Gravitational effects on inflaton decay at the onset of reheating

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学位論文

Gravitational effects on inflaton decay at the onset of reheating

（再加熱初期におけるインフラトン崩壊に対する重力の効果）

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Abstract

In the early universe, there exists a phase after the cosmic inflation called reheating, when the energy density of the inflaton is converted to that of light particles and these particles subsequently thermalize. At the onset of this reheating era, the oscillation of the inflaton generically induces oscillation modes in the expansion rate and the size of the universe due to the coupling between the inflaton and gravity, and these oscillation modes lead to the production of those particles coupled to gravity. The present thesis is devoted to the investigation of this production mechanism and its phenomenological consequences.

In the present thesis, we first point out that this mechanism exists even in the setup with theoretically minimal requirements, or the one where only the Einstein gravity and a canonical inflaton dominate the dynamics. This production process is interpreted as a two-body annihilation of the inflaton, and it brings about nonnegligible consequences to the present universe, for example as the dark matter. It is also shown that graviton production necessarily occurs by the same mechanism.

When the inflaton is nonminimally coupled to gravity, on the other hand, more violent oscillation of the expansion rate and nontrivial phases often emerge. In order to analyze the dynamics, a novel method is proposed in which an "adiabatic invariant" is constructed. This enables one to extract the oscillation modes in the expansion rate and to estimate the averaged expansion law of the universe, and as a consequence to understand the background dynamics of the models and the resulting particle production.

This method is applied to several observationally motivated models with the inflaton nonminimally coupled to gravity. Concretely, we analyze those models where the inflaton ϕ is coupled to the Ricci scalar R as $f(\phi)R$, and those where the inflaton is derivatively coupled to the Einstein tensor $G_{\mu\nu}$. In one example of the former, it is found that the oscillation mode of the expansion rate and the resulting particle production is interpreted as the emergence of a "decay" channel of the inflaton, in contrast to annihilation, which is strong enough to complete the inflaton decay. In another example of the former, it is pointed out that heavy particle production is possible during the nontrivial phase brought about by the nonminimal coupling to gravity. The latter case, on the other hand, turns out to have totally different features from the previous models. The violent oscillation of the expansion rate of the universe induces instabilities associated with the sound speed. In addition, we point out the possibility of graviton resonant production, which occurs if such a violent oscillation survives the backreaction from the (expectedly) explosive particle production triggered by the instability. These studies shed light on the rich phenomenology of gravitational couplings of the inflaton at the onset of reheating.

Contents

Part I

Review

Chapter 1

Introduction

Inflation [1–7], an accelerated expansion of the universe at its very beginning, is first recognized around 1980 and proposed as a possible solution to the horizon, flatness and monopole problems of the early universe. It explains the homogeneity and isotropy of the universe by setting the present whole observable universe in contact at the very beginning of its history. It also explains the flatness of the universe by wiping out all the curvature which might have existed at the beginning. What is surprising about inflation is that it explains the structure of the present universe as well [8]. The de Sitter phase gives to the scalar degree of freedom a "temperature" of roughly the expansion rate, and they become the seeds of galaxies after they re-enters the horizon.

In inflation, the interplay of the scalar field, inflaton, and gravity is crucial. The potential energy of the inflaton triggers the accelerated expansion of the geometry, and the expansion in turn gives an excitation to the inflaton to produce the seeds of the structure we observe at present. Such interplay of a scalar field and gravity has long been a subject of research in the context of the present cosmic expansion, or dark energy. All the possible form of interactions between a scalar field and gravity free from higher derivatives in equations of motion and from the resulting instability are first classified by Horndeski [9] in four dimensional spacetime, and later "re-discovered" in the studies of Galileon and its covariant extension [10–14], in a way extensible to an arbitrary dimensional spacetime. The knowledge obtained there was applied to the context of inflation, and it gave fruitful achievement in the field. Previous studies on noncanonical inflation models are classified in a clear-cut way [14], and the predictions of such inflation models have been calculated. In addition, inflation models within the standard model Higgs field [15, 16] such as Higgs G-inflation [17], Higgs inflation [18–20] and New Higgs inflation [21, 22] have been widely studied.

On the other hand, the energy density which we observe now is not made up of the inflaton condensation. Instead we know that Cosmic Microwave Background (CMB) [23,24], or thermally distributed photons with temperature $\sim$ 3K, fills the universe, and also that the abundance of light elements is consistent with thermally distributed plasma in the early universe. Therefore, assuming that inflation did occur, the energy stored in the inflaton condensation must be converted to that of light species and those species must be thermalized at some stage of the history of the universe. This process, called reheating [25, 26], was found to be totally different from what had been naively thought to be. One of such examples is preheating [27], where the decay of the inflaton to the light particles proceeds in a very short period of time due to resonance effects.

Then, what are the motivations to study the effects of gravity during this era? The answer is multifold. For one thing, gravity is the most universal force in nature. Gravity couples to every particle content unless it is Weyl invariant. In many studies of reheating, the coupling between the inflaton and light particles is often put by hand. However, before doing that, it seems reasonable to study the effects of this most universal type of coupling. For another thing, inflation and subsequent reheating era are presumably related to high energy phenomena beyond the standard model of particle physics. Gravity, especially gravitons, may play a crucial role in understanding these high energy physics in the future, due to the weakness of their interaction with other particles and the resulting suppression of information loss by scattering. One of the promising ways is the CMB B-mode, which observes the deflection pattern of the polarization of CMB photons caused by gravitational waves produced as quantum fluctuations during inflation. The gravitons produced during the reheating era may also be propagating around us, carrying the information of the very beginning of the universe. Therefore, revealing the nature of gravitons of reheating origin might contribute to our understanding of the unknown high energy physics in the future.

With these motivations, we study the effect of gravity on inflaton decay. The organization of the thesis is summarized below, and is illustrated in Fig. 1.1 as well.

- Part I consists of reviews.
  - In Chapter 2, Friedmann-Robertson-Walker cosmology is briefly summarized.
  - In Chapter 3, single-field slow roll inflation is summarized.
  - In Chapter 4, "Horndeski/Galileon theories" are introduced, which are general theories where a single scalar field and gravity are coupled so that higher derivatives do not appear in the equations of motion. Some inflation models based on these theories are explained. Gravitational effects on these models are studied in Part II.
  - In Chapter 5, methods of estimating particle production by oscillating background field are explained in detail, especially focusing on the correspondence between "narrow resonance" and perturbative decay. These methods are used in Chapter 7–9.
- Part II consists of original work.
  - In Chapter 6, "adiabatic invariant" in inflation models with Horndeski/Galileon theories is introduced. This quantity is used in Chapter 8–9 in order to extract the oscillation mode in the expansion rate of the universe, and to estimate the averaged expansion law of the universe.
  - In Chapter 7, gravitational effects in "minimal setup" are studied. Here "minimal setup" refers to the inflation model where Einstein gravity and a canonical inflaton dominate the dynamics.

  - In Chapter 8, gravitational effects in models with a nonminimal coupling of the inflaton to gravity are studied. Here "nonminimal coupling" refers to the one which does not exist in "minimal setup", within the theories explained in Chapter 4. Nonminimal coupling of the form $f(\phi)R$ is focused on here, where ϕ is the inflaton and R is the Ricci scalar.
  - In Chapter 9, gravitational effects in models with a nonminimal coupling of the inflaton to gravity are studied. Nonminimal coupling of the form $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is focused on here, where $G_{\mu\nu}$ is the Einstein tensor.
  - In Chapter 10, the thesis is summarized.
- Other related topics are summarized in the Appendix.

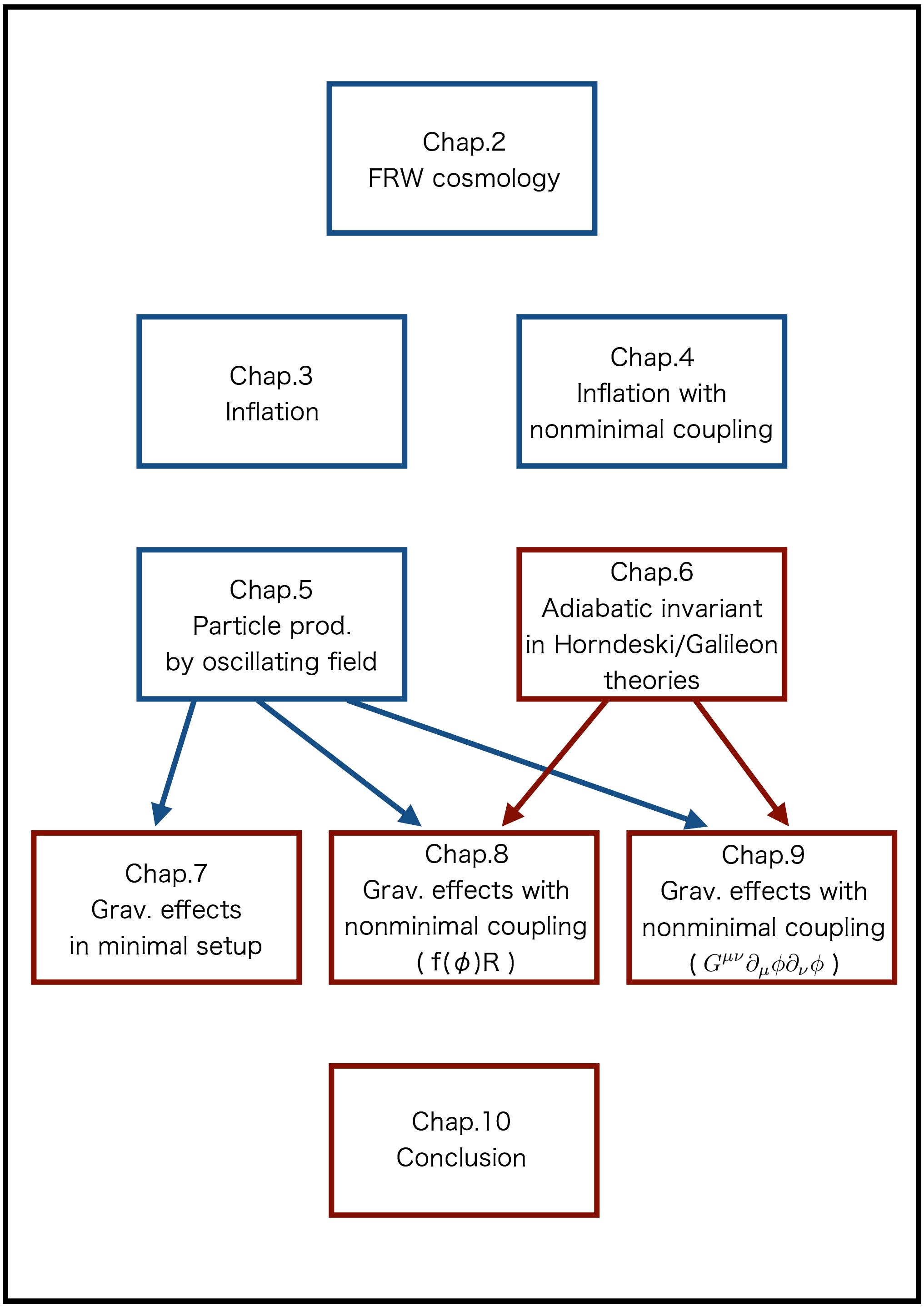

Figure 1.1: Organization of the thesis. Blue and red chapters correspond to review and original work, respectively. In Chapter 3 single-field inflation theories with Einstein gravity and a canonical inflaton are introduced, and in Chapter 4 they are generalized so that the inflaton nonminimally couples to gravity. The former setup is used in Chapter 7, while the latter is used in Chapter 8–9. The contents of Chapter 5 are used in Chapter 7–9, while those of Chapter 6 are used in Chapter 8–9.

Chapter 2

Friedmann-Robertson-Walker cosmology

General relativity [28], by far the best theory of gravity we have so far, is a classical theory which describes the gravitational dynamics as the geometry of spacetime. The geometry is determined by the metric $g_{\mu\nu}$ of the spacetime, which takes a simple form when one assumes the homogeneity and isotropy of the universe. This metric, now called Friedmann-Robertson-Walker (FRW) metric, is first assumed by Friedmann [29] in his models as a solution of the Einstein equation and is later derived from the argument on the homogeneity and isotropy by Robertson [30] and Walker [31]. Cosmology with this FRW metric has been quite successful in describing the evolution of the universe. In this chapter we briefly review the cosmology with FRW metric.

2.1 General relativity

We start with the Einstein-Hilbert action

$$S_G = \int d^4x \sqrt{-g}\, \frac{M_P^2}{2} \left(R - 2\Lambda \right), \tag{2.1.1}$$

where we use the natural unit $\hbar = c = 1$ throughout this thesis, and $M_P = 1/\sqrt{8\pi G} \simeq 2.4 \times 10^{18}$GeV is the reduced Planck mass. Here $g \equiv \det(g_{\mu\nu})$, and Λ is the cosmological constant. In addition, R is the Ricci tensor obtained from the Ricci tensor $R_{\mu\nu}$ as $R \equiv g^{\mu\nu} R_{\mu\nu}$. For the definition of the Ricci tensor, see Appendix A. Apart from this gravity action, we need a matter action in order to describe the observed universe

$$S_M = \int d^4x \sqrt{-g}\, \mathcal{L}_M. \tag{2.1.2}$$

The full action S is given by the sum $S = S_G + S_M$. In order to obtain the equation of motion we take variation with the metric $g^{\mu\nu}$[#1]. Using the formula in Appendix A, we have

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{M_P^2}{2}\left[R_{\mu\nu} - \frac{1}{2}\left(R - 2\Lambda\right)g_{\mu\nu}\right] + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \tag{2.1.3}$$

Here we define the energy-momentum tensor of matter as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{2.1.4}$$

and obtain the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{M_P^2}T_{\mu\nu}. \tag{2.1.5}$$

This can also be expressed with the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - Rg_{\mu\nu}/2$ as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{M_P^2}T_{\mu\nu}. \tag{2.1.6}$$

The covariant conservation of the energy-momentum tensor $\nabla^\mu T_{\mu\nu} = 0$ is guaranteed by Eq. (2.1.6) through the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ and the metric compatibility relation $\nabla^\alpha g_{\mu\nu} = 0$. For the definition of the covariant derivative ∇_μ, see Appendix A.

In the absence of matter, the Einstein equation describes two tensor degrees of freedom (DOF). The metric $g_{\mu\nu}$ contains 10 free components, since it is described as a 4×4 symmetric matrix. These are composed of 4 scalar, 4 vector and 2 tensor DOF. However, general coordinate transformation reduces the 10 to $10 - 4 = 6$. In addition, the Einstein equation (2.1.5) gives 4 constraint equations. This leaves only 2 tensor DOF left in the theory.

2.2 Friedmann-Robertson-Walker cosmology

Assuming the homogeneity and isotropy of the universe, all the possible form of the metric $g_{\mu\nu}$ reduces to the following

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{2.2.7}$$

where we have taken the coordinate to be (t, r, θ, ϕ), and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Also, a is called the scale factor of the uinverse. In addition, the universe is called open (closed) when $k > 0$ (< 0), while it is called flat for $k = 0$. The value of k can be normalized to $\pm1, 0$ by a proper rescaling of the radial coordinate.

[#1] Palatini formulation [32] is known as an alternative approach, where the metric $g_{\mu\nu}$ and the connection $\Gamma^\mu{}_{\nu\rho}$ are treated as independent quantities to take variation with. Though the metric formulation described in the text and this Palatini formulation give the same equation of motion in general relativity, they do not coincide with each other in other models such as $f(R)$ theories of gravity or Horndeski/Galileon theories described in Chapter 4. In this thesis we restrict ourselves within the metric formulation.

Substituting the metric (2.2.7) into the Einstein equation (2.1.5), we obtain Friedmann equation

$$H^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{\rho}{3M_P^2}, \tag{2.2.8}$$

and Raychaudhuri equation

$$2\dot{H} + 3H^2 + \frac{k}{a^2} - \Lambda = -\frac{p}{M_P^2}, \tag{2.2.9}$$

as the 00 and ii component of the Einstein equation. Here $H \equiv \dot{a}/a$ is called the Hubble parameter, and the energy density ρ and pressure p of matter are defined as

$$T^\mu{}_\nu = \mathrm{diag}(\rho, p, p, p). \tag{2.2.10}$$

The conservation of the energy momentum tensor $\nabla_\nu T^{\mu\nu} = 0$ reads

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.2.11}$$

2.3 Evolution of FRW universe

We briefly review the time evolution of the FRW universe with the particle content of the standard model (SM) as well as dark matter and dark energy, and with the baryon asymmetry observed. Let us suppose that the energy density ρ satisfies the following equation of state

$$w = \frac{p}{\rho}. \tag{2.3.12}$$

Here w is assumed to be constant. From the energy-momentum conservation (2.2.11) we have

$$\rho \propto a^{-3(1+w)}. \tag{2.3.13}$$

If the energy density consists of multiple components $\rho = \sum \rho_i$, each component ρ_i satisfies $\rho_i \propto a^{-3(1+w_i)}$ where w_i is the equation of state of that component. When a single component dominates the others, the Friedmann equation (2.2.8) gives

$$a(t) \propto \begin{cases} t^{2/3(1+w)} & (w \neq -1) \\ e^{Ht} & (w = -1) \end{cases}, \tag{2.3.14}$$

and

$$H = \begin{cases} \dfrac{2}{3(1+w)} \dfrac{1}{t} & (w \neq -1) \\ \text{const.} & (w = -1) \end{cases}. \tag{2.3.15}$$

The energy density of each component is often normalized by the critical energy density defined as $\rho_{\rm crit} \equiv 3M_P^2 H^2$. The normalized quantities are

$$\Omega_I \equiv \frac{\rho_I}{\rho_{\rm crit}}, \quad \Omega_k \equiv -\frac{k}{a^2 H^2}. \tag{2.3.16}$$

Here the cosmological constant Λ is regarded as a constant contribution (dark energy) $\rho_\Lambda = M_P^2 \Lambda$ to the energy density, and I runs over each contribution to the total energy density $\rho = \sum \rho_I$ including dark energy. From these definitions, we have

$$\sum_i \Omega_i = 1, \tag{2.3.17}$$

where i runs over I and k. Some comments are in order.

- Two types of contribution to the index I are important apart from dark energy: radiation ($I = R$, relativistic particles) and matter ($I = M$, nonrelativistic particles). Each satisfies the equation of state

  $$w_R = \frac{1}{3}, \quad w_M = 0. \tag{2.3.18}$$

  Also, the equation of state of dark energy is

  $$w_\Lambda = -1. \tag{2.3.19}$$

  The time or scale-factor dependence of radiation, matter and dark energy are obtained from Eqs. (2.3.13)–(2.3.15). The results are summarized in Table 2.1.

- From CMB observations [33] combined with other observations, ΛCDM model is known to be successful in describing the present universe. This model describes the universe with dark energy (denoted by Λ) and cold dark matter, in addition to radiation and baryons whose properties are well understood[#2]. Rough values of dominant components at present are

  $$\text{Baryon} \ : \ \Omega_b \simeq 0.05 \quad / \quad \text{Dark matter} \ : \ \Omega_c \simeq 0.25 \quad / \quad \text{Dark energy} \ : \ \Omega_\Lambda \simeq 0.05 \tag{2.3.20}$$

  Radiation energy density is estimated from the photon temperature $T_\gamma \simeq 2.7$K and assumed neutrino masses, using the formula for energy density in Appendix A. Also, radiation component other than SM is constrained to be $N_{\rm eff} \lesssim 0.3$, where $N_{\rm eff}$ roughly denotes the number of degrees of freedom of the extra component.

- Also from CMB [33] and other observations, the present curvature contribution to the Friedmann equation (2.2.8) is negligibly small:

  $$|\Omega_k| \lesssim 0.005. \tag{2.3.21}$$

  Since the curvature contribution to the expansion of the universe becomes smaller and smaller compared to radiation and matter as we go back in time, this suggests that the curvature is negligible from the distant past to the present epoch.

[#2] Except for the neutrino mass.

Table 2.1: Time and scale-factor dependence of the energy density, scale factor and the Hubble parameter of the universe dominated by a single component. For coherently oscillating scalar field, see Chapter 3.

	$\rho(a)$	$a(t)$	$H(t)$	w
Radiation	a^{-4}	$t^{\frac{1}{2}}$	$\frac{1}{2t}$	$\frac{1}{3}$
Matter	a^{-3}	$t^{\frac{2}{3}}$	$\frac{2}{3t}$	0
Dark (Vacuum) energy	const.	e^{Ht}	const.	-1
Coherent scalar field oscillating with $V \propto \phi^n$	$a^{-\frac{6n}{n+2}}$	$t^{\frac{n+2}{3n}}$	$\frac{n+2}{3n}\frac{1}{t}$	$\frac{n-2}{n+2}$

2.4 Brief thermal history of the universe

Assuming SM particle contents, dark matter and dark energy as constituents, and also assuming the baryon asymmetry observed $\eta \sim 10^{-9}$, the universe follows the thermal history described below.

- Electroweak phase transition ($T \sim 100\text{GeV}$)
  The standard model Higgs field acquires a nonzero expectation value, and the weak gauge bosons as well as quarks and leptons coupled to the Higgs field become massive.

- QCD phase transition ($T \sim 100\text{MeV}$)
  The universe experiences a phase transition associated with the chiral symmetry breaking and confinement. As a result, color nonsinglet states form baryons (three-quark bound states) and mesons (quark-antiquark bound states). The effective degrees of freedom g_* and g_{*s} significantly drop at this epoch.

- Neutrino decoupling ($T \sim 1\text{MeV}$)
  The neutrino scattering rate mediated by weak gauge bosons $\Gamma_\nu \sim n_\nu \sigma_\nu v_\nu \sim G_F^2 T^5$ falls below the expansion rate of the universe $H \sim T^2/M_P$. Here n_ν, σ_ν, v_ν and G_F are the number density, cross section, velocity of neutrinos and the Fermi constant, respectively. As a result, neutrinos decouples from the thermal plasma. Also, the freezeout of the weak interaction almost fixes the ratio of protons to neutrons except for the natural decay of neutrons. This gives the initial condition for the nucleosynthesis below.

- Nucleosynthesis ($T \sim 0.1\text{MeV}$)
  At the temperature slightly below 1 MeV these protons and neutrons form light elements, with the initial condition determined at the time of the freezeout of weak in-

teraction. Especially, almost all the remaining neutrons at the freezeout are absorbed into ^{4}He, and they determine the present Helium abundance.

- Matter-radiation equality ($T \sim 1$eV)
  The energy density of radiation is overtaken by that of matter. This occurs around the redshift $z \sim 3000$, where $1 + z \equiv a_0/a$.

- Recombination & Last scattering ($T \sim 1$eV)
  Electrons are captured by protons to form hydrogen atoms. As a result, the mean free path of photons exceeds the Hubble distance H^{-1}, and the universe becomes transparent. CMB photons today are typically scattered for the last time around this epoch, and therefore this epoch is called "Last scattering". This occurs around $z \sim 1000$.

- Dark energy domination
  Around $z \sim 1$ the universe is dominated by the unknown dark energy, and starts an accelerated expansion. The expansion rate today is roughly $H_0 \sim 70$km/s/Mpc.

Chapter 3

Inflation

Inflation [1–7], an accelerated expansion of the universe at the beginning of its history, now forms an indispensable part of cosmology. It is proposed as a solution to cosmological horizon, flatness and monopole problems, but nowadays its success lies in the generation of density perturbations which are consistent with observed power spectrum of galaxies and CMB.

In this chapter we first see the homogeneity, isotropy and flatness problem of the universe in Sec. 3.1. Then we review single-field slow-roll inflation in Sec. 3.2 as a solution to these problems, illustrating the background dynamics as well as the generation of density perturbations during inflation. The comparison between theoretical predictions and observations of the density perturbations is also given there.

3.1 Homogeneity, isotropy and flatness of the universe

Though FRW cosmology described in Chapter 2 is successful for example in reproducing the observed abundance of light elements, it does not answer to the question as to why the temperature of the photons is roughly the same everywhere and in every direction, or why the spacial curvature contribution is negligibly small. In the following we briefly see how serious these problems are.

3.1.1 Homogeneity and isotropy

What CMB observations told us is that the photon temperature T has only small fluctuations ΔT regardless of the direction they come from

$$\frac{\Delta T}{T} \sim 10^{-4}. \tag{3.1.1}$$

In order to see how strange this is, let us consider a test particle propagating with the speed of light. The coordinate distance travelled by the test particle from time t_1 to t_2 is given by

$$r(t_1, t_2) = \int_{t_1}^{t_2} \frac{dt}{a(t)}. \tag{3.1.2}$$

The physical distance d travelled by that particle measured at $t = t_2$

$$d(t_1, t_2) = a(t_2) r(t_1, t_2), \tag{3.1.3}$$

is called particle horizon. Assuming for simplicity radiation or matter dominated universe, the lower limit of the integral in Eq. (3.1.3) is negligible compared to the upper limit because $a \propto t^{1/2}$ or $t^{2/3}$ in these eras. Approximating the lower limit to be 0, one finds

$$d(0, t) = \begin{cases} H^{-1}(t) & \text{(Radiation dominated)} \\ 2H^{-1}(t) & \text{(Matter dominated)} \end{cases}. \tag{3.1.4}$$

If one takes into account that the time of last scattering is about $t_{\rm LS} \sim 10^{13}$s, photons cannot have covered the distance beyond $d(0, t_{\rm LS}) \sim H^{-1}(t_{\rm LS}) \sim 10^{-(4-5)} H_0^{-1}$ at the time of last scattering. Even after taking account of the expansion of the universe, the radius of last scattering patches at present is much below the present horizon size, $(a_0/a_{\rm LS}) d(0, t_{\rm LS}) \sim 10^{-2} H_0^{-1}$. This means that there exist $\sim (10^2)^3$ patches of the last scattering surface inside our observable universe. In the standard Big-Bang cosmology, it is not explained why the photons in these different patches have almost the same temperature at the time of last scattering. This is called the horizon problem.

3.1.2 Flatness

In Chapter 2, the smallness of the curvature contribution to the present cosmic expansion is given just as an observational fact. As mentioned there, this contribution becomes smaller and smaller compared to the radiation and matter energy density as we go back in time. For example, at the time of BBN

$$|\Omega_k(t_{\rm BBN})| \lesssim 10^{-16}, \tag{3.1.5}$$

must be realized. In the standard Big-Bang cosmology, the reason for this extreme smallness cannot be explained.

3.2 Slow-roll inflation

Inflation [1–7], an accelerated expansion of the universe caused by the potential energy of a scalar field called inflaton, is first proposed as a solution to cosmological horizon, flatness and monopole problems. Though the original model of Guth [3] had the problem of "graceful exit" [34, 35] from the expansion phase and of inhomogeneity and anisotropy resulting from bubble collisions, such problem was soon solved in "new inflation" [5, 6, 36, 37] and "chaotic inflation" [7] models. Amazingly, the density perturbation resulting from the quantum fluctuation during inflation [8, 38–41] was found to be consistent with the Harrison-Zel'dovich power spectrum [42, 43] necessary to produce observed galaxies, and it is also consistent with CMB observations.

In the following we first explain the background dynamics of inflation, which is called "slow-roll" of the inflaton. We then explain the generation of density perturbations during slow-roll inflation. Comparison between model predictions and CMB observations is also summarized.

3.2.1 Action and equations of motion

We consider the action $S = S_G + S_\phi$ with Einstein gravity and the inflaton ϕ

$$S_G = \int d^4x\sqrt{-g}\,\frac{M_P^2}{2}R, \tag{3.2.6}$$

$$S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - V(\phi)\right]. \tag{3.2.7}$$

Here $(\partial\phi)^2$ is a shorthand notation for $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, and the cosmological constant Λ is neglected since it is negligibly small compared to the energy scale of inflation. In the following we often refer to this setup with the Einstein gravity and the inflaton with a canonical kinetic term as "minimal setup". Taking variation with respect to the metric, one has the Einstein equation

$$G_{\mu\nu} = \frac{1}{M_P^2}T_{\mu\nu}^{(\phi)}, \tag{3.2.8}$$

with $T_{\mu\nu}^{(\phi)}$ being the energy momentum tensor of the inflaton ϕ

$$T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}}. \tag{3.2.9}$$

The variation with respect to ϕ gives the equation of motion for ϕ

$$-\Box\phi + V'(\phi) = 0, \tag{3.2.10}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ and the prime denotes the derivative with respect to ϕ.

Below we consider the background dynamics. We denote the background value of the inflaton by the same symbol ϕ as the full inflaton value for notational simplicity. Assuming the FRW metric with spacial curvature neglected

$$ds^2 = -dt^2 + a^2(t)dx^i dx^i, \tag{3.2.11}$$

we obtain the Friedmann and Raychaudhuri equations as in Chapter 2

$$H^2 = \frac{\rho_\phi}{3M_P^2}, \qquad 2\dot{H} + 3H^2 = -\frac{p_\phi}{M_P^2}, \tag{3.2.12}$$

where the energy density ρ_ϕ and pressure p_ϕ of the inflaton are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V, \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V. \tag{3.2.13}$$

In addition, the equation of motion for ϕ becomes

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0. \tag{3.2.14}$$

Here suppose that V is almost constant in time. The essence of inflation is that, such an (almost) constant value of the potential is regarded as an effective cosmological constant

$$\Lambda_\phi \equiv \frac{V(\phi)}{M_P^2}. \tag{3.2.15}$$

As we saw in Chapter 2, constant contribution to the energy density gives exponential expansion of the scale factor $a \propto e^{Ht}$. Since the curvature contribution to the Friedmann equation Eq. (2.2.8) is proportional to a^{-2}, this exponential expansion removes the curvature which might have existed at the beginning of inflation. In addition, this exponential expansion invalidates the assumption that the lower bound of the integration (3.1.2) is negligible. The physical distance traveled by the speed of light from the beginning of inflation to the present thus grows much larger than the present horizon size H_0^{-1}. In other words, inflation keeps all the observable universe today in contact at the beginning. These are how inflation solves the horizon and flatness problems illustrated in the previous section.

3.2.2 Slow-roll of the inflaton

One of the crucial requirements for inflation is that the potential is almost constant during the exponential expansion. Let us see more qualitatively what this means.

During the exponential expansion, the equation of state w takes roughly -1, see Table 2.1. Since the energy density and pressure of the inflaton field is given by Eq. (3.2.13), the equation of state is

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}. \tag{3.2.16}$$

Then one finds that w_ϕ being close to -1 means $\dot{\phi}^2 \ll V$. Thus one finds that the velocity of the inflaton field must not be large. This leads to the consideration on the slow-roll inflation, where the inflaton velocity and acceleration is small compared to the other terms in Friedmann equation (3.2.12) and ϕ's equation of motion (3.2.14)

$$\dot{\phi}^2 \ll V, \qquad |\ddot{\phi}| \ll |3H\dot{\phi}|. \tag{3.2.17}$$

With these conditions, Friedmann equation (3.2.12) and ϕ's equation of motion (3.2.14) become

$$3M_P^2 H^2 \simeq V, \tag{3.2.18}$$

$$3H\dot{\phi} \simeq -V'. \tag{3.2.19}$$

Here let us introduce quantitative measures of the slowness of the inflaton motion. We define potential slow-roll parameters as [44][#1]

$$\epsilon_V \equiv \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta_V \equiv M_P^2 \frac{V''}{V}, \qquad {}^n\beta_V \equiv M_P^2 \left[\frac{(V')^{n-1}V^{(n+1)}}{V^n}\right]^{\frac{1}{n}}. \tag{3.2.22}$$

[#1]Sometimes the following Hubble slow-roll parameters are also used [45, 46]

$$\epsilon_H \equiv 2M_P^2 \left(\frac{H'}{H}\right)^2 = -\frac{\dot{H}}{H^2}, \qquad \eta_H \equiv 2M_P^2 \frac{H''}{H} = -\frac{1}{2}\frac{\ddot{H}}{\dot{H}H}, \qquad {}^n\beta_H \equiv 2M_P^2 \left[\frac{(H')^{n-1}H^{(n+1)}}{H^n}\right]^{\frac{1}{n}}, \tag{3.2.20}$$

Note that ${}^1\beta_V$ coincides with η_V. Below we focus on the first two slow-roll parameters ϵ_V and η_V. The slow-roll conditions (3.2.17) are equivalent to the requirement that these parameters be much smaller than unity, since

$$\epsilon_V \sim M_P^2 \frac{H^2 \dot\phi^2}{V^2} \sim \frac{\dot\phi^2}{V} \ll 1, \tag{3.2.23}$$

$$\eta_V \sim M_P^2 \frac{\frac{d}{d\phi}(H\dot\phi)}{V} \sim M_P^2 \frac{(H\dot\phi)^{\cdot}}{V\dot\phi} \sim \frac{\dot\phi^2}{V} - \frac{\ddot\phi}{H\dot\phi} \ll 1, \tag{3.2.24}$$

where Friedmann equation, ϕ's equation of motion as well as Raychaudhuri equation (rewritten as $\dot H = -\dot\phi^2/2M_P^2$ with the help of Friedmann equation) are used. The definition of inflation, the accelerated expansion of the universe, means

$$\frac{\ddot a}{a} = (1-\epsilon_H)H^2 \simeq (1-\epsilon_V)H^2 > 0, \tag{3.2.25}$$

and therefore inflation ends at $\epsilon_H \simeq \epsilon_V \simeq 1$. Let us define a measure for the amount of inflation from time t to the end of inflation t_{end}. It is often given by the following e-folding number N

$$N(t) \equiv \ln \frac{a(t_{\text{end}})}{a(t)}. \tag{3.2.26}$$

The argument t during inflation has one-to-one correspondence to parameter k which satisfies relation $k = aH$, since a is an increasing function in time while H is roughly constant. Thus we sometimes write $N(t)$ as $N(k)$. This k is identified as the "comoving wavenumber" of perturbations later. With slow-roll approximations, e-folding number becomes

$$N = \int_t^{t_{\text{end}}} dt\, H = \int_\phi^{\phi_{\text{end}}} d\phi\, \frac{H}{\dot\phi} \simeq -\frac{1}{M_P^2}\int_\phi^{\phi_{\text{end}}} d\phi\, \frac{V}{V'} = \int_{\phi_{\text{end}}}^{\phi} d\phi\, \frac{1}{\sqrt{2\epsilon_V}}, \tag{3.2.27}$$

where ϕ and ϕ_{end} denote $\phi(t)$ and $\phi(t_{\text{end}})$, respectively. If we further assume that ϵ_V is roughly constant during inflation, above equation gives

$$N \simeq \frac{\Delta\phi}{\sqrt{2\epsilon_V}}, \tag{3.2.28}$$

where $\Delta\phi \equiv \phi - \phi_{\text{end}}$. All above is a short summary of the background dynamics during inflation. Below we discuss the generation of perturbations, or quantum fluctuations, during inflation.

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where the prime denotes the derivative with respect to ϕ, and ${}^1\beta_H$ coincides with η_H. Also, ${}^2\beta_H$ is often written as ζ_H. The relation to potential slow-roll parameters is given by

$$\epsilon_H \simeq \epsilon_V, \qquad \eta_H \simeq -\epsilon_V + \eta_V, \tag{3.2.21}$$

up to the first order in slow roll parameters.

3.2.3 Horizon crossing of peturbations

We define the "comoving wavenumber" $\mathbf{k}$ of perturbation $\delta(\mathbf{x})$ as the label of its Fourier component. The physical wavelength of the perturbation is given by k/a, where $k = |\mathbf{k}|$. Noting that two points away from each other by a physical distance of H^{-1}, or "horizon", go away at the speed of light, perturbations with mode k find their structures within causal region if $k > aH$. If this condition is satisfied the perturbation is said to be "inside the horizon", while in the opposite case $k < aH$ it is said to be "outside the horizon". During inflation, perturbations with different comoving wavenumbers get outside the horizon one after another. The time of this horizon-crossing differs from one comoving wavenumber to another, and in this sense k has one-to-one correspondence to the time of horizon-crossing.

Inflation ends when the slow roll parameter ϵ_V exceeds unity, as seen from Eq. (3.2.25). After inflation ends, the energy density of the inflaton field must be converted to that of light particles. Since the abundance light elements, which are produced around the time of BBN, and CMB observations are consistent with the FRW universe filled with thermal plasma, the produced light particles must be thermalized before BBN at latest. This process of inflaton decay and subsequent thermalization of produced particles is called reheating. In simplified models of reheating the inflaton starts to oscillate around the potential minimum after inflation, until the cosmic time reaches the inverse of the inflaton perturbative decay rate Γ_ϕ. The expansion of the universe during this era can be estimated by the equations of motion (3.2.12) and (3.2.14). Assuming that the inflaton is oscillating around the minimum of the potential

$$V(\phi) = \frac{\lambda}{n}\phi^n, \tag{3.2.29}$$

the expansion law of the universe is given in the last row of Table 2.1. Note that the Hubble parameter is now proportional to t^{-1}, not constant. In these simplified models, the light particles are assumed to thermalize as soon as they are produced. After that, the universe follows the FRW cosmology described in Chapter 2, thus connecting to the present universe. Here the time dependence of the horizon $H^{-1} \propto t$ is stronger than the evolution of the physical distance of perturbations $k/a \propto t^{1/2}$ (Radiation dominated) or $t^{2/3}$ (Matter dominated). Thus perturbations with fixed k find themselves "caught up" by the horizon, and then enters the horizon again. What we observe as the distribution of galaxies or fluctuations of CMB are these perturbations which exited the horizon during inflation and entered the horizon after inflation ends.

The value of e-folding number N with fixed t during inflation, or equivalently k, is constrained from observations. To see this, let us consider the ratio k/a_0H_0, where the subscript 0 denotes the present time. It is decomposed as

$$\frac{k}{a_0H_0} = \frac{a_kH_k}{a_0H_0} = \frac{a_k}{a_{\rm end}}\frac{a_{\rm end}}{a_{\rm reh}}\frac{a_{\rm reh}}{a_{\rm eq}}\frac{a_{\rm eq}}{a_0}\frac{H_k}{H_0}. \tag{3.2.30}$$

Here the subscript k means that the argument t corresponds to k, the subscript "end" denotes the end of inflation, and "reh" refers to the time of inflaton decay, or reheating. The e-folding number appears as the logarithm of the first factor in the RHS. Assuming that the universe has undergone an inflaton oscillation phase with quadratic potential from the end of inflation to the inflaton decay, and that radiation dominated era has continued all the way from the inflaton decay to the matter-radiation equality, we can invert Eq. (3.2.30) to obtain the expression for the e-folding number in terms of the inflaton potential V at each epoch [47]

$$N(k) \simeq 62 - \ln \frac{k}{a_0 H_0} - \ln \frac{10^{16}\ \text{GeV}}{V_k^{1/4}} + \ln \frac{V_k^{1/4}}{V_{\text{end}}^{1/4}} - \frac{1}{3} \ln \frac{V_{\text{end}}^{1/4}}{\rho_{\text{reh}}^{1/4}}. \tag{3.2.31}$$

The last three terms correspond to the uncertainty coming from various inflation and reheating models (or, the inflaton decay rate). The energy scale of reheating can be parametrized by the reheating temperature T_R, and the effective degrees of freedom g_* in the thermal bath:

$$\rho_{\text{reh}} = \frac{\pi^2}{30} g_* T_{\text{reh}}^4. \tag{3.2.32}$$

From Eqs. (3.2.31)–(3.2.32) one sees that e-folding number depends on the reheating temperature. In the simplest models of reheating the reheating temperature T_R is related to the perturbative decay rate of the inflaton Γ_ϕ, since the inflaton decays at $t \sim \Gamma_\phi^{-1}$ while t is written as $\sim H^{-1}$. From the Friedmann equation (3.2.12) and substituting ρ_ϕ with the energy density of radiation ρ_R which the inflaton has decayed into, one has

$$3M_P^2 \Gamma_\phi^2 = \rho_{\text{reh}}, \tag{3.2.33}$$

and thus

$$T_{\text{reh}} \simeq \mathcal{O}(0.1) \sqrt{M_P \Gamma_\phi}. \tag{3.2.34}$$

However, note that the dependence of $N(k)$ on the inflaton potential is not so strong because of the logarithm.

3.2.4 Power spectrum

Before discussing inflationary quantum fluctuations, we define the power spectrum. Suppose that $\delta(\mathbf{x})$ is a real quantity defined on each point of space. First, we perform Fourier transformation to $\delta(\mathbf{x})$

$$\delta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, \delta(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.2.35}$$

Then the power spectrum P_δ is defined as

$$\langle \delta(\mathbf{k}) \delta(\mathbf{k}') \rangle = P_\delta(k) (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}'), \tag{3.2.36}$$

where the bracket denotes quantum or position average, depending on the context. We further define the dimensionless power spectrum $\mathcal{P}_\delta$ as

$$\mathcal{P}_\delta(k) = \frac{k^3}{2\pi^2} P_\delta(k). \tag{3.2.37}$$

With this definition, the 2-point correlator in position space becomes

$$\langle \delta(\mathbf{x})\delta(\mathbf{x}')\rangle = \int \frac{d^3k}{(2\pi)^3}\, P_\delta(k) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \int_0^\infty d\ln k\, \mathcal{P}_\delta(k)\frac{\sin(k|\mathbf{x}-\mathbf{x}'|)}{k|\mathbf{x}-\mathbf{x}'|}. \tag{3.2.38}$$

Especially, this gives the squared average $\langle \delta^2(\mathbf{x})\rangle = \int d\ln k\, \mathcal{P}_\delta(k)$.

3.2.5 Generation of scalar and tensor perturbations

Let us first consider a de-Sitter background, where the scale factor is exponentially increasing with a constant rate H, i.e. $a = a_* e^{Ht}$ with a_* being some normalization constant. The relation between t and the conformal time $d\tau \equiv a^{-1}dt$ is given by

$$a_*\tau = -\frac{e^{-Ht}}{H}, \tag{3.2.39}$$

and therefore

$$\tau = -\frac{1}{aH}, \tag{3.2.40}$$

holds. The conformal time τ goes from $-\infty$ to -0 as t goes from $-\infty$ to ∞. We consider a light scalar field in this de-Sitter background

$$S_\chi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\chi)^2\right] = \int d\tau d^3x\, \frac{1}{2}a^2\left[\chi'^2 - (\partial_i\chi)^2\right]. \tag{3.2.41}$$

Here the conformal time is used in the last expression, and the prime denotes the derivative with respect to it. Later we identify this χ field as properly normalized scalar or tensor perturbations. The action for the canonically normalized field $\tilde{\chi} \equiv a\chi$ is

$$S_\chi = \int d\tau d^3x\, \frac{1}{2}\left[\tilde{\chi}'^2 + \frac{a''}{a}\tilde{\chi}^2 - (\partial_i\chi)^2\right]. \tag{3.2.42}$$

Here we perform Fourier transformation

$$\chi(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, \chi(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.2.43}$$

and expand $\tilde{\chi}$ in terms of creation and annihilation operators (of some fixed time):

$$\tilde{\chi}(\tau,\mathbf{k}) = \tilde{\chi}_k(\tau)a_{\mathbf{k}} + \tilde{\chi}_k^*(\tau)a^\dagger_{-\mathbf{k}}, \tag{3.2.44}$$

where $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ satisfy the commutation relation

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}'). \tag{3.2.45}$$

The canonical commutation relation for $\tilde{\chi}$ imposes the following condition on the mode function

$$\tilde{\chi}_k\tilde{\chi}_k^{*'} - \tilde{\chi}_k^*\tilde{\chi}_k' = 1. \tag{3.2.46}$$

The mode equation for $\tilde{\chi}_k$ becomes

$$\tilde{\chi}_k'' + \left(k^2 - \frac{a''}{a}\right)\tilde{\chi}_k = 0, \tag{3.2.47}$$

where the time dependent mass satisfies $a''/a = 2/\tau^2$. Requiring that there is only positive energy mode for $t \to -\infty$, we obtain the exact solution for Eq. (3.2.47) which satisfies the normalization condition (3.2.46)

$$\tilde{\chi}_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{3.2.48}$$

For $t \to \infty$, i.e. $\tau \to -0$, the dimensionless power spectrum for $\tilde{\chi}$ becomes

$$\mathcal{P}_{\tilde{\chi}} = \frac{k^3}{2\pi^2}|\tilde{\chi}_k|^2 = \frac{k^3}{2\pi^2}\frac{1}{2k^3\tau^2}. \tag{3.2.49}$$

Then the power spectrum for χ is given by

$$\mathcal{P}_\chi = a^{-2}\mathcal{P}_{\tilde{\chi}} = \left(\frac{H}{2\pi}\right)^2, \tag{3.2.50}$$

because of the relation between χ and $\tilde{\chi}$. Note that the value of the power spectrum significantly changes at $k\tau = -1$, or the time of the horizon crossing $k = aH$, while it is almost fixed after that.

We now move on to the production of scalar and tensor perturbations during inflation. However, before that, let us count the number of degrees of freedom in the system (3.2.6)–(3.2.7). As mentioned in Chapter 2, there are two physical tensor degrees of freedom in the absence of matter field. However, the inflaton field exists in the present case, which adds one additional scalar degree of freedom to the theory. These degrees of freedom may be easy to understand in Arnowitt-Deser-Misner (ADM) approach [48, 49] to general relativity, where all quantities are decomposed onto 3-dimensional hypersurfaces Σ_t at constant times t. In this approach, the metric is decomposed as

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{3.2.51}$$

where the perturbations α, β, ζ, β_{Ti} and h_{ij} are in

$$N = 1 + \alpha, \tag{3.2.52}$$

$$\beta_i \equiv \gamma_{ij}\beta^j = \partial_i\beta + \beta_{Ti}, \tag{3.2.53}$$

$$\gamma_{ij} = a^2e^{2\zeta}(e^h)_{ij}. \tag{3.2.54}$$

Here the vector part β_{Ti} and tensor part h_{ij} satisfy $\partial_i\beta_{Ti} = 0$ and $\partial_j h_{ij} = h_{ii} = 0$, respectively. The exponentiation of the tensor part is understood in a matrix sense. In the following we focus on the scalar and tensor part. In addition, the inflaton field can be decomposed into background and perturbation.

$$\phi = \bar{\phi} + \varphi. \tag{3.2.55}$$

Note that at this point we have already used part of gauge fixing to reduce the scalar DOF from 5 (4 from the metric and 1 from the inflaton) to 4. We still have one DOF to fix the gauge for the scalar perturbation, and the two gauges $\varphi = 0$ and $\zeta = 0$ are frequently used. We refer to the former as ζ gauge since ζ is nonzero in this gauge, and refer to the latter as φ gauge.

In this section we use ζ gauge. Two of the three remaining scalar perturbations α and β become auxiliary fields, and they can be eliminated from the action. The quadratic action for the scalar degree of freedom ζ becomes[#2]

$$\begin{aligned} S_\zeta &= \int d^4x \, a^3 \frac{\dot{\phi}^2}{H^2} \left[\frac{1}{2}\dot{\zeta}^2 - \frac{1}{2} a^{-2} (\partial_i \zeta)^2 \right] \\ &= \int d\tau d^3x \, a^2 \frac{\phi'^2}{\mathcal{H}^2} \left[\frac{1}{2}\zeta'^2 - \frac{1}{2}(\partial_i \zeta)^2 \right], \end{aligned} \tag{3.2.56}$$

where the prime denotes the derivative with respect to the conformal time τ, and $\mathcal{H} \equiv a'/a$. Notice that we have used ϕ instead of $\bar{\phi}$ for notational simplicity. This action leads to the equation of motion called Mukhanov-Sasaki equation [50, 51]

$$\zeta'' + \frac{2z'}{z}\zeta' - \partial_i^2 \zeta = 0. \tag{3.2.57}$$

where $z = a\phi'/\mathcal{H}$. Let us consider the normalized field $\tilde{\zeta}$ defined as

$$\tilde{\zeta} \equiv \frac{\phi'}{\mathcal{H}}\zeta. \tag{3.2.58}$$

The action becomes

$$S_\zeta = \int d\tau d^3x \, \frac{1}{2} a^2 \left[\tilde{\zeta}'^2 + (\epsilon_H - \eta_H + 2\epsilon_H^2 - 4\epsilon_H \eta_H + \eta_H^2 + \xi_H^2)\mathcal{H}^2 \tilde{\zeta}^2 - (\partial_i \tilde{\zeta})^2 \right]. \tag{3.2.59}$$

Note that the calculation is exact so far. For quasi de-Sitter limit, the time-dependent mass for $\tilde{\zeta}$ can be neglected. Then, from Eq. (3.2.50), the power spectrum for $\tilde{\zeta}$ becomes

$$\mathcal{P}_{\tilde{\zeta}} = \left(\frac{H}{2\pi} \right)^2. \tag{3.2.60}$$

In terms of ζ, we have

$$\mathcal{P}_\zeta = \left(\frac{H}{\dot{\phi}} \right)^2 \left(\frac{H}{2\pi} \right)^2. \tag{3.2.61}$$

For the tensor part, we substitute the metric

$$ds^2 = -dt^2 + a^2 (e^h)_{ij} dx^i dx^j, \tag{3.2.62}$$

[#2] See Chapter 4 for the quadratic action for ζ.

into the action. Also, we decompose gravitons into two polarization modes $\lambda = +, \times$ as

$$h_{ij}(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=+,\times} h^{(\lambda)}(t,\mathbf{k})\epsilon_{ij}^{(\lambda)}(\hat{\mathbf{k}})e^{i\mathbf{kx}}, \tag{3.2.63}$$

by the polarization tensor $\epsilon_{ij}^{(\lambda)}$ satisfying $\hat{k}_j\epsilon_{ij}^{(\lambda)}(\hat{\mathbf{k}}) = \epsilon_{ii}^{(\lambda)}(\hat{\mathbf{k}}) = 0$, $(\epsilon_{ij}^{(\lambda)}(\hat{\mathbf{k}}))^* = \epsilon_{ii}^{(\lambda)}(-\hat{\mathbf{k}})$ and $\epsilon_{ij}^{(\lambda)}(\hat{\mathbf{k}})\epsilon_{ij}^{(\lambda')}(-\hat{\mathbf{k}}) = \delta_{\lambda\lambda'}$ where $\hat{\mathbf{k}} \equiv \mathbf{k}/|\mathbf{k}|^2$, and then rescale the graviton field as $\tilde{h}^{(\lambda)} = (M_P/2)h_{ij}$ to find the action the same as that for two massless scalar fields

$$S_h = \int dt \int \frac{d^3k}{(2\pi)^3}\, a^3 \sum_{\lambda=+,\times} \frac{1}{2}\left[|\dot{\tilde{h}}^{(\lambda)}(t,\mathbf{k})|^2 - \frac{k^2}{a^2}|\tilde{h}^{(\lambda)}(t,\mathbf{k})|^2\right]. \tag{3.2.64}$$

Thus it has a spectrum

$$\mathcal{P}_{\tilde{h}^{(\lambda)}} = \left(\frac{H}{2\pi}\right)^2. \tag{3.2.65}$$

In the original field h, the spectrum becomes

$$\mathcal{P}_T = \left(\frac{2}{M_P}\right)^2 \sum_{\lambda=+,\times} \mathcal{P}_{\tilde{h}^{(\lambda)}} = \frac{8}{M_P^2}\left(\frac{H}{2\pi}\right)^2. \tag{3.2.66}$$

The ratio of $\mathcal{P}_T$ to $\mathcal{P}_\zeta$ is so-called tensor-to-scalar ratio

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} \simeq 16\epsilon_V. \tag{3.2.67}$$

The scale dependence for the ζ and h power spectra are calculated in the following way. We may evaluate the power spectrum $\mathcal{P}_\zeta$ and $\mathcal{P}_T$ at the time of horizon crossing $k = aH$. Then, due to the slow-roll of inflaton, the spectra has the wavenumber dependence[#3]

$$n_S - 1 \equiv \frac{d}{d\ln k}\ln \mathcal{P}_\zeta(k)\bigg|_{k=aH} = \frac{-4\epsilon_H + 2\eta_H}{1-\epsilon_H} \simeq -6\epsilon_V + 2\eta_V. \tag{3.2.70}$$

and

$$n_T \equiv \frac{d}{d\ln k}\ln \mathcal{P}_T(k)\bigg|_{k=aH} = \frac{-2\epsilon_H}{1-\epsilon_H} \simeq -2\epsilon_V. \tag{3.2.71}$$

[#3]For scalar mode,

$$n_S - 1 = \frac{dt}{d\ln(aH)}\frac{d}{dt}\ln\left[\left(\frac{H}{\dot{\phi}}\right)^2\left(\frac{H}{2\pi}\right)^2\right] = \left[H + \frac{\dot{H}}{H}\right]^{-1}\frac{d}{dt}\ln\left(\frac{H^2}{\epsilon_H}\right)$$
$$= [H(1-\epsilon_H)]^{-1}\left[-2H\epsilon_H - 2H(\epsilon_H - \eta_H)\right] = \frac{-4\epsilon_H + 2\eta_H}{1-\epsilon_H}. \tag{3.2.68}$$

For tensor mode,

$$n_T = \frac{dt}{d\ln(aH)}\frac{d}{dt}\ln\left[\left(\frac{H}{2\pi}\right)^2\right] = [H(1-\epsilon_H)]^{-1}\left[-2H\epsilon_H\right] = \frac{-2\epsilon_H}{1-\epsilon_H}. \tag{3.2.69}$$

3.2.6 Illustration with chaotic inflation

Let us consider the potential for chaotic inflation models [7]

$$V = \frac{\lambda}{n}\phi^n. \tag{3.2.72}$$

From the definition of potential slow-roll parameters (3.2.22), we have

$$\epsilon_V = \frac{n^2}{2}\frac{M_P^2}{\phi^2} = \frac{n}{4}\frac{1}{N}, \qquad \eta_V = n(n-1)\frac{M_P^2}{\phi^2} = \frac{n-1}{2}\frac{1}{N}, \tag{3.2.73}$$

where we used the expression for the e-folding number (3.2.27)

$$N \simeq \int d\phi\, \frac{1}{M_P^2}\frac{V}{V'} \simeq \frac{1}{2n}\frac{\phi^2}{M_P^2}. \tag{3.2.74}$$

Thus from Eqs. (3.2.67) and (3.2.70) we have

$$n_S - 1 = -6\epsilon_V + 2\eta_V = -\left(\frac{n}{2}+1\right)\frac{1}{N}, \qquad r = 16\epsilon_V = 4n\frac{1}{N}, \tag{3.2.75}$$

and

$$r = -\frac{8n}{n+2}(n_S - 1). \tag{3.2.76}$$

These predictions must be compared with observations. Fig. 3.1 is the observational results from PLANCK satellite [52], where the values n_S and r are plotted. Different colored blobs correspond to predictions from different inflaton potentials with the e-folding number $N = 50$ or 60.

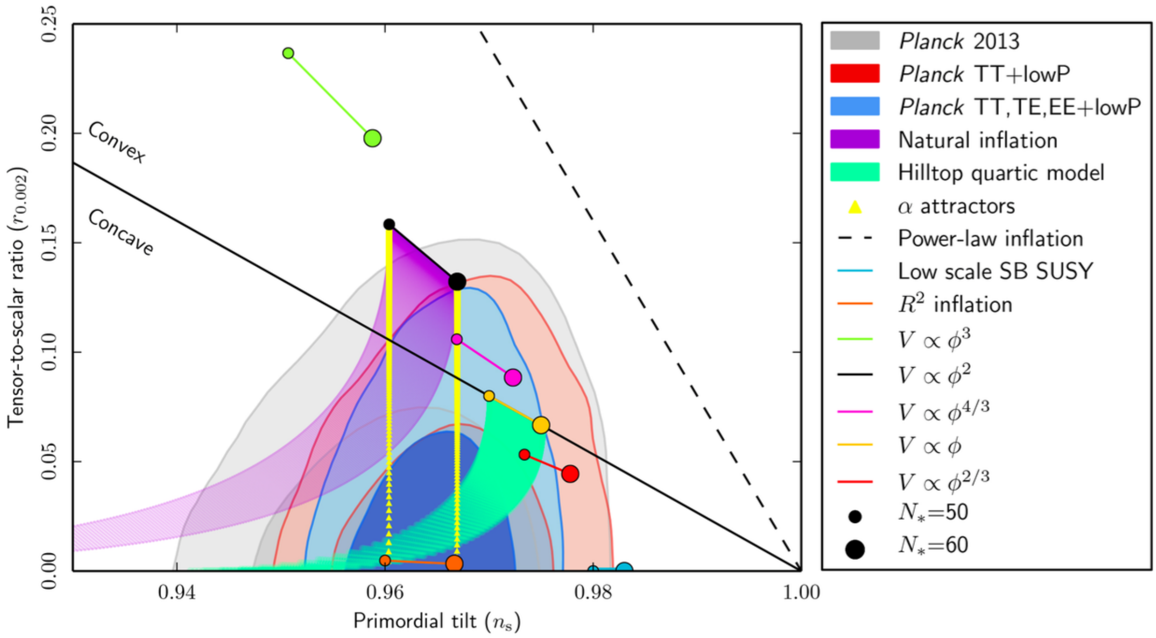

Figure 3.1: Constraints on n_s-r plane. This figure is taken from Ade *et al.* [52].

Chapter 4

Inflation with nonminimal couplings to gravity

As we saw in the previous chapter, inflation is an excellent mechanism to solve the horizon and flatness problem as well as to produce the seeds for galaxies in the present universe. In this scheme, the interplay between the inflaton and gravity is crucial. Then a natural question to ask would be whether the inflaton and gravity must be minimally introduced to the theory. In fact, attempts to couple the inflaton nonminimally to gravity have been made in some contexts.

In this chapter we review inflation theories where the inflaton and gravity are nonminimally implemented. We first give a brief review in Sec. 4.1 on the history of the theories with a scalar field and gravity coupled in a way free from higher order derivatives in the equations of motion. Next in Sec. 4.2 we write down the general action for such theories. Since part of such theories can be mapped to others by conformal transformation [53–55] or disformal transformation [56, 57], we briefly comment on these transformations as well. Then we see inflation models within these theories, identifying the scalar field as the inflaton. We first summarize general features of such inflation models and next see two examples, where the inflaton and gravity are coupled through the terms $\phi^2 R$ or $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. These two models are motivated in part by Higgs inflation [18–20] and New Higgs inflation [21, 22], respectively. Gravitational effects on inflaton decay during the oscillation regime in these models are analyzed in Chapter 8–9.

4.1 Brief historical introduction

General theories of a scalar field coupled to gravity which avoid such higher derivatives were first obtained by Horndeski [9][#1]. He started with a scalar and gravity in four dimensions from the outset, and obtained the general action which produces derivatives no more than second order in the equations of motion. Later this theory was rediscovered in a different

[#1] It is known [58] that the Hamiltonian constructed from Lagrangian with more than first order time derivatives which cannot be eliminated by integration by parts usually suffers from instability associated with negative energy unbounded below, called Ostrogradsky instability [59].

context, and named Galileon theories. The discovery of Galileon theories started with obtaining a general four-dimensional theory with a scalar in flat spacetime, which contains only second derivative terms in the equation of motion [10][#2]. One of the motivations for this kind of theory was that the Dvali-Gabadadze-Porrati (DGP) model [63], which had been studied in the context of the accelerated expansion of the present universe, has this feature in the so-called decoupling limit [64, 65][#3]. This feature is considered to the key to the Vainshtein mechanism [66], which allows an accelerated expansion of the universe at large scales while avoiding constraints from experiments in the scales of solar system. The name Galileon comes from the Galilean symmetry associated with this theory $\partial_\mu \pi \to \partial_\mu \pi + b_\mu$, with π being the scalar field. Soon the theory was extended to arbitrary dimensions [12], to covariant expression with derivatives equal or less than second order [13], and to p-forms [67], though the original Galilean symmetry was lost at the stage of covariantization. After that it was proven that this Galileon theory is equivalent to Horndeski theory [14]. We denote this theory with a single scalar field and gravity in four dimensional spacetime which contain derivatives no more than second order in the equations of motion as "Horndeski/Galileon theories" in the present thesis.

Horndeski/Galileon theories contain many inflation and dark energy models[#4]. For example, scalar-tensor theories including such as Jordan-Brans-Dicke theory [68, 69] used in extended inflation [70], as well as k-inflation [71, 72] and k-essence [73], kinetic gravity braided theories [74], running kinetic inflation [75, 76], (Higgs) G-inflation [17, 77], Higgs inflation [18–20], New Higgs inflation [21, 22] and so on. Below we write down the action of the theory.

4.2 Horndeski/Galileon theories

4.2.1 Action

The action for Horndeski/Galileon theory takes the following form [9, 13, 14]

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}, \tag{4.2.1}$$

[#2]The general action for a scalar in four-dimensional flat spacetime which contains no more than second order derivatives in the equations of motion was already known [60–62] in yet another context.

[#3]DGP model is a theory with 4D brane in 5D spacetime, and the scalar degrees of freedom corresponds to the bending mode of this brane. The decoupling limit denotes the limit where this scalar decouples from 4D gravity and vector modes. See the original paper [63] and the studies of the decoupling limit [64, 65] for details.

[#4]Galileons in flat spacetime include the decoupling limit of DGP models, massive gravity models and so on.

where $\mathcal{L} = \sum_{i=2}^{5} \mathcal{L}_i$ is given by

$$\mathcal{L}_2 = G_2(\phi, X), \tag{4.2.2}$$
$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \tag{4.2.3}$$
$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \tag{4.2.4}$$
$$\mathcal{L}_5 = G_5(\phi, X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]. \tag{4.2.5}$$

Here G_i's are arbitrary functions of ϕ and $X \equiv -g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi/2$ is the kinetic term of the scalar field. Also, $(\nabla_\mu\nabla_\nu\phi)^2 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi$ and $(\nabla_\mu\nabla_\nu\phi)^3 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\rho\phi\nabla_\rho\nabla^\mu\phi$. In addition, the subscript ϕ and X denotes the derivative with respect to that variable. The form of the Lagrangian presented here is used in the context of Galileon theories. These Lagrangians contain all the ones with a single scalar and gravity in four dimensions which contain derivatives no more than second order in the equations of motion. Here some comments are in order.

- $G_4 = M_P^2/2$ gives the Einstein-Hilbert term.
- The special case $G_3 = f(\phi)$ is equivalent to $K = -2Xf_\phi$, while $G_5 = -\phi$ is equivalent to $G_4 = X$. These relations are shown by partial integration in the action.

General covariant equations of motion in Horndeski/Galileon theories are calculated in [14, 78].

4.2.2 Conformal transformation

When G_4 depends only on ϕ and not X, the gravitational part of the system can be mapped to the system with the Einstein gravity and a canonical inflaton by conformal transformation [53–55] and a redefinition of the inflaton field

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{4.2.6}$$

where Ω is a nonvanishing and regular function. We refer to the transformed frame as the Einstein frame, while the original frame as the Jordan frame. The system described by $(\mathcal{M}, \tilde{g}_{\mu\nu})$, where $\mathcal{M}$ is the manifold, has the same causal structure as the original system $(\mathcal{M}, g_{\mu\nu})$. On the other hand, G_5-type coupling cannot be eliminated by this conformal transformation. As a simple example, let us consider the action of a canonical scalar field plus G_4-type coupling

$$G_2(\phi, X) = X - V(\phi), \qquad G_4(\phi, X) = f(\phi). \tag{4.2.7}$$

Using the formula presented in Appendix A, and taking the conformal factor to be $\Omega^2 = 2f/M_P^2$, we find

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[f(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \\ &= \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_P^2}{2}\tilde{R} - \left(\frac{1}{\Omega^2} + 6M_P^2(\ln\Omega)'^2\right)\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{V(\phi)}{\Omega^4}\right]. \end{aligned} \tag{4.2.8}$$

Here the prime denotes the derivative with respect to ϕ. This action can be made canonical by the redefinition of the scalar field

$$\frac{d\tilde{\phi}}{d\phi} = \sqrt{\frac{1}{\Omega^2} + 6M_P^2(\ln\Omega)'^2}. \tag{4.2.9}$$

The potential for $\tilde{\phi}$ becomes

$$\tilde{V}(\tilde{\phi}) \equiv \frac{V(\phi)}{\Omega^4}. \tag{4.2.10}$$

4.2.3 Disformal transformation

Disformal transformation is proposed in [56] as a generalization of the conformal transformation. It takes the form $\tilde{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\nabla_\mu\phi\nabla_\nu\phi$ in the context of Horndeski/Galileon theories, and the properties of the transformation in these theories are investigated in [57]. It was found that general forms of $A(\phi, X)$ and $B(\phi, X)$ do not preserve second-order property of the field equations. In order to retain this property, we have to restrict the transformation within

$$\tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\nabla_\mu\phi\nabla_\nu\phi. \tag{4.2.11}$$

The transformation properties of the coefficients in the Lagrangian (4.2.2)–(4.2.5) are written in the Appendix B of [57]. As to the elimination of the noniminimal coupling,

- The nonminimal coupling $G_4(\phi, X)$ can be eliminated by (4.2.11) only when

$$G_4(\phi, X) = (1 - C(\phi)X)^{1/2}, \tag{4.2.12}$$

  with $C(\phi)$ being an arbitrary function.

- On the other hand, the nonminimal coupling $G_5(\phi, X)$ cannot be eliminated.

4.2.4 Relation to Horndeski theory

As explained above, Galileon theory in four-dimensional curved spacetime is equivalent to Horndeski theory. This equivalence was first shown in [14]. Horndeski Lagrangian is given by

$$\begin{aligned} S = {} & \kappa_1\delta^{\alpha\beta\gamma}_{\mu\nu\rho}\nabla^\mu\nabla_\alpha\phi R_{\beta\gamma}{}^{\nu\rho} - \frac{4}{3}\kappa_{1,X}\delta^{\alpha\beta\gamma}_{\mu\nu\rho}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\nabla^\rho\nabla_\gamma\phi \\ & + \kappa_3\delta^{\alpha\beta\gamma}_{\mu\nu\rho}\nabla_\alpha\phi\nabla^\mu\phi R_{\beta\gamma}{}^{\nu\rho} - 4\kappa_{3,X}\delta^{\alpha\beta\gamma}_{\mu\nu\rho}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\nabla^\rho\nabla_\gamma\phi \\ & + (F + 2W)\delta^{\alpha\beta}_{\mu\nu}R_{\alpha\beta}{}^{\mu\nu} - 4F_{,X}\delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi \\ & - 3(2F_{,\phi} + 4W_{,\phi} + X\kappa_8)\nabla_\mu\nabla^\mu\phi + 2\kappa_8\delta^{\alpha\beta}_{\mu\nu}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi + \kappa_9. \end{aligned} \tag{4.2.13}$$

Here, $\kappa_i(\phi, X)$ with $i = 1, 3, 8, 9$ are arbitrary functions of ϕ and X, W is an arbitrary function of ϕ only, and $\delta^{\alpha_1\cdots\alpha_n}_{\mu_1\cdots\mu_n} = n!\delta^{[\alpha_1\cdots\alpha_n]}_{\mu_1\cdots\mu_n}$. Also, F is related to κ_i as $F_{,X} = \kappa_{1,\phi} - \kappa_3 - 2X\kappa_{3,X}$. This theory is equivalent to Galileon theory with the following identification

$$G_2 = \kappa_9 + X \int^X dX'(\kappa_{8,\phi} - 2\kappa_{3,\phi\phi}), \tag{4.2.14}$$

$$G_3 = 6(F + 2W)_{,\phi} + X\kappa_8 + 4X\kappa_{3,\phi} - \int^X dX'(\kappa_8 - 2\kappa_{3,\phi}), \tag{4.2.15}$$

$$G_4 = 2(F + 2W) + 2X\kappa_3, \tag{4.2.16}$$

$$G_5 = -4\kappa_1. \tag{4.2.17}$$

4.3 Inflation in Horndeski/Galileon theories – General discussion

In this section we summarize the background action and equations of motion, and explain the typical two types of inflation which occurs in the Horndeski/Galileon theories (4.2.1). We also summarize the sound speed for scalar and tensor perturbations. Note that we use ϕ to denote the background value $\bar{\phi}$ in this section.

4.3.1 Equations of motion in FRW background

Assuming the FRW metric with spacial curvature neglected, the background action becomes

$$\mathcal{L}_2 = G_2, \tag{4.3.18}$$

$$\mathcal{L}_3 = G_3(\ddot{\phi} + 3H\dot{\phi}), \tag{4.3.19}$$

$$\mathcal{L}_4 = -6H^2 G_4 - 6H\dot{\phi}G_{4\phi} + 12H^2 X G_{4X}, \tag{4.3.20}$$

$$\mathcal{L}_5 = -6H^2 X G_{5\phi} + 2H^3\dot{\phi}X G_{5X}. \tag{4.3.21}$$

In addition, the equations of motion reduces to the following form

- Friedmann equation

$$\mathcal{E} \equiv \sum_{i=2}^{5} \mathcal{E}_i = 0, \tag{4.3.22}$$

- Raychaudhuri equation

$$\mathcal{P} \equiv \sum_{i=2}^{5} \mathcal{P}_i = 0, \tag{4.3.23}$$

- ϕ's equation of motion

$$\Phi \equiv \sum_{i=2}^{5} \Phi_i = 0. \tag{4.3.24}$$

In the last expression Φ_i is defined as $\Phi_i \equiv a^{-3}(a^3\mathcal{J}_i)^{\cdot} - \mathcal{K}_{i\phi}$ with $\mathcal{J} \equiv \sum_{i=2}^{5}\mathcal{J}_i$ and $\mathcal{K} \equiv \sum_{i=2}^{5}\mathcal{K}_i$. Terms $\mathcal{L}_i$, $\mathcal{E}_i$, $\mathcal{P}_i$, $\mathcal{J}_i$ and $\mathcal{K}_i$ are summarized in Table 4.1. Note that the subscripts $\dot{H}, H, X, \ddot{\phi}, \dot{\phi}, \phi$ denote the derivative with respect to that quantity, and that $\mathcal{L}$ should be regarded as $\mathcal{L}(H, \dot{\phi}, \phi)$ when we use "Relations" in this table.

4.3.2 Potential-driven and kinetically-driven inflation

Starting with the equations of motion (4.3.22)–(4.3.24), it can be shown that there are two typical cases in inflation [14]

- Potential driven inflation
  This occurs when (some of) the arbitrary functions G_i have ϕ dependence. Neglecting terms including $\dot{\phi}$ in Friedmann and Raychaudhuri equations, it is shown that inflation occurs due to the following contribution of ϕ to the Hubble parameter
  $$H^2 \simeq -\frac{G_2(\phi,0)}{6G_4(\phi,0)}. \tag{4.3.25}$$
  This means that inflation occurs due to the same contribution as in the minimal setup, as one sees from the fact that $-G_2(\phi,0)$ and $G_4(\phi,0)$ coincides with the potential V and $M_P^2/2$ in that setup.

- Kinetically driven inflation
  This occurs when the action has a shift symmetry $\phi \to \phi + c$. In this case G_i cannot depend on ϕ, and therefore ϕ derivative of $\mathcal{L}$ vanishes in Eq. (4.3.24). As a result $\sum a^3\mathcal{J}_i$ conserves, and the theory has an attractor $\mathcal{J} \to 0$, along which $H = \text{const.}$ is realized. These features are found in G-inflation [77] and kinetic gravity braiding [74].

4.3.3 Inflationary predictions with modified setup

Let us consider the following noncanonical light scalar field

$$S_\chi = \int d^4x\,\frac{1}{2}a^3F^2\left[\dot{\chi}^2 - \frac{c_s^2}{a^2}(\partial_i\chi)^2\right] = \int d\tau d^3x\,\frac{1}{2}a^2F^2\left[\chi'^2 - c_s^2(\partial_i\chi)^2\right]. \tag{4.3.26}$$

Neglecting the time dependence of the sound speed c_s, we may rescale the time and position variables as $d\tilde{\tau} \equiv c_s^{3/2}d\tau$ and $d\tilde{x}^i \equiv c_s^{1/2}dx^i$, respectively. Defining $\tilde{\chi} \equiv F\chi$, we have

$$S_\chi = \int d\tilde{\tau}d^3\tilde{x}\,\frac{1}{2}a^2\left[\left(\frac{d\tilde{\chi}}{d\tilde{\tau}}\right)^2 + \frac{1}{F}\frac{d^2F}{d\tilde{\tau}^2}\tilde{\chi}^2 - \left(\frac{d\tilde{\chi}}{d\tilde{x}^i}\right)^2\right]. \tag{4.3.27}$$

Assuming that the mass term coming from the time dependence of F is small, one sees that $\tilde{\chi}$ and χ have the spectrum of inflationary origin

$$\mathcal{P}_{\tilde{\chi}} = \frac{1}{c_s^3}\left(\frac{H}{2\pi}\right)^2 \quad \to \quad \mathcal{P}_\chi = \frac{1}{F^2c_s^3}\left(\frac{H}{2\pi}\right)^2, \tag{4.3.28}$$

where the c_s^3 dependence comes from the redefinition of the time variable. This gives the leading modification to the power spectrum. Note that the value of the power spectrum changes at $c_sk = aH$, in contrast to $k = aH$ in the minimal setup.

4.3.4 Quadratic action for scalar and tensor perturbations

The quadratic action for generalized Galileon theories with FRW background is calculated in ζ gauge in [14]. Since the discussion on the sound speed is important in this thesis, we summarize their results. We also summarize tensor perturbations, which have no difference between ζ and ϕ gauges because of the gauge invariance. For the consistency check of the gauge independence of the sound speed, we also show the quadratic action in φ gauge in Appendix B. Non-gaussianities are calculated in e.g. [79]. The quadratic actions in ζ gauge become

$$S_S^{(\zeta)} = \int d^4x\, a^3 \Big[\Big(C_{(\alpha\alpha)}\alpha^2 + C_{(\alpha\dot\zeta)}\alpha\dot\zeta + C_{(\dot\zeta\dot\zeta)}\dot\zeta^2\Big) + a^{-2}\Big(D_{(\alpha\zeta)}\alpha\zeta_{,ii} + D_{(\zeta\zeta)}\zeta\zeta_{,ii} + D_{(\alpha\beta)}\alpha\beta_{,ii} + D_{(\dot\zeta\beta)}\dot\zeta\beta_{,ii}\Big)\Big]\,, \tag{4.3.29}$$

$$S_T = \int d^4x\, a^3 \Big[C_{(\dot h\dot h)}\dot h_{ij}^2 + a^{-2}D_{(hh)}h_{ij}h_{ij,kk}\Big]\,, \tag{4.3.30}$$

where $C_{(\alpha\alpha)}$, $C_{(\alpha\dot\zeta)}$, $C_{(\dot h\dot h)}$ and $D_{(hh)}$ are summarized in Table. 4.2. Here $C_{(\alpha\alpha)}$, $C_{(\alpha\dot\zeta)}$ and $C_{(hh)}$ are related to the background equations of motion as

$$C_{(\alpha\alpha)} = \frac{1}{2}H\mathcal{E}_H + \frac{1}{2}\dot\phi\mathcal{E}_{\dot\phi}, \qquad C_{(\alpha\dot\zeta)} = -\mathcal{E}_H, \qquad C_{(\dot h\dot h)} = \frac{1}{16}\mathcal{P}_{\dot H}. \tag{4.3.31}$$

In terms of the Lagrangian, they become

$$C_{(\alpha\alpha)} = \frac{1}{2}H^2\mathcal{L}_{HH} + H\dot\phi\mathcal{L}_{H\dot\phi} + \frac{1}{2}\dot\phi^2\mathcal{L}_{\dot\phi\dot\phi}, \qquad C_{(\alpha\dot\zeta)} = -H\mathcal{L}_{HH} - \dot\phi\mathcal{L}_{H\dot\phi}, \qquad C_{(\dot h\dot h)} = -\frac{1}{48}\mathcal{L}_{HH}. \tag{4.3.32}$$

Other coefficients are related to the ones above as follows

$$C_{(\dot\zeta\dot\zeta)} = -24C_{(\dot h\dot h)}, \quad D_{(\alpha\zeta)} = -16C_{(\dot h\dot h)}, \quad D_{(\zeta\zeta)} = -8D_{(hh)}, \quad D_{(\alpha\beta)} = -\frac{1}{3}C_{(\alpha\dot\zeta)}, \quad D_{(\zeta\beta)} = 16C_{(\dot h\dot h)}, \tag{4.3.33}$$

For the scalar perturbations, we must solve the constraint equations. The constraint equations for α and β give

$$(2C_{(\alpha\alpha)}\alpha + C_{(\alpha\dot\zeta)}\dot\zeta) + a^{-2}(D_{(\alpha\zeta)}\zeta_{,ii} + D_{(\alpha\beta)}\beta_{,ii}) = 0, \qquad D_{(\alpha\beta)}\alpha + D_{(\zeta\beta)}\dot\zeta = 0. \tag{4.3.34}$$

Using these, we have the following action for ζ

$$S_S^{(\zeta)} = \int d^4x\, a^3 \Big[C'_{(\dot\zeta\dot\zeta)}\dot\zeta^2 + a^{-2}D'_{(\zeta\zeta)}\zeta\zeta_{,ii}\Big]\,, \tag{4.3.35}$$

where $C'_{(\dot\zeta\dot\zeta)}$ and $D'_{(\zeta\zeta)}$ are

$$C'_{(\dot\zeta\dot\zeta)} = 36\frac{C_{(\alpha\alpha)}}{C^2_{(\alpha\dot\zeta)}}C^2_{(\dot h\dot h)} + 3C_{(\dot h\dot h)}, \qquad D'_{(\zeta\zeta)} = \frac{1}{a}\frac{d}{dt}\left(6a\frac{C^2_{(\dot h\dot h)}}{C_{(\alpha\dot\zeta)}}\right) - D_{(hh)}. \tag{4.3.36}$$

Table 4.1: Terms in the background Lagrangian and equations of motion[#5].

Terms		Explicit form	Relations
$\mathcal{L}$	$\mathcal{L}_2$	$+G_2$	
	$\mathcal{L}_3$	$+\ddot{\phi}G_3 + 3H\dot{\phi}G_3$ or $-X^{1/2}\int_0^X dX(X^{-1/2}G_{3\phi}) + 3H\dot{\phi}X^{-1/2}\int_0^X dX(X^{1/2}G_{3X})$	
	$\mathcal{L}_4$	$-6H^2G_4 - 6H\dot{\phi}G_{4\phi} + 12H^2XG_{4X}$	
	$\mathcal{L}_5$	$-6H^2XG_{5\phi} + 2H^3\dot{\phi}XG_{5X}$	
$\mathcal{E}$	$\mathcal{E}_2$	$-G_2 + 2XG_{2X}$	$-\mathcal{L} + H\mathcal{L}_H + \dot{\phi}\mathcal{L}_{\dot{\phi}}$
	$\mathcal{E}_3$	$-2XG_{3\phi} + 6H\dot{\phi}XG_{3X}$	
	$\mathcal{E}_4$	$+H^2(-6G_4 + 24XG_{4X} + 24X^2G_{4XX}) - H\dot{\phi}(6G_{4\phi} + 12XG_{4\phi X})$	
	$\mathcal{E}_5$	$-H^2X(18G_{5\phi} + 12XG_{5\phi X}) + H^3\dot{\phi}X(10G_{5X} + 4XG_{5XX})$	
$\mathcal{P}$	$\mathcal{P}_2$	$+G_2$	$-\dfrac{1}{3}(\mathcal{L}_H)^{\cdot} - H\mathcal{L}_H + \mathcal{L}$
	$\mathcal{P}_3$	$-2XG_{3\phi} - 2\ddot{\phi}XG_{3X}$	
	$\mathcal{P}_4$	$(4HG_4 + 2\dot{\phi}G_{4\phi} - 8HXG_{4X})^{\cdot} + (6H^2G_4 - 12H^2XG_{4X})$	
	$\mathcal{P}_5$	$(4HXG_{5\phi} - 2H^2\dot{\phi}XG_{5X})^{\cdot} + (6H^2XG_{5\phi} - 4H^3\dot{\phi}XG_{5X})$	
$\mathcal{J}$	$\mathcal{J}_2$	$+\dot{\phi}G_{2X}$	$\mathcal{L}_{\dot{\phi}}$
	$\mathcal{J}_3$	$-\dot{\phi}G_{3\phi} + 6HXG_{3X} - \frac{1}{2}\dot{\phi}X^{-1/2}\int_0^X dX(X^{-1/2}G_{3\phi})$	
	$\mathcal{J}_4$	$+H^2\dot{\phi}(6G_{4X} + 12XG_{4XX}) - 12HXG_{4\phi X}$	
	$\mathcal{J}_5$	$-H^2\dot{\phi}(6G_{5\phi} + 6XG_{5\phi X}) + H^3X(6G_{5X} + 4XG_{5XX})$	
$\mathcal{K}$	$\mathcal{K}_2$	$+G_{2\phi}$	$\mathcal{L}_\phi$
	$\mathcal{K}_3$	$-X^{1/2}\int_0^X dX(X^{-1/2}G_{3\phi\phi}) + 3H\dot{\phi}X^{-1/2}\int_0^X dX(X^{1/2}G_{3\phi X})$	
	$\mathcal{K}_4$	$-6H^2G_{4\phi} - 6H\dot{\phi}G_{4\phi\phi} + 12H^2XG_{4\phi X}$	
	$\mathcal{K}_5$	$-6H^2XG_{5\phi\phi} + 2H^3\dot{\phi}XG_{5\phi X}$	

[#5]Lagrangian $\mathcal{L}_3$ needs some explanation. Though it contains $\ddot{\phi}$ it can be eliminated from the Lagrangian by partial integration. For example, the term $\phi^m X^n \subset G_3$ gives

$$\ddot{\phi}\phi^m X^n = \frac{1}{2n+1}\phi^m\left(\dot{\phi}X^n\right)^{\cdot} \to \frac{1}{2n+1}\phi^m X^n\left(-2m\phi^{-1}X - 3H\dot{\phi}\right). \tag{4.3.37}$$

For general G_3, the expression after integration by parts becomes

$$\mathcal{L}_3 = -X^{1/2}\int_0^X dX(X^{-1/2}G_{3\phi}) + 3H\dot{\phi}X^{-1/2}\int_0^X dX(X^{1/2}G_{3X}). \tag{4.3.38}$$

This expression $\mathcal{L}(H, \dot{\phi}, \phi)$ must be used when using “Relations” in Table 4.1.

Table 4.2: Coefficients for tensor perturbations.

Tensor		Explicit form	Relations
$C_{(\dot{h}\dot{h})}$	$\mathcal{L}_2$		$\dfrac{1}{16}\mathcal{P}_{\dot{H}}$ or $-\dfrac{1}{48}\mathcal{L}_{HH}$
	$\mathcal{L}_3$		
	$\mathcal{L}_4$	$\frac{1}{4}(G_4 - 2XG_{4X})$	
	$\mathcal{L}_5$	$-\frac{1}{4}X(H\dot{\phi}G_{5X} - G_{5\phi})$	
$D_{(hh)}$	$\mathcal{L}_2$		
	$\mathcal{L}_3$		
	$\mathcal{L}_4$	$\frac{1}{4}G_4$	
	$\mathcal{L}_5$	$-\frac{1}{4}X(\ddot{\phi}G_{5X} + G_{5\phi})$	

Table 4.3: Coefficients for scalar perturbation in ζ gauge.

Scalar		Explicit form	Relations
$C_{(\alpha\alpha)}$	$\mathcal{L}_2$	$+X(K_X + 2XK_{XX})$	$\dfrac{1}{2}H\mathcal{E}_H + \dfrac{1}{2}\dot{\phi}\mathcal{E}_{\dot{\phi}}$ or $\dfrac{1}{2}H^2\mathcal{L}_{HH} + H\dot{\phi}\mathcal{L}_{H\dot{\phi}} + \dfrac{1}{2}\dot{\phi}^2\mathcal{L}_{\dot{\phi}\dot{\phi}}$
	$\mathcal{L}_3$	$-X(2G_{3\phi} + 2XG_{3\phi X}) + H\dot{\phi}X(12G_{3X} + 6XG_{3XX})$	
	$\mathcal{L}_4$	$+H^2(-6G_4 + 42XG_{4X} + 96X^2G_{4XX} + 24X^3G_{4XXX})$ $-H\dot{\phi}(6G_{4\phi} + 30XG_{4\phi X} + 12X^2G_{4\phi XX})$	
	$\mathcal{L}_5$	$-H^2X(36G_{5\phi} + 54XG_{5\phi X} + 12X^2G_{5\phi XX})$ $+H^3\dot{\phi}X(30G_{5X} + 26XG_{5XX} + 4X^2G_{5XXX})$	
$C_{(\alpha\dot{\zeta})}$	$\mathcal{L}_2$		$-\mathcal{E}_H$ or $-H\mathcal{L}_{HH} - \dot{\phi}\mathcal{L}_{H\dot{\phi}}$
	$\mathcal{L}_3$	$-6\dot{\phi}XG_{3X}$	
	$\mathcal{L}_4$	$+\dot{\phi}(6G_{4\phi} + 12XG_{4\phi X})$ $+H(12G_4 - 48XG_{4X} - 48X^2G_{4XX})$	
	$\mathcal{L}_5$	$+HX(36G_{5\phi} + 24XG_{5\phi X})$ $-H^2\dot{\phi}X(30G_{5X} + 12XG_{5XX})$	

4.4 Inflation in Horndeski/Galileon theories – Examples

In this section we consider two examples of inflation models with nonminimal coupling to gravity, and see the inflationary predictions. The beginning of the inflaton oscillation phase in these models is investigated in detail in Chapter 8–9.

4.4.1 Inflationary predictions with nonminimal coupling $\phi^2 R$

Coupling the inflaton nonminimally to gravity, especially to the Ricci scalar, has been considered in [18], and later studied in the context of identifying the inflaton with the standard model Higgs field [19]. This theory has recently been rephrased as "Higgs inflation" in [19,20,80]. Cosmological perturbations are calculated also in Jordan frame [81,82], though we proceed in the Einstein frame below.

The action we consider corresponds to

$$G_2 = X - V, \qquad G_4 = \frac{M_P^2}{2} + \frac{\xi}{2}\phi^2. \tag{4.4.39}$$

in the language of Eqs. (4.2.2)–(4.2.5). Here ξ is a model parameter. Assuming a monomial potential, the quartic one

$$V = \frac{\lambda}{4}\phi^4, \tag{4.4.40}$$

is the most interesting from the viewpoint of observations. The explicit form of the action is

$$S = \int d^4x\sqrt{-g}\left[\left(\frac{M_P^2}{2} + \frac{\xi}{2}\phi^2\right)R - \frac{1}{2}(\partial\phi)^2 - V\right]. \tag{4.4.41}$$

The system can be mapped onto the Einstein frame using conformal transformation. Defining the transformation as

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \qquad \Omega^2 = 1 + \frac{\xi\phi^2}{M_P^2}, \tag{4.4.42}$$

we have

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}\tilde{R} - \frac{1}{\Omega^4}\left(\Omega^2 + \frac{6\xi\phi^2}{M_P^2}\right)\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{\Omega^4}V\right] \tag{4.4.43}$$

$$\equiv \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \tilde{V}\right]. \tag{4.4.44}$$

Here we have dropped a total derivative term, which has no relation to the dynamics. The kinetic term for the inflaton is made canonical by the following redefinition

$$\frac{d\tilde{\phi}}{d\phi} = \sqrt{\frac{1}{\Omega^4}\left(\Omega^2 + \frac{6\xi^2\phi^2}{M_P^2}\right)}. \tag{4.4.45}$$

This redefinition gives the following approximate relation between $\tilde{\phi}$ and ϕ

$$\tilde{\phi} \simeq \begin{cases} \phi & (\phi \ll M_P/\xi) \\ \sqrt{\frac{3}{2}} M_P \ln \Omega^2 & (\phi \gg M_P/\xi) \end{cases} . \tag{4.4.46}$$

The potential for $\tilde{\phi}$ is given by $\tilde{V} \equiv V/\Omega^4$, which takes

$$\tilde{V} \simeq \frac{\lambda M_P^4}{4\xi^2} \left(1 - e^{-\alpha \frac{|\tilde{\phi}|}{M_P}}\right)^2 , \tag{4.4.47}$$

for $\phi \gg M_P/\xi$ (or equivalently $\tilde{\phi} \gg M_P/\xi$). Here $\alpha = \sqrt{2/3}$. Note that the potential $\tilde{V}$ becomes quartic for $\phi \ll M_P/\xi$, due to $\Omega^2 \simeq 1$ and $\tilde{\phi} \simeq \phi$.

Below we drop all the tildes for notational simplicity. The inflationary predictions of this model are calculated in the same way as in Chapter 3. Note that we do not have to care about the sound speed nor the overall factor illustrated in the previous section, because the system is now reduced to the one with the Einstein gravity and a canonical inflaton. First, the potential slow-roll parameters become

$$\epsilon_V = \frac{4}{3} \frac{1}{(e^{\alpha\phi/M_P} - 1)^2} \simeq \frac{3}{4} \frac{1}{N^2}, \qquad \eta_V = -\frac{4}{3} \frac{e^{\alpha\phi/M_P} - 2}{(e^{\alpha\phi/M_P} - 1)^2} \simeq -\frac{1}{N}. \tag{4.4.48}$$

Here we have rewritten these parameters in terms of the e-folding number

$$N \simeq \int_{\phi_{\rm end}}^{\phi} d\phi \, \frac{1}{M_P} \frac{1}{\sqrt{2\epsilon_V}} \simeq \frac{3}{4} e^{\alpha \frac{\phi}{M_P}} , \tag{4.4.49}$$

with $N \gg 1$. Since the amplitude of the scalar perturbation requires

$$\mathcal{P}_\zeta = \frac{V}{24\pi^2 M_P^4 \epsilon_V} = \frac{\lambda N^2}{72\pi^2 \xi^2} \simeq 2 \times 10^{-9}, \tag{4.4.50}$$

the parameter of the model ξ must be fixed to be $\mathcal{O}(10^4)$ for $\lambda = \mathcal{O}(1)$. The spectral tilt and tensor-to-scalar ratio is given by

$$n_S \simeq 1 - 6\epsilon_V + 2\eta_V \simeq 1 - \frac{2}{N}, \qquad r \simeq 16\epsilon_V \simeq \frac{12}{N^2}. \tag{4.4.51}$$

These predictions corresponds to the orange blobs in Fig. 3.1, which are well within the observationally allowed region.

4.4.2 Inflationary predictions with nonminimal coupling $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

The idea of coupling a scalar field to gravity in a derivative form is pioneered in [21], and recently rephrased in [22] and related works as "New Higgs Inflation", in the context of inflation with the standard model Higgs field. Since we focus on this derivative coupling in Chapter 9, we review the inflationary dynamics of the model. We refer to [83–86] for the cosmological perturbations in this model.

The model with nonminimal derivative coupling corresponds to the following choice in the language of Eqs. (4.2.2)–(4.2.5)

$$G_2 = X - V, \qquad G_4 = \frac{M_P^2}{2}, \qquad G_5 = -\frac{\phi}{2M^2}, \tag{4.4.52}$$

where M is a model parameter of mass dimension 1. The action of this system is

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}\left(g^{\mu\nu} - \frac{G^{\mu\nu}}{M^2}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{4.4.53}$$

which becomes at the background level

$$S = \int d^4x\, a^3\left[-3M_P^2H^2 + \left(1 + \frac{3H^2}{M^2}\right)\frac{\dot{\phi}^2}{2} - V(\phi)\right]. \tag{4.4.54}$$

Here note that the background relation $G^{00} = 3H^2$ is used. The nonminimal term $3H^2/M^2$ in front of the kinetic term of the inflaton works as an "enhanced friction" during inflation, as we will see below. In analogy with Eqs. (3.2.18)–(3.2.19), we impose slow-roll conditions to obtain

- Friedmann equation

$$3M_P^2H^2 = \left(1 + \frac{9H^2}{M^2}\right)\frac{\dot{\phi}^2}{2} + V \quad \rightarrow \quad 3M_P^2H^2 \simeq V, \tag{4.4.55}$$

- ϕ's equation of motion

$$a^{-3}\frac{d}{dt}\left[a^3\left(1 + \frac{3H^2}{M^2}\right)\dot{\phi}\right] + V' = 0 \quad \rightarrow \quad 3H\left(1 + \frac{3H^2}{M^2}\right)\dot{\phi} + V' \simeq 0. \tag{4.4.56}$$

We assume that the nonminimal term is effective $H^2/M^2 \gg 1$ during inflation. Note that the friction term in Eq. (4.4.56) is enhanced in this limit. This enhanced friction makes it possible to slow down the motion of the inflaton and suppress the tensor-to-scalar ratio, as we see from now. The redefinition which makes the Lagrangian of ϕ canonical

$$\mathcal{L}_\phi \simeq -\frac{3H^2}{2M^2}(\partial\phi)^2 - V(\phi) \simeq -\frac{V}{2M^2M_P^2}(\partial\phi)^2 - V(\phi) \equiv -\frac{1}{2}(\partial\phi_c)^2 - V(\phi_c), \tag{4.4.57}$$

also makes the equation of motion for ϕ canonical

$$3H\dot{\phi}_c + \frac{dV}{d\phi_c} \simeq 0. \tag{4.4.58}$$

Let us focus on a monomial potential

$$V = \frac{\lambda}{n}\phi^n. \tag{4.4.59}$$

Note that we sometimes write λ as m_ϕ^2 for $n = 2$ in the following. The canonical redefinition of the inflaton field is given by

$$\phi_c \equiv \left(\frac{\lambda}{n}\right)^{\frac{1}{2}} \frac{2}{n+2}\frac{\phi^{\frac{n+2}{2}}}{MM_P}, \qquad V(\phi_c) \equiv \left(\frac{\lambda}{n}\right)^{\frac{2}{n+2}} \left(\frac{n+2}{2}\right)^{\frac{2n}{n+2}} (MM_P)^{\frac{2n}{n+2}} \phi_c^{\frac{2n}{n+2}}, \tag{4.4.60}$$

The potential slow-roll parameters $\epsilon_{V_c}, \eta_{V_c}$ defined in terms of ϕ_c are now calculated by using Eq. (3.2.73) as,

$$\epsilon_{V_c} \equiv \frac{M_P^2}{2}\left(\frac{V_{\phi_c}}{V}\right)^2 = \frac{2n^2}{(n+2)^2}\frac{M_P^2}{\phi_c^2} = \frac{n}{2(n+2)}\frac{1}{N}, \tag{4.4.61}$$

$$\eta_{V_c} \equiv M_P^2 \frac{V_{\phi_c\phi_c}}{V} = \frac{2n(n-2)}{(n+2)^2}\frac{M_P^2}{\phi_c^2} = \frac{n-2}{2(n+2)}\frac{1}{N}, \tag{4.4.62}$$

where the subscript ϕ_c denotes the derivative with respect to it. Also, N is the e-folding number given by

$$N = \frac{n+2}{4n}\left(\frac{\phi_c}{M_P}\right)^2. \tag{4.4.63}$$

From these expressions it is obvious that the inflation ends at $\phi_c \sim M_P$.

The action for perturbations are calculated as

$$S_S = \int d^4x\, a^3 A^2 B \frac{\dot{\phi}^2}{H^2}\left[\frac{1}{2}\dot{\zeta}^2 - \frac{1}{2}\frac{c_s^2}{a^2}(\partial_i \zeta)^2\right], \tag{4.4.64}$$

$$S_T = \int d^4x\, a^3 \frac{M_P^2}{8}\left(1 - \frac{\epsilon}{2}\right)\left[\dot{h}_{ij}^2 - \frac{c_t^2}{a^2}(\partial_k h_{ij})^2\right], \tag{4.4.65}$$

with

$$c_s^2 = 1 + \frac{2}{B}\left[\frac{\epsilon}{1-\frac{\epsilon}{2}}\left(1 + \frac{H^2}{M^2}\frac{1}{A}\right) + \frac{3\dot{H}}{M^2}\right], \qquad c_t^2 = \frac{1+\frac{\epsilon}{2}}{1-\frac{\epsilon}{2}}, \tag{4.4.66}$$

$$A \equiv \frac{1-\frac{1}{2}\epsilon}{1-\frac{3}{2}\epsilon}, \qquad B \equiv 1 + \frac{3H^2}{M^2}\frac{1+\frac{3}{2}\epsilon}{1-\frac{1}{2}\epsilon}, \qquad \epsilon \equiv \frac{\dot{\phi}^2}{M^2 M_P^2}. \tag{4.4.67}$$

Here ϵ must not be confused with the slow-roll parameter ϵ_V and ϵ_H. Note that A, B, c_s^2 and c_t^2 all reduce to unity for $M \to \infty$, as they should be. Since ϵ is much smaller than unity because of the time derivative which appears in its definition, it does not affect inflationary predictions. However, in large friction limit B becomes much larger than unity due to the second term, and this brings about modifications in the inflationary predictions.

First let us consider the CMB normalization of the scalar perturbation, which determines one combination of the parameters λ and M. The amplitude of the scalar perturbation is calculated as

$$\mathcal{P}_\zeta \simeq \left(\frac{H}{\dot{\phi}_c}\right)^2 \left(\frac{H}{2\pi}\right)^2 \simeq \frac{V}{24\pi^2 \epsilon_{V_c} M_P^4} \tag{4.4.68}$$

$$\simeq \frac{1}{12\pi^2} n^{-\frac{2}{n+2}} (n+2)^{\frac{2(n+1)}{n+2}} N^{\frac{2(n+1)}{n+2}} \left(\frac{\lambda}{n} M^n \Big/ M_P^4\right)^{\frac{2}{n+2}}. \tag{4.4.69}$$

The normalization $\mathcal{P}_\zeta \simeq 2 \times 10^{-9}$ gives

$$\frac{m_\phi M}{M_P^2} \simeq 2 \times 10^{-10} \left(\frac{50}{N}\right)^{\frac{3}{2}} \quad (n=2), \qquad \frac{\lambda M^4}{M_P^4} \simeq 1 \times 10^{-31} \left(\frac{50}{N}\right)^5 \quad (n=4). \tag{4.4.70}$$

Next let us consider the tensor-to-scalar ratio and the spectral index. Because the tensor power spectrum is not much affected by the nonminimal coupling, and because the amplitude of the scalar perturbation is fixed as $\mathcal{P}_\zeta \simeq 2 \times 10^{-9}$, no significant change of $\mathcal{O}(H^2/M^2)$ does not occur to the tensor-to-scalar ratio. However, nonnegligible changes occur due to $B \gg 1$. Note that the canonical redefinition of the inflaton field $(3H^2/M^2)\dot{\phi}^2 \equiv \dot{\phi}_c^2$ makes the factor in the scalar quadratic action (4.4.64) the same as in the minimal setup. Therefore the prediction for the tensor-to-scalar ratio coincides with the one with a canonical inflaton with a canonical potential (4.4.60), the exponent of which is made smaller by the redefinition above. The spectral index n_s is also calculated using the same redefinition, and therefore

$$r \simeq 16\epsilon_{V_c} \simeq \frac{8n}{n+2}\frac{1}{N}, \qquad n_s - 1 \simeq -6\epsilon_{V_c} + 2\eta_{V_c} \simeq -\frac{2(n+1)}{n+2}\frac{1}{N}, \tag{4.4.71}$$

holds. As we have seen in Chapter 3, the tensor-to-scalar tend to be observationally safe for lower values of the potential exponent, and thus the nonminimal derivative coupling "improves" the prediction. These predictions are plotted in Fig. 4.1. One sees that the inflationary predictions tend to be consistent with observations in the presence of the nonminimal derivative coupling.

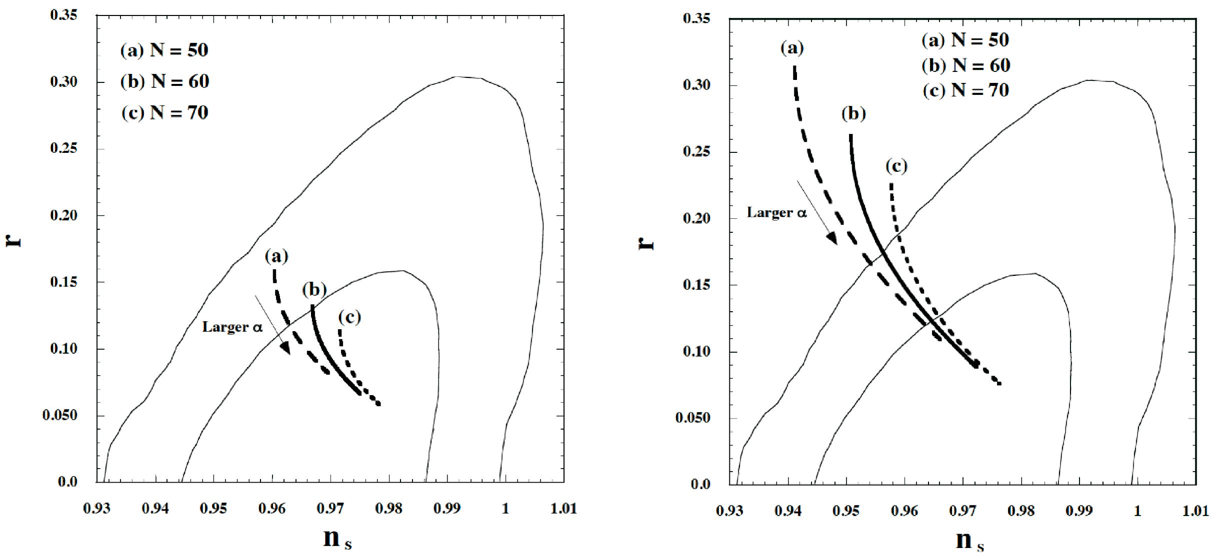

Figure 4.1: Inflationary predictions of nonminimal derivative coupling model for quadratic potential $m_\phi^2\phi^2/2$ (*left*) and quartic potential $\lambda\phi^4/4$ (*right*). Three lines correspond to $N = 50$, 60 and 70, respectively, and the parameter $\alpha \equiv m_\phi^2/M^2$ (*left*) and $\lambda \equiv (M_P/M)^2$ (*right*) are varied from 10^{-8} to 10^8. These figures are taken from Tsujikawa [85].

Chapter 5

Particle production by oscillating field

Conventional theory of reheating, first considered in [25, 26, 37], discusses the inflaton decay using perturbation theory. In this scheme, the inflaton decay occurs when the decay rate Γ_ϕ exceeds the expansion rate of the universe H, and the reheating temperature is estimated simply by the Friedmann equation as $T_R \sim \mathcal{O}(0.1)\sqrt{\Gamma_\phi M_P}$, assuming that the produced particles thermalize at once[#1].

However, later it was realized [88,89] that the coherently oscillating nature of the inflaton leads to much more interesting phenomena. What the authors found was that the coherent oscillation of the inflaton triggers an explosive resonant production of light particles, a phenomenon called "parametric resonance". The epoch of this resonant particle production was dubbed as "preheating" in [27], and has been a subject of interest since then.

In parametric resonance, the production of light particles is described by the Mathieu equation. The parameter space of this equation has instability bands as will be shown in Figs. 5.1–5.2 below, and totally different phenomena occur depending on the two parameter regions $q \lesssim 1$ and $q \gg 1$. In the following, we first explain the quantization of the produced light field, and then illustrate the parametric resonance for $q \lesssim 1$ (narrow resonance) and $q \gg 1$ (broad resonance) separately. We especially focus on the narrow resonance, which might occur through gravitational effects as we see in Chapter 7–9. We clarify that the narrow resonance can be interpreted as induced emission, in which the produced particles "induces" further production due to the bosonic nature of the particles. Throughout the chapter except for the last section, the produced light scalar field is denoted as χ with the action

$$S_M = \int d^4x \left[-\frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\chi^2(t)\chi^2 \right], \tag{5.0.1}$$

where $(\partial\chi)^2$ is a simplified notation for $g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$, and $m_\chi(t)$ is an oscillating mass of the χ field. For fermions induced emission does not occur due to Pauli blocking, and therefore we do not consider fermions here.

[#1] Of course, the inflaton decay (or damping) rate and the thermalization rate are totally different notions. This point is stressed in for example [87].

5.1 Quantization of the light field

We first expand χ in terms of its Fourier component

$$\chi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, \chi(t, \mathbf{k}). \tag{5.1.2}$$

The equation of motion for $\chi(t, \mathbf{k})$ is given by

$$\ddot{\chi}(t, \mathbf{k}) + \omega_k^2(t)\chi(t, \mathbf{k}) = 0, \tag{5.1.3}$$

where $\omega_k^2(t) = k^2 + m_\chi^2(t)$. The canonical commutation relation of $\chi(t, \mathbf{x})$

$$[\chi(t, \mathbf{x}), \dot{\chi}(t, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}'), \tag{5.1.4}$$

becomes

$$[\chi(t, \mathbf{k}), \dot{\chi}(t, -\mathbf{k}')] = i\delta^3(2\pi)^3(\mathbf{k} - \mathbf{k}'), \tag{5.1.5}$$

in momentum space. We define the following time-dependent creation and annihilation operators

$$a_{\mathbf{k}}(t) \equiv \frac{e^{i\int_0^t dt'\, \omega_k(t')}}{\sqrt{2\omega_k}} \left[\omega_k \chi(t, \mathbf{k}) + i\dot{\chi}(t, \mathbf{k})\right], \qquad a_{\mathbf{k}}^\dagger(t) \equiv \frac{e^{-i\int_0^t dt'\, \omega_k(t')}}{\sqrt{2\omega_k}} \left[\omega_k \chi(t, -\mathbf{k}) - i\dot{\chi}(t, -\mathbf{k})\right]. \tag{5.1.6}$$

These operators satisfy the commutation relation

$$[a_{\mathbf{k}}(t), a_{\mathbf{k}'}^\dagger(t)] = (2\pi)^3\delta(\mathbf{k} - \mathbf{k}'), \qquad [a_{\mathbf{k}}(t), a_{\mathbf{k}'}(t)] = [a_{\mathbf{k}}^\dagger(t), a_{\mathbf{k}'}^\dagger(t)] = 0, \tag{5.1.7}$$

and the following differential equation

$$\dot{a}_{\mathbf{k}}(t) = \frac{\dot{\omega}_k}{2\omega_k} e^{2i\int_0^t dt'\, \omega_k} a_{-\mathbf{k}}^\dagger(t), \qquad \dot{a}_{-\mathbf{k}}^\dagger(t) = \frac{\dot{\omega}_k}{2\omega_k} e^{-2i\int_0^t dt'\, \omega_k} a_{\mathbf{k}}(t). \tag{5.1.8}$$

Here the equation of motion (5.1.3) is used.

Next let us see how these creation and annihilation operators mix up as time evolves. We write

$$a_{\mathbf{k}}(t) = \alpha_k(t)a_{\mathbf{k}}(0) + \beta_k^*(t)a_{-\mathbf{k}}^\dagger(0), \qquad a_{-\mathbf{k}}^\dagger(t) = \alpha_k^*(t)a_{-\mathbf{k}}^\dagger(0) + \beta_k(t)a_{\mathbf{k}}(0). \tag{5.1.9}$$

Here α_k and β_k depend only on $k = |\mathbf{k}|$ for the reason which soon becomes clear. The initial condition is given by $\alpha_k(0) = 1$ and $\beta_k(0) = 0$. We substitute these expressions into Eq. (5.1.8) to get the differential equation for α_k and β_k

$$\dot{\alpha}_k = \frac{\dot{\omega}_k}{2\omega_k} e^{2i\int_0^t dt'\, \omega_k} \beta_k, \qquad \dot{\beta}_k = \frac{\dot{\omega}_k}{2\omega_k} e^{-2i\int_0^t dt'\, \omega_k} \alpha_k. \tag{5.1.10}$$

Since the initial condition and these equations do not depend on the direction of $\mathbf{k}$, the above statement about the momentum dependence of α_k and β_k is justified. Note here that the commutation relations (5.1.7) give the following condition for α_k and β_k

$$|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1. \tag{5.1.11}$$

Let us rewrite $\chi(t, \mathbf{k})$ in terms of these α_k and β_k. Inverting Eq. (5.1.6), we have

$$\begin{aligned}\chi(t, \mathbf{k}) &= \frac{1}{\sqrt{2\omega_k}} \left[a_{\mathbf{k}}(t) e^{-i\int_0^t dt'\,\omega_k} + a^\dagger_{-\mathbf{k}}(t) e^{i\int_0^t dt'\,\omega_k} \right] \\ &= \chi_k(t) a_{\mathbf{k}}(0) + \chi_k^*(t) a^\dagger_{-\mathbf{k}}(0).\end{aligned} \tag{5.1.12}$$

Here the mode function $\chi_k(t)$ is defined as

$$\chi_k(t) \equiv \frac{1}{\sqrt{2\omega_k}} \left[\alpha_k(t) e^{-i\int_0^t dt'\,\omega_k} + \beta_k(t) e^{i\int_0^t dt'\,\omega_k} \right]. \tag{5.1.13}$$

One can confirm that this mode function satisfies Eq. (5.1.3). In fact,

$$\dot{\chi}_k(t) = -i\sqrt{\frac{\omega_k}{2}} \left[\alpha_k(t) e^{-i\int_0^t dt'\,\omega_k} - \beta_k(t) e^{i\int_0^t dt'\,\omega_k} \right], \tag{5.1.14}$$

$$\ddot{\chi}_k(t) = -\frac{\omega_k^2}{\sqrt{2\omega_k}} \left[\alpha_k(t) e^{-i\int_0^t dt'\,\omega_k} + \beta_k(t) e^{i\int_0^t dt'\,\omega_k} \right], \tag{5.1.15}$$

and therefore

$$\ddot{\chi}_k + \omega_k^2 \chi_k = 0. \tag{5.1.16}$$

Here Eq. (5.1.10) is used. Note that the initial condition $\alpha_k(0) = 1$ and $\beta_k(0) = 0$ means

$$\chi_k(t \to 0) \simeq \frac{1}{\sqrt{2\omega_k}} e^{-i\int_0^t dt'\,\omega_k}, \qquad \dot{\chi}_k(t \to 0) \simeq -i\sqrt{\frac{\omega_k}{2}} e^{-i\int_0^t dt'\,\omega_k}. \tag{5.1.17}$$

Also note that the mode function satisfies

$$\chi_k(t)\dot{\chi}_k^*(t) - \chi_k^*(t)\dot{\chi}_k(t) = i, \tag{5.1.18}$$

due to Eq. (5.1.11)

Let us suppose that the system is in the vacuum at $t = 0$, which we denote as $|0_{t=0}\rangle$. The energy density is calculated as

$$\begin{aligned}\rho_\chi(t) &= \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \langle 0_{t=0}| \left[\frac{1}{2}|\dot{\chi}(t, \mathbf{k})|^2 + \frac{1}{2}\omega_k^2 |\chi(t, \mathbf{k})|^2 \right] |0_{t=0}\rangle - \rho_{\chi 0}(t) \\ &= \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2}|\dot{\chi}_k|^2 + \frac{1}{2}\omega_k^2 |\chi_k|^2 \right] - \rho_{\chi 0}(t),\end{aligned} \tag{5.1.19}$$

where the vacuum contribution

$$\rho_{\chi 0} = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k}{2}, \tag{5.1.20}$$

has been subtracted. The number density of χ particle is given by

$$n_\chi(t) = \int \frac{d^3k}{(2\pi)^3} f_\chi(t,k), \tag{5.1.21}$$

where

$$f_\chi(t,k) = \frac{1}{2\omega_k}(|\dot{\chi}_k(t)|^2 + \omega_k^2|\chi_k(t)|^2) - \frac{1}{2}, \tag{5.1.22}$$

is the occupation number of χ. Substituting Eqs. (5.1.13) and (5.1.14), we also have

$$f_\chi(t,k) = |\beta_k(t)|^2, \tag{5.1.23}$$

where Eq. (5.1.11) is used. Note that β_k is also written as

$$\beta_k(t) = \frac{1}{\sqrt{2\omega_k}} e^{-i\int_0^t dt'\,\omega_k} \left[\chi_k(t)\omega_k - i\dot{\chi}_k(t)\right]. \tag{5.1.24}$$

Therefore, all we have to do to know the amount of χ particle produced is to solve Eq. (5.1.16) with the initial condition $\alpha_k(0) = 1$ and $\beta_k(0) = 0$, and obtain the time dependence of χ_k or β_k.

5.2 Parametric resonance and Mathieu equation

Suppose that the oscillating mass $m_\chi(t)$ has the time dependence of sine- or cosine- type, and let us see the behavior of Eq. (5.1.16). In this case m_χ can be written as

$$m_\chi^2(t) = \overline{m_\chi^2} + \Delta m_\chi^2 \sin 2\Omega t. \tag{5.2.25}$$

With this parameterization, the mode equation (5.1.16) becomes

$$\ddot{\chi}_k + \left(k^2 + \overline{m_\chi^2} + \Delta m_\chi^2 \sin 2\Omega t\right)\chi_k = 0, \tag{5.2.26}$$

which reduces to the following Mathieu equation

$$\chi_k'' + \left[A - 2q\cos(2z)\right]\chi_k = 0. \tag{5.2.27}$$

Here

$$A = \frac{k^2 + \overline{m_\chi^2}}{\Omega^2}, \qquad q = \frac{\Delta m_\chi^2}{2\Omega^2}, \qquad z = \Omega t + \frac{\pi}{4}. \tag{5.2.28}$$

In some contexts of reheating, these parameters are related to the fundamental parameters in the Lagrangian. For example, suppose that χ has the interaction Lagrangian

$$\mathcal{L}_{\rm int} = \begin{cases} -g\sigma\phi\chi^2 \\ -\frac{1}{2}g^2\phi^2\chi^2 \end{cases}, \tag{5.2.29}$$

and suppose that ϕ is coherently oscillating with amplitude Φ and mass m_ϕ. In such cases ϕ may be treated as a classical oscillating background for χ, and the mode equation for χ_k becomes

$$\begin{cases} \ddot{\chi}_k + (k^2 + 2g\sigma\phi)\,\chi_k &= 0 \\ \ddot{\chi}_k + (k^2 + g^2\phi^2)\,\chi_k &= 0 \end{cases}, \tag{5.2.30}$$

and the parameters $\overline{m_\chi^2}$, Δm_χ^2 and Ω are given by

$$\overline{m_\chi^2} = \begin{cases} 0 \\ \frac{1}{2}g^2\Phi^2 \end{cases}, \qquad \Delta m_\chi^2 = \begin{cases} 2g\sigma\Phi \\ \frac{1}{2}g^2\Phi^2 \end{cases}, \qquad \Omega = \begin{cases} \frac{m_\phi}{2} \\ m_\phi \end{cases}. \tag{5.2.31}$$

The corresponding parameters in the Mathieu equation are written as

$$A = \begin{cases} \frac{4}{m_\phi^2}k^2 \\ \frac{1}{m_\phi^2}\left(k^2 + \frac{1}{2}g^2\Phi^2\right) \end{cases}, \qquad q = \begin{cases} \frac{4}{m_\phi^2}g\sigma\Phi \\ \frac{1}{4m_\phi^2}g^2\Phi^2 \end{cases}, \qquad z = \begin{cases} \frac{m_\phi t}{2} + \frac{\pi}{4} \\ m_\phi t + \frac{\pi}{4} \end{cases}. \tag{5.2.32}$$

Note that the parameter q is roughly the ratio of the amplitude of mass oscillation to the "parent" mass

$$q \sim \frac{\Delta m_\chi^2}{m_\phi^2}. \tag{5.2.33}$$

Also note that $A > 2q$ holds for quartic interaction case $\mathcal{L}_{\rm int} \sim \phi^2\chi^2$, while it does not hold for $\mathcal{L}_{\rm int} \sim \phi\chi^2$.

It is known that for some parameter regions of Eq. (5.2.27) there exist instabilities where χ_k grows as $\chi_k \propto e^{\mu z}$. Figs. 5.1–5.2 are the stability-instability chart of the Mathieu equation. The behavior of the solution to Eq. (5.2.27) differs for $q \lesssim 1$ and $q \gg 1$. The former, called "narrow resonance" [27, 88–92], has an analogy with perturbative decay as stressed in [91]. On the other hand, in the latter case various interesting phenomena occurs which have no analogy with perturbative decay, and this is called "broad resonance". Below we illustrate the behavior of the solutions in these two cases.

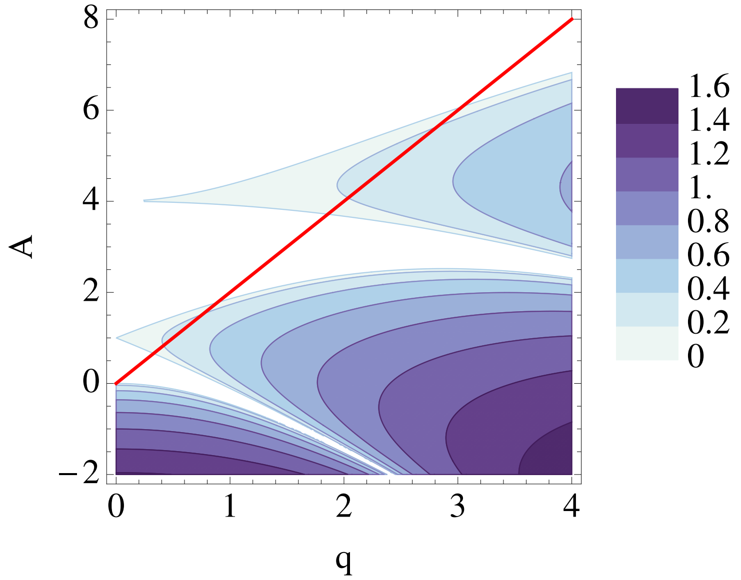

Figure 5.1: Stability-instability chart of Mathieu equation. The contour shows the value of characteristic exponent μ. The red line corresponds to $A = 2q$.

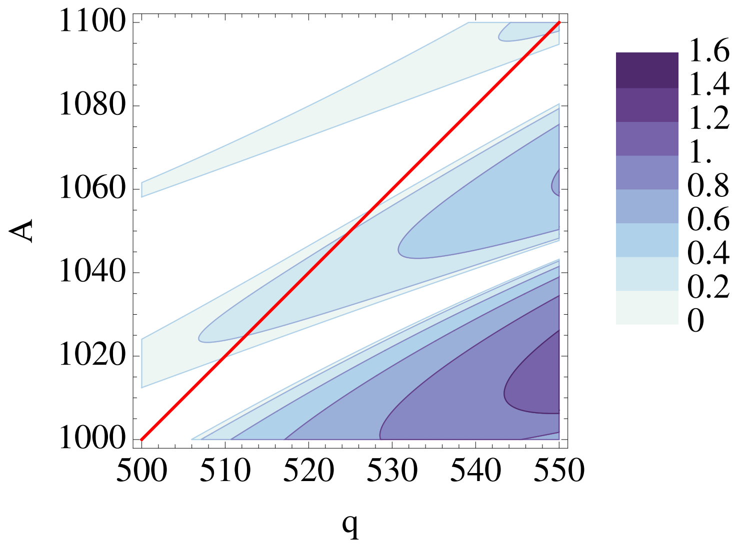

Figure 5.2: Same as Fig. 5.1 except for the plotted range.

5.3 Narrow resonance $q \lesssim 1$

In this section we consider $q \lesssim 1$. Let us first consider the beginning of the oscillation, when the coefficient α_k and β_k satisfy $\alpha_k \simeq 1$ and $\beta_k \ll 1$ for all k. With this assumption, β_k is approximately written from Eq. (5.1.10) as

$$\beta_k(t) \simeq \int_0^t dt' \, \frac{\dot{\omega}_k(t')}{2\omega_k(t')} e^{-2i\int_0^{t'} dt'' \, \omega_k(t'')} = \int_0^t dt' \, \frac{\frac{d}{dt'}\omega_k^2(t')}{4\omega_k^2(t')} e^{-2i\int_0^{t'} dt'' \, \omega_k(t'')}. \tag{5.3.34}$$

The intuition for this equation is as follows. The integral shows no rapid growth for almost all modes k, because the phase of the oscillation coming from $\frac{d}{dt'}\omega_k^2(t')$ and the one from the exponential does not cancel out. However, since $\frac{d}{dt'}\omega_k^2(t') \sim \sin 2\Omega t'$, the phase cancels out for the mode satisfying $\Omega \simeq \omega_k$, and a rapid growth occurs for that mode. Since the growth for the mode can be estimated as

$$\beta_k \sim \int_0^t dt' \, \frac{\Delta m_\chi^2 \Omega}{\Omega^2} \sim \frac{\Delta m_\chi^2}{\Omega} t \sim q\Omega t, \tag{5.3.35}$$

we must require $t \ll (q\Omega)^{-1} \equiv t_{\rm res}$ in order to have $\beta_k \ll 1$ for all k. Thus we consider $t \ll t_{\rm res}$ and $t \gtrsim t_{\rm res}$ separately in the following.

5.3.1 Case $t \ll t_{\rm res}$

Approximating ω_k to be constant except for the one which appears with time derivative in Eq. (5.3.34), and performing integration by parts, we have

$$\beta_k \simeq \frac{i}{2\omega_k} \int_0^t dt' \, m_\chi^2(t') e^{-2i\omega_k t'}. \tag{5.3.36}$$

Below we consider ω_k close to Ω

$$\omega_k = \Omega + \Delta\Omega. \tag{5.3.37}$$

Qualitative understanding

Let us first understand Eq. (5.3.36) qualitatively. For this ω_k, the phases coming from $m_\chi^2(t')$ and $e^{-2i\omega_k t'}$ cancel out for $t \lesssim 1/\Delta\Omega$, and β_k grows linearly during this period. However, the growth is weakened for $t \gtrsim 1/\Delta\Omega$, when the integrand begins to oscillate. In other words, the growth is significant for the mode satisfying[#2]

$$\Omega - \frac{1}{t} \lesssim \omega_k \lesssim \Omega + \frac{1}{t}, \tag{5.3.38}$$

and the growth rate is roughly

$$f_\chi(t) \sim \frac{(\Delta m_\chi^2)^2}{\Omega^2} t^2. \tag{5.3.39}$$

[#2] This may be understood as a realization of the uncertainty principle.

The growth for the total number density n_χ becomes

$$n_\chi(t) \sim \Omega^2 \cdot \frac{1}{t} \cdot \frac{(\Delta m_\chi^2)^2}{\Omega^2} t^2 \sim (\Delta m_\chi^2)^2 t, \tag{5.3.40}$$

where $\int d^3k \sim \int k^2 dk \sim \Omega^2 \cdot t^{-1}$ is taken into account. Sometimes the notion of "parent particle" is helpful for intuitive understanding. If the oscillation is due to a oscillating scalar field with mass m_ϕ and amplitude Φ, the decay rate of ϕ may be written as

$$\Gamma_{\phi\to\chi} \sim \frac{\dot{n}_\chi}{n_\phi} \sim \frac{(\Delta m_\chi^2)^2}{m_\phi \Phi^2} \sim \frac{q^2 m_\phi^3}{\Phi^2}. \tag{5.3.41}$$

The rough estimation (5.3.41) is consistent with the perturbative decay in vacuum. Suppose that the interaction Lagrangian (5.2.29) has the trilinear term $\sim g\sigma\phi\chi^2$, and that ϕ is coherently oscillating with amplitude Φ and frequency m_ϕ. Then, the interaction term may be regarded as

$$\mathcal{L}_{\rm int} \sim \Delta m_\chi^2 \frac{\phi}{\Phi} \chi^2, \tag{5.3.42}$$

and the perturbative decay rate can be estimated as

$$\Gamma_{\phi\to\chi} \sim \frac{1}{m_\phi} \left(\frac{\Delta m_\chi^2}{\Phi} \right)^2 \sim \frac{q^2 m_\phi^3}{\Phi^2}. \tag{5.3.43}$$

This is consistent with Eq. (5.3.41).

What should be stressed here is that, though the occupation number f_k grows as $\sim t^2$, the growth in the number density is linear in t because of the decrease in the peak width $\sim t^{-1}$, which is elucidated in [93].

Quantitative understanding

Next let us be more quantitative. Assuming $m_\chi^2(t) = \Delta m_\chi^2 \sin 2\Omega t$, Eq. (5.3.36) becomes

$$\beta_k \simeq -i \frac{\Delta m_\chi^2}{2k} \int_0^t dt' \ \sin 2\Omega t' e^{-2ikt'}. \tag{5.3.44}$$

Substituting this into Eq. (5.1.21), we have

$$n_\chi(t) \simeq \frac{(\Delta m_\chi^2)^2}{8\pi^2} \int dk \left| \int_0^t dt' \ \sin 2\Omega t' e^{-2ikt'} \right|^2 = \frac{(\Delta m_\chi^2)^2}{32\pi^2 \Omega} \int d(k/\Omega) \ I(2\Omega t, k/\Omega), \tag{5.3.45}$$

where

$$I(\tau, k_\Omega) \equiv \int_0^\tau d\tau' \int_0^\tau d\tau'' \ \sin\tau' \sin\tau'' e^{-ik_\Omega(\tau'-\tau'')}, \tag{5.3.46}$$

is plotted in Fig. 5.3. The behavior of the peak height $\propto \tau^2$ and that of the peak width $\propto \tau^{-1}$ are observed. After some calculation, n_χ becomes

$$n_\chi(t) = \frac{(\Delta m_\chi^2)^2}{32\pi} t \left(1 - \frac{\cos 2\Omega t \sin 2\Omega t}{2\Omega t}\right). \tag{5.3.47}$$

Neglecting the oscillating second term, which is much smaller than the first term for $\Omega t \gtrsim \mathcal{O}(1)$, we have

$$\dot{n}_\chi \simeq \frac{(\Delta m_\chi^2)^2}{32\pi}, \tag{5.3.48}$$

and the decay rate of ϕ

$$\Gamma_{\phi\to\chi} = -\frac{\dot{n}_\phi}{n_\phi} = \frac{\dot{n}_\chi/2}{n_\phi} \simeq \frac{1}{32\pi}\frac{(\Delta m_\chi^2)^2}{m_\phi \Phi^2}. \tag{5.3.49}$$

On the other hand, the perturbative decay rate with the interaction term

$$\mathcal{L}_{\rm int} = \frac{1}{2}\Delta m_\chi^2 \frac{\phi}{\Phi}\chi^2, \tag{5.3.50}$$

is given by

$$\Gamma_{\phi\to\chi} = \frac{(\Delta m_\chi^2/2\Phi)^2}{8\pi m_\phi}. \tag{5.3.51}$$

Then the growth of the number density of χ is given by

$$\dot{n}_\chi \simeq 2\Gamma_{\phi\to\chi} n_\phi = \frac{(\Delta m_\chi^2)^2}{32\pi}, \tag{5.3.52}$$

where the factor 2 comes from 2-body decay. Therefore, the growth of χ with oscillating classical background (5.3.48) coincides with the growth from perturbative decay (5.3.52).

5.3.2 Case $t \gtrsim t_{\rm res}$

After $t \gtrsim t_{\rm res}$, the solution to Eq. (5.2.27) begins to exponentiate for some values of A and q. The colored regions in Figs. 5.1–5.2 are called instability bands, where the solution to Eq. (5.2.27) exponentiate with the exponent shown in that figure, while such resonant effect does not occur in the white region. The plot of χ_k and f_χ for $A = 1$ and $\mu = 0.05$ is shown in Fig. 5.4. The most important band is $A \simeq 1$ and $q \ll 1$, where the expression for the exponent is known to be

$$\mu \simeq \sqrt{\left(\frac{q}{2}\right)^2 - \left(\sqrt{A} - 1\right)^2}. \tag{5.3.53}$$

We see that this exponential growth can be seen as an induced emission of the inflaton which occurs for $f_\chi \gg 1$. Let us again suppose that the interaction Lagrangian (5.2.29) has the

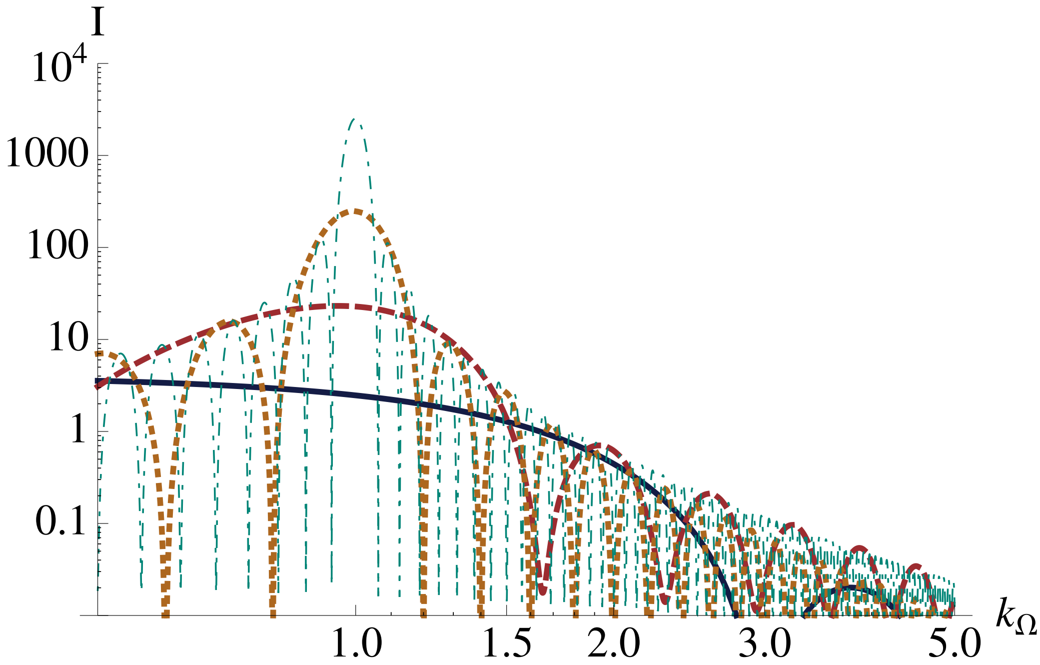

Figure 5.3: Function $I(\tau, k_\Omega)$. Blue, red, yellow and green lines correspond to $\tau = 10^{1/2}, 10^1, 10^{3/2}, 10^2$, respectively. Note that $k_\Omega = 1$ corresponds to $k = \Omega$. Also note that the peak hight goes as $\propto \tau^2$, while the peak width goes as $\propto \tau^{-1}$. These facts are consistent with Eqs. (5.3.38) and (5.3.39).

trilinear term $\sim g\sigma\phi\chi^2$, and that ϕ is coherently oscillating with amplitude Φ and mass m_ϕ. We focus on $A \simeq 1$, when the exponent becomes $\mu \simeq q/2$. If we consider the perturbative decay of inflaton, the decay rate is enhanced by the occupation number of the produced particle as

$$\dot{\rho}_\chi \simeq 2 f_\chi \Gamma_{\phi\to\chi} \rho_\phi. \tag{5.3.54}$$

In terms of number density, we have

$$\dot{n}_\chi \simeq 4 f_\chi \Gamma_{\phi\to\chi} n_\phi. \tag{5.3.55}$$

The occupation number f_χ may be estimated as follows. Produced χ particles have a momentum width Δk due to the coherent oscillation of the inflaton. The width is estimated as

$$\Delta k^2 \simeq 2\Delta m_\chi^2 \quad \rightarrow \quad \Delta k \simeq \frac{2\Delta m_\chi^2}{m_\phi}, \tag{5.3.56}$$

where $k \simeq m_\phi/2$ is used. This width relates the occupation number and the total number of χ particles as

$$\frac{k^2 \Delta k}{2\pi^2} f_\chi \simeq n_\chi \quad \rightarrow \quad f_\chi \simeq 4\pi^2 \frac{n_\chi}{m_\phi \Delta m_\chi^2}, \tag{5.3.57}$$

where $\int d^3k\ f_\chi \sim n_\chi$ is used. Also, the perturbative decay rate and number density are given by

$$\Gamma_{\phi\to\chi} \simeq \frac{1}{8\pi}\left(\frac{\Delta m_\chi^2}{2\Phi}\right)^2 \frac{1}{m_\phi} \tag{5.3.58}$$

$$n_\phi \simeq \frac{1}{2} m_\phi \Phi^2. \tag{5.3.59}$$

The former comes from $\mathcal{L}_{\rm int} = \Delta m_\chi^2(\phi/\Phi)\chi^2$. Eqs. (5.3.55), (5.3.57)–(5.3.59) and $q = 2\Delta m_\chi^2/m_\phi^2$ give

$$\dot{n}_\chi \simeq \frac{\pi}{8} q\ m_\phi n_\chi. \tag{5.3.60}$$

This means that the number of χ particle grows as $e^{\mu m_\phi t}$ with $\mu \simeq (\pi/8)q$, which is consistent with Eq. (5.3.53)[#3]. Thus, narrow resonance can be interpreted as a perturbative decay of the inflaton, with the Bose enhancement taken into account.

5.4 Broad resonance $q \gg 1$

When the oscillating mass term for the light field is much larger than the typical mass scale of the oscillation, i.e. $q \gg 1$, a more explosive production of the light particles occurs than in the case of $q \lesssim 1$ [27, 92]. This is due to the fact that almost all the momentum bands are within the instability band of Mathieu equation, as one sees from Figs. 5.1 and 5.2. This explosive production is important not only in understanding the inflaton decay process itself, but also in understanding baryogenesis [94, 95] and leptogenesis [96], production of supermassive dark matter [97, 98], production of gravitational waves [99–106] and so on.

There are two typical parameter regions in broad resonance. The first one $A > 2q$ is realized for the interaction $\mathcal{L}_{\rm int} \sim \phi^2\chi^2$. In this parameter region the production of χ particle occurs around the origin of the inflaton field, when the adiabaticity of χ field is violated. What is characteristic of this parameter region is that the mass of χ never becomes tachyonic $m_\chi^2(t) < 0$ for any t. On the other hand, for $A < 2q$, the mass term for χ goes negative for a some period of inflaton oscillation, and the particle production becomes more efficient due to the tachyonic effect. This occurs for example $\mathcal{L}_{\rm int} \sim \phi\chi^2$.

Below we summarize the behavior of the solution for $A > 2q$ and $A < 2q$ separately[#4].

[#3]If one take an average of Eq. (5.3.53) over the momentum in the first resonance band, the exponent coincides with each other. In fact,

$$\bar{\mu} \equiv \int_{1-\frac{q}{2}<\sqrt{A}<1+\frac{q}{2}} \frac{d^3k}{(2\pi)^3}\,\mu \Bigg/ \int_{1-\frac{q}{2}<\sqrt{A}<1+\frac{q}{2}} \frac{d^3k}{(2\pi)^3} = \frac{\pi}{8}q. \tag{5.3.61}$$

[#4]We briefly comment on fermions here. Though the resonance effect is inefficient for narrow region, it has been found that copious fermion production occurs for broad region [96, 107, 108]. The main difference from the bosonic case is

- The occupation number never exceeds 1 due to Pauli blocking.

5.4.1 $A > 2q$

We assume the interaction $\mathcal{L}_{\rm int} \sim \phi^2\chi^2$ with the inflaton oscillating in a quadratic potential. Fig. 5.5 is the plot of χ_k and f_χ for $A = 1040$ and $q = 520$. Below we summarize important points [92]:

- In most of the time during oscillation, the mass of χ is much larger than the inflaton mass.
- Particle production occurs when the inflaton passes through the origin of the potential, when the adiabaticity of the light field is violated

$$\left|\frac{\dot{\omega}_k}{\omega_k^2}\right| \gg 1, \tag{5.4.62}$$

  which means that the condition for particle production in WKB approximation is satisfied.
- The particle production in one oscillation of the inflaton field is significant, changing the number density of the light particles by an order of magnitude.
- The typical momentum k_* of the produced particles is

$$k_* \sim q^{1/4} m_\phi, \tag{5.4.63}$$

  which can be much larger than the one with perturbative decay of inflaton $k \sim m_\phi$.

5.4.2 $A < 2q$

For this parameter region, χ field becomes tachyonic during some period of inflaton oscillation. This tachyonic effect, combined with the production mechanism which also exists in non-tachyonic case, makes the particle production more efficient. A typical example is the case of interaction term $\mathcal{L}_{\rm int} \sim \phi\chi^2$. Particle production in such cases is estimated in [109–111]. In addition, such tachyonic effect occurs also when the inflaton has a negative coupling between to the light particle $\mathcal{L}_{\rm int} \sim -g^2\phi^2\chi^2$ with $g^2 < 0$. In this case, the naive estimation from the Mathieu equation does not apply since the χ field tends to follow the temporal minimum of the potential made by the inflaton oscillation. The analysis in such cases is carried out in [112]. Another important difference from $A > 2q$ case is that the produced particles have maximal momentum $k_* \sim q^{1/2} m_\phi$.

- The maximal momentum produced by the resonance is $k_* \sim q^{1/2} m_\phi$, in contrast to bosonic case with $\mathcal{L}_{\rm int} \sim -\phi^2\chi^2$, Eq. (5.4.63). Here q is the amplitude of the oscillating mass squared for fermions. This comes from the fact that the mass term for bosons never gets below the value at $\phi = 0$, while the fermion mass can vanish for some inflaton value, since the fermion total mass is a linear combination $m_\psi(t) = g\phi + m_\psi(0)$.

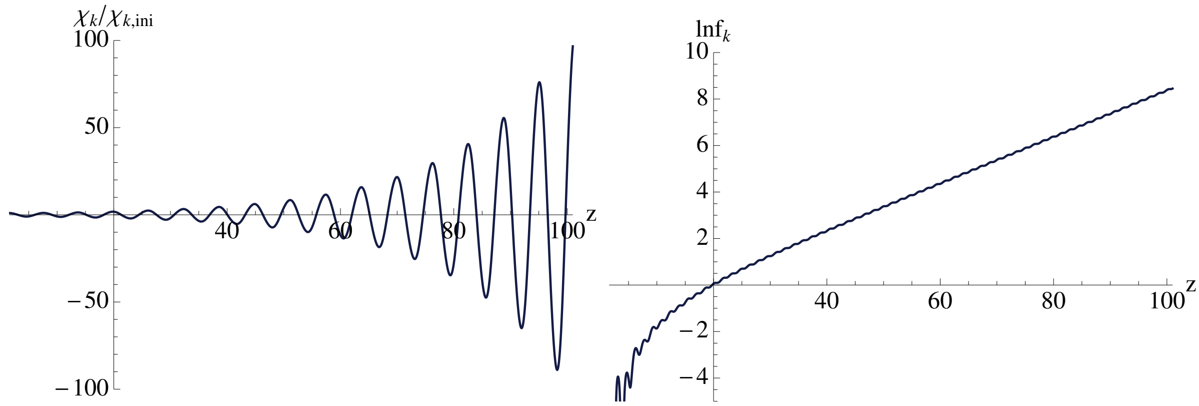

Figure 5.4: Plot of χ_k (left) and f_χ (right). Parameters are taken to be $A = 1$ and $q = 0.1$. Characteristic exponent is $\mu \simeq 0.05$ for these parameters.

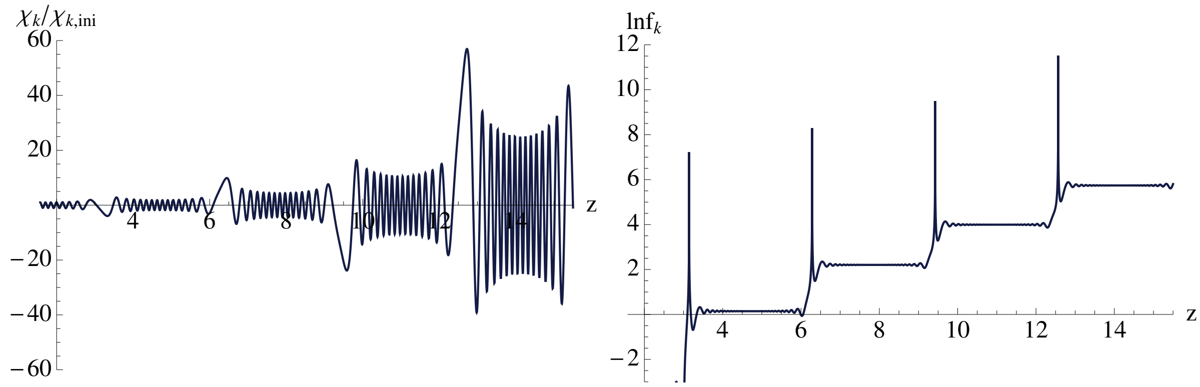

Figure 5.5: Plot of χ_k (left) and f_χ (right). Parameters are taken to be $A = 1040$ and $q = 520$. Characteristic exponent is $\mu \simeq 0.28$ for these parameters.

5.5 Effect of cosmic expansion

In the absence of cosmic expansion nor backreaction to the oscillating inflaton field, the resonant effect definitely occurs even for $q \lesssim 1$, if one waits until the produced particles accumulates above $f_\chi \sim 1$. However, in reality, cosmic expansion redshifts the momentum of the produced particles, shifting away the particles from the resonance band [113]. Such effect may invalidate the resonance for the narrow case, or makes the analysis more complicated in the broad case.

Below we summarize the condition for the resonance to occur in narrow case in the presence of cosmic expansion. We also mention the effect of cosmic expansion in broad case. We still neglect the backreaction of the produced particles to the inflaton.

5.5.1 Narrow resonance $q \lesssim 1$

As we saw above, the produced light particles have the physical momentum with width[#5]

$$k \sim m_\phi, \qquad \Delta k \sim q m_\phi. \tag{5.5.64}$$

The time Δt necessary for the cosmic expansion to remove these particles from the resonance band is estimated as

$$k\frac{\Delta a}{a} \sim \Delta k \quad \rightarrow \quad \Delta t \sim \frac{1}{H}\frac{\Delta k}{k}. \tag{5.5.65}$$

For the resonance to occur, the number density of the produced particles must exponentiate during this period. Therefore we need [92][#6]

$$\mu m_\phi \Delta t \gg 1 \quad \rightarrow \quad R_q \equiv q^2 \frac{m_\phi}{H} \gg 1. \tag{5.5.66}$$

Here we mention other conditions for the resonance [92]. First, if the perturbative decay rate $\Gamma_\phi^{(\text{other})}$ of the inflaton to other particles dominates over the decay into χ, no resonance occurs. Therefore we need

$$q m_\phi \gg \Gamma_\phi^{(\text{other})}. \tag{5.5.67}$$

Second, the decay of the produced χ particles must be inefficient compared to the resonance effect, since χ must remain in the resonance band. Thus we need

$$q m_\phi \gg \Gamma_\chi, \tag{5.5.68}$$

where Γ_χ denotes the decay rate of χ particles. Similar conditions apply for the interaction of χ particles which removes them from the resonance band.

[#5] Since we are now taking the cosmic expansion into account, these quantities should be written as k/a and $\Delta(k/a)$ in a strict sense, if we denote the comoving wavenumber by k.

[#6] When the oscillation timescale changes in time, we must take this effect into account as well. However, when the change in the oscillation timescale occurs due to the cosmic expansion, this effect gives roughly the same criteria as Eq. (5.5.66). The only case where we should be careful is when the shift in the oscillation mass scale (inverse of the oscillation timescale) is the same as the shift in the momentum of produced particles, i.e. $m_\phi \propto a^{-1}$.

5.5.2 Broad resonance $q \gg 1$

In the case of broad resonance, cosmic expansion brings about different effects from the ones we saw in narrow resonance. To understand what happens, let us consider $\mathcal{L}_{\rm int} \sim -\phi^2\chi^2$, when the parameter q is given by $q \sim \Phi^2/m_\phi^2$. We focus on $A < 2q < A + \sqrt{A}$, where resonance occurs most efficiently. For $q \gg 1$, this region belongs to the n-th resonance band with $n \sim \sqrt{A}$, which means $n \sim \sqrt{q}$. Since the amplitude Φ changes significantly during one oscillation, the number of the resonance band to which a particular comoving wavenumber belongs changes during one period of oscillation. In addition, more importantly, the mode is not necessarily "in phase" with the background oscillation at the time of particle production, i.e. $\phi \sim 0$. Therefore the number density significantly "increase" or "decrease" in one production, in a random way. This phenomenon is called "stochastic resonance" in [92], and an analytic method to investigate it is developed there. Analytic way of investigating the stochastic resonance is also studied in the context of understanding the parametric resonance in Schrödinger picture [114, 115]. In a nutshell, an explosive particle production is still possible because of the following reasons

- The number of χ increase or decrease in a random way in log-space, therefore the number of χ tends to increase after some production events.
- Though it is random whether the production event occurs "in phase" with the inflaton oscillation, about 75% of the particle production events at $\phi \sim 0$ leads to an increase in the number of χ particles.

5.6 Backreaction, end of preheating and thermalization

In the above discussion we treated the inflaton as an oscillating field in background, free from backreactions of the produced quanta. Though a detailed description of backreaction effects [92, 116] is beyond the scope of this thesis, the important thing to mention is the mass shift of the inflaton field. This effect can be understood if one uses the Hartree approximation in the equation of motion for the inflaton

$$\phi^2\chi^2 \to \left\langle \chi^2 \right\rangle \phi^2. \tag{5.6.69}$$

Also, the inflaton quanta φ gives a mass to the inflaton itself

$$\phi^4 \to 3 \left\langle \varphi^2 \right\rangle \phi^2, \tag{5.6.70}$$

In addition, the produced inflaton quanta change the mass of χ particles

$$\phi^2\chi^2 \to \left\langle \varphi^2 \right\rangle \chi^2. \tag{5.6.71}$$

One must take care that the timescale for the preheating and the subsequent thermalization can be totally different [87]. The thermalization must be achieved before the Big-Bang Nucleosynthesis $T \sim 1$MeV, in order to be consistent with observations of light elements.

5.7 Overall oscillation as mass oscillation

In discussing gravitational effects in Chapter 7–9, oscillations appear as an overall factor of a scalar field. Let us consider the action

$$S_M = \int d^4x \; F^2(\phi) \left[-\frac{1}{2}(\partial\chi)^2 \right]. \tag{5.7.72}$$

Here ϕ is treated as a background oscillating field, as before. Defining the canonically normalized field $\tilde{\chi} \equiv F(\phi)\chi$, the action becomes

$$S_M = \int d^4x \; \left[-\frac{1}{2}(\partial\tilde{\chi})^2 - \frac{1}{2}\frac{\partial^2 F}{F}\tilde{\chi}^2 \right], \tag{5.7.73}$$

and therefore the mode equation for $\tilde{\chi}$ is

$$\ddot{\tilde{\chi}}_k + \left(k^2 - \frac{\ddot{F}}{F} \right) \tilde{\chi}_k = 0. \tag{5.7.74}$$

Thus the oscillating field behaves as an oscillating mass term,

$$m_\chi^2(t) = \frac{\ddot{F}}{F}. \tag{5.7.75}$$

In fact, we have already encountered this kind of time-dependent mass in Eq. (3.2.47). This time-dependent mass gave the deviation from the plane wave solution in Eq. (3.2.48), producing the primordial fluctuations. In Chapter 7–9, we will see that oscillations in the time-dependent mass brings about particle production during the inflaton oscillation regime.

5.8 Summary

In this chapter we saw the effect of an oscillating field on particle production, especially focusing on narrow resonance. It is shown that narrow resonance is seen as perturbative decay of the parent field with Bose enhancement taken into account. We also summarized the effect of cosmic expansion. The main condition for the resonance to occur in narrow case is given by Eq. (5.5.66), which states that the induced emission must occur before the cosmic expansion removes the produced particle.

The main formulae we often refer to in the following chapters are

- Expression for Mathieu q parameter

$$q \sim \frac{\Delta m_\chi^2}{m_\phi^2}, \tag{5.8.76}$$

- Estimation for growth and decay rate

$$\dot{n}_\chi \sim (\Delta m_\chi^2)^2, \qquad \Gamma_{\phi\to\chi} \sim \frac{q^2 m_\phi^3}{\Phi^2}. \tag{5.8.77}$$

- Resonance condition

$$R_q \equiv q^2 \frac{m_\phi}{H} \gg 1. \tag{5.8.78}$$

Part II

Original work

Chapter 6

Adiabatic invariant in Horndeski/Galileon inflation theories

In potential-driven type inflation models of Horndeski/Galileon theories (see Chapter 4) violent oscillations of the Hubble parameter sometimes occur, and these oscillations make the analysis difficult. One of such examples is shown in the left panel of Fig. 6.1. Such oscillation is characterized by $\dot{H} \sim m_{\rm eff} H$, where $m_{\rm eff} \gg H$ is the effective mass scale (inverse of the oscillation timescale) of the inflaton. This is in contrast to $\dot{H} \sim H^2$ which occurs in the setup where only Einstein gravity and canonical inflaton exist.

Even in such setups where the Hubble parameter violently oscillates, there exists a quantity which has only a suppressed amplitude of oscillation. We call this quantity "adiabatic invariant". This quantity is useful in

- Extracting the oscillation of the Hubble parameter in terms of the inflaton field
- Estimating the oscillation-averaged expansion law of the universe

Both are important in understanding the background dynamics of the model. In addition, noting that the oscillation of the Hubble parameter, or equivalently the scale factor, couples to light particles, the extraction of the oscillation mode can be useful in estimating particle production caused by this oscillation.

In the following argument, we first state the setup, and next derive the adiabatic invariant. Then we show how to derive the averaged expansion law using the invariant. Lastly, we see some concrete examples to illustrate the procedure. It should be noted that this invariant is frequently used in the following analysis on the models with nonminimal coupling to gravity (Chapter 8–9). This chapter is based on the work [117] with Y. Ema, K. Mukaida and K. Nakayama.

6.1 Setup

The action we consider is that of Horndeski/Galileon theories with ϕ being identified as the inflaton

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}, \qquad \mathcal{L} = \sum_{i=2}^{5} \mathcal{L}_i, \tag{6.1.1}$$

where the Lagrangian $\mathcal{L}_i$ are given by Eqs. (4.2.2)–(4.2.5). We assume FRW metric with negligible spacial curvature

$$ds^2 = -dt^2 + a^2(t) dx^i dx^i. \tag{6.1.2}$$

Since we consider only the background dynamics, we denote the background inflaton field $\bar{\phi}$ simply by ϕ. The background action of the theory is given in Chapter 4 as

$$\mathcal{L}_2 = G_2, \tag{6.1.3}$$
$$\mathcal{L}_3 = G_3(\ddot{\phi} + 3H\dot{\phi}), \tag{6.1.4}$$
$$\mathcal{L}_4 = -6H^2 G_4 - 6HG_{4\phi}\dot{\phi} + 6H^2 G_{4X}\dot{\phi}^2, \tag{6.1.5}$$
$$\mathcal{L}_5 = -3H^2 G_{5\phi}\dot{\phi}^2 + H^3 G_{5X}\dot{\phi}^3, \tag{6.1.6}$$

after integration by parts. Though $\mathcal{L}_3$ contains $\ddot{\phi}$, it can be eliminated using integration by parts, as explained before. Therefore we write the background action as

$$S = \int d^4x\, a^3 \mathcal{L}(H, \dot{\phi}, \phi). \tag{6.1.7}$$

The background equations of motion of this system are

- Friedmann equation

$$\mathcal{L} - \dot{\phi}\mathcal{L}_{\dot{\phi}} - H\mathcal{L}_H = 0, \tag{6.1.8}$$

- Raychaudhuri equation

$$(\mathcal{L}_H)^{\cdot} + 3H\mathcal{L}_H - 3\mathcal{L} = 0, \tag{6.1.9}$$

- ϕ's equation of motion

$$\left(\mathcal{L}_{\dot{\phi}}\right)^{\cdot} + 3H\mathcal{L}_{\dot{\phi}} - \mathcal{L}_\phi = 0, \tag{6.1.10}$$

where the subscript denotes the derivative with respect to that quantity. Friedmann equation is obtained by introducing the lapse function N as $dt \to Ndt$, taking variation with respect to N and setting N to 1 because of the time reparameterization invariance. Note that we must take into account dt in d^4x, as well as time derivatives d/dt. Also, Raychaudhuri equation and ϕ's equation of motion are obtained by taking variation with respect to the scale factor a and the inflaton ϕ.

6.2 Adiabatic invariant

Now we consider the inflaton oscillation regime of potential-driven type inflation in Horndeski/Galileon theories, where the inflaton oscillates with the effective mass scale $m_{\rm eff}$, or the inverse of the oscillation timescale, $m_{\rm eff} \sim |\dot{\phi}/\phi|$. Here "$\sim$" means a typical value during oscillation. For example, if $\phi = \Phi \sin(m_\phi t)$, we have $|\dot{\phi}/\phi| = m_\phi |\cos(m_\phi t)/\sin(m_\phi t)|$ and we regard $\sin(m_\phi t)$ and $\cos(m_\phi t)$ as typically being unity to obtain $m_{\rm eff} \sim m_\phi$.

As we saw in Chapter 3, the inflaton oscillation regime typically starts when $m_{\rm eff} > H$ is realized. If one allows general couplings between the inflaton and gravity within Eqs. (6.1.3)–(6.1.6), there often appears an oscillation mode in the Hubble parameter which is correlated with the inflaton oscillation. Because of this correlation, the oscillating part of the Hubble parameter $\delta H \equiv H - \langle H \rangle$, where $\langle \cdots \rangle$ denotes time average over a timescale much longer than the oscillation timescale, satisfies $\dot{\delta H} \sim m_{\rm eff} \delta H$. If one assumes that the amplitude of the oscillation has the same order of magnitude as the Hubble parameter itself $\delta H \sim H$, then

$$\dot{H} \sim m_{\rm eff} H, \tag{6.2.11}$$

is satisfied. This equation characterizes the behavior of the Hubble parameter when a violent oscillation occurs to it.

On the other hand, we know that $\dot{H} \sim H^2$ is satisfied in the system where only the Einstein gravity and a canonical inflaton exist[#1]. Is there any quantity which satisfies this relation even when the inflaton is nonminimally coupled to gravity? In other words, we seek a quantity which satisfies

$$\dot{J} \sim HJ, \tag{6.2.12}$$

and it does exist. The argument is simple. First, Friedmann equation (6.1.8) implies $\mathcal{L} \sim H\mathcal{L}_H \sim \dot{\phi}\mathcal{L}_{\dot{\phi}}$[#2]. Then, Raychaudhuri equation (6.1.9) reads

$$(\mathcal{L}_H)^{\cdot} \sim H\mathcal{L}_H. \tag{6.2.13}$$

Thus the derivative of the Lagrangian with respect to the Hubble parameter $\mathcal{L}_H$ is what we need. We define J so that it reduces to the Hubble parameter in the system with Einstein gravity and a canonical inflaton

$$J \equiv -\frac{1}{6M_P^2}\mathcal{L}_H, \tag{6.2.14}$$

Since J is written in terms of the inflaton and the Hubble parameter in many models, it allows to extract the oscillation mode of the Hubble parameter by inverting the relation. In addition, the adiabatic invariant is useful in estimating the averaged cosmic expansion law.

[#1]From $\dot{H} = -\dot{\phi}^2/2M_P^2$ and the fact that the kinetic term $\dot{\phi}^2$ and potential term V have roughly the same order of magnitude (Virial theorem) during inflaton oscillation, one has $\dot{H} \sim (\dot{\phi}^2/2 + V)/M_P^2 \sim H^2$.

[#2]Exactly speaking, a weaker condition $\dot{\phi}\mathcal{L}_{\dot{\phi}} \lesssim H\mathcal{L}_H$ is sufficient, since the second and third term in Eq. (6.1.9) sum up to $-3\dot{\phi}\mathcal{L}_{\dot{\phi}}$ because of Eq. (6.1.8). The only case where this does not hold is when $\mathcal{L}$ and $\dot{\phi}\mathcal{L}_{\dot{\phi}}$ cancel out almost exactly in Eq. (6.1.8). This means that all dominant terms in the Lagrangian are linear in $\dot{\phi}$ and in addition that such terms do not couple to H. We do not consider such cases here.

However, before elucidating these points in the following sections, we briefly comment on the analogy with analytical mechanics. Let us consider a system with dynamical variable q

$$S = \int dt\ \mathcal{L}(\dot{q}, q), \tag{6.2.15}$$

which satisfies the Euler-Lagrange equation $(\mathcal{L}_{\dot{q}})^{\cdot} - \mathcal{L}_q = 0$. If $\mathcal{L}$ does not depend on q, there exists an invariant $\mathcal{L}_{\dot{q}}$. The quantity $\mathcal{L}_H$ is obtained with the identification of q with $\ln a$, though the explicit dependence of the integrand in (6.1.7) on the scale factor a prevents $\mathcal{L}_H$ from being an invariant in a strict sense.

In the following argument, we refer to a quantity Q which satisfies

$$\dot{Q} \sim HQ, \tag{6.2.16}$$

as "adiabatic invariant". The quantities $\mathcal{L}_H$ and J are adiabatic invariants, while the scale factor a is also an adiabatic invariant by the definition of the Hubble parameter.

6.3 Estimation of the cosmic expansion law

In order to obtain the expansion law, we first estimate the dependence of $\mathcal{L}_H$ on the scale factor. Taking an oscillation average in Friedmann equation (6.1.8),

$$\langle \mathcal{L} \rangle - \left\langle \dot{\phi} \mathcal{L}_{\dot{\phi}} \right\rangle - \langle H \mathcal{L}_H \rangle = 0. \tag{6.3.17}$$

Also, multiplying ϕ to ϕ's equation of motion (6.1.10) and taking an oscillation average, we obtain the Virial theorem

$$\left\langle \dot{\phi} \mathcal{L}_{\dot{\phi}} \right\rangle + \langle \phi \mathcal{L}_\phi \rangle \simeq 0. \tag{6.3.18}$$

Here suppose that $\mathcal{L}$ is decomposed into at most three terms which are proportional to $H\mathcal{L}_H$, $\phi\mathcal{L}_\phi$ and $\dot{\phi}\mathcal{L}_{\dot{\phi}}$, after neglecting small contributions in each regime under consideration. With Eqs. (6.3.17) and (6.3.18), this decomposition allows to express $\langle \mathcal{L} \rangle$ in terms of $\langle H\mathcal{L}_H \rangle$. Then, Eq. (6.1.9) is rewritten by taking an oscillation average as

$$\left\langle \dot{J} \right\rangle + c \langle H \rangle J \simeq 0, \tag{6.3.19}$$

with c being some constant which depends on the model and the regime under consideration. Here the oscillation average is not applied to J because it has only a small amplitude of oscillation, $\langle HJ \rangle = \langle H \rangle J$. One finds that

$$J \propto a^{-c}, \tag{6.3.20}$$

satisfies Eq. (6.3.19). In concrete models, $\langle H \rangle$ dependence of J is inferred from the dominant contributions in the Lagrangian. For example, if $\mathcal{L}$ has $-3M_P^2 H^2$ as one of the dominant contributions, then $\mathcal{L}_H \propto \langle H \rangle \propto a^{-c}$ is obtained. This determines the averaged expansion law $\langle H \rangle = 1/ct$. Concrete examples are shown below, as well as in Chapter 8–9, with numerical results.

6.4 Particle production

We give a brief comment on the usage of the adiabatic invariant as an estimator of gravitational particle production. As illustrated in Chapter 5, the overall oscillation of a scalar field contributes an oscillating mass to the canonically normalized field. When the overall factor is given by the scale factor, $\ddot{a}/a \sim \dot{H} \sim \delta\dot{H}$ contributes as an oscillating mass term. If we extract the oscillating part of the Hubble parameter in terms of the inflaton field ϕ, then the oscillating mass term $\delta\dot{H}\tilde{\chi}^2$, where $\tilde{\chi}$ is the canonically normalized field, is regarded as a coupling between the inflaton and the light field. This allows one to estimate the production rate of the χ field from the perturbative decay of the inflaton field, as long as the oscillation amplitude is small enough. This method is used in Chapter 8.

6.5 Examples

In this subsection we give some concrete examples. The first example has a coupling between the inflaton and gravity in a form

- $f(\phi)R$,

which is one of the examples in Horndeski/Galileon theories. We analyze this model in Chapter 8 as well. In addition, we briefly comment on

- $f(R)$ theories,

which is not in the Horndeski/Galileon Lagrangian (4.2.2)–(4.2.5). The adiabatic invariant can be defined in this case as well. One may skip the second example for the purpose of understanding Chapter 7–9.

In the first example we assume that the inflaton is oscillating around the minimum of the monomial potential

$$V(\phi) = \frac{\lambda}{n}\phi^n. \tag{6.5.21}$$

6.5.1 Example 1 : $f(\phi)R$ model

Let us consider the action

$$S = \int d^4x\sqrt{-g}\left[f(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{6.5.22}$$

The coupling $f(\phi)R$ reduces to the Einstein-Hilbert term for $f = M_P^2/2$. Below we often omit the argument ϕ for notational simplicity. We consider the case where f depends on the inflaton as

$$f = \frac{M_P^2}{2}\left(1 + f_1\frac{\phi}{M_P}\right), \tag{6.5.23}$$

with f_1 being some constant. Using the background relation $R = 12H^2 + 6\dot{H}$, and performing integration by parts, the background action reduces to

$$S = \int d^4x \, a^3 \mathcal{L}, \tag{6.5.24}$$

$$\mathcal{L} = -3M_P H f_1 \left(\dot{\phi} + H\phi\right) - 3M_P^2 H^2 + \frac{1}{2}\dot{\phi}^2 - V, \tag{6.5.25}$$

where the terms in the round bracket are the deviation from $f = M_P^2/2$. The adiabatic invariant reads

$$J \equiv -\frac{1}{6M_P^2}\mathcal{L}_H = H + \frac{f_1}{2M_P}\left(\dot{\phi} + 2H\phi\right), \tag{6.5.26}$$

which reduces to the Hubble parameter for $f_1 = 0$.

Fig. 6.1 is the result of numerical calculation with $f_1 = 0.2$. Blue lines show that Ht (and thus H) violently oscillates due to the coupling $f(\phi)R$, while J has only a small amplitude of oscillation as one sees from the red lines.

Note that the contribution from $\dot{\phi}$ is larger than that from $H\phi$ in Eq. (6.5.26) during inflaton oscillation regime[#3],

$$J \simeq H + \frac{f_1}{2M_P}\dot{\phi}. \tag{6.5.27}$$

Thus, the oscillating part of the Hubble parameter δH correlates with the inflaton field as

$$\delta H \simeq -\frac{f_1}{2M_P}\dot{\phi}. \tag{6.5.28}$$

See Chapter 8 for the numerical comparison between $\delta H \equiv H - \langle H \rangle$ and the formula (6.5.28).

Next let us estimate the averaged expansion law. From the argument above on the dominant contribution, the Lagrangian is approximated to be

$$\mathcal{L} \simeq -3M_P H f_1 \dot{\phi} - 3M_P^2 H^2 + \frac{1}{2}\dot{\phi}^2 - V. \tag{6.5.29}$$

Then it is decomposed as

$$\mathcal{L} \simeq \frac{1}{2}H\mathcal{L}_H + \frac{1}{2}\dot{\phi}\mathcal{L}_{\dot{\phi}} + \frac{1}{n}\phi\mathcal{L}_\phi. \tag{6.5.30}$$

Taking oscillation average, and using Eqs. (6.3.17) and (6.3.18), one has

$$\langle \mathcal{L} \rangle = \frac{2}{n+2}\langle H\mathcal{L}_H \rangle. \tag{6.5.31}$$

Substituting into oscillation-averaged Eq. (6.1.9), one sees

$$\left\langle \dot{J} \right\rangle + \frac{3n}{n+2}\langle H \rangle J = 0. \tag{6.5.32}$$

[#3] This is because $\dot{\phi} \sim m_{\rm eff}\phi$ and $m_{\rm eff}$ is larger than H in the inflaton oscillation regime.

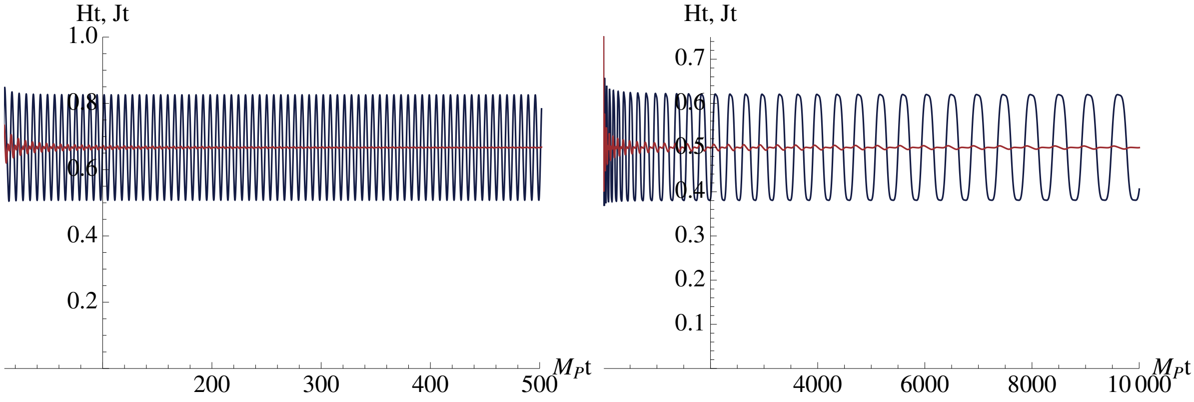

Figure 6.1: Time evolution of Ht (*blue*) and Jt (*red*) for $n = 2$ (*left*) and $n = 4$ (*right*). Parameters and initial conditions are $f_1 = 0.2$, $m_\phi = M_P$, $\phi_{\rm ini} = M_P$ and $M_P t_{\rm ini} = 1.72$ for the left panel, and $f_1 = 0.2$, $\lambda = 1$, $\phi_{\rm ini} = M_P$ and $M_P t_{\rm ini} = 2.2$ for the right panel. These parameters and initial conditions are the same in Fig. 8.1.

This determines the proportionality

$$J \propto a^{-\frac{3n}{n+2}}. \tag{6.5.33}$$

The averaged expansion law of the universe is estimated by using $J \propto \langle H \rangle$[#4] as

$$\langle H \rangle \propto a^{-\frac{3n}{n+2}} \quad \rightarrow \quad \langle H \rangle \simeq \frac{n+2}{3n}\frac{1}{t}. \tag{6.5.34}$$

Fig. 6.1 confirms this averaged expansion law for $n = 2, 4$. Note that this expansion law is the same as the one in the system with Einstein gravity and a canonical inflaton, to which we refer as "minimal setup". This is because the decomposition (6.5.30) is the same as in that system. Later we will see models where the averaged expansion law is modified from the minimal setup.

6.5.2 Example 2 : $f(R)$ theories

As the second example, let us consider the following action

$$S = \int d^4x \sqrt{-g}\, \frac{M_P^2}{2} f(R). \tag{6.5.35}$$

Though this action does not fall into the Horndeski/Galileon action (4.2.2)–(4.2.5), one finds that the adiabatic invariant exists in this case as well. We first rewrite the action using the "auxiliary field" method

$$S = \int d^4x \sqrt{-g}\, \frac{M_P^2}{2} \left[f(\phi) + F(\phi)(R - \phi) \right], \tag{6.5.36}$$

[#4] This is inferred from Eq. (6.5.27) in this concrete example, since $\left\langle \dot{\phi} \right\rangle$ vanishes.

where $F(\phi) \equiv f'(\phi)$, and note that the new field ϕ has no kinetic term and is an auxiliary field. Also note that the Lagrangian is now within Eqs. (4.2.2)–(4.2.5). Varying this action with respect to ϕ leads to the constraint equation

$$F'(\phi)(R-\phi)=0, \tag{6.5.37}$$

which gives $\phi = R$ if $F'(\phi) \neq 0$, and reduces the action (6.5.36) to the original action (6.5.35). The background Lagrangian of this system is

$$S=\int d^4x\, a^3 \frac{M_P^2}{2}\left[f(\phi)-F(\phi)\phi-6F(\phi)H^2-6\dot{F}(\phi)H\right], \tag{6.5.38}$$

which leads to the following equations of motion from Eqs. (6.1.8)–(6.1.9)

- Friedmann equation

$$\dot{F}+FH+\frac{1}{6H}(f-FR)=0. \tag{6.5.39}$$

- Raychaudhuri equation (after eliminating f using Eq. (6.5.39))

$$\ddot{F}-\dot{F}H+2F\dot{H}=0. \tag{6.5.40}$$

Note that ϕ is replaced by R after deriving these equations. Also note that R is understood as the background value $12H^2+6\dot{H}$. The adiabatic invariant is obtained as

$$\begin{aligned} J &= FH+\frac{1}{2}\dot{F} \\ &= \frac{1}{2}FH+\frac{1}{12H}(FR-f), \end{aligned} \tag{6.5.41}$$

where the Friedmann equation (6.5.39) is used in the last line[#5][#6].

Below we take the Starobinsky model $f(R)=R+R^2/6M^2$ as an example. We take the unit $M=1$ in the following. It is straightforward to recover M from dimensional argument. The equations of motion read

- Friedmann equation

$$\dot{R}=-H(R+3)+\frac{R^2}{12H}, \tag{6.5.42}$$

- Raychaudhuri equation[#7]

$$\ddot{R}+3H\dot{R}+R=0. \tag{6.5.43}$$

[#5] Though a term linear in $\dot{\phi}$ appears in the Lagrangian (6.5.38), this does not mean that the adiabatic invariant does not exist because the term is also proportional to H. See the footnote around Eq. (6.2.12).

[#6] In $f(R)$ theories, the adiabatic invariant does not reduce to H in $f=R$ limit.

[#7] This equation describes the propagation of the scalar degrees of freedom ($F(R)$, scalaron) of the system.

The adiabatic invariant is given by

$$J = \frac{H}{2}\left(1 + \frac{R}{3}\right) + \frac{R^2}{72H}. \tag{6.5.44}$$

This expression helps to understand that the system is actually a rotating system[#8].

Let us calculate the expansion law. First, note that the reheating era starts when the Einstein-Hilbert term dominates over the R^2 term. Then the dominant term in the action (6.5.38) is the third one, which has a contribution from the Einstein-Hilbert term. Note that the Einstein-Hilbert term cancels out in the first and second ones, and vanishes in the fourth one due to the time derivative. Therefore one has

$$\langle \mathcal{L} \rangle = \frac{1}{2} \langle H \mathcal{L}_H \rangle , \tag{6.5.47}$$

and, substituting this into Eq. (6.1.9),

$$\left\langle \dot{J} \right\rangle + \frac{3}{2} \langle H \rangle J = 0 \quad \to \quad J \propto a^{-3/2}. \tag{6.5.48}$$

The averaged cosmic expansion law becomes

$$\langle H \rangle = \frac{2}{3t}. \tag{6.5.49}$$

[#8]From the expression (6.5.44), it is found that the background dynamics of the Starobinsky model is equivalent to the following simple Hamilton system

$$\dot{A} = -B, \qquad \dot{B} = A - B^3, \qquad A \equiv \frac{R}{3\sqrt{2H}}, \qquad B \equiv \sqrt{2H}. \tag{6.5.45}$$

The reheating phase in the Starobinsky model corresponds to $A, B \ll 1$ (while the inflation phase corresponds to $A \gg B^3 \gg B \gg 1$), when Eq. (6.5.45) describes a rotating system modified by the small B^3 term. This expression is obtained by noting that the adiabatic invariant (6.5.44) consists of two dominant oscillating contributions $\sim H$ and $\sim R^2/H$. In fact,

$$J = \frac{1}{4}\left(A^2 + B^2 + AB^3\right). \tag{6.5.46}$$

In addition, it is clear that the Ricci scalar and Hubble parameter behave as $R \sim AB \sim \sin t \cos t$ and $H \sim B^2 \sim \cos^2 t$.

Chapter 7

Gravitational effects in Einstein gravity – canonical inflaton system

In the previous chapter we studied how to extract the oscillation mode of the Hubble parameter in the cases where the oscillation mode satisfies $\dot{H} \sim m_{\text{eff}} H$ with $m_{\text{eff}} \gg H$. This type of oscillation occurs when the inflaton is nonminimally coupled to gravity, as we will see in Chapter 8–9. However, before studying such models, we must understand the effects of gravity in minimal setup, where the system has only the Einstein gravity and a canonical inflaton. Even in this setup, the oscillation of the inflaton induces an oscillating mode to the Hubble parameter with $\dot{H} \sim H^2$, and this brings about particle production and nonnegligible cosmological consequences to the present universe.

Throughout the chapter, we consider the action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_M, \tag{7.0.1}$$

with monomial potential for simplicity

$$V(\phi) = \frac{\lambda}{n} \phi^n, \tag{7.0.2}$$

and the FRW metric

$$ds^2 = -dt^2 + a^2(t) dx^i dx^i, \tag{7.0.3}$$

where we neglect the spacial curvature. Here S_M is the matter action, and we assume that it does not contain the inflaton field ϕ explicitly. We also assume that the inflaton dominates the energy density of the universe. In other words, we consider the regime when the inflaton has just started to oscillate and decay.

Below we first consider the oscillation part of the Hubble parameter which appears in the background dynamics. Then we estimate how the oscillation causes the production of particles which are coupled to gravity. This chapter is based on the work [118] with Y. Ema, K. Mukaida and K. Nakayama.

7.1 Background dynamics

The background equations of motion derived from the action (7.0.1) are the Friedmann equation

$$3M_P^2H^2 = \rho_\phi, \tag{7.1.4}$$

and the equation of motion for ϕ

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0. \tag{7.1.5}$$

Here the prime denotes the derivative with respect to ϕ. With the potential (7.0.2), ϕ oscillates roughly with the inverse timescale given by the effective mass $m_{\rm eff} \equiv \sqrt{V'(\Phi)/\Phi}$. The energy-momentum conservation reads

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0, \tag{7.1.6}$$

where the energy density ρ_ϕ and pressure p_ϕ are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V, \qquad p_\phi = \frac{1}{2}\dot{\phi}^2 - V. \tag{7.1.7}$$

The conservation of the energy-momentum tensor is equivalent to the following equation for the time derivative of the Hubble parameter

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_P^2}. \tag{7.1.8}$$

First let us derive the oscillation averaged part of the Hubble parameter. Multiplying ϕ to Eq. (7.1.5) and taking oscillation average denoted by $\langle \ldots \rangle$ over the timescale much longer than the oscillation timescale, we have

$$\left\langle \dot{\phi}^2 \right\rangle \simeq n \left\langle V \right\rangle. \tag{7.1.9}$$

Here the surface term $\phi\dot{\phi}|_{t=t_i,t_f}$ is neglected, and $\left\langle H\phi\dot{\phi} \right\rangle$ is also neglected because it oscillates between positive and negative values to cancel out after taking an average. Eq. (7.1.9) is known as Virial theorem. Substituting this into time-averaged Eq. (7.1.6), we have

$$\frac{d}{dt}\left\langle \rho_\phi \right\rangle + \frac{6n}{n+2}H\left\langle \rho_\phi \right\rangle \simeq 0. \tag{7.1.10}$$

Here the time derivative on $\langle \rho_\phi \rangle$ is understood as the differentiation over the timescale longer than the oscillation timescale. Then we have $\langle \rho_\phi \rangle \propto a^{-6n/(n+2)}$ and $\langle H \rangle \propto a^{-3n/(n+2)}$, to obtain

$$\langle H \rangle \simeq \frac{n+2}{3n}\frac{1}{t}, \qquad \langle a \rangle \propto t^{(n+2)/3n}. \tag{7.1.11}$$

Extraction of the oscillation part of the Hubble parameter needs some calculation. We decompose the Hubble parameter as $H = \langle H \rangle + \delta H$, where $\langle H \rangle$ satisfies

$$\frac{d}{dt}\langle H \rangle = -\frac{3n}{n+2}\langle H \rangle^2 , \tag{7.1.12}$$

from Eq. (7.1.11). Then, from Eq. (7.1.8),

$$\begin{aligned}
\delta\dot{H} &\simeq \frac{3n}{n+2}\langle H \rangle^2 - \frac{\dot{\phi}^2}{2M_P^2} \\
&\simeq \frac{3n}{n+2}\frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right) - \frac{\dot{\phi}^2}{2M_P^2} \\
&= \frac{1}{n+2}\frac{1}{M_P^2}\left(-\dot{\phi}^2 + nV\right) .
\end{aligned} \tag{7.1.13}$$

In obtaining the second line, we used Eq.(7.1.4) and neglected the terms proportional to $\langle H \rangle \delta H$ and $(\delta H)^2$. This approximation is justified as long as the terms in the second line do not cancel out. Here we use Eq. (7.1.5) as $-\dot{\phi}^2 + nV = -d(\phi\dot{\phi})/dt - 3H\phi\dot{\phi}$ to obtain

$$\delta\dot{H} \simeq \frac{1}{n+2}\frac{1}{M_P^2}\left[-\frac{d}{dt}(\phi\dot{\phi}) - 3H\phi\dot{\phi}\right] . \tag{7.1.14}$$

Since the first term in the square bracket has the order $\sim M_P^2 H^2$ while the second term has $\sim (H/m_{\rm eff}) M_P^2 H^2 \ll M_P^2 H^2$, we may safely neglect the second term. Integrating both sides, we have

$$\delta H \simeq -\frac{1}{n+2}\frac{\phi\dot{\phi}}{M_P^2}. \tag{7.1.15}$$

Note that the relative oscillation amplitude falls as $\delta H/H \propto \Phi/M_P$, where $\Phi \propto t^{-2/n}$ denotes the amplitude of ϕ oscillation. Therefore the oscillation in the Hubble parameter is most efficient just after inflation. The oscillating part of the scale factor is approximately given by

$$a \simeq \langle a \rangle \left[1 - \frac{1}{2(n+2)}\frac{\phi^2 - \langle \phi^2 \rangle}{M_P^2}\right] , \tag{7.1.16}$$

where time dependence of the averaged part is $\langle a \rangle \propto t^{(n+2)/3n}$. The time evolution of H and δH are shown in Fig. 7.1. The blue and red lines in the right panel uses $\delta H = H - \langle H \rangle$ and the formula (7.1.15), respectively. One sees that they coincide well with each other.

7.2 Gravitational annihilation of inflaton

In the above argument it became clear that the Hubble parameter has an oscillation mode correlated with $\phi\dot{\phi}$. We consider below how the oscillation affects the production of light

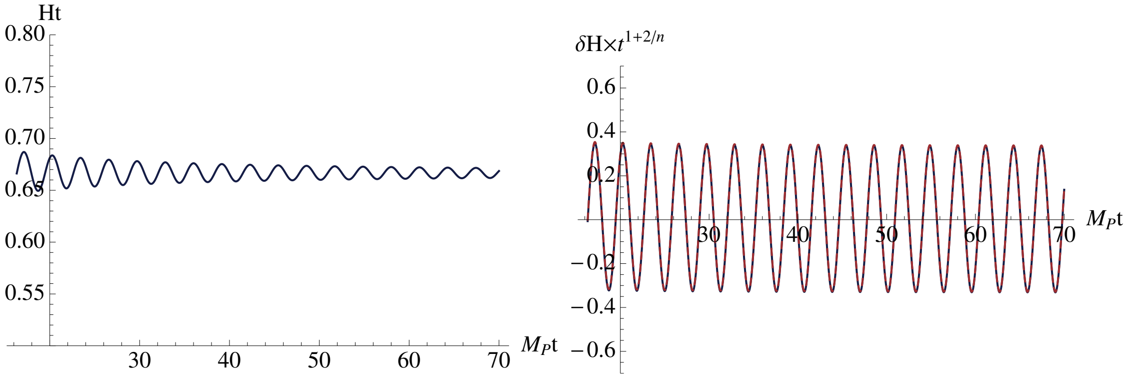

Figure 7.1: Time evolution of Ht (*left*) and $\delta Ht^{1+2/n}$ (*right*). The exponent of the potential and the initial values are taken to be $n = 2$, $\phi_{\rm ini} = 0.1$ and $M_P t_{\rm ini} = 16.32$. In the right panel, blue line uses $\delta H = H - \langle H \rangle$ with $\langle H \rangle$ given by Eq.(7.1.11) while red line uses the analytic formula (7.1.15).

scalar χ, fermion ψ and vector boson A_μ. However, before going into details of each particle species, we explain the relation of Weyl invariance with gravitational particle production.

The effect of the inflaton oscillation appears in the matter sector through background oscillation. The matter system can be described using the conformal time $d\tau \equiv a^{-1} dt$ as

$$S_M = \int d\tau d^3x \; a^4 \mathcal{L}_M. \tag{7.2.17}$$

Here suppose that the matter action has Weyl invariance, which transforms the metric and the fields as

$$g_{\mu\nu} \to e^{-2\sigma(x)} g_{\mu\nu}, \tag{7.2.18}$$

and

$$\chi \to e^{\sigma(x)} \chi, \qquad \psi \to e^{\frac{3}{2}\sigma(x)} \psi, \qquad A_\mu \to A_\mu, \tag{7.2.19}$$

by an arbitrary real function $\sigma(x)$. Since the metric we are considering now is conformal to $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$, the Weyl invariance means that the oscillation of the scale factor can be eliminated by some rescaling of the field. Below we see explicitly that this occurs for massless fermion and vector boson. For general argument on the properties of each component under Weyl transformation, see Appendix C.

7.2.1 Scalar

The action for scalar field is given by

$$S_S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} (\partial \chi)^2 - \frac{1}{2} m_\chi^2 \chi^2 \right]. \tag{7.2.20}$$

We assume $m_\chi \ll m_\phi$ for simplicity, and neglect the mass term in the following. The action is rewritten as

$$S_S = \int d\tau d^3x \; a^2 \left[\frac{1}{2}\chi'^2 - \frac{1}{2}(\partial_i\chi)^2\right]. \tag{7.2.21}$$

At this stage it is clear that ϕ explicitly couples to χ field, since

$$S_S = \int d\tau d^3x \; \langle a\rangle^2 \left[1 - \frac{1}{4}\frac{\phi^2 - \langle\phi^2\rangle}{M_P^2}\right]\left[\frac{1}{2}\chi'^2 - \frac{1}{2}(\partial_i\chi)^2\right], \tag{7.2.22}$$

from Eq. (7.1.16). The process of χ production may be called "gravitational annihilation," because square of the ϕ field is coupled to χ[#1]. The production rate of χ particle can be estimated from the argument in Chapter 5. The mode equation for the canonically normalized field $\tilde{\chi} \equiv a\chi$ becomes[#2]

$$\tilde{\chi}_k'' + \left(k^2 - \frac{a''}{a}\right)\tilde{\chi}_k = 0 \quad \rightarrow \quad \tilde{\chi}_k'' + \left[k^2 + \frac{1}{8}\frac{(\phi^2)''}{M_P^2}\right]\tilde{\chi}_k = 0. \tag{7.2.26}$$

The production rate of χ particle is given by[#3]

$$\dot{n}_\chi \simeq \frac{(\Delta m_\chi^2)^2}{32\pi}, \qquad \Delta m_\chi^2 = \frac{1}{4}\frac{m_\phi^2\Phi^2}{M_P^2}, \tag{7.2.27}$$

while the decay rate becomes[#4]

$$\Gamma_{\phi\to\chi} = -\frac{\dot{n}_\phi}{n_\phi} = \frac{\dot{n}_\chi}{n_\phi} \simeq \frac{1}{256\pi}\frac{m_\phi^3\Phi^2}{M_P^4}. \tag{7.2.28}$$

[#1] Note that such coupling does not arise if χ is conformally coupled to the Ricci scalar.

[#2] Seen in the coordinate (t, x_1, x_2, x_3), the canonical rescaling of χ field becomes $\tilde{\chi} = a^{3/2}\chi$, and the mode equation becomes

$$\ddot{\tilde{\chi}}_k + \left(\frac{k^2}{a^2} - \frac{3}{2}\dot{H} - \frac{9}{4}H^2\right)\tilde{\chi}_k = 0. \tag{7.2.23}$$

This seems to lead to a somewhat different production rate. However, we must take into account the oscillation of the wavenumber in this case. Substituting Eqs. (7.1.8) and (7.1.16), one has

$$\ddot{\tilde{\chi}}_k + \left[\frac{k^2}{\langle a\rangle^2}\left(1 + \frac{1}{4}\frac{\phi^2 - \langle\phi^2\rangle}{M_P^2}\right) + \frac{3}{4}\frac{\dot{\phi}^2}{M_P^2} - \frac{9}{4}H^2\right]\tilde{\chi}_k = 0. \tag{7.2.24}$$

Since the production occurs at $k/\langle a\rangle \simeq m_\phi$,

$$\ddot{\tilde{\chi}}_k + \left(m_\phi^2 + \frac{\dot{\phi}^2 - m_\phi^2\phi^2}{4M_P^2}\right)\tilde{\chi}_k = 0. \tag{7.2.25}$$

This gives the same production rate as Eq. (7.2.26).

[#3] We have neglected the change in the oscillation frequency of the inflaton seen in time coordinate τ, since m_ϕ is much larger than the Hubble parameter during the oscillation regime.

[#4] Though we use the word "decay" rate, the process is annihilation, strictly speaking.

Note that this production rate never exceeds the Hubble parameter $H \sim m_\phi \Phi/M_P$, therefore reheating is not completed by this effect. Therefore in the following we assume that the reheating process is completed by some other couplings of the inflaton with other particles. Taking into account that the gravitational annihilation is most efficient at the onset of inflaton oscillation, and evaluating the Hubble parameter as $3M_P^2 H_{\rm end}^2 \simeq m_\phi^2 \phi_{\rm end}^2/2$ where the subscript "end" denotes the time at the end of inflation, we may evaluate the number density of produced χ particles as

$$n_\chi(t_{\rm end}) \simeq \dot{n}_\chi(t_{\rm end}) \cdot H_{\rm end}^{-1} \simeq \frac{9}{128\pi} H_{\rm end}^3. \tag{7.2.29}$$

Let us check the possibility of parametric resonance. The condition for resonance is $R_q \equiv q^2 m_\phi / H \gg 1$ as we saw in Eq. (5.5.66), where q is given by

$$q \sim \frac{\Phi^2}{M_P^2}, \tag{7.2.30}$$

in the present case. Then one finds that $R_q \sim \Phi^3/M_P^3 \lesssim 1$, which means that no resonance occurs.

The abundance of the produced particles is estimated as follows. We first assume that χ is massive (though the mass is still negligible compared to the inflaton mass), and that it has such weak interaction that the produced particles never thermalizes. Denoting the end of reheating by the subscript "reh", the number density at that time is

$$n_\chi(t_{\rm reh}) = n_\chi(t_{\rm end}) \left(\frac{a_{\rm reh}}{a_{\rm end}}\right)^{-3} = n_\chi(t_{\rm end}) \left(\frac{H_{\rm reh}}{H_{\rm end}}\right)^2. \tag{7.2.31}$$

Estimating the end of reheating by the conventional expression $3M_P^2 H_{\rm reh}^2 = \frac{\pi^2}{30} g_* T_{\rm reh}^4$ where $T_{\rm reh}$ is the reheating temperature, and using the formula for the entropy density $s = \frac{2\pi^2}{45} g_{*s} T^3$, we obtain

$$\frac{\rho_\chi}{s} = m_\chi \frac{n_\chi}{s} \simeq 1 \times 10^{-9}\ {\rm GeV} \left(\frac{m_\chi}{10^6\ {\rm GeV}}\right) \left(\frac{T_{\rm reh}}{10^{10}\ {\rm GeV}}\right) \left(\frac{H_{\rm end}}{10^{14}\ {\rm GeV}}\right), \tag{7.2.32}$$

after χ becomes non-relativistic. Here $g_* = g_{*s}$ is assumed. If χ is stable over the present cosmic time, produced particles can be a candidate for the dark matter. On the other hand, this mechanism can bring about cosmological moduli problem if χ is identified as a moduli, a light field which decays into other particles only through Planck-suppressed interactions. Here we comment on other related works.

- In [119], the produced number of χ particles is estimated by approximating the evolution of the Hubble parameter by a transition from de Sitter stage to matter or radiation dominated stage[#5]. In other words, they smeared out the oscillation of the Hubble parameter, and their mechanism excites those particles with masses lighter than $H_{\rm end}$.

[#5] The effect of this adiabatic change in the Hubble parameter, or "gravitational particle production", on moduli production [120] or superheavy dark matter production [121] is estimated in the literature.

The number density found in [119] is of the same order as the number of χ particles estimated above (7.2.29). However, in our formulation it is clear that the oscillation of the Hubble parameter is actually correlated with that of the inflaton[#6], and it is also clear that those particles lighter than the inflaton mass, not the Hubble parameter, are excited.

- Light scalar fields develop quantum fluctuations during inflation, which effectively become a coherent oscillation mode because of the horizon exit of the fluctuations, and this coherent mode can be a dominant contribution to the χ abundance [120]. This production of χ's quantum fluctuations is often suppressed by Hubble-induced masses. However, our production mechanism excites χ particles as long as $m_\phi \gg H_{\rm end}$, which is satisfied by many inflation models including new inflation and hybrid inflation models.

Next let us consider the case where χ is massless. The fractional energy density of χ to that of radiation is estimated as

$$\frac{\rho_\chi}{\rho_R}(t = t_{\rm reh}) \simeq 2 \times 10^{-20} \left(\frac{m_\phi}{10^{13}\ {\rm GeV}}\right) \left(\frac{T_{\rm reh}}{10^{10}\ {\rm GeV}}\right)^{4/3} \left(\frac{H_{\rm end}}{10^{14}\ {\rm GeV}}\right)^{1/3}, \tag{7.2.33}$$

at the time of reheating. Here the SM value is assumed for g_*. Therefore, the energy density of χ is negligibly small and it does not contribute to the present dark radiation.

7.2.2 Fermion and vector boson

Next we consider the inflaton decay into fermion and vector boson. The action for massless fermion and vector boson is, respectively,

$$S_F = \int d^4x\, e \left[-\bar{\psi} \not{D} \psi\right], \tag{7.2.34}$$

$$S_V = \int d^4x \sqrt{-g} \left[-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}\right]. \tag{7.2.35}$$

Here we consider Abelian vector boson for simplicity. The conclusion is the same for non-Abelian case. The field strength of the gauge boson is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{7.2.36}$$

Note that the normalization is different from Appendix C. For fermion we must introduce the frame field ${e_\mu}^a$, which satisfies

$$g_{\mu\nu} = {e_\mu}^a \eta_{ab} {e_\nu}^b. \tag{7.2.37}$$

Here $\eta_{ab} = {\rm diag}(-1, +1, +1, +1)$ is the local Lorentz metric, and the Greek (Latin) indices are raised and lowered using $g_{\mu\nu}$ (η_{ab}). The spin connection and covariant derivative are defined as

$${\omega_\mu}^{ab} = 2 e^{\nu[a} \partial_{[\mu} {e_{\nu]}}^{b]} - e^{\nu[a} e^{b]\rho} e_{\mu c} \partial_\nu {e_\rho}^c, \tag{7.2.38}$$

[#6] The oscillation of the Hubble parameter is mentioned in [122], though its cosmological consequences are left undiscussed.

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}, \tag{7.2.39}$$

$$\not{D} = e^\mu{}_a \gamma^a D_\mu. \tag{7.2.40}$$

The γ matrices satisfy the $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and $\gamma^{ab} \equiv \gamma^{(a}\gamma^{b)}$. The bracket on indices denotes (anti)symmetrization $A_{(\mu}B_{\nu)} = (A_\mu B_\nu + A_\nu B_\mu)/2$ and $A_{[\mu}B_{\nu]} = (A_\mu B_\nu - A_\nu B_\mu)/2$.

From the argument at the beginning of this section, we choose $\mathrm{e}^{\sigma(x)} = a(t)$. The rescaled fermion field is

$$\tilde{\psi} \equiv a^{-3/2}\psi. \tag{7.2.41}$$

Substituting (7.2.41) into the action, we find

$$S_F = \int d\tau d^3x \left[-\bar{\tilde{\psi}}\delta^\mu{}_a \gamma^a \partial_\mu \tilde{\psi} \right]. \tag{7.2.42}$$

Note that the summation runs over (τ, x_1, x_2, x_3). Thus fermion is not coupled to the oscillation, and the production does not occur. For vector boson, the action is rewritten as

$$S_V = \int d\tau d^3x \left[-\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \right], \tag{7.2.43}$$

and the production does not occur. For massive fermion and vector boson particle production occurs, but the production rate is suppressed by the square of their masses and is generically small.

7.2.3 Graviton

Gravitons h_{ij} live in the ADM metric

$$ds^2 = -N^2 dt^2 + e^{2\zeta}\gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{7.2.44}$$

as

$$\gamma_{ij} = (e^h)_{ij}. \tag{7.2.45}$$

Here h_{ij} satisfies transverse and traceless condition $\partial_j h_{ij} = h_{ii} = 0$. Expanding the action up to quadratic order, we have the graviton action

$$S_G = \int d^4x\, a^3 \frac{M_P^2}{8} \left[\dot{h}_{ij}^2 - a^{-2}(\partial_k h_{ij})^2 \right]. \tag{7.2.46}$$

Decomposing h_{ij} as

$$h_{ij}(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=+,\times} h^{(\lambda)}(t, \mathbf{k})\epsilon_{ij}^{(\lambda)}(\hat{\mathbf{k}})e^{i\mathbf{k}\mathbf{x}}, \tag{7.2.47}$$

by the polarization tensor $\epsilon_{ij}^{(\lambda)}$ satisfying $\hat{k}_j \epsilon_{ij}^{(\lambda)}(\hat{\mathbf{k}}) = \epsilon_{ii}^{(\lambda)}(\hat{\mathbf{k}}) = 0$, $(\epsilon_{ij}^{(\lambda)}(\hat{\mathbf{k}}))^* = \epsilon_{ij}^{(\lambda)}(-\hat{\mathbf{k}})$ and $\epsilon_{ij}^{(\lambda)}(\hat{\mathbf{k}})\epsilon_{ij}^{(\lambda')}(-\hat{\mathbf{k}}) = \delta_{\lambda\lambda'}$ where $\hat{\mathbf{k}} \equiv \mathbf{k}/|\mathbf{k}|^2$, and rescaling the graviton field as $\tilde{h}^{(\lambda)} = (M_P/2)h_{ij}$, the action becomes the same as that for two massless scalar fields:

$$S_G = \int dt \int \frac{d^3k}{(2\pi)^3}\, a^3 \sum_{\lambda=+,\times} \frac{1}{2}\left[|\dot{h}^{(\lambda)}(t,\mathbf{k})|^2 - \frac{k^2}{a^2}|h^{(\lambda)}(t,\mathbf{k})|^2\right]. \tag{7.2.48}$$

Therefore the abundance of gravitons is estimated as twice Eq. (7.2.33). The present frequency is obtained just by redshifting the frequency at the time of production

$$f_{\mathrm{GW}} \simeq \frac{m_\phi}{2\pi}\frac{a_{\mathrm{end}}}{a_0} \simeq 2\times 10^5\ \mathrm{Hz}\left(\frac{m_\phi}{10^{13}\ \mathrm{GeV}}\right)\left(\frac{T_{\mathrm{reh}}}{10^{10}\ \mathrm{GeV}}\right)^{1/3}\left(\frac{H_{\mathrm{end}}}{10^{14}\ \mathrm{GeV}}\right)^{-2/3}. \tag{7.2.49}$$

Here the SM value is assumed for g_* and g_{*s}. At the peak frequency, the energy fraction of the graviton (per logarithmic frequency) to the present cosmic energy density is roughly estimated as $\Omega_{\mathrm{GW}} \sim \Omega_{R0}(\rho_{\mathrm{GW}}/\rho_R)$, and the present gravitational wave amplitude is below the sensitivities of proposed space-interferometer experiments [123, 124].

7.3 Conclusion

In this section, we estimated the gravitational effects on inflaton decay in a system with Einstein gravity and canonical inflaton. We found the following:

- Background analysis
  - The inflaton oscillation induces an oscillation mode in the Hubble parameter and the scale factor, which can be written explicitly using the inflaton as Eqs. (7.1.15) and (7.1.16).
  - Since this mode is correlated with the inflaton oscillation, it is different from the adiabatic (or smeared) change in the Hubble parameter considered in Ref. [119].
- Estimation of particle production
  - Production occurs unless the species is Weyl invariant. This means that light scalar particles and gravitons are produced, while massless fermions and massless vector bosons are not produced. Even if the latter species have masses, the production rate is generically small since the rate is proportional to the mass squared. Scalar particles are not produced if the scalar field is conformally coupled to gravity.
  - Produced light scalar particles can be the present dark matter depending on the parameters. This production mechanism can also be a new source for the moduli problem, since it excites those particles with masses smaller than the inflaton mass, not the Hubble parameter.
  - Though gravitons are produced, their contribution to the dark radiation is negligibly small. Detection by space-interferometer experiments also seems difficult.

Chapter 8

Gravitational effects on nonminimally coupled inflaton: $f(\phi)R$-type coupling

In the previous chapter we considered the minimal setup of the inflaton and gravity. There we found that the Hubble parameter has an oscillation mode correlated with the inflaton oscillation, though the amplitude is small $\dot{H} \sim H^2$. This oscillation mode leads to production of (canonical) scalar, graviton and massive fermion and vector boson.

When the inflaton is nonminimally coupled to gravity, more violent oscillation of the Hubble parameter is induced, as we will see below. The amplitude turns out to be $\dot{H} \sim m_{\text{eff}} H$ with m_{eff} being the effective mass scale (inverse of the oscillation timescale) of the inflaton, which can be much larger than the one found in the minimal setup. Particle production triggered by this oscillation is thus much larger than what we have previously found.

The model we consider in this chapter has $f(\phi)R$-type coupling. This corresponds to $\mathcal{L}_4$ in the language of Galileon/Horndeski theory, see Chapter 4. In the convention of Eqs. (4.2.2)–(4.2.5), we choose

$$G_2 = X - V, \qquad G_4 = f(\phi), \tag{8.0.1}$$

where X is the canonical kinetic term of the inflaton and V is the potential. The explicit form of the action we consider is

$$S = \int d^4x \sqrt{-g} \left[f(\phi)R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right] + S_M. \tag{8.0.2}$$

Here the potential is assumed to have a monomial form for simplicity

$$V = \frac{\lambda}{n}\phi^n. \tag{8.0.3}$$

For the nonminimal coupling $f(\phi)$, we especially consider the following two cases:

$$f \sim 1 + \phi, \qquad f \sim 1 + \phi^2. \tag{8.0.4}$$

Here coefficients on each term is omitted (i.e., the former means $f(\phi) = M_P^2/2 + f_1\phi$, and so on). The latter appears in Higgs inflation [18–20] for example, though we do not consider gauge couplings here. For the matter part, we do not introduce nonminimal couplings between the matter field and gravity throughout the chapter.

The main results we obtain are as follows:

- Case $f \sim 1 + \phi$
  - The oscillation mode of the scale factor, and thus the coupling of the inflaton to matter scalar field, is proportional to ϕ, not ϕ^2 as we found in the minimal setup. This coupling leads to "gravitational decay," in contrast to "gravitational annihilation" previously found, of the inflaton into scalar particles. The decay rate is found to be large enough to complete reheating.
- Case $f \sim 1 + \phi^2$
  - In the inflaton oscillation regime, two typical scale for the inflaton value appears: near the origin ($\phi \sim 0$) and far from the origin. At far from the origin the inflaton mass scale is suppressed, as we can infer from the fact that the inflaton potential is flattened during inflation if the system is seen in the Einstein frame (see Chapter 4). However, we will find that such suppression of the inflaton mass scale vanishes as the inflaton crosses the origin. The inflaton boosts up there, and this sudden change in the inflaton velocity triggers a spike-like feature in the derivative of the Hubble parameter. This "Hubble spike" brings about heavy particle production with masses which can be much larger than the GUT scale.

The dynamics which occurs in these two systems can also be seen in the Einstein frame, as explained in Chapter 4. Though the main analysis is carried out in Jordan frame, we provide interpretations from the Einstein frame as well.

The former part of this chapter (Sec. 8.1) is based on the work [118] with Y. Ema, K. Mukaida and K. Nakayama.

8.1 Case $f \sim 1 + \phi$

We first derive the background equations of motion and the adiabatic invariant introduced in Chapter 6. Next we show that "gravitational decay" occurs in this setup, with the oscillation part of the Hubble parameter and the scale factor correlating linearly with ϕ. Then we discuss particle production by this mechanism.

8.1.1 Background analysis

The background action is, using integration by parts,

$$\begin{aligned} S &= \int d^4x\, a^3 \left[f(12H^2 + 6\dot{H}) + \frac{1}{2}\dot{\phi}^2 - V \right] \\ &= \int d^4x\, a^3 \left[-6fH^2 - 6\dot{f}H + \frac{1}{2}\dot{\phi}^2 - V \right]. \end{aligned} \tag{8.1.5}$$

The background equations of motion are

- Friedmann equation
$$6H^2 f + 6H\dot{f} = \frac{1}{2}\dot{\phi}^2 + V, \tag{8.1.6}$$
- Raychaudhuri equation
$$2\ddot{f} + 4H\dot{f} + 2(3H^2 + 2\dot{H})f = -\frac{1}{2}\dot{\phi}^2 + V, \tag{8.1.7}$$
- ϕ's equation of motion
$$\ddot{\phi} + 3H\dot{\phi} + V' - 6(2H^2 + \dot{H})f' = 0. \tag{8.1.8}$$

Friedmann equation is derived by introducing the lapse function as $dt \to Ndt$, taking variation with respect to N and setting $N \to 1$ after that. Note that we must introduce the lapse function in $dt \subset d^4x$ as well. Also, Raychaudhuri equation is obtained by the variation with respect to the scale factor.

Here we assume that f can be expanded as
$$f = \frac{M_P^2}{2}\left(1 + f_1\frac{\phi}{M_P} + \cdots\right), \tag{8.1.9}$$
and consider up to the linear term in the following. The assumption $f(0) = M_P^2/2$ comes from the requirement that the gravity theory reduces to the Einstein gravity after ϕ settles down to the potential minimum[#1].

The adiabatic invariant in this system reads
$$J \equiv -\frac{1}{6M_P^2}\mathcal{L}_H = \frac{1}{M_P^2}(\dot{f} + 2fH). \tag{8.1.10}$$
Here the subscript H denotes the derivative with respect to H. Since we used $\mathcal{L} = \mathcal{L}(H, \phi, \dot{\phi})$ in deriving the invariant in Chapter 6, we used the expression in the second line of Eq. (8.1.5). From the assumption (8.1.9), we have
$$J = H + \frac{f_1}{2M_P}\left(\dot{\phi} + 2H\phi\right). \tag{8.1.11}$$
Since the first term in the bracket is typically larger than the second term after the onset of inflaton oscillation, we may safely neglect the latter. Using the fact that J has only a suppressed amplitude of oscillation $\dot{J} \sim HJ$, the oscillating mode in the Hubble parameter is extracted as[#2][#3]
$$\delta H \simeq -\frac{f_1}{2M_P}\dot{\phi}, \tag{8.1.15}$$

[#1] Strictly speaking, ϕ does not settle down to $\phi = 0$ since the coupling $f_1\phi R$ affects the dynamics of ϕ. However, this shift in ϕ's minimum can be neglected as the expansion of the universe decreases.

[#2] In $f_1 \to 0$ limit, this oscillation mode is dominated by the one studied in the previous chapter.

[#3] The oscillation mode of the Hubble parameter can also be derived by brute force, using f_1 as perturbation. Decomposing as $H = H_0 + H_1$ and $\phi = \phi_0 + \phi_1$, Friedmann equation reads
$$3M_P^2 H_0^2 = \rho_{\phi 0}, \tag{8.1.12}$$
$$6M_P^2 H_0 H_1 = \rho_{\phi 1} - 3M_P H_0 f_1(\dot{\phi}_0 + H_0\phi_0), \tag{8.1.13}$$

while the averaged part is estimated in Chapter 6 as

$$\langle H \rangle = \frac{n+2}{3n}\frac{1}{t}. \tag{8.1.16}$$

Integrating both hand sides of Eq. (8.1.15),

$$a \simeq \langle a \rangle \left(1 - \frac{f_1}{2}\frac{\phi}{M_P}\right). \tag{8.1.17}$$

Eqs. (8.1.15) and (8.1.17) show that both the Hubble parameter and scale factor are rapidly oscillating functions, with the oscillating parts correlating linearly with ϕ. Here note that the oscillation of the scale factor cancels out if one takes the combination $a^2 f$, which fact is important in discussing graviton production.

Fig. 8.1 is the result of numerical calculation with $n = 2$ and $n = 4$. Note that in the middle panels the blue line shows Ht while the red line shows $(\langle H \rangle + \delta H)t$ with Eqs. (8.1.15) and (8.1.16), thus proving the validity of this decomposition. Also note that Jt has only a small amplitude of oscillation compared to Ht. The bottom panels show the combination $a^2 t^{-2(n+2)/3n}$ and $a^2 f t^{-2(n+2)/3n}$. The oscillation cancels out in the combination $a^2 f$.

8.1.2 Gravitational decay of inflaton

We first consider the decay into a real scalar χ. For later purposes, we multiply an arbitrary function $h(\phi)$ to the kinetic term

$$S_S = \int d^4x \sqrt{-g}\, h(\phi) \left[-\frac{1}{2}(\partial\chi)^2\right]. \tag{8.1.18}$$

Here we neglected the small mass term of χ. Substituting the background expression into the gravity part, and using the conformal time $d\tau = a^{-1}dt$, we have

$$S_S = \int d\tau d^3x\, a^2 h(\phi) \left[\frac{1}{2}\chi'^2 - \frac{1}{2}(\partial_i\chi)^2\right]. \tag{8.1.19}$$

Let us expand h as

$$h = 1 + h_1 \frac{\phi}{M_P} + \cdots. \tag{8.1.20}$$

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where $\rho_{\phi 0} = \dot{\phi}_0^2/2 + V(\phi_0)$ and $\rho_{\phi 1} = \dot{\phi}_0\dot{\phi}_1 + V'(\phi_0)\phi_1$. We now show that the second term is dominant in the RHS of Eq. (8.1.13). Eliminating $\dot{H}$ from Eqs. (8.1.7)–(8.1.8), one has

$$\dot{\rho}_{\phi 1} + 6H_0\dot{\phi}_0\dot{\phi}_1 + \frac{\dot{\phi}_0^2}{2M_P^2 H_0}\rho_{\phi 1} = \frac{2f_1}{M_P}\rho_{\phi 0}\dot{\phi}_0. \tag{8.1.14}$$

It is easily seen that the first term dominates in the LHS. Thus the magnitude of $\rho_{\phi 1}$ is estimated as $\sim f_1\rho_{\phi 0}\phi_0/M_P$, which is suppressed compared to the second term in the RHS of Eq. (8.1.13). Thus one has $6M_P^2 H_0 H_1 \simeq -3M_P H_0 f_1 \dot{\phi}_0$, which gives Eq. (8.1.11).

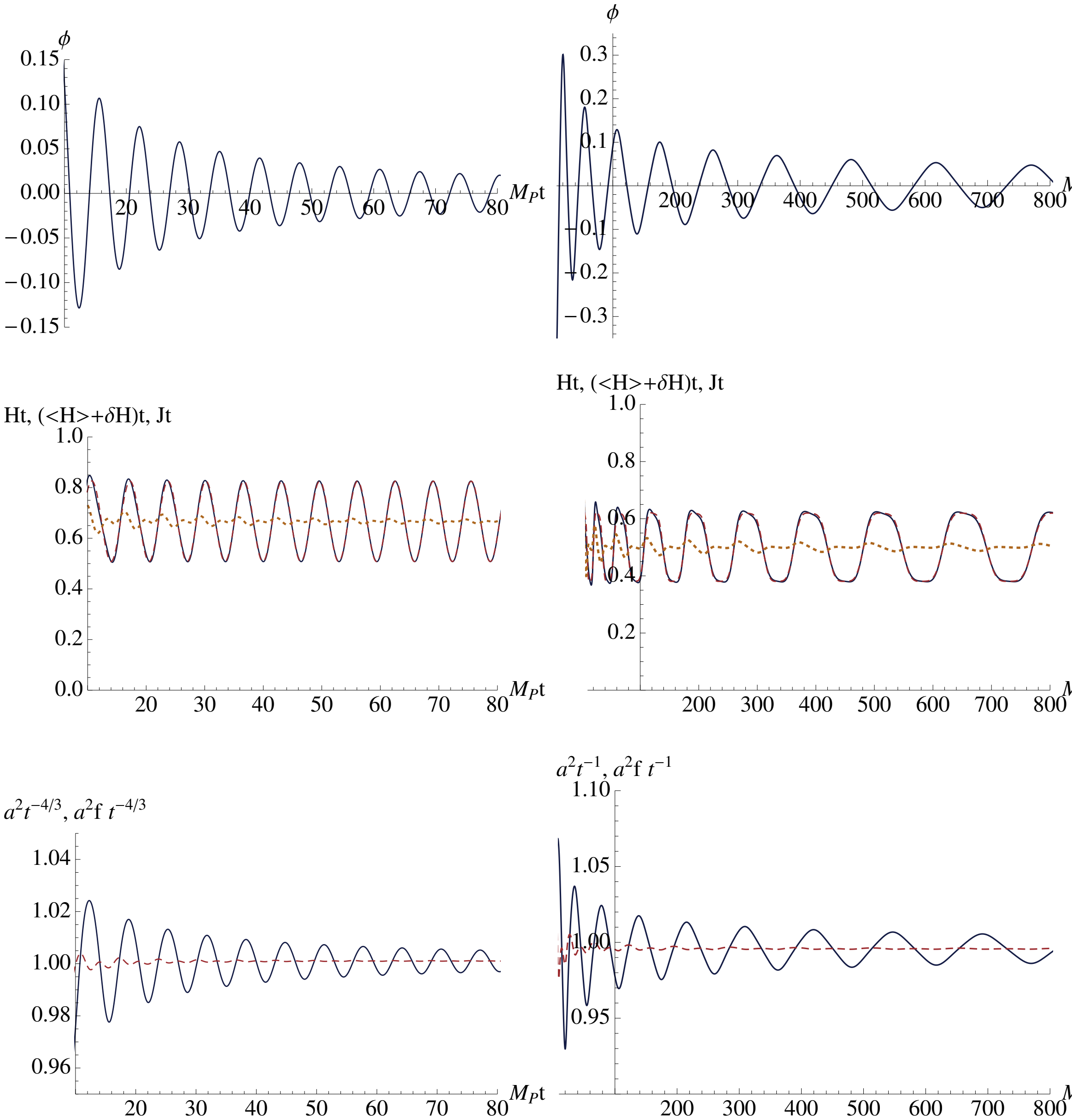

Figure 8.1: Time evolution of quantities for $n = 2$ (*left*) and $n = 4$ (*right*). Parameters and initial conditions are $f_1 = 0.2$, $m_\phi = M_P$, $\phi_{\rm ini} = M_P$ and $M_P t_{\rm ini} = 1.72$ for the left panels, and $f_1 = 0.2$, $\lambda = 1$, $\phi_{\rm ini} = M_P$ and $M_P t_{\rm ini} = 2.2$ for the right panels. (*Top*) Time evolution of ϕ. (*Middle*) Time evolution of Ht (*blue*), $(\langle H \rangle + \delta H)t$ (*red*) and Jt (*yellow*). In evaluating $\langle H \rangle + \delta H$, Eqs. (8.1.15) and (8.1.16) are used. (*Bottom*) Time evolution of $a^2 t^{-2(n+2)/3n}$ (*blue*) and $a^2 f t^{-2(n+2)/3n}$ (*red*).

Taking into account up to the linear part, the combination a^2h gives

$$a^2h \simeq \langle a\rangle^2 \left[1 + (h_1 - f_1)\frac{\phi}{M_P}\right]. \tag{8.1.21}$$

Therefore, the coupling between the inflaton and light scalar takes the form

$$S_S = \int d\tau d^3x \ \langle a\rangle^2 \left[1 + (h_1 - f_1)\frac{\phi}{M_P}\right]\left[\frac{1}{2}\chi'^2 - \frac{1}{2}(\partial_i\chi)^2\right]. \tag{8.1.22}$$

Note that the coupling is linear in ϕ, and thus we may call it "gravitational decay," in contrast to "gravitational annihilation" in the previous chapter. The overall function F defined in Eq. (5.7.72) becomes

$$F \simeq 1 + \frac{h_1 - f_1}{2}\frac{\phi}{M_P}. \tag{8.1.23}$$

In the following we consider $n = 2$ for simplicity. For $n \neq 2$ the procedure is basically the same if one uses the effective mass $m_{\text{eff}} \equiv \sqrt{V'(\Phi)/\Phi}$. Also we set $\langle a\rangle = 1$ since we focus on the inflaton oscillation much faster than the cosmic expansion. As the oscillating mass for χ is given by

$$m_\chi^2(\tau) = \frac{F''}{F} \simeq -\frac{h_1 - f_1}{2}\frac{m_\phi^2\phi}{M_P}, \tag{8.1.24}$$

the amplitude of the mass oscillation is

$$\Delta m_\chi^2 = \frac{|h_1 - f_1|}{2}\frac{m_\phi^2\Phi}{M_P}, \tag{8.1.25}$$

where Φ denotes the amplitude of the oscillation. Assuming that resonance does not occur, the production rate of χ particles is given by

$$\dot{n}_\chi = \frac{(\Delta m_\chi^2)^2}{32\pi}, \tag{8.1.26}$$

or equivalently, the decay rate of the inflaton into two χ's is given by the formula (5.3.49)

$$\Gamma_{\phi\to\chi} = \frac{1}{32\pi}\frac{(\Delta m_\chi^2)^2}{m_\phi\Phi^2} = \frac{(h_1 - f_1)^2}{128\pi}\frac{m_\phi^3}{M_P^2}. \tag{8.1.27}$$

This decay rate coincides with the result in [125], in which the rate is derived by calculating the coupling of the scalar perturbation to the light species χ, as we see below. It should be noticed that this constant decay rate exceeds the Hubble parameter as the amplitude Φ decreases, and therefore the completion of reheating only by gravitational effects is possible [125]. The result (8.1.26)–(8.1.27) must be compared to the production rate by "gravitational annihilation" (7.2.27)–(7.2.28). One finds that the former is larger by a factor $\sim (|h_1 - f_1|\Phi/M_P)^{-1}$.

The advantage of the method illustrated here is that it allows to discuss resonance effects. The Mathieu parameter q is estimated as

$$q \sim \frac{\Delta m_\chi^2}{m_\phi^2} \sim |h_1 - f_1| \frac{\Phi}{M_P}. \tag{8.1.28}$$

Then the resonance parameter R_q given in Eq. (5.5.66) is estimated as

$$R_q \equiv q^2 \frac{m_\phi}{H} \sim (h_1 - f_1)^2 \frac{\Phi}{M_P}. \tag{8.1.29}$$

Since we do not want the factor on the Ricci scalar to flip, we require $f_1 \Phi / M_P < 1$. Taking into account that $\Phi \sim M_P$ at the end of inflation, $f_1 \lesssim 1$ is required. Thus resonance through gravitational effects does not occur, or soon ceases even if it occurs at all.

Next let us consider fermion and vector boson.

$$S_F = \int d^4x \sqrt{-g} \left[-\bar{\psi} \not{D} \psi \right], \tag{8.1.30}$$

$$S_V = \int d^4x \sqrt{-g} \left[-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right]. \tag{8.1.31}$$

As explained in the previous chapter and Appendix C, gravitational coupling to these species can be eliminated by a proper redefinition of the field. Therefore, no gravitational decay into these particles if they are massless. Even when they are massive, the decay rate is suppressed by the mass squared of these particles.

Finally we discuss graviton production. Noting that the Ricci scalar contains the graviton kinetic term as

$$R \supset \frac{1}{4} \left[\dot{h}_{ij}^2 - a^{-2} (\partial_k h_{ij})^2 \right], \tag{8.1.32}$$

the graviton quadratic action is calculated as

$$S_G = \int d\tau d^3x \; a^2 f \frac{1}{4} \left[h_{ij}'^2 - (\partial_k h_{ij})^2 \right]. \tag{8.1.33}$$

Here it should be noted that the overall factor appears in the combination $a^2 f$. In this combination, the oscillation coming from the nonminimal coupling $f(\phi) R$ has already been shown to cancel out. Therefore no gravitational decay channel of the inflaton to gravitons, like $\phi (\partial h)^2$, emerges in $f(\phi) R$ models. However, it should be stressed that the production from the effect which already existed in the minimal setup still remains in the present setup as well.

8.1.3 Coupling of the scalar perturbation to the light particles

Here we comment on the coupling of the scalar perturbation, or the one scalar degree of freedom which consists of the inflaton perturbation and the metric scalar perturbation, to the light scalar particle χ [125]. The scalar perturbations of the metric is parameterized by the following ADM formalism

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{8.1.34}$$

Focusing on the scalar perturbations only, we adopt the following parameterization and gauge fixing condition[#4]

$$N = 1 + \alpha, \qquad \beta_i = 0, \qquad \gamma_{ij} = a^2 e^{2\zeta}\delta_{ij}. \tag{8.1.35}$$

Note that we must take account of the inflaton perturbation as well

$$\phi = \bar{\phi} + \varphi. \tag{8.1.36}$$

Here $\bar{\phi}$ (φ) denotes the homogeneous (inhomogeneous) part. We have changed the convention for ϕ, since we have denoted the homogeneous part of the inflaton field by ϕ so far. As the Einstein equation gives two constraint equations for these three scalar perturbations α, ζ and φ, two of them can be expressed by the remaining one degree of freedom.

We have to find those couplings which are linear in the scalar perturbation and quadratic in the χ field, in order to compare the result with the "gravitational decay" type coupling (8.1.22). For that purpose, it is enough to calculate the constraint equation up to first order in the scalar perturbations, or equivalently the quadratic action for the scalar perturbations. We may use the explicit formula in Table 4.2 or use the 3+1 decomposition. See Appendix D for 3 + 1 decomposition. Assuming that the inflaton is stuck satisfying the background equation of motion, the two constraint equations are solved to give the expression for α and ζ in terms of φ:

$$\alpha = \zeta = -\frac{f_1}{2}\frac{\varphi}{M_P}. \tag{8.1.37}$$

In the matter action, χ field is coupled to the scalar perturbations as follows

$$S_S = \int d^4x\, a^3 \left[\frac{1}{2}\left(h_1 \frac{\varphi}{M_P} + 3\zeta - \alpha \right)\dot{\chi}^2 - \frac{1}{2}a^{-2}\left(h_1 \frac{\varphi}{M_P} + \zeta + \alpha \right)(\partial_i \chi)^2 \right]. \tag{8.1.38}$$

Note that φ comes from the explicit coupling to $h(\phi)$ of the matter action (8.1.18), while ζ and α are understood as the inhomogeneous part of the scale factor a and the lapse function N, respectively. Substituting Eq. (8.1.37), the coupling of φ to χ is obtained as

$$S_S = \int d^4x\, a^3 \frac{h_1 - f_1}{2}\frac{\varphi}{M_P}\left[\dot{\chi}^2 - a^{-2}(\partial_i \chi)^2\right]. \tag{8.1.39}$$

This expression agrees with the $\phi(\partial\chi)^2$ coupling obtained from the analysis of homogeneous inflaton field, Eq. (8.1.22).

[#4]The present setup contains one scalar degree of freedom. The metric and the inflaton field contain four and one scalar degrees of freedom, respectively, but two of them can be dropped by the gauge fixing condition. In addition, the Einstein equation gives constraint equations for the two of the three remaining degrees of freedom. These three degrees of freedom are α, ζ and φ which appear here.

In addition, we see that $f(\phi)R$ does not induce the linear term of the scalar perturbation coupled to graviton quadratic terms, because the coupling

$$S_G = \int d^4x\, a^3 \frac{M_P^2}{4} \left[\frac{1}{2} \left(f_1 \frac{\varphi}{M_P} + 3\zeta - \alpha \right) \dot{h}_{ij}^2 - \frac{1}{2} \left(f_1 \frac{\varphi}{M_P} + \zeta + \alpha \right) a^{-2} (\partial_k h_{ij})^2 \right], \quad (8.1.40)$$

vanishes after the substitution of Eq. (8.1.22). This shows that the nonminimal coupling $f(\phi)R$ does not induce the inflaton decay into gravitons. However, it should be again noted that these still remains the gravitational annihilation type coupling $\phi^2(\partial h)^2$ found in Chapter 7.

8.1.4 Understanding in relation with the Einstein frame

The analysis so far made it clear that the inflaton decay into two χ's is not induced if $f = M_P^2 h/2$, which occurs if we identify χ as the gravition. This result is consistent with the understanding in the conformally transformed frame. As explained in Chapter 4, the system we are dealing with can be transformed to the Einstein frame by conformal transformation:

$$g_{E\mu\nu} \equiv \Omega^2 g_{\mu\nu}, \qquad \Omega^2 = \frac{2f}{M_P^2}. \quad (8.1.41)$$

Here the subscript E denotes "Einstein frame". The Ricci scalar transforms as

$$R = \Omega^2 \left(R_E + 6\Box_E \ln\Omega - 6 g_E^{\mu\nu} \partial_\mu \ln\Omega \partial_\nu \ln\Omega \right). \quad (8.1.42)$$

Using the transformation properties described in Appendix A, we may rewrite the action (8.0.2) for the inflaton and gravity as

$$S = \int d^4x \sqrt{-g_E} \left[\frac{M_P^2}{2} R_E - \frac{1}{2} g_E^{\mu\nu} \partial_\mu \phi_E \partial_\nu \phi_E - V_E(\phi_E) \right]. \quad (8.1.43)$$

Here we have defined the inflaton field and the potential in the Einstein frame as

$$\frac{d\phi_E}{d\phi} = \sqrt{\frac{1}{\Omega^2} + 6M_P^2 (\ln\Omega)'^2}, \qquad V_E = \frac{V}{\Omega^4}, \quad (8.1.44)$$

where the prime denotes the derivative with respect to ϕ. The redefinition affects the coupling to the light scalar field as

$$S_S = \int d^4x \sqrt{-g_E}\, \Omega^{-2} h \left[-\frac{1}{2} g_E^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \right]. \quad (8.1.45)$$

Thus the trilinear coupling of the inflaton to gravitons vanishes if $f = M_P^2 h/2$. This is consistent with the analysis in Jordan frame.

8.1.5 Conclusion

In this section we analyzed gravitational effects on inflaton decay in $f(\phi)R$ model with $f \sim 1 + \phi$. What we found is as follows:

- Background analysis
  - In this model, a more violent oscillation of the Hubble parameter occurs than in the minimal setup. The oscillation mode of the Hubble parameter and the scale factor can be extracted using the invariant introduced in Chapter 6, and we found that the latter is linearly dependent on the inflaton field.
- Particle production
  - The coupling between the inflaton and a canonical light scalar field is a trilinear one which linearly depends on the inflaton. This leads to "gravitational decay", which is more efficient than "gravitational annihilation" in the previous chapter. As a result, reheating can be completed by this coupling alone. Narrow resonance is inefficient.
  - For massless fermion and vector boson, gravitational decay does not occur as their coupling can be eliminated by the rescaling of the field. Even If they are massive, the decay rate is generically small since it is proportional to their masses squared.
  - For gravitons, gravitational decay does not occur because the oscillation in the scale factor is canceled out by the one in $f(\phi)$, though gravitational annihilation still occurs. This is consistent with the picture in the Einstein frame, where the inflaton is canonical and therefore only gravitational annihilation into gravitons is possible.

8.2 Case $f \sim 1 + \phi^2$

Next we consider $f(\phi)R$ type coupling with

$$f = \frac{M_P^2}{2} + \frac{\xi}{2}\phi^2. \tag{8.2.46}$$

In this model the most interesting value of the exponent of the potential is $n = 4$ from the viewpoint of observations, as discussed in Chapter 4. Therefore we consider

$$V = \frac{\lambda}{4}\phi^4. \tag{8.2.47}$$

This setup appears in Higgs inflation [18–20], for example. In order to make the prediction of the scalar perturbation in this model to be consistent with observations, we need $\xi \sim 10^4$ for $\lambda \sim 1$. Therefore we assume $\xi \gg 1$ in the following argument.

Below we first derive the background action, equations of motion and the adiabatic invariant. The equation of motion (8.2.53) derived with the help of the Friedmann and

Raychaudhuri equations is especially important in understanding the emergence of different phases (Phase 0–2). The peculiar behavior of the inflaton field and the Hubble parameter as well as the emergence of the constant mass scale in Phase 1 are illustrated in a simplified system, and the parameter dependence of these quantities is obtained. Next we point out that the averaged expansion law is in fact the one with quadratic potential, which is consistent with the emergence of the constant mass scale. All these results are interpreted from the Einstein frame. Lastly we give a simple estimation of gravitational particle production.

8.2.1 Background action, equations of motion and the adiabatic invariant

The background action and equations of motion are the same as Eqs. (8.1.5)–(8.1.8). After substituting Eq. (8.2.46) we have

$$S = \int d^4x\, a^3 \left[-3M_P^2 H^2 - 3\xi H^2\phi^2 - 6\xi H\phi\dot{\phi} + \frac{1}{2}\dot{\phi}^2 - V\right], \tag{8.2.48}$$

and

- Friedmann equation

$$3M_P^2H^2 + \xi\left(3H^2\phi^2 + 6H\phi\dot{\phi}\right) = \frac{1}{2}\dot{\phi}^2 + V, \tag{8.2.49}$$

- Raychaudhuri equation

$$\left(M_P^2 + \xi\phi^2\right)\left(2\dot{H} + 3H^2\right) + \xi\left(2\phi\ddot{\phi} + 2\dot{\phi}^2 + 4H\phi\dot{\phi}\right) = -\frac{1}{2}\dot{\phi}^2 + V. \tag{8.2.50}$$

- ϕ's equation of motion

$$\ddot{\phi} + 3H\dot{\phi} - \xi\left(6\dot{H} + 12H^2\right)\phi + V' = 0, \tag{8.2.51}$$

The adiabatic invariant is given by

$$J \equiv -\frac{1}{6M_P^2}\mathcal{L}_H = H + \xi\frac{H\phi^2}{M_P^2} + \xi\frac{\phi\dot{\phi}}{M_P^2}. \tag{8.2.52}$$

Also, the following equation obtained by eliminating $\dot{H}$ and H^2 from Eq. (8.2.51) using Eqs. (8.2.49)–(8.2.50) is helpful

$$\ddot{\phi} + 3H\dot{\phi} + m_{\text{eff}}^2\phi = 0, \qquad m_{\text{eff}}^2 = \frac{V'}{\phi} + \frac{\xi(1+6\xi)(\dot{\phi}^2 - V'\phi) + \xi(V'\phi - 4V)}{M_P^2 + \xi(1+6\xi)\phi^2}. \tag{8.2.53}$$

Note that the last terms in the numerator vanishes for the quartic potential (8.2.47).

We must mention previous studies here. Note that this expression for the equation of motion, as well as the blow-up of the inflaton motion at the origin and the spike-like feature of the derivative of the Hubble parameter as we see below, are already studied in [126][#5]. They studied the production of the inflaton quanta φ by this blow-up mechanism. Our novelties here are the analytic estimation of the blow-up of the inflaton velocity and the time derivative of the Hubble parameter, resulting phase classification, proof of the emergence of the expansion law with quadratic potential $\langle H \rangle = 2/3t$, estimation of gravitational heavy particle production and the interpretations from the Einstein frame, as we see below.

8.2.2 Phase classification

We show that the model has the following three phases in chronological order:

- Phase 0 ($M_P/\sqrt{\xi} \ll \Phi$) : Inflation,
- Phase 1 ($M_P/\xi \ll \Phi \ll M_P/\sqrt{\xi}$) : Oscillation phase with the nonminimal coupling being effective,
- Phase 2 ($\Phi \ll M_P/\xi$) : Einstein-Hilbert gravity and canonical inflaton system.

Inflation

Let us start with ϕ satisfying $\xi\phi^2 \gg M_P^2$ and see what happens. The dominant and next-to dominant terms in Friedmann equation (8.2.49) and ϕ's equation of motion (8.2.51) during inflation are

$$3M_P^2H^2 + \xi(3H^2\phi^2 + 6H\phi\dot{\phi}) \simeq \frac{\lambda}{4}\phi^4, \tag{8.2.54}$$

$$-\xi(6\dot{H} + 12H^2)\phi + \lambda\phi^3 \simeq 0. \tag{8.2.55}$$

The time evolution follows

$$\dot{H} \simeq -\frac{\lambda}{18\xi^2}M_P^2, \qquad \phi^2 \simeq \frac{12\xi}{\lambda}H^2 - \frac{M_P^2}{3\xi}. \tag{8.2.56}$$

In fact, these solutions satisfy the second equation, and also satisfy the first one with an error of $\mathcal{O}((\lambda/\xi^2)M_P^4)$ (use $\phi\dot{\phi} \simeq (12\xi/\lambda)H\dot{H}$). Note that the last term in the above ϕ-H relation is a small correction term for $\xi\phi^2 \gg M_P^2$. Therefore in the field value under consideration, ϕ and H follows $\dot{\phi} \simeq$ constant (< 0) and $\dot{H} \simeq$ constant (< 0). Also, it is confirmed that the terms in Eqs. (8.2.54)–(8.2.55) are lager than the neglected terms.

Phase 1 ($M_P/\xi \ll \Phi \ll M_P/\sqrt{\xi}$)

The validity of the solution (8.2.56) breaks down at $\xi\phi^2 \sim M_P^2$, when the dominant terms in Eq. (8.2.54) are comparable to the error of the solution $\mathcal{O}((\lambda/\xi^2)M_P^4)$. After ϕ drops below this value, ϕ oscillates around the minimum of the potential and we must take the full

[#5]The oscillation of the Hubble parameter is also studied in [127].

equations of motion (8.2.49)–(8.2.51) into account. The effective mass of the oscillation for the quartic potential (8.2.47) is

$$\ddot{\phi} + 3H\dot{\phi} + m_{\rm eff}^2 \phi = 0, \qquad m_{\rm eff}^2 = \frac{\xi(1+6\xi)\dot{\phi}^2 + \lambda M_P^2 \phi^2}{M_P^2 + \xi(1+6\xi)\phi^2}. \tag{8.2.57}$$

Note that the V'/ϕ contribution in Eq. (8.2.53) has almost canceled out with the second term to give $\lambda M_P^2 \phi^2$ term in the numerator.

Let us first consider the regime where the amplitude still satisfies $M_P^2 \ll \xi^2 \Phi^2$ (though $\xi \Phi^2 \ll M_P^2$), where Φ is the amplitude of ϕ oscillation. We call this Phase 1. In this phase, the system has several features as shown in Figs. 8.3–8.5

- The Hubble parameter violently oscillates with the amplitude comparable to itself (see Fig. 8.4).
- The Hubble parameter has two oscillation timescales (see the blow-ups in Fig. 8.4). The fast transition occurs when ϕ crosses the origin.

We elucidate this behavior in the following. What brings about this characteristic behavior is the denominator of $m_{\rm eff}^2$. Let us see the behavior neglecting the friction term, in order to focus on the timescale of a few oscillations of ϕ. Eq. (8.2.57) is rewritten as

$$\tilde{\phi}'' + \frac{\tilde{\phi}'^2 + \tilde{\phi}^2}{1+\tilde{\phi}^2}\tilde{\phi} = 0, \qquad \tilde{\phi} \equiv \frac{\sqrt{\xi(1+6\xi)}}{M_P}\phi, \qquad \tau \equiv \frac{\lambda^{1/2}}{\sqrt{\xi(1+6\xi)}} M_P t, \tag{8.2.58}$$

where the prime denotes the derivative with respect to τ. Since Eq. (8.2.58) is nonlinear, the solution behaves differently depending on the initial value of $\tilde{\phi}$, or the amplitude Φ. Fig. 8.2 is the numerical result with initial conditions $\tilde{\phi}_{\rm ini} = 1,\ 10^{1/2}$, $10,\ 10^{3/2}$ and $\tilde{\phi}'_{\rm ini} = 0$. Phase 1 corresponds to $\tilde{\phi}_{\rm ini} \gg 1$, and let us focus on this parameter region. The behavior of the solutions is understood as follows. First, for $\tilde{\phi} \gg 1$, we may neglect the constant in the denominator of the effective mass of $\tilde{\phi}$ to solve the differential equation as

$$\tilde{\phi} \simeq \pm \tilde{\phi}_{\rm ini} \sqrt{\left|\cos(\sqrt{2}\tau)\right|}, \tag{8.2.59}$$

where the sign flips when $\tilde{\phi}$ crosses 0. The fact that the oscillation frequency ~ 1 corresponds to the effective mass scale $m_{\rm eff}^2 \sim \lambda M_P^2/\xi^2$ in terms of ϕ. Therefore the typical velocity of the original ϕ field is

$$\dot{\phi}_{\rm non-origin} \sim \frac{\sqrt{\lambda}}{\xi}\Phi M_P. \tag{8.2.60}$$

As $\tilde{\phi}$ approaches the point $\tilde{\phi} \sim 1$, the above approximation about the denominator of the effective mass becomes unjustified. This corresponds to $\phi \sim M_P/\xi$ in the original inflaton field. Therefore we use the words "origin" and "non-origin" as follows

$$\text{origin} \quad : \quad |\phi| \lesssim \frac{M_P}{\xi}, \tag{8.2.61}$$

$$\text{non} - \text{origin} \quad : \quad |\phi| \gtrsim \frac{M_P}{\xi}. \tag{8.2.62}$$

From Eq. (8.2.59) one sees that $\tilde{\phi}$ enters the origin at $|\tau - \tau_\times| \sim \tilde{\phi}_{\rm ini}^{-2}$, where $\tau_\times$ is the time when $\tilde{\phi}$ crosses the origin. Then one finds that $\tilde{\phi}$ acquires a velocity of $\tilde{\phi}' \sim \tilde{\phi}_{\rm ini}^2 \gg 1$ around the time of crossing. It is estimated that $\tilde{\phi}$ crosses the origin with roughly this velocity[#6]. In terms of the original field ϕ, this means that it passes through the origin at

$$\dot{\phi}_{\rm origin} \sim \sqrt{\lambda}\Phi^2. \tag{8.2.65}$$

In other words, the typical velocity which occurs in the inflation model without the nonminimal coupling appears at the origin, though the oscillation mass scale is suppressed most of the time during oscillation. Eqs. (8.2.60) and (8.2.65) confirm in $\dot{\phi}_{\rm non-origin}$ and $\dot{\phi}_{\rm origin}$ during Phase 1 in Table 8.1. In addition, Eqs. (8.2.60) and (8.2.65) confirm the value of the Hubble parameter in Phase 1 in Table 8.1.

This blow-up of the velocity affects the derivative of the Hubble parameter, which is confirmed by the adiabatic invariant (8.2.52). The oscillation mode in the Hubble parameter can be extracted from Eq. (8.2.52) as

$$\delta H \simeq -\xi \frac{\phi\dot{\phi}}{M_P^2}. \tag{8.2.66}$$

Note that the second term in the RHS of Eq. (8.2.52) is smaller than the third term, irrespective of ϕ is at the origin or not. The magnitude of the oscillation is estimated as follows

$$\delta H \sim \begin{cases} \xi \dfrac{m_{\rm eff(non-origin)}\Phi^2}{M_P^2} \sim \dfrac{\sqrt{\lambda}\Phi^2}{M_P} \sim H & ({\rm non-origin}) \\ \xi \dfrac{(M_P/\xi)(\sqrt{\lambda}\Phi^2)}{M_P^2} \sim \dfrac{\sqrt{\lambda}\Phi^2}{M_P} \sim H & ({\rm origin}) \end{cases}. \tag{8.2.67}$$

Therefore, the oscillation mode of the Hubble parameter has the same order as the Hubble parameter itself. This estimation is consistent with the oscillation of the Hubble parameter in Fig. 8.4. Especially, since the inflaton passes through the origin $|\phi| \lesssim M_P/\xi$ with the timescale

$$\Delta t \sim \frac{M_P}{\xi}\Big/\dot{\phi}_{\rm origin} \sim \frac{M_P}{\xi}\Big/\sqrt{\lambda}\Phi^2, \tag{8.2.68}$$

the derivative of the Hubble parameter blows up at the origin

$$\dot{H}_{\rm origin} \sim \frac{H}{\Delta t} \sim \frac{\lambda\xi\Phi^4}{M_P^4}, \tag{8.2.69}$$

[#6] After $\tilde{\phi}^2$ drops below 1, the system is approximated by

$$\tilde{\phi}'' + \tilde{\phi}'^2\tilde{\phi} = 0. \tag{8.2.63}$$

Note that we have used the fact that $\tilde{\phi}'^2 \gg 1$ is already satisfied at $\tilde{\phi} \sim 1$. This differential equation is easily integrated by rewriting as $\tilde{\phi}''/\tilde{\phi}' + \tilde{\phi}'\tilde{\phi} = 0$ to give

$$\tilde{\phi}' = Ce^{-\tilde{\phi}^2/2}, \tag{8.2.64}$$

with C being the integration constant. This shows that $\tilde{\phi}'$ does not change much at $\tilde{\phi} = 0$ and $\tilde{\phi} = 1$.

Table 8.1: Parameter dependence of quantities in each phase. All quantities are written in the Planck unit $M_P = 1$.

Phase	Φ	H	$\dot{\phi}_{\text{non-origin}}$	$\dot{\phi}_{\text{origin}}$	$\dot{H}_{\text{non-origin}}$	$\dot{H}_{\text{origin}}$
Phase 0		$\sqrt{\lambda}/\xi$				
Phase 0 → 1	$1/\sqrt{\xi}$	$\sqrt{\lambda}/\xi$	$\sqrt{\lambda}/\xi^{3/2}$	$\sqrt{\lambda}/\xi$	λ/ξ^2	λ/ξ
Phase 1		$\sqrt{\lambda}\Phi^2$	$\sqrt{\lambda}\Phi/\xi$	$\sqrt{\lambda}\Phi^2$	$\lambda\Phi^2/\xi$	$\lambda\xi\Phi^4$
Phase 1 → 2	$1/\xi$	$\sqrt{\lambda}/\xi^2$	$\sqrt{\lambda}/\xi^2$	$\sqrt{\lambda}/\xi^2$	λ/ξ^3	λ/ξ^3
Phase 2		$\sqrt{\lambda}\Phi^2$	$\sqrt{\lambda}\Phi^2$	$\sqrt{\lambda}\Phi$	$\lambda\xi\Phi^4$	$\lambda\xi\Phi^4$

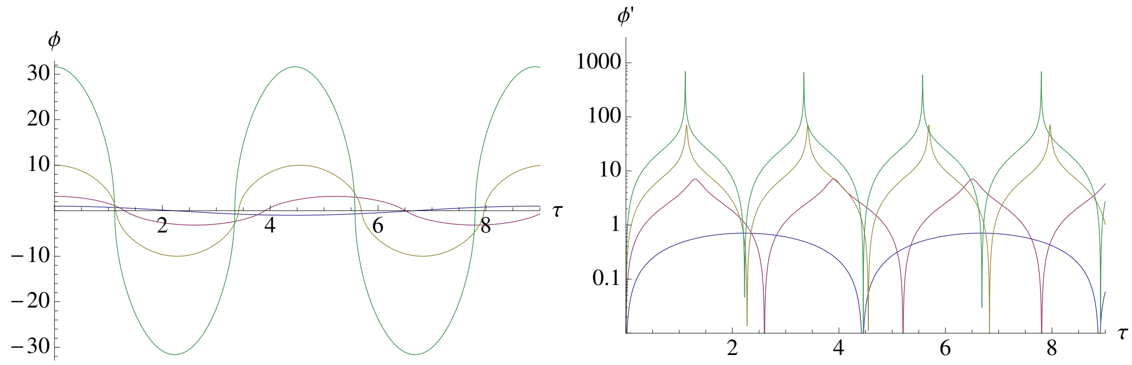

Figure 8.2: Plot of $\tilde{\phi}$ and $\tilde{\phi}'$ obeying the differential equation Eq. (8.2.58). In the right panel $|\tilde{\phi}'|$ is plotted. The initial conditions are set at $\tau = 0$ as $\tilde{\phi}_{\text{ini}} = 1$ (*blue*), $10^{1/2}$ (*red*), 10 (*yellow*), $10^{3/2}$ (*green*), while $\tilde{\phi}'_{\text{ini}} = 0$. The peaks in $\tilde{\phi}'$ appear for $\tilde{\phi}_{\text{ini}} \gg 1$. Also, the height of the peaks in $\tilde{\phi}'$ is proportional to $\tilde{\phi}^2_{\text{ini}}$.

compared to its value away from the origin

$$\dot{H}_{\text{non-origin}} \sim m_{\text{eff(non-origin)}} H \sim \frac{\lambda\Phi^2}{\xi}. \tag{8.2.70}$$

This estimation is consistent with Fig. 8.5.

Phase 2 ($\Phi \ll M_P/\xi$)

After the amplitude Φ drops below M_P/ξ, the nonminimal term becomes ineffective. This may be seen from the expression for the effective mass of the inflaton field (8.2.53), where the nonminimal term never dominates in the denominator in contrast to Phase 1. Therefore the dynamics is the same as in the minimal setup, where only Einstein gravity and a canonical inflaton exists.

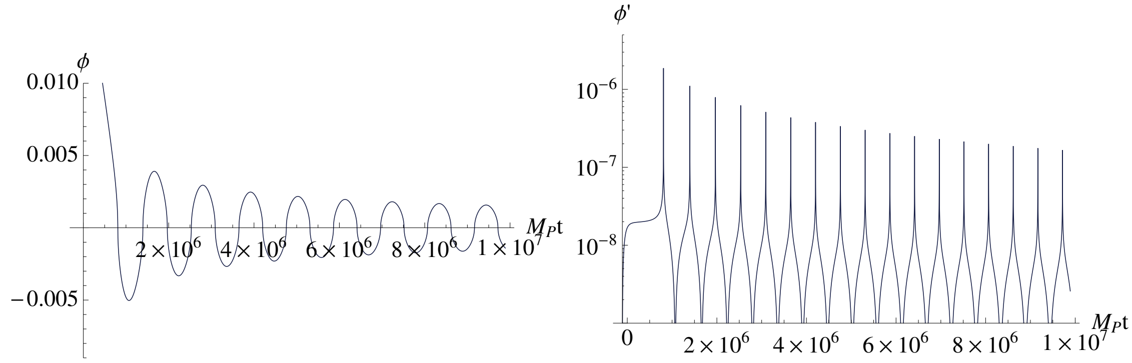

Figure 8.3: Plot of ϕ (*left*) and $\dot{\phi}$ (*right*) at the beginning of Phase 1. In the right panel $|\dot{\phi}|$ is plotted. Parameters are taken to be $\lambda = 0.01$, $\xi = 10^4$. Note that the features observed in the simplified system Fig. 8.2 are also observed in these panels.

8.2.3 Expansion law in Phase 1

The nontrivial phase which appears in this model is Phase 1. The adiabatic invariant can be used to derive the averaged expansion law in this phase. Since the dominant terms in Eq. (8.2.49) are $M_P^2 H^2$, $\xi H \phi \dot{\phi}$ and V in most of the time during oscillation[#7], the adiabatic invariant can be written as

$$\mathcal{L} \simeq \frac{1}{2} H \mathcal{L}_H + \frac{1}{4} \phi \mathcal{L}_\phi + \frac{1}{4} \dot{\phi} \mathcal{L}_{\dot{\phi}}. \tag{8.2.71}$$

As explained in Chapter. 6, what we need to estimate the expansion law is the Friedmann equation

$$\mathcal{L} - H \mathcal{L}_H - \dot{\phi} \mathcal{L}_{\dot{\phi}} = 0, \tag{8.2.72}$$

and the Virial theorem

$$\langle \phi \mathcal{L}_\phi \rangle + \left\langle \dot{\phi} \mathcal{L}_{\dot{\phi}} \right\rangle = 0. \tag{8.2.73}$$

Taking oscillation average in Eqs. (8.2.71)–(8.2.72), and taking Eq. (8.2.73) into account, we obtain

$$\langle \mathcal{L} \rangle = \frac{1}{2} \langle H \mathcal{L}_H \rangle . \tag{8.2.74}$$

Since the time evolution of the adiabatic invariant is given by (see Eq. (6.1.9))

$$\dot{\mathcal{L}}_H + 3H \mathcal{L}_H - 3\mathcal{L} = 0, \tag{8.2.75}$$

[#7] When ϕ crosses the origin, i.e. $\xi \phi \ll M_P$, the term $-3M_P^2 H^2$ in the action cannot be negligible and this expression does not seem hold. However, since this occurs instantly compared to the slow dynamics in $\xi \phi \gg M_P$, it does not affect the estimation of the expansion law.

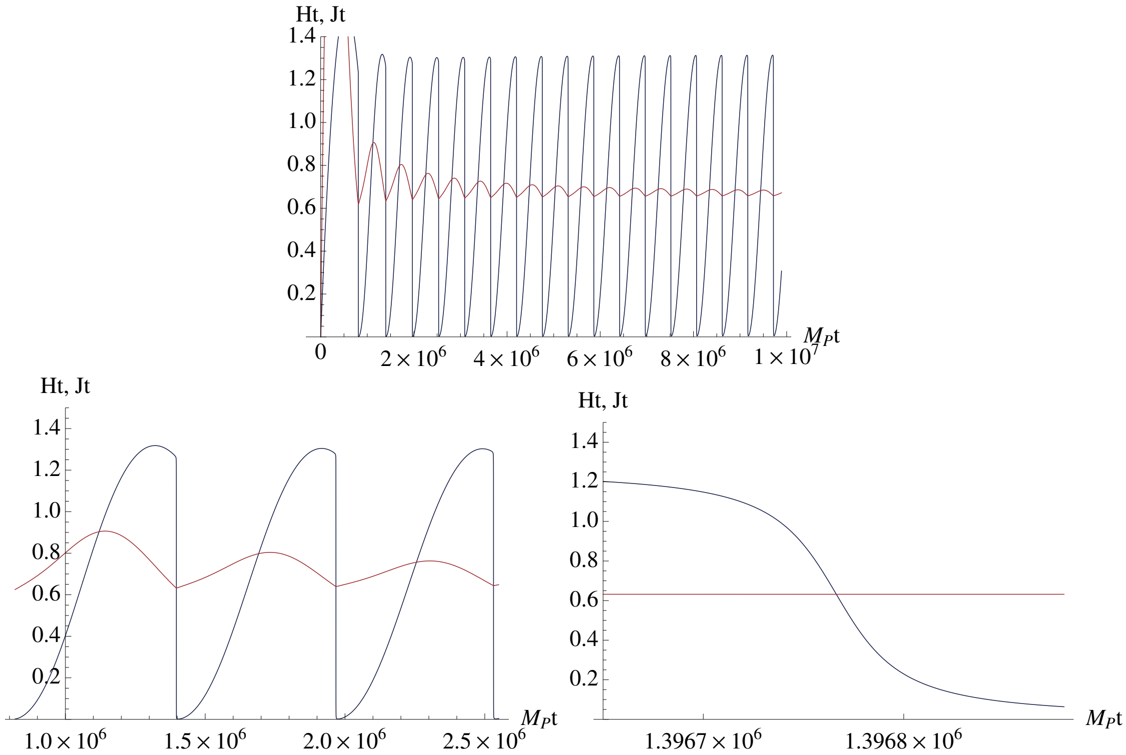

Figure 8.4: Plot of Ht (*blue*) and Jt (*red*) is shown. Bottom panels are blow-ups of the top panel. Parameters are the same as in Fig. 8.3. Note the scale of the horizontal axis.

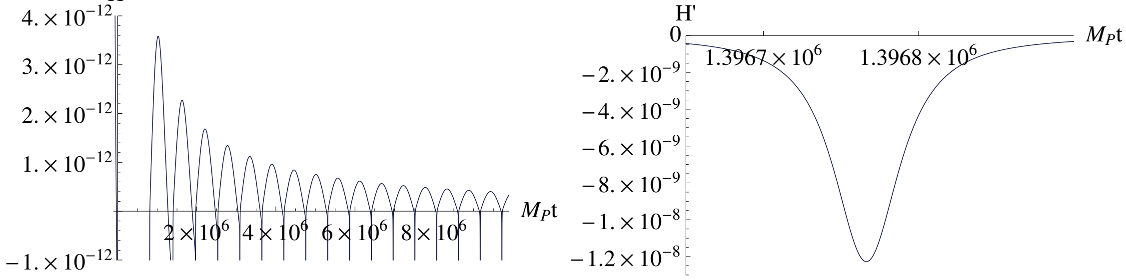

Figure 8.5: Plot of $\dot{H}$ for positive region (*left*) and negative region (*right*) at the beginning of Phase 1. Note the difference in the amplitude of the positive peaks ($\dot{H} \sim 10^{-12}$) and negative peaks ($\dot{H} \sim 10^{-8}$). Also note that the positive peaks of $\dot{H}$ are proportional to t^{-1}, which is consistent with Table. 4.1.

we have

$$\left\langle \dot{\mathcal{L}}_H \right\rangle + \frac{3}{2} \langle H \rangle \mathcal{L}_H = 0. \tag{8.2.76}$$

Here we do not put the oscillation average $\langle \cdots \rangle$ to $\mathcal{L}_H$ because it is an adiabatic invariant. Therefore we obtain

$$\mathcal{L}_H \propto a^{-3/2}. \tag{8.2.77}$$

Since $\mathcal{L}_H \propto \langle H \rangle$, we have

$$\langle H \rangle = \frac{2}{3t}. \tag{8.2.78}$$

Note that this expansion law is the same as that in matter domination phase, or oscillation phase with quadratic potential, not quartic. This may be understood as the result of the constant mass scale $m_{\mathrm{eff(non-origin)}} \sim \sqrt{\lambda} M_P/\xi$ explained above. The numerical calculation in Fig. 8.4 confirms this averaged expansion law. This behavior is also explained by the analysis in the Einstein frame below.

8.2.4 Understanding in relation with Einstein frame

Let us understand the dynamics in Einstein frame[#8]. As explained in Chapter. 4, the relation between the metric in Jordan and Einstein frames is given by

$$g_{E\mu\nu} = \Omega^2 g_{\mu\nu}, \qquad \Omega^2 = 1 + \frac{\xi\phi^2}{M_P^2}. \tag{8.2.79}$$

Also, the relation between the inflaton field with canonical kinetic term in both frames is given by

$$\phi_E \simeq \begin{cases} \phi & (\phi \ll M_P/\xi), \\ \sqrt{\frac{3}{2}} M_P \ln \Omega^2 & (\phi \gg M_P/\xi), \end{cases} \tag{8.2.80}$$

The potential for ϕ_E is given by

$$V_E \simeq \begin{cases} \dfrac{\lambda}{4}\phi_E^4 & (\phi_E \ll M_P/\xi), \\ \dfrac{\lambda M_P^4}{4\xi^2}\left(1 - e^{-\alpha\phi_E/M_P}\right)^2 & (\phi_E \gg M_P/\xi), \end{cases}, \tag{8.2.81}$$

[#8]Notice that the time variables in Jordan and Einstein frames are different, although the conformal transformation itself is just the redefinition of the metric and does not change the time variable. This is because we set the lapse functions N_E and N to be 1 in both frames, thanks to the time reparameterization invariance. However, since the conformal transformation relates the lapse function in each frame as $N_E = \Omega N$, this is impossible in principle. In other words, if we set the lapse functions to be unity in both frames, the time derivative in Einstein frame must be interpreted as the time derivative with respect to $dt_E = \Omega^{-1} dt$. However, this is negligible since the deviation of Ω from 1 is $\lesssim 1$ in the inflaton oscillation regime.

where $\alpha = \sqrt{2/3}$.

Seen in this frame, ϕ_E starts to oscillate when it rolls down to $\phi_E \sim M_P$ as in many chaotic inflation models. Then one sees that the latter expression of Eq. (8.2.80) still applies at the onset of oscillation $\phi \sim M_P/\sqrt{\xi}$. Phase 1 in the previous subsections corresponds to this regime, when the inflaton ϕ_E has already started to oscillate but the map between ϕ_E and ϕ is still given by the latter expression of Eq. (8.2.80). The oscillation mass scale during this phase is given by the constant $m_E \sim \sqrt{\lambda} M_P/\xi$, which is the same as $m_{\text{eff(non-origin)}}$ in Phase 1 in the Jordan frame analysis. Also note that the appearance of constant mass scale implies the expansion law $H = 2/3t$, which is also consistent with the previous analysis.

Next let us focus on the peculiar behavior of the Hubble parameter observed in Jordan frame. Since Eq. (8.2.79) means

$$a_E = \Omega a, \tag{8.2.82}$$

the relation between the Hubble parameter in both frames is

$$H_E \simeq H + \frac{\dot{\Omega}}{\Omega}. \tag{8.2.83}$$

See the previous footnote for the approximate equality. Note that H_E has almost no oscillation $\dot{H}_E \sim H_E^2$, as we saw in Chapter 7. What caused the oscillation in H is in the last term of Eq. (8.2.83). The magnitude of this term is estimated as

$$\frac{\dot{\Omega}}{\Omega} \sim \begin{cases} \dfrac{\xi \phi_E \dot{\phi}_E}{M_P^2} & (\phi_E \ll M_P/\xi) \\ \dfrac{\dot{\phi}_E}{M_P} & (\phi_E \gg M_P/\xi) \end{cases}. \tag{8.2.84}$$

Therefore, the derivative of the Hubble parameter has the order of

$$\dot{H} \sim \frac{d}{dt}\left(\frac{\dot{\Omega}}{\Omega}\right) \sim \begin{cases} \dfrac{\lambda \Phi_E^2}{\xi} \sim \dfrac{\lambda \xi \Phi^4}{M_P^2} & (\phi_E \ll M_P/\xi) \\ \dfrac{\lambda \phi_E M_P}{\xi^2} \sim \dfrac{\lambda \Phi^2}{\xi} & (\phi_E \gg M_P/\xi) \end{cases}, \tag{8.2.85}$$

where we used $\dot{\phi}_E \sim m_E \Phi_E$ and $\Phi_E \sim \xi \Phi^2/M_P$. The estimation of Eq. (8.2.85) is consistent with Table. 4.1.

8.2.5 Estimation of particle production

Let us estimate the gravitational production of a canonical light scalar field χ

$$S_M = \int d^4x \sqrt{-g}\left[-\frac{1}{2}(\partial\chi)^2\right]. \tag{8.2.86}$$

The mode equation for the canonically normalized field $\tilde{\chi} \equiv a^{-3/2}\chi$ becomes

$$\ddot{\tilde{\chi}}_k + \left(\frac{k^2}{a^2} - \frac{3}{2}\dot{H} - \frac{9}{4}H^2\right)\tilde{\chi}_k = 0. \tag{8.2.87}$$

Below we consider the effect of the oscillating Hubble parameter on χ particle production in Phase 1. Since $\dot{H} \gtrsim H^2$ holds after the onset of the oscillation, we take only $\dot{H}$ into account. As explained above, $\dot{H}$ has two typical inverse timescales with which it changes, $m_{\text{non-origin}}$ and m_{origin}. We denote them as m_f (fast) and m_s (slow) for notational simplicity. For each mode, the timescale and the amplitude of the time-depending mass of χ field is given by

- Slow mode: $m_s \sim \sqrt{\lambda} M_P/\xi$, $\dot{H} \sim \lambda \Phi^2/\xi$,
- Fast mode: $m_f \sim \sqrt{\lambda}\xi\Phi^2/M_P$, $\dot{H} \sim \lambda\xi\Phi^4/M_P^2$.

Note that m_s refers to "oscillation" timescale, while m_f refers to the typical timescale of the "spike" in the derivative of the Hubble parameter.

For the slow mode, the behavior of $\dot{H}$ is oscillation-like and we may use the argument in Chapter 5. The q parameter for the slow mode is

$$q_s \sim \frac{\dot{H}}{m_s} \sim \frac{\sqrt{\lambda}\Phi^2}{M_P^2} \lesssim \frac{\sqrt{\lambda}}{\xi} \ll 1. \tag{8.2.88}$$

Here the inequality is evaluated at the beginning of Phase 1. Also, the resonance condition (5.5.66) is

$$R_{qs} \equiv q_s^2 \frac{m_s}{H} \sim \left(\frac{\dot{H}}{m_s}\right)^2 \frac{m_s}{H} \sim \frac{\lambda \Phi^2}{\xi M_P^2} \lesssim \frac{\lambda}{\xi^2} \ll 1. \tag{8.2.89}$$

The resulting decay rate of ϕ field is found to be small

$$\Gamma_{\phi\to\chi} \sim \frac{m_s \dot{n}_\chi}{\rho_\phi} \sim \frac{m_s \dot{H}^2}{\lambda\Phi^4} \sim \frac{\lambda^{3/2}}{\xi^3} M_P. \tag{8.2.90}$$

For the fast mode, the behavior of $\dot{H}$ is spike-like rather than oscillation-like. Let us estimate the amount of particle produced by one spike. For that purpose, we return to the expression for β_k (5.3.34)

$$\beta_k \simeq \int_0^t dt' \, \frac{\frac{d}{dt'}\omega_k^2(t')}{4\omega_k^2(t')} e^{-2i\int_0^{t'} dt'' \omega_k(t'')}, \tag{8.2.91}$$

where

$$\omega_k^2(t) = k^2 + m_\chi^2(t). \tag{8.2.92}$$

Here we simply parameterize the time-dependent mass by

$$m_\chi^2(t) = \begin{cases} \Delta m_\chi^2 \sin^2(m_f t) & (0 < t < \pi/m_f) \\ 0 & (\text{otherwise}) \end{cases}, \tag{8.2.93}$$

with Δm_χ^2 being the height of the spike, $\Delta m_\chi^2 \sim \lambda\xi\Phi^4/M_P^2$. Note that Δm_χ^2 is much smaller than m_f^2. We now show that particles up to $k \sim m_f$ is produced. Focusing on wavenumbers satisfying $k^2 \gg \Delta m_\chi^2$, we may approximate β_k as

$$\begin{aligned}
\beta_k &\simeq \frac{2m_f \Delta m_\chi^2}{k^2} \int_0^{\pi/m_f} dt\ \sin(m_f t)\cos(m_f t) e^{-2ikt} \\
&= \frac{\Delta m_\chi^2}{2k^2(1-k^2/m_f^2)} \left(1 - e^{-2\pi ik/m_f}\right) \\
&\simeq \mathcal{O}(1) \times \begin{cases} \dfrac{\Delta m_\chi^2}{k m_f} & (k \ll m_f) \\ \dfrac{m_f^2 \Delta m_\chi^2}{k^4} & (k \gg m_f) \end{cases}.
\end{aligned} \tag{8.2.94}$$

Thus, in terms of the number density per logarithmic wavenumber $k^3|\beta_k|^2$, the contribution from $k \sim m_f$ dominates. Note that this indicates that particles with masses $\sim m_f$ can be copiously produced. The number and energy density are estimated as

$$n_\chi \sim \frac{(\Delta m_\chi^2)^2}{m_f} \sim \frac{\lambda^{3/2}\xi\Phi^6}{M_P^3} \lesssim \frac{\lambda^{3/2}}{\xi^2} M_P^3, \tag{8.2.95}$$

$$\rho_\chi \sim m_f n_\chi \sim \frac{\lambda^2\xi^2\Phi^8}{M_P^4} \lesssim \frac{\lambda^2}{\xi^2} M_P^4, \tag{8.2.96}$$

and thus can be nonnegligible, though we must take into account the production of the inflaton quanta and backreactions in order to obtain the exact amount of particles produced.

8.2.6 Conclusion

In this section we analyzed gravitational effects on inflaton decay in $f(\phi)R$ model with $f \sim 1 + \phi^2$. What we found is as follows:

- Background analysis
  - The model has three different phases in chronological order:
    Phase 0($M_P/\sqrt{\xi} \ll \Phi$) / Phase 1($M_P/\xi \ll \Phi \ll M_P/\sqrt{\xi}$) / Phase 2($\Phi \ll M_P/\xi$). Here Φ is the amplitude of ϕ oscillation.
  - Inflation occurs during Phase 0, while the inflaton oscillates during Phase 1 and 2. In terms of the Einstein frame, Phase 1 corresponds to the era when the inflaton oscillates but the map between the two frames is still given by the second line of Eq. (4.4.46). On the other hand, the map is given by the first line of Eq. (4.4.46) in Phase 2.
  - In Phase 1, the effective mass scale (inverse of the oscillation timescale) of the inflaton is not given by a naive estimation from the potential (8.2.47), but is substantially suppressed. On the other hand, the dynamics is the same as in the minimal setup in Phase 2.

- In Phase 1, the Hubble parameter oscillates with an amplitude the same order as itself: $\delta H \sim H$. The oscillation has two different inverse time scales, m_s (slow) and m_f (fast). The slow scale correlates with the inflaton oscillation timescale (suppressed as mentioned above), while the fast scale corresponds to the timescale with which the inflaton passes through the origin $|\phi| \lesssim M_P/\xi$.

- Particle production
  - Particle production by the slow scale is inefficient, while the fast scale causes particle production up to the mass $\sim \sqrt{\lambda} M_P$, because of the spike-like features in the derivative of the Hubble parameter.

Chapter 9

Gravitational effects on nonminimally coupled inflaton: $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$-type coupling

As we saw in Chapter 4, there are basically two types of nonminimal coupling between a scalar field and gravity which avoid higher derivatives in the equations of motion. One is the coupling to the Ricci scalar, and the other is the one to the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - Rg_{\mu\nu}/2$. The model we consider in this section belongs to the latter. In the language of Horndeski/Galileon theories (4.2.2)–(4.2.5),

$$G_2 = X - V, \qquad G_4 = \frac{M_P^2}{2}, \qquad G_5 = -\frac{\phi}{2M^2}, \tag{9.0.1}$$

where X and V are the kinetic term and potential of the inflaton ϕ, and M is a parameter with mass dimension one. The explicit form of the action is

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}\left(g^{\mu\nu} - \frac{G^{\mu\nu}}{M^2}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + S_M, \tag{9.0.2}$$

where we assume that the potential is monomial for simplicity

$$V = \frac{\lambda}{n}\phi^n. \tag{9.0.3}$$

The difference of this system from the minimal setup, where the Einstein gravity and the canonical inflaton determines the dynamics, is the derivative coupling $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/M^2$. In the context of inflation, the effect of this nonminimal derivative coupling is clear. It makes the potential "shallower", suppressing the predicted value of the tensor-to-scalar ratio, see Chapter 4. On the other hand, for the matter part, we do not introduce nonminimal couplings throughout the chapter.

We analyze the inflaton oscillation regime of this model. The nonminimal coupling $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/M^2$ takes the value $3H^2\dot{\phi}^2/M^2$ at the background level, and the magnitude of the dimensionless combination H^2/M^2 determines whether nontrivial dynamics occurs to this model.

We summarize the main results we obtain from now:

- Background analysis
  - The background evolution of the model has three phases in chronological order: Phase 0 / Phase 1 / Phase 2.
    Inflation occurs during Phase 0, while the inflaton oscillates during Phase 1 & 2.
  - The difference in Phase 1 and 2 comes from the nonminimal contribution H^2/M^2 to the kinetic term, which is $\gg 1$ during Phase 1 while it is $\ll 1$ during Phase 2. Violent oscillation of the Hubble parameter occurs in correlation with the inflaton oscillation during Phase 1, while the dynamics is basically the same as the minimal setup in Phase 2 except for the beginning of this phase.
- Instability in perturbations
  - In contrast to $f(\phi)R$-type models, $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$-type model suffers from an instability of the scalar perturbation, which is triggered by the violent oscillation of the Hubble parameter.
  - The instability is called "gradient instability", in which the sound speed squared of the scalar perturbation becomes negative. Due to the instability, scalar perturbations with higher momentum modes are more likely to grow, and the background oscillation is likely to cease because the explosive particle production carries away the energy. However, we need UV completion of the theory in order to give conclusive remarks on this point.
- Particle production
  - If we ASSUME that the background oscillation somehow survives the gradient instability, resonant production occurs to gravitons, while no such production occurs to a canonical scalar field.

This chapter is based on the works [128] with K. Mukaida and K. Nakayama and [129] with Y. Ema, K. Mukaida and K. Nakayama.

9.1 Background analysis

We first see the background dynamics, neglecting particle production. The actual dynamics of the model including particle production can be different from the time evolution obtained here because the produced particles carry away the energy from the background oscillation. However, we have to see the background evolution in the first place in order to estimate the particle production caused by the background dynamics.

First we derive the background action, equations of motion and the adiabatic invariant. A qualitative understanding of the time evolution is shown at the same time. Next we explain the three phases which appears in this model. Our main focus is Phase 1, when the inflaton oscillates with the nonminimal coupling $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/M^2$ being effective.

9.1.1 Action, EOM and adiabatic invariant

The background action is derived from Eq. (9.0.2) as

$$S = \int d^4x \, a^3 \left[-3M_P^2 H^2 + \left(1 + \frac{3H^2}{M^2}\right)\frac{\dot{\phi}^2}{2} - V(\phi) \right], \tag{9.1.4}$$

where we denote the background part of the inflaton by the same symbol ϕ as the full inflaton field. Assuming the FRW metric with negligible curvature, the equations of motion are

- Friedmann equation

$$3M_P^2 H^2 = \rho_\phi \tag{9.1.5}$$

- Raychaudhuri equation

$$M_P^2(2\dot{H} + 3H^2) = -p_\phi. \tag{9.1.6}$$

- ϕ's equation of motion

$$\left(1 + \frac{3H^2}{M^2}\right)\ddot{\phi} + 3H\left(1 + \frac{3H^2}{M^2} + \frac{2\dot{H}}{M^2}\right)\dot{\phi} + V' = 0, \tag{9.1.7}$$

where the energy density and pressure are given by[#1]

$$\rho_\phi = \left(1 + \frac{9H^2}{M^2}\right)\frac{\dot{\phi}^2}{2} + V, \tag{9.1.8}$$

$$p_\phi = \left(1 - \frac{3H^2}{M^2}\right)\frac{\dot{\phi}^2}{2} - V - \frac{1}{M^2}\frac{d}{dt}(H\dot{\phi}^2). \tag{9.1.9}$$

Friedmann equation (9.1.5) is derived by introducing the lapse function as $dt \to N dt$, taking variation with respect to N and setting N to 1. Also, Raychaudhuri equation (9.1.6) and ϕ's equation of motion (9.1.7) are derived by the variation with respect to the scale factor and ϕ, respectively. Note that Eq. (9.1.6) can also be obtained from the other two equations. Also, $\dot{H}$ is calculated by eliminating $\ddot{\phi}$ from the time derivative of Eqs. (9.1.5) and (9.1.6):

$$\frac{\dot{H}}{M^2} = -\frac{(1+3h^2)(1+9h^2)\frac{\epsilon}{2} + h\frac{\dot{V}}{M^3 M_P^2}}{(1+3h^2) - (1-9h^2)\frac{\epsilon}{2}}, \tag{9.1.10}$$

[#1] This definition of the energy density and pressure is rather artificial, because the inflaton and gravity are mixed in this model. This definition is made from the viewpoint of regarding the nonminimal coupling as a correction for the kinetic term of the inflaton.

where

$$h \equiv \frac{H}{M}, \qquad \epsilon \equiv \frac{\dot{\phi}^2}{M^2 M_P^2} < \frac{2}{3}. \tag{9.1.11}$$

are introduced for notational simplicity. The inequality $\epsilon < 2/3$ holds because the Friedmann equation is written as

$$3M_P^2 H^2 \left(1 - \frac{3}{2}\epsilon\right) = \frac{\dot{\phi}^2}{2} + V, \tag{9.1.12}$$

where the RHS is positive. In the actual dynamics ϵ almost hits the upper bound 2/3 for small M[#2]. This may be understood in comparison with the case where the nonminimal coupling is absent. Without the nonminimal coupling, $\dot{\phi}^2$ increases up to the value $\sim \lambda\Phi^n$ as the inflaton rolls down to the potential minimum, where Φ is the amplitude of ϕ oscillation. However, the condition (9.1.11) prohibits the inflaton velocity $\dot{\phi}$ from blowing up beyond MM_P. In this sense the parameter M works as a regulator for the inflaton velocity.

Let us understand the dynamics of the model qualitatively. During inflation, Eq. (9.1.7) reduces to

$$3H\left(1 + \frac{3H^2}{M^2}\right)\dot{\phi} + V' \simeq 0. \tag{9.1.13}$$

Note that $\dot{H}$ is negligible compared to H^2 during inflation, though this statement does not hold in the inflaton oscillation regime. Eq. (9.1.13) should be compared with the inflaton dynamics in minimal setup, Eq. (3.2.19). One finds that the nonminimal contribution effectively lowers the potential by a factor of $\sim H^2/M^2$ if $H/M \gg 1$. On the other hand, the oscillation of the inflaton field is basically determined by the following terms

$$\left(1 + \frac{3H^2}{M^2}\right)\ddot{\phi} + V' \simeq 0, \tag{9.1.14}$$

as long as we consider the dynamics over a timescale not much longer than the oscillation timescale. Thus the potential for the inflaton is again effectively lowered by a factor $\sim H^2/M^2$ if $H/M \gg 1$. From these considerations, we define the effective mass of the inflaton

$$m_{\text{eff}} \equiv \begin{cases} \dfrac{M}{H}\sqrt{\dfrac{V'(\Phi)}{\Phi}} = \dfrac{M}{H}\sqrt{\lambda}\Phi^{\frac{n}{2}-1} & \left(\dfrac{H}{M} \gg 1\right) \\ \sqrt{\dfrac{V'(\Phi)}{\Phi}} = \sqrt{\lambda}\Phi^{\frac{n}{2}-1} & \left(\dfrac{H}{M} \ll 1\right) \end{cases}, \tag{9.1.15}$$

where Φ denotes the amplitude of ϕ oscillation. Inflation occurs when the Hubble parameter dominates over this effective mass $H > m_{\text{eff}}$, while the oscillation regime occurs when it drops below the mass $H < m_{\text{eff}}$. Notice that the combination $m_{\text{eff}}\Phi \propto \Phi^{n/2}/H$ remains constant in time for $H/M \gg 1$, because the Friedmann equation (9.1.12) reads $M_P^2 H^2 \sim \lambda\Phi^n$. Therefore

[#2] When M is smaller than M_c defined below.

m_{eff} is an increasing function of time during this regime. We sometimes write m_{eff} as m_ϕ for $n = 2$.

The oscillation of the Hubble parameter in this model, which we will see below, may be understood with the help of the adiabatic invariant

$$J \equiv -\frac{1}{6M_P^2}\mathcal{L}_H = H\left(1 - \frac{\dot{\phi}^2}{2M_P^2 M^2}\right). \tag{9.1.16}$$

Taking into account that J has only small oscillations $\dot{J} \sim HJ$, one sees that the oscillation of the Hubble parameter correlates with $\dot{\phi}^2$. In addition, from Eq. (9.1.11) one finds that the quantity inside the parenthesis oscillates between 2/3 and 1. This feature is observed in the numerical results shown below. Note that this means that the oscillation amplitude of the Hubble parameter has the same order of magnitude as the Hubble parameter itself, $\delta H \sim H$.

9.1.2 Phase analysis

We first classify the phases of the model into Phase 0–2 formally:

- Phase 0 : Inflation
- Phase 1 : Oscillation with $H^2/M^2 > 1$
- Phase 2 : Oscillation with $H^2/M^2 < 1$

It depends on the parameter M whether Phase 1 exists or not, while Phase 2 exists for any value of M after the Hubble parameter sufficiently decreases. To find the condition for Phase 1 to exist, let us consider Phase 0 and see the condition for the parameter M to satisfy $H^2/M^2 > 1$ at the end of it. We assume $H^2/M^2 > 1$ during inflation and derive a consistency condition. Since $3M_P^2 H^2 \simeq V$ in the inflationary regime, the inflaton slow-roll (9.1.13) is written as

$$3H\,\frac{3H^2}{M^2}\dot{\phi} + V' \simeq 0 \quad \rightarrow \quad 3H\,\frac{V}{M^2 M_P^2}\dot{\phi} + V' \simeq 0. \tag{9.1.17}$$

The inflaton slow-roll (9.1.17) is made canonical

$$3H\dot{\phi}_c + \frac{dV}{d\phi_c} \simeq 0, \tag{9.1.18}$$

by the field redefinition

$$\phi_c \equiv \left(\frac{\lambda}{n}\right)^{\frac{1}{2}} \frac{2}{n+2}\frac{\phi^{\frac{n+2}{2}}}{MM_P}, \qquad V(\phi_c) \equiv \left(\frac{\lambda}{n}\right)^{\frac{2}{n+2}} \left(\frac{n+2}{2}\right)^{\frac{2n}{n+2}} (MM_P)^{\frac{2n}{n+2}} \phi_c^{\frac{2n}{n+2}}. \tag{9.1.19}$$

Note that this redefinition is equivalent to the one which makes the Lagrangian canonical

$$\mathcal{L} \simeq -\frac{3H^2}{2M^2}(\partial\phi)^2 - V(\phi) \simeq -\frac{1}{2}(\partial\phi_c)^2 - V(\phi_c). \tag{9.1.20}$$

Also note that the field redefinition reads $\phi_c \sim (H/M)\phi$. The slow-roll parameters are estimated as

$$\epsilon_{V_c} \equiv \frac{1}{2} M_P^2 \left(\frac{V_{\phi_c}}{V} \right)^2 = \frac{2n^2}{(n+2)^2} \frac{M_P^2}{\phi_c^2} = \frac{n}{2(n+2)} \frac{1}{N}, \tag{9.1.21}$$

$$\eta_{V_c} \equiv M_P^2 \frac{V_{\phi_c \phi_c}}{V} = \frac{2n(n-2)}{(n+2)^2} \frac{M_P^2}{\phi_c^2} = \frac{n-2}{2(n+2)} \frac{1}{N}, \tag{9.1.22}$$

where the subscript ϕ_c denotes the derivative with respect to it, and the e-folding N is given by

$$N = \frac{n+2}{4n} \frac{\phi_c^2}{M_P^2}. \tag{9.1.23}$$

Evaluating the end of slow-roll by $\epsilon_{V_c} = 1$, we have the value of the inflaton at that time

$$\phi_{c,\mathrm{end}} = \frac{\sqrt{2}n}{n+2} M_P. \tag{9.1.24}$$

Requiring that the nonminimal contribution H/M is still effective at the end of slow-roll, i.e. $V(\phi_{c,\mathrm{end}})/M^2 M_P^2 > 1$, the parameter M must be smaller than the critical value

$$M_c \equiv \left(\frac{\lambda}{n} \right)^{\frac{1}{2}} \left(\frac{\sqrt{2}n}{n+2} \right)^{\frac{n}{2}} M_P^{\frac{n}{2}-1}, \tag{9.1.25}$$

while in the opposite case Phase 2 begins as soon as Phase 0 ends. Thus we are led to classify two cases:

- Case A : $M > M_c$

  Time evolution follows

  - Phase 0 : Inflation ($H < m_{\mathrm{eff}}$)
  - Phase 2 : Oscillation ($H < m_{\mathrm{eff}}$) with $H/M < 1$

- Case B : $M < M_c$

  Time evolution follows

  - Phase 0 : Inflation ($H > m_{\mathrm{eff}}$)
  - Phase 1 : Oscillation ($H < m_{\mathrm{eff}}$) with $H/M > 1$
  - Phase 2 : Oscillation ($H < m_{\mathrm{eff}}$) with $H/M < 1$

If we keep particle production in mind, what we are interested in is the oscillation of the Hubble parameter. It satisfies $\dot{H} \sim H^2$ for the minimal setup, while the nonminimal coupling sometimes brings about a violent oscillation $\dot{H} \sim m_{\mathrm{eff}} H$, as we saw in $f(\phi)R$-type models.

The derivative of the Hubble parameter in the present model is estimated from Eq. (9.1.10) as[#3]

$$\frac{\dot{H}}{M^2} \sim \begin{cases} \dfrac{m_{\rm eff} H}{M^2} & \text{(Phase1)} \\ \dfrac{H^2}{M^2} + \dfrac{m_{\rm eff} H^3}{M^4} & \text{(Phase2)} \end{cases} . \tag{9.1.28}$$

In Case A, where only Phase 2 exists at the inflaton oscillation regime, the first term dominates in the second line of Eq. (9.1.28)[#4]. Therefore the dynamics is the same as the minimal setup, $\dot{H} \sim H^2$. In Case B, on the other hand, the dynamics in Phase 2 needs some explanation. In this case the second term dominates for some period after Phase 2 sets in. Therefore, strictly speaking, Phase 2 must be further classified by which term dominates in Eq. (9.1.28) (Phase 2-2 $\to$ 2-3). In addition, the value of $\dot{H}/M^2$ itself must be compared to unity because of the reason which soon becomes clear (Phase 2-1 $\to$ 2-2). This transition occurs during the second term dominates the expression (9.1.28). However, we restrict ourselves just to mention Phase 2 in footnote [#5], since the most interesting regime is Phase 1. The behavior of the Hubble parameter during Phase 1 is $\dot{H} \sim m_{\rm eff} H$, as expected. The oscillation amplitude of $\dot{H}/M^2$ far exceeds unity, because of the hierarchy $m_{\rm eff} > H > M$.

Fig. 9.1 shows the time evolution of this model obtained by numerical calculation. The parameters are chosen to satisfy the condition for Phase 1 to exist. The violent oscillation of the Hubble parameter is observed in Phase 1, while in Phase 2 it approaches the same value as predicted in the minimal setup. Note that the fractional amplitude of oscillation is consistent with the estimation obtained by the adiabatic invariant below Eq. (9.1.16). Also note that the oscillation timescale, the inverse of $m_{\rm eff}$, gets shorter and shorter in Phase 1. This is because $m_{\rm eff}$ is an increasing function of time in this phase.

[#3]For Phase 1,

$$\frac{\dot{H}}{M^2} \sim \frac{h^4 \epsilon + h \frac{\dot{V}}{M^3 M_P^2}}{h^2} \sim \frac{h^4 + h \frac{m_{\rm eff} V}{M^3 M_P^2}}{h^2} \sim h^2 + h \frac{m_{\rm eff}}{M} \sim h \frac{m_{\rm eff}}{M}, \tag{9.1.26}$$

where in the second equality we used the fact that ϵ is not much smaller than 1 in this regime, since the kinetic energy from the nonminimal coupling $\sim \dot{\phi}^2 H^2/M^2$ has roughly the same order as $M_P^2 H^2$. The last equality follows from $H < m_{\rm eff}$.
For Phase 2,

$$\frac{\dot{H}}{M^2} \sim \frac{\epsilon + h \frac{\dot{V}}{M^3 M_P^2}}{1} \sim h^2 + h^3 \frac{m_{\rm eff}}{M}, \tag{9.1.27}$$

where $\epsilon \sim h^2$ is used in the second equality.

[#4]This is because the ratio of the two terms $m_{\rm eff} H/M^2$ is bounded above by its value at the beginning of inflaton oscillation $H \sim m_{\rm eff}$, and therefore satisfies $m_{\rm eff} H/M^2 \lesssim H^2/M^2 \lesssim 1$. Note that $m_{\rm eff}$ is a decreasing function of time in Case A.

[#5]Since $m_{\rm eff} H/M^2 \gg 1$ and $H^2/M^2 \ll 1$ in Eq. (9.1.28), the transition Phase 2-1 $\to$ 2-2 ($\dot{H}/M^2 \sim 1$) occurs when $m_{\rm eff} H^3/M^4$ dominates. Therefore this transition is before the transition Phase 2-2 $\to$ 2-3, when $H^2/M^2 \sim m_{\rm eff} H^3/M^4$.

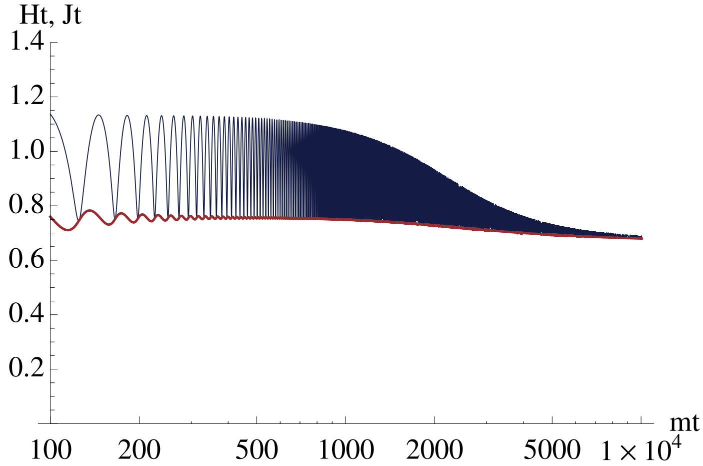

Figure 9.1: Plot of Ht (blue) and Jt (red). Parameter is taken to be $M/m_\phi = 10^{-3}$, and initial conditions are $\phi_{\rm ini} = \sqrt{M/m_\phi}M_P$, $\dot{\phi}_{\rm ini} = 0$ at $m_\phi t_{\rm ini} = 50$. Ht oscillates violently while Jt has only a suppressed amplitude of oscillation. Also, the fractional amplitude of oscillation confirms the estimation below Eq. (9.1.16).

Table 9.1: Typical values of H, $\dot{H}$ and $|c_s^2|$ during each phase in Case B. Phase 0, 1 and 2 refers to inflation, inflaton oscillation with $H > M$, inflaton oscillation with $H < M$, respectively. Phase 2-1 → 2-2 corresponds to the transition $\dot{H}/M^2 \gg 1 \to \dot{H}/M^2 \sim 1$, while Phase 2-2 → 2-3 corresponds to the transition when the first term starts to dominate in the second line of Eq. (9.1.28).

Phase	H	$\dot{H}$	$\lvert c_s^2\rvert$	Explanation
Phase 0				Inflation
Phase 0 → 1	$m_{\rm eff}$	$m_{\rm eff}H$	$\gg 1$	Inflation → Oscillation
Phase 1			$\gg 1$	Oscillation with $H > M$
Phase 1 → 2	M	$m_{\rm eff}H$	$\gg 1$	$H > M \to H < M$
Phase 2-1			$\gg 1$	Oscillation with $H < M$
Phase 2-1 → 2-2	$(M^4/m_{\rm eff})^{1/3}$	M^2	$\gg 1 \to \sim 1$	Oscillation with $H < M$
Phase 2-2			~ 1	Oscillation with $H < M$
Phase 2-2 → 2-3	$M^2/m_{\rm eff}$	$M^2 \to H^2$	~ 1	Oscillation with $H < M$
Phase 2-3			~ 1	Oscillation with $H < M$

9.1.3 Expansion law

The expansion law in Phase 1 is derived by the method illustrated in Chapter 6. Since the nonminimal coupling is effective during this phase, the Lagrangian is approximately written as

$$\mathcal{L} \simeq \frac{1}{2} H \mathcal{L}_H + \frac{1}{n} \phi \mathcal{L}_\phi. \tag{9.1.29}$$

Apart from this, what we need to derive the averaged expansion law are the Friedmann equation

$$\mathcal{L} - H\mathcal{L}_H - \dot{\phi}\mathcal{L}_{\dot{\phi}} = 0, \tag{9.1.30}$$

and the Virial theorem

$$\langle \phi \mathcal{L}_\phi \rangle + \left\langle \dot{\phi} \mathcal{L}_{\dot{\phi}} \right\rangle = 0, \tag{9.1.31}$$

where the subscript denotes the derivative with respect to that quantity, and the angular bracket denotes oscillation average. From these three equations we have

$$\langle \mathcal{L} \rangle = \frac{n+2}{2n+2} \langle H \mathcal{L}_H \rangle . \tag{9.1.32}$$

Since the time evolution of the adiabatic invariant is given by

$$\dot{\mathcal{L}}_H + 3H\mathcal{L}_H - 3\mathcal{L} = 0, \tag{9.1.33}$$

we have

$$\left\langle \dot{\mathcal{L}}_H \right\rangle + \frac{3n}{2n+2} \langle H \rangle \mathcal{L}_H = 0, \tag{9.1.34}$$

after eliminating $\langle \mathcal{L} \rangle$. Then

$$\mathcal{L}_H \propto a^{-\frac{3n}{2n+2}}. \tag{9.1.35}$$

Since $\mathcal{L}_H$ is proportional to $\langle H \rangle$[#6], the expansion law can be derived as

$$\langle H \rangle = \frac{2n+2}{3n} \frac{1}{t}. \tag{9.1.36}$$

Fig. 9.1 confirms this estimation.

On the other hand, the expansion law in Phase 2 is given by the prediction with a canonical inflaton,

$$H = \frac{n+2}{3n} \frac{1}{t}. \tag{9.1.37}$$

[#6] This may be understood from Eq. (9.1.16) as well, where the second term in the RHS oscillates only between 0 and 1/3.

9.2 Perturbation analysis

As we have seen, the nonminimal coupling causes a violent oscillation in the Hubble parameter $\dot{H} \sim m_{\text{eff}} H$ in Phase 1. Let us consider the effect of this oscillation mode on the scalar and tensor perturbations. We refer to [83–86] for the cosmological perturbations in this model.

9.2.1 Scalar perturbation

We use the ADM metric

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{9.2.38}$$

Focusing on scalar perturbations we adopt the decomposition and gauge condition

$$N = 1 + \alpha, \qquad \beta_i = \gamma_{ij}\beta^i = \partial_i \beta, \qquad \gamma_{ij} = a^2 e^{2\zeta}\delta_{ij}, \tag{9.2.39}$$

and

$$\phi = \bar{\phi}. \tag{9.2.40}$$

Note that the gauge condition we adopt here differs from the one we adopt in Sec. 8.1, where ψ is set to 0 while $\varphi \equiv \phi - \bar{\phi}$ is nonvanishing. Calculating the quadratic action and deriving the two constraint equations for the three perturbations#7, we obtain the relation among the perturbations

$$\alpha = \frac{A}{H}\dot{\zeta}, \qquad \beta = -\frac{A}{H}\zeta + \xi, \qquad \partial_i^2 \xi = a^2 \frac{\dot{\phi}^2}{2M_P^2 H^2}\frac{A^2 B}{1 - \frac{\epsilon}{2}}\dot{\zeta}, \tag{9.2.41}$$

where

$$A \equiv \frac{1 - \frac{1}{2}\epsilon}{1 - \frac{3}{2}\epsilon}, \qquad B \equiv 1 + \frac{3H^2}{M^2}\frac{1 + \frac{3}{2}\epsilon}{1 - \frac{1}{2}\epsilon}, \tag{9.2.42}$$

are quantities which approach 1 for $M \to \infty$. Note that in Eq. (9.2.41) ξ is an auxiliary quantity, which should not be confused with the nonminimal coupling ξ in Chapter 8. After substituting these relations into the quadratic action, we obtain the action for ζ

$$S_\zeta = \int d^4x \, a^3 A^2 B \frac{\dot{\phi}^2}{H^2}\left[\frac{1}{2}\dot{\zeta}^2 - \frac{1}{2}\frac{c_s^2}{a^2}(\partial_i \zeta)^2\right], \tag{9.2.43}$$

where the sound speed squared c_s^2 is given by

$$c_s^2 = 1 + \frac{2}{B}\left[\frac{\epsilon}{1 - \frac{\epsilon}{2}}\left(1 + \frac{H^2}{M^2}\frac{1}{A}\right) + \frac{3\dot{H}}{M^2}\right]. \tag{9.2.44}$$

#7Refer to Table 4.2, or Table B.1–B.3, or the discussion based on $3+1$ decomposition in Appendix D for the derivation of these constraint equations.

Note here that the combination A^2B in the action (9.2.43) is positive definite because $\epsilon < 2/3$, and therefore no ghost mode appears in the scalar perturbation. However, one can see that another type of instability could appear in this action. It comes from the term proportional to $\dot{H}/M^2$ in Eq. (9.2.44). Though this quantity is smaller than unity in Case A, in the other case it oscillates with amplitude much larger than unity, taking both positive and negative values[#8]. In addition, from $\dot{H} \sim m_{\rm eff} H \gg H^2$, one sees that this term dominates other terms in the square bracket. Therefore we arrive at the following observation for the scalar sound speed squared.

- Case A

  From $\dot{H} \sim H^2$, we have

$$c_s^2 = 1 + \mathcal{O}\left(\frac{H^2}{M^2}\right). \tag{9.2.45}$$

  Since $H < M$, no gradient instability $c_s^2 < 0$ appears.

- Case B

  In Phase 1, the term $\dot{H}/M^2$ dominates the sound speed squared

$$c_s^2 \sim \frac{\dot{H}}{H^2} \qquad (M < H), \tag{9.2.46}$$

  and it oscillates between positive and negative values. Even in Phase 2 such behavior occurs, during the period when the second term in Eq. (9.1.28) exceeds unity (and thus dominates over the first term)

$$c_s^2 \sim \frac{\dot{H}}{M^2} \qquad \left((M^4/m_{\rm eff})^{1/3} < H < M\right). \tag{9.2.47}$$

  Here the lower bound for H is derived by comparing the second term in Eq. (9.1.28) with 1. To infer the effects of gradient instability, let us consider a simplified toy model

$$S = \int d^4x \left[\frac{1}{2}\dot{\zeta}^2 - \frac{c_s^2}{2}(\partial_i \zeta)^2\right], \tag{9.2.48}$$

  with $c_s^2 < 0$. The mode equation for ζ field and its solution become

$$\ddot{\zeta}_k + c_s^2 k^2 \zeta_k = 0 \quad \rightarrow \quad \zeta_k \propto e^{\pm|c_s|kt}. \tag{9.2.49}$$

  Therefore the scalar perturbation is likely to blow up. In addition, the growth rate is more likely to be enhanced for the modes with larger k. This requires the knowledge of UV completion of the model in order to discuss the consequences of gradient instability. Though the true dynamics when this gradient instability occurs is beyond the scope

[#8] It is known that gradient instability also occurs in other models of Horndeski/Galileon theories [130].

of this thesis, we can infer that the timescale for blowup is much shorter than the oscillation period of the inflaton. Taking k to be the strong coupling scale of the model $k \sim (M_P H^2)^{1/3}$ [22], substituting the expression $|c_s| \sim \dot{H}/H^2 \sim m_{\rm eff}/H$ for Phase 1, the blowup timescale for this mode becomes

$$t_{\rm blow} \sim \frac{1}{|c_s|k} \sim \frac{1}{m_{\rm eff}} \left(\frac{H}{M_P}\right)^{1/3}. \tag{9.2.50}$$

Note that this timescale is already below $m_{\rm eff}^{-1}$ at the end of inflation, and in addition it drops faster than $m_{\rm eff}^{-1}$. The analysis after the blowup of ζ is difficult because of the nonlinearity.

To summarize, gradient instability occurs in Case B while no such behavior occurs in Case A. This means that Case A is required in order to avoid the instability at the end of inflation[#9].

9.2.2 Tensor perturbation

We again use the ADM metric (9.2.38), where the gravitons live in the spacial metric γ_{ij} as

$$\gamma_{ij} = a^2 (e^h)_{ij}, \tag{9.2.51}$$

where h_{ij} satisfies the transverse and traceless condition $\partial_j h_{ij} = h_{ii} = 0$. From the expansion of the Ricci scalar and the Einstein tensor up to second order in tensor perturbation

$$R = 6(2H^2 + \dot{H}) + \frac{1}{4}\left[\dot{h}_{ij}^2 - a^{-2}(\partial_k h_{ij})^2\right], \tag{9.2.52}$$

$$G^{00} = 3H^2 - \frac{1}{8}\left[\dot{h}_{ij}^2 + a^{-2}(\partial_k h_{ij})^2\right], \tag{9.2.53}$$

we obtain the quadratic action for gravitions

$$S_h = \int d^4x \, a^3 \frac{M_P^2}{8} \left[\left(1 - \frac{\epsilon}{2}\right)\dot{h}_{ij}^2 - a^{-2}\left(1 + \frac{\epsilon}{2}\right)(\partial_k h_{ij})^2\right]. \tag{9.2.54}$$

Note that the sign difference in the factors $1 \pm \epsilon/2$ comes from the opposite signs in Eqs. (9.2.52) and (9.2.53) between $\dot{h}_{ij}^2$ and $(\partial_k h_{ij})^2$. One may note two things

- Though the two factors $1 \pm \epsilon/2$ oscillate, no ghost (negative sign for $\dot{h}_{ij}^2$) nor gradient instability (positive sign for $(\partial_k h_{ij})^2$) appears because $\epsilon < 2/3$.
- Superluminality ($c_s^2 > 1$) appears, because $\epsilon > 0$.

[#9] Note that this by no means states that the existence of instability itself contradicts with the evolution of the universe or present cosmological observations. The gradient instability may turn out not to be harmful after we understand the physics of the instability. However, it is beyond the scope of the thesis to make a conclusive remark on this point.

9.3 Particle production

Let us discuss particle production by the oscillating Hubble parameter in the present model. We focus only on the production of scalar particles and gravitons, since that for fermion and massless vector boson are suppressed by their masses, as we saw in Chapter 7. Since the dynamics differs between Case A and B, we discuss both cases separately. It must be noted that in Case B there exists gradient instability, on the consequences of which we avoided to make conclusive remarks. The estimation for the particle production below may be invalidated in Case B if the particles produced by this instability carry away the energy from the background oscillation and the oscillation soon disappears. In other words, the analysis for Case B below is based on the ASSUMPTION that the violent oscillation of the Hubble parameter somehow survives the gradient instability.

Before going into details of each particle species, we must consider what is the good measure for particle production, especially for $H/M \gg 1$. The production rate of light particles $\dot{n}_\chi$ is a good one, because the increase in the number density of χ particle is well-defined. However, if one uses the decay rate of the inflaton field, ambiguity arises as to what is the number or energy density of the field. Since the potential energy of the inflaton field in the present model remains unchanged from the minimal setup, we may use it to normalize the energy density of the produced particles. For that purpose we extend the definition of the canonical inflaton field for both $H \gg M$ and $H \ll M$[#10]

$$\phi_c \equiv \begin{cases} \dfrac{H}{M}\phi & \left(\dfrac{H}{M} \gg 1\right) \\ \phi & \left(\dfrac{H}{M} \ll 1\right) \end{cases}. \tag{9.3.55}$$

With the definition of m_{eff} (9.1.15) the potential energy of the inflaton field becomes

$$V(\Phi) \sim m_{\text{eff}}^2 \Phi_c^2, \tag{9.3.56}$$

where Φ_c is the amplitude of ϕ_c oscillation. From this definition of the canonical inflaton field and the fact that the energy of the produced particles is centered around $\sim m_{\text{eff}}$, we adopt

$$\Gamma_{\phi\to\chi} \sim \frac{\dot{\rho}_\chi}{\rho_\phi} \sim \frac{m_{\text{eff}}\dot{n}_\chi}{m_{\text{eff}}^2\Phi_c^2} \sim \frac{\dot{n}_\chi}{m_{\text{eff}}\Phi_c^2}, \tag{9.3.57}$$

as the good measure for the inflaton decay rate. This is equivalent to call $m_{\text{eff}}\Phi_c^2$ the inflaton number density n_ϕ.

9.3.1 Scalar particle production

We consider the canonical light scalar field χ

$$S_S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\chi)^2\right]. \tag{9.3.58}$$

[#10] This definition differs by some $\mathcal{O}(1)$ factor from the previous one for $H \gg M$, but we do not need to care about that in the following discussion.

Taking only the background part of the metric, we have

$$S_S = \int d\tau d^3x \, a^2 \left[\frac{1}{2}\chi'^2 - \frac{1}{2}(\partial_i \chi)^2 \right]. \tag{9.3.59}$$

From the discussion in Chapter 5, the canonical field $\tilde{\chi} \equiv a^{3/2}\chi$ has an oscillating mass term $m_\chi^2 \sim \ddot{a}/a \sim \dot{H}$. Here we took into account only $\dot{H}$ and neglected H^2, since latter is at most comparable to the former. The amplitude of the oscillation Δm_χ^2 and the resonance parameter q is estimated as

$$\Delta m_\chi^2 \sim \dot{H}, \qquad q^{(\chi)} \sim \frac{\Delta m_\chi^2}{m_{\text{eff}}^2} \sim \frac{\dot{H}}{m_{\text{eff}}^2}, \tag{9.3.60}$$

where $\dot{H}$ should be understood to symbolically denote the amplitude of the oscillation of $\dot{H}$. The production rate is estimated as

$$\dot{n}_\chi \sim (\Delta m_\chi^2)^2 \sim \dot{H}^2, \tag{9.3.61}$$

$$\Gamma_{\phi\to\chi} \sim \frac{\dot{n}_\chi}{n_\phi} \sim \frac{q^{(\chi)2} m_{\text{eff}}^3}{\Phi_c^2}, \tag{9.3.62}$$

as explained in Chapter 5. Note that m_ϕ and Φ in that section is now replaced by m_{eff} and Φ_c. Also note that the subscript $\phi \to \chi$ does not refer to the number of particles present in the process. Since $q^{(\chi)}$ is at most H/m_{eff}, the resonance condition R_q given in Eq. (5.5.66) is given by

$$R_q \equiv q^{(\chi)2} \frac{m_{\text{eff}}}{H} < \frac{H}{m_{\text{eff}}}. \tag{9.3.63}$$

Therefore no resonance occurs for scalar particles. Below we summarize the production and decay rate written in terms of m_{eff}, Φ, M and M_P.

- Case A

  Since $\dot{H} \sim H^2$, the production rate (9.3.62) is estimated as

  $$\Gamma_{\phi\to\chi} \sim \frac{m_{\text{eff}}^3 \Phi^2}{M_P^4}, \tag{9.3.64}$$

  which is the same rate as is found in Eq. (7.2.28). Thus the production of χ reduces to the minimal setup in Case A.

- Case B

Using Eqs. (9.1.28) and (9.3.60), the production rate (9.3.62) is estimated as

$$\Gamma_{\phi\to\chi} \sim \begin{cases} \dfrac{m_{\text{eff}} M^2}{\Phi^2} & \text{Phase 1 } (M < H) \\ \dfrac{m_{\text{eff}}^7 \Phi^4}{M^4 M_P^6} & \text{Phase } 2-1 \text{ \& } 2-2 \ (M^2/m_{\text{eff}} < H < M) \\ \dfrac{m_{\text{eff}}^3 \Phi^2}{M_P^4} & \text{Phase } 2-3 \ (H < M^2/m_{\text{eff}}) \end{cases} \cdot \tag{9.3.65}$$

9.3.2 Graviton production

For graviton production, there are two contributions: the oscillation in the scale factor (or Hubble parameter and its derivative), and the direct coupling between $\dot{\phi}^2$ and graviton quadratic term, see the action (9.2.54). Since the former gives the same contribution as the scalar particle production, we only take the latter into account. We consider the action

$$S_h = \int d^4x \left[\frac{1}{2}\left(1 - \frac{\epsilon}{2}\right) \dot{h}_{ij}^2 - \frac{1}{2}\left(1 + \frac{\epsilon}{2}\right) (\partial_k h_{ij})^2. \right], \tag{9.3.66}$$

where $\epsilon \equiv \dot{\phi}^2/M^2 M_P^2 < 2/3$. Here we rescaled gravitons as $M_P h_{ij}/2 \to h_{ij}$. By the following redefinition of the time variable, this action is made to have the same form as Eq. (5.7.72)

$$dt' = \sqrt{\frac{1+\frac{\epsilon}{2}}{1-\frac{\epsilon}{2}}} dt, \tag{9.3.67}$$

$$S_h = \int dt' d^3x \sqrt{1 - \frac{\epsilon^2}{4}} \left[\frac{1}{2}\left(\frac{\partial h_{ij}}{\partial t'}\right)^2 - \frac{1}{2} (\partial_k h_{ij})^2 \right]. \tag{9.3.68}$$

Then we may estimate the amplitude of the oscillating mass from Eq. (5.7.75), and

$$\Delta m_h^2 \sim \frac{d^2}{dt'^2}\left(1 - \frac{\epsilon^2}{4}\right)^{1/4} \Big/ \left(1 - \frac{\epsilon^2}{4}\right)^{1/4} \sim \frac{d^2}{dt'^2}\epsilon^2 \sim m_{\text{eff}}^2 \epsilon^2, \tag{9.3.69}$$

where we used the fact that ϵ is a rapidly oscillating function with inverse timescale m_{eff}[#11]. Note that we have neglected all the numerical coefficients. Then we have

$$q_{\text{dir}}^{(h)} \sim \frac{\Delta m_h^2}{m_{\text{eff}}^2} \sim \epsilon^2. \tag{9.3.70}$$

[#11] The oscillation of ϵ must be evaluated by the time variable t', not t, because the action is now written in terms of t'. However, the difference between dt' and dt is as most $\sim \sqrt{2}$ as seen from Eq. (9.3.67), and therefore we may safely neglect the difference here.

Here "dir" denotes that the contribution comes from the direct nonminimal coupling $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, not from the oscillation of the overall scale factor as we saw in χ production. It should be noted here that $\epsilon \sim 1$ during Phase 1 while it drops after the onset of Phase 2. Since the oscillating scale factor also contributes to the resonance parameter $q^{(h)}$, and this contribution is the same as $q^{(\chi)}$, we may write

$$q^{(h)} = \max\left(q^{(\chi)}, q_{\rm dir}^{(h)}\right). \tag{9.3.71}$$

The decay rate into gravitons is estimated from

$$\Gamma_{\phi\to h} \sim \frac{q^{(h)2} m_{\rm eff}^3}{\Phi_c^2}. \tag{9.3.72}$$

We summarize graviton production.

- Case A

  In this case $q^{(\chi)}$ dominates over $q_{\rm dir}^{(h)}$[#12], and the decay rate is the same as that into scalar particles, Eq. (9.3.64). Resonance does not occur because of the same reason as χ production in Case A.

- Case B

  In Phase 1, $q_{\rm dir}^{(h)}$ dominates $q^{(\chi)}$, and is kept to be ~ 1 since $\epsilon \sim 1$. The resonance condition reads

$$R_q \sim q^{(h)2}\frac{m_{\rm eff}}{H} > 1, \tag{9.3.74}$$

  and therefore resonant graviton production may occur. If such resonance occurs, the picture for the perturbative decay rate in vacuum does not hold, as explained in Chapter 5. However, it should be noted that this argument is based on the assumption that the gradient instability of the scalar perturbation does not spoil the background inflaton oscillation. After the model enters Phase 2, the decay rate becomes

$$\Gamma_{\phi\to h} \sim \begin{cases} \dfrac{m_{\rm eff}^{11}\Phi^6}{M^8 M_P^8} & \text{Phase } 2-1 \ \& \ 2-2 \ (M^2/m_{\rm eff} < H < M) \\ \dfrac{m_{\rm eff}^{3}\Phi^2}{M_P^4} & \text{Phase } 2-3 \ (H < M^2/m_{\rm eff}) \end{cases}, \tag{9.3.75}$$

  where the former comes from $q^{(h)} \sim q_{\rm dir}^{(h)}$ while the latter from $q^{(h)} \sim q^{(\chi)}$.

[#12] Because

$$\frac{q_{\rm dir}^{(h)}}{q^{(\chi)}} \sim \frac{H^2 m_{\rm eff}^2}{M^4}, \tag{9.3.73}$$

is already below unity at the onset of oscillation $H \sim m_{\rm eff}$.

9.4 Conclusion

In this section we analyzed gravitational effects on inflaton decay in a model with nonminimal derivative coupling $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. What we found is as follows.

- Background analysis
  - The background evolution of the model has three phases in chronological order: Phase 0 / Phase 1 / Phase 2.
    Inflation occurs during Phase 0, while the inflaton oscillates during Phase 1 & 2.
  - The difference in Phase 1 and 2 comes from the nonminimal contribution H^2/M^2 to the kinetic term, which is $\gg 1$ during Phase 1 while it is $\ll 1$ during Phase 2. Violent oscillation of the Hubble parameter occurs in correlation with the inflaton oscillation during Phase 1, while the dynamics is basically the same as the minimal setup in Phase 2 except for the beginning of this phase.
- Instability in perturbations
  - In contrast to $f(\phi)R$-type models, $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$-type model suffers from an instability of the scalar perturbation, which is triggered by the violent oscillation of the Hubble parameter.
  - The instability is called "gradient instability", in which the sound speed squared of the scalar perturbation becomes negative. Due to the instability, scalar perturbations with higher momentum modes are more likely to grow, and the background oscillation is likely to cease because the explosive particle production carries away the energy. However, we need UV completion of the theory in order to give conclusive remarks on this point.
- Particle production
  - If we ASSUME that the background oscillation somehow survives the gradient instability, resonant production occurs to gravitons, while no such production occurs to a canonical scalar field.

Chapter 10

Conclusion

In the present thesis, we have investigated gravitational effects on inflaton decay during the inflaton oscillation regime after inflation. Inflaton oscillation generally excites oscillation modes to the expansion rate (Hubble parameter) and the size (scale factor) of the universe, due to its coupling to gravity. These oscillation modes lead to light particle production, which may occur with resonance effects. We have investigated this process by using the method of analyzing particle production due to oscillating field illustrated in Chapter 5. We first focused on the simplest setup, where the theory contains the Einstein gravity, an inflaton with canonical kinetic term and potential, and the matter fields.

- Gravitational effects in minimal setup (Chapter 7)
  Even in this simplest setup, such oscillation modes exist with the amplitude $\dot{H} \sim H^2$, with H being the Hubble parameter, and they excite light scalar particles with masses up to the mass of the inflaton. Especially, it has been pointed out that such particle production is regarded as annihilation of the inflaton, and also that these excited particles can be a candidate for the dark matter, or a source for the cosmological moduli problem. Gravitons, or gravitational waves, are also found to be produced, though the amplitude seems too small to observe.

Next we focused on the cases where the inflaton is nonminimally coupled to gravity. In Chapter 4 we explained the classification of the theories with a scalar field and gravity coupled in a way free from higher derivatives and the resulting instability. We considered inflation theories within these theories, and proposed a quantity to analyze them.

- Adiabatic invariant in Horndeski/Galileon inflation theories (Chapter 6)
  When the inflaton is nonminimally coupled to gravity, a violent oscillation of the Hubble parameter $\dot{H} \sim m_{\rm eff} H$, with $m_{\rm eff}$ being the effective mass of the inffaton, is often induced. Even in such cases there exists a quantity which has a suppressed amplitude of oscillation $\dot{J} \sim HJ$. We illustrated the way to construct it within the theories mentioned above, and showed that it is useful in at least two ways: extraction of the oscillation mode of the Hubble parameter, and estimation of the cosmic expansion law.

After constructing the adiabatic invariant, we focused on nonminimal couplings of the form $f(\phi)R$ and $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$, which are motivated also by observations.

- Gravitational effects in $f(\phi)R$ theories (Chapter 8)
  We analyzed two cases, $f \sim 1 + \phi$ and $f \sim 1 + \phi^2$.

  - Case $f \sim 1 + \phi$
    In this case a violent oscillation $\dot{H} \sim m_{\rm eff} H$ occurs. By using the adiabatic invariant it was shown the oscillation mode in the Hubble parameter and the scale factor is linear in ϕ, in contrast to the minimal setup. Thus the production process of light scalar is regarded as the decay, not annihilation, of the inflaton. The decay rate is shown to be large enough to complete the inflaton decay, though resonance effects are ineffective. We also investigated graviton production. Though f oscillates, this oscillation does not bring about graviton production beyond the amount predicted in the minimal setup, because the oscillation of f cancels out with that in the scale factor. This result is consistent with that of analysis in the conformally transformed frame (Einstein frame).
  - Case $f \sim 1 + \phi^2$
    We focused on large coupling case $\mathcal{L} \supset \xi \phi^2 R$ with $\xi \gg 1$, with potential $\sim \lambda \phi^4$. In this case the Hubble parameter again violently oscillates, with two different oscillation timescales. Particle production due to the slow timescale is found to be ineffective, while the fast scale, which manifests itself as a spike in the derivative of the Hubble parameter, leads to heavy particle production up to mass $\sim \sqrt{\lambda} M_P$. Interpretation from the Einstein frame was also given there.

- Gravitational effects in $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ theories (Chapter 9)
  It was shown that a violent oscillation of the Hubble parameter $\dot{H} \sim m_{\rm eff} H$ occurs in this case as well, with the fractional oscillation amplitude $\delta H / H_{\rm max} \simeq 1/3$ being predicted by the adiabatic invariant. It has been found that the sound speed squared of the scalar perturbation becomes negative in this model, which makes a clear difference from $f(\phi)R$ models. The resulting instability (gradient instability) needs the knowledge of UV completion, which is beyond the scope of the thesis. Though the gradient instability may soon damp out the oscillation of the Hubble parameter, we estimated particle production assuming that the oscillation somehow survives the instability. With this assumption, it has been found that resonant graviton production is possible in this model.

To summarize, though the full understanding of the observable consequences of gravitational effects on inflaton decay as mentioned in the introduction is still on the way, we believe that these studies will give a useful tool to analyze them and will shed light on the various phenomenology of gravitational effects at the onset of the reheating era.

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Appendix A

Supplemental material for GR & FRW cosmology

A.1 Geometric quantities

Covariant derivative of a tensor quantity is defined as

$$\begin{aligned}\nabla_\mu T^{\alpha_1\alpha_2\ldots}{}_{\beta_1\beta_2\ldots} = \partial_\mu T^{\alpha_1\alpha_2\ldots}{}_{\beta_1\beta_2\ldots} \\ + \Gamma^{\alpha_1}{}_{\mu\gamma} T^{\gamma\alpha_2\ldots}{}_{\beta_1\beta_2\ldots} + \Gamma^{\alpha_2}{}_{\mu\gamma} T^{\alpha_1\gamma\ldots}{}_{\beta_1\beta_2\ldots} + \ldots \\ - \Gamma^{\gamma}{}_{\mu\beta_1} T^{\alpha_1\alpha_2\ldots}{}_{\gamma\beta_2\ldots} - \Gamma^{\gamma}{}_{\mu\beta_2} T^{\alpha_1\alpha_2\ldots}{}_{\beta_1\gamma\ldots} - \cdots\,,\end{aligned} \tag{A.1.1}$$

where the Christoffel symbol is given by

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha} \right). \tag{A.1.2}$$

Riemann tensor, Ricci tensor and Ricci scalar are given by

$$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu{}_{\nu\sigma,\rho} - \Gamma^\mu{}_{\nu\rho,\sigma} + \Gamma^\mu{}_{\alpha\rho}\Gamma^\alpha{}_{\nu\sigma} - \Gamma^\mu{}_{\alpha\sigma}\Gamma^\alpha{}_{\nu\rho}, \tag{A.1.3}$$

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \tag{A.1.4}$$

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{A.1.5}$$

A.2 Variation with respect to metric

The variation with respect to the metric gives

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\; \delta g^{\mu\nu} = \frac{1}{2}\sqrt{-g}\; g^{\mu\nu}\; \delta g_{\mu\nu}, \tag{A.2.6}$$

$$\delta\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\delta g_{\alpha\nu;\rho} + \delta g_{\alpha\rho;\nu} - \delta g_{\nu\rho;\alpha} \right), \tag{A.2.7}$$

$$\delta R_{\mu\nu} = (\delta\Gamma^\alpha{}_{\mu\nu})_{;\alpha} - (\delta\Gamma^\alpha{}_{\alpha\mu})_{;\nu}, \tag{A.2.8}$$

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \tag{A.2.9}$$

A.3 Expressions with FRW metric

With the FRW metric

$$ds^2 = -N^2 dt^2 + a^2 \left(\frac{dr^2}{1-kr^2} + d\Omega^2 \right), \tag{A.3.10}$$

The nonzero components of the Christoffel symbol are

$$\Gamma^0{}_{00} = \frac{\dot{N}}{N}, \qquad \Gamma^i{}_{j0} = \frac{\dot{a}}{a}\delta_{ij}, \qquad \Gamma^0{}_{ij} = \frac{a\dot{a}}{N^2}\delta_{ij} \tag{A.3.11}$$

The nonzero components of the Riemann tensor, Ricci tensor and Ricci scalar are

$$R^{0i}{}_{0j} = \frac{1}{N^2}\frac{\ddot{a}}{a} - \frac{\dot{N}}{N^2}\frac{\dot{a}}{a}, \qquad R^{ij}{}_{ij} = \frac{\dot{a}^2}{N^2 a^2} + \frac{k}{a^2}, \tag{A.3.12}$$

$$R_{00} = -\frac{3}{N}\frac{\ddot{a}}{a} + 3\frac{\dot{N}}{N}\frac{\dot{a}}{a}, \qquad R_{ij} = a^2 \left[\frac{1}{N^2}\frac{\ddot{a}}{a} + \frac{2}{N^2}\frac{\dot{a}^2}{a^2} - \frac{\dot{N}}{N^3}\frac{\dot{a}}{a} + \frac{2k}{a^2} \right], \tag{A.3.13}$$

$$R = 6 \left(\frac{1}{N^2}\frac{\ddot{a}}{a} + \frac{1}{N^2}\frac{\dot{a}^2}{a^2} - \frac{\dot{N}}{N^3}\frac{\dot{a}}{a} + \frac{k}{a^2} \right). \tag{A.3.14}$$

A.4 Conformal transformation

We follow [131]. We denote the spacetime dimension by D. The angular bracket below denotes antisymmetrization $A_{[\mu\nu]} = (A_{\mu\nu} - A_{\nu\mu})/2$. Conformal transformation is one which changes the length, preserving causal structure. We define the transformation as

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{A.4.15}$$

where Ω is a nonvanishing regular function. Then the relation between the two frames is given by

$$\tilde{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + \delta^\mu{}_\nu \nabla_\rho(\ln\Omega) + \delta^\mu{}_\rho \nabla_\nu(\ln\Omega) - g_{\nu\rho}\nabla^\mu(\ln\Omega), \tag{A.4.16}$$

$$\begin{aligned}\tilde{R}^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} &+ 2\delta^\mu{}_{[\nu}\nabla_{\rho]}\nabla_\sigma(\ln\Omega) - 2g_{\sigma[\nu}\nabla_{\rho]}\nabla^\mu(\ln\Omega) \\ &+ 2\delta^\mu{}_{[\nu}\nabla_{\rho]}(\ln\Omega)\nabla_\sigma(\ln\Omega) - 2g_{\sigma[\nu}\nabla_{\rho]}(\ln\Omega)\nabla^\mu(\ln\Omega) \\ &- 2\delta^\mu{}_{[\nu}g_{\rho]\sigma}g^{\alpha\beta}\nabla_\alpha(\ln\Omega)\nabla_\beta(\ln\Omega),\end{aligned} \tag{A.4.17}$$

$$\begin{aligned}\tilde{R}_{\mu\nu} = R_{\mu\nu} &- (D-2)\nabla_\mu\nabla_\nu(\ln\Omega) - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta(\ln\Omega) \\ &+ (D-2)\nabla_\mu(\ln\Omega)\nabla_\nu(\ln\Omega) - (D-2)g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha(\ln\Omega)\nabla_\beta(\ln\Omega),\end{aligned} \tag{A.4.18}$$

$$\tilde{R} = \Omega^{-2}\left[R - 2(D-1)\Box(\ln\Omega) - (D-1)(D-2)g^{\alpha\beta}\nabla_\alpha(\ln\Omega)\nabla_\beta(\ln\Omega) \right]. \tag{A.4.19}$$

The invariant generalization of the Klein-Gordon equation is

$$\Box\phi - \frac{D-2}{4(D-1)}R\phi = 0. \tag{A.4.20}$$

Especially, for $D = 4$

$$\begin{aligned}\tilde{R} &= \Omega^{-2}\left[R - 6\Box(\ln\Omega) - 6g^{\alpha\beta}\nabla_\alpha(\ln\Omega)\nabla_\beta(\ln\Omega)\right] \\ &= \Omega^{-2}\left[R - \frac{6\Box\Omega}{\Omega}\right] = \Omega^{-2}\left[R - \frac{12\Box\sqrt{\Omega}}{\sqrt{\Omega}} - 3g^{\alpha\beta}\nabla_\alpha(\ln\Omega)\nabla_\beta(\ln\Omega)\right],\end{aligned} \tag{A.4.21}$$

holds.

A.5 Equilibrium thermodynamics

The number density, energy density and pressure of a species with mass m and chemical potential μ is

$$n = g\int\frac{d^3p}{(2\pi)^3}f(\mathbf{p}) = \frac{g}{2\pi^2}\int_m^\infty\frac{(E^2-m^2)^{1/2}}{e^{(E-\mu)/T}\pm 1}EdE, \tag{A.5.22}$$

$$\rho = g\int\frac{d^3p}{(2\pi)^3}E(\mathbf{p})f(\mathbf{p}) = \frac{g}{2\pi^2}\int_m^\infty\frac{(E^2-m^2)^{1/2}}{e^{(E-\mu)/T}\pm 1}E^2dE, \tag{A.5.23}$$

$$p = g\int\frac{d^3p}{(2\pi)^3}\frac{|\mathbf{p}|^2}{3E(\mathbf{p})}f(\mathbf{p}) = \frac{g}{6\pi^2}\int_m^\infty\frac{(E^2-m^2)^{3/2}}{e^{(E-\mu)/T}\pm 1}dE. \tag{A.5.24}$$

Especially, in high temperature limit $T \gg m, \mu$, they take the following form

$$n = \frac{\zeta(3)}{\pi^2}gT^3, \quad \rho = \frac{\pi^2}{30}gT^4, \quad p = \frac{\rho}{3}, \tag{A.5.25}$$

for bosons, and

$$n = \frac{3}{4}\frac{\zeta(3)}{\pi^2}gT^3, \quad \rho = \frac{7}{8}\frac{\pi^2}{30}gT^4, \quad p = \frac{\rho}{3}, \tag{A.5.26}$$

for fermions. Here $\zeta(3) \simeq 1.20206...$ is the zeta function.

Another useful quantity is the entropy density $s \equiv S/V$. From the second law of thermodynamics

$$TdS = d(\rho V) + pdV = d[(\rho + p)V] - Vdp, \tag{A.5.27}$$

and the relation between the energy density and pressure

$$dp = \frac{\rho + p}{T}dT, \tag{A.5.28}$$

which is derived from the integrability condition $\partial^2 S/\partial T\partial V = \partial^2 S/\partial V\partial T$, we have

$$s = \frac{\rho + p}{T}. \tag{A.5.29}$$

It becomes in high-temperature limit $T \gg m, \mu$

$$s = \frac{2\pi^2}{45} g T^3, \tag{A.5.30}$$

for bosons, and

$$s = \frac{7}{8} \frac{2\pi^2}{45} g T^3, \tag{A.5.31}$$

for fermions.

We often parameterize the energy density and entropy density of the thermal bath by the effective relativistic degrees of freedom g_* and g_{*s} as

$$\rho = \frac{\pi^2}{30} g_*(T) T^4, \quad s = \frac{2\pi^2}{45} g_{*s}(T) T^3, \tag{A.5.32}$$

where

$$g_*(T) = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4, \tag{A.5.33}$$

$$g_{*s}(T) = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3. \tag{A.5.34}$$

Here T and T_i denote the temperature of photons and the species i, respectively, g_i is the degrees of freedom of the species i, and the summation runs for relativistic species $T_i \gg m_i$. We took into account the possibility for the species i to have a different temperature than that of photons. In the Standard Model (SM) of particle physics, g_* and g_{*s} both take the value of 106.75 at high temperature $T \gtrsim \mathcal{O}(100)$ GeV.

	SU(3)	SU(2)	Anti	Other factors	f/b
Quark (L)	3	2	2	3 (generation)	7/8
Quark (R)	3	2	2	3 (generation)	7/8
Lepton (L)	1	2	2	3 (generation)	7/8
Lepton (R)	1	1	2	3 (generation)	7/8
SU(3) gauge boson	8	1	1	2 (polarization)	1
SU(2) gauge boson	1	3	1	2 (polarization)	1
U(1) gauge boson	1	1	1	2 (polarization)	1
Higgs	1	2	2	1	1

Table A.1: SM particle contribution to the effective degrees of freedom g_* and g_{*s}. "f/b" denotes the factor one should take into account in calculating g_* or g_{*s} depending on whether the species is fermion or boson.

Appendix B

Quadratic action of Horndeski/Galileon theories

In Chapter 4 we saw the quadratic action and the sound speed for Horndeski/Galileon theories in ζ gauge. In this appendix we show the quadratic action calculated in φ gauge for consistency check.

$$S_S^{(\varphi)} = \int d^4x \; a^3 \left[\left(C_{(\alpha\alpha)}\alpha^2 + C_{(\alpha\varphi)}\alpha\varphi + C_{(\alpha\dot{\varphi})}\alpha\dot{\varphi} + C_{(\varphi\varphi)}\varphi^2 + C_{(\dot{\varphi}\dot{\varphi})}\dot{\varphi}^2 \right) \right. \\ \left. + a^{-2}\left(D_{(\alpha\beta)}\alpha\beta_{,ii} + D_{(\alpha\varphi)}\alpha\varphi_{,ii} + D_{(\varphi\beta)}\varphi\beta_{,ii} + D_{(\dot{\varphi}\beta)}\dot{\varphi}\beta_{,ii} + D_{(\varphi\varphi)}\varphi\varphi_{,ii} \right) \right] . \tag{B.0.1}$$

The coefficients are summarized in Table B.1–B.3. Note that the subscripts $\dot{H}, H, X, \ddot{\phi}, \dot{\phi}, \phi$ denote the derivative with respect to that quantity, and that $\mathcal{L}$ should be regarded as $\mathcal{L}(H, \dot{\phi}, \phi)$ when we use "Relations" in these tables.

Since $\beta_{,ii}$ appears in the above action only up to linear order, the constraint equation for $\beta_{,ii}$ can easily be solved:

$$\alpha = -\frac{D_{(\varphi\beta)}}{D_{(\alpha\beta)}}\varphi - \frac{D_{(\dot{\varphi}\beta)}}{D_{(\alpha\beta)}}\dot{\varphi}. \tag{B.0.2}$$

Substituting this into the action, and doing integration by parts, we have the sound speed:

$$c_s^2 = \frac{-(D_{(\alpha\varphi)}D_{(\varphi\beta)}/D_{(\alpha\beta)}) + (D_{(\alpha\varphi)}D_{(\dot{\varphi}\beta)}/2D_{(\alpha\beta)})^{\cdot} + HD_{(\alpha\varphi)}D_{(\dot{\varphi}\beta)}/2D_{(\alpha\beta)} + D_{(\varphi\varphi)}}{(C_{(\alpha\alpha)} - C_{(\alpha\dot{\varphi})})D_{(\dot{\varphi}\beta)}/D_{(\alpha\beta)} + C_{(\dot{\varphi}\dot{\varphi})}}. \tag{B.0.3}$$

Table B.1: Coefficients for perturbations in φ gauge.

Scalar		Explicit form	Relations
$C_{(\alpha\alpha)}$	$\mathcal{L}_2$	$+X(G_{2X}+2XG_{2XX})$	$\frac{1}{2}H\mathcal{E}_H+\frac{1}{2}\dot{\phi}\mathcal{E}_{\dot{\phi}}$
	$\mathcal{L}_3$	$-X(2G_{3\phi}+2XG_{3\phi X})+H\dot{\phi}X(12G_{3X}+6XG_{3XX})$	or
	$\mathcal{L}_4$	$+H^2(-6G_4+42XG_{4X}+96X^2G_{4XX}+24X^3G_{4XXX})$ $-H\dot{\phi}(6G_{4\phi}+30XG_{4\phi X}+12X^2G_{4\phi XX})$	$\frac{1}{2}H^2\mathcal{L}_{HH}+H\dot{\phi}\mathcal{L}_{H\dot{\phi}}+\frac{1}{2}\dot{\phi}^2\mathcal{L}_{\dot{\phi}\dot{\phi}}$
	$\mathcal{L}_5$	$-H^2X(36G_{5\phi}+54XG_{5\phi X}+12X^2G_{5\phi XX})$ $+H^3\dot{\phi}X(30G_{5X}+26XG_{5XX}+4X^2G_{5XXX})$	
$C_{(\alpha\varphi)}$	$\mathcal{L}_2$	$+G_{2\phi}-2XG_{2\phi X}$	$-\mathcal{E}_\phi$
	$\mathcal{L}_3$	$+2XG_{3\phi\phi}-6H\dot{\phi}XG_{3\phi X}$	or
	$\mathcal{L}_4$	$+H^2(6G_{4\phi}-24XG_{4\phi X}-24X^2G_{4\phi XX})$ $+H\dot{\phi}(6G_{4\phi\phi}+12XG_{4\phi\phi X})$	$\mathcal{L}_\phi-H\mathcal{L}_{H\phi}-\dot{\phi}\mathcal{L}_{\dot{\phi}\phi}$
	$\mathcal{L}_5$	$+H^2X(18G_{5\phi\phi}+12XG_{5\phi\phi X})$ $-H^3\dot{\phi}X(10G_{5\phi X}+4XG_{5\phi XX})$	
$C_{\alpha\dot{\varphi}}$	$\mathcal{L}_2$	$-\dot{\phi}(G_{2X}+2XG_{2XX})$	$-\mathcal{E}_{\dot{\phi}}$
	$\mathcal{L}_3$	$+\dot{\phi}(2G_{3\phi}+2XG_{3\phi X})$ $-HX(18G_{3X}+12XG_{3XX})$	or
	$\mathcal{L}_4$	$+H(6G_{4\phi}+48XG_{4\phi X}+24X^2G_{4\phi XX})$ $-H^2\dot{\phi}(18G_{4X}+72XG_{4XX}+24\dot{\phi}X^2G_{4XXX})$	$-H\mathcal{L}_{H\dot{\phi}}-\dot{\phi}\mathcal{L}_{\dot{\phi}\dot{\phi}}$
	$\mathcal{L}_5$	$+H^2\dot{\phi}(18G_{5\phi}+42XG_{5\phi X}+12X^2G_{5\phi XX})$ $-H^3X(30G_{5X}+40XG_{5XX}+8X^2G_{5XXX})$	

Table B.2: Coefficients for perturbations in φ gauge (cont.).

Scalar		Explicit form	Relations
$C_{(\varphi\varphi)}$	$\mathcal{L}_2$	$+\frac{1}{2}G_{2\phi\phi} - \frac{1}{2}(\ddot{\phi} + 3H\dot{\phi})G_{2\phi X} - XG_{2\phi\phi X} - \ddot{\phi}XG_{2\phi XX}$	$-\dfrac{1}{2}\Phi_{\phi}$ or $-\dfrac{1}{2}\left[(\mathcal{L}_{\dot{\phi}\phi})^{\cdot} + 3H\mathcal{L}_{\dot{\phi}\phi} - \mathcal{L}_{\phi\phi}\right]$
	$\mathcal{L}_3$	$+(\ddot{\phi} + 3H\dot{\phi})G_{3\phi\phi} - (3\dot{H}X + 3H\ddot{\phi}\dot{\phi} + 9H^2X)G_{3\phi X}$ $+XG_{3\phi\phi\phi} + (\ddot{\phi}X - 3H\dot{\phi}X)G_{3\phi\phi X} - 3H\ddot{\phi}\dot{\phi}XG_{3\phi XX}$	
	$\mathcal{L}_4$	$+(3\dot{H} + 6H^2)G_{4\phi\phi} - (6\dot{H}H\dot{\phi} + 3H^2\ddot{\phi} + 9H^3\dot{\phi})G_{4\phi X}$ $+(6\dot{H}X + 9H\ddot{\phi}\dot{\phi} + 18H^2X)G_{4\phi\phi X}$ $-(12\dot{H}H\dot{\phi}X + 24H^2\ddot{\phi}X + 18H^3\dot{\phi}X)G_{4\phi XX}$ $+6H\dot{\phi}XG_{4\phi\phi\phi X} + (6H\ddot{\phi}\dot{\phi}X - 12H^2X^2)G_{4\phi\phi XX}$ $-12H^2\ddot{\phi}X^2G_{4\phi XXX}$	
	$\mathcal{L}_5$	$+(6\dot{H}\dot{\phi} + 3H\ddot{\phi} + 9H^2\dot{\phi})G_{5\phi\phi}$ $-(9\dot{H}H^2X + 3H^3\ddot{\phi}\dot{\phi} + 9H^4X)G_{5\phi X}$ $+3H^2XG_{5\phi\phi\phi} + (6\dot{H}H\dot{\phi}X + 15H^2\ddot{\phi}X + 7H^3\dot{\phi}X)G_{5\phi\phi X}$ $-(6\dot{H}H^2X^2 + 7H^3\ddot{\phi}\dot{\phi}X + 6H^4X^2)G_{5\phi XX}$ $+6H^2X^2G_{5\phi\phi\phi X} + (6H^2\ddot{\phi}X^2 - 2H^3\dot{\phi}X^2)G_{5\phi\phi XX}$ $-2H^3\ddot{\phi}\dot{\phi}X^2G_{5\phi XXX}$	
$C_{(\dot{\varphi}\dot{\varphi})}$	$\mathcal{L}_2$	$+\frac{1}{2}G_{2X} + XG_{2XX}$	$\dfrac{1}{2}\Phi_{\ddot{\phi}}$ or $\dfrac{1}{2}\mathcal{L}_{\dot{\phi}\dot{\phi}}$
	$\mathcal{L}_3$	$-G_{3\phi} + 3H\dot{\phi}G_{3X} - XG_{3\phi X} + 3H\dot{\phi}XG_{3XX}$	
	$\mathcal{L}_4$	$+3H^2G_{4X} - 9H\dot{\phi}G_{4\phi X} + 24H^2XG_{4XX}$ $-6H\dot{\phi}XG_{4\phi XX} + 12H^2X^2G_{4XXX}$	
	$\mathcal{L}_5$	$-3H^2G_{5\phi} + 3H^3\dot{\phi}G_{5X}$ $-15H^2XG_{5\phi X} + 7H^3\dot{\phi}XG_{5XX}$ $-6H^2X^2G_{5\phi XX} + 2H^3X^2G_{5XXX}$	

Table B.3: Coefficients for perturbations in φ gauge (cont.).

Scalar		Explicit form	Relations
$D_{(\alpha\beta)}$	$\mathcal{L}_2$		
	$\mathcal{L}_3$	$+2\dot{\phi}XG_{3X}$	
	$\mathcal{L}_4$	$+H(-4G_4+16XG_{4X}+16X^2G_{4XX})-\dot{\phi}(2G_{4\phi}+4XG_{4\phi X})$	
	$\mathcal{L}_5$	$-HX(12G_{5\phi}+8XG_{5\phi X})+H^2\dot{\phi}X(10G_{5X}+4XG_{5XX})$	
$D_{(\alpha\varphi)}$	$\mathcal{L}_2$		$-D_{(\dot{\varphi}\beta)}$
	$\mathcal{L}_3$	$+2XG_{3X}$	
	$\mathcal{L}_4$	$-(2G_{4\phi}+4XG_{4\phi X})+H\dot{\phi}(4G_{4X}+8XG_{4XX})$	
	$\mathcal{L}_5$	$-H^2\dot{\phi}(4G_{5\phi}+4XG_{5\phi X})+H^2X(6G_{5X}+4XG_{5XX})$	
$D_{(\varphi\beta)}$	$\mathcal{L}_2$	$+\dot{\phi}G_{2X}$	
	$\mathcal{L}_3$	$-2\dot{\phi}G_{3\phi}+6HXG_{3X}$	
	$\mathcal{L}_4$	$+2\dot{\phi}G_{4\phi\phi}-H(2G_{4\phi}+20XG_{4\phi X})+H^2\dot{\phi}(6G_{4X}+12XG_{4XX})$	
	$\mathcal{L}_5$	$+4HXG_{5\phi\phi}-H^2\dot{\phi}(6G_{5\phi}+8XG_{5\phi X})+H^3X(6G_{5X}+4XG_{5XX})$	
$D_{(\dot{\varphi}\beta)}$	$\mathcal{L}_2$		$-D_{(\alpha\varphi)}$
	$\mathcal{L}_3$	$-2XG_{3X}$	
	$\mathcal{L}_4$	$+(2G_{4\phi}+4XG_{4\phi X})-H\dot{\phi}(4G_{4X}+8XG_{4XX})$	
	$\mathcal{L}_5$	$+H\dot{\phi}(4G_{5\phi}+4XG_{5\phi X})-H^2X(6G_{5X}+4XG_{5XX})$	
$D_{(\varphi\varphi)}$	$\mathcal{L}_2$	$+\frac{1}{2}G_{2X}$	
	$\mathcal{L}_3$	$-G_{3\phi}+XG_{3\phi X}+(\ddot{\phi}+2H\dot{\phi})G_{3X}+\ddot{\phi}XG_{3XX}$	
	$\mathcal{L}_4$	$+(2\dot{H}+3H^2)G_{4X}-(3\ddot{\phi}+6H\dot{\phi})G_{4\phi X}+(4\dot{H}X+6H\ddot{\phi}\dot{\phi}+10H^2X)G_{4XX}$ $-2XG_{4\phi\phi X}+(-2X\ddot{\phi}+4H\dot{\phi}X)G_{4\phi XX}+4H\ddot{\phi}\dot{\phi}XG_{4XXX}$	
	$\mathcal{L}_5$	$-(2\dot{H}+3H^2)G_{5\phi}+(2\dot{H}H\dot{\phi}+2H^3\dot{\phi}+H^2\ddot{\phi})G_{5X}$ $-(2\dot{H}X+5H^2X+4H\ddot{\phi}\dot{\phi})G_{5\phi X}+(2\dot{H}H\dot{\phi}X+5H^2\ddot{\phi}X+2H^3\dot{\phi}X)G_{5XX}$ $-2H\dot{\phi}XG_{5\phi\phi X}+(2H^2X^2-2H\ddot{\phi}\dot{\phi}X)G_{5\phi XX}+2H^2\ddot{\phi}X^2G_{5XXX}$	

Appendix C

Weyl transformation

In this appendix, we explicitly show how the action for scalar, fermion and vector boson transforms under Weyl transformation. Throughout this appendix (and also the main text) we do not consider the effect of tortion. The action for scalar, fermion and vector boson is given by

$$S_S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \frac{1}{2}\xi\chi^2 R\right], \tag{C.0.1}$$

$$S_F = \int d^4x\; e\left[-\bar{\psi}\not{D}\psi\right], \tag{C.0.2}$$

$$S_V = \int d^4x\sqrt{-g}\left[-\frac{g_V^2}{4}g^{\mu\rho}g^{\nu\sigma}F^i_{\mu\nu}F^i_{\rho\sigma}\right]. \tag{C.0.3}$$

Here we included the nonminimal coupling $\chi^2 R$ in the scalar action for the reason which soon becomes clear. Also note the sign of the scalar kinetic term. The field strength of the gauge boson is given by

$$F_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ig_V f^{ijk}A^j_\mu A^k_\nu, \tag{C.0.4}$$

Here g_V is the gauge coupling and f^{ijk} is the structure constant. On the other hand, when we discuss fermions in curved spacetime, we must introduce the frame field (also called vielbein or tetrad) ${e_\mu}^a$. Mathematically, the frame field is obtained by diagonalizing the metric tensor $g_{\mu\nu}$ by an orthogonal matrix ${e_\mu}^a$ as

$$g_{\mu\nu} = {e_\mu}^a\eta_{ab}{e_\nu}^b, \tag{C.0.5}$$

Here $\eta_{ab} = \mathrm{diag}(-1,+1,+1,+1)$ is the local Lorentz metric. Defining the Greek (Latin) indices to be raised and lowered by $g_{\mu\nu}$ (η_{ab}), we have

$${e^\mu}_a g_{\mu\nu}{e^\nu}_b = \eta_{ab}. \tag{C.0.6}$$

The volume element e is given by $e \equiv \det({e_\mu}^a) = \sqrt{-g}$, where the last equality follows from Eq. (C.0.5). Using the frame field, the covariant derivative of fermions is written as

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}, \tag{C.0.7}$$

where the spin connection is given by

$$\omega_\mu{}^{ab} = 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\rho}e_{\mu c}\partial_\nu e_\rho{}^c, \tag{C.0.8}$$

and $\gamma^{ab} \equiv \gamma^{(a}\gamma^{b)}$. The bracket on indices denotes (anti)symmetrization $A_{(\mu}B_{\nu)} = (A_\mu B_\nu + A_\nu B_\mu)/2$ and $A_{[\mu}B_{\nu]} = (A_\mu B_\nu - A_\nu B_\mu)/2$. Note that this expression for the spin connection is valid only when there is no torsion. Also, the γ matrices satisfy

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{C.0.9}$$

Note that the bracket used for (anti)commutation relation does not include the overall factor 1/2. In addition, the slashed object $\not{D}$ is

$$\not{D} = e^\mu{}_a\gamma^a D_\mu. \tag{C.0.10}$$

Now we show the Weyl transformation properties of these actions. Weyl transformation, or local scale transformation, is implemented as

$$e_\mu{}^a \to e^{-\sigma(x)}e_\mu{}^a, \qquad g_{\mu\nu} \to e^{-2\sigma(x)}g_{\mu\nu}, \tag{C.0.11}$$

while the fields transform as

$$\chi \to e^{\sigma(x)}\chi, \qquad \psi \to e^{\frac{3}{2}\sigma(x)}\psi, \qquad A_\mu \to A_\mu. \tag{C.0.12}$$

For scalar field, we use the result in Appendix A. Each term transforms as

$$\sqrt{-g} \to e^{-4\sigma(x)}\sqrt{-g}, \tag{C.0.13}$$

$$\frac{1}{2}(\partial\phi)^2 \to \frac{1}{2}e^{2\sigma(x)}g^{\mu\nu}(\partial_\mu\phi + \phi\partial_\mu\sigma)(\partial_\nu\phi + \phi\partial_\nu\sigma), \tag{C.0.14}$$

$$\phi^2 \to e^{2\sigma(x)}\phi, \tag{C.0.15}$$

$$R \to e^{2\sigma(x)}\left[R + 6\Box\sigma - 6(\partial\sigma)^2\right], \tag{C.0.16}$$

therefore

$$S_S \to \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi + \phi\partial_\mu\sigma)(\partial_\nu\phi + \phi\partial_\nu\sigma) + \frac{1}{2}\xi\phi^2\left[R + 6\Box\sigma - 6(\partial\sigma)^2\right]\right] \tag{C.0.17}$$

Using integration by parts as $\sqrt{-g}\phi^2\Box\sigma \sim -\sqrt{-g}\phi\nabla^\mu(\phi^2)\nabla_\mu\sigma$, one finds that the action is invariant only for $\xi = 1/6$.

For the fermion, the Weyl invariance is proved as follows [132]. First let us rewrite the action as

$$S_F = \int d^4x\, e\left[-\frac{1}{2}\bar{\psi}\left(\not{D} - \overleftarrow{\not{D}}\right)\psi\right], \tag{C.0.18}$$

where

$$\overleftarrow{\not{D}} = \left(\overleftarrow{\partial}_\mu - \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)e^\mu{}_c\gamma^c. \tag{C.0.19}$$

Using the transformation (C.0.11), one sees that

$$\omega_\mu{}^{ab} \to \omega_\mu{}^{ab} + \partial_\nu \sigma \left(-e_\mu{}^a e^{\nu b} + e_\mu{}^b e^{\nu a}\right). \tag{C.0.20}$$

Then, with (C.0.12), one finds that the terms from ∂_μ in $\not{D}$ operating on $e^{\frac{3}{2}\sigma(x)}$ cancel out while the second term in Eq. (C.0.20) seems to remain in the transformed action

$$S_F \to S_F + \int d^4x\, e \left[-\frac{1}{8}\bar{\psi} e^\mu{}_c \partial_\nu \sigma \left(-e_{\mu a} e^\nu{}_b + e_{\mu b} e^\nu{}_a\right) \{\gamma^{ab}, \gamma^c\} \psi\right]. \tag{C.0.21}$$

However, noting that

$$\frac{1}{2}\{\gamma^{ab}, \gamma^c\} = \gamma^{abc} \equiv \gamma^{[a}\gamma^b\gamma^{c]}, \tag{C.0.22}$$

is totally asymmetric in the three indices, the second term in Eq. (C.0.21) vanishes since it is proportional to $\gamma^{abc}\eta_{ac}$ or $\gamma^{abc}\eta_{bc}$. Thus, the action for massless fermion is invariant under Weyl transformation.

The invariance for massless vector boson is obvious from the action (C.0.3) and the transformation (C.0.11)–(C.0.12). Also note that mass terms for scalar, fermion and vector boson all violate Weyl invariance.

Appendix D

3+1 decomposition

Detailed explanation of $3+1$ decomposition in general relativity is found in e.g. [133]. In this appendix we first summarize the basics of $3+1$ decomposition referring to [133], and then apply the decomposition to the models we considered in the main text.

D.1 Basics

D.1.1 Embedding, push-forward, pull-back

We consider a spacetime $(\mathcal{M}, \boldsymbol{g})$ with $\mathcal{M}$ being a four-dimensional real smooth manifold and $\boldsymbol{g}$ being a Lorentzian metric. We denote by $\mathcal{T}_p(\mathcal{M})$ and $\mathcal{T}_p^*(\mathcal{M})$ the tangent and cotangent space at $p \in \mathcal{M}$. We also denote by $\mathcal{T}(\mathcal{M})$ and $\mathcal{T}^*(\mathcal{M})$ the space of smooth vector fields and that of 1-forms. In addition, the spacetime connection is denoted by $\boldsymbol{\nabla}$. The scalar product of two vectors by the metric is denoted by the round bracket as

$$\forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}), \quad (\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu} u^\mu u^\nu. \tag{D.1.1}$$

On the other hand, angular brackets are used to denote the action of linear forms on vectors

$$\forall(\omega, \boldsymbol{v}) \in \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}), \quad \langle \boldsymbol{\omega}, \boldsymbol{v} \rangle = \omega_\mu v^\mu. \tag{D.1.2}$$

The directional covariant derivative $\boldsymbol{\nabla}_{\boldsymbol{u}}$ is defined so that the components of the tensor $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{T}$ where

$$\boldsymbol{T} = T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} \boldsymbol{e}_{\alpha_1} \otimes \cdots \otimes \boldsymbol{e}_{\alpha_p} \otimes \boldsymbol{e}^{\beta_1} \otimes \cdots \otimes \boldsymbol{e}^{\beta_q} \tag{D.1.3}$$

become $u^\mu \nabla_\mu T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q}$.

An image of 3-dimensional manifold $\hat{\Sigma}$ onto $\mathcal{M}$ is called embedding

$$\Phi \; : \; \hat{\Sigma} \quad \longrightarrow \quad \mathcal{M}, \tag{D.1.4}$$

and is denoted by $\Sigma = \Phi(\hat{\Sigma})$ here. Embedding also maps a vector $\boldsymbol{u}$ on $\hat{\Sigma}$ onto $\mathcal{M}$, and this map is called "push-forward"

$$\begin{array}{rccc} \Phi_* \; : & \mathcal{T}_p(\hat{\Sigma}) & \longrightarrow & \mathcal{T}_p(\mathcal{M}) \\ & \boldsymbol{u} & \longmapsto & \Phi_* \boldsymbol{u}. \end{array} \tag{D.1.5}$$

This embedding induces a "pull-back" Φ^* of a linear form $\boldsymbol{\omega}$ on $\mathcal{M}$ onto $\hat{\Sigma}$

$$\begin{array}{rcl} \Phi^* \;:\; \mathcal{T}_p^*(\mathcal{M}) & \longrightarrow & \mathcal{T}_p^*(\hat{\Sigma}) \\ \boldsymbol{\omega} & \longmapsto & \Phi^*\boldsymbol{\omega}. \end{array} \tag{D.1.6}$$

This pull-back is naturally extended to multi-linear forms. For example, the action of the n-linear form $\boldsymbol{T}$ on vectors becomes

$$\forall(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_n)\in\boldsymbol{T}_p(\hat{\Sigma})^n,\quad \Phi^*\boldsymbol{T}(\boldsymbol{u}_1,\cdots,\boldsymbol{u}_n)=\boldsymbol{T}(\Phi_*\boldsymbol{u}_1,\cdots,\Phi_*\boldsymbol{u}_n). \tag{D.1.7}$$

D.1.2 Induced metric, intrinsic & extrinsic curvature

The induced metric γ on Σ, or the 3-metric, is the pull-back of the metric $\boldsymbol{g}$

$$\boldsymbol{\gamma}\equiv\Phi^*\boldsymbol{g}. \tag{D.1.8}$$

In components, this states that $\gamma_{ij}=g_{ij}$, where the Latin indices run through $1,2,3$. The (torsion-free) covariant derivative on Σ must satisfy

$$\boldsymbol{D}\boldsymbol{\gamma}=0. \tag{D.1.9}$$

The Riemann curvature tensor is defined as

$$\begin{array}{rcl} {}^4\boldsymbol{R} \;:\; \mathcal{T}^*(\mathcal{M})\times\mathcal{T}(\mathcal{M})^3 & \longrightarrow & C^\infty(\mathcal{M},\mathbb{R}) \\ (\boldsymbol{\omega},\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}) & \longmapsto & \langle\omega,\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{w}-\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}\rangle\,. \end{array} \tag{D.1.10}$$

In components,

$$\forall\boldsymbol{u}\in\mathcal{T}(\mathcal{M}),\quad (\nabla_\alpha\nabla_\beta-\nabla_\beta\nabla_\alpha)u^\gamma={}^4R^\gamma{}_{\mu\alpha\beta}u^\mu. \tag{D.1.11}$$

The 3-dimensional Riemann tensor ${}^3\boldsymbol{R}$ has a similar expression to above equation

$$\forall\boldsymbol{u}\in\mathcal{T}(\hat{\Sigma}),\quad (D_iD_j-D_jD_i)u_k={}^3R^k{}_{lij}u^l. \tag{D.1.12}$$

This 3-dimensional Riemann tensor is called intrinsic curvature. On the other hand, there exists a quantity which parameterizes how the manifold $\hat{\Sigma}$ is bending when embedded in $\mathcal{M}$. This quantity is called extrinsic curvature. To define it, we first define normal vector to the hypersurface Σ. With appropriate assumptions on the hypersurface, there exists a 1-form $\boldsymbol{d}t$ which satisfies

$$\forall\boldsymbol{u}\in\mathcal{T}_p(\hat{\Sigma}),\quad\langle\boldsymbol{d}t,\Phi_*\boldsymbol{u}\rangle=0. \tag{D.1.13}$$

The dual to $\boldsymbol{d}t$ is written as $\vec{\boldsymbol{\nabla}}t$. We normalize $\vec{\boldsymbol{\nabla}}t$ to define the normal vector $\boldsymbol{n}$

$$\boldsymbol{n}=-N\vec{\boldsymbol{\nabla}}t,\qquad N\equiv\left[-(\vec{\boldsymbol{\nabla}}t,\vec{\boldsymbol{\nabla}}t)\right]^{-1/2}. \tag{D.1.14}$$

This normal vector satisfies $(\boldsymbol{n}, \boldsymbol{n}) = -1$. We also define a normal vector $\boldsymbol{m}$ as

$$\boldsymbol{m} \equiv N\boldsymbol{n}, \tag{D.1.15}$$

which satisfies $(\boldsymbol{m}, \boldsymbol{m}) = -N^2$. Next we construct the Weingarten map as

$$\begin{array}{rccc} \boldsymbol{\chi} \;:\; & \mathcal{T}_p(\Sigma) & \longrightarrow & \mathcal{T}_p(\Sigma) \\ & \boldsymbol{u} & \longmapsto & \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n}. \end{array} \tag{D.1.16}$$

This map is well-defined because $[\boldsymbol{n}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n}] = 0$ is confirmed. Also, the Weingarten map is self-adjoint

$$\forall(\boldsymbol{u}, \boldsymbol{v}) \in \mathcal{T}_p(\Sigma) \times \mathcal{T}_p(\Sigma), \quad (\boldsymbol{u}, \boldsymbol{\chi}(\boldsymbol{v})) = (\boldsymbol{v}, \boldsymbol{\chi}(\boldsymbol{u})). \tag{D.1.17}$$

Note that, since $\boldsymbol{u} \in \mathcal{T}_p(\Sigma)$ is mapped onto $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n} \in \mathcal{T}_p(\Sigma)$ by the Weingarten map, the inner product above is automatically taken with respect to the induced metric $\boldsymbol{\gamma}$. The extrinsic curvature tensor is defined as a bilinear form

$$\begin{array}{rccc} \boldsymbol{K} \;:\; & \mathcal{T}_p(\Sigma) \times \mathcal{T}_p(\Sigma) & \longrightarrow & \mathbb{R} \\ & (\boldsymbol{u}, \boldsymbol{v}) & \longmapsto & -(\boldsymbol{u}, \boldsymbol{\chi}(\boldsymbol{v})). \end{array} \tag{D.1.18}$$

The extrinsic curvature K is defined as the trace of the extrinsic curvature tensor.

D.1.3 Orthogonal projector

The orthogonal projector refers to the map

$$\begin{array}{rccc} \vec{\gamma} \;:\; & \mathcal{T}_p(\mathcal{M}) & \longrightarrow & \mathcal{T}_p(\Sigma) \\ & \boldsymbol{u} & \longmapsto & \boldsymbol{u} + (\boldsymbol{n}, \boldsymbol{u})\boldsymbol{n}. \end{array} \tag{D.1.19}$$

Note that this map is in the inverse direction to the push-forward $\Phi_* : \mathcal{T}_p(\Sigma) \to \mathcal{T}_p(\mathcal{M})$[#1]. We may also construct a map with the inverse direction to pull-back

$$\begin{array}{rcccccc} \vec{\gamma}^*_{\mathcal{M}} \;:\; & \mathcal{T}^*_p(\Sigma) & \longrightarrow & \mathcal{T}^*_p(\mathcal{M}) & & & \\ & \boldsymbol{\omega} & \longrightarrow & \vec{\gamma}^*_{\mathcal{M}}(\boldsymbol{\omega}) \;:\; & \mathcal{T}_p(\mathcal{M}) & \longrightarrow & \mathbb{R} \\ & & & & \boldsymbol{u} & \longmapsto & \boldsymbol{\omega}(\vec{\gamma}(\boldsymbol{u})). \end{array} \tag{D.1.20}$$

These definitions are extended to apply to $\mathcal{T}_p(\mathcal{M})^n$ or $\mathcal{T}^*_p(\Sigma)^n$. Note that these maps are constructed with the help of the normal vector $\boldsymbol{n}$, while push-forward and pull-back are defined without the notion of the normal vector. With above definitions, we may extend the induced metric to be defined in $\mathcal{T}_p(\mathcal{M})$ as $\boldsymbol{\gamma} \equiv \vec{\gamma}^*_{\mathcal{M}}\boldsymbol{\gamma}$. In components, the induced metric is written as

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{D.1.21}$$

[#1] Here and in the following we implicitly identify Σ and $\hat{\Sigma}$ through embedding Φ.

The extrinsic curvature is also extended to be $\boldsymbol{K} \equiv \vec{\gamma}^*_{\mathcal{M}}\boldsymbol{K}$. Below we denote by $\vec{\gamma}^*_{\mathcal{M}}\boldsymbol{T}$ an arbitrary tensor $\boldsymbol{T}$ on Σ extended onto $\mathcal{M}$, with indices for covariant forms extended by $\vec{\gamma}^*_{\mathcal{M}}$ while indices for contravariant tensors extended by push-forward Φ_*. In addition, we define orthogonal projection operation $\vec{\gamma}^*$ as

$$\forall \boldsymbol{T} \in \mathcal{T}_p(\mathcal{M})^p \times \mathcal{T}_p^*(\mathcal{M})^q, \quad (\vec{\gamma}^*\boldsymbol{T})^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \gamma^{\mu_1}{}_{\alpha_1}\cdots\gamma^{\mu_p}{}_{\alpha_p}\gamma^{\beta_1}{}_{\nu_1}\cdots\gamma^{\beta_q}{}_{\nu_q}T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}. \tag{D.1.22}$$

With these definitions, the action of three-dimensional covariant derivative $\boldsymbol{D}$ on an arbitrary tensor $\boldsymbol{T}$ on Σ is related to the four-dimensional covariant derivative as

$$\vec{\gamma}_{\mathcal{M}}\boldsymbol{D}\boldsymbol{T} = \vec{\gamma}^*[\boldsymbol{\nabla}(\vec{\gamma}^*_{\mathcal{M}}\boldsymbol{T})]. \tag{D.1.23}$$

In the LHS, the tensor $\boldsymbol{T}$ is taken derivative in three-dimensional sense, and then extended onto $\mathcal{M}$. On the other hand, in the RHS, the tensor $\boldsymbol{T}$ is first extended onto $\mathcal{M}$ and then taken derivative in four-dimensional sense, and finally projected by the orthogonal projector[#2]. Thus we can extend the three-dimensional covariant derivative to be defined in four-dimension, $\boldsymbol{D} = \vec{\gamma}^*\boldsymbol{\nabla}$. In components,

$$D_\mu T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = \gamma^{\alpha_1}{}_{\gamma_1}\cdots\gamma^{\alpha_p}{}_{\gamma_p}\gamma^{\delta_1}{}_{\beta_1}\cdots\gamma^{\delta_q}{}_{\beta_q}\nabla_\mu T^{\gamma_1\cdots\gamma_p}{}_{\delta_1\cdots\delta_q}. \tag{D.1.24}$$

D.1.4 Coordinate choice, lapse, shift

So far we have considered only one slice in the manifold $\mathcal{M}$. Let us consider many of such slices which cover the entire manifold $\mathcal{M}$

$$\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t. \tag{D.1.25}$$

This family of (spacelike) hypersurfaces is called foliation or slicing. The normal vector $\boldsymbol{n}$ defined above is extended to all slices, and its norm N is called the lapse function. Also, the normalized vector $\boldsymbol{m}$ is called the time evolution vector. We identify the real variable t as the time coordinate, while we assign the spatial coordinates x^i to each slice. Then there exist a natural basis of $\mathcal{T}_p(\mathcal{M})$ associated with the coordinate $(x^\mu) = (t, x^i)$. These basis vectors are written as

$$\boldsymbol{\partial}_t \equiv \frac{\partial}{\partial t}, \quad \boldsymbol{\partial}_i \equiv \frac{\partial}{\partial x^i}. \tag{D.1.26}$$

Though the time vector $\boldsymbol{\partial}_t$ and time evolution vector $\boldsymbol{m}$ both satisfy

$$\langle \boldsymbol{d}t, \boldsymbol{\partial}_t\rangle = 1, \quad \langle \boldsymbol{d}t, \boldsymbol{m}\rangle = 1, \tag{D.1.27}$$

they do not necessarily coincide with each other. This is because, plainly stated, the value of the spatial coordinates do not necessarily remain constant as we move in the direction normal to a slice Σ_t. The shift vector β is defined as the difference between the two vectors

$$\boldsymbol{\partial}_t \equiv \boldsymbol{m} + \boldsymbol{\beta}. \tag{D.1.28}$$

This $\boldsymbol{\beta}$ satisfies $\langle \boldsymbol{d}t, \boldsymbol{\beta}\rangle = 0$, and hence $(\boldsymbol{n}, \boldsymbol{\beta}) = 0$.

[#2] The final result does not depend on how we extend $\vec{\gamma}^*_{\mathcal{M}}\boldsymbol{T}$ outside Σ when we take the covariant derivative.

D.1.5 Lie derivative

When taking a derivative of tensor fields on the manifold $\mathcal{M}$, one cannot naively compare the tensor fields at two different neighboring points. Instead one may introduce a "flow" along which the tensor field is transported, and take the derivative by comparing two tensor fields at the same point. This is so called Lie derivative. It is defied without covariant derivative, and the explicit form takes

$$\mathcal{L}_{\boldsymbol{u}} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = u^\mu \partial_\mu T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} - \sum_{i=1}^{p} T^{\alpha_1\cdots\alpha_{i-1}\nu\alpha_{i+1}\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\partial_\nu u^{\alpha_i} + \sum_{i=1}^{q} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_{i-1}\nu\beta_{i+1}\cdots\beta_q}\partial_{\beta_i} u^\nu. \tag{D.1.29}$$

The partial derivatives in the above expression can be replaced by covariant derivative

$$\mathcal{L}_{\boldsymbol{u}} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} = u^\mu \nabla_\mu T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q} - \sum_{i=1}^{p} T^{\alpha_1\cdots\alpha_{i-1}\nu\alpha_{i+1}\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}\nabla_\nu u^{\alpha_i} + \sum_{i=1}^{q} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_{i-1}\nu\beta_{i+1}\cdots\beta_q}\nabla_{\beta_i} u^\nu. \tag{D.1.30}$$

For vector fields, Lie derivative coincides with the commutator $(\mathcal{L}_{\boldsymbol{u}}\boldsymbol{v})^\mu = u^\nu\partial_\nu v^\mu - v^\nu\partial_\nu u^\mu$.

D.2 Geometric relations

For our purpose, it is enough to decompose the quantities geometrically (without using the Einstein equation). We summarize geometric relations below.

D.2.1 Relations

With the definitions above, projections of the four-dimensional Riemann or Ricci tensor become

- Gauss relation

$$\gamma^\mu{}_\alpha\gamma^\beta{}_\nu\gamma^\gamma{}_\rho\gamma^\delta{}_\sigma\,{}^4R^\alpha{}_{\beta\gamma\delta} = {}^3R^\mu{}_{\nu\rho\sigma} + K^\mu{}_\rho K_{\nu\sigma} - K^\mu{}_\sigma K_{\rho\nu}, \tag{D.2.31}$$

- Contracted Gauss relation

$$\gamma^\alpha{}_\mu\gamma^\beta{}_\nu\,{}^4R_{\alpha\beta} + \gamma_{\mu\alpha}n^\beta\gamma^\gamma{}_\nu n^{\delta\,4}R^\alpha{}_{\beta\gamma\delta} = {}^3R_{\mu\nu} + KK_{\mu\nu} - K_{\mu\alpha}K_\nu{}^\alpha, \tag{D.2.32}$$

- Scalar Gauss relation

$${}^4R + 2n^\mu n^{\nu\,4}R_{\mu\nu} = {}^3R + K^2 - K_{\mu\nu}K^{\mu\nu}, \tag{D.2.33}$$

- Codazzi relation

$$\gamma^{\mu}{}_{\alpha} n^{\beta} \gamma^{\gamma}{}_{\rho} \gamma^{\delta}{}_{\sigma} {}^{4}R^{\alpha}{}_{\beta\gamma\delta} = D_{\sigma} K^{\mu}{}_{\rho} - D_{\rho} K^{\mu}{}_{\sigma}, \tag{D.2.34}$$

- Contracted Codazzi relation

$$\gamma^{\alpha}{}_{\mu} n^{\nu 4} R_{\mu\nu} = D_{\mu} K - D_{\alpha} K^{\alpha}{}_{\mu}. \tag{D.2.35}$$

On the other hand, projection onto other directions give

- Ricci equation

$$\gamma_{\mu\alpha} n^{\beta} \gamma^{\gamma}{}_{\nu} n^{\delta 4} R^{\alpha}{}_{\beta\gamma\delta} = K_{\mu\alpha} K_{\nu}{}^{\alpha} + \frac{1}{N} D_{\mu} D_{\nu} N + \frac{1}{N} (\mathcal{L}_{\boldsymbol{m}} \boldsymbol{K})_{\mu\nu}. \tag{D.2.36}$$

Other useful relations are

- Lie derivative of the extrinsic curvature

$$\mathcal{L}_{\boldsymbol{m}} K = N n^{\mu} \nabla_{\mu} K = N[K^2 + \nabla_{\mu}(n^{\mu} K)], \tag{D.2.37}$$

- Lie derivative of the (inverse) induced metric and orthogonal projector

$$\mathcal{L}_{\boldsymbol{m}} \gamma_{\mu\nu} = -2NK_{\mu\nu}, \qquad \mathcal{L}_{\boldsymbol{m}} \gamma^{\mu}{}_{\nu} = 0, \qquad \mathcal{L}_{\boldsymbol{m}} \gamma^{\mu\nu} = 2NK^{\mu\nu}. \tag{D.2.38}$$

- Relation between the extrinsic curvature and the normal vector

$$K_{\mu\nu} = -\nabla_{\nu} n_{\mu} - a_{\mu} n_{\nu}, \qquad K = -\nabla_{\mu} n^{\mu}, \tag{D.2.39}$$

- Four-acceleration

$$\boldsymbol{a} \equiv \nabla_{\boldsymbol{n}} \boldsymbol{n}, \qquad a_{\mu} = D_{\mu} \ln N. \tag{D.2.40}$$

- 3-dimensional covariant derivative of the lapse function

$$D_{\mu} D^{\mu} N = \nabla_{\mu}(n^{\nu} \nabla_{\nu} n^{\mu}), \tag{D.2.41}$$

From above equations, we obtain the $3+1$ decomposition of the Ricci scalar[#3]

$${}^{4}R = {}^{3}R + K^2 + K_{\mu\nu} K^{\mu\nu} - \frac{2}{N} \mathcal{L}_{\boldsymbol{m}} K - \frac{2}{N} D_{\mu} D^{\mu} N. \tag{D.2.42}$$

[#3]Eliminate $\gamma n \gamma n^4 R$ term from Eqs. (D.2.32) and (D.2.36), contract the two indices in the resulting equation, and combine it with Eq. (D.2.33) to eliminate $\gamma_{\mu\nu}$ and $n_{\mu} n_{\nu}$ using $g_{\mu\nu} = \gamma_{\mu\nu} + n_{\mu} n_{\nu}$.

D.3 Expressions with ADM metric

With the line element

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{D.3.43}$$

the metric and its inverse become

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \frac{1}{N^2}\begin{pmatrix} -1 & \beta^i \\ \beta^i & -\beta^i\beta^j + N^2\gamma^{ij} \end{pmatrix}, \tag{D.3.44}$$

Here we have used $\beta_i = \gamma_{ij}\beta^j$. Also, the normal vector and time-evolution vector take

$$n^\mu = \frac{1}{N}(1, -\beta^i), \qquad n_\mu = (-N, 0), \qquad m^\mu = (1, -\beta^i), \qquad m_\mu = (-N^2, 0). \tag{D.3.45}$$

Focusing on scalar and tensor perturbations, we decompose the quantities in the metric as

$$N = 1 + \alpha, \qquad \beta_i = \partial_i\beta + \beta_{Ti}, \qquad \gamma_{ij} = a^2 e^{2\zeta}(e^h)_{ij}, \tag{D.3.46}$$

with $\beta_{Ti,i} = 0$ and $h_{ii} = h_{ij,j} = 0$. The intrinsic and extrinsic curvature which appear in the $3+1$ decomposition is expressed as[#4]

$$ {}^3R^i{}_{jkl} = \Gamma^i{}_{jl,k} - \Gamma^i{}_{jk,l} + \Gamma^i{}_{mk}\Gamma^m{}_{jl} - \Gamma^i{}_{ml}\Gamma^m{}_{jk}, \tag{D.3.49}$$

$$K_{ij} = -\frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i\beta_j - D_j\beta_i\right), \tag{D.3.50}$$

where the covariant derivative is calculated with the connection

$$\Gamma^i{}_{jk} = \frac{1}{2}\gamma^{il}(\gamma_{lj,k} + \gamma_{lk,j} - \gamma_{jk,l}). \tag{D.3.51}$$

Some examples are

$$ {}^3R = a^{-2}\left[-4\zeta_{,ii} - 2\zeta_{,i}^2 + 8\zeta\zeta_{,ii} + 4\zeta_{,ij}h_{ij} - \frac{1}{4}h_{ij,k}^2 + \frac{1}{2}\zeta h_{ij,k}^2 - 2\zeta_{,k}h_{ij}h_{jk,i} - 2\zeta_{,ik}h_{ij}h_{jk}\right], \tag{D.3.52}$$

$$E = -3H + (\beta^i_{,i} - 3\dot{\zeta}) + 3H\beta^i\zeta_{,i}, \tag{D.3.53}$$

$$\begin{aligned} E_{ij}E^{ij} = 3H^2 &- 2H(\beta^i_{,i} - 3\dot{\zeta}) \\ &+ \frac{1}{4}(\beta^i_{,j} + \beta^j_{,i})^2 - 6H\beta^i\zeta_{,i} - 2\beta^i_{,i}\dot{\zeta} + 3\dot{\zeta}^2 - \beta^i_{,j}\dot{h}_{ij} + \frac{1}{4}\dot{h}_{ij}^2 \\ &+ \frac{1}{2}\left[-\beta^k\dot{h}_{ij}h_{ij,k} - (\beta^i_{,k} - \beta^k_{,i})h_{ij}\dot{h}_{jk}\right], \end{aligned} \tag{D.3.54}$$

up to second order in scalar and tensor perturbations.

[#4] From the time evolution of the induced metric (D.2.38),

$$\dot{\gamma}_{ij} - (\mathcal{L}_{\boldsymbol{\beta}}\boldsymbol{\gamma})_{ij} = -2NK_{ij}. \tag{D.3.47}$$

Noting that $\boldsymbol{D\gamma} = 0$, we have

$$(\mathcal{L}_{\boldsymbol{\beta}}\boldsymbol{\gamma})_{ij} = D_i\beta_j + D_j\beta_i. \tag{D.3.48}$$

D.4 Models

Lastly, we perform 3 + 1 decomposition to the models we used in Chapter 8 and 9, and calculate the constraint equations for the scalar perturbations.

D.4.1 $f(\phi)R$-type coupling

The action is given by

$$S = \int d^4x\sqrt{-g}\left[f(\phi)\,{}^4R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{D.4.55}$$

Using Eqs. (D.2.37) and (D.2.42),

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[f\left({}^3R - K^2 + K_{\mu\nu}K^{\mu\nu} - 2\nabla_\mu(n^\mu K) - \frac{2}{N}D_\mu D^\mu N\right)\right. \\ &\qquad\qquad \left. -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V\right] \\ &= \int d^4x\sqrt{\gamma}\left[Nf\left({}^3R - K^2 + K_{ij}K^{ij}\right) + 2f'K\left(\dot{\phi} - \beta^i\varphi_{,i}\right) - 2ND_iD^if\right. \\ &\qquad\qquad \left. +\frac{1}{2N}\left((\dot{\phi} - \beta^i\varphi_{,i})^2 - N^2\gamma^{ij}\varphi_{,i}\varphi_{,j}\right) - NV\right]. \end{aligned} \tag{D.4.56}$$

The constraint equations for N and β^i become

- Constraint equation for N

$$\begin{aligned} f\left({}^3R + K^2 - K_{ij}K^{ij}\right) - \frac{2}{N}f'K(\dot{\phi} - \beta^i\varphi_{,i}) - 2D_iD^if \\ - \frac{1}{2N^2}(\dot{\phi} - \beta^i\varphi_{,i})^2 - \frac{1}{2}\gamma^{ij}\varphi_{,i}\varphi_{,j} - V = 0, \end{aligned} \tag{D.4.57}$$

- Constraint equation for β^i

$$2D_j\left[f(K\gamma^j{}_i - K^j{}_i)\right] - \frac{2}{N}f'K\varphi_{,i} - 2D_i\left[\frac{1}{N}f'(\dot{\phi} - \beta^j\varphi_{,j})\right] - \frac{1}{N}(\dot{\phi} - \beta^j\varphi_{,j})\varphi_{,i} = 0. \tag{D.4.58}$$

Substituting

$$f = \frac{M_P^2}{2}\left(1 + f_1\frac{\varphi}{M_P}\right), \tag{D.4.59}$$

adopting $\beta = 0$ gauge and taking up to first order in perturbations,

- Constraint equation for N

$$\frac{M_P^2}{2}\left(-4a^{-2}\zeta_{,ii}-4H(3H\alpha-3\dot{\zeta})\right)+3M_PH^2f_1\varphi+3M_PHf_1\dot{\phi}$$
$$-a^{-2}M_Pf_1\varphi_{,ii}-V'(\bar{\phi})\varphi=0, \tag{D.4.60}$$

- Constraint equation for β^i

$$\left[M_P^2(2H\alpha_{,i}-2\dot{\zeta}_{,i})-2M_PHf_1\varphi_{,i}\right]+3M_PHf_1\varphi_{,i}-M_Pf_1\dot{\varphi}_{,i}=0. \tag{D.4.61}$$

With the background equation of motion $V'(\bar{\phi})=6M_PH^2f_1$, the solution is obtained as

$$\alpha=\zeta=-\frac{f_1}{2M_P}\varphi. \tag{D.4.62}$$

D.4.2 $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$-type coupling

We take $\varphi=0$ gauge. Then, the action becomes

$$S=\int dx^4\sqrt{-g}\left[\frac{M_P^2}{2}\,{}^4R-\left(g^{00}-\frac{G^{00}}{M^2}\right)\frac{\dot{\bar{\phi}}^2}{2}-V\right]. \tag{D.4.63}$$

Using Eqs. (D.2.37) and (D.2.42), and also using Eq. (D.2.33), we rewrite the action as[#5]

$$S=\int d^4x\sqrt{\gamma}\left[{}^3R\left(\frac{M_P^2N}{2}+\frac{\dot{\bar{\phi}}^2}{4M^2N}\right)+\left(K_{ij}K^{ij}-K^2\right)\left(\frac{M_P^2N}{2}-\frac{\dot{\bar{\phi}}^2}{4M^2N}\right)+\frac{\dot{\bar{\phi}}^2}{2N}-NV\right]. \tag{D.4.66}$$

This action leads to the constraint equations

- Constraint equation for N

$${}^3R\left(\frac{M_P^2}{2}-\frac{\dot{\phi}^2}{4M^2N^2}\right)+\left(K_{ij}K^{ij}-K^2\right)\left(-\frac{M_P^2}{2}+\frac{3\dot{\phi}^2}{4M^2N^2}\right)-\frac{\dot{\phi}^2}{2M_P^2N^2}-V=0. \tag{D.4.67}$$

[#5] From Eq. (D.2.33),

$${}^4R^{00}=\frac{1}{2N^2}\left(-{}^4R+{}^3R+K^2-K_{\mu\nu}K^{\mu\nu}\right), \tag{D.4.64}$$

is found. Then,

$$G^{00}=R^{00}-\frac{1}{2}g^{00}\,{}^4R=\frac{1}{2N^2}\left({}^3R+K^2-K_{\mu\nu}K^{\mu\nu}\right), \tag{D.4.65}$$

is obtained.

- Constraint equation for β^i

$$D_i\left[\left(\frac{M_P^2N}{2}-\frac{\dot{\phi}^2}{4M^2N}\right)\left(K^i{}_j-\gamma^i{}_jK\right)\right]=0. \tag{D.4.68}$$

To first order, the relations among α, β and ζ are given by

$$\alpha=\frac{A}{H}\dot{\zeta},\quad \beta=-\frac{A}{H}\zeta+\xi,\quad \partial_i^2\xi=a^2\frac{\dot{\phi}^2}{2M_P^2H^2}\frac{A^2B}{1-\frac{1}{2}\epsilon}\dot{\zeta}, \tag{D.4.69}$$

where

$$A=\frac{1-\frac{1}{2}\epsilon}{1-\frac{3}{2}\epsilon},\qquad B=1+\frac{3H^2}{M^2}\frac{1+\frac{3}{2}\epsilon}{1-\frac{1}{2}\epsilon},\qquad \epsilon\equiv\frac{\dot{\phi}^2}{M^2M_P^2}. \tag{D.4.70}$$