Scalar potentials and weak gravity

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À mes parents

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Contents

Introduction

Modern physics is a young science, but it may be interesting to look at its short history. At the end of the XIXth century, most if not all physicists believed that the entire physical phenomena were understood and the physical experiments that, at this stage, were not well understood will be soon explained using the same formalism. The future has of course proved them wrong, and the experimental data which, at that time, were still waiting for an explanation have opened the way to all the physics of the XXth century, namely the General Relativity on one side and the Quantum Mechanics, followed by the Quantum Field Theory on the other side.

During the XXth century, improvements of our knowledge of Quantum Field Theories and their phenomenological implications, linked to huge progress in the experiment side, lead to the edification of a very impressive phenomenological model, the Standard Model. This model explains with a tremendous accuracy a lot of experimental results obtained during the last decades. In 2012, the announcement of the discovery of the Higgs boson, then a missing piece of the SM, can be seen as a major achievement of theoretical and experimental physics of the last half-century. But in the same time, there are some phenomena for which physics explanations have not been found yet. For some astronomical observation an unknown Dark Matter has been invoked. Also, for example, the nature of masses of the neutrinos, and the protection of the Higgs boson mass under the quantum corrections remain open issues.

However, if in a first approach one can establish a parallel between today and where the physicist were one century ago, the situation is, when looking at the details, very different. First, we believe that the Standard Model (SM) is an Effective Field Theory, and so it is understood that this theory is not valid at all energy scale, but has a maximal energy scale, its cut-off, beyond which the model has to be modified. Besides, if the SM gives a very good understanding of the three fundamental interactions between particles, it doesn't take into account a quantum description of gravity. The construction of a complete quantum gravity theory has not yet been achieved. Proposals for high energy extensions of the SM can be divided more or less in two categories. On one side, there are models adding one or several fields to the standard model ones, but always in the framework of an EFT. The cut-off is pushed to higher energies. These models aim to explain some experimental observations that the SM doesn't take into account. On the other side, there are a few proposals of UV-completion, complete theoretical frames valid for all energy, and in particular including a theoretical description of quantum gravity. This is the case of the String Theory. Since they are defined at all energies, they could *a priori* make predictions for the EFT at experimentally accessible energies.

Here, we will discuss some present approaches to discriminate Effective Field Theories, using the terminology introduced a few years ago. The EFT consistent with a UV completion form the Landscape, and those that cannot arise from a quantum gravity theory lie in the Swampland. We are interested in the goal and different results of conjectures that have been put forward within this line of thoughts. We will focus on a particular conjecture, namely the Weak Gravity Conjecture (WGC). This states that gravity is weaker than gauge forces. We will see how it is possible to implement it, in a quite formal and proper way for a single abelian gauge field. This simple case, has not found an important phenomenological impact. However, several authors have extended this conjecture, and we will present some of possible ways here. These include for example to the case of multiples abelian gauge fields, or the presence of a massless scalar field. Though they don't change the spirit of the conjecture, the required modifications are important.

The WGC has not found many phenomenological applications, with the exception of [46]. This will be presented in chapter 2. It concerns the case of $U(1)$ mixing. Indeed, in the case of tiny mixing, the WGC gives an upper bound on the scale of new physics. We will also see that a tiny mixing can be obtained in a low string scale framework, advocating for the use of the WGC in this model.

The following chapters focus on the exploration of another form of the WGC, one comparing scalar interactions to gravitational ones ([45]). In particular, dealing with scalar self-interactions turns not to be a so simple extension of the WGC presented in the first chapter. We will present a formulation based on the computation of forces in the non-relativistic limit. We will investigate different scalar potentials, having in mind, in particular, cosmological applications. We will also study the implication of the derived constraints for the case of several scalars, or moduli. We will look at how these constraints evolve under the dimensional reduction, and the effect of a potential that stabilizes the scalar fields. For the case where the scalar is a dilaton, adding a potential can lead allow non-asymptotically flat black hole solution for the Einstein-Maxwell-dilaton equation of motion. This leads us therefore to study the behaviour of the the WGC for non asymptotically flat space ([44]).

The second, independent part of the thesis, concerns different subjects.

The chapter 6 is dedicated to the study of a new possibility of signal for indirect detection of spin-3/2 particles [47]. Assumed to couple only through a gravitational portal, these are difficult to detect. But 2016 has inaugurated a new way to probe gravitational interactions, namely through detection of gravitational waves. We have looked for physical processes where the presence of spin-3/2 leaves an imprint in the production of such signals. This happens for instance during preheating, at the end of the inflation, where stochastic gravitational waves can be copiously produced. We will present the computation of the spectrum of gravitational waves produced by a gas of spin-3/2 particle.

The last chapter 7 is devoted to the study of a peculiar model of Higgs alignment

([49]). In model with more than one Higgs doublet, the Higgs boson observed at the LHC is one eigenstate of the mass matrix. But the experimental constraints require that this scalar is also to a very good approximation in the direction of the doublet acquiring a vacuum expectation value. This feature is denoted "alignment". We will look at a particular model where a peculiar $SU(2)$ symmetry predicts the alignment at tree level. Even broken, this symmetry allows to keep small the misalignment induced by loop corrections.

CHAPTER 1

An introduction to the Swampland and the Weak Gravity Conjecture

1.1 The Weak Gravity Conjecture

The Weak Gravity Conjecture is a relation between the gravity and the other forces at the level of the Effective Field Theory (EFT). Before presenting the Weak Gravity Conjecture (WGC), we will give a short presentation of the frame in which this conjecture has been elaborated, which is called the Swampland.

1.1.1 The Swampland

It is important for physicists to test their theories. Sometimes, experimental data are available, and can be confronted with theory. Then, one either modifies the theory or rejects it if it doesn't agree with the data. Other times one makes new predictions and some new experiment needs to be build to probe a new range of energy, or precision for example, in which the theory predicts some specific features. Often the result of the experiment can be quite long to come. Meanwhile, it is important to understand how data that can be explained in different ways, could allow to reject or accept, with some limitations, the theory.

Today, no clear experimental results can be used to infirm or confirm, with a sufficient acceptability, all the different models or theories put forward. There are different situations to manage. Some promising experimental results may not reach the targeted precision required to be confirmed as a true deviation. Signals at the LHC, for example, that look promising for a while might turn to be only statistical fluctuations. Even if we consider that some experimental data are close to the target precision and might turn to be a sign of new physics, they might be explained by many different models, so seem to be today the B anomalies. In another case, we have a lot of hints for the need of a new theory to explain the experimental results, but the data are not sufficiently clear to give an accurate direction for how to build a theory. This is the case of Dark Matter and inflation. We seem to have today a situation with some observations that need to be explained, and many theories to "falsify", but not yet enough experimental data to help in getting satisfactory answers. One important feature of our proposed theoretical models is that there are EFTs: they are defined only for a specific range of energy.

We can adopt the following point of view. We expect that an EFT that describes some experiments has a good UV completion. That is to say there is a theory valid at arbitrarily high energies, and the EFT is its limit at low energies, lower than a given cut-off energy scale. We have so a new questions to address. Instead of focusing solely at experimental signatures of our models, we can also look at the existence or not of an UV complete extension. This is an old question but that seems to have new developments in recent years. Considering all known UV-complete models, one looks for the common characteristics present at their low energy limits. If they seem quite general, they can be conjectured as conditions to be respected by any EFT. One can so reject some EFT not from their experimental signatures, but by the fact that they are not well UV-completed. Of course, it is of the most important to prove under which condition the conjectures are valid.

We have so defined the Swampland, presented in [216]. Let's call the ensemble of all possible EFT the landscape. Giving some specific theories UV-complete, one can look at the limit of low energy of these theories in order to find some shared features between these models. It is possible to consider that these characteristics are some conjectures. If an EFT doesn't respect these conjectures, it is considered it belongs to the Swampland. When we have defined the Swampland, there is a lot of questions arising, and which are under investigation, for example did we see all the admitted EFT when we look at the low-energy-limit of the UV-complete theories, which is a problem known under the name of lamppost. One general question is which kind of UV-complete theories we want to take? Generally, it's the different models of String Theory which are used, but one can generalize this principle using some physical features which are independent of the specific content of the theory, as unitarity of the S-matrix for example, which is a physical criteria independent of a specific theory.

Since its elaboration in 2005, a lot of different Swampland conjectures have been obtained, looking at different quantities. A way through the different conjectures is given in [190, 62]. We will present some of the most important ones in the following.

We can begin with the absence of (continuous and exact) global symmetries. This conjecture was elaborated first because in a string theory perturbations frame, all the global symmetries are gauged or broken. So this is an illustration of the principle of the Swampland. Since in our UV theory, it seems that all the global symmetries are gauged or broken, this will also be the case in the low energy limit of these theories, and so we demand that all the exact global symmetries are gauged or broken in the EFT. But it is interesting to see this argument from another point of view, looking at black holes. We know that if there are no gauged symmetries, the Schwarzchild black hole is the unique solution of the Einstein equations. But if we have a global symmetry, we can have a particle charged under this symmetry, and this particle can fall in the black hole. When the black hole will evaporate, since we cannot see from the outside if it is charged or not, the Hawking radiation will be neutral, and so at the end we have what is called a black hole *remnant*. Since we can add in the black hole as many particles as we want, we obtain at the end all the remnants possible, so an infinity of different states with a possibly small mass, which is a sign that the theory is not valid. This argument works only for continuous symmetry, and requires also that the symmetry is exact. The main interest of this explanation is the link that it creates between black holes physics and the Swampland. As any theory of quantum gravity should manage the existence of black holes, black holes physics can be used to infer some conjecture about the EFT deriving from quantum gravity without any assumption on the underlying theory we look at. We will use the black holes physics in the following.

If we turn now to gauged symmetries, the first thing we can infer is on the presence of charged states in the theory. If we look at the string theory spectrum under the compactification, we see that all charges of the gauge fields are present in the theory. This can be promoted to a conjecture, called the completeness conjecture. In [33], it was shown that there is a link between the completeness conjecture and the global symmetry one, in the sense that in a few models, all the charges of the spectrum are required to completely break a global symmetry. This conjecture is not the only one we can do in this set-up, but we will come back after on the conjectures about gauge theory and quantum gravity.

We will now take a look at some conjectures on the fields themselves. One conjecture is that the fields of the theory cannot take any transplanckian value. This is not actually a conjecture, it is more a requisite of the EFT. Since we know that at such a scale (or before at the string scale) the EFT falls down and we have to take into account the all UV theory, it is normal to require that in the domain of the EFT, the field cannot have excursion outside the domain of validity. But we can try to refine such a demand, in a more instructive way. One way to see that our theory loses its sensitivity is the apparition of a infinite tower of massless states in the spectrum. So one way to reformulate the conjecture would be to postulate that when some fields approach transplanckian value, we can see, in the theory, an infinite tower of massless states coming up, signifying the breakdown of the theory. When we look at supersymmetry or string theory, the moduli can play the role of such fields. Indeed, in a simple model of compactification where there is a dilaton (or radion, see appendix A) ϕ, the Kaluza-Klein states masses are under the form $m \sim m_0 e^{-\alpha\phi}$, where α is a constant of order one, $\alpha \sim \mathcal{O}(1)$ in Planck units. The moduli has the behaviour we are looking for. Generally there is not only one moduli, but a lot of moduli and all these moduli form a manifold. On this manifold we can define a distance $d(\Phi_1, \Phi_2)$ between two points, and so the Swampland Distance Conjecture will postulate that when this distance becomes infinite (or of the order of the Planck scale when we put the right normalization), there is the apparition of a tower of massless states with a mass scaling as $m_0 e^{-\alpha d(\Phi_1,\Phi_2)}$. This conjecture is quite complex and depend on the properties of the scalar manifold and its metric, and has been extensively studied (see [188, 166, 35, 135, 149, 173, 60]) and generalized especially in a supersymmetric set-up where there is more information on the behaviour of the moduli manifold.

Until now these Swampland conjectures are not very constraining on the phenomenological point of view, at least in an immediate way. But there exist some conjectures much more constraining. One of these examples is the de Sitter (dS) swampland conjecture studied in [185, 122, 99, 15, 182, 187, 100, 64, 202, 81]. It is based on the fact that by now, it is quite difficult to build up dS vacua in string theory, and the possibilities found are under discussion, in particular due to instabilities that can appear, and their behaviour under corrections. Based on some hints from the string theory, some authors have postulated that it is not possible to form such vacua in string theory, and translated it in conditions under the scalar potential V, which are

$$|\vec{\nabla}V| \geq \frac{cV}{M_{Pl}} \qquad (1.1.1)$$
$$\min(\nabla_i\nabla_j V) \leq -\frac{c'V}{M_{Pl}^2}$$

Such formulae have profound implications in phenomenology, as they are very constraining for the inflation, forbidding a large domain of different models.

The de Sitter Swampland conjecture is looking at some cosmological models. Another conjecture looking at cosmological considerations is the Transplanckian Censorship Conjecture (TCC) . The problem this conjecture is dealing with is the fact that during the inflation, some sub-planckian modes, i.e. modes under the Planck Scale can, during the inflation period, be larger than the Hubble radius and so become classical and freeze. This means that quantum features of the theory are accessible at the classical level. In order to avoid this problem, it was postulated in [35] that such a situation cannot arrive in a coherent description of quantum gravity. This is the TCC, and can be used to constrain models of inflation in the early Universe or today.

There are a lot of others conjectures, for example cobordism conjecture, or Non-SUSY AdS vacua conjecture, or some conjectures on the fermions part of the spectrum. Our purpose in this short presentation is not to be exhaustive, but just to give a first taste of the goal and some aspects of the Swampland program. Conjectures that we have presented here will be present, or at least mentioned in the following parts of this thesis. But among all these conjectures, there is one that we haven't presented yet, and we will now turn to it. This conjecture is the Weak Gravity Conjecture (WGC).

1.1.2 The Weak Gravity Conjecture : definition

As indicated by its name, the WGC postulates that the gravity is the weakest of the force. However, to just postulate this is not very instructive, and we have to put it in a more formal way. The first proposal of this conjecture and its formalization was done in [29], and the arguments that we will present for the definition and derivation of the WGC are presented in this paper. Let's suppose that there is a gauged $U(1)$ symmetry, with a gauge coupling g. The WGC indicates that there must exist a particle with mass m and charge q verifying

$$\sqrt{2}gq \geq \frac{m}{M_{Pl}} \qquad (1.1.2)$$

where M_{Pl} is the reduced Planck mass defined by $M_{Pl} = \frac{1}{\sqrt{8\pi G_N}}$, G_N being the Newton constant in four dimensions.

For a first check, we can look at the SM to see if there is a particle verifying this constraint. Taking the electron, we have $|gq| \simeq 0.3$, and $m_e = 511$ keV. We will use also $M_{Pl} \simeq 2.10^{18}$ GeV. With these values the bound (1.1.2) gives us $0.42 > 2.5\ 10^{-19}$. The electron is so a particle that verifies largely the WGC bound. Actually, all the fundamental particles of the SM verify this conjecture. The question is now, how can we derive this conjecture from a Swampland point of view?

One way to derive it is to use black holes physics arguments. Since we are looking for charged objects, we will use the Reissner-Nordström metric, which have the following form

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2\mathrm{d}\Omega_2^2. \tag{1.1.3}$$

When $M > Q$, this solution has two horizons, and describe a black hole. When $M = Q$, the two horizons are combined, and we have what we call an *extremal* black hole with only one horizon. With $M < Q$, there is no horizon, the solution is a naked singularity and is considered as a non-physical solution. So let's take an extremal black hole, with $M = Q$. Normally, due to Hawking evaporation, all the black holes should disappear after what could be a long time. But since this black hole is charged, it has to emit charged radiation (in other words charged particles) in order to conserve the total charge of the system. Hence the black hole will emit a particle with charge q and mass m, and the final black hole state will have a mass $M' \leq M - m$ and a charge $Q' = Q - q$, as the particle emitted momentum takes a part of the energy of the black hole. Since we want the final black hole to be a physical state, we have to impose $M' \geq Q'$. We obtain $M - m \geq Q - q$ and as the initial black hole was extremal, we derive $m \leq q$. But the parameters M and Q defined in (1.1.3) are not well normalized. We call it in the following mass and charge in geometrical unit. The physical mass and charge of the black hole are given by

$$M = \frac{\tilde{M}}{8\pi M_{Pl}^2} \qquad \text{and} \qquad Q^2 = \frac{g^2\tilde{Q}^2}{32\pi^2 M_{Pl}^2}. \tag{1.1.4}$$

Using these relations, we obtain for the extremality condition $M = Q$ the physical condition $\tilde{M} = \sqrt{2}(g\tilde{Q})M_{Pl}$. This is converted for the particle state as $\frac{m}{M_{Pl}} \leq \sqrt{2}gq$, which is the equation (1.1.2).

A first comment can be given at this stage, on the features of the state respecting the equation (1.1.2). In the derivation with this black hole argument, we justify why such a state with a charge to mass ratio greater than one should exist, but there is no constraint on which state on the spectrum it has to be. A question that one can ask is if this state is the state of the theory spectrum with minimal charge, the state which has the minimal mass over all the charged states, or the state with the biggest charge of mass ratio. The first proposition is excluded because spectra of some string theories don't respect it. The second condition implies the third, and in [29], it was argued that it is the third proposed state that has to enter in the conjecture. It means that we require the biggest charge to mass ratio in the theory spectrum to be bigger than one.

We can note also that this result was obtained with the dimension four operators, without taking into account any corrections from the higher-dimensional operators. If we want the extreme case, which corresponds in this set-up to $M = Q$, to be always the same, we need that the corrections decrease the ratio $\frac{M}{Q}$ below one for the extremal black hole. In the case where the corrections increase the ratio $\frac{M}{Q}$, the bound we obtained for the ratio of the WGC state will be modified and lowered. Instead, when the ratio decreased, this means that the extremal black holes become unstable, and decay in smaller black holes. There is no need to modifiy the WGC bound. These questions of higher-order corrections are extensively studied (see for example [37]), and it seems that using some positivity bounds on scattering amplitudes from this higher-order terms, one can constrain them to be such that the ratio is always decreasing. So the WGC seems robust to the higher-order corrections.

In the Swampland program, when we have a conjecture, we have to see if it possible to derive it from a UV-theory, and generally in the string theory frame. It seems that the WGC is obeyed in such theories, but there is a complication with respect to this simple case. First there is in general not a simple $U(1)$, but a product of different $U(1)$ in the theory. There is also the presence of moduli, which can play a role and require a modification of the WGC. But before looking at the modifications induced by adding these fields, let's see the WGC from another point of view, which is called the magnetic Weak Gravity Conjecture (mWGC), presented in [29].

To derive the mWGC, we use the duality between magnetic and electric coupling. We have $g_{mag} = \frac{1}{g_{elec}}$. Using the WGC, we can constrain the mass of a magnetic monopole to be less than this coupling, as the WGC

$$m_{mag} \leq g_{mag} q M_{Pl} = \frac{1}{g_{elec}} q M_{Pl} \tag{1.1.5}$$

What is interesting with the magnetic monopole, is the fact that its mass is not a free parameter of the model. Actually, the mass is proportional to the energy stored in the magnetic field, and this is lineary divergent. Since we have a cut-off in our EFT, we can say that $m_{mag} \sim g_{mag}^2 \Lambda$, for example the formula for the monopole mass in the SM is given at leading order by $\frac{M_W}{\alpha_{em}}$. Replacing it in (1.1.5), using a unit charged state ($q = 1$), we obtain a bound on the cut-off of the theory

$$\Lambda \lesssim g_{elec} M_{Pl} \tag{1.1.6}$$

This is interesting, because *a priori* there is no bound on the cut-off scale. Knowing the UV-theory, we can give a value at which the EFT construction will break down, but knowing only the parameter of the low energy limit, it is no possible to build up the value of the energy-scale. Here, one obtains an upper-limit on the cut-off knowing only the parameter of the EFT. But the interest of this bound has to be moderated by two remarks. First, this is only an upper-limit. The cut-off of the theory could be at lower scale. If we take $U(1)_{em}$ of the SM, we have $g \sim \frac{1}{10}$, and so the upper limit is close to 10^{17} GeV, which is larger than the electroweak symmetry breaking scale. As we can see from the SM example, in the case where the coupling is not very small, the scale at which the coupling will appear is quite large, as we will see in the following. This bound is so not very constraining. Besides, even if for "natural" values of the coupling this cut-off is large, it is not a sign of the apparition of the UV-theory. Indeed, we can see from its construction that this cut-off is not linked to the UV-theory, as the string scale for example. It just a sign of new physics, but the new physics is not determined. It can be that the $U(1)$ we are looking for is embedded in a $SU(2)$ group for example, or a larger symmetry group. But it doesn't mean that this scale is obviously the string scale. It can be, but it's not an obligation.

This scale gives us also a new argument against the existence of global symmetry. Indeed, as presented in [141, 33, 192], since the value of the gauge coupling is linked to the apparition of new physics, we cannot take the gauge coupling to 0. This argument prevents to go continuously from a gauged group to a global one. This shows another link between two different conjectures of the Swampland, as the link between completeness conjecture and absence of global symmetries.

1.2 Developments of the Weak Gravity Conjecture

Now that we have given a first definition of the WGC, we can look at some refinements that we can introduce in this set-up. Generally, in phenomenological model, we will not have a single $U(1)$. We can so begin with the extension of the WGC for the case of multiple $U(1)$s.

1.2.1 Case of multiple $U(1)$s

In order to take into account multiple $U(1)$s, we have to look at the black hole solutions in this scheme. In order to do this, we will follow the discussion of [76]. The first thing to say is that we consider N gauged unbroken $U(1)$s. The gauge bosons are massless, and so if there is a gauge kinetic mixing, we can perform a $SO(N)$ rotation of the gauge fields such that this kinetic mixing disappears. We will so considerate in the discussion that the $U(1)$s are independent. The first generalisation one can think of is the fact that it exists a particle with mass m and charge q_i such that, if we think at the ratio of charge over the mass as a vector, we have to consider that the norm of this vector is greater than one. So we can think at this formula as

$$\sum_i \frac{(g_i q_i)^2}{m^2} > 1, \tag{1.2.1}$$

in Planck units. When we reduce this formula to one gauge field, we recover the previous formula (1.1.2), so this can be a good generalisation. However, in the previous section, the WGC was derived by imposing the possibility to all black holes to decay. Let's assume that a particle with mass m and charge q_i verifying (1.2.1) is present in the spectrum of the theory, and take a black hole with charge Q_i and mass M, extremal. In this case, the extremality of the black hole means that $\sum_i Q_i^2 = M^2$. We will also ask that the charges Q_i of the black hole verify $\sum_i q_i Q_i = 0$, where q_i are the charges of the particle we define just above. Because of this property, the black hole cannot emit the particle that we have defined, and so this black hole, which is extremal, is stable, which is the problem we want to avoid. We have so to search for another form of the WGC.

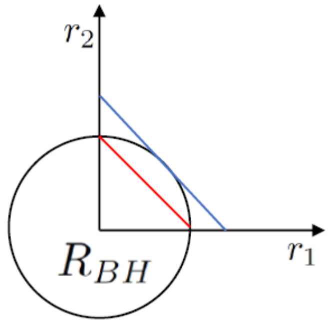

Figure 1.1 – Representation of (one part of) the convex Hull and the black hole mass to charge ratio in the case of 2 $U(1)$s

Let's take a black hole with charges Q_i and mass M, extremal. We want that this black hole can emit a particle and that the final black hole state is extremal or subextremal. But if we look at only one particle, we have seen that the WGC we can infer from this state doesn't prevent the stability of some others extremal black holes. We will take so as many independent particles as there are $U(1)$s (taking into account that we have to add the anti-particles). Independent means that their charges q_i^a and q_i^b for a particle a and b verify the relations $\sum_i q_i^a q_i^b = 0$. We can imagine for simplicity that each state is only charged under one gauge field, but this is not necessary. We will also consider that the black hole decays completely in the different particles, and we call n^a the number of particles a, and m^a their masses. The conservation of charge and energy gives us relations between the masses and the charges, $M > \sum_a n^a m^a$ and $Q_i = \sum_a n^a q_i^a$. We can rewrite these two equations as,

$$\frac{Q_i}{M} = \sum_a \frac{n^a m^a}{M} \frac{q_i^a}{m^a} \qquad \text{with} \qquad \sum_a \frac{n^a m^a}{M} < 1$$

So if we look at the space of the gauge charges, there is a simple geometrical interpretation of the relation we derive. Let's call $\vec{R} = \frac{1}{M}(Q_i)$, $\vec{r^a} = \frac{1}{m^a}(q_i^a)$ and $c^a = \frac{n^a m^a}{M}$. We have so

$$\vec{R} = \sum_a c^a \vec{r^a} \qquad \text{with} \qquad \sum_a c^a < 1 \tag{1.2.2}$$

This means that the charge to mass ratio of the black hole belongs to the convex hull defined by the charge of mass ratios of the different particle species. The biggest charge to mass ratio for a black hole is obtained for the extremal case, for which we have $\sum_i Q_i^2 = M^2$, which is the definition of the unit sphere. Hence we can formulate the WGC in this way. If the theory contains N gauged unbroken $U(1)$s, there must exist at least N different particles, independently charged such that the convex hull defined by their charge to mass ratio includes the unit sphere. We see that thanks to the presence of the anti-particles, we can cover the entire unit sphere.

We present one part of the convex hull for the case of two $U(1)$s in Figure 1.1. As explained a few lines above, the other part of the convex hull is obtained by the fact that anti-particles with $-r_{1/2}$ are also present in the spectrum. If we have just required that the two particles have a charge to mass ratio equals to 1, as with only one gauge field, we would obtain the red curves, and so some black hole states, above the red line, would be stable. Requiring that the unit sphere belongs to the convex hull of the WGC state raises to a stronger constraint since the states have a charge to mass ratio corresponding to the extremities of the blue curve.

We see that adding at least one abelian gauge group in the theory complicates a little bit the conjecture, in two ways. The number of states required by the conjecture is bigger, and the constraint of the charge to mass ratio is also increased.

1.2.2 A dilatonic WGC

Another case we want to explore is the case of massless scalars. Indeed, in string theory, the spectrum includes some (and possibly a huge number of) massless scalars, without potential, called moduli. The goal of the Swampland conjecture is to see what are the EFT compatible with a string theory in the UV, so one needs to include such scalars in the conjecture. In order to do that, we will look at the simplest model containing such particle, which is the following Lagrangian

$$\mathcal{L} = \sqrt{-g}\left(-R + 2\partial_\mu\phi\partial^\mu\phi + e^{-2\alpha\phi}F^2\right) \tag{1.2.3}$$

The coefficient α depends on the particle we look for. In the case of the string theory, this particle is called dilaton, and $\alpha = 1$. A similar Lagrangian can be obtained from the dimensional reduction as presented in appendix A. In this case, the coefficient α takes the value $\alpha = \sqrt{3}$.

If we want to obtain a WGC applied at this particular case, the more natural way is to search the existence of black holes with such a Lagrangian. Such a black hole solution was found in [125, 121], and has the following form

$$\begin{cases} \mathrm{d}s^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}\mathrm{d}t^2 + \frac{1}{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}}}\mathrm{d}r^2 + r^2\left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\mathrm{d}\Omega_2^2, \\ e^{2\alpha\phi} = e^{2\alpha\phi_0}\left(1-\frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \\ F = \frac{Qe^{2\alpha\phi_0}}{r}\,\mathrm{d}t\wedge\mathrm{d}r, \end{cases} \tag{1.2.4}$$

where r_+ and r_- are two integration constants, and ϕ_0 is the value of the dilaton field at the infinity. We can also define a scalar charge D for the black hole, computed as the integral over a two sphere at infinity $D = \frac{1}{4\pi}\lim_{r\to\infty}\int d^2\Sigma^\mu\nabla_\mu\phi$, or, equivalently, through the expansion $\phi = \phi_0 - \frac{D}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)$ at large r. The relation between the physical quantities and the integration constants can be easily computed and gives

$$\begin{cases} 2M = r_+ + \frac{1-\alpha^2}{1+\alpha^2}r_-, \\ Q^2e^{2\alpha\phi_0} = \frac{r_+r_-}{1+\alpha^2}, \\ D = \frac{\alpha}{1+\alpha^2}r_-. \end{cases} \tag{1.2.5}$$

It is interesting to make a comment now on the integration constants r_+ and r_-.

Inverting the relation defined above gives

$$\begin{cases} r_+ = M \pm \sqrt{M^2 - (1-\alpha^2)Q^2 e^{2\alpha\phi_0}} \\ r_- = \frac{(1+\alpha^2)Q^2 e^{2\alpha\phi_0}}{M \pm \sqrt{M^2-(1-\alpha^2)Q^2 e^{2\alpha\phi_0}}}. \end{cases} \tag{1.2.6}$$

Taking $Q = 0$, in order the metric (1.2.4) to recover the Schwarzschild metric we have to take $r_- = 0$ and not $r_+ = 0$, and we will consider in the following only the $+$ sign for the definition of r_+ and r_-.

D can now be expressed in functions of M and Q as

$$D = \alpha \frac{Q^2 e^{2\alpha\phi_0}}{M + \sqrt{M^2 - (1-\alpha^2)Q^2 e^{2\alpha\phi_0}}}. \tag{1.2.7}$$

The physical mass is not modified, and is given by (1.1.4). However, the gauge coupling constant is given in (1.2.3) by the coupling term between $F_{\mu\nu}$ and the dilaton ϕ. We will so define the physical charge as

$$Q^2 = \frac{\tilde{Q}^2}{32\pi^2 M_{Pl}^2}. \tag{1.2.8}$$

From (1.2.4), we see immediately that there is only one horizon, contrary to the Reissner-Nordström solution. Indeed, if r_+ is a horizon, r_- is no longer one, but a singularity as soon as $\alpha \neq 0$. This can be seen from the coefficient in front of the $\mathrm{d}\Omega_2^2$ in the metric. This coefficient cancels for $r = r_-$, which means that at this point the volume of the the slice is 0, which is a sign of a singularity. This can be seen also from the Ricci scalar, which is divergent at $r = r_-$.

Since we have only one horizon, it is not obvious that there will be an extremal solution as in the Reissner-Nordström case. We could think that we are more in the situation of a Schwarschild solution. However, looking at the metric (1.2.4), it is possible to see that, when $r_- > r_+$, there is no longer a horizon, and there is a naked singularity. Besides, it was shown (see [191]) that the inner horizon of the Reissner-Nordström black hole is unstable and becomes a singularity when adding perturbations. It seems so that we can think at the limit $r_- = r_+$ as an extremal limit. In function of the physical parameters M and Q, this limit can be rewritten using (1.2.5) in

$$Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2. \tag{1.2.9}$$

We can see directly from the Lagrangian (1.2.3) that the $e^{2\alpha\phi_0}$ corresponds to the coupling constant square, with A_μ rescaled by a factor 2. Using the proper normalization for M and Q, we can write the dilatonic WGC as the presence in the spectrum of a state satisfying the relation

$$(gq)^2 = \frac{(1+\alpha^2)}{2} \frac{m^2}{M_{Pl}^2}. \tag{1.2.10}$$

So the WGC that we can write in the presence of a dilaton field is quite the same as the WGC written only in presence of a gauge field, but there is a new term, proportional to α^2. It is the contribution of the dilaton field. This conjecture was first presented in [147], with p-Branes, but the solutions for a p-Brane can be derived from the one of a black hole, so we prefer present the conjecture in this way.

1.2.3 Different Lattice and Tower Weak Gravity Conjectures

We have now to recall the goal of the Swampland. The idea is to look at which Effective Field Theories can be derived at the low energy limit of a UV theory, namely a string theory. One important point for string theory models is the fact that some dimensions of the theory are compactified. Since we want the WGC to be a property coming from the UV theory, and since the UV theory is defined with more dimensions than our world at low energy, it could be interesting to look at the behaviour of the WGC under the compactification of one or several dimensions.

This was for example done in [147] and leads to interesting results. When we compactify a dimension around a circle, we can observe the apparition of new fields, and in particular we obtain a new gauge field, with a new $U(1)$ symmetry which is a graviphoton, and a massless scalar that we will call dilaton in the following even if the proper term is more radion. This is presented in details in the appendix A for the compactification of a scalar, and will be useful in the following discussion. So if we have in dimension $D+1$ a theory with one gauge boson, we are let in dimension D with two gauge fields.

However, as we have seen before, the WGC with several gauge fields is quite different from the one with a single $U(1)$. The question is, imposing the WGC as presented in the first part of this chapter in dimension $D+1$ is sufficient or not to obtain the WGC in dimension D after compactification?

The case we are looking at corresponds to two $U(1)$s, one present in dimension $D+1$, and the other one coming from the compactification. In the section 1.2.2, we have seen that the dilaton coupling to a gauge field modifies the form of the WGC. In the case of interest, the dilaton coming from the dimensional reduction couples only to one gauge field. So, instead of having, as in 1.2.1, the extremality region to be the unit sphere, the extremality region of the black holes is an ellipsoid, since the WGC condition that one demands for each separate $U(1)$ is different. But it is possible to redefine the charge under the $U(1)$ coupling to the dilaton such that the ellipsoid corresponds to the unit sphere, so we can keep the unit sphere in order to look at the behaviour of the WGC under dimensional reduction.

In dimension $D+1$, we take one state with mass m and charge q, which satisfies the WGC, i.e. (1.1.2) . After dimensional reduction, there is a tower of Kaluza-Klein (KK) states, where the charge under the $U(1)$ coming from the $D+1$ theory is always the same, but with a term added in the mass, and a charge under the graviphoton, terms given in (A.2.13). We will have so for the state K_n

$$m_n^2 = m^2 + C_1 \left(\frac{n}{L}\right)^2 \quad q_n^0 = q, q_n^1 = C_2 \frac{n}{L} \tag{1.2.11}$$

where n is an integer, L the compactification radius, 0 represents the gauge field coming from the $D+1$ theory and 1 represents the graviphoton. C_1 and C_2 are constants useless for our present discussion. These states will all lie outside the unit disk. But as we have seen in Figure 1.1, even if the states verify the WGC, the convex hull they form will not verify the multiple $U(1)$s conjecture.

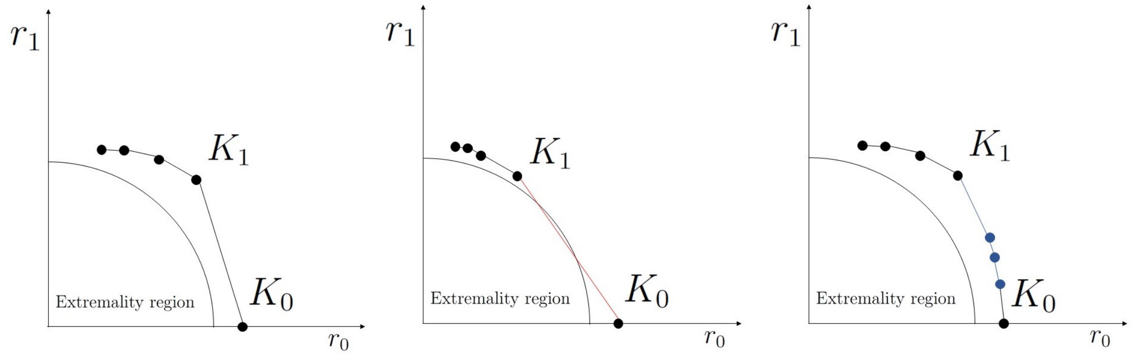

Figure 1.2 – The representations of the charge of mass ratios of the KK state for a given radius compactification, and the situation when we increase the radius compactification (centre). The situation when we impose the Lattice WGC is presented in the right figure

This is what happens in the case of the KK states. When we decrease the compactification radius L, the charge under $U(1)^1$ and the mass of all states K_n except K_0 increase, but q^0 is always the same, so the charge to mass ratio r_1 is quite the same, but the charge to mass ratio r_0 decreases, except for the K_0 states which will be at the same place on the r_0 axis. There is a value of the compactification radius such that the first state K_1 will be to close from the r_1 axis and the line between K_0 and K_1 will cross the extremality region of the black hole. This means that the convex hull of the states of the theory will not contain the black hole region. This is shown in the Figure 1.2.

So the WGC seems to be unstable under the dimensional reduction, since the states that we obtain from the compactification are not sufficient to ensure the multiple $U(1)$s conjecture, which is the form of the WGC for several gauge fields.

In order to solve this problem, it was proposed in [147] to enforce the WGC by asking that, in each point of the charge lattice there is a massive state such that its charge to mass ratio is at least equal to the value of the charge of mass ratio for an extremal solution with the same charge. So we are let with an infinity of states, and in particular with states with very large charge and mass. If we look at the result of this conjecture in the example we took, this means that we have in dimension $D+1$ an infinite number of state with large charge and mass, and when we will compactify, the Kaluza-Klein states corresponding to these states will be closer from the state K_0. These states are the blue ones in the right picture of 1.2. They will be between what we called before K_0 and K_1, and so even when we will decrease the compactification radius, as there is an infinity of them, the convex hull formed by all these states will always contains the extremality region, and the WGC will be stable by dimensional reduction.

Actually, this conjecture is very strong, since it requires a state with a charge to mass ratio greater than one for each point of the charge lattice. This conjecture is so strong that it is not verified in some cases of compactification. It was so relaxed in [148], in what is called a sublattice WGC. Instead of asking a superextremal state at each point of the charge lattice, they asked that the superextremal states are present only in a sublattice. The fact that this sublattice is infinite preserves the property of stability under the dimensional reduction, and the relaxation of the conjecture is enough to pass all the compactification schemes which violate the Lattice WGC.

A last possibility offered by the dimensional reduction is to look at the scattering amplitudes obtained by integrating out the massive KK-fields coming from the compactification, and to use unitarity and positivity constraints on the amplitudes to infer some information about the charge of mass ratio and the WGC. This was done in [14], and they showed the need of an infinite tower of states satisfying the WGC in $D+1$ dimensions in order to conserve the WGC under dimensional reduction. But an infinite tower and a sublattice of the charge lattice are not the same thing. An infinite tower is a weaker restriction. However, since the two conditions are obtained from two different aspects, one from the point of view of the charge to mass ratio constraints, the other from the properties of the scattering amplitudes, it seems difficult to discriminate them. In the spirit of the Swampland program, the way to obtain this discrimination is to find a string theory model in which exists an infinite tower of states, which not form a sub lattice. It is not clear if such a model exists or not today.

When we studied the conjecture in a fixed dimension frame, the number of states required by the WGC is finite (1, or more, if we work with multiple $U(1)$s). The most important fact about the dimensional reduction is the requirement that we need an infinite number of states to satisfy the WGC.

1.2.4 Repulsive Force Conjecture

In the previous subsection, we have introduced the KK states. We will now use these states in another way. The first thing we can note is that these states saturate the bound in (1.2.9). So we know states that saturate the dilatonic WGC. But if we compute the different forces between two copies of the same KK-states, which are in this simple example three: the dilatonic, the gravitational and the "electromagnetic", as presented in chapter 4, we see that the sum of the forces is null.

This was already noticed in [29], for one singe gauge field. The WGC equation (1.1.2) can be seen as the existence of a particle such that the repulsive gauge force between two copies of this particle is greater than the gravitational attractive forces. This conjecture was extended in [189] in the case of massless scalars, and extensively studied and formally put as a Swampland conjecture in [151].

This is called now the Repulsive Force Conjecture (RFC). The first constraint is to look only at the long-range forces of the theory. The long range forces are due to three different types of particles: the "photons", the graviton and massless scalars. We can so write the sum of the forces for two particles 1 and 2 in d dimensions as

$$F = \frac{g^{ab} q_{1a} q_{2b}}{r^{d-2}} - \frac{G_N m_1 m_2}{r^{d-2}} - \frac{k^{ij} \mu_{1i} \mu_{2j}}{r^{d-2}}. \tag{1.2.12}$$

In this formula, g^{ab} represents the coupling of the different gauge fields, and k^{ij} is the metric that stands for the interactions between the massless scalar fields. Now that we have fixed the forces we are looking at, the conjecture states that there must exist at least one particle such that the sum of the repulsive forces is greater than the attractive one. We see that for two copies of the same particles, the attractive forces are those mediated by the massless scalars and the gravity, and the repulsive ones are the gauge forces. This particle is called *self-repulsive*. Actually, looking at stability of the conjecture under dimensional reduction, the conjecture will be modified as the WGC, and the RFC states that there exist a sublattice of the charge lattice such that in each point of this sublattice a *self-repulsive* particle is present.

When studying different models of compactification of string theories, the authors of [151] show that even if the sublattice WGC and sublattice RFC are always respected in these different frames, there are some cases where the saturation of the two conjectures are not fulfilled by the same states. It seems that, contrary to the case of the presence only of the gravity and the "electromagnetic" forces, where the two conjectures are the same, when adding massless scalars, the two conjectures have not a common behaviour.

In this chapter, we have briefly presented the Swampland program and its goal. We focused on a specific conjecture called the Weak Gravity Conjecture. In the simple case of a single gauge field, the WGC can be found through different points of views. For example from the side of the black holes physics, looking for the decay of extremal black holes, or from the point of view of Field Theory, looking at the preeminence of the repulsive forces over the attractive ones. However, to conserve the spirit of the Swampland, it can be interesting to look at changes in such a conjecture induced by adding some ingredients in the theory (more gauge fields, massless scalars, ...). The different schemes used to derive the WGC lead to the proposal of several "WGC inspired" constraints. These different points of views are also interesting for who want to look at their implications for phenomenology and stability or existence in UV-theory frame.

CHAPTER 2

WGC and U(1) mixing

As we have seen in the previous chapter, the Weak Gravity Conjecture seems not very useful for phenomenology with the Standard Model (SM) spectrum. However, there are some special cases where this conjecture can be applied and leads to interesting results. In this short chapter, we will present one of these cases, which corresponds to a model where there is a mixing between different $U(1)$ gauge group. In particular, the case of tiny mixing has witnessed a recent surge of interest, following the results announced by XENON1T [27]. The collaboration has reported an excess between 1 and 7 keV, close to the lower threshold of the experiment, with a peak around $2-3$ keV. However, we have to be careful, because the significance of this excess could melt with a re-analysis of the signal. This could either follow from an accumulation of more data or from more thorough searches for evidences of contamination of the apparatus by some impurities as for the tritium hypothesis suggested by the collaboration in [27]. In the meantime, the possibility that it could be a signal of new physics does not seem excluded, and so we will examine in the following what the WGC can say of such a tiny mixing.

Looking at the XENON1T data, a possible fit in terms of dark photons coupled to the SM through a kinetic mixing portal [153, 186, 90, 94] was analysed in [12, 7, 78]. While solar emitted dark photons are not favoured, a scenario where light dark photons with masses of $2-4$ keV are absorbed by the xenon seems to correctly reproduce the excess, though with a reduced significance due to a look elsewhere effect. This can be achieved for a tiny visible-dark photon kinetic mixing parameter in the range

$$\epsilon \simeq \mathcal{O}(10^{-16} - 10^{-15}) \tag{2.0.1}$$

which is in agreement with the upper bound limit given by XENON1T on ϵ, that also claims a 3σ significance for a 2.3 keV dark photon over the background. This was argued in [183] to lead to the correct result for the dark photon relic density. The dark photon in XENON1T can also appear as a vector portal for fermionic or bosonic dark matter, where, depending on the model, the mixing can take different values. In [32, 167, 222], mixing parameters of order $\epsilon \simeq \mathcal{O}(10^{-4}), \mathcal{O}(10^{-7})$ or $\mathcal{O}(10^{-10})$, with, respectively, an order $\mathcal{O}(\mathrm{GeV})$ massive dark photon in the first two cases and a massless one in the last one have been advocated. We discuss below a possible origin of such mixing parameter, especially for challenging tiny values, where we find that the WGC allows to hope for an accompanying signal at collider experiments.

We focus on the sector of the low energy Effective Field Theory describing the $U(1)$ gauge groups representations and interactions. One of the two, $U(1)_v$, is called visible as we have in mind hypercharge or electromagnetism. Another, $U(1)_d$, corresponds to an extra factor we call "dark" $U(1)$, having in mind an hidden sector. It is straightforward to generalize to cases with more abelian gauge groups. The associated gauge fields and gauge fields strengths are denoted as $A^{\mu}_{(v)}, F^{\mu\nu}_{(v)}$ and $A^{\mu}_{(d)}, F^{\mu\nu}_{(d)}$, respectively. The corresponding two-derivative Lagrangian reads:

$$\mathcal{L} \supset -\frac{1}{4}F^{\mu\nu}_{(v)}F_{(v)\mu\nu} - \frac{1}{4}F^{\mu\nu}_{(d)}F_{(d)\mu\nu} - \frac{\epsilon_{vd}}{2}F^{\mu\nu}_{(v)}F_{(d)\mu\nu} + g_v J^{\mu}_{(v)}A_{(v)\mu} + g_d J^{\mu}_{(d)}A_{(d)\mu} \,. \tag{2.0.2}$$

For massless visible and dark photons, this mixing in the two-derivative Lagrangian can be eliminated by performing the appropriate rotation. When the $U(1)_d$ gauge boson acquires a mass, through a Stueckelberg or Higgs mechanisms, the mixing has physical implications. The visible and dark photons couple in the new basis to the currents J^{μ}_v and J^{μ}_d through:

$$\mathcal{L} \supset \left[\frac{g_d}{\sqrt{1-\epsilon_{vd}^2}}J^{\mu}_{(d)} - \frac{\epsilon_{vd}g_v}{\sqrt{1-\epsilon_{vd}^2}}J^{\mu}_{(v)}\right]A_{(d)\mu} + g_v J^{\mu}_{(v)}A_{(v)\mu} \,, \tag{2.0.3}$$

thus implying that the visible matter is charged under the dark gauge symmetry with charge $\sim \epsilon_{vd}g_v$.

It is most natural to assume that the dark $U(1)$ mass and mixing vanish in the fundamental theory at the ultra-violet (UV) cut-off and are generated at lower energies. The mixing can be generated at one loop by states with masses m_i and charges $(q_v^{(i)}, q_d^{(i)})$ under $(U(1)_v, U(1)_d)$. It is then given by:

$$\epsilon_{vd} = \frac{g_v g_d}{16\pi^2}\sum_i q_v^{(i)} q_d^{(i)} \ln\frac{m_i^2}{\mu^2}, \tag{2.0.4}$$

where μ^2 is the renormalization scale. In the case of the hyper-charge $U(1)_v \equiv U(1)_Y$ we have $g_v = g'$ and $q_v^{(i)} = Y^{(i)}$, while $g_v = g'\cos\theta_w$ and $q_v^{(i)} = q_{em}^{(i)}$ the electrical charge for $U(1)_v \equiv U(1)_{em}$.

In order to generate such a small mixing as the one required by XENON1T results, we either require the dark photon coupling to be appropriately small, a cancellation in the one-loop logarithms, or appeal to higher order non-renormalizable operators. The cancellation can be partial, for instance between particles with (order one) charges $(q_v^{(i)}, q_d^{(i)})$ and $(q_v^{(j)}, q_d^{(j)} = -q_d^{(i)})$ and masses m_i and m_j with $m_j = m_i + \Delta m_{ij}$. For $\Delta m_{ij} \ll m_i$, we have an approximation:

$$\epsilon_{vd} \sim \frac{g_v g_d}{16\pi^2}\frac{\Delta m_{ij}}{m_i} \,. \tag{2.0.5}$$

For complete cancellation, this one loop contribution is replaced by higher loop ones. However, gravitational loops are expected to show up at some order and lead to a lower bound. It was shown in [124] that this is expected at six loop order giving rise to an $\epsilon_{vd} \gtrsim \mathcal{O}(10^{-13})$ for a *bona fide* four dimensional theory. We shall discuss below the first alternative of a tiny dark sector coupling.

To look at such a value, the quantity we have to compute is the mixing given by (2.0.4). In order to do so, we need an explicit model with a knowledge of all the spectrum. But even if we have not such a model, we can give a first taste of the value of this mixing by looking at which states can contribute to the sum. Following the Completeness Conjecture presented in chapter 1, all the sets of $U(1)$ charges are present in the theory without stating anything about their masses. But we have seen in the same chapter that there are some proposals following the WGC that state that an infinity of states in the charge lattice, forming a sub-lattice or not are present in the theory. These states will contribute to generating a mixing between the $U(1)$s. For two $U(1)$s, the masses of the charged particles can be expressed as $m = c\sqrt{(g_v q_v)^2 + (g_d q_d)^2}$, where $c < 1$ is a state-dependent constant. For the following discussion, we use two integers i and j for the visible and dark charge of the particle respectively, as would be for quantized charges forming a lattice in charge space. The equation (2.0.4) then becomes in this scheme

$$\epsilon_{vd} = \frac{g_v g_d}{16\pi^2} \sum_{i,j} q_i q_j \ln\left(\frac{c^{(i,j)}\left[(g_v q_i)^2 + (g_d q_j)^2\right]}{\mu^2}\right) \tag{2.0.6}$$

Though the number of states is infinite, we include in the loop only states below the cut-off. If a particle with charge (q_i, q_j) is in the spectrum, there are also particles with charge $(q_i, -q_j)$, $(-q_i, q_j)$ and $(-q_i, -q_j)$ giving

$$\epsilon_{vd} \simeq \frac{g_v g_d}{16\pi^2} \sum_{i,j} q_i q_j \ln\left(\frac{c^{(i,j)} c^{(-i,-j)}}{c^{(-i,j)} c^{(i,-j)}}\right) \tag{2.0.7}$$

which as a result of the diverse cancellation between different contributions, could remain small (for typical sizes, see for example discussion in [94]).

The most relevant facet of the WGC for this work is the magnetic WGC presented in the previous chapter. The statement is that it exists a cut-off scale Λ_{UV} such that $\Lambda_{UV} \lesssim g M_{Pl}$. We have seen that this argument is linked with the absence of global symmetry, which was presented as another Swampland conjecture. We can use this to generalize, as done by [150], the magnetic WGC to the case with multiple $U(1)$ gauge groups. We require that none of the gauge symmetry factors should turn into a continuous global symmetry by taking the corresponding coupling to vanish. This implies that a tiny value of the dark photon gauge coupling, introduced to make the mixing tiny, requires a UV cut-off at most of order $\Lambda_{UV} \lesssim g_d M_{Pl}$. This is sensibly lower than M_{Pl} and could have important consequences in phenomenology and cosmology.

Starting from (2.0.4), we identify the visible photon with the SM photon, i.e. $g_v = e \sim 0.3$, and the logarithm to be $\mathcal{O}(1-10)$, then

$$\epsilon_{vd} \sim \frac{g_v g_d}{16\pi^2} \sim \mathcal{O}(10^{-3} - 10^{-2}) g_d \qquad \Rightarrow \qquad g_d \sim \mathcal{O}(10^2 - 10^3)\epsilon_{vd} \tag{2.0.8}$$

As explained before, the WGC does not provide information on the new physics required at $\Lambda_{UV} \lesssim g_d M_{Pl}$. A simple possibility is that the $U(1)_d$ becomes part of some non-abelian gauge group $SU(2)_D$ with field strength $F^{\mu\nu}_{(D)}$ broken by a vacuum expectation value (vev) of $\langle\Sigma\rangle = v \simeq \Lambda_{UV}/g_d$ of a field in the adjoint representation. One could then induce a contribution ϵ^{NR}_{vd} to the kinetic mixing through the effective non-renormalizable operator (see e.g. [106]):

$$\frac{c^{NR}}{M_{Pl}} Tr\left[\Sigma F^{\mu\nu}_{(D)}\right] F_{(v)\mu\nu} \quad \Rightarrow \quad \epsilon^{NR}_{vd} \simeq \frac{c^{NR}\ v}{M_{Pl}} \tag{2.0.9}$$

where c^{NR} is a constant. For this contribution to remain sub-leading, we require:

$$c^{NR}\ v \lesssim \epsilon_{vd} M_{Pl} \quad \Rightarrow \quad c^{NR}\ v \lesssim 10^{-3}\ g_d\ M_{Pl}, \tag{2.0.10}$$

which for $\epsilon_{vd} \sim 10^{-15}$ gives $c^{NR}\ v \lesssim$ TeV.

Kinetic mixing might also arise from D-terms in supersymmetric theories through effective operators [40, 130]:

$$\frac{D^2}{\Lambda_D^4} F^{\mu\nu}_{(d)} F_{(v)\mu\nu} \tag{2.0.11}$$

that are expected to be very small. For example, they can be suppressed by the value of the ratio SM Higgs vev over the scale Λ_D for hypercharge D-term or through powers of the dark sector coupling for the dark $U(1)$ D-term.

In the following we will use (2.0.8) to compute g_d from ϵ. A value of $\epsilon_{vd} \sim 10^{-15}$ as in (2.0.1) would require $g_d \sim \mathcal{O}(10^{-13} - 10^{-12})$. The WGC implies then that the theory has a UV cut-off:

$$\Lambda_{UV} \lesssim g_d M_{Pl} \sim \mathcal{O}(10^2 - 10^3)\text{TeV}. \tag{2.0.12}$$

Therefore, new physics must appear below energies of order $\mathcal{O}(100)$ TeV. Such physics could be accessible at future experiments at collider, such as the 100 TeV Future Circular Collider (FCC).

We have obtained a scale for the apparition of a new physics. However, this bound is obtained from the WGC, which is a conjecture from the Swampland. Following the Swampland program, such a scenario is consistent with quantum gravity only if it could arise from a string theory model. We will discuss now one possible venue for realizing this UV completion in a string theory. We do not attempt an explicit string model building which is beyond the scope of this work. We contemplate the possibility that a hierarchy $g_d \ll g_v$ is obtained through the suppression of g_d by the volume of the internal compactified space.

More precisely, we consider a scenario where we start from ten-dimensional type IIB string theory compactified on a six-dimensional space of volume $V_6 \equiv (2\pi R)^6$. The four-dimensional reduced Planck mass M_{Pl} is related to the string scale mass M_s and string coupling g_s through (multiplied by 2 for type I strings):

$$M_{Pl}^2 = \frac{R^6 M_s^6}{2\pi g_s^2} M_s^2 \tag{2.0.13}$$

The visible $U(1)_v$ is taken to live on a D5-brane wrapping a small two-dimensional cycle of approximate string size of volume $(2\pi r)^2 \gtrsim 4\pi^2 M_s^{-2}$. Then, the visible coupling reads:

$$g_v^2 = \frac{2\pi g_s}{r^2 M_s^2} \simeq 2\pi g_s \tag{2.0.14}$$

The dark $U(1)_d$ is instead chosen to live on a D9-brane wrapping the whole six-dimensional compact space and its gauge coupling is given by:

$$g_d^2 = \frac{2\pi g_s}{R^6 M_s^6} \tag{2.0.15}$$

Then, we get:

$$\epsilon_{vd} \sim \frac{g_v g_d}{16\pi^2} \sim \frac{g_s}{8\pi R^3 M_s^3} \quad \Rightarrow \quad \epsilon_{vd} \sim \frac{1}{\sqrt{128\pi^3}} \frac{M_s}{M_{Pl}} \sim 10^{-2} \frac{M_s}{M_{Pl}} \tag{2.0.16}$$

thus

$$\epsilon_{vd} \sim 10^{-15} \quad \Rightarrow \quad M_s \sim \mathcal{O}(100)\text{TeV} \tag{2.0.17}$$

This is merely two orders of magnitude above the proposals of TeV strings for solving the hierarchy problem [18, 20, 22, 28, 19, 175, 92, 93, 42, 197]. Note that our analysis is similar to the analysis performed in [130]. However there is a notable difference in that we impose that the $U(1)_d$ dark propagates in the whole large dimensions, thus six in this example, therefore we have considered D5-D9 branes instead of D3-D7 , leading to different results, and in particular allowing smaller values of the mixing. The Dp-D(p-4) set-up is enforced by supersymmetry, but, in the case of low string scale, we could have taken instead, without change in our results, a non-supersymmetric configuration of D3-D9 branes, our world being non-supersymmetric at least up to TeV energy scales. However, one should keep in mind that some of the non-supersymmetric configurations tend to fall in the Swampland [188].

The above scenario implies the appearance of large extra-dimensions at a scale of order:

$$\frac{1}{R} = \left(\frac{M_s}{\sqrt{8\pi} M_{Pl}} \frac{1}{\alpha_{YM}} \right)^{1/3} M_s \tag{2.0.18}$$

where we have identified the tree-value of the SM gauge couplings as $\alpha_{YM} = g_s/2$. Taking an approximate value for $\alpha_{YM} \sim 1/25$, we get

$$\frac{1}{R} \sim \mathcal{O}(10)\ \text{GeV} \tag{2.0.19}$$

Though these values of the compactification energy scale might seem low, they are not experimentally excluded. Gauge bosons propagate in these extra dimensions, in addition to the gravitons. However, in contrast to the case in [18, 20, 22, 21, 3, 23], these are Kaluza-Klein excitations of the dark $U(1)$ with tiny couplings. It is the production of a huge number of them that will compensate the coupling strong suppression. They can be observed as missing energy at collider experiments in particular at a 100 TeV collider.

We can express the string mass scale and the compactification radius as a fonction of g_d and M_{Pl} :

$$M_s \sim \sqrt{g_s} g_d M_{Pl} \quad \text{and} \quad \frac{1}{R} \sim \frac{g_d^{\frac{4}{3}}}{(8\pi)^{\frac{1}{6}}} M_{Pl} \lesssim g_d M_{Pl} \tag{2.0.20}$$

The most stringent bound on ϵ today is $\epsilon \sim \mathcal{O}(10^{-16})$ (see e.g. [107]), with a mass for the dark photon around the keV. For this case, one obtains for the string mass scale $M_s \sim 10^4$ GeV, and $\frac{1}{R} \sim 0.1 - 1$ GeV. Smaller values of ϵ cannot be obtained through our simple large extra-dimension setup as they will conflict with the current experimental limit on the string scale. For different values of the dark photon mass, the constraints on ϵ are weaker, and consequently the extra-dimension and string mass scales are set at higher energies. Taking the three values mentioned above we can have for $\epsilon \sim 10^{-10}$, $\epsilon \sim 10^{-7}$ and $\epsilon \sim 10^{-4}$, respectively, a string mass scale and an inverse compactification radius of order $M_s \sim 10^{10}$ GeV and $\frac{1}{R} \sim 10^7$ GeV, $M_s \sim 10^{13}$ GeV and $\frac{1}{R} \sim 10^{11}$ GeV, and finally $M_s \sim 10^{16}$ GeV and $\frac{1}{R} \sim 10^{15}$ GeV.

The intermediate scale $\sim 10^{11}$ GeV has diverse motivations [65, 42]. It also corresponds to the energy where the SM quartic Higgs coupling vanish, thus a scale where new degrees of freedom might be expected. Though we cannot proceed to the same string embedding as we have done above, for kinetic mixing as small as $\epsilon \sim 10^{-23}$ the WGC requires new physics around the scale $\Lambda \sim 1 - 10$ MeV, that could then in turn be constrained by the Big Bang Nucleosynthesis.

There are some comments to do about the application of the WGC that we have presented in this chapter.

In our way to generate the large hierarchy between the two couplings g_v and g_d, we have assumed the existence of small cycle with a size of order of the string scale inside six large compact dimensions. This is not the case in the simplest toroidal compactifications and requires some warping. Thus, the KK excitations of the dark photon are not expected in general to exhibit the same spectrum as in the simplest case. However, assuming that the rough behaviour of the density of states goes with the energy as E^6/M_s^6, a sizable value of the effective coupling between SM states and the dark photons is reached only at energies of the order of M_s.

In most phenomenological applications, the dark $U(1)$ is massive. The WGC that we have presented in chapter 1 concerns massless $U(1)$ gauge bosons. For instance, in the case of a massive $U(1)$, the charge is not conserved and there is no problem of remnants as charged black holes decay. However, one may argue that if the weak gravity states masses m_{WGC} are much bigger than the dark photon mass m_{γ_d}, the massive case is a (Higgs) phase of the same theory and remains in the landscape. Moreover, the comparison of gravity and gauge forces should be done at energies of order m_{WGC} and makes sense in the region $m_{\gamma_d} \ll m_{WGC}$. Finally, we have explicitly illustrated the WGC prediction for the UV cut-off of the theory by a type IIB string scenario that we do not expect to break down because of an infrared Higgsing of the $U(1)$. In fact, [150] have argued, through the explicit investigations of the properties of the WGC charge lattice, that the bounds used here on the mass, combination of charges ratios and ultraviolet cut-off of the theory remain true. A detailed discussion of the expected masses for the dark photon in different string settings is provided elsewhere [13].

To conclude, we will recapitulate the work done in this chapter. Even if in general the WGC does not seem useful for phenomenology, we found that there is some place in the space of physics models where the WGC can have a utility, in particular when we are looking at some models of dark matter. Looking at models with a dark photon, we have pointed out the amusing coincidence that the observation of kinetic mixing between this dark photon and the ordinary one would suggest new physics at scales that should be probed by a future collider.

CHAPTER 3

Revisiting the Scalar Weak Gravity Conjecture

In the chapter 1, we have presented a lot of different aspects of the Weak Gravity Conjecture, as the magnetic WGC, or the modification one should do when adding several gauge fields or massless scalars for example. But in all these different set-ups, there is no adding of a potential for the scalars. In this chapter, we will propose an extension of the WGC to the case of scalars with potential, and see what are the consequences of such a conjecture.

The rest of the chapter is organized as follows. In the first section we will present some attempts to obtain a Scalar WGC. In Section 2, we formulate the constraint of dominance of scalar interactions with respect to the gravitational ones for the case of a single massive scalar field self-interacting. We illustrate the constraint by the simplest example of a single real field with a cubic and quartic potential. A few other examples are studied in section 3. Those include the quartic complex potential, the axion, the exponential and the Starobinsky potential. In the section 4, we discuss an extension to moduli and massless scalars. Section 5 presents our conclusions.

3.1 Towards a Scalar Weak Gravity Conjecture

We begin by note that going beyond gauge fields and writing a conjecture similar to the WGC one for scalar fields, possibly complementary to Swampland conjectures, is not straightforward. First, there is no such obvious arguments on decay of black holes that can be used to induce the form of the conjecture. Second, to test in all generality different scalar conjectures in a quantum gravity theory is not easy. The scalar sector of the theory is very sensitive to the supersymmetry breaking. Implementing supersymmetry breaking in a string theory and extracting the full corrections to the scalar potential of a single real field in flat space-time is a non trivial problem. Moreover, supersymmetric models involve complex scalars, and it is not evident how to disentangle all the facets of constraints applying on one real scalar. With the lack of non-supersymmetric string theory examples, one is lead to postulate some form of the scalar conjecture and evaluate it by investigating the consequences. The hope is that even this modest trial and error method will turn out to be useful and will allow us to shed some light on the landscape of the Effective Field Theories coupled to gravity. This way of proceeding applies to the conjectures discussed below.

A Scalar Weak Gravity Conjecture (SWGC) was investigated in [189] and can be seen as a special case of the RFC. In the context of the RFC, the scalar field is massless and one is interested in the long range interactions it mediates. In an attempt to retrieve the Swampland Distance Conjecture mass formulae, it was proposed that:

$$g^{ij}\partial_i m \partial_j m \geq m^2 \tag{3.1.1}$$

where $\partial_i m \equiv \partial m/\partial\phi_i$ is the derivative of the mass term m with respect to the scalar field ϕ_i and g^{ij} is the appropriate metric on the manifold of fields. In a footnote of [189], it was also mentioned that, looking at different forms of the equalities satisfied by the central charge in $\mathcal{N}=2$, another possible form of the conjecture could have been:

$$g^{ij}\partial_i\partial_j m^2 \geq g^{ij}\partial_i m\partial_j m + m^2. \tag{3.1.2}$$

But we face immediately to a puzzle. The constraint (3.1.1) does not involve repulsive interactions and as such cannot be considered as a realization of the RFC, since the RFC states that the repulsive forces are greater than the attractive ones. It seemed strange in the RFC set-up discussed in [189], as scalar mediated forces are attractive, and the possibility (3.1.2) was not pursued any further, with the exception of a few comments in [85]. It was somehow dismissed due to the lack of simple physical interpretation.

All these considerations led to the proposal of another form of the conjecture for scalar fields in [128]: the mass m of an interacting scalar field satisfies the bound [172]:

$$m^2\frac{\partial^2}{\partial\phi^2}\left(\frac{1}{m^2}\right) \geq \frac{1}{{M_{Pl}}^2} \tag{3.1.3}$$

This was obtained by modifying by a factor 2 and an additional four-point contact interaction the inequality (3.1.1) expressed as derivatives of the scalar potential. This form of the conjecture was motivated by a set of implications [128, 172, 170, 208, 16], some of which might be of phenomenological importance. However, it raises some questions about its origin and the meaning of the corresponding inequality. As a consequence of the (3.1.3), for states with a mass depending on the scalar ϕ, the equality in (3.1.3) is reached for

$$m^2(\phi) = \frac{m_0^2}{Ae^{-\phi} + Be^{\phi}} \tag{3.1.4}$$

where A and B are integration constants. Through the identification $e^{-\phi} = R^2$, the result of (3.1.4) has been interpreted in [128] as an indication of the extended nature of the fundamental states.

Taken as such, the above proposals were dismissed in [118], because of inconsistent implications for simple scalar potentials, and it was instead suggested that scalar particles should be subject to constraints in such a way that they would not form bound states with size smaller than their Compton wavelength. No generic alternative formulation for these constraints on the scalar potential was proposed.

In this work, we will postulate that in the appropriate low energy limit, for the leading interaction, the gravitational contribution must be sub-leading. For particular scalar fields, we will propose an explicit set-up, based on the computation of four-point functions, for comparing the different interactions. The resulting inequalities will reproduce different forms of the Swampland conjectures, and, in a particular case, the inequality will be saturated for masses of the form (3.4.11):

$$m_X^2(\phi) = m_-^2 e^{-2\phi} + m_+^2 e^{2\phi} \,. \tag{3.1.5}$$

instead of (3.1.4).

Now that we have presented different proposals for a Scalar Weak Gravity Conjecture, let's develop more what we want to study.

3.2 Scalar vs Gravity in the non-relativistic regime

One interesting feature of the WGC is that it can be, at least in the flat space, expressed in different terms. In chapter 1, we have presented refinements of the WGC, obtained using the decay or the extremality of black holes, but one can also see the WGC as a statement that for any abelian gauge symmetry $U(1)$ there is at least one state with gauge self-interaction stronger than the gravitational one. It is this point of view which is emphasized by the RFC conjecture, as presented above. The goal of this chapter here is to give a possibility of the extension of this conjecture to the case of scalars fields.

To begin with, we will start with the case of a single self-interacting *massive* scalar field. There is no clear statement that we can obtain for black holes physics, or RFC. We will so just postulate that for this scalar field the self-interaction is stronger than the gravitational one.

This assertion calls for a few immediate remarks. First, instead of the case of the RFC where all the forces are long-range, here we will have forces mediated by massive particles, and we need to specify at which scale the different interactions are computed and compared. We will take the scale to be of order of the mass of the self-interacting particle. This is consistent with the fact that the WGC makes statements about properties of EFT. When we look at these energy scales, we can take as a good approximation the non-relativistic theory. This means, for example, that in scattering processes the particle number is conserved. We shall therefore investigate the strength of the interactions by computing the simplest scattering processes. Precisely, we will compare the four-point amplitude contribution of the scalar self-interaction versus the gravitational one.

Since we work in the non-relativistic limit, we will keep only the leading order in $1/c^2$. The gravitational forces are then expected to be well described by the Newtonian potential. Higher order corrections, as those given by the Einstein–Infeld–Hoffman Lagrangian, will be neglected. In practice, instead of dealing with the potential in coordinates space, we will work in the Fourier-transform space by computing the scattering amplitudes. The dominance of scalar self-interaction means in particular that all the higher dimensional non-renormalizable interactions suppressed by higher powers of the Planck mass should be subdominant and may be neglected. We will see below that this preeminence can happen to be violated in isolated regions of size $\frac{\Delta\phi^2}{m^2} \sim \frac{m^2}{\tilde{M}_{Pl}^2}$ where the interactions can switch nature between attractive and repulsive.

We first investigate the simplest case of cubic and quartic potential, in order to see if there is a proper form that we can obtain from such a basic potential. We will discuss other forms of scalar potentials, with a phenomenological interest in the next section.

Let's consider a real scalar ϕ with the potential:

$$V(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{\mu}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4. \tag{3.2.1}$$

In string theory, our fiducial quantum gravity theory, all the low energy parameters are field dependent. For simplicity we will consider here that the other scalar fields are fixed to their vacuum value and decouple from the dynamics of the low energy effective action under scrutiny. At energy scales $E \sim m_0$, as we have said the theory is non-relativistic and can be described by the corresponding limit. We study fluctuations around $\phi = 0$ and make the field redefinition:

$$\phi(x) = \frac{1}{\sqrt{2m_0}}\left(\psi(\mathbf{x},t)e^{-im_0t} + \psi^*(\mathbf{x},t)e^{im_0t}\right) \tag{3.2.2}$$

where the phase e^{-im_0t} is introduced to take into account the leading m_0 term in the non-relativistic limit expansion $E \simeq m_0 + \mathbf{p^2}/2m_0$ where $\mathbf{p}$ is the particle three dimensional momentum. The denominator $\sqrt{2m_0}$ comes from the different normalization in relativistic and non-relativistic quantum mechanics.
The potential for the non-relativistic field ψ should be of the form

$$V_{eff}\left(\psi\psi^*\right) = m_0\psi\psi^* + \frac{\tilde{\lambda}}{16m_0^2}\left(\psi\psi^*\right)^2. \tag{3.2.3}$$

We now want to relate the single non-relativistic coupling $\tilde{\lambda}$ with the coefficients of the relativistic potential. We identify the low energy limit of the $2 \to 2$ scattering in the ϕ description with the corresponding scattering of four ψ states. This leads trivially to $\lambda = \tilde{\lambda}$ when $\mu = 0$ in (3.2.1). In the case where $\mu \neq 0$, we will have to take into account the contributions to the $2 \to 2$ scattering from the exchange of a virtual ϕ. We have in this case three diagrams, one for each channel, as shown in Figure 3.1. We can compute the non-relativistic limit of each one of them. This is obtained requiring $s - 4m_0^2 \ll m_0^2$, where $s = (p_1 + p_2)^2$ is the usual Mandelstam variable and p_1, p_2 the four-momenta of the initial states. We also have $t = -\frac{1}{2}(s - 4m_0^2)(1 - \cos(\theta))$ and $u = -\frac{1}{2}(s - 4m_0^2)(1 + \cos(\theta))$, θ being the angle between the in-going and out-going particles momenta in the centre of mass frame. This basic computation yields the s-channel contribution as:

$$(-i\mu)^2\frac{i}{s - m_0^2} = \frac{-i\mu^2}{3m_0^2} + \mathcal{O}\left(\frac{s - 4m_0^2}{m_0^2}\right), \tag{3.2.4}$$

and the t-channel as:

$$(-i\mu)^2\frac{i}{t - m_0^2} = \frac{i\mu^2}{m_0^2} + \mathcal{O}\left(\frac{s - 4m_0^2}{m_0^2}\right). \tag{3.2.5}$$

Finally, the u-channel contribution is the same as the t-channel one. Summing up the three contributions we obtain $i\frac{5}{3}\frac{\mu^2}{m_0^2}$, so that the effective four-point self-interaction coupling in the non-relativistic limit is:

$$\tilde{\lambda} = \lambda - \frac{5}{3}\frac{\mu^2}{m_0^2}. \tag{3.2.6}$$

In computing the gravitational interaction, we have assumed $m_0^2 > 0$. Both attractive and repulsive forces can be obtained from the quartic self-interaction, through the choice of $\lambda < 0$ and $\lambda > 0$ respectively. On the other hand, the trilinear term always leads to an attractive force in a $2 \to 2$ states scattering. However, when $\lambda < 0$ the stability of the potential means that additional non-renormalizable terms are important and should be taken into account. In the case of $\lambda > 0$, eq (3.2.6) shows the competition between the attractive and repulsive interactions in the non-relativistic limit. The resulting sign of $\tilde{\lambda}$ tells us about the attractive or repulsive nature of the effective interaction and, in the case where they are in competition, which one of the two terms dominate at energies $E \sim m_0$.

As said previously, in the WGC, the gauge and gravity scalar forces have similar dependence in the distance between the scattering particles at leading order. In this case, there are two type of corrections. The one from the evolution of gauge coupling with energy and the other from post-Newtonian effects. But for the scalar interactions the situation is not the same, in particular because the range of the forces will depend on the mass of the mediators. In the non-relativistic limit, the scalar potential is approximated by a delta distribution in space while the gravitational potential is Newtonian. A point-like interaction arises from integrating out massive mediators. In the infrared, at energies below the mass scale, the gravitational scattering exhibits a divergence coming from the t and u channels. This divergence is the one which came from the long-range character of the gravitational force. Obviously, to compare a Newtonian potential at long distance with the strength of the scalar localised interaction is not very instructive. It is essential in the comparison to fix the energy scale, and naturally it is given by the mass of the scalar particle, and consider the gravitational scattering in the s-channel at $s \sim 4m_0^2$.

Requiring that gravity is the weakest force at low energy amounts then to impose:

$$\left|\tilde{\lambda}\right| = \left|\lambda - \frac{5}{3}\frac{\mu^2}{m_0^2}\right| \geq \frac{m_0^2}{M_{Pl}^2}. \tag{3.2.7}$$

We have put an absolute value on the left hand side so that it holds independently of the sign of the self-interaction. Note also that, in the spirit of [29, 174, 83], the quantity $\sqrt{|\tilde{\lambda}|}M_{Pl}$, could be interpreted as an ultra-violet cut-off scale dictated by quantum gravity. In particular, this means that both the limits $\lambda \to 0$ and $\mu \to 0$ cannot be taken simultaneously. Cancellation of the two terms in $\tilde{\lambda}$, as we said, might encode the change of nature of the scalar interactions on a region of the phase space that need to be studied case by case.

Below, we will work in more generic field background values and potentials, there-

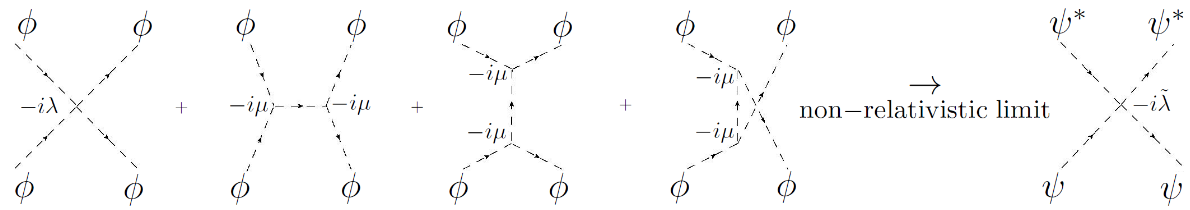

Figure 3.1 – The identification of the $2 \to 2$ scattering in the non-relativistic theory coming from the corresponding scattering in the relativistic case.

fore we will impose a stronger condition

$$4m_0^2 \left| \frac{\partial^4 V_{eff}}{\partial^2\psi \partial^2\psi^*} \right|_{\psi=0} \geq \frac{\tilde{c}}{M_{Pl}^2} \left| \frac{\partial^2 V_{eff}}{\partial\psi \partial\psi^*} \right|_{\psi=0}^2 \tag{3.2.8}$$

and take the order one constant $\tilde{c}$ to be $\tilde{c} = 1$, which amounts to redefine the Planck mass to $\tilde{M}_{Pl}$. The r.h.s. of (3.2.8) represents the gravitational attractive interaction between the two particles only when we work at the minimum of the potential and the squared mass is positive defined.

We focus now on the simplest case $\mu = 0$ and investigate the relative strengths of self-interaction and gravitational one when ϕ sweeps the range of possible values. For this purpose we consider small perturbations $\delta\phi$, corresponding to the above ψ, around background values ϕ. We expand

$$V(\phi+\delta\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + m_0^2\phi\delta\phi + \frac{\lambda}{3!}\phi^3\delta\phi + \frac{1}{2}\left(m_0^2 + \frac{\lambda}{2}\phi^2\right)(\delta\phi)^2 + \frac{\lambda}{3!}\phi(\delta\phi)^3 + \frac{\lambda}{4!}(\delta\phi)^4. \tag{3.2.9}$$

From (3.2.9), we can immediately read the mass term, the cubic and the quartic couplings for $\delta\phi$ and the effective quartic coupling in the non-relativistic limit. Those are given by:

$$m_{\delta\phi}^2(\phi) = m_0^2 + \frac{\lambda}{2}\phi^2, \qquad \mu_{\delta\phi} = \lambda\phi, \qquad \lambda_{\delta\phi} = \lambda \qquad \tilde{\lambda} = \lambda - \frac{5}{3}\frac{\lambda^2\phi^2}{m_0^2 + \lambda/2\phi^2}. \tag{3.2.10}$$

We restrict to the case with $m_0^2, \lambda > 0$ to explicitly exhibit the competition between the attractive and repulsive terms. Requiring gravity to be the weakest force leads to

$$\left| \lambda - \frac{5}{3}\frac{\lambda^2\phi^2}{m_0^2 + \frac{\lambda}{2}\phi^2} \right| \geq \frac{1}{\tilde{M}_{Pl}^2}\left(m_0^2 + \frac{\lambda}{2}\phi^2\right). \tag{3.2.11}$$

The term inside the absolute value of (3.2.11) vanishes for $\phi^2 = \frac{6}{7}\frac{m_0^2}{\lambda}$. The cubic term dominates above this turning point, a region where the interaction is attractive. The quartic one dominates instead below the turning point, making the scalar interaction repulsive.

We first investigate the $\phi^2 \leq \frac{6}{7}\frac{m_0^2}{\lambda}$ region where (3.2.11) reads

$$\phi^4 + \left(4\frac{m_0^2}{\lambda} + \frac{14}{3}\tilde{M}_{Pl}^2\right)\phi^2 + 4\frac{m_0^4}{\lambda^2} - 4\tilde{M}_{Pl}^2\frac{m_0^2}{\lambda} \leq 0. \tag{3.2.12}$$

Assuming $\lambda \geq \frac{m_0^2}{\tilde{M}_{Pl}^2}$ and discarding the solutions with $\phi^2 < 0$, this is verified inside the region

$$0 \leq \phi^2 \leq -2\frac{m_0^2}{\lambda} - \frac{7}{3}\tilde{M}_{Pl}^2 + \frac{7}{3}\tilde{M}_{Pl}^2\sqrt{1 + \frac{120}{49}\frac{m_0^2}{\lambda \tilde{M}_{Pl}^2}}. \tag{3.2.13}$$

At the first order in $\frac{1}{\tilde{M}_{Pl}^2}$, this is obtained:

$$\phi^2 \lesssim \frac{6}{7}\frac{m_0^2}{\lambda} - \frac{600}{343}\frac{m_0^4}{\lambda^2}\frac{1}{\tilde{M}_{Pl}^2} \tag{3.2.14}$$

which exhibits a small region of order $\tilde{M}_{Pl}^{-2}$ below the critical value where gravity is stronger than quartic scalar self-interaction.

For $\lambda \leq \frac{m_0^2}{\tilde{M}_{Pl}^2}$, the turning point happens at a scale $\phi^2 \sim \frac{m_0^2}{\lambda} \geq \tilde{M}_{Pl}^2$ and, as the inequality would not be solved for $\phi^2 \leq \frac{6}{7}\frac{m_0^2}{\lambda}$, this would translate in gravity being stronger than scalar interactions all the way up to the Planck scale.

Let's now turn to the case $\phi^2 \geq \frac{6}{7}\frac{m_0^2}{\lambda}$. There, the inequality translates into

$$\phi^4 + \left(4\frac{m_0^2}{\lambda} - \frac{14}{3}\tilde{M}_{Pl}^2\right)\phi^2 + 4\frac{m_0^4}{\lambda^2} + 4\tilde{M}_{Pl}^2\frac{m_0^2}{\lambda} \leq 0. \tag{3.2.15}$$

At leading order in $\frac{m_0^2}{\tilde{M}_{Pl}^2}$, the region where the inequality is verified is given by

$$\frac{6}{7}\frac{m_0^2}{\lambda} + \frac{600}{343}\frac{m_0^2}{\lambda^2}\frac{m_0^2}{\tilde{M}_{Pl}^2} \lesssim \phi^2 \lesssim \frac{14}{3}\tilde{M}_{Pl}^2 - \frac{6}{7}\frac{m_0^2}{\lambda} + \mathcal{O}(\tilde{M}_{Pl}^{-2}) \tag{3.2.16}$$

In conclusion, up to the Planck scale, the gravity seems to dominate only around the special value $\phi^2 = \frac{6}{7}\frac{m_0^2}{\lambda}$ in a symmetric interval of radius $\Delta\phi^2 \sim \frac{m_0^4}{\tilde{M}_{Pl}^2}$. It would be interesting to investigate, for explicit examples of quantum gravity, if the theory can be insensitive to such small field excursion regions, but this goes beyond the scope of this work.

3.3 Single Scalar Field Potentials

In this section, we would like to investigate what the implications of requiring gravity to be weaker than the scalar field self-interactions in the non-relativistic limit are on different potentials of phenomenological interest. More precisely, we will consider *very slowly rolling fields*, having in mind possible cosmological applications. The idea is to impose the condition (3.2.8) and extract its implications for the involved scales and couplings.

3.3.1 The Mexican Hat or Higgs-like Quartic Potential

We will begin the discussion with the quartic scalar potential

$$V(\phi, \bar{\phi}) = -m^2\bar{\phi}\phi + \lambda(\bar{\phi}\phi)^2. \tag{3.3.1}$$

with $\lambda > 0$, insuring stability, and $m^2 > 0$.

For our calculation it is convenient to use the parametrization $\phi(x) = \frac{1}{\sqrt{2}}\rho(x)e^{i\pi(x)}$. This potential develops a minimum at $\rho^2 = \frac{m^2}{\lambda}$. We note that this theory has a global $U(1)$ symmetry. We have already explained that quantum gravity requires that either the symmetry is gauged or broken, since there is no possibility of global symmetry. However, the term which breaks explicitly the symmetry might be sub-leading to the quartic self-interaction considered here. We will consider that the gauge group is spontaneously broken at the minimum, and we call $\pi(x)$ the associated Goldstone boson. The final mass of $\pi(x)$ depends on details of the complete theory. It might be generated by the higher order terms breaking the global symmetry, terms that we have neglected. If we consider the other possibility offered by the Swampland, it could also be that the $U(1)$ symmetry is gauged. In this case, $\pi(x)$ gives rise to the longitudinal mode of the massive gauge boson. We will focus here only on the field $\rho(x)$ which plays in the latter case the role of the Higgs field.

We consider a small perturbation $\delta\rho(x)$ around a background value $\rho(x)$. The expansion of the potential, up to $\mathcal{O}(\delta\rho^4)$, reads:

$$V(\rho+\delta\rho) \simeq -\frac{1}{2}m^2\rho^2 + \frac{\lambda}{4}\rho^4 + (\lambda\rho^3 - m^2\rho)\delta\rho + \frac{1}{2}(3\lambda\rho^2 - m^2)\delta\rho^2 + \lambda\rho\delta\rho^3 + \frac{\lambda}{4}\delta\rho^4. \quad (3.3.2)$$

The expansion gives us the effective mass term, trilinear and quartic couplings of $\delta\rho(x)$. There are $m_{\delta\rho}^2 = 3\lambda\rho^2 - m^2$, $\mu_{\delta\rho} = 6\lambda\rho$, $\lambda_{\delta\rho} = 6\lambda$, respectively. The $\delta\rho(x)$ resulting quartic self-interaction $\tilde{\lambda}$ at low energies can now be computed to be

$$\tilde{\lambda} = 6\lambda - \frac{60\lambda^2\rho^2}{3\lambda\rho^2 - m^2} = -6\lambda\frac{(m^2 + 7\lambda\rho^2)}{3\lambda\rho^2 - m^2}. \quad (3.3.3)$$

Vanishing self-interaction, i.e. a null value for $\tilde{\lambda}$, corresponds to $m^2\lambda + 7\lambda^2\rho^2 = 0$. This is obviously never satisfied here.

We discard the region $\rho^2 < \frac{m^2}{3\lambda}$ where the effective mass of $\delta\rho(x)$ is either tachyonic or vanishing, though we have checked that the inequality (3.2.8) is satisfied in this case.

We will investigate the region $m_{\delta\rho}^2 > 0$, i.e. $\rho^2 > \frac{m^2}{3\lambda}$. We have:

$$9\frac{\lambda^2}{\tilde{M}_{Pl}^2}\rho^4 - \left(6\lambda\frac{m^2}{\tilde{M}_{Pl}^2} + 42\lambda^2\right)\rho^2 + \frac{m^4}{\tilde{M}_{Pl}^2} - 6m^2\lambda \leq 0. \quad (3.3.4)$$

Discarding the region $\rho^2 \in \left[0, \frac{m^2}{3\lambda}\right]$ as discussed above, the inequality is satisfied for:

$$\frac{m^2}{3\lambda} < \rho^2 \leqslant \frac{14}{3}\tilde{M}_{Pl}^2 + \frac{17}{21}\frac{m^2}{\lambda} + \mathcal{O}(\tilde{M}_{Pl}^{-2}) \quad (3.3.5)$$

It is worth mentioning that at the minimum, where $\rho^2 = \frac{m^2}{\lambda} \equiv v$, we get $\tilde{\lambda} = -24\lambda$, and the conjecture is then verified in the case:

$$\lambda \geq \frac{1}{12}\frac{m^2}{\tilde{M}_{Pl}^2} \sim 10^{-17} \Leftrightarrow v^2 \leq 12\tilde{M}_{Pl}^2 \sim 10^{37}GeV^2, \quad (3.3.6)$$

where we have taken m to be the electroweak scale. A condition for the respect of the conjecture is that the vev of the scalar fields doesn't take transplanckian values.

3.3.2 Axion-like Potential

We will now consider the case of the axion potential:

$$V(\phi) = \mu^4 \left(1 - \cos\left(\frac{\phi}{f_a}\right)\right). \tag{3.3.7}$$

Once again, expanding this potential around a fixed value ϕ_0 and excluding points where $\cos\left(\frac{\phi_0}{f_a}\right) = 0$, in which the state becomes massless and our non-relativistic limit no more applies, we obtain up to fourth order in $\delta\phi$:

$$\begin{aligned} V(\phi) \simeq \;& \mu^4 \left[1 - \cos\left(\frac{\phi_0}{f_a}\right) + \sin\left(\frac{\phi_0}{f_a}\right)\frac{\delta\phi}{f_a} + \frac{1}{2}\cos\left(\frac{\phi_0}{f_a}\right)\frac{(\delta\phi)^2}{f_a^2}\right. \\ & \left. - \frac{1}{3!}\sin\left(\frac{\phi_0}{f_a}\right)\frac{(\delta\phi)^3}{f_a^3} - \frac{1}{4!}\cos\left(\frac{\phi_0}{f_a}\right)\frac{(\delta\phi)^4}{f_a^4}\right], \end{aligned} \tag{3.3.8}$$

We can extract the different couplings needed: $\tilde{\lambda} = -\frac{1}{f_a^4}\left(\cos\left(\frac{\phi_0}{f_a}\right) + \frac{5}{3}\frac{\sin^2(\phi_0/f_a)}{\cos(\phi_0/f_a)}\right)$. Requiring gravity to be the weakest force leads to

$$\frac{1}{f_a^2}\left|\cos\left(\frac{\phi_0}{f_a}\right) + \frac{5}{3}\frac{\sin^2\left(\frac{\phi_0}{f_a}\right)}{\cos\left(\frac{\phi_0}{f_a}\right)}\right| \geq \frac{1}{\tilde{M}_{Pl}^2}\left|\cos\left(\frac{\phi_0}{f_a}\right)\right|, \tag{3.3.9}$$

which yields

$$\frac{1}{f_a^2}\left|1 + \frac{5}{3}\tan^2\left(\frac{\phi_0}{f_a}\right)\right| \geq \frac{1}{\tilde{M}_{Pl}^2}. \tag{3.3.10}$$

We have expanded around a generic background value ϕ_0 thus this inequality leads to:

$$f_a^2 \leq \tilde{M}_{Pl}^2 \tag{3.3.11}$$

We therefore retrieve the Axion Weak Gravity Conjecture, which requires an axion decay constant lower the Planck scale [29, 203, 179, 63, 146, 88, 144, 31, 204, 161, 169, 156, 145, 143, 86]. Note that, in the r.h.s. of (3.3.9), we have taken the absolute value of the squared mass term. Here we see the inequality as taken on derivatives of the potential since the squared mass can be negative.

3.3.3 Inverse power-law effective scalar potential

Another scalar potential is the inverse power-law one, frequently used in cosmological applications. It reads

$$V(\phi) = M^{4+p}\phi^{-p}, \tag{3.3.12}$$

where $p > 0$ is a constant and M sets the energy scale. In the general case, we expand the potential as a Taylor series

$$\begin{aligned} \frac{1}{M^{4+p}}V(\phi_0 + \delta\phi) \;=\;& \phi_0^{-p} - p\phi_0^{-p-1}\delta\phi + \frac{p(p+1)}{2}\phi_0^{-p-2}(\delta\phi)^2 - \frac{p(p+1)(p+2)}{3!}\phi_0^{-p-3}(\delta\phi)^3 \\ & + \frac{p(p+1)(p+2)(p+3)}{4!}\phi_0^{-p-4}(\delta\phi)^4. \end{aligned} \tag{3.3.13}$$

The effective quartic interaction in the non-relativistic limit is given by

$$\tilde{\lambda} = -\frac{p(p+1)(p+2)}{3}(2p+1)\phi_0^{-p-4}\,. \tag{3.3.14}$$

The gravitational interaction will thus be weaker than the scalar self-interaction in the non-relativistic limit if

$$\frac{p(p+1)(p+2)}{3}(2p+1)\left|\phi_0^{-p-4}\right| \geq \frac{p(p+1)}{\tilde{M}_{Pl}^2}\left|\phi_0^{-p-2}\right|. \tag{3.3.15}$$

which is satisfied for

$$\phi_0^2 \leq \frac{(p+2)(2p+1)}{3}\tilde{M}_{Pl}^2, \tag{3.3.16}$$

therefore forbidding large transplanckian excursions.

3.3.4 Exponential Scalar Potential

Another popular class of scalar potentials is represented by sums of exponential functions. We focus here on the simplest case

$$V(\phi) = \Lambda_0 e^{-\lambda\phi/f}. \tag{3.3.17}$$

The expansion around a background value ϕ_0 reads

$$V(\phi_0+\delta\phi) = \Lambda_0 e^{-\lambda\phi_0/f}\left[1 - \lambda\frac{\delta\phi}{f} + \frac{1}{2}\lambda^2\left(\frac{\delta\phi}{f}\right)^2 - \frac{1}{3!}\lambda^3\left(\frac{\delta\phi}{f}\right)^3 + \frac{1}{4!}\lambda^4\left(\frac{\delta\phi}{f}\right)^4\right] \tag{3.3.18}$$

and the self-interaction of the scalar field in the non-relativistic limit is encoded in the $\tilde{\lambda}$ quartic coupling

$$\tilde{\lambda} = \Lambda_0 e^{-\lambda\phi_0/f}\left(\frac{\lambda^4}{f^4} - \frac{5}{3}\frac{\lambda^4}{f^4}\right) = -\frac{2}{3}\frac{\lambda^4}{f^4}\Lambda_0 e^{-\lambda\phi_0/f}. \tag{3.3.19}$$

Application of our bound is straightforward and yields the following inequality

$$\frac{2}{3}\frac{\lambda^2}{f^2} \geq \frac{1}{\tilde{M}_{Pl}^2}, \tag{3.3.20}$$

The weak gravity regime under scrutiny is realized for scalars with an exponential potential as long as their scale does not exceed the Planck one, with

$$f^2 \leq \frac{2}{3}\lambda^2\tilde{M}_{Pl}^2. \tag{3.3.21}$$

This bound still allows for a cosmological expansion (see e.g. [214]), but is in conflict with the requirement obtained in [4], as we will discuss below.

Let's consider the case of a double exponential potential

$$V(\phi) = \Lambda_1 e^{-\lambda_1\phi/f} + \Lambda_2 e^{-\lambda_2\phi/f}, \tag{3.3.22}$$

with the assumption $\lambda_1 \sim \lambda_2$. We develop each exponential as in (3.3.18) to get

$$\tilde{\lambda} = \Lambda_1 \frac{\lambda_1^4}{f^4} e^{-\lambda_1 \phi_0/f} + \Lambda_2 \frac{\lambda_2^4}{f^4} e^{-\lambda_2 \phi_0/f} - \frac{5}{3} \frac{1}{f^4} \frac{\left(\Lambda_1 \lambda_1^3 e^{-\lambda_1 \phi_0/f} + \Lambda_2 \lambda_2^3 e^{-\lambda_2 \phi_0/f}\right)^2}{\Lambda_1 \lambda_1^2 e^{-\lambda_1 \phi_0/f} + \Lambda_2 \lambda_2^2 e^{-\lambda_2 \phi_0/f}}, \tag{3.3.23}$$

which can be rewritten as

$$\tilde{\lambda} = -\frac{1}{f^4} \frac{\frac{2}{3}\Lambda_1^2 \lambda_1^6 e^{-2\lambda_1 \phi_0/f} + \frac{2}{3}\Lambda_2^2 \lambda_2^6 e^{-2\lambda_2 \phi_0/f} + \Lambda_1 \Lambda_2 \lambda_1^2 \lambda_2^2 \left(\frac{10}{3}\lambda_1 \lambda_2 - \lambda_1^2 - \lambda_2^2\right) e^{-(\lambda_1+\lambda_2)\phi_0/f}}{\Lambda_1 \lambda_1^2 e^{-\lambda_1 \phi_0/f} + \Lambda_2 \lambda_2^2 e^{-\lambda_2 \phi_0/f}}. \tag{3.3.24}$$

The analysis of this constraint on a double exponential is somehow quite involved, and not useful here to discuss in full generality. In the case where $\lambda_1^2 + \lambda_2^2 \leq \frac{10}{3}\lambda_1\lambda_2$, all three terms in the numerator have the same sign. For $\Lambda_{1,2} > 0$ ($\Lambda_{1,2} < 0$) the scalar self-interaction is attractive (repulsive). The condition for gravity to be the weakest force reads

$$\begin{aligned} I(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, f) \quad &= \quad \lambda_1^4 \Lambda_1^2 \left(\frac{2}{3}\frac{\lambda_1^2}{f^2} - \frac{1}{\tilde{M}_{Pl}^2}\right) e^{-2\lambda_1 \phi_0/f} + \lambda_2^4 \Lambda_2^2 \left(\frac{2}{3}\frac{\lambda_2^2}{f^2} - \frac{1}{\tilde{M}_{Pl}^2}\right) e^{-2\lambda_2 \phi_0/f} \\ &\quad + \Lambda_1 \Lambda_2 \lambda_1^2 \lambda_2^2 \left(\frac{10/3 \lambda_1 \lambda_2 - \lambda_1^2 - \lambda_2^2}{f^2} - \frac{2}{\tilde{M}_{Pl}^2}\right) e^{-(\lambda_1+\lambda_2)\phi_0/f} \\ &\geq 0 \end{aligned} \tag{3.3.25}$$

It is verified for mass scales not exceeding the value $f^2 \sim \frac{2}{3}\lambda_{1,2}^2 \tilde{M}_{Pl}^2$.

3.3.5 Starobinsky Potential

The power-law and the exponential potentials are frequently used in early Universe cosmology. We investigate here the implications of (3.2.8) for the Starobinsky's potential [211].

We consider the potential:

$$V(\phi) = \Lambda^4 \left(1 - e^{-\sqrt{2/3}\phi/\tilde{M}_{Pl}}\right)^2 \tag{3.3.26}$$

and expand it around a background field value ϕ_0, and study the leading order contribution to the quartic self-interaction perturbation $\delta\phi = \phi - \phi_0$. The non-relativistic regime quartic coupling $\tilde{\lambda}$ is given by:

$$\frac{\tilde{M}_{Pl}^4}{\Lambda^4}\tilde{\lambda} \quad = \quad \frac{-\frac{256}{27} e^{-4\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} + \frac{80}{27} e^{-3\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} - \frac{16}{27} e^{-2\sqrt{2/3}\phi_0/\tilde{M}_{Pl}}}{2e^{-2\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} - e^{-\sqrt{2/3}\phi_0/\tilde{M}_{Pl}}}. \tag{3.3.27}$$

The weakness of the gravitational interaction reads now

$$\left|-\frac{16}{9} e^{-2\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} + \frac{5}{9} e^{-\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} - \frac{1}{9}\right| \geq \left| e^{-2\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} - e^{-\sqrt{2/3}\phi_0/\tilde{M}_{Pl}} + \frac{1}{4}\right|, \tag{3.3.28}$$

where we have put the absolute value on the r.h.s. to stress its positivity even if it is useless, being the square of a real quantity. Nevertheless, we still should study the sign and the strength of the l.h.s. of (3.3.28). The term inside the absolute value is always negative, meaning the scalar interaction is always attractive. So we can just drop the absolute values in (3.3.28). Simple algebra finally leads us to the conclusion that gravity is weaker than the scalar self-interaction if

$$\phi_0 \leq \sqrt{\frac{3}{2}} \ln\left(\frac{14}{\sqrt{51}-4}\right) \tilde{M}_{Pl} \sim 2\tilde{M}_{Pl}. \tag{3.3.29}$$

The coefficient in front of $\tilde{M}_{Pl}$ in the above equation is of order 1. Slightly before reaching this scale, we would encounter tachyonic modes for $\phi \sim \sqrt{\frac{3}{2}} \ln(2)\tilde{M}_{Pl}$. In this Starobinsky's model, self-interactions are strong enough to keep gravity the weakest force all the way up to the Planck scale.

3.3.6 Weak Gravity and Quintessence

The last scalar potentials that we have presented are used in theoretical models for inducing cosmic acceleration. In this case, the fields ϕ is called the quintessence field. We discuss in the following some direct implications of our constraints, and their relations to the others constraints on the model parameters derived from the experimental data, or others Swampland conjectures.

We will begin with a simple presentation of the quintessence. The late time cosmic acceleration that we can observe today may be understood in terms of a cosmological constant, in the context of the ΛCDM. It is also possible to describe this expansion using a dynamical scalar field, slowly rolling towards the minimum of its potential [198, 220]. This scalar field is called the quintessence field. One parameter driving the expansion is the equation of state, and more precisely the ratio pressure/energy density w. In the ΛCDM case, this quantity is fixed to the value $w = -1$. On the contrary, in the quintessence models, it is promoted to a dynamical variable [67]. The Swampland criteria seems to be in favor of the latter scenario, that, with parameters tightened by the current observations, may fit into the program (see [4]). In this context, for the dark energy to take over the control of the expansion of the Universe at late times, the quintessence field needs to be very light, with mass of order the Hubble parameter as measured today $m \lesssim H_0 \sim 10^{-33} eV$. The corresponding potential is unknown. That's why forms similar to those studied above have been considered. A review of the different potentials can be found in [214]. Requirements for the evolution equations of a scalar field ϕ to have a fixed point realizing the desired equation of state can be expressed as

$$\begin{cases} w_{eff} \equiv \frac{\rho_\phi + \rho_m}{P_\phi + P_m} = w_\phi > -\frac{1}{3}; \\ \Omega_\phi \equiv \frac{\rho_\phi}{3M_{Pl}^2 H^2} = 1, \end{cases} \tag{3.3.30}$$

where we denote with the subscript m the matter contribution, and ϕ for the quintessence one, and [82]:

$$\left(M_{Pl}\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \lambda^{*2} < 2. \tag{3.3.31}$$

Obviously, $w_\phi \equiv \frac{P_\phi}{\rho_\phi} = \frac{\dot\phi/2-V(\phi)}{\dot\phi/2+V(\phi)}$, leads to different dynamics for the different potentials.

We can first examine the axion potential. It gives what is called the thawing solution. The field and its corresponding equation of state are almost constant in the early cosmological era, with $w_\phi = -1$, and then starts to evolve after the mass drops below the Hubble parameter, leading to $w_\phi \geq -1$ [119, 68]. The axion shift symmetry might allow to tame loop corrections. The condition (3.3.31) reads then

$$\sin^2\left(\frac{\phi}{f_a}\right) < 2\frac{f_a^2}{M_{Pl}^2}\left(1+\cos\left(\frac{\phi}{f_a}\right)\right)^2. \tag{3.3.32}$$

The first thing we can note is that for this cosmological application, we have taken, as in [119], the potential to be $V(\phi) = \mu^4\left(1+\cos\left(\frac{\phi}{f_a}\right)\right)$. This corresponds to a shift of the minimum in (3.3.7), and so as no consequences for the analysis performed in section 3.3.2. We can use this result in the case we look at here.

As it is easy to see, the requirements (3.3.32) and (3.3.11) will not give the same bound. In particular, one can have an axion as a quintessence field without requiring that the gravity is the weakest force. However, it is important to note that requiring (3.3.11) will not prevent the possibility to this axionic field to be the quintessence field, as long as the variation of the field are small compare to the decay constant. Observational constraints allow this model to be used for quintessence with $w_0 \in]-1,-0.7[$, w_0 being today's value.

The use of a power law potential for the quintessence field gives rise to the tracking solution [221, 212]. This allows for a cosmic evolution from the so-called scaling fixed point $(x,y) = \left(\sqrt{\frac{3}{2}}\frac{1+w_m}{\lambda}, \sqrt{\frac{3}{2}\frac{1-w_m{}^2}{\lambda^2}}\right)$, with $x = \frac{\dot\phi}{\sqrt{6}M_{Pl}H}$ and $y = \frac{\sqrt{V(\phi)}}{\sqrt{3}M_{Pl}H}$, where matter dominates, to the fixed point $(x,y) = (\lambda^*/\sqrt{6}, \sqrt{1-\lambda^{*2}/6})$, where the cosmic acceleration can be realized [82]. The behaviour of the equation of state is opposite to the previous case, as w slowly decreases with the evolution. Equation (3.3.31) gives

$$\phi^2 > \frac{1}{2}p^2 M_{Pl}^2, \tag{3.3.33}$$

Unless the p parameter is tuned to be very small, this calls for transplanckian values of the field, as we should have expected since the potential is monotonically decreasing to reach its asymptotic value $V = 0$ at infinity. Together with our constraint of weak gravity $\phi^2 \leq \frac{(p+2)(2p+1)}{3}M_{Pl}^2$, this leads to:

$$\frac{(p+2)(2p+1)}{3} > \frac{p^2}{2}, \tag{3.3.34}$$

which is valid for all positive powers. It seems so at a first level that the constraints we obtained from requiring the gravity to be the weakest force are compatible with the constraints obtained from the expansion of the Universe. However, the applicability of the effective field theory treatment at transplanckian scales is for the least questionable. Besides, observations have led to constrain the tracker equation of state so tightly that the current accepted range of value for the exponent p is very restricted. Indeed, the upper bound on p was argued to be $p < 0.107$ in [77], or $p < 0.17$ in [163], so that positive integers should be excluded, making it difficult to realize power law potentials within the observational bounds in particle physics models.

We will now take a look at the single exponential potential. It is a popular model as the cosmological evolution is there described by a closed system of equation [82, 114]. However, the fact that λ^* is constant in this case leads to strongly constrain this potential. It is realized again in the fixed point mentioned above but to be reached from the trivial fixed point $(x, y) = (0, 0)$ [214]. In particular, the transition from the more interesting scaling fixed point $(x, y) = \left(\sqrt{\frac{3}{2}}\frac{1+w_m}{\lambda}, \sqrt{\frac{3}{2}\frac{1-w_m{}^2}{\lambda^2}}\right)$ is forbidden. This can be circumvented by taking the case of a double exponential potential, as in (3.3.22). The solution which is realized in this case is a tracking one with constant Ω_ϕ[34].

We recall that the constraints we obtained requiring that the gravity is weaker that the scalar interactions can be written for such a potential under the form (3.3.20) :

$$\lambda^2 \frac{M_{Pl}^2}{f^2} > \frac{3}{2}. \tag{3.3.35}$$

But in order to use this potential for a quintessence field, in particular for the epoch of cosmic acceleration to be realized we require instead

$$\lambda^2 \frac{M_{Pl}^2}{f^2} < 2. \tag{3.3.36}$$

As we see, this seems to leave a short window for both the weakness of gravity and the period of cosmic accelerated expansion to be realized through an exponential potential.

An instructive point of these type of potentials is that they have also been constrained with current observations in the interest of other Swampland conjectures, namely the de Sitter and the TCC conjectures [4, 35]. It was argued in [4] that we should have for an exponential potential $\lambda^* = \lambda \frac{M_{Pl}}{f_a} \leq 0.6$. This was devised to be in agreement with the de Sitter conjecture with the constant c appearing in (1.1.1) bounded to be $c \leq 0.6$. This bound is sensitive to uncertainties in the data as was investigated in e.g. [152, 5]. This seems to leave as the only viable conclusion that an exponential quintessence model can only lead to fifth force interactions weaker than gravity. We can add another exponential term in order to obtain a double exponential potential. such a potential is usually devised to respect both constraints coming from big-bang nucleosynthesis and cosmic acceleration. As such, one exponent, λ_1, is taken to give $\lambda_1 \frac{M_{Pl}}{f} \sim 1 - 10$, while the second is expected to take over at late times and respects the same bounds as those for the single exponent [4, 77, 34]. In this case, the weak gravity may be realized in the early Universe as long as the double exponential is concerned, but at late time, one faces the same strong constraints as discussed above.

However, [217] has hinted to the possibility that dark matter-dark energy coupling may relax constraints on λ. Hence the difficulty to have a clear and complete answer on the respect or not of our requirement by the quintessence models.

3.4 Multiple Scalar and Moduli Fields

In the previous section, we considered one scalar field, with different potential used for phenomenological applications, and we looked for the consequences or the implications of requiring that the gravitational forces are sub-dominant over the scalar ones. But in most of theoretical models, there is more than one scalar field. We will now turn to more complex situations, taking into account multiple scalar fields. When we have several scalar fields, the preeminence of the scalar interaction over the gravitational one has to be formulated in more general terms to account for these cases. In particular, we need to specify what are the processes we should consider to compare scalar and gravitational interactions.

In the case of multiple scalars, we will assume that *in the appropriate low energy limit, for the leading interaction, the gravitational contribution must be sub-leading*. The focus on the leading scalar interaction can be seen as parallel to constraining the biggest ratio q/m in the WGC, as was postulated in the first definition of the WGC.

To illustrate the meaning of this statement, we will first consider the case of a massive scalar X, taken to be complex for simplicity. The leading interaction is given by the Yukawa coupling to another real scalar field ϕ and is described by:

$$\mathcal{L}_{int} = \mu\phi|X|^2 + \cdots \tag{3.4.1}$$

where the dots stand for sub-leading higher order terms. We can write the potential as:

$$V(X,\phi) = m_X^2(\phi)\ |X|^2, \qquad \mu = \partial_\phi m_X^2 \tag{3.4.2}$$

The preeminence of scalar interactions must be taken at the mass scale $\sim\ 2m_X$ and reads then:

$$|\partial_\phi m_X| \geq \frac{m_X}{\tilde{M}_{Pl}} \tag{3.4.3}$$

We can square the above three-point amplitudes on each side, $2X \to \phi$ on the left and $2X \to G$, on the right side, where G is the graviton. The comparison concerns then two $XX^* \to XX^*$ processes, at the energy scale m_X, one through scalar and the other through graviton exchange. This leads to the following potentials for X:

$$V_{scalar}(r) = -\frac{\mu^2}{4m_X^2 r}, \qquad V_{grav}(r) = -\frac{m_X^2}{\tilde{M}_{Pl}^2 r} \tag{3.4.4}$$

Now, both scalar and gravitational interactions have similar dependence in the inter-particles distance and the comparison is straightforward:

$$\frac{\mu^2}{4m_X^2} \geq \frac{m_X^2}{\tilde{M}_{Pl}^2} \tag{3.4.5}$$

which can be written:

$$\partial_\phi m_X \partial_\phi m_X \geq \frac{m_X^2}{\tilde{M}_{Pl}^2} \tag{3.4.6}$$

In the extremal case saturating the above inequality, the solution is given by:

$$m_X^2(\phi) = m_0^2\ e^{\pm 2\phi/\tilde{M}_{Pl}}. \tag{3.4.7}$$

It is interesting to note that the spectrum of the states that saturates the inequality (3.4.6) respect what we called before in the chapter 1 the Swampland Distance Conjecture (SDC), since the states becomes light when the field ϕ approaches the Planck mass. A second observation that we can make is that the inequality (3.4.6) has been proposed by [189] in order to retrieve (3.4.7). This inequality was also discussed by [189, 208, 123] with different motivations.

Let's now move forward and consider another case: a massless complex modulus field Φ, therefore with vanishing potential. We assume again that the theory contains at least one complex scalar field X such that the mass of X and its different couplings are functions of ϕ. For simplicity, we also assume that X has no tadpole and its vacuum expectation value vanishes, $\langle X \rangle = 0$. Under these assumptions, the scalar potential then takes the form:

$$V(X,\Phi) = m_X^2(\Phi)|X|^2 + \cdots \qquad m_X^2 = m_{X0}^2 + \lambda_\Phi |\Phi|^2 + \cdots \tag{3.4.8}$$

where

$$\lambda_\Phi = \partial_\Phi \partial_{\bar{\Phi}} m_X^2(\Phi, \bar{\Phi}) \tag{3.4.9}$$

represents now the leading non-gravitational interaction of Φ. Here, m_{X0}^2 is a contribution to the squared mass independent of Φ, but depending on other fields while λ_Φ gives a scalar four-point interaction term of Φ and X obtained by expanding (3.4.8) in powers of Φ and $\bar{\Phi}$. The weakness of gravitational interaction becomes a statement comparing on one side the annihilation of two X states into two Φ state (and vice-versa) and on the other side the same channel through graviton exchange, both taken at the threshold energy scale $\sim 2m_X$.

As the modulus is massless, the gravitational interaction gets an enhancing factor of 2 compared to the massive case, analogous to the case of the gravitational deflection of light. In this case, the statement that the gravitational interaction is weaker reads:

$$\partial_\Phi \partial_{\bar{\Phi}} m_X^2 \geq 2 \frac{m_X^2}{\tilde{M}_{Pl}^2} \tag{3.4.10}$$

We can make a comment here on the case of real fields. If the fields are real, the inequality reads $g^{ij}\partial_i \partial_j m_X^2 \geq 2n m_X^2/\tilde{M}_{Pl}^2$ where g^{ij} and n are the metric in the manifold and the number of moduli fields. The dots in (3.4.8) include Φ^2 and $\bar{\Phi}^2$ as required to recover the case of real fields scattering and account for an extra factor of 2.

If the state X has a self-quartic interaction, then we will also have to check a similar constraint on the self coupling $|\tilde{\lambda}_4|\tilde{M}_{Pl}^2 \geq m_X^2$.

The extremal case corresponds to the case of equality in (3.4.10). It is solved for:

$$m_X^2(\Phi, \bar{\Phi}) = m_-^2 e^{-\sqrt{2}\frac{\Phi+\bar{\Phi}}{\tilde{M}_{Pl}}} + m_+^2 e^{\sqrt{2}\frac{\Phi+\bar{\Phi}}{\tilde{M}_{Pl}}} \tag{3.4.11}$$

This is not the most general solution. We can add some terms functions of $|\Phi|$ and also the same terms present in (3.4.11) but function of $i(\Phi - \bar{\Phi})$, corresponding to the imaginary part. We will not consider the term proportional to the modulus of $|\Phi|$ to focus on reproducing the toroidal compactification scheme. Besides, as the potential (3.4.8) and the equation (3.4.10) are symmetric under the exchange of the real and imaginary part, The fact to focus on the real part of the field only, i.e. KK and winding excitation corresponds to choose only one of the torus dimensions.

We can use the following parametrization:

$$\Phi = \frac{1}{\sqrt{2}}(\phi + i\chi), \qquad e^{\sqrt{2}\frac{\Phi+\bar{\Phi}}{M_{Pl}}} = e^{2\frac{\phi}{M_{Pl}}}, \qquad \text{and} \qquad e^{\frac{\phi}{M_{Pl}}} = R \tag{3.4.12}$$

then:

$$m_X^2(R) = \frac{m_-^2}{R^2} + m_+^2 R^2 \tag{3.4.13}$$

which is the well known formula for string states squared masses with the $\frac{m_-^2}{R^2}$ as the low energy Kaluza-Klein states that we have presented in the chapter 1. But there is also another part in the formula, namely the term in $m_+^2 R^2$. This term corresponds to winding modes, and such modes are typical to extended objects, strings, winding around a compactified dimension. The (3.4.11) differs sensibly from (3.1.3) as it extremizes a different inequality.

Note that in the statement about the preeminence of the scalar interaction, the two fields Φ and X play a symmetric role.

Now, consider the case where the field ϕ is a modulus appearing only as a parameter in the couplings of the massive scalar X ($\langle X \rangle = 0$), through

$$V(X,\phi) = m_X^2(\phi)X^2 + \sum_{n\geq 4} \lambda_n(\phi)X^n \tag{3.4.14}$$

Then, the condition (3.4.3) can be written as:

$$\left.\frac{|\partial_\phi V(X,\phi)|}{V}\right|_{X=0} \geq \frac{\sqrt{\tilde{c}}}{M_{Pl}} \tag{3.4.15}$$

while the condition (3.4.10) reads now:

$$\left.\frac{|\partial_\phi \partial_{\bar{\phi}} V(X,\phi)|}{V}\right|_{X=0} \geq \frac{2\tilde{c}}{M_{Pl}^2} \tag{3.4.16}$$

where we note the similarity with the Refined de Sitter Conjectures presented in chapter 1, with the subtlety that in (3.4.16) the second derivative has to be negative for recovering exactly the de Sitter conjecture. The same treatment can be applied at the case of several moduli, and we will recover also the general formula (1.1.1).

We have presented in chapter 1 different point of view to approach the WGC. A popular way to look at the Weak Gravity Conjecture rests on the fact that the equality in (1.1.2) relates to the BPS states relation. In [189], it was suggested to use the identity satisfied by the central charge in $\mathcal{N} = 2$ supersymmetry [111]

$$g^{i\bar{j}} D_i \bar{D}_{\bar{j}} |Z|^2 = g^{i\bar{j}} D_i Z \bar{D}_{\bar{j}} \bar{Z} + n|Z|^2. \tag{3.4.17}$$

This formula is a geometric expression, not dependent of the value of the central charge, but just on the fact that it is holomorphic. It can be used to extract a bound on the mass m thanks to the fact that in the BPS case $|Z| = m$. We have:

$$g^{ij}\partial_i\partial_j m^2 \geq g^{ij}\partial_i m \partial_j m + nm^2 \tag{3.4.18}$$

with derivatives are with respect to scalar fields, and g_{ij} is the corresponding metric of the scalars manifold. Here, we would like to contemplate a different possibility. Following [111], the right hand side of (3.4.17) is identified with the scalar potential of the black hole solution, and it was shown that it implies that at the critical point the potential satisfies (in reduced Planck mass units):

$$\partial_i \partial_{\bar{j}} V \Big|_{critical} = 2 G_{i\bar{j}} V_{critical} \tag{3.4.19}$$

We would like to contemplate here the possibility to extend this relation, beyond its derivation in the $\mathcal{N} = 2$ world, to

$$|\partial_i \partial_{\bar{j}} V| \geq cV \tag{3.4.20}$$

as given by (3.4.16). Along this line of thought, we note the similarity of (3.4.10), up to a factor 2 due to the masslessness of our field Φ, and the equation [111]:

$$\partial_i \partial_{\bar{j}} m(\Phi, \bar{\Phi}, p, q) \Big|_{critical} = \frac{1}{2} G_{i\bar{j}}(\Phi, \bar{\Phi}) \; m(\Phi, \bar{\Phi}, p, q)_{critical} \tag{3.4.21}$$

where $\Phi, \bar{\Phi}$ are moduli fields, p, q electrical and magnetic charges, m is the black hole mass and $G_{i\bar{j}}$ is the scalar metric on the moduli space.

Finally, let's comment that while supersymmetry was not explicitly invoked here, it might be required to insure the stability of some flat directions, therefore moduli fields, when radiative corrections are taken into account.

3.5 Conclusions

In contrast with the WGC, there is no obvious, no totally convincing road towards uncovering a law governing the scalar potential in quantum gravity. The main ideas have been reviewed in the first section. Their variety can be considered as an evidence both for the difficulty and risks in writing such constraints and for the interest in investigating their implications.

We postulate that in the appropriate low energy limit, for the leading interaction, the gravitational contribution must be sub-leading. Such a statement is hollow if one does not specify which process is concerned and the energy scale at which the interaction strengths are compared. We provided answers for these questions for some cases and found that we retrieve some forms of the Swampland conjectures.

The constraint (3.2.8) differs from previous proposed inequalities. Indeed, The RFC conjecture that we have presented before focused on massless scalars and postulate that the repulsive interactions are stronger than gravitational one. But in our case the scalar mediated interaction is attractive. Strictly speaking, the logic behind their inequalities would lead to (3.4.6) but with an opposite sign for the r.h.s. part. While the logic in this work differs, in the massless case (3.4.5) agrees with one of the proposals of [189], that was also discussed further in [128, 208, 123]. This is all but surprising as the different arguments were put such as one recovers the SDC, which corresponds to the ubiquitous Kaluza-Klein states present in String theory compactifications. Our analysis differs also in the fact that we have also considered self-interacting scalars but only focused on the case of neutral states.

The conjecture presented in [128] leads to an inequality that would constrain in qualitatively similar manner attractive self-interactions for a massive particle (non-tachyonic), but we were not able to recover their coefficients for the different contributions. Moreover, the field dependence of the extremal states squared mass (3.1.4) differs sensibly from our result (3.4.11).

The main playground for testing different conjectures about quantum gravity is string compactifications and their effective supergravity theories. While they represent an opportunity to put the conjecture on firm grounds (see [73] for a recent proposal), one should be able to disentangle what is due to generic quantum gravity from what is due to supersymmetry, other symmetries or just consistency of the precise string theory compactification. Here, we have kept the analysis on a very basic level which we believe is sufficient to stress the main points.

We end by mentioning two immediate remarks. For the Standard Model Higgs scalar, as mentioned in the previous chapter, the running quartic coupling vanishes at energies of order 10^{11} GeV [89]. We should therefore contemplate this intermediate energy scale as an ultra-violet cut-off. Scalar interactions determine the behaviour of spherically symmetric cosmological clumps. The size and dynamics of these objects is different depending on the quartic self-interaction coupling λ. For the case of repulsive complex scalars, massive boson stars, with masses comparable to the fermionic ones, are allowed only when the relevant *relativistic* parameter $\lambda M_{Pl}^2/m^2$ is big [80]. This is a prediction of the WGC discussed here.

Going through the implications of our weak gravity requirement we recovered, in the corresponding cases and forms, some of the Swampland program expectations: the Axion Weak Gravity Conjecture, the Swampland Distance Conjecture, the string Kaluza-Klein and winding modes mass formula and the Swampland de Sitter Conjecture. This is perhaps a hint of the existence of a more general formula that can encompass all the Swampland conjectures.

CHAPTER 4

Scalar Weak Gravity Conjecture and dimensional reduction

In the previous chapter, we have presented a new way to formulate the Scalar Weak Gravity Conjecture. But as we have seen in chapter 1, it can be interesting to look at the behaviour of such conjecture under dimensional reduction, even if there is no reference to a specific string theory frame. Indeed, since the Swampland program is based on an UV theory with compactified dimensions, it seems quite natural that a Swampland conjecture should be preserved in a way during such a process. This is the spirit of the Lattice WGC and its derivatives that we have presented in chapter 1. One interest of such a computation is also the appearance of a dilaton field, and since our computation is at short distances, we will look at the changes induced if this field acquires a mass.

In a first section, we will look at the dimensional reduction of a free scalar, looking at the forces, and at a WGC inspired conjecture with the corresponding KK state. In a second section, we will add scalar interactions, and see how the SWGC evolves in the resulting compactified theory. In a third section, a mass for the dilaton is added, to observe the changes in the SWGC. We will conclude in a fourth section.

4.1 Lower dimensional EFT and scattering amplitudes

The compactification process for a theory with a scalar is presented in appendix A. As we will be mainly interested in this chapter at the computation of vertices and amplitudes in a field theory setting, we will express our results in the $(+,-,...,-)$ signature, so that it will be immediate to use the standard tools and recognize known results. Writing the dilaton around a generic background value as $\phi_0+\phi$ we will study the field theory defined by the action (A.2.9):

$$\mathcal{S}_f = \int \mathrm{d}^D x \sqrt{(-)^{D-1} g} \Bigg\{ \frac{R}{2\kappa^2} + \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}} \sum_{m=0}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \frac{1}{M_{Pl}^{(D-2)/2}} \right)^m \frac{\phi^m}{m!} F^2$$

$$+\frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0+\sum_{n=1}^{\infty}\left(\partial_\mu\varphi_n\partial^\mu\varphi_n^*-\frac{n^2}{L^2}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\sum_{m=0}^{\infty}\left(2\sqrt{\frac{D-1}{D-2}}\frac{1}{M_{Pl}^{(D-2)/2}}\right)^m\frac{\phi^m}{m!}\varphi_n\varphi_n^*\right)$$
$$+\sum_{n=1}^{\infty}\left(i\frac{\sqrt{2}}{M_{Pl}^{\frac{D-2}{2}}}\frac{n}{L}A^\mu\left(\partial_\mu\varphi_n\varphi_n^*-\varphi_n\partial_\mu\varphi_n^*\right)+\frac{2}{M_{Pl}^{D-2}}\frac{n^2}{L^2}A_\mu A^\mu\varphi_n\varphi_n^*\right)\Bigg\}. \tag{4.1.1}$$

4.1.1 Vertices

From the above action, we can immediately derive Feynman rules needed to study the perturbative regime of the theory.

Starting from the bottom, we see in the last line the usual structure for the 3 and 4-point vertices minimally coupling a complex scalar field with a $U(1)$ gauge field. The charge appearing in the well known expression of such vertices will be

$$gq_n=\frac{\sqrt{2}}{M_{Pl}^{(D-2)/2}}\frac{n}{L}e^{\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}.$$

For $m\neq 0$, the last term of the second line leads us to an interaction between the K.K. modes and the dilaton. More specifically, the m-th term in the sum defines a $(2+m)$-point function with m dilatons whose vertex is readily seen to be given by

$$\mathcal{U}(\varphi_n,\varphi_n^*,\phi^m)=-i\left(2\sqrt{\frac{D-1}{D-2}}\frac{1}{M_{Pl}^{(D-2)/2}}\right)^m e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{n^2}{L^2}.$$

Finally, the m-th term in the sum in front of the F^2 one in the first line gives us a vertex with m dilaton and two gauge fields expressed by

$$\mathcal{U}\left(A_\mu(p_1),A_\nu(p_2),\phi^m\right)=-i\left(-2\sqrt{\frac{D-1}{D-2}}\frac{1}{M_{Pl}^{(D-2)/2}}\right)^m e^{-2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(p_1\cdot p_2 g_{\mu\nu}-p_{1\,\nu}p_{2\,\mu}\right).$$

The apparent inversion of the momenta's indices ($A_\mu(p_1),A_\nu(p_2)$ give $p_{1\,\nu}p_{2\,\mu}$) comes from the structure of $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$.

We have to add at these vertices discussed here the D-dimensional gravitational ones. As presented in appendix A, all the interacting terms in this set-up where there is no scalar interactions are of gravitational origin. We will keep this expression to those arising from perturbation of the D-dimensional metric. The explicit form of such terms has been derived in appendix A up to second order in $1/M_{Pl}^{(D-2)/2}$ and gives rise to the usual Feynman rules for real and complex, charged scalar fields in a weak gravity regime.

4.1.2 Force between K.K. states

As a first check of the theory defined by (4.1.1), we can compute the force between two copies of the same K.K. state. We have said in chapter 1 that the forces should cancel. We will see that it is effectively the case.

So, let us consider the tree-level $2 \to 2$ scattering defined by

$$\begin{aligned}
&\langle \varphi_n(p_3)\varphi_n(p_4)|\varphi_n(p_1)\varphi_n(p_4)\rangle = \\
&ig^2q_n^2\left(\frac{p_1\cdot(p_2+p_4)+p_3\cdot(p_2+p_4)}{t}+\frac{p_1\cdot(p_2+p_3)+p_4\cdot(p_2+p_3)}{u}\right) \\
&-4i\frac{D-1}{D-2}e^{4\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{n^4/L^4}{M_{Pl}^{D-2}}\left(\frac{1}{t}+\frac{1}{u}\right)-4i\frac{D-3}{D-2}\frac{m_n^4}{M_{Pl}^{D-2}}\left(\frac{1}{t}+\frac{1}{u}\right),
\end{aligned} \tag{4.1.2}$$

where we find in the first line the contribution from the gauge mediated amplitude, and in second line the two contributions coming from the dilaton and the graviton mediated ones, respectively. The D-dependent factor in front of the gravitational term is explicitly derived in the Appendix B.

Taking now the non-relativistic limit (NR) as defined by $\frac{s-4m^2}{m^2} \ll 1$, $\frac{t+m^2}{m^2} \ll 1$ and $\frac{u+m^2}{m^2} \ll 1$, and plugging in the expressions for the charge and the mass (A.2.13) we obtain

$$\begin{aligned}
&\langle \varphi_n(p_3)\varphi_n(p_4)|\varphi_n(p_1)\varphi_n(p_4)\rangle_{NR} = \\
&i\left(8\frac{m_n^4}{M_{Pl}^{D-2}}-4\frac{D-1}{D-2}\frac{m_n^4}{M_{Pl}^{D-2}}-4\frac{D-3}{D-2}\frac{m_n^4}{M_{Pl}^{D-2}}\right)\left(\frac{1}{t}+\frac{1}{u}\right)=0,
\end{aligned} \tag{4.1.3}$$

where the subscript NR means that the above mentioned non-relativistic limit has been taken. The position space potential between two particles is solely determined by the Fourier transform of the t-channel amplitude, through a matching with the Born approximation for scattering amplitudes in non-relativistic quantum mechanics, but we have displayed here both the t and u-channels for completeness, as they share the same expression.

What we see here is that it is the relation between the charge and the mass (A.2.14) that ensures the precise cancellation between the three forces. Indeed the value $g^2q_n^2 = 2\frac{m_n^2}{M_{Pl}^{D-2}}$ is a threshold for the mass above which the attraction dominates over the repulsion. This is exactly the statement used in chapter 1 to introduce the Repulsive Force Conjecture. If we compare this result to the situation where the dilaton field is not present, which is just the generalization of (1.1.2) at D dimensions $g^2q_n^2 = \frac{D-3}{D-2}\frac{m_n^2}{M_{Pl}^{D-2}}$, we have added a term which can be considered as "the scalar charge" of the field. This scalar charge is defined by the coupling of the KK-states with the dilaton through the mass-dependence under this field. If we have considered a different system with a different coupling between our external states and the dilaton, the no force condition would not take the same form. So it seems that the relation of the forces cancellation is related to the structure one obtains under the $\mathbb{S}^1$ compactification.

On the other hand, the first threshold bound discussed, $g^2q^2 = 2\frac{m^2}{M_{Pl}^{D-2}}$ is the extremality case of the formula (1.2.10) with the peculiar value $\alpha^2 = 3$. We have seen that this constraint was obtained in the context of Reissner-Nordström black hole in the presence of a dilatonic coupling for the $U(1)$ gauge field on the form $e^{2\gamma\phi}F^2$. The reason why this single condition gives both the black hole evaporation and the attractive overall behaviour is readily found looking at the action (4.1.1): the factor in front of the F^2, the γ entering in the inequality, is the same as the one determining the coupling of the K.K. states to the dilaton (with a $-$ sign in the right place as we identify the gauge coupling g^2 with the inverse of this factor). If one chose a different coupling between the dilaton and the states of the theory, while keeping the same coupling between the gauge field and the dilaton, the condition of null forces would be different, and will not coincide with the extremality bound that will be unchanged. This is a illustration of the fact, expressed in [151] and presented in the chapter 1, that the RFC and the WGC considered as the limit of extremality for black holes are not equivalent in generic model.

Now that we have seen how the forces cancel in the case of the KK-states arising from the compactification of a massless scalar field, we will see how this is modified when adding some scalar interactions. But before studying this, we will make use of this minimal theory here to investigate the proposal recently put forward in [129].

4.1.3 Pair Production and the Weak Gravity

In this reference, it was observed a connection between the different forms of the WGC and the pair production of the WGC states from the massless force carriers of the theory under consideration. More specifically, it was observed that, in a $U(1)$ gauge theory, requiring that the square of the amplitude for the pair production of massive states (scalars or fermions) from a pair of photons is greater than the production of the same massive states from a pair of graviton corresponds to impose the WGC bound in the threshold limit. This is due to the fact that both the amplitudes roughly reduce to the square of the couplings with the good numerical factors. The same idea, applied to a theory with one massive field and n scalar massless force carriers (e.g. n moduli), was shown to correctly reproduce the relevant form of the Scalar WGC that we have showed in the previous chapter. In order to see whether this approach can give successful insights into some developments of the WGC, in particular the dilatonic WGC, we will look at its behaviour when we mix scalars and gauge massless bosons. The KK states can be useful in this set-up.

According to (4.1.1), the non-gravitational production of a pair $|\varphi_n\varphi_n^*\rangle$ can come from different channels: a pair of photons $\langle\gamma,\gamma|$, a pair of dilatons $\langle\phi,\phi|$, and a mixed state of a dilaton and a photon $\langle\phi,\gamma|$. Turning on gravity side, there is also a pair production from mixed states, as $\langle g,\gamma|$ or $\langle g,\phi|$. We can even consider the insertion of a graviton mediator in the $\langle\gamma,\gamma|$ and $\langle\phi,\phi|$. However, in the formulation of [129], one should consider only the production from a pair of gravitons $\langle g,g|$, and the mixed cases are not considered. We will follow this prescription and compare pure gravitational versus pure non gravitational processes.

Let us consider then the scattering $\langle massless, massless|\varphi_n\varphi_n^*\rangle$. If we treat the asymptotic states as classical, we should perform the sum of the diagrams involving different initial states as a sum of probabilities, as one does, for example, when performing the sum over internal degrees of freedom. The Pair Production Weak Gravity requirement in this scheme would take the form

$$\Big| \langle A_\mu(p_1), A_\nu(p_2)|\varphi_n\varphi_n^*\rangle \Big|^2 + \Big| \langle \phi(p_1), \phi(p_2)|\varphi_n\varphi_n^*\rangle \Big|^2 + \Big| \langle A_\mu(p_1), \phi(p_2)|\varphi_n\varphi_n^*\rangle \Big|^2 \geq 3\Big| \langle g_{\mu\nu}(p_1), g_{\rho\sigma}(p_2)|\varphi_n\varphi_n^*\rangle \Big|^2, \quad (4.1.4)$$

where these amplitudes are taken at the threshold limit defined by $\frac{s-4m^2}{m^2} \ll 1$, $\frac{t+m^2}{m^2} \ll 1$ and $\frac{u+m^2}{m^2} \ll 1$.

The contributions to the photon production are the usual ones coming from the minimal coupling to the $U(1)$ gauge field plus an s-channel term mediated by the dilaton, so that

$$\begin{aligned}\langle A_\mu(p_1), A_\nu(p_2)|\varphi_n\varphi_n^*\rangle =& 4ig^2q_n^2\epsilon_\mu(p_1)\epsilon_\nu(p_2)\left(\frac{p_3^\mu p_4^\nu}{t-m_n^2} + \frac{p_4^\mu p_3^\nu}{u-m_n^2} + \frac{1}{2}g^{\mu\nu}\right) \\ &- \frac{4i}{M_{Pl}^{D-2}}\frac{D-1}{D-2}\frac{n^2}{L^2}\epsilon_\mu(p_1)\epsilon_\nu(p_2)\frac{p_1\cdot p_2 g^{\mu\nu} - p_1^\nu p_2^\mu}{s}.\end{aligned} \quad (4.1.5)$$

Using $\frac{n}{L} = m_n e^{-\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}$ one then gets, in the above mentioned threshold limit,

$$\begin{aligned}&\Big| \langle A_\mu(p_1), A_\nu(p_2)|\varphi_n\varphi_n^*\rangle \Big|^2 = \\ &Dg^4q_n^4 + \frac{(D-1)^2}{D-2}\frac{m_n^4}{M_{Pl}^{2(D-2)}}e^{-4\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}} - 4(D-1)\frac{m_n^4}{M_{Pl}^{2(D-2)}}e^{-2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}.\end{aligned} \quad (4.1.6)$$

The dilaton production is given by the same diagrams considered in [129] and gives a null result in the threshold limit:

$$\begin{aligned}\langle \phi(p_1), \phi(p_2)|\varphi_n\varphi_n^*\rangle =& -4i\frac{D-1}{D-2}e^{4\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{n^4}{L^4}\left(\frac{1}{t-m_n^2} + \frac{1}{u-m_n^2}\right) \\ &- 4i\frac{D-1}{D-2}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{n^2}{L^2} \\ &\lim_{\text{Threshold}} \longrightarrow 0.\end{aligned} \quad (4.1.7)$$

Finally, the mixed photon-dilaton production receives contribution from the three s,t and u-channels, reading

$$\langle A_\mu(p_1), \phi(p_2)|\varphi_n\varphi_n^*\rangle =$$

$$\epsilon_\mu(p_1)\Bigg\{-2\sqrt{\frac{D-1}{D-2}}\frac{gq_n}{M_{Pl}^{(D-2)/2}}e^{-2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(p_1\cdot(p_1+p_2)g^{\mu\rho}-p_1^\rho p_2^\mu\right)(p_{3\rho}-p_{4\mu})\frac{i}{s}$$
$$+4i\sqrt{\frac{D-1}{D-2}}\frac{gq_n}{M_{Pl}^{(D-2)/2}}\frac{n^2}{L^2}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\frac{p_3^\mu}{t-m_n^2}-\frac{p_4^\mu}{u-m_n^2}\right)\Bigg\}. \tag{4.1.8}$$

We then obtain for $\Big|\langle A_\mu(p_1),\phi(p_2)|\varphi_n\varphi_n^*\rangle\Big|^2$:

$$4\frac{D-1}{D-2}\frac{g^2q_n^2}{M_{Pl}^{(D-2)/2}}\Bigg\{e^{-4\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}(p_1\cdot p_2g^{\mu\rho}-p_1^\rho p_2^\mu)(p_1\cdot p_2g_\mu^\sigma-p_1^\sigma p_{2\,\mu})\frac{(p_{3\,\rho}-p_{4\,\rho})(p_{3\,\sigma}-p_{4\,\sigma})}{s^2}$$
$$+4m_n^4\left(\frac{p_3^\mu}{t-m_n^2}-\frac{p_4^\mu}{u-m_n^2}\right)\left(\frac{p_{3\,\mu}}{t-m_n^2}-\frac{p_{4\,\mu}}{u-m_n^2}\right)+$$
$$-4m_n^2e^{-2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}(p_1\cdot p_2g^{\mu\rho}-p_1^\rho p_2^\mu)\frac{p_{3\,\rho}-p_{4\,\rho}}{s}\left(\frac{p_{3\,\mu}}{t-m_n^2}-\frac{p_{4\,\mu}}{u-m_n^2}\right)\Bigg\}. \tag{4.1.9}$$

In the threshold limit, this gives also a null contribution. As such, we will be left in (4.1.4) with only the term coming from the photon production. We see then that the saturation one would expect to be verified by KK states cannot be realized in the proposed setup. In other words, the KK states are not extremal for the Pair Production conjecture. It seems that this conjecture has a different behaviour that the WGC or the RFC when we mix scalar force and gauge force carriers.

4.2 Compactification of scalar interactions

We have investigated, in the previous sections, properties of a higher dimensional free scalar under the $\mathbb{S}^1$ compactification. We will now make a step forward and consider a $(D+1)$-dimensional self-interacting scalar field, building thus a setup where, ultimately, we wish to study the evolution of the Scalar WGC proposed in the previous chapter.

4.2.1 Φ^2, Φ^3 and Φ^4 operators

Before computing the different contributions, the first thing to do is to derive the behaviour of scalar interactive terms under the dimensional reduction process. Accordingly, let us consider now quadratic, cubic and quartic operators for our higher dimensional theory. These terms are the one we consider in the first presentation of the conjecture. Adding the interacting terms in (A.2.1) we write

$$\mathcal{S}_{int}=\int \mathrm{d}^{D+1}x\sqrt{(-1)^D\hat{g}}\;-\frac{1}{2}\hat{m}^2\hat{\Phi}^2+\frac{\hat{\mu}}{3!}\hat{\Phi}^3-\frac{\hat{\lambda}}{4!}\hat{\Phi}^4. \tag{4.2.1}$$

In $D+1$ dimensions, the $\hat{m}$ operator has mass dimension 1, as in any dimension, the $\hat{\mu}$ one has dimensions $3-\frac{D+1}{2}$ and finally λ has dimension $4-(D+1)$. Using the decomposition (A.2.4), we obtain that the mass term compactifies as

$$\int \mathrm{d}^{D+1}x\sqrt{(-1)^D\hat{g}}\frac{1}{2}\hat{m}^2\hat{\Phi}^2 \Rightarrow \int \mathrm{d}^D x\sqrt{(-1)^{D-1}\hat{g}}\,\frac{1}{2}\hat{m}^2\left(\varphi_0^2+2\sum_{n=1}^{\infty}\varphi_n\varphi_n^*\right). \tag{4.2.2}$$

The $(D+1)$-dimensional mass manifests itself unchanged as a D-dimensional mass for all the Fourier modes. Applying the same decomposition (A.2.4) to the $\hat{\Phi}^3$ term we get after compactification

$$\int \mathrm{d}^{D+1}x\sqrt{(-1)^D\hat{g}}\,\frac{\hat{\mu}}{3!}\hat{\Phi}^3 \Rightarrow \int \mathrm{d}^D x\sqrt{(-1)^{D-1}\hat{g}}\,\frac{\hat{\mu}}{3!\,\sqrt{2\pi L}}\sum_{n,m=-\infty}^{+\infty}\varphi_n\varphi_m\varphi_{n+m}^*. \tag{4.2.3}$$

Although the sum over the n and m indices seems to leave us with some complicate interactions between the different KK states, the expression can actually be simplified. Rearranging the different terms we can in a first time isolate all the contributions with a φ_0

$$\sum_{n,m=-\infty}^{+\infty}\varphi_n\varphi_m\varphi_{n+m}^* = \varphi_0^3+6\varphi_0\sum_{n=1}^{\infty}\varphi_n\varphi_n^* + \sum_{\substack{n,m=-\infty\\ n\neq0,m\neq0,n+m\neq0}}^{\infty}\varphi_n\varphi_m\varphi_{n+m}^* \tag{4.2.4}$$

The last term contains yet possible complicate contributions, and we will now simplify it. Let's define

$$S=\sum_{\substack{n,m=-\infty\\ n\neq0,m\neq0,n+m\neq0}}^{\infty}\varphi_n\varphi_m\varphi_{n+m}^*$$

We can divide S into its different contributions, looking at different sets of m and n.

$$\begin{aligned}S=&\sum_{n>0}^{\infty}\varphi_n\varphi_n\varphi_{2n}^*+\sum_{m>n>0}\varphi_n\varphi_m\varphi_{n+m}^*+\sum_{n>m>0}\varphi_n\varphi_m\varphi_{n+m}^*+\sum_{n>m>0}\varphi_n\varphi_{-m}\varphi_{n-m}^*\\ &+\sum_{m>n>0}\varphi_n\varphi_{-m}\varphi_{n-m}^*+\sum_{m>n>0}\varphi_{-n}\varphi_{-m}\varphi_{-n-m}^*+\sum_{n>0}\varphi_{-n}\varphi_{-n}\varphi_{-2n}^*+\sum_{n>m>0}\varphi_{-n}\varphi_{-m}\varphi_{-n-m}^*\\ &+\sum_{n>m}\varphi_{-n}\varphi_m\varphi_{m-n}^*+\sum_{m>n>0}\varphi_{-n}\varphi_m\varphi_{m-n}^*\end{aligned}$$

Relabelling $m\leftrightarrow n$ and using $\varphi_{-n}=\varphi_n^*$ we obtain

$$S=\sum_{n>0}^{\infty}(\varphi_n\varphi_n\varphi_{2n}^*+\varphi_{-n}\varphi_{-n}\varphi_{-2n}^*)+2\sum_{m>n>0}\left(\varphi_n\varphi_m\varphi_{m+n}^*+\varphi_n\varphi_m^*\varphi_{n-m}^*+\varphi_n^*\varphi_m\varphi_{m-n}^*+\varphi_n^*\varphi_m^*\varphi_{m+n}\right)$$

Looking at one term of the sum, we can rewrite

$$\begin{aligned}\sum_{m>n>0}\varphi_n\varphi_m^*\varphi_{n-m}^* &= \sum_{n>0}\sum_{m=u+n}\varphi_n\varphi_{u+n}^*\varphi_{-u}^* = \sum_{u,n>0}\varphi_n\varphi_u\varphi_{n+u}^*\\ &= \sum_{u>n>0}\varphi_n\varphi_u\varphi_{u+n}^*+\sum_{n>0}\varphi_n\varphi_n\varphi_{2n}^*+\sum_{n>u>0}\varphi_n\varphi_u\varphi_{u+n}^*\end{aligned}$$

We can inject this decomposition in the definition of S and we get

$$\begin{aligned} S =& 4\sum_{m>n>0}\varphi_n\varphi_m\varphi^*_{m+n} + 4\sum_{m>n>0}\varphi^*_n\varphi^*_m\varphi_{n+m} + 2\sum_{n>0}\varphi_n\varphi_n\varphi^*_{2n} + 2\sum_{n>0}\varphi^*_n\varphi^*_n \\ &+ 2\sum_{n>m>0}\varphi_n\varphi_m\varphi_{m+n} + 2\sum_{n>m>0}\varphi^*_n\varphi^*_m\varphi_{n+m} + \sum_{n>0}\varphi_n\varphi_n\varphi^*_{2n} + \sum_{n>0}\varphi^*_n\varphi^*_n\varphi_{2n} \end{aligned}$$

Relabelling once again $m \leftrightarrow n$, this becomes

$$S = 3\sum_{n>0}(\varphi_n\varphi_n\varphi^*_{2n} + \varphi^*_n\varphi^*_n\varphi_{2n}) + 6\sum_{m>n>0}(\varphi_n\varphi_m\varphi_{m+n} + \varphi^*_n\varphi^*_m\varphi_{n+m})$$

Using this result, the cubic interactions are given by

$$\sum_{n,m=-\infty}^{+\infty}\varphi_n\varphi_m\varphi^*_{n+m} = \varphi_0^3 + 6\varphi_0\sum_{n=1}^{\infty}\varphi_n\varphi^*_n + 3\sum_{m,n>0}^{\infty}\left(\varphi_n\varphi_m\varphi^*_{n+m} + \varphi^*_n\varphi_m * \varphi_{n+m}\right) \quad (4.2.5)$$

The first thing we can note is the coupling constant governing the strenght of the three point interactions between the K.K. states is $\mu \equiv \frac{\hat{\mu}}{\sqrt{2\pi L}}$, matching the expectation one could have from naive dimensional analysis. From a low-energy observer point of view, we can say that the $\mathbb{S}^1$ compactification process augments the mass dimension of three point couplings by a factor $1/2$.

Turning now to the $\hat{\Phi}^4$ term , it gives after dimensional reduction :

$$\int \mathrm{d}^{D+1}x\sqrt{(-1)^D\hat{g}}\,\frac{\hat{\lambda}}{4!}\hat{\Phi}^4 \Rightarrow \int \mathrm{d}^D x\sqrt{(-1)^{D-1}\hat{g}}\,\frac{1}{4!}\frac{\hat{\lambda}}{2\pi L}\sum_{m,n,p=-\infty}^{+\infty}\varphi_m\varphi_n\varphi_p\varphi^*_{m+n+p}. \quad (4.2.6)$$

We can make analogous manipulation to the sum and finally obtain

$$\begin{aligned} \sum_{m,n,p=-\infty}^{+\infty}\varphi_m\varphi_n\varphi_p\varphi^*_{m+n+p} =& \varphi_0^4 + 12\varphi_0^2\sum_{n=1}^{\infty}\varphi_n\varphi^*_n + 12\varphi_0\sum_{m,n=1}^{\infty}\left(\varphi_m\varphi_m\varphi^*_{n+m} + \varphi^*_m\varphi^*_n\varphi_{n+m}\right) \\ &+ 12\sum_{n=1}^{\infty}\varphi_n\varphi^*_n\sum_{m=1}^{\infty}\varphi_m\varphi^*_m + 4\sum_{m,n,p=1}^{\infty}\left(\varphi_m\varphi_n\varphi_p\varphi^*_{m+n+p} + \varphi^*_m\varphi^*_n\varphi^*_p\varphi_{m+n+p}\right) \\ &+ 6\sum_{\substack{m,n,p=1 \\ m\neq p,n\neq p;m+n>p}}^{\infty}\varphi_m\varphi_n\varphi^*_p\varphi^*_{n+m-p}. \end{aligned} \quad (4.2.7)$$

As for the $\hat{\Phi}^3$ term, we see that all the different four-point interactions between K.K. states are governed by the coupling $\lambda \equiv \frac{\hat{\lambda}}{2\pi L}$. This is also the result we expect from naive dimensional analysis observing that the mass dimension of the operator is augmented by a factor 1.

At this point, the only thing from (4.2.1) that we still need to develop in terms of D-dimensional quantities is the determinant of the metric, so that all these interactive terms discussed here will be accompanied with an exponential coupling to the dilaton. Putting all the ingredients we have developed so far together, we can thus write the complete action as

$$\begin{aligned}
\mathcal{S} &= \mathcal{S}_f + \mathcal{S}_{int} \\
&= \int \mathrm{d}^D x \sqrt{(-)^{D-1} g} \Bigg\{ \frac{R}{2\kappa^2} + \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}} \frac{\phi}{M_{Pl}^{(D-2)/2}}} F^2 + \frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0 \\
&- \frac{1}{2} e^{\frac{2}{\sqrt{(D-1)(D-2)}} \frac{\phi}{M_{Pl}^{(D-2)/2}}} \hat{m}^2 \varphi_0^2 \\
&+ \sum_{n=1}^{\infty} \left[\partial_\mu\varphi_n\partial^\mu\varphi_n^* - \left(\frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}} \frac{\phi}{M_{Pl}^{(D-2)/2}}} + e^{\frac{2}{\sqrt{(D-1)(D-2)}} \frac{\phi}{M_{Pl}^{(D-2)/2}}} \hat{m}^2 \right) \varphi_n\varphi_n^* \right. \\
&+ \sum_{n=1}^{\infty} \left(i \frac{\sqrt{2}}{M_{Pl}^{\frac{D-2}{2}}} \frac{n}{L} A^\mu \left(\partial_\mu\varphi_n\varphi_n^* - \varphi_n\partial_\mu\varphi_n^* \right) + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n\varphi_n^* \right) \\
&+ e^{\frac{2}{\sqrt{(D-1)(D-2)}} \frac{\phi}{M_{Pl}^{(D-2)/2}}} \left[\frac{\mu}{3!}\varphi_0^3 - \frac{\lambda}{4!}\varphi_0^4 + \mu\varphi_0 \sum_{n=1}^{\infty} \varphi_n\varphi_n^* - \frac{\lambda}{2}\varphi_0^2 \sum_{n=1}^{\infty} \varphi_n\varphi_n^* \right. \\
&- \frac{\lambda}{2}\varphi_0 \sum_{m,n=1}^{\infty} \left(\varphi_m\varphi_m\varphi_{n+m}^* + \varphi_m^*\varphi_n^*\varphi_{n+m} \right) + \frac{\mu}{2} \sum_{n,m=1}^{\infty} \varphi_n\varphi_m\varphi_{n+m}^* + \varphi_n^*\varphi_m^*\varphi_{n+m} \\
&- \frac{\lambda}{3!} \sum_{m,n,p=1}^{\infty} \left(\varphi_m\varphi_n\varphi_p\varphi_{m+n+p}^* + \varphi_m^*\varphi_n^*\varphi_p^*\varphi_{m+n+p} \right) - \frac{\lambda}{2} \sum_{n=1}^{\infty} \varphi_n\varphi_n^* \sum_{m=1}^{\infty} \varphi_m\varphi_m^* \\
&\left.\left.- \frac{\lambda}{4} \sum_{\substack{m,n,p=1 \\ m\neq p, n\neq p; m+n>p}}^{\infty} \varphi_m\varphi_n\varphi_p^*\varphi_{n+m-p}^* \right]\right] \Bigg\},
\end{aligned} \tag{4.2.8}$$

where we have to keep in mind that the dilaton should again be developed around a vacuum value ϕ_0 as previously done. As we see, the above action gives rise to quite a large number of couplings. We can see that the mass term for the KK states is shifted compared to the previous case, since the tree-level mass term of the φ_0 field is $m_0^2 \equiv e^{\frac{2}{\sqrt{(D-1)(D-2)}} \frac{\phi_0}{M_{Pl}^{(D-2)/2}}} \hat{m}^2$ and that of the nth K.K. state is

$$m_n^2 \equiv \frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}} \frac{\phi_0}{M_{Pl}^{(D-2)/2}}} + e^{\frac{2}{\sqrt{(D-1)(D-2)}} \frac{\phi_0}{M_{Pl}^{(D-2)/2}}} \hat{m}^2.$$

4.2.2 Scattering Amplitudes, SWGC and RFC

With the action written in terms of D-dimensional quantities, we can now proceed towards our goal, and as in the previous chapter, compare the different forces.

We will begin by computing the $\varphi_0\varphi_0 \to \varphi_0\varphi_0$ scattering, and look at the different contributions for the forces.

In the previous section, when the KK states were coming from a massless scalar field, the field φ_0 was a massless neutral scalar field, with no interactions with the dilaton. The addition of $\mathcal{S}_{int}$ in the action changes the situation, because φ_0 field couples to

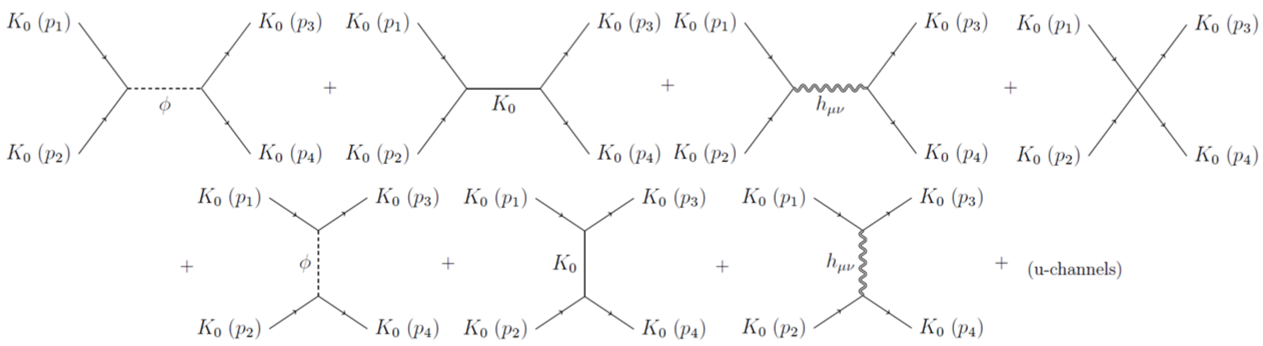

Figure 4.1 – All the channels present in the $\varphi_0\varphi_0$ scattering

the dilaton. Besides, it is again uncharged under the $U(1)$ of the theory. It seems so that this state is a good playground to test the SWGC proposed in the previous chapter.

The diagrams intervening in the scattering are the usual s,t and u channels coming from the φ_0^3, the first order development of the exponential coupling to the mass term and gravity, and the contact one due to the φ_0^4. They are presented in the Figure 4.1, where the $+u$-channels stands for the u channels that we did not explicitly write as they can be inferred from the t-channel ones.

We will look at the non-relativistic limit of the tree-level amplitude, which takes the form

$$\begin{aligned}\langle\varphi_0\varphi_0|\varphi_0\varphi_0\rangle = {}& ie^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\left[e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\frac{5}{3}\frac{\mu^2}{m_0^2}-\lambda\right]\\ &-\frac{i}{(D-1)(D-2)}\frac{m_0^2}{M_{Pl}^{D-2}}-4\frac{i}{(D-1)(D-2)}\frac{m_0^4}{M_{Pl}^{D-2}}\left(\frac{1}{t}+\frac{1}{u}\right)\\ &+i\frac{D-1}{D-2}\frac{m_0^2}{M_{Pl}^{D-2}}-4i\frac{D-3}{D-2}\frac{m_0^4}{M_{Pl}^{D-2}}\left(\frac{1}{t}+\frac{1}{u}\right).\end{aligned} \tag{4.2.9}$$

We recognize, in each line, respectively, the different contributions from the self interaction, the dilaton and gravity. As we have presented in the previous chapter, in our set-up, we decided to separate the comparison between the long-range and the short range forces. The long ranges forces correspond to the t and u poles in (4.2.9). In the case we look at, as in the case of the previous chapter the overall long range force will of course be attractive, as since the charge of φ_0 is null in our model, and the dilatonic forces is attractive. Following again the previous chapter, we will only keep the short range interaction term in the non-relativistic limit. These terms give

$$\begin{aligned}\langle\varphi_0\varphi_0|\varphi_0\varphi_0\rangle_{CT} &= ie^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\left(\frac{5}{3}\frac{\mu^2}{\hat{m}^2}-\lambda-\frac{1}{(D-1)(D-2)}\frac{\hat{m}^2}{M_{Pl}^{D-2}}+\frac{D-1}{D-2}\frac{\hat{m}^2}{M_{Pl}^{D-2}}\right)\\ &= i\frac{e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}}{2\pi L}\left(\frac{5}{3}\frac{\hat{\mu}^2}{\hat{m}^2}-\hat{\lambda}+2\pi L\frac{D}{D-1}\frac{\hat{m}^2}{M_{Pl}^{D-2}}\right).\end{aligned} \tag{4.2.10}$$

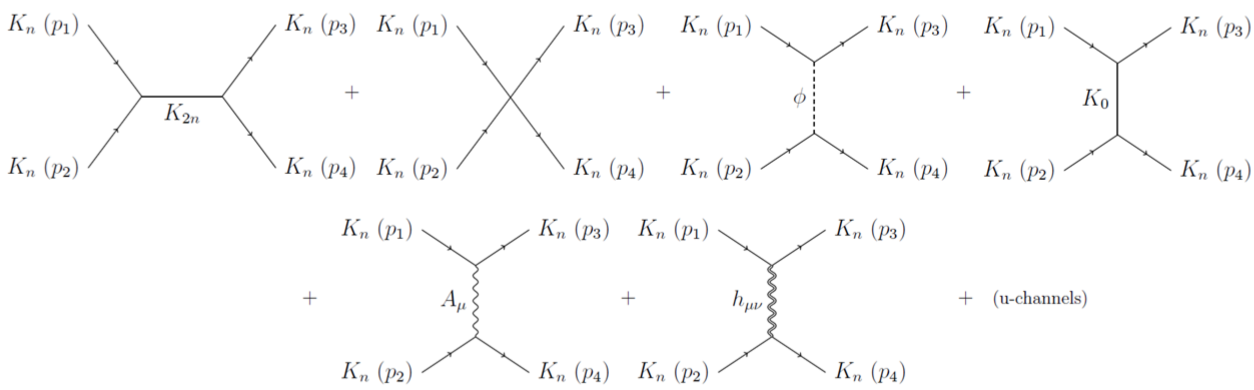

Figure 4.2 – Diagrams present in the $\varphi_n\varphi_n$ scattering.

Using (A.1.25) we can identify $\frac{M_{Pl}^{D-2}}{2\pi L}$ as the $D+1$-dimensional Planck mass $\hat{M}_{Pl}^{D-1}$. Moreover, the last term can be rewritten as $\frac{(D+1)-1}{(D+1)-2}\frac{\hat{m}_0^2}{\hat{M}_{Pl}^{D-1}}$. This is exactly what one would obtain computing the gravitational s-channel contribution to the $\hat{\Phi}\hat{\Phi} \to \hat{\Phi}\hat{\Phi}$ scattering in $D+1$ dimensions. We can thus conclude that, if one asks, in $D+1$ dimensions, that the scalar interactions of the field $\hat{\Phi}$ are dominant with respect to gravity, the scalar interactions of its zero mode KK state φ_0 will then automatically be dominant with respect to both the gravitational and dilatonic contributions in D dimensions.

But φ_0 is not the only state present in the theory, and as we did in the section 4.1.2, we can turn to the computation of $\varphi_n\varphi_n \to \varphi_n\varphi_n$. In this case where we have added interacting terms, the scattering receives contribution from the t and u-channels of gauge boson, dilaton, graviton and φ_0 exchange, from the s-channel exchange of a φ_{2n} particle and from a 4-point contact term. These diagrams are presented in Figure 4.2 . We again directly show the form of the amplitude in the non-relativistic limit:

$$\langle\varphi_n\varphi_n|\varphi_n\varphi_n\rangle = ie^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\mu^2\left(\frac{2}{m_0^2}-\frac{1}{4m_n^2-m_{2n}^2}\right) - i\lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}$$
$$+\,i\left(\frac{1}{t}+\frac{1}{u}\right)\left(4g^2q_n^2m_n^2 - 4\frac{D-3}{D-2}\frac{m_n^4}{M_{Pl}^{D-2}} - (\partial_\phi m_n^2)^2\right) \tag{4.2.11}$$

with

$$\partial_\phi m_n^2 = \frac{1}{M_{Pl}^{(D-2)/2}}\left(\frac{2}{\sqrt{(D-1)(D-2)}}e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\hat{m}^2 + 2\sqrt{\frac{D-1}{D-2}}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{n^2}{L^2}\right). \tag{4.2.12}$$

In this case, there is no gauge, gravitational or dilatonic term that collapse into a short range one. As such, there is no information about the relative strengths of the interactions that we can deduce looking at the terms effectively appearing as contact ones. The relation between m_n and m_{2n} gives $4m_n^2 - m_{2n}^2 = 3m_0^2$. Thus, the scalar contact terms for the $\varphi_n\varphi_n \to \varphi_n\varphi_n$ scattering are just the same as those we have seen in the $\varphi_0\varphi_0 \to \varphi_0\varphi_0$ one. One could say that they trivially dominate.

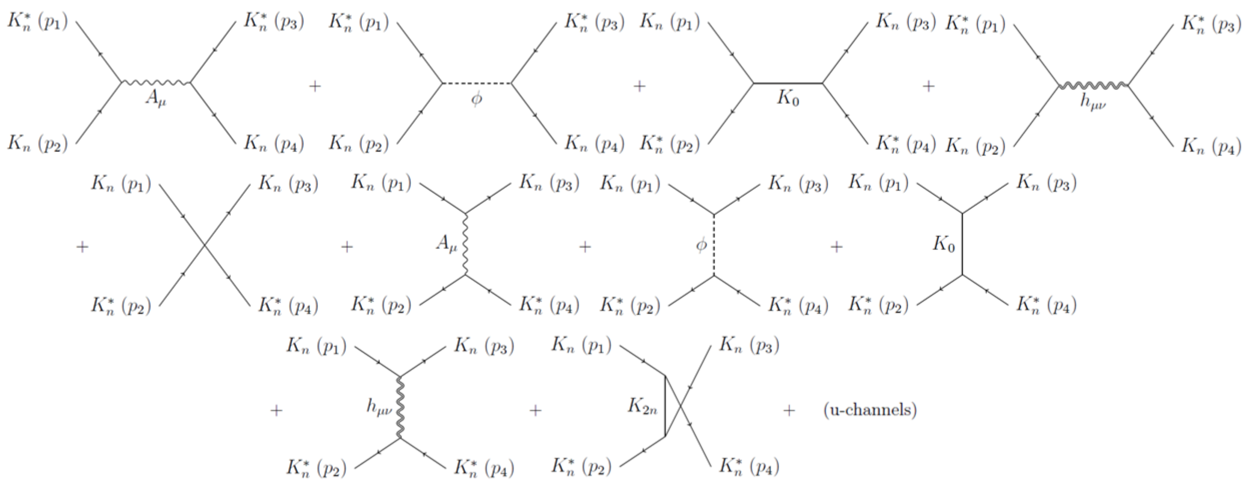

Figure 4.3 – Diagrams from the $\varphi\varphi^*$ scattering

This scattering presents also long range interactions, which explicitly shows the conflict between attractive and repulsive terms, since the φ_n is charged under the gauge group of the theory. We can so apply the formula of the RFC presented in the chapter 1, with the specific form of the scalar coupling here which is proportional to the derivative of the mass. This gives

$$g^2q^2 \geq \frac{D-3}{D-2}\frac{m^2}{M_{Pl}^{D-2}} + \frac{(\partial_\phi m^2)^2}{4m^2}, \tag{4.2.13}$$

The only difference with the result presented in section 4.1.2 comes from the additional mass term, that in turn generates a new three point coupling to the dilaton. Developing (4.2.11) one can see that the force is always an attractive one. The mass term added in the $D+1$ dimension theory spoils the no-force property verified in the case $\hat{m} = 0$ studied in section 4.1.2. This is expected, since in $D+1$ dimension there is just the gravity as long range force, which is an attractive one.

We will see what happens for the $\varphi_n\varphi_n^* \to \varphi_n\varphi_n^*$ scattering. It receives contributions from the s and t-channels of gauge, dilaton, gravitational and φ_0 self mediated diagrams, from the φ_{2n} self mediated u-channel and from a contact four point term, as presented in Figure 4.3.

The structure of the amplitude in the non-relativistic limit is then

$$\begin{aligned}\langle\varphi_n\varphi_n^*|\varphi_n\varphi_n^*\rangle &= i\mu^2 e^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\frac{1}{m_0^2}+\frac{1}{m_{2n}^2}-\frac{1}{4m_n^2-m_0^2}\right) - i\lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}} \\ &- i\frac{(\partial_\phi m_n^2)^2}{4m_n^2} + i\frac{D-1}{D-2}\frac{m_n^2}{M_{Pl}^{(D-2)/2}} - 4ig^2q_n^2\frac{m_n^2}{t} - i\frac{(\partial_\phi m_n^2)^2}{t} - 4i\frac{D-3}{D-2}\frac{m_n^4}{M_{Pl}^{(D-2)/2}}\frac{1}{t}.\end{aligned} \tag{4.2.14}$$

The long range $1/t$ force is, of course, attractive, but this scattering is not constrained by the the RFC. Looking at the short range interaction, there are many contributions.

To compare the different contributions to the short range interactions, we begin by rewriting

$$\frac{1}{m_0^2}+\frac{1}{m_{2n}^2}-\frac{1}{4m_n^2-m_0^2}=\frac{1}{m_0^2}\left(1+\frac{1}{1+4\frac{n^2/L^2}{\hat{m}^2}e^{2\sqrt{\frac{D-2}{D-1}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}-\frac{1}{3+4\frac{n^2/L^2}{\hat{m}^2}e^{2\sqrt{\frac{D-2}{D-1}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}\right)$$
$$=\frac{1}{m_0^2}\left(1+\frac{2}{3+16\frac{n^2/L^2}{\hat{m}^2}e^{2\sqrt{\frac{D-2}{D-1}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(1+\frac{n^2/L^2}{\hat{m}^2}e^{2\sqrt{\frac{D-2}{D-1}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\right)}\right). \tag{4.2.15}$$

We see that, when $n=0$, this correctly reproduces the effective result obtained for the φ_0 self interaction with the factor $\frac{5}{3}$. We can also develop m_n^2 to obtain

$$\frac{D-1}{D-2}\frac{m_n^2}{M_{Pl}^{D-2}}-\frac{(\partial_\phi m_n^2)^2}{4m_n^2}= \tag{4.2.16}$$
$$\frac{1}{M_{Pl}^{D-2}}\left(\frac{\frac{D}{D-1}\hat{m}^4e^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}+2\hat{m}^2\frac{n^2}{L^2}e^{\frac{2D}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}{\hat{m}^2e^{\frac{2}{\sqrt{(D-1)(D-2)}}}+\frac{n^2}{L^2}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}\right).$$

Combining these results, the comparison of the forces in the way of the previous sections gives, for the $\varphi_n\varphi_n^* \to \varphi_n\varphi_n^*$ scattering

$$\left|\hat{\mu}^2\left(1+\frac{2}{3+16\frac{n^2/L^2}{\hat{m}^2}e^{2\sqrt{\frac{D-2}{D-1}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(1+\frac{n^2/L^2}{\hat{m}^2}e^{2\sqrt{\frac{D-2}{D-1}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\right)}\right)-\hat{\lambda}\right|\geq$$
$$\geq\frac{1}{\hat{M}_{Pl}^{D-1}}\left(\frac{\frac{D}{D-1}\hat{m}^4e^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}+2\hat{m}^2\frac{n^2}{L^2}e^{\frac{2D}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}{\hat{m}^2e^{\frac{2}{\sqrt{(D-1)(D-2)}}}+\frac{n^2}{L^2}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}\right) \tag{4.2.17}$$

with n=0 we find again (4.2.10). But as soon as we look at heavier KK states, the r.h.s increases (for $D\geq 4$), and so there is a value of n for which this formula is not verified anymore, at fixed compactification radius. This comes just from the fact that the scalar interaction terms, in contrast to the mass of the KK states, don't have a contribution proportional to n^2. In the previous chapter, we looked at the scattering of $\varphi_n\varphi_n^* \to \varphi_n\varphi_n^*$ but comparing the forces mediated by the moduli and the gravitational forces. Here, we see that comparing the scalar forces to the dilatonic and gravitational ones doesn't lead to instructive results.

To conclude, it seems that if one imposes that the scalar self-interactions are dominant compared to gravity in $D+1$ dimensions, this property is also true for the lowest scalar field of the KK spectrum. For the others KK states, the fact that they are charged under the graviphoton leads to non-instructive results in the case of two same-particles scattering. It seems that the hierarchy between the strength of the short ranges forces of a neutral scalar is not modified by a dimensional reduction.

4.3 Massive dilaton

In a phenomenological theory, the dilaton has to be stabilized, i.e. to acquire a potential. We don't look at a specific model of stabilization, we just consider that the dilaton ϕ has a mass m_ϕ and a vev ϕ_0. When the dilaton acquires a mass, it cannot be a part of the RFC defined in the chapter 1. However, it can modify or give new contributions to short range forces that we analyzed in the previous section. We want to look at these modifications of the long and short range forces, in order to see how the behaviour of the conjectures under the dimensional reduction is changed. The scattering amplitudes of interest in this case are the $\varphi_0\varphi_0 \to \varphi_0\varphi_0$ and the $\varphi_n\varphi_n \to \varphi_n\varphi_n$ ones, when the non-relativistic limit has been taken.

The $\varphi_0\varphi_0 \to \varphi_0\varphi_0$ scattering amplitude (4.2.9) with a massive dilaton becomes, in the non-relativistic limit $s - 4m_0^2 \ll m_0^2$, $t \ll m_0^2$ and $u \ll m_0^2$:

$$\begin{aligned}\mathcal{A}(\varphi_0\varphi_0 \to \varphi_0\varphi_0) = {} & ie^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\left[e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\frac{5}{3}\frac{\mu^2}{m_0^2} - \lambda\right] \\ & - 4\frac{i}{(D-1)(D-2)}\frac{m_0^4}{M_{Pl}^{D-2}}\frac{1}{s-m_\phi^2} \\ & - 4\frac{i}{(D-1)(D-2)}\frac{m_0^4}{M_{Pl}^{D-2}}\left(\frac{1}{t-m_\phi^2} + \frac{1}{u-m_\phi^2}\right) \\ & + i\frac{D-1}{D-2}\frac{m_0^2}{M_{Pl}^{D-2}} - 4i\frac{D-3}{D-2}\frac{m_0^4}{M_{Pl}^{D-2}}\left(\frac{1}{t} + \frac{1}{u}\right). \end{aligned} \tag{4.3.1}$$

Since the mass of the dilaton is not fixed by any model here, it is possible to scan all the range of mass, and to compare it to the other mass scale m_0 present in the theory.

- When $m_\phi \ll m_0$, dilatonic interactions effectively seem infinite range ones compared to φ_0 self interactions. Accordingly, the physical picture does not change much from the $m_\phi = 0$ one.
- For $m_\phi \lesssim m_0$, m_ϕ^2 dominates with respect to t and u and the dilatonic terms appear point-like. A rough approximation to the dilatonic + gravitational contributions to such contact terms, when $m_\phi \simeq m_0$, is

$$\frac{20}{3}\frac{i}{(D-1)(D-2)}\frac{m_0^2}{M_{Pl}^{D-2}} + i\frac{D-1}{D-2}\frac{m_0^2}{M_{Pl}^{D-2}} = i\frac{D^2-2D+\frac{23}{3}}{(D-1)(D-2)}\frac{m_0^2}{M_{Pl}^{D-2}}. \tag{4.3.2}$$

It is immediate to verify that, for $D \geq 4$, $\left|\frac{D}{D-1}\right| < \left|\frac{D^2-2D+\frac{11}{3}}{(D-1)(D-2)}\right|$, where $\frac{D}{D-1}$ was previously found to be the strength of the two combined interactions in the $m_\phi = 0$ case. Requiring the dominance of the short range scalar forces in dimension $D+1$ doesn't lead necessarily to the same result for D dimensional theory.

- As m_ϕ increases, dilatonic t and u-channels are more and more suppressed. The s-channel, instead, grows until it meets the resonance, and then decreases again. In this range of mass, the gravitational contribution dominates and once again, it is not possible to ensure the domination of the short range scalar forces compared to the gravitational ones.
- Increasing further the mass of the dilaton, the dilatonic forces are more and more suppressed, leaving only the gravitational contribution. However, the coefficient in front of the gravitational forces is $\frac{D-1}{D-2}$. In $D+1$ dimension, this coefficient is $\frac{D}{D-1}$, which is smaller than $\frac{D-1}{D-2}$. Similarly to the previous cases, the requirement of scalar forces dominance in $D+1$ dimensions cannot ensure the same happens in D dimensions.

This shows that, when the mass of the dilaton is smaller than the mass of the lightest KK states, the request that scalar self interactions dominate in $D+1$ dimensions is sufficient to ensure the same property is verified by its 0^{th} KK mode in D dimensions. But as soon as the mass of the dilaton is close to the value of the mass of this state, just requiring the dominance of short scalar forces is not sufficient anymore, and a more careful treatment is required in order to obtain an more accurate result.

The $\varphi_n\varphi_n \to \varphi_n\varphi_n$ scattering amplitude with the massive dilaton reads

$$\begin{aligned}
&\mathcal{A}(\varphi_n\varphi_n \to \varphi_n\varphi_n) = \\
&- ie^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\mu^2\left(\frac{1}{s-m_{2n}^2}+\frac{1}{t-m_0^2}+\frac{1}{u-m_0^2}\right) - i\lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}} \\
&- i(\partial_\phi m_n^2)^2\left(\frac{1}{t-m_\phi^2}+\frac{1}{u-m_\phi^2}\right) + i\left(\frac{1}{t}+\frac{1}{u}\right)\left(4g^2q_n^2m_n^2 - 4\frac{D-3}{D-2}\frac{m_n^4}{M_{Pl}^{D-2}}\right). \quad (4.3.3)
\end{aligned}$$

For later convenience, we define here $\bar{m}_n^2 \equiv e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{n^2}{L^2}$, so that $m_n^2 = m_0^2 + \bar{m}_n^2$.

The difference from the previous case comes from the presence of two scales m_0 and m_n instead of one. It requires to distinguish the case of a large hierarchy $m_0^2 \ll m_n^2$ from that of a small one, $m_0^2 \lesssim m_n^2$ ($m_0^2 \geq \bar{m}_n^2$). Besides, since the external states are now φ_n, the non-relativistic limit corresponds to $s - 4m_n^2 \ll m_n^2$, $t \ll m_n^2$ and $u \ll m_n^2$.

Starting with $m_0^2 \ll m_n^2$ there are different regimes to look at.

- For $m_\phi^2 \lesssim m_0^2 \ll m_n^2$, dilatonic interactions, as well as t, u-channels self-ones, look long range. As for $m_\phi = 0$, the long range interactions do not satisfy the RFC.

  Contact terms arise only from the mediated s-channel and the 4-point self-interactions and may show the above mentioned s-channel enhancement.

- In the intermediate region $m_0^2 \ll m_\phi^2 \lesssim m_n^2$, the long range contribution to the amplitude can be rewritten as

$$\begin{aligned}&\mathcal{A}^{LR}(\varphi_n\varphi_n \to \varphi_n\varphi_n) = \\ &-ie^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\mu^2\left[\frac{1}{t}\left(1+\frac{m_0^2}{t}+\mathcal{O}\left(\frac{m_0^4}{t^2}\right)\right)+\frac{1}{u}\left(1+\frac{m_0^2}{u}++\mathcal{O}\left(\frac{m_0^4}{u^2}\right)\right)\right] \\ &+i\left(\frac{1}{t}+\frac{1}{u}\right)\left(4g^2q_n^2m_n^2-4\frac{D-3}{D-2}\frac{m_n^4}{M_{Pl}^{D-2}}\right). \end{aligned} \tag{4.3.4}$$

  Application of the RFC to this amplitude amounts to require, at leading order in $\frac{m_0^2}{m_n^2}$, where $m_n^2 \simeq \bar{m}_n^2$ and $g^2q_n^2 \simeq 2\frac{m_n^2}{M_{Pl}^{D-2}}$,

$$\frac{D-1}{D-2}\frac{m_n^2}{M_{Pl}^{D-2}} \geq e^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{\mu^2}{4m_n^2}. \tag{4.3.5}$$

  Turning now to the contact terms, from (4.3.3) we have

$$\begin{aligned}\mathcal{A}^{CT}(\varphi_n\varphi_n \to \varphi_n\varphi_n) =& -ie^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\mu^2\frac{1}{s-m_{2n}^2} - i\lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}} \\ &+i\frac{(\partial_\phi m_n^2)^2}{m_\phi^2}\left(2+\frac{t}{m_\phi^2}+\frac{u}{m_\phi^2}+\mathcal{O}\left(\frac{t^2}{m_\phi^4},\frac{u^2}{m_\phi^4}\right)\right). \end{aligned} \tag{4.3.6}$$

  At leading order in $\frac{t,u}{m_\phi^2}$, short range subdominance of gravity can be obtained if

$$e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\left|e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{D-2}}}\mu^2\frac{1}{s-m_{2n}^2}+\lambda\right| \geq 8\frac{D-1}{D-2}\frac{m_n^4}{m_\phi^2 M_{Pl}^{D-2}}. \tag{4.3.7}$$

  Assuming the short range gravitational subdominance to be satisfied in $D+1$ dimensions, it would be tempting to conclude that the above relation is verified.

- For a dilaton mass verifying $m_0^2 \ll m_n^2 \ll m_\phi^2$, the expansion of the amplitudes is the same as in the last point. The r.h.s. of (4.3.7) is more and more suppressed, making it even more likely for the short-range subdominance to be verified.

The analysis has to be repeated for $m_0^2 \lesssim m_n^2$.

- When $m_\phi^2 \ll m_0^2 \lesssim m_n^2$, the situation is similar to that of a massless dilaton delineated in section 4.2.2.

- In the regime $m_\phi^2 \lesssim m_0^2 \lesssim m_n^2$, all terms in (4.3.3) can be treated as contact ones, except for the graviton and gauge mediated diagrams.

  Starting with the long range interactions, the RFC would mandate the inequality

$$g^2q_n^2 \geq \frac{D-3}{D-2}\frac{m_n^2}{M_{Pl}^{D-2}} \Longrightarrow \frac{D-1}{D-2}\frac{\bar{m}_n^2}{M_{Pl}^{D-2}} \geq \frac{D-3}{D-2}\frac{m_0^2}{M_{Pl}^{D-2}} \tag{4.3.8}$$

which finally gives, for the nth K.K. mode to feel an overall repulsive long range force

$$m_0^2 \leq \frac{D-1}{D-3}\frac{n^2}{L^2}e^{2\sqrt{\frac{D-2}{D-1}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}. \tag{4.3.9}$$

The RFC can be accommodated in the parametric region defined by the equation above. However, m_0^2 should not be too small, otherwise we fall back to the previous case, where other attractive forces come into play.

The contact terms are given by

$$\begin{aligned}
&\mathcal{A}^{CT}(\varphi_n\varphi_n \to \varphi_n\varphi_n) \simeq i\frac{5}{3}\frac{\mu^2}{m_0^2}e^{\frac{4}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{\frac{D-2}{2}}}} - i\lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{\frac{D-2}{2}}}} + 2i\frac{(\partial_\phi m_n^2)^2}{m_\phi^2}\\
&\to i\left(\frac{5}{3}\frac{\mu^2}{\hat{m}^2} - \lambda\right)e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{\frac{D-2}{2}}}}\\
&+ i\frac{8}{M_{Pl}^{(D-2)}}\frac{\left(\frac{\hat{m}^2}{\sqrt{(D-1)(D-2)}}e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}} + \sqrt{\frac{D-1}{D-2}}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{n^2}{L^2}\right)^2}{m_\phi^2}.
\end{aligned} \tag{4.3.10}$$

The usual condition for the short range subdominance is now fulfilled when

$$\begin{aligned}
&\left|\frac{5}{3}\frac{\mu^2}{\hat{m}^2} - \lambda\right| \geq\\
&\frac{8}{M_{Pl}^{(D-2)}}\frac{\frac{\hat{m}^4}{(D-1)(D-2)}e^{\frac{2}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}} + \frac{\hat{m}^2\frac{n^2}{L^2}}{D-2}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}} + \frac{D-1}{D-2}\frac{n^4}{L^4}e^{2\frac{2D-3}{\sqrt{(D-1)(D-2)}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}{m_\phi^2}.
\end{aligned} \tag{4.3.11}$$

Taking $m_\phi^2 \simeq m_0^2$, it reduces to

$$\left|\frac{5}{3}\frac{\mu^2}{\hat{m}^2} - \lambda\right| \geq \frac{8}{M_{Pl}^{(D-2)}}\left(\frac{\hat{m}^2}{(D-1)(D-2)} + \frac{n^2}{L^2}\frac{e^{2\sqrt{\frac{D-2}{D-1}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}{D-2} + \frac{D-1}{D-2}\frac{n^4/L^4}{\hat{m}^2}e^{4\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\right). \tag{4.3.12}$$

Assuming $\left|\frac{5}{3}\frac{\mu^2}{\hat{m}^2} - \lambda\right| \geq \frac{D-2}{D-1}\frac{\hat{m}^2}{M_{Pl}^{(D-2)}}$ is not sufficient to ensure validity of (4.3.12). Indeed, to be verified up to a certain n, this bound requires to start at least with $\hat{m} > \frac{1}{L}$. In any case, there will always be an n^* such that $\forall n \geq n^*$ the inequality is not valid. Considering a lighter dilaton, $m_\phi^2 \leq m_0^2$, the r.h.s. of (4.3.11) increases, making it more complicated to realize the inequality.

- When $m_\phi^2 > m_0^2$, the results are those presented just above. In (4.3.11), the dilatonic interactions get more and more suppressed, until a region of parameters is reached where the self interaction dominance is restored.

Contrary to the φ_0 scattering, for the $n \neq 0$ KK states the situation is more intricated. Requiring short range subdominance in $D+1$ dimensions is not sufficient in the whole parametric space to ensure the same is valid in D dimensions. It seems rather interesting that the conditions for such short range subdominance are deluded in the same parametric region where the long range force can be made repulsive.

4.4 Conclusion

The dimensional reduction gives a good playground to discuss conjectures inspired by the Swampland program. In this chapter we have looked at the dimensional reduction of a scalar, with or without self-interactions. We see that, in this case, the Pair Production conjecture is satisfied by the KK states, but not saturated. It seems so that this conjecture is not exactly the same as the RFC or the WGC.

In the case of the compactification of a scalar with self-interactions, we see that the lightest state of the KK spectrum has an interesting feature. Requiring that the short range self-interactions of the scalar are dominant compared to the gravity, the KK lightest state verify the same property, adding the dilaton interaction to the gravitational one. This property is also verified if the dilaton is stabilized by a potential, as long as the mass of the dilaton is smaller than the one of the scalar. For larger mass, one has to look more carefully at the details of the model.

Looking at the other states of the KK spectrum, the conclusions are less clear. Requiring the SWGC to be verified at $D+1$ dimensions is not sufficient to conclude at the dominance of the scalar forces over the gravitational ones in D dimensions, even when the dilaton is stabilized.

As said in the previous chapter, there is not a straightforward way to obtain a relation between scalar forces and gravity, and all the possibilities are not equivalent. For a neutral scalar, comparing the short range scalar forces with gravity can lead to interesting results, and is well-behaved under dimensional reduction in the sense we describe above. For charged scalars, the situation is more complex. The simplest way to confirm these results is to look at specific models of scalar potential in a String Theory frame, but one has to be careful, and correctly separates the model specificities from the quantum gravity features.

CHAPTER 5

Dilatonic WGC in (Anti-)de Sitter space

In the two previous chapters, we have devoted our attention to a comparison of the forces for a self-interacting scalar, leading to what we call a Scalar WGC. But as we have seen in chapter 1, it can be interesting to use black holes physics to obtain conjectures like WGC ones. A peculiar case we look at in the previous chapter is when the dilaton is stabilized. It can be interesting to look at such a case under the view of black hole physics, because it corresponds to a generalisation of the dilatonic WGC that we have presented in the chapter 1. However, when we add a potential to the dilaton, we have to be careful, because such a potential will modify the equation of motions, and in particular can modify the asymptotic behaviour of the metric.

In general we are interested in metrics with peculiar behaviours at the infinity, which are (Anti-)de-Sitter ((A)dS) or flat ones. The fact that the solution with a specified potential has an asymptotic limit corresponding to one of the three cases mentioned above is not trivial. In particular, it was shown in [194], in the case of a simple Liouville potential for the dilaton, of the form $\Lambda e^{\alpha\phi}$, that there is no possibility to obtain black holes asymptotically (A)dS.

However, the case dS is interesting, since it represents the current evolution of our Universe. Even if we have seen that there are some conjectures about the exclusion of dS vacua in the Swampland program, these ones are always under investigation. Conserving the spirit of the Swampland program, we can consider that our Universe is described by an "approximate" background of string theory. We can so, in this scheme, try to obtain a conjecture for a dS background. An attempt of this was done in [24], without a dilaton. They used the Reissner-Nordström-de Sitter solution, and the behaviour of its horizons when changing M and Q. The case AdS presents also an interest for string theory and AdS/CFT correspondence, and it is extensively studied. Some aspects of the asymptotically Anti-de Sitter metric were discussed for the case of AdS_5 in [103] and for AdS_4 in some limits by [132]. A conjecture for dilatonic black holes asymptotically AdS was presented in [155].

In a first section, we will review quickly the work done in [24], and look at possible solutions of dilatonic black holes with a potential, requiring the solution to be asymptotically dS. The horizons of charged dilatonic de Sitter black hole are described in Section 2. The same thing is done for asymptotically AdS solution in a third section. The issue of attractive and repulsive forces, in the case of asymptotically flat space-time, are analyzed in Section 4. We present our conclusion in section 5.

5.1 dS-WGC conjecture and dS dilatonic black hole solutions

We begin with a review of the previous work on a dS-WGC, and on the solutions for a dilatonic dS black hole. In all this chapter, we will use the signature (-,+,+,+).

5.1.1 A dS-WGC conjecture

We will follow in this subsection the work of [24]. If we look at the metric of Reissner-Nordström-de Sitter solution, it has the form

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 d\Omega_2^2 \tag{5.1.1}$$

with

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2. \tag{5.1.2}$$

The horizons radius corresponds to solutions of the equation $f(r) = 0$. Multiplying by r^2, this is a quartic polynomial, but one solution is always negative. Thus, there are at most three horizons. Requiring a de-Sitter asymptotic behaviour instead of a flat one gives one more horizon. This can be seen already in the case of a Scwarzschild-de Sitter solution. There are in this case two horizons, called the event horizon and the cosmological one. The singularity is protected by an event horizon only if $M^2 \leq \frac{1}{27H^2}$, where H is the Hubble parameter. However, when this condition is saturated, the two horizons are degenerate; increasing again M, they become complex. We have to keep in mind that in asymptotically dS space-times, we should expect to find upper bounds on the black hole size, translating the fact that, to have a healthy solution, the event horizon should not exceed the cosmological radius. Otherwise, we say that the de Sitter space-time has been "completely eaten up" by the black hole.

Coming back to the charged case, there are three horizons: an inner Cauchy horizon, an event horizon, and a cosmological one. But in the (Q, M) plane, this solution with three horizons is not always possible. There are roughly two ways to get out of that region: either by increasing the mass at fixed charge, or by increasing the charge at fixed mass. In the first case, when we increase the mass, the event and cosmological horizon gets closer and closer, until they merge. Above this extremal value, g_{00} still have one root, corresponding to the inner horizon. However, the crossing of the event and cosmological horizon should be interpreted as the black hole eating the whole dS space. If we look at the other case, increasing the charge at fixed mass, we observe that it is the Cauchy and the BH horizons that tend to merge. After we reach the merging point, we are in a condition where the electromagnetic energy density is too strong and prevent the formation of an event horizon. However, a cosmological horizon is still present. Again, in this region $g_{00} = 0$ presents only one solution, but it is different from the previous one we encountered. Although several regions where roots of g_{00} become complex can be present, they are not of the same nature and one should be careful in giving a physical interpretation to each of them.

These two conditions are given by two equations in the Q, M plane at fixed $l = \frac{\sqrt{3}}{\Lambda}$:

$$M_{\pm}^2 = \frac{1}{54l}\left[l(l^2 + 36Q^2) \pm (l^2 - 12Q^2)^{\frac{3}{2}}\right] \tag{5.1.3}$$

This formula is not valid for large charge, but it exists a continuation of the solution for larger Q. The dS WGC that one can make in the case where this formula is defined corresponds to require $M < M_-$. It is the region where Q is large enough to forbid the existence of an event horizon.

5.1.2 A dilatonic (A-)dS black-hole solution

We will now add a dilaton, and a potential in order to obtain an asymptotically dS solution. The action becomes

$$\mathcal{S} = \int d^4x\sqrt{-g}\frac{1}{2\kappa^2}\left(R - 2\left(\partial\phi\right)^2 - V(\phi) - e^{-2\alpha\phi}F^2\right). \tag{5.1.4}$$

In the following, we will use a static, spherically symmetric solution to the related Einstein's equations, which was found in [120], reading

$$\mathrm{d}s^2 = -\left[\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \mp H^2r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]\mathrm{d}t^2 \tag{5.1.5}$$

$$+\left[\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \mp H^2r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\right]^{-1}\mathrm{d}r^2 \tag{5.1.6}$$

$$+ r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}\mathrm{d}\Omega_2^2, \tag{5.1.7}$$

$$\tag{5.1.8}$$

where H^2 is the Hubble parameter. As we have said, the potential is dictated by the equation of motions, and for this solution it has the form

$$V(\phi) = \frac{2}{3}\frac{\Lambda}{1+\alpha^2}\left(\alpha^2(3\alpha^2 - 1)e^{-2\frac{\phi-\phi_0}{\alpha}} + (3-\alpha^2)e^{2\alpha(\phi-\phi_0)} + 8\alpha^2 e^{\alpha(\phi-\phi_0)-\frac{\phi-\phi_0}{\alpha}}\right) \tag{5.1.9}$$

where Λ is the cosmological constant defined by $H^2 = \frac{|\Lambda|}{3}$. The solution for $\phi(r)$ and $F(r)$, given in (1.2.4), and the relation between the constants r_-, r_+ and the physical parameters M, Q, (1.2.5) are unchanged. Depending on the sign of Λ we have either an asymptotically dS (upper sign in (5.1.5)) or AdS (lower sign) space-time. This solution has also been obtained in [103] for the AdS case, and for both asymptotic behaviours in [178]. Comparing to the dilatonic case with an asymptotic flat space presented in the chapter 1, we can notice that r_+ is no longer a horizon, instead r_- is still a singularity.

5.2 Study of the horizons of dilatonic dS black holes

Now we have all the elements to study the solutions of black holes, looking at the number of horizons in function of M and Q. We will separate the case in function of the value of α, since the behaviour of the solutions will also depend on this parameter.

5.2.1 $\alpha = 1$

The $\alpha = 1$ case allows for explicit expressions of the black hole horizons. The metric can be written as

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} - r(r-2D)H^2\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} - r(r-2D)H^2\right)^{-1}\mathrm{d}r^2 + r(r-2D)\mathrm{d}\Omega_2^2. \tag{5.2.1}$$

D is related to M and Q through

$$D = \frac{Q^2 e^{2\phi_0}}{2M} \tag{5.2.2}$$

and $r = r_- = 2D$ is a singular surface. The horizons correspond to the loci of the roots of the polynomial of degree 3 in r:

$$P(r) = H^2 r^3 - 2DH^2 r^2 - r + 2M \tag{5.2.3}$$

Their explicit expressions are not very illuminating. We find more instructive, in particular for discussing below $\alpha \neq 1$, to provide a description of the behaviour of the roots as functions of M and D.

First, note that $P(r) \underset{r\to+\infty}{\to} +\infty$, and can have two extrema $R_- < R_+$ given by the roots of $P'(r) = 3H^2r^2 - 4DH^2 r - 1$. As $R_- R_+ = -1$, $R_- < 0$ while $R_+ = \frac{2}{3}D + \frac{1}{6}\sqrt{16D^2 + \frac{12}{H^2}} > 0$. We are interested only in solutions of $P(r) = 0$ in the region $r > 2D$ outside the singularity. Therefore, we will discuss the signs of $P(2D)$ and, when $R_+ > 2D$, $P(R_+)$.

The case of $R_+ < 2D$, i.e. $D^2H^2 > \frac{1}{4}$, corresponds to $r_-^2 H^2 > 1$, which means that the radius of the singular surface is greater than the Hubble one. We discard this situation as unphysical. In any case, no black hole solutions can arise there: when $P(2D) < 0$ one root is present, otherwise the polynomial is always positive $\forall r > 2D$. We restrict in the following to $R_+ > 2D$.

Next, note that when $P(2D) = 2(M - D) \leq 0$, P only has one root. If $P(2D) > 0$, there can be 0,1 or 2 roots, depending on the sign of P at the minimum R_+. Studying this case, i.e. $M \geq D$, we have

$$P(R_+) = -\frac{16}{27}D^3H^2 - \frac{2}{3}D + 2M - \sqrt{4D^2 + \frac{3}{H^2}}\left(\frac{8}{27}D^2H^2 + \frac{2}{9}\right). \tag{5.2.4}$$

If $-\frac{16}{27}D^3H^2 - \frac{2}{3}D + 2M$ is negative, the sign of $P(R_+)$ is fixed to be negative, and there are two zeros for P. In order to further investigate the sign of $P(R_+)$, it is helpful to consider the function

$$\begin{cases} U(D) \equiv -\frac{16}{27}D^3H^2 - \frac{2}{3}D + 2M \\ D < D_1 \to U(D) > 0 \\ D = D_1 \to U(D) = 0 \\ D > D_1 \to U(D) < 0 \end{cases} \tag{5.2.5}$$

We have $U(0) = 2M > 0$ and U is decreasing with D. There is so one solution D_1 such that $U(D_1) = 0$. For $D > D_1$, U is negative and P has two zeros. The region $D < D_1$, where $U(D)$ is positive, needs further investigation.

It is easier for the rest of the computation, with $D < D_1$, to reformulate the zeros of $P(R_+)$ as the zeros of a simpler function:

$$\begin{aligned} P(R_+) = 0 \Leftrightarrow \left(-\frac{16}{27}D^3H^2 - \frac{2}{3}D + 2M\right)^2 &= \left(4D^2 + \frac{3}{H^2}\right)\left(\frac{8}{27}D^2H^2 + \frac{2}{9}\right)^2 \\ \Leftrightarrow -\left(\frac{4}{3}\right)^3 H^2MQ(D) = 0, \end{aligned}$$

where Q is a function of D defined as

$$Q(D) = D^3 + \frac{1}{16H^2M}D^2 + \frac{9}{8H^2}D - \frac{27M}{16H^2} + \frac{1}{16H^4M}, \tag{5.2.6}$$

so that $P(R_+) < 0$ when $Q(D) > 0$. Q is an increasing function for positive D. The sign of $Q(0) = -\frac{27M}{16H^2} + \frac{1}{16H^4M}$ discriminates between two cases. If $Q(0) > 0$, $Q(D)$ is positive for all positive D. If $Q(0) < 0$, there is one D_0 such that $Q(D_0) = 0$:

$$\begin{cases} D < D_0 \to Q(D) < 0 \Rightarrow P(R_+) > 0 \Rightarrow P(r) \neq 0,\ \forall r \in \mathbb{R}^+ \\ D = D_0 \to Q(D) = 0 \\ D > D_0 \to Q(D) > 0 \Rightarrow P(R_+) < 0. \end{cases}$$

Imposing the necessary condition $D^2H^2 < \frac{1}{4}$, we shall now group all cases. There are three possibilities corresponding to 0, 1 or 2 roots.

- For P to have 2 roots, the first condition to be satisfied is $P(2D) \geq 0$, i.e. $D \leq M$. If $D > D_1$, P has two roots without the need for further investigations. If $D < D_1$, P has also two roots when $M^2H^2 < \frac{1}{27}$. Finally, when $M^2H^2 > \frac{1}{27}$, P has two roots if the additional condition $Q(D) > 0 \Leftrightarrow D > D_0$ is verified.

- There are two scenarios where P has one root. The first corresponds to $P(2D) < 0$, ie $D \geq M$. The second possibility is $P(2D) \geq 0$ and $P(R_+) = 0$, which can be expressed as $M \geq D$ and $D = D_0$ with $D < D_1$. This case is the limit where the two horizons defined just above coincide.

- Finally, P does not have roots corresponds when $D < M$, $D < D_1$, $D < D_0$ and $M^2H^2 > \frac{1}{27}$.

All the cases listed above depend on the values of D_0 and D_1. Those are given in terms of M as roots of the polynomials Q and U. More compact expressions, can be given using the variables $Y = DH$ and $X = MH$:

$$Y_0 = -\frac{1}{48X} + \frac{1}{48}\left(-\frac{1}{X^3} - \frac{2^4.3^3.5}{X} + 2^7.3^6X + \frac{48\sqrt{3}}{X^2}\sqrt{1 + 2^2.3^4X^2 + 2^4.3^7X^4 + 2^6.3^9X^6}\right)^{\frac{1}{3}}$$
$$-\frac{16}{3}\left(\frac{27}{8} - \frac{1}{256X^2}\right)\left(-\frac{1}{X^3} - \frac{2^4.3^3.5}{X} + 2^7.3^6X + \frac{48\sqrt{3}}{X^2}\sqrt{1 + 2^2.3^4X^2 + 2^4.3^7X^4 + 2^6.3^9X^6}\right)^{-\frac{1}{3}}$$
$$Y_1 = \frac{3}{2\sqrt{6}}\left(-3\sqrt{6}X + \sqrt{1 + 54X^2}\right)^{-\frac{1}{3}} - \frac{3}{2\sqrt{6}}\left(-3\sqrt{6}X + \sqrt{1 + 54X^2}\right)^{\frac{1}{3}}$$

Actually, the value of Y_0 presented just above is complex for $X < \frac{1}{12\sqrt{6}}$. In that range of parameters, of the three roots of $Q(D)$, it is another one which is real, corresponding to Y_0 with an absolute value taken on the factors elevated to the $\pm\frac{1}{3}$ power and on the factor $\frac{27}{8} - \frac{1}{256X^2}$. However, as we are only interested in $D > 0$ and D_0 which is positive for $X > \frac{1}{\sqrt{27}}$, the expression for Y_0 is real in the whole range of interest for D and the absolute values are of no use.

The different cases for the black hole horizons are represented graphically in Figure 5.1. Instead of D, we used the electric charge Q (actually, Q really is Qe^{ϕ_0}). The green curve represents $D = D_0$, while the yellow one is $M = D$. The function D_1, represented by the dashed blue curve, is below $M = D$ for $D^2H^2 < \frac{1}{4}$ so that, according to our previous findings, it plays no role in the separation of the different regimes. In the region between the green and the yellow curves, black hole solutions with two horizons are found. For $Q = 0$, the discriminant between solutions with two and zero horizons is $M = \frac{1}{\sqrt{27}H}$, as it should. Solutions describing a singularity with a cosmological horizon are found below the yellow curve. Finally, the red curve is defined by $2DH = 1$. On its right, the radius of the singularity is greater than the Hubble one.

To illustrate the solution, we now follow two straight horizontal lines, like the green and yellow dashed ones, in Figure 5.1, with $MH < \frac{1}{\sqrt{27}}$ in one case and $\frac{1}{\sqrt{27}} < MH \leq \frac{1}{2}$ in the other.

- $\boldsymbol{MH < \frac{1}{\sqrt{27}}}$. At $Q = 0$ there are two horizons: the event and the cosmological horizon. Increasing Q, the radius of the cosmological horizon and of the

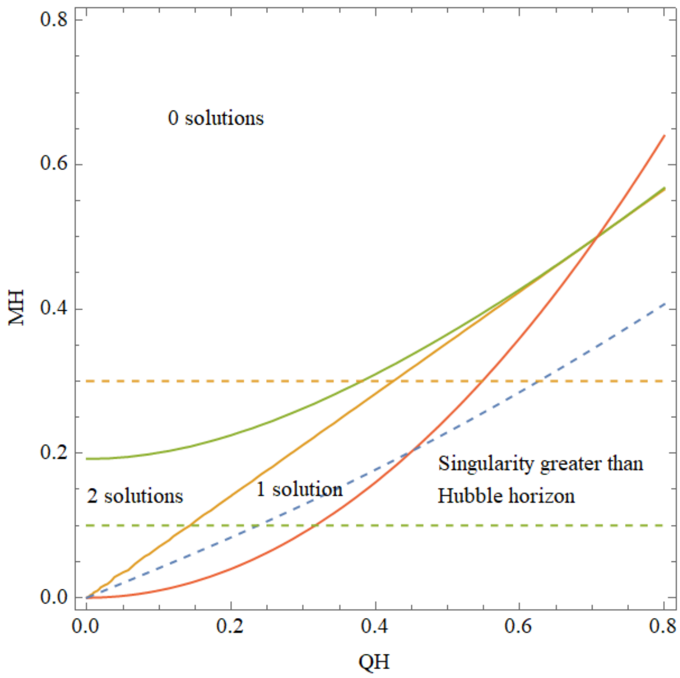

Figure 5.1 – Number of horizons of the $\alpha = 1$ de Sitter black hole as a function of MH and QH. The green curve represents HD_0, the yellow one the limit $\sqrt{2}MH = Qe^{\alpha\phi_0}H$, and the red one $D^2H^2 = \frac{1}{4}$. Dotted lines are for intermediate steps and discussions in the text.

singularity increase while that of the event horizon decreases until the value $Qe^{\phi_0} = \sqrt{2}M$ is reached. Here, the event horizon coincides with the singularity. For $Qe^{\phi_0} > \sqrt{2}M$, only the cosmological horizon surrounds $r_s = \frac{Q^2e^{2\alpha\phi_0}}{M}$. The radius of the singular surface increases with Q until it coincides the Hubble radius when $Q^2e^{2\alpha\phi_0} = \frac{M}{H}$.

- $\boldsymbol{\frac{1}{\sqrt{27}} < MH < \frac{1}{2}}$. With $MH < \frac{1}{2}$, at $Q = 0$ no horizons are present. Increasing Q the situation doesn't change until Q such that $\frac{Q^2e^{2\alpha\phi_0}}{2M} = D_0$ is reached: at this point, one horizon appears. Here, two roots of g_{00} coincide, meaning that the event and cosmological horizons have the same size. As Q further grows the two horizons disentangle, the radius of the event horizon shrinks, while that of the cosmological horizon increases. Continuing to increase Q, the analysis is the same as in the previous point: when the condition $Qe^{\alpha\phi_0} = \sqrt{2}M$ is reached, the singularity coincide with the event horizon, and for greater charges the solutions only show a cosmological horizon.

When $MH = \frac{1}{2}$, the region with two horizons disappears. At the point (QH, MH) defined by $MH = \frac{1}{2}$ and $Qe^{\alpha\phi_0}H = \sqrt{2}MH = \frac{1}{\sqrt{2}}$, the green, yellow and red curves meet. It means that the two roots of g_{00}, the singularity and the Hubble horizon coincide, corresponding to the case where $P(2D) = P(R_+) = 0$ with $R_+ = 2D$. This point defines the maximal mass and charge for which a black hole solution exists. Keeping M fixed, larger charges allow for the presence of a cosmological horizon, with the singularity bigger than the Hubble surface.

For $MH > \frac{1}{2}$, no black hole solution is possible: the singularity is either completely naked, when $Qe^{\alpha\phi_0} < \sqrt{2}M$, or shielded by a cosmological horizon when $Qe^{\alpha\phi_0} \geq \sqrt{2}M$, with the latter coinciding with the singularity when the inequality is saturated. The condition for the singularity to be bigger than the Hubble horizon is now met before this last one.

Following the arguments of [24] to infer the WGC condition from the absence of event horizons shielding the singularity, the WGC would require the existence of a state with mass m and charge q, in geometrized units, satisfying $qe^{\phi_0} > \sqrt{2}m$. This corresponds to the dilatonic WGC bound in asymptotically flat space-time, as discussed above. Thus, for $\alpha = 1$, the dilatonic WGC seems to be insensitive to the presence of a cosmological constant.

5.2.2 $\alpha > 1$

We first study the $\alpha \to \infty$ limit, where one should recover the Schwarzschild-de Sitter solution, and then look at the generic $\alpha > 1$ case.

In the $\alpha \to \infty$ limit the metric reads

$$\mathrm{d}s^2 = -\left[\frac{1-\frac{r_+}{r}}{1-\frac{r_-}{r}} - H^2r^2\left(1-\frac{r_-}{r}\right)^2\right]\mathrm{d}t^2+\left[\frac{1-\frac{r_+}{r}}{1-\frac{r_-}{r}} - H^2r^2\left(1-\frac{r_-}{r}\right)^2\right]^{-1}\mathrm{d}r^2+r^2\left(1-\frac{r_-}{r}\right)^2\mathrm{d}\Omega_2^2. \tag{5.2.7}$$

To study the horizons of the above metric, we need to find the roots of the polynomial

$$G(r) \equiv H^2(r-r_-)^3 - (r-r_+) = 0. \tag{5.2.8}$$

G has two extrema: a minimum at $r = r_- + \frac{1}{\sqrt{3}H}$ and a maximum at $r = r_- - \frac{1}{\sqrt{3}H}$. The latter is inside the singular surface. The knowledge of the values on the singular surface, $G(r_-)$, and at its minimum, $G(r_- + \frac{1}{\sqrt{3}H})$, allows to find the number of roots of G. We have

$$\begin{cases} G(r_-) = r_+ - r_- \\ G(r_- + \frac{1}{\sqrt{3}H}) = r_+ - r_- - \frac{2}{\sqrt{27}H}. \end{cases} \tag{5.2.9}$$

For $0 < r_+ - r_- < \frac{2}{\sqrt{27}H}$, the singularity is protected by two horizons: the event and the cosmological horizons. When $r_+ - r_- = \frac{2}{\sqrt{27}H}$, the two horizons merge. Above, neither the event nor the cosmological horizon are present. At $r_+ = r_-$, the event horizon coincides with the singularity. Using (1.2.5) one obtains, for $\alpha \to \infty$, $r_+ - r_- = 2M$. Thus, we discard the $r_+ < r_-$ region as corresponding to negative masses. In the $\alpha \to \infty$ limit of (5.1.5) the Schwarzschild-de Sitter solution is recovered: the discriminant between a naked and a shielded singularity is the sign of $M - \frac{1}{\sqrt{27}H}$.

Now, we consider the general metric (5.1.5) and define a new function F that vanishes for the same values of r than g_{00}:

$$F(r) \equiv r - r_+ - H^2 r^3 \left(1 - \frac{r_-}{r}\right)^{\frac{3\alpha^2-1}{\alpha^2+1}} . \tag{5.2.10}$$

To investigate the solutions of $F(r) = 0$, we divide F into the sum of two contributions: $A(r) \equiv r - r_+$, and $B(r) \equiv H^2 r^3 \left(1 - \frac{r_-}{r}\right)^{\frac{3\alpha^2-1}{\alpha^2+1}}$.

The intersection points of the two curves defined by A and B give the zeros of F. We carry the analysis in two regions of the parameter space:

- For $r_+ < r_-$, $A(r_-) > 0$ and the two curves always cross in one point. Accordingly there is, in this case, only one zero, corresponding to the cosmological horizon.
- For $r_+ > r_-$, $A(r_-) < 0$ and there are either two, one or zero solutions depending on the location of the point r_0 where $B'(r_0) = 1$. $B(r_0) \leq A(r_0)$ corresponds to the case where the function has two zeros, coalescing into one when the equality is satisfied. $B(r_0) > A(r_0)$ will determine the horizon-less regime where the dS space has been "completely eaten up" by the black hole.

We consider, from now on, $r_+ \geq r_-$. To discriminate between the different regimes we just described, we proceed in the following way.

First, we observe that the limit for the two zeros to collapse into one is obtained where $A(r)$ and $B(r)$ are tangent, thus $F(r_0) = 0$ and $F'(r_0) = 0$ $(B'(r_0) = 1)$. The solutions of this system are always two as the equation $F(r_0) = 0$, with the prior $F'(r_0) = 0$, reduces to a quadratic equation for r_0.

Consider $r_{0\pm}$ two functions of $r_\pm$ given by:

$$r_{0\pm} = \frac{(3-\alpha^2)r_- + 3(1+\alpha^2)r_+}{4(1+\alpha^2)} \pm \sqrt{\left(\frac{(3-\alpha^2)r_- + 3(1+\alpha^2)r_+}{4(1+\alpha^2)}\right)^2 - 2\frac{r_+ r_-}{\alpha^2+1}} \tag{5.2.11}$$

When

$$\begin{cases} F(r_{0\pm}) = 0 \\ F'(r_{0\pm}) = 0 \end{cases} \tag{5.2.12}$$

the event and cosmological horizons coincide.

As $F(r) \to -\infty$ for $r \to \infty$, starting with $F(r_-) < 0$, if $F(r)$ takes a positive value at some coordinate value this ensures that it crosses twice the abscissa axis thus allowing the existence of two horizons. Therefore, when $r_{0\pm}$ are both greater than r_-, $r_+ \geq r_-$, the black hole solution exists in the parameter region of (r_+, r_-) where $F(r_{0+}) > 0$ and $F(r_{0-}) < 0$.

Next, note that for $\alpha > 1$, $r_{0-} < r_-$ does not intervene. Using $F(r_-) < 0$, the region where two horizons are present is defined by $F(r_{0+}) > 0$. the lower boundary of the region for the existence of a black hole solution is thus given by $F(r_{0+}) = 0$. In terms of M and Q, $F(r_{0+}) < 0$ translates to

$$\left((1-2\alpha^2)M + (\alpha^2-2)\sqrt{M^2 - (1-\alpha^2)Q^2 e^{2\alpha\phi_0}} + \sqrt{P(M, Q, \alpha, \phi_0)}\right)$$

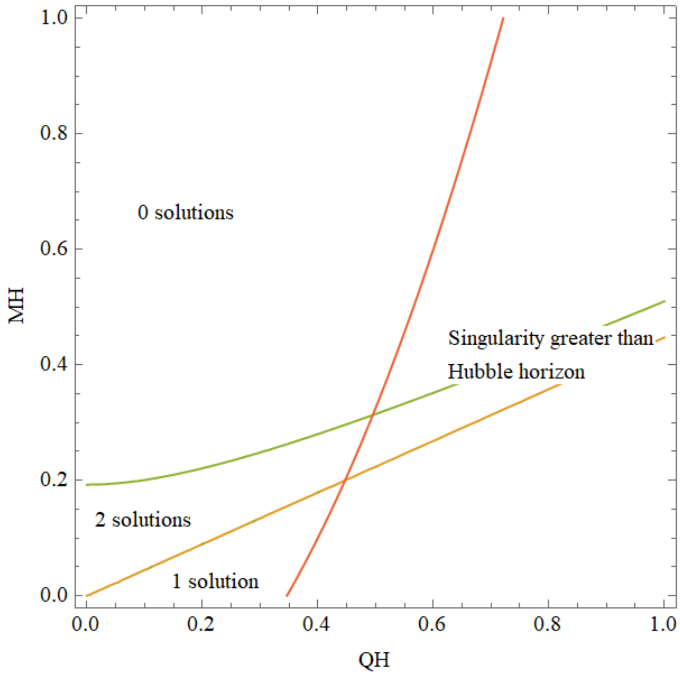

Figure 5.2 – Number of horizons of the $\alpha > 1$ de Sitter black hole as a function of MH and QH. The green curve represents $F(r_{0+}) = 0$, the yellow one the limit $Q^2 = (1+\alpha^2)M^2$, and the red one $r_- H = 1$. We chose for the illustration $\alpha = 2$ and $\phi_0 = 0$.

$$- H^2 \left((10+9\alpha^2)M - (4+3\alpha^2)\sqrt{M^2 - (1-\alpha^2)Q^2 e^{2\alpha\phi_0}} + \sqrt{P(M,Q,\alpha,\phi_0)} \right)^{\frac{3\alpha^2-1}{1+\alpha^2}} \times$$
$$\left((4+3\alpha^2)M - \sqrt{M^2 - (1-\alpha^2)Q^2 e^{2\alpha\phi_0}} + \sqrt{P(M,Q,\alpha,\phi_0)} \right)^{\frac{4}{1+\alpha^2}} > 0, \tag{5.2.13}$$

where $P(M,Q,\alpha,\phi_0)$ is defined by

$$\begin{aligned} P(M,Q,\alpha,\phi_0) =& (17+24\alpha^2+9\alpha^4)M^2 - (9+15\alpha^2+8\alpha^4)Q^2 e^{2\alpha\phi_0} \\ & - (8+6\alpha^2)M\sqrt{M^2-(1-\alpha^2)Q^2 e^{2\alpha\phi_0}}. \end{aligned} \tag{5.2.14}$$

Figure 5.2 presents the results of the discussion above. In analogy to the case $\alpha = 1$, we have added a further constraint for the singularity to be inside the Hubble horizon, $r_- < \frac{1}{H}$.

When the black hole charge vanishes, the equation $F(r_{0+}) = 0$ reduces to $M = \frac{1}{\sqrt{27}H}$. Consider in this figure a point in the region corresponding to a black hole with two horizons and vary the charge or the mass:

- Increasing the mass, the event horizon reaches the cosmological one for the black hole mass M such that $F(r_{0+}) = 0$. Beyond this value, the singularity is naked.
- Increasing the charge instead we encounter, at some point, the line $Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2$, where the event horizon and the singular surface r_- coalesce. Continuing to increase the charge, the electromagnetic energy density becomes strong enough to prevent the formation of an event horizon. The WGC states are expected to have mass and charge in this region of parameters.

The WGC would then require the existence of a state with $q^2 e^{2\alpha\phi_0} > (1+\alpha^2)m^2$, as in the case $\alpha = 1$. Although the presence of a cosmological constant changes the form of $g_{00}(r)$, in the $\alpha > 1$ case the weak gravity bound would take the same form as in asymptotically flat space-time.

5.2.3 Some comments on the $\alpha < 1$ case

For $\alpha < 1$, both the terms in g_{00} given by:

$$g_{00}(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} - H^2 r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}$$

go to zero when $r \to r_-$. For $\alpha^2 = \frac{1}{3}$, the r_- dependence factorizes and g_{00} can be written as

$$g_{00}(r) = -\left(1 - \frac{r_-}{r}\right)^{\frac{1}{2}}\left(1 - \frac{r_+}{r} - H^2 r^2\right), \tag{5.2.15}$$

where the second factor takes the form of the g_{00} of a Schwarzschild-de Sitter metric with mass $M^* \equiv \frac{r_+}{2}$. The relative importance of the two terms in g_{00} depends on whether $\alpha^2 < \frac{1}{3}$ or $\alpha^2 > \frac{1}{3}$. *A priori*, we may expect a dilatonic-like black hole behaviour for $\frac{1}{\sqrt{3}} < \alpha < 1$, similar to the $\alpha > 1$, while a different, de Sitter-like black hole, behaviour for $\alpha < \frac{1}{\sqrt{3}}$. We thus split the $\alpha < 1$ analysis in three parts: $\frac{1}{\sqrt{3}} < \alpha < 1$, $\alpha = \frac{1}{\sqrt{3}}$ and $\alpha < \frac{1}{\sqrt{3}}$.

There is another comment to add. For $0 < \alpha < 1$, looking at (1.2.6), one can see that there is a region of parameters Q and M such that r_+, r_- and the metric becomes complex. It happens for $Q^2 < (1-\alpha^2)M^2$. We will have to discuss this region in the following.

5.2.4 $\frac{1}{\sqrt{3}} < \alpha < 1$

Factorizing the $\left(1 - \frac{r_-}{r}\right)$ factor, we study the zeros of F defined in (5.2.10), with $3\alpha^2 - 1 > 0$. The only difference with the $\alpha > 1$ case comes from the convergence of the first term in g_{00} when $r \to r_-$. The convergence is to 0^+ when $r_+ > r_-$ and to 0^- when $r_+ < r_-$. Concretely, this does not affect the zeros of F, and thus the results obtained in the case $\alpha \geq 1$. In the (Q, M) plane, the boundaries of the region allowing black holes is still given by $Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2$ and $F(r_{0+}) = 0$. Note that the most involved part of the analysis in the case $\alpha > 1$ was for the situation $r_+ > r_-$ and used $r_{0-} < r_-$. While in the region $\frac{1}{\sqrt{3}} < \alpha < 1$, r_{0-} can become greater than r_-, this only happens when $r_+ < r_-$ and therefore does not modify that analysis. Black hole arguments would again indicate for the WGC the existence of a particle satisfying $q^2 e^{2\alpha\phi_0} > (1+\alpha^2)m^2$. As long as the second term in g_{00} (*de Sitter-like*) is sub-dominant, the transition between black holes and singularities (with a cosmological horizon) seems to happen in the same parametric region as in asymptotically flat space-time.

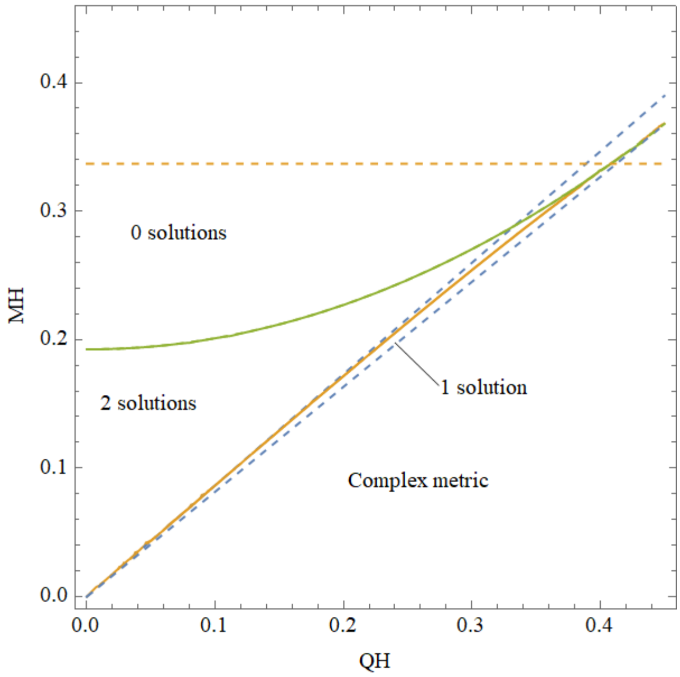

Figure 5.3 – $\alpha^2 = \frac{1}{3}$ case. Blue dotted curves correspond to $Q^2 = \frac{4}{3}M^2$ and $Q^2 = \frac{3}{2}M^2$. Green curve represents (5.2.18) and yellow one corresponds to (5.2.19).The yellow dotted line represents the maximal black hole mass: $M_{\max} = \frac{7}{12\sqrt{3}H}$

5.2.5 $\alpha = \frac{1}{\sqrt{3}}$

The $\alpha = \frac{1}{\sqrt{3}}$ case allows for explicit expressions of the horizons and can be studied in full details. The second factor in (5.2.15) can be seen as the time component of a Schwarzschild-de Sitter metric with an effective mass $M^* \equiv r_+/2$. This factor has two zeros for $r_+ < \frac{2}{\sqrt{27}H}$ and none for $r_+ > \frac{2}{\sqrt{27}H}$. The roots of the polynomial $P(r) \equiv r - r_+ - H^2 r^3$ are:

$$\begin{cases} r_c = \frac{1}{H}\left(\frac{\left(\frac{2}{3}\right)^{1/3}}{\left(-9r_+H+\sqrt{3}\sqrt{-4+27r_+^2H^2}\right)^{1/3}} + \frac{\left(-9r_+H+\sqrt{3}\sqrt{-4+27r_+^2H^2}\right)^{1/3}}{2^{1/3}3^{2/3}}\right) \\ r_h = -\frac{1}{H}\left(\frac{\left(\frac{2}{3}\right)^{1/3}e^{-i\pi/3}}{\left(-9r_+H+\sqrt{3}\sqrt{-4+27r_+^2H^2}\right)^{1/3}} + \frac{\left(-9r_+H+\sqrt{3}\sqrt{-4+27r_+^2H^2}\right)^{1/3}e^{i\pi/3}}{2^{1/3}3^{2/3}}\right) \end{cases} \tag{5.2.16}$$

with r_c, r_h the cosmological and the event horizons respectively. The third root of P is negative thus of no physical interest. In fact, one can see from the coefficients of P that the product of the roots is $-\frac{r_+}{H^2} < 0$ and their sum is null. So, there are either two real positive roots and a negative one (corresponding to the case where we have two horizons) or two complex conjugate and a negative root (corresponding to the case where no horizon is present). The transition between these two regimes happens when the horizons coincide, $r_c = r_h$, i.e. for $r_+ = \frac{2}{\sqrt{27}H}$. We also require that these roots are located outside the singular surface located at r_-. In order to study the behaviour of the roots of g_{00}, we consider the equations:

$$\begin{cases} r_c - r_- = 0 \\ r_h - r_- = 0 \\ r_+ = \frac{2}{\sqrt{27}H}. \end{cases} \tag{5.2.17}$$

There is an equivalence between the condition $r_c \geq r_-$, $r_+ < \frac{2}{\sqrt{27}H}$ and $F(r_{0+}) \geq 0$, and between $r_h \geq r_-$, $r_+ = \frac{2}{\sqrt{27}H}$ and $F(r_{0-}) \leq 0$.

To express these conditions in terms of the physical variables (Q, M) we use the relations $r_+ = M + \sqrt{M^2 - \frac{2}{3}Q^2 e^{2\alpha\phi_0}}$ and $r_- = \frac{4}{3}\frac{Q^2}{M+\sqrt{M^2-\frac{2}{3}Q^2 e^{2\alpha\phi_0}}}$. Redefining $\hat{M} \equiv MH$ and $\hat{Q} \equiv e^{\alpha\phi_0}QH$, the conditions then take the following form

$$r_c - r_- = 0 \Leftrightarrow \frac{\left(\frac{2}{3}\right)^{1/3}}{\left(-9\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)+\sqrt{3}\sqrt{-4+27\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}} + \frac{\left(-9\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)+\sqrt{3}\sqrt{-4+27\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3}\,3^{2/3}} = \frac{4\hat{Q}^2}{3\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}, \tag{5.2.18}$$

$$r_h - r_- = 0 \Leftrightarrow \frac{\left(\frac{2}{3}\right)^{1/3} e^{i\pi/3}}{\left(-9\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)+\sqrt{3}\sqrt{-4+27\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}} + \frac{e^{-i\pi/3}\left(-9\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)+\sqrt{3}\sqrt{-4+27\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3}\,3^{2/3}} = \frac{4\hat{Q}^2}{3\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}. \tag{5.2.19}$$

and

$$r_+ = \frac{2}{\sqrt{27}H} \Leftrightarrow \hat{M} = \frac{1}{\sqrt{27}} + \frac{\sqrt{3}\hat{Q}^2}{2}. \tag{5.2.20}$$

It is possible to identify a *triple point* $(\hat{Q}, \hat{M}) = \left(\frac{1}{\sqrt{6}}, \frac{7}{12\sqrt{3}}\right)$ where the regions with two, one and zero solutions meet. We now have all the elements to show the different region of solutions, as displayed in the Figure 5.3.

- Restricting to masses below the *triple point* bound, thus $\hat{M} < \frac{7}{12\sqrt{3}}$, the upper bound on the mass allowing the two-horizons solution is given by $r_+ = \frac{2}{\sqrt{27}H}$. It is represented by the green curve, $\hat{M} = \frac{1}{\sqrt{27}} + \frac{\sqrt{3}\hat{Q}^2}{2}$, from $\left(0, \frac{1}{\sqrt{27}}\right)$ up to the triple point.
- The lower bound on this black hole region is given by (5.2.19) and it is represented in yellow in Figure 5.3. It corresponds to the limit where the event horizon coincides with the singularity. Note that the lower bound yellow curve starts at $(0, 0)$ and crosses the upper bound green curve at the *triple point*. After that, the yellow curve runs above the green one and does not bound any physical region.

The graphical representation of the two bounds reveals that for masses $M > \frac{7}{12\sqrt{3}H}$ the event horizon cannot form: this is the maximal mass above which asymptotically de Sitter black hole solutions are no more possible (yellow dashed line). Accordingly, this point corresponds to a maximal charge $Q_{\max} = \frac{1}{\sqrt{6}H}$.

- Singularities with only a cosmological horizon are found in the domain given by the union of: (1) the region between the yellow and the lower dashed blue curve from the origin up to the *triple point*, (2) the region between the green and the same dashed blue curve, now above it. On this dashed curve $Q^2 e^{2\alpha\phi_0} = \frac{3}{2}M^2$ marks the limit of definition of the metric.

The green curve (defined by $r_c = r_-$) crosses the blue one, corresponding to $Q^2 e^{2\alpha\phi_0} = \frac{3}{2}M^2$ in the point $(Qe^{\alpha\phi_0}, M) = \left(\frac{\sqrt{3}}{4H}, \frac{1}{2\sqrt{2}H}\right)$. This is a point of maximal charge and mass. Above it, the green curve delimiting singularities with and without cosmological horizon cannot be drawn: either it is not defined, or it lies inside the inaccessible region (complex metric).

To confirm that (5.2.19) can be seen as a WGC bound, one can look at its behaviour when $H \to 0$. To look at this limit, let us rewrite r_h, in the region where it is real, as

$$r_h = \frac{2}{\sqrt{3}H} \sin\left(\frac{\theta}{3}\right), \tag{5.2.21}$$

where the angle θ is defined by $\sin(\theta) = \frac{3\sqrt{3}}{2} r_+ H$ and $\cos(\theta) = \sqrt{1 - \frac{27}{4} r_+^2 H^2}$. In the limit $H \ll \frac{1}{r_+}$, one obtains

$$r_h = r_+ + H^2 r_+^3 + \mathcal{O}(r_+^3 H^4). \tag{5.2.22}$$

Looking at $r_h - r_- = 0$, replacing r_+ and r_- by their definition in function of M and Q given by (1.2.6), and looking at the expansion of Q in powers of H one obtains the equation

$$Q^2 e^{(2/\sqrt{3})\phi_0} = \frac{4}{3}M^2 + \frac{4^3}{3^4} M^4 H^2 + \mathcal{O}(M^6 H^4). \tag{5.2.23}$$

In the limit $H \to 0$, the bound given by (5.2.19) reduces to (1.2.9).

In conclusion, for $\alpha = \frac{1}{\sqrt{3}}$, the study of horizons of the dilatonic de Sitter black hole would rather suggest (5.2.19) as a WGC bound than (1.2.9).

5.2.6 $\alpha < \frac{1}{\sqrt{3}}$

As shown above, when $\alpha = \frac{1}{\sqrt{3}}$, $r_+ = r_-$ is no longer the black hole extremality condition, as it was for all cases with $\alpha > \frac{1}{\sqrt{3}}$. In the following, we will see that this remains true for $\alpha < \frac{1}{\sqrt{3}}$.

The first difference one can observe with respect to previous cases is the change in the behaviour of the derivative of g_{00} in a neighbourhood of the singularity r_-. Leading terms are given by

$$\begin{cases} \partial_r(g_{00}) \underset{r\to r_-}{\sim} -\frac{1-\alpha^2}{1+\alpha^2}\frac{r_-}{r^2}\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{-2\frac{\alpha^2}{1+\alpha^2}} & \text{for } \alpha > \frac{1}{\sqrt{3}} \\ \partial_r(g_{00}) \underset{r\to r_-}{\sim} 2\frac{\alpha^2}{1+\alpha^2}H^2 r_-\left(1-\frac{r_-}{r}\right)^{-\frac{1-\alpha^2}{1+\alpha^2}} & \text{for } \alpha < \frac{1}{\sqrt{3}}. \end{cases} \tag{5.2.24}$$

When $\alpha < \frac{1}{\sqrt{3}}$, close to the singularity the sign of the derivative is independent of r_+ and is always positive, with $g_{00}(r) \underset{r\to r_-}{\to} 0^+$. This combined with the asymptotic value $g_{00} \to +\infty$ when $r \to \infty$, implies that the metric exhibits a horizon only if the parameters in $g_{00}(r)$ are such that the function is decreasing in an interval to reach a negative minimum. In this situation, there are two horizons, coincident when the minimum of g_{00} is 0. This leads to a first conclusion:

- In the parametric (QH, MH) space, a singularity surrounded only by a cosmological horizon can only appear on a curve, rather than in a portion of the plane as happened for all cases with $\alpha \geq \frac{1}{\sqrt{3}}$.

Remember that, in contrast with the asymptotically flat case, here both $r_+ > r_-$ and $r_+ \leq r_-$ are now allowed. The method used above to investigate the limits between regions with different behaviours of the horizons was valid only for $r_+ > r_-$ but can now be extended to all the situations. We proceed thus by using the function F defined in (5.2.10), and its decomposition into A and B. For $\alpha^2 < \frac{1}{3}$, $B \to +\infty$ when $r \to r_-$. In a neighbourhood of r_- we always have $B(r) > A(r)$.

We notice that the condition $F(r_{0-}) < 0$ is not very illuminating in this case of $\alpha^2 < 1/3$. Indeed, for $r_+ > r_-$, $F(r_{0-})$ is always negative while for $r_+ \leq r_-$, expanding (5.2.11), we have $r_{0-} \leq r_-$, i.e. r_{0-} lies inside the singular surface and $F(r_{0-}) < 0$ should not be considered.

Therefore:

- The condition for the existence of black hole solutions is given by $F(r_{0+}) > 0$. When $F(r_{0+}) = 0$, the event and cosmological horizons coincide. $F(r_{0+}) < 0$ defines naked singularities with no cosmological horizon.

As in the case $\alpha = \frac{1}{\sqrt{3}}$, there is a maximal mass, above which there is no black hole solution. This mass corresponds to the point where the curve defined by $F(r_{0+}) = 0$ cross the line defined by $Q^2 e^{2\alpha\phi_0}(1-\alpha^2) = M^2$. It is given by

$$M_{max} = \frac{1}{2\sqrt{2}H}\left(\frac{1-3\alpha^2}{2(1-\alpha^2)}\right)^{\frac{1-3\alpha^2}{2(1+\alpha^2)}} \tag{5.2.25}$$

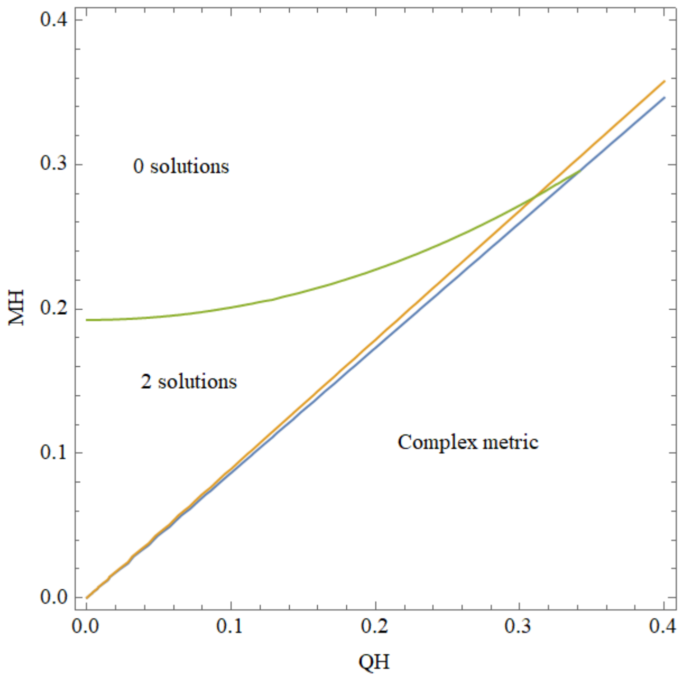

Figure 5.4 – Number of horizons for $\alpha < \frac{1}{\sqrt{3}}$, here illustrated by the value $\alpha = \frac{1}{2}$. The green curve represents $F(r_{0+}) = 0$ and gives an upper bound on the mass. The yellow line represents $Q^2 = (1+\alpha^2)M^2$, that does not play anymore the same role for $\alpha < \frac{1}{\sqrt{3}}$. The blue one is $Q^2 = \frac{M^2}{1-\alpha^2}$. In the region between the green and the blue curves, cosmological and event horizons are present. Below the blue one, r_+, r_- and the metrics become complex valued.

The behaviour of the horizons for the asymptotically de Sitter metric for $\alpha < 1/\sqrt{3}$ is described in figure 5.4 where we have taken, for an explicit illustrative example, $\alpha = 1/2$.

In figure 5.4, the region of the (QH, MH) plane with two horizons shows an upper bound represented by the green curve, $F(r_{0+}) = 0$. On the green curve, the event and cosmological horizons coincide. The lower bound is given by the blue curve, where $Q^2 e^{2\alpha\phi_0}(1-\alpha^2) = M^2$ and the metric is on the verge of becoming complex. Plots of g_{00} reveal that, on this line, the event horizon and the singularity are still far apart. Approaching this line from above (the black hole solution region), we see that the event horizon and the singularity get closer but never touch. Expected solutions with only the cosmological horizon seem to be hidden inside the inaccessible region. The maximal mass for the black hole, corresponding to the crossing point of the green and blue curves are given by $M_{max}H = 2^{\frac{-8}{5}} 3^{\frac{-1}{10}} \simeq 0.3$.

We can compare with the RN-dS black hole solution studied in [201, 24], corresponding here to $\alpha = 0$. In that case the (QH, MH) plane shows a central region with three horizons surrounded by two regions with one horizon. One of them is attained, in parametric space, after the event horizon has reached the cosmological one, and is interpreted as a dS space "eaten" by the black hole. The other, related in [24] to dS-WGC states, is beyond the locus of the coincidence of the inner and event horizons. For $0 < \alpha < 1/\sqrt{3}$, there are two horizons in a central region and zero horizons outside.

Strictly speaking, $r = r_-$ is a zero of g_{00} for all $\alpha \neq 0$ but the inner horizon is traded for a singularity. The instability of Cauchy horizons (that we mentioned in chapter 1 for Reissner-Nordström black hole, but which is also present for Reissner-Nordström de Sitter black hole, see [154]), may provide an additional motivation towards their identification. However, we have observed that for $\alpha \to 0$, while the two cosmological and event horizons tend to their corresponding surfaces in RN-dS, the singularity r_- seems not numerically to coincide exactly with the inner horizon but lies slightly above: it is no more a singularity neither a horizon. In fact, in this limit, the expression of $r_- = r_-(Q, M)$, given in (1.2.6), takes the same form as the RN black hole inner horizon and becomes trivially this surface when $H \to 0$.

Finally, note that for $\alpha = 0$ the parametric equations $F(r_{0\pm}) = 0$ reproduce the relations separating the regions with different horizons in [24]

$$\begin{cases} F(r_{0-}) = 0 \underset{\alpha=0}{\Leftrightarrow} M_-^2 = \frac{1}{54l}[l(l^2+36Q^2) - (l^2-12Q^2)^{\frac{3}{2}}] \\ F(r_{0+}) = 0 \underset{\alpha=0}{\Leftrightarrow} M_+^2 = \frac{1}{54l}[l(l^2+36Q^2) + (l^2-12Q^2)^{\frac{3}{2}}] \end{cases} \tag{5.2.26}$$

with $l = \frac{1}{H}$. It is M_-^2, and thus $F(r_{0-})$, that marks the transition between black holes and naked singularities with cosmological horizon. However, the solution of $F(r_{0-}) = 0$ cannot be used for $0 < \alpha < 1/\sqrt{3}$, as the metric is complex in that region. Note that in all previous literature, because the asymptotically flat metric always shows a naked singularity before turning complex, this region was simply ignored.

5.3 AdS case

Changing the sign of the H^2 terms in the metric (5.1.5) allows to obtain a particular class of dilatonic asymptotically AdS black hole solutions. For completeness, we will investigate the phase space exhibiting the behaviour of the horizons as one varies α, M and Q using the same method as for the de Sitter case.

We start by briefly recalling the Reissner-Nordström AdS case as it will correspond to the $\alpha \to 0$ limit. The time component of the metric is $g_{00}(r) = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} + H^2r^2\right)$. Its roots are given by those of the polynomial $G(r) \equiv H^2r^4 + r^2 - 2Mr + Q^2$. They are two (degenerate in the extremal case) real positive roots as long as

$$M^2 \leq \frac{1}{54}\left(36Q^2 - \frac{1}{H^2} + \frac{(1+12H^2Q^2)^{\frac{3}{2}}}{H^2}\right). \tag{5.3.1}$$

Returning to the case $\alpha \neq 0$, the presence of horizons can be inspected through the study of the zeros of the function:

$$F_{AdS}(r) \equiv r - r_+ + H^2r^3\left(1 - \frac{r_-}{r}\right)^{\frac{3\alpha^2-1}{1+\alpha^2}} = -r\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} g_{00}(r). \tag{5.3.2}$$

This turns out to be much simpler to study than the corresponding dS function F. It is indeed straightforward to see that whenever $r_+ < r_-$, $F_{AdS}(r) > 0$ for all $r \in [r_-, \infty[$ and so the function cannot have zeros. It will prove useful to split F_{AdS} into the sum of the two contributions $A_{AdS}(r) \equiv r - r_+$, a straight line, and $B_{AdS}(r) \equiv -H^2 r^3 \left(1 - \frac{r_-}{r}\right)^{\frac{3\alpha^2-1}{1+\alpha^2}}$ which is always negative. In this way, the problem is again recast in terms of the intersection points of A_{AdS} and B_{AdS}. We split the discussion into three parts depending on the value of α.

$\underline{\boldsymbol{\alpha^2 > \frac{1}{3}}}$ When approaching the singularity, B_{AdS} goes to 0^-. As a consequence, the curves defined by A_{AdS} and B_{AdS} have either one intersection point when $r_+ > r_-$ ($A_{AdS}(r_-) < 0$), or no intersections at all when $r_+ < r_-$ ($A_{AdS}(r_-) > 0$). We conclude that the discriminant between the black hole regime and the naked singularity is given by $r_+ = r_-$ i.e. $Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2$.

$\underline{\boldsymbol{\alpha^2 = \frac{1}{3}}}$ B_{AdS} does not depend on r_-: this is again related to a factorization in the metric as we have seen in the dS case. As such, F_{AdS} now corresponds to the Schwarzschild-AdS polynomial $F_{AdS}(r) = r - r_+ + H^2 r^3$, whose real root is given by

$$r_h = \frac{1}{H}\left(-\frac{\left(\frac{2}{3}\right)^{1/3}}{\left(9r_+H + \sqrt{3}\sqrt{4+27r_+^2H^2}\right)^{1/3}} + \frac{\left(9r_+H + \sqrt{3}\sqrt{4+27r_+^2H^2}\right)^{1/3}}{2^{1/3}3^{2/3}}\right). \tag{5.3.3}$$

There is also two complex conjugate non-physical roots. r_h is real and positive for any value $r_+ \in \mathbb{R}^+$. Accordingly, the condition for the singularity to be shielded by the horizon is just $r_h \geq r_-$, which reads

$$\frac{\left(9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3}\,3^{2/3}} - \frac{\left(\frac{2}{3}\right)^{1/3}}{\left(9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}} \geq \frac{4\hat{Q}^2}{3\left(M + \sqrt{M^2 - \frac{2}{3}\hat{Q}^2}\right)}. \tag{5.3.4}$$

In the $(\hat{Q}, \hat{M})$ space this gives a lower bound on the mass which is a little higher than the asymptotically flat case: $\hat{Q}^2 > \frac{4}{3}\hat{M}^2$, as shown in Figure 5.5 (*left* panel). The $H \to 0$ limit of (5.3.4) is again given by (5.2.23) with the change of sign in front of the H^2 term.

$\underline{\boldsymbol{\alpha^2 < \frac{1}{3}}}$ This is again the most intricate parametric region. Here, B_{AdS} diverges to $-\infty$ when $r \to r_-$. For $r_+ < r_-$, since A_{AdS} is positive for all $r \geq r_-$, and so is the difference $A_{AdS} - B_{AdS}$, no horizon can ever be present.

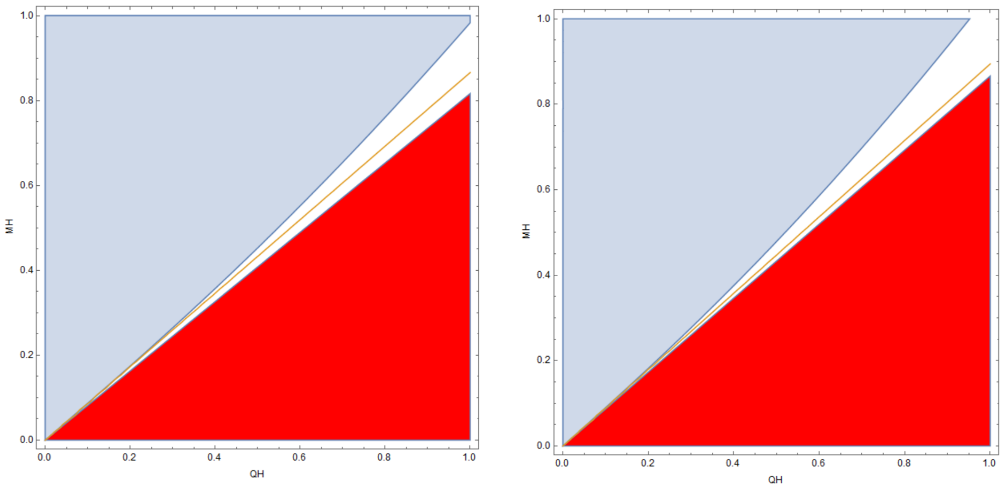

Figure 5.5 – The dilatonic AdS black hole case. The *left panel* describes the phase diagram for the $\alpha^2 = \frac{1}{3}$, the *right panel* shows the explicit example of $\alpha^2 = \frac{1}{4}$ to illustrate the situation for $\alpha^2 < \frac{1}{3}$. In both cases, the blue region corresponds to black hole solutions, the yellow line is the flat-space discriminant between shielded and naked singularities (playing no role here but shown for comparison) and the red region is the inaccessible region where the metric becomes complex.

On the other hand, when $r_+ > r_-$ we have $A_{AdS}(r) < 0$ for all $r \in [r_-, r_+[$, so the combination $A_{AdS} - B_{AdS}$ could result to be negative there. As B_{AdS} is a concave function of r, two roots will be present when A_{AdS} and B_{AdS} intersect, collapsing to one when they are tangent, and zero otherwise. The region of parameters allowing the presence of a horizon, black hole solution region, is obtained as in the dS case by solving the combined equations $F_{AdS}(r) = 0$ and $F'_{AdS}(r) = 0$. The solutions to this system are the same $r_{0\pm}$ found in (5.2.11). In the (r_+H, r_-H) plane, $F_{AdS}(r_{0+})$ is always null or positive, leading to no constraint in practice. As a consequence:

- The condition for the singularity to be shielded can be expressed as $F_{AdS}(r_{0-}) \leq 0$, with the equality being satisfied by extremal black holes with coincident horizons.

This leads to a lower bound on the mass, that lies above the flat-space one ($Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2$), as illustrated in the example of figure 5.5 (*right* panel).

For $\alpha^2 = \frac{1}{3}$, the lower bound on the mass coincides with the one obtained above by simply requiring $r_h - r_- \geq 0$, the set of curves are continuously connected. The presence of black holes with two horizons is a new characteristic that was not present for $\alpha > \frac{1}{\sqrt{3}}$.

As in the dS case, we verify again the equivalence between the limit $\lim_{\alpha \to 0} F_{AdS}(r_{0-}) \leq 0$ and (5.3.1).

Note that the singularity at $r = r_-$ changes its nature: from a space-like one, as it happens when it is behind the single $\alpha^2 > \frac{1}{3}$ horizon, to a time-like one. This is dictated by the derivative of $g_{00}(r)$ that diverges now to $-\infty$ for $r \to r_-^+$. The $\alpha^2 < \frac{1}{3}$ AdS black holes are the only ones where the singularity can be avoided: $r = r_-$ is not in the future light-cone of all the observers that crosses the horizons, g_{00} becomes time-like again before reaching it, as already observed by [132]. Of all the setups we studied, this is the only case where $\lim_{r \to r_-^+} g_{00} < 0$ in a black hole parametric region.

This is similar to what happens in the Reissner Nordstrom AdS metric for the $r = 0$ singularity. Varying α, starting with $\alpha^2 > \frac{1}{3}$, we encounter at $\alpha^2 = \frac{1}{3}$ a transition from Schwarzschild-AdS like black holes, with only one horizon and a space-like singularity, to Reissner-Nordstrom AdS like ones, with two horizons and a time-like singularity. For all $\alpha \neq 0$, the singularity at $r = r_-$ resembles here the singularity at $r = 0$ of $\alpha = 0$. Note that we do not encounter the issue of a complex metric, in contrast with the dS case, as the naked singularity bound is reached for values $M^2 > (1 - \alpha^2)Q^2$.

It can be interesting to look at the first correction to the flat space-time condition $r_- = r_+$ for small H. For $H = 0$, $F(r_{0-}) = 0$ reduces to $r_{0-} = r_+$ which is equivalent to $r_- = r_+$.

In order to find the first term in the expansion in H, we set

$$r_- = r_+ + c r_+^{\gamma+1} H^\gamma + o(H^\gamma), \tag{5.3.5}$$

where the constants c and γ have to be fixed.

From (5.3.5), it is possible to express r_{0-} as $r_{0-} = r_+ + \frac{(1+\alpha^2)}{2(1-\alpha^2)} c r_+^{\gamma+1} H^\gamma + o(H^\gamma)$. Requiring $F(r_{0-}) = 0$ at first order gives:

$$\gamma = \frac{1+\alpha^2}{1-\alpha^2} \qquad \text{and} \qquad \frac{1+\alpha^2}{2(1-\alpha^2)} c + \left[\frac{3\alpha^2-1}{2(1-\alpha^2)} c\right]^{\frac{3\alpha^2-1}{1+\alpha^2}} = 0. \tag{5.3.6}$$

For $\alpha < \frac{1}{\sqrt{3}}$, c is single valued, negative, with the limit $c \to -2$ when $\alpha \to 0$ and $c \to -1$ for $\alpha \to \frac{1}{\sqrt{3}}$.

Plugging the relation between (r_-, r_+) and (Q, M) given by (1.2.6) in (5.3.5), we can look for the corresponding relation $Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2 + bM^{2+\delta}H^\delta$, that defines the boundary of the black hole region. It is possible to determine the constants δ and b: $\delta = \gamma$ and $b = \alpha^2(1+\alpha^2)^{\frac{2}{1-\alpha^2}} c$. Thus, the constraint $F(r_{0-}) = 0$ can be expanded for $H \to 0$ as

$$Q^2 e^{2\alpha\phi_0} = (1+\alpha^2)M^2 + \alpha^2(1+\alpha^2)^{\frac{2}{1-\alpha^2}} c\, M^{\frac{3-\alpha^2}{1-\alpha^2}} H^{\frac{1+\alpha^2}{1-\alpha^2}} + o(H^{\frac{1+\alpha^2}{1-\alpha^2}}). \tag{5.3.7}$$

We see that for $\alpha = \frac{1}{\sqrt{3}}$, it reduces to

$$Q^2 e^{2\alpha\phi_0} = \frac{4}{3} M^2 - \frac{4^3}{3^4} M^4 H^2 + o(H^2). \tag{5.3.8}$$

It is the same equation as for the dS case, with a difference of sign. for $\alpha \to 0$, the power of H tends to 1, but the coefficient in front gives 0. This is coherent with [201], since there is no term in the development in H.

5.4 Test particles in charged dilatonic black hole metric

As we have seen in the previous chapters, it is possible to formulate the WGC, for abelian gauge fields as a condition on long range interaction between charged states.

When adding scalar interactions, it leads to the RFC. The states under consideration can be elementary in the theory, but also solitonic as D-branes. We therefore wish to study the non-relativistic, large distance, leading interactions between the charged black holes. The latter, seen at very large distances and interacting through gravitons, gauge bosons and scalar fields with large wavelengths compared to their typical size, i.e. their horizon radius, look like point particles. The challenge here is to identify the expression of the scalar charge and the associated coupling for these states.

It appears instructive to first consider the simpler case of a test particle submitted to the forces generated by a black hole. Also, taking in our computations the limit $H = 0$, allows to compare with the available results of explicit amplitudes computation.

5.4.1 Large distance action of the dilatonic black holes on a test particle

In our effective theory description, the scalar charge of the point-like particle with respect to a field ϕ appears encoded in the field-dependent mass $m(\phi)$. This in turn will be translated into three-point couplings in field theory context, as we shall discuss later. The action is given by

$$S_m = \int \mathrm{d}\tau \left(-m(\phi)\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + e^{\alpha\phi_0} q A_\mu \dot{x}^\mu\right), \tag{5.4.1}$$

where x^μ represent the particle's coordinates and the dot indicates a derivative with respect to the proper time τ. The last term is the coupling to the abelian gauge field. To be consistent with the previous part of this work, we use for the charge and mass of the particle the geometrized ones, obtained from the physical mass and charge through (1.1.4) and (1.2.8). The equation of motions yield the geodesic equation

$$-m(\phi)\left(\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho\right) + e^{\alpha\phi_0} q F^\mu_{\ \rho}\dot{x}^\rho - \frac{dm(\phi)}{d\phi}\left(\dot{x}^\mu\dot{x}^\rho\partial_\rho\phi - \dot{x}^\rho\dot{x}_\rho\partial^\mu\phi\right) = 0, \tag{5.4.2}$$

where the Γs denote the Christoffel symbols and $F_{\mu\nu}$ is the abelian field strength. The last term comes from the ϕ-dependence of the mass and provides an additional contribution to the geodesics equation that can be interpreted in terms of a scalar force. Here, we are mainly interested by the last term, interpreted in terms of a scalar force.

To study the motion of a test particle in the space-time defined by the metric (5.1.5), we first rewrite the Lagrangian as

$$\mathcal{L} = -m(\phi)\sqrt{f(r)\dot{t}^2 - \frac{\dot{r}^2}{f(r)} - r^2 g(r)\dot{\theta}^2 - r^2 g(r)\sin^2\theta\,\dot{\varphi}^2} - \frac{e^{2\alpha\phi_0} q Q}{r}\dot{t}, \tag{5.4.3}$$

where the gauge field was chosen as $A = \left(-\frac{Q}{r}, 0, 0, 0\right)$, $\tilde{Q}$ being the charge of the black hole defined in (1.1.4). In (5.4.3), we have introduced two functions f and g:

$$\begin{cases} f(r) \equiv \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} \\ g(r) \equiv \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}}, \end{cases} \tag{5.4.4}$$

It corresponds to the solution presented in chapter 1 in order to define the dilatonic WGC.

The spherical symmetry allows us to restrict the analysis to the equatorial plane $\theta = \frac{\pi}{2}$. The two Killing vectors ∂_t and ∂_φ, correspond the two conserved quantities E and L, constants proportional to the energy and angular momentum as measured at infinity, respectively. They are given by:

$$\begin{cases} E = -\frac{\partial \mathcal{L}}{\partial \dot{t}} = m(\phi) f(r)\dot{t} + \frac{e^{2\alpha\phi_0} qQ}{r} \\ L = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m(\phi) r^2 g(r)\dot{\varphi}, \end{cases} \tag{5.4.5}$$

In (5.4.5) we have used the normalization $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$ whose explicit form reads

$$-f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2 g(r)\dot{\varphi}^2 = -1, \tag{5.4.6}$$

or, using (5.4.5),

$$-\frac{1}{m^2(\phi)f(r)}\left(E - \frac{e^{2\alpha\phi_0}qQ}{r}\right)^2 + \frac{\dot{r}^2}{f(r)} + \frac{L^2}{m^2(\phi)r^2 g(r)} = -1. \tag{5.4.7}$$

Restricting to radial paths and null angular momentum L, this gives

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = -f(r) + \frac{1}{m^2(\phi)}\left(E - \frac{e^{2\alpha\phi_0}qQ}{r}\right)^2. \tag{5.4.8}$$

After putting the equation in the form $\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}(r) = 0$, one can read the forces from the $\frac{1}{r}$ coefficient in $V_{\text{eff}}(r)$, the Newtonian approximation being recovered for large distance ($r \gg r_-$). In this limit, using (5.1.5), the leading order of f and m are

$$f(r) = 1 - \frac{1}{r}\left(r_+ + \frac{1-\alpha^2}{1+\alpha^2}r_-\right) + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{5.4.9}$$

$$m^2(\phi) = m^2\left(\phi_0 - \frac{\alpha}{1+\alpha^2}\frac{r_-}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right) = m^2(\phi_0) - \left.\frac{\mathrm{d}m^2}{\mathrm{d}\phi}\right|_{\phi_0}\frac{\alpha}{1+\alpha^2}\frac{r_-}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{5.4.10}$$

Together with the identification in (1.2.5), this gives

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \frac{\frac{E^2}{m_0^2} - 1}{2} + \frac{M}{r} + \frac{E^2}{m_0^2}\frac{\frac{m_0'}{m_0}D}{r} - \frac{E}{m_0}\frac{\frac{e^{\alpha\phi_0}q}{m_0}e^{\alpha\phi_0}Q}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{5.4.11}$$

where the $'$ stands for the derivative with respects to ϕ and the subscripts 0 denote quantities evaluated at $\phi = \phi_0$.n (5.4.11), M, Q and D are the mass, the charge and scalar charge of the black holes expressed in geometrical units related as in (1.2.7). The E/m_0 factors should be seen as relativistic corrections, intrinsically present in the GR framework, and kicking in for $v/c \sim 1$. In the classical limit, the potential V_{pp} felt by a point-particle takes the form

$$V_{pp} = m_0 V_{\text{eff}}(r) = -\frac{m_0 M + m_0' D - e^{2\alpha\phi_0}qQ}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{5.4.12}$$

We have therefore shown that the forces felt by a test particle match the result for the non-relativistic limit of the $2 \to 2$ scattering amplitude between this test particle and a state with the gauge charge, scalar charge and mass of the black hole.

Before turning to the question of how to extend this picture to the black hole itself, we give here the final expression for the second order term in the expansion of $V_{\text{eff}}(r)$

$$\begin{aligned} V^{(2)}_{\text{eff}}(r) = -\frac{1}{2r^2}\Bigg[& \frac{e^{2\alpha\phi_0}q^2}{m_0^2}e^{2\alpha\phi_0}Q^2 - (1-\alpha^2)\left(e^{2\alpha\phi_0}Q^2 - D^2\right) - \frac{1}{2}\frac{E^2}{m_0^2}D^2\frac{{m''_0}^2}{m_0^2} \\ & + \frac{E^2}{m_0^2}\frac{1+\alpha^2}{\alpha}D^2\frac{m'_0}{m} - 4\frac{E}{m_0}\frac{e^{2\alpha\phi_0}q}{m_0}QD\frac{m'_0}{m_0} + 4\frac{E^2}{m_0^2}D^2\left(\frac{m'_0}{m_0}\right)^2\Bigg], \end{aligned} \tag{5.4.13}$$

This generalizes the $1/r^2$ term one finds in the Reissner-Nordström case:

$$V^{(2)}_{\text{eff}}(r) = -\frac{1}{2r^2}\left(\frac{e^{2\alpha\phi_0}q^2}{m^2}e^{2\alpha\phi_0}Q^2 - e^{2\alpha\phi_0}Q^2\right) \tag{5.4.14}$$

that is recovered in the limit $\alpha \to 0$ (thus $D \to 0$), giving a non vanishing contribution even for purely radial motion.

The difference for the dilatonic black hole compared to the Reissner-Nordström solution, for the overall potential V_{eff} is that the sub leading contributions defined above are parts of a formally infinite expansion in powers of $\frac{r_-}{r}$, with r_- the singularity. As we approach it, higher orders will grow and the whole expansion needs to be taken into account. Contrary to Schwarzschild or Reissner-Nordström solutions, there is no fixed-order dominating term whose sign determines whether the overall effective potential is attractive or repulsive around the singularity.

5.4.2 Forces between two point-like states with black holes charges

We will investigate now the interaction between two point-like states both describing the black hole type solutions. These states will be characterized by their mass, their charge and their coupling to the dilaton ϕ. We will pursue this description in region of parameters of the solution even beyond the extremal black hole limit, therefore point-like states not corresponding to black holes anymore, in an attempt to get some indication of what happens where the metric becomes complex.

The question of how to associate the parameters of a dilatonic black hole to a particle state was addressed in [160, 164] for the case of an asymptotically flat space-time solution.

The black hole parameters (say its ADM mass, gauge charge and scalar charge) are defined at infinity. As such, for a point particle to effectively describe this black hole, its charge q, mass $m(\phi)$ and first derivative $m'(\phi)$ observed at infinity must satisfy the conditions

$$\begin{cases} m(\phi_0) = M = \frac{1}{2}\left(r_+ + \frac{1-\alpha^2}{1+\alpha^2} r_-\right) \\ q = Q = \sqrt{\frac{r_+ r_-}{1+\alpha^2}} e^{-\alpha\phi_0} \\ m'(\phi)\big|_{\phi_0} = D = \frac{\alpha}{1+\alpha^2} r_-, \end{cases} \tag{5.4.15}$$

where ϕ_0 is the asymptotic value of ϕ at infinity.

In order to obtain an explicit expression for the scalar charge/scalar coupling of the point-like black hole approximation, we consider the relation (1.2.7) and express it as:

$$D = \frac{\alpha}{1-\alpha^2}\left(M - \sqrt{M^2 - (1-\alpha^2) Q^2 e^{2\alpha\phi_0}}\right) . \tag{5.4.16}$$

We consider that the point-like state lives in a region where $\phi_0 \simeq \phi$, generated by the other (distant) black hole, and therefore has a coupling to the dilaton given by:

$$\frac{\mathrm{d}m}{\mathrm{d}\phi} = \frac{\alpha}{1-\alpha^2}\left(m(\phi) - \sqrt{m^2(\phi) - (1-\alpha^2) q^2 e^{2\alpha\phi}}\right) . \tag{5.4.17}$$

As was shown in [160, 164], the useful parameters to describe the scalar interactions are

$$\gamma(\phi) \equiv \frac{\mathrm{d}}{\mathrm{d}\phi} \ln m(\phi) \quad \text{and} \quad \beta(\phi) \equiv \frac{\mathrm{d}\gamma(\phi)}{\mathrm{d}\phi}. \tag{5.4.18}$$

and the mass $m(\phi)$ can be expanded around a background value $\bar{\phi}$ as

$$m(\phi) = m(\bar{\phi})\left(1 + \gamma(\bar{\phi})(\phi - \bar{\phi}) + \frac{1}{2}\left(\gamma^2(\bar{\phi}) + \beta(\bar{\phi})\right)(\phi - \bar{\phi})^2 + \mathcal{O}\left((\phi - \bar{\phi})^3\right)\right) . \tag{5.4.19}$$

Using (5.4.17) one obtains

$$\begin{cases} \gamma(\phi) = \frac{\alpha}{1-\alpha^2}\left(1 - \sqrt{1 - (1-\alpha^2)\frac{q^2}{m^2(\phi)} e^{2\alpha\phi}}\right) \\ \beta(\phi) = \frac{\alpha^2}{1-\alpha^2}\frac{q^2 e^{2\alpha\phi}}{m^2(\phi)}\left(1 - \frac{\alpha^2}{\sqrt{1-(1-\alpha^2)\frac{q^2}{m^2(\phi)} e^{2\alpha\phi}}}\right) . \end{cases} \tag{5.4.20}$$

With these formulae at hand, we can now extend the analysis of one black hole and a test particle case to the present case with two black holes.

For $\alpha = 1$, it is easy to see that an explicit solution is $m(\phi) = \sqrt{\mu^2 + \frac{q^2 e^{2\phi}}{2}}$, where μ is an integration constant. It is useful for the discussion below to recall that geometrized units have been used so far and that the ϕ field here is dimensionless (see (A.1.26)). In terms of physical quantities, this translates into

$$m(\phi) = \sqrt{\mu^2 + M_P^2 q^2 e^{\sqrt{2}\frac{\phi}{M_P}}}, \tag{5.4.21}$$

where, although we have used the same notation for simplicity, the quantities should now be understood to be the physical ones.

The tree-level t-channel contribution to the $2 \to 2$ scattering amplitude of 2 such states with same charge q and mass $m(\phi)$ reads:

$$\mathcal{A} = \frac{4m^2}{t}\left(q^2 e^{\sqrt{2}\frac{\phi}{M_{Pl}}} - (\partial_\phi m)^2 - \frac{1}{2}\frac{m^2}{M_{Pl}^2}\right)\Bigg|_{\bar{\phi}} = \frac{4m^2}{t}\left(q^2 e^{\sqrt{2}\frac{\phi}{M_{Pl}}} - \frac{1}{2}\frac{M_{Pl}^2}{m^2}q^4 e^{2\sqrt{2}\frac{\phi}{M_{Pl}}} - \frac{1}{2}\frac{m^2}{M_{Pl}^2}\right)\Bigg|_{\bar{\phi}}, \tag{5.4.22}$$

where the bar from now on indicates quantities evaluated at the background value $\bar{\phi}$. The two states being the same, they have the same asymptotic value of ϕ thus $\bar{\phi} = \phi_0$. We however keep the bar notation here.

The amplitude can then be put in the simple form

$$\mathcal{A} = -2\frac{M_P^2}{t}\left(\frac{\bar{m}^2}{M_P^2} - q^2 e^{\sqrt{2}\frac{\bar{\phi}}{M_P}}\right)^2 = -\frac{2\mu^4}{M_P^2\, t} \tag{5.4.23}$$

from which it is straightforward to observe that it vanishes for $q^2 e^{\sqrt{2}\frac{\bar{\phi}}{M_P}} = \frac{\bar{m}^2}{M_P^2}$ $(\mu = 0)$ and is always negative otherwise. The no-force condition is readily seen to correspond, once we revert again to geometrized units, to the black hole extremality

$$q^2 e^{2\bar{\phi}} = 2\bar{m}^2. \tag{5.4.24}$$

As such, for $\alpha = 1$, the leading classical force between two particles with charge q and mass (5.4.21) giving an effective description of a pair of the same black hole is always attractive and vanishes only for extremal states.

Turning now to generic values of α, $\gamma(\phi)$ can be rewritten in terms of physical quantities as

$$\gamma(\phi) = \frac{\alpha}{1-\alpha^2}\left(1 - \sqrt{1 - 2M_P^2(1-\alpha^2)\frac{q^2}{m^2(\phi)}e^{\sqrt{2}\alpha\frac{\phi}{M_P}}}\right). \tag{5.4.25}$$

The tree-level contribution to the force between two states given by the coefficient of the t-channel pole

$$\mathcal{A}^{\text{t-pole}} = 4e^{\sqrt{2}\alpha\frac{\bar{\phi}}{M_P}} q^2 \bar{m}^2 - 2\frac{\bar{m}^4}{M_P^2} - 2\frac{\alpha^2}{(1-\alpha^2)^2}\frac{\bar{m}^4}{M_P^2}\left(1 - \sqrt{1 - 2M_P^2(1-\alpha^2)\frac{q^2}{\bar{m}^2}e^{\sqrt{2}\alpha\frac{\phi}{M_P}}}\right)^2 \tag{5.4.26}$$

The resulting behaviour of $\mathcal{A}$ is represented by the blue curve in figure 5.6. We observe again that the overall force between two particles is always attractive, even beyond the point $q^2 e^{\sqrt{2}\alpha\frac{\bar{\phi}}{M_P}} = \frac{1+\alpha^2}{2}\frac{\bar{m}^2}{M_P^2}$ corresponding to extremality, which is again found to be the only point where the force vanishes. Contrary to the Reissner-Nordstrom case, the dilatonic coupling does not allow repulsive forces beyond extremality: increasing q at fixed m, the scalar force grows at least as strong as the gauge one. As observed in [160] and later shown in [75], in the asymptotic $\phi \to \infty$ limit, the solution to (5.4.17) takes the form $\sqrt{\frac{1+\alpha^2}{2}}\frac{m(\phi)}{M_P} = qe^{\frac{\alpha}{\sqrt{2}}\frac{\phi}{M_P}}$. We note here that it coincides with the

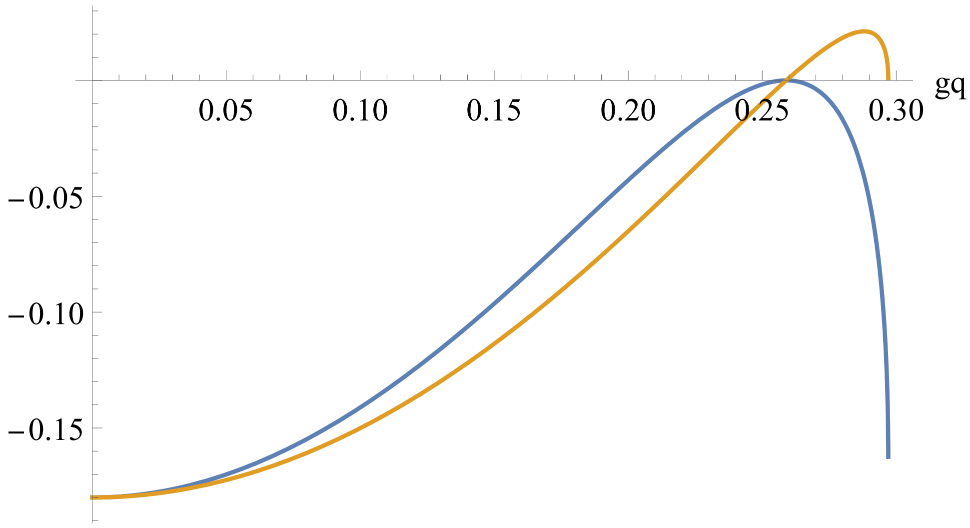

Figure 5.6 – Comparison of the leading overall forces for the scattering of two point-like states approximating far-away black holes in blue, and one Kaluza-Klein-like state and one similar to the black hole in yellow. Here, $\alpha = 0.7$, and $m = 0.3$ in Planck units.

extremal relation.

We now discuss the force felt by a particle of charge q and mass $m(\phi) = m_A e^{\alpha\phi}$ (in geometrized units). Our result for the effective potential for the motion of such a particle in the asymptotically flat background metric, when applied for this case, agrees with [209]. We will address here the attractive or repulsive nature of the leading interaction, the $1/r$ contribution, for the case of our point-like states.

The effective potential takes the form (5.4.12), with $m' = \alpha m$. If we choose the two, particle and black hole, states such that they share the same mass ($m(\bar{\phi}) = M$) and charge ($q = Q$), we find that the overall force, proportional to $M^2 + \alpha M D - e^{2\alpha\bar{\phi}} Q^2$, as shown by the yellow curve in figure 5.6, vanishes at two points:

- the point defining the extremality condition

$$M^2 = \frac{e^{2\alpha\bar{\phi}} Q^2}{1+\alpha^2} \tag{5.4.27}$$

- and the point

$$M^2 = (1-\alpha^2) e^{2\alpha\bar{\phi}} Q^2 \tag{5.4.28}$$

  where the metric is on the verge of becoming complex.

This remains valid as long as $\frac{m}{q} = \frac{M}{Q}$.

Denoting the point-like state approximating a black hole as S, we now turn to the computation of the amplitude $S \to S\phi\phi$ for the emission of a pair of dilatons due to the couplings in (5.4.19). It takes the form

$$\mathcal{A}(S \to S\phi\phi) = -2i \frac{\bar{m}^4}{M_P^2} \bar{\gamma}^2 \left(\frac{1}{t - \bar{m}^2} + \frac{1}{u - \bar{m}^2} \right) - i \frac{\bar{m}^2}{M_P^2} \left(2\bar{\gamma}^2 + \bar{\beta} \right), \tag{5.4.29}$$

where $\bar{\beta}$ is now written in physical units as the other quantities. At threshold, $t, u = -\bar{m}^2$, and this simplifies to

$$\mathcal{A}(S \to S\phi\phi) = -i\frac{\bar{m}^2}{M_P^2}\bar{\beta} = -2i\frac{\alpha^2}{1-\alpha^2}q^2 e^{\sqrt{2}\alpha\frac{\bar{\phi}}{M_P}}\left(1 - \frac{\alpha^2}{\sqrt{1-(1-\alpha^2)2M_P^2\frac{q^2}{\bar{m}^2}e^{\sqrt{2}\alpha\frac{\bar{\phi}}{M_P}}}}\right). \tag{5.4.30}$$

For any finite value $\alpha \geq 1$, the amplitude is finite as well and reduces to $\mathcal{A} = -2iq^2 e^{\sqrt{2}\alpha\frac{\bar{\phi}}{M_P}}$ for $\alpha = 1$. However, for $\alpha < 1$ the amplitude diverges at

$$q^2 e^{\sqrt{2}\alpha\frac{\phi}{M_P}} = \frac{1}{2(1-\alpha^2)}\frac{\bar{m}^2}{M_P^2}, \tag{5.4.31}$$

corresponding to the largest charge before the metric becomes complex. It is also easily verified that the amplitude (5.4.30) vanishes when the no force (extremality) condition (5.4.27) is met.

5.5 Conclusion

We give here an overview of the results obtained above for the existence of horizons in the black hole solution (5.1.5). We also attempt to infer from them new bounds for the dilatonic Weak Gravity Conjecture (dWGC). We will keep separate the discussions about asymptotically flat, AdS and dS space-time. For AdS, this extends previous analysis of the dilatonic black hole solution performed in [103, 132] (and of course for $\alpha = 0$ in [201]). The authors of [132] focused on the region $r_+ \gg r_-$. Our analysis of the dS solution is to our knowledge new. For the WGC in this case we follow [24], where the dS-WGC bound was conjectured to be set by the boundary, in the (QH, MH) plane, between the black hole solution region exhibiting both an event and a cosmological horizon and the naked singularity region with only a cosmological horizon.

The asymptotically AdS BH and AdS-dWGC:

- For $\alpha > \frac{1}{\sqrt{3}}$:

  The black hole solutions exhibit only one (event) horizon. It is located outside the singular surface as long as

  $$Q^2 e^{2\alpha\phi_0} < (1+\alpha^2)M^2 \tag{5.5.1}$$

  and the two surfaces coincide when the inequality turns to equality. The dWGC condition is the same as in the asymptotically flat-space one.

- When $\alpha = \frac{1}{\sqrt{3}}$:

The black hole solutions still possess only one horizon. The coincidence of that horizon with the singularity is no more obtained for $Q^2e^{2\alpha\phi_0} = (1+\alpha^2)M^2$ but for a smaller charge now, saturating the inequality

$$\frac{\left(9\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)+\sqrt{3}\sqrt{4+27\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3}\,3^{2/3}} - \frac{\left(\frac{2}{3}\right)^{1/3}}{\left(9\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)+\sqrt{3}\sqrt{4+27\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}} \geq \frac{4\hat{Q}^2}{3\left(M+\sqrt{M^2-\frac{2}{3}\hat{Q}^2}\right)}. \tag{5.5.2}$$

The expansion for small H (large $L = \frac{1}{H}$, with L the AdS length scale) gives

$$Q^2e^{(2/\sqrt{3})\phi_0} \leq \frac{4}{3}M^2 - \frac{4^3}{3^4}M^4H^2 + o(H^2). \tag{5.5.3}$$

which reproduces the flat space-time case for $H \to 0$.

- For $0 < \alpha < \frac{1}{\sqrt{3}}$:

  The black holes have both an inner and an outer horizon. The extremality condition is now obtained when the two horizons coincide and is expressed as

$$F_{AdS}(r_{0-}) = 0 \tag{5.5.4}$$

  where F_{AdS} is defined by (5.3.2) and (5.2.11). Here the two coincident horizons are located outside the singularity. The expansion for small H gives

$$Q^2e^{2\alpha\phi_0} = (1+\alpha^2)M^2 + \alpha^2(1+\alpha^2)^{\frac{2}{1-\alpha^2}}cM^{\frac{3-\alpha^2}{1-\alpha^2}}H^{\frac{1+\alpha^2}{1-\alpha^2}} + o(H^{\frac{1+\alpha^2}{1-\alpha^2}}), \tag{5.5.5}$$

  where c is defined in (5.3.6). From this expansion one can see that the condition tends to the flat space one in the limit $H \to 0$. Black hole states solve $F_{AdS}(r_{0-}) < 0$.

- For $\alpha = 0$:

  This is the well studied case of charged AdS without dilaton.

A WGC bound can be identified as the requirement of the presence of a state with a charge Q and a mass M that verifies an inequality opposite to the ones above respected by black holes with a horizon.

In [155], the classical decay of (near)-extremal solutions through the charged superradiance mechanism was used to obtain a WGC bound in asymptotically AdS spacetime with a dilaton. The conjecture requires the existence of a state with mass m and charge q solving

$$q \geq \frac{\Delta}{\mu}, \qquad \text{with} \qquad \Delta = \frac{3H}{2} + \sqrt{\frac{9H^2}{4} + m^2}, \qquad \mu = \frac{Q}{r_+}. \tag{5.5.6}$$

where Δ is the minimum frequency of a scalar perturbation in AdS and μ the difference between the component A_t of the gauge field at infinity and on the horizon for extremal solutions. The condition for the onset of superradiance was obtained considering the leading order in the horizon radius for small black hole and the equation $Q^2e^{2\alpha\phi_0} = (1+\alpha^2)M^2$ to define extremal states. With these assumptions, eq. (5.5.6) reads

$$q \geq \Delta\sqrt{1+\alpha^2}. \tag{5.5.7}$$

Here, for $\alpha \leq 1/\sqrt{3}$, we found that the extremality condition receives corrections. Accordingly, the bound from (5.5.6) becomes

$$q \geq \frac{r_{0-}}{\sqrt{r_+ r_-}}\Delta\sqrt{1+\alpha^2}, \tag{5.5.8}$$

where r_+ and r_- are related by the condition $F_{AdS}(r_{0-}) = 0$, $r_{0-} = r_{0-}(r_+, r_-)$. Again, F_{AdS} is defined by (5.3.2) and r_{0-} in (5.2.11). For $r_+ \geq r_-$, and thus for extremal solutions, $r_{0-} < \sqrt{r_+ r_-}$, so that the bound (5.5.8) is weaker than (5.5.7). Using (5.2.11) and the expansion of the extremality condition for small H (5.3.5), this gives at leading order:

$$q \geq \Delta\sqrt{1+\alpha^2}\left(1 + \frac{\alpha^2 c}{1-\alpha^2} r_+^\gamma H^\gamma + o(r_+^\gamma H^\gamma)\right). \tag{5.5.9}$$

where $\gamma = (1+\alpha^2)/(1-\alpha^2)$ and c is a constant solution of the equation given in (5.3.6). Note that the expression of the minimum frequency in AdS might also receive which we expect to be sub-leading (for RN-AdS, we have $\omega = H\Delta + o(r_h H^2)$).

The asymptotically dS-BH and dS-dWGC

- For $\alpha > \frac{1}{\sqrt{3}}$, the condition for the event horizon to coincide with the singularity is given by $Q^2e^{2\alpha\phi_0} = (1+\alpha^2)M^2$. It is again the same as in both asymptotically flat and AdS space.

- When $\alpha = \frac{1}{\sqrt{3}}$, the extremal solution solutions solve

$$\frac{\left(\frac{2}{3}\right)^{1/3} e^{-i\pi/3}}{\left(-9\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right) + \sqrt{3}\sqrt{-4 + 27\left(\hat{M} + \sqrt{\hat{M}^2 - \frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}$$

$$+\frac{e^{i\pi/3}\left(-9\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)+\sqrt{3}\sqrt{-4+27\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}\right)^{1/3}}{2^{1/3}\,3^{2/3}}$$
$$=-\frac{4\hat{Q}^2}{3\left(\hat{M}+\sqrt{\hat{M}^2-\frac{2}{3}\hat{Q}^2}\right)}. \tag{5.5.10}$$

This condition can be expanded for small H as

$$Q^2e^{(2/\sqrt{3})\phi_0}=\frac{4}{3}M^2+\frac{4^3}{3^4}M^4H^2+\mathcal{O}(M^6H^4), \tag{5.5.11}$$

showing that it allows for slightly greater charges than the corresponding flat space limit. It goes to the flat space-time condition for $H\to 0$.

- For $0<\alpha<\frac{1}{\sqrt{3}}$:

  For a real valued metric, one has always either two horizons or, trivially for huge masses, a naked singularity. We have found no region of parameters with a (real) metric exhibiting only a naked singularity with a cosmological horizon.

  Note that in both the $\alpha\to 0$ and $\alpha\to\frac{1}{\sqrt{3}}$ limits, we retrieve the $\alpha=0$ and $\alpha=\frac{1}{\sqrt{3}}$ transition from the black hole to a naked singularity with the cosmological horizon happening for $F(r_{0-})=0$ defined by (5.2.10) and (5.2.11). We discuss further this case below.

- For $\alpha=0$

  The phases of the dS-RN black hole metric are described in details in [201, 24]. The condition for the existence of the black hole is

$$Q^2\leq M^2+M^4H^2+\mathcal{O}(M^6H^4) \tag{5.5.12}$$

  with $\quad M^2H^2\leq\frac{2}{27}.$

The appearance of a complex valued metric leaves the lower bound on the black holes' masses for the $0<\alpha<\frac{1}{\sqrt{3}}$ asymptotically dS solutions undefined. Because previous literature focused on the asymptotically flat metric, and this one always shows a naked singularity before reaching the complex valued metric region, this was not investigated. Here, however, the condition for metric becoming complex,

$$Q^2e^{2\alpha\phi_0}\geq\frac{M^2}{1-\alpha^2} \tag{5.5.13}$$

can be reached within the domain of the black hole solution region and represents a new bound. We might consider that (5.5.13) represents a new WGC bound for this domain of dilaton couplings but further investigation is needed to confirm this.

CHAPTER 6

Spin 3/2 and gravitational waves

6.1 Introduction

The existence of gravitational waves was postulated a short time after the first article on General Relativity, but it took quite a century to discover it experimentally [1, 2]. The main reason explaining the long time between the theoretical prediction and its experimental confirmation is the weakness of the gravitational waves signal. This feature, and so the need to a very precise and complex experiment which can detect a tiny signal, explains also that we can detect, today, only astrophysical signals. These astrophysical signals are the merge of two heavy objects, as black holes, or neutron stars. The detection of this signal with such sources are also a successful test of General Relativity [66, 219], since the shape and the frequency of the signal corresponded with a great precision to what was expected by the General Relativity computation. When the two objects are too close, the action of the gravity is too strong and one need some numerical computation. The simulations are sufficiently precise to give quite a good approximation of the parameters of the black holes or the neutron stars (masses, spins before and after the merge). The fact that we can derive physical parameters from the observation of this gravitational waves is an incentive to go beyond these astrophysical signals, and try to observe cosmological signals from source of the early Universe, for example [210]. The problem is the same as for the astrophysical sources, that is to say the strength of the signal; but some questions are specific to this signal, as the frequency. The LIGO/VIRGO collaboration was designed to have the best precision at the frequency of black holes merges. This kind of experiment has not the sufficient precision to detect signals with an average frequency away of the range of astrophysical sources (close to 100 Hz). The question is to build some experiment with such a precision. There are today some proposals of apparatus like [10], which can detect corresponding gravitational waves from cosmological sources at low frequency; but it exists cosmological sources that lead to signals at higher frequencies. Example of such gravitational waves are the ones produced during the preheating era [168, 213]. Since the current experiments cannot reach a sufficient sensibility in the range considered, new experiments need to be designed to detect them (e.g. [206, 131, 142, 207, 199, 157]).

One of the interests of the gravitational waves is their sensitivity to process we can't look at with current signals and experiments. In order to illustrate this, we will focus on a specific type of particle. For many years, and until quite recently, we have discovered particles with spin 0, 1/2, 1 and 2. But we haven't seen yet fundamental spin-3/2 particles, and the question is can we detect if such particles exist or not? We know that composite states are produced as hadrons, with a total spin of 3/2. A simple question to begin with is so is there any possibility to construct a simple theory of spin-3/2 interacting with known particles, in order to be detected easily? One can try, for example, to couple this particle to an electromagnetic field. Doing this, a difficulty will appear, known as Velo-Zwanziger problem [218], which corresponds to the fact that the theory loses causality because the equations of motion have solutions which are supraluminal. The Lagrangian leading to equation of motion is not physical, and it is not a good description for this particle. This problem is quite general when looking at spin 3/2, and is difficult to settle.

In order to solve this problem, a solution could be to look at non-minimal couplings, and try to couple to others forces. One is then led to either consider nonminimal couplings of spin-3/2 states with gravitational interactions, and with no electromagnetic interactions. In a supersymmetric theory, it is well known that the longitudinal modes can be understood as due to a super-Higgs mechanism. If one looks at a nonunitary gauge and in the supersymmetric phase, the longitudinal mode is a well-behaved fermion that has a causal behaviour. So it seems thanks to the coupling between the spin-3/2 and the gravity one can escape the Velo-Zwanziger problem. Since the supersymmetry is used in order to eliminate the causality violation, supergravity and string theory are safe of this problem, see for example [195, 196]. In such theories, the spin-3/2 is called the gravitino, and it's this particle we will look at in this work, with the Rarita-Schwinger Lagrangian to describe it.

But one problem with the gravitino is that such a particle has only gravitational interactions. Since there is no interaction with others particles of the Standard Model, it is difficult if not impossible to produce it in collider experiments ; besides, its mass can be quite large (if one considers the case where the longitudinal modes are coming from a super-Higgs mechanism, the mass of the gravitinos is linked to the scale of the supersymmetry breaking, which can be quite large). So even if it can be produced, we are not sure to find it in the current or next colliders. On the other way, we know that it is quite difficult to discriminate a particle with only gravitational interactions with only astrophysical observations. The quest of the dark matter illustrates this point perfectly. So we have to search for experiments and observations that are sensitive to such gravitational interactions, and in which the spin have a detectable impact. Since the gravitational waves are a door to new signals and processes, we can try to see if there is a production process of gravitational waves with some features specific to the spin of the particles involved.

As we have said, there are many ways of producing gravitational waves, and one can find production processes on all the spectrum, with different shape and strength.

In this work, we try to find an observation where the spin of the particles plays an important role, so we will focus in one way of producing gravitational waves from systems of quantum gases with large anisotropic stress tensor. These quantum gases produce stochastic gravitational waves [116, 69]. One question to ask is when and where such gases with a peculiar stress tensor can be present in the history of the Universe. One possibility is the preheating. During this phase of the Universe, particles are violently produced far from thermal equilibrium [165], and this leads to huge non diagonal term in the stress tensor, and so to the production of gravitational waves. What kind of particles are produced during this phase? There is no natural restriction which can occur, and so we can infer that all types of elementary particles are created. Previous studies have focused on bosonic sources including both scalars and gauge bosons [101, 97, 98] as well as Dirac spin-1/2 fermions [105, 115]. In this work, we are interested to see if such signal is possible with spin-3/2 particles, and so we will study the gravitational wave signals coming from non-adiabatic spin-3/2 gases.

When we study this gravitational waves production from decay of particle during preheating, we can investigate the following features: shape, amplitude, and peak frequency. The question is always: can these quantities be sensitive to the spin of the particles produced? We have so to answer to two questions: Does the signal coming from spin-3/2 production is different from what we obtain with others spins, and is these signal characteristics are detectable for future experiments ? The interest of this work is double. As we have already said, we haven't detected signal of supersymmetry in colliders yet. Since the energy in the early Universe is much larger than the one we can achieve in current colliders, looking for signatures of spin-3/2 fundamental particles in the early Universe could answer at the existence or not of the supersymmetry. But the study of gravitational waves from spin-3/2 particles can also give us clues in the quest of the role of supersymmetry in the early Universe cosmology.

This chapter will be organized as follows. Section 2 reviews the basics facts about Rarita-Schwinger description of spin-3/2 fields, in order to be able to compute the stress energy tensor. Section 3 presents the computation of the spectrum of energy density of gravitational waves per frequency interval. It contains the main results of this work: the master formula for the estimate of produced gravitational waves from both the transverse and longitudinal modes. We show an enhancement of the latter compared to expected signal from spin-1/2 fermions. A quantitative evaluation requires explicit models where wave functions of the produced spin-3/2 states can be computed. We illustrate our results in a simple model in section 4. Our results are briefly summarized in the conclusion.

6.2 The Rarita-Schwinger fields

Before studying the production of the gravitational waves, we need to specify our description of the spin-3/2, and to derive some quantities useful for the rest of the computation. In order to describe a spin-3/2 field, one starts with a field ψ_μ in the spinor-vector representation of the Lorentz group that obeys a Dirac equation. Using representations of $SU(2)_L \times SU(2)_R$ for Weyl spinors, this is obtained from the tensor product of spin representations as

$$(\frac{1}{2},\frac{1}{2})\otimes(\frac{1}{2},0)=\frac{1}{2}\oplus(1\otimes\frac{1}{2})=\frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}\,, \tag{6.2.1}$$

We have so in this description one spin-3/2, but also two spin 1/2. To get rid of these degrees of freedom, two constraints have to be imposed. These constraints project out the two additional spin-1/2 representations. In flat space time, the spin-3/2 field ψ_μ obeys then the equations

$$\begin{aligned}(i\partial\!\!\!/ - m_{3/2})\psi_\mu &= 0, &(6.2.2)\\ \gamma^\mu\psi_\mu &= 0, &(6.2.3)\\ \partial^\mu\psi_\mu &= 0. &(6.2.4)\end{aligned}$$

We have so an equation of motion and two projective equations. These three equations can be obtained from the Rarita-Schwinger Lagrangian

$$\mathcal{L}=-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma-\frac{1}{4}m_{3/2}\bar{\psi}_\mu[\gamma^\mu,\gamma^\nu]\psi_\nu\,. \tag{6.2.5}$$

As usual, we will define the corresponding field in the momentum space. The solution $\tilde{\psi}^\mu$ to the above equations of motion reads:

$$\tilde{\psi}^\mu_{\vec{p},\lambda}=\sum_{s=\pm1,l=\pm1,0}\langle 1,\frac{1}{2},l,\frac{s}{2}|\frac{3}{2},\lambda\rangle\mathbf{u}_{\vec{p},\frac{s}{2}}\epsilon^\mu_{\vec{p},l}, \tag{6.2.6}$$

where $\langle 1,\frac{1}{2},l,\frac{s}{2}|\frac{3}{2},\lambda\rangle$ are the Clebsch-Gordan coefficients in the decomposition (6.2.1) in the standard notation. The $\epsilon^\mu_{\vec{p},l}$ and $u_{\vec{p},\frac{s}{2}}$ are normalized solutions of massive spin-1 and spin-1/2 fields equations. Explicit expressions of the decomposition can be found in [30, 181, 51]. In the rest of the computation, we will need the stress energy tensor of the spin-3/2 field. In order to obtain an easier formula, we will use the identity

$$\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\sigma=-i\gamma^{[\mu\nu\rho]}. \tag{6.2.7}$$

The Lagrangian (6.2.5) can be so rewritten as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\bar{\psi}_\mu(i\partial\!\!\!/ - m_{3/2})\psi^\mu-\frac{i}{2}\bar{\psi}_\mu(\gamma^\mu\partial^\nu+\gamma^\nu\partial^\mu)\psi_\nu\\ &+ \frac{i}{2}\bar{\psi}_\mu\gamma^\mu\partial\!\!\!/\gamma^\nu\psi_\nu+\frac{1}{2}m_{3/2}\bar{\psi}_\mu\gamma^\mu\gamma^\nu\psi_\nu. \end{aligned}\tag{6.2.8}$$

We can now apply the formula to compute the stress tensor, which gives us

$$\begin{aligned}T_{\alpha\beta} &= \frac{e_{c\alpha}}{2e}\frac{\delta(e\mathcal{L})}{\delta e_c^\beta}+(\alpha\leftrightarrow\beta)\\ &= \frac{i}{4}\bar{\psi}_\mu\gamma_{(\alpha}\partial_{\beta)}\psi^\mu-\frac{i}{4}\bar{\psi}_\mu\gamma_{(\alpha}\partial^\mu\psi_{\beta)}+h.c.. \end{aligned}\tag{6.2.9}$$

In the last step we have used the equations of motion and constraints (6.2.2) - (6.2.4) to eliminate irrelevant terms, and simplify again the form of this stress tensor. Finally, we will have to use the quantization of the spin-3/2. The gravitino is a Majorana fermion, and so we can quantize it as

$$\psi^{\mu}(\vec{x},t) = \sum_{\lambda=\pm\frac{3}{2},\pm\frac{1}{2}} \int \frac{d\vec{p}}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \{\hat{a}_{\vec{p},\lambda}\tilde{\psi}^{\mu}_{\vec{p},\lambda}(t) + \hat{a}^{\dagger}_{-\vec{p},\lambda}\tilde{\psi}^{\mu C}_{\vec{p},\lambda}(t)\}, \tag{6.2.10}$$

where the annihilation and creation operators are time independent and satisfy

$$\{\hat{a}_{\vec{p},\lambda}, \hat{a}^{\dagger}_{\vec{p}',\lambda'}\} = (2\pi)^3 \delta_{\lambda,\lambda'} \delta^{(3)}(\vec{p}-\vec{p}'). \tag{6.2.11}$$

Now that we have defined all the quantities we need, we can focus on the gravitational waves production.

6.3 Gravitational wave production

Our goal here is to compute the spectrum of energy density of gravitational waves produced by a gas of spin-3/2 states. In the rest of this work we will consider wavelengths in the subhorizon limit, and so we will neglect the effects of curvature and torsion, which is normally different from 0 in the presence of fermions.

6.3.1 General results

We will begin by the description of the gravitational waves. Since the gravitational waves we look for are produced during the preheating, for the propagation of this waves we need to take into account the expansion of the Universe. We will describe the gravitational waves as linear tensor perturbations, here in the transverse-traceless (TT) gauge, of the Friedman-Robertson-Walker (FRW) metric,

$$ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j], \tag{6.3.1}$$

where τ is the conformal time. Since we look for linear perturbations, in order to obtain the gravitational wave equation of motion, we just need to look at the linear perturbation part of Einstein equations

$$\ddot{h}_{ij} + 2\mathcal{H}\dot{h}_{ij} - \nabla h_{ij} = 16\pi G \Pi^{TT}_{ij}, \tag{6.3.2}$$

where the dot (.) stands for the derivative with respect to the conformal time τ, $\mathcal{H} = \frac{\dot{a}}{a}$ is then the comoving Hubble rate, and Π^{TT}_{ij} is the TT part of the anisotropic stress tensor, since we work in this special gauge. This projector can be quite difficult to apply since it is non local. To avoid this complex manipulation, we perform a Fourier transform of the stress tensor $T_{\mu\nu}$ in terms of comoving wave number $\vec{k}$. Then $-\nabla$ gives $k^2 = |\vec{k}|^2$ and we can write

$$\Pi^{TT}_{ij}(\vec{k},t) = \Lambda_{ij,lm}(\hat{\vec{k}})(T^{lm}(\vec{k},t) - \mathcal{P}g^{lm}), \tag{6.3.3}$$

where $\mathcal{P}$ is the background pressure and $\Lambda_{ij,lm}$ is the TT projection tensor which is a function of the comoving wave number $\vec{k}$:

$$\Lambda_{ij,lm}(\hat{\vec{k}}) \equiv P_{il}(\hat{\vec{k}})P_{jm}(\hat{\vec{k}}) - \frac{1}{2}P_{ij}(\hat{\vec{k}})P_{lm}(\hat{\vec{k}}), \qquad P_{ij}(\hat{\vec{k}}) = \delta_{ij} - \hat{\vec{k}}_i\hat{\vec{k}}_j. \tag{6.3.4}$$

We assume the stochastic gravitational background to be isotropic, stationary, and Gaussian, therefore completely specified by its power spectrum. For the subhorizon modes $k \gg \mathcal{H}$, the spectrum of energy density per logarithmic frequency interval can be written as [97]

$$\frac{d\rho_{GW}}{d\text{log}k}(k,t) = \frac{2Gk^3}{\pi a^4(t)}\int_{t_I}^{t} dt' \int_{t_I}^{t} dt'' a(t')a(t'')\text{cos}[k(t'-t'')]\Pi^2(k,t',t''), \tag{6.3.5}$$

where $\Pi^2(k,t',t'')$ is the unequal-time correlator of Π_{ij}^{TT} defined as

$$\langle \Pi_{ij}^{TT}(\vec{k},t)\Pi^{TTij}(\vec{k}',t')\rangle \equiv (2\pi)^3\Pi^2(k,t,t')\delta^{(3)}(\vec{k}-\vec{k}'), \tag{6.3.6}$$

and $\langle ... \rangle$ denotes ensemble average. It is this object that we will characterized in the following.

The equation (6.3.5) is an integral over time. To make it non-zero with massive particles, we expect the time dependence of the wave function to vary nonadiabatically with frequencies which we discuss in the next section. In order to neglect the curvature, we will restrict ourselves to a situation where $m_{3/2} \gg \mathcal{H}$. This hypothesis is useful, as we can use the flat limit quantization of the spin 3/2 (6.2.10) that we wrote in the previous section. Always in order to simplify the computation, we will put the time dependence only in the spinor part, writing the spinor wave functions as functions of time. We will keep the vector polarisations $\epsilon^{\mu}_{\vec{p},l}$ constant. We can write the coefficients in front of the annihilation and creation operators appearing in (6.2.10) under the form

$$\tilde{\psi}^{\mu}_{\vec{p},\lambda}(t) = \sum_{s=\pm1,l=\pm1,0} \langle 1,\frac{1}{2},l,\frac{s}{2}|\frac{3}{2},\lambda\rangle \epsilon^{\mu}_{\vec{p},l}\mathbf{u}^{(|\lambda|)}_{\vec{p},\frac{s}{2}}(t), \tag{6.3.7}$$

where we defined

$$\mathbf{u}^{(|\lambda|)T}_{\vec{p},\frac{s}{2}}(t) = \begin{pmatrix} u^{(|\lambda|)}_{\vec{p},+}(t)\chi_s^T(\vec{p}) \\ s\, u^{(|\lambda|)}_{\vec{p},-}(t)\chi_s^T(\vec{p}) \end{pmatrix}, \tag{6.3.8}$$

expressed in terms of the (scalar) wave function $u^{(|\lambda|)}_{\vec{p},\pm}(t)$ and the two-component normalized eigenvectors $\chi_s(\vec{p})$ of the helicity operator. Now that we have defined this wave function, we can try to see if this is well defined and leads to correct value for the occupation number for example.

In order to do this, we first consider the Hamiltonian of the fields, which is the space integral of the T^{00} component of the stress tensor (6.2.9),

$$\begin{aligned} H(t) &= \int d\vec{x}\, T^{00}(\vec{x},t) \\ &= \int d\vec{x}\, \frac{i}{4}\bar{\psi}_{\mu}(\vec{x},t)\gamma^0\partial_t\psi^{\mu}(\vec{x},t) + h.c. \\ &= \int d\vec{x}\, \frac{i}{4}\bar{\psi}^{(\frac{1}{2})}(\vec{x},t)\gamma^0\partial_t\psi^{(\frac{1}{2})}(\vec{x},t) + \frac{i}{4}\bar{\psi}^{(\frac{3}{2})}(\vec{x},t)\gamma^0\partial_t\psi^{(\frac{3}{2})}(\vec{x},t) + h.c., \end{aligned} \tag{6.3.9}$$

where in the second line the second term of (6.2.9) vanishes since we can do the integral by part, leading to the constraint (6.2.4). In the last line, we used the property $\epsilon^{\mu}_{\vec{p},l}\epsilon^{*}_{\mu\vec{p},l'} =$

$\delta_{l,l'}$ and $\chi_s^\dagger(\vec{p})\chi_{s'}(\vec{p}) = \delta_{s,s'}$. The two spinors are defined as

$$\psi^{(|\lambda|)}(\vec{x},t) = \sum_{s=\pm1} \int \frac{d\vec{p}}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \{\hat{a}_{\vec{p},\lambda} \mathbf{u}^{(|\lambda|)}_{\vec{p},\frac{s}{2}}(t) + \hat{a}^\dagger_{-\vec{p},\lambda} \mathbf{v}^{(|\lambda|)}_{\vec{p},\frac{s}{2}}(t)\}. \tag{6.3.10}$$

Substituting (6.3.10) into (6.3.9) does not give a diagonal form in terms of annihilation and creation operators. Thus the quantity from this operators are not well defined. To settle this problem we need to perform the Bogoliubov transformation

$$\hat{\hat{a}}_{\vec{p},\lambda}(t) \;=\; \alpha^{(|\lambda|)}_{\vec{p}}(t)\, \hat{a}_{\vec{p},\lambda} + \beta^{(|\lambda|)}_{\vec{p}}(t)\, \hat{a}^\dagger_{-\vec{p},\lambda},$$

in order to make the Hamiltonian (6.3.9) diagonal,

$$H(t) = \int \frac{d\vec{p}}{(2\pi)^3} \sqrt{m_{3/2}^2 + p^2} \sum_{\lambda=\pm\frac{1}{2},\pm\frac{3}{2}} \hat{\hat{a}}^\dagger_{\vec{p},\lambda}(t)\, \hat{\hat{a}}_{\vec{p},\lambda}(t), \tag{6.3.11}$$

where $p = |\vec{p}|$ and $\alpha^{(|\lambda|)}_{\vec{p}}(t)$, $\beta^{(|\lambda|)}_{\vec{p}}(t)$ are complex numbers satisfying $|\alpha^{(|\lambda|)}_{\vec{p}}(t)|^2 + |\beta^{(|\lambda|)}_{\vec{p}}(t)|^2 = 1$. In the Heisenberg picture, the expectation value is defined by projecting the time-dependent operator on the initial vacuum $|0\rangle$ that corresponds to vanishing number density. Using $n^{(\lambda)}_{\vec{p}}(t) = \hat{\hat{a}}^\dagger_{\vec{p},\lambda}(t)\, \hat{\hat{a}}_{\vec{p},\lambda}(t)$ leads to the occupation number

$$\begin{aligned} \langle 0|n^{(\lambda)}_{\vec{p}}(t)|0\rangle &= |\beta^{(|\lambda|)}_{\vec{p}}(t)|^2 \\ &= \frac{\sqrt{m_{3/2}^2+p^2} - p\,\mathrm{Re}(u^{(|\lambda|)*}_{\vec{p},+}(t)\, u^{(|\lambda|)}_{\vec{p},-}(t)) - m_{3/2}\,(1 - |u^{(|\lambda|)}_{\vec{p},+}(t)|^2)}{2\sqrt{m_{3/2}^2+p^2}}. \end{aligned} \tag{6.3.12}$$

We also get a time-dependent physical vacuum satisfying

$$\hat{\hat{a}}_{\vec{p},\lambda}(t)|0_t\rangle = 0. \tag{6.3.13}$$

Now that we defined proper creation and annihilation operator, we can come back to our problem and consider the sources of the gravitational waves. Plugging the mode decomposition (6.2.10) into (6.3.3) leads to

$$\Pi^{TT}_{ij}(\vec{k},t) \;=\; \frac{1}{4}\Lambda_{ij,lm} \int \frac{d\vec{p}}{(2\pi)^3} \{\hat{\Pi}^{lm}(\vec{p},t) + h.c.\}, \tag{6.3.14}$$

where $\vec{k}$ is the momentum mode of the gravitational wave and

$$\begin{aligned} \hat{\Pi}^{lm}(\vec{p},t) \;=\; & \left[\hat{a}_{-\vec{p},\lambda}\bar{\tilde{\psi}}^{\mu C}_{\vec{p},\lambda} + \hat{a}^\dagger_{\vec{p},\lambda}\bar{\tilde{\psi}}^{\mu}_{\vec{p},\lambda}\right] \gamma^{(l}\partial^{m)} \left[\hat{a}_{\vec{p}+\vec{k},\lambda'}\tilde{\psi}_{\mu\vec{p}+\vec{k},\lambda'} + \hat{a}^\dagger_{-\vec{p}-\vec{k},\lambda'}\tilde{\psi}^C_{\mu\vec{p}+\vec{k},\lambda'}\right] \\ & - \left[\hat{a}_{-\vec{p},\lambda}\bar{\tilde{\psi}}^{\mu C}_{\vec{p},\lambda} + \hat{a}^\dagger_{\vec{p},\lambda}\bar{\tilde{\psi}}^{\mu}_{\vec{p},\lambda}\right] \gamma^{((l}\partial_\mu \left[\hat{a}_{\vec{p}+\vec{k},\lambda'}\tilde{\psi}^{m)}_{\vec{p}+\vec{k},\lambda'} + \hat{a}^\dagger_{-\vec{p}-\vec{k},\lambda'}\tilde{\psi}^{m)C}_{\vec{p}+\vec{k},\lambda'}\right]. \end{aligned} \tag{6.3.15}$$

Notice that (6.3.4) implies that $\Lambda_{ij,lm}k_l = \Lambda_{ij,lm}k_m = 0$, which removes the linear k dependence from ∂_m in the first line of (6.3.15), similar to the case of scalars or spin-1/2 fermions [105]. However, in the second line of (6.3.15), ∂_μ leads to nonvanishing k_μ contracting with $\epsilon^\mu_{\vec{p},m}$, which is an important property of spin-3/2 gases.

The annihilation and creation operators lead to $2^4 = 16$ combinations among which only one contributes to nontrivial results:

$$\langle 0|\hat{a}_{-\vec{p},\lambda}\hat{a}_{\vec{k}+\vec{p},\kappa}\hat{a}^{\dagger}_{\vec{q},\lambda'}\hat{a}^{\dagger}_{\vec{k}'-\vec{q},\kappa'}|0\rangle = (2\pi)^6\delta^{(3)}(\vec{k}-\vec{k}')\{\delta^{(3)}(\vec{k}+\vec{p}-\vec{q})\delta_{\lambda,\kappa'}\delta_{\kappa,\lambda'} - \delta^{(3)}(\vec{p}+\vec{q})\delta_{\lambda,\lambda'}\delta_{\kappa,\kappa'}\}. \quad (6.3.16)$$

The two terms inside the brackets in (6.3.16) come from the Majorana nature, assumed for the spin-3/2 fields, and lead to the same results.

It is convenient to define

$$\vec{p}' = \vec{p} + \vec{k}. \quad (6.3.17)$$

We now turn to the unequal-time correlator and write it in terms of 4-spinors,

$$\Pi^2(k,t,t') = 2\int \frac{d\vec{p}}{(2\pi)^3}\left[\bar{\mathbf{v}}^{(|\lambda|)}_{\vec{p},\frac{s}{2}}(t)\Delta^{\lambda s,\lambda' s'}_{ij}(t)\mathbf{u}^{(|\lambda'|)}_{\vec{p}',\frac{s'}{2}}(t)\right]\left[\bar{\mathbf{u}}^{(|\lambda'|)}_{\vec{p}',\frac{r'}{2}}(t')\Delta^{\lambda r,\lambda' r'}_{ij}(t')^*\mathbf{v}^{(|\lambda|)}_{\vec{p},\frac{r}{2}}(t')\right], \quad (6.3.18)$$

where $\mathbf{v}^{(|\lambda|)}_{\vec{p},\frac{r}{2}} = i\gamma^0\gamma^2\bar{\mathbf{u}}^{|\lambda|T}_{\vec{p},\frac{r}{2}}$ and

$$\begin{aligned}\Delta^{\lambda s,\lambda' s'}_{ij}(t) &= \frac{1}{4}\Lambda_{ij,lm}\langle 1,\frac{1}{2},r,\frac{s}{2}|\frac{3}{2},\lambda\rangle\langle 1,\frac{1}{2},r',\frac{s'}{2}|\frac{3}{2},\lambda'\rangle \times \\ &\{2\epsilon_{\mu\mathbf{p},r}\epsilon^{\mu}_{\mathbf{p}',r'}\, p^{(l}\gamma^{m)} - \epsilon_{\mu\mathbf{p},r}p'^{\mu}\epsilon^{(l}_{\mathbf{p}',r'}\gamma^{m)} - \epsilon_{\mu\mathbf{p}',r'}p^{\mu}\epsilon^{(l}_{\mathbf{p},r}\gamma^{m)}\}.\end{aligned} \quad (6.3.19)$$

Normally, the helicity λ, λ' can take the values $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$ indifferently. For the following discussion we define, omitting the different indices for simplicity.

$$\Delta_1 = \frac{1}{2}\Lambda_{ij,lm}\langle 1,\frac{1}{2},r,\frac{s}{2}|\frac{3}{2},\lambda\rangle\langle 1,\frac{1}{2},r',\frac{s'}{2}|\frac{3}{2},\lambda'\rangle\epsilon_{\mu\mathbf{p},r}\epsilon^{\mu}_{\mathbf{p}',r'}\, p^{(l}\gamma^{m)}$$

$$\Delta_2 = -\frac{1}{4}\Lambda_{ij,lm}\langle 1,\frac{1}{2},r,\frac{s}{2}|\frac{3}{2},\lambda\rangle\langle 1,\frac{1}{2},r',\frac{s'}{2}|\frac{3}{2},\lambda'\rangle(\epsilon_{\mu\mathbf{p},r}p'^{\mu}\epsilon^{(l}_{\mathbf{p}',r'}\gamma^{m)} + \epsilon_{\mu\mathbf{p}',r'}p^{\mu}\epsilon^{(l}_{\mathbf{p},r}\gamma^{m)})$$

We have so

$$\Delta^{\lambda s,\lambda' s'}_{ij} = \Delta_{1ij}^{\lambda s,\lambda' s'} + \Delta_{2ij}^{\lambda s,\lambda' s'} \quad (6.3.20)$$

In our case, the gravitational waves are produced mainly by relativistic states, and in this limit the different helicity states are not produced with the same mechanism (see e.g., for gravitinos [162, 127, 126]). Hence we will not consider in the rest of the computation crossing helicity terms, and we separate the calculation into two parts, $\lambda, \lambda' = \pm\frac{3}{2}$ and $\lambda, \lambda' = \pm\frac{1}{2}$.

Before entering in the details of the computation, let's put some conventions for our different vectors and spinors. The comoving wave number $\vec{k}$ is taken in the z direction. from this we can define the vector $\vec{p}$ (and the four-vector p) by

$$p^{\mu} = \begin{pmatrix} E \\ |\vec{p}|\sin\theta\cos\phi \\ |\vec{p}|\sin\theta\sin\phi \\ |\vec{p}|\cos\theta \end{pmatrix} \quad (6.3.21)$$

Since we have $\vec{p}' = \vec{p}+\vec{k}$, we can define also $\vec{p}'$ but there are in the same plane, so $\phi = \phi'$. We will use the following basis for the polarisations

$$\epsilon_k^\mu(p) = \begin{pmatrix} 0 \\ -k\cos\theta\cos\phi + i\sin\phi \\ -k\cos\theta\sin\phi - i\cos\phi \\ k\sin\theta \end{pmatrix}, \; \epsilon_0^\mu(p) = \begin{pmatrix} |\vec{p}| \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix} \tag{6.3.22}$$

with $k = \pm 1$.
The last quantity we have to define is the basis for the helicity spinors used in (6.3.8). We will take

$$\chi^+ = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix} \text{ and } \chi^- = \begin{pmatrix} -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\cos\theta/2 \end{pmatrix} \tag{6.3.23}$$

Now that we have defined all the quantities needed for the rest of this work, we can come back to the computation of the unequal time correlator.

6.3.2 Helicity $\pm\frac{3}{2}$

We first consider the case of $\lambda, \lambda' = \pm\frac{3}{2}$. Such a restriction can be thought as working in the massless limit for the spin-3/2 state. For a gravitino, this is the high energy limit before the spontaneous breaking of supersymmetry. The mode decomposition (6.3.7) reads

$$\tilde{\psi}^\mu_{\vec{p},\pm\frac{3}{2}}(t) = \epsilon^\mu_{\vec{p},\pm 1}\, \mathbf{u}^{(3/2)}_{\vec{p},\pm\frac{1}{2}}(t). \tag{6.3.24}$$

We will forget the $\left(\frac{3}{2}\right)$ exponent in the rest of this part. We will recover it at the end, when presenting the result. Our goal is to calculate the corresponding unequal-time correlator (6.3.18). Since we have a product of two Δ, using (6.3.20) we can have 4 different contributions coming from the product of Δ_1 and Δ_2 . The crossing term are the same, and so we are let with three parts to compute.

- We begin with the product $\Delta_1\Delta_1$.

The product of the polarisation can be computed independently from the product of the spinor part. We obtain using the definition above

$$\epsilon_\mu^{\vec{p}\,r}\epsilon^{\mu\,\vec{p}'\,s} = \frac{1}{2}(sr\cos(\theta-\theta') - 1) \tag{6.3.25}$$

Using (6.3.25) we can simplify the unequal time correlator to obtain

$$\Pi_1^2 = \frac{1}{2}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\Lambda_{ijab}\bar{\mathbf{v}}_r^{\vec{p}}\gamma_{(i}p_{j)}\mathbf{u}_s^{\vec{p}'}\bar{\mathbf{u}}_s^{\vec{p}'}\gamma_{(a}p_{b)}\mathbf{v}_r^{\vec{p}}\frac{1}{4}(sr\cos(\theta-\theta')-1)^2 \tag{6.3.26}$$

W can now focus on the second part of the computation, involving the spin $1/2$ part of our decomposition of the spin $3/2$. The quantity we are interested in is the trace

$$T = \mathrm{Tr}(\mathbf{v}_r^{\vec{p}}\bar{\mathbf{v}}_r^{\vec{p}}\gamma_{(i}p_{j)}\mathbf{u}_s^{\vec{p}'}\bar{\mathbf{u}}_s^{\vec{p}'}\gamma_{(a}p_{b)}) \tag{6.3.27}$$

To compute this trace we need to express the spinors $\mathbf{v}$ and $\mathbf{u}$. We can rewrite them under the form

$$\mathbf{u}_s^{\vec{k}} = u_{\vec{k},a}\chi^s \otimes P^a \quad \mathbf{v}_s^{\vec{k}} = au^*_{\vec{k},a}\tilde{\chi}^s \otimes P^a \tag{6.3.28}$$

Where $P^{\pm}$ is just a projector on respectively the first and the second coordinate, and $\tilde{\chi}$ is just defined by $\tilde{\chi} = -i\sigma_2\chi$. Using these definitions, we can now return to the computation of the trace,

$$\begin{aligned}
T &= \mathrm{Tr}\left(adu^*_{\vec{p},-a}\tilde{\chi}^r \otimes P^a u_{\vec{p},-d}\tilde{\chi}^{r^\dagger} \otimes P^{d^T}\gamma_0\gamma_{(i}p_{j)}u_{\vec{p}',b}\chi'^s \otimes P^b u^*_{\vec{p}',c}\chi'^{s^\dagger} \otimes P^{c^T}\gamma_0\gamma_{(\alpha}p_{\beta)}\right)\\
&= adu^*_{\vec{p},-a}u_{\vec{p},-d}u_{\vec{p}',b}u^*_{\vec{p}',c}\mathrm{Tr}\Big((\tilde{\chi}^r \otimes P^a)(\tilde{\chi}^{r^\dagger} \otimes P^{d^T})(1 \otimes \sigma_3)(p_{(j}\sigma_{i)} \otimes -i\sigma_2)\\
&\qquad (\chi'^s \otimes P^b)(\chi'^{s^\dagger} \otimes P^{c^T})(1 \otimes \sigma_3)(p_{(\beta}\sigma_{\alpha)} \otimes -i\sigma_2)\Big)\\
&= adu^*_{\vec{p},-a}u_{\vec{p},-d}u_{\vec{p}',b}u^*_{\vec{p}',c}\mathrm{Tr}\left((\tilde{\chi}^{r^\dagger} \otimes P^{d^T}\sigma_3)(p_{(j}\sigma_{i)}\chi'^s \otimes -i\sigma_2 P^b)(\chi'^{s^\dagger} \otimes P^{c^T}\sigma_3)(p_{(\beta}\sigma_{\alpha)}\tilde{\chi}^r \otimes -i\sigma_2 P^a)\right)\\
&= adu^*_{\vec{p},-a}u_{\vec{p},-d}u_{\vec{p}',b}u^*_{\vec{p}',c}\mathrm{Tr}\left(\tilde{\chi}^{r^\dagger}p_{(j}\sigma_{i)}\chi'^s\chi'^{s^\dagger}p_{(\beta}\sigma_{\alpha)}\tilde{\chi}^r \otimes P^{d^T}\sigma_3(-i\sigma_2)P^bP^{c^T}\sigma_3(-i\sigma_2)P^a\right)\\
&= adu^*_{\vec{p},-a}u_{\vec{p},-d}u_{\vec{p}',b}u^*_{\vec{p}',c}\mathrm{Tr}(\tilde{\chi}^{r^\dagger}p_{(j}\sigma_{i)}\chi'^s\chi'^{s^\dagger}p_{(\beta}\sigma_{\alpha)}\tilde{\chi}^r)\delta_{d,-b}\delta_{c,-a}\\
&= adu^*_{\vec{p},-a}u_{\vec{p},-d}u_{\vec{p}',-d}u^*_{\vec{p}',-a}\mathrm{Tr}(\tilde{\chi}^r\tilde{\chi}^{r^\dagger}p_{(j}\sigma_{i)}\chi'^s\chi'^{s^\dagger}p_{(\beta}\sigma_{\alpha)})
\end{aligned}$$

We will find again this trace in the rest of the computation. In order to perform the calculation of the trace, we will define the matrix

$$M_{i\alpha} = \mathrm{Tr}(\tilde{\chi}^r\tilde{\chi}^{r^\dagger}\sigma_i\chi'^s\chi'^{s^\dagger}\sigma_\alpha) \tag{6.3.29}$$

To simplify this matrix, we need the following relations on the spinors,

$$\chi^s\chi^{s^\dagger} = \frac{1}{2}(\mathbb{1} + sn.\sigma) \quad \tilde{\chi}^r\tilde{\chi}^{r^\dagger} = \frac{1}{2}(\mathbb{1} - rn.\sigma) \tag{6.3.30}$$

Where we have defined n as the unit vector in the $\vec{p}$ direction, and σ are the Pauli matrices. We can now simplify $M_{i\alpha}$

$$\begin{aligned}
M_{i\alpha} &= \frac{1}{4}\mathrm{Tr}((\mathbb{1} - rn.\sigma)\sigma_i(\mathbb{1} + sn'.\sigma)\sigma_\alpha)\\
&= \frac{1}{4}(\sigma_i\sigma_\alpha - rn.\sigma\sigma_i\sigma_\alpha + s\sigma_i n'.\sigma\sigma_\alpha - srn.\sigma\sigma_i n'.\sigma\sigma_\alpha)\\
&= \frac{1}{4}(2\delta_{i\alpha} - 2rn_k i\epsilon_{ki\alpha} + 2sn'_k i\epsilon_{ik\alpha} - 2srn_k n'_l(\delta_{ki}\delta_{l\alpha} - \delta_{kl}\delta_{i\alpha} + \delta_{k\alpha}\delta_{li}))\\
&= \frac{1}{2}(\delta_{i\alpha} - i\epsilon_{ki\alpha}(n_k r + sn'_k) - sr(n_i n'_\alpha - \delta_{i\alpha}n.n' + n_\alpha n'_i))
\end{aligned}$$

We have to remember that there is a projector in the unequal-time operator, so we will compute in all generality the term $\Lambda_{ij\alpha\beta}X_{(j}M_{i)(\alpha}Y_{\beta)}$, where X and Y are general vectors. we recall the definition of Λ,

$$\Lambda_{ij\alpha\beta} = P_{i\alpha}P_{j\beta} - P_{ij}P_{\alpha\beta} \text{ with } P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j \tag{6.3.31}$$

$\hat{k}$ is the unit vector pointing in the direction of the vector k. We have so

$$\Lambda_{ij\alpha\beta}X_{(j}M_{i)(\alpha}Y_{\beta)} = \frac{1}{4}(P_{i\alpha}P_{j\beta} - \frac{1}{2}P_{ij}P_{\alpha\beta})(X_jM_{i\alpha}Y_\beta + X_jM_{i\beta}Y_\alpha + X_iM_{j\alpha}Y_\beta + X_iM_{j\beta}Y_\alpha)$$

$$
\begin{aligned}
&= \frac{1}{4}(P_{i\alpha}P_{j\beta}X_jY_\beta M_{i\alpha} + P_{i\alpha}P_{j\beta}X_jY_\alpha M_{i\beta} + P_{i\alpha}P_{j\beta}X_iY_\beta M_{j\alpha} + P_{i\alpha}P_{j\beta}X_iY_\alpha M_{j\beta} \\
&\quad - \frac{1}{2}(P_{ij}P_{\alpha\beta}X_jY_\beta M_{i\alpha} + P_{ij}P_{\alpha\beta}X_jY_\alpha M_{i\beta} + P_{ij}P_{\alpha\beta}X_iY_\beta M_{j\alpha} + P_{ij}P_{\alpha\beta}X_iY_\alpha M_{j\beta})) \\
&= \frac{1}{2}(P_{i\alpha}P_{j\beta}X_jY_\beta M_{i\alpha} + P_{i\alpha}P_{j\beta}X_jY_\alpha M_{i\beta} - P_{ij}P_{\alpha\beta}X_jY_\beta M_{i\alpha}) \\
&= \frac{1}{2}(P_{i\alpha}P_{j\beta}X_jY_\beta M_{i\alpha} + P_{i\alpha}P_{j\beta}X_jY_\alpha M_{i\beta} - P_{i\alpha}P_{j\beta}X_\alpha Y_\beta M_{ij}) \\
&= \frac{1}{2}P_{i\alpha}P_{j\beta}X_jY_\beta M_{i\alpha}
\end{aligned}
$$

Where we have used the fact that P and M are symmetric matrices in the three last equations. We have parametrized $\vec{k}$ following the z direction, so P has the simple form

$$
P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.3.32}
$$

and we can rewrite the equation,

$$
\Lambda_{ij\alpha\beta}X_{(j}M_{i)(\alpha}Y_{\beta)} = \frac{1}{2}(X_1Y_1 + X_2Y_2)(M_{11} + M_{22}) \tag{6.3.33}
$$

In the case we look at, we have seen that the spin 1 part contains no term proportional to one or other helicity s or r. Since we sum over these parameters to compute the unequal time operator, we can limit ourselves at the part of the M matrices proportional to the product sr. We can also restrict the result (6.3.33) to the case $X = Y = p$ and so we obtain

$$
\Lambda_{ij\alpha\beta}X_{(j}M_{i)(\alpha}Y_{\beta)} = \frac{1}{2}(p_1^2 + p_2^2)(1 + sr\cos\theta\cos\theta') \tag{6.3.34}
$$

Using the definition of p we have

$$
p_1^2 + p_2^2 = p^2 - p_3^2 = p^2(1 - \cos(\theta)^2) = p^2\sin(\theta)^2
$$

To sum up,

$$
\begin{aligned}
T &= ad u^*_{\vec{p},-a}u_{\vec{p},-d}u_{\vec{p}',-d}u^*_{\vec{p}',-a}\frac{1}{2}p^2\sin^2\theta(1 + sr\cos\theta\cos\theta') \\
&= W_{\vec{k},\vec{p}}W^*_{\vec{k},\vec{p}}\frac{1}{2}p^2\sin^2\theta(1 + sr\cos\theta\cos\theta')
\end{aligned} \tag{6.3.35}
$$

where we have defined

$$
W^{(|\lambda|)}_{\vec{k},\vec{p}}(t) = u^{(|\lambda|)}_{\vec{p},+}(t)u^{(|\lambda|)}_{\vec{p}',+}(t) - u^{(|\lambda|)}_{\vec{p},-}(t)u^{(|\lambda|)}_{\vec{p}',-}(t) \tag{6.3.36}
$$

in order to isolate kinematic factors from parts containing the wave functions. The partial unequal time correlator becomes

$$
\begin{aligned}
\Pi_1^2 &= \frac{1}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2}W_{\vec{k},\vec{p}}W^*_{\vec{k},\vec{p}}p^2\sin^2\theta(1 + sr\cos\theta\cos\theta')\frac{1}{4}(sr\cos(\theta - \theta') - 1)^2 \\
&= \frac{1}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2}W_{\vec{k},\vec{p}}W^*_{\vec{k},\vec{p}}p^2\sin^2\theta(1 + \cos(\theta - \theta')^2 - 2\cos(\theta)\cos(\theta')\cos(\theta - \theta'))
\end{aligned}
$$

$$= \frac{1}{4} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} W_{\vec{k},\vec{p}} W^*_{\vec{k},\vec{p}} p^2 \sin^2\theta (1 - \cos(\theta - \theta')\cos(\theta + \theta'))$$

We have conserved only the even power of sr, because the odd powers are cancelled when we perform the sum over s and r. We recall also that there is no final dependence on p' and θ' as these are expressed before integration as functions of p, k, and θ,

$$p' = \sqrt{p^2 + k^2 + 2kp\cos\theta}, \qquad \theta' = \arccos(\frac{p\cos\theta + k}{\sqrt{p^2 + k^2 + 2kp\cos\theta}}). \tag{6.3.37}$$

This computation is just one part of the unequal time correlator.

- We will compute now the second part of this quantity, $\Delta_2\Delta_2$.

This term is namely

$$\frac{1}{8}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\Lambda_{ijab}\left(\bar{\mathbf{v}}_l^{\vec{p}}(\epsilon_\mu^{\vec{p}\,l} p'^\mu \gamma_{(i}\epsilon_{j)}^{\vec{p}'\,r} + \epsilon_{(i}^{\vec{p}\,l}\gamma_{j)} p^\mu \epsilon_\mu^{\vec{p}'\,r})\mathbf{u}_r^{\vec{p}'}\bar{\mathbf{u}}_r^{\vec{p}'}(\epsilon_{(a}^{*\vec{p}'\,r}\gamma_{b)} p'^\nu \epsilon_\nu^{*\vec{p}\,l} + p^\nu \epsilon_\nu^{*\vec{p}'\,r}\gamma_{(a}\epsilon_{b)}^{*\vec{p}\,l})\mathbf{v}_l^{\vec{p}}\right) \tag{6.3.38}$$

Using the fact that $\vec{k}$ is along the z axis, and $\epsilon_\mu^{\vec{p}\,l} p^\mu = 0$, we can compute

$$\epsilon_\mu^{\vec{p}\,l} p'^\mu = \epsilon_\mu^{\vec{p}\,l} k^\mu = \frac{1}{\sqrt{2}} lk\sin(\theta)$$
$$\epsilon_\mu^{\vec{p}'\,r} p^\mu = -\epsilon_\mu^{\vec{p}'\,r} k^\mu = -\frac{1}{\sqrt{2}} rk\sin(\theta')$$

Using these relations we simplify the unequal time correlator,

$$\Pi_2^2 = \frac{1}{8}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\Lambda_{ijab}\left[\frac{\bar{\mathbf{v}}_l^{\vec{p}}}{\sqrt{2}}(lk\sin\theta\gamma_{(i}\epsilon_{j)}^{\vec{p}'\,r} - rk\sin\theta'\epsilon_{(i}^{\vec{p}\,l}\gamma_{j)})\mathbf{u}_r^{\vec{p}'}\bar{\mathbf{u}}_r^{\vec{p}'}(lk\sin\theta\epsilon_{(a}^{*\vec{p}'\,r}\gamma_{b)} - rk\sin\theta'\gamma_{(a}\epsilon_{b)}^{*\vec{p}\,l})\frac{\mathbf{v}_l^{\vec{p}}}{\sqrt{2}}\right]$$

We can use results derived in the $\Delta_1\Delta_1$ computation also in this second part. Indeed, Π_2^2 can be rewritten as

$$\Pi_2^2 = \frac{1}{8}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{\Lambda_{ijab}}{2} W_{\vec{k},\vec{p}} W^*_{\vec{k},\vec{p}}\left(lk\sin\theta\epsilon_{(i}^{\vec{p}'\,r} - rk\sin\theta'\epsilon_{(i}^{\vec{p}\,l}\right) M_{j)(a}\left(lk\sin\theta\epsilon_{b)}^{*\vec{p}'\,r} - rk\sin\theta'\epsilon_{b)}^{*\vec{p}\,l}\right) \tag{6.3.39}$$

Using now the equation (6.3.33), one obtains

$$\begin{aligned}\Pi_2^2 = \frac{1}{8}\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{1}{4} W_{\vec{k},\vec{p}} W^*_{\vec{k},\vec{p}} \Big[& l^2k^2\sin^2\theta(\epsilon_1^{\vec{p}',r}\epsilon_1^{*\vec{p}',r} + \epsilon_2^{\vec{p}',r}\epsilon_2^{*\vec{p}',r}) \\ & -rlk^2\sin\theta\sin\theta'(\epsilon_1^{\vec{p}',r}\epsilon_1^{*\vec{p},l} + \epsilon_2^{\vec{p}',r}\epsilon_2^{*\vec{p},l} + \epsilon_1^{\vec{p},l}\epsilon_1^{*\vec{p}',r} + \epsilon_2^{\vec{p},l}\epsilon_2^{*\vec{p}',r}) \\ & +r^2k^2\sin^2\theta'(\epsilon_1^{\vec{p},l}\epsilon_1^{*\vec{p},l} + \epsilon_2^{\vec{p},l}\epsilon_2^{*\vec{p},l})\Big](1 + rl\cos\theta\cos\theta').\end{aligned}$$

The sum over the polarisations appearing here is given by

$$\epsilon_1^{\vec{p}',r}\epsilon_1^{*\vec{p}',r} + \epsilon_2^{\vec{p}',r}\epsilon_2^{*\vec{p}',r} = \frac{1}{2}(1 + r^2\cos^2\theta') \ , \ \epsilon_1^{\vec{p},l}\epsilon_1^{*\vec{p},l} + \epsilon_2^{\vec{p},l}\epsilon_2^{*\vec{p},l} = \frac{1}{2}(1 + l^2\cos^2\theta) \ , \tag{6.3.40}$$

$$\epsilon_1^{\vec{p},l}\epsilon_1^{*\vec{p'},r}+\epsilon_2^{\vec{p},l}\epsilon_2^{*\vec{p'},r}=\frac{1}{2}(1+rl\cos\theta\cos\theta')\ .$$

Summing over the spin indices yields

$$\Pi_2^2=\frac{1}{8}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{1}{2}W_{\vec{k},\vec{p}}W^*_{\vec{k},\vec{p}}\left[k^2\sin^2\theta(1+\cos^2\theta')+k^2\sin^2\theta'(1+\cos^2\theta)-4k^2\sin\theta\sin\theta'\cos\theta\cos\theta'\right]. \tag{6.3.41}$$

Eventually, we simplify this expression by using some relations between the vectors

$$k^2\sin^2\theta=p'^2\sin^2(\theta-\theta')\ ,\ k^2\sin^2\theta'=p^2\sin^2(\theta'-\theta)\ ,\ p'\sin\theta'=p\sin\theta \tag{6.3.42}$$

and we obtain

$$\Pi_2^2=\frac{1}{8}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}W_{\vec{k},\vec{p}}W^*_{\vec{k},\vec{p}}\sin^2(\theta-\theta')(k^2+p^2\sin^2\theta). \tag{6.3.43}$$

- The last contribution to add is the crossing term $\Delta_1\Delta_2$.

This contribution is written as

$$\Pi_3^2=-\frac{1}{4}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\Lambda_{ijab}\Big[\bar{v}_l^{\vec{p}}(\epsilon_\mu^{\vec{p},l}\epsilon^{\mu\ \vec{p'},s}\gamma_{(i}p_{j)}u_r^{\vec{p'}}\bar{u}_r^{\vec{p'}}(\epsilon^{*\vec{p'}\ s}_{(a}\gamma_{b)}p'^\nu\epsilon_\nu^{*\vec{p},l}+\epsilon_\nu^{*\vec{p'},s}p^\nu\gamma_{(a}\epsilon^{*\vec{p}\ l}_{b)})v_l^{\vec{p}}$$
$$+\bar{v}_l^{\vec{p}}(\epsilon_\mu^{\vec{p},l}\gamma_{(i}p'^\mu\epsilon^{\vec{p'},s}_{j)}+\epsilon^{\vec{p},l}_{(i}p^\mu\gamma_{j)}\epsilon_\mu^{\vec{p'},s})u_r^{\vec{p'}}\bar{u}_r^{\vec{p'}}(\epsilon_\nu^{*\vec{p'}\ s}\epsilon^{*\nu\ \vec{p},l}\gamma_{(a}p_{b)})v_l^{\vec{p}}\Big]. \tag{6.3.44}$$

This time we can use

$$\epsilon_\mu^{\vec{p},l}\epsilon^{\mu\ \vec{p}'r}=\frac{1}{2}(rl\cos(\theta-\theta')-1)=\epsilon_\nu^{*\vec{p'}l}\epsilon^{*\nu\ \vec{p},r} \tag{6.3.45}$$

and write $\Pi_3^2=-\frac{1}{4}\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}W_{\vec{k},\vec{p}}W^*_{\vec{k},\vec{p}}\Lambda_{ijab}S_{ijab}$

$$S_{ijab}=-\frac{1}{2\sqrt{2}}\Big((sl\cos(\theta-\theta')-1)p_{(i}M_{j)(a}(\epsilon^{*\vec{p'}\ s}_{b)}lp'\sin(\theta'-\theta)+\epsilon^{*\vec{p}\ l}_{b)}(sp\sin(\theta'-\theta)$$
$$+(lp'\sin(\theta'-\theta)\epsilon^{\vec{p'}s}_{(i}+sp\sin(\theta-\theta')\epsilon^{\vec{p}l}_{(i})M_{j)(a}p_{b)}(sl\cos(\theta'-\theta)-1)\Big) \tag{6.3.46}$$

Using the same formula as in the two previous cases, one gets

$$\Lambda_{ijab}S_{ijab}=$$
$$\frac{1}{2\sqrt{2}}\Big((sl\cos(\theta-\theta')-1)(lp'\sin(\theta'-\theta)(p_1\epsilon_1^{*\vec{p'}s}+p_2\epsilon_2^{*\vec{p'}s})+sp\sin(\theta-\theta')(p_1\epsilon_1^{*\vec{p}l}+p_2\epsilon_2^{*\vec{p}l})$$
$$+(sl\cos(\theta'-\theta)-1)(lp'\sin(\theta'-\theta)(p_1\epsilon_1^{\vec{p'}s}+p_2\epsilon_2^{\vec{p'}s})+sp\sin(\theta-\theta')(p_1\epsilon_1^{\vec{p}l}+p_2\epsilon_2^{\vec{p}l})\Big)$$
$$\times\frac{1}{2}(1+sl\cos\theta\cos\theta')$$

Summing over s and l, $\Lambda_{ijab}S_{ijab}$ reduces to

$$\Lambda_{ijab}S_{ijab}=\big(\cos(\theta'-\theta)(pp'\sin(\theta'-\theta)\cos\theta'\sin\theta+p^2\sin(\theta-\theta')\sin\theta\cos\theta)$$

$$- \cos\theta\cos\theta'(pp'\sin(\theta'-\theta)\cos\theta'\sin\theta + p^2\sin(\theta-\theta')sin\theta\cos\theta)$$
$$= -\, kp\sin(\theta+\theta')\sin\theta'\sin^2\theta.$$

We have so

$$\Pi_3^2 = \frac{1}{4}\int \frac{d^3\vec{p}}{(2\pi)^3} W_{\vec{k},\vec{p}} W^*_{\vec{k},\vec{p}} kp\sin(\theta+\theta')\sin\theta'\sin^2\theta. \tag{6.3.47}$$

Now that we have compute the three contribution to the unequal time correlator, we can add them and get the total unequal time correlator for the helicity $\pm\frac{3}{2}$

$$\Pi^2_{\frac{3}{2}}(k,t,t') = \Pi_1^2 + \Pi_2^2 + \Pi_3^2$$
$$= \int \frac{d^3\vec{p}}{(2\pi)^3} W^{(\frac{3}{2})}_{\vec{k},\vec{p}}(t) W^{(\frac{3}{2})*}_{\vec{k},\vec{p}}(t') \left[\frac{1}{4}p^2\sin^2\theta(1-\cos(\theta-\theta')\cos(\theta+\theta')) \right.$$
$$\left. + \frac{1}{8}\sin^2(\theta-\theta')(k^2+p^2\sin^2\theta) + \frac{1}{4}kp\sin(\theta+\theta')\sin\theta'\sin^2\theta\right]$$

Using $pk\sin(\theta-\theta')\sin^2\theta\sin\theta' = k^2\sin^2\theta\sin^2\theta'$ leads us to the final result

$$\Pi^2_{\frac{3}{2}}(k,t,t') \;=\; \frac{1}{32\pi^2}\int dp\, d\theta\; K^{(\frac{3}{2})}(p,k,\theta,m_{3/2})\;\; W^{(\frac{3}{2})}_{\vec{k},\vec{p}}(t) W^{(\frac{3}{2})*}_{\vec{k},\vec{p}}(t'), \tag{6.3.48}$$

with

$$K^{(\frac{3}{2})}(p,k,\theta,m_{3/2}) \;=\; p^2k^2\{5\sin^3\theta\sin^2\theta' + \sin^2(\theta-\theta')\sin\theta\} + 4p^4\sin^4\theta\sin\theta'. \tag{6.3.49}$$

6.3.3 Helicity $\pm\frac{1}{2}$

We will now turn to $\lambda = \pm\frac{1}{2}$. The computation is the same in spirit, but the mode decomposition (6.3.7) is more involved,

$$\tilde{\psi}^{\mu}_{\vec{p},\pm\frac{1}{2}}(t) = \sqrt{\frac{2}{3}}\epsilon^{\mu\,0\,\vec{p}}\,\mathbf{u}^{(\frac{1}{2})}_{\vec{p},\pm\frac{1}{2}}(t) + \sqrt{\frac{1}{3}}\epsilon^{\mu\,\pm1\,\vec{p}}\,\mathbf{u}^{(\frac{1}{2})}_{\vec{p},\mp\frac{1}{2}}(t). \tag{6.3.50}$$

Since both $\epsilon^{\mu}_{\vec{p},0}$ and $\epsilon^{\mu}_{\vec{p},\pm1}$ appear, we have $2^4 = 16$ helicity combinations in the four-point correlation functions. But since we look at the relativistic limit, we can simplify a little bit the computation. In the relativistic limit $p \gg m_{3/2}$, one could expand $\epsilon^{\mu}_{\vec{p},0}$,

$$\epsilon^{\mu}_{\vec{p},0} = \frac{1}{m_{3/2}}(p, \sqrt{p^2+m^2_{3/2}}\hat{\vec{p}}) = \frac{p^{\mu}}{m_{3/2}} + \frac{m_{3/2}}{2p}(-1,\hat{\vec{p}}) + O(\frac{m^2_{3/2}}{p^2}). \tag{6.3.51}$$

Thus, in the relativistic regime, we expect the leading order result to be obtained by replacing $\epsilon^{\mu}_{\vec{p},0} \to \frac{p^{\mu}}{m_{3/2}}$. However, correlators with the four $\epsilon^{\mu}_{\vec{p},0}$ inside (6.3.18) replaced by $\frac{p^{\mu}}{m_{3/2}}$ vanish. The dominant contribution comes then from terms in in (6.3.18) where two of the four $\epsilon^{\mu}_{\vec{p},r}$ are $\epsilon^{\mu}_{\vec{p},0}$. Notice that the two $\epsilon^{\mu}_{\vec{p},0}$ can't be inside the same $\hat{\Pi}^{lm}(\vec{p},t)$ since the leading order of the stress tensor in (6.3.15) would vanish. Thus there are

$\binom{4}{2} - 2 = 4$ such polarisation combinations in all. These polarisation combinations are

$$\epsilon^{\,r\,\vec{p}}\epsilon^{0\,\vec{p}'}\epsilon^{*0\,\vec{p}'}\epsilon^{*\,r\,\vec{p}},\ \epsilon^{\,0\,\vec{p}}\epsilon^{s\,\vec{p}'}\epsilon^{*s\,\vec{p}'}\epsilon^{*\,0\,\vec{p}} \text{ and } \epsilon^{\,0\,\vec{p}}\epsilon^{s\,\vec{p}'}\epsilon^{*\,0\,\vec{p}'}\epsilon^{*r\,\vec{p}},\ \epsilon^{\,r\,\vec{p}}\epsilon^{0\,\vec{p}'}\epsilon^{*\,s\,\vec{p}'}\epsilon^{*0\,\vec{p}}. \tag{6.3.52}$$

We have combined the helicity combination by two because the results are quite similar for each pair considered. As for the helicity 3/2 part, we can separate the three different contributions coming from the definition of $\Delta_{ij}^{\lambda s,\lambda' s'}$ in (6.3.19).

- We will follow the same path as before, and begin with the $\Delta_1\Delta_1$ contribution.

The difference comes from the different polarisation combinations.
The first polarisation gives

$$\frac{2}{9}\epsilon^{\mu\,r\,\vec{p}}\epsilon^{0\,\vec{p}'}_{\mu}\epsilon^{*0\,\vec{p}'}_{\nu}\epsilon^{*\nu\,r\,\vec{p}}\bar{\mathbf{v}}^{\vec{p}}_{-r}\gamma_{(i}p_{j)}\mathbf{u}^{\vec{p}'}_{s}\bar{\mathbf{u}}^{\vec{p}'}_{s}\gamma_{(a}p_{b)}\mathbf{v}^{\vec{p}}_{-r} \tag{6.3.53}$$

We have already computed the spinor part (using a symmetry $r \leftrightarrow -r$) which is given in (6.3.35) (with different wave factors), so the only thing to compute here is the helicity part which is given by

$$\epsilon^{\mu\,r\,\vec{p}}\epsilon^{0\,\vec{p}'}_{\mu}\epsilon^{*0\,\vec{p}'}_{\nu}\epsilon^{*\nu\,r\,\vec{p}} = |\epsilon^{r\,\vec{p}}_{\mu}\epsilon^{\mu\,0\,\vec{p}'}|^2 = \frac{E'^2}{2m^2_{3/2}}\sin^2(\theta-\theta') \tag{6.3.54}$$

where E' is the energy of the gravitino with momemtum $\vec{p}'$. Gathering the contribution we obtain for the first polarisation combination

$$\begin{aligned} &\frac{2}{9}\frac{E'^2}{2m^2_{3/2}}\sin^2(\theta-\theta')\frac{1}{2}(1+sr\cos\theta\cos\theta')p^2\sin^2\theta \\ &= \frac{2E'^2}{9m^2_{3/2}}p^2\sin^2(\theta-\theta')\sin^2\theta \\ &\simeq \frac{2p'^2}{9m^2_{3/2}}p^2\sin^2(\theta-\theta')\sin^2\theta \end{aligned} \tag{6.3.55}$$

where we summed over the helicity s, r. In the last equations we have omitted the wave factor $W^{(\frac{1}{2})}$ but the product of the two wave factors is present as the integral over $\vec{p}$.
For the second combination of the first pair, we have

$$\frac{2}{9}\epsilon^{\mu\,r\,\vec{p}}\epsilon^{0\,\vec{p}'}_{\mu}\epsilon^{*0\,\vec{p}'}_{\nu}\epsilon^{*\nu\,r\,\vec{p}}\bar{\mathbf{v}}^{\vec{p}}_{r}\gamma_{(i}p_{j)}\mathbf{u}^{\vec{p}'}_{-s}\bar{\mathbf{u}}^{\vec{p}'}_{-s}\gamma_{(a}p_{b)}\mathbf{v}^{\vec{p}}_{r} \tag{6.3.56}$$

The spinor part is always the same and the polarisation part gives

$$\epsilon^{\mu\,0\,\vec{p}}\epsilon^{s\,\vec{p}'}_{\mu}\epsilon^{*\nu\,s\,\vec{p}'}\epsilon^{*0\,\vec{p}}_{\nu} = |\epsilon^{0\,\vec{p}}_{\mu}\epsilon^{\mu\,s\,\vec{p}'}|^2 = \frac{E^2}{2m^2_{3/2}}\sin^2(\theta-\theta')$$

and so the final result is, always omitting the integral and the wave factor product

$$\frac{2}{9m^2_{3/2}}p^4\sin^2(\theta-\theta')\sin^2(\theta) \tag{6.3.57}$$

which is quite similar to the first contribution, except that $\vec{p}$ is exchanged with $\vec{p'}$, remembering that $p\sin\theta = p'\sin\theta'$.
Looking now at the second pair of helicity, the first contribution gives

$$\frac{2}{9}\epsilon^{\mu\,0\,\vec{p}}\epsilon^{s\,\vec{p'}}_{\mu}\epsilon^{*\nu\,0\,\vec{p'}}\epsilon^{*r\,\vec{p}}_{\nu}\bar{\mathbf{v}}^{\vec{p}}_{r}\gamma_{(i}p_{j)}\mathbf{u}^{\vec{p'}}_{-s}\bar{\mathbf{u}}^{\vec{p'}}_{s}\gamma_{(a}p_{b)}\mathbf{v}^{\vec{p}}_{-r} \tag{6.3.58}$$

We have for the helicity product

$$\epsilon^{\mu\,0\,\vec{p}}\epsilon^{s\,\vec{p'}}_{\mu} = \frac{E}{\sqrt{2}m_{3/2}}s\sin(\theta-\theta')\ ,\ \epsilon^{*\nu\,0\,\vec{p'}}\epsilon^{*r\,\vec{p}}_{\nu} = \frac{E'}{\sqrt{2}m_{3/2}}r\sin(\theta-\theta')\ ,$$

This time the spinor part is different from the one of the helicity 3/2, but we can compute it following the same path we follow for the helicity 3/2 case. Using (6.3.28) we simplify the trace

$$\mathrm{Tr}(\bar{\mathbf{v}}^{\vec{p}}_{r}\gamma_{(i}p_{j)}\mathbf{u}^{\vec{p'}}_{-s}\bar{\mathbf{u}}^{\vec{p'}}_{s}\gamma_{(a}p_{b)}\mathbf{v}^{\vec{p}}_{-r}) = W_{\vec{k},\vec{p}}W^{*}_{\vec{k},\vec{p}}\mathrm{Tr}(\tilde{\chi}^{-r}\tilde{\chi}^{r^\dagger}p_{(j}\sigma_{i)}\chi^{-s}\chi^{s^\dagger}p_{(a}\sigma_{b)}). \tag{6.3.59}$$

We define so a new matrix

$$N_{ib} = \mathrm{Tr}(\tilde{\chi}^{-r}\tilde{\chi}^{r^\dagger}\sigma_i\chi^{-s}\chi^{s^\dagger}\sigma_b) \tag{6.3.60}$$

Using (6.3.23) we have

$$\chi^{\pm}\chi^{\mp^\dagger} = \begin{pmatrix} -\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \pm e^{i\Phi}\cos^2\frac{\theta}{2} \\ \mp e^{i\Phi}\sin^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} \end{pmatrix} \tag{6.3.61}$$

We can express directly the formula for $\tilde{\chi}$: $\tilde{\chi}^{\pm}\tilde{\chi}^{\mp^\dagger} = -\chi^{\mp}\chi^{\pm^\dagger}$, and write (6.3.61) as $\chi^{\pm}\chi^{\mp^\dagger} = b^{\pm}.\sigma$ (where σ are always Pauli matrices), one obtains

$$N_{ib} = -\mathrm{Tr}(b^r.\sigma\sigma_i b'^s.\sigma\sigma_b) = -2(b^r_i b'^s_b - b^r.b'^s\delta_{ib} + b^r_b b'^s_i). \tag{6.3.62}$$

We recall that in the formula (6.3.19), we have to apply a projector Λ_{ijab}, and following the computation in the helicity 3/2 part which are valid in this case since N is symmetric,

$$\begin{aligned} S &= \Lambda_{ijab}X_{(a}N_{b)(i}Y_{j)} \\ &= \frac{1}{2}(X_1Y_1+X_2Y_2)(N_{11}+N_{22}) = -2(X_1Y_1+X_2Y_2)(b^r_1b'^s_1 - b^r.b'^s + b^r_2b'^s_2) \\ &= 2(X_1Y_1+X_2Y_2)b^r_3b'^s_3. \end{aligned} \tag{6.3.63}$$

From (6.3.61), we have $b^{\pm}_3 = \frac{1}{2}\sin\theta$, thus

$$S = \frac{1}{2}(X_1Y_1+X_2Y_2)\sin\theta\sin\theta'. \tag{6.3.64}$$

In the part of the computation we are looking at now, $X = Y = p$ and so the spinor part is proportional to

$$(p_1^2+p_2^2)\sin\theta\sin\theta' = p^2\sin^3\theta\sin\theta'.$$

The product of the polarisations in (6.3.58) is proportional to rs and the spinor part is independent of the helicities, so when we perform the sum over r, s, the result is 0.
For the last polarisation combination we obtain

$$\frac{2}{9}\epsilon^{\mu\, r\, \vec{p}}\epsilon_{\mu}^{0\, \vec{p}'}\epsilon^{*\nu\, s\, \vec{p}'}\epsilon_{\nu}^{*0\, \vec{p}}\bar{\mathbf{v}}_r^{\vec{p}}\gamma_{(i}p_{j)}\mathbf{u}_s^{\vec{p}'}\bar{\mathbf{u}}_{-s}^{\vec{p}'}\gamma_{(a}p_{b)}\mathbf{v}_{-r}^{\vec{p}} \tag{6.3.65}$$

The product of the polarisations is also proportional to rs, and the spinor part is the same as the previous case, so the contribution in this case is also 0.

- The second term is the $\Delta_2\Delta_2$ contribution.

For the first polarisation case given in (6.3.52), this contribution is proportional to

$$\frac{2}{9}\bar{\mathbf{v}}_{-l}^{\vec{p}}\left(\epsilon_{\mu}^{l\, \vec{p}}p'^{\mu}\gamma_{(i}\epsilon_{j)}^{0\, \vec{p}'}+\epsilon_{(i}^{l\, \vec{p}}\gamma_{j)}p^{\mu}\epsilon_{\mu}^{0\, \vec{p}'}\right)\mathbf{u}_s^{\vec{p}'}\bar{\mathbf{u}}_s^{\vec{p}'}\left(\epsilon_{(a}^{*0\, \vec{p}'}\gamma_{b)}p'^{\nu}\epsilon_{\nu}^{*l\, \vec{p}}+\epsilon_{\nu}^{*0\, \vec{p}'}p^{\nu}\gamma_{(a}\epsilon_{b)}^{*l\, \vec{p}}\right)\mathbf{v}_{-l}^{\vec{p}}. \tag{6.3.66}$$

Using $\epsilon_{\mu}^{l\, \vec{p}}p'^{\mu}=-\frac{1}{\sqrt{2}}lp'\sin(\theta'-\theta)$ and $\epsilon_{\mu}^{0\, \vec{p}'}p^{\mu}=\frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)-1)$, and multiplying by Λ_{ijab} we get, omitting the wave-factors

$$\begin{aligned}\frac{2}{9}&\left(-\frac{1}{\sqrt{2}}p'l\sin(\theta'-\theta)\epsilon_{(i}^{0\, \vec{p}'}+\frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)-1)\epsilon_{(i}^{l\, \vec{p}}\right)M_{j)(a}\times\\&\left(-\frac{1}{\sqrt{2}}p'l\sin(\theta'-\theta)\epsilon_{b)}^{*0\, \vec{p}'}+\frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)-1)\epsilon_{b)}^{l\, \vec{p}}\right)\end{aligned} \tag{6.3.67}$$

Using equation (6.3.34), and the following relations on the polarisations: $\epsilon_1^{0\, \vec{p}'}\epsilon_1^{*0\, \vec{p}'}+\epsilon_2^{0\, \vec{p}'}\epsilon_2^{*0\, \vec{p}'}=\frac{E'^2}{m_{3/2}^2}\sin^2\theta$, $\epsilon_1^{l\, \vec{p}}\epsilon_1^{*l\, \vec{p}}+\epsilon_2^{l\, \vec{p}}\epsilon_2^{*l\, \vec{p}}=\frac{1}{2}(1+\cos^2\theta)$, and $\epsilon_1^{0\, \vec{p}'}\epsilon_1^{*l\, \vec{p}}+\epsilon_2^{0\, \vec{p}'}\epsilon_2^{*l\, \vec{p}}=-\frac{E'l}{\sqrt{2}m_{3/2}}\cos\theta\sin\theta'$ it reduces to

$$\begin{aligned}\frac{1}{9}(1+rl\cos\theta\cos\theta')&\left[\frac{E'^2p'^2}{2m_{3/2}^2}\sin^2\theta'\sin^2(\theta'-\theta)+\frac{pp'^2E'}{m_{3/2}^2}\sin(\theta'-\theta)(\cos(\theta'-\theta)-1)\cos\theta\sin\theta'\right.\\&\left.+\frac{(pp')^2}{2m_{3/2}^2}(\cos(\theta'-\theta)-1)^2(1+\cos^2\theta)\right].\end{aligned} \tag{6.3.68}$$

Summing over l and s we obtain eventually this polarisation contribution to be

$$\frac{2}{9m_{3/2}^2}\left(p'^2(p'\sin\theta'\sin(\theta'-\theta)+p\cos\theta(\cos\theta'-\theta)-1))^2+p^2p'^2(\cos\theta'-\theta)-1)^2\right). \tag{6.3.69}$$

The second polarisation contribution is given by

$$\frac{2}{9}\bar{\mathbf{v}}_l^{\vec{p}}\left(\epsilon_{\mu}^{0\, \vec{p}}p'^{\mu}\gamma_{(i}\epsilon_{j)}^{s\, \vec{p}'}+\epsilon_{(i}^{0\, \vec{p}}\gamma_{j)}p^{\mu}\epsilon_{\mu}^{s\, \vec{p}'}\right)\mathbf{u}_s^{\vec{p}'}\bar{\mathbf{u}}_s^{\vec{p}'}\left(\epsilon_{(a}^{*r\, \vec{p}'}\gamma_{b)}p'^{\nu}\epsilon_{\nu}^{*0\, \vec{p}}+\epsilon_{\nu}^{*r\, \vec{p}'}p^{\nu}\gamma_{(a}\epsilon_{b)}^{*0\, \vec{p}}\right)\mathbf{v}_{-l}^{\vec{p}}. \tag{6.3.70}$$

We can see directly from this expression that, as in the $\Delta_1\Delta_1$ part, the result will be the same as for the previous polarisation contribution, exchanging $\vec{p}$ and $\vec{p'}$.
We can pass to the third polarisation contribution which is written

$$\frac{2}{9}\bar{\mathbf{v}}_l^{\vec{p}}\left(\epsilon_{\mu}^{0\, \vec{p}}p'^{\mu}\gamma_{(i}\epsilon_{j)}^{r\, \vec{p}'}+\epsilon_{(i}^{0\, \vec{p}}\gamma_{j)}\epsilon_{\mu}^{r\, \vec{p}'}p^{\mu}\right)\mathbf{u}_{-r}^{\vec{p}'}\bar{\mathbf{u}}_{-r}^{\vec{p}'}\left(\epsilon_{(a}^{*0\, \vec{p}'}\gamma_{b)}p'^{\nu}\epsilon_{\nu}^{*l\vec{p}}+\epsilon_{\nu}^{*0\, \vec{p}'}p^{\nu}\gamma_{(a}\epsilon_{b)}^{*l\, \vec{p}'}\right)\mathbf{v}_{-l}^{\vec{p}}. \tag{6.3.71}$$

We compute the contraction between the polarisations and the momentum $\epsilon_\mu^{0\,\vec{p}} p'^\mu = \epsilon_\mu^{0\,\vec{p}'} p^\mu = \frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)-1)$, $\epsilon_\mu^{r\,\vec{p}'} p^\mu = -\frac{1}{\sqrt{2}} rpsin(\theta-\theta')$ and $\epsilon_\nu^{*l\vec{p}} p'^\nu = -\frac{1}{\sqrt{2}} lp'\sin(\theta'-\theta)$ and we get, always omitting the wave factors

$$\frac{2}{9}(\frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)\epsilon_{(i}^{r\,\vec{p}'} - \frac{1}{\sqrt{2}} rp\sin(\theta-\theta')\epsilon_{(i}^{0\,\vec{p}})N_{j)(a}\times$$
$$(-\frac{1}{\sqrt{2}} lp'\sin(\theta'-\theta)\epsilon_{b)}^{*0\,\vec{p}'} + \frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)-1)\epsilon_{b)}^{*l\,\vec{p}}) \tag{6.3.72}$$

When we apply the projector we use the formula (6.3.64), and we need some relations of polarisations contractions :

$$\epsilon_1^{r\,\vec{p}'}\epsilon_1^{*l\,\vec{p}} + \epsilon_2^{r\,\vec{p}'}\epsilon_2^{*l\,\vec{p}} = \frac{1}{2}(1+lr\cos\theta\cos\theta')\ ,\ \epsilon_1^{r\,\vec{p}'}\epsilon_1^{*0\,\vec{p}'} + \epsilon_2^{r\,\vec{p}'}\epsilon_2^{*0\,\vec{p}'} = \frac{-r}{\sqrt{2}m_{3/2}}E'\sin\theta'\cos\theta'$$
$$\epsilon_1^{0\,\vec{p}}\epsilon_1^{*l\,\vec{p}} + \epsilon_2^{0\,\vec{p}}\epsilon_2^{*l\,\vec{p}} = \frac{-l}{\sqrt{2}m_{3/2}}E\sin\theta\cos\theta\ ,\ \epsilon_1^{0\,\vec{p}}\epsilon_1^{*0\,\vec{p}'} + \epsilon_2^{0\,\vec{p}}\epsilon_2^{*0\,\vec{p}'} = \frac{EE'}{2m_{3/2}^2}\sin\theta\sin\theta'.$$

The polarisations contribution becomes

$$\frac{1}{9}\sin\theta\sin\theta'\Bigg[\left(\frac{pp'}{m_{3/2}}\right)^2(\cos(\theta'-\theta)-1)^2\frac{1}{2}(1+rl\cos\theta\cos\theta')$$
$$+\frac{rl}{2m_{3/2}^2}pp'^2E'\sin\theta'\cos\theta'\sin(\theta'-\theta)(\cos(\theta'-\theta)-1)$$
$$+\frac{rl}{2m_{3/2}^2}p^2Ep'\sin\theta\cos\theta\sin(\theta-\theta')(\cos(\theta'-\theta)-1)$$
$$-\frac{rl}{2m_{3/2}^2}pEp'E'\sin(\theta-\theta')^2\sin\theta\sin\theta'\Bigg]$$

summing over l, r ends to

$$\frac{2}{9}\left(\frac{pp'}{M}\right)^2\sin\theta\sin\theta'(\cos(\theta'-\theta)-1)^2. \tag{6.3.73}$$

The fourth polarisation will give the same result, as there is an exchange over $\vec{p}\leftrightarrow\vec{p'}$ and the result above is invariant under this transformation.

- We are let with the crossing term $\Delta_1\Delta_2$.

The expression to compute with the first polarisation combination of (6.3.52) is now

$$\frac{2}{9}\bar{\mathbf{v}}_{-l}^{\vec{p}}\left[\epsilon_\mu^{l\,\vec{p}}\gamma_{(i}p_{j)}\epsilon^{\mu\,0\,\vec{p}'}\mathbf{u}_r^{\vec{p}'}\bar{\mathbf{u}}_r^{\vec{p}'}(\epsilon_{(a}^{*0\,\vec{p}'}\gamma_b p'^\nu\epsilon_\nu^{*l\,\vec{p}} + \epsilon_\nu^{*0\,\vec{p}'}p^\nu\gamma_{(a}\epsilon_{b)}^{*l\,\vec{p}})\right. \tag{6.3.74}$$
$$\left.+(\epsilon_\mu^{l\,\vec{p}}p'^\mu\gamma_{(i}\epsilon_{j)}^{0\,\vec{p}'} + \epsilon_{(i}^{l\,\vec{p}}\gamma_{j)}p^\mu\epsilon_\mu^{0\,\vec{p}'})\mathbf{u}_r^{\vec{p}'}\bar{\mathbf{u}}_r^{\vec{p}'}\epsilon_\nu^{*0\,\vec{p}'}\gamma_{(a}p_{b)}\epsilon^{*\nu\,l\,\vec{p}}\right]\mathbf{v}_{-l}^{\vec{p}}. \tag{6.3.75}$$

This time, the polarisations contractions give

$$\epsilon_\mu^{l\,\vec{p}}\epsilon^{\mu\,0\,\vec{p}'} = \frac{lE'}{2m_{3/2}}\sin(\theta-\theta')\ ,\ \epsilon_\nu^{*l\,\vec{p}}p'^\nu = -\frac{lp'}{\sqrt{2}}\sin(\theta'-\theta)\ \text{and}\ \epsilon_\nu^{*0\,\vec{p}'}p^\nu = \frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)-1)$$

and we obtain

$$\frac{2}{9}\left[\frac{lE'}{2m_{3/2}}\sin(\theta-\theta')p_{(i}M_{j)(a}\left(-\frac{lp'}{\sqrt{2}}\sin(\theta'-\theta)\epsilon^{*0\,\vec{p}'}_{b)}+\frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)-1)\epsilon^{*l\,\vec{p}}_{b)}\right)\right.$$
$$\left.+\left(-\frac{lp'}{\sqrt{2}}\sin(\theta'-\theta)\epsilon^{0\,\vec{p}'}_{(i}+\frac{pp'}{m_{3/2}}(\cos(\theta'-\theta)-1)\epsilon^{l\,\vec{p}}_{(i}\right)M_{j)(a}\frac{lE'}{2m_{3/2}}\sin(\theta-\theta')p_{b)}\right].$$

In this case, to apply (6.3.34) we will use the polarisations contractions $p_1\epsilon^{*0\,\vec{p}'}_1+p_2\epsilon^{*0\,\vec{p}'}_2=p_1\epsilon^{0\,\vec{p}'}_1+p_2\epsilon^{0\,\vec{p}'}_2=\frac{pE'}{m_{3/2}}\sin\theta\sin\theta'$ and $p_1\epsilon^{*l\,\vec{p}}_1+p_2\epsilon^{*l\,\vec{p}}_2=p_1\epsilon^{l\,\vec{p}}_1+p_2\epsilon^{l\,\vec{p}}_2=-\frac{lp}{\sqrt{2}}\sin\theta\cos\theta$, giving

$$\frac{2}{9}\left[\frac{E'^2pp'}{2m^2_{3/2}}\sin(\theta'-\theta)^2\sin\theta\sin\theta'-\frac{p^2E'p'}{2m^2_{3/2}}\sin(\theta-\theta')\sin\theta\cos\theta(\cos(\theta'-\theta)-1)\right](1+rl\cos\theta\cos\theta')$$
$$=\frac{4}{9m^2_{3/2}}\left[p'^3p\sin^2(\theta-\theta')\sin\theta\sin\theta'-p^2p'^2\sin(\theta-\theta')(\cos(\theta'-\theta)-1)\sin\theta\cos\theta\right] \tag{6.3.76}$$

after we performed the sum over rl.
The second polarisations contribution will give the same result exchanging once again $\vec{p}\leftrightarrow\vec{p}'$, and so we can go to the third contribution which is given by

$$\frac{2}{9}\left[\epsilon^{0\,\vec{p}}_{\mu}\epsilon^{\mu\,r\,\vec{p}'}\epsilon^{*0\,\vec{p}'}_{(a}\epsilon^{*l\,\vec{p}}_{\nu}\bar{\mathbf{v}}^{\vec{p}}_{l}\gamma_{(i}p_{j)}\mathbf{u}^{\vec{p}'}_{-r}\bar{\mathbf{u}}^{\vec{p}'}_{r}\gamma_{b)}p'^{\nu}\mathbf{v}^{\vec{p}}_{-l}+\epsilon^{0\,\vec{p}}_{\mu}\epsilon^{\mu\,r\,\vec{p}'}\epsilon^{*0\,\vec{p}'}_{\nu}\epsilon^{*l\,\vec{p}}_{(a}\bar{\mathbf{v}}^{\vec{p}}_{l}\gamma_{(i}p_{j)}\mathbf{u}^{\vec{p}'}_{-r}\bar{\mathbf{u}}^{\vec{p}'}_{r}\gamma_{b)}p^{\nu}\mathbf{v}^{\vec{p}}_{-l}\right.$$
$$\left.+\epsilon^{0\,\vec{p}}_{\mu}\epsilon^{r\,\vec{p}'}_{(i}\epsilon^{*0\,\vec{p}'}_{\nu}\epsilon^{*\nu\,l\,\vec{p}}\bar{\mathbf{v}}^{\vec{p}}_{l}p'^{\mu}\gamma_{j)}\mathbf{u}^{\vec{p}'}_{-r}\bar{\mathbf{u}}^{\vec{p}'}_{r}\gamma_{(a}p_{b)}\mathbf{v}^{\vec{p}}_{-l}+\epsilon^{0\,\vec{p}}_{(i}\epsilon^{r\,\vec{p}'}_{\mu}\epsilon^{*0\,\vec{p}'}_{\nu}\epsilon^{*l\,\vec{p}}_{(a}\bar{\mathbf{v}}^{\vec{p}}_{l}\gamma_{j)}p^{\mu}\mathbf{u}^{\vec{p}'}_{-r}\bar{\mathbf{u}}^{\vec{p}'}_{r}\gamma_{b)}p^{\nu}\mathbf{v}^{\vec{p}}_{-l}\right]$$
$$=\frac{2}{9}\left[-\frac{rlEp'}{2m_{3/2}}\sin^2(\theta-\theta')p_{(i}N_{j)(a}\epsilon^{*0\,\vec{p}'}_{b)}+\frac{rpEp'}{\sqrt{2}m^2_{3/2}}\sin(\theta-\theta')(\cos(\theta-\theta')-1)p_{(i}N_{j)(a}\epsilon^{*l\,\vec{p}}_{b)}\right.$$
$$\left.+\,sin(\theta-\theta')(\cos(\theta-\theta')-1)\left(\frac{lE'pp'}{\sqrt{2}m^2_{3/2}}\epsilon^{r\,\vec{p}'}_{(i}N_{j)(a}p_{b)}-\frac{rp^2p'}{\sqrt{2}m_{3/2}}\epsilon^{0\,\vec{p}}_{(i}N_{j)(a}\epsilon^{*l\,\vec{p}}_{b)}\right)\right].$$

We can see from this formula that all the four different terms are proportional to the product lr and so, when summing, will give a 0 contribution; the same is true for the fourth polarisation. Eventually, this concludes the computation.

We have now to gather all the results, taking into account the wave factors, integrals and numerical factors that we have omitted. The part coming from the third and fourth polarisation contributions in (6.3.52) is quite simple, because only the $\Delta_2\Delta_2$ term is non-zero in this case and it gives

$$\Pi^2_1=\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}W^{(\frac{1}{2})}_{\vec{k},\vec{p}}W^{*(\frac{1}{2})}_{\vec{k},\vec{p}}\frac{1}{18}\frac{p^2p'^2}{m^2_{3/2}}(\cos(\theta-\theta')-1)^2\sin\theta\sin\theta'. \tag{6.3.77}$$

For the first and second polarisation, since the result are equivalent, we will just look at the first contribution. There are three term to sum up which gives

$$\Pi^2_2=\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}W^{(\frac{1}{2})}_{\vec{k},\vec{p}}W^{*(\frac{1}{2})}_{\vec{k},\vec{p}}\frac{1}{9m^2_{3/2}}\left[p'^4\sin^2\theta\sin^2(\theta'-\theta)+\right. \tag{6.3.78}$$

$$+\frac{1}{4}\left(p'^2(p'\sin\theta'\sin(\theta'-\theta)+p\cos\theta(\cos\theta'-\theta)-1))^2+p^2p'^2(\cos\theta'-\theta)-1)^2\right)$$
$$-\left(p'^3p\sin^2(\theta-\theta')\sin\theta\sin\theta'-p^2p'^2\sin(\theta-\theta')(\cos(\theta'-\theta)-1)\sin\theta\cos\theta\right)\Big]$$
$$=\int\frac{d^3\vec{p}}{(2\pi)^3}W^{(\frac{1}{2})}_{\vec{k},\vec{p}}W^{*(\frac{1}{2})}_{\vec{k},\vec{p}}\frac{1}{36m^2_{3/2}}\left((\cos\theta-\cos\theta')^2+(\cos(\theta'-\theta)^2-1)^2\right) \tag{6.3.79}$$

Since this result is symmetric under the exchange $\vec{p}\leftrightarrow\vec{p'}$, we can use the same for the second polarisation contribution and we get

$$\Pi^2_{\frac{1}{2}}(k,t,t')\simeq\frac{1}{2\pi^2}\int_{p,p'\gg m_{3/2}}dp\,d\theta\ \ K^{(\frac{1}{2})}(p,k,\theta,m_{3/2})\ \ W^{(\frac{1}{2})}_{\vec{k},\vec{p}}(t)W^{(\frac{1}{2})*}_{\vec{k},\vec{p}}(t')\,. \tag{6.3.80}$$

with

$$K^{(\frac{1}{2})}(p,k,\theta,m_{3/2}) = \frac{1}{36m^2_{3/2}}p^4p'^2\sin\theta\{(\cos\theta-\cos\theta')^2+4\sin^4(\frac{\theta-\theta'}{2})(1+\sin\theta\sin\theta')\} + \cdots, \tag{6.3.81}$$

The dots are here to recall that we omit in all the computation the terms subleading and proportional to $m_{3/2}$ in $\cdots$.

6.3.4 Summary and interpretation

We have computed the unequal time correlator in the previous section, and there is just a few steps before we obtain the spectrum of energy density of the gravitational wave production. But before gathering the result and write it in a practical way, there are some remarks we can do.
First, we can see that the two formulas we wrote, (6.3.48) and (6.3.80) present a UV divergence. An important step before extracting a quantitative result is to remove these divergences in the momentum integral. The regularized operator is built from the non-regularized one by subtracting the zero point fluctuations. Since at each time t the physical vacuum is different, we should use the time-dependent vacuum defined in (6.3.13),

$$\begin{aligned}\langle O(t)\rangle_{reg} &\equiv \langle 0|O(t)|0\rangle-\langle 0_t|O(t)|0_t\rangle\\ &= \langle 0|O(t)-\tilde{O}(t)|0\rangle.\end{aligned} \tag{6.3.82}$$

In the second line, we introduced an operator $\tilde{O}(t)$ in which all the fields are defined after Bogoliubov transformations. We follow [105] where it was proposed, for an operator formed by products of several bilinear spinor fields, that the regularized operator can be written by simply dressing the wave functions by the occupation number,

$$\tilde{u}^{(|\lambda|)}_{\vec{p},\pm}=\sqrt{2}|\beta^{(|\lambda|)}_{\vec{p}}|u^{(|\lambda|)}_{\vec{p},\pm}. \tag{6.3.83}$$

Through the use of the regularized wave functions,

$$\tilde{W}^{(|\lambda|)}_{\vec{k},\vec{p}}(t)=2|\beta^{(|\lambda|)}_{\vec{p}}(t)||\beta^{(|\lambda|)}_{\vec{p'}}(t)|\,\{u^{(|\lambda|)}_{\vec{p},+}(t)u^{(|\lambda|)}_{\vec{p'},+}(t)-u^{(|\lambda|)}_{\vec{p},-}(t)u^{(|\lambda|)}_{\vec{p'},-}(t)\}, \tag{6.3.84}$$

we get an effective ultraviolet cutoff as particles are not excited when occupation numbers vanish.
Now that we have correctly define our wave-function in order to have proper result, we can take a look at the factors $p^2/m_{3/2}^2$ in the spin-1/2 helicity. These factors are those expected for longitudinal modes following the equivalence theorem. The integral in (6.3.80) has only contributions from the relativistic regime where the equivalence theorem for spin-3/2 massive states shows that the couplings of their helicity-1/2 components are enhanced with respect to the helicity-3/2 ones [109, 72, 176] by factors of $p/m_{3/2}$. Knowing that, we expect the helicity-1/2 components to produce stronger gravitational wave signals. One can compare $K^{(\frac{1}{2})}$ (6.3.81) to $K^{(\frac{3}{2})}$ (6.3.49) and the dependence for spin-1/2 fermions. The latter was found in [105] to scale like $p^4 \sin^3\theta$.

It may be easier to understand the equivalence theorem when the spin-3/2, here the gravitino, acquires a mass through a super-Higgs mechanism. Imposing cancellation of the vacuum energy allows one to identify the scale of supersymmetry breaking as $\sqrt{F} = \sqrt{\sqrt{3}m_{3/2}M_{Pl}}$. The power law behaviour is then valid for momenta in the range $m_{3/2} \ll p \ll \sqrt{\sqrt{3}m_{3/2}M_{Pl}}$. Discussion of the necessity of this UV cutoff using a bottom-up approach for massive Rarita-Schwinger fields minimally coupled to gravity can also be found for example in [196]. Therefore, in our computation the maximum energy scale for p and p', which corresponds to the vanishing occupation number through the regularization process (6.3.84), is required to be below this cutoff. Indeed, in the example of the next section, we would see that the nonadiabatic production of fermions forms a Fermi sphere whose radius k_F is related to the mass of the scalar field source, and thus to the symmetry breaking scale.

Finally, we can obtain the density energy spectrum by plugging (6.3.80) into the subhorizon spectrum (6.3.5). Taking into account the background evolution, we get

$$\frac{d\rho_{GW}}{d\log k}(k,t) \simeq \frac{Gk^3}{\pi^3 a^4(t)} \int dp\, d\theta\, K^{(\frac{1}{2})}(p,k,\theta,m_{3/2})\, \{|I_c(k,p,\theta,t)|^2 + |I_s(k,p,\theta,t)|^2\}, \tag{6.3.85}$$

where

$$I_c(k,p,\theta,t) = \int_{t_i}^{t} \frac{dt'}{a(t')} \cos(kt')\, \tilde{W}^{(\frac{1}{2})}_{\vec{k},\vec{p}}(t'), \qquad I_s(k,p,\theta,t) = \int_{t_i}^{t} \frac{dt'}{a(t')} \sin(kt')\, \tilde{W}^{(\frac{1}{2})}_{\vec{k},\vec{p}}(t') \tag{6.3.86}$$

parametrize the spectrum of helicity-1/2 component. Then, (6.3.85) is the master equation for gravitational waves produced from nonadiabatic spin-3/2 gases.

6.4 Spin-3/2 state produced during preheating and gravitational wave spectrum

In the previous section, we have obtained the dependence of the spectrum in the kinematic variable, but there is always a time-dependence to compute, which depends on the mode of production of our particles. This is encoded in the wave functions $u^{(|\lambda|)}_{\vec{p},\pm}$. We need a model for this quantity in order to extract quantitative results for the expected gravitational waves spectrum. We have so to fix a model for the production of gravitinos during preheating since it is our case of interest. We will choose a toy model, which has the advantage to give explicit results.

Processes that produce gravitinos in the early Universe can be separated in two classes: thermal and nonthermal. Unless they are more massive than the reheating temperature, gravitinos can always be thermally produced in scattering of the particles in the thermal bath [180, 58, 205, 48, 96]. But this process is adiabatic. A nonthermal case is provided for example when gravitinos are produced during preheating by the conversion of the energy stored in the coherently oscillating scalar field. As this process is nonadiabatic, it provides a possible framework for production of gravitational waves, and so for our computation. For our purpose, we consider the case where supersymmetry breaking by the F terms always dominates the one from the inflaton (i.e., the curvature). This ensures that $\mathcal{H} \ll m_{3/2}$, which allows one to neglect curvature when discussing the production of gravitational waves as we have assumed to derive the power spectrum.

As we have seen in the results of the previous section, the signal of the gravitational waves is stronger for the helicity $\pm 1/2$ of the gravitinos. Thus, we will consider the case of generation of gravitational waves by longitudinal modes. In a FRW background, the corresponding equations of motion can be written in the form [162, 127]

$$[i\gamma^0\partial_0 - a\, m_{3/2} + (A + iB\gamma^0)\vec{p}\cdot\gamma] \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = 0, \qquad (6.4.1)$$

where we have omitted the label $\frac{1}{2}$ and a is the conformal factor in (6.3.1). The functions A and B satisfy $A^2 + B^2 = 1$.

As the initial condition, $u^{(|\lambda|)}_{\vec{p},\pm}$ satisfies the vanishing occupation number condition in (6.3.12). Since the wave function is isotropic, without losing generality, we take the momentum $\vec{p}$ to lie along the z direction. Following [127], we define

$$A + iB = \exp(2i \int \theta(t)dt), \qquad \text{and} \qquad f(t)_\pm = \exp(\mp i \int \theta(t)dt) u_\pm\,, \qquad (6.4.2)$$

and the equation of motion (6.4.1) becomes

$$\ddot{f}_\pm + [p^2 + (\theta + m_{3/2}a)^2 \pm i(\dot{\theta} + \dot{m_{3/2}a})] f_\pm = 0. \qquad (6.4.3)$$

In this equation, we can see that θ plays the role of a source for our wave-functions. We will give a simple example when this θ is non-zero, and coming from the oscillation of a scalar field. This model is the Polonyi model studied in [104]. This model is supersymmetric, and the corresponding Kähler potential and superpotential were chosen to be

$$\mathcal{K} = |z|^2 - \frac{|z|^4}{\Lambda^2}, \qquad (6.4.4)$$

$$\mathcal{W} = \mu^2 z + \mathcal{W}_0, \qquad (6.4.5)$$

where z is the Polonyi field. An estimate of the mass order near the minimum is

$$m_{3/2} \simeq \frac{\mu^2}{\sqrt{3}M_{Pl}^2} \simeq \frac{\mathcal{W}_0}{M_{Pl}^2}, \qquad m_z \simeq 2\sqrt{3}\frac{m_{3/2}M_{Pl}}{\Lambda}. \qquad (6.4.6)$$

Here $\simeq$ means, in particular, that we neglect the higher order terms suppressed by M_{Pl} and tune the cosmological constant to almost 0. Requiring $\Lambda \ll M_{Pl}$ leads to $m_z \gg m_{3/2}$. It is also assumed that the F term of z does not contribute to the Hubble expansion but is large enough to lead to a gravitino mass that satisfies

$$\mathcal{H} \ll m_{3/2} \ll m_z, \tag{6.4.7}$$

in agreement with previous section assumptions; in particular, the background curvature can be neglected and we can use the flat space quantization. What is interesting with this simple model is that even with this assumption on the range of mass of the particles and the curvature, it was shown in [104] that the Polonyi model contains a nontrivial source term $\theta(t)$ in (6.4.3) to produce helicity-1/2 gravitino,

$$\theta(t) = -\frac{a m_z^2 \delta z}{2\sqrt{3} m_{3/2} M_{Pl}} = -\frac{a m_z^2 \delta z}{2F}, \tag{6.4.8}$$

where $\delta z = z - z_0$ is the displacement of z from its value z_0 at the minimum of the the scalar potential and $F = \sqrt{3} m_{3/2} M_{Pl}$ is the supersymmetry breaking scale.

Now that we have the source term, we can use it in the equations of motion (6.4.3) in order to estimate the production of longitudinal modes of the gravitinos. This modes couple to δz through the θ term. The equation of motion becomes

$$\ddot{f}_\pm + [k^2 + (a\, m_{3/2} - \frac{a m_z^2 \delta z}{2F})^2 \mp i \frac{a m_z^2 \dot{\delta z}}{2F}] f_\pm = 0. \tag{6.4.9}$$

When we look at (6.4.9) for helicity-1/2 gravitinos, we see that this form is quite the same as the one of spin-1/2 fermions produced nonadiabatically from Yukawa coupling with a coherently oscillating scalar, with a quadratic potential for this scalar. Thus one expects the spectrum of helicity-1/2 gravitino and (6.3.86) in this model to be similar to the spin-1/2 fermion cases considered in [105]. The effective Yukawa coupling $\tilde{y}$, required to be smaller than 1 by unitarity, reads

$$\tilde{y} = \frac{m_z^2}{2F}. \tag{6.4.10}$$

According to [133, 134], the fermion production in this case is expected to fill up a Fermi sphere with comoving radius,

$$k_F \sim (a/a_I)^{1/4} q^{1/4} m_z, \qquad q \equiv \frac{\tilde{y}^2 z_I^2}{m_z^2}, \tag{6.4.11}$$

where q is the resonance parameter and z_I is the initial vacuum expectation value (i.e., where inflation ends) of the Polonyi field. Outside the Fermi sphere, the occupation number decreases exponentially. Thus one can expect that the peak of the gravitational wave spectrum corresponds to the radius of Fermi-sphere $k_p \sim k_F$, which in the present case leads to the characteristic frequency,

$$f_p \simeq 6 \cdot 10^{10} \tilde{y}^{\frac{1}{2}} \text{Hz}. \tag{6.4.12}$$

One can see from above and (6.4.10) that the validity of the Effective Field Theory requires the peak frequency to be below 10^{10} Hz. The amplitude at the peak of the gravitational wave spectrum can also be estimated by taking the result of a spin 1/2 field and multiplying it by the enhancement factor $(\frac{k_F}{m_{3/2}})^2$,

$$\begin{aligned} h^2\Omega_{GW}(f_p) &\simeq 2.5\cdot 10^{-12}(\frac{m_z^2}{z_I M_{Pl}})^2(\frac{a_*}{a_I})^{\frac{1}{2}}q^{\frac{3}{2}}(\frac{k_F}{m_{3/2}})^2 \\ &= 3\cdot 10^{-11}\tilde{y}^6(\frac{z_I}{m_z})^2(\frac{a_*}{a_I}) \\ &\simeq 3\cdot 10^{-10}(\frac{f_p}{6\cdot 10^{10}\text{Hz}})^{12}(\frac{z_I}{m_z})^2, \end{aligned} \quad (6.4.13)$$

where a_I and a_* are the scale factor at initial time and the end of gravitational wave production respectively. We assumed an order 10 increase for scale factor in the last line. The relation between the amplitude and the peak frequency is shown in Figure (6.2). If we take $z_I = 10^{-3}\, M_{Pl}, m_z = 10^{10}$ GeV, the amplitude at peak frequency $3\cdot 10^9$ Hz is $7.3\cdot 10^{-16}$. For a peak appearing at lower frequency, the amplitude is too small to be observed due to the power 12 in (6.4.13). For very large value of $\frac{z_I}{m_z}$, one can consider that the low frequency tail gets enhanced enough to become observable at lower frequency detector.

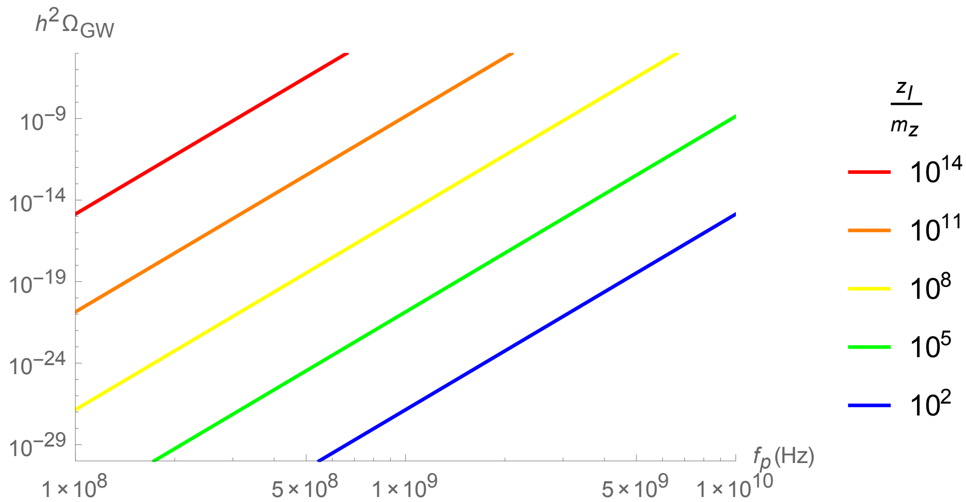

Figure 6.1 – The peak amplitude of gravitational wave according to (6.4.13).

We said that the equation of motion for the wave function was close to the one of a spin 1/2. But we want to see if there is a difference between the power spectrum of a gravitino and the one of a spin 1/2. Near the peak, the dominant part of $K^{(\frac{1}{2})}$ (6.3.81) scales as $p^4\frac{k^2}{m_{3/2}^2}$, which makes the spectrum (6.3.85) go as k^5, as can be seen on Figure [].

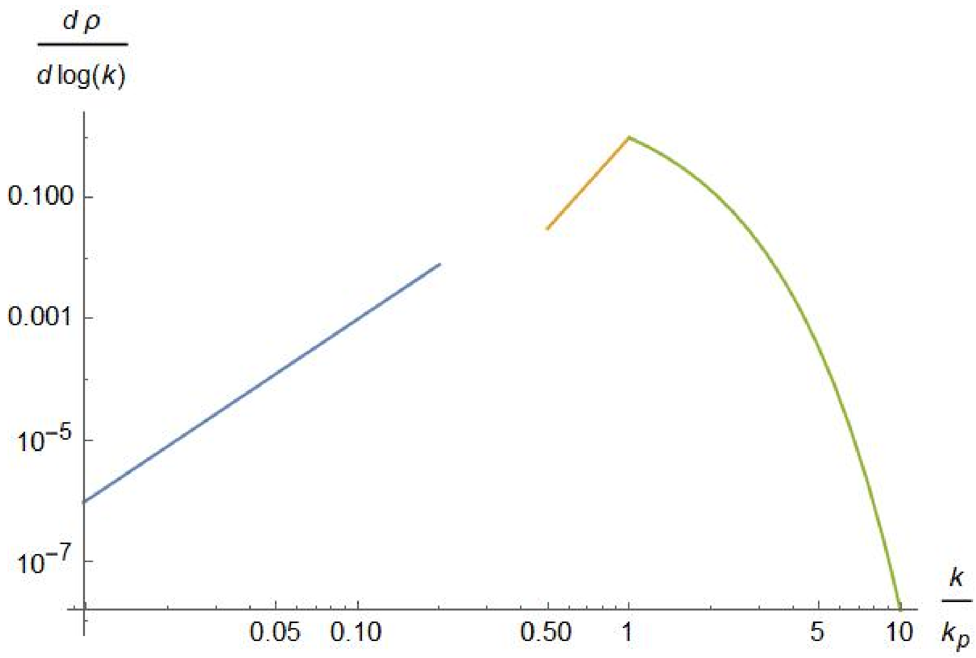

Figure 6.2 – schematic illustration of the normalized power spectrum in function of k. The small k behavior proportional to k^3 is in blue, the behavior close to the peak in k^5 is in yellow, and the exponential decreasing behavior in green.

This behaviour is different from the one of a spin 1/2. However, when we look at lower frequency, the spectrum becomes k^3, thus scaling again like the spin $1/2$. These two different scaling behaviours are a feature of gravitational wave spectrum produced from spin-3/2 particles. Interestingly, the difference of scaling between the spin 1/2 and the gravitino is close to the peak, where it is easier to detect these gravitational waves, facilitating the discrimination between the different spins. Our calculations are made in a simple model, but since the peak was due to the presence of a filled-up Fermi-sphere, the result can be thought as quite model independent.

6.5 Conclusion

The gravitational wave signals from spin-3/2 fermions are especially interesting since the latter are the only missing piece in the nature with spin between 0 and 2. Moreover, their presence can be a smoking gun for the existence of supersymmetry and a clue for its role in the early Universe. The nonadiabatic production of helicity-1/2 gravitino in a simple model takes the similar form as spin-1/2 fermions nonadiabatically produced from coherently oscillating scalars with quadratic potential. Thus it fills up a Fermi sphere in the occupation number. The corresponding comoving radius governs the position of the peak frequency of the gravitational waves. Their spectrum has two main differences compared to the one from spin-1/2 fermions. The first one is the order of the amplitude, which gets enhanced by a factor of $\frac{k_F^2}{m_{3/2}^2}$. The second is given by the stress tensor of spin-3/2 state, which contains a term proportional to the gravitational wave mode $\vec{k}$, while for scalars and spin-1/2 fermions, the $\vec{k}$ dependence is projected out by the projector $\Lambda_{ij,lm}(\vec{k})$ on traceless-transverse modes. Thus one could expect a k^5 dependence for the gravitational wave spectrum near the peak. The observed window for the gravitational waves, like most preheating scenarios, lies at very high frequency, around 10^9 Hz in our simple example, which calls for the design of new experiments like [206]. The value of the frequency of the peak is dependent of the inflation model, so it is possible that some more accurate models can manage a smaller value of the frequency of the peak, keeping the strength at the same level, and so facilitating the detection of such gravitational waves.

CHAPTER 7

Higgs *alignment* in 2HDM

7.1 Introduction

The previous chapter was dedicated to a possibility to detect an elementary particle of spin 3/2, which has not been observed yet. But if we look at the known particle spectrum in the Standard Model (SM), there is one thing we can notice. In contrast to fermions and vectors, there is only one known fundamental spin zero particle in Nature: the SM Higgs boson. However, a lot of theoretical models require additional fundamental scalars, as in Early Universe cosmological and supersymmetric models. If we focus on the latter, we will have two new types of scalars: those which are the supersymmetric partner of the SM fermions, and some additional Higgs scalars appearing in their electroweak symmetry breaking sectors. The first thing we can do with these new scalars is to search the constraints on the mixings of these particles with the observable Higgs. By now the LHC experiments data are putting very strong constraints on these mixings. In order to manage the new scalars with these observations, we have to set that the observed Higgs, which is an eigenstate of the scalars mass matrix, is *aligned* with the direction acquiring a non-zero vev. One simple way to achieve this is to make all the additional scalars heavy enough, and thus decoupling them from the theory at low energy. But a more interesting option is that alignment emerges as a consequence of specific patterns of the model. The benefits are that less constraints on masses allow to keep new scalars within the reach of future searches at the LHC, but we have to found a specific model in which this phenomenon emerges. This is possible, and was obtained in [17] and further discussed in [102, 54].

The effective low energy scalar potential of [17], studied in details in [36], corresponds to a peculiar case of a Two Higgs Doublet Model (2HDM). There is a large broad of 2HDM (see for example [137, 61, 95]). In these types of models, symmetries play a great role (e.g. [87, 159, 112]), and in particular for some peculiar cases they imply alignment without decoupling [136, 57, 171]. Unfortunately, these symmetries have been quoted to lead to problematic phenomenological consequences, as massless quarks [113]. But even if these symmetries are broken or not present in the model, alignment without decoupling remains viable. This situation was discussed for example in [55, 56, 71, 70, 138] for the MSSM and NMSSM. But in order to obtain this alignment, we need an ad-hoc specific choice of the model parameters, and this is not totally satisfactory. However, it exists some models where the alignment is present without symmetry and without any "tuning" of parameters, as in the work already mentioned [17]. In this model, the alignment at tree-level is a prediction, and survive with a impressive precision when radiative corrections, up to two-loop, are taken into account [54]. The question is so when does this alignment come from? This chapter is dedicated to the question of uncovering the symmetry at the origin of this automatic Higgs alignment.

The crucial ingredient in [17] is the presence of $\mathcal{N} = 2$ extended supersymmetry. This $\mathcal{N} = 2$ supersymmetry enables us to have the Higgs alignment. However, it is difficult to build up a model with only $\mathcal{N} = 2$ supersymmetry, that is to say the case where this symmetry is present on the entire spectrum. Models where $\mathcal{N} = 2$ supersymmetry acts on the whole SM states suffer from the non-chiral nature of quarks and leptons [108, 91], and so are not relevant for phenomenology. To overcome this issue, that is to say allowing both $\mathcal{N} = 2$ supersymmetry and chirality, we can add some ingredients in the model, inspired by superstring theory. In this set-up, orbifold fixed points and brane localizations enable to construct models where different parts preserve different amounts of supersymmetries. It is such a construction which is used in [17] : the (non-chiral) gauge and Higgs states appear in a $\mathcal{N} = 2$ supersymmetry sector while the matter states, quarks and leptons, appear in an $\mathcal{N} = 1$ sector and so are chiral. An important feature of such constructions considered here [26, 25, 17, 6, 36] is that gauginos have Dirac masses instead of Majorana ones [110, 193, 140, 117, 39]. Besides, these $\mathcal{N} = 2$ extended models have implication for Higgs boson physics as discussed in [36, 40, 11, 41, 79, 43, 52, 158, 53, 38, 177, 59, 215, 74, 84].

If we can exhibit in such a model an alignment, it is not totally satisfactory to explain it only by the presence of $\mathcal{N} = 2$ extended supersymmetry because this is realized only at a very high energy, the fundamental scale of the theory. However, the alignment is present even at low energy, where the supersymmetry is no longer present. In order to explain the origin of such an alignment, we have to look at some symmetry manifest in the scalar potential of 2HDM. In this chapter, we will exhibit the relevant symmetry.

This chapter is organized as follows. In Section 2, the main ingredients of the model are presented succinctly. This allows to define the notation used through this work. In section 3, we show that the potential can be written as a sum of two $SU(2)_R$ R-symmetry singlet representations. We also show how this implies alignment. In section 4, we review how higher order corrections induce a small misalignment. Section 5 presents our conclusions.

7.2 Short presentation of the model

As was said in the introduction, the model presents a $\mathcal{N} = 2$ part including the gauge fields and the Higgs. This structure of the gauge sector implies the presence of chiral superfields in the adjoint representations of SM gauge group, representing the superpartners of the SM fields. These are a singlet $\mathbf{S}$ and an $SU(2)$ triplet $\mathbf{T}$. We note that in the model studied in [54], there is also an octet $\mathbf{O}$. This field will introduce new soft breaking terms, and bring contributions to loop computations. In the following, we will focus only on the electroweak sector, and so forget the presence of this octet.

We define

$$S = \frac{S_R + iS_I}{\sqrt{2}} \tag{7.2.1}$$

$$\mathbf{T} = \frac{1}{2}\begin{pmatrix} T_0 & \sqrt{2}T_+ \\ \sqrt{2}T_- & -T_0 \end{pmatrix}, \quad T_i = \frac{1}{\sqrt{2}}(T_{iR} + iT_{iI}) \quad \text{with} \quad i = 0, +, - \tag{7.2.2}$$

In this work we will use two specificities present in this model. As we have said in the introduction, the adjoint field will promote the gauginos from Majorana to Dirac fermions, but they will also generate new Higgs interactions. These two contributions will be visible when we write the superpotential of this model:

$$\begin{aligned} W = \quad & \sqrt{2}\,\mathbf{m}^{\alpha}_{1D}\mathbf{W}_{1\alpha}\mathbf{S} + 2\sqrt{2}\,\mathbf{m}^{\alpha}_{2D}\mathbf{tr}\left(\mathbf{W}_{2\alpha}\mathbf{T}\right) + \frac{M_S^2}{2}\mathbf{S}^2 + \frac{\kappa}{3}\mathbf{S}^3 + M_T\,\mathrm{tr}(\mathbf{T}\mathbf{T}) \\ & + \mu\,\mathbf{H_u}\cdot\mathbf{H_d} + \lambda_S S\,\mathbf{H_u}\cdot\mathbf{H_d} + 2\lambda_T\,\mathbf{H_d}\cdot\mathbf{T}\mathbf{H_u}\,, \end{aligned} \tag{7.2.3}$$

where the Dirac masses can be read by taking $\mathbf{m}_{\alpha iD} := \theta_\alpha m_{iD}$ where θ_α are the Grassmanian superspace coordinates. We want also to break supersymmetry at low energy, and in order to implement this condition in the model, we will define the soft terms appearing in the Higgs and adjoint scalar sectors. For simplicity we chose them to be real, and we get

$$\begin{aligned} \mathcal{L}_{\text{soft}} =& m^2_{H_u}|H_u|^2 + m^2_{H_d}|H_d|^2 + B\mu(H_u\cdot H_d + \text{h.c}) \\ & + m_S^2|S|^2 + 2m_T^2\mathbf{tr}(T^\dagger T) + \frac{1}{2}B_S\left(S^2 + h.c\right) + B_T\left(\mathbf{tr}(TT) + h.c.\right) \\ & + A_S\left(SH_u\cdot H_d + h.c\right) + 2A_T\left(H_d\cdot TH_u + h.c\right) + \frac{A_\kappa}{3}\left(S^3 + h.c.\right) + A_{ST}\left(S\mathrm{tr}(TT) + h.c\right). \end{aligned} \tag{7.2.4}$$

Considering specific scenarios for generation of the above soft-terms lead us to make some assumption about their relative sizes. In the case of gauge mediation, one is first tempted to consider secluded supersymmetry breaking and the mediators sectors to have an $\mathcal{N} = 2$ structure [108]. This would allow to preserve the underlying R-symmetry. Unfortunately, it leads to tachyonic directions for the adjoint scalars. To overcome this, we must restrict to particular $\mathcal{N} = 1$ breaking and mediating sectors [39, 50, 84] (see also [11, 184, 8, 9]). As a consequence, the $SU(2)_R$ structure is not preserved by the quadratic part of the scalar potential given by the soft breaking terms. Furthermore, we will consider that the remaining $U(1)_R$ symmetry is broken by the presence of a non-vanishing $B\mu$ term (keeping zero the coefficients of supersymmetric terms as M_S or M_T). We also take $\kappa = 0$ for simplicity. This avoids then the introduction of extra-doublets as in the MRSSM [11] and allows us to consider an effective

2HDM. Note also that the trilinear terms in the last line of (7.2.4) will be neglected here, as they were shown to be generically small [39, 50].

Now that we have discussed the size of the soft term in the potential, we can minimize it. This minimization was discussed in [36]. An effective 2HDM is obtained by taking the limit $m_S^2, m_T^2 \gg m_Z^2$ and keeping the Dirac masses m_{iD} as well as $m_{H_u}^2, m_{H_d}^2, B\mu$ and μ small. Note that $B\mu$ measures the size of $\mathcal{N}=1$ $U(1)_R$ symmetry breaking. The effective theory just above the electroweak scale is a 2HDM with a set of light charginos and neutralinos. The minimization of the corresponding potential was discussed in [36]. We restrict to the case of CP neutral vacuum, i.e. $H_{dI}^0 = H_{uI}^0 = 0$ which implies also $S_I = T_I = 0$. Dropping the obvious 0 indices for neutral components, we define:

$$
\begin{aligned}
M_Z^2 &= \frac{g_Y^2 + g_2^2}{4} v^2 , \qquad v \simeq 246\text{GeV} && (7.2.5)\\
< H_{uR} > &= v s_\beta, \qquad < H_{dR} >= v c_\beta, && (7.2.6)\\
< S_R > &= v_s , \qquad < T_R >= v_t && (7.2.7)
\end{aligned}
$$

where:

$$
\begin{aligned}
c_\beta &\equiv \cos\beta, \qquad s_\beta \equiv \sin\beta, \qquad t_\beta \equiv \tan\beta , \qquad 0 \leqslant \beta \leqslant \frac{\pi}{2}\\
c_{2\beta} &\equiv \cos 2\beta, \qquad s_{2\beta} \equiv \sin 2\beta && (7.2.8)
\end{aligned}
$$

Also, the leading-order squared-masses for the real part of the adjoint fields are given by [52]:

$$
m_{SR}^2 = m_S^2 + 4m_{DY}^2 + B_S, \qquad m_{TR}^2 = m_T^2 + 4m_{D2}^2 + B_T . \qquad (7.2.9)
$$

Note that as we have

$$
\begin{aligned}
v_s &\simeq \frac{v^2}{2m_{SR}^2} \left[\ \ g_Y m_{1D} c_{2\beta} + \sqrt{2}\mu\lambda_S \right]\\
v_t &\simeq \frac{v^2}{2m_{TR}^2} \left[-g_2 m_{2D} c_{2\beta} - \sqrt{2}\mu\lambda_T \right] , && (7.2.10)
\end{aligned}
$$

the bounds on the expectation value of the Dirac Gauginos-triplet come from the electroweak precision data, i.e. the contribution to the ρ parameter, remains acceptable in our scenario as we keep m_{iD} and μ an order of magnitude smaller than m_{SR} and m_{TR}: the light charginos and neutralinos are in the sub-TeV energy region accessible to future searches.

7.3 R-symmetric Higgs alignment

Now that we have presented succinctly the model, we will reformulate the quantities defined in the previous section in order to highlight the symmetry. Indeed, this model has a $U(1)_R \times SU(2)_R$ global R-symmetry. The $U(1)_R$ is the R-symmetry present at the level of $\mathcal{N}=1$ and plays no role in what follows. Using

$$
\Phi_2 = H_u, \qquad \Phi_1^i = -\epsilon_{ij}(H_d^j)^* \Leftrightarrow \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix} = \begin{pmatrix} \Phi_1^0 \\ -(\Phi_1^+)^* \end{pmatrix} \qquad (7.3.1)
$$

the two Higgs doublets can be assembled into one hypermultiplet $(\Phi_1, \Phi_2)^T$ in the fundamental representation of the $SU(2)_R$ R-symmetry. The coupling of vector multiplet fields to hypermultiplet leads then to the presence of additional terms in the superpotential,

$$W_{\text{Higgs}} \supset \lambda_S \mathbf{S}\, \mathbf{\Phi_2}^\dagger \cdot \mathbf{\Phi_1} + 2\lambda_T\, \mathbf{\Phi_1}^\dagger \cdot \mathbf{T\Phi_2} \tag{7.3.2}$$

and $\mathcal{N} = 2$ supersymmetry requires

$$\lambda_S = \frac{1}{\sqrt{2}} g_Y, \qquad \lambda_T = \frac{1}{\sqrt{2}} g_2 \tag{7.3.3}$$

where g_Y and g_2 stand for the hypercharge and $SU(2)$ gauge couplings, respectively.

The integration out of adjoint scalars leads to a potential for the Higgs fields that corresponds to a peculiar 2HDM. We can parametrize a generic 2HDM under the form:

$$V_{EW} = V_{2\Phi} + V_{4\Phi} \tag{7.3.4}$$

where

$$\begin{aligned} V_{2\Phi} &= m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c}] \\ V_{4\Phi} &= \frac{1}{2}\lambda_1 (\Phi_1^\dagger \Phi_1)^2 + \frac{1}{2}\lambda_2 (\Phi_2^\dagger \Phi_2)^2 \\ &\quad + \lambda_3 (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_4 (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \\ &\quad + \left[\frac{1}{2}\lambda_5 (\Phi_1^\dagger \Phi_2)^2 + [\lambda_6 (\Phi_1^\dagger \Phi_1) + \lambda_7 (\Phi_2^\dagger \Phi_2)] \Phi_1^\dagger \Phi_2 + \text{h.c} \right], \end{aligned} \tag{7.3.5}$$

from which we can write down

$$m_{11}^2 = m_{H_d}^2 + \mu^2, \qquad m_{22}^2 = m_{H_u}^2 + \mu^2, \qquad m_{12}^2 = B\mu. \tag{7.3.6}$$

The parameters λ_i can be decomposed as their leading order tree-level values and corrections due to loops $\delta\lambda_i^{(rad)}$, but we will have to add a threshold contribution coming from the integrating out of heavy states $\delta\lambda_i^{(tree)}$,

$$\lambda_i = \lambda_i^{(0)} + \delta\lambda_i^{(tree)} + \delta\lambda_i^{(rad)} \tag{7.3.7}$$

whose values were computed in [36, 53].

In our minimal model:

$$\lambda_5 = \lambda_6 = \lambda_7 = 0. \tag{7.3.8}$$

while

$$\begin{aligned} \lambda_1^{(0)} = \lambda_2^{(0)} &= \frac{1}{4}(g_2^2 + g_Y^2) \\ \lambda_3^{(0)} &= \frac{1}{4}(g_2^2 - g_Y^2) + 2\lambda_T^2 \qquad \xrightarrow{\mathcal{N}=2} \qquad \frac{1}{4}(5g_2^2 - g_Y^2) \end{aligned}$$

$$\lambda_4^{(0)} = -\frac{1}{2}g_2^2 + \lambda_S^2 - \lambda_T^2 \qquad \xrightarrow{\mathcal{N}=2} \qquad -g_2^2 + \frac{1}{2}g_Y^2 \tag{7.3.9}$$

At this leading order, the quartic part of the potential can then be put in the form:

$$\begin{aligned} V_{4\Phi} &= \frac{3g_2^2}{8}\left[(\Phi_1^\dagger\Phi_1) + (\Phi_2^\dagger\Phi_2)\right]^2 + \frac{(-2g_2^2+g_Y^2)}{8}\left[\left((\Phi_1^\dagger\Phi_1) - (\Phi_2^\dagger\Phi_2)\right)^2 + 4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\right] \\ &= \lambda_{|0_1,0>}|0_1,0\rangle \quad + \quad \lambda_{|0_2,0>}|0_2,0\rangle \end{aligned} \tag{7.3.10}$$

where we recognize between brackets the two invariant combinations under $SU(2)_R$. These quartic terms, thus product of four fields, are the combination of the two $SU(2)_R$ doublets giving singlet irreducible representations. In the last equality, they are written in the standard spin representation notation $|l,m>$ with l the spin and m its projection along the z axis. They were classified in [159]. However, we found a minus sign for the $|0_2,0\rangle$, compensated by a minus sign in $\lambda_{|0_2,0>}$ compared to the previous cited classification. We can write:

$$|0_1,0\rangle \;=\; \tfrac{1}{2}\left[(\Phi_1^\dagger\Phi_1) + (\Phi_2^\dagger\Phi_2)\right]^2, \tag{7.3.11}$$

$$|0_2,0\rangle \;=\; -\tfrac{1}{\sqrt{12}}\left[\left((\Phi_1^\dagger\Phi_1) - (\Phi_2^\dagger\Phi_2)\right)^2 + 4(\Phi_2^\dagger\Phi_1)(\Phi_1^\dagger\Phi_2)\right] \tag{7.3.12}$$

while

$$\lambda_{|0_1,0>} \;=\; \frac{\lambda_1+\lambda_2+2\lambda_3}{4} = \frac{3g_2^2}{4} \tag{7.3.13}$$

and

$$\lambda_{|0_2,0>} = -\frac{\lambda_1+\lambda_2-2\lambda_3+4\lambda_4}{4\sqrt{3}} = -\frac{\sqrt{3}(-2g_2^2+g_Y^2)}{4}\,. \tag{7.3.14}$$

Therefore, the $SU(2)_R$ R-symmetry acts here as an $SU(2)$ Higgs family symmetry [87, 159]. The potential contains only terms that are invariant (singlet) under $SU(2)_R$.

For the case of CP conserving Lagrangian under consideration, there are two CP even scalars with squared-mass matrix

$$\mathcal{M}_h^2 = \begin{pmatrix} Z_1 v^2 & Z_6 v^2 \\ Z_6 v^2 & m_A^2 + Z_5 v^2 \end{pmatrix}. \tag{7.3.15}$$

The coefficient Z_6, present in the off-diagonal terms of the mass-matrix is a measure of the displacement from the alignment, and so we will try to give a value at this coefficient in our model.

Using the notation $\lambda_{345} \equiv \lambda_3 + \lambda_4 + \lambda_5$, we have

$$\begin{aligned} Z_1 &= \lambda_1 c_\beta^4 + \lambda_2 s_\beta^4 + \frac{1}{2}\lambda_{345}s_{2\beta}^2, \qquad \xrightarrow{\mathcal{N}=2} \qquad \frac{1}{4}(g_2^2+g_Y^2) \\ Z_5 &= \frac{1}{4}s_{2\beta}^2\left[\lambda_1+\lambda_2-2\lambda_{345}\right]+\lambda_5 \qquad \xrightarrow{\mathcal{N}=2} \qquad 0 \end{aligned}$$

$$Z_6 = -\frac{1}{2}s_{2\beta}\left[\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2 - \lambda_{345}c_{2\beta}\right] \quad \xrightarrow{\mathcal{N}=2} \quad 0. \tag{7.3.16}$$

while the pseudoscalar mass m_A is given by

$$m_A^2 = -\frac{m_{12}^2}{s_\beta c_\beta} - \lambda_5 v^2 \quad \xrightarrow{\mathcal{N}=2} \quad -\frac{m_{12}^2}{s_\beta c_\beta} \tag{7.3.17}$$

Denoting by M_Z and M_W the Z and W boson masses respectively, we find that the $SU(2)_R$ symmetric potential leads to a diagonal squared-mass matrix with eigenvalues:

$$\begin{aligned} m_h^2 &= \frac{1}{4}(g_2^2 + g_Y^2)v^2 = M_Z^2 \\ m_H^2 &= m_A^2 \end{aligned} \tag{7.3.18}$$

and a charged Higgs of mass

$$m_{H^+}^2 = \frac{1}{2}(\lambda_5 - \lambda_4)v^2 + m_A^2 \quad \xrightarrow{\mathcal{N}=2} \quad \frac{1}{2}(g_2^2 - \frac{1}{2}g_Y^2)v^2 + m_A^2 = 3M_W^2 - M_Z^2 + m_A^2. \tag{7.3.19}$$

As expected, we find that $Z_6 = 0$, i.e. there is an Higgs alignment in this model. It is a well known fact (e.g. [136]) that Z_6 can be expressed, as we have written above as functions of only the quartic potential parameters, or as function of mass parameters. The different expressions are related by minimization conditions. We would like now to explain what this implies in our specific model for the mass parameters, and can explain the cancellation of Z_6.

Let's turn now to the quadratic part of the potential. It can be written as:

$$\begin{aligned} V_{2\Phi} \;=\; & \frac{m_{11}^2 + m_{22}^2}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\left[(\Phi_1^\dagger\Phi_1) + (\Phi_2^\dagger\Phi_2)\right] \\ & + \frac{m_{11}^2 - m_{22}^2}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\left[(\Phi_1^\dagger\Phi_1) - (\Phi_2^\dagger\Phi_2)\right] \\ & - [m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c}] \end{aligned} \tag{7.3.20}$$

Only the first line corresponds to the $SU(2)_R$ invariant term. One can be tempted so to impose a complete symmetry of the potential. Imposing a Higgs family symmetry would have required that both coefficients of the two non-invariant operators to vanish, therefore $m_{11}^2 = m_{22}^2$ and $m_{12} = 0$. This is problematic in our model. First, it implies $m_A^2 = 0$. But there is also another reason why we do not want to impose invariance of the quadratic potential under $SU(2)_R$: the mass terms in the potential are controlled by supersymmetry breaking effects. We can take a look at one possible scenario of supersymmetry breaking as the gauge mediation one. In this case, An $\mathcal{N} = 2$ supersymmetry breaking and messengers sectors has been shown to lead to tachyonic masses for the adjoint scalars S and T [39]. Avoiding this instability requires peculiar structure of these sectors that is not compatible with the R-symmetry. So the presence of the non-invariant $SU(2)_R$ is required for the self-consistency of our model.

However, it is always possible to obtain some conditions on the coefficients of the quadratic terms from the minimization of the potential. The corresponding equations take the form (e.g. [139]):

$$\begin{aligned} 0 &= m_{11}^2 - t_\beta m_{12}^2 + \frac{1}{2} v^2 c_\beta^2 (\lambda_1 + 3\lambda_6 t_\beta + \lambda_{345} t_\beta^2 + \lambda_7 t_\beta^3) \\ 0 &= m_{22}^2 - \frac{1}{t_\beta} m_{12}^2 + \frac{1}{2} v^2 s_\beta^2 (\lambda_2 + 3\lambda_7 \frac{1}{t_\beta} + \lambda_{345} \frac{1}{t_\beta^2} + \lambda_6 \frac{1}{t_\beta^3}) \end{aligned} \quad (7.3.21)$$

Plugging the values of $\lambda_i^{(0)}$ in (7.3.21) leads to the equations:

$$0 = m_{11}^2 - t_\beta m_{12}^2 + \frac{1}{8}(g_2^2 + g_Y^2) v^2 \quad (7.3.22)$$

$$0 = m_{22}^2 - \frac{1}{t_\beta} m_{12}^2 + \frac{1}{8}(g_2^2 + g_Y^2) v^2 \quad (7.3.23)$$

Subtraction of one of the equations from the other one leads (for $s_{2\beta} \neq 0$) to

$$0 = \frac{1}{2}(m_{11}^2 - m_{22}^2) s_{2\beta} + m_{12}^2 c_{2\beta} \equiv Z_6 v^2 \quad (7.3.24)$$

So, even if the quadratic term are not invariant under $SU(2)_R$, we see that values of the quartic couplings imply that the $SU(2)_R$ violating terms are such that their combination contributing to Z_6 vanishes. Therefore, we have shown that the constraint of $SU(2)_R$ invariance of the quartic part of the potential is sufficient to insure an automatic alignment without decoupling. On the other hand, we also found that given m_{11}^2, m_{22}^2 and m_{12}^2 the potential minimization equation fixes β such that an alignment is obtained. This is different from the [136, 57] where β remains arbitrary.

These results also allow us to understand why the simple identification of couplings in the Higgs couplings with their $\mathcal{N} = 2$ expected values, as it was attempted for the MRSSM in [54], fails to achieve alignment. There, the integration out of the additional doublets breaks the $SU(2)_R$ R-symmetry explicitly at tree level, and the $SU(2)_R$ symmetry of the quartic terms in the potential is required to obtain the alignment.

7.4 R-symmetry breaking and misalignment

We have shown how the alignment is enforced by the $SU(2)_R$ symmetry of the quartic potential. However, it is clear that the leading order values $\lambda_i^{(0)}$ received corrections from many sources that do not respect the $SU(2)_R$ symmetry. It was numerically checked, computing two-loop quantum corrections, that the alignment remains amazingly true to a very high precision when taking into account the full corrections [54]. We would like to investigate how these sub-leading terms affect not only the numerical value but the $SU(2)_R$ group theory structure of the scalar potential. The quartic scalar potential can then be written as:

$$V_{4\Phi} = \sum_{i,j} \lambda_{|i,j>} \times |i,j\rangle \quad (7.4.1)$$

where $|i,j\rangle$ are irreducible representations of $SU(2)_R$.

The alignment is measured by computing the off-diagonal squared-mass matrix element

$$Z_6 = -\frac{1}{2} s_{2\beta} \left[\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2 - \lambda_{345} c_{2\beta} \right] \tag{7.4.2}$$

Interestingly, this can be recasted as

$$Z_6 = \frac{1}{2} s_{2\beta} \left[\sqrt{2}\lambda_{|1,0>} - \sqrt{6}\lambda_{|2,0>} c_{2\beta} + (\lambda_{|2,-2>} + \lambda_{|2,+2>}) c_{2\beta}. \right] \tag{7.4.3}$$

with, using the notation of [159]:

$$\begin{aligned} |1,0\rangle &= \tfrac{1}{\sqrt{2}} \left[(\Phi_2^\dagger \Phi_2) - (\Phi_1^\dagger \Phi_1) \right] \left[(\Phi_1^\dagger \Phi_1) + (\Phi_2^\dagger \Phi_2) \right] \\ |2,0\rangle &= \tfrac{1}{\sqrt{6}} \left[(\Phi_1^\dagger \Phi_1)^2 + (\Phi_2^\dagger \Phi_2)^2 - 2(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) - 2(\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1) \right] \\ |2,+2\rangle &= (\Phi_2^\dagger \Phi_1)(\Phi_2^\dagger \Phi_1) \\ |2,-2\rangle &= (\Phi_1^\dagger \Phi_2)(\Phi_1^\dagger \Phi_2) \end{aligned} \tag{7.4.4}$$

The coefficients that play a role in the misalignment (7.4.3) are given as function of λ_i by:

$$\begin{aligned} \lambda_{|1,0>} &= \frac{\lambda_2 - \lambda_1}{2\sqrt{2}} \xrightarrow{\text{leading order}} 0 \\ \lambda_{|2,0>} &= \frac{\lambda_1 + \lambda_2 - 2\lambda_3 - 2\lambda_4}{\sqrt{24}} \xrightarrow{\text{leading order}} -\frac{1}{\sqrt{24}} \left[(2\lambda_S^2 - g_Y^2) + (2\lambda_T^2 - g_2^2) \right] \\ \lambda_{|2,+2>} &= \frac{\lambda_5^*}{2} \xrightarrow{\text{leading order}} 0, \quad \lambda_{|2,-2>} = \frac{\lambda_5}{2} \xrightarrow{\text{leading order}} 0. \end{aligned} \tag{7.4.5}$$

In the previous section, we have found that breaking of the $SU(2)_R$ invariance in the quartic term of the scalar potential is necessary in order to have misalignment. We can see directly here that the conservation of the $U(1)$ subgroup of $SU(2)_R$ is not sufficient for alignment as we have contribution from $|i,0\rangle$ combinations. Here, we have $\lambda_5 = 0$ thus there is no contribution from $|2,\pm 2>$. The breaking of $SU(2)_R$ symmetry leads then to a contribution to the Z_6 parameter of order:

$$\delta Z_6^{(tree)} = \frac{1}{2} s_{2\beta} \left[\sqrt{2}\delta\lambda_{|1,0>} - \sqrt{6}\delta\lambda_{|2,0>} c_{2\beta} \right] \tag{7.4.6}$$

where $\delta\lambda_{|i,0>}$ are contributions generated by higher order corrections to the tree-level $\lambda_{|i,0>}^{(0)}$. As presented in (7.3.7), there are two types of contributions, one from the integration out of heavy particles, and one from the loops. We will examine both in the following.

First, the parameters λ_i receive tree-level correction from the threshold when the adjoint scalars are integrated out while the Higgs μ-term and the Dirac masses m_{1D}, m_{2D} for the hypercharge $U(1)$ and weak interaction $SU(2)$ are small but not zero:

$$\delta\lambda_1^{(tree)} \simeq -\frac{\left(g_Y m_{1D} - \sqrt{2}\lambda_S \mu\right)^2}{m_{SR}^2} - \frac{\left(g_2 m_{2D} + \sqrt{2}\lambda_T \mu\right)^2}{m_{TR}^2}$$

$$
\begin{aligned}
\delta\lambda_2^{(tree)} &\simeq -\frac{\left(g_Y m_{1D}+\sqrt{2}\lambda_S\mu\right)^2}{m_{SR}^2}-\frac{\left(g_2 m_{2D}-\sqrt{2}\lambda_T\mu\right)^2}{m_{TR}^2}\\
\delta\lambda_3^{(tree)} &\simeq \frac{g_Y^2 m_{1D}^2-2\lambda_S^2\mu^2}{m_{SR}^2}-\frac{g_2^2 m_{2D}^2-2\lambda_T^2\mu^2}{m_{TR}^2}\\
\delta\lambda_4^{(tree)} &\simeq \frac{2g_2^2 m_{2D}^2-4\lambda_T^2\mu^2}{m_{TR}^2},
\end{aligned}
\tag{7.4.7}
$$

The effect on the quartic scalar potential can be written:

$$
\delta V_{4\Phi}^{(tree)} = \delta\lambda_{|0_1,0>}^{(tree)}|0_1,0\rangle+\delta\lambda_{|0_2,0>}^{(tree)}|0_2,0\rangle+\delta\lambda_{|1,0>}^{(tree)}|1,0\rangle+\delta\lambda_{|2,0>}^{(tree)}|2,0\rangle\,. \tag{7.4.8}
$$

The two singlet coefficients get corrections

$$
\delta\lambda_{|0_1,0>}^{(tree)} \simeq -2\lambda_S^2\frac{\mu^2}{m_{SR}^2}-g_2^2\frac{m_{2D}^2}{m_{TR}^2} \tag{7.4.9}
$$

$$
\delta\lambda_{|0_2,0>}^{(tree)} \simeq \frac{1}{\sqrt{3}}\left[g_Y^2\frac{m_{1D}^2}{m_{SR}^2}-2g_2^2\frac{m_{2D}^2}{m_{TR}^2}+6\lambda_T^2\frac{\mu^2}{m_{TR}^2}\right] \tag{7.4.10}
$$

and, by themselves do not contribute to misalignment. The breaking of the $SU(2)_R$ symmetry shows up through the appearance of new terms in the scalar potential:

$$
\begin{aligned}
\delta\lambda_{|1,0>}^{(tree)} &\simeq 2g_2\lambda_T\frac{m_{2D}\mu}{m_{TR}^2}-2g_Y\lambda_S\frac{m_{1D}\mu}{m_{SR}^2}\\
&\simeq \sqrt{2}g_2^2\frac{m_{2D}\mu}{m_{TR}^2}-\sqrt{2}g_Y^2\frac{m_{1D}\mu}{m_{SR}^2}\\
\delta\lambda_{|2,0>}^{(tree)} &\simeq \sqrt{\frac{2}{3}}\left[g_Y^2\frac{m_{1D}^2}{m_{SR}^2}+g_2^2\frac{m_{2D}^2}{m_{TR}^2}\right]
\end{aligned}
\tag{7.4.11}
$$

that preserve the subgroup $U(1)'_R$, as expected since the scalar potential results from integrating out the adjoints which have zero $U(1)'_R$ charge. For a numerical estimate, we take the example of values used in [54] with $m_{SR} \simeq m_{TR} \simeq 5$ TeV, $m_{1D} \simeq m_{1D} \simeq \mu \simeq 500$ GeV, $g_Y \simeq 0.37$ and $g_2 \simeq 0.64$. This gives

$$
\delta\lambda_{|1,0>}^{(tree)} \simeq 4\times 10^{-3}, \qquad \delta\lambda_{|2,0>}^{(tree)} \simeq 4.5\times 10^{-3} \tag{7.4.12}
$$

which show that the contribution to Z_6 is very small and they will not be discussed further below.

The second source of misalignment that we will consider is the contributions from quantum corrections. Radiative corrections to different couplings are generated when supersymmetry is broken, inducing mass splitting between scalars and fermionic partners. This happens for instance through loops of the adjoint scalar fields S and T^a. However, these scalars are singlets under the $SU(2)_R$ symmetry and at leading order, when their couplings λ_S and λ_T are given by their $\mathcal{N}=2$ values, their interactions with the two Higgs doublets preserve $SU(2)_R$. Therefore we do not expect them to lead to any contribution to Z_6. In fact, explicit calculations of these loop diagrams were performed in (3.5) of [54] and it was found that when summed up their contribution to Z_6

cancels out. This unexpected result is now easily understood as the consequence of the $SU(2)_R$ symmetry.

Let's denote the auxiliary fields D^a for the gauge fields A^a and F^a_Σ for the corresponding adjoint scalars $\Sigma^a \in \{S, T^a\}$ of $U(1)_Y$ and $SU(2)$. Then the set:

$$(F^a_\Sigma \ , \ D^a \ , \ F^{a*}_\Sigma) \tag{7.4.13}$$

forms a triplet of $SU(2)_R$. This enforces the equalities $\lambda_S = g_Y/\sqrt{2}$ and $\lambda_T = g_2/\sqrt{2}$ in (7.3.3) whose violation by quantum effects translates into breaking of $SU(2)_R$. First, consider the correction due to running of the above couplings. This accounts for the violation of $\mathcal{N} = 2$ induced relations (7.3.3) due to radiative corrections from $\mathcal{N} = 1$ chiral matter. As λ_1 and λ_2 are affected in the same way, we have $\delta\lambda^{(2\to1)}_{|01,0>} = 0$, and using (7.4.5), we get:

$$\begin{aligned} \delta Z_6^{(2\to1)} &= -\frac{\sqrt{6}}{2}\, s_{2\beta}\, c_{2\beta}\, \delta\lambda^{(2\to1)}_{|2,0>} \\ &= -\frac{1}{2}\,\frac{t_\beta(t_\beta^2-1)}{(t_\beta^2+1)^2}\,\left[(2\lambda_S^2 - g_Y^2) + (2\lambda_T^2 - g_2^2)\right] \end{aligned} \tag{7.4.14}$$

Another source of misalignment comes from the $\mathcal{N} = 1 \to \mathcal{N} = 0$ mass splitting in chiral superfields. The $SU(2)_R$ symmetry is broken by the different Yukawa couplings to the two Higgs doublets, which for $t_\beta \sim \mathcal{O}(1)$ will be dominated by the top Yukawa. The stops contributes mainly by correcting λ_2.

$$\delta\lambda_2 \sim \frac{3y_t^4}{8\pi^2}\log\frac{m_{\tilde{t}}^2}{Q^2} \tag{7.4.15}$$

where y_t, $m_{\tilde{t}}$ are the top Yukawa and stop masses, respectively, while Q is the renormalization scale. This leads to a $\delta Z_6^{(1\to0)}$ induced by $\delta\lambda^{(stops)}_{|1,0>} \sim \sqrt{3}\delta\lambda^{(stops)}_{|2,0>}$ in (7.4.6). Thus, chiral matter through their Yukawa couplings contribute to Z_6 with both of $|1,0\rangle$ and $|2,0\rangle$ combinations of doublets. The contributions $\delta Z_6^{(1\to0)}$ and $\delta Z_6^{(2\to1)}$ have similar strength but opposite sign so that they lead to a small misalignment compatible with LHC bounds as was shown in [54] by explicit computation.

What is interesting to note is that the origin of misalignment is not the quadratic terms breaking $SU(2)_R$. In section 3, we explained how this breaking of $SU(2)_R$ does not imply loss of alignment but fixes $\tan\beta$. The misalignment comes from the presence of chiral matter and the most important single contribution arises from a large top Yukawa coupling, which modifies the quartic terms in the potential.

7.5 Conclusions

When we look at 2HDM, the nature of the fields is largely constraint by the experiments. Indeed, the LHC experiments data require one of the Higgs squared-mass matrix eigenstates to be aligned with the SM-like direction. One has to explain the origin of this peculiar alignment. A case of interest is when this alignment comes without decoupling. The additional scalars in the electroweak sector are then subject to milder constraints, and they might have masses in an energy range that can be reached in future LHC searches, leaving open the possibility of new discovery of fundamental scalars. Such an alignment was not only achieved but also predicted at tree-level in [17]. This success calls for understanding the main mechanism behind it. We have shown in this chapter that this mechanism is an $SU(2)_R$ R-symmetry that acts as a Higgs family symmetry in the quartic scalar potential and enforces an automatic alignment. Another new result of this work is that we have written the CP-even Higgs squared-mass matrix off-diagonal element as a linear combination of the coefficients of non-singlet of $SU(2)_R$ representations. These are generated in the quartic potential by tree level threshold and loop corrections. Their numerical values have been computed in [54] where it was proven that the alignment is preserved to an unexpected precision level. A new way of expressing the observables is presented here. We have found that using $SU(2)_R$ symmetry allows to shed light on some of the existing results. For instance, we understand the origin of the cancellation between loop contributions of the adjoint scalars. When interested by alignment, only some contributions need to be evaluated or computed explicitly in the future: those contributing to coefficients of particular combinations of terms in the potential that break the $SU(2)_R$ symmetry. This sheds light not only on how the alignment is realized here, but also on why $\mathcal{N} = 2$ realizations of other models as the MRSSM attempted in [54] have not been successful. The extension of the quartic potential to an $\mathcal{N} = 2$ sector with a particular attention to preserving the $SU(2)_R$ symmetry at tree-level can be a way to pursue in order to implement alignment in extended Higgs sector models.

APPENDIX A

Dimensional Reduction of massless scalar field on a circle

In this appendix, we will present the dimensional reduction on a circle from a dimension $D+1$ to a dimension D. This is important for the analysis of the Scalar Weak Gravity Conjecture that we did in the chapter 4.

Notation

We will work in $D+1$ dimensions, with the signature $(-,+,...,+)$. All the quantities defined in $D+1$ dimensions will be denoted with a hat, for example $\hat{R}$ the $D+1$-dimensional Ricci scalar. We use latin letters for the $D+1$ dimensional indices, as M, N, greek ones μ, ν for uncompactified dimensions and z for the compactified one. In the same spirit, A, B will be used to define the local frame in $D+1$ dimensions, a, b for the uncompactified ones and $\tilde{z}$ for the compactified one. We use, in general dimension d, $\kappa_d^2 = 8\pi G_d = \frac{1}{M_{P,d}^{d-2}}$, where G_d is the d-dimensional Newton constant and $M_{P,d}$ the d-dimensional reduced Planck mass.

A.1 Gravitational action

We will discuss in this first section the compactification of pure $(D+1)$-dimensional gravity to D dimensions on a circle. We will present two closely related methods to perform the computations.

Direct computation in the Einstein frame

We start with the gravitational action in $D+1$ dimension,

$$\mathcal{S} = \frac{1}{2\hat{\kappa}^2} \int \mathrm{d}^{D+1}x \sqrt{-\hat{g}} \hat{R}, \tag{A.1.1}$$

where the constant $\hat{\kappa}$ is understood to be the $D+1$-dimensional one. We assume here and for the following the so called cylinder condition

$$\partial_z \hat{g} = 0. \tag{A.1.2}$$

One could consider non-vanishing dependence under the additional dimension, but we will focus on the simplest possibility, and use (A.1.2). We write the metric in the form

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{pmatrix} \tag{A.1.3}$$

with ϕ, A_μ and $g_{\mu\nu}$ D-dimensional fields independent of the z coordinate, as dictated by (A.1.2). The inverse metric is then

$$\hat{g}^{MN} = \begin{pmatrix} e^{-2\alpha\phi} g^{\mu\nu} & -e^{-2\alpha\phi} A_\mu \\ -e^{-2\alpha\phi} A_\nu & e^{-2\beta\phi} + e^{-2\alpha\phi} A_\rho A^\rho \end{pmatrix}. \tag{A.1.4}$$

To describe the physics observed by a D-dimensional observer, we should express the action completely in terms of the D-dimensional fields. We will use the Cartan formalism to perform the dimensional reduction, and so we define the vielbeins $\hat{e}^A_M$ with coordinates

$$\hat{e}^a_\mu = e^{\alpha\phi} e^a_\mu\,,\ \hat{e}^a_z = 0\,,\ \hat{e}^{\tilde{z}}_\mu = e^{\beta\phi} A_\mu \text{ and } \hat{e}^{\tilde{z}}_z = e^{\beta\phi}, \tag{A.1.5}$$

leading to the forms

$$\hat{e}^a = e^{\alpha\phi} e^a \text{ and } \hat{e}^{\tilde{z}} = e^{\beta\phi}(A_\mu \mathrm{d}x^\mu + \mathrm{d}z).$$

To find the spin connection, we use the first Cartan structure equation

$$\mathrm{d}\hat{e}^A + \hat{\omega}^A{}_B \wedge \hat{e}^B = 0. \tag{A.1.6}$$

Plugging (A.1.5) in it and using the antisymmetry of the spin connection we find

$$\hat{\omega}^{ab} = \omega^{ab} + \alpha e^{-\alpha\phi}(\partial^b\phi \hat{e}^a - \partial^a\phi \hat{e}^b) + \frac{1}{2} e^{(\beta-2\alpha)\phi} F^{ba} \hat{e}^{\tilde{z}} \tag{A.1.7}$$

$$\hat{\omega}^{a\tilde{z}} = \frac{1}{2} e^{(\beta-2\alpha)\phi} F_b{}^a \hat{e}^b - \beta e^{-\alpha\phi} \partial^a\phi \hat{e}^{\tilde{z}}, \tag{A.1.8}$$

where the two-form F as usual is $F = dA$. The second Cartan equation gives us the Ricci form in terms of the spin connection as

$$\hat{R}^M_N = \mathrm{d}\hat{\omega}^M_N + \hat{\omega}^M_K \wedge \hat{\omega}^K_N \tag{A.1.9}$$

The two form components expressed in the usual frame are the Riemann tensor components, so that one can deduce the Ricci scalar from them. We have for the $M = a$, $N = z$ component

$$\hat{R}^a_{\tilde{z}} = \mathrm{d}\hat{\omega}^a_{\tilde{z}} + \hat{\omega}^a_d \wedge \hat{\omega}^d_{\tilde{z}}. \tag{A.1.10}$$

Finally we obtain

$$\hat{R}^a_{\tilde{z}} = e^{-2\alpha\phi}\left[(\alpha\beta - \beta^2)\partial_c\phi\partial^a\phi - \beta\partial_c\partial^a\phi\right]\hat{e}^c \wedge \hat{e}^{\tilde{z}} - \frac{1}{4} e^{2(\beta-2\alpha)\phi} F_d{}^a F_b{}^d \hat{e}^b \wedge \hat{e}^{\tilde{z}} - \beta e^{-\alpha\phi}\partial^d\phi\omega^a_d \wedge \hat{e}^{\tilde{z}} \tag{A.1.11}$$

$$
\begin{aligned}
&- \alpha\beta e^{-2\alpha\phi}(\partial_b\phi\partial^b\phi\hat{e}^a \wedge \hat{e}^{\tilde{z}} - \partial^a\phi\partial_d\phi\hat{e}^d \wedge \hat{e}^{\tilde{z}}) \\
&+ \frac{1}{2}e^{(\beta-3\alpha)\phi}\left[(\beta - 2\alpha)\partial_c\phi F_b{}^a + \partial_c F_b{}^a\right]\hat{e}^c \wedge \hat{e}^b - \frac{1}{2}e^{(\beta-2\alpha)\phi}F_b{}^a\omega^b_{\ d} \wedge \hat{e}^d \\
&- \frac{1}{2}e^{(\beta-3\alpha)\phi}\left(\alpha F_b{}^a\partial_d\phi + \beta\partial^a\phi F_{bd}\right)\hat{e}^b \wedge \hat{e}^d \\
&+ \frac{1}{2}e^{(\beta-2\alpha)\phi}F_b{}^d\omega^a_{\ d} \wedge \hat{e}^b + \frac{\alpha}{2}e^{(\beta-3\alpha)\phi}\left(\partial_d\phi F_b{}^d\hat{e}^a \wedge \hat{e}^b - \partial^a\phi F_b{}^d\hat{e}_d \wedge \hat{e}^b\right).
\end{aligned}
$$

From this, one can then extract the $\hat{R}^a_{\ \tilde{z}c\tilde{z}}$ and $\hat{R}^a_{\ \tilde{z}bc}$ components of the Riemann tensor:

$$
\hat{R}^a_{\ \tilde{z}c\tilde{z}} = e^{-2\alpha\phi}\left[(2\alpha\beta - \beta^2)\partial_c\phi\partial^a\phi - \beta\partial_c\partial^a\phi - \alpha\beta\partial_d\phi\partial^d\phi\delta^a_c\right] - \frac{1}{4}e^{2(\beta-2\alpha)\phi}F_d{}^aF_c{}^d - \beta e^{-2\alpha\phi}\partial^d\phi\omega^a_{\ dc} \tag{A.1.12}
$$

$$
\begin{aligned}
\hat{R}^a_{\ \tilde{z}bc} = \frac{1}{2}e^{(\beta-3\alpha)\phi}\big[&(\beta - \alpha)\partial_c\phi F^a_{\ b} + \partial_c F^a_{\ b} - \beta\partial^a\phi F_{bc} + \alpha\partial_d\phi F_c{}^d\delta^a_b - \alpha\partial^a\phi F_{cb} - F_k{}^a\omega^k_{\ cb} + F_c{}^d\omega^a_{\ db} \\
&- \{b \leftrightarrow c\}\big]
\end{aligned} \tag{A.1.13}
$$

We thus obtain two of the components of the Ricci tensor in the local frame, given as

$$
\hat{R}_{\tilde{z}\tilde{z}} = \hat{R}^a_{\ \tilde{z}a\tilde{z}} = \beta e^{-2\alpha\phi}\left\{[(2 - D)\alpha - \beta](\partial\phi)^2 - \Box\phi\right\} + \frac{1}{4}e^{2(\beta-2\alpha)\phi}F_{ad}F^{ad} \tag{A.1.14}
$$

and

$$
\hat{R}_{\tilde{z}c} = \hat{R}^a_{\ \tilde{z}ac} = -\frac{1}{2}e^{(\beta-3\alpha)\phi}\left\{\nabla_a F^a_{\ c} + [3\beta - (4 - D)\alpha]\partial_a\phi F^a_{\ c}\right\} \tag{A.1.15}
$$

with $(\partial\phi)^2 = \partial_d\phi\partial^d\phi$, $\Box\phi = \nabla_d\nabla^d\phi = \nabla_d(\partial^d\phi) = \partial_d\partial^d\phi + \omega^d_{\ ad}\partial^a\phi$ and $\nabla_a F^a_{\ c} = \partial_a F^a_{\ c} - \omega^a_{\ da}F_c{}^d + \omega^k_{\ ca}F_k{}^a$.
To find the missing components of the Ricci tensor we again apply the Cartan's second equation to

$$
\hat{R}^a_{\ b} = \mathrm{d}\hat{\omega}^a_{\ b} + \hat{\omega}^a_{\ d} \wedge \hat{\omega}^d_{\ b} + \hat{\omega}^a_{\ \tilde{z}} \wedge \hat{\omega}^{\tilde{z}}_{\ b}, \tag{A.1.16}
$$

giving

$$
\begin{aligned}
\hat{R}^a_{\ b} = &\frac{1}{2}\Big\{ - \partial_c\left[e^{(\beta-2\alpha)\phi}F^a_{\ b}\right]e^{-\alpha\phi}\hat{e}^c \wedge \hat{e}^{\tilde{z}} - \beta e^{(\beta-3\alpha)\phi}F^a_{\ b}\,\partial_d\phi\ \hat{e}^d \wedge \hat{e}^{\tilde{z}} - e^{(\beta-2\alpha)\phi}F^d_{\ b}\,\omega^a_{\ d} \wedge \hat{e}^{\tilde{z}} \\
&- \alpha e^{(\beta-3\alpha)\phi}F^d_{\ b}\,\partial_d\phi\ \hat{e}^a \wedge \hat{e}^{\tilde{z}} + \alpha e^{(\beta-3\alpha)\phi}F^d_{\ b}\,\partial^a\phi\ \hat{e}_d \wedge \hat{e}^{\tilde{z}} + e^{(\beta-2\alpha)\phi}F^a_{\ d}\,\omega^d_{\ b} \wedge \hat{e}^{\tilde{z}} \\
&+ \alpha e^{(\beta-3\alpha)\phi}F^a_{\ d}\,\partial_b\phi\ \hat{e}^d \wedge \hat{e}^{\tilde{z}} - \alpha e^{(\beta-3\alpha)\phi}F^a_{\ d}\,\partial^d\phi\ \hat{e}_b \wedge \hat{e}^{\tilde{z}} - \beta e^{(\beta-3\alpha)\phi}F^a_{\ l}\,\partial_b\phi\ \hat{e}^l \wedge \hat{e}^{\tilde{z}} \\
&+ \beta e^{(\beta-3\alpha)\phi)}\partial^a\phi\,F_{bk}\ \hat{e}^k \wedge \hat{e}^{\tilde{z}}\Big\} \\
&+ \mathrm{d}\omega^a_{\ b} + \alpha e^{-\alpha\phi}\partial_c(e^{-\alpha\phi}\partial_b\phi)\,\hat{e}^c \wedge \hat{e}^a - \alpha^2 e^{-2\alpha\phi}\partial_b\phi\,\partial_d\phi\ \hat{e}^a \wedge \hat{e}^d - \alpha e^{-\alpha\phi}\partial_c(e^{-\alpha\phi}\partial^a\phi)\,\hat{e}^c \wedge \hat{e}_b \\
&+ \alpha e^{-\alpha\phi}\partial^a\phi\,\omega_{bd} \wedge \hat{e}^d + \alpha^2 e^{-2\alpha\phi}\partial^a\phi\,\partial_d\phi\ \hat{e}_b \wedge \hat{e}^d - \frac{1}{4}e^{2(\beta-2\alpha)\phi}F^a_{\ b}\,F_{kl}\ \hat{e}^k \wedge \hat{e}^l + \omega^a_{\ d} \wedge \omega^d_{\ b} \\
&- \alpha e^{-\alpha\phi}\partial^d\phi\,\omega^a_{\ d} \wedge \hat{e}_b - \alpha e^{-\alpha\phi}\partial_d\phi\,\omega^d_{\ b} \wedge \hat{e}^a + \alpha e^{-\alpha\phi}\partial^a\phi\,\omega^d_{\ b} \wedge \hat{e}_d + \alpha^2 e^{-2\alpha\phi}\partial_d\phi\,\partial_b\phi\ \hat{e}^a \wedge \hat{e}^d \\
&- \alpha^2 e^{-2\alpha\phi}(\partial\phi)^2\,\hat{e}^a \wedge \hat{e}_b + \alpha^2 e^{-2\alpha\phi}\partial^a\phi\partial^d\phi\hat{e}_d \wedge \hat{e}_b - \frac{1}{4}e^{2(\beta-2\alpha)\phi}F^a_{\ l}F_{bk}\hat{e}^l \wedge \hat{e}^k,
\end{aligned} \tag{A.1.17}
$$

where the basis elements with a lower index are used as a shortcut for the contraction with local Minkowski metric, as in $\hat{e}_b = \eta_{bc}\hat{e}^c$. With the same reasoning as before one can extract now the $\hat{R}^a_{\ bc\tilde{z}}$ and $\hat{R}^a_{\ bcd}$ components of the form and use that to compute the $\hat{R}_{bd}$ component of the Ricci tensor as

$$\begin{aligned}\hat{R}_{bd} &= \hat{R}^{\tilde{z}}_{\ b\tilde{z}d} + \hat{R}^a_{\ bad} \\ &= e^{-2\alpha\phi}\left\{R_{bd} + \nabla_d(\partial_b\phi)\left[(2-D)\alpha - \beta\right] - \alpha\Box\phi\,\eta_{bd} + \partial_d\phi\,\partial_b\phi\left[\alpha^2(D-2) + 2\alpha\beta - \beta^2\right]\right. \\ &\left. +(\partial\phi)^2\eta_{bd}\left[(2-D)\alpha^2 - \alpha\beta\right]\right\} - \frac{1}{2}e^{2(\beta-2\alpha)\phi}F^a_{\ b}F_{ad}\,,\end{aligned} \tag{A.1.18}$$

where R_{bd} is the D-dimensional Ricci tensor. We eventually get $\hat{R}$:

$$\begin{aligned}\hat{R} &= \eta^{bd}\hat{R}_{bd} + \hat{R}_{\tilde{z}\tilde{z}} \\ &= e^{-2\alpha\phi}\left\{R + [2(1-D)\alpha - 2\beta]\,\Box\phi + \left[(D-2)(1-D)\alpha^2 + 2\beta(2-D)\alpha - 2\beta^2\right](\partial\phi)^2\right\} \\ &- \frac{1}{4}e^{2(\beta-2\alpha)\phi}F_{ab}F^{ab}.\end{aligned} \tag{A.1.19}$$

Finally, the determinant of the metric $\hat{g}$ in $D+1$ dimensions is given by

$$\hat{g} = e^{(2D\alpha - 2\beta)\phi}g, \tag{A.1.20}$$

where g is the determinant of the D-dimensional metric, and thus plugging (A.1.20) and (A.1.19) in (A.1.1) we obtain

$$\begin{aligned}\mathcal{S} = \frac{1}{2\hat{\kappa}^2}\int \mathrm{d}^{D+1}x\ \ e^{(D\alpha+\beta)\phi}\sqrt{-g}\Big\{e^{-2\alpha\phi}\Big[R + \big[2(1-D)\alpha - 2\beta\big]\Box\phi \\ + \left[{\scriptstyle (D-2)(1-D)\alpha^2 + 2\beta\left((2-D)\alpha-\beta\right)}\right](\partial\phi)^2\Big] - \frac{1}{4}e^{2(\beta-2\alpha)\phi}F^2\Big\}.\end{aligned} \tag{A.1.21}$$

We can now proceed to fix the constants. In order to get rid of the exponential in front of the D-dimensional Ricci scalar, that will allow us to get a canonical term for D-dimensional gravity, we require

$$(D-2)\alpha + \beta = 0. \tag{A.1.22}$$

Accordingly, the scalar kinetic term reads $(D-2)(1-D)\alpha^2(\partial\phi)^2$. For it to be canonical we thus fix the constant α to the value

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}. \tag{A.1.23}$$

Since all fields are independent of z, we can perform the integration over this dimension, that we take as compact in the shape of a circle of radius L. Of course, if the additional dimension needs to be hidden for low energy-observers, it should be defined on distances that could only be resolved with higher energies. In that case, we should thus expect the radius L to be small. We can then rewrite the action

$$\mathcal{S} = \frac{2\pi L}{2\hat{\kappa}^2}\int \mathrm{d}^Dx\sqrt{-g}\left(R - 2\alpha\Box\phi - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{2(1-D)\alpha\phi}F^2\right). \tag{A.1.24}$$

The negative sign in front of the kinetic term for the scalar field is the correct one as we are working with the $(-,+,...,+)$ signature. The coefficient in front of R now reads $\frac{2\pi L}{2\hat{\kappa}^2}$. We thus define the D-dimensional constant κ in terms of the $(D+1)$-dimensional $\hat{\kappa}$ as

$$\frac{1}{\kappa^2}=\frac{2\pi L}{\hat{\kappa}^2}\Longrightarrow M_{Pl}^{D-2}=2\pi L\,\hat{M}_{Pl}^{D-1} \tag{A.1.25}$$

Besides, from (A.1.3) we see that the ϕ and A_μ fields that we have used so far are dimensionless. The physical dimension-full fields, that we call $\tilde{\phi}$ and $\tilde{A}_\mu$, are obtained once we reabsorb the $1/2\kappa^2$ pre-factor in front of their kinetic terms as

$$\tilde{\phi}=\frac{\phi}{\sqrt{2}\kappa}=\frac{M_{Pl}^{\frac{D-2}{2}}}{\sqrt{2}}\phi\,;\qquad \tilde{A}_\mu=\frac{A_\mu}{\sqrt{2}\kappa}=\frac{M_{Pl}^{\frac{D-2}{2}}}{\sqrt{2}}A_\mu. \tag{A.1.26}$$

In terms of D-dimensional quantities, the action of $(D+1)$-dimensional pure gravity with one compact dimension in the shape of a circle reads:

$$\mathcal{S}=\int \mathrm{d}^Dx\sqrt{-g}\left(\frac{R}{2\kappa^2}-2\alpha\Box\tilde{\phi}-\frac{1}{2}(\partial\tilde{\phi})^2-\frac{1}{4}e^{2\sqrt{2}(1-D)\alpha\kappa\tilde{\phi}}\tilde{F}^2\right). \tag{A.1.27}$$

The second term is a total derivative and, as such, we will not display it in the following. If the gauge field that we obtain under compactification is usually called graviphoton, and the name of the scalar is radion. This name is due to the fact that the scalar field vev is fixing the length of the compactified dimension. However, a scalar with the same coupling but with a different value of α can appear in different theory, with a different name. We will choose this more generic name and call this scalar a dilaton. In the rest of the work, we will remove the tilde and denote physical fields without it. In the remainder of this section we will instead discuss the second method advocated at the beginning of it.

Computation via Weyl rescaling

In the computation described above, we used a particular form of the metric that brought us all the way down to the action in (A.1.27), that defines an Einstein-Maxwell-scalar theory in the Einstein frame. Of course, one could ask whether this ad-hoc form of the metric could have been guessed from first principles and what happens if one uses a different metric. Here we try to fill this gap by showing how the same computation can be done using the Weyl transformations relating different, physically equivalent, frames.

As a starter, we should compute the variation of the Ricci tensor under a Weyl rescaling of the metric $g\to\bar{g}e^{-2\omega(x)}$ in d generic dimensions.

The vielbeins transforms accordingly as $e^a=e^{-\omega(x)}\bar{e}^a$. Using the above mentioned Cartan's equations we can derive the following transformation rules for the spin connection and the Ricci form

$$\begin{cases}\omega^a_{\ b}=\bar{\omega}^a_{\ b}-\partial_b\omega\,\bar{e}^a+\partial^a\omega\,\bar{e}_b\\ R^a_{\ b}=\frac{1}{2}\bar{R}^a_{\ b}+\partial_c\partial^a\omega\,\bar{e}^c\wedge\bar{e}_b+\partial^d\omega\,\bar{\omega}^a_{\ d}\wedge\bar{e}_b+\partial^a\omega\partial^d\omega\,\bar{e}_d\wedge\bar{e}_b-\frac{1}{2}\partial_d\omega\partial^d\omega\,\bar{e}^a\wedge\bar{e}_b-\{^a\leftrightarrow{}_b\}.\end{cases} \tag{A.1.28}$$

In the same spirit as before, we can now extract from the components of the form the Ricci tensor

$$R_{bd} = R^a_{bad} = e^{2\omega}\left\{\bar{R}_{bd} + (d-2)\nabla_b\nabla_d\omega + \nabla_a\nabla^a\omega\,\eta_{bd} + (d-2)\partial_b\omega\partial_d\omega - (d-2)\partial_a\omega\partial^a\omega\,\eta_{bd}\right\} \tag{A.1.29}$$

and the Ricci scalar

$$R = e^{2\omega}\left(\bar{R} + 2(d-1)\Box\omega + (d-2)(1-d)(\partial\omega)^2\right). \tag{A.1.30}$$

With this result in mind, we can now take the $D+1$-dimensional metric of generic form as

$$\hat{g}_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}. \tag{A.1.31}$$

The corresponding vielbeins read

$$\hat{e}^a = \tilde{e}^a \text{ and } \hat{e}^{\tilde{z}} = A + dz, \text{ with } A = A_\mu \mathrm{d}x^\mu. \tag{A.1.32}$$

The spin connection and the Ricci form associated are

$$\hat{\omega}^{\tilde{z}}_{\ a} = \frac{1}{2}F_{ab}\tilde{e}^b \text{ and } \hat{\omega}^a_{\ b} = \tilde{\omega}^a_{\ b} + \frac{1}{2}F_b{}^a\hat{e}^{\tilde{z}} \tag{A.1.33}$$

$$\hat{R}^a_{\ \tilde{z}} = \frac{1}{2}\left(\partial_c F_b{}^a\tilde{e}^c\wedge\tilde{e}^b - F_b{}^a\tilde{\omega}^b_{\ d}\wedge\tilde{e}^d + F_l{}^c\tilde{\omega}^a_{\ c}\wedge\tilde{e}^l + \frac{1}{2}F_c{}^aF_l{}^c\tilde{e}^{\tilde{z}}\wedge\tilde{e}^l\right) \tag{A.1.34}$$

$$\hat{R}^a_{\ b} = \tilde{R}^a_{\ b} + \frac{1}{2}\partial_c F_b{}^a\tilde{e}^c\wedge\hat{e}^{\tilde{z}} + \frac{1}{2}F_d{}^a\hat{e}^{\tilde{z}}\wedge\tilde{\omega}^d_{\ b} + \frac{1}{2}F_b{}^d\tilde{\omega}^a_{\ d}\wedge\hat{e}^{\tilde{z}} + \frac{1}{4}F_b{}^aF_{cd}\tilde{e}^c\wedge\tilde{e}^d - \frac{1}{4}F_l{}^aF_{db}\tilde{e}^l\wedge\tilde{e}^d \tag{A.1.35}$$

The Ricci scalar is then

$$\hat{R} = \tilde{R} - \frac{1}{4}F^2. \tag{A.1.36}$$

With the metric in (A.1.31) the determinant $\hat{g}$ is $\hat{g} = \tilde{g}$.
We now perform a first Weyl rescaling of the $D+1$-dimensional metric from $\hat{g}$ to $\hat{g}' = e^{2\omega(x)}\hat{g}$, where we take the function $\omega(x)$ to depend only on the non compact coordinates to preserve the cylinder condition (A.1.2). Using the relation $\sqrt{-\hat{g}} = e^{-(D+1)\omega(x)}\sqrt{-\hat{g}'}$ and (A.1.30) for $d = D+1$ we obtain

$$\sqrt{-\hat{g}}\hat{R} = e^{(1-D)\omega}\sqrt{(-1)^D\hat{g}'}\left(\tilde{R} - \frac{1}{4}\tilde{F}^2 + 2D\tilde{\Box}\omega - D(D-1)(\tilde{\partial}\omega)^2\right), \tag{A.1.37}$$

where D-dimensional quantities have been explicitly displayed with the symbol on top. As we see, the D-dimensional gravitational term is not in the Einstein frame. We thus perform another Weyl rescaling to get rid of the exponential, this time on the lower dimensional metric. Writing the transformation as $\tilde{g} \to e^{-2\omega'(x)}g$ we get

$$\begin{aligned}\sqrt{-\hat{g}}\hat{R} = {} & e^{(1-D)\omega+(2-D)\omega'}\sqrt{-g}\Big(R + 2(D-1)\Box\omega' + (1-D)(D-2)(\partial\omega')^2\Big) \\ & + e^{(1-D)\omega-D\omega'}\sqrt{-g}\left(-\frac{1}{4}F^2 + 2D\tilde{\Box}\omega - D(D-1)(\tilde{\partial}\omega)^2\right)\end{aligned} \tag{A.1.38}$$

$$= \sqrt{-g}\left\{e^{(1-D)\omega+(2-D)\omega'}\left(R + 2(D-1)\Box\omega' + (1-D)(D-2)(\partial\omega')^2\right.\right.$$
$$\left.\left.+2D\Box\omega - D(D-1)(\partial\omega)^2 - \frac{1}{4}e^{-2\omega'}F^2\right)\right\},$$

where in the last line we have written all the operators with respect to the new metric g (or vielbein basis). Asking for the exponential in front of the Ricci scalar to vanish means that

$$(1-D)\omega(x) + (2-D)\omega'(x) = 0 \longrightarrow \omega(x) = -\frac{2-D}{1-D}\omega'(x). \tag{A.1.39}$$

We see that if our transformations should bring us to the Einstein frame, the functions defining the two rescalings performed need to be proportional to one another. If, accordingly, we now define $\omega(x) \equiv \beta\phi(x)$ and $\omega'(x) = (\alpha-\beta)\phi(x)$, we see that the action defined by (A.1.38) takes the form (A.1.21), thus justifying a posteriori the computation with the metric (A.1.3).

A.2 Free higher dimensional scalar field

We now move forward and start adding some particle content in our $D+1$-dimensional theory in the form of a free real massless scalar field, so that the action looks

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_{\Phi}, \tag{A.2.1}$$

where $\mathcal{S}_{EH}$ is defined in (A.1.1) and

$$\mathcal{S}_{\Phi} = -\int \mathrm{d}^{D+1}x\,\sqrt{-\hat{g}}\,\frac{1}{2}\hat{g}^{MN}\partial_M\hat{\Phi}\partial_N\hat{\Phi} \tag{A.2.2}$$

Here we take the real scalar field Φ to be single valued at any point in space-time. Calling x the D non compact coordinates and z the compact one, this physical requirement fixes the periodicity along the compactified dimension:

$$\hat{\Phi}(x, z + 2\pi L) = \hat{\Phi}(x, z). \tag{A.2.3}$$

This in turn allows us to write the field Φ in the Fourier basis as

$$\hat{\Phi}(x, z) = \frac{1}{\sqrt{2\pi L}}\sum_{n=-\infty}^{+\infty}\varphi_n(x)e^{\frac{inz}{L}}, \tag{A.2.4}$$

where the Fourier components only depends on the x-coordinate, thus realizing the periodic z-dependence through the basis elements.
Plugging the explicit form of the metric given in (A.1.4) together with this decomposition in the action (A.2.2) we find

$$\mathcal{S}_{\Phi} = -\frac{1}{2}\int \mathrm{d}^{D+1}x\sqrt{-g}\sum_{n,m=-\infty}^{+\infty}\frac{e^{i\frac{(n+m)z}{L}}}{2\pi L}\left(g^{\mu\nu}\partial_\mu\varphi_n\partial_\nu\varphi_m - 2i\frac{\sqrt{2}m}{LM_{Pl}^{\frac{D-2}{2}}}A^\mu\partial_\mu\varphi_n\varphi_m\right.$$

$$
- \frac{2}{M_{Pl}^{D-2}} \frac{nm}{L^2} A_\mu A^\mu \varphi_n \varphi_m - \frac{nm}{L^2} e^{-2\sqrt{2}(\beta-\alpha)\frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_m \Bigg) \tag{A.2.5}
$$

The integration over the compactified dimension will give a term in $2\pi L\, \delta_{n,-m}$ that we can then use to perform the sum over one of the two variables, say m, and the action is thus written

$$
\mathcal{S}_\Phi = -\frac{1}{2} \int \mathrm{d}^D x \sqrt{-g} \sum_{n=-\infty}^{+\infty} \Bigg(g^{\mu\nu} \partial_\mu \varphi_n \partial_\nu \varphi_{-n} + 2i \frac{\sqrt{2}n}{L M_{Pl}^{\frac{D-2}{2}}} A^\mu \partial_\mu \varphi_n \varphi_{-n} + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_{-n}
$$

$$
+ \frac{n^2}{L^2} e^{-2\sqrt{2}(\beta-\alpha)\frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_{-n} \Bigg). \tag{A.2.6}
$$

The reality condition $\Phi(x,z) = \Phi^*(x,z)$ straightforwardly implies $\varphi_{-n} = \varphi_n^*$. For the 0-mode φ_0 this in turn implies that such mode is real.
To rewrite the action (A.2.6) in a more canonical and elegant way we should first observe that all the bilinears $\varphi_n \varphi_{-n} = \varphi_n \varphi_n^*$ are symmetric under the transformation $n \to -n$. This straightforwardly applies to all but one term, giving

$$
\sum_{n=-\infty}^{+\infty} \frac{1}{2} \Bigg(g^{\mu\nu} \partial_\mu \varphi_n \partial_\nu \varphi_{-n} + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_{-n} + \frac{n^2}{L^2} e^{-2\sqrt{2}(\beta-\alpha)\frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_{-n} \Bigg) =
$$

$$
= \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_0 \partial_\nu \varphi_0 + \sum_{n=1}^{\infty} \Bigg(g^{\mu\nu} \partial_\mu \varphi_n \partial_\nu \varphi_n^* + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* + \frac{n^2}{L^2} e^{-2\sqrt{2}(\beta-\alpha)\frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_n^* \Bigg). \tag{A.2.7}
$$

For the remaining term, we see that under $n \to -n$ it transforms as $n\, A^\mu \partial_\mu \varphi_n \varphi_n^* \to -n\, A^\mu \partial_\mu \varphi_n^* \varphi_n$. Grouping the $(n, -n)$ terms two by two we thus get

$$
\sum_{n=-\infty}^{+\infty} i \frac{\sqrt{2}n}{L M_{Pl}^{\frac{D-2}{2}}} A^\mu \partial_\mu \varphi_n \varphi_{-n} = \sum_{n=1}^{\infty} i \frac{\sqrt{2}n}{L M_{Pl}^{\frac{D-2}{2}}} A^\mu \left(\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^* \right), \tag{A.2.8}
$$

which together with the $A_\mu A^\mu$ term previously shown forms the well-known minimal coupling dictated by the gauge principle for a complex scalar field. The action (A.2.1) finally reads

$$
\mathcal{S} = \int \mathrm{d}^D x \sqrt{-g} \Bigg\{ \frac{R}{2\kappa^2} - \frac{1}{2} (\partial\phi)^2 - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}}\frac{\phi}{M_{Pl}^{(D-2)/2}}} F^2 - \frac{1}{2} \partial_\mu \varphi_0 \partial^\mu \varphi_0 \tag{A.2.9}
$$

$$
- \sum_{n=1}^{\infty} \Bigg(\partial_\mu \varphi_n \partial^\mu \varphi_n^* + \frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi}{M_{Pl}^{(D-2)/2}}} \varphi_n \varphi_n^* \Bigg)
$$

$$
- \sum_{n=1}^{\infty} \Bigg(i \frac{\sqrt{2}n}{L M_{Pl}^{\frac{D-2}{2}}} A^\mu \left(\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^* \right) + \frac{2}{M_{Pl}^{D-2}} \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* \Bigg) \Bigg\}, \tag{A.2.10}
$$

where we have chosen the positive root for α (A.1.23).
We will now make some comments about this formula. The first thing we can note is that a $D+1$-dimensional system with gravity and a real massless scalar field is described, in terms of D-dimensional quantities accessible to a low energy observer, as a rather involved system. There is the presence of the graviphoton and the dilaton, while the higher dimensional real massless scalar compactifies down to a lower dimensional real massless scalar and an infinite tower of massive complex scalars which are the Kaluza-Klein states (KK-states) minimally coupled to the D-dimensional gauge field. The ϕ-dependent mass of such fields in turn determines a coupling with the dilaton.

Besides, φ_0 represents a mode with no momentum along the compact dimension. It does not feel the presence of the additional dimension as given by the components $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$, that we have written in terms of A_μ and ϕ. The φ_n's, on the other hand, describe modes with non vanishing momentum along the compact dimension, and thus couple to A_μ and ϕ. The periodicity condition required in (A.2.3) implies the quantization of such momentum $p_z \sim \frac{n}{L}$, $n \in \mathbb{N}$, that will then appear as a mass in lower dimensions, which can be seen writing propagator with $P = (p, p_z)$ as

$$\frac{i}{P^2} = \frac{i}{p^2 + p_z^2} \tag{A.2.11}$$

Coming back to (A.1.27), we see that the gauge coupling g is

$$g^2 = e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi}{M_{Pl}^{(D-2)/2}}}. \tag{A.2.12}$$

For each KK mode, we can now express both the mass and the charge and get

$$gq_n = \frac{\sqrt{2}n}{LM_{Pl}^{(D-2)/2}} e^{\sqrt{\frac{D-1}{D-2}}\frac{\phi}{M_{Pl}^{(D-2)/2}}} \text{ and } m_n = \frac{n}{L} e^{\sqrt{\frac{D-1}{D-2}}\frac{\phi}{M_{Pl}^{(D-2)/2}}}. \tag{A.2.13}$$

This shows that, in any space-time dimension D, the mass and the charge of the KK modes are related to each other as

$$(gq_n)^2 = 2\frac{m_n^2}{M_{Pl}^{D-2}}. \tag{A.2.14}$$

The charge to mass ratio of the KK-states is so equal to the bound (1.2.9) with $\alpha = \sqrt{3}$ in 4-dimensions.

A.3 Compactification via the higher dimensional gravitational vertices

In the previous subsections we have been able to derive the effective description of a $D+1$-dimensional theory of gravity and a massless scalar from the perspective of a D-dimensional observer. In particular, we have obtained the lower dimensional theory expressing the whole action in terms of fields defined in D dimensions. All the interactive terms for the KK modes of the scalar field displayed in (A.2.9) should be

of gravitational origin and as such should come from the gravitational vertices of the higher dimensional scalar. We should thus in principle be able to derive all the lower dimensional vertices involving scalars from the corresponding $(D+1)$-dimensional gravitational ones. We can immediately see from (A.2.9) that, in the process, we will need to go up to second order in $\frac{1}{M_{Pl}^{(D-1)/2}}$. To do so, we will first verify the perturbativity of the metric (A.1.3).
So let's start again with the metric (A.1.3), that we rewrite here for the sake of clarity:

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & e^{2\beta\phi} \end{pmatrix}.$$

What we need to do now is to develop this metric around a background, and then, using the conventions in (A.1.1), verify whether it is possible or not to write this metric as a formal development $\hat{g}_{MN} = \hat{\zeta}_{MN} + 2\hat{\kappa}\hat{h}_{MN} + 4\hat{\kappa}^2\hat{f}_{MN} + o(\hat{\kappa}^3) = \hat{\zeta}_{MN} + 2\frac{\hat{h}_{MN}}{\hat{M}_{Pl}^{(D-1)/2}} + 4\frac{\hat{f}_{MN}}{M_{Pl}^{D-1}} + \mathcal{O}\left(\frac{1}{\hat{M}_{Pl}^{3(D-1)/2}}\right)$, with $\hat{\kappa}^2\hat{f}_{MN} \ll \hat{\kappa}\hat{h}_{MN} \ll 1 \; \forall M, N$. Usually the perturbative expansion for gravity is set up around a Minkowski background. This need not be true in all cases as the dilaton ϕ may take a vev., and $\beta \neq \alpha$, so that in principle $\hat{\zeta}$ is not even proportional to $\hat{\eta}$. The lower dimensional metric is undetermined, and as such we can arbitrarily set up its perturbative expansion to be around the Minkowski background as $g_{\mu\nu} = \eta_{\mu\nu} + 2\frac{h_{\mu\nu}}{M_{Pl}^{(D-2)/2}} + 4\frac{f_{\mu\nu}}{M_{Pl}^{D-2}} + \mathcal{O}\left(\frac{1}{M_{Pl}^{3(D-/2}}\right)$, where here η is the Minkowski metric, as usual. Using now (A.1.25) and inserting the physical fields (A.1.26) we can write:

$$\begin{aligned}
\hat{g}_{MN} &= \begin{pmatrix} e^{2\sqrt{2}\alpha\frac{\phi}{M_{Pl}^{(D-2)/2}}}\left(\eta_{\mu\nu}+2\frac{h_{\mu\nu}}{M_{Pl}^{(D-2)/2}}+4\frac{f_{\mu\nu}}{M_{Pl}^{D-2}}+\ldots\right)+\frac{2}{M_{Pl}^{D-2}}e^{2\sqrt{2}\beta\frac{\phi}{M_{Pl}^{(D-2)/2}}}A_\mu A_\nu & \frac{\sqrt{2}}{M_{Pl}^{(D-2)/2}}e^{2\sqrt{2}\beta\frac{\phi}{M_{Pl}^{(D-2)/2}}}A_\mu \\ \frac{\sqrt{2}}{M_{Pl}^{(D-2)/2}}e^{2\sqrt{2}\beta\frac{\phi}{M_{Pl}^{(D-2)/2}}}A_\nu & e^{2\sqrt{2}\beta\frac{\phi}{M_{Pl}^{(D-2)/2}}} \end{pmatrix} \\
&= \begin{pmatrix} e^{2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\eta_{\mu\nu} & 0 \\ 0 & e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}} \end{pmatrix} + \frac{2}{\sqrt{2\pi L}\hat{M}_{Pl}^{(D-1)/2}}\begin{pmatrix} e^{2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\sqrt{2}\alpha\phi\,\eta_{\mu\nu}+h_{\mu\nu}\right) & e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{A_\mu}{\sqrt{2}} \\ e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{A_\nu}{\sqrt{2}} & e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\sqrt{2}\beta\phi \end{pmatrix} \\
&\quad + \frac{4}{2\pi L\,\hat{M}_{Pl}^{D-1}}\begin{pmatrix} e^{2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\alpha^2\phi^2\eta_{\mu\nu}+\sqrt{2}\alpha\phi\,h_{\mu\nu}+f_{\mu\nu}\right)+e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{A_\mu A_\nu}{2} & e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\beta\phi A_\mu \\ e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\beta\phi A_\nu & e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\beta^2\phi^2 \end{pmatrix} \\
&\quad + \mathcal{O}\left(\frac{1}{(2\pi L)^{3/2}\hat{M}_{Pl}^{3(D-1)/2}}\right),
\end{aligned} \tag{A.3.1}$$

where we have developed the dilaton around a generic background value as $\phi_0 + \phi$. We immediately see that the background is different from Minkowski except for the case $\phi_0 = 0$ since we have

$$\hat{\zeta}_{MN} = \begin{pmatrix} e^{2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\eta_{\mu\nu} & 0 \\ 0 & e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}} \end{pmatrix}. \tag{A.3.2}$$

Nevertheless, we can say that, as long as the fields do not develop trans-Planckian values, our metric is perturbative in the sense of a Planck suppressed series around a background, even if the background is not flat.
We are now allowed to proceed in our declared goal of computing the lower dimensional vertices from the higher dimensional ones. In order to do so, we should first fix the form of the terms in the perturbative series up to the second order. Let's write, in generic dimension d, the perturbation

$$\begin{cases} g_{\mu\nu} = \zeta_{\mu\nu} + 2\kappa h_{\mu\nu} + 4\kappa^2 f_{\mu\nu} + \mathcal{O}(\kappa^3) \\ g^{\mu\nu} = \zeta^{\mu\nu} + 2\kappa t^{\mu\nu} + 4\kappa^2 l^{\mu\nu} + \mathcal{O}(\kappa^3). \end{cases} \tag{A.3.3}$$

The relation $g_{\mu\rho}g^{\rho\nu} \equiv \delta^\nu_\mu$ reads

$$\begin{aligned} &\left(\zeta_{\mu\rho} + 2\kappa h_{\mu\rho} + 4\kappa^2 f_{\mu\rho}\right)\left(\zeta^{\rho\nu} + 2\kappa t^{\rho\nu} + 4\kappa^2 l^{\rho\nu}\right) = \\ &\delta^\nu_\mu + 2\kappa\left(\zeta^{\rho\nu}h_{\mu\rho} + \zeta_{\mu\rho}t^{\rho\nu}\right) + 4\kappa^2\left(\zeta^{\rho\nu}f_{\mu\rho} + \zeta_{\mu\rho}l^{\rho\nu} + h_{\mu\rho}t^{\rho\nu}\right) + \mathcal{O}\left(\kappa^3\right) \equiv \delta^\nu_\mu, \end{aligned} \tag{A.3.4}$$

and thus

$$\begin{cases} t^{\mu\nu} = -h^{\mu\nu} \\ l^{\mu\nu} + f^{\mu\nu} = h^\mu_\rho h^{\rho\nu}, \end{cases} \tag{A.3.5}$$

where it is understood that the indices are raised and lowered with the background metric ζ. We find here the first and second order equations relating the perturbation of the metric and its inverse. It is common to find, in the literature, the particular choice $f_{\mu\nu} = 0$, $l^{\mu\nu} = h^\mu_\rho h^{\rho\nu}$ (with a minkowskian background). This is perfectly fine when one works with generic, non specified perturbations. In our case, however, as we already have imposed the specific form of the metric, we cannot use such convention but need to develop the gravitational formalism with the specific h, t, f, l dictated by our metric.
To go further, we now need to express the determinant of the metric. Defining $\kappa' \equiv 2\kappa$ and using g as the matrix representation of the metric $\sqrt{-\det(g)} = \exp\left(\frac{1}{2}\mathrm{tr}\,\log(g)\right) = \exp\left\{\frac{1}{2}\mathrm{tr}\left[\zeta + \kappa'\zeta^{-1}h + \kappa'^2\zeta^{-1}f - \frac{\left(\zeta^{-1}\kappa' h + \kappa'^2\zeta^{-1}f\right)^2}{2}\right] + \mathcal{O}(\kappa'^3)\right\}$ we have

$$\begin{aligned} &\sqrt{-\det(g)} = \\ &\sqrt{-\det(\zeta)}\left(1 + \frac{\kappa'}{2}\mathrm{tr}(\zeta^{-1}h) + \frac{\kappa'^2}{2}\mathrm{tr}(\zeta^{-1}f) - \frac{\kappa'^2}{4}\mathrm{tr}((\zeta^{-1}h)^2) + \frac{\kappa'^2}{8}(\mathrm{tr}(\zeta^{-1}h))^2 + \mathcal{O}(\kappa'^3)\right). \end{aligned} \tag{A.3.6}$$

Let's now apply all of this to the $(D+1)$-dimensional action $\mathcal{S}_\Phi$ (A.2.2), recovering the usual notation g for the determinant. We obtain

$$\begin{aligned} &\sqrt{-\hat{g}}\mathcal{L}_\Phi = -\sqrt{-\hat{g}}\frac{1}{2}\hat{g}^{MN}\partial_M\hat{\Phi}\partial_N\hat{\Phi} \Longrightarrow \\ &-\sqrt{-\hat{\zeta}}\left(1 + \frac{\hat{\kappa}'}{2}\hat{h}^M_M + \frac{\hat{\kappa}'^2}{2}\hat{f}^M_M - \frac{\hat{\kappa}'^2}{4}\hat{h}^{MN}\hat{h}_{MN} + \frac{\hat{\kappa}'^2}{8}(\hat{h}^M_M)^2\right)\left(\hat{\zeta}^{MN} - \hat{\kappa}'\hat{h}^{MN} + \hat{\kappa}'^2\hat{l}^{MN}\right)\frac{1}{2}\partial_M\hat{\Phi}\partial_N\hat{\Phi} \\ &= -\sqrt{-\hat{\zeta}}\left[\frac{1}{2}\partial_M\hat{\Phi}\partial^M\hat{\Phi} + \frac{\hat{\kappa}'}{2}\hat{h}^{MN}\left(\partial_M\hat{\Phi}\partial_N\hat{\Phi} - \frac{1}{2}\hat{\eta}_{MN}\partial_P\hat{\Phi}\partial^P\hat{\Phi}\right)\right. \end{aligned}$$

$$-\frac{\hat{\kappa}'^2}{2}\left(\hat{l}^{MN}-\frac{1}{2}\hat{h}^{MN}\hat{h}^P_P\right)\partial_M\hat{\Phi}\partial_N\hat{\Phi}-\frac{\hat{\kappa}'^2}{4}\left(\hat{f}^P_P-\frac{1}{2}\hat{h}_{MP}\hat{h}^{PM}+\frac{1}{4}(\hat{h}^P_P)^2\right)\partial_M\hat{\Phi}\partial^M\hat{\Phi}\Bigg]. \tag{A.3.7}$$

As we can see, at first order we recover the formula

$$-\sqrt{-\hat{\zeta}}\frac{\hat{\kappa}'}{2}\hat{h}^{MN}T^{\hat{\Phi}}_{MN}=-\sqrt{-\hat{\zeta}}\frac{\hat{h}^{MN}}{\hat{M}_{Pl}^{(D-1)/2}}T^{\hat{\Phi}}_{MN}$$

(the different sign in the definition of the stress-energy tensor with respect to the one usually displayed coming from the signature), while at second order we now have the generic structure of the interaction. Of course, the above development is rather independent of the particular system we are studying or the dimension.
As we can see from (A.3.1), raising the indices with ζ to stay at lowest order we have in our case

$$\hat{h}^{MN}=\frac{1}{\sqrt{2\pi L}}\begin{pmatrix} e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\sqrt{2}\alpha\phi\,\eta^{\mu\nu}+h^{\mu\nu}\right) & e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{A^\mu}{\sqrt{2}} \\ e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{A^\nu}{\sqrt{2}} & e^{-2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\sqrt{2}\beta\phi\end{pmatrix}, \tag{A.3.8}$$

and we can thus write, using $\sqrt{-\hat{\zeta}}=e^{\sqrt{2}(D\alpha+\beta)\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}$,

$$\begin{aligned}-\sqrt{2\pi L}\sqrt{-\hat{\zeta}}\,\hat{h}^{MN}\hat{T}^{\hat{\Phi}}_{MN}=&\,e^{\sqrt{2}((D-2)\alpha+\beta)\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\sqrt{2}\alpha\phi\,\eta^{\mu\nu}+h^{\mu\nu}\right)\left(\partial_\mu\hat{\Phi}\partial_\nu\hat{\Phi}-\frac{1}{2}\eta_{\mu\nu}\partial_P\hat{\Phi}\partial^P\hat{\Phi}\right)\\&+\sqrt{2}e^{\sqrt{2}((D-2)\alpha+\beta)\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}A^\mu\partial_\mu\hat{\Phi}\partial_z\hat{\Phi}\\&+e^{\sqrt{2}(D\alpha-\beta)\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\sqrt{2}\beta\phi\left(\partial_z\hat{\Phi}\partial_z\hat{\Phi}-\frac{1}{2}\partial_P\hat{\Phi}\partial^P\hat{\Phi}\right).\end{aligned} \tag{A.3.9}$$

Inserting now the Fourier decomposition (A.2.4) and performing the integration over the z variable we get, for the D-dimensional action,

$$\begin{aligned}&e^{\sqrt{2}\frac{((D-2)\alpha+\beta)\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\sqrt{2}\alpha\phi\,\eta^{\mu\nu}+h^{\mu\nu}\right)\Bigg[\partial_\mu\varphi_0\partial_\nu\varphi_0+\sum_{n=1}^{\infty}\partial_\mu\varphi_n\partial_\nu\varphi_n^*+\partial_\nu\varphi_n\partial_\mu\varphi_n^*\\&-\eta_{\mu\nu}\left(\frac{1}{2}\partial_\rho\varphi_0\partial^\rho\varphi_0+\sum_{n=1}^{\infty}\partial_\mu\varphi_n\partial^\mu\varphi_n^*+\frac{n^2}{L^2}\varphi_n\varphi_n^*\right)\Bigg]-e^{\sqrt{2}\frac{((D-2)\alpha+\beta)\phi_0}{M_{Pl}^{(D-2)/2}}}\sqrt{2}A^\mu\sum_{n=1}^{\infty}i\frac{n}{L}\left(\partial_\mu\varphi_n\,\varphi_n^*-\varphi_n\,\partial_\mu\varphi_n^*\right)\\&+e^{\sqrt{2}\frac{(D\alpha-\beta)\phi_0}{M_{Pl}^{(D-2)/2}}}\sqrt{2}\beta\phi\left(\sum_{n=1}^{\infty}\frac{n^2}{L^2}\varphi_n\varphi_n^*-\frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0-\sum_{n=1}^{\infty}\partial_\mu\varphi_n\partial^\mu\varphi_n^*\right)\end{aligned} \tag{A.3.10}$$

The second term in the first line is ($-$) the stress-energy tensor for φ_0 and φ_n. We have thus the D-dimensional gravitational interaction plus a bunch of other interactions. More specifically, this simplifies to

$$
\begin{aligned}
&-\sqrt{2\pi L}\sqrt{-\hat{\zeta}}\,\hat{h}^{MN}\hat{T}^{\hat{\Phi}}_{MN} = \qquad\qquad (A.3.11)\\
&-h^{\mu\nu}T^{(\varphi_0,\varphi_n)}_{\mu\nu} - i\sqrt{2}A^{\mu}\sum_{n=1}^{\infty}\frac{n}{L}\left(\partial_\mu\varphi_n\,\varphi_n^* - \varphi_n\,\partial_\mu\varphi_n^*\right) + \frac{(2-D)\,\alpha-\beta}{\sqrt{2}}\phi\partial_\mu\varphi_0\partial^\mu\varphi_0\\
&+\sqrt{2}\left[(2-D)\,\alpha-\beta\right]\phi\sum_{n=1}^{\infty}\partial_\mu\varphi_n\partial^\mu\varphi_n^* - \sqrt{2}\,(\beta-D\alpha)\,e^{\sqrt{2}\frac{(D\alpha-\beta)\phi_0}{M_{Pl}^{(D-2)/2}}}\phi\sum_{n=1}^{\infty}\frac{n^2}{L^2}\varphi_n\varphi_n^*\\
=&-h^{\mu\nu}T^{(\varphi_0,\varphi_n)}_{\mu\nu} - i\sqrt{2}A^{\mu}\sum_{n=1}^{\infty}\frac{n}{L}\left(\partial_\mu\varphi_n\,\varphi_n^* - \varphi_n\,\partial_\mu\varphi_n^*\right) - 2\sqrt{\frac{D-1}{D-2}}e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\phi\sum_{n=1}^{\infty}\frac{n^2}{L^2}\varphi_n\varphi_n^*,
\end{aligned}
$$

where we have used $\beta = -(D-2)\alpha$ and $\alpha = \frac{1}{\sqrt{2(D-1)(D-2)}}$. Finally, if one wants to write it properly, using again (A.1.25) we can conclude

$$
\begin{aligned}
-\frac{\hat{h}^{MN}}{\hat{M}_{Pl}^{(D-1)/2}}\hat{T}^{\hat{\Phi}}_{MN} = &-\frac{h^{\mu\nu}}{M_{Pl}^{(D-2)/2}}T^{(\varphi_0,\varphi_n)}_{\mu\nu} - i\sqrt{2}\frac{A^{\mu}}{M_{Pl}^{(D-2)/2}}\sum_{n=1}^{\infty}\frac{n}{L}\left(\partial_\mu\varphi_n\,\varphi_n^* - \varphi_n\,\partial_\mu\varphi_n^*\right)\\
&-2\sqrt{\frac{D-1}{D-2}}\frac{e^{2\sqrt{\frac{D-1}{D-2}}\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\phi}{M_{Pl}^{(D-2)/2}}\sum_{n=1}^{\infty}\frac{n^2}{L^2}\varphi_n\varphi_n^*. \qquad (A.3.12)
\end{aligned}
$$

The above result exactly matches what one obtains at order $\mathcal{O}\left(M_{Pl}^{(2-D)/2}\right)$ from (A.2.9). As we see, if one were to stop at first order, he would miss the gauge invariance of the theory with respect to the lower dimensional gauge field. Put in different words, the gauge invariance of the D-dimensional theory being a realization of the general coordinate invariance of the $D+1$-dimensional one along the compact direction, consistency of the theory forces us to go to second order.
To do so, let's first identify $\hat{f}_{MN}$ from (A.3.1):

$$
\hat{f}_{MN} = \frac{1}{2\pi\,L}\begin{pmatrix} e^{2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\alpha^2\phi^2\eta_{\mu\nu}+\sqrt{2}\alpha\phi\,h_{\mu\nu}+f_{\mu\nu}\right)+e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{A_\mu A_\nu}{2} & e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\beta\phi A_\mu \\ e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\beta\phi A_\nu & e^{2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\beta^2\phi^2 \end{pmatrix}. \qquad (A.3.13)
$$

With this result, $\hat{l}^{MN}$ will be given by

$$
\hat{l}^{MN} = \hat{h}^{MP}\hat{h}_P^N - \hat{f}^{MN} \qquad (A.3.14)
$$

Using (A.3.13) and (A.3.8) one obtains

$$
\hat{l}^{MN} = \frac{1}{2\pi\,L}\begin{pmatrix} e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\alpha^2\phi^2\eta_{\mu\nu}+\sqrt{2}\alpha\phi\,h_{\mu\nu}+l^{\mu\nu}\right) & e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\alpha\phi A_\mu+\frac{1}{\sqrt{2}}h^{\mu\rho}A_\rho\right) \\ e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\alpha\phi A^\nu+\frac{1}{\sqrt{2}}h^\nu_\rho A^\rho\right) & e^{-2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\beta^2\phi^2+e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{1}{2}A_\rho A^\rho \end{pmatrix}. \qquad (A.3.15)
$$

As suggested by (A.3.7), we need to construct, with these ingredients, the following quantities:

$$\hat{l}^{MN}-\frac{1}{2}\hat{h}^P_P\hat{h}^{MN}=\frac{1}{2\pi L}\begin{pmatrix} e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\alpha^2\phi^2\eta^{\mu\nu}+\sqrt{2}\alpha\phi\, h^{\mu\nu}+l^{\mu\nu}\right) & e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\alpha\phi A^\mu+\frac{1}{\sqrt{2}}h^{\mu\rho}A_\rho \\ e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\alpha\phi A^\nu+\frac{1}{\sqrt{2}}h^\nu_\rho A^\rho & e^{-2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\beta^2\phi^2+e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{1}{2}A_\rho A^\rho \end{pmatrix}$$
$$-\frac{\frac{D}{\sqrt{2}}\alpha\phi+\frac{h^\rho_\rho}{2}+\frac{\beta\phi}{\sqrt{2}}\beta\phi}{2\pi L}\begin{pmatrix} e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\sqrt{2}\alpha\phi\,\eta^{\mu\nu}+h^{\mu\nu}\right) & e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{A^\mu}{\sqrt{2}} \\ e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{A^\nu}{\sqrt{2}} & e^{-2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\sqrt{2}\beta\phi \end{pmatrix} \tag{A.3.16}$$

getting

$$\frac{e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}}{2\pi L}\begin{pmatrix} -\alpha^2\phi^2\eta^{\mu\nu}-\frac{\alpha}{\sqrt{2}}\phi\, h^\rho_\rho\eta^{\mu\nu}+l^{\mu\nu}-\frac{h^\rho_\rho h^{\mu\nu}}{2} & \frac{1}{\sqrt{2}}\left(h^{\mu\rho}A_\rho-\frac{1}{2}h^\rho_\rho A^\mu\right) \\ \frac{1}{\sqrt{2}}\left(h^\nu_\rho A^\rho-\frac{1}{2}h^\rho_\rho A^\nu\right) & e^{-2\sqrt{2}(\beta-\alpha)\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(D(D-2)\alpha^2\phi^2+\sqrt{2}\alpha h^\rho_\rho\phi\right)+\frac{1}{2}A_\rho A^\rho \end{pmatrix}. \tag{A.3.17}$$

For the second term of the second order expression of (A.3.7), we have

$$\hat{f}^P_P-\frac{1}{2}\hat{h}_{MP}\hat{h}^{PM}+\frac{1}{4}(\hat{h}^P_P)^2=D\alpha^2\phi^2+\sqrt{2}\alpha\phi h^\mu_\mu+f^\mu_\mu+\frac{1}{2}e^{2\sqrt{2}(\beta-\alpha)\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}A^2+\beta^2\phi^2 \tag{A.3.18}$$
$$-\frac{1}{2}\left((2D\alpha^2+\beta^2)\phi^2+2\sqrt{2}\alpha\phi h^\mu_\mu+h^{\mu\nu}h_{\mu\nu}+e^{2\sqrt{2}(\beta-\alpha)\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}A^2\right)$$
$$+\frac{1}{4}\left(2(D\alpha+\beta)^2\phi^2+(h^\mu_\mu)^2+2\sqrt{2}(D\alpha+\beta)\phi h^\mu_\mu\right)$$
$$=f^\mu_\mu-\frac{1}{2}h^{\mu\nu}h_{\mu\nu}+\frac{1}{4}(h^\mu_\mu)^2+\frac{1}{2}(D\alpha+\beta)^2\phi^2+\frac{1}{\sqrt{2}}(D\alpha+\beta)\phi h^\mu_\mu$$

summing up the two contributions of the second order with the Φ derivative in (A.3.7), one obtains

$$\frac{1}{2\pi L}\Bigg\{e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(-\alpha^2\phi^2\eta^{\mu\nu}-\frac{\alpha}{\sqrt{2}}\phi\, h^\rho_\rho\eta^{\mu\nu}+l^{\mu\nu}-\frac{h^\rho_\rho h^{\mu\nu}}{2}\right)\frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi \tag{A.3.19}$$
$$+\sqrt{2}e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(h^{\mu\rho}A_\rho-\frac{1}{2}h^\rho_\rho A^\mu\right)\frac{1}{2}\partial_\mu\Phi\partial_z\Phi$$
$$+\left(e^{-2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}(D(D-2)\alpha^2\phi^2+\sqrt{2}\alpha h^\rho_\rho\phi)+\frac{1}{2}e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}A_\rho A^\rho\right)\frac{1}{2}(\partial_z\Phi)^2\Bigg\}$$
$$+\frac{1}{2}\left(f^\mu_\mu-\frac{1}{2}h^{\mu\nu}h_{\mu\nu}+\frac{1}{4}(h^\mu_\mu)^2+\frac{1}{2}(D\alpha+\beta)^2\phi^2+\frac{1}{\sqrt{2}}(D\alpha+\beta)\phi h^\mu_\mu\right)\times$$
$$\left(e^{-2\sqrt{2}\alpha\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi+e^{-2\sqrt{2}\beta\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\frac{1}{2}(\partial_z\Phi)^2\right)$$

Adding the contribution from the determinant, and as for the first order, using (A.2.4) and integrating over the z dimension, one obtains for the second order interaction, defining $J_{\mu,n} = (\varphi_n \partial_\mu \varphi_n^* - \varphi_n^* \partial_\mu \varphi_n)$,

$$\frac{1}{2}\partial_\mu\varphi_0\partial_\nu\varphi_0 \left[\left(\frac{f^\rho_\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h^\rho_{\ \rho})^2}{8} + \frac{1}{2}\left(D^2\alpha^2 + 2D\beta\alpha + \beta^2 - 4D\alpha^2 - 4\beta\alpha + 4\alpha^2\right)\phi^2 \right.\right. \tag{A.3.20}$$
$$\left.\left. + \frac{1}{2}((D-2)\alpha+\beta)\phi h^\rho_{\ \rho}\right)\eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h^\rho_{\ \rho}h^{\mu\nu}\right]$$
$$+\sum_{n=1}^{\infty}\partial_\mu\varphi_n\partial_\nu\varphi_n^* \left[\left(\frac{f^\rho_\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h^\rho_{\ \rho})^2}{8} + \frac{1}{2}\left(D^2\alpha^2 + 2D\beta\alpha + \beta^2 - 4D\alpha^2 - 4\beta\alpha + 4\alpha^2\right)\phi^2 \right.\right.$$
$$\left.\left. + \frac{1}{2}((D-2)\alpha+\beta)\phi h^\rho_{\ \rho}\right)\eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h^\rho_{\ \rho}h^{\mu\nu}\right]$$
$$-\sum_{n=1}^{\infty}\frac{n^2}{L^2}|\varphi_n|^2\left[A^2 + e^{\sqrt{2}(D\alpha-\beta)\frac{\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\frac{f^\rho_\rho}{2} - \frac{h^{\rho\sigma}_{\rho\sigma}}{4} + \frac{(h^\rho_{\ \rho})^2}{8} + \left(\frac{1}{2}(D\alpha+\beta) - \beta\right)\phi h^\rho_{\ \rho}\right.\right.$$
$$\left.\left. + \frac{1}{2}(D^2\alpha^2 + 2D\alpha\beta + \beta^2 - 4D\alpha\beta - 4\beta^2 + 4\beta^2)\phi^2\right)\right]$$
$$+\sum_{n=1}^{\infty} i\frac{n}{L}h^{\rho\sigma}A_\rho J_{\sigma,n} + i\frac{n}{L}A^\rho J_{\rho,n}\left(-\frac{h^\sigma_{\ \sigma}}{2} - ((D-2)\alpha+\beta)\phi\right)$$

We can simplify the expression with the relation between β and α that we have already used to eliminate the exponential factor in front of the interaction term. In particular, the term in ϕ^2 or ϕh^ρ_ρ will disappear, and one obtains

$$\frac{1}{2}\partial_\mu\varphi_0\partial_\nu\varphi_0 \left[\left(\frac{f^\rho_\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h^\rho_{\ \rho})^2}{8}\right)\eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h^\rho_{\ \rho}h^{\mu\nu}\right] \tag{A.3.21}$$
$$+\sum_{n=1}^{\infty}\partial_\mu\varphi_n\partial_\nu\varphi_n^* \left[\left(\frac{f^\rho_\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h^\rho_{\ \rho})^2}{8}\right)\eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h^\rho_{\ \rho}h^{\mu\nu}\right]$$
$$-\sum_{n=1}^{\infty}\frac{n^2}{L^2}|\varphi_n|^2 e^{\frac{2\sqrt{2}(D-1)\alpha\phi_0}{M_{Pl}^{(D-2)/2}}}\left(\frac{f^\rho_\rho}{2} - \frac{h^{\rho\sigma}_{\rho\sigma}}{4} + \frac{(h^\rho_{\ \rho})^2}{8}\right)$$
$$-\sum_{n=1}^{\infty}\frac{n^2}{L^2}|\varphi_n|^2\left(A^2 + e^{\frac{2\sqrt{2}(D-1)\alpha\phi_0}{M_{Pl}^{(D-2)/2}}}\left(2(D-1)^2\alpha^2\phi^2 + (D-1)\alpha\phi h^\rho_{\ \rho}\right)\right) - i\frac{n}{L}A^\rho\frac{h^\sigma_{\ \sigma}}{2}J_{\rho,n} + i\frac{n}{L}h^{\rho\sigma}A_\rho J_{\sigma,n}$$

This formula calls for some remarks. The term which multiplies the $\frac{1}{2}\partial_\mu\varphi_0\partial_\nu\varphi_0$ is exactly the term obtained from the perturbation of the metric at second order. It corresponds then to the interaction term between two gravitons and two massless scalars. The same term is present for the φ_n field with an additional contribution from the mass term. We also see several terms of interaction between the A_μ fields, the graviton and the φ_n. There is no interaction between the gauge field and the φ_0 since the latter is not charged under the gauge symmetry. The interaction term between two gauge fields and two scalars fields was, at first order, the term missing in order to achieve the gauge invariance of the interaction. We could expect an exponential factor to multiply it, but we should recall that in our theory this exponential term is present in front of the kinetic term of the A_μ, and this is why it is not present here when we only look at the interaction. However, the expected exponential term is present in front of the interaction term between two dilatons ϕ and the mass term of φ_n. This computation of the interaction in the compactified theory gives the same result as the one we did in the section A.1. So we can interpret our interaction with the scalar ϕ and the gauge field A_μ as a facet of gravitational interaction.

APPENDIX B

Gravitational D-dependent vertex

In chapter 4, we exhibit a factor $\frac{D-3}{D-2}$ in the gravitational force. This factor can be explained from the gravitational propagator, which takes in D dimensions the form

$$\mathcal{P}^{\alpha\beta\gamma\delta} = \frac{-i}{p^2}\left[\frac{1}{2}(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma}) - \frac{1}{D-2}\eta^{\alpha\beta}\eta^{\gamma\delta}\right]. \tag{B.0.1}$$

However, we will try to explain it from a different point of view, following [200]. It explains why, in the context of the Newtonian limit of GR in D spacetime dimensions, the $\frac{D-3}{D-2}$ factor comes about in the gravitational force with the usual normalization conventions for the Planck scale $\mathcal{S} = \int d^Dx\sqrt{-g}\frac{R}{2\kappa^2}$ and $\kappa^2 = M_{Pl}^{2-D}$.
Let's start with the action for a real scalar field

$$\mathcal{S} = \int d^Dx\sqrt{(-)^{D-1}g}\left(\frac{R}{2\kappa^2} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right). \tag{B.0.2}$$

Variation of the action with respect to the metric gives us the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu} \Longrightarrow R_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \frac{1}{D-2}g_{\mu\nu}T^\rho_\rho\right). \tag{B.0.3}$$

If we now develop the metric around a Minkowski background as $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} + \mathcal{O}(\kappa^2)$ and make the usual definition $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\rho_\rho$ we obtain, in the de Donder gauge $\partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\mu_\mu = 0$,

$$\partial^2 h_{\mu\nu} = -\frac{1}{2}\left(T_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}T^\rho_\rho\right) \Longleftrightarrow \partial^2\bar{h}_{\mu\nu} = -\frac{1}{2}T_{\mu\nu}. \tag{B.0.4}$$

This equation tells us that the field respecting the wave equation with $T_{\mu\nu}$ as a source term is rather $\bar{h}_{\mu\nu}$ than $h_{\mu\nu}$. With these results at hand, the quantization procedure is more naturally applied to the $\bar{h}$ field, that will thus represent the gravitational degrees of freedom of our quantum theory.

In terms of the development of the metric we just displayed, we have seen that the first order interaction is given by

$$\sqrt{(-)^{D-1}g}\,\mathcal{L}_{int} = -\kappa h^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi - \frac{\eta_{\mu\nu}}{2}\left(\partial_\rho\phi\partial^\rho\phi - m^2\phi^2\right)\right]. \tag{B.0.5}$$

Inverting the relation $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\rho_\rho$ we can obtain h in terms of $\bar{h}$ as $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}\bar{h}^\rho_\rho$ and then

$$\sqrt{(-)^{D-1}g}\,\mathcal{L}_{int} = -\kappa\bar{h}^{\mu\nu}\left(\partial_\mu\phi\partial_\nu\phi - \frac{\eta_{\mu\nu}}{D-2}m^2\phi^2\right). \tag{B.0.6}$$

To derive the propagator for this field $\bar{h}_{\mu\nu}$ requires to take the Lagrangian of the perturbation $h_{\mu\nu}$, with the gauge fixing term corresponding to the de Donder gauge. It reads (forgetting the κ coefficient in front)

$$\mathcal{L} = \frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu} + \frac{1}{4}h^\rho_\rho\Box h^\rho_\rho. \tag{B.0.7}$$

With the relation defined above $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}\bar{h}^\rho_\rho$, one can express the Lagrangian for $\bar{h}$

$$\mathcal{L} = \frac{1}{2}\left(\bar{h}_{\mu\nu}\Box\bar{h}^{\mu\nu} - \frac{1}{D-2}\bar{h}^\rho_\rho\Box\bar{h}^\rho_\rho\right) = \frac{1}{2}\bar{h}_{\mu\nu}\mathcal{O}^{\mu\nu\alpha\beta}\bar{h}_{\alpha\beta} \tag{B.0.8}$$

where the operator $\mathcal{O}$ is defined by

$$\mathcal{O}^{\mu\nu\alpha\beta} = \Box\left(\frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) - \frac{1}{D-2}\eta^{\mu\nu}\eta^{\alpha\beta}\right). \tag{B.0.9}$$

The propagator of $\bar{h}$ is defined by taking inverse of the Fourier transform operator of $\mathcal{O}$, which gives

$$\mathcal{P}^{\mu\nu\alpha\beta} = \frac{-i}{2p^2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}\right). \tag{B.0.10}$$

With these interactive term and propagator, looking at the t-channel to avoid lengthy expressions, the tree-level $2 \to 2$ scattering then reads

$$\begin{aligned}
&\langle\phi(p_3)\phi(p_4)|\phi(p_1)\phi(p_2)\rangle = \\
&-\kappa^2\left(p_{1\mu}p_{3\nu} + p_{1\nu}p_{3\mu} - 2\eta_{\mu\nu}\frac{m^2}{D-2}\right) i\frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}}{2t}\left(p_{2\rho}p_{4\sigma} + p_{2\sigma}p_{4\rho} - 2\eta_{\rho\sigma}\frac{m^2}{D-2}\right) \\
&= -\frac{\kappa^2}{t}\Bigg[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) - 2(p_1\cdot p_3)(p_2\cdot p_4) - 2m^2(p_1\cdot p_3 + p_2\cdot p_4) \\
&\qquad - 2\frac{D}{(D-2)}m^4\Bigg].
\end{aligned} \tag{B.0.11}$$

In the non-relativistic limit we finally obtain

$$\langle\phi(p_3)\phi(p_4)|\phi(p_1)\phi(p_2)\rangle_{NR} = -i\frac{\kappa^2}{t}2m^4\left(4 - 1 - \frac{D}{D-2}\right)$$

$$= -4\kappa^2 \frac{m^4}{t} \frac{D-3}{D-2}. \tag{B.0.12}$$

In terms of the Planck mass, the non-relativistic limit of the $2 \to 2$ scattering mediated by gravitons, when one adds s and u-channels, takes the form

$$\langle \phi(p_3)\phi(p_4)|\phi(p_1)\phi(p_2)\rangle_{NR} = -4i \frac{D-3}{D-2} \frac{m^4}{M_{Pl}^{D-2}} \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right), \tag{B.0.13}$$

and, factorizing the $4m^2$ expressing the usual QFT normalization over the non-relativistic one, the position space potential between two identical particles of mass m reads accordingly $V(r) = -\frac{D-3}{D-2} \frac{m^2}{M_{Pl}^{D-2}} \frac{1}{r}$.

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