Categorified symmetries [*]

Urs Schreiber [†]
Department Mathematik, Universität Hamburg
Bundesstrasse 55, D-20146 GERMANY

Zoran Škoda [‡]
Division of Theoretical Physics Institut Rudjer Bošković
P.O.Box 180, HR-10002 Zagreb, CROATIA

Abstract

Quantum field theory allows more general symmetries than groups and Lie algebras. For instance quantum groups, that is Hopf algebras, have been familiar to theoretical physicists for a while now. Nowdays many examples of symmetries of categorical flavor – categorical groups, groupoids, Lie algebroids and their higher analogues – appear in physically motivated constructions and faciliate constructions of geometrically sound models and quantization of field theories. Here we consider two flavours of categorified symmetries: one coming from noncommutative algebraic geometry where varieties themselves are replaced by suitable categories of sheaves; another in which the gauge groups are categorified to higher groupoids. Together with their gauge groups, also the fiber bundles themselves become categorified, and their gluing (or descent data) is given by nonabelian cocycles, generalizing group cohomology, where ∞-groupoids appear in the role both of the domain and the coefficient object. Such cocycles in particular represent higher principal bundles, gerbes, – possibly equivariant, possibly with connection – as well as the corresponding *associated* higher vector bundles. We show how the Hopf algebra known as the Drinfeld double arises in this context.

1. Introduction

This is an overview for a general audience of mathematical physicists of (some appearances of) categorified symmetries of geometrical spaces and symmetries of constructions related to physical theories on spaces. Our main emphasis is on geometric and physical motivation, and the kind of

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[†] e-mail address: urs.schreiber@googlemail.com

[‡] e-mail address: zskoda@irb.hr

mathematical structures involved. Sections 2-4 treat examples in noncommutative geometry, while 5-9 study nonabelian cocycles motivated in physics, with a short content outline for block 6-9 at the end of section 5.

1.1. Categories and generalizations

We assume the reader is familiar with basics of the theory of categories, functors and sheaves, as the mathematical physics community has adopted these by now. At a few places for instance we use (co)limits in categories. The concept of a category $\mathcal{C}$ is often extended in several directions [2, 4, 24]. Instead of a *set* $\mathcal{C}_1 = \mathrm{Ob}\mathcal{C}$ of objects and set $\mathcal{C}_0 = \mathrm{Mor}\mathcal{C}$ of morphisms, with the usual operations (assignment of identity $i : X \mapsto \mathrm{id}_X$ to X; domain (source) and codomain (target) maps $s, t : \mathcal{C}_1 \to \mathcal{C}_0$; composition of composable pairs of morphism $\circ : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$) one defines an **internal category** in some ambient category $\mathcal{A}$ by specifying *objects* $\mathcal{C}_0, \mathcal{C}_1$ in $\mathcal{A}$, together with morphisms $i, s, t, \circ$ as above and satisfying analogous relations; **internal groupoids** in addition have the inverse-assigning morphism $(\cdot)^{-1} : \mathcal{C}_1 \to \mathcal{C}_1$ satisfying the usual properties. For instance smooth groupoids (Lie groupoids) are internal groupoids in the category of manifolds [2, 14, 23, 40]). A category may be given additional structure, e.g. a **monoidal category** is equipped with tensor (monoidal) products and tensor unit object (cf. [4, 24, 29] and section 3.). Given a monoidal category $\mathcal{D}$, a **$\mathcal{D}$-enriched category** $\mathcal{C}$ has a set of objects, but each set of morphisms $\mathrm{hom}_{\mathcal{C}}(A, B)$ is replaced by an object D in $\mathcal{D}$; it is required that the composition be a monoidal functor. In particular $\mathcal{D}$ may be the category of small categories, in which case a $\mathcal{D}$-enriched category is precisely a 2-category: it has morphisms between morphisms. This process may be iterated and leads to n-categories of various flavour, with n-morphisms or *n-cells*. It is natural to weaken the associativity conditions for compositions of k-cells for $0 < k < n$. This weakening is difficult to deal with, but often required by applications. A strict $(n+1)$-category is the same as nCat-enriched category where nCat is the category of strict n-categories and strict n-functors.

1.2. Basic idea of descent

Suppose we are given a geometric space and its decomposition in pieces with some intersections, e.g. an open cover of a manifold. The manifold can be reconstructed as the disjoint union modulo the identification of points in the pairwise intersections. For this we need to specify the identifications explicitly, and they may be considered as additional data. Suppose we now want to glue not the underlying sets, but some structures above, e.g. vector bundles. Bundles on each open set U_α of the cover form a category Vec_α and there are restriction functors from Vec_α to the 'localized' category of bundles on $U_{\alpha\beta} := U_\alpha \cap U_\beta$. A global bundle F is determined by its restrictions F_α to each open set U_α of the cover, together with identifications $(F_\alpha)|_{U_{\alpha\beta}} \overset{f_{\alpha\beta}}{\cong} (F_\beta)|_{U_{\alpha\beta}}$ via some isomorphisms $f_{\alpha\beta}$. These isomorphisms satisfy the *cocycle condition* $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ and $f_{\alpha\alpha} = \mathrm{id}$. The data $\{F_\alpha, f_{\alpha\beta}\}$ are called descent data ([19, 9, 47]). Equivalence classes of descent data are cohomology classes (with values in the automorphism group of the typical fiber) and they correspond to isomorphism classes bundles over the base space. There are vast generalizations of this theory, cf. [19, 9, 46, 36, 40, 41]. Gluing categories of quasicoherent sheaves/modules (see section 2. on their role) over noncommutative (NC) localizations which replace open sets ([34, 33, 44]) is a standard tool in NC geometry. Localization functors Q_α for different α, usually do not commute, what may be pictured as a noncommutativity of intersections of 'open sets". In order to reconstruct the module from its localizations (restrictions to localized "regions") we need match at *both* consecutive localizations, $Q_\alpha Q_\beta$ and $Q_\beta Q_\alpha$.

2. From noncommutative spaces to categories

By a noncommutative (NC) space ([14, 34, 44]) we mean any object for which geometrical intuition is available and whose description is given by the data pertaining to some geometrical objects living on the 'space'. Suppose we measure observable corresponding to some property depending on a local position in space. If the position changes from one part to another part of a space, we get different measurements, thus the measurements are expected to be functions of the local position. If the space is made out of points and we can make measurements closely about each point, then we get a function on the underlying set of points. This corresponds to the observables on phase space in classical physics; the quantum physics and noncommutative geometry mean that we can not decompose some 'spaces' to points, hence we can not really construct set-theoretic functions. Still, one can often localize observables to some geometrical 'parts' if not points.

The most standard case is when the 'space' is represented by a $\mathbb{C}$-valued algebra A. If A is commutative then the points of the space correspond to the characters (nonzero homomorphisms $\chi : A \to \mathbb{C}$), or equivalently, to maximal ideals $I = \mathrm{Ker}\chi$ in A. Knowing all functions at all points, physically means being able to measure all local quantities, and mathematically expresses the Gel'fand-Naimark theorem: from the C^*-algebra of continuous $\mathbb{C}$-valued functions on a compact Hausdorff space we can reconstruct back the space as the Gel'fand spectrum of A ([14, 23]). The Gel'fand spectrum can be constructed for NC algebras as well, but in that process we lose information and get smaller commutative algebras – the spectrum is roughly extracting the points, together with some topology, and there are not sufficiently many points to determine the NC space. Instead one is trying to express the geometrical and physical constructions we need in terms of algebra A, at least for good A-s of physical interest. This strategy usually works e.g. for small NC deformations of commutative algebras. Thus such a *quantum algebra* is by physicists usually called a NC space. We emphasise that there are more general NC spaces and more general types of their description.

We often know how the local coordinate charts look like and glue them together. The global ring is in principle sufficient information in C^*-algebraic framework, but many constructions are difficult as one has to make correct choices in operator analysis. Thus sometimes one resorts to algebraic geometry, that is algebras of regular (polynomial) functions; but even commutative algebraic variety/scheme X is not always determined by its ring of global regular functions $\mathcal{O}(X)$. Even if it is, we may find convenient to glue together more complicated objects (say fiber bundles) over the space from pieces. One way or another, we need to glue the spaces represented by algebras of functions to an object which will not loose information as the global coordinate ring sometimes does. If we have some sort of a cover of the space by collection of open sets U_i where on each U_i the algebra of functions determines the space, then having all of them together conserves all local information; moreover we should be able to pass to other open sets. Thus one needs a correspondence which to every open set gives an algebra of observables, that is some **sheaf** of functions in the case of commutative space; to do the same for fibre bundles means that we need to do the same for sheaves of sections of other bundles. It seems reasonable to take *a* **category of all sheaves** *of suitable kind on the space as a replacement of space.* This point of view in geometry was advocated by A. Grothendieck in 1960-s (geometry of toposes). Gabriel-Rosenberg's theorem states that every algebraic scheme X (typical geometrical space in algebraic geometry) can be reconstructed, up to an isomorphism of schemes, from the abelian category $\mathbf{Qcoh}_X$ of quasicoherent sheaves on X ([34]). For noetherian schemes a smaller (tensor abelian) subcategory $\mathbf{Coh}_X$ of coherent sheaves is enough, and in some cases even its derived category $D^b(\mathbf{Coh}_X)$ in the sense of homological algebra ([31]).

Examples suggest that instead of small deformations of commutative algebras of functions, we may consider deformations and similarly behaved analogues of categories $\mathbf{Qcoh}_X$. Principal examples appeared related to mirror symmetry. Mirror symmetry is a duality involving two Calabi-Yau 3-folds X and Y, saying that $N = 2$ SCFT-s A-model on X and B-model on Y and viceversa, A-model on Y and B-model on X are (nontrivially) equivalent as $N = 1$ SCFT-s (the difference at $N = 2$ level is in a ± 1 eigenvalue of an additional $U(1)$-symmetry operator, what is physically not distinguishable). In 1994, Maxim Kontsevich (M.K.) proposed the homological mirror symmetry conjecture [27], which is an equivalence of A_∞-categories related to topological A- and B-models. In A-model, the A_∞-category involved is the *Fukaya category* defined in terms of symplectic geometry on X (Y), and B-model is the A_∞-enhancement of the derived category of coherent sheaves on Y (X). M.K. also suggested a definition of a category of B-branes in N=2 *Landau-Ginzburg models* ([21, 26]) which have very similar structure to, but are *different* from, the derived categories of coherent sheaves on quasiprojective varieties. There are known relations between Hochschild cohomology (expressed in terms of $D^b(\mathbf{Coh}_X)$) and n-point correlation functions in the corresponding SCFT. Around 2003, M.K. and, independently K. Costello, found a way to go back and reconstruct SCFT from sufficiently good, but abstract A_∞-categories [15, 26], where 'good' involves generalizations of certain properties of (A_∞ enhancement of) $D^b(\mathbf{Coh}_X)$ where X is a Calabi-Yau variety. This shows that indeed physically relevant generalizations and deformations of varieties of complex algebraic geometry may come out of generalizations of algebraic geometry in terms of categories of sheaves and their abstract generalizations.

3. Monoidal categories as symmetries of NC spaces

The role of **symmetry** objects extends to the NC world: they help us singling out good candidates for the underlying space-time of a theory, and one employs the covariance properties of the tensors built out of field variables when constructing model Lagrangeans. In QFT, one wants not only that the fields form a representation of a symmetry algebra, but also to describe the second quantized systems, where the Hilbert space $\mathcal{H}$ is replaced by its exponent – the direct sum of n-particle Hilbert spaces, for all n, which are (in bosonic case, for simplicity) the symmetric powers of 1-particle Hilbert space $\mathcal{H}$. Thus the symmetry has to be defined on tensor products of representation spaces (classical example: addition of angular momenta of subsystems). **Hopf algebras** have the structure sufficient to define the tensor product of representations and the dual representations ([28, 29]), and each finite group gives rise to a Hopf algebra ("group algebra") with the same representation theory. Locally compact groups considered in axiomatic QFT, may also be generalized to Hopf algebra-like structures called locally compact quantum groups. If the underlying space is undeformed and in 4D, the axiomatic QFT actually proves that the full symmetry is decribed by a locally compact group, but in dimension 2, exotic braiding symmetries and quantum groups are allowed; in NC case a generic model will have a nonclassical symmetry. The symmetry algebra is here understood in usual sense – consisting of all observables which commute with the Hamiltonian. The natural Hamiltonians preserve the symmetries of the underlying space geometry, but there are often other symmetries which are not the symmetries of the underlying space; there are also hidden symmetries not seen at Hamiltonian level, but only in solutions.

Now we want to discuss the geometrical symmetries of "bare" underlying noncommutative space. For this we need to discuss more carefully a role of Hopf algebras. Recall ([29]) that a Hopf algebra H is an associative algebra equipped with some other maps including coassociative coproduct, which is an algebra map $\Delta : H \to H \otimes H$. Starting with any (say finite) group one may form its group algebra $\mathbb{C}G$, which is a Hopf algebra whose representations coincide with the representations of the group. On the other hand, the group itself may be replaced by a suitable algebra of functions $\mathcal{O}(G)$ on it, and then the corepresentations of $\mathcal{O}(G)$ (linear maps $\rho : V \to V \otimes \mathcal{O}(G)$ with $(\rho \otimes \mathrm{id})\rho = (\mathrm{i}d \otimes \Delta)\rho)$ will correspond to the representations of G. $\mathcal{O}(G)$ is commutative, and one may consider noncommutative Hopf algebras instead. Mathematically, replacing the commutative algebras, by the noncommutative, one should change the tensor product of commutative algebras (a categorical coproduct in the category of commutative algebras) by the so-called free product of noncommutative algebras, in all considerations. This would yield straightforward transfer of many constructions and their properties. However, examples of Hopf algebras with respect to the categorical coproduct are just few, while usual NC Hopf algebras with respect to $\otimes$ (e.g. quantum groups) are abundant in physical applications.

Recall that in commutative case, A-Mod is equivalent to $\mathbf{Qcoh}_{\mathrm{Spec}\,A}$, and that we took a viewpoint that the categories like $\mathbf{Qcoh}_X$ are representing spaces. H-Mod and H-Comod where H is a Hopf algebra are rigid monoidal categories. A **monoidal category** is a category equipped with a bifunctor $\otimes$ (monoidal or tensor product), which is associative up to coherent isomorphisms $M \otimes (N \otimes P) \cong (M \otimes N) \otimes P$, this category has a unit object $\mathbf{1}$ (satisfying $\mathbf{1} \otimes M \cong M \cong M \otimes \mathbf{1}$) and this category has dual objects M^* with usual properties (rigidity/autonomous category). Not only $\mathbf{Qcoh}_X$ remembers scheme X (Gabriel-Rosenberg theorem), also in favorite cases H-Mod as a rigid monoidal category remembers the underlying Hopf algebra (or Hopf algebroids, appearing as symmetries of inclusions of factors [8], relevant to CFT): this is an aspect of so-called Tannakian duality used widely in physics, e.g. the Doplicher-Roberts duality dealing with reconstruction of a QFT in 4d out of knowledge of full symmetry algebra is also a form of Tannaka reconstruction theorem; some reconstructions in CFT are as well ([29], Ch.9).

The reason why the usual Hopf algebras geometrically still fit into NC world is that the Hopf actions $H \otimes A \to A$ (i.e. when A is H-module algebra [29]) or Hopf coaction $A \to A \otimes H$ (A is H-comodule algebra), give rise to an action of the **monoidal category** of H-comodules (1st case) or H-modules (2nd case) on A-Mod (cf.[45] for recipes how to induce the categorical actions in these cases). Hopf coaction is hence replaced by action bifunctor $\Diamond : A\text{-Mod} \times H\text{-Mod} \to A\text{-Mod}$. The action axiom is the mixed associativity with product $\otimes$ in H-Mod, namely $M \Diamond (N \Diamond P) \cong (M \otimes N) \Diamond P$. Replacing the Hopf algebra H by its monoidal category of left modules H-Mod, we can as well, replace the Hopf coaction of H on A by the corresponding action $\Diamond$ of H-Mod. In some cases, we have no choice but to talk about categorical actions instead of Hopf (co)actions, e.g. if we want to globalize the action of Hopf algebra to nonaffine noncommutative varieties, then the latter may not be represented by a single algebra, but rather by gluing data for several of them. As the (co)action usually does not make the pieces invariant one really needs to talk about an action of Hopf algebra on the entire category of sheaves glued from pieces. But such action has no sense literally, unless we replace the Hopf algebra by its category of modules as well.

4. Application to Hopf algebraic coherent states

There is a projective operator valued measure on the space of coherent states (CS) which integrates to a constant operator: the CS are not mutually orthogonal but they are still involved in a resolution of unity operator formula. The Schrödinger equation can therefore be written in CS representation. Perelomov CS minimize generalized (covariant) uncertainty relations and transform in an appropriate covariant manner. Tensor operators of various “spin” may be treated simulteneously by forming CS operators, what is useful for discussing QFT on homogeneous spaces. I. Todorov with collaborators ([20, 37]) has been taking advantage of CS in formulating gauged WZNW models in Hamiltonian formalism; but their CS are attached to quantum groups (cf. [28] for variants of Hopf algebras in 2dCFT context) whose general and, particularly, geometric theory was lacking; the open problem was to extend the ”projective CS measure” to the quantum group case (existing formulas in simple cases in literature were just formal identities and usually the claimed invariance is incorrect). Motivated by

([20, 37]), one of us has shown in [43] that using NC localization and gluing one can study the geometry of line bundles over the quantum group homogeneous spaces and express the correct algebraic conditions for the analogues of Perelomov CS and of the invariant "projective" CS measure. Local coordinates on quantum G/B are constructed from the coinvariants (under the quantum B_q) in coaction-compatible localized charts on G_q as we earlier introduced in [42].

5. Higher gauge theories

An n-groupoid is a n-category in which all k-cells for all $1 \leq k \leq n$ are invertible (depending on the choice of context, this means strictly invertible or weakly invertible, i.e. up to higher cells). An n-group is a one-object n-groupoid. Smooth n-group(oid)s appear as analogues of Lie gauge groups for parallel transport along higher dimensional surfaces. One can build a theory of bundles with total space (which is now replaced by smooth n-category), possesing local trivialization and differential forms which are analogues of connection forms; two-torsors of [30], principal bigroupoid 2-bundles of [5] and gerbes [9] are examples. The cocycle data of a gerbe may be used to twist usual bundles and constructions with bundles, e.g. to get twisted K-theory. Instead of looking at the total space and differential forms, one may instead consider the effect of parallel transport to the points in a typical fiber. Thus an n-bundle with connection gets replaced by transport n-functor from some groupoid corresponding to the path geometry of the underlying space (fundamental n-groupoid, path n-groupoid) to the symmetry object of the fiber. The same formalism may incorporate gluing of (hyper)covers, by replacing the path groupoids with Čech n-groupoids of hypercovers: corresponding n-functors into symmetry n-group(oid) are the appropriate cocycles. On the other hand, the fiber bundles may be pulled back from the universal bundle over the classifying space of the group; the classifying space $\mathcal{B}G$ of an n-group corresponds to regarding the n-group G as a one-object $(n+1) - groupoid$ $\mathbf{B}G$. Thus in the next few sections we view cocycles as some weak maps into $\mathbf{B}G$. In general we find useful to employ some abstract homotopy theory of a certain model of ∞-categories described in section 6. to express what sort of "weak maps" the cocycles really are (for more details see [36]). We present two collections of definitions, one encoding the theory of nonabelian cocycles and ω-bundles in section 7., the other describing the quantum theory of the corresponding σ-models in section 8.. Some examples and applications are in section 9..

6. ∞-Categories and Homotopy Theory

We model ∞-categories as *strict* ∞-categories, usually called $\underline{\omega\text{-categories}}$. This choice turns out to be not only convenient but remarkably sufficient in our applications. In particular, we can translate back and forth between simplicial sets and ω-categories by means of a fixed cosimplicial ω-categeory, i.e. a functor $O : \Delta \to \omega\mathsf{Categories}$ from the simplicial category Δ.

Given that we obtain an $\underline{\omega\text{-nerve}}$ functor $N : \omega\mathsf{Categories} \to \mathsf{SimplicialSets}$

by

$$N(C):\ \Delta^{\mathrm{op}} \xrightarrow{O^{\mathrm{op}}} \omega\mathsf{Categories}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-,C)} \mathsf{Sets}$$

and its left adjoint $F : \mathsf{SimplicialSets} \to \omega\mathsf{Categories}$ given by the coend formula

$$F(S^\bullet) := \int^{[n]\in\Delta} S^n \cdot O([n]).$$

Ross Street defined ([46]) a particular cosimplical ω-category O called the orientals, for which 'the n-th oriental' $O([n])$ is the ω-category free on a single n-morphism of the shape of an n-simplex. To obtain more inverses, we can alternatively use the unorientals ([36]), for which $O([n])$ is the ω-category with n-objects, 1-morphisms are finite sequences of these objects, 2-morphisms are finite sequences of such finite sequences, and so on.

Particular important ω-categories are the ω-groupoids, forming a full subcategory $\omega\mathsf{Groupoids} \hookrightarrow \omega\mathsf{Categories}$, of those ω-categories whose all cells have strict inverses under all composition operations. Brown and Higgins proved that the data of ω-groupoids are equivalent to crossed complexes of groupoids, which are a non-abelian and many-object generalization of complexes of abelian groups. This equivalence enables usage of (generalizations of) tools familiar from homological algebra to the setup of ω-categories.

The category $\omega\mathsf{Categories}$ is equipped with the Crans-Gray tensor product, which is the extension to ω-categories of the tensor product on globular sets; the latter is induced via Day convolution from the addition of natural numbers. This means that the Crans-Gray tensor product is dimension raising analogously to the cartesian product on topological spaces:

for instance the tensor product of the interval ω-category $I = \{\ a \longrightarrow b\ \}$ with itself is the ω-category free on a single directed square

$$I \otimes I = \left\{ \begin{tikzcd} (a,a) \arrow[r] \arrow[d] & (a,b) \arrow[d] \arrow[dl, Rightarrow, shorten=10pt] \\ (b,a) \arrow[r] & (b,b) \end{tikzcd} \right\}.$$

Moreover, $\omega\mathsf{Categories}$ is biclosed with respect to this monoidal structure. We are particularly concerned with the internal hom-ω-categories of the form $A^I := \mathrm{hom}(I, A)$ which satisfy $\mathrm{Hom}(X \otimes I, A) \simeq \mathrm{Hom}(X, A^I)$, where the set in question here is the set of lax *transformations*

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=F}] \arrow[r, bend right=50, "g"'{name=G}] \arrow[from=F, to=G, Rightarrow, "\eta"] & A \end{tikzcd} \Leftrightarrow \begin{tikzcd} X \arrow[r, "\eta"] \arrow[rr, bend left=40, "f\times g"] & A^I \arrow[r, "d_0\times d_1"] & A \times A \end{tikzcd}$$

or *directed right homotopies* between ω-functors from X to A.

The 1-category $\omega\mathsf{Categories}$ is really an ∞-structure itself as it is remembered by a *model category structure* carried by it, due to [22], with respect to which the acyclic fibrations or hypercovers $f : C \overset{\simeq}{\twoheadrightarrow} D$ are those ω-functors which are k-surjective for all $k \in \mathbb{N}$, meaning that the universal dashed morphism in

$$\begin{tikzcd}
C_{k+1} \arrow[dr, dashed] \arrow[drr, bend left, "f_{k+1}"] \arrow[ddr, bend right, "s\times t"'] & & \\
& (f_k \times f_k)^* C_{k+1} \arrow[r] \arrow[d] & D_{k+1} \arrow[d, "s\times t"] \\
& C_k \times C_k \arrow[r, "f_k \times f_k"'] & D_k \times D_k
\end{tikzcd}$$

is epi, for all k. The weak equivalences $f : C \xrightarrow{\simeq} D$ are those ω-functors where these dashed morphisms become epi after projecting onto ω-equivalence classes of $(k+1)$-morphisms.

Using this we define an ω-anafunctor from an ω-category X to an ω-category A to be a span

$$\begin{tikzcd}
 & & & \hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & A \\
(g :\ X \arrow[r, "|" marking] & A\ ) & := & X &
\end{tikzcd}$$

whose left leg is a hypercover (the terminology follows [25, 6]). In the context of $\omega\mathsf{Groupoids}$ such ω-anafunctors represent morphisms in the homotopy category $[g] \in \mathrm{Ho}(X, A)$ which allows us to regard g as a cocycle in nonabelian cohomology on the ω-groupoid X with coefficients in the ω-groupoid A. Cocycles are regarded as distinct only up to refinements of their covers. This makes their composition by pullbacks

$$\begin{tikzcd}
 & & & & & g^*\hat{A} \arrow[r] \arrow[d, two heads, "\simeq"] & \hat{A} \arrow[r, "r"] \arrow[d, two heads, "\simeq"] & A' \\
(\ X \arrow[r, "g", "|" marking] & A \arrow[r, "r", "|" marking] & A'\ ) & := & & \hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & A & \\
 & & & & & X & &
\end{tikzcd}$$

well defined (noticing that acyclic fibrations are closed under pullback) and associative.

Definition 6..1 *We write* **Ho** *for the corresponding category of ω-anafunctors,*

$$\mathbf{Ho}(C,D) := \mathrm{colim}_{\hat{C}\in \mathsf{Hypercovers}(C)} \mathrm{Hom}(\hat{C},D)\,.$$

(This is to be contrasted with the true homotopy category Ho, which is obtained by further dividing out homotopies.)

While cocycles in nonabelian cohomology are morphisms in **Ho**, coboundaries should be morphisms between these morphisms. Hence **Ho** is to be thought of as enriched over $\omega\mathsf{Categories}$.

Definition 6..2 *Bifunctor* hom : $\mathbf{Ho}^{\mathrm{op}} \times \mathbf{Ho} \to \omega\mathsf{Categories}$ *is given on objects by* $\mathrm{hom}(C,D) := F(\mathrm{Hom}(C \otimes O([\bullet]), D))\,.$

7. Nonabelian cohomology, higher vector bundles and background fields

We consider fiber bundles whose fibers are ω-categories on which an ω-group G acts in a prescribed way. Here we conceive ω-groups as the hom-objects of one-object ω-groupoids.

Definition 7..1 (ω-group) *Given a one-object ω-groupoid* $\mathbf{B}G$ *the pullback*

$$\begin{tikzcd} G \arrow[r] \arrow[d] & (\mathbf{B}G)^I \arrow[d, "(d_0\times d_1)"] \\ \mathrm{pt} \arrow[r] & \mathbf{B}G \times \mathbf{B}G \end{tikzcd}$$

is the corresponding <u>*ω-group*</u>*.*

For X an ω-groupoid – addressed as <u>target space</u> – and for G an ω-group as above – the <u>gauge ω-group</u> or <u>structure ω-group</u> G – and given an ω-category F – the ω-category of <u>typical fibers</u> – together with a morphism $\rho:\ \mathbf{B}G \longrightarrow F$ into a pointed codomain, $\mathrm{pt}_F : \mathrm{pt} \to F$ – which we address as a <u>representation</u> – of G, we can speak of

- G-cocycles g on G;
- the <u>G-principal ω-bundle</u> $P := g^*\mathbf{E}G$ on X classified by these;
- the <u>ρ-associated</u> ω-bundles $V := g^*\rho^*\mathbf{E}F$
- the collection $\Gamma(V)$ of <u>sections</u> of V.

7.1. Principal ω-bundles

Definition 7..2 (universal G-principal ω-bundle) *The universal G-principal ω-bundle* $\mathbf{E}G \twoheadrightarrow \mathbf{B}G$ *is given by the pullback*

$$\begin{tikzcd}
\mathbf{E}G \arrow[r, two heads, "\simeq"] \arrow[d] \arrow[dd, bend right] & \mathrm{pt} \arrow[d] \\
(\mathbf{B}G)^I \arrow[r, two heads, "\simeq", "d_0"'] \arrow[d, "d_1"', "\simeq"] & \mathbf{B}G \\
\mathbf{B}G
\end{tikzcd} .$$

Proposition 7..3 *The morphism* $\mathbf{E}G \twoheadrightarrow \mathbf{B}G$ *defined this way is indeed a fibration and its kernel is G: we have a short exact sequence*

$$\begin{tikzcd}
G \arrow[r, hook, "i"] & \mathbf{E}G \arrow[r, two heads, "p"] & \mathbf{B}G
\end{tikzcd} .$$

Proof. That p is a fibration follows from a lemma in [13]. To see that G is indeed the kernel of this fibration, consider the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & \mathbf{E}_{\mathrm{op}}G \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\mathbf{E}G \arrow[r] \arrow[d] & \mathbf{B}G^I \arrow[r, "d_1"] \arrow[d, "d_0"] & \mathbf{B}G \\
\mathrm{pt} \arrow[r] & \mathbf{B}G
\end{tikzcd} .$$

The right and bottom squares are pullback squares by definition. Moreover, G is by definition 7..1 the total pullback

$$\begin{tikzcd}
G \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{pt} \arrow[d] \\
& \mathbf{B}G^I \arrow[r, "d_1"] \arrow[d, "d_0"] & \mathbf{B}G \\
\mathrm{pt} \arrow[r] & \mathbf{B}G
\end{tikzcd} .$$

Therefore also the top left square exists and is a pullback itself and hence so is the pasting composite of the two top squares. This says that i is the kernel of p.

Definition 7..4 (G-principal ω-bundles) *For X an ω-groupoid and G an ω-group, there is for every $\mathbf{B}G$-cocycle on X represented by an ω-anafunctor $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} \mathbf{B}G$ the corresponding G-principal ω-bundle $\pi_g : P \twoheadrightarrow X$ classified by g given by the pullback diagram*

$$\begin{tikzcd}
g^*\mathbf{E}G \arrow[r] \arrow[d, two heads] \arrow[dd, two heads, bend right] & \mathbf{E}G \arrow[d, two heads] \\
\hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & \mathbf{B}G \\
X &
\end{tikzcd} .$$

For $n \leq 2$ this way of describing (universal) principal n-bundles was described in [32].

Theorem 7..5 *If G is a group or strict 2-group, this definition of G-principal bundles is equivalent to the definitions in [6, 5, 48].*

Remarks.

- This statement involves higher categorical equivalences: for G a 2-group and $g : X \mathrel{\mkern3mu|\mkern-12mu\longrightarrow} \mathbf{B}G$ a cocycle, the pullback $g^*\mathbf{E}G$ is a priori a 2-groupoid, whereas in the literature on 2-bundles one expects this total space to be a 1-groupoid. But this desired 1-groupoid is obtained by dividing out 2-isomorphisms in $g^*\mathbf{E}G$ and the result is weakly equivalent to the original 2-groupoid $g^*\mathbf{E}G \overset{\simeq}{\twoheadrightarrow} (g^*\mathbf{E}G)_\sim$.
- For the purposes of this article we are glossing over the *internalization* of the entire setup from a context internal to Sets to a context internal to a category Spaces, for instance of topological spaces or of suitable generalized smooth spaces. The above statements generalize to such internal contexts by suitably lifting the model structure on $\omega\mathsf{Groupoids}$ to the structure of a *category of fibrant objects* on $\omega\mathsf{Groupoids}(\mathsf{Spaces})$. Further discussion of this point is relegated to [36].

7.2. Associated ω-bundles

Associated ω-bundles similarly arise from pullback along more general cocycles with prescribed factorization through $\mathbf{B}G$. For that purpose let F be some ω-category (*not* necessarily an ω-groupoid). An ω-anafunctor

$$\rho : \mathbf{B}G \mathrel{\mkern3mu|\mkern-12mu\longrightarrow} F$$

may be addressed as an ω-group cocycle with values in F. If F is equipped with a point, $\mathrm{pt} \xrightarrow{\mathrm{pt}_F} F$, we can address such a morphism ρ also as a representation of G. In analogy to the universal G-principal ω-bundle from definition 7..2 we obtain the universal F-bundle (with respect to the chosen point pt_F) as a pullback from the point:

Definition 7..6 (universal F-bundle) *For F an ω-category with chosen point $\mathrm{pt} \xrightarrow{\mathrm{pt}_F} F$ the universal F-bundle $\mathbf{E}F \twoheadrightarrow F$ is the pullback*

$$\begin{tikzcd}
\mathbf{E}F \arrow[r, two heads] \arrow[d] \arrow[dd, bend right, two heads] & \mathrm{pt} \arrow[d, "\mathrm{pt}_F"] & . \\
F^I \arrow[r, two heads, "d_0"] \arrow[d, two heads, "d_1"] & F & \\
F & &
\end{tikzcd}$$

Definition 7..7 (associated F-bundle) *Given a representation morphism $\rho : \mathbf{B}G \to F$ we accordingly address the pullback*

$$\begin{tikzcd}
\rho^*\mathbf{E}F \arrow[r] \arrow[d, two heads] & \mathbf{E}F \arrow[d, two heads] \\
\mathbf{B}G \arrow[r, "\rho"] & F
\end{tikzcd}$$

as the F-bundle ρ-associated to the universal G-bundle. Correspondingly the further pullback along a g-cocycle

$$\begin{tikzcd}
g^*\rho^*\mathbf{E}F \arrow[r] \arrow[d, two heads] \arrow[dd, bend right, two heads] & \rho^*\mathbf{E}F \arrow[r] \arrow[d, two heads] & \mathbf{E}F \arrow[d, two heads] \\
\hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & \mathbf{B}G \arrow[r, "\rho"] & F \\
X & &
\end{tikzcd}$$

is the F-bundle ρ-associated to the specific G-principal bundle $g^*\mathbf{E}G$.

7.3. Sections of associated ω-bundles

Definition 7..8 (section) *A section σ of a ρ-associated ω-bundle $V := \rho^* g^*\mathbf{E}F$ coming from a cocycle $X \xrightarrow{g} \mathbf{B}G$ is a lift of the cocycle through $\rho^*\mathbf{E}F \twoheadrightarrow \mathbf{B}G$ or equivalently a morphism from the trivial F-bundle with fiber pt_F to V*

$$\Gamma(V) := \left\{ \begin{tikzcd} & \rho^*\mathbf{E}F \arrow[d, two heads] \\ X \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & X \arrow[dl] \arrow[dr, "\mathrm{Id}"] & \\ \mathrm{pt} \arrow[rr, Rightarrow, dashed, "\sigma"] \arrow[dr, "\mathrm{pt}_F"'] & & X \arrow[dl, "\rho\circ g"] \\ & F & \end{tikzcd} \right\}$$

Proposition 7..9 *These two characterizations of sections are indeed equivalent.*

Proof. First rewrite

$$\left\{ \begin{tikzcd} & X \arrow[dl] \arrow[d, "g"] \\ \mathrm{pt} \arrow[r, Rightarrow, "\sigma"'] \arrow[dr, "\mathrm{pt}_F"'] & \mathbf{B}G \arrow[d, "\rho"] \\ & F \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & \mathrm{pt} \arrow[r, "\mathrm{pt}_F"] & F \\ X \arrow[ur] \arrow[rr, dashed, "\sigma"] \arrow[dr, "g"'] & & F^I \arrow[u, "d_0"'] \arrow[d, "d_1"'] \\ & \mathbf{B}G \arrow[r, "\rho"] & F \end{tikzcd} \right\}$$

using the characterization of right (directed) homotopies by the (directed) path object F^I. Using the universal property of $\mathbf{E}F$ as a pullback this yields

$$\cdots \simeq \left\{ \begin{tikzcd} & & \mathbf{E}F \arrow[d, two heads] \\ X \arrow[urr, dashed, "\sigma"] \arrow[r, "g"'] & \mathbf{B}G \arrow[r, "\rho"'] & F \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & \rho^*\mathbf{E}F \arrow[d, two heads] \\ X \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \right\}.$$

8. Quantization and quantum symmetries

We want to think of an associated ω-bundle $V \twoheadrightarrow X$ as a *background field* (a generalization of an electromagnetic field) on X to which a higher dimensional *fundamental brane* – such as a *particle*, a *string* or a *membrane* – propagating on X may *couple*. If a piece of *worldvolume* of this fundamental brane is modeled by an ω-category Σ then, following [16], we say that

- the space of fields over Σ is $C_\Sigma := \mathrm{hom}(\Sigma, X)$, the "space" of maps from the worldvolume to target space X;

- the space of states over Σ is the space of sections $\Gamma(\tau_\Sigma V)$ of the background field V *transgressed* to the space of fields.

We now give a diagrammatic definition of transgression and of such spaces of states.

8.1. Transgression of cocycles to mapping spaces

Following [40], we identify transgression to mapping spaces with the internal hom applied to cocycles:

Definition 8..1 (transgression of cocycles) *For $X \overset{\rho\circ g}{-\!\!\!|\!\!\!\rightarrow} F$ a cocycle classifying a ρ-associated ω-bundle on X, and for Σ any other ω-groupoid, we say that the transgression $\tau_\Sigma(\rho\circ g)$ of $\rho\circ g$ to X^Σ is its value under the internal hom in* **Ho**, *definition 6..2:*

$$\tau_\Sigma(\rho\circ g) := \mathrm{hom}(\Sigma, \rho\circ g) : \mathrm{hom}(\Sigma, X) \to \mathrm{hom}(\Sigma, F)\,.$$

8.2. Background field and space of states

Definition 8..2 *A background structure for a σ-model is*

- *an ω-groupoid X called target space;*
- *an ω-group G, called the gauge group;*
- *a representation* $$\begin{tikzcd} \rho^*\mathbf{E}G \arrow[r] \arrow[d] & \mathbf{E}F \arrow[d] & \\ \mathbf{B}G \arrow[r, "\rho"] & F & \mathrm{pt} \arrow[l, "\mathrm{pt}_F"'] \end{tikzcd}$$ *called the matter content;*
- *the ω-bundle $g^*\rho^*\mathbf{E}G$ which is ρ-associated to a G-principal bundle on* X $$\begin{tikzcd} \hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & \mathbf{B}G \\ X & \end{tikzcd}$$ *, called the background field.*

Definition 8..3 *Given a background structure (X, g, ρ) and for Σ any other ω-groupoid we say that*

- $\mathbf{Ho}(\Sigma, X)$ *is the space of fields over Σ of the σ-model defined by the background structure;*
- $\mathbf{Ho}(\Sigma, g) : \mathbf{Ho}(\Sigma, X) \to \mathbf{Ho}(\Sigma, \mathbf{B}G)$ *is the action functional of the Σ-model over Σ.*

8.3. Branes and bibranes

From the second part of definition 7..8 one sees that spaces of states, being spaces of sections, are given by certain morphisms between background fields pulled back to spans/correspondences of target spaces. From the diagrammatics this has an immediate generalization, which leads to the notion of *branes* and *bibranes*.

Definition 8..4 (branes and bibranes) *A brane for a background structure $(X, \rho \circ g)$ is a morphism $\iota : Q \to X$ equipped with a section of the background field pulled back to Q, i.e. a transformation*

$$\begin{tikzcd} & Q \arrow[dl] \arrow[dr, "\iota"] & \\ \mathrm{pt} \arrow[rr, phantom, "\overset{V}{\Rightarrow}"] \arrow[dr, "\mathrm{pt}_F"'] & & X \arrow[dl, "\rho\circ g"] \\ & F & \end{tikzcd}.$$

More generally, given two background structures (X, g, ρ) and (X', g', ρ), a bibrane between them is a span

$$\begin{tikzcd} & Q \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\ X & & X' \end{tikzcd}$$

equipped with a transformation

$$\begin{tikzcd} & Q \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\ X \arrow[rr, phantom, "\overset{V}{\Rightarrow}"] \arrow[dr, "\rho\circ g"'] & & X' \arrow[dl, "\rho'\circ g'"] \\ & F & \end{tikzcd}.$$

Bibranes may be composed –"fused" – along common background structures $(X, \rho\circ g)$: the composite or *fusion* of a bibrane V on Q with a bibrane V' on Q' is the bibrane $V \cdot V'$ given by the diagram

$$\begin{tikzcd} & Q \times_{X'} Q \arrow[dl] \arrow[dr] & \\ X \arrow[rr, Rightarrow, "s^*V\cdot t^*V'"] \arrow[dr, "\rho\circ g"'] & & X \arrow[dl, "\rho''\circ g''"] \\ & F & \end{tikzcd} \quad := \quad \begin{tikzcd} & & Q \times_{X'} Q' \arrow[dl, "s"'] \arrow[dr, "t"] & & \\ & Q \arrow[dl] \arrow[dr] & & Q' \arrow[dl] \arrow[dr] & \\ X \arrow[rr, Rightarrow, "V"] \arrow[drr, "\rho\circ g"'] & & X' \arrow[rr, Rightarrow, "V'"] \arrow[d, dashed, "\rho'\circ g'"] & & X'' \arrow[dll, "\rho''\circ g''"] \\ & & F & & \end{tikzcd}$$

If Q carries further structure, the fused bibrane on $Q \times_{t,s} Q$ may be pushed down again to Q, such as to produce a monoidal structure on bibranes on Q. Consider therefore a category $Q \overset{s}{\underset{t}{\rightrightarrows}} X$ *internal* to ω-groupoids, equivalently a monad in the bicategory of spans internal to $\omega\mathsf{Groupoids}$, with composition operation the morphism of spans

$$\begin{tikzcd}
 & & Q \times_{t,s} Q \arrow[dl] \arrow[dr] \arrow[ddd, bend left, "\mathrm{comp}"] & & \\
 & Q \arrow[dl, "s"'] \arrow[dr, "t"] & & Q \arrow[dl, "s"'] \arrow[dr, "t"] & \\
X & & X & & X \\
 & & Q \arrow[ull, "s"] \arrow[urr, "t"'] & &
\end{tikzcd} .$$

Definition 8..5 (monoidal structure on bibranes) *Given an internal category as above, and given an F-cocycle $g : X \to F$, the composite of two bibranes* $\begin{tikzcd} & Q \arrow[dl] \arrow[dr] & \\ X \arrow[dr, "g"'] & \overset{V,W}{\Rightarrow} & X \arrow[dl, "g"] \\ & F & \end{tikzcd}$ *on Q is the result of first forming their composite bibrane on on $Q \times_{t,s} Q$ and then pushing that forward along* comp*:*

$$V \star W := \int_{\mathrm{comp}} (s^* V) \cdot (t^* W) \,.$$

Here for finite cases, which we concentrate on, push-forward is taken to be the right adjoint to the pullback in a proper context.

Remarks. Notice that branes are special cases of bibranes and that bibrane composition restricts to an action of bibranes on branes. Also recall that the sections of a cocycle on X are the same as the branes of this cocycle for $\iota = \mathrm{Id}_X$.

The idea of bibranes was first formulated in [18] in the language of modules for bundle gerbes. We show in section 9.4. how this is reproduced within the present formulation.

9. Examples and applications

We start with some simple applications to illustrate the formalism and then exhibit some useful constructions in the context of finite group QFT.

9.1. Ordinary vector bundles

Let G be an orinary group, hence a 1-group, and denote by $F :=$ Vect the 1-category of vector spaces over some chosen ground field k. A linear representation ρ of G on a vector space V is indeed the same thing as a functor $\rho : \mathbf{B}G \to \mathrm{Vect}$ which sends the single object of $\mathbf{B}G$ to V.

The canonical choice of point $\mathrm{pt}_F : \mathrm{pt} \to \mathrm{Vect}$ is the ground field k, regarded as the canonical 1-dimensional vector space over itself. Using this we find from definition 7..6 that the *universal* Vect-*bundle* is $\mathbf{E}\mathrm{Vect} = \mathrm{Vect}_*$, the category of *pointed* vector spaces with $\mathrm{Vect}_* \twoheadrightarrow \mathrm{Vect}$ the canonical forgetful functor. Using this one finds from definition 7..7 that the ρ-associated vector bundle to the universal G-bundle is $V//G \twoheadrightarrow \mathbf{B}G$, where $V//G := \left(\begin{tikzcd} V \times G \arrow[r, shift left, "p_1"] \arrow[r, shift right, "\rho"'] & V \end{tikzcd} \right)$ is the action groupoid of G acting on V, the weak quotient of V by G.

For $g : \begin{tikzcd} X \arrow[r, "g", "|" marking] & \mathbf{B}G \end{tikzcd}$ a cocycle describing a G-principal bundle and for V the corresponding ρ-associated vector bundle according to definition 7..7, one sees that sections $\sigma \in \Gamma(V)$ in the sense of definition 7..8 are precisely sections of V in the ordinary sense.

9.2. Group algebras and category algebras from bibrane monoids

In its simplest version the notion of monoidal bibranes from section 8.3. reproduces the notion of *category algebra* $k[C]$ of a category C, hence also that of a *group algebra* $k[G]$ of a group G. Recall that the category algebra $k[C]$ of C is defined to have as underlying vector space the span of C_1, $k[C] = \mathrm{span}_k(C_1)$, where the product is given on generating elements $f, g \in C_1$ by

$$f \cdot g = \begin{cases} g \circ f & \text{if the composite exists} \\ 0 & \text{otherwise} \end{cases}$$

To reproduce this as a monoid of bibranes in the sense of section 8.3., take the category of fibers in the sense of section 7.2. to be $F = \mathrm{Vect}$ as in section 9.1.. Consider on the space (set) of objects, C_0, the trivial line bundle given as an F-cocycle by $i : C_0 \longrightarrow \mathrm{pt} \xrightarrow{\mathrm{pt}_k} \mathrm{Vect}$. An element in the monoid of bibranes for this trivial line bundle on the span given by the source and target map $\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd}$ is a transformation of the form

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 \arrow[dr, "i"'] \arrow[rr, Rightarrow, "V"] & & C_0 \arrow[dl, "i"] \\ & \mathrm{Vect} & \end{tikzcd}$$

. In terms of its components this is canonically identified with a function $V : C_1 \to k$ from the space (set) of morphisms to the ground field and every such function gives such a transformation. This identifies the C-bibranes with functions on C_1.

Given two such bibranes V, W, their product as bibranes is, according to definition 8..5, the push-forward along the composition map on C of the function on the space (set) of composable morphisms

$$C_1 \times_{t,s} C_1 \to k$$

$$(\xrightarrow{f}\xrightarrow{g}) \mapsto V(f) \cdot W(g)\,.$$

This push-forward is indeed the product operation on the category algebra.

9.3. Monoidal categories of graded vector spaces from bibrane monoids

The straightforward categorification of the discussion of group algebras in section 9.2. leads to bibrane monoids equivalent to monoidal categories of graded vector spaces.

Let now $F := 2\mathrm{Vect}$ be a model for the 2-category of 2-vector spaces. For our purposes and for simplicity, it is sufficient to take $F := \mathbf{B}\mathrm{Vect} \hookrightarrow 2\mathrm{Vect}$, the 2-category with a single object, vector spaces as morphisms with composition being the tensor product, and linear maps as 2-morphisms. This can be regarded as the full sub-2-category of 2Vect on 1-dimensional 2-vector spaces. And we can assume $\mathbf{B}\mathrm{Vect}$ to be strictified.

Then bibranes over G for the trivial 2-vector bundle on the point, i.e. transformations of the form

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ \mathrm{pt} \arrow[rr, Rightarrow] \arrow[dr] & & \mathrm{pt} \arrow[dl] \\ & \mathbf{B}\mathrm{Vect} & \end{tikzcd}$$

canonically form the category Vect^G of G-graded vector spaces. The fusion of such bibranes reproduces the standard monoidal structure on Vect^G.

9.4. Twisted vector bundles

The ordinary notion of a brane in string theory is: for an abelian gerbe $\mathcal{G}$ on target space X a map $\iota : Q \to X$ and a $\mathrm{PU}(n)$-principal bundle on Q whose lifting gerbe for a lift to a $U(n)$-bundle is the pulled back gerbe $\iota^*\mathcal{G}$. Equivalently: a twisted $U(n)$-bundle on Q whose twist is $\iota^*\mathcal{G}$. Equivalently: a gerbe module for $\iota^*\mathcal{G}$.

We show how this is reproduced as a special case of the general notion of branes from definition 8..4, see also [41].

The bundle gerbe on X is given by a cocycle $g : \begin{tikzcd} X \arrow[r, "|" marking] & \mathbf{BB}U(1) \end{tikzcd}$. The coefficient group has a canonical representation $\rho : \mathbf{B}^2U(1) \to F := \mathbf{B}\mathrm{Vect} \hookrightarrow 2\mathrm{Vect}$ on 2-vector spaces (as in section 9.3.) given by

$$\rho : \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}", ""{name=U, below}] \arrow[r, bend right=50, "\mathrm{Id}"', ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D, "c\in U(1)"] \end{tikzcd} \mapsto \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathbb{C}", ""{name=U, below}] \arrow[r, bend right=50, "\mathbb{C}"', ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D, "\cdot c"] \end{tikzcd} .$$

See also [41, 38].

By inspection one indeed finds that branes in the sense of diagrams

$$\begin{tikzcd} & Q \arrow[dl] \arrow[dr, "\iota"] & \\ \mathrm{pt} \arrow[rr, Rightarrow, "V"] \arrow[dr, "\mathrm{pt}_F"'] & & X \arrow[dl, "\rho\circ g"] \\ & \mathbf{B}\mathrm{Vect} & \end{tikzcd}$$

are canonically identified with twisted vector bundles on

Q with twist given by the $\iota^* g$: the naturality condition satisfied by the components of V is

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, Rightarrow, "\mathrm{Id}"] & \\ \mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] \arrow[d, "(\pi_1^*E)_y"'] & {} & \mathbb{C} \arrow[d, "\pi_3^*E_y"] \arrow[dll, Rightarrow, "\pi_{13}^*g_{\mathrm{tw}}(y)"] \\ \mathbb{C} \arrow[rr, "\mathbb{C}"] & & \mathbb{C} \end{tikzcd} = \begin{tikzcd} & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[dd, "(\pi_2^*E)_y"] \arrow[dddl, Rightarrow, "\pi_{12}^*g_{\mathrm{tw}}(y)"'] & \\ \mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[dd, "\pi_1^*E_y"'] & & \mathbb{C} \arrow[dd, "(\pi_3^*E_y)"] \arrow[dl, Rightarrow, "\pi_{23}^*g_{\mathrm{tw}}(y)"'] \\ & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, Rightarrow, "\cdot g(y)"] & \\ \mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] & {} & \mathbb{C} \end{tikzcd}$$

, for all $y \in Y \times_X Y \times_X Y \times_X Y$ in the triple fiber product of a local-sections admitting map $\pi : Y \to X$ whose simplicial nerve $Y^\bullet$, regarded as an ω-category, provides the cover for the ω-anafunctor $X \overset{\simeq}{\twoheadleftarrow} Y^\bullet \xrightarrow{g} \mathbf{B}^2 U(1)$ representing the gerbe. See [41] for details. $E \to Y$ is the vector bundle on the cover encoded by the transformation V. The above naturality diagram says that its transition function g_{tw} satisfies the usual cocycle condition for a bundle only up to the twist given by the gerbe g: if $Y \to X$ is a cover by open subsets $Y = \sqcup_i U_i$, then the above diagram is equivalent to the familiar equation

$$(g_{\mathrm{tw}})_{ij}(g_{\mathrm{tw}})_{jk} = (g_{\mathrm{tw}})_{ik} \cdot g_{ijk} \,.$$

In this functorial cocyclic form twisted bundles on branes were described in [39, 41].

9.5. Dijkgraaf-Witten theory

Dijkgraaf-Witten theory [17] is the σ-model which in our terms is specified by the data

- target space $X = \mathbf{B}G$, the one-object groupoid corresponding to an ordinary 1-group G;
- background field $\alpha : \mathbf{B}G \to \mathbf{B}^3 U(1)$, a group 3-cocycle on G.

9.5.1. The 3-cocycle

Indeed, we can understand group cocycles precisely as ω-anafunctors $\mathbf{B}G \overset{\simeq}{\twoheadleftarrow} Y \xrightarrow{\alpha} \mathbf{B}^n U(1)$. This is described in [11]. Here it is convenient to take Y to be essentially the free ω-category on the nerve of $\mathbf{B}G$, i.e. $Y := F(N(\mathbf{B}G))$, but with a few formal inverses thrown in to ensure that we have an acyclic fibration to $\mathbf{B}G$:

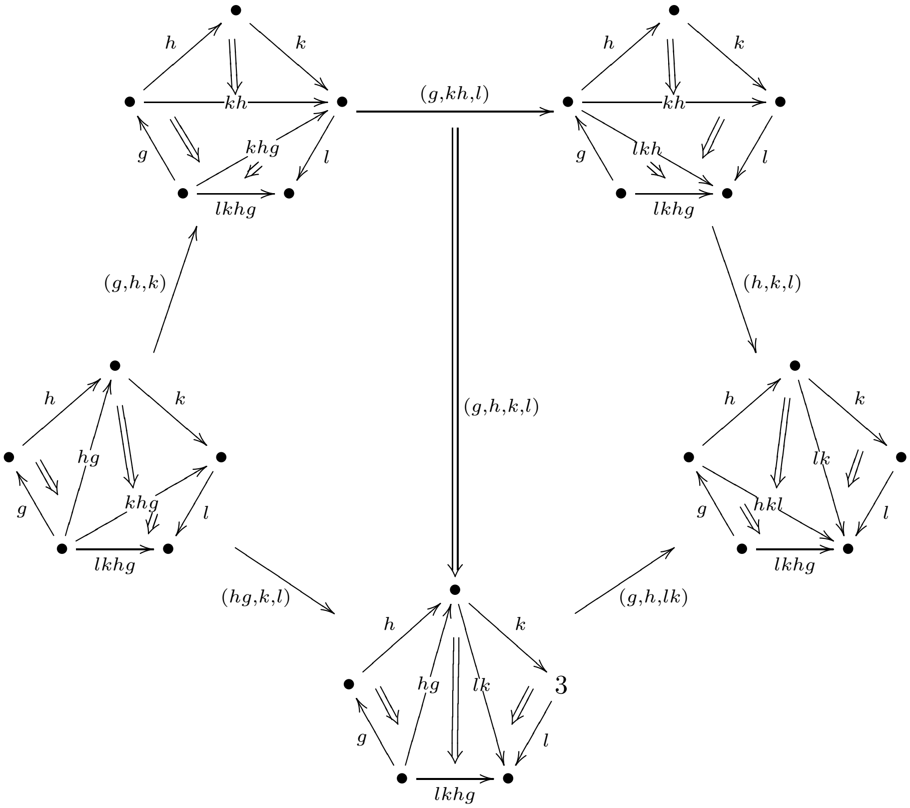
1-morphisms of Y are finite sequences of elements of G; 2-morphisms are freely generated as pasting diagrams from the 2-morphisms of the form

$$\left\{\begin{tikzcd} & \bullet \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur, "g"] \arrow[rr, "hg"'] & {} & \bullet \end{tikzcd}\right\}$$

together with their formal inverses. 3-morphisms in Y are freely generated as pasting diagrams from the 3-morphisms of the form

$$\left\{\begin{tikzcd}
\bullet \arrow[r, "h"] & \bullet \arrow[dd, "k"] & & \bullet \arrow[r, "h"] \arrow[ddr, "kh" description] & \bullet \arrow[dd, "k"] \\
 & {} \arrow[l, phantom, "\Downarrow"] & {} \arrow[r, "{(g,h,k)}"] & {} \arrow[r, phantom, "\Downarrow"] & {} \\
\bullet \arrow[uu, "g"] \arrow[uur, "hg" description] \arrow[r, "khg"'] & \bullet & & \bullet \arrow[uu, "g"] \arrow[r, "khg"'] & \bullet
\end{tikzcd}\right\}$$

together with their formal inverses. 4-morphisms in Y are freely generated from pasting diagrams of 4-morphisms of the form



together with their formal inverses.

The ω-functor $\alpha : Y \to \mathbf{B}^3U(1)$ has to send the generating 3-morphisms (g,h,k) to a 3-morphism in $\mathbf{B}^3U(1)$, which is an element $\alpha(g,h,k) \in U(1)$.

In addition, it has to map the generating 4-morphisms between pasting diagrams of these 3-morphisms to 4-morphisms in $\mathbf{B}^3U(1)$. Since there are only identity 4-morphisms in $\mathbf{B}^3U(1)$ and since composition of 3-morphisms in $\mathbf{B}^3U(1)$ is just the product in $U(1)$, this says that α has to satisfy the equations

$$\forall g,h,k,l \in G: \ \alpha(g,h,k)\alpha(g,kh,l)\alpha(h,k,l) = \alpha(hg,k,l)\alpha(g,h,lk)$$

in $U(1)$. This identifies the ω-functor α with a group 3-cocycle on G. Conversely, every group 3-cocycle gives rise to such an ω-functor and one can check that coboundaries of group cocycles correspond precisely to transformations between these ω-functors. Notice that α uniquely extends to the additional formal inverses of cells in Y which ensure that $Y \xrightarrow{\simeq} \mathbf{B}G$ is indeed an acyclic fibration. For instance the 3-cell

$$\left\{ \begin{tikzcd} \bullet \arrow[r, "h"] & \bullet \arrow[d, "k"] & & \bullet \arrow[r, "h"] \arrow[dr, "kh"] & \bullet \arrow[d, "k"] \\ \bullet \arrow[u, "g"] \arrow[ur, "hg"] \arrow[r, "khg"'] & \bullet \arrow[r, phantom, "\xrightarrow{(g,h,k)'}"] & {} & \bullet \arrow[u, "g"] \arrow[r, "khg"'] & \bullet \end{tikzcd} \right\}$$

has to go to $\alpha(g,h,k)^{-1}$.

9.5.2. Chern-Simons theory

The details of extension of this framework to ω-categories internal to smooth spaces is beyond the scope of this article. In light of the previous section 9.5. it should be however noted that in terms of nonabelian cocycles the appearance of Chern-Simons theory is formally essentially the same as that of Dijkgraaf-Witten theory:

if we take BG to be a smooth model of the classifying space of G-principal bundles, then a smooth cocycle $BG \longrightarrow \mathbf{B}^3U(1)$, i.e. an ω-anafunctor internal to (suitably generalized) smooth spaces is precisely the cocycle for a 2-gerbe, i.e. a line 3-bundle. In nonabelian cohomology, the difference between group cocycles and higher bundles is no longer a conceptual difference, but just a matter of choice of target "space" ω-groupoid.

9.6. Transgression of DW theory to loop space: the twisted Drinfeld double

Proposition 9..1 *The background field α of Dijkgraaf-Witten theory transgressed according to definition 8..1 to the mapping space of parameter space $\Sigma := \mathbf{B}\mathbb{Z}$ – a combinatorial model of the circle –*

$$\tau_{\mathbf{B}\mathbb{Z}}\alpha := \hom(\mathbf{B}\mathbb{Z}, \alpha)_1 : \Lambda G \to \mathbf{B}^2U(1)$$

is the groupoid 2-cocycle known as the twist of the Drinfeld double ([10, 29]):

$$(\tau_{\mathbf{B}\mathbb{Z}}\alpha) : (\ x \xrightarrow{g} gxg^{-1} \xrightarrow{h} (hg)x(hg)^{-1}\ ) \mapsto \frac{\alpha(x,g,h)\ \alpha(g,h,(hg)x(hg)^{-1})}{\alpha(h,gxg^{-1},g)} .$$

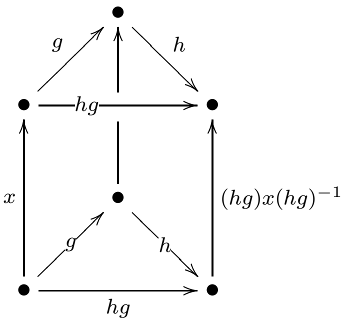
Proof. According to definition 6..2 the transgressed functor is obtained on 2-cells as the composition of ω-anafunctors $\mathbf{B}\mathbb{Z} \overset{(x,g,)}{\longmapsto} \mathbf{B}G \overset{\alpha}{\longmapsto} \mathbf{B}^3U(1)$, given by

$$\begin{tikzcd}
(x,g,h)^*Y \arrow[r] \arrow[d, two heads, "\simeq"] & Y \arrow[r, "\alpha"] \arrow[d, two heads, "\simeq"] & \mathbf{B}^3U(1)\ , \\
\mathbf{B}\mathbb{Z}\otimes O([2]) \arrow[r, "(x,g,h)"] & \mathbf{B}G &
\end{tikzcd}$$

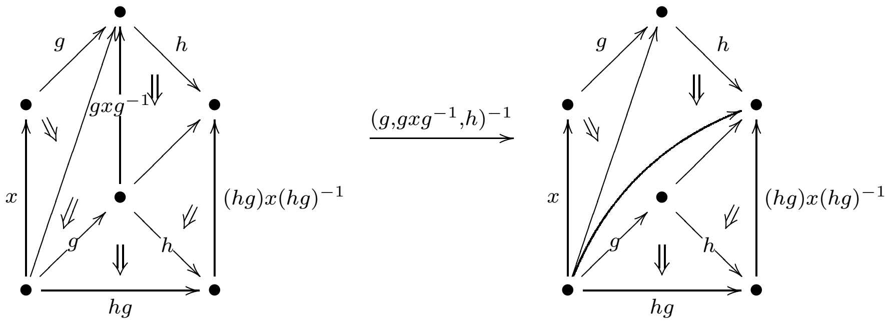
where (x,g,h) denotes a 2-cell in ΛG

$$\begin{tikzcd}
& gxg^{-1} \arrow[dr, "h"] & \\
x \arrow[ur, "g"] \arrow[rr, "hg"'] & & (hg)x(hg)^{-1}
\end{tikzcd}$$

which comes from a prism



in $\mathbf{B}G$. The 2-cocycle $\tau_{\mathbf{B}\mathbb{Z}}\alpha$ evidently sends this to the evaluation of α on a 3-morphism in the cover Y filling this prism. One representation of such a 3-morphism, going from the back and rear to the top and front of this prism, is



$$\xrightarrow{(g,h,(hg)x(hg)^{-1})} \begin{tikzcd} & \bullet \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur, "g"] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[dr, phantom, "\Downarrow"] & \bullet \\ & \bullet \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[uu, "x"] \arrow[uuur] \arrow[uurr, bend left] \arrow[ur, "g"] \arrow[rr, bend left=70, "hg"] \arrow[rr, "hg"'] & {} & \bullet \arrow[uu, "(hg)x(hg)^{-1}"'] \end{tikzcd} \xrightarrow{=} \begin{tikzcd} & \bullet \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur, "g"] \arrow[d, phantom, "\Downarrow"] & {} & \bullet \\ \bullet \arrow[u, "x"] \arrow[uur] \arrow[urr, bend left] \arrow[rr, "hg"'] & & \bullet \arrow[u, "(hg)x(hg)^{-1}"'] \end{tikzcd}$$

$$\xrightarrow{(x,g,h)} \begin{tikzcd} & \bullet \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[ur, "g"] \arrow[rr, "hg"] \arrow[dr, phantom, "\Downarrow"] & {} & \bullet \\ & {} \arrow[d, phantom, "\Downarrow"] & \\ \bullet \arrow[uu, "x"] \arrow[uurr, "hgx"] \arrow[rr, "hg"'] & {} & \bullet \arrow[uu, "(hg)x(hg)^{-1}"'] \end{tikzcd} .$$

This manifestly yields the cocycle as claimed.

9.6.1. The Drinfeld double modular tensor category from DW bibranes

Let again $\rho : \mathbf{B}^2U(1) \to 2\mathrm{Vect}$ be the representation of $\mathbf{B}U(1)$ from section 9.3. and let $\tau_{\mathbf{B}\mathbb{Z}}\alpha : \Lambda G \to \mathbf{B}^2U(1)$ be the 2-cocycle obtained in section 9.6. from transgression of a Dijkgraaf-Witten line 3-bundle on $\mathbf{B}G$ and consider the the ρ-associated 2-vector bundle $\rho \circ \tau_{\mathbf{B}\mathbb{Z}}\alpha$ corresponding to that. Its sections according to definition 7..8 form a category $\Gamma(\tau_{\mathbf{B}\mathbb{Z}}\alpha)$.

Corollary 9..2 *The category $\Gamma(\tau_{\mathbf{B}\mathbb{Z}}\alpha)$ is canonically isomorphic to the representation category of the α-twisted Drinfeld double of G.*

Proof. Follows by inspection of our definition of sections applied to this case and using the relation established in 9.6. between nonabelian cocycles and the ordinary appearance of the Drinfeld double in the literature.

In the case that α is trivial, the representation category of the twisted Drinfeld double is well known to be a modular tensor category. We show in the next section how the fusion tensor product on this category is reproduced from a monoid of bibranes on ΛG.

9.6.2. Fusion tensor product from fusion of bibranes on 2-groups

Consider any 2-group $\mathbf{B}G_2 := (G \ltimes H \overset{p_1}{\underset{(\mathrm{Id}\cdot\delta)}{\rightrightarrows}} G \longrightarrow \mathrm{pt})$.

Pullback to the single object of $\mathbf{BE}\mathbb{Z}$ yields a canonical morphism from the *disk-space* $DG_2 := \hom(\mathbf{BE}\mathbb{Z}, \mathbf{B}G_2)$ to $\mathbf{B}G$, $p : DG_2 \to \mathbf{B}G$ which inherits from the 2-group the structure of a category internal to groupoids in that on the span

$$\begin{tikzcd} & DG_2 \arrow[dl, "p"'] \arrow[dr, "p"] & \\ \mathbf{B}G & & \mathbf{B}G \end{tikzcd}$$

there is induced the structure of a monad from the horizontal composition in G_2. Notice that DG_2 is very similar to but in general slightly different from the action groupoid $H//G$ obtained from the canonical action of G on H in a 2-group. Both coincide in the special case that $G_2 = \mathbf{E}G$, so that $H = G$. In this case the morphism p exhibits DG_2 as the action groupoid (as in section 9.1.) of G acting on itself by the adjoint action.

For $\mathbf{B}G \to 2\mathrm{Vect}$ the trivial gerbe, the transformations

$$\begin{tikzcd} & DG_2 \arrow[dl] \arrow[dr] & \\ \mathbf{B}G \arrow[rr, Rightarrow] \arrow[dr] & & \mathbf{B}G \arrow[dl] \\ & 2\mathrm{Vect} & \end{tikzcd}$$

are representations of DG_2 on vector spaces. In [7] representations of $H//G$ were considered and shown to always form pre-modular tensor categories except in the case $G_2 = \mathbf{E}G$, in which case $H//G = \Lambda G$ is again the loop groupoid from section 9.6. whose representations actually form a modular tensor category. Moreover, in precisely this case also $DG_2 = \Lambda G$, so that, by the above, bibranes on DG_2 become representations of ΛG.

Direct check shows that the fusion product of bibranes using the internal category structure on DG_2 from 8..5 reproduces the familiar fusion tensor product on representations of ΛG, hence of the Drinfeld double.

10. Conclusion

We argued that symmetries assembled into categories and higher analogues help systematic and uniform treatment of many phenomena in noncommutative geometry, geometry and physics. Emphasis has been put on monoidal categories acting on categories of sheaves in NC geometry; and on higher cocycle for ω-categories where the latter may be very generally defined in terms of abstract homotopy theory of ω-categories. We sketched here generalized notions of backgrounds and (bi)branes for σ-models. We currently study nonabelian cocycles, sections and bibranes in a more satisfactory, while more general, context of enriched homotopical categories with path objects.

Let us list some related topics not touched here. Some σ-models and couplings can be defined using infinitesimal versions of gauge n-groupoids. E.g. a remarkable AKSZ construction [1] utilizes essentially Lie algebroids as gauge "Lie algebras". The relation between higher groupoids and L_∞-

algebroids (particularly "integration") is an active area of research (cf. its role in our context in [36]).

With actions of higher groups, notions of equivariance for categorified objects (e.g. gerbes) under usual or higher groups need some treatment. The first author has studied Z_2-equivariant gerbes as an expression of so-called Yandl structures in CFT; and the second author studied 2-equivariant object in 2-fibered categories (presented at WAGP06, Vienna 2006; the basic definition is sketched in [45]).

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References

[1] M. Alexandrov, A. Schwarz, O. Zaboronsky, M. Kontsevich, *The geometry of the master equation and TQFT*, Int. J. Mod Phys. A **12**, 7 (1997), 1405–1429.

[2] J. Baez and A. Lauda, *Higher-dimensional algebra V: 2-groups*, Theory Appl. Categ. **12** (2004), 423–491, [`arXiv:math.QA/0307200`].

[3] J. Baez and M. Shulman, *Lectures on n-categories and cohomology*, [`arXiv:math.CT/0608420`].

[4] J. Baez, A. Crans, *Higher-dimensional algebra VI: Lie 2-algebras*, Theory Appl. Categ. **12** (2004), 492–528, [`arXiv:math.QA/0307263`].

[5] I. Baković, *Bigroupoid 2-bundles*, LMU thesis, 2008; *The simplicial interpretation of bigroupoid 2-torsors*, [`arXiv:0902.3436`].

[6] T. Bartels, *2-Bundles*, [`arXiv:math/0410328`] [math.CT].

[7] P. Bantay, *Characters of crossed modules and premodular categories* [`arXiv:math.QA/0512542`].

[8] G. Böhm, K. Szlachányi, *Hopf algebroid symmetry of abstract Frobenius extensions of depth 2*, Comm. Alg. **32**, 11, 4433–4464 (2004)

[9] L. Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque **225** (1994); J. Giraud, *Cohomologie non abélienne*, Grundl. Math. Wiss. 179, Springer 1971; T. Beke, *Higher Čech theory*, *K*-Theory **32** 4 (2004); 293–322; I.Moerdijk, *Introduction to the language of stacks and gerbes*, [`arXiv:math.AT/0212266`].

[10] R. Dijkgraaf, V. Pasquier, P. Roche, *Quasi Hopf algebras, group cohomology and orbifold models*, Nucl. Phys. B Proc. Suppl. **18B** 60–72 (1991); S.Majid, *Quantum double for quasiHopf algebras*, Lett. Math. Phys. 45 (1998) 1–9; P. Bressler, *Levels and characters*, lecture at NCG III, Mittag-Leffler May 19, 2004; S. Willerton, *The twisted Drinfeld double of a finite group via gerbes and finite groupoids* [`arXiv:math/0503266`]; V.Hinich, *Drinfeld double for orbifolds*, [`arXiv:math.QA/0511476`].

[11] R. Brown, P. Higgins, R. Sivera, *Nonabelian algebraic topology*, book draft.

[12] R. Brown, M. Golasiński, *A model structure on the homotopy theory of crossed complexes*, Cahier Top. Géom. Diff. Catég. **30** (1) (1989) 61–82

[13] R. Brown, *Abstract homotopy Theory and generalized sheaf cohomology*, Trans. Am. Math. Soc. **186** (1973), 419–458

[14] A. Connes, *Noncommutative geometry*, Academic Press, San Diego 1994.

[15] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214, [`arXiv:math/0412149`] [math.QA].

[16] D. Freed, *Higher Algebraic Structures and Quantization*, Comm. Math. Phys. **159** (1994) 343–398 [`arXiv:hep-th/9212115`].

[17] D. Freed, *Chern-Simons theory with finite gauge group*, Comm. Math. Phys. **156** (1993) 435–472 [`arXiv:hep-th/9111004`]

[18] J. Fuchs, C. Schweigert, K. Waldorf, *Bi-branes: Target space geometry for world sheet topological defects*, J. Geom. Phys. **58** (2008) 576–598 [`arXiv:hep-th/0703145`]

[19] A. Grothendieck et al., *Revêtements étales et groupe fondamental*, Séminaire de Géometrie Algébrique du Bois Marie 1960–1961 (SGA 1), LNM **224**, Springer (1971), retyped as: [`arXiv:math.AG/0206203`].

[20] L. K. Hadjiivanov, R. R. Paunov, I. T. Todorov, $\mathfrak{U}_q$*–covariant oscillators and vertex operators*, J. Math. Phys. **33** (4), pp. 1379–1394, April 1992.

[21] A. M. Kapustin, D. O. Orlov, *Lectures on mirror symmetry, derived categories and D-branes*, Uspehi Mat. Nauk **59**, 5(2004), 101–134 (Rus. Math. Surveys **59**, 5(2004), 907–940), [`arXiv:math.AG/0308173`]; *Remarks on A-branes, mirror symmetry, and the Fukaya category*, J. Geom. Phys. **48**:84 (2003) [`arXiv:hep-th/0109098`].

[22] Y. Lafont, F. Metayer, and K. Worytkiewicz, *A folk model structure on omega-cat*, [`arXiv:0712.0617`] [math.CT]; F. Metayer, *Cofibrant complexes are free*, [`arXiv:math.CT/0701746`].

[23] N. P. Landsman, *Mathematical topics between classical and quantum mechanics*, Springer, New York 1998. xx+529 pp.

[24] T. Leinster, *Higher categories, higher operads*, London Math. Soc. LNS **298**, Cambridge Univ. Press 2004, [`arXiv:math.CT/0305049`].

[25] M. Makkai, *Avoiding the axiom of choice in general category theory*, J. Pure Appl. Algebra **108** (1996), no. 2, 109–173, [`http://www.math.mcgill.ca/makkai/anafun`].

[26] L. Katzarkov, M. Kontsevich, T. Pantev, *Hodge theoretic aspects of mirror symmetry*, [`arXiv:0806.0107`].

[27] M. Kontsevich, *Homological algebra of mirror symmetry*, Proc. ICM Zurich 1994, Vol. 1, 2, 120–139, Birkhäuser, Basel, 1995.

[28] G. Mack, V. Schomerus, *Quasihopf quantum symmetry in quantum theory*, Nucl. Phys. B370 (1992) 185–230.

[29] S. Majid, *Foundations of quantum group theory*, CUP 1995, 2000.

[30] L. Mauri, M. Tierney, *Two descent, two-torsors and local equivalence*, J. Pure Appl. Algebra **143**, 1-3 (1999), 313–327.

[31] D. O. Orlov, *Derived categories of coherent sheaves, and motives.* Uspehi M. N. **60**, 6 (2005), no. 6(366), 231–232 (Russ. Math. Surveys **60**, 6 (2005), 1242–1244).

[32] D. Roberts, U. Schreiber, *The inner automorphism 3-group of a strict 2-group*, J. Homotopy Relat. Struct. **3** (2008) no. 1, 193–244, [`arXiv:0708.1741`] [math.CT].

[33] A. L. Rosenberg, *Noncommutative schemes*, Comp. Math. **112** (1998), 93–125;

[34] A. L. Rosenberg, *Topics in noncommutative algebraic geometry, homological algebra and K-theory*, preprint MPIM2008-57, at `http://www.mpim-bonn.mpg.de`

[35] H. Sati, U. Schreiber and J. Stasheff, *L_∞-connections and applications to String- and Chern-Simons n-transport*, in *Recent Developments in QFT*, eds. B. Fauser et al., Birkhäuser, Basel (2008), [`arXiv:0801.3480`] [math.DG].

[36] H. Sati, U. Schreiber, Z. Škoda, D. Stevenson, *Twisted differential nonabelian cohomology: Twisted (n-1)-brane n-bundles and their Chern-Simons (n+1)-bundles with characteristic (n+2)-classes*, [`http://www.math.uni-hamburg.de/home/schreiber/nactwist.pdf`]

[37] H. Sazdjian, Y. S. Stanev, I. T. Todorov, *$SU(3)$-coherent state operators and invariant correlation functions and their quantum group counterparts*, J. Math. Phys., **36** (1995), 2030–2052.

[38] U. Schreiber, *AQFT from n-extended FQFT*, [`arXiv:0806.1079`]

[39] U. Schreiber *Quantum 2-states: Sections of 2-vector bundles*, talk at *Higher categories and their applications*, Fields Institute, Jan. 2007, [`http://www.math.uni-hamburg.de/home/schreiber/atd.pdf`]

[40] U. Schreiber, K. Waldorf, *Parallel transport and functors*, [`arXiv:0705.0452`]; sequel *Smooth functors vs. differential forms*, [`arXiv:0802.0663`].

[41] U. Schreiber and K. Waldorf, *Connections on nonabelian gerbes and their holonomy*, [`arXiv:0808.1923`] [math.DG].

[42] Z. Škoda, *Localizations for construction of quantum coset spaces*, Banach Center Publications **61** (2003), 265–298 [`math.QA/0301090`].

[43] Z. Škoda, *Coherent states for Hopf algebras*, Letters in Mathematical Physics **81**, 1, (2007) 1–17 [`math.QA/0303357`].

[44] Z. Škoda, *Noncommutative localization in noncommutative geometry*, London Math. Soc. Lecture Note Series 330, ed. A. Ranicki; pp. 220–313, Cambridge Univ. Press, Cambridge, 2006, [`math.QA/0403276`].

[45] Z. Škoda, *Some equivariant constructions in noncommutative algebraic geometry*, Georgian Math. J. (accepted) [`arXiv:0811.4770`].

[46] R. Street, *The algebra of oriented simplexes*, J. Pure Appl. Alg. **49**, 3 (1987) 283–335; R. Street, *Categorical and combinatorial aspects of descent theory*, Appl. Categ. Structures **12** (2004), no. 5–6, 537–576, [`arXiv:math.CT/0303175`].

[47] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, 1–104, Math. Surveys Monogr. 123, Amer. Math. Soc. Providence, RI, 2005, [`arXiv:math.AG/0412512`].

[48] C. Wockel, *A global perspective to gerbes and their gauge stacks*, [`arXiv:0803.3692`]