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Physics in Six Universal Dimensions — Effective Models with Spherical Orbifolds

Abstract : *During my internship at the Institut de Physique Nucléaire de Lyon, I studied the possibility of constructing Universal Extra Dimension models using spherical orbifolds as extra space. The main interest of such models is the possibility to have a dark matter candidate among the Kaluza-Klein excitations of the Standard Model fields if the extra-dimensions have a residual symmetry after the orbifolding. I found the symmetries of all seven infinite families of orbifolds using a computer program I wrote and found that none of the spaces with a residual symmetry can include massless chiral fermions naturally. To solve this issue, I investigated the popular idea of having a gauge field with a monopole background on the sphere counter the effect of the spin connexion and allow phenomenologically interesting fermions to exist. This gauge field cannot be one of the Standard Model's and has to be hidden by a spontaneous breaking and I studied numerically the Higgs mechanism in the monopole background to show that there exist potentials with reasonable parameters that allow a breaking where the extra gauge field complies with the experimental constraints.*

Keywords : *Universal extra-dimensions, sphere, orbifolds, LHC*

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Introduction

The Standard Model of particle physics has very well known and advertised shortcomings. Some issues are theoretical, like the hierarchy problem, or the question of the metastability of the Higgs vacuum, while some others are experimental, like the absence of a convincing description for neutrino masses and of a dark matter candidate, but they all point to the idea that our understanding of particle physics is yet incomplete and that new phenomena should be expected to show up in experiments.

A large number of models has been proposed as solutions to part or all of these problems, like those based on supersymmetry or extra-dimensions. Many such models predict the existence of new massive particles, one of which is sometimes used to provide a dark matter candidate by making it stable with a new symmetry (the so-called R-parity or KK-parity). However, this symmetry is most often ad-hoc and little motivated, which spoils the niceness of the underlying new principle.

This is why some recent constructions seek to describe dark matter using an actual built-in symmetry [3, 9, 16]. The underlying idea is that of Universal Extra-Dimensions: all fields are allowed to propagate in all the dimensions and will manifest themselves as Kaluza-Klein towers in four dimensions. If a symmetry of the extra-space provides the conservation of some kinematic quantity in the extra-dimensions, some decays between modes will be forbidden, and a heavy stable excitation could exist and be a dark matter candidate

It would of course be a pain to try and solve the equations of motions for field propagation on arbitrary spaces, so the most fashionable way of having nice extra dimensions is to use orbifolds of simple spaces. Orbifolds are spaces that look locally like a symmetric space of constant curvature but do not have the same global properties. Hence wave-equations will be the same and due to the global conditions, their solutions will be a subset of those in the original space.

In 5D, the nature of the space is constrained by the necessity of having massless chiral fermions and it is not compatible with having dark matter. Things are less constraining in 6D and orbifolds have been studied systematically in the flat case [17] where only one geometry is acceptable and has been thoroughly investigated[3, 4]. For the spherical case, some models have been built [9, 16], and use a combination of orbifolding and background gauge fields to obtain a realistic spectrum, but no complete study has been performed.

The original aim of my internship at the Institut de Physique Nucléaire de Lyon was to try and look systematically at how a Universal-Extra-Dimensional model could be built on spherical orbifolds and in which cases such a model could be convicing. My work is organized as follows: after introducing the necessary notions for working on spherical orbifolds, I explain the method I used to obtain their properties in a systematic way, which allowed to determine those that have a residual symmetry compatible with a dark matter candidate. I then expose how orbifolding could be expected to modify the fermion spectrum and proceed to show that in fact none of the orbifolds favorable for dark matter allow the existence of massless chiral fermions. This led me to try to exploit the gauge background idea exposed in [9, 16], with the aim of hiding the new gauge bosons in a more convincing way that those exposed in the literature. I devote some time to useless attempts made at constructing models based on a paper with incorrect results and finish this section with the only working idea of which we could) think, *i.e.* breaking the gauge with a Higgs. This breaking in a non-zero gauge background is treated numerically and is shown to provide a viable model with respect to existing experimental limits.

Preliminary Section: 6D Physics and Spherical Orbifolds

1.1 Physics in $\mathbb{M}_4 \times S^2$

1.1.1 Geometry

Orbifolding affects global topological properties and discrete objects at the boundaries. However, local properties, including differential geometry, are conserved and it is hence useful to study how physics is modified by adding a sphere to the usual Minkowski space. But first, let us recall the basic properties on the sphere (color conventions are defined in the appendix):

- a sphere is a compact manifold with a definite length scale, which we shall take to be R, its radius,
- the usual parametrization is done in terms of the colatitude θ and the longitude ϕ,
- the longitude is ill-defined at the poles and it is impossible to have well defined coordinates over the whole sphere. For our purpose, it will be sufficient to use gauge transformations on two different patches to keep objects from having definition problems at one pole when necessary (*i.e.* most of the time we do not care about that and keep the naive parametrization and gauge).
- there are three geometric objects of interest to us:

$$\begin{array}{ccc} \text{the metric } g_{\alpha\beta} & \text{the zweibein } e^{\alpha}_{a} & \text{the spin connexion form } \Omega \\ \begin{pmatrix} -R^2 & 0 \\ 0 & -R^2\sin^2\theta \end{pmatrix} & \begin{pmatrix} R & 0 \\ 0 & R\sin\theta \end{pmatrix} & \frac{i}{2}\cos\theta \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix} \mathrm{d}\phi \end{array}$$

The next three subsection will consist in expliciting the spectra of the sphere coordinate-dependent part of the kinetic operator for $6D$ fields on $\mathbb{M}_4 \times S^2$. I will keep things sketchy for scalars and gauge bosons, which are well treated in the literature, while I will put more detail for fermions, as I did this calculation myself along the lines suggested in [9], which provide a more natural framework than previous, *brute force* calculations [1].

1.1.2 Scalars

Scalars fields have a very simple Kaluza-Klein expansion, given that the corresponding kinetic operator is the square angular momentum:

$$\overline{\phi}\partial_{\alpha}\partial^{\alpha}\phi = \overline{\phi}L^2\phi \tag{1.1}$$

Whence it is clear that Kaluza-Klein eigenmodes are spherical harmonics and that a mode (l, m) will have an extra squared mass $\frac{l(l+1)}{R^2}$ from the kinetic term.

1.1.3 Gauge Fields

In six dimensions, gauge fields have six components. The reduction is thus less straightforward than for scalars because the two extra-dimensional components cannot be embedded into the gauge vectors in the effective theory. As in the flat case [3], these will in fact behave like scalars with no zero mode, half of which will be Goldstone bosons and be eaten away, as a calculation in general ξ-gauge reveals [16]. The physical spectrum is:

4D vector A_μ	**scalar** ϕ_A
$\forall l \geq 0$	$\forall l \geq 1$
$2l+1$ modes	$2l+1$ modes
$M^2 = \frac{l(l+1)}{R^2}$	$M^2 = \frac{l(l+1)}{R^2}$

1.1.4 Fermions

We have to deal with two unusual features of fermions in our case:

- in six dimensions, Dirac fermions have 8-components and Γ-matrices are 2×2 blocks containing the usual γ-matrices. With the explicit representations of [16], a 6D spinor is written as a sum of two chiral spinors:

$$\Psi = \Psi_+ + \Psi_- = \begin{pmatrix} \chi_+ \\ 0 \\ 0 \\ \overline{\eta}_+ \end{pmatrix} + \begin{pmatrix} 0 \\ \overline{\eta}_- \\ \chi_- \\ 0 \end{pmatrix} \tag{1.2}$$

  Where the 2-component elements are Weyl spinors. This will be a source of difficulty given that a 6D-chiral theory will have what will appear as both left and right chiralities in the 4D effective theory.

- spinors are defined in local Minkowski space so that to accommodate for curvature, one has to use both a spin connexion to have a covariant derivative and the sechsbein to couple derivatives on the manifold to Γ-matrices on the tangent space. In compact form, the Dirac equation is written:

$$e^M_A \Gamma^A D_M \Psi = 0 \tag{1.3}$$

The S^2 of the kinetic term mixes the two Weyl components in each chirality, which gives the following form for the Dirac equation:

$$\partial_0 \overline{\eta}_- + \vec{\sigma}\cdot\vec{\partial}\overline{\eta}_- - \frac{\overline{\eth}_{\frac{1}{2}}}{R}\chi_- = 0 \quad \partial_0 \overline{\chi}_- - \vec{\sigma}\cdot\vec{\partial}\overline{\chi}_+ + \frac{\eth_{\frac{1}{2}}}{R}\overline{\eta}_+ = 0 \tag{1.4}$$

$$\partial_0 \overline{\eta}_+ + \vec{\sigma}\cdot\vec{\partial}\overline{\eta}_+ - \frac{\eth_{-\frac{1}{2}}}{R}\chi_+ = 0 \quad \partial_0 \overline{\chi}_+ - \vec{\sigma}\cdot\vec{\partial}\overline{\chi}_+ + \frac{\overline{\eth}_{\frac{1}{2}}}{R}\overline{\eta}_+ = 0 \tag{1.5}$$

Where $\eth_s = -(\partial_\theta + i\csc\theta\partial_\phi - s\cot\theta)$ is the spin-raising operator for spin-weighted spherical harmonics ${}_sY^l_m$ (see [6] for a concise and self-contained account for physicists, and [19] for a more mathematical description). Its complex-conjugate is of course the spin-lowering operator so that it seems totally natural to use these functions to expand the Weyl components in the following fashion:

$$\chi_+(x^M) = \sum_{l=1/2}^{\infty} \sum_{m=-l}^{l} \chi_+(x^\mu)\ {}_{-\frac{1}{2}}Y^l_m(\theta,\phi) \qquad \overline{\eta}_+(x^M) = \sum_{l=1/2}^{\infty} \sum_{m=-l}^{l} \overline{\eta}_+(x^\mu)\ {}_{\frac{1}{2}}Y^l_m(\theta,\phi) \tag{1.6}$$

$$\chi_-(x^M) = \sum_{l=1/2}^{\infty} \sum_{m=-l}^{l} \chi_-(x^\mu)\ {}_{\frac{1}{2}}Y^l_m(\theta,\phi) \qquad \overline{\eta}_- = \sum_{l=1/2}^{\infty} \sum_{m=-l}^{l} \overline{\eta}_-(x^\mu)\ {}_{-\frac{1}{2}}Y^l_m(\theta,\phi) \tag{1.7}$$

Using $\eth_s Y^l_m = \sqrt{l(l+1) - s(s+1)}\ {}_{s+1}Y^l_m$ and $\overline{\eth}_s Y^l_m = -\sqrt{l(l+1) - s(s-1)}\ {}_{s-1}Y^l_m$, we get the following mass terms for the effective theory:

$$\sqrt{l(l+1) + 1/4}\,(\eta_-\chi_- + \overline{\chi}_-\overline{\eta}_-)\,, \quad \sqrt{l(l+1) + 1/4}\,(\eta_+\chi_+ + \overline{\chi}_+\overline{\eta}_+) \tag{1.8}$$

Which shows that even starting from a chiral theory in 6D, not only do we get both left and right Weyl fermions, but they also get a non zero mass. This is why having massless chiral fermions is such a challenge in this setup.

1.2 Orbifolds

1.2.1 General Principle

An orbifold is the coset space of a manifold with constant curvature by the equivalence relation given by one of its discrete symmetry groups. It is thus the space of the orbits under the group (whence the name) and a simple representation can be given by considering a tile of the paving provided by the group and specifying the "boundary conditions", since boundaries of the tile can either be actual boundaries of the orbifold or just be connected to another point.

This second case is often illustrated in the most famous examples of orbifolds, which include the flat torus and the Möbius strip. For example, a torus is obtained by taking the quotient of the real plane by two linearly independent translations, which leaves a square with each boundary glued to its opposite, as shown in Fig.1.1

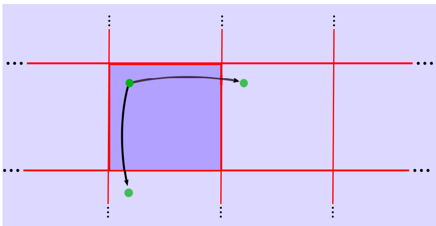


Figure 1.1: Quotienting the real plane into a torus: all green points are identified. Note that the resulting space cannot easily be embedded isometrically in 3D space (though it is possible [2])

If many well-known orbifolds actually are manifolds, the mathematical interest for orbifolds arose historically from the study of singular points, which appear naturally in this framework. For instance, a cone can be constructed out of the plane, by using the group of n-folds rotations around a point, which amounts to cutting out $\frac{n-1}{n}^{\text{th}}$ of the plane and glueing the remaining edges as in Fig.1.2.

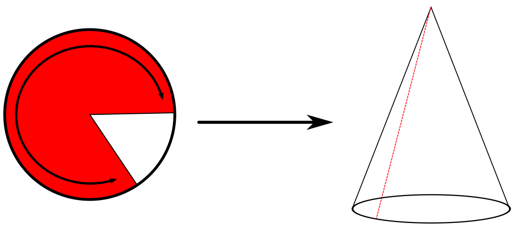


Figure 1.2: Making a cone by glueing edges. The resulting space is no longer a manifold.

It is the points that are left unchanged by one of the group transformation that will be singular, but in the vicinity of the other points, the local geometric properties are conserved, whence we will have the metric inherited from the original space and all the differential geometry will be unchanged.

Differential geometry seems to have little to say about the singular points, where there is no tangent space, but we can actually determine the curvature at these points: consistency will require that it be a delta function whose pre-factor can be calculated. An example of this is provided by *rugbyball* spaces [5] which are the analog of the conical case on the sphere (identify by n-fold rotations around a given axis). Carroll *et al.* argue that this space is still topologically a sphere and so should have Euler characteristic 2. This allows to compute the extra curvature that has to be added on the conical points at the poles.

Singularities are generally the kind of features that physicists tend to try and avoid, as they are usually the mark of a theory taken out of its validity domain and hence not a good starting point for a model. This is however not true in string theory, which is well-defined on orbifolds even with conical singularities. Given that many extra-dimensional models appeal to string theory as the most reasonable ultra-violet completion of their effective field theory, this is good enough a reason to try and exploit the richness of the orbifold framework, without which having reasonable features such as chiral massless fermions seems to be an impossible enterprise.

Since we keep most of the original differential geometry, it is easy to understand that a solution of the equations of motion for any field on an orbifold will be a solution on the original space by mapping the field in a consistent way on the tiling. Seeing things the other way around, the solutions of the equations of motion on the orbifold are a subset of those on the manifold, characterized by a symmetry property. A nice and simple way of doing this is to ask that some representation of the orbifolding group be a symmetry of the set of fields in the theory. This kind of procedure is often used to change the spectrum of the theory or break gauge symmetries [12], and we will try to apply it to get chiral fermions in our model and to embed our theory into a unified group broken at very high scale by the geometry.

1.2.2 Orbifolds on S^2

The complete set of orbifolds on the sphere is well-known by mathematicians [8]. They come in 8 families:

- cyclic groups C_n of n-fold rotational symmetry around Oz,
  - their pyramidal variants C_{nv} which include n additional vertical mirror planes (except for $n = 2$ where the two planes are identical),
  - other "horizontal" variants C_{nh} which include one horizontal mirror plane,
- *spiegel* groups S_{2n} of n-fold improper rotation symmetries around Oz. Note that $S_{2n+1} = C_{(2n+1)v}$,
- dihedral groups D_n, which include $\frac{2\pi}{n}$ rotations around Oz and n π rotation axes in the Oyx plane.
  - prismatic variants D_{nh} which have an additional Oxy mirror symmetry,
  - anti-prismatic variants D_{nd} which have n additional mirror planes containing Oz and the line bisecting a pair of adjacent rotation axes,
- polyhedral groups, based on the symmetries of regular polyhedra.

For reasons of simplicity, we will focus our work on the seven infinite families. The next section describes how we determined their symmetry properties using a computer program.

Properties of Spherical Orbifolds

2.1 General Principle

The original goal of my internship was to determine which geometries could be promising for the construction of a UED model to make predictions for LHC physics and dark matter. The first step in this process was to classify the S^2 orbifolds according to their properties:

- residual symmetries, which are important to have a Kaluza-Klein excitation play the role of dark matter
- boundaries and singularities, where interactions can be localized and could break the residual symmetries
- topological properties such as orientability and simple-connectedness.

Determining in a systematic and complete fashion all the residual symmetries seemed rather long and error-prone, so we decided that an automation would be a good way to do it, all the more as having an algorithm for this process would be a lot more interesting for hyperbolic spaces – which could be another interesting field to explore – where geometrical intuition is often less developed.

The classification algorithm relies on the observation that symmetries of orbifolds are subgroups of $O(3)$ and that their action on the whole sphere has to be compatible with the transformations induced by the orbifolding group. This means that a symmetry S of the orbifold has to respect equivalence relation defined by the quotient:

$$x \sim y \Rightarrow Sx \sim Sy \tag{2.1}$$

This means that for any g in the orbifold group, there is a g' such that $SgS^{-1} = g'$, which is the characterization property that we used in our algorithm.

2.2 Algorithm

The automation program was written in `Mathematica` [24] to keep equation solving exact. An implementation of the 7 infinite families of discrete subgroups of $O(3)$ is provided at [25], which I used. For a given orbifolding group G, the algorithm goes along the following steps:

- Define a generic $O(3)$ matrix $A = \epsilon R(\theta, \phi, \psi)$ $(\epsilon = \pm 1)$
- For all g, $g' \in G$, find the conditions in (θ, ϕ, ψ) to solve $SgS^{-1} = g'$
- For all permutations σ of the elements of G, check wether the intersection of all the corresponding conditions is compatible. This step has some technical subtleties because of the redundant description of rotations in terms of angles which leads to errors when the residual symmetry is continuous: the transformation matrix still has free parameters that can be redefined by some offset which has to be accounted for.

This algorithm is rather naive and not very efficient. Some technical tricks had to be used to deal with excessive memory usage which still are a problem when dealing with groups of order > 6. This however should not be a problem because it is sufficient to deal with generators, which are generally in low number for useful groups. Its main advantage on the other hand is that it is absolutely independent of the S^2 framework and could be adapted to others, among which the hyperbolic plane H^2 is probably the most attractive example.

2.3 The Limits of Laziness

One very important feature when building a model on an orbifold is understanding the structure of its singular points because these points will be needed to harbor localized counter-terms which will regularize some of the theory's divergences. This is not viewed as a flaw as it would be expected that the theory has branes at these points [5, 11] where some interactions could indeed be localized [23]. However, if a singular object is not left invariant by the residual symmetry – *i.e.* it is not transformed within its orbit but into the orbit of another point – the symmetry that protects the decays of the dark matter candidate might be spoiled since there is no reason why different singular points should have the same couter-terms.

It is easy for `Mathematica` to find the points that are left invariant by some of the orbifold group elements: just take a point of S^2 parametrized by unspecified coordinates (x, y, z) and solve $gX = X$. What needs to be done then is to find wether the residual symmetries transform the singular points within their orbit if they are isolated points or within the orbit of the singular object they belong to if they are in a boundary. This latter case is rather non-trivial because of technical reasons such as the separation of singular objects into parts (*e.g.* a singular circle is often separated into the two curves $y = \pm\sqrt{1-x^2}$), so its implementation was delayed to until it is felt necessary (maybe for the hyperbolic case). It is then up to the user to see how the singular points or objects found by the program behave under the residual symmetries.

Topological properties of the orbifold were also determined by hand. Orientability simply amounts to identifying wether the orbifold group contains a parity transformation so it is anyway not difficult. Multiple connexity was studied diagrammatically as depicted in Fig.A.1 and Fig.A.2 of Appendix B.

2.4 Results

Group	Symmetry	Group	Symmetry
C_n	$O(2)_z$	D_{nd}	None
C_{nh}	$O(2)_z$	D_{nv}	None
C_{nv}	None	S_2	$O(3)$
D_n	$M(Oxz),\ M(Oxy)$	$S_{2n},\ n > 1$	$\langle O(2)_z, M(Oxy)\rangle$

Table 2.1: Residual symmetries

Group	Point	Order	Group	Point	Order
C_n	North Pole	n	C_{nh}	North Pole	n
	South Pole	n		Equator Segment	2
C_{nv}	North Pole	2n	D_{nd}	North Pole	n
	South Pole	2n		Equator Midpoint	2
	Eastern Meridian	2		Western Equator End	2
	Western Meridian	2		Eastern Equator End	2
D_n	North Pole	n	D_{nv}	North Pole	n
	South Pole	n		Eastern Edge	2
	Western Equator End	2		Southern Edge	2
	Eastern Equator End	2		Western Edge	2
S_2	None	N/A	S_{2n}	North Pole	n

Table 2.2: Singular objects

Massless Fermions – Round One: Orbifolding

3.1 Minimal Example

To avoid dealing with the ills of curved spaces, I will illustrate how orbifolding allows to have a chiral effective theory in 5D with an extra circle, where there is no Weyl representation of spinors.

In 5D, fermions are represented as in 4D and the γ-matrix algebra is complemented by γ^5. The gross idea is that this leaves no room for defining a chiral projector and thus there is no such thing as a 5D chiral theory: a fermion Ψ always has a non-zero left and right part.

However, there are successful examples of five dimensional models that reproduce the features of the Standard Model in 4D such as Randall-Sundrum models. These models use the orbifolding of S^1 into a segment. The associated transformation ($\phi \leftrightarrow -\phi$) is just a $\mathcal{P}$ in the fifth dimension and so will act on a fermion Ψ as $\Psi \to \pm\gamma^5\Psi$. By imposing the intrinsic parity $\pm$ of the field, we get rid of one chirality of the zero mode of the Kaluza-Klein expansion because both are constant.

3.2 Transforming Fermions in Curved Space

Let us start again from the general Dirac equation:

$$e^M_A \Gamma^A D_M \Psi = 0 \tag{3.1}$$

This equation is covariant under any diffeomorphism, and the geometric quantities absorb all the dependence in coordinate changes so that in principle the spinor is a scalar under manifold coordinate transformations. However, one is often led to make a change of coordinates in the local Minkowski space to compensate for changes in the manifold coordinates so that one can compare spinors before and after the transformation locally in the same basis.

The zweibein on the sphere is expressed as:

$$e^\alpha_a = \begin{pmatrix} R & 0 \\ 0 & R\sin\theta \end{pmatrix} \tag{3.2}$$

Where the colors correspond to the two vectors $\boxed{e^\alpha_\theta}$ and $\boxed{e^\alpha_\phi}$ which are the object that are transformed like vectors under a manifold coordinate change.

Hence, under a reflection $R_{\mathcal{M}}$ with respect to the equatorial plane,

$$R_{\mathcal{M}} \cdot e_a^\alpha (R_{\mathcal{M}} \cdot (\theta,\ \phi)) = \begin{pmatrix} -R & 0 \\ 0 & R\sin\theta \end{pmatrix} \neq e_a^\alpha (R_{\mathcal{M}} \cdot (\theta,\ \phi)) \tag{3.3}$$

Which means that Ψ after and before the rotation is not expressed in the same basis in the point $R_{\mathcal{M}} \cdot (\theta,\ \phi)$. We have to change coordinates in the tangent space to map the zweibein back to its original value if we want to compare Ψ before and after the rotation locally. For this we will do a reflection $R_{\mathbb{T}}$ in the tangent space, hence now treating $\boldsymbol{e_A^\theta}$ and $\boldsymbol{e_A^\phi}$ as vectors.

$$\begin{aligned} R_{\mathbb{T}} \cdot R_{\mathcal{M}} \cdot e_a^\alpha (R_{\mathcal{M}} \cdot (\theta,\ \phi)) &= \begin{pmatrix} \mathrm{R} & 0 \\ 0 & R\sin\theta \end{pmatrix} \\ &= e_a^\alpha (R_{\mathcal{M}} \cdot (\theta,\ \phi)) \end{aligned} \tag{3.4}$$

The transformation $R_{\mathcal{M}}$ does not have any effect on Ψ because it lives on the tangent space. However, $R_{\mathbb{T}}$ acts on the tangent space coordinates and hence transforms the spinor.

There are two types of spinors transformations that are of interest:

- $\mathcal{P}$ transformations around a given tangent space unit vector x^A [22]:

$$\boxed{\Psi \to \not{x}\Psi} \tag{3.5}$$

- Rotations that mix two coordinates A and B:

$$\boxed{\Psi \to e^{i\alpha\Sigma^{AB}}\Psi} \tag{3.6}$$

In our case, we want to change the orientation of the θ coordinate in the tangent space. The relevant operator is : $\Gamma^\theta\Gamma^7$ since Γ^7 changes the sign of all coordinates five times and Γ^θ changes a sixth time all coordinates except θ.

We can now study wether orbifolding allows us to have massless chiral fermions.

3.3 Application to C_n, C_{nh}, S_n and D_n

Fermions are expanded in terms of their 4 Weyl components. In the absence of background gauge field, their S^2 eigenfunctions are spin-$\frac{\pm 1}{2}$ spherical harmonics, which means that their lowest mode is twice degenerate $(m = \pm\frac{1}{2})$. The S^2 curvature couples modes with same $6D$ chirality but opposite Weyl chirality and manifests itself as a coupling between modes with same j, m numbers in the KK-expansion.

3.3.1 Cyclic group: S^2/C_n

These groups include only parity-conserving transformations, which means that $6D$-chiral fermions are allowed in this theory. We will hence start with a Ψ_+ field and see how orbifolding will affect the KK-decomposition.

We first need to see the action of the generator of the group over the zweibein, which is easy: the transformation is trivial. Hence, orbifolding can only change the field by some phase whose n$^{\text{th}}$ power is -1, since we are dealing with spinors.

It seems pretty straightforward to try and set this phase to $e^{i\frac{\pi}{n}}$. If we now remember that a m-mode spherical harmonic of any spin transforms as $e^{im\alpha}$ under an α rotation about the Oz axis, we see that for both $4D$ chiralities, this eliminates the $(j = 1/2, m = -1/2)$ mode. A Oz rotation is useful to get rid of the degeneracy, but since it acts in the same way for fields coupled by a mass term, this will not allow any kind of $4D$ chirality on its own.

3.3.2 "Horizontal" cyclic groups: S^2/C_{nh}

These groups are just C_n with an extra reflection about the equator plane. This makes a significant change since a reflection is a parity transformation: the orbifold is non-orientable and we have to add a mirror $\Psi_{\text{-}}$ to the theory. In reality this does not change the number of degrees of freedom since the orbifolding through the reflection will identify each $+$ component with a $-$ component. The C_n subgroup can be used in the exact same way as before to kill the degeneracy.

Hence as before, we are left with massive vector-like fermions. The extra generator came along with a doubling of the degrees of freedom so that its effect was balanced and did not bring an extra condition to have phenomenologically interesting fermions.

3.3.3 Dihedral groups: S^2/D_n

Dihedral groups have two generators: a Oz n-fold rotation and a Ox 2-fold rotation. The Oz rotation still generates a C_n subgroups and allows us to eliminate the m degeneracy, which is a first step. Let us now turn to the Ox rotation.

To map the zweibein back to its configuration after the Ox rotation, we need to invert the signs of θ and ϕ on the tangent space, which is going to be problematic: it changes the sign of the m number. This is incompatible with eliminating one of the modes with rotations unless the "parity" associated with the transformations is represented by a complex conjugation and some matrix, but a theory that is not $\mathcal{C}$-invariant, like the Standard Model, will not comply with this identification.

3.3.4 Spiegel Groups: S^2/S_n

These group have only one generator, which changes parity. Hence, it will be impossible to kill the multiple extra degrees of freedom of the lowest mode using this group.

All the groups that leave a residual symmetry to the orbifold fail to generate a theory with massless chiral fermions. The usual procedure used in 5D and 6D flat cases is not applicable here because the spin connexion is adding an extra difficulty. Hence, a second approach which aims directly at suppressing the effect of the spin connexion has become rather popular in UED models and the next section summarizes our attempts at using this idea.

Massless Fermions – Round Two: Disconnecting Gauge Field

4.1 Disconnecting the Spin-Connexion

4.1.1 A Gauge Background to Hide Curvature

The whole of the problem with our fermion spectrum comes from the spin-connexion in the covariant derivative. Many models of UEDs making use of the sphere rely on the observation that the relative signs are different between the Weyl components for both chiralities:

$$\Omega|_{\Psi_\pm} \propto \pm \cos\theta \gamma^5 \mathrm{d}\phi \tag{4.1}$$

Which means that having a background gauge field of the form: $A_0 \cos\theta \mathrm{d}\phi$ would solve our problems provided fermions have the right charges. Let us consider a 6D-chiral field Ψ_+ with charge q under this gauge. This changes the kinetic term in the following way:

$$\partial_0 \overline{\eta}_+ + \vec{\sigma}\cdot\vec{\partial}\overline{\eta}_+ - \frac{\eth_{qA_0\text{-}\frac{1}{2}}}{R}\chi_+ = 0 \quad \partial_0 \overline{\chi}_+ - \vec{\sigma}\cdot\vec{\partial}\overline{\chi}_+ + \frac{\overline{\eth}_{qA_0+\frac{1}{2}}}{R}\overline{\eta}_+ = 0 \tag{4.2}$$

We are thus led to setting $qA_0 = 1/2$ and to perform this expansion:

$$\chi_+(x^M) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \chi_+(x^\mu) Y^l_m(\theta,\phi) \qquad \overline{\eta}_+(x^M) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \overline{\eta}_+(x^\mu)_1 Y^l_m(\theta,\phi) \tag{4.3}$$

Which amounts to having $\overline{\eta}^0_{0+}$ have 0 mass while χ_+ only has massive modes: we have one massless chiral fermion and a tower of massive vector-like fermions !

4.1.2 Randjbar Daemi-Salam-Strathdee Spontaneous Compactification

The first question that comes to mind when looking at such a proposal is “why would such a background exist in the first place ?”. Randjbar Daemi, Salam and Strathdee [20] provide a partially convincing answer: to be stable, our 6D space needs it. More precisely, we need the field strength form to be:

$$F = -\frac{n}{2gR^2} R\mathrm{d}\theta \wedge R\sin\theta \mathrm{d}\phi \tag{4.4}$$

$$\Rightarrow A = \frac{n}{2g}\cos\theta \mathrm{d}\phi + \mathrm{d}f \tag{4.5}$$

The reason why this is not so convincing is that there is only one value of the radius such that the 4D space is flat:

$$R^2 = \frac{n^2\kappa^2}{8g^2}, \tag{4.6}$$

(with κ the 6D gravitational coupling constant) and that this seemingly has to be fine tuned, as other values work but give curved 4D spaces. We will however not focus much on the gravity part of our model as we will consider orbifolds with singular points, but we can note that localized interactions in these points could pick out the right boundary conditions for the field to choose the nice case.

Note that on the other hand, this value of the background will make the coupling constant disappear in the fermion derivative and hence limit the amount of fine-tuning necessary for the cancellation to work: it depends only on the eigenvalue of the field under the relevant gauge generator. Asking that fermions are transformed consistently under this gauge forces n to be an integer.

Fixing the background field to this value also allows to get an estimate of the coupling constant from a hand-waving argument. The 6D κ should be linked to the 4D effective Newton constant by $\kappa^2/4\pi R^4 \simeq 8G\pi/R^2$ as gravitational forces at distances of the order of R should convert from the usual G/r^2 law to a $1/r^4$ law. Taking this as an actual equality, we get the effective 4D coupling constant of the new gauge:

$$g_{4D} = \frac{g}{\sqrt{4\pi}R} = \frac{n\sqrt{G}}{2\sqrt{4\pi}R} \simeq 2 \times 10^{-8} \tag{4.7}$$

4.2 How to Hide The Extra Fields ?

4.2.1 (Unconvincing) Existing Proposals

Let us push forward with our model without minding the fine-tuning of our parameters. What happens to the excitations of our background field ? If it is abelian, its background value does not change the quadratic part of the Lagrangian and the expansion should be exactly the same as described in the first section. Here we hit a difficulty: there is a zero mode. This field cannot be the electromagnetic field because the charge assignment would be inconsistent, so we have an extra long distance force, which, even with its low coupling, is incompatible with sub-millimeter gravity experiments [10].

The existing literature proposes ways to hide this field:

- with only the Standard Model fermions, the theory is anomalous and thus is broken anyway [16]. This is rather annoying because it destroys much of the predictivity of the model due to the disparition of the Ward identities which limit the number of effective operators the theory could include.
- a model on the real projective plane tries giving an unsual symmetry to their theory, with a modified 6D-CP transformation between the SM fermions and mirror fermions of oppositde SM charges but identical extra gauge charges [9]. This works at the classical level but since the symmetry is put there by hand, it has no reason not to be broken by quantum effects.

4.2.2 Self-Hiding Fields: A Nice (and Wrong) Idea

What then would happen if we gave this kind of background value to a non-abelian gauge field ? The claim of [7] is that the 4D vectors that have a charge under the direction which the vacuum chooses get a mass and that a similar process happens for scalars, except that the $(1, 1)$ mode gets a negative squared mass: it could be a Higgs and it has a charge under the background direction, which sounds nice.

The argument relies upon a calculation of the mass terms diagonal in the expansion. However the mass terms also introduce mixings, but the implicit claim of the authors is that this is negligible. I tested it numerically for the vector case, in which this is quite true: Fig.4.1 depicts the mixing between the (0,0) mode and the (2k,0) (the only ones to which it couples), and shows that it is small.

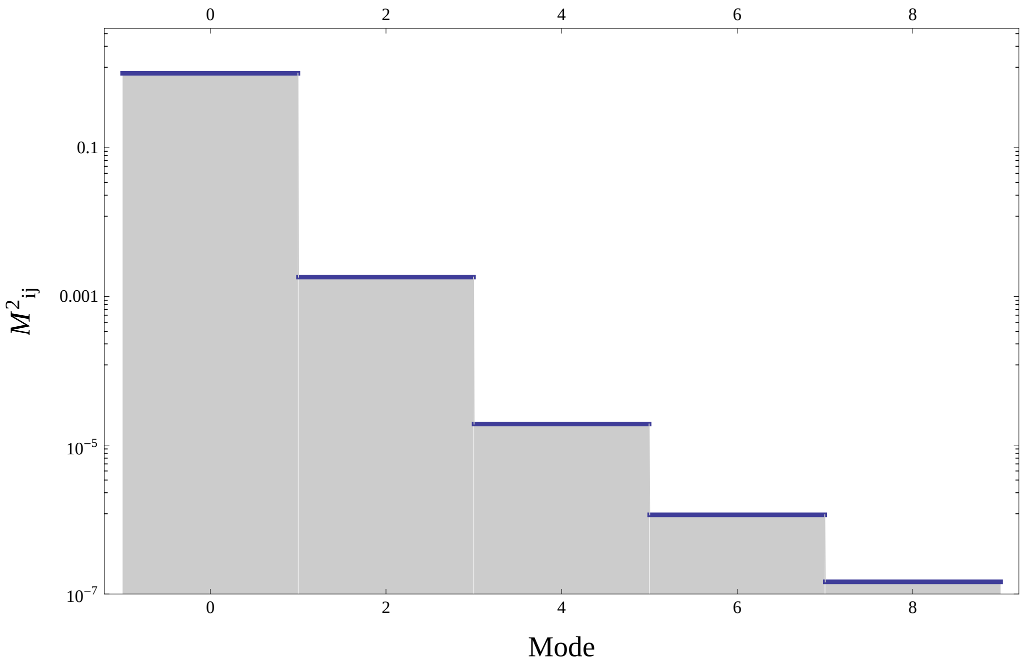


Figure 4.1: Square of the projection of the (0,0) mode onto the the first (2k,0) modes and the vector mass spectrum

The physical scalar spectrum is rather a mess to figure out. Chiang *et al.* do not make a distinction between physical scalars and Goldstone bosons and do not include gauge-fixing terms that could explicit this. My attempt at resolving this issue showed that the gauge fixing should be modified to account for derivative couplings between all the scalars and the gauge fields, which also includes a position-dependent, non-diagonal factor so that it was not clear how this picture would hold using the approximately diagonal Kaluza-Klein expansion in terms of spherical harmonics.

The issue was in fact solved in the litterature in a paper from 1977 [13] which simply finds an actual diagonalization of the full kinetic operators and shows that both gauge scalars and vectors charged under the background get a large positive mass ($\sim 1/R$) and that this works as well for extra scalar field, so that breaking the symmetry with a Higgs requires a large negative bulk mass squared.

We built attempts at using the idea of the Chiang *et al.* paper before realizing it was based on a wrong setup, so the next subsections detail what our ideas were and how they turned out when correctly worked out.

4.2.3 Had It Worked... Adding a $SU(2)$

The minimal way to have a non-abelian field do the expected work is to use a $SU(2)$ gauge symmetry. Given our expectations, we would want our standard model fermions to be in the "isospin-up" component of a doublet and set $n = 2$. This requires adding another "isospin-down" partner which is going to have only massive particles in its expansion. This is not a problem phenomenologically and this would work fine. Let us sketch how, from [7], we expected things to work this way:

- the background value A_B sets in one direction, say σ^3
- the vectors along $\sigma^{1,2}$ both get a mass from the term $A_i \epsilon^{i3k} A_B^2 \epsilon_{j3k} A^j$, leaving only a $U(1)$ symmetry along σ^3,
- scalars in directions $\sigma^{1,2}$ gets a negative mass. A quartic coupling could be obtained, from radiative corrections for example [14],

- the scalars get a vaccuum expectation value and break the remaining $U(1)$ is broken down.

As a result: $SU(2)$ with one background could break itself totally.

In fact, the spectrum arising from the 6D $SU(2)$ with a background on σ^3 is:

- For each $j \geq 2$, $2(2j+1)$ vectors with masses $\sqrt{j(j+1)-2}$
- For each $j \geq 0$, $(2j+1)$ vectors with masses $\sqrt{j(j+1)}$. This includes a 0-mode massles vector with $U(1)$ symmetry.
- For each $j \geq 2$, $2(2j+1)$ scalars with masses $\sqrt{j(j+1)-2}$
- For each $j \geq 1$, $(2j+1)$ scalars with masses $\sqrt{j(j+1)}$.

Which shows that $SU(2)$ in fact fails to hide itself. We could hope to hide this remaining $U(1)$ away using a Higgs boson but then we would gain nothing compared to having an abelian monopole with a Higgs boson, which we will describe in a few sections.

4.2.4 Had It Worked... $SO(10)$ GUT

Orbifold can be used as a way to break gauge symmetries, with some limitations. The rank of a group can only be reduced if the matter content is invariant under a change of sign of its coupling constant [12]. Since this will not be the case here we can only hope to use orbifolding to break Grand Unified Theories into the semi-simple group of our effective models and not actually hide the extra gauge away. This GUT framework will not provide a lot of extra predictions for collider physics, since the mass scales at which new degrees of freedom should manifest themselves is the scale at which the geometry stablilizes, which one can expect to be a reasonable fraction of the 6D Planck energy, $\sim 10^7$ TeV. However, if this model had been interesting for describing physics at the TeV scale, this could have been a nice feature giving more precision concerning a hypothetical UV-completion.

In the case of our $(SU(3)_c \times SU(2)_L \times U(1)_Y)_{\text{SM}} \times SU(2)_{\text{disconnect}}$, we should look for a rank 5 group among simple Lie group. The first to come to mind is $SU(6)$ but its breakings are not compatible with our expectations [21] so we discard it. Nevertheless, $SO(10)$ has the right breaking pattern and we can fit Standard Model fermions into some of its irreductible representations:

f	$G_{\text{SM}} \times SU(2)_{\text{disconnect}}$ Irrep	$\subset SO(10)$ Irrep
L	$(1, 2, -3, 2)$	144
e_R	$(1, 1, -6, 2)$	126
Q_L	$(3, 2, 1, 2)$	144
u_R	$(3, 1, 4, 2)$	210
d_R	$(3, 1, -2, 2)$	125

To illustrate how this breaking could happen, let us consider a C_n orbifolding and let us define $z = e^{2\pi/n}$. We take as representation of the gauge group the matrices that conserve the scalar product given by the matrix: $\tau_1 \otimes \tau_1 \otimes \tau_1 \otimes \mathbb{I}_4$, whare τ_1 is the first Pauli matrix and $\mathbb{I}_1$ is the 4×4 identity matrix. We ask that the action of the generator g of the orbifolding group acts on the $SO(10)$ gauge field in the following way:

$$g \cdot (A_\mu(x)) = M(z) A_\mu(gx) M^\dagger(z) \tag{4.8}$$

where $M(z) = \begin{pmatrix} \mathbb{I}_4 & 0 & 0 \\ 0 & z\mathbb{I}_2 & 0 \\ 0 & 0 & z^2\mathbb{I}_4 \end{pmatrix}$, which will preserve $SU(3) \times U(1) \times SO(4) \sim SU(3) \times U(1) \times SU(2) \times SU(2)$.

4.2.5 Had It Worked... Electroweak-Disconnexion Unification in $SU(3)$

Since giving a background to one component of a gauge field amounts to giving mass to all the gauge fields that have mass under this field, it could have been a nice idea to fit the electroweak group with this background into $SU(3)$, especially as we expected scalars charged under the background to get a vacuum expectation value. Hence we could have had the following pattern:

- The background has a shape to accomodate a breaking to $SU(2) \times U(1)$: $A_B \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
- If the process described in [7] was correct, the massless vectors would organize as:

$$\frac{W^i_\mu}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} + \frac{B_\mu}{2\sqrt{3}} \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -2 \end{pmatrix} \tag{4.9}$$

- while the scalars that get a negative mass zero mode are organized as follow:

$$\begin{pmatrix} 0 & 0 & \phi^1 \\ 0 & 0 & \phi^2 \\ \phi^{1\dagger} & \phi^{2\dagger} & 0 \end{pmatrix} \tag{4.10}$$

  Which fall precisely into the representation one would expect for a Standard Model Higgs. If it did get a negative mass, it would provide a nice model of electroweak symmetry breaking, even though the current LHC data would probably prove it irrelevant for a direct application to electroweak physics. Instead we get 8 towers of massive scalars, 2 towers of massive vectors and 4 towers of vectors with a massless zero mode, which by itself will not suffice for having a realistic low energy spectrum.

4.2.6 Abelian Higgs Mechanism in the Background

As we mentionned in the part describing our attempt at having $SU(2)$ break itself completely thanks to its background value, we can hope to break the extra gauge field by using a Higgs mecanism. To this effect we will no longer rely on non-abelian gauge interactions and we can keep the simple $U(1)$ version. We then add a bulk Higgs lagrangian:

$$\mathcal{L}_H = |D^M\Phi|^2 + \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 \tag{4.11}$$

And we will need to minimize the associated hamiltonian to find the vacuum expectation value. Since the background gauge field will result in an extra θ-dependent mass term, the vacuum expectation value will most likely be θ-dependent as well. Hence, we will have to take into account the derivative part of the hamiltonian, contrary to the Standard Model case. Since this problem implies minimizing a non-linear functional, it seemed hard to solve it analytically.

A first idea was to compare the derivative term to the potential (bulk Higgs + gauge background-induced) term in hope that the first would prove negligible and that minimizing the second one would suffice to have a nice enough approximation. If that approximation is true, the Higgs vacuum value in the north patch is expected to be:

$$v = \sqrt{\frac{|\mu^2 - \left(\frac{\cos\theta - 1}{2R^2 \sin\theta}\right)^2|}{\lambda}} \tag{4.12}$$

The vacuum value expressed here depends on the background gauge field in a gauge that gives a regular function in for $\theta \to 0$. We take this expression to be valid on the North hemisphere and we can switch to a South pole-compatible gauge by replacing $(\cos\theta - 1)$ by $(\cos\theta + 1)$, which is indeed a gauge transformation. Upon doing the operation $\theta \to \pi/2 - \theta$, we go from the North patch to the South patch so v is even and it suffices to compare the potential and kinetic terms in the North hemisphere.

To make the comparison in a somewhat representative scenario, I used sensible scales from standard Higgs theory: without the background gauge, the mass of the physical boson would be $\sqrt{2}\mu$, which we would expect to be around the scale of the extra dimension, so we set $(R\mu)^2 = 1.2$, and we would want the phase transition (given typically by $v_0 = \frac{\mu}{\sqrt{\lambda}}$) to happen a slightly higher energy scale, so $5/R$ for the example. In this framework, we get a point-by-point ratio as depicted in Fig.4.2 and a ratio of the integrated energies around 16%. This ratio is pretty big but falls very fast when μ grows, which can be seen in Fig.4.3 so that setting $\mu = 2/R$ is sufficient to have a satisfying approximation. Let us however work in our original settings and see how having a rather big kinetic energy changes the configuration.

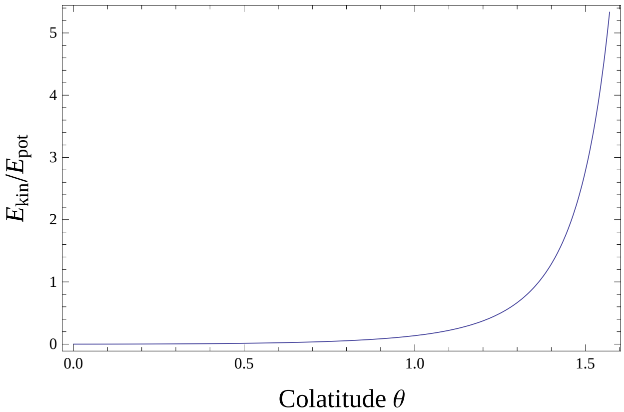


Figure 4.2: Point-by-point ratio of the kinetic energy to the potential energy in the minimum of the potential

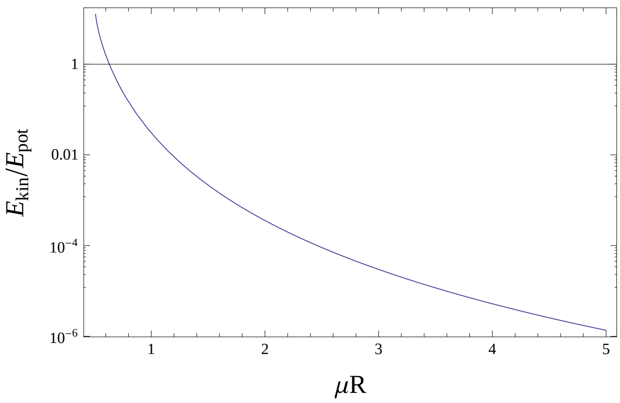


Figure 4.3: Ratio of the total kinetic energy to the total potential energy in the minimum of the potential

The first thing that could be done was to see wether minimizing the energy amounts to changing the profile of v much. To do so, I set up a very naive random optimization code in `C` starting from the no-kinetic-energy configuration and modifying a small portion of the points by a small amount if that resulted in a smaller energy at each iteration. This program is unstable, non-optimized and cannot be trusted to find the actual global minimum of the energy, all the more as its convergence is slow. It can show however that a configuration is far from minimal by making it evolve a lot. Fig.4.4 shows the evolution of the Higgs profile as a function of the colatitude for a growing number of iterations, which demonstrates that the original background is definitely not a good approximation of the vacuum state.

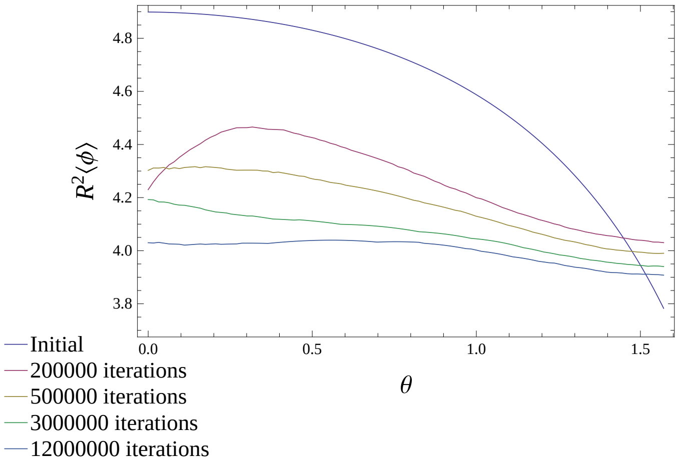


Figure 4.4: Optimization of the profile. This shows that the original profile is a poor approximation here.

To find a better approximation of the Higgs profile, I used a minimization with respect to its Fourier coefficients. The minimization has a clear minimum (see Fig.4.5) that is nearly constant, as depicted in Fig.4.6, which illustrates the important role of the derivative term in this minimization.

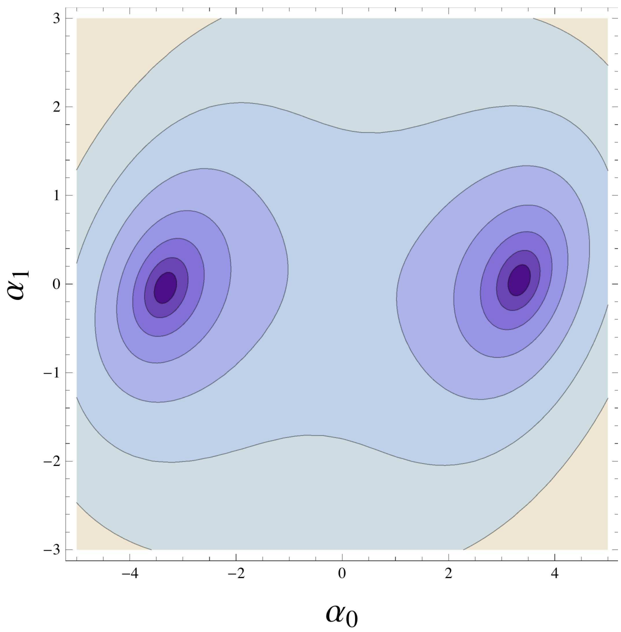


Figure 4.5: Energy landscape in the plane of the first two Fourier. A flat non-zero profile is clearly favoured

Now that we see that this expansion is appropriate to extract information about the Higgs vacuum value, we can try and apply these results to see what physical outcome can be expected. We can first compute the energy of the system at the minimum (Fig.4.7(a)) to get an idea of the phase transition energy and we can compute the effective 4D mass added to all components of the vector tower. If the Higgs profile is constant, which is well-verified for a wide variety of parameters, as shown in Fig.4.2.6, then the Higgs interacts only with the gauge vector 0-mode and gives it a mass of $g\sqrt{\int \mathrm{d}\Omega \frac{\langle\phi\rangle^2}{4\pi}}$ (the factor of 4π comes from the 0-mode normalization to 1). The mass of the vector boson is depicted in Fig.4.7(b) as a function of μ for several values of λ.

The resulting mass for the vector boson is pretty small since g is very small, but as far as micrometer-scale gravity measurement are concerned, the constraint is no longer an issue. For example, setting $RM/g = 10$ gives a mass of 20 keV, whence the extra force has a range of around 1 nm. The associated limit in interaction

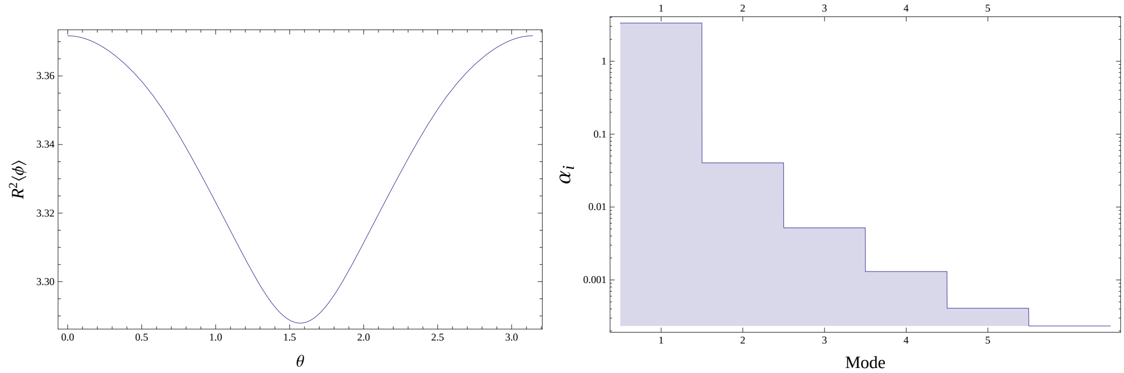
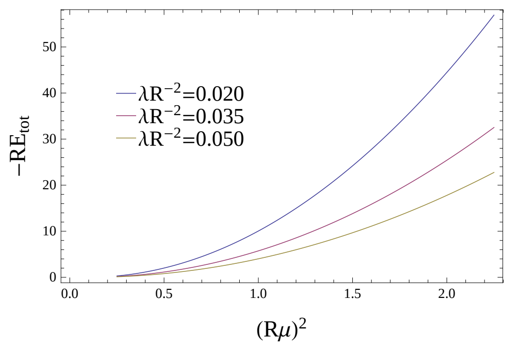
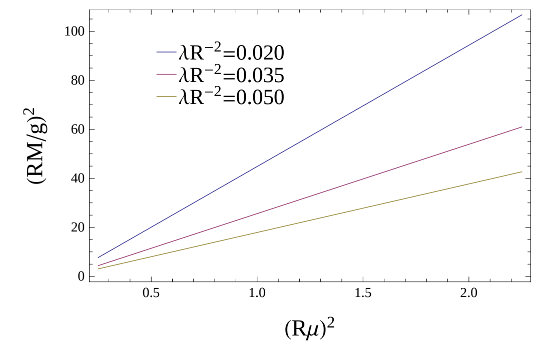


Figure 4.6: Higgs vacuum profile on the sphere and its spectrum



(a) Total energy of the Higgs sector as a function of μ



(b) Mass of the zero-mode vector boson as a function of μ

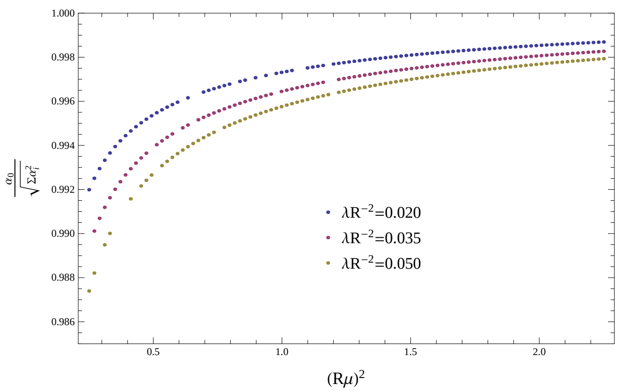


Figure 4.7: Normalized projection of the Higgs profile onto the constant Fourier component. This shows that the profile stays nearly constant

strength is 10^8 that of gravitational interactions [15], whilst that at hand is only around 10^2 times bigger than gravitation, so that this symmetry breaking hides satisfyingly the extra gauge boson in micrometer-range gravity tests.

Conclusion

If this study has shown something about models with curved extra-dimensions, it is that obtaining realistic physics is a hard problem because the natural fermion spectrum has many more low-energy features than in the flat case. Orbifolding is not a sufficiently constraining method for twisting this spectrum and it seems that for non-localized fermions to have a massless, chiral zero-mode, the only convincing approach is to have a background gauge-field that could counter the effect of the spin-connexion. This gauge field is motivated by RSS spontaneous compactification, which stabilizes the space $\mathbb{M}_4 \times S^2$ and fixes the new gauge coupling as a function of the sphere radius and the 6D Planck mass but requires fine-tuning to have a flat 4D space. Despite the small coupling that this mechanism imposes to the gauge field, sub-milimeter gravity experiments rule out such an infinite-range new force. Attemps have been made to break the associated gauge but to our knowledge the only convincing way is our proposed Higgs mechanism in the background, which can give a keV-ish mass to the gauge boson, which is low but sufficient to reach the domain compatible with gravity tests.

This model however spoils some of the simplicity of formulating the Standard Model in a UED framework and having KK excitations provide a natural dark matter candidate. Besides the extra-dimensions, new fields have to be added and the extra Higgs fields means having an extra energy scale on top of that of the sphere volume. This however can be modulated by the fact that setting the Higgs-associated scales close to the compactification scale yields a – so-far – realistic model.

One cannot however yet claim that this model is interesting for LHC physics or cosmology. Some work needs to be done at tree-level to determine couplings and see for example how KK excitations could intervene in loops to modify LHC observables, like Higgs production and decay [18]. Other developments would require looking at 1-loop corrections to the masses of the first levels of Kaluza-Klein excitations to determine the splitting between tree-level degenerate modes, which would allow to determine the lightest Kaluza-Klein particle, the potential dark matter candidate.

Exploring this model could be a nice continuation of my internship, as well as investigating the possibility offered by hyperbolic orbifolds, and if it does prove promising, I hope to be able to complete this work in the beginning of my thesis next September.

Appendix

A - Conventions

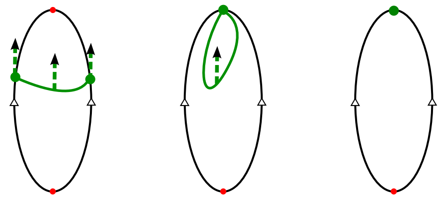
Given that dealing with fermions in curved space is not standard (or is for me at least), I decided to make derivations and explanations more clear by using colors. We have the following conventions:

- upper-case letters are for coordinates on the whole space
- greek (latin), middle-of-alphabet letters are for 4D manifold (local Minkowski) coordinates
- greek (latin), beginning-of the alphabet letters are for 2D manifold (local Minkowski) coordinates
- local Minkowski objects and coordinates indices are in “warm colors” (orange, yellow, red)
- manifold objects and coordinate indices are in “cold colors” (blue, green, purple)

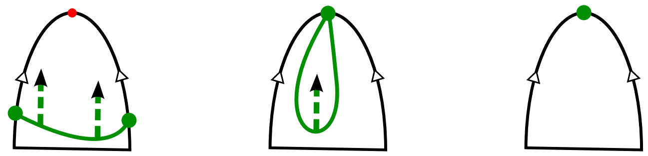
  There is somewhat redundant information with colors but it helped me have a clearer view of where each index belongs.

B - Holes in Orbifolds

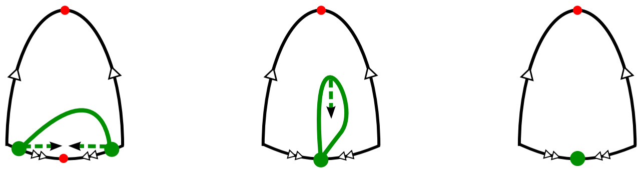
This appendix displays graphical and sketchy proofs of the simple connexity of orbifold on C_n, C_{nh} and D_n (those with a residual symmetry) by showing how loops can be deformed to a single point. This is done on the fundamental tile of the group, which comes with peridodic "gluing" of some pairs of boundaries. Only loops passing through one such boundary are non-trivial and explicited.



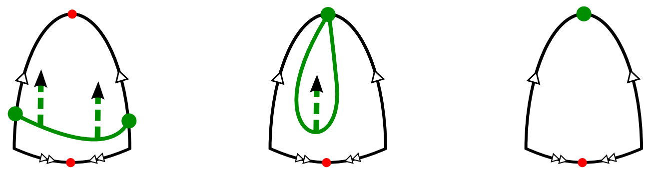
(a) C_n



(b) C_{nh}



(c) $D_n(1)$



(d) $D_n(2)$

Figure A.1: Simply connected spaces with a residual symmetry

On the other hand, S_n groups are multply-connected and I exhibit below an example of a loop not homotopic to identity.

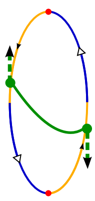


Figure A.2: Homotopically non-trivial loop

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