Quantum Conformal Algebra with Central Extension

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The structure of quantum group has appeared originally in the studies of the properties of the Yang-Baxter equation [1,2]. Subsequently, a mathematical formulation of quantum groups was given in terms of Hopf algebras [3,4] and the q-deformation of enveloping algebras [5,6]. Deformation of the Heisenberg group of a simple harmonic oscillator yields a q-oscillator while for the case of the spin we have the $SU(2)_q$ group [7,8]. Deformation of superalgebras, as well as infinite demensional algebras, such as Virasoro algebra have also been studied [9,10,11]. There is also an intimate connection between quantum groups as dynamical groups in lattice models [12], non commutative geometry [13] and conformal field theories [14] which will not be discussed in this review. Here we will give a short presentation of the basic notions and some applications of the quantum groups in quantum optics [15], in the quantum conformal algebra with central extension [16] and in the q-deformed analogue of the KdV equation [17].

Let us consider the generators $J_3, J_\pm$ of the $SU(2)_q$ algebra with commutation relations (CR):

$$[J_3, J_\pm] = \pm J_\pm , \qquad (1)$$

$$[J_+, J_-] = [2J_3] , \qquad (2)$$

where we use the notation $[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}$ and $q = e^\hbar$, and so $[x] = \frac{\sin \hbar \hbar x}{\sin \hbar \hbar}$.

The Casimir operator is given by $C_2(q) = [J_3 + \frac{1}{2}]^2 + J_- J_+ = [J_3 - \frac{1}{2}]^2 + J_- J_+ = J(J+1)$ where J is defined in terms of $C_2(q)$ and for given Casimir

index j we have a $2j + 1$-dimensional matrix realization,

$$J_3 = \sum_{m=j}^{j} m|m><m| \qquad (3)$$

and

$$J_\pm = \sum_{m=-j}^{j} \sqrt{[j \mp m][j \pm m + 1]} \, |m \pm 1><m| . \quad (4)$$

Obviously the limit $\hbar \to 0$, $q \to 1$ restores the CR of the ordinary angular momentum algebra. This limit becomes more transparent writting [11]

$$J_+ = \sqrt{\frac{[J_3 - J][J_3 - 1 - J]}{[J_3 + J][J_3 - 1 - J]}} S_+ , \quad J_- = (J_+)^\dagger ,$$

$$J_3 = S_3 , \qquad (5)$$

where $S_3, S_\pm$ are the usual $SU(2)$ generators. This factorization of the deformation of the algebra exibits the nature of the deformation as a non unitary tranformation of the S_3, $S_\pm$ generators to the deformed ones J_3, $J_\pm$. Also, we notice that we deformed the algebra asymmetically, namely $S_\pm \to J_\pm$ but $S_3 \to J_3 \equiv S_3$. From this point on a whole representation theory is been constructed for $SU(2)_q$: tensor product, comultiplication, Clebsh-Gordon coefficients etc. which we will not enter here [3,4]. The most natural way to define the q-oscillator and its deformed CR is by the method of group contraction. Indeed as is the case for the non-deformed Heisenberg-Weyl algebra, which can be obtained from the contraction of the $SU(2)$ algebra, similarly here contraction of the $SU(2)_q$ yields the quantum Heisenberg-Weyl algebra. We define the contration limit as, $j \to \infty$, $(q \geq 1)$ and $q^j \to \infty$, while

$[j] = \frac{q^j - q^{-j}}{q - q^{-1}} \rightarrow q^j(q - q^{-1})$. Then contracting the generators we have $h_{\pm} \equiv \frac{J_{\pm}}{\sqrt{[2j]}}$ and $h_3 \equiv J_3 + 1$ which yields for the CR:

$$[h_+, h_-] = \frac{[2h_3 - 2j\mathbf{1}]}{[2j]} \qquad (6)$$

and

$$[h_3, h_{\pm}] = h_{\pm} . \qquad (7)$$

In matrix form,

$$h_- = \begin{bmatrix} 0 & \sqrt{[1]} & & & \\ & 0 & q^{-1/2}\sqrt{[2]} & & \\ & & 0 & q^{-1}\sqrt{[3]} & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix}$$

$$= q^{-h_3/2}a(q) = q^{1/2}a(q)q^{-h_3/2} , \qquad (8)$$

where we have introduced the annihilation operator $a(q)$ by

$$a(q) = \sum_{n=0}^{\infty} \sqrt{[n]} \, |n - 1>_q \; _q<n| . \qquad (9)$$

Finally, the contraction of the commutator in eq. (6) gives,

$$qq^{-h_3/2}a^\dagger a q^{-h_3/2} - q^{-h_3/2}aa^\dagger q^{-h_3/2} = -q^{-2h_3} , \; (10)$$

which is written as,

$$aa^\dagger - qa^\dagger a = q^{-N} , \qquad (11)$$

where $N = \sum_{n=0}^{\infty} n|n>_q \; _q<n|$ is the contraction limit of the h_3 operator and it stands for the number operator of the q-oscillator. The dynamics of such a q-oscillator and its physical meaning is still an open problem [18]. The CR in eq. (11) can be written in a more familiar form:

$$bb^\dagger - q^2 b^\dagger b = 1 , \qquad (12)$$

$$cc^\dagger - c^\dagger c = q^{-2N} , \qquad (13)$$

where $b \equiv q^{N/2}a$ and $c = q^{-\frac{N}{2}}a$. Finally, a deformed, non-deformed relation similar to that in eq.(5) exists for the oscillator,

$$a \equiv \left(\frac{[N+1]}{N+1}\right)^{\frac{1}{2}}\beta \; ; \; a^\dagger \equiv \beta^\dagger\left(\frac{[N+1]}{N+1}\right)^{\frac{1}{2}} , \quad (14)$$

where $[\beta, \beta^\dagger] = 1$ and $a^\dagger a = [N]$ and $aa^\dagger = [N+1]$.

The deformed oscillators are now particularly useful in studing the deformation of several algebras of interest which posses bosonic (Schwinger-like) realizations. With applications in the Conformal Field Theory in mind one can study deformations of the Virasoro algebra generalizing its usual bosonic realization to one with deformed Bose operators [10,11]:

$$L_n = (a^\dagger)^{n+1}a , \qquad (15)$$

with C.R. in the centreless case:

$$[L_n, L_m]_{q^{m-n}} = q^{-N+m}[m - n]L_{n+m} , \qquad (16)$$

with the definition $[A, B]_{(p,q)} \equiv pAB - qBA$. Any central extension of the q-Virasoro algebra should satisfy the deformed Jacobi identity, which in its most general form is:

$$[A, [B, C]_{(q_1, q_1^{-1})}]_{(\frac{q_3}{q_2}, \frac{q_2}{q_3})} + [B, [C, A]_{(q_2, q_2^{-1})}]_{(\frac{q_1}{q_3}, \frac{q_3}{q_1})}$$
$$+ [C, [A, B]_{(q_3, q_3^{-1})}]_{(\frac{q_2}{q_1}, \frac{q_1}{q_2})} = 0 \qquad (17)$$

where q_1, q_2, q_3 arbitrary numbers.

At this stage, and in order to be able to evaluate the q-commutators in the Jacobi identity, we impose the invariance of the $a(q)$, $a(q)^\dagger$, and $L_n(q)$, $N(q)$ on the substitution $q \rightarrow q^{-1}$. Then we define a new set of generators $\hat{L}_n(q) = q^{-N}L_n$ with C.R.:

$$[\hat{L}_n, \hat{L}_m]_{(q^{m-n}, q^{n-m})} = [m - n]\hat{L}_{n+m} , \qquad (18)$$

for which follows that $\hat{L}_n(q^{-1}) = q^{2N}\hat{L}_n(q)$, which turns the above q-commutator to a usual one:

$$[\hat{L}_n, \hat{L}_m] = [m - n]q^{-2N+n+m}\hat{L}_{n+m} . \qquad (19)$$

To gain full generality we treat the last commutator between generators of the q-Virasoro algebra in the abstract level and we will define the "number operator" N by:

$$N = -\frac{1}{2\ln q}\ln[1 - (q - q^{-1})\hat{L}_0] . \qquad (20)$$

Assuming the eq. (18) as valid in the abstract level as well, we now consider the central extension of the q-deformed Virasoro algebra as follows:

$$[\hat{L}_n, \hat{L}_m] = [m - n]q^{-2N+n+m}\hat{L}_{n+m} + c(n, m, q), \; (21)$$

which reads using (18):

$$[\hat{L}_n, \hat{L}_m]_{(q^{m-n}, q^{n-m})} = [m - n]\hat{L}_{n+m} + c(n, m, q). \; (22)$$

A straight forward calculation shows that the usual, as well as the q-Jacobi identity, is satisfied if the central term obeys the equations [16]:

$$\frac{[k-m]}{[n-m-k]}([n]q^{-4N}c(n+m,n,q^{-1}) \qquad (23)$$

$$+[m+k-2n]c(m+k,n,q)) + cycl.\ perm. = 0$$

and

$$c(n,0,q) = c(n,0,q^{-1}) = 0 \ . \qquad (24)$$

Equation (22) in the non-deformed limit $q \to 1$ reduces to the usual constraint imposed on the central term for the usual Virasoro, however its solution (λ arbitrary),

$$c(n,m) \equiv c(n,m,q) = c(n,m,q^{-1})$$
$$+\lambda \cdot [2n]\delta_{n+m,0} \ , \qquad (25)$$

does not. This is due to the fact that in the r.h.s. of eq.(19) the existence of the operator N forbids the removal of terms such as $c(1,1)$ in the central extension as is happend in the non-deformed problem. Thus, the algebra in eq.(19) with N defined as in eq.(20) and with the central term given by (24) appears to be the only admissible q-deformed Virasoro algebra with the central extension if one requires the fulfilmnet of the general Jacobi identity. In case one removes the requirement of general Jacobi identity but instead of it introduces a cubic equality containing only q-commutators, one obtains the central term in the form [19,20]

$$c(n.m) = (c_1[n] + c_2\frac{[n-1][n][+1]}{2})\delta_{n+m,0} \ ,$$

with two arbitrary parameters c_1 and c_2, which in the $q \to 1$ limit recovers the usual non-deformed central term.

We now turn to an application of the q-oscillators which allow us to probe the changes on the dynamics of the deformation [15]. We consider a fundamental quantum optical system, the Jaynes-Cummings Model (JCM), which describes the coupling of a single bosonic mode (laser field) with a two-level atom during the passing of a beam of such atoms through a cavity. For the coherent excitation of the atoms we consider the JCM Hamiltonian with intensity-dependent coupling, which physically signifies the fact that the strength of the laser-atom interaction is analogous to the number of photons in the cavity:

$$H_{int} = \lambda(\sqrt{N}\beta^\dagger\sigma^- + \beta\sqrt{N}\sigma^+) \ , \qquad (26)$$

where N is the bosonic number operator ($[\beta,\beta^\dagger] = 1$) and $\sigma^\pm$ are the Pauli matrices. Here we have also assume valid the Rotating Wave Approximation (RWA). Utilizing the Holstein-Primakoff (HP) realization of the $su(1,1)$ algebra we can write the Hamiltonian as:

$$H_{int} = \lambda(L_+\sigma^- + L_-\sigma^+) \ , \qquad (27)$$

where $L_+ = \sqrt{N}a^\dagger$, $L_- = a\sqrt{N}$, $L_0 = N + \frac{1}{2}$ the $su(1,1)$ generator. The deformation of this model formally is consisted in replacing the usual bosons, $\beta,\beta^\dagger$, by deformed bosons $a,a^\dagger$, (see eq.(11)). To see what physically such replacement means we write the deformed interaction Hamiltonian:

$$H_{int}^{(q)} = \lambda(\sqrt{[N]}a^\dagger\sigma^- + a\sqrt{[N]}\sigma^+) \ , \qquad (28)$$

which using the q-analogue of HP realization for the $su(1,1)_q$ algebra:

$$K_+ = \sqrt{[N]}a^\dagger = L_+\frac{[N+1]}{N+1} \ , \ K_- = a\sqrt{[N]}$$
$$= \frac{[N+1]}{N+1}L_- \ , \ K_0 = L_0 \ , \qquad (29)$$

can be written as:

$$H_{int}^{(q)} = \lambda(K_+\sigma^- + K_-\sigma^+) = \lambda(L_+\sigma^-\frac{[N+1]}{N+1}$$
$$+\frac{[N+1]}{N+1}L_-\sigma^+) \ . \ (30)$$

The last form of the Hamiltonian shows that the deformation of bosons introduces a q-dependence in the coupling constant in addition to its dependence from the field intensity. Also it is clear that the dynamical algebra of the model from $su(1,1)\oplus su(2)$ becomes after the deformation the quantum algebra $su(1,1)_q \oplus su(2)$.

The unitary evolution is given by exponentiation of the Hamiltonian $U(t) = \exp\left(-itH_{int}^{(q)}\right)$ which gives

$$U(t) = \begin{pmatrix} \cos(\lambda t\sqrt{K_-K_+}) & -i\frac{\sin(\lambda t\sqrt{K_-K_+})}{\sqrt{K_-K_+}}K_- \\ -iK_+\frac{\sin(\lambda t\sqrt{K_-K_+})}{\sqrt{K_-K_+}} & \cos(\lambda t\sqrt{K_+K_-}) \end{pmatrix} ,$$

$$(31)$$

with $K_+K_- = [N]^2$ and $K_-K_+ = [N+1]^2$.

Choosing the initial condition $|\psi(0) >= |+ > \otimes |\psi >$ we calculated the evolution of the population inversion,

$$< \sigma_3(t) >= \sum_{n=0}^{\infty} \cos(2\lambda t[n+1]) \; | < \psi|n > |^2 \; . \quad (32)$$

If we specify the initial state of the field $|\psi >$ as a q-deformed coherent state [8], $|\psi >\equiv |\alpha >_q = (\exp_q(-|\alpha|^2))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}}|n >$ then we obtain

$$< \sigma_3(t) >= (\exp_q |\alpha|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \cos(2\lambda t[n+1]) \; \frac{|\alpha|^{2n}}{[n]!} \; ,$$

$$(33)$$

where the q-analogue of the exponential appears and $[n]! = [1][2]\cdots[n]$. The deformed quantities in the sum forbid the exact summation and a numerical calculation shows that periodicity for $q = 1$ is progressively destroyed for $q > 1$ values. For details see [15].

As a final application, we discuss briefly the deformed Korteweg-de Vries equation. A realization in terms of currents of the q-deformed Virasoro algebra (eq.(22)) with central extension (eq.(25)) is achieved by defining [17],

$$u(x) = \sum_{n=-\infty}^{\infty} \hat{L}_n e^{-inx} \; , \quad (34)$$

and choosing $q = e^{-i\varepsilon}$, where ε is an arbitrary real number. The C.R. (22) induces the following commutator between the currents:

$$\frac{1}{i}[u(x), u(q)] = P \cdot \delta(x - y)$$

$$= \frac{1}{2 \cdot \sin \varepsilon}\{e^{2\varepsilon \partial_x}u(x) - u(x)e^{-2\varepsilon \partial_x}\}q^{-2N}\delta(x - y)$$

$$+c \; \frac{\sin h(2\varepsilon \partial_x)}{\sin \varepsilon} \; \delta(x - y) \; . \quad (35)$$

By choosing the Hamiltonian as

$$H = \frac{1}{2} \int dx u^2(x) \; , \quad (36)$$

the corresponding q-KdV system appears as [17]:

$$u_t \cdot \sin \varepsilon = \frac{1}{2}(e^{2\varepsilon \partial_x}u(x) - u(x)e^{-2\varepsilon \partial_x})q^{-2N}u(x)$$

$$+c \sin h(2\varepsilon \partial_x)u(x) \; . \quad (37)$$

This equation simplifies by making the transformation to a new current, $u(x) = q^{2N}w(x)$ for which

we obtain by discretization the nonlinear difference-differential equations:

$$2w_{n_t} \cdot \sin \varepsilon = w_{n+1}^2 - w_n w_{n-1} + 2c \; w_{n+1} - 2cw_{n-1} \; ,$$

$$(38)$$

where $w_{n\pm 1} = w(x \pm 2\varepsilon, t)$. This systen is easily seen to reduce to the usual KdV equation in the non-deformed $\varepsilon \to 0$ limit.

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DISCUSSION

Q. **A. LeClair** *(Cornell Univ.)*: Does the quantum Virasoro algebra have a closed $SU(1,1)_q$ sub-algebra?

A. **M. Chaichian**: No. Contrary to the usual case of Virasoro algebra in which $SU(1,1) \subset$ Virasoro, in the q case the $SU(1,1)_q$ is not the subalgebra of q-Virasoro.