Journal of Physics: Conference Series

PAPER • OPEN ACCESS

New class of entropy-power-based uncertainty relations

To cite this article: P Jizba *et al* 2017 *J. Phys.: Conf. Ser.* **880** 012054

IOP Conf. Series: Journal of Physics: Conf. Series **880** (2017) 012054 doi:10.1088/1742-6596/880/1/012054

New class of entropy-power-based uncertainty relations

P Jizba[1], **A Hayes**[2] **and JA Dunningham**[2]

[1] FNSPE, Czech Technical University in Prague, Břehová 7, 115 19 Praha 1, Czech Republic
[2] Department of Physics and Astronomy, University of Sussex, Brighton BN1 9QH, UK

E-mail: p.jizba@fjfi.cvut.cz; Anthony.Hayes@sussex.ac.uk; J.Dunningham@sussex.ac.uk

Abstract. We use the concept of entropy power to introduce a new one-parameter class of information-theoretic uncertainty relations. This class constitutes an infinite hierarchy of uncertainty relations, which allows to determine the shape of the underlying information-distribution function by measuring the relevant entropy powers. The efficiency of such uncertainty relations in quantum mechanics is illustrated with two examples: superpositions of two squeezed states and the Lévy-type heavy-tailed wave function. Improvement over both the variance-based and Shannon entropy based uncertainty relations is demonstrated in both these cases.

1. Introduction

In his 1948 seminal paper, Shannon laid down the foundations of modern communication theory currently known as information theory [1]. He was instrumental in pointing out that, in contrast to discrete signals or messages where information is quantified via (Shannon's) entropy, the case with continuous variables is less satisfactory. The continuous version of Shannon's entropy (SE) — the so-called differential entropy — may take negative values [1, 2] and so it does not enjoys the same status as its discrete-variable counterpart. To solve a range of communication theoretic problems related to continuous cases Shannon shifted the emphasis from the differential entropy to entropy power (EP). The EP represents the variance of a would-be Gaussian random variable with the same differential entropy as the random variable under investigation. In this connection we remind that the concept of EP was used by Shannon [1] in order to formulate information theory for continuous random variables. Since then, the EP has proved to be essential in number of information-theoretic applications ranging from interference channels to secrecy capacity [3, 4, 5, 6]. It has also lead to new advances in information parametric statistics [7] and network information theory. Apart from its rôle in information theory, the EP has found wide use in pure mathematics, namely in theory of inequalities [8] and mathematical statistics [9, 10, 11].

Information theory now extends far beyond the realm of communications and the same principles and concepts can be employed in applications that include statistical physics, biological science and quantum mechanics [12]. In this paper we focus on the application of the EP to quantum-mechanical uncertainty relations (URs). In essence, quantum-mechanical URs place fundamental limits on the accuracy with which one is able to know the values of different physical quantities. In the 1920s, Kennard and independently Robertson and Schrödinger reformulated original Heisenberg's UR in terms of variances of the observables [13, 14, 15]. In 1959, Stam [16] conjectured that the EP could be used to obtain Heisenberg's UR. This conjecture was bolstered in [17] by showing that the usual Schrödinger–Robertson variance-based URs (VURs) [14, 15] can be derived from entropic URs. VURs are useful and widely applied but have two major restrictions: Firstly, the product of the conjugate variances is a single number and so can only give partial information about the underlying states; secondly, variances are only useful concepts for well-behaved bell-like distributions. For heavy-tailed or multimodal distributions, the variances can be large or even infinite, making VURs ill-suited or even useless.

Here we show that Stam conjectured UR is just a member of a one-parameter class of EP-based inequalities, all of which stem from yet another important information measure, namely the differential Rényi entropy (RE) [18, 19, 20] and the ensuing Rényi entropy power (REP). We prove that this class constitutes an infinite hierarchy of higher-order cumulant URs, which allows one in principle to reconstruct the underlying information-distribution function in a process akin to quantum state tomography [21] using EPs in the place of the usual measurements.

For a better understanding of the upcoming results, we begin in Section 2 by introducing the concept of Rényi's EP. By virtue of the Beckner–Babenko theorem we derive in Section 3 the one-parameter hierarchy of entropy-power-based URs. After this we introduce, in Section 4 the concepts of information distribution and show how cumulants of the information distribution can be obtained from the knowledge of EPs. With cummulants at hand one is (in principle) able to reconstruct the underlying information distribution by means of the reconstruction theorem. Section 5 is devoted to the extension of REPURs to the quantum-mechanical setting. In Section 6 we illustrate the inner workings of the REPURs obtained on the two quantum-mechanical examples. Various remarks and generalizations are addressed in the concluding section.

2. Differential entropies and entropy powers

Let $\boldsymbol{X}$ be a random vector in $\mathbb{R}^d$ with the probability density function (PDF), $\wp$. The *differential entropy* $H(\boldsymbol{X})$ of $\boldsymbol{X}$ is defined as [1]

$$H(\boldsymbol{X}) \ = \ -\int_{\mathbb{R}^d} \wp(\boldsymbol{x}) \log_2 \wp(\boldsymbol{x}) \ d\boldsymbol{x} \, . \tag{1}$$

The discrete version of (1) is nothing but the SE [1] and in such a case it represents an average number of binary questions that are needed to reveal the value of $\boldsymbol{X}$. Strictly speaking, $H(\boldsymbol{X})$ defined in (1) is not a proper entropy but rather an information gain [2, 18].

The entropy power $N(\boldsymbol{X})$ of $\boldsymbol{X}$ is the unique number such that [1, 22]

$$H(\boldsymbol{X}) \ = \ H(\boldsymbol{X}_G) \ , \tag{2}$$

where $\boldsymbol{X}_G$ is a Gaussian random vector with zero mean and variance equal to $N(\boldsymbol{X})$, i.e., $\boldsymbol{X}_G \sim \mathcal{N}(\mathbf{0}, N(\boldsymbol{X})\mathbf{1}_{d\times d})$. Eq.(2) can be equivalently rewritten in the form

$$H(\boldsymbol{X}) \ = \ H\Big(\sqrt{N(\boldsymbol{X})} \cdot \boldsymbol{G}\Big) \, , \tag{3}$$

with $\boldsymbol{G}$ representing a Gaussian random vector with zero mean and unit covariance matrix. The solution of (3) has the form [1]

$$N(\boldsymbol{X}) \ = \ \frac{2^{\frac{2}{d} \, H(\boldsymbol{X})}}{2\pi e} \, . \tag{4}$$

When the Shannon differential entropy is measured in *nats* then for the EP we have

$$N(\boldsymbol{X}) \;=\; \frac{1}{2\pi e}\exp\left(\frac{2}{d}\,H(\boldsymbol{X})\right). \tag{5}$$

Let us now assume that $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are two independent random vectors of finite variance. The entropy power (4) [as well as (5)] satisfies the following important *entropy power inequality* [16, 20]

$$N(\boldsymbol{X}_1 + \boldsymbol{X}_2) \;\geq\; N(\boldsymbol{X}_1) \;+\; N(\boldsymbol{X}_2)\,, \tag{6}$$

where the equality holds if and only if $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are multivariate normal random variables with proportional covariance matrices [1]. In general, inequality (6) does not hold when $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are discrete random variables and the differential entropy $H(\boldsymbol{X})$ is replaced with the (discrete) Shannon entropy.

The differential RE $I_p(\boldsymbol{X})$ of $\boldsymbol{X}$ is defined as [2, 18, 19]

$$I_p(\boldsymbol{X}) \;=\; \frac{1}{(1-p)}\log_2\left(\int_M d\boldsymbol{x}\,\wp^p(\boldsymbol{x})\right)\,, \tag{7}$$

where the index $p \in \mathbb{R}^+$. By virtue of L'Hospital's rule we obtain that $\lim_{p\to 1} I_p(\boldsymbol{X}) = H(\boldsymbol{X})$. As was the case with H, I_p is also additive for independent events [2]. The p-th Rényi entropy power $N_p(\boldsymbol{X})$ is defined as the solution of the equation

$$I_p\left(\boldsymbol{X}\right) \;=\; I_p\left(\sqrt{N_p(\boldsymbol{X})}\cdot \boldsymbol{G}\right)\,, \tag{8}$$

where $\boldsymbol{G}$ represents a Gaussian random vector with zero mean and unit covariance matrix. This equation was studied in [23, 24] where it was shown that the only class of solution of (8) is

$$N_p(\boldsymbol{X}) \;\;=\;\; \frac{1}{2\pi}p^{-p'/p}\exp\left(\frac{2}{d}\,I_p(\boldsymbol{X})\right), \tag{9}$$

with $1/p + 1/p' = 1$ and $p \in \mathbb{R}^+$. So, p' and p are Hölder conjugates. In addition, one can check that $\lim_{p\to 1_+} N_p(\boldsymbol{X}) = N(\boldsymbol{X})$. For explicitness sake we consider here *nats* as units of information. We might also note in passing that from (9) follows that $N_p(\sigma\boldsymbol{G}) = \sigma^2$, i.e. for Gaussian processes the EP is simply the variance σ^2. When $\boldsymbol{X}_G$ represents a random Gaussian vector of zero mean and covariance matrix $\mathbb{K}_{ij}$, then $N_p(\boldsymbol{X}_G) = [\det(\mathbb{K}_{ij})]^{1/d} \equiv |\mathbb{K}|^{1/d}$. Importantly, since the REs are measurable qunatities [25, 26], the associated REPs are experimentally accessible. For some recent applications of the REs in quantum theory see, e.g., Refs. [27, 28, 29].

3. Entropy-power-based Uncertainty Relations

In the following we will need the so-called Beckner–Babenko theorem [30, 31].

Beckner–Babenko Theorem: Let

$$f^{(2)}(\boldsymbol{x}) = \int_{\mathbb{R}^d} e^{2\pi i \boldsymbol{x}.\boldsymbol{y}}\; f^{(1)}(\boldsymbol{y})\; d\boldsymbol{y}\,,$$

then for $p \in [1,2]$

$$|(p')^{d/2}|^{1/p'}\|f^{(2)}\|_{p'} \;\leq\; |p^{d/2}|^{1/p}\|f^{(1)}\|_p\,, \tag{10}$$

where p and p' are the Hölder conjugates and

$$\|g\|_p \equiv \left(\int_{\mathbb{R}^d} |g(\boldsymbol{y})|^p \, d\boldsymbol{y}\right)^{1/p}, \tag{11}$$

for any $g \in L^p(\mathbb{R}^d)$. Of course, the role of $f^{(1)}$ and $f^{(2)}$ may be interchanged in the inequality (10). An elementary proof can be found, e.g., in [23]. Inequality (10) is saturated only for Gaussian functions [31, 32].

Anticipating quantum-mechanical applications we define the square-root density likelihood as $\sqrt{\wp(\boldsymbol{y})} \equiv |f(\boldsymbol{y})|$. After simple algebra we recast (10) in the form [23]

$$\left(\int_{\mathbb{R}^d} [\wp^{(2)}(\boldsymbol{y})]^{(1+t)} \, d\boldsymbol{y}\right)^{1/t} \left(\int_{\mathbb{R}^d} [\wp^{(1)}(\boldsymbol{x})]^{(1+r)} \, d\boldsymbol{x}\right)^{1/r} \leq [2(1+t)]^d \, |t/r|^{d/2r}. \tag{12}$$

Here, $r = p/2-1$ and $t = p'/2-1$. Because $1/p+1/p' = 1$ we have the constraint $t = -r/(2r+1)$. Since $p \in [1, 2]$ one has $r \in [-1/2, 0]$ and $t \in [0, \infty)$. Eq. (12) can be equivalently written as

$$I_{1+t}(\wp^{(2)}) \ + \ I_{1+r}(\wp^{(1)}) \ \geq \ \frac{1}{r}\log_2[2(1+r)]^{d/2} \ + \ \frac{1}{t}\log_2[2(1+t)]^{d/2}. \tag{13}$$

Note, in particular, that in the limit $t \to 0_+$ and $r \to 0_-$ this boils down to

$$H(\wp^{(2)}) \ + \ H(\wp^{(1)}) \ \geq \ \log_2\left(\frac{e}{2}\right)^d, \tag{14}$$

which is just the classical Hirschman conjecture for Shannon's differential entropies [17, 33]. However, the semidefiniteness of I_p makes the URs (13) unsatisfactory. Fortunately, we can use the positiveness of REPs and rewrite (13) as

$$N_{1+t}(\wp^{(2)})N_{1+r}(\wp^{(1)}) \ \equiv \ N_{p/2}(\boldsymbol{X})N_{q/2}(\boldsymbol{Y}) \ \geq \ \frac{1}{16\pi^2}, \tag{15}$$

where $q \equiv p'$ and the REs involved are measured in bits. This is a one-parameter family of inequalities (p and q are the Hölder conjugates). In contrast to (13), the r.h.s. of (15) represents a *universal* lower bound independent of t and r. Note that when $\boldsymbol{X}$ is Gaussian, then $\boldsymbol{Y}$ is also Gaussian and (15) reduces to

$$|\mathbb{K}_{\boldsymbol{X}}|^{1/d}|\mathbb{K}_{\boldsymbol{Y}}|^{1/d} \ = \ \frac{1}{16\pi^2}. \tag{16}$$

Here, $|\mathbb{K}_{\boldsymbol{X}}|$ and $|\mathbb{K}_{\boldsymbol{Y}}|$ are determinants of the respective covariance matrices. The equality follows from the saturation of the inequality (10) by Gaussian functions.

By assuming that a PDF has a finite covariance matrix $(\mathbb{K}_{\boldsymbol{X}})_{ij}$ then

$$N(\boldsymbol{X}) \ \leq \ |\mathbb{K}_{\boldsymbol{X}}|^{1/d} \ \leq \ \sigma^2_{\boldsymbol{X}}, \tag{17}$$

with equality in the first inequality if and only if $\boldsymbol{X}$ is a Gaussian vector, and in the second if and only if $\boldsymbol{X}$ has covariance matrix that is proportional to the identity matrix. The proof of (17) is based on the non-negativity of the Kullback–Leibler divergence and can be found, e.g. in [23, 34, 35]. Inequality (17) immediately gives

$$\sigma^2_{\boldsymbol{X}}\sigma^2_{\boldsymbol{Y}} \ \geq \ |\mathbb{K}_{\boldsymbol{X}}|^{1/d}|\mathbb{K}_{\boldsymbol{Y}}|^{1/d} \geq \ N(\boldsymbol{X})N(\boldsymbol{Y}) \ \geq \ \frac{1}{16\pi^2}, \tag{18}$$

which saturates only for Gaussian (respective white) random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$. Note, that when $(\mathbb{K}_{\boldsymbol{X}})_{ij}$ and $(\mathbb{K}_{\boldsymbol{Y}})_{ij}$ exist then (18) automatically implies the conventional Robertson–Schrödinger VUR. Since the VUR is implied by the Shannon EPUR alone, a question arises; in what sense is the general set of inequalities (15) more informative than the special case $r = t = 0$?

IOP Conf. Series: Journal of Physics: Conf. Series **880** (2017) 012054 doi:10.1088/1742-6596/880/1/012054

4. Reconstruction theorem

To guide our intuition and to show the conceptual underpinning for REPURs (12) we first note that the differential RE can be written as ($\mathbb{E}[\cdots]$ denotes the mean value)

$$I_p(\boldsymbol{X}) \; = \; \frac{1}{(1-p)} \log_2 \mathbb{E}\left[2^{(1-p)i_{\boldsymbol{X}}}\right] . \tag{19}$$

Here $i_{\boldsymbol{X}}(\boldsymbol{x}) \equiv -\log_2 \wp(\boldsymbol{x})$ is the information in $\boldsymbol{x}$ (with respect to the PDF $\wp(\boldsymbol{x})$). From (19), the differential RE can be viewed as a reparametrized version of the *cumulant generating function* of the information random variable $i_{\boldsymbol{X}}(\boldsymbol{X})$. The ensuing cumulant expansion is

$$pI_{1-p}(\boldsymbol{X}) \; = \; \log_2 e \sum_{n=1}^{\infty} \frac{\kappa_n(\boldsymbol{X})}{n!} \left(\frac{p}{\log_2 e}\right)^n \; = \; pH(\boldsymbol{X}) \; + \; \log_2 e \sum_{n=2}^{\infty} \frac{\kappa_n(\boldsymbol{X})}{n!} \left(\frac{p}{\log_2 e}\right)^n , \tag{20}$$

where $\kappa_n(\boldsymbol{X}) \equiv \kappa_n(i_{\boldsymbol{X}})$ denotes the n-th cumulant of $i_{\boldsymbol{X}}(\boldsymbol{X})$ (in units of $bits^n$). From (20) it follows that REPs can be written in terms of κ_n's. In fact, N_p's of order $p > 0$ can, in principle, uniquely determine the underlying information PDF (for the proof see Ref. [36]). So, the REPURs of different orders provide additional structural constraints between $\wp^{(1)}$ and $\wp^{(2)}$ which cannot be seen with the VUR or Shannon entropy UR alone. In this connection we list some further salient results [36]:

a) Only Gaussian PDFs saturate all REPURs. REPURs with $r = -1/2$ can be saturated with a wider class of PDFs. b) When $\wp(\boldsymbol{x})$ is close to (or equimeasurable with) a Gaussian PDF then only N_p's with p's in a neighborhood of 1 are needed. The closer the shape is to the Gaussian PDF, the smaller neighborhood of 1 needed. c) For heavy-tailed PDFs, we need to know all REPURs to estimate the shape of the underlying information PDF. d) The non-linear nature of the RE emphasizes the more probable parts of the PDF (typically the middle parts) for Rényi's index $p > 1$ while for $p < 1$ the less probable parts of the PDF (typically the tails) are accentuated. So, when the accentuated parts in $|\psi|^2$ and $|\hat{\psi}|^2$ are close to Gaussian PDF sectors, the associated REPUR will approach its lower bound. In the asymptotic regime when $r = -1/2$, the saturation of the REPUR means that the peak of $\wp^{(1)}$ and tails of $\wp^{(2)}$ are Gaussian, though both $\wp^{(1)}$ and $\wp^{(2)}$ might be non-Gaussian.

5. REPUR in Quantum Mechanics

Let us consider state vectors that are Fourier transform duals – the most prominent example being *phase quadratures* with the configuration and momentum space wave functions as special examples. In the latter case there is a reciprocal relation between $\psi(\mathbf{x})$ and $\hat{\psi}(\mathbf{p})$, namely

$$\psi(\boldsymbol{x}) \; = \; \int_{\mathbb{R}^d} e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}\, \hat{\psi}(\boldsymbol{p}) \, \frac{d\boldsymbol{p}}{(2\pi\hbar)^{d/2}} . \tag{21}$$

The Riesz–Fischer equality [37] guarantees mutual normalization $\|\psi\|_2 = \|\hat{\psi}\|_2 = 1$. Let us define

$$f^{(2)}(\boldsymbol{x}) \; = \; (2\pi\hbar)^{d/4}\psi(\sqrt{2\pi\hbar}\,\boldsymbol{x}) , \qquad f^{(1)}(\boldsymbol{p}) \; = \; (2\pi\hbar)^{d/4}\hat{\psi}(\sqrt{2\pi\hbar}\,\boldsymbol{p}) . \tag{22}$$

The factor $(2\pi\hbar)^{d/4}$ ensures that the new functions are normalized (in sense of $\| \ldots \|_2$) to unity. With these we have the same structure of the Fourier transform as in the Beckner–Babenko theorem. Consequently we can write the associated RE-based URs (13) in the form

$$I_{1+t}(|\psi|^2) \; + \; I_{1+r}(|\hat{\psi}|^2) \; \geq \; \frac{1}{r}\log_2\left(\frac{1+r}{\pi\hbar}\right)^{d/2} \; + \; \frac{1}{t}\log_2\left(\frac{1+t}{\pi\hbar}\right)^{d/2} , \tag{23}$$

where we have employed of the identity

$$I_p(|f^{(1)}|^2) \ = \ I_p(|\hat{\psi}|^2) - \frac{d}{2}\log_2(2\pi\hbar)\,, \tag{24}$$

(and similarly for $f^{(2)}$). Relation (24) just states that the scaled PDFs $|f^{(1)}|^2$ and $|f^{(2)}|^2$ obtained from (22) are less peaked (and hence less informative) than the original PDFs $|\hat{\psi}|^2$ and $|\psi|^2$, respectively. Consequently, we increase our ignorance when passing from $\hat{\psi}$ to $f^{(1)}$, and from ψ to $f^{(2)}$. In terms of the REP we can recast (23) into the form [cf. Eq. (15)]

$$N_{1+t}(|\psi|^2)N_{1+r}(|\hat{\psi}|^2) \ \geq \ \frac{\hbar^2}{4}\,. \tag{25}$$

This is akin to the Robertson–Schrödinger VUR, but is now a family of relations parametrised by t each having the same universal lower bound $\hbar^2/4$. It should be noted that the familiar VUR follows directly from Shannon's EPUR alone since

$$\sigma_x^2\sigma_p^2 \ \geq \ N_1(|\psi|^2)N_1(|\hat{\psi}|^2) \ \geq \ \frac{\hbar^2}{4}\,. \tag{26}$$

In the special case of Gaussian PDFs, the whole family reduces to the single relation

$$\sigma_x^2\sigma_p^2 \ = \ N_{1+t}(|\psi_G|^2)N_{1+r}(|\hat{\psi}_G|^2) \ = \ \frac{\hbar^2}{4}\,, \tag{27}$$

which is nothing but the familiar coherent-state VUR.

6. Applications in Quantum Mechanics

As a first example we consider an optical state that is pertinent to quantum metrology [38]. It consists of a superposition of a vacuum $|0\rangle$ and a squeezed vacuum $|z_\zeta\rangle$ which has the form $|\psi_\zeta\rangle = \mathcal{N}\left(|0\rangle + |z_\zeta\rangle\right)$, with the normalization factor $\mathcal{N} = 1/\sqrt{2+2(\cosh\zeta)^{-1/2}}$, and

$$|z_\zeta\rangle \ = \ \sum_{m=0}^{\infty}(-1)^m\frac{\sqrt{(2m)!}}{2^m m!}\left[\frac{(\tanh\zeta)^m}{\sqrt{\cosh\zeta}}\right]\,|2m\rangle\,, \tag{28}$$

where $|2m\rangle$ are even-number energy eigenstates and $\zeta \in \mathbb{R}$ is the squeezing parameter. If we rewrite $|\psi_\zeta\rangle$ in the basis of the eigenstates of the position and momentum *quadrature operators*

$$\hat{X} \ = \ \sqrt{\frac{\hbar}{2\omega}}(\hat{a} \ + \ \hat{a}^\dagger), \quad \hat{P} \ = \ -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a} \ - \ \hat{a}^\dagger)\,, \tag{29}$$

(ω is the optical frequency and $\hat{a}$ and $\hat{a}^\dagger$ are the photon ladder operators), we get for the PDFs

$$\begin{aligned}|\psi_\zeta(x)|^2 \ &= \ \sqrt{\frac{\omega}{\pi\hbar}}\left|\exp\left(-\frac{\omega x^2}{2\hbar}\right) \ + \ e^{\zeta/2}\exp\left(-\frac{\omega e^{2\zeta}x^2}{2\hbar}\right)\right|^2,\\ |\hat{\psi}_\zeta(p)|^2 \ &= \ \frac{1}{\sqrt{\pi\hbar\omega}}\left|\exp\left(-\frac{p^2}{2\hbar\omega}\right) + e^{-\zeta/2}\exp\left(-\frac{e^{-2\zeta}p^2}{2\hbar\omega}\right)\right|^2.\end{aligned} \tag{30}$$

Here the normalization factor $\mathcal{N}^2$ was dropped out. Relations (30) can be now employed to calculate the product $N_{1+t}(x)N_{1+r}(p)$ for different values of t. What we find is that the lower bound $\hbar^2/4$ is saturated for both $N_\infty(x)N_{1/2}(p)$ and $N_{1/2}(x)N_\infty(p)$ irrespective of the squeezing parameter ζ. From our foregoing analysis of REPURs this is easy to understand because the infinite and half indices of the EPs focus on the peak and tails of the PDF, respectively and from (30) we see that both the x and p PDFs are Gaussian in the tails as well as at the peaks (i.e., at $x = p = 0$). A REPUR is saturated only when the RE-accentuated sectors in both dual PDFs are Gaussian [36]. On the other hand, it is also clear that both PDFs (30) as a whole are highly non-Gaussian. We would therefore not expect REPURs with different indices to saturate the bound.

It is also interesting to compute the VUR for the state $|\psi_\zeta\rangle$. The associated variances read

$$\begin{aligned}\langle(\Delta X)^2\rangle_\zeta &= \mathcal{N}^2\frac{\hbar}{\omega}\left[\frac{1}{2}(1+e^{-2\zeta}) \;+\; \sqrt{\text{sech}\zeta}\,(1-\tanh\zeta)\right],\\ \langle(\Delta P)^2\rangle_\zeta &= \mathcal{N}^2\hbar\omega\left[\frac{1}{2}(1+e^{2\zeta}) \;+\; \sqrt{\text{sech}\zeta}\,(1+\tanh\zeta)\right].\end{aligned} \tag{31}$$

Note that for $\zeta = 0$, we have $\langle(\Delta X)^2\rangle_0\langle(\Delta P)^2\rangle_0 = \hbar^2/4$, i.e. the VUR is saturated. This is not surprising because the vacuum $|\psi_0\rangle = |0\rangle$ is the Glauber coherent state. However, as the squeezing parameter ζ is increased the product blows up rapidly, which makes the VUR uninformative. So the set of REPURs outperforms both the Shannon EPUR and the VUR by providing more information on the structural features of $|\psi_\zeta\rangle$ via the related PDFs (e.g., Gaussian peaks and tails in p-x quadratures).

We note that the conventional VUR does not pose any restriction on the variance of the observable whose conjugate observable has a PDF with infinite covariance matrix. So, such a state is maximally uncertain. In contrast to this, the set of related REPURs brings considerably more information about the structure of these states. To illuminate this point we discuss in our second example a power-law tail wave packet (PLTWP). PTLWPs are emblematic examples of quantum states with anomalous behavior during their temporal evolution [39]. For definiteness we will consider the PLTWP of the form

$$\psi(x) \;=\; \sqrt{\frac{\gamma}{\pi}}\sqrt{\frac{1}{\gamma^2+(x-m)^2}} \quad\Rightarrow\quad |\psi(x)|^2 \;=\; \frac{\gamma}{\pi}\,\frac{1}{\gamma^2+(x-m)^2}\,. \tag{32}$$

The latter is the Cauchy PDF with a scale parameter γ and median m. The Fourier transform gives

$$\hat{\psi}(p) \;=\; e^{-imp/\hbar}\sqrt{\frac{2\gamma}{\pi^2\hbar}}\,K_0(\gamma|p|/\hbar) \quad\Rightarrow\quad |\hat{\psi}(p)|^2 \;=\; \frac{2\gamma}{\pi^2\hbar}\,K_0^2(\gamma|p|/\hbar)\,, \tag{33}$$

where K_0 is the modified Bessel function. With these we can directly write two emblematic REPURs

$$N_1(|\hat{\psi}|^2)N_1(|\psi|^2) \;=\; 0.0052\,\hbar^2\pi^4 \;>\; \hbar^2/4\,, \tag{34}$$

$$N_{1/2}(|\hat{\psi}|^2)N_\infty(|\psi|^2) \;=\; \frac{\hbar^2}{4}\,. \tag{35}$$

Note also that $\langle(\Delta p)^2\rangle_\psi = \hbar^2\pi/16c^2$ and $\langle(\Delta x)^2\rangle_\psi \to \infty$ (the latter behavior is typical for many PLTWPs), and so the Schrödinger–Robertson's VUR is completely uninformative. What can we conclude from (34)–(35)? First, the REPUR (35) is saturated. This implies that the peak part of $|\psi|^2$ and the tail part of $|\hat{\psi}|^2$ are Gaussian (as can be directly checked). Shannon's EPUR (34) implies: a) the involved PDFs are not Gaussian, b) in contrast to other REPURs it quantifies only shape structures of PDFs but is γ insensitive [36], c) from (14) [cf. also (24)] the lower bound of Hirschman's UR is $\log_2(\pi\hbar e)$ while (34) gives $\log_2(\pi\hbar e) + 0.5141$, so one could still gain 0.5141 bits of information should the system by prepared in a Gaussian state. Finally, we note that $N_\infty(|\hat{\psi}|^2) = 0$ and $N_{1/2}(|\psi|^2) \to \infty$, hence the related REPUR is indeterminate. This behavior is easy to understand. For a strongly leptokurtic PDF (such as $|\psi|^2$) $N_{1/2}$ emphasizes the very flat power-law tails of $|\psi|^2$, and hence $N_{1/2}$ represents the variance of a very flat Gaussian PDF. Similarly, N_∞ accentuates only the peak part of $|\hat{\psi}|^2$ that is sharply (almost δ-function) peaked, and so N_∞ represents the variance of the Gaussian PDF with zero spread. In principle it is possible to deduce from REPs also the scaling characteristics for PLTWPs. This fact was discussed in Ref. [36].

7. Conclusions and perspectives

Here we have introduced a new one-parameter class of REP-based URs for pairs of observables in an infinite-dimensional Hilbert space. On the conceptual side, the infinite tower of inequalities obtained possess a clear advantage over the single VUR by revealing the finer structure of the underlying PDFs further to their standard deviations. This was demonstrated on two relevant quantum mechanical examples.

On application side the proposed REPURs could be directly relevant, e.g., in

a) *quantum-state diagnostics, quantum metrology and measurement*:

- whenever Heisenberg (variance based)URs are inconvenient (uninformative), e.g., when state probability distributions (PDFs) are multimodal or do not have finite variances (heavy tailed PDFs).
- respective members in the tower of URs represent higher-order statistics (i.e., beyond simple variances) restrictions between conjugated-state PDFs. So by knowing one entropy power one can restrict the shape of the conjugated-state PDF via reconstruction theorem
- the more entropy powers are measured or controlled the more structural details about conjugated-state PDF can be obtained (number and height of peaks, tails asymptotic behavior, etc.). This is done via information scan and reconstruction theorem
- designing quantum states (or better their associated PDFs) via the reconstruction theorem. This may give a lot of insight into the structure of quantum states spectrum and so enable states to be tailored for specific metrology tasks.

b) *foundational and conceptual issues*:

- black hole mass-temperature (Hawking) relation can be obtained from the Heisenberg (variance-based) UR via Susskind–Adler mechanism. Mass-temperature relation could thus get corrections when the higher-order statistics restrictions are taken into account.
- origin and distribution of the structure in the universe is attributed to seeding quantum inhomogeneities formed during and after inflation. Usual picture is that these originate from density fluctuations controlled by Heisenberg URs. If the higher-order statistics URs are employed, the power spectrum of the density fluctuations will get corrections. This might better fit the experimentally observed spectrum (the Planck satellite and WMAP data).
- on more speculative vein, one can study the quantum vacuum structure (e.g., in QCD) when the higher-order correlations in energy fluctuations are taken into account.
- REPURs succinctly grasp the otherwise cumbersome higher-order cumulant URs but at the same time both the form and intuition behind the entropy-power based URs are very close to the usual Heisenberg UR.

Most of these issues are presently under active investigation.

Acknowledgments
It is pleasure to acknowledge helpful feedbacks from R Rivers and I Białynicki-Birula. P.J. was supported by the Czech Science Foundation Grant No. 17-33812L. J.D. acknowledges support from DSTL and the UK EPSRC through the NQIT Quantum Technology Hub (Grant No. EP/M013243/1).

References

[1] Shannon C E 1948 *A mathematical theory of communication Bell Syst. Tech. J.* **27** 379; 623
[2] Jizba P and Arimitsu T 2004 *Ann. Phys.* **312** 17
[3] Liu R, Liu T, Poor H V and Shamai S 2010 *IEEE Transactions on Information Theory* **56** 1865
[4] Costa M H M 1985 *IEEE Transactions on Information Theory* **31** 607
[5] Polyanskiy Y and Wu Y 2015 *Preprint* `arXiv:1504.04419`
[6] Bagherikaram G, Motahari A S and Khandani A K 2013 *IEEE Transactions on Information Theory* **59** 2673
[7] Frieden B R 2004 *Science from Fisher Information: A Unification* (Cambridge: Cambridge Univ. Press)
[8] Courtade T A 2016 *Preprint* `arXiv:1602.03033`
[9] Barron A 1986 *Ann. Probab.* **14** 336
[10] Artstein S, Ball K M, Barthe F and Naor A 2004 *Probabil. Theory Rel. Fields* **129** 381
[11] Johnson O and Barron A 2004 *Probabil. Theory Rel. Fields* **129** 391
[12] see, e.g., Roederer J 2005 *Information and Its Role in Nature* (Berlin: Springer)
[13] Kennard E H 1927 *Zeitschrift für Physik* **44** 326
[14] Schrödinger E 1930 *Sitzungsber. Preuss. Acad. Wiss.* **24** 296
[15] Robertson H P 1929 *Phys. Rev.* **34** 163
[16] Stam A 1959 *Inform. Contr.* **2** 101
[17] Białynicki-Birula I and Mycielski J 1975 *Commun. Math. Phys.* **44** 129
[18] Rényi A 1970 *Probability Theory* (Amsterdam: North-Holland)
[19] Rényi A 1976 *Selected Papers of Alfred Rényi* Vol.2 (Budapest: Akadémia Kiado)
[20] Cover T and Thomas J A 1991 *Elements of Information Theory* (London: Wiley)
[21] Lvovsky A I and Raymer M G 2009 *Rev. Mod. Phys.* **81** 299
[22] Costa M H 1985 *IEEE Trans. Inform. Theory* **31** 751
[23] Jizba P., Dunningham J A and Joo J 2015 *Annals of Physics* **355** 87
[24] Gardner R J 2002 *Bull. Amer. Math. Soc.* **39** 355
[25] Campbell L L 1965 *Inf. Control* **8** 423
[26] Jizba P and Arimitsu T 2004 *Phys. Rev.* **E69** 026128
[27] Bacco D et all 2013 *Nat. Commun.* **4** 2363
[28] Müller-Lennert M et all 2013 *J. Math. Phys.* **54** 122203
[29] Coles P J et all 2012 *Phys. Rev. Lett.* **108** 210405
[30] Beckner W 1975 *Ann. of Math.* **102** 159
[31] Babenko K I 1962 *Amer. Math. Soc. Transl.* **44** 115
[32] Lieb E H 1990 *Invent. Math.* **102** 179
[33] Hirschman I I 1957 *Am. J. Math.* **79** 152
[34] Dembo A and Cover T M 1991 *IEEE Trans. Inf. Theory* **37** 1501
[35] Rioul O 2011 *IEEE Trans. Inf. Theory* **57** 33
[36] Jizba P, Ma Y, Hayes A and Dunningham J A 2016 *Phys. Rev.* E **93** 060104
[37] Hardy G, Littlewood J E and Pólya G 1952 *Inequalities* (Cambridge: Cambridge Univ. Press)
[38] Knott P A, Cooling J P, Hayes A, Proctor T J and Dunningham J A 2016 *Phys. Rev.* A **93** 033859
[39] Lillo F and Mantegna R N 2000 *Phys. Rev. Lett.* **84** 1061