1}^{k} \delta^{4|4}\Big(\sum_{i=1}^{n} t_{ai}\mathcal{W}_i\Big), \tag{2.7.27}$$

where the dual superconformal symmetry is manifest. The integration variables t_{ai} are again a $(k \times n)$ matrix of complex parameters.

The equivalence of the two formulations (2.7.26) and (2.7.27) was shown in [233], through a change of variables. Therefore, since each of the formulas has a different superconformal symmetry manifest, they both possess an invariance under the Yangian $Y(PSL(4|4))$. The Yangian symmetry of these formulas was explicitly demonstrated in [232] by directly applying the Yangian level-one generators to the Grassmannian integrand to yield a total derivative

$$J^{(1)\mathcal{A}}{}_{\mathcal{B}} K = d\Omega^{\mathcal{A}}{}_{\mathcal{B}} \tag{2.7.28}$$

which guarantees that $\mathcal{L}$ is invariant when the contour is closed.

Now let see if it is possible to modify the above form in some way that preserves the property (2.7.28) by considering the same integral as before but with an extra arbitrary function $f(t)$ on the Grassmannian in the integrand,

$$\tilde{\mathcal{L}}_{n,k} = \int \frac{D^{k(n-k)}t}{\mathcal{M}_1 \dots \mathcal{M}_n} f(t) \prod_a \delta^{m|m}\Big(\sum_{i=1}^n t_{ai} \mathcal{W}_i\Big). \tag{2.7.29}$$

Preserving the property (2.7.28) leads to the following condition after many computations [232],

$$\sum_{a,j} O_{bl,aj} \frac{\partial}{\partial t_{aj}} f(t) = 0, \tag{2.7.30}$$

where the matrix O satisfies

$$\det O = [\mathcal{M}_1 \dots \mathcal{M}_n]^2. \tag{2.7.31}$$

Since the matrix O is generically invertible we can multiply (2.7.30) by the inverse of

O and deduce that $f(t)$ must be constant almost everywhere. In principle the function $f(t)$ can have discontinuities across the hyperplanes defined by the vanishing of the minors $\mathcal{M}_p$ but the only continuous function allowed is a constant function.

This shows that Yangian symmetry fixes uniquely the structure of the Grassmannian integral, which produces leading singularities of $N^{k-2}MHV$ scattering amplitudes on a suitable closed contour.

Chapter 3

Wilson Loops and Duality

3.1 Duality at Strong Coupling

We will review the scattering amplitudes at strong coupling through AdS/CFT and its duality to the Wilson loops in this section [59, 60, 236**?**].

The AdS/CFT correspondence [6] relates the $4D$ $\mathcal{N} = 4$ SYM theory with type IIB string theory on $AdS_5 \times S^5$, by the identification of gauge-invariant operators and string states, as well as symmetries between the two theories. This correspondence then provides a way to study the strong coupling regime of $\mathcal{N} = 4$ SYM. In the limit of large N_c (color number) and large 't Hooft coupling λ, the string theory lives on a weakly curved space that it could be described by a weakly-coupled worldsheet sigma-model. Then by appending an open string sector to closed string theory in $AdS_5 \times S^5$, gluon scattering amplitude could be computed in terms of correlation functions of vertex operators: one simply computes the transition amplitude between some in and out asymptotic states. To describe scattering amplitudes of $\mathcal{N} = 4$ SYM fields, these states must be located at spatial infinity in the directions parallel to the boundary of the AdS space. As usual, two-dimensional conformal invariance on the string worldsheet allows a description of the asymptotic states in terms of local vertex operators inserted on the boundary of the worldsheet.

Since scattering amplitudes of colored objects are not well defined in the conformal theory it is necessary to introduce an infrared regulator. The answer we obtain will depend on the regularization scheme. Once we compute a well defined (IR safe) physical observable the IR regulator will drop out. In terms of the gravity dual, we will use a IR regulator, which is a D-brane that extends along the worldvolume directions but is localized in the radial direction.

We will start with the AdS_5 metric

$$ds^2 = R^2 \frac{d\mathbf{y}_{3+1}^2 + dz^2}{z^2} \quad . \tag{3.1.1}$$

and we place a $D3$-brane at some fixed and large value of the radial coordinate $z = z_{IR} \gg R$. Such D-branes arise, if we go to the Coulomb branch of the theory.

The asymptotic states are open strings that end on the D-brane. We then scatter these open strings and study the scattering at fixed angles and very high momentum. Such amplitudes were studied in flat space by Gross and Mende, where an important feature is that amplitudes at high momentum transfer are dominated by a saddle point of the classical action. Thus, in order to compute our amplitude we simply have to compute a solution of the classical action of a classical string in AdS.

Let consider a metric of the form

$$ds^2 = h(z)^2(dy_\mu dy^\mu + dz^2) \quad . \tag{3.1.2}$$

On an Euclidean worldsheet, the T-dual coordinates x^μ are defined by

$$\partial_\alpha x^\mu = ih^2(z)\epsilon_{\alpha\beta}\partial_\beta y^\mu \quad . \tag{3.1.3}$$

Defining the radial coordinate

$$r = \frac{R^2}{z} \tag{3.1.4}$$

leads to the metric

$$ds^2 = \frac{R^2}{r^2}\left(dx_\mu dx^\mu + dr^2\right) \quad , \tag{3.1.5}$$

which is identical to (3.1.1) except that its boundary is located at $r = 0$ (where $z = \infty$).

The transformation (3.1.3) has another effect: the zero-mode of the field y corresponding to the momentum k^μ (and described by a local vertex operator) is replaced by a "winding" mode of the field x implying that the difference between the two endpoints of the string obeys

$$\Delta x^\mu = 2\pi k^\mu \quad , \tag{3.1.6}$$

We then construct the boundary of the worldsheet as follows:

- For each particle of momentum k^μ, draw a segment joining two points separated by $\Delta x^\mu = 2\pi k^\mu$.

- Concatenate the segments according to the insertions on the disk (corresponding to a particular color ordering or to a particular ordering of vertex operators on the original boundary)

- As gluons are massless, the segments are light-like. Due to momentum conservation, the diagram is closed.

As the infrared regulator is removed when $z_{IR} \to \infty$ in the original coordinates, the boundary of the worldsheet moves towards the boundary of the T-dual metric, at $r = 0$. The solution we want, living at values of $r > r_{IR}$, is a surface which at $r_{IR} = R^2/z_{IR}$ ends on the above light-like polygon.

The standard prescription [99, 100] implies that the leading exponential behavior of the $n-$point scattering amplitude is given by the area A of the minimal surface that ends on a sequence of light-like segments on the boundary of 3.1.5

$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi}A(k_1,\ldots,k_n)} \; . \tag{3.1.7}$$

The area $A(k_1, ..., k_n)$ contains the kinematic information about the momenta through its boundary conditions.

By construction, to leading order in the strong coupling expansion, the computation of scattering amplitudes becomes formally equivalent to that of the expectation value of a Wilson loop given by a sequence of light-like segments. This establish the duality between gluon scattering amplitudes and Wilson loops at strong coupling. However the study of this duality at different aspects and higher orders is still an open problem.

Using this procedure, Alday and Maldacena [59] was able to derive the expression for the four-gluon scattering amplitude at strong coupling as below

$$\mathcal{A} = e^{iS} = \exp\left[iS_{div} + \frac{\sqrt{\lambda}}{8\pi}\left(\log\frac{s}{t}\right)^2 + \tilde{C}\right] \ , \tag{3.1.8}$$

$$\tilde{C} = \frac{\sqrt{\lambda}}{4\pi}\left(\frac{\pi^2}{3} + 2\log 2 - (\log 2)^2\right) \tag{3.1.9}$$

$$S_{div} = 2S_{div,s} + 2S_{div,t} \ , \tag{3.1.10}$$

$$iS_{div,s} = -\frac{1}{\epsilon^2}\frac{1}{2\pi}\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon}\frac{1}{4\pi}(1-\log 2)\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} \ . \tag{3.1.11}$$

and $S_{div,t}$ is given by a similar expression with $s \to t$.

We can see that the singularity structure and the momentum dependent finite piece of this result are precisely the expression predicted by the ABDK/BDS ansatz, equations (1.2.69) and (1.2.70):

$$\mathcal{A} \sim (\mathcal{A}_{div,s})^2 (\mathcal{A}_{div,t})^2 \exp\left\{\frac{f(\lambda)}{8}(\ln s/t)^2 + \frac{4\pi^2}{3} + C(\lambda)\right\} \tag{3.1.12}$$

$$\mathcal{A}_{div,s} = \exp\left\{-\frac{1}{8\epsilon^2}f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon}g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right\} \ . \tag{3.1.13}$$

However, the constant finite pieces were later shown to not agree with the ABDK/BDS ansatz in [60] when these authors calculate the n-point scattering amplitude and take the limit $n \to \infty$. This is the first evidence that the ABDK/BDS needs to be corrected.

Let us summarize now what we have learned about the computation of the amplitude: In the classical limit, a gluon scattering amplitude at strong coupling corresponds to a minimal surface that ends on the AdS_5 boundary on a special null polygonal contour, which is constructed as follows. When we consider a color-ordered amplitude involving n particles with null momenta $(k_1, \ldots, k_n)$, we get the contour specified by its ordered vertices $(x_1, \ldots, x_n)$ with $x_i^\mu - x_{i-1}^\mu = k_i^\mu$. Then the computation of gluon scattering amplitudes becomes identical to the computation of a Wilson loop with this contour. And as mentioned in chapter 2, under a certain combination of bosonic and fermionic T-dualities, the scattering amplitudes in the full superstring theory on $AdS_5 \times S^5$ are mapped into Wilson loops in the same $AdS_5 \times S^5$ dual theory. This duality also maps the dual superconformal symmetry of scattering amplitudes of original theory into the conventional superconformal symmetry of Wilson loops of the dual model.

Alday, Maldacena, Sever and Vieira [235] have shown how to compute this area of the minimal surface as a function of the conformal cross ratios characterizing the polygon at the boundary. Their method uses integrability of the sigma model in the following way. First they define a family of flat connections with a spectral parameter θ. Sections of this flat connection will be used to define solutions which depend on the spectral parameter θ. Then they define a set cross ratios $Y_k(\theta)$. They find a functional Y system that constrains the θ dependence of the functions Y_k. For n-point amplitudes, this system has $3(n-5)$ integration constants which come in when we specify the boundary conditions for $\theta \to \pm\infty$. We can restate these functional equations in terms of integral equations, where the $3(n-5)$ parameters appear explicitly. These integral equations have the form of Thermodynamic Bethe Ansatz (TBA) equations. Schematically they are

$$\log Y_k(\theta) = -m_k \cosh\theta + c_k + K_{k,s} \star \log(1 + Y_s) \ , \qquad (3.1.14)$$

where m_k, c_k are the $3(n-5)$ mentioned parameters and $K_{r,s}$ are some kernels. And the area can be expressed in terms of the TBA free nergy of the system as

$$\text{Area} = \int \frac{d\theta}{2\pi}\, m_k \cosh\theta\ \log(1+Y_k(\theta))\ . \tag{3.1.15}$$

Evaluating Y_k at $\theta = 0$ we get the physical values of the cross ratios. The other values of θ are a one parameter family of cross ratios , which give the same value for the area. Changing θ generates a symmetry of the problem.

Motivated by the relation between Wilson loops and scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills, there has been many efforts on the computation of the Wilson loops in different contexts. Notable one is the use of an Operator Product Expansion (OPE)-like expansion for light-like polygonal Wilson loops in [234] that is valid for any conformal gauge theory, for any coupling and in any dimension. They consider the case when several successive lines of the polygon are becoming aligned, which corresponds to a collinear or multi-collinear limit. The OPE expansion is performed by picking two non-consecutive null lines in the polygonal Wilson loop. This divides the Wilson loop into a "top" part and a "bottom" part with states propagating between the two. The state that propagates contains a flux tube going between the two selected null lines. The states consist of excitations of this flux tube. These states can also be understood as excitations around high spin operators. The spectrum of states is continuous and consist of many particles propagating along the flux tube. In $\mathcal{N} = 4$ super Yang-Mills these particles have a calculable dispersion relation. Then the OPE expansion leads to predictions for the Wilson loop expectation values. Namely, it implies constraints on the subleading terms in the collinear limit of the remainder function, which is the function containing the conformal invariant information of the Wilson loop expectation value. These predictions have been checked both at strong coupling and at two loops at weak coupling.

3.2 Duality at Weak Coupling

The infinite sequence of n-point planar maximally helicity violating (MHV) amplitudes in $\mathcal{N}=4$ super-Yang-Mills theory (SYM) has a remarkably simple structure. Due to supersymmetric Ward identities [143, 144, 145, 146], at any loop order L, the amplitude can be expressed as the tree-level amplitude, times a scalar, helicity-blind function $\mathcal{M}_n^{(L)}$:

$$\mathcal{A}_n^{(L)} \; = \mathcal{A}_n^{\text{tree}}\, \mathcal{M}_n^{(L)}. \tag{3.2.1}$$

In [133], ABDK discovered an intriguing iterative structure in the two-loop expansion of the MHV amplitudes at four points. This relation can be written as

$$\mathcal{M}_4^{(2)}(\epsilon) - \frac{1}{2}\Big(\mathcal{M}_4^{(1)}(\epsilon)\Big)^2 \;=\; f^{(2)}(\epsilon)\mathcal{M}_4^{(1)}(2\epsilon) + C^{(2)} + \mathcal{O}(\epsilon)\ , \tag{3.2.2}$$

where IR divergences are regulated by working in $D = 4 - 2\epsilon$ dimensions (with $\epsilon < 0$),

$$f^{(2)}(\epsilon) \;=\; -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2\ , \tag{3.2.3}$$

and

$$C^{(2)} \;=\; -\frac{1}{2}\zeta_2^2\ . \tag{3.2.4}$$

The ABDK relation (3.2.2) is built upon the known exponentiation of infrared divergences [147, 148], which guarantees that the singular terms must agree on both sides of (3.2.2), as well as on the known behavior of amplitudes under collinear limits [149, 150]. The (highly nontrivial) content of the ABDK relation is that (3.2.2) holds exactly as written at $\mathcal{O}(\epsilon^0)$. However, ABDK observed that the $\mathcal{O}(\epsilon)$ terms do not satisfy the same iteration relation [133].

In [133], it was further conjectured that (3.2.2) should hold for two-loop amplitudes with an arbitrary number of legs, with the same quantities (3.2.3) and (3.2.4) for any n. In the five-point case, this conjecture was confirmed first in [134] for the parity-even part of the two-loop amplitude, and later in [84] for the complete amplitude. Notice that for the iteration to be satisfied parity-odd terms that enter on the left-hand side of the relation must cancel up to and including $\mathcal{O}(\epsilon^0)$ terms, since the right-hand side is parity even up this order in ϵ. This has been checked and confirmed at two-loop order for five and six particles [84, 92, 151]. This is also crucial for the duality with Wilson loops (discussed below) which by construction cannot produce parity-odd terms at two loops.

It has been found that starting from six particles and two loops, the ABDK/BDS ansatz (3.2.2) needs to be modified by allowing the presence of a remainder function $\mathcal{R}_n$ [142],

$$\mathcal{M}_n^{(2)}(\epsilon) - \frac{1}{2}\Big(\mathcal{M}_n^{(1)}(\epsilon)\Big)^2 \;=\; f^{(2)}(\epsilon)\mathcal{M}_n^{(1)}(2\epsilon) \,+\, C^{(2)} \,+\, \mathcal{R}_n \;+\, E_n(\epsilon)\;, \qquad (3.2.5)$$

where $\mathcal{R}_n$ is ϵ-independent and E_n vanishes as $\epsilon \to 0$. We parameterize the latter as

$$E_n(\epsilon) = \epsilon\, \mathcal{E}_n + \mathcal{O}(\epsilon^2)\;. \qquad (3.2.6)$$

In the next next subsections, we will discuss in detail $\mathcal{E}_n$ for $n = 4, 5$ where we find a remarkable relation to the same quantity calculated from the Wilson loop. Hitherto this relation was only expected to hold for the finite parts of the remainder $\mathcal{R}_n$ [1].

As we have seen in the previous section, Alday and Maldacena addressed the problem of calculating scattering amplitudes at strong coupling in $\mathcal{N} = 4$ SYM using the AdS/CFT correspondence. Their remarkable result showed that the planar amplitude at strong coupling is calculated by a Wilson loop

$$W[\mathcal{C}_n] \ := \ \mathrm{Tr}\,\mathcal{P}\exp\left[ig \oint_{\mathcal{C}_n} d\tau \ \dot{x}^{\mu}(\tau) A_{\mu}(x(\tau))\right] \ , \tag{3.2.7}$$

whose contour $\mathcal{C}_n$ is the n-edged polygon obtained by joining the light-like momenta of the particles following the order induced by the color structure of the planar amplitude. At strong coupling the calculation amounts to finding the minimal area of a surface ending on the contour $\mathcal{C}_n$ embedded at the boundary of a T-dual AdS_5 space [59]. Shortly after, it was realized that the very same Wilson loop evaluated at weak coupling reproduces all one-loop MHV amplitudes in $\mathcal{N} = 4$ SYM [129, 130]. The conjectured relation between MHV amplitudes and Wilson loops found further strong support by explicit two loop calculations at four [131], five [132] and six points [142, 151]. In particular, the absence of a non-trivial remainder function in the four- and five-point case was later explained in [132] from the Wilson loop perspective, where it was realized that the BDS ansatz is a solution to the anomalous Ward identity for the Wilson loop associated to the dual conformal symmetry [140].

The Wilson loop in (3.2.7) can be expanded in powers of the 't Hooft coupling

$a := g^2N/(8\pi^2)$ as

$$\langle W[\mathcal{C}_n]\rangle := 1 + \sum_{l=1}^{\infty} a^l W_n^{(l)} := \exp \sum_{l=1}^{\infty} a^l w_n^{(l)} . \tag{3.2.8}$$

Note that the exponentiated form of the Wilson loop is guaranteed by the non-Abelian exponentiation theorem. The $w_n^{(l)}$ are obtained from "maximally non-Abelian" subsets of Feynman diagrams contributing to the $W_n^{(l)}$ and in particular from (3.2.8) we find

$$w_n^{(1)} = W_n^{(1)} , \qquad w_n^{(2)} = W_n^{(2)} - \frac{1}{2} (W_n^{(1)})^2 . \tag{3.2.9}$$

The UV divergences of the n-gon Wilson loop are regulated by working in $D = 4+2\epsilon$ dimensions with $\epsilon < 0$. The one-loop Wilson loop $w_n^{(1)}$ times the tree-level MHV amplitude is equal to the one-loop MHV amplitude, calculated by using the unitarity-based approach [**?**], up to a regularization-dependent factor. This implies that non-trivial remainder functions can only appear at two and higher loops. At two loops, we define the remainder function $\mathcal{R}_n^{\rm WL}$ for an n-sided Wilson loop as[1]

$$w_n^{(2)}(\epsilon) = f_{\rm WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{\rm WL}^{(2)} + \mathcal{R}_n^{\rm WL} + E_n^{\rm WL}(\epsilon) , \tag{3.2.10}$$

where

$$f_{\rm WL}^{(2)}(\epsilon) := f_0^{(2)} + f_{1,\rm WL}^{(2)}\epsilon + f_{2,\rm WL}^{(2)}\epsilon^2 . \tag{3.2.11}$$

Note that $f_0^{(2)} = -\zeta_2$, which is the same as on the amplitude side, while $f_{1,\rm WL}^{(2)} = G_{\rm eik}^{(2)} = 7\zeta_3$ [153]. In [152], the four- and five-edged Wilson loops were cast in the

[1]We expect a remainder function at every loop order l and the corresponding equations would be $w_n^{(l)}(\epsilon) = f_{\rm WL}^{(l)}(\epsilon)\, w_n^{(1)}(l\epsilon) + C_{\rm WL}^{(l)} + \mathcal{R}_{n,\rm WL}^{(l)} + E_{n,\rm WL}^{(l)}(\epsilon)$.

form (3.2.10) and by making the natural requirements

$$\mathcal{R}_4^{\rm WL} \;=\; \mathcal{R}_5^{\rm WL} \;=\; 0 \;, \tag{3.2.12}$$

this allowed for a determination of the coefficients $f^{(2)}_{2,\rm WL}$ and $C^{(2)}_{\rm WL}$. The results found in[152], are[2]

$$f^{(2)}_{\rm WL}(\epsilon) \;=\; -\zeta_2 \;+\; 7\zeta_3\,\epsilon \;-\; 5\zeta_4\,\epsilon^2 \;, \tag{3.2.13}$$

and

$$C^{(2)}_{\rm WL} \;=\; -\frac{1}{2}\zeta_2^2 \;. \tag{3.2.14}$$

As noticed in [152], there is an intriguing agreement between the constant $C^{(2)}_{\rm WL}$ and the corresponding value of the same quantity on the amplitude side.

What has been observed so far is a duality between Wilson loops and amplitudes up to finite terms. In turn this can be reinterpreted as an equality of the corresponding remainder functions

$$\mathcal{R}_n \;=\; \mathcal{R}_n^{\rm WL} \;. \tag{3.2.15}$$

An alternative interpretation of the duality in terms of certain ratios of amplitudes (Wilson loops) has been given recently in [154].

A consequence of the precise determination of the constants $f^{(2)}_{2,\rm WL}$ and $C^{(2)}_{\rm WL}$ is that no additional constant term is allowed on the right hand side of (3.2.15). For the same reason, the Wilson loop remainder function must then have the same collinear

[2] The $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ coefficients of $f^{(2)}_{\rm WL}(\epsilon)$ had been determined earlier in [131].

limits as its amplitude counterpart, i.e.

$$\mathcal{R}_n^{\mathrm{WL}} \to \mathcal{R}_{n-1}^{\mathrm{WL}} \,, \tag{3.2.16}$$

with no extra constant on the right hand side of (3.2.16).

3.2.1 At One-Loop Level for All n

One-Loop Amplitudes

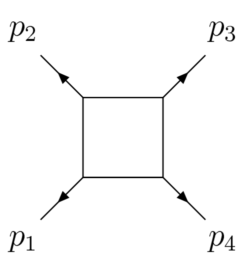
We begin with the one-loop amplitudes, for which analytic results can be given to all orders in ϵ.

Following the conventions of [133], the one-loop four-point amplitude may be expressed as [155]

$$\mathcal{M}_4^{(1)} = -\frac{1}{2} st I_4^{(1)} \tag{3.2.17}$$

where $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ are the usual Mandelstam variables and $I_4^{(1)}$ is the massless scalar box integral

$I_4^{(1)} =$



$$= \frac{e^{\epsilon\gamma}}{i\pi^{D/2}} \int d^D p \frac{1}{p^2 (p-p_1)^2 (p-p_1-p_2)^2 (p+p_4)^2}, \tag{3.2.18}$$

which we have written out in order to emphasize the normalization convention (followed throughout this section) that each loop momentum integral carries an overall factor of $e^{\epsilon\gamma}/i\pi^{D/2}$. The integral may be evaluated explicitly (see for example [156]) in terms of the ordinary hypergeometric function ${}_2\mathrm{F}_1$, leading to the exact expression

$$\mathcal{M}_4^{(1)} = -\frac{e^{\epsilon\gamma}}{\epsilon^2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[(-s)^{-\epsilon}{}_2\mathrm{F}_1(1,-\epsilon,1-\epsilon,1+s/t)+(s\leftrightarrow t)\right], \tag{3.2.19}$$

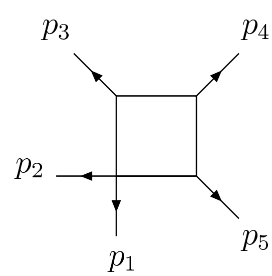
valid to all orders in ϵ. We will always be studying the amplitude/Wilson loop duality in the fully Euclidean regime where all momentum invariants such as s and t are negative. The formula (3.2.19) applies in this regime as long as we are careful to navigate branch cuts according to the rule

$$(-z)^{-\epsilon}\,{}_2\mathrm{F}_1(-\epsilon,-\epsilon,1-\epsilon,1+z) := \lim_{\varepsilon\to 0}\mathrm{Re}\left[\frac{{}_2\mathrm{F}_1(-\epsilon,-\epsilon,1-\epsilon,1+z+i\varepsilon)}{(-z+i\varepsilon)^{\epsilon}}\right] \tag{3.2.20}$$

when $z > 0$.

Five-point loop amplitudes $\mathcal{M}_5^{(L)}$ contain both parity-even and parity-odd contributions after dividing by the tree amplitude as in (3.2.1). The parity-even part of the one-loop five-point amplitude is given by

$$\mathcal{M}_{5+}^{(1)} = -\frac{1}{4}\sum_{\text{cyclic}} s_3 s_4 I_5^{(1)}, \qquad I_5^{(1)} = \tag{3.2.21}$$



,

where $s_i = (p_i + p_{i+1})^2$ and the sum runs over the five cyclic permutations of the external momenta p_i. This integral can also be explicitly evaluated (see for example [156]), leading to the all-orders in ϵ result

$$\begin{aligned}\mathcal{M}_{5+}^{(1)} = -\frac{e^{\epsilon\gamma}}{\epsilon^2}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{2}\sum_{\text{cyclic}}\Bigg[&\left(-\frac{s_1-s_4}{s_3s_4}\right)^{\epsilon}{}_2\mathrm{F}_1(-\epsilon,-\epsilon,1-\epsilon,1-\frac{s_3}{s_1-s_4})\\ &+\left(-\frac{s_1-s_3}{s_3s_4}\right)^{\epsilon}{}_2\mathrm{F}_1(-\epsilon,-\epsilon,1-\epsilon,1-\frac{s_4}{s_1-s_3})\\ &-\left(-\frac{(s_1-s_3)(s_1-s_4)}{s_1s_3s_4}\right)^{\epsilon}{}_2\mathrm{F}_1(-\epsilon,-\epsilon,1-\epsilon,1-\frac{s_3s_4}{(s_1-s_3)(s_1-s_4)})\Bigg],\end{aligned} \tag{3.2.22}$$

again keeping in mind (3.2.20).

One-Loop Wilson Loops

The one-loop Wilson loop was found in [130] for any number of edges and to all orders in the dimensional regularization parameter ϵ. It is obtained by summing over diagrams with a single gluon propagator stretching between any two edges of the Wilson loop polygon. Diagrams with the propagator stretching between adjacent

$\text{cusp} := \Gamma(1+\epsilon)e^{\epsilon\gamma}\times\left(-\frac{1}{2\epsilon^2}(-s_i)^{-\epsilon}\right)$ $\qquad$ $\text{finite} := \Gamma(1+\epsilon)e^{\epsilon\gamma}\times\mathcal{F}_\epsilon$

Figure 3.1: *One-loop Wilson loop diagrams. The expression of* $\mathcal{F}_\epsilon$ *is given in* (C.0.12) *of equivalently in* (C.0.14).

edges p_i and p_{i+1} are known as cusp diagrams, and give the infrared-divergent terms in the Wilson loop, proportional to $(-2p_i\cdot p_{i+1})^{-\epsilon}/\epsilon^2 = (-s_i)^{-\epsilon}/\epsilon^2$.

On the other hand, diagrams for which the propagator stretches between two non-adjacent edges are finite. Their contribution to the Wilson loop can be found to all orders in ϵ and is (up to an ϵ-dependent factor) precisely equal to the finite part of a two-mass easy or one-mass box function [130] (for details see appendix C). The general n-point one loop amplitude is given by the sum over precisely these two-mass easy and one-mass box functions [**?**] to $\mathcal{O}(\epsilon^0)$.[3] Thus we conclude that the Wilson loop is equal to the amplitude at one loop for any n up to finite order in ϵ only (and up to a kinematic independent factor).

However at four and five points a much stronger statement can be made. The four-point amplitude and the parity-even part of the five-point amplitude are both given by the sum over zero- and one-mass boxes *to all orders in* ϵ. Thus the Wilson loop correctly reproduces these one-loop amplitudes to all orders in ϵ. Using the results in appendix C, we find that the four-point Wilson loop (in a form which is manifestly real in the Euclidean regime $s, t < 0$) is given by

$$\begin{aligned} W_4^{(1)} &= \Gamma(1+\epsilon)e^{\epsilon\gamma}\Big\{ -\frac{1}{\epsilon^2}\left[(-s)^{-\epsilon}+(-t)^{-\epsilon}\right] + \mathcal{F}_\epsilon(s,t,0,0) + \mathcal{F}_\epsilon(t,s,0,0)\Big\} \\ &= \Gamma(1+\epsilon)e^{\epsilon\gamma}\Bigg\{ -\frac{1}{\epsilon^2}\left[(-s)^{-\epsilon}+(-t)^{-\epsilon}\right] \\ &+\frac{1}{\epsilon^2}\left(\frac{u}{st}\right)^{\epsilon}\left[\left(\frac{t}{s}\right)^{\epsilon} {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;-t/s) + \left(\frac{s}{t}\right)^{\epsilon} {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;-s/t) - 2\pi\epsilon\cot(\epsilon\pi)\right]\Bigg\} . \end{aligned} \tag{3.2.23}$$

Note in particular the additional cotangent term explained in detail at the end of appendix C. The generic form of the function $\mathcal{F}_\epsilon$ is given in (C.0.12) of equivalently

[3] The all-orders in ϵ n-point amplitude contains new integrals contributing at $\mathcal{O}(\epsilon)$.

in (C.0.14).

For the five-point amplitude we display a new form which has a simple analytic continuation in all kinematical regimes and also a very simple expansion in terms of Nielsen polylogarithms (see (C.0.11)). It is given in terms of ${}_3\mathrm{F}_2$ hypergeometric functions and is derived in detail in appendix C:

$$\begin{aligned}
W_5^{(1)} =&\sum_{i=1}^{5}\Gamma(1+\epsilon)e^{\epsilon\gamma}\Big[-\frac{1}{2\epsilon^2}(-s_i)^{-\epsilon}+\mathcal{F}_\epsilon(s_i,s_{i+1},s_{i+3},0)\Big]\\
=&\sum_{i=1}^{5}\Gamma(1+\epsilon)e^{\epsilon\gamma}\Bigg\{-\frac{1}{2\epsilon^2}(-s_i)^{-\epsilon}\\
&-\frac{1}{2}\left(\frac{s_{i+3}-s_i-s_{i+1}}{s_is_{i+1}}\right)^{\epsilon}\Big[\tfrac{s_{i+3}-s_i}{s_{i+1}}\,{}_3\mathrm{F}_2\Big(1,1,1+\epsilon;2,2;\tfrac{s_{i+3}-s_i}{s_{i+1}}\Big)\\
&+\tfrac{s_{i+3}-s_{i+1}}{s_i}\,{}_3\mathrm{F}_2\Big(1,1,1+\epsilon;2,2;\tfrac{s_{i+3}-s_{i+1}}{s_i}\Big)\\
&+\frac{H_{-\epsilon}}{\epsilon}-\tfrac{(s_{i+3}-s_i)(s_{i+3}-s_{i+1})}{s_is_{i+1}}\,{}_3\mathrm{F}_2\Big(1,1,1+\epsilon;2,2;\tfrac{(s_{i+3}-s_i)(s_{i+3}-s_{i+1})}{s_is_{i+1}}\Big)\Big]\Bigg\}
\end{aligned} \tag{3.2.24}$$

where H_n is the n^{th}-harmonic number. Using hypergeometric identities one can show that (up to the prefactor) the four- and five-sided Wilson loops (3.2.23), (3.2.24) are equal to the four-point and the (parity-even part of the) five-point amplitudes of (3.2.19) and (3.2.22).

The precise relation between the Wilson loop and the amplitude is

$$W_4^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\mathcal{M}_4^{(1)}\,, \qquad W_5^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\mathcal{M}_{5+}^{(1)}\,, \tag{3.2.25}$$

where $\mathcal{M}_4^{(1)}$ is the one-loop four-point amplitude and $\mathcal{M}_{5+}^{(1)}$ is the parity-even part of the five-point amplitude.

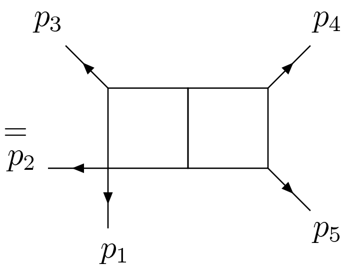
3.2.2 At Two-Loop Level for $n = 4, 5$

The main result of this subsection is that for $n = 4, 5$ the relation between amplitudes and Wilson loops continues to hold for terms of order ϵ^1. In particular we find

$$\mathcal{E}_4^{(2)} = \mathcal{E}_{4,\mathrm{WL}}^{(2)} - 3\zeta_5 \ , \tag{3.2.26}$$

$$\mathcal{E}_5^{(2)} = \mathcal{E}_{5,\mathrm{WL}}^{(2)} - \frac{5}{2}\zeta_5 \ . \tag{3.2.27}$$

Note that these results have been obtained (semi-)numerically with typical errors of 10^{-8} at $n = 4$ and 10^{-4} for $n = 5$. Details of the calculations are presented in the remaining subsections of this chapter. More precisely $\mathcal{E}_4^{(2)}$ is known analytically [39], while the analytic evaluation of $\mathcal{E}_{4,\mathrm{WL}}^{(2)}$ is discussed in appendix D. At five points all results are numerical and furthermore on the amplitude side we only considered the parity-even terms. It is an interesting open question whether the parity-odd terms cancel at $\mathcal{O}(\epsilon)$ as they do at $\mathcal{O}(\epsilon^0)$ [84].

$I^{(2)a} =$ 

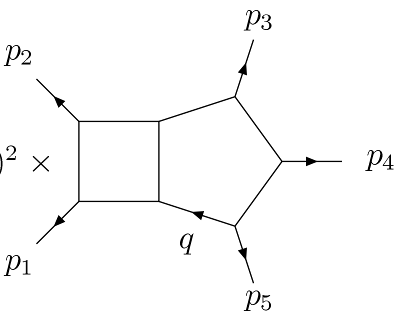
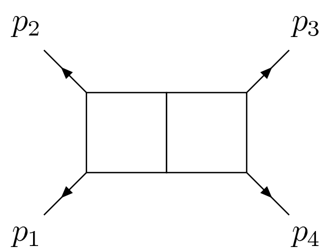
$I_5^{(2)d} = (q - p_1)^2 \times$ 

Figure 3.2: *Integrals appearing in the amplitude $\mathcal{M}_{5+}^{(2)}$. Note that $I_5^{(2)d}$ contains the indicated scalar numerator factor involving q, one of the loop momenta.*

Two-Loop Amplitudes

The two-loop four-point amplitude is expressed as [80]

$$\mathcal{M}_4^{(2)} = \frac{1}{4} s^2 t I_4^{(2)} + (s \leftrightarrow t), \qquad I_4^{(2)} = \tag{3.2.28}$$



which may be evaluated analytically through $\mathcal{O}(\epsilon^2)$ using results from [39] (no all-orders in ϵ expression for the double box integral is known), from which we find

$$\begin{aligned} \mathcal{E}_4 = 5\,\mathrm{Li}_5(-x) - 4L\,\mathrm{Li}_4(-x) + \frac{1}{2}(3L^2 + \pi^2)\,\mathrm{Li}_3(-x) - \frac{L}{3}(L^2 + \pi^2)\,\mathrm{Li}_2(-x) \\ - \frac{1}{24}(L^2 + \pi^2)^2 \log(1 + x) + \frac{2}{45}\pi^4 L - \frac{39}{2}\zeta_5 + \frac{23}{12}\pi^2\zeta_3, \end{aligned} \tag{3.2.29}$$

where $x = t/s$ and $L = \log x$. A comment is in order here: In order to be able to present the amplitude remainder (3.2.29) in this form, we have pulled out a factor of $(st)^{-L\epsilon/2}$ from each loop amplitude $\mathcal{M}_4^{(L)}$. This renders the amplitudes, and hence the ABDK remainder $E_4(\epsilon)$, dimensionless functions of the single variable x. We perform this step in the four-point case only, where we are able to present analytic results for the amplitude and Wilson loop remainders.

The parity-even part of the two-loop five-point amplitude involves the two integrals shown in Figure 3.2, in terms of which [84, 134?]

$$\mathcal{M}_{5+}^{(2)} = \frac{1}{8} \sum_{\text{cyclic}} \Big(s_3 s_4^2 I^{(2)a} + (p_i \to p_{6-i}) \Big) + s_1 s_3 s_4 I^{(2)d}, \tag{3.2.30}$$

where $s_i = (p_i + p_{i+1})^2$. To evaluate this amplitude to $\mathcal{O}(\epsilon)$ we must resort to a numerical calculation using Mellin-Barnes parameterizations of the integrals (which may be found for example in [134]), which we then expand through $\mathcal{O}(\epsilon)$, simplify, and numerically integrate with the help of the `MB`, `MBresolve`, and `barnesroutines` programs [157, 158], In this manner we have determined the $\mathcal{O}(\epsilon)$ contribution $\mathcal{E}_5^{(2)}$ to the five-point ABDK relation numerically at a variety of kinematic points. The results are displayed in Table 3.1.

Two-Loop Wilson Loops

At two-loop order, the n-point Wilson loop is given by a sum over six different types of diagrams. These are described in general for polygons with any number of edges in [152] and are displayed for illustration below.

The computation of the four-point two-loop Wilson loop up to $\mathcal{O}(\epsilon^0)$ was first performed in [131]. In appendix **??** we display all the contributing diagrams for this case and give expressions for these to all orders in ϵ in all cases except for the “hard"

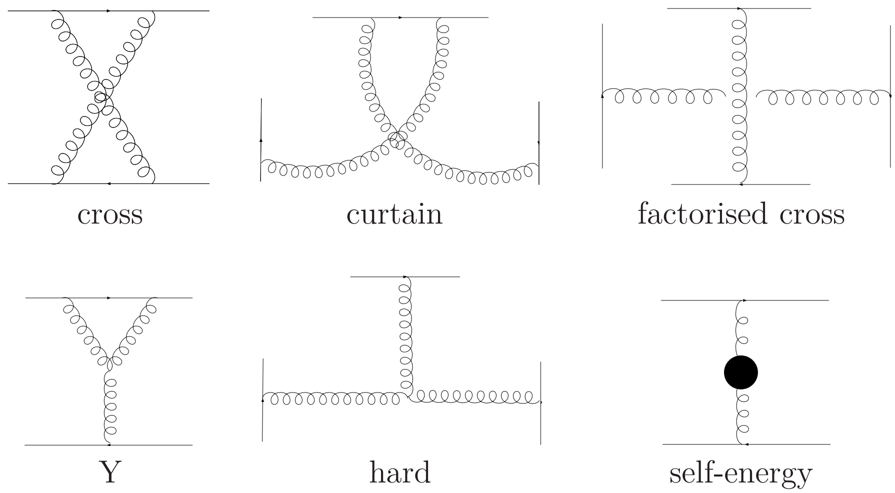


Figure 3.3: *The six different diagram topologies contributing to the two-loop Wilson loop. For details see [152].*

diagram, which we give up to and including terms of $\mathcal{O}(\epsilon)$. Summing up the contributions from all these diagrams we obtain the result for the two-loop four-point Wilson loop to $\mathcal{O}(\epsilon)$. This is displayed in (3.2.31) of the next subsection.

The five-point two-loop Wilson loop was calculated up to $\mathcal{O}(\epsilon^0)$ in [132]. In order to obtain results at one order higher in ϵ we have proceeded by using numerical methods. In particular we have used Mellin-Barnes techniques to evaluate and expand all the two-loop integrals of Figure 3.3. This is described in more detail in Section 3.2.2.

The Complete Two-Loop Wilson Loop at Four Points Here is our final result for the four-point Wilson loop at two loops expanded up to and including terms of $\mathcal{O}(\epsilon)$:

$$w_4^{(2)} = \mathcal{C} \times \left[(-s)^{-2\epsilon} + (-t)^{-2\epsilon}\right] \times \left[\frac{w_2}{\epsilon^2} + \frac{w_1}{\epsilon} + w_0 + w_{-1}\epsilon + \mathcal{O}(\epsilon^2)\right], \quad (3.2.31)$$

where

$$
\begin{aligned}
w_2 &= \frac{\pi^2}{48} \,, && (3.2.32)\\
w_1 &= -\frac{7\zeta_3}{8} \,, && (3.2.33)\\
w_0 &= -\frac{\pi^2}{48}\left(\log^2 x + \pi^2\right) + \frac{\pi^4}{144} \;=\; -\frac{\pi^2}{48}\left(\log^2 x + \frac{2}{3}\pi^2\right) \,, && (3.2.34)\\
w_{-1} &= -\frac{1}{1440}\Big[- 46\pi^4 \log x - 10\pi^2 \log^3 x + 75\pi^4 \log(1+x) + 90\pi^2 \log^2 x \log(1+x) \\
&\quad +15 \log^4 x \log(1+x) + 240\pi^2 \log x \,\mathrm{Li}_2(-x) + 120 \log^3 x \,\mathrm{Li}_2(-x) \\
&\quad -300\pi^2 \mathrm{Li}_3(-x) - 540 \log^2 x \,\mathrm{Li}_3(-x) + 1440 \log x \,\mathrm{Li}_4(-x) \\
&\quad -1800 \mathrm{Li}_5(-x) - 1560\pi^2\zeta_3 - 1260 \log^2 x \,\zeta_3 + 5940\zeta_5\Big] \,, && (3.2.35)
\end{aligned}
$$

and

$$
\mathcal{C} := 2\Big[\Gamma(1+\epsilon)e^{\gamma\epsilon}\Big]^2 \;=\; 2\left(1 + \zeta_2\epsilon^2 - \frac{2}{3}\zeta_3\epsilon^3\right) \;+\; \mathcal{O}(\epsilon^4) \;. \qquad (3.2.36)
$$

We recall that $x = t/s$.

We would like to point out the simplicity of our result (3.2.31) – specifically, (3.2.32)–(3.2.35) are expressed only in terms of standard polylogarithms. Harmonic polylogarithms and Nielsen polylogarithms are present in the expressions of separate Wilson loop diagrams, as can be seen in appendix A, but cancel after summing all contributions.

The $\mathcal{O}(\epsilon)$ Wilson Loop Remainder Function at Four Points Using the result (3.2.31) and the one-loop expression for the Wilson loop, one can work out the expression for the remainder function at $\mathcal{O}(\epsilon)$, as defined in (3.2.5) and (3.2.6). Our result is

$$
\begin{aligned}
\mathcal{E}_{4,\mathrm{WL}} \;=\;& \frac{1}{360}\Big[16\pi^4 \log x - 15\pi^4 \log(1+x) - 30\pi^2 \log^2 x \log(1+x) \\
& -15 \log^4 x \log(1+x) - 120\pi^2 \log x \mathrm{Li}_2(-x) - 120 \log^3(x) \mathrm{Li}_2(-x) \\
& +180\pi^2 \mathrm{Li}_3(-x) + 540 \log^2 x \mathrm{Li}_3(-x) - 1440 \log(x) \mathrm{Li}_4(-x) \\
& +1800 \mathrm{Li}_5(-x) + 690\pi^2 \zeta_3 - 5940\zeta_5\Big] , \qquad (3.2.37)
\end{aligned}
$$

where we recall that $\mathcal{E}_{n,\mathrm{WL}}$ is related to the quantity E_n introduced in (3.2.5) and (3.2.6). Remarkably, (3.2.37) does not contain any harmonic polylogarithms. We will compare the Wilson loop remainder (3.2.37) to the corresponding amplitude remainder (3.2.29) in Section 3.2.2.[4]

The $\mathcal{O}(\epsilon)$ Wilson Loop at Five Points and the Five-Point Remainder Function For the five-point amplitude and Wilson loop at two loops we resort to completely numerical evaluation of the contributing integrals, and a comparison of the remainder functions is then performed. We postpone this discussion to section 3.2.2.

Mellin-Barnes Integration

The two-loop five-point Wilson loop and amplitude have been numerically evaluated by means of the Mellin-Barnes (MB) method using the `MB` package [157] in

[4] Similarly to what was done for the amplitude remainder (3.2.29), in arriving at (3.2.37) we have pulled out a factor of $(st)^{-\epsilon/2}$ per loop in order to obtain a result which depends only on the ratio $x := t/s$.

MATHEMATICA. At the heart of the method lies the Mellin-Barnes representation

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{2\pi i}\frac{1}{\Gamma(\lambda)}\int_{-i\infty}^{+i\infty} dz\, \frac{X^z}{Y^{\lambda+z}}\Gamma(-z)\Gamma(\lambda+z). \tag{3.2.38}$$

We will use the integral representation for the hard diagram of the Wilson loop as an example in order to describe the procedure we followed. The integral for the specific diagram shown in Figure 3.4 has the expression

$$\begin{aligned} &f_{\mathrm{H}}(p_1,p_2,p_3;Q_1,Q_2,Q_3) \\ &\quad = \frac{1}{8}\frac{\Gamma(2+2\epsilon)}{\Gamma(1+\epsilon)^2}\int_0^1 (\prod_{i=1}^3 d\tau_i)\int_0^1 (\prod_{i=1}^3 d\alpha_i)\delta(1-\sum_{i=1}^3\alpha_i)(\alpha_1\alpha_2\alpha_3)^\epsilon \frac{\mathcal{N}}{\mathcal{D}^{2+2\epsilon}}. \end{aligned} \tag{3.2.39}$$

We write the numerator and denominator as a function of the momentum invariants, i.e. squares of sums of consecutive momenta,

$$\begin{aligned} \mathcal{D} = -\alpha_1\alpha_2 \Big[&(p_1+Q_3+p_2)^2(1-\tau_1)\tau_2 + (p_1+Q_3)^2(1-\tau_1)(1-\tau_2) \\ &+(Q_3+p_2)^2\tau_1\tau_2 + Q_3^2\tau_1(1-\tau_2)\Big] + \text{cyclic}(1,2,3), \end{aligned} \tag{3.2.40}$$

$$\begin{aligned} \mathcal{N} = 2\,&[2(p_1p_2)(p_3Q_3) - (p_2p_3)(p_1Q_3) - (p_1p_3)(p_2Q_3)]\,\alpha_1\alpha_2 \\ &+ 2(p_1p_2)(p_3p_1)\,[\alpha_1\alpha_2(1-\tau_1) + \alpha_3\alpha_1\tau_1] + \text{cyclic}(1,2,3), \end{aligned} \tag{3.2.41}$$

where

$$
\begin{aligned}
2p_ip_{i+1} &= -(p_i+Q_{i+2})^2 + Q_{i+2}^2 - (Q_{i+2}+p_{i+1})^2 + (Q_i+p_{i+2}+Q_{i+1})^2, \\
2p_iQ_i &= -(p_i+Q_{i+2}+p_{i+1})^2 + (Q_{i+2}+p_{i+1})^2 \\
&\quad - (p_{i+2}+Q_{i+1}+p_i)^2 + (p_{i+2}+Q_{i+1})^2, \\
2p_iQ_j &= (p_i+Q_j)^2 - Q_j^2. \qquad (3.2.42)
\end{aligned}
$$

By means of the substitution $\alpha_1 \to 1-\tau_4$, $\alpha_2 \to \tau_4\tau_5$ and $\alpha_3 \to \tau_4(1-\tau_5)$, we eliminate one integration and the delta function to get a five-fold integral over $\tau_i \in [0,1]$. Next, we obtain an MB representation using the generalisation of (3.2.38)

$$
\frac{1}{(\sum_{s=1}^m X_s)^\lambda} = \frac{1}{(2\pi i)^{m-1}} \frac{1}{\Gamma(\lambda)} \left(\prod_{s=1}^{m-1} \int_{-i\infty}^{+i\infty} dz_s \right) \frac{\prod_{s=1}^{m-1} X_s^{z_s} \Gamma(-z_s)}{X_m^{\lambda+\sum_{s=1}^{m-1} z_s} \Gamma(\lambda + \sum_{s=1}^{m-1} z_s)}, \qquad (3.2.43)
$$

which introduces $m-1$ MB integration variables z_s, where m is the number of terms in the denominator. At this point, the integrations over the τ_i's can be easily performed by means of the substitution

$$
\int_0^1 dx\, x^\alpha (1-x)^\beta = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,. \qquad (3.2.44)
$$

We are now left with an integrand that is an analytic function containing powers of the momentum invariants $(-s_{ij})^{f(\{z_s\},\epsilon)}$ and Gamma functions $\Gamma(g(\{z_i\},\epsilon))$, where f and g are linear combinations of the z_s's and ϵ. In order to perform the MB integrations, one has to pick appropriate contours, so that for each z_s the $\Gamma(\cdots + z_s)$ poles are to the left of the contour and the $\Gamma(\cdots - z_s)$ poles are to the right.

At this point we use various `Mathematica` packages to perform a series of operations

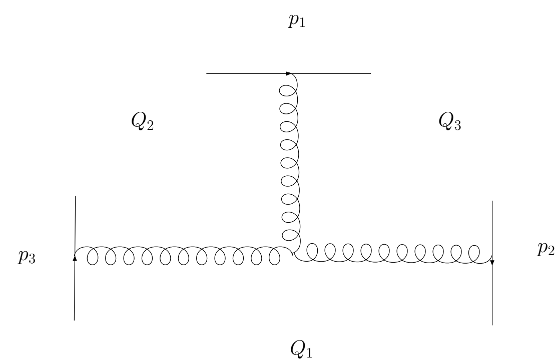


Figure 3.4: *The hard diagram corresponding to* (3.2.39).

in an automated way to finally obtain a numerical expression at specific kinematic points. We will briefly summarise the steps followed, while for more details we refer the reader to the references documenting these packages and references therein. Using the `MBresolve` package [158], we pick appropriate contours and resolve the singularity structure of the integrand in ϵ. The latter involves taking residues and shifting contours, and is essential in order to be able to Laurent expand the integrand in ϵ. Using the `barnesroutines` package [157, 158], we apply the Barnes lemmas, which in general generate more integrals but decrease their dimensionality, leading to higher precision results. Finally, using the `MB` package [157] we numerically integrate at specific Euclidean kinematic points to obtain a numerical expression. While all manipulations of the integrals and the expansion in ϵ are performed in `Mathematica`, the actual numerical integration for each term is performed using the `CUBA` routines [159] for multidimensional numerical integration in `FORTRAN`. The high number of diagrams, and number of integrals for each diagram, makes the task of running the `FORTRAN` integrations ideal for parallel computing.

Comparison of the Remainder Functions

Four-point Amplitude and Wilson Loop Remainders The remainder functions for the four-point amplitude and Wilson loops are given in (3.2.29) and (3.2.37), respectively. From these relations, it follows that the difference of remainders is a constant, x-independent term:

$$\mathcal{E}_4 = \mathcal{E}_{4,\mathrm{WL}} - 3\,\zeta_5 \ , \tag{3.2.45}$$

as anticipated in (3.2.26).

We would like to stress that this is a highly nontrivial result since there is no reason a priori to expect that the four-point remainder on the amplitude and Wilson loop side, (3.2.29) and (3.2.37) respectively, agree (up to a constant shift). For example, anomalous dual conformal invariance is known to determine the form of the four- and five-point Wilson loop only up to $\mathcal{O}(\epsilon^0)$ terms [132], but does not constrain terms which vanish as $\epsilon \to 0$. The expressions we derived for the amplitude and Wilson loop four-point remainders at $\mathcal{O}(\epsilon)$ are also pleasingly simple, in that they only contain standard polylogarithms.

Five-point Amplitude and Wilson Loop Remainders We have numerically evaluated both the five-point two-loop amplitude and Wilson loop up to $\mathcal{O}(\epsilon)$ at 25 Euclidean kinematic points, i.e. points in the subspace of the kinematic invariants with all $s_{ij} < 0$. The choice of these points and the values of the remainder functions $\mathcal{E}_5^{(2)}$, $\mathcal{E}_{5,\mathrm{WL}}^{(2)}$ at $\mathcal{O}(\epsilon)$ together with the errors reported by the CUBA numerical integration library [159] appear in Table 3.1, while in Figures 3.5 and 3.6 we plot both

#	$(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})$	$\mathcal{E}_5^{(2)}$	$\mathcal{E}_{5,\mathrm{WL}}^{(2)}$
1	$(-1,-1,-1,-1,-1)$	-8.463173 ± 0.000047	-5.8705280 ± 0.0000068
2	$(-1,-1,-2,-1,-1)$	-8.2350 ± 0.0024	-5.64560 ± 0.00063
3	$(-1,-2,-2,-1,-1)$	-7.7697 ± 0.0026	-5.17647 ± 0.00076
4	$(-1,-2,-3,-4,-5)$	-6.234809 ± 0.000032	-3.642125 ± 0.000018
5	$(-1,-1,-3,-1,-1)$	-8.2525 ± 0.0027	-5.65919 ± 0.00097
6	$(-1,-2,-1,-2,-1)$	-8.142702 ± 0.000023	-5.5500050 ± 0.0000092
7	$(-1,-3,-3,-1,-1)$	-7.6677 ± 0.0034	-5.0784 ± 0.0013
8	$(-1,-2,-3,-2,-1)$	-6.8995 ± 0.0029	-4.31395 ± 0.00093
9	$(-1,-3,-2,-5,-4)$	-6.9977 ± 0.0031	-4.40806 ± 0.00099
10	$(-1,-3,-1,-3,-1)$	-8.2759 ± 0.0025	-5.69086 ± 0.00085
11	$(-1,-4,-8,-16,-32)$	-8.7745 ± 0.0078	-6.1825 ± 0.0051
12	$(-1,-8,-4,-32,-16)$	-11.991985 ± 0.000089	-9.398659 ± 0.000084
13	$(-1,-10,-100,-10,-1)$	-2.914 ± 0.022	-0.300 ± 0.010
14	$(-1,-100,-10,-100,-1)$	-3.237 ± 0.011	-0.6648 ± 0.0028
15	$(-1,-1,-100,-1,-1)$	-12.686 ± 0.014	-10.108 ± 0.010
16	$(-1,-100,-1,-100,-1)$	-14.7067 ± 0.0077	-12.1136 ± 0.0071
17	$(-1,-100,-100,-1,-1)$	-182.32 ± 0.11	-179.722 ± 0.039
18	$(-1,-100,-10,-100,-10)$	-6.3102 ± 0.0062	-3.7281 ± 0.0013
19	$\left(-1,-\frac{1}{4},-\frac{1}{9},-\frac{1}{16},-\frac{1}{25}\right)$	-19.0031 ± 0.0077	-16.4136 ± 0.0021
20	$\left(-1,-\frac{1}{9},-\frac{1}{4},-\frac{1}{25},-\frac{1}{16}\right)$	-15.1839 ± 0.0046	-12.5995 ± 0.0016
21	$\left(-1,-1,-\frac{1}{4},-1,-1\right)$	-9.7628 ± 0.0028	-7.17588 ± 0.00079
22	$\left(-1,-\frac{1}{4},-\frac{1}{4},-1,-1\right)$	-9.5072 ± 0.0036	-6.9186 ± 0.0014
23	$\left(-1,-\frac{1}{4},-1,-\frac{1}{4},-1\right)$	-12.6308 ± 0.0031	-10.04241 ± 0.00083
24	$\left(-1,-\frac{1}{4},-\frac{1}{9},-\frac{1}{4},-1\right)$	-11.0200 ± 0.0056	-8.4281 ± 0.0030
25	$\left(-1,-\frac{1}{9},-\frac{1}{4},-\frac{1}{9},-1\right)$	-19.1966 ± 0.0070	-16.6095 ± 0.0043

Table 3.1: $\mathcal{O}(\epsilon)$ *five-point remainders for amplitudes* ($\mathcal{E}_5^{(2)}$) *and Wilson loops* ($\mathcal{E}_{5,\mathrm{WL}}^{(2)}$).

#	$(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})$	$\mathcal{E}_5^{(2)} - \mathcal{E}_{5,\mathrm{WL}}^{(2)}$	$\lvert\mathcal{E}_5^{(2)} - \mathcal{E}_{5,\mathrm{WL}}^{(2)} + \frac{5}{2}\zeta_5\rvert/\sigma$
1	$(-1,-1,-1,-1,-1)$	-2.592645 ± 0.000048	6.8
2	$(-1,-1,-2,-1,-1)$	-2.5894 ± 0.0025	1.2
3	$(-1,-2,-2,-1,-1)$	-2.5932 ± 0.0027	0.32
4	$(-1,-2,-3,-4,-5)$	-2.592697 ± 0.000036	10
5	$(-1,-1,-3,-1,-1)$	-2.5933 ± 0.0028	0.35
6	$(-1,-2,-1,-2,-1)$	-2.592697 ± 0.000025	15
7	$(-1,-3,-3,-1,-1)$	-2.5893 ± 0.0036	0.82
8	$(-1,-2,-3,-2,-1)$	-2.5856 ± 0.0030	2.2
9	$(-1,-3,-2,-5,-4)$	-2.5897 ± 0.0032	0.82
10	$(-1,-3,-1,-3,-1)$	-2.5851 ± 0.0026	2.8
11	$(-1,-4,-8,-16,-32)$	-2.5920 ± 0.0093	0.034
12	$(-1,-8,-4,-32,-16)$	-2.59333 ± 0.00012	8.3
13	$(-1,-10,-100,-10,-1)$	-2.614 ± 0.024	0.89
14	$(-1,-100,-10,-100,-1)$	-2.572 ± 0.011	1.9
15	$(-1,-1,-100,-1,-1)$	-2.578 ± 0.017	0.80
16	$(-1,-100,-1,-100,-1)$	-2.593 ± 0.010	0.071
17	$(-1,-100,-100,-1,-1)$	-2.60 ± 0.11	0.039
18	$(-1,-100,-10,-100,-10)$	-2.5820 ± 0.0063	1.6
19	$\left(-1,-\frac{1}{4},-\frac{1}{9},-\frac{1}{16},-\frac{1}{25}\right)$	-2.5894 ± 0.0080	0.36
20	$\left(-1,-\frac{1}{9},-\frac{1}{4},-\frac{1}{25},-\frac{1}{16}\right)$	-2.5844 ± 0.0049	1.6
21	$\left(-1,-1,-\frac{1}{4},-1,-1\right)$	-2.5869 ± 0.0029	1.9
22	$\left(-1,-\frac{1}{4},-\frac{1}{4},-1,-1\right)$	-2.5886 ± 0.0038	0.96
23	$\left(-1,-\frac{1}{4},-1,-\frac{1}{4},-1\right)$	-2.5884 ± 0.0032	1.2
24	$\left(-1,-\frac{1}{4},-\frac{1}{9},-\frac{1}{4},-1\right)$	-2.5919 ± 0.0064	0.064
25	$\left(-1,-\frac{1}{9},-\frac{1}{4},-\frac{1}{9},-1\right)$	-2.5870 ± 0.0082	0.65

Table 3.2: *Difference of the five-point amplitude and Wilson loop two-loop remainder functions at $\mathcal{O}(\epsilon)$, and its distance from $-\frac{5}{2}\zeta_5 \sim -2.592319$ in units of σ, the standard deviation reported by the CUBA numerical integration package [159].*

remainders for all kinematic points. We have calculated the difference between the amplitude and Wilson loop remainders, see Table 3.2 and Figure 3.7. Remarkably, this difference also appears to be constant (within our numerical precision) as in the four-point case, and hence we conjecture that

$$\mathcal{E}_5^{(2)} = \mathcal{E}_{5,\mathrm{WL}}^{(2)} - \frac{5}{2}\zeta_5 \ . \tag{3.2.46}$$

It is also intriguing that the constant difference is fit very well by a simple rational multiple of ζ_5, rather than a linear combination of ζ_5 and $\zeta_2\zeta_3$ as would have been allowed more generally by transcendentality.

In the last column of Table 3.2 we give the distance of our results from this conjecture in units of their standard deviation.

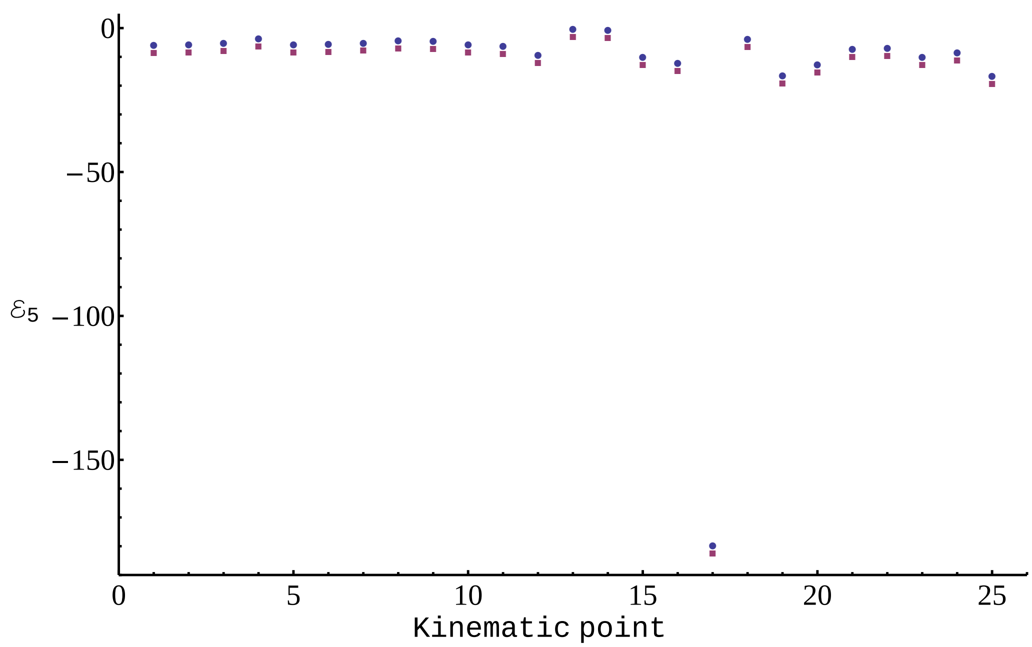


Figure 3.5: *Remainder functions at $\mathcal{O}(\epsilon)$ for the amplitude (circle) and the Wilson loop (square).*

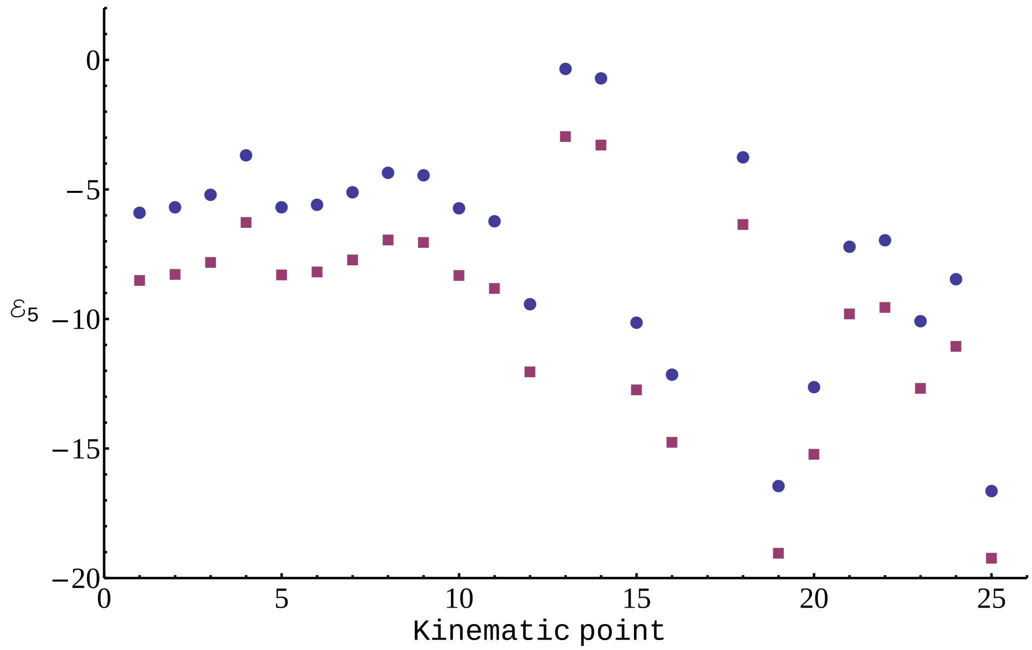


Figure 3.6: *Remainder functions at $\mathcal{O}(\epsilon)$ for the amplitude (circle) and the Wilson loop (square). In this Figure we have eliminated data point 17 and zoomed in on the others.*

For kinematic points 1, 4 and 6 we have evaluated the remainder functions with even higher precision, and found agreement with the conjecture to 4 digits. A remark is in order here. By increasing the precision, the mean value of the difference of remainders approaches the conjectured value, but we notice that in units of σ it drifts away from it, hinting at a potential underestimate of the errors. To test our error estimates we used the remainder functions at $\mathcal{O}(\epsilon^0)$, that are known to vanish. Our analysis confirmed that, as we increase the desired precision, the actual precision of the mean value does increase, but on the other hand reported errors tend to become increasingly underestimated.

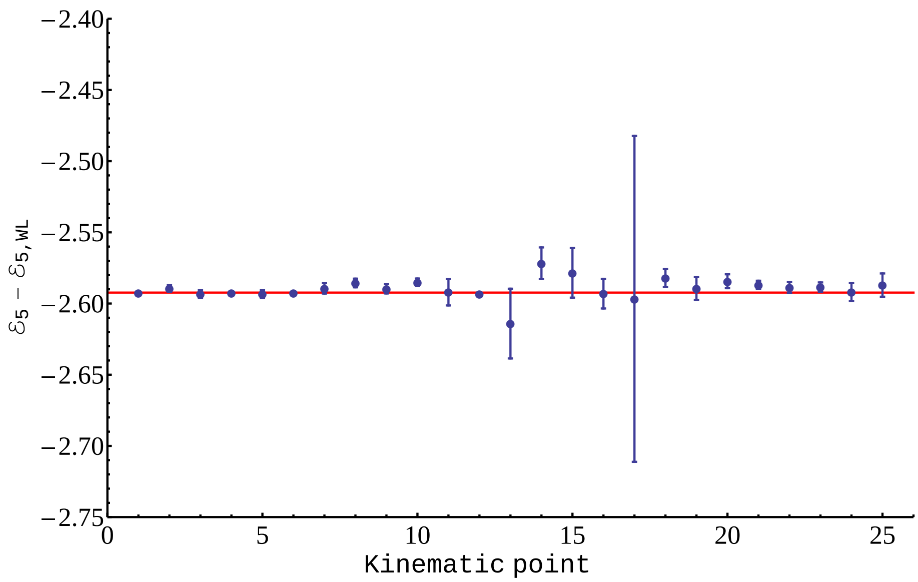


Figure 3.7: *Difference of the remainder functions* $\mathcal{E}_5^{(2)} - \mathcal{E}_{5,\mathrm{WL}}^{(2)}$.

Chapter 4

Scattering Amplitudes in Super Gravity

4.1 Relations to Super Yang-Mills

4.1.1 KLT Relations

In this section, we will review how tree amplitudes in gravity can be expressed in terms of tree amplitudes in gauge theory as bilinear combinations, following [239]. The reason is that, by using generalized unitarity, we will be able to chop the gravity loop amplitudes up into products of gravity trees. Then we can use the gravity-gauge relations to write everything in terms of products of gauge-theory trees, products which actually appear in cuts of gauge loop amplitudes. In this way, multi-loop gauge amplitudes provide the information needed to construct multi-loop gravity amplitudes.

The original gravity-gauge tree amplitude relations were found by Kawai, Lewellen and Tye [240], who recognized that the world-sheet integrands needed to compute tree-level amplitudes in the closed type II superstring theory were essentially the square of the integrands appearing in the open-superstring tree amplitudes. KLT represented the closed-string world-sheet integrals over the complex plane as products of contour integrals, and then deformed the contours until they could be identified as integrals for open-string amplitudes, thus deriving relations between closed- and open-string tree amplitudes.

Because the low-energy limit of the perturbative sector of the closed type II superstring in $D = 4$ is $\mathcal{N} = 8$ supergravity, and that of the open superstring is $\mathcal{N} = 4$ SYM, as the string tension goes to infinity the KLT relations express any $\mathcal{N} = 8$ supergravity tree amplitude in terms of amplitudes in $\mathcal{N} = 4$ SYM.

The KLT relations for $\mathcal{N} = 8$ supergravity amplitudes are bilinear in the $\mathcal{N} = 4$ SYM amplitudes, for two complementary reasons: (1) Integrals over the complex plane naturally break up into pairs of contour integrals, and (2) the $\mathcal{N} = 8$ supergravity Fock space naturally factors into a product of "left" and "right" $\mathcal{N} = 4$ SYM Fock spaces,

$$[\mathcal{N} = 8] \;=\; [\mathcal{N} = 4]_L \otimes [\mathcal{N} = 4]_R \,. \tag{4.1.1}$$

With the definition of the supergravity tree amplitudes $\mathcal{M}_n^{tree}$ as

$$\mathcal{M}_n^{tree}(\{k_i\}) \;=\; \left(\frac{\kappa}{2}\right)^{n-2} M_n^{tree}(1, 2, \ldots, n)\,, \tag{4.1.2}$$

where the gravitational coupling $\kappa^2 = 32\pi^2 G_N$, the first few KLT relations have the form as follows,

$$M_3^{tree}(1,2,3) = i\,A_3^{tree}(1,2,3)\tilde{A}_3^{tree}(1,2,3)\,, \tag{4.1.3}$$

$$M_4^{tree}(1,2,3,4) = -is_{12}\,A_4^{tree}(1,2,3,4)\,\tilde{A}_4^{tree}(1,2,4,3)\,, \tag{4.1.4}$$

$$M_5^{tree}(1,2,3,4,5) = is_{12}s_{34}\,A_5^{tree}(1,2,3,4,5)\tilde{A}_5^{tree}(2,1,4,3,5)\;+\;\mathcal{P}(2,3)\,, \tag{4.1.5}$$

$$\begin{aligned} M_6^{tree}(1,2,3,4,5,6) &= -is_{12}s_{45}\,A_6^{tree}(1,2,3,4,5,6) \\ &\quad \times\Big[s_{35}\,\tilde{A}_6^{tree}(2,1,5,3,4,6)+(s_{34}+s_{35})\,\tilde{A}_6^{tree}(2,1,5,4,3,6)\Big] \\ &\quad +\mathcal{P}(2,3,4)\,, \end{aligned} \tag{4.1.6}$$

where $s_{ij} \equiv (k_i + k_j)^2$, and "$\mathcal{P}$" indicates a sum over the $m!$ permutations of the m arguments of $\mathcal{P}$. Here A_n^{tree} indicates a tree amplitude for which the external states are drawn from the left-moving Fock space $[\mathcal{N}=4]_L$ in the tensor product (4.1.1), while $\tilde{A}_n^{tree}$ denotes an amplitude from the right-moving copy $[\mathcal{N}=4]_R$.

Now we will illustrate how KLT relation can be used to compute $\mathcal{N}=8$ supergravity amplitudes from $\mathcal{N}=4$ SYM amplitudes by a two-loop example.

The full two-loop four-point amplitude in $\mathcal{N}=4$ SYM is given by

$$\begin{aligned} A_4^{(2)} &= -s_{12}s_{23}A_4^{tree}\Big[C_{1234}^{P}\,s_{12}\,\mathcal{I}_4^{(2),P}(s_{12},s_{23})+C_{1234}^{\mathrm{NP}}\,s_{12}\,\mathcal{I}_4^{(2),NP}(s_{12},s_{23}) \\ &\qquad +\mathcal{P}(2,3,4)\Big]\,, \end{aligned} \tag{4.1.7}$$

where $\mathcal{I}_4^{(2),(P,NP)}$ are the scalar planar and non-planar double box integrals , and $C_{1234}^{(P,NP)}$ are color factors constructed from structure constant vertices.

The complete two-loop four-point amplitude in $\mathcal{N}=8$ supergravity is found simply by squaring the prefactors in 4.1.7 (and removing the color factors):

$$\begin{aligned} M_4^{(2)} &= -i(s_{12}s_{23}A_4^{tree})^2\Big[s_{12}^2\,\mathcal{I}_4^{(2),P}(s_{12},s_{23}) + s_{12}^2\,\mathcal{I}_4^{(2),NP}(s_{12},s_{23}) + \mathcal{P}(2,3,4)\Big] \\ &= s_{12}s_{23}s_{13}M_4^{tree}\Big[s_{12}^2\,\mathcal{I}_4^{(2),P}(s_{12},s_{23}) + s_{12}^2\,\mathcal{I}_4^{(2),NP}(s_{12},s_{23}) + \mathcal{P}(2,3,4)\Big]. \end{aligned} \tag{4.1.8}$$

4.1.2 BCJ Color Duality

Here we will review a conjectured duality between color and kinematics, discovered by Bern-Carrasco-Johansson (BCJ) [241]. To understand the proposed duality, we first rearrange an L-loop amplitude with all particles in the adjoint representation into the form,

$$(-i)^L \mathcal{A}_n^{loop} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}\,, \tag{4.1.9}$$

where the sum runs over the set of n-point L-loop diagrams with only cubic vertices, S_j are the symmetry factors, c_i are the color factors and n_i are kinematic numerator factors.

According to the color-kinematics duality proposal of [241], arrangements of the diagrammatic numerators in 4.1.9 exist such that they satisfy equations in one-to-one correspondence with the color Jacobi identities. That is, for every color Jacobi identity we have a relation between kinematic numerators,

$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k\,. \tag{4.1.10}$$

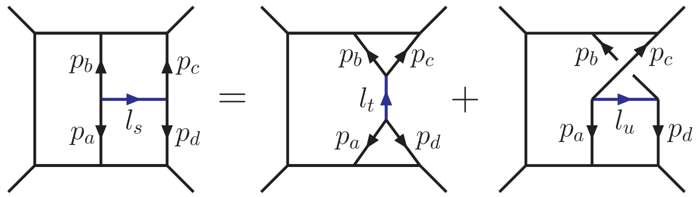


Figure 4.1: A numerator duality relation at three loops. The relation is either for the color factors or for the diagram numerators.

For example, as illustrated in Fig. 4.1, the numerators of the three displayed diagrams satisfy a similar equation as satisfied by the color factors of the diagram.

Perhaps more remarkable than the duality itself, is a related conjecture that once the gauge-theory amplitudes are arranged into a form satisfying the duality (4.1.10), corresponding gravity amplitudes can be obtained simply by taking a double copy of gauge-theory numerator factors [241],

$$(-i)^{L+1}\mathcal{M}_n^{loop} \;=\; \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \nu_j}{\prod_{\alpha_j} p_{\alpha_j}^2}\,, \qquad (4.1.11)$$

where the ν_i represent numerator factors of a second gauge theory amplitude, the sum runs over the same set of diagrams as in 4.1.9. We suppressed the gravitational coupling constant in this expression.

This is expected to hold in a large class of gravity theories, including theories that are the low-energy limits of string theories. It should also hold in pure gravity, but in this case extra projectors would be required to remove the unwanted states arising in the direct product of two pure Yang-Mills theories. At tree level ($L = 0$), this double-copy property is closely related to the KLT relations between gravity and gauge theory as mentioned above.

4.2 Tree Formula for Graviton Scattering

The past several years have witnessed tremendous progress in our understanding of the mathematical structure of scattering amplitudes, particularly in maximally supersymmetric theories. It is easy to argue that the seeds of this progress were sown over two decades ago by the discovery [164, 165] of the stunningly simple formula (Here in this section we use calligraphic letters $\mathcal{A}$, $\mathcal{M}$ to denote superspace amplitudes with the overall delta-function of supermomentum conservation suppressed.)

$$\mathcal{A}^{\text{MHV}}(1, \ldots, n) = \frac{1}{\langle 1\,2\rangle\langle 2\,3\rangle \cdots \langle n\,1\rangle} \tag{4.2.1}$$

for the MHV color-ordered subamplitude for n-gluon scattering. The importance of this formula goes far beyond simply knowing the answer for a certain scattering amplitude, which one may or may not be particularly interested in. Rather, the mere existence of such a simple formula for something which would normally require enormously tedious calculations using traditional Feynman diagram techniques suggests firstly that the theory must possess some remarkable and deeply hidden mathematical structure, and secondly that if one actually is interested in knowing the answer for a certain amplitude it behooves one to discover and understand this structure. In other words, the formula (4.2.1) is as important psychologically as it is physically, since it provides strong motivation for digging more deeply into scattering amplitudes.

Much of the progress on gluon amplitudes can be easily recycled and applied to graviton amplitudes due ultimately to the KLT relations [166], as mentioned above, which roughly speaking state that "gravity is Yang-Mills squared".

There are several indications that maximal supergravity may be an extraordinarily remarkable theory [168, 169, 170, 171, 172, 173, 174, 175, 176, 177], and possibly even ultraviolet finite [178, 179, 180, 181, 182, 185], but our feeling is that even at tree level we are still far from fully unlocking the structure of graviton amplitudes.

The original BGK (Berends, Giele and Kuijf) formula for the n-graviton MHV amplitude [186] is now over 20 years old. For later convenience we review here a different form due to Mason and Skinner [187], who proved the equivalence of the original BGK formula to the expression

$$\mathcal{M}_n^{\text{MHV}} = \sum_{P(1,\ldots,n-3)} \frac{1}{\langle n\, n{-}2\rangle\langle n{-}2\, n{-}1\rangle\langle n{-}1\, n\rangle} \frac{1}{\langle 1\, 2\rangle \cdots \langle n\, 1\rangle} \prod_{k=1}^{n-3} \frac{[k|p_{k+1} + \cdots + p_{n-2}|n{-}1\rangle}{\langle k\, n{-}1\rangle}\,, \tag{4.2.2}$$

where the sum indicates a sum over all $(n-3)!$ permutations of the labels $1, \ldots, n-3$ and we use the convention

$$[a|p_i + p_j + \cdots |b\rangle = [a\, i]\langle i\, b\rangle + [a\, j]\langle j\, b\rangle + \cdots\,. \tag{4.2.3}$$

The fact that any closed form expression exists at all for this quantity, the calculation of which would otherwise be vastly more complicated even than the corresponding one for n gluons, is an amazing achievement. Nevertheless the formula has some features which strongly suggest that it is not the end of the story.

First of all, the formula (4.2.2) does not manifest the requisite permutation symmetry of an n-graviton superamplitude. Specifically, any superamplitude $\mathcal{M}_n$ must be fully symmetric under all $n!$ permutations of the labels $1, \ldots, n$ of the external particles, but only an S_{n-3} subgroup of this symmetry is manifest in (4.2.2) (several formulas which manifest a slightly larger S_{n-2} subgroup are known [188, 189]). Of course one can check, numerically if necessary, that (4.2.2) does in fact have this symmetry, but it is far from obvious. Moreover, even the S_{n-3} symmetry arises in a somewhat contrived way, via an explicit sum over permutations. Undoubtedly the summand in (4.2.2) contains redundant information which is washed out by taking the sum. This situation should be contrasted with that of Yang-Mills theory, where (4.2.1) is manifestly invariant under the appropriate dihedral symmetry group (not the full permutation group, due to the color ordering of gluons).

Secondly, one slightly disappointing feature of all previously known MHV formulas including (4.2.2) is the appearance of "$\cdots$", which indicates that a particular cyclic ordering of the particles must be chosen in order to write the formula, even though a graviton amplitude ultimately cannot depend on any such ordering since gravitons do not carry any color labels. This vestigial feature usually traces back to the use of the KLT relations to calculate graviton amplitudes by recycling gluon amplitudes.

An important feature of graviton amplitudes is that they fall off like $1/z^2$ as the supermomenta of any two particles are taking to infinity in a particular complex direction (see [188, 190, 191, 192, 193, 194, 195] and [173] for the most complete treatment), unlike in Yang-Mills theory where the falloff is only $1/z$ [196]. It has been argued [193] that this exceptionally soft behavior of graviton tree amplitudes is of direct importance for the remarkable ultraviolet cancellations in supergravity loop amplitudes [178, 197, 198, 199, 200].

The $1/z^2$ falloff of (4.2.2) is manifest for each term separately inside the sum over permutations. Two classes of previously known formulas for the n-graviton MHV amplitude are: those like (4.2.2) which manifest the $1/z^2$ falloff but only S_{n-3} symmetry, and others (see for example [188, 189]) which have a larger S_{n-2} symmetry but only manifest falloff like $1/z$. In the latter class of formulas the stronger $1/z^2$ behavior arises from delicate and non-obvious cancellations between various terms in the sum. This is both a feature and a bug. It is a feature because it implies the existence of linear identities (which have been called bonus relations in [201]) between individual terms in the sum which have proven useful, for example, in establishing the equality of various previously known but not obviously equivalent formulas [201]. But it is a bug because it indicates that the S_{n-2}-invariant formulas contain redundant information distributed amongst the various terms in the sum. The bonus relations allow one to squeeze this redundant information out of any S_{n-2}-invariant formula at the cost of reducing the manifest symmetry to S_{n-3}.

It is difficult to imagine that it might be possible to improve upon the Parke-Taylor formula (4.2.1) for the n-gluon MHV amplitude. However, for the reasons just reviewed, we feel that (4.2.2) cannot be the end of the story for gravity. Ideally one would like to have a formula for n-graviton scattering that (1) is manifestly S_n symmetric without the need for introducing an explicit sum over permutations to impose the symmetry *vi et armis*; (2) makes no vestigial reference to any cyclic ordering of the n gravitons, and (3) manifests $1/z^2$ falloff term by term, making it unsqueezable by the bonus relations.

In this section we present and prove the "tree formula" (4.2.4) for the MHV scattering amplitude which addresses the second and third points but only manifests S_{n-2} symmetry [2].

There is an ansatz for the MHV graviton amplitude presented in section 6 of [231] which upon inspection is immediately seen to share the nice features of the tree formula. In fact, although terms in the two formulas are arranged in different ways (labeled tree diagrams versus Young tableaux), it is not difficult to check that their content is actually identical. Interestingly the formula of [231] was constructed with the help of "half-soft factors" similar in idea to the "inverse soft limits" which appeared much more recently in [203]. Our work establishes the validity of the ansatz conjectured in [231] and demonstrates that it arises naturally in twistor space.

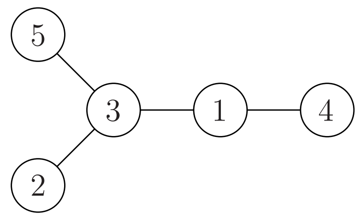
4.2.1 The MHV Tree Formula

Statement of the Tree Formula

Here we introduce a formula for the n-graviton MHV scattering amplitude which we call the "tree formula" since it consists of a sum of terms, each of which is conveniently represented by a tree diagram [2]. The tree formula manifests an S_{n-2} subgroup of the full permutation group. For the moment we choose to treat particles $n-1$ and n as special. With this arbitrary choice the formula is:

$$\mathcal{M}_n^{\mathrm{MHV}} = \frac{1}{\langle n-1\, n\rangle^2} \sum_{\mathrm{trees}} \left(\prod_{\mathrm{edges}\, ab} \frac{[a\, b]}{\langle a\, b\rangle} \right) \left(\prod_{\mathrm{vertices}\, a} \left(\langle a\, n-1\rangle \langle a\, n\rangle\right)^{\deg(a)-2} \right). \quad (4.2.4)$$

To write down an expression for the n-point amplitude one draws all inequivalent connected tree graphs with vertices labeled $1, 2, \ldots, n-2$. (It was proven by Cayley that there are precisely $(n-2)^{n-4}$ such diagrams.) For example, one of the 125 labeled tree graphs contributing to the $n = 7$ graviton amplitude is



According to (4.2.4) the value of a diagram is then the product of three factors:

1. an overall factor of $1/\langle n{-}1\, n\rangle^2$,
2. a factor of $[a\, b]/\langle a\, b\rangle$ for each propagator connecting vertices a and b, and
3. a factor of $(\langle a\, n{-}1\rangle\langle a\, n\rangle)^{\deg(a)-2}$ for each vertex a, where $\deg(a)$ is the degree of the vertex (the number of edges attached to it).

An alternate description of the formula may be given by noting that a vertex factor of $\langle a\, n-1\rangle\langle a\, n\rangle$ may be absorbed into each propagator connected to that vertex. This leads to the equivalent formula

$$\mathcal{M}_n^{\mathrm{MHV}} = \frac{1}{\langle n{-}1\, n\rangle^2}\left(\prod_{a=1}^{n-2}\frac{1}{(\langle a\, n{-}1\rangle\langle a\, n\rangle)^2}\right)\sum_{\text{trees}}\prod_{\text{edges}\, ab}\frac{[a\, b]}{\langle a\, b\rangle}\langle a\, n{-}1\rangle\langle b\, n{-}1\rangle\langle a\, n\rangle\langle b\, n\rangle. \tag{4.2.5}$$

Examples

We will illustrate the tree formula in some examples, having some convincingness by seeing the formula in action here for small n and by noting that it has the correct soft limits for all n, as we discuss shortly.

For each of the trivial cases $n = 3, 4$ there is only a single tree diagram,

$$\mathcal{M}_3^{\text{MHV}} = \textcircled{1} = \frac{1}{(\langle 1\,2\rangle\langle 1\,3\rangle\langle 2\,3\rangle)^2} \tag{4.2.6}$$

and

$$\mathcal{M}_4^{\text{MHV}} = \textcircled{1}\!-\!\!-\!\textcircled{2} = \frac{[1\,2]}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 1\,4\rangle\langle 2\,3\rangle\langle 2\,4\rangle\langle 3\,4\rangle^2} \tag{4.2.7}$$

respectively, which immediately reproduce the correct expressions.

For $n = 5$ there are three tree diagrams

$$\begin{aligned} \textcircled{1}\!-\!\!-\!\textcircled{2}\!-\!\!-\!\textcircled{3} &= \frac{[1\,2][2\,3]}{\langle 1\,2\rangle\langle 1\,4\rangle\langle 1\,5\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 4\,5\rangle^2} \\ \textcircled{1}\!-\!\!-\!\textcircled{3}\!-\!\!-\!\textcircled{2} &= \frac{[1\,3][2\,3]}{\langle 1\,3\rangle\langle 1\,4\rangle\langle 1\,5\rangle\langle 2\,3\rangle\langle 2\,4\rangle\langle 2\,5\rangle\langle 4\,5\rangle^2} \\ \textcircled{2}\!-\!\!-\!\textcircled{1}\!-\!\!-\!\textcircled{3} &= \frac{[1\,2][1\,3]}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 2\,4\rangle\langle 2\,5\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 4\,5\rangle^2} \end{aligned} \tag{4.2.8}$$

which can easily be verified by hand to sum to the correct expression. Agreement between the tree formula and other known formulas such as (4.2.2) may be checked numerically for slightly larger values of n by assigning random values to all of the spinor helicity variables. A simple implementation of the tree formula in the Mathematica symbolic computation language is given as follows:

```
Needs["Combinatorica‘"];
MHV[n_Integer]/;n>4 := 1/ket[n-1,n]^2 1/(Times @@ ((ket[n-1,#] ket[n,#])^2 & /@ Range[n-2])) ((Times @@ (Transpose[#]/.{a___,1,b___,1,c___} :> prop[Length[{a}]+1,Length[{a,b}]+2])) & /@ IncidenceMatrix /@ CodeToLabeledTree /@ Flatten[Outer[List,Sequence @@ Table[Range[n-2],{n-4}]],n-5]) /. prop[a_,b_] ->
```

```
bra[a,b]/ket[a,b] ket[n-1,a] ket[n-1,b] ket[n,a] ket[n,b];
```

Here we use the notation $\texttt{ket}[a,b] = \langle a\,b\rangle$ and $\texttt{bra}[a,b] = [a\,b]$. The (trivial) cases $n = 3, 4$ must be handled separately.

Relation to Other Known Formulas

The MHV tree formula is evidently quite different in form from most other expressions in the literature. In particular, no reference at all is made to any particular ordering of the particles (there is no vestigial "$\cdots$"), and the manifest S_{n-2} arises not because of any explicit sum over $P(1,\ldots,n-2)$ but rather from the simple fact that the collection of labeled tree diagrams has a manifest S_{n-2} symmetry. In our view these facts serve to highlight the essential "gravitiness" of the formula, in contrast to expressions such as (4.2.2) which are ultimately recycled from Yang-Mills theory.

One interesting feature of the MHV tree formula is that it is, in a sense, minimally non-holomorphic. Graviton MHV amplitudes, unlike their Yang-Mills counterparts, do not depend only the holomorphic spinor helicity variables λ_i. The tree formula packages all of the non-holomorphicity into the $[a\,b]$ factors associated with propagators in the tree diagrams. Each diagram has a unique collection of propagators and a correspondingly unique signature of $[\,]$'s, which only involve $n-2$ of the n labels.

Like the MHV tree formula, the Mason-Skinner formula (4.2.2) (unlike most other formulas in the literature, including the original BGK formula) has non-holomorphic dependence on only $n-2$ variables. In our labeling of (4.2.2) we see that $\widetilde{\lambda}_{n-1}$ and

$\widetilde{\lambda}_n$ do not appear at all. Of course we do not mean to say that $\mathcal{M}$ is "independent" of these two variables since there is a suppressed overall delta function of momentum conservation $\delta^4(\sum_i \lambda_i \widetilde{\lambda}_i)$ which one could use to shuffle some $\widetilde{\lambda}$'s into others. Rather we mean that the tree and MS formulas have the property that all appearance of two of the $\widetilde{\lambda}$'s has already been completely shuffled out.

It is an illuminating exercise to attempt a direct term-by-term comparison of the MHV tree formula with the MS formula (4.2.2). For the first non-trivial case $n = 5$ the MS formula provides the two terms

$$\frac{[2\,3][1|p_2 + p_3|4\rangle}{\langle 1\,2\rangle\langle 1\,4\rangle\langle 1\,5\rangle\langle 2\,3\rangle\langle 2\,4\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 4\,5\rangle^2} - \frac{[1\,3][2|p_1 + p_3|4\rangle}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 1\,4\rangle\langle 2\,4\rangle\langle 2\,5\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 4\,5\rangle^2}. \tag{4.2.9}$$

If we now expand out the bracket $[a|p_i + p_j|b\rangle = [a\,i]\langle i\,b\rangle + [a\,j]\langle j\,b\rangle$ then we find four terms: one of them is proportional to $[1\,2][2\,3]$ and is identical to the first line in (4.2.8), another proportional to $[1\,2][1\,3]$ is identical to the last line in (4.2.8). The remaining two terms are both proportional to $[1\,3][2\,3]$ and may be combined as

$$\frac{[1\,3][2\,3]\left(\langle 1\,3\rangle\langle 2\,5\rangle - \langle 1\,5\rangle\langle 2\,3\rangle\right)}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 1\,4\rangle\langle 1\,5\rangle\langle 2\,3\rangle\langle 2\,4\rangle\langle 2\,5\rangle\langle 3\,5\rangle\langle 4\,5\rangle^2} \tag{4.2.10}$$

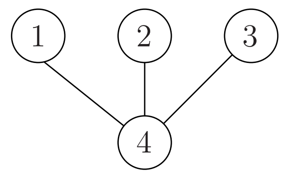
which with the help of a Schouten identity we recognize as precisely the second line in (4.2.8).

And we will give one more final example. Expanding the MS formula for $n = 6$ into $\langle\,\rangle$'s and $[\,]$'s yields a total of 36 terms. For example there are 6 terms proportional to the antiholomorphic structure $[1\,4][2\,4][3\,4]$, totalling

$$\frac{[1\,4][2\,4][3\,4]\langle 4\,5\rangle}{\langle 1\,5\rangle\langle 2\,5\rangle\langle 3\,5\rangle\langle 4\,6\rangle\langle 5\,6\rangle^2}\left[\frac{1}{\langle 1\,2\rangle\langle 1\,6\rangle\langle 2\,3\rangle\langle 3\,4\rangle}+\frac{1}{\langle 1\,2\rangle\langle 1\,4\rangle\langle 2\,3\rangle\langle 3\,6\rangle}-\frac{1}{\langle 1\,3\rangle\langle 1\,6\rangle\langle 2\,3\rangle\langle 2\,4\rangle}\right.$$
$$\left.-\frac{1}{\langle 1\,3\rangle\langle 1\,4\rangle\langle 2\,3\rangle\langle 2\,6\rangle}-\frac{1}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 2\,6\rangle\langle 3\,4\rangle}-\frac{1}{\langle 1\,2\rangle\langle 1\,3\rangle\langle 2\,4\rangle\langle 3\,6\rangle}\right]. \quad (4.2.11)$$

After repeated use of Schouten identities this amazingly collapses to the single term

$$\frac{[1\,4][2\,4][3\,4]\langle 4\,5\rangle\langle 4\,6\rangle}{\langle 1\,4\rangle\langle 1\,5\rangle\langle 1\,6\rangle\langle 2\,4\rangle\langle 2\,5\rangle\langle 2\,6\rangle\langle 3\,4\rangle\langle 3\,5\rangle\langle 3\,6\rangle\langle 5\,6\rangle^2} =$$



(4.2.12)

We believe that these examples are representative of the general case. Expanding out all of the brackets in the n-graviton MS formula generates a total of $[(n-3)!]^2$ terms, but there are only $(n-2)^{n-4}$ possible distinct antiholomorphic signatures. Collecting terms with the same signature and repeatedly applying Schouten identities should collapse everything into the terms generated by the MHV tree formula. Note that this is a huge simplification: $(n-2)^{n-4}$ is smaller than $[(n-3)!]^2$ by a factor that is asymptotically n^n. We certainly do not have an explicit proof of this cancellation; instead we are relying on fact that the MS formula and the tree formula are separately proven to be correct in order to infer how the story should go.

To conclude this discussion we should note that we are exploring here only the structure of the various formulas, not making any claims about the computational complexity of the MHV tree formula as compared to (4.2.2) or any other known formula. No practical implementation of the MS formula would proceed by first splitting all of the brackets as we have outlined. Indeed a naive counting of the number of terms, $(n-3)!$ in (4.2.2) versus $(n-2)^{n-4}$ for the tree formula, suggests that for computational purposes the former is almost certainly the clear winner despite the conceptual strengths of the latter.

Soft Limit of the Tree Formula

Let us consider for a moment the component amplitude

$$M(1^+,\ldots,(n-2)^+,(n-1)^-,n^-) = \langle n-1\, n\rangle^8 \mathcal{M}_n^{\mathrm{MHV}} \tag{4.2.13}$$

with particles $n-1$ and n having negative helicity. The universal soft factor for gravitons is [186, 202]

$$\lim_{p_1\to 0} \frac{M(1^+,\ldots,(n-2)^+,(n-1)^-,n^-)}{M(2^+,\ldots,(n-2)^+,(n-1)^-,n^-)} = \sum_{i=2}^{n-2} g(i^+), \quad g(i^+) = \frac{\langle i\, n-1\rangle}{\langle 1\, n-1\rangle}\frac{\langle i\, n\rangle}{\langle 1\, n\rangle}\frac{[1\, i]}{\langle 1\, i\rangle}. \tag{4.2.14}$$

It is simple to see that the MHV tree formula satisfies this property: the tree diagrams which do not vanish in the limit $p_1 \to 0$ are those in which vertex 1 is connected by a propagator to a single other vertex i. Such diagrams remain connected when vertex 1 is chopped off, leaving a contribution to the $n-1$-graviton amplitude times the indicated factor $g(i^+)$.

Thinking about this process in reverse therefore suggests a simple interpretation of (4.2.14) in terms of tree diagrams—it is a sum over all possible places i where the vertex 1 may be attached to the $n-1$-graviton amplitude. This structure is exactly that of the "inverse soft factors" suggested recently in [203], and we have checked that the MHV tree formula may be built up by recursively applying the rule proposed there.

4.2.2 The MHV Tree Formula in Twistor Space

Before turning to the formal proof of the tree formula in the next section, here we work out the link representation of the MHV graviton amplitude in twistor space, which was one of the steps which led to the discovery of the tree formula. Two papers [204, 205] have recently constructed versions of the BCF on-shell recursion relation directly in twistor space variables. We follow the standard notation where μ, $\widetilde{\mu}$ are respectively Fourier transform conjugate to the spinor helicity variables λ, $\widetilde{\lambda}$, and assemble these together with a four-component Grassmann variable η and its conjugate $\widetilde{\eta}$ into the 4|8-component supertwistor variables

$$\mathcal{Z} = \begin{pmatrix} \lambda \\ \mu \\ \eta \end{pmatrix}, \qquad \mathcal{W} = \begin{pmatrix} \widetilde{\mu} \\ \widetilde{\lambda} \\ \widetilde{\eta} \end{pmatrix}. \tag{4.2.15}$$

In the approach of [205], in which variables of both chiralities $\mathcal{Z}$ and $\mathcal{W}$ are used simultaneously, an apparently important role is played by the link representation which expresses an amplitude $\mathcal{M}$ in the form

$$\mathcal{M}(\mathcal{Z}_i, \mathcal{W}_J) = \int dc\, U(c_{iJ}, \lambda_i, \widetilde{\lambda}_J) \exp\left[i \sum_{i,J} c_{iJ} \mathcal{Z}_i \cdot \mathcal{W}_J \right]. \tag{4.2.16}$$

Here one splits the n particles into two groups, one of which (labeled by i) one chooses to represent in $\mathcal{Z}$ space and the other of which (labeled by J) one chooses to represent in $\mathcal{W}$ space. The integral runs over all of the aptly-named link variables c_{iJ} and we refer to the integrand $U(c_{iJ}, \lambda_i, \widetilde{\lambda}_J)$ as the link representation of $\mathcal{M}$. It was shown in [205] that the BCF on-shell recursion in twistor space involves nothing more than a simple integral over $\mathcal{Z}$, $\mathcal{W}$ variables with a simple (and essentially unique) measure factor.

The original motivation for our investigation was to explore the structure of link representations for graviton amplitudes. We will always adopt the convenient convention of expressing an N^{k}MHV amplitude in terms of $k+2$ $\mathcal{Z}$ variables and $n-k-2$ $\mathcal{W}$ variables. The three-particle MHV and $\overline{\text{MHV}}$ amplitudes

$$U_3^{\text{MHV}} = \frac{|\langle 1\,2\rangle|}{c_{13}^2 c_{23}^2}, \qquad U_3^{\overline{\text{MHV}}} = \frac{|[1\,2]|}{c_{31}^2 c_{32}^2} \tag{4.2.17}$$

seed the on-shell recursion, which is then sufficient (in principle) to determine the link representation for any desired amplitude.

For example, the four-particle amplitude is the sum of two contributing BCF diagrams

$$U_4^{\text{MHV}} = \frac{\langle 1\,2\rangle[3\,4]}{c_{13}^2 c_{24}^2 c_{12:34}} + \frac{\langle 1\,2\rangle[3\,4]}{c_{13}^2 c_{24}^2 c_{14} c_{23}} \tag{4.2.18}$$

where we use the notation

$$c_{i_1 i_2 : J_1 J_2} = c_{i_1 J_1} c_{i_2 J_2} - c_{i_1 J_2} c_{i_2 J_1}. \tag{4.2.19}$$

Remarkably the two terms in (4.2.18) combine nicely into the simple result presented already in [205]:

$$U_4^{\rm MHV} = \frac{\langle 1\,2\rangle[3\,4]}{c_{13}c_{14}c_{23}c_{24}c_{12:34}}. \tag{4.2.20}$$

This simplification seems trivial at the moment but it is just the tip of an iceberg. For larger n the enormous simplifications discussed in the previous section, which are apparently non-trivial in physical space, occur automatically in the link representation.

For example the five particle MHV amplitude is the sum of three BCF diagrams,

$$U_5^{\rm MHV} = \left\{ \frac{|\langle 1\,2\rangle|[4\,5](c_{24}[3\,4]+c_{25}[3\,5])}{c_{13}c_{23}c_{14}c_{25}^2c_{12:34}c_{12:45}} + (3 \leftrightarrow 4)\right\} + \frac{|\langle 1\,2\rangle|[3\,4](c_{24}[4\,5]+c_{23}[3\,5])}{c_{13}c_{14}c_{15}c_{23}c_{24}c_{25}^2c_{12:34}} \tag{4.2.21}$$

which nicely simplifies to

$$\frac{1}{|\langle 1\,2\rangle|}U_5^{\rm MHV} = \frac{[3\,4][4\,5]}{c_{13}c_{15}c_{23}c_{25}c_{12:34}c_{12:45}} + \frac{[3\,5][4\,5]}{c_{13}c_{14}c_{23}c_{24}c_{12:35}c_{12:45}} + \frac{[3\,4][3\,5]}{c_{14}c_{15}c_{24}c_{25}c_{12:34}c_{12:35}}. \tag{4.2.22}$$

This expression already exhibits the structure of the MHV tree formula (except that here particles 1 and 2 are singled out, and the vertices of the trees are labeled by $\{3,4,5\}$).

Subsequent investigations for higher n reveal the general pattern which is as follows. Returning to the convention where particles $n-1$ and n are treated as special, the link representation for any desired MHV amplitude may be written down by drawing all tree diagrams with vertices labeled by $\{1,\ldots,n-2\}$ and then assigning

1. an overall factor of $\langle n-1\,n\rangle \mathrm{sign}(\langle n-1\,n\rangle)^n$,

2. for each propagator connecting nodes a and b, a factor of $[a\,b]/c_{n-1,n:a,b}$,

3. for each vertex a, a factor of $(c_{n-1,a}c_{n,a})^{\deg(a)-2}$, where $\deg(a)$ is the degree of the vertex labeled a.

It is readily verified by direct integration over the link variables that these rules are precisely the link-space representation of the physical space rules for the MHV tree formula given in the previous section.

4.2.3 Proof of the MHV Tree Formula

Here we present a proof of the MHV tree formula. One way one might attempt to prove the formula would be to show directly that it satisfies the BCF on-shell recursion relation [196, 209] for gravity [188, 190, 192], but the structure of the formula is poorly suited for this task. Instead we proceed by considering the usual BCF deformation of the formula $M_n^{\rm MHV}$ by a complex parameter z and demonstrating that $M_n^{\rm MHV}(z)$ has the same residue at every pole (and behavior at infinity) as the similarly deformed graviton amplitude, thereby establishing equality of the two for all z.

In this section we return to singling out particles 1 and 2, letting the vertices in the tree diagrams carry the labels $\{3,\ldots,n\}$. Then the MHV tree formula (4.2.4) can be written as

$$M_n^{\rm MHV} = \langle 1\,2\rangle^6 \sum_{\rm trees} \frac{[\,]\cdots[\,]}{\langle\,\rangle\cdots\langle\,\rangle} \prod_{a=3}^{n} \left(\langle 1\,a\rangle\langle 2\,a\rangle\right)^{\deg(a)-2} \tag{4.2.23}$$

(note that we continue to work with the component amplitude (4.2.13)) where the factors $[\,]\cdots[\,]/\langle\,\rangle\cdots\langle\,\rangle$ associated with the propagators of a diagram are independent of 1 and 2. Let us now make the familiar BCF shift [196]

$$\lambda_1 \to \lambda_1(z) = \lambda_1 - z\lambda_2, \qquad \widetilde{\lambda}_2 \to \widetilde{\lambda}_2(z) = \widetilde{\lambda}_2 + z\widetilde{\lambda}_1 \tag{4.2.24}$$

which leads to the z-deformed MHV tree formula

$$M_n^{\text{MHV}}(z) = \langle 1\,2\rangle^6 \sum_{\text{trees}} \frac{[\,]\cdots[\,]}{\langle\,\rangle\cdots\langle\,\rangle} \prod_{a=3}^{n} [(\langle 1\,a\rangle - z\langle 2\,a\rangle)\langle 2\,a\rangle]^{\deg(a)-2}\,. \tag{4.2.25}$$

Here we are in a position to observe a nice fact: since each tree diagram is connected, the degrees satisfy the sum rule

$$\sum_{a=3}^{n}(\deg(a) - 2) = -2, \tag{4.2.26}$$

which guarantees that each individual term in (4.2.25) manifestly behaves like $1/z^2$ at large z. This exceptionally soft behavior of graviton amplitudes is completely hidden in the usual Feynman diagram expansion.

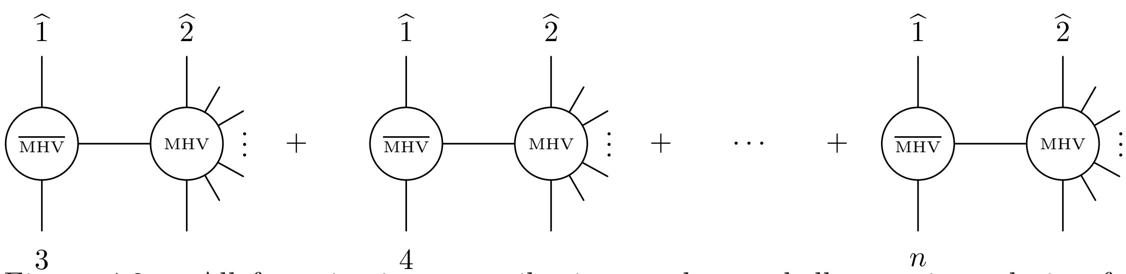


Figure 4.2: All factorizations contributing to the on-shell recursion relation for the n-point MHV amplitude. Only the first diagram contributes to the residue at $z = \langle 1\,3\rangle/\langle 2\,3\rangle$.

A complex function of a single variable which vanishes at infinity is uniquely determined by the locations of its poles as well as its residues. Having noted that (4.2.25) has the correct behavior at large z, we can conclude the proof of the MHV tree formula by demonstrating that (4.2.25) has precisely the expected residues at all of its poles. In order to say what the expected residues are we shall use induction on n. As discussed above the tree formula is readily verified for sufficiently small n, so let us assume that it has been established up through $n-1$. We can then use BCF on-shell recursion (whose terms are displayed graphically in 4.2) to determine what the residues in the deformed n-point amplitude ought to be.

Without loss of generality let us consider just the pole at $z = z_3 \equiv \langle 1\,3\rangle/\langle 2\,3\rangle$. The only tree diagrams which contribute to the residue at this pole are those with $\deg(3) = 1$, meaning that the vertex labeled 3 is connected to the rest of the diagram by a single propagator. Chopping off vertex 3 gives a subdiagram with vertices labeled $\{4, \ldots, n\}$. Clearly all diagrams which contribute to this residue can be generated by first considering the collection of tree diagrams with vertices labeled $\{4, \ldots, n\}$ and then attaching vertex 3 in all possible ways to the $n-3$ vertices of the subdiagram. We therefore have

$$M_n^{\mathrm{MHV}}(z) \sim \langle 1\,2\rangle^6 \sum_{\text{subdiagrams}} \frac{[\,]\cdots[\,]}{\langle\,\rangle\cdots\langle\,\rangle} \left(\sum_{b=4}^{n} \frac{[3\,b]}{\langle 3\,b\rangle} \langle \hat{1}\,b\rangle\langle 2\,b\rangle \right) \frac{1}{\langle \hat{1}\,3\rangle\langle 2\,3\rangle} \prod_{a=4}^{n} \left(\langle \hat{1}\,a\rangle\langle 2\,a\rangle \right)^{\deg(a)-2} \tag{4.2.27}$$

where $\sim$ denotes that we have dropped terms which are nonsingular at $z = z_3$, the sum over b runs over all the places where vertex 3 can be attached to the subdiagram, and $[\,]\cdots[\,]/\langle\,\rangle\cdots\langle\,\rangle$ indicates all edge factors associated the subdiagram, necessarily independent of 3. Using the Schouten identity we find that $\langle \hat{1}\,b\rangle = \langle 1\,2\rangle\langle b\,3\rangle/\langle 2\,3\rangle$

so we have after a couple of simple steps (and using (4.2.26))

$$M_n^{\mathrm{MHV}}(z) \sim \langle 1\,2\rangle^6 \frac{[1\,3]}{\langle 1\,3\rangle - z\langle 2\,3\rangle} \sum_{\text{subdiagrams}} \frac{[\,]\cdots[\,]}{\langle\,\rangle\cdots\langle\,\rangle} \prod_{a=4}^{n} \left(\langle 2\,a\rangle\langle 3\,a\rangle\right)^{\deg(a)-2}. \quad (4.2.28)$$

On the other hand we know from the on-shell recursion for the n-point amplitude that the residue at $z = z_3$ comes entirely from the first BCFW diagram in 4.2, whose value is

$$M_3^{\overline{\mathrm{MHV}}}(z_3) \times \frac{1}{P^2(z)} \times M_{n-1}^{\mathrm{MHV}}(z_3) \quad (4.2.29)$$

where

$$P(z) = p_1 + p_3 - z\lambda_2\widetilde{\lambda}_1. \quad (4.2.30)$$

Assuming the validity of the MHV tree formula for the $n-1$-point amplitude on the right, the expression (4.2.29) evaluates to

$$\frac{[\widehat{P}\,3]^6}{[3\,1]^2[1\,\widehat{P}]^2} \times \frac{1}{[1\,3](\langle 1\,3\rangle - z\langle 2\,3\rangle)} \times \langle \widehat{P}\,2\rangle^6 \sum_{\text{subdiagrams}} \frac{[\,]\cdots[\,]}{\langle\,\rangle\cdots\langle\,\rangle} \prod_{a=4}^{n} \Big(\langle \widehat{P}\,a\rangle\langle 2\,a\rangle\Big)^{\deg(a)-2} \quad (4.2.31)$$

where $\widehat{P} = P(z_3)$. After simplifying this result with the help of (4.2.30) we find precise agreement with (4.2.27), thereby completing the proof of the MHV tree formula.

4.2.4 Discussion and Open Questions

The tree formula introduced in here has several conceptually satisfying features and almost completely fulfills the wish-list outlined at the beginning. It appears to be a genuinely gravitational formula, rather than a recycled Yang-Mills result. Is it, finally, the end of the story for the the MHV amplitude, as the Parke-Taylor formula (4.2.1) surely is for the n-gluon MHV amplitude?

Among the wish-list items the MHV tree formula fails only in manifesting the full S_n symmetry. Of course it is possible that there simply does not exist any natural more primitive formula which manifests the full symmetry. It is not obvious how one could go about constructing such a formula, but we can draw some encouragement and inspiration from the recent paper [210] which demonstrates how to write manifestly dihedral symmetric formulas for NMHV amplitudes in Yang-Mills theory as certain volume integrals in twistor space. Different ways of dividing the volume into tetrahedra give rise to apparently different but equivalent formulas for NMHV amplitudes. The same goal can apparently also be achieved by writing the amplitude as a certain contour integral where different choices of contour produce different looking but actually equivalent formulas [211, 212]. Perhaps in gravity even the MHV amplitude needs to be formulated in a way which is fundamentally symmetric but which nevertheless requires choosing two of the n gravitons for special treatment.

In Yang-Mills theory the only formula we know of which manifests the full dihedral symmetry for all superamplitudes is the connected prescription [213, 214, 215, 216, 217] which follows from Witten's formulation of Yang-Mills theory as a twistor string theory [218]. Perhaps finding fully S_n symmetric formulas for graviton superamplitudes requires the construction of an appropriate twistor string theory for supergravity, an important question in its own right which has attracted some attention [187, 219, 220, 221, 222]. An important motivation for Witten's twistor string theory was provided by Nair's observation [223] that the Parke-Taylor formula (4.2.1) could be computed as a current algebra correlator in a WZW model. The BGK formula (essentially (4.2.2)) can similarly be related to current correlators and vertex operators in twistor space [224], but we hope that the new MHV tree formula might provide a more appropriate starting point for this purpose and perhaps shed some more light on a twistor-string-like description for supergravity.

Another obvious avenue for future research is to investigate whether any of the advances made here can be usefully applied to non-MHV amplitudes. Unfortunately we have not yet found any very nice structure in the link representation for non-MHV graviton amplitudes. Recently in [226] it was demonstrated how to solve the on-shell recursion for all tree-level supergraviton amplitudes, following steps very similar to those which were used to solve the recursion for supersymmetric Yang-Mills [225]. In [226] a crucial role was played by what was called the graviton subamplitude, which is the summand of an n-particle graviton amplitude inside a sum over $(n-2)!$ permutations. The decomposition of every amplitude into its subamplitudes allowed for a very efficient application of the on-shell recursion since the same two legs could be singled out and shifted at each step in the recursion. Unfortunately there is no natural notion of a subamplitude for the MHV tree formula, making it very poorly suited as a starting point for attempting to solve the on-shell recursion. In our view the fact that the tree formula apparently can neither be easily derived from BCF, nor usefully used as an input to BCF, suggests the possible existence of some kind of new rules for the efficient calculation of more general gravity amplitudes.

The arrangement of supergravity amplitudes into ordered subamplitudes also proved very useful in [227, 228] for the purpose of expressing the coefficients of one-loop supergravity amplitudes in terms of one-loop Yang-Mills coefficients. It would certainly be very interesting to see if any of aspects of the MHV tree formula could be useful for loop amplitudes in supergravity, if at least as input for unitarity sums [229, 230].

4.3 Finiteness of $\mathcal{N} = 8$ Super Gravity

We will discuss briefly the finiteness of $\mathcal{N} = 8$ supergravity in this section, following the work of [197, 199, 200, 239]. Although a construction of an ultraviolet-finite, point-like quantum field theory of gravity in four dimensions is unknown, it has not been shown such a construction is impossible. It is well known that point-like quantum gravity is non-renormalizable by power counting, due to the dimensionful nature of Newton's constant, $G_N = 1/M_{\rm Pl}^2$. So far, the fact that no known symmetry can fix the divergences leads to the widespread belief of a requirement for new physics in the ultraviolet limit. String theory cures these divergences by introducing a new length scale, related to the string tension, at which particles are no longer point-like. Some open question arise naturally that: Whether a non-point-like theory is actually necessary for perturbative finiteness? Could a point-like theory of quantum gravity with enough symmetry have an ultraviolet-finite perturbative expansion? Are there any imaginable point-like completions for $\mathcal{N} = 8$ supergravity? Maybe the only completion is string theory; or maybe this cannot happen because of the impossibility of decoupling non-perturbative string states not present in $\mathcal{N} = 8$ supergravity.

There are many other proposals for making sense of quantum gravity with point-like particles. For example, the asymptotic safety program proposes that the Einstein action for gravity flows in the ultraviolet to a nontrivial, Lorentz-invariant fixed point. It has also been suggested by Horava that the ultraviolet theory could break Lorentz invariance.

The on-shell ultraviolet divergences of $\mathcal{N} = 8$ supergravity, *i.e.* those which cannot be removed by field redefinitions, can be probed by studying the ultraviolet behavior of multi-loop on-shell amplitudes for graviton scattering. Such scattering amplitudes would be very difficult to compute in a conventional framework using Feynman diagrams. However, tree amplitudes in gravity can be expressed in terms of tree amplitudes in gauge theory, by making use of the KLT and BCJ relations [240, 241]. Loop amplitudes can be constructed efficiently from tree amplitudes via generalized unitarity, particularly in theories with maximal supersymmetry. Using these methods, the four-graviton amplitude in $\mathcal{N} = 8$ supergravity has been computed at up to four loops.

In every explicit computation to date, through four loops as mentioned above, the ultraviolet behavior of $\mathcal{N} = 8$ supergravity has proven to be no worse than that of $\mathcal{N} = 4$ super-Yang-Mills theory. On the other hand, there are several recent arguments in favor of the existence of a seven-loop counterterm of the form $\mathcal{D}^8 R^4$ that leads to divergences. This makes the study of finiteness of $\mathcal{N} = 8$ supergravity is still quite open and interesting with the need of further investigation.

As a reminder, even suppose that $\mathcal{N} = 8$ supergravity turns out to be finite to all orders in perturbation theory, this result still would not prove that it is a consistent theory of quantum gravity at the non-perturbative level. There are at least two reasons to think that it might need a non-perturbative ultraviolet completion [239]:

1. The (likely) $L!$ or worse growth of the coefficients of the order L terms in

the perturbative expansion, which for fixed-angle scattering, would imply a non-convergent behavior $\sim L!\,(s/M_{\rm Pl}^2)^L$.

2. The fact that the perturbative series seems to be $E_{7(7)}$ invariant, while the mass spectrum of black holes is non-invariant.

Chapter 5

Conclusion

In this dissertation, we have studied the scattering amplitudes, especially their symmetries and dualities, in the planar limit of $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ supergravity theories.

Firstly, we study the dual superconformal symmetry in $\mathcal{N} = 4$ SYM. We have classified all four-point dual conformal Feynman diagrams through four loops. In addition to the previously known $(1, 1, 2, 8)$ integrals that contribute to the dimensionally-regulated on-shell amplitude respectively at $(1, 2, 3, 4)$ loops, we find $(0, 0, 2, 9)$ new dual conformal integrals that vanish on-shell in $D = 4-2\epsilon$ but not off-shell in $D = 4$. There are also $(0, 0, 0, 11)$ dual conformal diagrams that diverge in four dimensions even when taken off-shell and therefore do not give rise to true dual conformal integrals. We then addressed the problem of evaluating new off-shell integrals in four dimensions. Of the total number $(1, 1, 4, 17)$ of such integrals, explicit results for

$(1, 1, 2, 5)$ have appeared previously in [245, 264]. We find Mellin-Barnes representations for an additional $(0, 0, 2, 8)$ integrals, including two pairs related by new "magic identities", and evaluate their infrared singularity structure explicitly. Evaluation of the remaining $(0, 0, 0, 4)$ integrals $\mathcal{I}^{(4)f}$, $\mathcal{I}^{(4)f'}$, $\mathcal{I}^{(4)e2'}$, and $\mathcal{I}^{(4)c'}$ is left for future work.

Secondly, we investigate the Wilson loops in $\mathcal{N} = 4$ SYM at both strong and weak coupling, and their duality to MHV gluon scattering amplitudes. To our pleasant surprise we find that at two loops: the agreement between the $n = 4$ and the parity-even part of the $n = 5$ amplitude and the corresponding Wilson loop continues to hold at $\mathcal{O}(\epsilon)$ up to an additive constant which can be absorbed into various structure functions. Let us emphasize that this is a rather striking result which cannot reasonably be called a coincidence: at this order in ϵ the amplitudes and Wilson loops we compute depend on all of the kinematic variables in a highly nontrivial way, involving polylogarithmic functions of degree 5. It would be very interesting to continue exploring this miraculous agreement and to understand the reason behind it. Dual conformal invariance cannot help in this regard since the symmetry is explicitly broken in dimensional regularization so it cannot say anything about terms of higher order in ϵ, but of course as mentioned above already at $\mathcal{O}(\epsilon^0)$ there must be some mechanism beyond dual conformal invariance at work.

Finally, we study the graviton scattering amplitudes in $\mathcal{N} = 8$ supergravity. We present and prove a formula for the MHV scattering amplitude of n gravitons at tree level. Some of the more interesting features of the formula, which set it apart as being significantly different from many more familiar formulas, include the absence of any vestigial reference to a cyclic ordering of the gravitons—making it in a sense a truly gravitational formula, rather than a recycled Yang-Mills result, and the fact that it simultaneously manifests both S_{n-2} symmetry as well as large-z behavior that is $\mathcal{O}(1/z^2)$ term-by-term, without relying on delicate cancellations. The formula is seemingly related to others by an enormous simplification provided by $\mathcal{O}(n^n)$ iterated Schouten identities, but our proof relies on a complex analysis argument rather than such a brute force manipulation. We also find that the formula has a very simple link representation in twistor space, where cancellations that are non-obvious in physical space become manifest.

Appendix A

Conventions

We use the two-component spinor formalism as follows.

The dotted and undotted spinor indices are raised and lowered as follows:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \,, \qquad \bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{\dot{\beta}} \;; \tag{A.0.1}$$

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \,, \qquad \bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}} \tag{A.0.2}$$

where the antisymmetric ϵ symbols have the properties:

$$\epsilon_{12} = \epsilon_{\dot{1}\dot{2}} = -\epsilon^{12} = -\epsilon^{\dot{1}\dot{2}} = 1 \,, \qquad \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma \,, \qquad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}} \tag{A.0.3}$$

The convention for the contraction of a pair of spinor indices is

$$\psi^\alpha\lambda_\alpha \equiv \langle\psi\lambda\rangle \,, \qquad \bar{\chi}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}} \equiv [\bar{\chi}\bar{\rho}] \tag{A.0.4}$$

Two-component spinors satisfy the cyclic identity

$$\langle\psi\lambda\rangle\chi_\alpha + \langle\lambda\chi\rangle\psi_\alpha + \langle\chi\psi\rangle\lambda_\alpha = 0 \tag{A.0.5}$$

which simply means that the antisymmetrization over three two-component indices vanishes identically.

The sigma matrices σ^μ are defined as follows:

$$(\sigma^\mu)_{\alpha\dot\alpha} = (1, \vec\sigma)_{\alpha\dot\alpha}\,, \qquad (\tilde\sigma^\mu)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta} = (1, -\vec\sigma)^{\dot\alpha\alpha} \tag{A.0.6}$$

and have the basic properties:

$$\begin{aligned} &\sigma^\mu\tilde\sigma^\nu = \eta^{\mu\nu} - i\sigma^{\mu\nu}\,, \qquad \tilde\sigma^\mu\sigma^\nu = \eta^{\mu\nu} - i\tilde\sigma^{\mu\nu}\,, \\ &(\sigma^\mu)_{\alpha\dot\alpha}(\tilde\sigma_\mu)^{\dot\beta\beta} = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}\,, \qquad (\sigma_\mu)_{\alpha\dot\alpha}(\tilde\sigma^\nu)^{\dot\alpha\alpha} = 2\delta^\nu_\mu\,, \\ &\sigma^{\mu\nu} = -\sigma^{\nu\mu}\,, \qquad \tilde\sigma^{\mu\nu} = -\tilde\sigma^{\nu\mu}\,, \qquad (\sigma^{\mu\nu})_\alpha{}^\alpha = (\tilde\sigma^{\mu\nu})_{\dot\alpha}{}^{\dot\alpha} = 0 \end{aligned} \tag{A.0.7}$$

A four-vector x^μ can be written as a two-component bispinor:

$$x_{\alpha\dot\alpha} = x^\mu(\sigma_\mu)_{\alpha\dot\alpha}\,, \qquad x^{\dot\alpha\alpha} = x^\mu\tilde\sigma^{\dot\alpha\alpha}_\mu\,, \qquad x^\mu = \frac{1}{2}x^{\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\alpha} \tag{A.0.8}$$

$$x^2 = x^\mu x_\mu = \frac{1}{2}x^{\dot\alpha\alpha}x_{\alpha\dot\alpha}\,, \qquad x_{\alpha\dot\alpha}\; x^{\dot\alpha\beta} = x^2\delta^\beta_\alpha\,, \qquad x^{\dot\alpha\alpha}\; x_{\alpha\dot\beta} = x^2\delta^{\dot\alpha}_{\dot\beta} \tag{A.0.9}$$

$$(x^{-1})_{\alpha\dot\alpha} = \frac{x_{\alpha\dot\alpha}}{x^2}\,, \qquad (x^{-1})_{\alpha\dot\alpha}x^{\dot\alpha\beta} = \delta^\beta_\alpha\,, \qquad x^{\dot\alpha\alpha}(x^{-1})_{\alpha\dot\beta} = \delta^{\dot\alpha}_{\dot\beta} \tag{A.0.10}$$

Some short-hand notations:

$$\langle p|x_{mn}x_{kl}|q\rangle = \lambda_p^{\alpha}(x_{mn})_{\alpha\dot{\alpha}}(x_{kl})^{\dot{\alpha}\beta}\lambda_{q\,\beta} = -\langle q|x_{kl}x_{mn}|p\rangle$$

$$\langle px_{mn}|q] = \lambda_p^{\alpha}(x_{mn})_{\alpha\dot{\alpha}}\tilde{\lambda}_q^{\dot{\alpha}}\,, \qquad \text{etc.} \tag{A.0.11}$$

Appendix B

Conventional and Dual Superconformal Generators

In this appendix we review the conventional and dual representations of the superconformal algebra [127]. We also list here the commutation relations of the algebra $U(2,2|4)$ which would be useful to work with.

The Lorentz generators $\mathbb{M}_{\alpha\beta}$, $\overline{\mathbb{M}}_{\dot{\alpha}\dot{\beta}}$ and the $SU(4)$ generators $\mathbb{R}^{A}{}_{B}$ act canonically on the remaining generators carrying Lorentz or $SU(4)$ indices. The dilatation $\mathbb{D}$ and hypercharge $\mathbb{B}$ act via

$$[\mathbb{D}, \mathbb{J}] = \dim(\mathbb{J}), \qquad [\mathbb{B}, \mathbb{J}] = \text{hyp}(\mathbb{J}). \tag{B.0.1}$$

The non-zero dimensions and hypercharges of the various generators are

$$\begin{aligned} &\dim(\mathbb{P}) = 1, \qquad \dim(\mathbb{Q}) = \dim(\overline{\mathbb{Q}}) = \tfrac{1}{2}, \qquad \dim(\mathbb{S}) = \dim(\overline{\mathbb{S}}) = -\tfrac{1}{2} \\ &\dim(\mathbb{K}) = -1, \qquad \mathrm{hyp}(\mathbb{Q}) = \mathrm{hyp}(\overline{\mathbb{S}}) = \tfrac{1}{2}, \qquad \mathrm{hyp}(\overline{\mathbb{Q}}) = \mathrm{hyp}(\mathbb{S}) = -\tfrac{1}{2}. \end{aligned} \tag{B.0.2}$$

The remaining non-trivial commutation relations are,

$$\begin{aligned} &\{\mathbb{Q}_{\alpha A}, \overline{\mathbb{Q}}^B_{\dot\alpha}\} = \delta^B_A \mathbb{P}_{\alpha\dot\alpha}, \qquad \{\mathbb{S}_{\alpha A}, \overline{\mathbb{S}}^B_{\dot\alpha}\} = \delta^B_A \mathbb{K}_{\alpha\dot\alpha}, \\ &[\mathbb{P}_{\alpha\dot\alpha}, \mathbb{S}^A_\beta] = \epsilon_{\dot\alpha\dot\beta}\overline{\mathbb{Q}}^A_{\dot\alpha}, \qquad [\mathbb{K}_{\alpha\dot\alpha}, \mathbb{Q}_{\beta A}] = \epsilon_{\alpha\beta}\overline{\mathbb{S}}^A_{\dot\alpha}, \\ &[\mathbb{P}_{\alpha\dot\alpha}, \overline{\mathbb{S}}_{\dot\beta A}] = \epsilon_{\dot\alpha\dot\beta}\mathbb{Q}_{\alpha A}, \qquad [\mathbb{K}_{\alpha\dot\alpha}, \overline{\mathbb{Q}}_{\dot\beta A}] = \epsilon_{\dot\alpha\dot\beta}\mathbb{S}_{\alpha A}, \\ &[\mathbb{K}_{\alpha\dot\alpha}, \mathbb{P}^{\beta\dot\beta}] = \delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}\mathbb{D} + \mathbb{M}_\alpha{}^\beta\delta^{\dot\beta}_{\dot\alpha} + \overline{\mathbb{M}}_{\dot\alpha}{}^{\dot\beta}\delta^\beta_\alpha, \\ &\{\mathbb{Q}_{\alpha A}, \mathbb{S}^B_\beta\} = \epsilon_{\alpha\beta}\mathbb{R}^B{}_A + \mathbb{M}_{\alpha\beta}\delta^B_A + \epsilon_{\alpha\beta}\delta^B_A(\mathbb{D} + \mathbb{C}), \\ &\{\overline{\mathbb{Q}}^A_{\dot\alpha}, \overline{\mathbb{S}}_{\dot\beta B}\} = \epsilon_{\dot\alpha\dot\beta}\mathbb{R}^A{}_B + \overline{\mathbb{M}}_{\dot\alpha\dot\beta}\delta^A_B + \epsilon_{\dot\alpha\dot\beta}\delta^A_B(\mathbb{D} - \mathbb{C}). \end{aligned} \tag{B.0.3}$$

For convenience, we will use the following shorthand notation:

$$\partial_{i\alpha\dot\alpha} = \frac{\partial}{\partial x_i^{\alpha\dot\alpha}}, \qquad \partial_{i\alpha A} = \frac{\partial}{\partial \theta_i^{\alpha A}}, \qquad \partial_{i\alpha} = \frac{\partial}{\partial \lambda_i^\alpha}, \qquad \partial_{i\dot\alpha} = \frac{\partial}{\partial \bar\lambda_i^{\dot\alpha}}, \qquad \partial_{iA} = \frac{\partial}{\partial \eta_i^A}. \tag{B.0.4}$$

Now we will give the generators of the conventional superconformal symmetry, using lower case characters:

$$
\begin{array}{rclrcl}
p^{\alpha\dot{\alpha}} & = & \sum_i \lambda_i^{\alpha}\bar{\lambda}_i^{\dot{\alpha}}, & k_{\alpha\dot{\alpha}} & = & \sum_i \partial_{i\alpha}\partial_{i\dot{\alpha}}, \\
\overline{m}_{\dot{\alpha}\dot{\beta}} & = & \sum_i \bar{\lambda}_{i(\dot{\alpha}}\partial_{i\dot{\beta})}, & m_{\alpha\beta} & = & \sum_i \lambda_{i(\alpha}\partial_{i\beta)}, \\
d & = & \sum_i[\frac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \frac{1}{2}\bar{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} + 2], & r^A{}_B & = & \sum_i[\eta_i^A\partial_{iB} - \frac{1}{4}\eta_i^C\partial_{iC}], \\
q^{\alpha A} & = & \sum_i \lambda_i^{\alpha}\eta_i^A, & \bar{q}_A^{\dot{\alpha}} & = & \sum_i \bar{\lambda}_i^{\dot{\alpha}}\partial_{iA}, \\
s_A^{\alpha} & = & \sum_i \partial_{i\alpha}\partial_{iA}, & \bar{s}_{\dot{\alpha}}^A & = & \sum_i \eta_i^A\partial_{i\dot{\alpha}}.
\end{array}
$$

$$
c = \sum_i [1 + \tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} - \tfrac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} - \tfrac{1}{2}\eta_i^A\partial_{iA}]\,. \tag{B.0.5}
$$

We can construct the generators of dual superconformal transformations by starting with the standard chiral representation and extending the generators so that they commute with the constraints,

$$
(x_i - x_{i+1})_{\alpha\dot{\alpha}} - \lambda_{i\,\alpha}\,\tilde{\lambda}_{i\,\dot{\alpha}} = 0, \qquad (\theta_i - \theta_{i+1})_{\alpha}^A - \lambda_{i\alpha}\eta_i^A = 0. \tag{B.0.6}
$$

By construction they preserve the surface defined by these constraints, which is where the amplitude has support.

The generators of dual superconformal symmetry are:

$$P_{\alpha\dot{\alpha}} = \sum_i \partial_{i\alpha\dot{\alpha}}, \tag{B.0.7}$$

$$Q_{\alpha A} = \sum_i \partial_{i\alpha A}, \tag{B.0.8}$$

$$\overline{Q}^A_{\dot{\alpha}} = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha\dot{\alpha}} + \eta_i^A \partial_{i\dot{\alpha}}], \tag{B.0.9}$$

$$M_{\alpha\beta} = \sum_i [x_{i(\alpha}{}^{\dot{\alpha}}\partial_{i\beta)\dot{\alpha}} + \theta^A_{i(\alpha}\partial_{i\beta)A} + \lambda_{i(\alpha}\partial_{i\beta)}], \tag{B.0.10}$$

$$\overline{M}_{\dot{\alpha}\dot{\beta}} = \sum_i [x_{i(\dot{\alpha}}{}^{\alpha}\partial_{i\dot{\beta})\alpha} + \bar{\lambda}_{i(\dot{\alpha}}\partial_{i\dot{\beta})}], \tag{B.0.11}$$

$$R^A{}_B = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha B} + \eta_i^A\partial_{iB} - \tfrac{1}{4}\delta^A_B\theta_i^{\alpha C}\partial_{i\alpha C} - \tfrac{1}{4}\eta_i^C\partial_{iC}], \tag{B.0.12}$$

$$D = \sum_i [x_i^{\alpha\dot{\alpha}}\partial_{i\alpha\dot{\alpha}} + \tfrac{1}{2}\theta_i^{\alpha A}\partial_{i\alpha A} + \tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \tfrac{1}{2}\bar{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}}], \tag{B.0.13}$$

$$C = \sum_i \tfrac{1}{2}[\lambda_i^{\alpha}\partial_{i\alpha} - \bar{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}} - \eta_i^A\partial_{iA}], \tag{B.0.14}$$

$$S^A_{\alpha} = \sum_i [\theta^B_{i\alpha}\theta_i^{\beta A}\partial_{i\beta B} - x_{i\alpha}{}^{\dot{\beta}}\theta_i^{\beta A}\partial_{\beta\dot{\beta}} - \lambda_{i\alpha}\theta_i^{\gamma A}\partial_{i\gamma} - x_{i+1\,\alpha}{}^{\dot{\beta}}\eta_i^A\partial_{i\dot{\beta}} + \theta^B_{i+1\,\alpha}\eta_i^A\partial_{iB}], \tag{B.0.15}$$

$$\overline{S}_{\dot{\alpha}A} = \sum_i [x_{i\dot{\alpha}}{}^{\beta}\partial_{i\beta A} + \bar{\lambda}_{i\dot{\alpha}}\partial_{iA}], \tag{B.0.16}$$

$$K_{\alpha\dot{\alpha}} = \sum_i [x_{i\alpha}{}^{\dot{\beta}}x_{i\dot{\alpha}}{}^{\beta}\partial_{i\beta\dot{\beta}} + x_{i\dot{\alpha}}{}^{\beta}\theta^B_{i\alpha}\partial_{i\beta B} + x_{i\dot{\alpha}}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot{\beta}}\bar{\lambda}_{i\dot{\alpha}}\partial_{i\dot{\beta}} + \bar{\lambda}_{i\dot{\alpha}}\theta^B_{i+1\,\alpha}\partial_{iB}], \tag{B.0.17}$$

and the hypercharge B,

$$B = \sum_i \tfrac{1}{2}[\theta_i^{\alpha A}\partial_{i\alpha A} + \lambda_i^{\alpha}\partial_{i\alpha} - \bar{\lambda}_i^{\dot{\alpha}}\partial_{i\dot{\alpha}}] \tag{B.0.18}$$

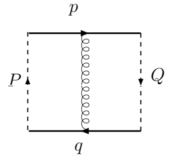
If we restrict the dual generators $\bar{Q}, \bar{S}$ to the on-shell superspace they become identical to the conventional generators $\bar{s}, \bar{q}$.

Appendix C

Details of One-Loop Calculation of Wilson Loops

In this appendix, we derive a new expression for the all-orders in ϵ one-loop finite Wilson loop diagrams, and hence also a new expression for the all orders in ϵ (finite part of the) $2me$ box function. We also improve a previous expression for use in all kinematical regimes.

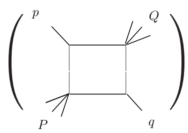
A general one-loop Wilson loop diagram is given by the following integral



$$\begin{aligned} &:= \Gamma(1+\epsilon)e^{\epsilon\gamma} \times \mathcal{F}_\epsilon(s,t,P^2,Q^2) \\ &= \Gamma(1+\epsilon)e^{\epsilon\gamma} \times \int_0^1 d\tau \int_0^1 d\sigma \frac{u/2}{\{-[P^2+\sigma(s-P^2)+\tau(t-Q^2+\sigma\tau u)]-i\varepsilon\}^{1+\epsilon}}\ . \end{aligned} \tag{C.0.1}$$

Here we have defined $s = (p+P)^2$, $t = (p+Q)^2$ and $u = P^2 + Q^2 - s - t$. The relation between this Wilson loop diagram and the corresponding $2me$ box function is [130]

finite part of



$$= e^{\epsilon\gamma}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \times \mathcal{F}_\epsilon(s,t,P^2,Q^2) \ . \tag{C.0.2}$$

Notice that we have included an infinitesimal negative imaginary part $-i\varepsilon$ in the denominator which dictates the analytic properties of the integral. This has the opposite sign to the one expected from a propagator term in a Wilson loop in configuration space. On the other hand it has the correct sign for the present application, namely for the duality with amplitudes [162]. One simple way to deal with this is simply to add an identical positive imaginary part to all kinematical invariants

$$s \to s + i\varepsilon, \quad t \to t + i\varepsilon, \quad P^2 \to P^2 + i\varepsilon, \quad Q^2 \to Q^2 + i\varepsilon \ . \tag{C.0.3}$$

We will assume this in the following.

Changing variables to $\sigma' = \sigma - (P^2 - t)/u$ and $\tau' = \tau - (P^2 - s)/u$ and then dropping the primes, this becomes

$$\mathcal{F}_\epsilon(s,t,P^2,Q^2) = \int_{\frac{t-P^2}{u}}^{\frac{Q^2-s}{u}} d\sigma \int_{\frac{s-P^2}{u}}^{\frac{Q^2-t}{u}} d\tau \frac{u/2}{\left[-(a^{-1}+\sigma\tau u)\right]^{1+\epsilon}} \tag{C.0.4}$$

$$= -\frac{1}{2\epsilon}\int_{\frac{t-P^2}{u}}^{\frac{Q^2-s}{u}} \frac{d\sigma}{\sigma} \ \frac{1}{\left[-(a^{-1}+\sigma\tau u)\right]^{\epsilon}}\Bigg|_{\tau=(s-P^2)/u}^{\tau=(Q^2-t)/u} , \tag{C.0.5}$$

where $a = u/(P^2Q^2 - st)$. Note that the analytic continuation of a implied by (C.0.3) is $a \to a - i\varepsilon$. Now we split the integration into two parts,

$$\int_{\frac{t-P^2}{u}}^{\frac{Q^2-s}{u}} = -\int_0^{\frac{t-P^2}{u}} + \int_0^{\frac{Q^2-s}{u}} , \tag{C.0.6}$$

and rescale the integration variable so that it runs between 0 and 1 in each case. It is important to split the integral in this way since there is a singularity at $\sigma = 0$ which one must be very careful when integrating over. We obtain in this way

$$\begin{aligned}\mathcal{F}_\epsilon(s,t,P^2,Q^2) =& \frac{(-a)^\epsilon}{2\epsilon}\int_0^1 \frac{d\sigma}{\sigma}\left\{\frac{1}{[1-(1-aP^2)\sigma]^\epsilon} + \frac{1}{[1-(1-aQ^2)\sigma]^\epsilon}\right. \\ & \left. - \frac{1}{[1-(1-as)\sigma]^\epsilon} - \frac{1}{[1-(1-at)\sigma]^\epsilon}\right\} ,\end{aligned} \tag{C.0.7}$$

where $a = u/(P^2Q^2 - st)$. Now each of the four terms by itself is divergent (even for $\epsilon \neq 0$), only the sum gives a finite integral. A straightforward way of regulating each term individually is to simply subtract $1/\sigma$ from each term in the integrand, thus removing the divergence at $\sigma = 0$. We thus have

$$\mathcal{F}_\epsilon(s,t,P^2,Q^2) \;=\; \frac{(-a)^\epsilon}{2}\Big[f(1-aP^2)+f(1-aQ^2)-f(1-as)-f(1-at)\Big] , \tag{C.0.8}$$

where

$$f(x) \;=\; \frac{1}{\epsilon}\int_0^1 d\sigma \frac{(1-x\sigma)^{-\epsilon}-1}{\sigma} = \frac{1}{\epsilon}\int_0^x d\sigma \frac{(1-\sigma)^{-\epsilon}-1}{\sigma} . \tag{C.0.9}$$

The problem becomes that of finding the integral $f(x)$. It has two equivalent forms, both given in terms of hypergeometric functions. The first form is given by

$$f(x) \;=\; x \times {}_3\mathrm{F}_2(1,1,1+\epsilon;2,2;x) = \sum_{n=1}^{\infty} \epsilon^n S_{1\,n+1}(x)\ . \tag{C.0.10}$$

Notice the very simple expansion in terms of Nielsen polylogarithms. The second form is

$$f(x) \;=\; -\frac{1}{\epsilon^2}\left[(-x)^{-\epsilon}{}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;1/x) + \epsilon\log x\right] + \text{constant}\ , \tag{C.0.11}$$

where the constant is there to make $f(0) = 0$, and is not important since it will cancel in $\mathcal{F}_\epsilon$.

We thus arrive at two different forms for the Wilson loop diagram. The first form is

$$\begin{aligned}&\mathcal{F}_\epsilon(s,t,P^2,Q^2)\\ =&\frac{(-a)^\epsilon}{2}\Big[(1-aP^2){}_3\mathrm{F}_2(1,1,1+\epsilon;2,2;1-aP^2) + (1-aQ^2){}_3\mathrm{F}_2(1,1,1+\epsilon;2,2;1-aQ^2)\\ &\quad - (1-as){}_3\mathrm{F}_2(1,1,1+\epsilon;2,2;1-as) - (1-at){}_3\mathrm{F}_2(1,1,1+\epsilon;2,2;1-at)\Big]\ ,\end{aligned} \tag{C.0.12}$$

and it is manifestly finite. Furthermore, since ${}_3\mathrm{F}_2(1,1,1+\epsilon;2,2;x) = \mathrm{Li}_2(x)/x$, this form directly leads to the expression derived in [163**?**] for the finite $2me$ box function,

$$\mathcal{F}_{\epsilon=0}(s,t,P^2,Q^2) \;=\; \frac{1}{2}\left[\mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at)\right]\ . \tag{C.0.13}$$

We also notice that the simple expansion of (C.0.10) gives a correspondingly simple expansion for the Wilson loop diagram in terms of Nielsen polylogarithms.

The more familiar looking second form for the two-mass easy box function is (see (A.13) of [?])

$$
\begin{aligned}
\mathcal{F}_\epsilon(s,t,P^2,Q^2) \;&= -\frac{1}{2\epsilon^2}\times \\
\Bigg[\left(\frac{a}{1-aP^2}\right)^\epsilon {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;1/(1-aP^2)) &+ \left(\frac{a}{1-aQ^2}\right)^\epsilon {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;1/(1-aQ^2)) \\
- \left(\frac{a}{1-as}\right)^\epsilon {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;1/(1-as)) &- \left(\frac{a}{1-at}\right)^\epsilon {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;1/(1-at)) \\
+\, \epsilon(-a)^\epsilon \Big(\log(1-aP^2) + \log(1-aQ^2) &- \log(1-as) - \log(1-at) \Big) \Bigg] \,.
\end{aligned}
\tag{C.0.14}
$$

This second form was derived in [130?] except for the last line which is an additional correction term needed to obtain the correct analytic continuation in all regimes. The identity

$$
\frac{(1-aP^2)(1-aQ^2)}{(1-as)(1-at)} \;=\; 1\,, \tag{C.0.15}
$$

implies that if all the arguments of the logs are positive then this additional term vanishes, but for example if we have $1-aP^2, 1-aQ^2 > 0$ and $1-as, 1-at < 0$ then the additional term gives (taking care of the appropriate analytic continuation in (C.0.3)) $\mathrm{sgn}(a)2\pi i(-a)^\epsilon/\epsilon$. This becomes important when considering this expression at four and five points in the Euclidean regime.

For applications in this chapter we are interested in taking either one massive leg massless (for the five-point case) or both massive legs massless (for the four-point case). Using the first expression for the finite Wilson loop diagram in terms of ${}_3\mathrm{F}_2$

functions and using that ${}_3\mathrm{F}_2(1,1,1+\epsilon;2,2;1) = \frac{-\psi(1-\epsilon)-\gamma}{\epsilon} = -\frac{H_{-\epsilon}}{\epsilon}$ (where $\psi(x)$ is the digamma function, γ is Euler's constant and H_n is the harmonic number of n), we obtain the four- and five-point one-loop Wilson loop expressions of (3.2.23) and (3.2.24).

For completeness we also consider the limit with $P^2 = Q^2 = 0$ using the second expression for the finite diagram (C.0.14), since this and similar expressions have been used throughout appendix A. When $P^2 = Q^2 = 0$, we have $a = 1/s + 1/t$, $1 - as = -s/t$ and $1 - at = -t/s$ and using

$$ {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;1) = \epsilon\pi\csc(\epsilon\pi) \ , \tag{C.0.16} $$

we get

$$ \begin{aligned} &\mathcal{F}_\epsilon(s,t,0,0) = -\frac{1}{2\epsilon^2}\times \\ &\Bigg[-\left(\frac{u}{st}\right)^\epsilon \left(\frac{t}{s}\right)^\epsilon {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;-t/s) - \left(\frac{u}{st}\right)^\epsilon \left(\frac{s}{t}\right)^\epsilon {}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;-s/t) \\ &+ 2(a)^\epsilon\epsilon\pi\csc(\epsilon\pi) + \epsilon(-a)^\epsilon\Big(-\log(1-as) - \log(1-at)\Big)\Bigg] \ . \end{aligned} \tag{C.0.17} $$

We wish to know this in the Euclidean regime in which $s, t, a < 0$. The first line is then manifestly real, whereas the second gives

$$ 2\pi\epsilon(-a)^\epsilon e^{-i\epsilon\pi}\csc(\epsilon\pi) + 2\pi\epsilon i(-a)^\epsilon = 2\pi\epsilon(-a)^\epsilon\cot(\epsilon\pi) \ . \tag{C.0.18} $$

This is the form we will use for the one-loop Wilson loop.

Appendix D

Details of the Two-Loop Four-Point Wilson Loop to All Orders in ϵ

In this appendix, we present the results for the separate classes of Wilson loop diagrams contributing to a four-point loop. In all cases (with the exception of the "hard" diagram) our results are valid to all orders in the dimensional regularization parameter ϵ. These expressions are given in terms of hypergeometric functions. We also expand these up to $\mathcal{O}(\epsilon)$ with the help of the mathematica packages `HPL` and `HypExp` [160, 161].

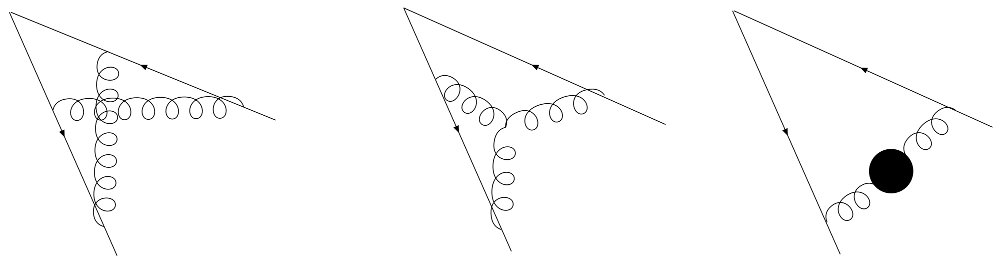


Figure D.1: *The two-loop cusp corrections. The second diagram appears with its mirror image where two of the gluon legs of the three-point vertex are attached to the other edge; these two diagrams are equal. The blue bubble in the third diagram represents the gluon self-energy correction calculated in dimensional reduction.*

D.1 Two-loop Cusp Diagrams

The total contributions of all diagrams that cross a single cusp (the crossed diagram across a cusp, the self-energy diagram across the cusp and the vertex across the cusp, as depicted in Figure D.1) is easily seen to be[1]

$$(-s)^{-2\epsilon}\ \frac{1}{16\epsilon^4}\left[\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)}-1\right]\ . \tag{D.1.1}$$

Adding the contributions for the four cusps, we obtain

$$\mathcal{W}_{\text{cusp}} = \left[(-s)^{-2\epsilon}+(-t)^{-2\epsilon}\right]\frac{1}{8\epsilon^4}\left[\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)}\ -\ 1\right] \tag{D.1.2}$$

$$= \left[(-s)^{-2\epsilon}+(-t)^{-2\epsilon}\right] \times\left[\frac{1}{\epsilon^2}\frac{\pi^2}{24}-\frac{1}{\epsilon}\frac{\zeta_3}{4}+\frac{\pi^4}{80}-\frac{\epsilon}{12}\left(\pi^2\zeta_3+9\zeta_5\right)+\mathcal{O}(\epsilon^2)\right]. \tag{D.1.3}$$

[1]In this and the following formulae, a factor of $\mathcal{C}$ is suppressed in each diagram, where $\mathcal{C}$ is defined in (3.2.36).

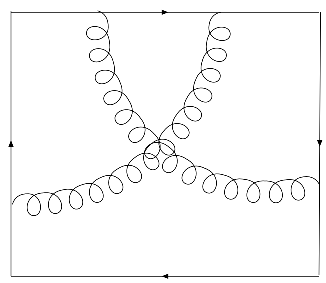


Figure D.2: *One of the four curtain diagrams. The remaining three are obtained by cyclic permutations of the momenta.*

D.2 The Curtain Diagram

The contribution of all four curtain diagrams is

$$
\begin{aligned}
\mathcal{W}_{\text{curtain}} &= (st)^{-\epsilon}\left(-\frac{1}{2\epsilon^4}\right)\left[1-\frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\right] \\
&= \left[(-s)^{-2\epsilon}+(-t)^{-2\epsilon}\right]\left[-\frac{1}{\epsilon^2}\frac{\pi^2}{24}-\frac{1}{\epsilon}\frac{\zeta_3}{2}-\frac{\pi^4}{160}+\frac{\pi^2}{48}\log^2 x\right. \\
&\left.-\epsilon\Big(-\frac{1}{4}\zeta_3\log^2 x+\frac{3}{2}\zeta_5-\frac{\pi^2}{12}\zeta_3\Big)+\mathcal{O}(\epsilon^2)\right].
\end{aligned}
\tag{D.2.1}
$$

D.3 The Factorized Cross Diagram

The factorized cross diagram is given by the product of two finite one-loop Wilson loop diagrams, expressed each by (3.2.23)

$$
\frac{1}{2\epsilon^2}\left(\frac{st}{u}\right)^{-\epsilon}\left[x^{-\epsilon}{}_2\mathrm{F}_1(\epsilon,\epsilon,1+\epsilon,-1/x)+x^{\epsilon}{}_2\mathrm{F}_1(\epsilon,\epsilon,1+\epsilon,-x)-2\pi\epsilon\cot(\epsilon\pi)\right].
$$

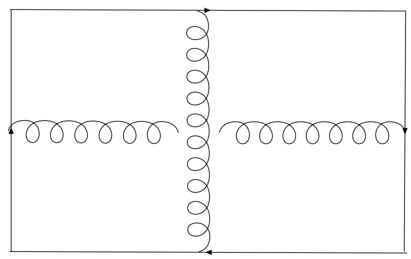


Figure D.3: *One of the four factorized cross diagram.*

The result for the factorized cross is therefore

$$\begin{aligned} -\frac{1}{8\epsilon^4}\left(\frac{st}{u}\right)^{-2\epsilon}&\left[{}_2\mathrm{F}_1(\epsilon,\epsilon;1+\epsilon;-x)x^{\epsilon}+{}_2\mathrm{F}_1\left(\epsilon,\epsilon;1+\epsilon;-\frac{1}{x}\right)x^{-\epsilon}-2\pi\epsilon\cot(\epsilon\pi)\right]^2 \\ =&\left[(-s)^{-2\epsilon}+(-t)^{-2\epsilon}\right]\times\left(g_0+g_{-1}\epsilon+\mathcal{O}(\epsilon^2)\right), \qquad \text{(D.3.1)}\end{aligned}$$

with

$$g_0 = -\frac{1}{64}\left(\log^2 x+\pi^2\right)^2, \qquad \text{(D.3.2)}$$

$$\begin{aligned} g_{-1} =& \frac{1}{192}\left(\log^2 x+\pi^2\right)\left[\log^3 x - 6\log(x+1)\log^2 x - 12\mathrm{Li}_2(-x)\log x\right. , \\ &+\left. 3\pi^2\log x - 6\pi^2\log(x+1) + 12\,\mathrm{Li}_3(-x) - 12\zeta_3\right]. \qquad \text{(D.3.3)}\end{aligned}$$

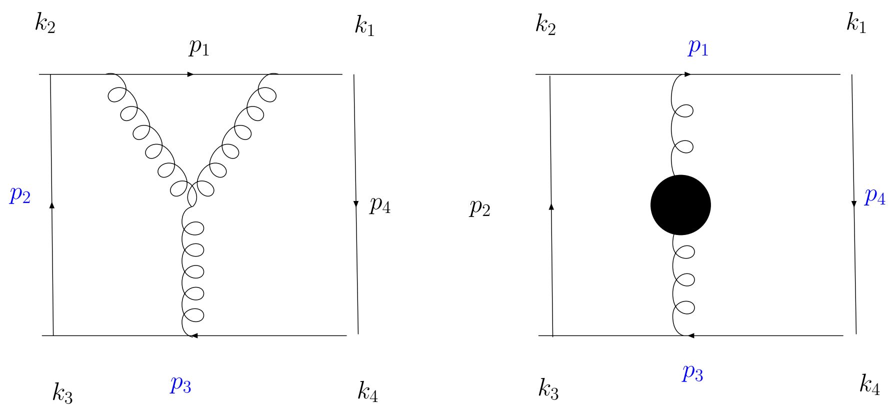


Figure D.4: *The Y diagram together with the self-energy diagram. The sum of these two topologies gives a maximally transcendental contribution.*

D.4 The Y Diagram

The diagrams in Figure D.4 correspond to the following contribution to the two-loop Wilson loop:

$$\begin{aligned} \mathcal{W}_{\rm Y} \;=\; & -\frac{1}{16\epsilon}\,\frac{\Gamma(1+2\epsilon)}{\Gamma(1+\epsilon)^2}\;u \\ & \times \int_0^1 d\tau_1 \int_0^1 d\tau_2 \Big[2I(z_1(\tau_1),z_2(\tau_2),z_2(\tau_2)) - I(z_1(\tau_1),z_2(\tau_2),k_1) - I(z_1(\tau_1),z_2(\tau_2),k_2)\Big]\;, \end{aligned} \tag{D.4.1}$$

where $z_1(\tau_1) = k_3 - p_3\tau_1,\; z_2(\tau_2) = k_1 - p_1\tau_2$, and

$$I(z_1,z_2,z_3) \;=\; \int_0^1 d\sigma(\sigma(1-\sigma))^{\epsilon}\Big[-(-z_1+\sigma z_2+(1-\sigma)z_3)^2+i0\Big]^{-1-2\epsilon}\;, \tag{D.4.2}$$

where z_2, z_3 must be light-like. The evaluation of (D.4.1) gives

$$\begin{aligned}\mathcal{W}_{\mathrm{Y}} &= \left(\frac{st}{u}\right)^{-2\epsilon}\frac{1}{64\epsilon^4}\Bigg[-2(x+1)^{-2\epsilon}\frac{\Gamma(1+2\epsilon)\Gamma(-\epsilon+1)}{\Gamma(1+\epsilon)} \\ &+ 4x^{\epsilon}\frac{\Gamma(-\epsilon+1)^2}{\Gamma(-2\epsilon+1)}\,{}_2\mathrm{F}_1(\epsilon,1+2\epsilon;1+\epsilon;-x)+x^{2\epsilon}\,{}_2\mathrm{F}_1\left(2\epsilon,2\epsilon;1+2\epsilon;-x\right) \\ &- 4\pi\epsilon\cot(2\pi\epsilon)+\Gamma(1+2\epsilon)\Gamma(-2\epsilon+1)\;+\;x\leftrightarrow\frac{1}{x}\Bigg]\end{aligned}$$

We multiply by four to obtain the contribution of all such diagrams. Then the expansion of this contribution in ϵ begins at $\mathcal{O}(\epsilon^{-1})$,

$$\left[(-s)^{-2\epsilon}+(-t)^{-2\epsilon}\right]\times\left[\frac{c_1}{\epsilon}+c_0+c_{-1}\epsilon+\mathcal{O}(\epsilon^2)\right], \tag{D.4.3}$$

where

$$\begin{aligned}c_1 &= -\frac{1}{48}\Big[\log^3 x-6\log(x+1)\log^2 x-12\mathrm{Li}_2(-x)\log x+3\pi^2\log x) \\ &\quad -6\pi^2\log(x+1+12\mathrm{Li}_3(-x)-12\zeta_3\Big],\end{aligned} \tag{D.4.4}$$

$$\begin{aligned}c_0 &= \frac{1}{960}\Big[5\log^4 x-40\log(x+1)\log^3 x+120\log^2(x+1)\log^2 x+10\pi^2\log^2 x \\ &\quad -120\pi^2\log(x+1)\log x+480\log(x+1)\mathrm{Li}_2(-x)\log x-240\,\mathrm{Li}_3(-x)\log x \\ &\quad +480S_{1,2}(-x)\log x-240\zeta_3\log x+120\pi^2\log^2(x+1)-480\,\log(x+1)\mathrm{Li}_3(-x) \\ &\quad +480\mathrm{Li}_4(-x)-480S_{2,2}(-x)+480\log(x+1)\zeta_3+\pi^4\Big],\end{aligned} \tag{D.4.5}$$

$$
\begin{aligned}
c_{-1} =& \frac{1}{240}\Big[-2\log^5 x + 10\log(x+1)\log^4 x + 10\log^2(x+1)\log^3 x + 20\mathrm{Li}_2(-x)\log^3 x \\
&- 5\pi^2\log^3 x - 20\log^3(x+1)\log^2 x - 30\zeta_3\log^2 x + 30\pi^2\log^2(x+1)\log x \\
&- 120\log^2(x+1)\mathrm{Li}_2(-x)\log x + 120\log(x+1)\mathrm{Li}_3(-x)\log x - 120\mathrm{Li}_4(-x)\log x \\
&+ 120 S_{2,2}(-x)\log(x) - 240\log(x+1)S_{1,2}(-x)\log x - 240 S_{1,3}(-x)\log x \\
&+ 120\log(x+1)\zeta_3\log x + 4\pi^4\log x - 20\pi^2\log^3(x+1) \\
&- 8\pi^4\log(x+1) + 120\log^2(x+1)\mathrm{Li}_3(-x) \\
&- 40\mathrm{Li}_2(-x)\mathrm{Li}_3(-x) - 240\log(x+1)\mathrm{Li}_4(-x) + 240\mathrm{Li}_5(-x) + 240\log(x+1)S_{2,2}(-x) \\
&+ 40H_{2,3}(-x) + 120H_{3,2}(-x) + 40\mathrm{Li}_2(-x)S_{1,2}(-x) - 40H_{2,1,2}(-x) \\
&- 120H_{2,2,1}(-x) - 240\zeta_5 - 120\log^2(x+1)\zeta_3 - 30\pi^2\zeta_3\Big]\,.
\end{aligned}
\tag{D.4.6}
$$

D.5 The Half-Curtain Diagram

We now consider the “half-curtain" diagram, whose contribution to the Wilson loop

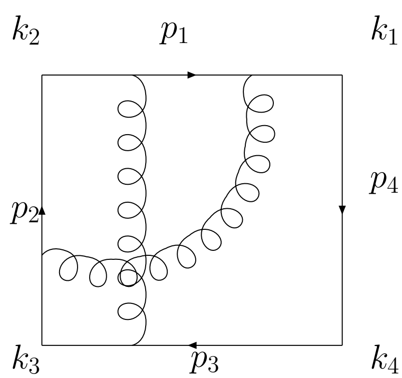


Figure D.5: *Diagram of the half-curtain topology.*

is

$$\mathcal{W}_{\rm hc}(x) = -\frac{1}{8}\int_0^1 d\sigma\, d\rho\, d\tau_1 \int_{\tau_1}^1 d\tau_2 \frac{s}{(-s\sigma\tau_2)^{1+\epsilon}} \frac{u}{(-s\tau_1 - t\rho - u\rho\tau_1)^{1+\epsilon}}$$
$$= \frac{1}{8}\left(\frac{st}{u}\right)^{-2\epsilon}(1+x)^{-\epsilon}\int_0^1 d\sigma \int_1^{-1/x} da \int_1^{-x} db \int_{\frac{1-b}{1+x}}^1 d\tau_2 \frac{1}{(\sigma\tau_2)^{1+\epsilon}} \frac{1}{(1-ab)^{1+\epsilon}}\,, \tag{D.5.1}$$

where we have changed variables in the second line to $a = 1+u\rho/s$, and $b = 1+u\tau_1/t$.

The evaluation of this diagram and of that with $s \leftrightarrow t$ leads to

$$\begin{aligned}
&\mathcal{W}_{\rm hc}(x) + \mathcal{W}_{\rm hc}(1/x) = \frac{1}{8\epsilon^4}\left(\frac{st}{u}\right)^{-2\epsilon} \\
&\times \Big[2\pi\epsilon\cot(2\pi\epsilon) - \frac{1}{2}x^{-2\epsilon}\,{}_2\mathrm{F}_1(2\epsilon, 2\epsilon; 1+2\epsilon; -1/x) \\
&- \frac{1}{2}x^{2\epsilon}\,{}_2\mathrm{F}_1(2\epsilon, 2\epsilon; 1+2\epsilon; -x) \\
&+ \Big[(1+x)^{-\epsilon} + (1+1/x)^{-\epsilon}\Big]\Big(x^{-\epsilon}\,{}_2\mathrm{F}_1(\epsilon, \epsilon; 1+\epsilon; -1/x) \\
&+ x^{\epsilon}\,{}_2\mathrm{F}_1(\epsilon, \epsilon; 1+\epsilon; -x) - 2\pi\epsilon\cot(\pi\epsilon)\Big)\Big]\,.
\end{aligned} \tag{D.5.2}$$

This can be expanded in ϵ using

$$\begin{aligned}
{}_2\mathrm{F}_1(\epsilon, \epsilon; 1+\epsilon; x) \;&=\; 1 + \epsilon^2 \mathrm{Li}_2(x) - \epsilon^3\Big[\mathrm{Li}_3(x) - S_{12}(x)\Big] \\
&+\; \epsilon^4\Big[\mathrm{Li}_4(x) - S_{22}(x) + S_{13}(x)\Big] + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{D.5.3}$$

The contribution of all diagrams of the half-curtain type is obtained by multiplying (D.5.2) by a factor of four. One obtains thus

$$\Big[(-s)^{-2\epsilon} + (-t)^{-2\epsilon}\Big] \times \; \Big[\frac{d_1}{\epsilon} + d_0 + d_{-1}\epsilon + \mathcal{O}(\epsilon^2)\Big]\,, \tag{D.5.4}$$

where

$$d_1 = 2c_1 \; , \tag{D.5.5}$$

$$d_0 = -3c_0 - \frac{1}{64}\left(3\pi^2 - \log^2 x\right)\left(\log^2 x + \pi^2\right) \; , \tag{D.5.6}$$

$$\begin{aligned} d_{-1} &= \frac{7}{2}c_{-1} \\ &- \frac{1}{96}\Big[-\log^5(x) + 6\log(x+1)\log^4 x + 12\mathrm{Li}_2(-x)\log^3(x) - 3\pi^2\log^3 x \\ &+ 6\pi^2\log(x+1)\log^2 x - 12\mathrm{Li}_3(-x)\log^2 x - 30\zeta_3\log^2 x - 42\pi^2\zeta_3\Big] \; , \end{aligned} \tag{D.5.7}$$

and c_j are the coefficients for the Y diagram, given in (D.4.4)–(D.4.6).

D.6 The Cross Diagram

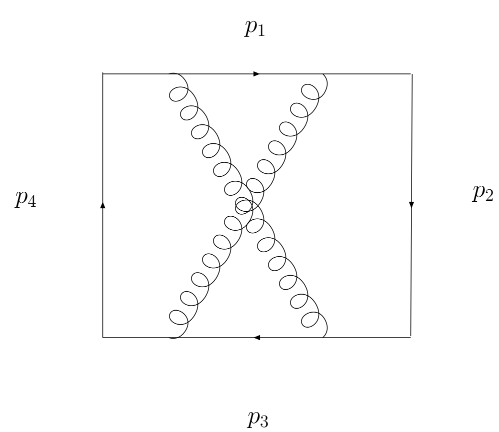


Figure D.6: *One of the cross diagrams. As before, the remaining three can be generated by cyclic permutations of the momentum labels.*

We now consider the cross diagram, whose expression is given by

$$\begin{aligned}\mathcal{W}_{\rm cr}(x) &= -\frac{1}{8}\int_0^1 d\tau_1\, d\sigma_1 \int_0^{\tau_1} d\tau_2 \int_0^{\sigma_1} \tau_2 \frac{u}{(-s\sigma_1 - t\tau_2 - u\sigma_1\tau_2)^{1+\epsilon}} \frac{u}{(-s\sigma_2 - t\tau_1 - u\sigma_2\tau_1)^{1+\epsilon}} \\ &= -\frac{1}{8}\left(\frac{st}{u}\right)^{-2\epsilon} \int_{-1/x}^{1} da_1 \int_{-x}^{1} db_1 \int_1^{a_1} da_2 \int_1^{b_1} db_2 \frac{1}{(1-a_1b_2)^{1+\epsilon}} \frac{1}{(1-a_2b_1)^{1+\epsilon}}\ ,\end{aligned} \tag{D.6.1}$$

for which we find

$$\begin{aligned}\mathcal{W}_{\rm cr}(x) = -&\frac{1}{8\epsilon^4}\left(\frac{st}{u}\right)^{-2\epsilon} \\ &\Big\{ -\frac{1}{4}\,{}_3\mathrm{F}_2(2\epsilon, 2\epsilon, 2\epsilon; 1+2\epsilon, 1+2\epsilon; -x)x^{2\epsilon} \\ &+ \frac{\Gamma(-\epsilon+1)^2\,{}_3\mathrm{F}_2(2\epsilon, \epsilon, \epsilon; 1+\epsilon, 1+\epsilon; -x)x^{\epsilon}}{\Gamma(-2\epsilon+1)} \\ &+ \frac{1}{4}\Big[\,{}_2\mathrm{F}_1(\epsilon, \epsilon; 1+\epsilon; -x)x^{\epsilon} + {}_2\mathrm{F}_1\Big(\epsilon, \epsilon; 1+\epsilon; -\frac{1}{x}\Big)x^{-\epsilon} \\ &- 2\pi\epsilon\cot(\epsilon\pi)\Big]^2 - \epsilon^2\pi\cot(2\pi\epsilon)\left(\psi(2\epsilon)+\gamma\right) - \pi^2\epsilon^2\cot(\pi\epsilon)^2 \\ &+ \Big(x \leftrightarrow \frac{1}{x}\Big)\Big\}\ .\end{aligned} \tag{D.6.2}$$

Notice the presence of the one-loop finite diagram squared. Expanding this and multiplying by a factor of two to account for all diagrams leads to the following result,

$$\mathcal{W}_{\rm cr}(x) \;=\; \left[(-s)^{-2\epsilon} + (-t)^{-2\epsilon}\right] \times \left[f_0 + f_{-1}\epsilon + \mathcal{O}(\epsilon^2)\right]\,, \tag{D.6.3}$$

where

$$f_0 \;=\; -\frac{1}{192}(\pi^2+\log^2 x)^2\;, \tag{D.6.4}$$

$$\begin{aligned} f_{-1} \;=\; -\frac{1}{2880}\Big[& -3\log^5 x + 30\log(x+1)\log^4 x + 180\mathrm{Li}_2(-x)\log^3 x \\ & -20\pi^2\log^3 x + 60\pi^2\log(x+1)\log^2 x - 900\mathrm{Li}_3(-x)\log^2 x - 540\,\zeta_3\log^2 x \\ & +180\pi^2\mathrm{Li}_2(-x)\log x + 2880\mathrm{Li}_4(-x)\log x \\ & -49\pi^4\log x + 30\pi^4\log(x+1) - 420\pi^2\mathrm{Li}_3(-x) - 4320\,\mathrm{Li}_5(-x) \\ & +4320\zeta_5 - 1020\,\pi^2\zeta_3\Big]\;. \end{aligned} \tag{D.6.5}$$

D.7 The Hard Diagram

The generic n-point hard diagram topology is depicted in Figure 3.4. In the four-point case, one diagram is obtained from Figure 3.4 by simply setting $Q_2 = Q_3 = 0$, and $Q_1 = p_4$. There are four such diagrams, obtained by cyclic rearrangements of the momenta. We have evaluated the hard diagrams using Mellin-Barnes, arriving at the following result:

$$W_{\mathrm{hard}} \;=\Big[(-s)^{-2\epsilon}+(-t)^{-2\epsilon}\Big]\times\;\Big[\frac{h_2}{\epsilon^2}\;+\;\frac{h_1}{\epsilon}\;+\;h_0\;+\;h_{-1}\epsilon+\mathcal{O}(\epsilon^2)\Big]\;, \tag{D.7.1}$$

where

$$h_2 = \frac{\pi^2}{48} \,, \tag{D.7.2}$$

$$h_1 = -\left(c_1 + \frac{\zeta_3}{8}\right) \,, \tag{D.7.3}$$

$$h_0 = -2c_0 - \frac{1}{192}\left(\log^2 x + \pi^2\right)^2 - \frac{1}{48}\pi^2\left(\log^2 x + \pi^2\right) + \frac{31}{1440}\pi^4 \tag{D.7.4}$$

$$\begin{aligned} h_{-1} = -\frac{1}{480}\Big[&200H_{2,3}(-x) + 600H_{3,2}(-x) - 200H_{2,1,2}(-x) - 600H_{2,2,1}(-x) \\ &+ 200\zeta_3\mathrm{Li}_2(-x) - 200\mathrm{Li}_2(-x)\mathrm{Li}_3(-x) - 200\mathrm{Li}_2(-x)\mathrm{Li}_3(x+1) + 1320\mathrm{Li}_5(-x) \\ &+ 20\mathrm{Li}_2(-x)\log^3(x) + 60\mathrm{Li}_3(-x)\log^2(x) - 600\mathrm{Li}_2(-x)\log^2(x+1)\log x \\ &- 600\mathrm{Li}_2(x+1)\log^2(x+1)\log x + 100\mathrm{Li}_2(-x)\log(-x)\log^2(x+1) \\ &+ 600\mathrm{Li}_3(-x)\log^2(x+1) + 20\pi^2\mathrm{Li}_2(-x)\log(x) + 600\mathrm{Li}_3(-x)\log(x+1)\log(x) \\ &- 600\mathrm{Li}_4(-x)\log(x) + 1200\mathrm{Li}_4(x+1)\log x + 200\mathrm{Li}_2(-x)\mathrm{Li}_2(x+1)\log(x+1) \\ &- 1200\mathrm{Li}_4(-x)\log(x+1) + 600S_{2,2}(-x)\log x + 1200S_{2,2}(-x)\log(x+1) \\ &- 240\zeta_3\log^2 x - 600\zeta_3\log^2(x+1) - 600\zeta_3\log(x+1)\log x - 2\log^5 x \\ &+ 5\log(x+1)\log^4 x - 400\log(-x)\log^3(x+1)\log x - 100\pi^2\log^3(x+1) \\ &- 40\pi^2\log(x+1)\log^2 x + 150\pi^2\log^2(x+1)\log x + 50\log^2(x+1)\log^3 x \\ &- 100\log^3(x+1)\log^2 x + 7\pi^4\log x - 35\pi^4\log(x+1) - 1020\zeta_5 - 320\pi^2\zeta_3\Big] \,. \end{aligned} \tag{D.7.5}$$

The analytical evaluation of this diagram up to $\mathcal{O}(\epsilon^0)$ was obtained in [**?**]. Our evaluation of the $\mathcal{O}(\epsilon^0)$ terms agrees precisely with that of [152] (and with [**?**] up to a constant term). The evaluation of the $\mathcal{O}(\epsilon)$ term is new and has been performed numerically. We have then compared the entire Wilson loop expansion to the analytic expression for the amplitude remainder given in (3.2.29), finding the relation (3.2.26).

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