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Reconstructing $f(T)$ gravity and exploring the torsion driven warm inflationary cosmology

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Abstract The current paper reports an investigation of a warm inflationary scenario in the context of $f(T)$ gravity for a spatially flat FLRW universe. In our model, inflation is driven purely by the torsional sector of $f(T)$ gravity, without introducing any additional scalar fields. We focus on the high dissipative regime ($R \gg 1$), reconstruct the Hubble parameter as a function of the e-folding number N, and derive the slow-roll parameters $\varepsilon_1(N)$ and $\varepsilon_2(N)$. The study has encapsulated the dynamics of inflation and its duration under strong dissipation. The dissipative coefficient Γ is modeled with a temperature-dependent power-law form, linking the inflationary dynamics to thermal corrections and the particle content of the early universe. The analysis has affirmed that the torsion-induced energy density ρ_T successfully transitions to radiation energy density ρ_{rad}, facilitating a graceful exit from inflation. Finally, we have validated our model by comparing the scalar spectral index and tensor-to-scalar ratio with Planck 2018 results, demonstrating consistency within observational bounds. Additionally, it is verified that the thermal domination condition $T_*/H > 1$ and the torsion dominance condition $\rho_T/\rho_{rad} > 1$ are satisfied.

Contents

1 Introduction

The warm inflation (WI) paradigm [1–4] is among the most appealing of the various ideas that sought to incorporate coherent inflationary dynamics inside an explicit quantum field theory realization. Warm inflation investigates how the vacuum energy that propels inflation must finally be transformed into radiation, which is typically made up of a range of particle species, so making inflationary dynamics intrinsically a multi-field problem. The conventional slow-roll inflation model distinguishes between two distinct time periods: expansion and reheating. First, the cosmos is thought to be in a super-cooled phase due to inflation's exponential expansion. The cosmos is then warmed up after that. In such a scenario, two outcomes occur. First, the inflaton's quantum fluctuations are left to produce the necessary density perturbations in this frigid world. Second, a temporally localized process that swiftly disperses enough vacuum energy for warming is needed to overcome the temperature cliff following expansion [2].

The main focus of inflationary dynamics is the growth of a scalar field, which carries the major energy of the universe during inflation and interacts with other fields. In the warm inflation picture, interactions not only modify the scalar field

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effective potential through quantum corrections, but they also cause fluctuation and dissipation effects. In the standard inflation [5–8] picture, it is implicitly assumed that these interactions have no effect. All three of these effects are undoubtedly caused by interactions in condensed matter systems (some examples are given in [9]). Furthermore, the system as a whole would attempt to spread the available energy equally, and the scalar field would desire to dissipate its energy to other fields from the standpoint of statistical mechanics. In the end, the question requires a comprehensive dynamical computation [10].

Inflation causes the universe to expand almost exponentially, which causes all species' number densities to decrease and the universe to become supercooled. Upon the termination of inflation, the universe experiences a phase of reheating during which the inflaton oscillates and releases its energy as particles [11]. This scenario is known as "Cold Inflation". However, the warm inflation dynamics offer some intriguing aspects. In warm inflation the coupling of the inflaton with other fields during and after inflation is taken into account. As a result of this interaction energy transfers from the inflation field to the radiation field keeping the universe warm. Therefore, in warm inflation the reheating mechanism is not needed. Another difference between warm and cold inflation is fluctuation. In cold inflation, fluctuations are quantum [12–15] type whereas in warm inflation both thermal fluctuation (T_*) and quantum fluctuation exists and the dominance of thermal fluctuations is contingent upon the condition $T_* > H$ [16–21].

Curvature is not a special way of describing gravity. Torsion can be used as an alternative to Riemannian geometry for the geometrical description of gravity [22,23]. The generalization of the concept of teleparallel gravity [24–30] has drawn interest in recent years. Spin is generally considered as a source of torsion. Several forms of torsion tensor can be araised from such a source that gives different modification to energy–momentum tensor [31]. Many researcher accept the idea that torsion might have influenced the dynamics of the early universe. Importantly, torsion naturally contributes repulsive terms to energy–momentum tensor allowing singular free cosmological models [32,33]. Teleparallel gravity is constructed from a skew-symmetric Weitzenbock connection whereas general relativity is constructed using a symmetric Levi Civita connection. The specific modified theory of gravity that has grabbed the attention of cosmologists is $f(T)$ teleparallel gravity [24–30]. Similar to $f(R)$ gravity, which is based on the formulation where the lagrangian of the gravitational field equations is a function of the Ricci scalar R of the underlying geometry, $f(T)$ gravity. A linear $f(T)$ is equivalent to GR. Einstein first proposed teleparallel theory to integrate gravity and electromagnetics [34,35]. $f(T)$ teleparrel gravity is proved to be useful in context of cosmological and astrophysical applications [36,37]. Particularly, various bounce and inflationary scenarios have been studied in $f(T)$ gravity framework [38–43].

In this paper we will investigate the warm inflationary scenario in the context of $f(T)$ teleparallel gravity. The manuscript is arranged in the following way: in Sect. 2, we introduce the theoretical background of $f(T)$ gravity and its relevance. Section 3 elaborates the fundamental mechanisms needed to achieve an inflationary phase. Subsequently, we analyze the specific realization of inflationary behavior under $f(T)$ gravity in Sect. 4, that is followed by an in-depth treatment of high dissipative effects in warm inflation scenarios in Sect. 5. Section 6 evaluates the model's predictions in the n_s–r parameter space, offering a direct comparison with observational bounds. The necessary criteria for sustaining inflation are detailed in Sect. 7. In Sect. 8, we have compared with existing literature to highlight how the present study realizes torsion-driven inflation in $f(T)$ gravity without an inflaton scalar field or reheating phase, leading to a warm inflationary scenario with coexisting thermal and quantum fluctuations, and we conclude with a summary and outlook in Sect. 9.

2 Theoretical backgraoung of $f(T)$ gravity and its cosmological implications

Recently, a new modified theory of gravity known as the $f(T)$ theory of gravity-also known as "gravity with torsion" [28,31,44]-has been presented as an analogy to the $f(R)$ theory of gravity . It aims to explain the current accelerated expansion without utilizing dark energy. Teleparallelism employs a vierbein field as a dynamical entity $e_k(x^\nu)$ (k = 0, 1, 2, 3) that serves as an orthonormal basis for the tangent space at each point x^ν of the manifold: $e_k.e_l = \zeta_{kl}$, where $\zeta_{kl} = diag(1, -1, -1, -1)$. The dual vierbein yields the metric tensor as $g_{\nu\mu} = \zeta_{kl}\ e^k_\nu(x)\ e^l_\mu(x)$. Instead of using the curvature defined by the Levi–Civita connection to describe gravity in teleparallel gravity, one might investigate the model of torsion using the curvature-less Weitzenbock connection, whose non-null torsion is

$$T^\beta_{\nu\mu} = e^\beta_k(\partial_\nu e^i_\mu - \partial_\mu e^i_\nu) \tag{1}$$

All of the information on the gravitational field is contained in this tensor. Its dynamical equations for the vierbein entail the Einstein equations for the metric, and the TEGR Lagrangian is constructed using the torsion Eq. (1). The teleparallel Lagrangian is

$$T = S_\gamma{}^{\nu\mu}\ T^\gamma{}_{\nu\mu} \tag{2}$$

where

$$S_\gamma{}^{\nu\mu} = \frac{1}{2}\left(K^{\nu\mu}{}_\gamma + \delta^\mu_\gamma T^{\rho\mu}{}_\rho - \delta^\nu_\gamma T^{\rho\nu}{}_\rho\right) \tag{3}$$

and $K^{\nu\mu}_{\gamma}$ is known as contorsion tensor which is given by

$$K^{\nu\mu}{}_{\gamma} = -\frac{1}{2}\left(T^{\nu\mu}{}_{\gamma} - T^{\nu\mu}{}_{\gamma} - T_{\gamma}{}^{\mu\nu}\right) \tag{4}$$

Promoting teleparallel Lagrangian density as a function of T, the action reads as

$$I = \frac{1}{16\pi G}\int d^4x \; e \; f(T) \;\; + I_m \tag{5}$$

where $e = det(e^k_\nu)$=$\sqrt{-g}$. The field equation obtained by the variation of the action with respect to the vierbein is

$$\frac{1}{2}K^2 \, e_k{}^{\gamma} \, T_{\gamma}{}^{\mu} = \left[e^{-1}\partial_\nu(e\; S_k{}^{\nu\mu})) - e^{\beta}_k \, T^{\gamma}{}_{\nu\beta}\, S_{\gamma}{}^{\mu\nu}\right] f_T + S_k{}^{\nu\mu}\partial_\nu \; f_{TT} + \frac{1}{4}e^{\nu}_k f(T) \tag{6}$$

where $K^2 = 8\pi G$, $f_T = \frac{df}{dT}$, $f_{TT} = \frac{d^2 f}{dT^2}$, $S_k{}^{\nu\mu} = e_k{}^{\gamma} S_{\gamma}{}^{\nu\mu}$ and $T_{\mu\nu}$ is the energy–momentum tensor.

We assume that our universe is homogeneous and isotropic. Therefore

$$e^i_\mu = diag(1, a(t), a(t), a(t)) \tag{7}$$

where $a(t)$ is the scale factor. Now combining Eqs. (1)–(4) we obtain

$$T = -6H^2 \tag{8}$$

H in Eq. (8) is known as Hubble parameter which is denoted by $H = \frac{\dot{a}}{a}$. The substitution of the vierbein Eq. (7) in Eq. (6) yields (k=$\mu = 0$)

$$12H^2 f_T + f = 2K\rho \tag{9}$$

and (k=$\mu = 1$) yields

$$48H^2\dot{H} f_{TT} - (12H^2 + 4\dot{H}) f_T - f = 2K^2 p \tag{10}$$

In Eqs. (9) and (10) ρ and p represent total energy density and pressure, and these two equations can be rewritten as [45,46]

$$\rho_T = \frac{1}{16\pi G}\left[2T f_T - f(T) - T\right] \tag{11}$$

$$p_T = -\frac{1}{16\pi G}\left[-8\dot{H}T f_{TT} + (2T - 4\dot{H}) f_T - f + 4\dot{H} - T\right] \tag{12}$$

where ρ_T and p_T are energy and pressure due to torsion contribution. It may be noted that in the limit of $f(T) = T$, one can get $\rho_T = 0$ as $f_T = 1$, ensuring consistency with General Relativity at low-energy scales. Furthermore, an effective torsion equation of state can be derived from the relation $w_T = \frac{p_T}{\rho_T}$.

3 Mechanism and key ingredients of the inflationary dynamics

In physical cosmology, the Friedmann equations are a system of equations that control cosmic expansion in homogeneous and isotropic universe models within the general relativity framework. To formulate the Friedmann equations, we begin with the Einstein field equations in a spatially flat Friedmann–Robertson–Walker (FRW) universe. The metric is given by $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$, where $a(t)$ is the scale factor. The two fundamental equations of cosmology i.e., the Friedmann equations, govern the evolution of the system of the universe. The Friedmann equations are read as,

$$H^2 = \frac{1}{3M_P{}^2}\left(\rho_{rad} + \rho_{in}\right) \tag{13}$$

$$\dot{H} = -\frac{1}{2M_P{}^2}\left[(\rho_{rad} + \rho_{in}) + (p_{rad} + p_{in})\right] \tag{14}$$

where suffix "rad" indicates the energy density of radiation and "in" represents the energy density of the inflaton field. Both the inflaton field and the radiation fluid contribute to the overall energy density in this framework. The conservation equation takes the form,

$$\dot{\rho}_{in} + 3H\left(\rho_{in} + p_{in}\right) = -\Gamma\left(\rho_{in} + p_{in}\right), \tag{15}$$

$$\dot{\rho}_{rad} + 3H\left(\rho_{rad} + p_{rad}\right) = \Gamma\left(\rho_{in} + p_{in}\right), \tag{16}$$

where Γ is the dissipation coefficient, and it can be either constant, dependent on either the temperature (T_{rad}) or the scalar field, or dependent on both. Slow-roll parameters play an important role in inflationary dynamics. The first slow-roll parameter is defined by,

$$\varepsilon_1 = -\frac{\dot{H}}{H^2} \tag{17}$$

and the next slow-roll parameters are defined by

$$\varepsilon_{n+1} = \frac{\dot{\varepsilon_n}}{H\epsilon_n} \tag{18}$$

In warm inflation, there is another type of slow-roll parameter, which is defined by

$$\vartheta = \frac{\dot{\Gamma}}{H\Gamma} \tag{19}$$

The parameter ϑ represents the evolution of the dissipation coefficient during inflation.

In inflationary cosmology, an explicit exponential term can, to a certain extent, indicate the evolution of the patch of the inflated universe. By definition Hubble rate of expansion can be expressed as $H = \frac{\dot{a}}{a}$ whose solution obviously reads as $a(t) = a_0 \int_{t_{initial}}^{t} H(\eta)\, d\eta$ Generally, the time integral of the Hubble rate is called the number of e-fold and is defined by

$$N = \int_{t_initial}^{t_{end}} H(\eta)\, d\eta \tag{20}$$

4 Inflationary dynamics in the $f(T)$ gravity framework

We know that the $f(T)$ gravity framework is a modification of teleparallel gravity and offers an alternative way to explain the universe's accelerated expansion without invoking dark energy [47,48]. Unlike $f(R)$ gravity, $f(T)$ theories retain second-order field equations, making the analysis of cosmic dynamics more tractable [37]. Exploration of inflation within the $f(T)$ framework is useful because it gives the scope for novel inflationary scenarios. In such scenarios, the torsion, rather than curvature, is responsible for the early universe's expansion. Such models can lead to distinctive observational signatures, including modified tensor-to-scalar ratios and non-standard consistency relations [49,50]. Therefore, understanding inflation in the $f(T)$ framework broadens the landscape of viable cosmological theories and provides testable predictions for upcoming observations. In $f(T)$ gravity energy density is given by

$$\rho_T = \frac{1}{16\pi G}[2Tf_T - f(T) - T] \tag{21}$$

We consider inflation in the framework of $f(T)$ gravity. Therefore $\rho_{in} = \rho_T$ and Eq. (13) takes the form (considering $M_p = 1$)

$$H^2 = \frac{1}{3}(\rho_T + \rho_{rad}) \tag{22}$$

During warm inflation radiation is quasi stable i.e $\dot{\rho}_{rad} << H\rho_{rad}$. Now, quasi-stable production of heat is applied to Eq. (16) and the equation reads as

$$4H\rho_{rad} = \Gamma(\rho_T + p_T) \tag{23}$$

$$\rho_{rad} = -\frac{3}{2}\left(\frac{R}{1+R}\right)\dot{H} \tag{24}$$

where R is termed as dissipative parameter and defined as $R = \frac{\Gamma}{3H}$. The dissipative coefficient Γ regulates the inflaton field's damped evolution. There exist two kinds of scenarios depending upon R. $R > 1$ corresponds to a strong dissipative process, $R < 1$ corresponds to a weak dissipative process and $R \leq 1$ goes back to ordinary inflation case. ρ_{rad} is the energy density due to torsion. Therefore $\dot{H}$ should be negative to become $\dot{\rho}_{rad}$ positive. Substituting Eq. (24) in Eq. (15) we obtain ρ_T as

$$\rho_T = 3H^2 + \frac{3}{2}\left(\frac{R}{1+R}\dot{H}\right) \tag{25}$$

We choose a particular form of $f(T)$ to derive ρ_T and the chosen model is $f(T) = \alpha T + \beta T^2$. Before proceeding further, let us elaborate the rationale behind such a choice of $f(T)$ model. It may be noted that the choice of the functional form $f(T) = \alpha T + \beta T^2$ chosen here is motivated by its simplicity and ability to capture leading-order deviations from teleparallel equivalent of general relativity (TEGR), which corresponds to the linear term αT. The quadratic correction term βT^2 introduces modifications relevant at high-energy scales, such as those encountered in the early universe, and has been widely studied in the context of inflationary cosmology. This form allows analytical tractability while retaining rich phenomenology to model early-universe acceleration. In support of the chosen form $f(T) = \alpha T + \beta T^2$, we refer to studies of [51,52] where asymptotic solutions of a homogeneous and isotropic universe governed by a quadratic form of $f(T)$ gravity have been derived. These works demonstrate that higher-order torsion terms can naturally lead to a late-time accelerated expansion in the Friedmann–Robertson–Walker (FRW) framework, with the scalar torsion T evolving as a function of cosmic time t. In an extensive review, Cai et al. [37] thoroughly discussed quadratic forms of $f(T)$ in an elaborate manner. Now with the chosen form of $f(T)$ Eq. (25) reduces to

$$\rho_T = 3(1-\alpha)H^2 + \frac{\beta(2n-1)}{2}(-6)^n H^{2n} \tag{26}$$

Substituting Eq. (27) in Eq. (25) time derivative of Hubble parameter is obtained as

$$\dot{H} = \frac{2}{3}\left(\frac{R}{1+R}\right)\left[(A-3)H^2 + BH^{2n}\right] \tag{27}$$

where $A = 3(1-\alpha)$ and $B = \frac{\beta(2n-1)}{2}(-6)^n$. The first slow roll parameter ε_1 is derived using Eq. (27) which takes the form

$$\varepsilon_1 = -\frac{2}{3}\left(\frac{1+R}{R}\right)[(A-3) + BH^{2n-2}] \tag{28}$$

Now using the relation $\varepsilon_2 = \frac{\dot{\varepsilon}_1}{H\epsilon_1}$ we obtain the second slow-roll parameter as

$$\varepsilon_2 = \frac{\left[\frac{1+R}{R}\left(B(2n-2)H^{2n-3}\dot{H}\right) - \frac{\dot{R}}{R^2}\left((A-3) + BH^{2n-2}\right)\right]}{H\left[(A-3) + BH^{2n-2}\right]} \tag{29}$$

The connection between the energy density of the radiation and the universal temperature (T_*)is provided by the hydrodynamic description of the radiation [53]

$$\rho_{rad} = C_* T_*^4 \tag{30}$$

where C_* is the Stephen–Boltzman constant and is defined by $C_* = \frac{\pi^2 g_*}{30}$. g_* is the number of degrees of freedom of the radiation field [54]. Combining Eqs. (22) and (30) the temperature can be written as

$$T_*^4 = \frac{1}{C_*}\left[(3-A)H^2 - BH^{2n}\right] \tag{31}$$

The dissipation coefficient Γ can be considered as a constant; however, in a broader perspective, it can be considered as a function of temperature. Therefore, a power law form of temperature is considered i.e

$$\Gamma = \Theta T_*^m \tag{32}$$

where Θ is constant. Combining Eqs. (31) and (32) γ is derived as a function of Hubble parameter

$$\Gamma = \Theta\left[\frac{1}{C_*}((3-A)H^2 - BH^{2n})\right]^{\frac{m}{4}} \tag{33}$$

This form of Γ presented above in (33), indicates that Γ is influenced by two different powers of the Hubble parameter H. Firstly, a standard quadratic dependence (H^2) and secondly, a higher-order term (H^{2n}), accompanied by constants A, B, and the exponent n. If we have a further insight into the expression, we observe that the expression $(3-A)H^2$ may correspond to standard Friedmann-like evolution, while the BH^{2n} is a correction that introduces nonlinearity, that might be possibly associated with modifications to general relativity, higher-order curvature corrections, or exotic matter effects. We can further infer that the presence of two different scalings with H allows Γ to be enriched with a better dynamical behavior, especially at different epochs of the universe. This is because of the fact that, depending on whether H is large (early universe) or small (late universe), the scalings would change accordingly. If $n > 1$, the H^{2n} term would be large and hence it would dominate at early times, and if $n < 1$, it would become significant for the late times. Thus, Eq. (33) suggests that Γ plays a crucial role in describing deviations from simple cosmological evolution, possibly tied to particle production, modified gravity, or dynamical dark energy scenarios.

5 Inflationary dynamics in high dissipative regime

For warm inflationary models in the high dissipative regime, the dissipation coefficient satisfies $\Gamma \gg 3H$. This leads to significant interest for warm inflationary models as a significant alternative to standard cold inflation scenarios [2,55]. In this regime, the inflaton field interacts strongly with other fields, continuously converting part of its energy into a thermal radiation bath during inflation. This mechanism alleviates the need for extremely flat potentials, as the friction induced by dissipation effectively slows the inflaton's motion even for steeper potentials [56]. In addition to this, during primordial density perturbations, the thermal fluctuations dominate over quantum fluctuations and this leads to distinct predictions for the scalar power spectrum and a naturally suppressed tensor-to-scalar ratio [54]. According to [57], the presence of a sustained thermal bath throughout inflation facilitates a smooth transition to the radiation-dominated era, potentially avoiding the complexities associated with the reheating process . Apart from the phenomenological advantages, high dissipative inflation paves avenues to the embedding of inflation within the frameworks of realistic particle physics in which the strong interactions and thermal effects are naturally present [58]. Overall, considering the inflationary dynamics in the high dissipative regime enriches the inflationary models and offers novel observational signatures that, in the future, can be tested with future cosmological data.

In this article, we use the high dissipative region for our warm inflation. As we know, when $\Gamma >> 3H$ i.e $R >> 1$, the effect of warm inflation would be dominant enough to give a smooth transition into the "graceful exit". Therefore in the rest of our work, we will take $R >> 1$. Imposing this condition first slow-roll parameter is obtained as (from Eq. (30))

$$\varepsilon_1 = -\frac{2}{3}\left[(A-3) + BH^{2n-2}\right] \tag{34}$$

Also, in order for inflation to end soon enough so that it does not affect the baryogenesis process, we have tight constraints on the e-folding numbers. So, as a consequence, we have written H as a function of the e-folding number. Using Eqs. (27) and (20) the Hubble parameter is obtained as

$$H(N) = \left(\frac{e^{\frac{4(2A-3)(1-n)N}{3}} - B}{(2A-3)}\right)^{\frac{1}{2-2n}} \tag{35}$$

In the previous section, we have seen that the slow roll parameters ε_1, ε_2, and ϑ are functions of the Hubble parameter. We have obtained the Hubble parameter as a function of the e-folding number (in Eq. (35)). Therefore, the parameters in terms of e-folding number N are obtained as

$$\varepsilon_1(N) = \frac{4e^{4N(-1+n)(-1+2\alpha)}(-1+2\alpha)}{2e^{4N(-1+n)(-1+2\alpha)} + \left((-6)^n - (-3)^n 2^{1+n} n\right)\beta} \tag{36}$$

$$\varepsilon_2(N) = \frac{4(-1+n)\left(-(-6)^n + (-3)^n 2^{1+n} n\right)(-1+2\alpha)\beta}{-2e^{4N(-1+n)(-1+2\alpha)} + (-6)^n(-1+2n)\beta} \tag{37}$$

$$\vartheta(N) = \frac{(-3+2A)P^{\frac{1}{2(-1+n)}}\left(-3+A+BP^{\frac{1}{-1+n}}\left(P^{\frac{1}{2-2n}}\right)^{2n} n\right)N}{3\left(-1+Be^{\frac{4}{3}(-3+2A)N(-1+n)}\right)\left(-3+A+BP^{\frac{1}{-1+n}}\left(P^{\frac{1}{2-2n}}\right)^{2n}\right)} \tag{38}$$

where $P = \left(\frac{-B+e^{-\frac{4}{3}(-3+2A)N(-1+n)}}{-3+2A}\right)$.

The set of intricate equations presented, namely $\varepsilon_1(N)$, $\varepsilon_2(N)$, and $\vartheta(N)$, are evidently functions dependent on the parameter N, with additional parameters such as α, β, A, B, and n that are controlling their behavior. Equation (36) for $\varepsilon_1(N)$ exhibits an exponential dependence on N through terms involving e^{4N} in the numerator of the respective equation, suggesting rapid growth or decay depending on the sign of the exponent that may result from the behaviour of $(-1+n)(-1+2\alpha)$. The polynomial factors $(-1+n)$ and $(-1+2\alpha)$ indicate symmetry breaking or critical points when $n = 1$ or $\alpha = 1/2$, which could be pertaining to phase transitions or bifurcations in physical or dynamical systems [59,60]. Similarly, $\varepsilon_2(N)$ in Eq. (37) exhibits a balance between exponential terms and algebraic corrections using β, indicating regimes where the exponential dominates versus where β-corrections become significant. Let us now have a look into the form of $\vartheta(N)$ in Eq. (38). We observe that it is involving powers of P and is described by an exponential relation. Referring to [61,62], we can say that the existence of $P = \left(\frac{-B+e^{-\frac{4}{3}(-3+2A)N(-1+n)}}{-3+2A}\right)$ increased to non-zero powers of n implies the existence of scaling rules or self-similar structures. The complicated dependence on N through P indicates that $\vartheta(N)$ may result in cumulative or integrated effects over the evolution of N, consistent with behaviors observed in systems exhibiting intermediate asymptotics, as indicated in [63]. Overall, these expressions seem to arise from a system where exponential dynamics are modulated by nonlinear interactions, possibly hinting at applications in cosmology [64], phase transitions [65], or nonlinear field theories [66]. To have a further insight into the behaviour of the above-mentioned parameters in terms of e-folding number N, let us have an asymptotic analysis. If we consider N to be moderately large, the exponential term in P keeps decaying, assuming $(-3+2A)(-1+n) > 0$. Thus, $P \to \frac{-B}{-3+2A} = \frac{B}{3-2A}$. If we look into the behavior of $\varepsilon_1(N)$ we find that as N increases, the dominant terms in the numerator and denominator of $\varepsilon_1(N)$ are proportional to e^{4N}, allowing lower-order terms to be neglected. Hence, we find $\varepsilon_1(N) \to \frac{4(-1+n)(-1+2\alpha)(-1+2\alpha)}{2(-1+n)(-1+2\alpha)} = 2(-1+2\alpha)$.. On the other hand, the denominator of $\varepsilon_2(N)$ gets an exponential increase with N, while the numerator remains finite. Consequently, $\varepsilon_2(N) \to 0$. The behavior of $\vartheta(N)$ depends on the asymptotic form of P and the exponentially growing terms. Provided the dominant exponential contributions in the numerator and denominator cancel appropriately, $\vartheta(N)$ approaches a finite limit determined by the parameters A, B, n, and m. Otherwise, $\vartheta(N)$ may diverge. Hence, we can conclude the following limiting cases for N tending to be significantly large.

$\varepsilon_1(N) \to 2(-1+2\alpha)$;
$\varepsilon_2(N) \to 0$;
$\vartheta(N) \to$ finite or divergent depending on parameters

Table 1 demonstrates the numerical behavior of the first slow-roll parameter ε_1 as a function of N. We have taken fixed values of $\alpha = 1.22$, $n = -1$, and $\beta = -7.5$ for the Table. It is observed that ε_1 decreases rapidly as N increases, exhibiting an exponential suppression. It is apparent that during horizon crossing ε_1 is very very less than one. Moreover at $N = 0$ i.e at the end of inflation $\varepsilon_1 \approx 1$ which ensures the end of inflation.

The specific values of model parameters such as $\beta = -11/3$ and $n = 2$ were chosen for pictorial presentation in Fig. 1. The purpose behind the choices of these values were to check the predictions for the model under study for its consistency in terms of the scalar spectral index (n_s) and tensor-to-scalar ratio (r) with respect to the latest observational constraints. The pictorial presentation in Fig. 1 makes it apparent that the trajectory of the $n_s - r$ curve for increasing e-folding number N lies inside the observationally acceptable region and hence it comes out that the model that we are discussing can get hold of the early universe inflationary phase. In Table 1 we have mentioned values for the first slow-roll parameter ϵ_1 pertaining to different e-folding numbers N, where we have a set of fixed parameters, namely, $\alpha = 1.22$, $n = -1$, and $\beta = -7.5$. The purpose of demonstrating this table is to view the behavior of ϵ_1 in the context of our model for the dynamics of inflation and the graceful exit. Table 1 shows that ϵ_1 gradually decays as the e-folding

Table 1 The table shows numerical values of ε_1 for different e-folding numbers

N	0	10	20	30	40	50	60	70
α	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22
n	− 1	− 1	− 1	− 1	− 1	− 1	− 1	− 1
β	− 7.5	− 7.5	− 7.5	− 7.5	− 7.5	− 7.5	− 7.5	− 7.5
ε_1	1.0017	1.43×10^{-50}	1.33×10^{-100}	1.24×10^{-150}	1.15×10^{-200}	1.07×10^{-250}	1.004×10^{-300}	9.36×10^{-351}

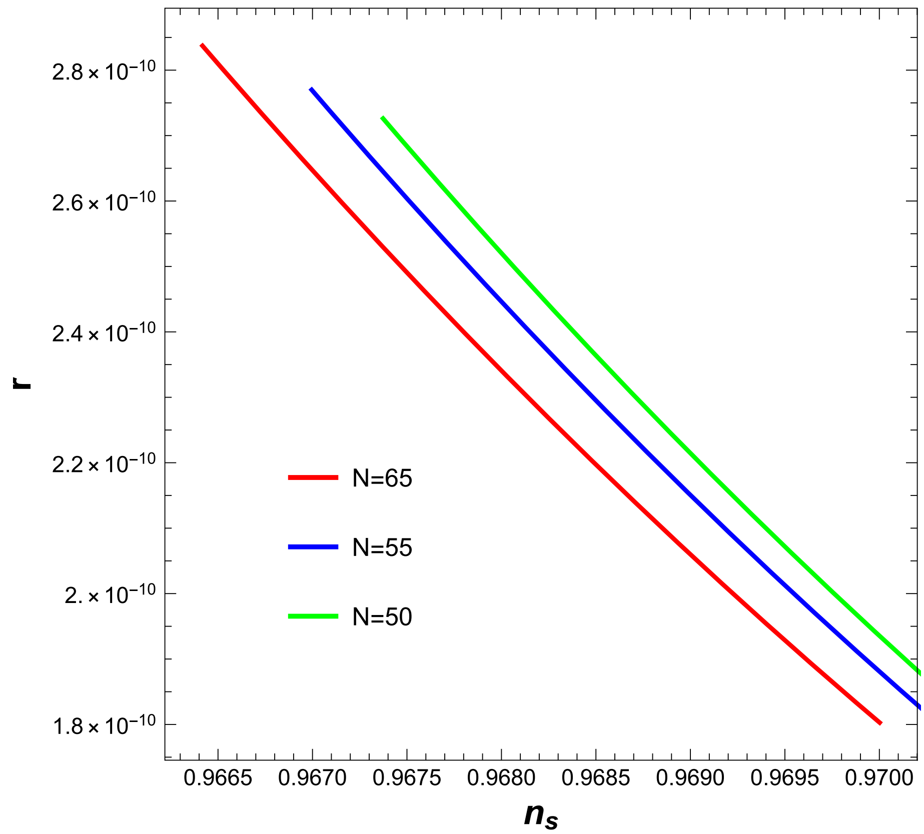

Fig. 1 Scalar spectral index (n_s) versus tensor-to-scalar ratio (r) for our torsion-driven warm inflationary model in $f(T)$ gravity. The trajectory, for increasing e-folding number N (50 (green), 55 (blue), 65 (red))), is plotted with fixed parameters $\beta = -11/3$, $n = 2$, $C_* = 70$, $\Theta = 0.5$ and $m = -1$. This figure visually confirms that the model presented here has viability for inflationary dynamics

number N increases. This behaviour makes it apparent that the slow-roll conditions ($\epsilon_1 \ll 1, \epsilon_2 \ll 1$) (for $N \neq 0$) are satisfied during the inflationary epoch. The graceful exit from inflation is important for warm inflationary scenario. Let us now comment on the possibility of end of inflation as can be visualized in Table 1. At $N = 0$, which corresponds to the end of the inflationary period, $\epsilon_1 \approx 1.0017$. The condition $\epsilon_1 \approx 1$ (or $\epsilon_1 \geq 1$) . We observe this to be violation of the slow-roll condition. This implies that the inflationary phase is terminated. From this, we can interpret that in Table 1, there is an indication of a transition to a radiation-dominated era.

6 Analysis of inflationary dynamics through the $n_s - r$ plot

We note that in order for our model to be phenomenologically sound, we need to find some measurable quantities that we could tally against the observations. We note that the amplitude of scalar perturbation (P_s^*), scalar spectral index (n_s), and scalar-to-tensor ratio (r) are good indications for these.

We first note that for warm inflation, the physical parameters are temperature-dependent, so the scalar perturbations are also temperature-dependent. We note that in the article [67] Taylor and Berera have explicitly calculated all the phenomenological parameters, such as mentioned above, from the first principle. They first take the metric perturbation in the isocurvature gauge and then also explicitly show the tensor perturbations. In order to include the effect of temperature, they have used the fact for radiation in the blackbody, the energy density is proportional to T_r^4, and during the graceful exit, that is when the inflation is ending, and the universe is entering a radiation-dominated era one would expect the equality of scalar field and radiation. Under such conditions, it is possible to show that the amplitude of the scalar perturbation acquires a Bose-Einstein-like distribution due to the

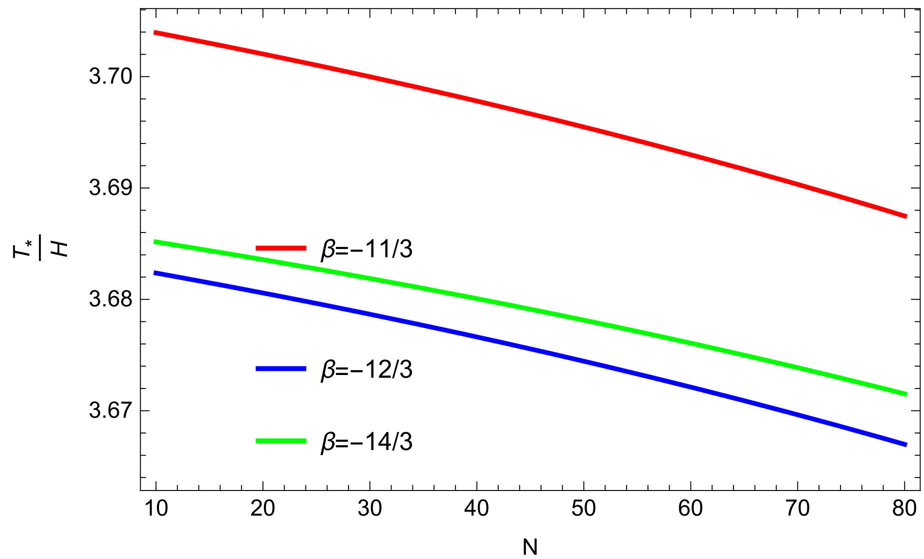

Fig. 2 Plot of T_*/H versus e-folding number N for different values of the model parameter β. The parameters for this plot are fixed as: $\alpha = 0.5008$, $n = 2$, $C_* = 70$, and $m = -1$. This figure illustrates that $T_* > H$. This provides insights into the inflationary dynamics under the chosen parameter set

photon's presence in the radiation bath. Similar to that, other quantities follow. They have also shown that for polynomial and exponential potential, the results are in very good agreement with the currently observed anisotropy in the CMB. As a consequence, we can see that P_s^* can be given as [68,69]

$$P_s^* = \frac{H^2}{8\pi^2 M_p^2 \varepsilon_1}\left[1 + 2n_{BE} + \frac{2\sqrt{3}\pi R}{\sqrt{3+4\pi R}}\frac{T_*}{H}\right]G(R) \quad (39)$$

where G(R) is given by [70,71]

$$G(R) = 1 + 0.0185R^{2.315} + 0.335R^{1.364} \quad (40)$$

Note that the Bose-Einstein distribution (n_{BE}) comes because of a radiation bath during inflation. The Bose-Einstein distribution n_{BE} is given by $\frac{1}{\left(e^{\frac{H}{T_{in}}}-1\right)}$. Here T_{in} is inflaton fluctuation. Moreover there is no requirement that T_{in} should be equal to radiation temperature [71].

We note that the spectral index formula in terms of wave number (k) is given by

$$n_s - 1 = \frac{d\, ln(P_s^*)}{d\, ln(k)} \quad (41)$$

Finally, it has been shown that in the same isocurvature gauge, one can use the tensor perturbation formula to calculate the tensor-to-scalar ratio

$$r = \frac{P_t^*}{P_s^*} \quad (42)$$

where P_t^* denotes the amplitude of the tensor perturbation, which is defined by $P_t^* = \frac{2H^2}{\pi^2 {M_p}^2}$ [68,69]. With correct observation, we have a very sharp bound on the or spectral index $(n_s\ )$ from PLANCK data. Observational evidence indicates that the scalar spectral index should lie within the range 0.96 to 0.9684, and the upper limit of the parameter, the tensor-to-scalar ratio, is $r < 0.064$ [70] Now computing Eqs. (39), (40), (41), (42) the spectral index(n_s) and tensor-to-scalar ratio is obtained as [71]

$$n_s - 1 = 1.815\varepsilon_1 + 3.815\vartheta - \varepsilon_2 \quad (43)$$

and

$$r = 16\varepsilon_1\left(\sqrt{3}\pi\frac{T_*}{H}0.0185R^{2.815}\right)^{-1} \quad (44)$$

In Fig. 1, we have plotted the spectral index (n_s) versus the tensor-to-scalar ratio (r) for different values of the e-folding number N. It is evident from the plot that the model predictions are in good agreement with the latest observational constraints. The trajectory of the curve for increasing N moves within the allowed region of the n_s–r plane, suggesting that the underlying inflationary dynamics are consistent with current cosmological observations such as those reported by *Planck* and *BICEP/Keck* experiments. This favorable behavior highlights the viability of the model in describing the early universe inflationary phase.

7 Condition for satisfying inflationary behavior

The key idea of warm inflation is that the inflation is not isolated. The inflaton field continuously interacts with other fields and dissipates energy. In warm inflation, there are two fundamental conditions. One is whether thermal fluctuation is greater than quantum fluctuation, i.e, $T_* > H$, and the other is whether energy is transferred from the inflaton field to the radiation field.

Figure 2 illustrates that during horizon crossing, thermal fluctuation dominates quantum fluctuations i.e the condition

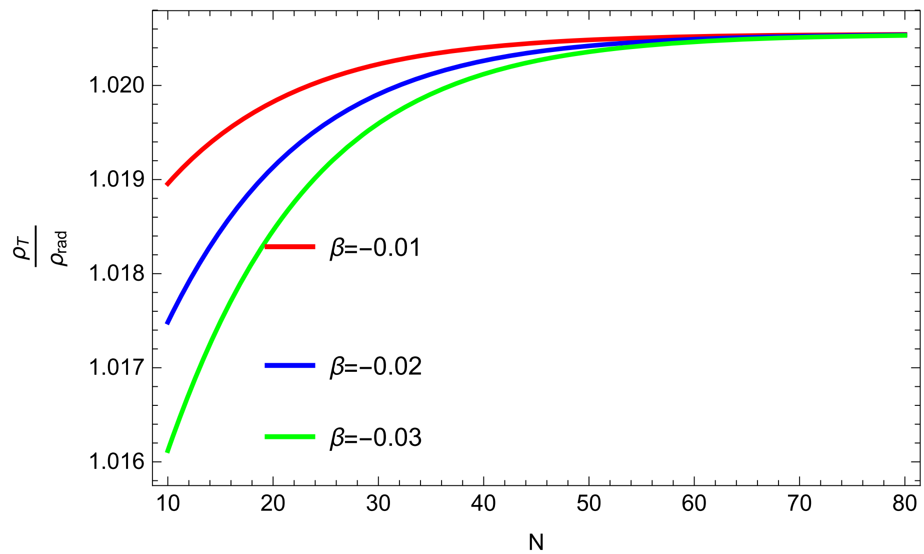

Fig. 3 Plot of ρ_T/ρ_{rad} versus e-folding number N for different values of the model parameter β. The parameters for this plot are fixed as: $\alpha = 0.494$, $n = -1.001$. This figure illustrates that $\rho_T/\rho_{rad} > 1$. This provides insights into the inflationary dynamics under the chosen parameter set

$T_*/H > 1$ is verified. Let us have some further insight into the Fig. 2. Following [72–74] we can say that this condition $\frac{T_*}{H} > 1$ resembles a critical indicator for the fact that the perturbation spectrum is not purely of quantum origin, as in standard cold inflation, but is instead thermally sourced. The dominance of thermal over quantum fluctuations implies the presence of a radiation bath with sufficient temperature to influence the dynamics of the inflaton or the background scalar field driving the evolution [2]. In scenarios such as warm inflation [2], this condition leads to a modified scalar power spectrum of the form [54,56]

$$P_{\mathcal{R}}(k) \propto \left(\frac{T}{H}\right)^p H^2,$$

where p is model-dependent and reflects the dissipative strength of the thermal bath. The consequence of $T_*/H > 1$ is not limited to amplitude modifications but may also impact the spectral index, non-Gaussianities, and the tensor-to-scalar ratio, leading to observationally distinguishable features [54]. Moreover, in pre-inflationary or bounce scenarios where thermal relics persist, such a condition can suggest a departure from vacuum initial conditions, affecting the interpretation of CMB constraints on primordial perturbations [75]. Therefore, in view of the studies mentioned above, we can say that the verification of $T_*/H > 1$ at horizon crossing, as shown in Fig. 2, supports the thermal origin hypothesis for perturbations in the early Universe and demands a re-evaluation of the standard quantum-dominated perturbation theory.

Figure 3 illustrates that during horizon crossing, energy contribution due to torsion is greater than energy contribution due to radiation i.e the condition $\rho_T/\rho_{rad} > 1$ is verified. Moreover it is apparent that with the decrease of e-folding number, the ratio is decreasing i.e there remain an energy transformation from ρ_T to ρ_{rad}. This behavior emphasizes on the significant role of spacetime torsion in the early scenario of the universe. In particular, in models inspired by Einstein–Cartan theory and extensions of GR, where torsion naturally arises from intrinsic spin or non-Riemannian geometries [76,77], the above-mentioned significance is important. Moreover, this figure further implies that as the e-folding number decreases, the ratio ρ_T/ρ_{rad} correspondingly decreases. This suggests a dynamical transfer of energy from the torsion sector to the radiation sector:

$$\frac{d}{dN}\left(\frac{\rho_T}{\rho_{rad}}\right) < 0, \tag{45}$$

where $N = \ln a$ is the number of e-folds. Such a pattern implies that torsion that is dominant at the time of horizon crossing, gradually becomes less dominant as radiation is regenerated or amplified. It may be noted that this pertains to approaching the later stages of inflation or the onset of reheating. Furthermore, this transition is potentially through particle creation or torsion-induced reheating mechanisms [78,79]. Following [80,81], we can comment that this energy transformation could be indications for entropy generation, and reheating efficiency.

8 Comparative comments with scalar field-driven inflationary models

In the present study, we inve elaborated how inflation is being realized purely from the torsional sector of $f(T)$ gravity. This automatically, eliminated the requirement for an inflaton scalar field. Existing literatures show that inflationary models come from a scalar field that drives expansion. In the current work, the torsion-induced energy density ρ_T naturally generates the scope for an inflationary phase through modified teleparallel gravity. The evolution equations involve a temperature-dependent dissipation coefficient. This leads to a transition from inflation to a radiation-dominated epoch. This mechanism allows the universe to have accelerated expansion without introduction of a separate reheating phase, and this is making inflation a consequence of the geometric properties of spacetime in $f(T)$ gravity in the current study.

The work presented in the current study is in contrast with the scalar field-driven warm inflation models, which naturally depend on an inflaton scalar field for cosmological expansion [2,56,82]. Furthermore, this torsion-driven model eliminates the necessity for a distinct reheating phase, a characteristic feature of warm inflation, where continuous energy dissipation from the driving mechanism to the radiation bath maintains a warm universe throughout inflation [83,84]. This mechanism differs from cold inflation, where the universe must be thermalized through a separate post-inflation reheating process [85,86]. In terms of the origin of cosmological perturbations, this model suggests that thermal and quantum fluctuations coexist [87]. The observed perturbation spectrum is thermally sourced, with thermal fluctuations predominating during the inflationary epoch when the condition $T_* > H$ (temperature greater than the Hubble rate) is met [54,88]. This is in complete contrast to cold inflationary models in which primordial perturbations are seeded solely by quantum fluctuations [73,89].

9 Concluding remarks

Inflation causes the universe to expand almost exponentially, which causes all species densities to decrease and the universe to become supercooled. In cold inflation, the inflation field slowly rolls down, and at the end of inflation, it goes into a reheating era where it decays into radiation and matter. In contrast, during warm inflation there exist a continuous reaction between radiation and the inflation field, causing an energy transfer between the inflaton and radiation field. Therefore the reheating mechanism is not further needed in warm inflation.

We would also like to note that here the inflation (or the early de-Sitter type solutions) have been generated from the $f(T)$ gravity, so we have not used any additional scalar fields to drive the inflation, so after the inflation ends the ρ_T would transfer the energy to ρ_{rad} and reach the radiation dominated phase. In this article, we have explored the warm inflationary scenario for the FLRW Universe in the background of $f(T)$ gravity. The warm inflationary scenario is generally considered for the weak and strong dissipative regime. Our work is restricted to a high dissipative regime ie $R >> 1$. Imposing this condition we have reconstructed the Hubble parameter in terms of the e-folding number N. Consequently, the slow roll parameters are obtained in terms of the e-folding number N.

In Sect. 4, we first discussed that the dissipative coefficient Γ regulates the damped evolution of the inflaton field during warm inflation. Depending on the ratio $R = \Gamma/3H$, distinct scenarios arise as follows: $R > 1$ implies a strong dissipative regime; $R < 1$ implies a weak dissipative regime, and $R \leq 1$ implies that the dynamics approach the standard cold inflation scenario. The radiation energy density ρ_{rad}, sourced by torsion effects, increases if $\dot{H} < 0$, implying that a negative $\dot{H}$ is necessary for $\dot{\rho}_{rad} > 0$. Substituting the explicit form of ε_1 (as derived earlier) and applying the chain rule, we obtain an expression for ε_2 that captures the contribution from both the evolution of the Hubble parameter and the dissipative effects encoded in R [2,37,54,57,90–93]. While the dissipation coefficient Γ can be treated as a constant, it is often more realistic to consider it as a temperature-dependent quantity. We adopted a commonly used assumption is a power-law dependence of the form $\Gamma = \Theta T^m$ with Θ as a constant and m as a model-dependent exponent. Substituting the temperature expression into this form, the dissipation coefficient is obtained in terms of the Hubble parameter. In Eq. (33), the coefficients A and B arise from the specific form of the $f(T)$ gravity model, with $A = 3(1-\alpha)$ and $B = \frac{\beta(2n-1)}{2}(-6)^n$. The factor $C_* = \frac{\pi^2 g_*}{30}$ is the radiation constant determined by the effective number of relativistic degrees of freedom, g_*. This formulation effectively connects the dissipative dynamics of warm inflation to the underlying modified gravity and particle content of the early universe.

In Sect. 5, in our analysis, we focus on the high dissipative regime of warm inflation, where the dissipation coefficient Γ dominates over the Hubble friction term. Specifically, we consider the condition $\Gamma \gg 3H$, which implies that the dissipation ratio $R = \Gamma/3H$ satisfies $R \gg 1$. This assumption ensures that the warm inflationary effects are significant enough to allow a smooth transition into the reheating phase, commonly referred to as the "graceful exit" from inflation. Under the high dissipative condition, the first slow-roll parameter simplifies significantly. Imposing $R \gg 1$, we obtain the slow-roll parameters. To study the evolution of inflation with respect to the number of e-folds N, we express the slow-roll parameters ε_1 and ε_2 as functions of N. These parameters provide crucial information about the inflationary dynamics and the duration of the inflationary phase. The expression of $\varepsilon_1(N)$, in which α, β, and n are model parameters, the exponential dependence on N reflects the sensitivity of ε_1 to the number of e-folds in the high dissipative regime.The second slow-roll parameter $\varepsilon_2(N)$ provides insights into the variation of the first slow-roll parameter with respect to the number of e-folds N. It plays a vital role in determining the running of the scalar spectral index and the dynamics of inflation [94]. For the model under consideration, $\varepsilon_2(N)$ takes the form, which reflects the combined influence of the number of e-folds N, the model parameters n, α, and the $f(T)$ gravity correction parameter β. The exponential function's existence in the denominator exhibits how inflation is time-dependent as it increases. Significant effects from the improved torsion-based gravity model are present in the numerator. The rate at which inflation slows down is generally determined by the form of $\varepsilon_2(N)$, which also affects important observational quantities like the scalar spectral index and its variation. This study has its inspiration from [37,54,90,92,95–98]. Moreover, we have shown that the first slow roll parameter ε_1 is nearly equal to 1 at $N = 0$ which confirms the end of inflation (Table 1).

In Sect. 6, we compare the theoretical inflationary model with observational data to measure the accuracy and consistency of the model. To validate our model with observational data, the scalar spectral index and tensor-to-scalar ratio are calculated, and it is apparent from Fig. 1 that our model comes with a good agreement with Planck data [71]. At the final stage (in Sect. 7), we verify the fundamental condition for inflation i.e $T_*/H > 1$ and $\rho_T > \rho_{rad}$. In Fig. 2 we have shown the behavior of T_*/H and the condition $T_*/H > 1$ is satisfied. This condition guarantees that the thermal fluctuations dominate over the quantum fluctuations. Also we have plotted the behavior of ρ_T/ρ_{rad} in Fig. 3. This figure illustrates that the ratio is high at the beginning of inflation. However, it slows down as it approaches the inflation. This is consistent with the radiation dominated epoch at high redshifts with gradual decay to dark energy dominance at lower redshifts, and hence it supports the evolution of the universe through the various epochs.

As future study, we propose to explore warm inflation under the purview of more generalized models including $f(T, B)$ or $f(T, \mathcal{T})$ gravity incorporation with modified gravity frameworks and holographic principle [99]. Furthermore, inclusion of quantum corrections or non-equilibrium thermodynamic effects can help us further in studying dissipation. A comparative study with scalar field–driven warm inflation models such as that in [57], where dissipation arises from inflaton couplings to a thermal bath, could further distinguish the geometric origin of inflation proposed in this paper from field-theoretic mechanisms. This could open up new avenues for detecting observational signatures unique to torsion-based cosmologies.

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Code Availability Statement My manuscript has no associated code/software. [Author's comment: Code/Software sharing not applicable to this article as no code/software was generated or analysed during the current study.]

References

1. A. Berera, L.Z. Fang, Phys. Rev. Lett. **74**(11), 1912 (1995). https://doi.org/10.1103/PhysRevLett.74.1912
2. A. Berera, Phys. Rev. Lett. **75**(18), 3218 (1995). https://doi.org/10.1103/PhysRevLett.75.3218
3. A. Berera, M. Gleiser, R.O. Ramos, Phys. Rev. Lett. **83**(2), 264 (1999). https://doi.org/10.1103/PhysRevLett.83.264
4. V. Kamali, M. Motaharfar, R.O. Ramos, Universe **9**(3), 124 (2023). https://doi.org/10.3390/universe9030124
5. D. Kazanas, Astrophys. J. **347**, 74–86 (1989)
6. A.A. Starobinsky, Phys. Lett. B **91**, 99 (1980)
7. A.H. Guth, Phys. Rev. D **23**, 347 (1981)
8. A.D. Linde, Phys. Lett. B **108**, 389 (1982)
9. U. Weiss, Quantum dissipative system. 5th Edn. World Scientific Publishing Co. Pte. Ltd., University of Stuttgart, Germany, p. 608 (2021). https://doi.org/10.1142/12402
10. A. Berera, Universe **9**(6), 272 (2023). https://doi.org/10.3390/universe9060272
11. L. Kofman, A.D. Linde, A.A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994)
12. A. Riotto, Inflation and the theory of cosmological perturbations, astroparticle physics and cosmology, in *Proceedings: Summer School, Trieste, Italy, Jun 17–Jul 5 2002, ICTP Lecture Notes in Series 14, 317* (2003). arXiv:hep-ph/0210162
13. D. Baumann, TASI lectures on inflation. arXiv preprint arXiv:0907.5424
14. S. Weinberg. Cosmology. Oxford University Press, Oxford, UK (2008)
15. D.H. Lyth, A.R. Liddle, The primordial density perturbation: Cosmology, inflation and the origin of structure hardcover, 1st edn. Cambridge University Press (2009)
16. L.M. Hall, I.G. Moss, A. Berera, Phys. Rev. D **69**, 083525 (2004)
17. A. Berera, Nucl. Phys. B **585**, 666 (2000)
18. I.G. Moss, C. Xiong, J. Cosmol. Astropart. Phys. **04**, 007 (2007)
19. C. Graham, I.G. Moss, J. Cosmol. Astropart. Phys. **07**, 013 (2009)
20. R.O. Ramos, L. da Silva, J. Cosmol. Astropart. Phys. **03**, 032 (2013)
21. M. Bastero-Gil, A. Berera, R.O. Ramos, J. Cosmol. Astropart. Phys. **07**, 030 (2011)
22. K. Hayashi, T. Shirafuji, Phys. Rev. D **19**, 3524 (1979)
23. K. Hayashi, T. Shirafuji, Phys. Rev. D **24**, 3312 (1981)
24. R. Ferraro, F. Fiorini, Phys. Rev. D **75**, 084031 (2007). arXiv:gr-qc/0610067
25. G.R. Bengochea, R. Ferraro, Phys. Rev. D **79**, 124019 (2009). arXiv:0812.1205

26. R. Ferraro, F. Fiorini, Phys. Rev. D **78**, 124019 (2008). arXiv:0812.1981
27. E.V. Linder, Phys. Rev. D **81**, 127301 (2010). arXiv:1005.3039
28. R. Myrzakulov, Eur. Phys. J. C **71**, 1752 (2011). arXiv:1006.1120 [gr-qc]
29. S. Nojiri, S.D. Odintsov, Phys. Rev. D **85**, 104036 (2012)
30. L. Iorio, E.N. Saridakis, MNRAS **427**, 1555 (2012)
31. S. Capozziello, G. Lambiase, C. Stornaiolo, Ann. Phys. **10**, 713 (2001)
32. A.J. Fennelly, L.L. Smalley, Phys. Lett. **129A**, 195 (1988)
33. I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro, Phys. Lett. **162B**, 92 (1985)
34. A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. **217**(1930), 401 (1928)
35. Y.C. Ong, K. Izumi, J.M. Nester, P. Chen, arXiv:1303.0993v1
36. S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys. Rep. **692**, 1–104 (2017). arXiv:1705.11098
37. Y.-F. Cai, S. Capozziello, M. De Laurentis, E.N. Saridakis, Rep. Prog. Phys. **79**, 106901 (2016). arXiv:1511.07586
38. K. Bamba, G.G.L. Nashed, W. El Hanafy, S.K. Ibraheem, Phys. Rev. D **94**(8), 083513 (2016). arXiv:1604.07604
39. Y.-F. Cai, S.-H. Chen, J.B. Dent, S. Dutta, E.N. Saridakis, CQG **28**(28), 215011 (2011). arXiv:1104.4349
40. K. Izumi, Y.C. Ong, JCAP **1306**, 029 (2013). arXiv:1212.5774
41. M. Ghosh, S. Chattopadhyay, Int. J. Mod. Phys. D **32**(10), 2350066 (2023)
42. A. Awad, W. El Hanafy, G.G.L. Nashed, S.D. Odintsov, V.K. Oikonomou, JCAP **07**, 026 (2018)
43. R. Ferraro, F. Fiorini, Phys. Lett. B **702**(1), 75–80 (2011)
44. K. Bamba, C.Q. Geng, C.C. Lee, L.W. Luo, JCAP **2011**(01), 021 (2011)
45. K. Karami, S. Asadzadeh, A. Abdolmaleki, Z. Safari, Phys. Rev. D **88**(8), 084034 (2013)
46. K. Karami, A. Abdolmaleki, JCAP **2012**(04), 007 (2012)
47. G.R. Bengoche, R. Ferraro, Phys. Rev. D **79**, 124019 (2009)
48. E.V. Linder, Phys. Rev. D **81**, 127301 (2010)
49. A. Awad, W. El Hanafy, G.G.L. Nashed, E.N. Saridakis, JCAP **02**, 052 (2018)
50. M. Krššák, R.J. van den Hoogen, J. Pereira, C.G. Boehmer, A. Coley, Class. Quantum Gravity **36**, 183001 (2019)
51. G.L. Nashed, Gen. Relativ. Gravit. **47**, 1–14 (2015)
52. G.G. Nashed, Astrophys. Space Sci. **357**, 1–9 (2015)
53. M. Turner, *The Early Universe by Edward Kolb* (Westview Press, Boulder, 1994)
54. L.M.H. Hall, I.G. Moss, A. Berera, Phys. Rev. D **69**, 083525 (2004). arXiv:astro-ph/0305015
55. A. Berera, L.-Z. Fang, Phys. Rev. Lett. **74**, 1912 (1995)
56. A.N. Taylor, A. Berera, Phys. Rev. D **62**, 083517 (2000)
57. A. Berera, Nucl. Phys. B **585**, 666 (2000). arXiv:hep-ph/9904409
58. M. Bastero-Gil, A. Berera, R.O. Ramos, JCAP **09**, 033 (2009)
59. S.H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (Westview Press, Boulder, 2015)
60. H.E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 1971)
61. G.I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics* (Cambridge University Press, Cambridge, 1996)
62. D. Sornette, *Critical Phenomena in Natural Sciences: Chaos, Fractals, Self-Organization and Disorder* (Springer, Berlin, 2000)
63. C.M. Bender, S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978)
64. P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, 1993)
65. N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Boston, 1992)
66. S. Weinberg, *The Quantum Theory of Fields, Volume 2: Modern Applications* (Cambridge University Press, Cambridge, 1996)
67. A.N. Taylor, A. Berera, Phys. Rev. D **62**(8), 083517 (2000)
68. M. Bastero-Gil, A. Berera, R.O. Ramos, J.G. Rosa, Phys. Rev. Lett. **117**, 151301 (2016)
69. A. Berera, J. Mabillard, M. Pieroni, R.O. Ramos, JCAP **07**, 021 (2018)
70. M. Bastero-Gil, A. Berera, R.O. Ramos, JCAP **07**, 030 (2011)
71. A. Mohammadi, Phys. Rev. D **104**, 123538 (2021)
72. V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, Phys. Rep. **215**, 203 (1992)
73. A.R. Liddle, D.H. Lyth, *Cosmological Inflation and Large-Scale Structure* (Cambridge University Press, Cambridge, 2000)
74. A. Berera, Phys. Rev. Lett. **75**, 3218 (1995)
75. R.H. Brandenberger, Int. J. Mod. Phys. Conf. Ser. **01**, 67 (2010)
76. I.L. Shapiro, Phys. Rep. **357**, 113 (2002). https://doi.org/10.1016/S0370-1573(01)00030-8
77. N.J. Popławski, Phys. Lett. B **694**, 181 (2010). https://doi.org/10.1016/j.physletb.2010.09.056
78. R. Kuhfuss, J. Nitsch, Gen. Relativ. Gravit. **18**, 1207 (1986). https://doi.org/10.1007/BF00763434
79. D. Palle, Nuovo Cim. B **114**, 853 (1999). https://doi.org/10.1007/BF02907478
80. D. Puetzfeld, Y.N. Obukhov, Phys. Rev. D **78**, 121501 (2008). https://doi.org/10.1103/PhysRevD.78.121501
81. R.T. Hammond, Rep. Prog. Phys. **65**, 599 (2002). https://doi.org/10.1088/0034-4885/65/5/201
82. I.G. Moss, C. Xiong, arXiv:hep-ph/0603266
83. A. Berera, I.G. Moss, R.O. Ramos, Rep. Prog. Phys. **72**(2), 026901 (2009)
84. M. Bastero-Gil, A. Berera, R.O. Ramos, JCAP **2011**(07), 030 (2011)
85. L. Kofman, A. Linde, A.A. Starobinsky, Phys. Rev. Lett. **73**(24), 3195 (1994)
86. B.A. Bassett, S. Tsujikawa, and D. Wands, Rev. Mod. Phys. **78**(2), 537–589 (2006). https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.78.537
87. M. Bastero-Gil, A. Berera, R.O. Ramos, J.G. Rosa, J. High Energy Phys. **2018**(2), 1–22 (2018)
88. S. Bartrum, M. Bastero-Gil, A. Berera, R. Cerezo, R.O. Ramos, J.G. Rosa, Phys. Lett. B **732**, 116–121 (2014)
89. V.F. Mukhanov, Zh. Eksp. Teor. Fiz **94**(1) (1988)
90. R. Herrera, M. Olivares, N. Videla, Phys. Rev. D **82**, 063513 (2010). arXiv:1007.0776 [gr-qc]
91. M.R. Setare, V. Kamali, Phys. Rev. D **88**, 083524 (2013). arXiv:1310.5096 [hep-th]
92. N. Aghanim et al. (Planck Collaboration), Astron. Astrophys. **641**, A6 (2020). arXiv:1807.06209 [astro-ph.CO]
93. E.W. Kolb, M.S. Turner, *The Early Universe* (Addison-Wesley, Boston, 1990)
94. J.Q. Xia, X. Zhang, Phys. Lett. B **656**(1–3), 19–25 (2008)
95. S.D. Odintsov, V.K. Oikonomou, Phys. Lett. B **797**, 134874 (2019). arXiv:1908.07555 [gr-qc]
96. N.J. Poplawski, Phys. Lett. B **694**, 181–185 (2010)
97. I.G. Moss, C. Xiong, JCAP **023**, 0811 (2008)
98. C. Gao, F. Wu, X. Chen, Y.G. Shen, Phys. Rev. D **79**, 043511 (2009). arXiv:0712.1394 [astro-ph]
99. L.N. Granda, A. Oliveros, Phys. Lett. B **669**, 275–277 (2008). arXiv:0810.3149 [gr-qc]