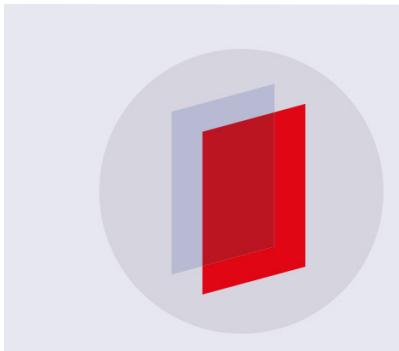


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# Sasaki-Einstein space $T^{1,1}$ , transverse Kähler-Ricci flow and Sasaki-Ricci soliton

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**Abstract.** We examine the compact Sasaki manifolds in view of transverse Kähler geometry and transverse Kähler-Ricci flow. In particular we study the transverse Kähler geometry of the five-dimensional Sasaki-Einstein space  $T^{1,1}$ . For this purpose a set of local holomorphic coordinates is introduced and a Sasaki-analogue of the Kähler potential is produced. There are considered deformations of the contact structure fixing the Reeb foliation while varying the Kähler metric on the transverse holomorphic structure.

## 1. Introduction

Sasakian geometry is an important odd-dimensional counterpart of Kähler geometry. Recently, the subject of Sasakian geometry has gathered a great deal of interest in mathematics [1, 2] and theoretical physics in connection with the AdS/CFT correspondence [3].

AdS/CFT correspondence, in the most interesting dimensions, provides a duality between field theories and string theories on  $AdS_5 \times M$  where supersymmetry requires the five dimensional Euclidean manifold  $M$  to have a real Killing spinor [4]. In particular Sasaki-Einstein manifolds  $M$  whose metric cone  $C(M)$  are Calabi-Yau find applications in string theory where they provide explicit tests of AdS/CFT duality. New classes of Sasaki-Einstein structures on  $S^2 \times S^3$ , denoted  $Y^{p,q}$ , have been constructed in [5] which contain the homogeneous space  $T^{1,1} = Y^{1,0}$  as a special case [6].

Sasaki manifolds have a one dimensional foliation, called the Reeb foliation, which has a transverse Kähler structure. The Sasaki-Ricci flow was introduced in [7] to study the existence of Sasaki-Einstein metrics. The Sasaki-Ricci flow is just a transverse Kähler-Ricci flow which deforms the transverse Kähler structure.

The aim of this paper is to investigate the transverse Kähler-Ricci flow in the case of the Sasaki-Einstein space  $T^{1,1}$ . We consider the deformation of the Sasaki structure fixing the Reeb foliation, while varying the Kähler metric on the transverse holomorphic structure and, as a result, the contact structure.

The paper is organized as follows. In the next Section we review Sasaki geometry and describe the transverse Kähler geometry. In Section 3 we investigate the transverse Kähler structure of the Sasaki space  $T^{1,1}$ . In order to describe the transverse geometry and its deformation we introduce local holomorphic coordinates. The Sasaki-Einstein space represents a steady soliton for the transverse Kähler-Ricci flow. In the last Section we provide some closing remarks.

## 2. Sasakian manifolds and Sasaki-Ricci flow

Here we recall the definitions and main facts about Sasakian structures and their deformations. For more details we refer to [1].

### 2.1. Contact structures and Sasaki manifolds

Let  $(M, g^M)$  be a  $2n+1$  dimensional manifold,  $\nabla^M$  the Levi-Civita connection of the Riemannian metric  $g^M$ , and let  $R^M(X, Y)$  and  $\text{Ric}^M$  respectively denote the Riemannian curvature and Ricci tensor of  $\nabla^M$ .

By a *contact manifold* we mean a  $C^\infty$  manifold  $M^{2n+1}$  together with a 1-form  $\eta$  such that  $\eta \wedge (\frac{1}{2}d\eta)^n \neq 0$ . There is a canonical vector field  $\xi$  called the characteristic vector field or Reeb vector field defined by the contractions (interior products):

$$\iota_\xi \eta = 1 \quad \text{and} \quad \iota_\xi d\eta = 0. \quad (1)$$

$\eta$  defines a  $2n$ -dimensional vector bundle  $\mathcal{D}$  over  $M$  where at each point  $p \in M$  the fiber  $\mathcal{D}_p$  of  $\mathcal{D}$  is given by

$$\mathcal{D}_p = \ker \eta_p. \quad (2)$$

The tangent bundle of  $M$  may be decomposed into

$$TM = \mathcal{D} \oplus L_\xi, \quad (3)$$

where  $L_\xi$  is the trivial bundle generated by the Reeb vector field  $\xi$ .

There also exists a  $(1, 1)$ -tensor field  $\Phi$  defined by

$$\Phi(X) = \nabla_X \xi, \quad (4)$$

for any smooth vector field  $X$  on  $M$ . The tensor field  $\Phi$  satisfies

$$\Phi^2 = -I + \eta \otimes \xi, \quad (5)$$

and

$$g^M(\Phi X, \Phi Y) = g^M(X, Y) - \eta(X)\eta(Y), \quad (6)$$

for any pair of vector fields  $X$  and  $Y$ .

$(M, g^M)$  is said to be a *Sasaki manifold* if the cone manifold

$$(C(M), \bar{g}) = (\mathbb{R}_+ \times M, dr^2 + r^2 g^M), \quad (7)$$

is Kähler. For a Sasaki manifold an equivalent condition holds:

$$R^M(X, \xi)Y = g^M(\xi, Y)X - g^M(X, Y)\xi, \quad (8)$$

for any pair of vector fields  $X$  and  $Y$  on  $M$ .

A Sasaki manifold  $(M, \xi, \eta, \Phi, g^M)$  is a Sasaki-Einstein manifold if the metric  $g^M$  is Einstein, i.e.

$$\text{Ric}^M = c g^M, \quad (9)$$

for some constant  $c$ . Taking into account (8) we have  $c = 2n > 0$ .

$\mathcal{D}$  it is naturally endowed with both a complex structure  $\Phi|_{\mathcal{D}}$  and a symplectic structure  $d\eta$ .  $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$  gives  $M$  a transverse Kähler structure with Kähler form  $\frac{1}{2}d\eta$  and the transverse metric  $g^T$  is

$$g^T(X, Y) = d\eta(X, \Phi Y), \quad (10)$$

which is related to the Sasakian metric  $g^M$  by

$$g^M = g^T + \eta \otimes \eta. \quad (11)$$

For the transverse metric  $g^T$  one can define a connection  $\nabla^T$  on  $\mathcal{D}$  which is the unique torsion free such that  $\nabla^T g^T = 0$ . One can check that

$$\text{Ric}^T(X, Y) = \text{Ric}^M(X, Y) + 2g^T(X, Y). \quad (12)$$

For a Sasaki-Einstein manifold, we get that the transverse metric is also Einstein

$$\text{Ric}^T = 2(n+1)g^T. \quad (13)$$

We may introduce a foliation chart  $\{U_\alpha\}$  on  $M$ , where each  $U_\alpha$  is of the form  $U_\alpha = I \times V_\alpha$  with  $I \subset \mathbb{R}$  an open interval, and  $V_\alpha \subset \mathbb{C}^n$ .

Let  $(x, z^1, \dots, z^n)$  be the local holomorphic coordinates on  $U_\alpha$  such that the Reeb vector field is  $\xi = \partial_x$  and  $z^1, \dots, z^n$  are local holomorphic coordinates on  $V_\alpha$ . Moreover it is possible to introduce a real-valued function  $K$  on  $U_\alpha$  such that  $\xi(K) = 0$  and [8]

$$\eta = dx + i \sum_{j=1}^n K_{,j} dz^j - i \sum_{j=1}^n K_{,\bar{j}} d\bar{z}^j, \quad (14)$$

$$d\eta = -2i \sum_{j,k=1}^n K_{,j\bar{k}} dz^j \wedge d\bar{z}^k, \quad (15)$$

$$g^T = 2 \sum_{j,k=1}^n K_{,j\bar{k}} dz^j d\bar{z}^k, \quad (16)$$

$$\Phi = -i \sum_{j=1}^n [(\partial_j - iK_{,j}\partial_x) \otimes dz^j] + i \sum_{j=1}^n [(\partial_{\bar{j}} + iK_{,\bar{j}}\partial_x) \otimes d\bar{z}^j]. \quad (17)$$

The function  $K$  is a Sasakian analogue of the Kähler potential for the Kähler geometry. The fundamental two-form of the transverse Kähler manifold is given by

$$\Omega = -i \sum_{j,k=1}^n K_{,j\bar{k}} dz^j \wedge d\bar{z}^k = \frac{1}{2} d\eta. \quad (18)$$

Note that the Sasakian potential is not unique, but admit a Kähler transformation

$$K(z, \bar{z}) \longrightarrow K(z, \bar{z}) + f(z^j) + \bar{f}(\bar{z}^j), \quad (19)$$

where  $f(z)$  and  $\bar{f}(\bar{z})$  are arbitrary holomorphic and anti-holomorphic functions.

It has been often remarked that Kähler geometry is connected with a  $U(1)$  gauge theory whose strength is identified with the Kähler form [9]. Indeed the Kähler form (18) can be written as

$$\Omega = d\mathcal{A} \quad \text{and} \quad A = \frac{i}{2}(\bar{\partial} - \partial)K(z, \bar{z}), \quad (20)$$

where the exterior differential operator is given by  $d = \partial + \bar{\partial}$  with  $\partial = dz^j \frac{\partial}{\partial z^j}$  and  $\bar{\partial} = d\bar{z}^j \frac{\partial}{\partial \bar{z}^j}$ . The one form  $\mathcal{A}$  is associated with  $U(1)$  gauge fields and the transformation (19) corresponds to the gauge transformation

$$\mathcal{A} \longrightarrow \mathcal{A} + \Lambda, \quad (21)$$

where  $\Lambda = \frac{i}{2}(\bar{f}(\bar{z}) - f(z))$ .

## 2.2. Deformations of Sasaki structures

We shall consider deformations of a Sasaki structure fixing the Reeb field  $\xi$  and varying the 1-form  $\eta$ .

For the beginning we introduce the *basic r-forms*  $\alpha$  on  $M$  which satisfy

$$\iota_\xi \alpha = 0 \quad , \quad \mathcal{L}_\xi \alpha = 0 , \quad (22)$$

where  $\mathcal{L}_\xi$  is the Lie derivative with respect to the vector field  $\xi$ . In particular a function  $f$  is basic if and only if  $\xi(f) = 0$ . We observe that  $d\eta$  is a basic 2-form, though  $\eta$  is not a basic form.

In the system of coordinates  $(x, z^1, \dots, z^n)$  given above, a basic  $r$ -form of type  $(p, q)$ ,  $r = p+q$  has the form

$$\alpha = \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q} , \quad (23)$$

where  $\alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}$  does not depend on  $x$ .

The space of global basic  $r$ -forms will be denoted by  $\Lambda_B^r(M)$  and the transverse complex structure  $\Phi|_{\mathcal{D}}$  allows us to decompose

$$\Lambda_B^r \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda_B^{p,q} . \quad (24)$$

The exterior derivative maps basic forms to basic forms and it is justified to define  $d_B = d|_{\Lambda_B^r}$  so that

$$d_B : \Omega_B^r \rightarrow \Omega_B^{r+1} , \quad (25)$$

and the subcomplex  $(\Lambda_B^*(M), d_B)$  is called *basic de Rham complex*. One can also construct the basic Dolbeault operators  $\partial_B$  and  $\bar{\partial}_B$  with the usual properties on complex manifolds

$$\begin{aligned} \partial_B : \Lambda_B^{p,q} &\rightarrow \Lambda_B^{p+1,q} , \\ \bar{\partial}_B : \Lambda_B^{p,q} &\rightarrow \Lambda_B^{p,q+1} , \end{aligned} \quad (26)$$

and  $d_B = \partial_B + \bar{\partial}_B$ . For what follows it is useful to introduce the operator  $d_B^c = \frac{i}{2}(\bar{\partial}_B - \partial_B)$ .

A particular basic cohomology class of  $H_B^{1,1}(S)$  is the *first basic Chern class*, defined by the basic class  $c_1^B(M) := [\frac{1}{2\pi}\rho^T]_B$  where  $\rho^T = \text{Ric}^T(\Phi \cdot, \cdot)$  is the transverse Ricci form. Moreover the real first Chern class  $c_1(\mathcal{D})$  of the vector bundle  $\mathcal{D}$  vanishes if, and only if, there exists  $a \in \mathbb{R}$  such that  $c_1^B(M) = [a d\eta]_B$  [1, 10]. We shall apply the general results to the Sasaki-Einstein space  $T^{1,1}$  which is toric and has the basic first Chern class positive  $c_1^B > 0$  and  $c_1(\mathcal{D}) = 0$  [11].

By means of a certain basic function  $\varphi$  we introduce the following deformations of the contact form

$$\tilde{\eta} = \eta + d_B^c \varphi . \quad (27)$$

This deformation modifies the transverse form

$$d\tilde{\eta} = d\eta + d_B d_B^c \varphi , \quad (28)$$

in the same transverse Kähler class  $c_1^B = \kappa[\frac{1}{2}d\eta]_B$ . The other tensors vary as follows

$$\begin{aligned} \tilde{\Phi} &= \Phi - \xi \otimes (d_B^c \varphi) \circ \Phi , \\ \tilde{g} &= d\tilde{\eta} \circ (\text{Id} \otimes \Phi) + \tilde{\eta} \otimes \tilde{\eta} . \end{aligned} \quad (29)$$

For such deformations, we consider the following flow  $(\xi, \eta(t), \Phi(t), g(t))$  with initial data  $(\xi, \eta(0), \Phi(0), g(0)) = (\xi, \eta, \Phi, g^M)$ .

The Sasaki-Ricci flow is defined by

$$\frac{\partial g^T}{\partial t} = -\text{Ric}_{g(t)}^T + \kappa g^T(t). \quad (30)$$

In what follows we take  $\kappa = 2n + 2$  in agreement with the normalization of the Sasaki-Einstein metric (13).

A Sasakian structure  $(M, \xi, \eta, \Phi, g^M)$  with a Hamiltonian holomorphic vector field  $X$  is called a *transverse Kähler-Ricci soliton* or *Sasaki-Ricci soliton* if [11]

$$\text{Ric}^T - (2n + 2)g^T = \mathcal{L}_X g^T. \quad (31)$$

In particular any Kähler-Einstein metric is a steady soliton with  $X = 0$  [12].

### 3. The Sasaki-Ricci flow on $T^{1,1}$ manifold

The metric on the homogeneous toric Sasaki-Einstein space  $T^{1,1} = S^2 \times S^2$  is [13, 6]

$$ds^2(T^{1,1}) = \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2. \quad (32)$$

Here  $\theta_i, \phi_i, i = 1, 2$  are the usual coordinates on two round  $S^2$  spheres and the angle  $\psi \in [0, 4\pi)$  parametrizes the  $U(1)$  fiber over  $S^2 \times S^2$ .

The global defined contact 1-form  $\eta$  is

$$\eta = \frac{1}{3}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2), \quad (33)$$

and the corresponding Reeb vector is

$$\xi = 3 \frac{\partial}{\partial \psi}. \quad (34)$$

Using the contact form  $\eta$  (33), the transverse metric  $g^T$  (11) is

$$g^T = \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2). \quad (35)$$

To investigate the leaf space of the foliation induced by the Reeb field, we have to parametrize the foliation chart  $\{U_\alpha\}$  on  $T^{1,1}$  where each  $U_\alpha = I \times V_\alpha$  with  $I \subset \mathbb{R}$  and  $V_\alpha \subset \mathbb{C}^2$  as described in the previous Section.

Let the coordinates  $(x, z^1, z^2)$  where  $\xi = \partial_x$  and  $(z^1, z^2)$  are complex coordinates on  $V_\alpha$ . Taking into account the form (34) of the Reeb vector field we choose the real coordinate

$$x = \frac{1}{3}\psi. \quad (36)$$

Concerning the complex coordinates  $(z^1, z^2)$ , it proves that the complex coordinates on the metric  $C(T^{1,1})$  [14] are not helpful. Unfortunately it seems that there is no direct procedure to get the complex coordinates to describe the transverse Kähler geometry of those of the metric cone.

The natural way to find the complex coordinates  $(z^1, z^2)$  is to take into account the fact that  $T^{1,1}$  is a  $U(1)$  bundle over  $S^2 \times S^2$ , the  $U(1)$  fiber being parametrized by the real coordinate  $x$  (36). On each  $S^2$  sphere the complex coordinate  $z^j$  is related to the spherical coordinates by

$$z^j = \tan \frac{\theta_j}{2} e^{i\phi_j}, \quad j = 1, 2, \quad (37)$$

and the standard  $S^2$  metric is

$$ds_j^2 = d\theta_j^2 + \sin^2 \theta_j d\phi_j^2 = 4 \frac{dz^j d\bar{z}^j}{(1 + z^j \bar{z}^j)^2}, \quad j = 1, 2 \quad (\text{no summation}). \quad (38)$$

In terms of the complex coordinates (37), the Sasakian potential generating the transverse metric  $g^T$  (35), according to (16), could be chosen to be of the form

$$K = \frac{1}{3} \sum_{j=1}^2 \log(1 + z^j \bar{z}^j). \quad (39)$$

However this potential does not provide the contact form (33). This deficiency can be corrected using the fact that the Sasakian potential is defined up to a gauge transformation (19). We shall perform a gauge transformation with a function

$$f(z^j) = -\frac{1}{6} \log z^j, \quad j = 1, 2, \quad (40)$$

such that Sasakian potential becomes

$$K = \frac{1}{3} \sum_{j=1}^2 \log(1 + z^j \bar{z}^j) - \frac{1}{6} \sum_{j=1}^2 \log(z^j \bar{z}^j). \quad (41)$$

One can check that this potential provides the contact  $\eta$  (33) according to (14).

Concerning the transverse Kähler-Ricci flow, we note that the Sasaki space  $T^{1,1}$  is Einstein and, according to (31), represents a steady soliton with the Hamiltonian holomorphic vector field  $X = 0$ .

In general a complex vector field  $X$  on  $M$ , commuting with  $\xi$ , is called Hamiltonian vector field if its projection  $d\pi_\alpha(X)$  is a holomorphic vector field on  $V_\alpha$  and the basic function  $u_X := i\eta(X)$  is such that [11, 10]:

$$\bar{\partial}_B u_X = -\frac{i}{2} \iota_X d\eta. \quad (42)$$

The basic function  $u_X$  is called a Hamiltonian function. It is worth noting that the Hamiltonian function  $u_X$  is a constant if the holomorphic vector field  $X$  is proportional to the Reeb vector field  $\xi$ .

In the system of coordinates  $(x, z^1, z^2)$  on  $U_\alpha$ , a vector field can be written as

$$X = a \frac{\partial}{\partial x} + \sum_{j=1}^2 X^j \frac{\partial}{\partial z^j}, \quad (43)$$

where  $X^j$  are local holomorphic basic functions. With this parametrization

$$u_X = i\eta(X) = i \left( a + i \sum_{j=1}^2 X^j K_{,j} \right), \quad (44)$$

is a basic function if  $a$  and  $X^j$  do not depend on  $x$ . Therefore the vector  $X$  can be written as

$$X = \left( \eta(X) - i \sum_{j=1}^2 X^j K_{,j} \right) \frac{\partial}{\partial x} + \sum_{j=1}^2 X^j \frac{\partial}{\partial z^j}. \quad (45)$$

Moreover, condition (42) is satisfied since

$$\bar{\partial}_B u_X = - \sum_{j,k=1}^2 X^j K_{j\bar{k}} d\bar{z}^k = -\frac{i}{2} \iota_X d\eta. \quad (46)$$

Finally we mention that by a deformation (27) of the contact form, the Hamiltonian function  $u_X$  is deformed to

$$\tilde{u}_X = i\tilde{\eta}(X) = u_X + X\varphi. \quad (47)$$

#### 4. Conclusions

In the last time, there has been many efforts to extend the symplectic Hamiltonian mechanics to contact Hamiltonian mechanics. These attempts are motivated by the necessity to describe the systems with dissipation or time dependent Hamiltonians [15]. In the contact geometry the Poisson brackets are no more available and the analogue of Hamiltonian vector fields will be the contact Hamiltonian vector fields.

On the other hand the contact geometry has proved its usefulness in gauge theories of gravity, black holes in higher dimensions, string theories. A prominent role is played by the Sasaki-Einstein manifolds in connection with the AdS/CFT correspondence. The integrability in contact geometry was introduced in [16, 17] and after that the subject has seen a large development, see, e.g., [18, 19, 20].

A Sasakian structure sits between two Kähler structures, namely the one on its metric cone and the one on the normal bundle of its Reeb foliation. In this paper we examined the Kähler structure of the transverse Kähler geometry and considered possible deformations of the contact structures.

We exemplified the general results in the case of the five-dimensional Sasaki-Einstein space  $T^{1,1}$ . We introduced local holomorphic coordinates and constructed the Sasakian potential, analogous to the Kähler potential.

It is worth extending the study of the transverse Kähler structure to the five-dimensional Sasaki-Einstein spaces  $Y^{p,q}$  as well as higher-dimensional contact spaces. Starting with a Sasaki soliton and using the Sasaki-Ricci flow it would be interesting to generate new contact structures.

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