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A four-dimensional generalization of Misner space in curved spacetime

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**OPEN ACCESS****PAPER****A four-dimensional generalization of Misner space in curved spacetime****RECEIVED**
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**Abstract**

We present a topologically trivial nonvacuum solution of the Einstein's field equations in curved spacetime with stress-energy tensor Type I fluid, satisfying the energy conditions. The metric admits closed timelike curves which appear after a certain instant of time, and the spacetime is a four-dimensional generalization of flat Misner space in curved spacetime.

1. Introduction

There are many examples of solution to the Einstein's field equations containing closed timelike curves (CTC) exist in general relativity. The closed timelike curves are the worldline of a physical observer that loop back on itself, i.e., an observer could return to his/her own's past. A closed timelike curve allows time travel, in the sense that an observer which travels on a trajectory in spacetime along this curve, return to an event which coincides with the departure. This fact violates the causality condition, opening time travel paradoxes (grandfather paradox) in classical general relativity. Hawking motivated by this proposed a Chronology Protection Conjecture (CPC) [1] which states that the laws of physics will always prevent a spacetime to form CTC. However, the general proof of the Chronology Protection Conjecture has not yet existed. Therefore, constructing such spacetimes with CTC satisfying the different energy condition cannot be discard/ruled out easily, because these spacetime satisfied the Einstein field equations. Few examples of spacetimes with CTC are in [2–6]. Some well-known vacuum spacetimes (e.g. [7, 8] and references therein) are also admits CTC. The CTC spacetimes are categorize in two way. First one being an eternal time machine spacetime in which CTC either form everywhere or pre-exist (e.g. [9, 10]). Second one being a time machine spacetime, where CTC appears after a certain instant of time in a causally well-behaved manner (e.g. [7, 8, 11]). However, some known solutions of the field equations with CTC are considered unphysical because of their unrealistic or exotic matter-energy source which are violate the weak energy condition (WEC). The time machine models discussed in [12, 13] violate the Weak energy condition, and the model [14] violates the strong energy condition (SEC). In addition, few other solutions of the field equations with CTC does not admit a partial Cauchy surface (an initial spacelike hypersurface) (e.g. [15]) and/or CTC come from infinity (e.g. [16, 17]). The non-spherical gravitational collapse solutions [18–21] with a naked singularity also admits CTC.

In the context of CTC, the Misner space metric in 2D is a prime example, where CTC evolve smoothly from an initial conditions in a causally well-behaved manner. The Misner space metric in 2D [22] is given by

$$ds_{\text{Misn}}^2 = -2 dt d\psi - t d\psi^2, \quad (1)$$

where $-\infty < t < \infty$, and the ψ coordinate is periodic. The Misner space metric in 2D is a flat space and regular everywhere. The curves defined by $t = t_0 > 0$, where t_0 is a constant being timelike provided $ds^2 < 0$ and closed (periodicity of ψ coordinate), are formed closed timelike curves. Levanony *et al* [23] generalized this flat Misner space in three and four-dimensional flat space. Li [24] constructed a Misner-like anti-de Sitter (AdS) metric in four-dimensional curved spacetime.

In this article, we attempt to construct a four-dimensional curved spacetime, not necessarily flat space, a generalization of the 2D Misner space. This curved spacetime is a nonvacuum solution of the field equations with suitable stress-energy tensor, satisfying the different energy conditions. Moreover, the four-dimensional

curved spacetime admits CTC which appear after a certain instant of time in a causally well-behaved manner, and may represent a time machine spacetime.

2. A four-dimensional curved spacetime

We attempt to construct a four-dimensional metric in curved spacetime given by

$$ds^2 = e^{-f(x,y)}(dx^2 + dy^2) - 2 dt d\psi - t d\psi^2, \quad (2)$$

where $f(x, y)$ is an arbitrary function of x, y . Here ψ coordinate is periodic, that is, each ψ is identified with $\psi + \psi_0$ for a certain parameter $\psi_0 > 0$. The ranges of the other coordinates are $-\infty < t, x, y < \infty$. The metric has signature $(-, +, +, +)$ and the determinant of the metric tensor $g_{\mu\nu}$ is

$$\det g = -e^{-2f(x,y)}. \quad (3)$$

The non-zero components of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R are

$$R_{xx} = R_{yy} = \frac{1}{2}(f_{,xx} + f_{,yy}), \quad R = e^{f(x,y)}(f_{,xx} + f_{,yy}) \quad (4)$$

where coma denotes derivative w. r. t. the argument.

The Kretschmann scalar of the spacetime is

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = e^{2f(x,y)}(f_{,xx} + f_{,yy})^2. \quad (5)$$

The non-zero components of the Riemann tensor $R_{\mu\nu\rho\sigma}$ are

$$R_{xyxy} = R_{yxyx} = -R_{xyyx} = -R_{yxx} = \frac{1}{2} e^{-f(x,y)}(f_{,xx} + f_{,yy}), \quad (6)$$

while rest are all vanishes.

For an example, we choose the following function for $f(x, y)$ given by

$$f(x, y) = \frac{1}{2}(x^2 + y^2), \quad (f_{,xx} + f_{,yy}) > 0. \quad (7)$$

Noted that if one choose the function $f(x, y) = \frac{1}{2}(x^2 - y^2)$, then the presented spacetime represents a flat space.

For the metric (2), a closed curve of constant t, x, y is spacelike provided $ds^2 > 0$ for $t = t_0 < 0$, and null curve provided $ds^2 = 0$ for $t = t_0 = 0$ which serve as the Chronology horizon. In addition, there are timelike curves provided $ds^2 < 0$ for $t = t_0 > 0$ and being closed (due to periodicity of ψ), formed closed timelike curves. These timelike closed curves evolve from an initial spacelike $t = \text{const} < 0$ hypersurface since $g^{tt} = t < 0$ and therefore, t is a time coordinate. A hypersurface $t = \text{const}$ is spacelike provided $g^{tt} < 0$ for $t = t_0 < 0$, timelike provided $g^{tt} > 0$ for $t = t_0 > 0$, and null provided $g^{tt} = 0$ for $t = t_0 = 0$. Noted that for constant x and y , the four-dimensional curved spacetime (2) reduces to the Misner space metric in 2D. Since few components of the Riemann tensor from (6) are non-zero. Therefore, the presented metric is a four-dimensional generalization of the Misner space in curved spacetime, satisfying the energy conditions which we shall discussed below next.

2.1. Stress-energy tensor: Type I fluid

First we construct a set of null tetrad vectors $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$ [25] for the metric (2). These are

$$\begin{aligned} k_\mu &= (0, 0, 0, 1), & l_\mu &= \left(1, 0, 0, \frac{t}{2}\right), \\ m_\mu &= \frac{e^{-\frac{f(x,y)}{2}}}{\sqrt{2}}(0, 1, i, 0), & \bar{m}_\mu &= \frac{e^{-\frac{f(x,y)}{2}}}{\sqrt{2}}(0, 1, -i, 0), \end{aligned} \quad (8)$$

where $i = \sqrt{-1}$. The set of tetrad vectors are such that the metric tensor for the spacetime (2) can be expressed as

$$g_{\mu\nu} = -k_\mu l_\nu - l_\mu k_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu. \quad (9)$$

The tetrad vectors (8) are null and orthogonal except that $k_\mu l^\mu = -1$ and $m_\mu \bar{m}^\mu = 1$.

Using the tetrad vectors we calculate the five Weyl scalars and these are

$$\Psi_2 = -\frac{1}{12} e^{f(x,y)}(f_{,xx} + f_{,yy}), \quad \Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4. \quad (10)$$

Thus the presented metric is clearly of type D in the Petrov classification scheme.

An orthonormal tetrad frame $\mathbf{e}_{(a)} = \{\mathbf{e}_{(0)}, \mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \mathbf{e}_{(3)}\}$ in terms of tetrad vectors can be expressed as

$$\begin{aligned}\mathbf{e}_{(0)} &= \mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{k} + \mathbf{l}), & \mathbf{e}_{(1)} &= \mathbf{w} = \frac{1}{\sqrt{2}}(\mathbf{m} + \bar{\mathbf{m}}), \\ \mathbf{e}_{(2)} &= \zeta = \frac{-i}{\sqrt{2}}(\mathbf{m} - \bar{\mathbf{m}}), & \mathbf{e}_{(3)} &= \mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{k} - \mathbf{l}),\end{aligned}\quad (11)$$

with the normalization conditions

$$u_\mu u^\mu = -1, \quad v_\mu v^\mu = 1 = w_\mu w^\mu = \zeta_\mu \zeta^\mu. \quad (12)$$

Therefore, the metric tensor $g_{\mu\nu}$ (9) can be expressed as

$$g_{\mu\nu} = -u_\mu u_\nu + v_\mu v_\nu + w_\mu w_\nu + \zeta_\mu \zeta_\nu. \quad (13)$$

The Einstein's field equations (taking cosmological term $\Lambda = 0$) are given by

$$G_{\mu\nu} = T_{\mu\nu} \quad \text{or} \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \quad (14)$$

where $G_{\mu\nu}$ is the Einstein tensor, and $T_{\mu\nu}$ is the stress-energy tensor. Here units are chosen such that $c = 1$ and $8\pi G = 1$. In terms of the Ricci tensor, the field equations can be written as

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T, \quad (15)$$

where $T_\mu^\mu = T$ is the trace of stress-energy tensor. Considering the stress-energy tensor Type II fluid (or radiation fields and null dust fluid) [26–31] given by

$$T_{\mu\nu} = \mu k_\mu k_\nu + (\rho + p)(k_\mu l_\nu + l_\mu k_\nu) + p g_{\mu\nu}, \quad (16)$$

where μ as the radiation energy density, ρ as the null string energy density, and p as the null string pressure.

For $\mu = 0$, the stress-energy tensor (16) corresponds to Type I fluid. For $\mu = 0$ and $p = 0$, it represents null string dust [29, 30]. Therefore, we have

$$T_{\mu\nu} = \rho(k_\mu l_\nu + l_\mu k_\nu), \quad T = T_\mu^\mu = -2\rho, \quad (17)$$

where $T_{\mu\nu} l^\mu k^\nu = \rho = T_{\mu\nu} k^\mu l^\nu$.

Projecting the stress-energy tensor (17) onto the orthonormal tetrad basis (11), one will find that $T_{(a)(b)} \equiv e_{(a)}^\mu e_{(b)}^\nu T_{\mu\nu}$ takes the following form

$$T_{(a)(b)} = \text{diag}(\rho, 0, 0, -\rho), \quad a, b = 0, 1, 2, 3, \quad (18)$$

which belongs to Type I fluid [26].

Using the tetrad vectors (8), one can easily show that the stress-energy tensor (17) satisfy the field equations (15) provided the null string energy-density is

$$\rho = \frac{1}{2} e^{f(x,y)} (f_{,xx} + f_{,yy}) = \frac{1}{2} e^{\frac{1}{2}(x^2+y^2)} > 0. \quad (19)$$

For studying the energy conditions, we consider the four-velocity vector \mathbf{U} for a time-like observer

$$U^\mu = \hat{\alpha} u^\mu + \hat{\beta} v^\mu + \hat{\gamma} w^\mu + \hat{\delta} \eta^\mu, \quad (20)$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ are arbitrary constants [31]. The timelike four-velocity vector U^μ is subjected to the condition that

$$U_\mu U^\mu = -\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2 + \hat{\delta}^2 < 0. \quad (21)$$

The stress-energy tensor (17) in terms of the orthonormal tetrad (11) is

$$T_{\mu\nu} = \rho(u_\mu u_\nu - v_\mu v_\nu). \quad (22)$$

The pressureless Type I fluid (17) satisfy the following energy conditions [25, 26, 31], namely,

- (i) the Weak Energy Condition: $T_{\mu\nu} U^\mu U^\nu \Rightarrow \rho > 0$.
- (ii) the Strong Energy Condition: $(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) U^\mu U^\nu \Rightarrow \rho > 0$.
- (iii) the Dominant Energy Condition: $T_{\mu\nu} U^\mu U^\nu \Rightarrow \rho > 0$, and $X_\mu = T_{\mu\nu} U^\nu$ such that $X_\mu X^\mu < 0$ implies $\rho^2 > 0$.

3. Conclusions

A topologically trivial four-dimensional curved spacetime, nonvacuum solution of the Einstein's field equations, was presented. The spacetime is free-from curvature singularities, and content null string fluid (or Type I fluid with zero pressure) as the matter-energy source, obeying the different energy conditions. The metric is of type D in the Petrov classification scheme, and may represent a time machine model in the sense that closed timelike curves develop at some particular moment from an initial spacelike hypersurface in a causally well-behaved manner. The presented metric is a generalization of 2D Misner space in four-dimensional curved spacetime.

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