Perturbation theory of N point mass gravitational lens

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Abstract

We make the first systematic attempt to determine using perturbation theory the positions of images by gravitational lensing due to arbitrary number of coplanar masses without any symmetry on a plane, as a function of lens and source parameters. We present a method of Taylor-series expansion to solve the lens equation under a small mass-ratio approximation. The advantage of our method is that it allows a systematic iterative analysis. We determine image positions for binary lens systems up to the third order in mass ratios and for arbitrary N point masses up to the second order. The number of the images that admit a small mass-ratio limit is less than the maximum number. It is suggested that positions of extra images could not be expressed as Maclaurin series in mass ratios. Magnifications are finally discussed.

1 Introduction

Gravitational lensing has become one of important subjects in modern astronomy and cosmology. It plays crucial roles as gravitational telescopes in various fields ranging from extra-solar planets to dark matter and dark energy at cosmological scales. This work focuses on gravitational lensing due to a N-point mass system. Definitely it is a challenging problem to express the image positions as functions of lens and source parameters.

There are several motivations for this work. One is that gravitational lensing offers a tool of discoveries and measurements of planetary systems (Schneider and Weiss 1986, Mao and Paczynski 1991, Gould and Loeb 1992, Bond et al. 2004, Beaulieu et al. 2006), compact stars, or a cluster of dark objects, which are difficult to probe with other methods. Gaudi et al. (2008) have recently found an analogy of the Sun-Jupiter-Saturn system by lensing. Efficient methods for producing light curves beyond binary cases are preferred.

Another motivation is to pursue a transit between a particle method and a fluid (mean field) one. For microlensing studies, particle methods are employed, because the systems consist of stars, planets or MACHOs. In cosmological lensing, on the other hand, light propagation is considered for the gravitational field produced by inhomogeneities of cosmic fluids, say galaxies or large scale structures of our Universe. It seems natural, though no explicit proof has been given, that observed quantities computed by continuum fluid methods will agree with those by discrete particle ones in the limit $N \to \infty$, at least on average, where N is the number of particles.

Galois showed that the fifth-order and higher polynomials cannot be solved algebraically. Hence, no formula for quintic equations is known.

In this work, we present a method of Taylor-series expansion to solve the lens equation under a small mass-ratio approximation.

2 Complex Formalism

We consider a lens system with N point masses. The mass and two-dimensional location of each body is denoted as M_i and the vector $\boldsymbol{E}_i$, respectively. For the later convenience, let us define the angular size of the Einstein ring as

$$\theta_E = \sqrt{\frac{4GM_{tot}D_{\rm LS}}{c^2 D_{\rm L} D_{\rm S}}}, \tag{1}$$

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where G is the gravitational constant, c is the light speed, M_{tot} is the total mass $\sum_{i=1}^{N} M_i$ and $D_{\rm L}$, $D_{\rm S}$ and $D_{\rm LS}$ denote distances between the observer and the lens, between the observer and the source, and between the lens and the source, respectively. In the unit normalised by the angular size of the Einstein ring, the lens equation becomes

$$\boldsymbol{\beta} = \boldsymbol{\theta} - \sum_{i}^{N} \nu_i \frac{\boldsymbol{\theta} - \boldsymbol{e}_i}{|\boldsymbol{\theta} - \boldsymbol{e}_i|^2}, \tag{2}$$

where $\boldsymbol{\beta} = (\beta_x, \beta_y)$ and $\boldsymbol{\theta} = (\theta_x, \theta_y)$ denote the vectors for the position of the source and image, respectively and we defined the mass ratio and the angular separation vector as $\nu_i = M_i/M_{tot}$ and $\boldsymbol{e}_i = \boldsymbol{E}_i/\theta_E = (e_{ix}, e_{iy})$.

Bourassa, Kantowski and Norton (1973), Bourassa and Kantowski (1975) introduced a complex notation to describe gravitational lensing. In a formalism based on complex variables, two-dimensional vectors for the source, lens and image positions are denoted as $w = \beta_x + i\beta_y$, $z = \theta_x + i\theta_y$, and $\epsilon_i = e_{ix} + ie_{iy}$, respectively. By employing this formalism, the lens equation is rewritten as

$$w = z - \sum_{i}^{N} \frac{\nu_i}{z^* - \epsilon_i^*}, \tag{3}$$

where the asterisk $*$ means the complex conjugate. The lens equation is non-analytic because it contains both z and z^*.

3 Perturbation

The lens equation is written as

$$C(z, z^*) = \sum_{k=2}^{N} \nu_k D_k(z^*), \tag{4}$$

where $C(z, z^*)$ was defined by Eq. (5) and we defined

$$C(z, z^*) = w - z + \frac{1}{z^*}, \tag{5}$$

$$D_k(z^*) = \frac{1}{z^*} - \frac{1}{z^* - \epsilon_k^*}. \tag{6}$$

We seek a solution in expansion series as

$$z = \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} \cdots \sum_{p_N=0}^{\infty} (\nu_2)^{p_2} (\nu_3)^{p_3} \cdots (\nu_N)^{p_N} z_{(p_2)(p_3)\cdots(p_N)}, \tag{7}$$

where $z_{(p_2)(p_3)\cdots(p_N)}$ is a constant to be determined iteratively.

It is convenient to normalise the perturbed roots in the units of the zeroth-order one as

$$\sigma_{(p_2)(p_3)\cdots(p_N)} = \frac{z_{(p_2)(p_3)\cdots(p_N)}}{z_{(0)\cdots(0)}}. \tag{8}$$

Here, we shall give perturbative roots when the zeroth order roots are not located at the lens position. Please see Asada (2009) for the special case when the zeroth order roots are located at the lens position.

3.1 0th order ($z_{(0)\cdots(0)} \neq \epsilon_i$ for $i = 1, \cdots, N$)

Zeroth-order solution is given as

$$z_{(0)\cdots(0)} = Aw, \tag{9}$$

where, we defined

$$A = \frac{1}{2}\left(1 \pm \sqrt{1 + \frac{4}{ww^*}}\right). \tag{10}$$

3.2 1st order ($z_{(0)\cdots(0)} \neq \epsilon_i$ for $i = 1, \cdots, N$)

We obtain the first order solution as

$$\begin{aligned} & z_{(0)\cdots(1_k)\cdots(0)} \\ = & \frac{b_{(0)\cdots(1_k)\cdots(0)} - a_{(0)\cdots(1_k)\cdots(0)} b^*_{(0)\cdots(1_k)\cdots(0)}}{1 - a_{(0)\cdots(1_k)\cdots(0)} a^*_{(0)\cdots(1_k)\cdots(0)}}. \end{aligned} \tag{11}$$

Here, we defined

$$a_{(0)\cdots(1_k)\cdots(0)} = \frac{1}{(z^*_{(0)\cdots(0)})^2}, \tag{12}$$

$$b_{(0)\cdots(1_k)\cdots(0)} = \frac{\epsilon_k^*}{z^*_{(0)\cdots(0)}(z^*_{(0)\cdots(0)} - \epsilon_k^*)}. \tag{13}$$

3.3 2nd order ($z_{(0)\cdots(0)} \neq \epsilon_i$ for $i = 1, \cdots, N$)

There are two types of second-order solutions as shown below. First, we obtain

$$\begin{aligned} & z_{(0)\cdots(2_k)\cdots(0)} \\ = & \frac{b_{(0)\cdots(2_k)\cdots(0)} - a_{(0)\cdots(2_k)\cdots(0)} b^*_{(0)\cdots(2_k)\cdots(0)}}{1 - a_{(0)\cdots(2_k)\cdots(0)} a^*_{(0)\cdots(2_k)\cdots(0)}}. \end{aligned} \tag{14}$$

Here, we defined

$$a_{(0)\cdots(2_k)\cdots(0)} = \frac{1}{(z^*_{(0)\cdots(0)})^2}, \tag{15}$$

$$b_{(0)\cdots(2_k)\cdots(0)} = -D_{k(0)\cdots(1_k)\cdots(0)} + \frac{(\sigma^*_{(0)\cdots(1_k)\cdots(0)})^2}{z^*_{(0)\cdots(0)}}. \tag{16}$$

Next, let us assume $k < \ell$. We obtain

$$\begin{aligned} & z_{(0)\cdots(1_k)\cdots(1_\ell)\cdots(0)} \\ = & \frac{b_{(0)\cdots(1_k)\cdots(1_\ell)\cdots(0)} - a_{(0)\cdots(1_k)\cdots(1_\ell)\cdots(0)} b^*_{(0)\cdots(1_k)\cdots(1_\ell)\cdots(0)}}{1 - a_{(0)\cdots(1_k)\cdots(1_\ell)\cdots(0)} a^*_{(0)\cdots(1_k)\cdots(1_\ell)\cdots(0)}}. \end{aligned} \tag{17}$$

Here, we defined

$$a_{(0)\cdots(1_k)\cdots(1_\ell)\cdots(0)} = \frac{1}{(z^*_{(0)\cdots(0)})^2}, \tag{18}$$

$$\begin{aligned} b_{(0)\cdots(1_k)\cdots(1_\ell)\cdots(0)} = & -D_{k(0)\cdots(1_\ell)\cdots(0)} - D_{\ell(0)\cdots(1_k)\cdots(0)} \\ & + \frac{2\sigma^*_{(0)\cdots(1_k)\cdots(0)} \sigma^*_{(0)\cdots(1_\ell)\cdots(0)}}{z^*_{(0)\cdots(0)}}. \end{aligned} \tag{19}$$

4 Conclusion

Under a small mass-ratio approximation, we have developed a perturbation theory of N coplanar (in the thin lens approximation) point-mass gravitational lens systems without symmetries on a plane. The system can be separated into a single mass lens as a background and its perturbation due to the remaining point masses.

The number of the images that admit the small mass-ratio limit is less than the maximum number. This suggests that the other images do not have the small mass limit. Therefore, it is conjectured that positions of the extra images could not be expressed as Maclaurin series in mass ratios. This is a topic of future work.

There are possible applications along the course of the perturbation theory of N point-mass gravitational lens systems. For instance, it will be interesting to study lensing properties such as magnifications by using the functional form of image positions. Furthermore, the validity of the present result may be limited in the weak field regions. It is important also to extend the perturbation theory to a domain near the strong field.

References

[1] Asada H., 2009, MNRAS, *in press* (ArXiv:0809.4122)

[2] Beaulieu J. P., et al., 2006, Nature, 439, 437

[3] Bond I. A., et al., 2004, ApJ, 606, L155

[4] Bourassa R. R., Kantowski R., Norton T. D., 1973, ApJ, 185, 747

[5] Bourassa R. R., Kantowski R., 1975, ApJ, 195, 13

[6] Gaudi B. S., et al., 2008, Science, 319, 927

[7] Gould A., Loeb A., 1992, ApJ, 396, 104

[8] Mao S., Paczynski B., 1991, ApJ, 374, 37L

[9] Schneider P., Weiss A., 1986, A&A. 164, 237