The $\mathfrak{psu}(1,1|2)$ Spin Chain of $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

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Abstract

Persuasive evidence indicates that planar $\mathcal{N}=4$ supersymmetric Yang-Mills theory is integrable, which implies that the superconformal symmetry of $\mathcal{N}=4$ SYM is enhanced by infinitely many conserved charges. Importantly, integrability has enabled recent impressive tests of the AdS/CFT gauge/string duality. Seeking a more complete understanding of gauge theory integrability, in this dissertation we use spin chain methods for perturbative $\mathcal{N}=4$ SYM to study intricate symmetries related to integrability.

We restrict our analysis to the $\mathfrak{psu}(1,1|2)$ sector. Although it is conceptually simpler than the full theory, this sector retains essential features. We construct a novel infinite-dimensional symmetry from nonlocal products of spin chain symmetry generators. This symmetry explains a very large degeneracy of the spectrum of anomalous dimensions. An investigation of next-to-leading order quantum corrections follows. Using only constraints from superconformal invariance and from basic properties of Feynman diagrams, we find simple iterative expressions for the next-to-leading order symmetry generators, including the two-loop dilatation generator of the $\mathfrak{psu}(1,1|2)$ sector. This solution's spectrum provides strong evidence for two-loop integrability. Finally, we prove two-loop integrability for the $\mathfrak{su}(2|1)$ subsector by constructing the next-to-leading order $\mathfrak{su}(2|1)$ Yangian symmetry. The corrections to the Yangian generators are built naturally out of the quantum corrections to superconformal symmetry generators, which suggests that the construction can be generalized to higher loops and larger sectors.

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Contents

Introduction

The Standard Model of particle physics describes many aspects of nature with impressive precision. In practice, this success relies largely on the applicability of Feynman diagram perturbation theory, which in turn depends on two key properties of the Standard Model. First, the coupling constant for the electroweak force is small due to the small fine-structure constant. Second, for the strong force, described by the nonabelian $SU(3)$ gauge theory, asymptotic freedom insures the weakening of the effective coupling at short distances or for high-momentum interactions. On the other hand, for strong coupling, successful direct QCD calculations are notoriously limited. Instead, there is diverse support for a qualitative picture of string-like flux tubes connecting confined quarks. For instance, lattice calculations reveal that the potential energy required to separate a quark-antiquark pair increases linearly with distance. This harmonizes with accelerator string-like observations: the mass squared of the lightest hadron for a given spin depends linearly on the spin.

A more precise relationship between gauge theories and string theories is suggested by a famous result of 't Hooft [1]. For a nonabelian gauge theory with N colors, one can obtain a genus expansion by interpreting Feynman diagrams as two-dimensional surfaces. This expansion proceeds in powers of $1/N$, and is of the same form as the genus expansions characteristic of string theory. This suggestion of gauge/string duality is realized through the the AdS/CFT correspondence [2]. In the most symmetric case, this states that

$\mathcal{N}=4$ supersymmetric Yang-Mills Theory is dual to type IIB string theory on $AdS_5 \times S^5$. Here the string coupling constant is given by $\lambda/(4\pi N)$, where the 't Hooft coupling constant is $\lambda = g_{\rm YM}^2 N$. Additionally, the "coupling constants" at fixed genus are related as $\sqrt{\lambda} = R^2/\alpha'$, where R is the radius of S^5 and $1/\alpha'$ gives the string tension. It follows that for fixed genus, strong coupling in one theory is equivalent to the weak coupling in the dual theory. While the AdS/CFT correspondence provides a breakthrough in our understanding of gauge (and string) theories, its usefulness is limited still by the challenge of computing beyond the leading orders in perturbation theory on either side of the duality. Developments in recent years have revealed special properties of AdS/CFT that hold promise for overcoming this challenge.

Before discussing these recent developments, let us examine the gauge theory in more detail. $\mathcal{N}=4$ supersymmetric Yang-Mills theory [3] describes interactions among a gauge field, six scalars, and their superpartners, four Dirac fermions. Unlike QCD, all of the fields transform in the adjoint of the gauge group. It is exactly scale invariant, including quantum corrections [4]. This scale invariance is part of a larger conformal symmetry, which combines with supersymmetry to form the full superconformal symmetry of $\mathcal{N}=4$ SYM, corresponding to the group PSU$(2,2|4)$. This superconformal symmetry is very constraining. An especially important consequence for this work is that, up to normalization, the two-point functions of local operators are fixed by a single quantity: the anomalous dimensions of the local operators[1]. Anomalous dimensions will be a major focus of this dissertation, and, as we will see, despite the constraints from superconformal symmetry computing anomalous dimensions is generically very nontrivial.

There is large experimental and theoretical motivation to study anomalous dimensions

[1] While the coupling constant is not renormalized in $\mathcal{N}=4$ SYM, local operators are renormalized.

in gauge theories. Anomalous dimensions are central to computations for deep inelastic scattering experiments, which played a key role in establishing QCD as the theory of the strong force. At leading order, deep inelastic scattering of electrons and hadrons behaves according to the parton model. Constituent quarks carry a definite fraction of the momentum of the scattered hadron. However, this Bjorken scaling receives corrections that depend logarithmically on the momentum scale of the scattering. The logarithmic corrections are controlled by the anomalous dimensions of the operators that appear in the operator product expansion of quark electromagnetic currents [5]. Anomalous dimensions are also especially important in a (super)conformal theory such as $\mathcal{N} = 4$ SYM. As mentioned above, up to normalization, the two-point function of an operator is fixed by its anomalous dimension. Since a conformal field theory is completely described by the correlation functions of its local operators, the set of anomalous dimensions is the natural first set of quantities used to describe the theory; the anomalous dimensions of local operators are the spectrum of the theory. The AdS/CFT correspondence makes it even more natural to view the anomalous dimensions as the spectrum. They are mapped by the duality to energies of quantum string states on $AdS_5 \times S^5$.

The recent tremendous progress on anomalous dimensions began with Berenstein, Maldacena and Nastase's study of strings rotating rapidly in a circle within the S^5 of $AdS_5 \times S^5$ [6]. Conveniently, the string theory spectrum is exactly solvable for the angular momentum $J \to \infty$ [7]. This yields a simple AdS/CFT prediction for the strong-coupling limit of anomalous dimensions of gauge theory operators with a large R charge[2]. Also, Gubser, Klebanov, and Polyakov observed that more general string states with large quantum numbers (solitons) could be studied with semiclassical methods [8], giving further predictions

[2] R symmetry corresponds to a SU(4) subgroup of the superconformal group.

for strong-coupling gauge theory. Many studies of quantum corrections to the spectrum of anomalous dimensions (string energies) followed, including planar [9] and nonplanar corrections [10]. For reviews see [11, 12].

This new interest in anomalous dimensions of $\mathcal{N} = 4$ SYM led to the discovery of the integrability of the planar theory. Minahan and Zarembo [13] found that the one-loop planar dilatation generator for the $SO(6)$ sector of scalar fields was isomorphic to the Hamiltonian of an integrable spin chain. The dilatation generator acts on local operators, and its eigenvalues are the anomalous dimensions. The isomorphism works as follows. In the planar limit, the dilatation generator does not mix fields inside different traces, so we can focus on the single-trace operators. Because of the cyclicity of the trace, we can view single-trace operators of the $SO(6)$ sector as cyclic one-dimensional lattices or chains, with each site of the chain occupied by a $SO(6)$ "spin" corresponding to the 6 possible scalars. Then, the dilatation generator mixes the different spin chain states. Since we are interested in its eigenvalues, the anomalous dimensions, it is natural to think of it as the Hamiltonian[3]. An important aspect of integrability is that there are infinitely many charges that commute with the Hamiltonian and with each other. Given the tight constraints of supersymmetry, it is perhaps not surprising that the complete one-loop dilatation operator [14] is also integrable [15], giving a novel $\mathfrak{psu}(2,2|4)$ spin chain.

It should be noted that integrability had been discovered previously in sectors of one-loop planar QCD both for scattering [16] and anomalous dimensions calculations [17]. However, in those cases, as well as for the complete one-loop dilatation generator of $\mathcal{N} = 4$ SYM, the Hamiltonian only contains nearest neighbor interactions. At higher loops, the dilatation generator instead acts on an increasing number of adjacent sites. Since integrable systems

[3] We will therefore use "anomalous part of the dilatation generator" and "Hamiltonian" interchangeably.

typically only involve nearest neighbor or pairwise interactions, it is remarkable that the integrability is preserved by the higher order corrections. The first evidence for this was obtained using a very useful property of the gauge theory. $\mathcal{N} = 4$ SYM has sectors, or sets of operators that are closed to all orders in perturbation theory under the action of the dilatation generator. Furthermore, some of these sectors are compact: They are composed of only finitely many types of fields. Compactness enabled calculations of the three-loop dilatation generator for the $\mathfrak{su}(2)$ and $\mathfrak{su}(2|3)$ sectors [18, 19], which provided evidence of perturbative integrability.

A tremendously valuable consequence of integrability is that the spectrum can be obtained by solving a set of algebraic equations, the Bethe equations (for an introduction see [20]). These equations reflect simple scattering of spin chain excitations such that the spin chain S-matrix for multiple particle scattering factorizes into products of two-particle scattering. The one-loop Bethe equations were obtained simultaneously with the observation of integrability [13, 15]. As emphasized in [21], assuming integrability, the quickest route to the higher loop corrections to the anomalous dimensions is through the two-particle S-matrix and the Bethe equations that directly follow. This bypasses the seemingly intractable problem of computing the dilatation generator beyond low orders in compact sectors. Following previous results for higher loop corrections to the Bethe equations for various sectors [22, 21], an all-loop asymptotic Bethe ansatz for the full gauge theory was found in [23].

The Bethe equations make possible very impressive tests of the AdS/CFT correspondence, finally overcoming the challenge of comparing one theory at weak coupling with the other at strong coupling. Developments on integrability in the string theory proceeded in parallel to the progress on the gauge theory integrability. Classical string theory in

$AdS_5 \times S^5$ was shown to be integrable [24, 25], integrability was used to solve the classical spectrum in terms of algebraic curves [26], and Bethe ansätze for quantum strings were proposed [27, 21, 23]. In fact, because the Bethe equations follow from the two-particle S-matrix, Beisert was able to prove that (asssuming integrability) supersymmetry fixes the gauge/string Bethe equations up to a phase function [28]. Furthermore, requiring crossing symmetry, which should be a consequence of two-dimensional Lorentz invariance of the string world sheet[4], leads to another condition on this phase [29]. This condition has been solved [30, 31] using an ansatz introduced earlier in [27, 32]. Including this solution for the phase function[5], the AdS/CFT Bethe equations interpolate between the gauge and string theory consistently with all known perturbative field theory and string theory calculations, including recent impressive tests at four loops in the gauge theory [33] and at one loop in the string theory [34].

It should be emphasized that beyond leading order on both sides of the duality, integrability is only an assumption. It has not been tested beyond four loops in the gauge theory or beyond one loop in the string theory (the successful interpolation between the theories is very strong evidence though). The lack of proof of integrability provides one motivation for continuing to look at the perturbative corrections to the dilatation generator, despite the great simplicity and success of the Bethe ansatz approach. In this work, we will focus on the dilatation generator and other local and nonlocal spin chain symmetry generators. We will find multiple benefits to this research path. In addition to laying the groundwork for proving perturbative integrability of the spin chain, we will uncover the spin chain origin of

[4]Much recent work on strings in $AdS_5 \times S^5$ has used light-cone quantization, which is inconsistent with manifest Lorentz invariance of the world sheet. However, it is very possible that a different quantization method would preserve manifest Lorentz invariance, at the expense of introducing ghosts.

[5]The crossing symmetry equation has many solutions, and additional insights from the gauge and string theory were used to identify what appears to be "the solution."

a previously unexplained Bethe ansatz symmetry. Furthermore, we will find iterative structure to perturbative corrections. This raises the possibility of escaping the weak-coupling regime through a more complete understanding of the pattern to the corrections.

Our study will be restricted to the $\mathfrak{psu}(1,1|2)$ sector because it has complexity well-balanced between realistic features and simplifications. Features of the full theory shared by this sector include noncompactness (an infinite number of types of fields) and a Hamiltonian that is a nonseparable part of the supersymmetry algebra. However, unlike the full theory, the Hamiltonian preserves the number of spin chain sites. In the full theory there are dynamic interactions [19] that change the length of the spin chain. For instance, the one and one-half loop (complete) dilatation generator has interactions that replace two fermionic fields with three bosonic fields. Still, while dynamic generators are not part of the ordinary $\mathfrak{psu}(1,1|2)$ algebra, the embedding of this sector within $\mathcal{N}=4$ SYM insures that there are four hidden supercharges (forming a $\mathfrak{psu}(1|1)^2$ algebra) that increase or decrease the length of the spin chain by one unit. These hidden supercharges will play a prominent role in much of this work. Most of this work will be restricted to the planar theory, for which integrability is present.

We start in Chapter 1 with a brief review of the origin of the $\mathcal{N}=4$ SYM spin chain, the restriction to sectors of the theory such as the $\mathfrak{psu}(1,1|2)$ sector, and the leading-order $\mathfrak{psu}(1,1|2)$ spin chain representation. The study of the quantum corrections begins in Chapter 2. After reviewing the one-loop Bethe equations for the sector, we use them to describe a very large degeneracy of this sector, first observed in [23]. To explain the degeneracy, we construct the one-half loop hidden supersymmetry generators mentioned above, and use them to build a nonlocal infinite dimensional symmetry that commutes with all of the $\mathfrak{psu}(1,1|2)$ generators.

In Chapter 3, we solve the symmetry constraints for the dilatation generator and other local spin chain symmetry generators up to two loops. Furthermore, we confirm that the anomalous dimensions of this solution are in agreement with rigorous field theory calculations and the predictions of the Bethe ansatz. At two loops, spin chain interactions act on up to three adjacent sites of the spin chain, so this would seem to be an intractable problem. However, we have introduced a new approach to deal with this challenge, generalizing the methods of [19]. We include an auxiliary generator and write the perturbative corrections iteratively in terms of leading order symmetry generators. Using the properties of the auxiliary generator and the leading order symmetry algebra relations, it is then straightforward to prove that the corrections satisfy the symmetry algebra. It should be possible to extend this method to higher orders.

Finally, in Chapter 4 we consider the problem of proving integrability. In typical integrable systems, with a Hamiltonian built from nearest neighbor interactions, there is also a R-matrix. Roughly, the R-matrix describes scattering between adjacent spin chain excitations. The R-matrix is a function of a spectral parameter u. The expansion around $u = 0$ yields an infinite family of commuting local charges, including the Hamiltonian. The expansion around $u = \infty$ instead yields an infinite-dimensional associative Hopf algebra called a Yangian [35]. At leading order, [36, 37] applied this formalism to construct the Yangian symmetry. However, because of the long-ranged interactions that appear at higher orders, it is a priori unclear how to apply the R-matrix formalism beyond leading order. As we show in Chapter 4, there is a natural way to construct the Yangian symmetry using the perturbative corrections to the local symmetry generators. Because of the complexity of the Yangian symmetry, we restrict to the compact $\mathfrak{su}(2|1)$ subsector of the $\mathfrak{psu}(1,1|2)$ sector. By constructing the two-loop Yangian symmetry, we show both that this sector is integrable at two loops and lay the foundation for generalizations to higher orders and larger sectors.

The Conclusions include a discussion of directions for further research. A number of central but very technical topics are included in the appendices. Appendix A discusses an oscillator realization of the classical superconformal symmetry generators and their leading quantum corrections. Appendix B includes interesting and useful multilinear invariants of the $\mathfrak{psu}(1,1|2)$ algebra, and Appendix C gives a proof that the nonlocal symmetry described in Chapter 2 commutes with the local $\mathfrak{psu}(1,1|2)$ symmetry (at leading order). The last two appendices, D and E, include details of the proof that the perturbative corrections obtained in Chapter 3 satisfy the symmetry algebra.

This thesis is based largely on published works [38, 39] and work with Niklas Beisert that will appear [40].

Chapter 1

The $\mathcal{N} = 4$ super Yang-Mills spin chain and the $\mathfrak{psu}(1,1|2)$ sector

In this chapter we describe the path from $\mathcal{N} = 4$ supersymmetric Yang-Mills theory to a $\mathfrak{psu}(1,1|2)$ spin chain. We present important properties of the spin chain and of the spin chain symmetry generators.

We begin with a review of $\mathcal{N} = 4$ SYM and its superconformal symmetry, and in Section 1.2 we discuss anomalous dimensions and two-point functions, as well as the dilatation generator. This leads us to a spin chain description for local operators, and we explain field theoretic constraints on the spin chain symmetry generators in Section 1.3. We present the $\mathfrak{psu}(1,1|2)$ sector in Section 1.4, and its subsectors in the final section. In between, we discuss the leading order representation for the $\mathfrak{psu}(1,1|2)$ generators (Section 1.5), an external $\mathfrak{su}(2)$ automorphism (Section 1.6), and a hidden $\mathfrak{psu}(1|1)^2$ symmetry (Section 1.7).

1.1 $\mathcal{N} = 4$ SYM and superconformal symmetry

We start by introducing the fields and Lagrangian for $\mathcal{N} = 4$ supersymmetric Yang-Mills Theory [3]. The fields include the gauge field $\mathcal{A}_\mu$, six scalars $\Phi^{ab} = -\Phi^{ba}$, and Weyl fermions and conjugate Weyl fermions, $\Psi_{\alpha b}$ and $\dot{\Psi}^b_{\dot{\alpha}}$. The Greek dotted and undotted indices take values $1,2$ and correspond to $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$ Lorentz symmetry. Latin indices take values $1,2,3,4$, and correspond to an internal $\mathfrak{su}(4)$ $\mathfrak{R}$ symmetry. All of the fields take values in the adjoint of the gauge group, which for this work will be $\mathrm{U}(N)$. The covariant derivative $\mathcal{D}_{\dot\alpha\beta}$ acts on any field X as

$$\mathcal{D}_{\dot\alpha\beta}X = \tfrac{1}{2}\sigma^{\mu}_{\dot\alpha\beta}\left(\partial_\mu X - ig\left[\mathcal{A}_\mu, X\right]\right). \tag{1.1}$$

The Pauli matrices σ^μ transform between spinor and vector spacetime indices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

As usual, the field strength is the commutator of covariant derivatives,

$$\mathcal{F}_{\mu\nu} = ig^{-1}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig[\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{1.3}$$

In order to write the Lagrangian in a more unified notation, we write the field strength in terms of two bispinors $\mathcal{F}_{\alpha\beta}$ and $\mathcal{F}_{\dot\alpha\dot\beta}$,

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \frac{1}{2\sqrt{2}}\sigma_\mu^{\alpha\dot\gamma}\varepsilon_{\dot\gamma\dot\delta}\sigma_\nu^{\beta\dot\delta}\mathcal{F}_{\alpha\beta} + \frac{1}{2\sqrt{2}}\sigma_\mu^{\gamma\dot\alpha}\varepsilon_{\gamma\delta}\sigma_\nu^{\delta\dot\beta}\dot{\mathcal{F}}_{\dot\alpha\dot\beta}, \\ \mathcal{F}_{\alpha\beta} &= \frac{1}{2\sqrt{2}}\sigma^\mu_{\alpha\dot\gamma}\varepsilon^{\dot\gamma\dot\delta}\sigma^\nu_{\beta\dot\delta}\mathcal{F}_{\mu\nu}, \quad \dot{\mathcal{F}}_{\dot\alpha\dot\beta} = \frac{1}{2\sqrt{2}}\sigma^\mu_{\gamma\dot\alpha}\varepsilon^{\gamma\delta}\sigma^\nu_{\delta\dot\beta}\mathcal{F}_{\mu\nu}. \end{aligned} \tag{1.4}$$

The ε-tensors are $\mathfrak{su}(2)$-invariant and antisymmetric with $\varepsilon^{12} = \varepsilon_{12} = 1$, and they are used to raise or lower indices. According to our convention, the first index of the ε tensors is the summation variable for raising or lowering. Note that there are 16 bosonic and 16 fermionic physical degrees of freedom, with this equality guaranteed by supersymmetry.

To simplify the presentation of symmetries, we write the Lagrangian completely in bispinor notation

$$\mathcal{L} = \mathrm{Tr}\left\{ \frac{1}{4}(\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta} + \dot{\mathcal{F}}^{\dot\alpha\dot\beta}\dot{\mathcal{F}}_{\dot\alpha\dot\beta}) + \frac{1}{4}\mathcal{D}^{\dot\alpha\beta}\Phi^{ab}\mathcal{D}_{\dot\alpha\beta}\Phi_{ab} + \dot\Psi^{a}_{\dot\alpha}\mathcal{D}^{\dot\alpha\beta}\Psi_{\beta a} \right.$$
$$\left. -\frac{1}{4}g^2\,[\Phi^{ab},\Phi^{cd}][\Phi_{ab},\Phi_{cd}] + g\,\Psi^{\alpha}_{a}[\Phi^{ab},\Psi_{\alpha b}] + g\,\dot\Psi^{\dot\alpha a}[\Phi_{ab},\dot\Psi^{b}_{\dot\alpha}] \right\}. \tag{1.5}$$

To obtain the scalars Φ^{ab} with raised indices, we use the $\mathfrak{su}(4)$-invariant antisymmetric tensor ε^{abcd},

$$\Phi^{ab} = \tfrac{1}{2}\varepsilon^{abcd}\Phi_{cd}. \tag{1.6}$$

Now we turn to the symmetries of the Lagrangian, which are the basis for this work. It is not difficult to see that the $\mathcal{N}=4$ SYM action is classically conformally invariant. Recalling that the gauge field and scalars have scaling dimension one and the fermions have dimension 3/2, we observe that g is dimensionless. Integrating $\mathcal{L}$ over 4-dimensional spacetime thus yields a scale-invariant action. Combined with the Poincaré symmetry, this scale invariance extends to classical conformal symmetry. Classically conformal theories, such as QCD with massless quarks, typically have conformal symmetry broken by a nonzero beta function. Remarkably, $\mathcal{N}=4$ SYM has a vanishing beta function, and is conformal at the quantum level [4]. Supersymmetry is essential for the survival of conformal symmetry, and supersymmetry combines with the conformal symmetry to form the larger superconformal symmetry of $\mathcal{N}=4$ SYM.

Much of this work will entail studying how quantum corrections realize superconformal symmetry, so we will spend the rest of this section discussing this symmetry. We will use the local properties of the $\mathcal{N}=4$ superconformal group, so we focus on $\mathfrak{psu}(2,2|4)$, its Lie algebra. First we discuss the elements of the algebra and their commutation relations. The algebra includes generators corresponding to the manifest symmetries of the Lagrangian given above, Lorentz generators $\mathfrak{L}^{\alpha}_{\beta}$ and $\dot{\mathfrak{L}}^{\dot\alpha}_{\dot\beta}$, and $\mathfrak{R}$ symmetry generators $\mathfrak{R}^{a}_{b}$. The remaining bosonic generators are translations $\mathfrak{P}^{\alpha\dot\beta}$, special conformal transformations or boosts $\mathfrak{K}_{\alpha\dot\beta}$, and the dilatation generator $\mathfrak{D}$. They complete the conformal (sub-)algebra, $\mathfrak{so}(4,2)$. Finally, the supersymmetry generators are $\mathfrak{Q}^{\alpha}_{b}$, $\dot{\mathfrak{Q}}^{\dot\alpha b}$, and their conjugates $\mathfrak{S}^{b}_{\alpha}$, $\dot{\mathfrak{S}}_{\dot\alpha b}$.

In Appendix A.1 we show how this algebra can be simply described using the oscillator formalism. Here we just give a brief sketch of the commutation relations. Under commutation with Lorentz and $\mathfrak{R}$ symmetry generators, the symmetry generators' indices transform canonically. A generator's commutator with the dilatation generator just gives its dimension, $\pm\frac{1}{2}$ for the supercharges and ± 1 for the translations and special conformal transformations. Schematically, the nonvanishing commutators[1] between supercharges is

$$\{\mathfrak{Q},\dot{\mathfrak{Q}}\} \sim \mathfrak{P}, \qquad \{\mathfrak{S},\dot{\mathfrak{S}}\} \sim \mathfrak{K},$$

$$\{\mathfrak{Q},\mathfrak{S}\} \sim \mathfrak{R}+\mathfrak{L}+\mathfrak{D}, \qquad \{\dot{\mathfrak{Q}},\dot{\mathfrak{S}}\} \sim \mathfrak{R}+\dot{\mathfrak{L}}+\mathfrak{D}.$$

These commutators fix those that involve $\mathfrak{P}$ or $\mathfrak{K}$.

Later, we will study how the symmetry generators rotate local operators. The classical superconformal variations suggest how this could work since they move between fields. For instance, the translation variations act as derivatives, as usual. We have, suppressing g-independent coefficients,

$$\mathfrak{P}_{\alpha\dot\beta}\,\Phi_{cd} \sim \mathcal{D}_{\alpha\dot\beta}\Phi_{cd}, \quad \mathfrak{P}_{\alpha\dot\beta}\,\mathcal{D}_{\gamma\dot\delta} \sim g\,\varepsilon_{\alpha\gamma}\,\dot{\mathcal{F}}_{\dot\beta\dot\delta} + g\,\varepsilon_{\dot\beta\dot\delta}\,\mathcal{F}_{\alpha\gamma}. \tag{1.7}$$

Then adding the supersymmetry variations enables us to use the symmetry generators to step between all of the fields. For example,

$$\mathfrak{Q}^{a}_{\beta}\,\Phi_{cd} \sim \delta^{a}_{[c}\,\Psi_{|\beta|d]}, \quad \mathfrak{Q}^{a}_{\beta}\,\Psi_{\gamma d} \sim \delta^{a}_{d}\,\mathcal{F}_{\beta\gamma} + g\,\varepsilon^{aefg}\varepsilon_{\beta\gamma}\,[\Phi_{de},\Phi_{fg}]. \tag{1.8}$$

[1]Note that, for simplicity, throughout this work we call both commutators and anticommutators, commutators; of course, the commutator of two fermionic generators is actually an anticommutator, as reflected by the use of curly brackets.

Note that the $\mathcal{O}(g)$ terms are the ones with ε tensors. The full set of variations for the super Poincaré algebra are reviewed in Chapter 1 of [41], and the remaining variations are given by conjugation. Of course, using these superconformal variations one can check that the Lagrangian (1.5) has this symmetry classically.

This large symmetry makes possible major simplifications in the classification of local operators of $\mathcal{N} = 4$ SYM. Since local operators form highest weight representations of the superconformal algebra, we now explain this type of representation. Such an (irreducible) representation is specified by a highest weight state[2], which is annihilated by all of the raising operators or positive roots ($\mathfrak{K}$, $\mathfrak{S}$, $\dot{\mathfrak{S}}$, $\mathfrak{L}^1_2$, $\dot{\mathfrak{L}}^1_2$, and $\mathfrak{R}^a_b$ for $a < b$) and is an eigenstate of all of the Cartan generators. Since the raising operators lower or preserve the dimension of a state, a highest weight state has the lowest dimension within its representation. The lowering generators or negative roots ($\mathfrak{P}$, $\mathfrak{Q}$, $\dot{\mathfrak{Q}}$, $\mathfrak{L}^2_1$, $\dot{\mathfrak{L}}^2_1$, and $\mathfrak{R}^a_b$ for $a > b$), increase or preserve the dimension and fill out the remaining states of the irreducible representation. The Cartan generators are the diagonal components of the Lorentz and $\mathfrak{R}$ symmetry generators, as well as $\mathfrak{D}$.

An important example of a $\mathfrak{psu}(2,2|4)$ highest weight representation is given by the single physical fields, rotated by the classical ($g = 0$) superconformal variations or generators. The highest weight field is Φ_{34}, and acting with all of the lowering generators yields an infinite-dimensional representation or module, corresponding to the fields (suppressing indices),

$$\mathcal{D}^k\Phi,\ \mathcal{D}^k\Psi,\ \mathcal{D}^k\dot{\Psi},\ \mathcal{D}^k\mathcal{F},\ \mathcal{D}^k\dot{\mathcal{F}}, \tag{1.9}$$

for all nonnegative integers k. $\mathcal{D}^k$ represents a product of covariant indices with symmetrized-traceless indices. Using the oscillator formalism of Appendix A.1, it is straightforward to

[2] We will also call local operators "states" when we think of them as transforming under $\mathfrak{psu}(2,2|4)$

verify that this is an irreducible highest weight module of the classical $\mathfrak{psu}(2,2|4)$ symmetry.

Classically, a single-trace local operator built from n fields decomposes into components in the direct sum of representations that results from the tensor product of n copies of the single-field module. We will be interested in how these representations are deformed by quantum corrections. For this purpose, we now turn to an examination of quantum correction to scaling dimensions.

1.2 Anomalous dimensions and the dilatation generator

Since $\mathcal{N}=4$ SYM is an interacting quantum field theory, local operators' dimensions receive quantum corrections, the anomalous dimensions. As we will see, this is consistent with quantum (super)conformal symmetry. By definition, the dilatation generator has eigenvalues equal to the scaling dimensions. Since it is a nonseparable part of the symmetry algebra, this implies that the $\mathfrak{psu}(2,2|4)$ generators act on local operators in a g-dependent way, but still satisfy the algebra,

$$[\mathfrak{J}^A(g), \mathfrak{J}^B(g)\} = f^{AB}{}_C \mathfrak{J}^C(g), \tag{1.10}$$

where $f^{AB}{}_C$ are the $\mathfrak{psu}(2,2|4)$ structure constants.

Let us discuss what would be involved in a direct field theory computation of anomalous dimensions. This will suggest the usefulness of focusing on the dilatation generator. In a conformal theory, the anomalous dimension, δD of a local operator appears simply in the two-point function. For instance, for a Lorentz and $\mathfrak{R}$ symmetry scalar local operator $\mathcal{O}(x)$,

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{C}{|x_1 - x_2|^{2(D_0+\delta D)}}, \tag{1.11}$$

where D_0 is the classical scaling dimension of $\mathcal{O}$. We briefly recall how this is treated in perturbation theory. When one computes perturbative quantum corrections to a two-point function such as (1.11), one finds divergences that need to be regularized. This leads to the introduction of an energy scale, μ. Generically, to cancel divergences one needs to renormalize operators as

$$\mathcal{O}(x) \mapsto Z(\mu, g)\mathcal{O}(x). \tag{1.12}$$

In fact, canceling divergences leads to

$$Z(\mu, g) = z(g)\mu^{-\delta D(g)}. \tag{1.13}$$

$z(g)$ is scheme-dependent, but $\delta D(g)$ is not. Expanding (1.11) for the renormalized operators, and making the g-dependence explicit, we obtain

$$\langle Z\mathcal{O}(x_1)Z\mathcal{O}(x_2)\rangle = \frac{C(g)(z(g))^2\mu^{-2\,\delta\mathfrak{D}(g)}}{|x_1 - x_2|^{2(D_0+\delta D(g))}} = \frac{C(g)(z(g))^2}{|x_1 - x_2|^{2D_0}}e^{-2\,\delta D(g)\log(\mu|x_1-x_2|)}. \tag{1.14}$$

Typically, $D(g)$, $C(g)$, and $z(g)$, will receive an infinite sequence of perturbative corrections,

$$\delta D(g) = \sum_{m=2}^{\infty}\delta D_m g^m, \quad C(g) = \sum_{m=0}^{\infty} C_m g^m \quad z(g) = \sum_{m=0}^{\infty} z_m g^m \tag{1.15}$$

which can be combined easily with the Taylor series expansion for the exponential function to obtain the full perturbative series for the two-point function. In practice, of course, one rarely computes beyond the first few orders.

We have made an important simplifying assumption. A local operator with definite scaling dimension for $g = 0$ typically does not have definite scaling dimension once perturbative corrections are included. Equivalently, the renormalization factor Z becomes a matrix, and we have

$$\mathcal{O}_i(x) \mapsto \sum_j Z_{ij}(\mu, g)\mathcal{O}_j(x) \tag{1.16}$$

If Z_{ij} is nonzero, $\mathcal{O}_i$ and $\mathcal{O}_j$ must share the same $\mathfrak{R}$ symmetry and Lorentz symmetry quantum numbers, since these symmetries are manifestly preserved during renormalization.

Furthermore, one can choose a regularization scheme such that the $\mathcal{O}_i$ also share the same classical dimension. Once one rotates to a basis in which the matrix Z_{ij} is diagonal, one can than proceed from (1.12), where now $\mathcal{O}$ is a new basis element and Z is the corresponding diagonal component of the (diagonalized) Z-matrix.

To isolate the anomalous dimension from the scheme-dependent part of Z and the normalization of two-point functions, it makes sense to switch focus from two-point functions to the dilatation generator [42, 18]. Substituting state notation for operators,

$$\mathcal{O}_i \mapsto |X_i\rangle \tag{1.17}$$

we see that the dilatation generator acts as a matrix as well,

$$\mathfrak{D}|X_i\rangle = \mathfrak{D}_0|X_i\rangle + \delta\mathfrak{D}|X_i\rangle = D_0|X_i\rangle + \sum_j \delta D_{ij}|X_j\rangle. \tag{1.18}$$

It follows directly from the properties of the Z_{ij} that $\delta\mathfrak{D}$ commutes with the classical dilation generator, $\mathfrak{D}_0$ and the Lorentz and R symmetry generators. Because the dilatation generator is part of the $\mathfrak{psu}(2,2|4)$ algebra, there will be many further constraints following from (1.10). For instance, since we choose a regularization scheme such that the dilatation generator commutes with its classical part, all descendants (after including quantum corrections) have the same anomalous dimension [41], or equivalently

$$[\mathfrak{J}, \delta\mathfrak{D}] = 0, \tag{1.19}$$

for any $\mathfrak{psu}(2,2|4)$ generator $\mathfrak{J}$.

Once we think of the dilatation generator as an (infinite-dimensional) matrix, we need to specify the states upon which it acts. As we will discuss in the next section, we will be interested in single-trace local operators. Any trace of a product including covariant derivatives, field strengths, scalars, or fermions is gauge invariant. Naively, we might think that we could use a basis of all such products. However, this would not be a linearly independent set because the equations of motion and the Bianchi identities relate different elements. In particular, using Bianchi identities we can reduce to products only including symmetrized covariant derivatives, and using the equations of motion we can eliminate all products where indices are contracted between different fields. This leads to a simple basis: the set of operators of the form

$$\mathrm{Tr}(X_1(x)X_2(x)\dots X_n(x)), \quad X_i \in \{\mathcal{D}^k\mathcal{F}_{\alpha\beta}, \mathcal{D}^k\dot{\mathcal{F}}_{\dot{\alpha}\dot{\beta}}, \mathcal{D}^k\Phi_{nm}, \mathcal{D}^k\Psi_{\alpha b}, \mathcal{D}^k\dot{\Psi}^b_{\dot{\alpha}}\} \tag{1.20}$$

where $\mathcal{D}^k$ is short-hand for a product of k-covariant derivatives with symmetrized-traceless indices. We recognize the set of X_i as the single field highest weight representation discussed above (1.9). From now on, we will suppress the argument x, as all fields will always act at the same point, and the value of x is irrelevant to anomalous dimensions and operator mixing.

We close this section by giving a concrete example of the action of the dilation generator, illustrating our notation and a simple example of mixing at $\mathcal{O}(g^2)$ between two states of classical dimension 6,

$$\begin{aligned}
\mathfrak{D}_2|\Phi_{34}\Phi_{34}\Phi_{34}\Psi_{14}\Psi_{14}\rangle &= 8|\Phi_{34}\Phi_{34}\Phi_{34}\Psi_{14}\Psi_{14}\rangle - 4|\Phi_{34}\Phi_{34}\Psi_{14}\Phi_{34}\Psi_{14}\rangle, \\
\mathfrak{D}_2|\Phi_{34}\Phi_{34}\Psi_{14}\Phi_{34}\Psi_{14}\rangle &= -4|\Phi_{34}\Phi_{34}\Phi_{34}\Psi_{14}\Psi_{14}\rangle + 12|\Phi_{34}\Phi_{34}\Psi_{14}\Phi_{34}\Psi_{14}\rangle.
\end{aligned} \tag{1.21}$$

Note that the subscript again denotes the power of g which accompanies this contribution to the dilatation generator. Diagonalizing this results in the states (labeled by their one-loop anomalous dimension)

$$\begin{aligned}
|10+2\sqrt{5}\rangle &\propto |\Phi_{34}\Phi_{34}\Phi_{34}\Psi_{14}\Psi_{14}\rangle - \frac{1+\sqrt{5}}{2}|\Phi_{34}\Phi_{34}\Psi_{14}\Phi_{34}\Psi_{14}\rangle, \\
|10-2\sqrt{5}\rangle &\propto |\Phi_{34}\Phi_{34}\Phi_{34}\Psi_{14}\Psi_{14}\rangle - \frac{1-\sqrt{5}}{2}|\Phi_{34}\Phi_{34}\Psi_{14}\Phi_{34}\Psi_{14}\rangle.
\end{aligned} \tag{1.22}$$

1.3 Basic properties of the spin chain representation

Since we have begun to specify single-trace local operators simply as a sequence of fundamental fields, we can think of this system as a spin chain. Each fundamental field corresponds to a type of "spin", and each "site" of the spin chain is occupied by one of these spins. Since these spins transform under $\mathfrak{psu}(2,2|4)$, this is a $\mathfrak{psu}(2,2|4)$ spin chain. Furthermore, because the fields appear inside a trace, we should identify states cyclically. That is

$$\begin{aligned}|X_1X_2\dots X_m\dots X_n\rangle &\cong (-1)^{X_1(X_2\dots X_n)}|X_2\dots X_m\dots X_nX_1\rangle \\ &\cong (-1)^{(X_1\dots X_m)(X_{m+1}\dots X_n)}|X_{m+1}\dots X_nX_1\dots X_m\rangle, \end{aligned} \tag{1.23}$$

where $(-1)^{AB}$ is -1 when both A and B are fermionic and 1 otherwise.

Alternatively, instead of using cyclic identification, we can think of cyclic states as belonging to the physical subspace of the set of periodic spin chain states. To specify this physical subspace we need to introduce the shift operator $\mathfrak{U}$, which shifts the spins by one site to the right,

$$\mathfrak{U}|X_1X_2\dots X_m\dots X_n\rangle = (-1)^{X_1(X_2\dots X_n)}|X_2\dots X_m\dots X_nX_1\rangle. \tag{1.24}$$

Then the physicality constraint for a periodic spin chain state $|X\rangle$ is simply

$$\mathfrak{U}|X\rangle = |X\rangle. \tag{1.25}$$

Later, we will focus on the $\mathfrak{psu}(2,2|4)$ cyclic spin chain rarely discussing its $\mathcal{N}=4$ SYM origin. However, $\mathcal{N}=4$ SYM imposes further important constraints on the spin chain that we will use throughout the work. We now explain four of these: connectedness, range of interactions, the planar limit, and parity.

Connectedness. Because only connected Feynman diagrams contribute to perturbative computations of the anomalous dimensions, the symmetry generators act on the states through connected interactions. These interactions can be written in terms of fields X and variations of fields $\frac{\delta}{\delta X}$, and connectedness just means that these $\mathfrak{u}(n)$ valued fields and variations can be written as a product inside a single trace. An important exception to this appears at higher orders when interactions can wrap around an entire state. We will discuss these wrapping interactions further in Chapter 3.

Range of interactions. Simple counting of powers of g shows how many sites are involved in a symmetry generator interaction at order g^{2l}, which we will call l loops. First, since propagators connect two fields, we must have

$$3V_3 + 4V_4 + E = 2I, \tag{1.26}$$

where V_n are the number of n-field vertices, E includes the the total number of initial and final fields (or fields and variations of fields), and I is the number of propagators. Also, since propagators introduce momentum integrals while vertices and fields each provide a momentum constraint,

$$L = I - V_3 - V_4 - E + 1, \tag{1.27}$$

where L is the number of momentum integrals or loops (different from l) and we have taken into account overall momentum conservation. Combining these equations, we obtain

$$V_3 + 2V_4 = 2l = E - 2 + 2L. \tag{1.28}$$

We have used that 3-field vertices have coefficient proportional to g and 4-field vertices have coefficients proportional to g^2. From this we infer that at l loops, interactions involve at most $2l+2$ total initial and final sites. For instance, the one-loop dilation generator acts on a total of at most 4 sites and at two loops it acts on a total of at most 6 sites.

Planar limit. Famously, 't Hooft [1] showed that perturbative gauge theory expansions can be organized as genus expansions, with leading terms corresponding to planar (genus 0) Feynman graphs. For this work, we will only need the consequence that taking $N \to \infty$ restricts to interactions involving adjacent sites. To see this, we rescale the coupling constant:

$$g \mapsto \frac{g_{\mathrm{YM}}\sqrt{N}}{4\pi}. \tag{1.29}$$

Combining this with a redefinition of the fields $\{\mathcal{F}, \dot{\mathcal{F}}, \mathit{\Phi}, \mathit{\Psi}, \dot{\mathit{\Psi}}\}$ abbreviated by X,

$$X \mapsto \frac{4\pi X}{\sqrt{N}}, \tag{1.30}$$

we obtain the action

$$S = \frac{N}{8\pi^2}\int \mathrm{d}^4x\, \mathcal{L}. \tag{1.31}$$

It follows that Feynman diagram vertices have coefficients proportional to N, and propagators are proportional to N^{-1}. Since the adjoint representation can be written as the product of a fundamental and antifundamental representation, It is possible to use double lines for propagators, where each line connects one fundamental gauge index to an antifundamental index, as in Figure 1.1. Then every closed loop within a Feynman diagram corresponds to a contracted gauge group delta function, $\delta^{\mathfrak{a}}_{\mathfrak{a}} = N$. It follows that the N-dependence of a Feynman graph is N^{F+V-I}, where F is the number of loops or faces within the graph and V is the number of vertices. Now, we apply Euler's Theorem to the (connected) Feynman diagram for a correlation function of two local operators drawn on the minimal possible genus G surface. We have $V + (F+T) = I + 2 - 2G$, where T are the number of traces

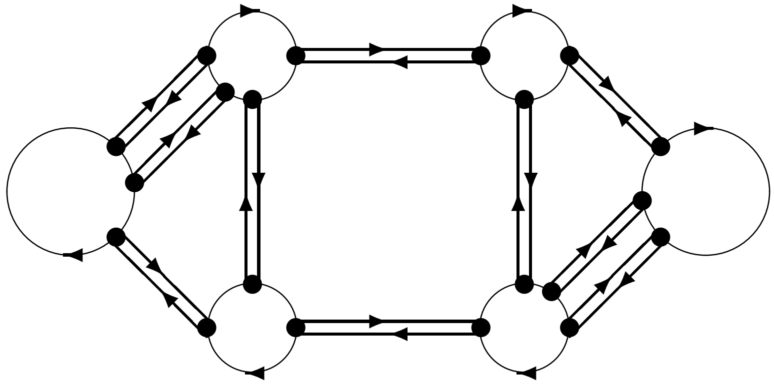

Figure 1.1: A planar diagram using double lines for propagators. Here there are ten propagators, four vertices (two three-field and two four-field), six faces (one face encloses the entire graph), and two local operator traces.

(faces) within the local operators. So the N-dependence is N^{2-2G-T}. The leading contribution for $N \to \infty$ comes from correlators between single-trace operators (total $T = 2$) and $G = 0$. These genus 0 diagrams correspond to spin chain interactions that act upon adjacent sites.

Parity The charge conjugation symmetry of $\mathcal{N} = 4$ SYM acts by sending any (matrix) field X to $-X^*$, which equals $-X^{\mathrm{T}}$ for a U(N) gauge group. In terms of the spin chain description, this symmetry reverses the order of the fields up to signs, and therefore we will call it parity, $\mathfrak{p}$:

$$\mathfrak{p}|X_1 \ldots X_L\rangle = (-1)^{n_f(n_f-1)/2+L}|X_L \ldots X_1\rangle, \tag{1.32}$$

where n_f is the number of fermionic fields among the X_i. The inclusion of the length L in the exponent insures that, including length-changing quantum corrections, all of the $\mathfrak{psu}(2,2|4)$ generators are parity even (commute with $\mathfrak{p}$).

1.4 The restriction to the $\mathfrak{psu}(1,1|2)$ sector

As discussed in the Introduction, we will focus on the $\mathfrak{psu}(1,1|2)$ sector in this work. This yields a more manageable spin chain system that still captures essential features of the full $\mathfrak{psu}(2,2|4)$ spin chain. In this section, first we explain how to restrict consistently to the $\mathfrak{psu}(1,1|2)$ sector. Then we discuss the $\mathfrak{psu}(1,1|2)$ generators in detail. Such restrictions to subsectors were fully classified in [14].

The restriction can be described very concisely. The states of the $\mathfrak{psu}(1,1|2)$ sector satisfy two equalities:

$$D_0 = L^1_1 - L^2_2 - 2R^4_4,$$
$$D_0 = \dot{L}^1_1 - \dot{L}^2_2 + 2R^3_3. \tag{1.33}$$

The ordinary font denotes eigenvalues, e.g. a state characterized by L^1_1 satisfies

$$\mathfrak{L}^1_1|X\rangle = L^1_1|X\rangle. \tag{1.34}$$

Since the dilatation generator commutes with its classical part and the manifest symmetries, it follows that the dilatation generator will only mix these states to all orders in perturbation theory. Using the explicit algebra given in Appendix A.1, one finds that generators that have undotted and dotted Lorentz indices equal to 1, and $\mathfrak{R}$ symmetry indices 1 or 2 commute with the conditions (1.33). This set of generators includes supersymmetry generators

$$\mathfrak{Q}^1_1,\ \mathfrak{Q}^2_1,\ \mathfrak{S}^1_1,\ \mathfrak{S}^1_2,\ \dot{\mathfrak{Q}}_{11},\ \dot{\mathfrak{Q}}_{12},\ \dot{\mathfrak{S}}^{11},\ \dot{\mathfrak{S}}^{12}, \tag{1.35}$$

as well as the bosonic generators

$$\mathfrak{R}^1_2,\ \mathfrak{R}^2_1,\ \mathfrak{P}_{11},\ \mathfrak{K}^{11}. \tag{1.36}$$

In fact, when the dilatation generator and

$$\mathfrak{R}^0 = \mathfrak{R}^1_1 - \mathfrak{R}^2_2 \tag{1.37}$$

are included, this forms a closed $\mathfrak{psu}(1,1|2)$ subalgebra. We will discuss its algebra relations further below. First, we note that it is now easy to identify the states of this subsector. The field Φ_{23} can appear within states in this sector since it has dimension 1 and has eigenvalues $R^4_4 = -1/2$ and $R^3_3 = 1/2$. Then, acting with the classical $\mathfrak{psu}(1,1|2)$ generators we can generate all states built out of

$$\mathcal{D}^k_{11}\Phi_{23},\, \mathcal{D}^k_{11}\Phi_{13},\, \mathcal{D}^k_{11}\Psi_{13},\, \mathcal{D}^k_{11}\dot{\Psi}^4_1. \tag{1.38}$$

In fact, no other fields can appear in states in this sector, though to show this it is easiest to use the oscillator formalism described in Appendix A.1.

An important property of this sector relates to $\mathcal{L}$, the length operator, which multiplies a state composed of L fundamental fields by L. In this sector it satisfies

$$\mathcal{L} = \mathfrak{L}^1_1 - \dot{\mathfrak{L}}^1_1 - 2\,\mathfrak{R}^4_4. \tag{1.39}$$

It follows that the quantum corrections to the dilatation generator in this sector preserve the length of the spin chain which is not true for the full $\mathcal{N} = 4$ SYM spin chain [19]. Furthermore, using the superconformal algebra one can verify that (1.39) implies that the correction to all of the $\mathfrak{psu}(1,1|2)$ generators preserve length. Since we concluded from power counting of Feynman diagrammatics that an interaction involving n sites appears at $\mathcal{O}(g^{n-2})$, we see that the quantum corrections to symmetry generators in this sector expand in even powers of g.

Since the $\mathfrak{psu}(1,1|2)$ sector will be the focus of much of this work, we introduce a convenient notation for the generators of the algebra. The R symmetry generators form a $\mathfrak{su}(2)$ subalgebra. We raise indices so that we have $\mathfrak{R}^{ab} = \mathfrak{R}^{ba}$ $(a,\, b = 1, 2)$,

$$\mathfrak{R}^{ab} = \varepsilon^{ac}\mathfrak{R}^b{}_c. \tag{1.40}$$

Similarly, the momentum, special conformal transformation, and dilatation generators form an $\mathfrak{su}(1,1)$ subalgebra. For these generators, we use $\mathfrak{J}^{\alpha\beta} = \mathfrak{J}^{\beta\alpha}$,

$$\mathfrak{J}^{++} = \mathfrak{P}_{11}, \qquad \mathfrak{J}^{--} = \mathfrak{K}^{11}, \qquad \mathfrak{J}^{+-} = \tfrac{1}{2}\mathfrak{D} + \tfrac{1}{2}\mathfrak{L}^1_1 + \tfrac{1}{2}\dot{\mathfrak{L}}^1_1. \tag{1.41}$$

Since the supercharges carry $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ charge, we assign new symbols for them as well, with a second $\pm$ index corresponding to the original distinction between $\mathfrak{Q}$ or $\mathfrak{S}$ (and between $\dot{\mathfrak{Q}}$ or $\dot{\mathfrak{S}}$).

$$\begin{aligned} \mathfrak{Q}^{a+} &= \mathfrak{Q}^a_1, & \dot{\mathfrak{Q}}^{a+} &= \varepsilon^{ab}\dot{\mathfrak{Q}}_{1b}, \\ \dot{\mathfrak{Q}}^{a-} &= \dot{\mathfrak{S}}^{a1}, & \mathfrak{Q}^{a-} &= \varepsilon^{ab}\mathfrak{S}^1_b. \end{aligned} \tag{1.42}$$

Next we present the algebra relations. The $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ algebra relations take the simple form

$$\begin{aligned} [\mathfrak{R}^{ab}, \mathfrak{R}^{cd}] &= \varepsilon^{cb}\mathfrak{R}^{ad} - \varepsilon^{ad}\mathfrak{R}^{cb}, \\ [\mathfrak{J}^{\alpha\beta}, \mathfrak{J}^{\gamma\delta}] &= \varepsilon^{\gamma\beta}\mathfrak{J}^{\alpha\delta} - \varepsilon^{\alpha\delta}\mathfrak{J}^{\gamma\beta}. \end{aligned} \tag{1.43}$$

The supercharges transform canonically with respect to these rank-1 subalgebras,

$$\begin{aligned} [\mathfrak{R}^{ab}, \mathfrak{Q}^{c\delta}] &= \tfrac{1}{2}\varepsilon^{ca}\mathfrak{Q}^{b\delta} + \tfrac{1}{2}\varepsilon^{cb}\mathfrak{Q}^{a\delta}, & [\mathfrak{R}^{ab}, \dot{\mathfrak{Q}}^{c\delta}] &= \tfrac{1}{2}\varepsilon^{ca}\dot{\mathfrak{Q}}^{b\delta} + \tfrac{1}{2}\varepsilon^{cb}\dot{\mathfrak{Q}}^{a\delta}, \\ [\mathfrak{J}^{\alpha\beta}, \mathfrak{Q}^{c\delta}] &= \tfrac{1}{2}\varepsilon^{\delta\alpha}\mathfrak{Q}^{c\beta} + \tfrac{1}{2}\varepsilon^{\delta\beta}\mathfrak{Q}^{c\alpha}, & [\mathfrak{J}^{\alpha\beta}, \dot{\mathfrak{Q}}^{c\delta}] &= \tfrac{1}{2}\varepsilon^{\delta\alpha}\dot{\mathfrak{Q}}^{c\beta} + \tfrac{1}{2}\varepsilon^{\delta\beta}\dot{\mathfrak{Q}}^{c\alpha}. \end{aligned} \tag{1.44}$$

Finally, the nonvanishing commutators of the supercharges are

$$\begin{aligned} \{\mathfrak{Q}^{a+}, \mathfrak{Q}^{b-}\} &= -\mathfrak{R}^{ab}, & \{\dot{\mathfrak{Q}}^{a+}, \dot{\mathfrak{Q}}^{b-}\} &= \mathfrak{R}^{ab}, \\ \{\mathfrak{Q}^{a+}, \dot{\mathfrak{Q}}^{b\gamma}\} &= \varepsilon^{ab}\mathfrak{J}^{+\gamma}, & \{\mathfrak{Q}^{a-}, \dot{\mathfrak{Q}}^{b\gamma}\} &= -\varepsilon^{ab}\mathfrak{J}^{-\gamma}. \end{aligned} \tag{1.45}$$

1.5 The $\mathcal{O}(g^0)$ representation

In this section we will explain how the generators act on spin chain states at leading order. This will provide the basis for later analysis of additional symmetries of this sector (starting in the next section). Since the leading order interactions act on only one (initial and final) spin chain site, we can write the actions explicitly. To simplify these expressions we introduce a notation for the fields of this sector,

$$|\phi_a^{(n)}\rangle, \qquad |\psi^{(n)}\rangle, \qquad |\dot{\psi}^{(n)}\rangle, \tag{1.46}$$

where n is a nonnegative integer, and these are identified with the previous notation as

$$|\phi_a^{(n)}\rangle \simeq \frac{1}{n!}\,\mathcal{D}_{11}^n \Phi_{a3}, \qquad |\psi^{(n)}\rangle \simeq \frac{1}{n!\sqrt{n+1}}\,\mathcal{D}_{11}^n \Psi_{13}, \qquad |\dot{\psi}^{(n)}\rangle \simeq \frac{1}{n!\sqrt{n+1}}\,\mathcal{D}_{11}^n \dot{\Psi}_1^4. \tag{1.47}$$

It is straightforward to apply the representation given in Appendix A.1 to obtain the classical action of the generators on the fields. The $\mathfrak{R}$ symmetry generators act canonically on the bosonic doublet of states (to all orders)

$$\mathfrak{R}^{ab}|\phi_c^{(n)}\rangle = \delta_c^{\{a}\varepsilon^{b\}d}|\phi_d^{(n)}\rangle. \tag{1.48}$$

The $\mathfrak{J}$ act on the states by changing the index n (the number of derivatives) by up to one unit

$$\begin{aligned}
\mathfrak{J}_{(0)}^{++}|\phi_a^{(n)}\rangle &= (n+1)|\phi_a^{(n+1)}\rangle, & \mathfrak{J}_{(0)}^{++}|\psi^{(n)}\rangle &= \sqrt{(n+1)(n+2)}|\psi^{(n+1)}\rangle,\\
\mathfrak{J}_{(0)}^{+-}|\phi_a^{(n)}\rangle &= (n+\tfrac{1}{2})|\phi_a^{(n)}\rangle, & \mathfrak{J}_{(0)}^{+-}|\psi^{(n)}\rangle &= (n+1)|\psi^{(n)}\rangle,\\
\mathfrak{J}_{(0)}^{--}|\phi_a^{(n)}\rangle &= n|\phi_a^{(n-1)}\rangle, & \mathfrak{J}_{(0)}^{--}|\psi^{(n)}\rangle &= \sqrt{n(n+1)}|\psi^{(n-1)}\rangle.
\end{aligned} \tag{1.49}$$

The action of the $\mathfrak{J}$ on the $\dot{\psi}$ takes the same form as its action on the undotted ψ. Finally, the supercharges act as

$$\begin{aligned}
\mathfrak{Q}_{(0)}^{a+}|\phi_c^{(n)}\rangle &= -\sqrt{n+1}\,\delta_c^a|\dot{\psi}^{(n)}\rangle, & \mathfrak{Q}_{(0)}^{a+}|\psi^{(n)}\rangle &= \sqrt{n+1}\,\varepsilon^{ad}|\phi_d^{(n+1)}\rangle,\\
\dot{\mathfrak{Q}}_{(0)}^{a+}|\phi_c^{(n)}\rangle &= \sqrt{n+1}\,\delta_c^a|\psi^{(n)}\rangle, & \dot{\mathfrak{Q}}_{(0)}^{a+}|\dot{\psi}^{(n)}\rangle &= \sqrt{n+1}\,\varepsilon^{ad}|\phi_d^{(n+1)}\rangle,\\
\mathfrak{Q}_{(0)}^{a-}|\phi_c^{(n)}\rangle &= \sqrt{n}\,\delta_c^a|\psi^{(n-1)}\rangle, & \mathfrak{Q}_{(0)}^{a-}|\dot{\psi}^{(n)}\rangle &= \sqrt{n+1}\,\varepsilon^{ad}|\phi_d^{(n)}\rangle,\\
\dot{\mathfrak{Q}}_{(0)}^{a-}|\phi_c^{(n)}\rangle &= -\sqrt{n}\,\delta_c^a|\dot{\psi}^{(n-1)}\rangle, & \dot{\mathfrak{Q}}_{(0)}^{a-}|\psi^{(n)}\rangle &= \sqrt{n+1}\,\varepsilon^{ad}|\phi_d^{(n)}\rangle.
\end{aligned} \tag{1.50}$$

One can now check explicitly that the representation described by (1.48)-(1.50) satisfies the $\mathfrak{psu}(1,1|2)$ algebra, (1.43)-(1.45). For example, using (1.48) and (1.50) we find

$$\begin{aligned}\{\dot{\mathfrak{Q}}^{1+}_{(0)},\dot{\mathfrak{Q}}^{1-}_{(0)}\}|\phi_c^{(n)}\rangle &= (-\sqrt{n}\,\delta^1_c)\dot{\mathfrak{Q}}^{1+}_{(0)}|\dot{\psi}^{(n-1)}\rangle + (\sqrt{n+1}\,\delta^1_c)\dot{\mathfrak{Q}}^{1-}_{(0)}|\psi^{(n+1)}\rangle \\ &= (-\sqrt{n}\,\delta^1_c)(\sqrt{n}\,\varepsilon^{1f})|\phi_f^{(n)}\rangle + (\sqrt{n+1}\,\delta^1_c)(\sqrt{n+1}\,\varepsilon^{1f})|\phi_f^{(n)}\rangle \\ &= -n\,\delta^1_c\varepsilon^{1f}|\phi_f^{(n)}\rangle + (n+1)\,\delta^1_c\varepsilon^{1f}|\phi_f^{(n)}\rangle \\ &= \delta^1_c\varepsilon^{1f}|\phi_f^{(n)}\rangle = \delta_c^{\{1}\varepsilon^{1\}d}|\phi_d^{(n)}\rangle \\ &= \mathfrak{R}^{11}|\phi_c^{(n)}\rangle. \end{aligned} \tag{1.51}$$

This agrees with (1.45). A complete check of the algebra involves many other commutators and fermionic initial states, but these calculations proceed similarly (and can be done much quicker with the help of a computer).

The actions we have presented are easily generalized to spin chains of arbitrary length: The symmetry generators act as a tensor product. For instance,

$$\dot{\mathfrak{Q}}^{1+}_{(0)}|\phi_1^{(3)}\psi^{(5)}\dot{\psi}^{(2)}\phi_2^{(4)}\rangle = 2|\psi^{(3)}\psi^{(5)}\dot{\psi}^{(2)}\phi_2^{(4)}\rangle - \sqrt{3}|\phi_1^{(3)}\psi^{(5)}\phi_2^{(3)}\phi_2^{(4)}\rangle \tag{1.52}$$

$\dot{\mathfrak{Q}}^{1+}_{(0)}$ acts on the first site to yield the first term on the right side, and its action on the third site gives the second term ($\dot{\mathfrak{Q}}^{1+}_{(0)}$ has vanishing action on the fields of the second and fourth sites). Note that we have included a minus sign for the crossing of a fermionic site by a fermionic generator. For convenient generalization, we give a more formal definition of a generator $\mathfrak{M}$ acting on a state $|X\rangle$ of length L,

$$\mathfrak{M}|X\rangle = \sum_{i=0}^{L-1}\mathfrak{U}^{-i}\mathfrak{M}(1)\mathfrak{U}^{i}|X\rangle. \tag{1.53}$$

The argument 1 for $\mathfrak{M}$ just indicates that it acts only on the first site. The shift operators insure that the generator acts homogeneously on the chain and accounts for fermion statistics.

We close this section with a discrete symmetry of the representation that will be useful. We define hermitian conjugation to simply switch initial and final states. With this definition, hermitian conjugation transforms this representation as follows,

$$(\mathfrak{Q}^{a+})^\dagger = \varepsilon_{ad}\mathfrak{Q}^{d-}, \qquad (\dot{\mathfrak{Q}}^{a+})^\dagger = -\varepsilon_{ad}\dot{\mathfrak{Q}}^{d-},$$
$$(\mathfrak{R}^{ab})^\dagger = -\varepsilon_{ac}\varepsilon_{bd}\mathfrak{R}^{cd}, \qquad (\mathfrak{J}^{++})^\dagger = \mathfrak{J}^{--}. \tag{1.54}$$

Also, $\mathfrak{J}^{+-}$ is hermitian, $(\mathfrak{J}^{+-})^\dagger = \mathfrak{J}^{+-}$. The perturbative corrections that we will find will usually preserves this simple hermitian structure, though this is just a convention.

1.6 The $\mathfrak{su}(2)$ automorphism

The expressions for the leading order representation (1.48)-(1.50) are symmetric, up to a few minus signs, under switching undotted and dotted fermions and supercharges. However, this is more than just a discrete symmetry. In fact, this representation has an outer $\mathfrak{su}(2)$ automorphism, see e.g. [43]. This additional symmetry will play an important role in our analysis of Chapters 2 and 3. To make this automorphism manifest, we replace the dotted/undotted notation with a Gothic index which can take the values '<', '>'. Now the supercharges are written as

$$\mathfrak{Q}^{a+>} = \mathfrak{Q}^{a+} = \mathfrak{Q}^a_1, \qquad \mathfrak{Q}^{a+<} = \dot{\mathfrak{Q}}^{a+} = \varepsilon^{ab}\dot{\mathfrak{Q}}_{1b},$$
$$\mathfrak{Q}^{a->} = \dot{\mathfrak{Q}}^{a-} = \dot{\mathfrak{S}}^{a1}, \qquad \mathfrak{Q}^{a-<} = \mathfrak{Q}^{a-} = \varepsilon^{ab}\mathfrak{S}^1_b. \tag{1.55}$$

With this new notation, the algebra relations (1.44-1.45) can be written much more compactly. The commutators with $\mathfrak{R}$ and $\mathfrak{J}$ are

$$[\mathfrak{R}^{ab}, \mathfrak{Q}^{c\delta\mathfrak{e}}] = \tfrac{1}{2}\varepsilon^{ca}\mathfrak{Q}^{b\delta\mathfrak{e}} + \tfrac{1}{2}\varepsilon^{cb}\mathfrak{Q}^{a\delta\mathfrak{e}},$$
$$[\mathfrak{J}^{\alpha\beta}, \mathfrak{Q}^{c\delta\mathfrak{e}}] = \tfrac{1}{2}\varepsilon^{\delta\alpha}\mathfrak{Q}^{c\beta\mathfrak{e}} + \tfrac{1}{2}\varepsilon^{\delta\beta}\mathfrak{Q}^{c\alpha\mathfrak{e}}, \tag{1.56}$$

and we can write the commutators between supercharges in one expression,

$$\{\mathfrak{Q}^{a\beta\mathfrak{c}}, \mathfrak{Q}^{d\epsilon\mathfrak{f}}\} = \varepsilon^{\beta\epsilon}\varepsilon^{\mathfrak{c}\mathfrak{f}}\mathfrak{R}^{da} - \varepsilon^{ad}\varepsilon^{\mathfrak{c}\mathfrak{f}}\mathfrak{J}^{\beta\epsilon} + \varepsilon^{ad}\varepsilon^{\beta\epsilon}\mathfrak{C}^{\mathfrak{c}\mathfrak{f}}. \tag{1.57}$$

For completeness, we have introduced a maximal set of three central charges $\mathfrak{C}^{\mathfrak{a}\mathfrak{b}} = \mathfrak{C}^{\mathfrak{b}\mathfrak{a}}$. In the case of the gauge theory spin chain they act trivially. Also, with this notation hermitian conjugation (1.54) acts on the supercharges as

$$(\mathfrak{Q}^{a+\gamma})^\dagger = -\varepsilon_{ad}\varepsilon_{\gamma\zeta}\mathfrak{Q}^{d-\zeta} \tag{1.58}$$

The simplifications continue when we turn to the leading order representation (1.48)-(1.50). Again to make the automorphism manifest, we label the fermionic fields with this index, $|\psi_{\mathfrak{a}}\rangle$,

$$|\psi^{(n)}_{>}\rangle = |\psi^{(n)}\rangle \simeq \frac{1}{n!\sqrt{n+1}}\,\mathcal{D}^n_{11}\Psi_{13}, \qquad |\psi^{(n)}_{<}\rangle = |\dot\psi^{(n)}\rangle \simeq \frac{1}{n!\sqrt{n+1}}\,\mathcal{D}^n_{11}\dot\Psi^4_1. \tag{1.59}$$

The action of the $\mathfrak{J}$ on fermions can now be written all at once,

$$\begin{aligned} \mathfrak{J}^{++}_{(0)}|\psi^{(n)}_{\mathfrak{a}}\rangle &= \sqrt{(n+1)(n+2)}|\psi^{(n+1)}_{\mathfrak{a}}\rangle, & \mathfrak{J}^{+-}_{(0)}|\psi^{(n)}_{\mathfrak{a}}\rangle &= (n+1)|\psi^{(n)}_{\mathfrak{a}}\rangle, \\ \mathfrak{J}^{--}_{(0)}|\psi^{(n)}_{\mathfrak{a}}\rangle &= \sqrt{n(n+1)}|\psi^{(n-1)}_{\mathfrak{a}}\rangle, \end{aligned} \tag{1.60}$$

while the action of the supercharges can be written using $\delta^{\mathfrak{a}}_{\mathfrak{b}}$ and $\varepsilon^{\mathfrak{a}\mathfrak{b}}$,

$$\begin{aligned} \mathfrak{Q}^{a+\mathfrak{b}}_{(0)}|\phi^{(n)}_c\rangle &= \sqrt{n+1}\,\delta^a_c\varepsilon^{\mathfrak{b}\mathfrak{d}}|\psi^{(n)}_{\mathfrak{d}}\rangle, & \mathfrak{Q}^{a+\mathfrak{b}}_{(0)}|\psi^{(n)}_{\mathfrak{c}}\rangle< &= \sqrt{n+1}\,\delta^{\mathfrak{b}}_{\mathfrak{c}}\varepsilon^{ad}|\phi^{(n+1)}_d\rangle, \\ \mathfrak{Q}^{a-\mathfrak{b}}_{(0)}|\phi^{(n)}_c\rangle &= \sqrt{n}\,\delta^a_c\varepsilon^{\mathfrak{b}\mathfrak{d}}|\psi^{(n-1)}_{\mathfrak{d}}\rangle, & \mathfrak{Q}^{a-\mathfrak{b}}_{(0)}|\psi^{(n)}_{\mathfrak{c}}\rangle &= \sqrt{n+1}\,\delta^{\mathfrak{b}}_{\mathfrak{c}}\varepsilon^{ad}|\phi^{(n)}_d\rangle. \end{aligned} \tag{1.61}$$

The existence of the $\mathfrak{su}(2)$ automorphism follows from these expressions. Explicitly, we introduce generators of the automorphism $\mathfrak{B}^{\mathfrak{a}\mathfrak{b}}$. They rotate the fermions as

$$\mathfrak{B}^{\mathfrak{a}\mathfrak{b}}|\psi^{(n)}_{\mathfrak{c}}\rangle = \delta^{\{\mathfrak{a}}_{\mathfrak{c}}\varepsilon^{\mathfrak{b}\}\mathfrak{d}}|\psi^{(n)}_{\mathfrak{d}}\rangle, \tag{1.62}$$

and satisfy

$$[\mathfrak{B}^{\mathfrak{ab}}, \mathfrak{B}^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}\mathfrak{B}^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\mathfrak{B}^{\mathfrak{cb}}. \tag{1.63}$$

One can then check that the leading order representation (1.61) satisfies the following canonical relations,

$$[\mathfrak{B}^{\mathfrak{ab}}, \mathfrak{Q}^{c\delta\mathfrak{e}}] = \tfrac{1}{2}\varepsilon^{\mathfrak{ea}}\mathfrak{Q}^{c\delta\mathfrak{b}} + \tfrac{1}{2}\varepsilon^{\mathfrak{eb}}\mathfrak{Q}^{c\delta\mathfrak{a}}. \tag{1.64}$$

Also, the $\mathfrak{R}$ and $\mathfrak{J}$ commute with the $\mathfrak{B}$. Again for completeness, we include the commutators with the "central charges" (which vanish for the gauge theory)

$$[\mathfrak{B}^{\mathfrak{ab}}, \mathfrak{C}^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}\mathfrak{C}^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\mathfrak{C}^{\mathfrak{cb}}. \tag{1.65}$$

The $\mathfrak{C}^{\mathfrak{ab}}$ now become a spin-1 triplet under this outer $\mathfrak{su}(2)$, i.e. they are not central for the maximally extended algebra. This maximally extended algebra including $\mathfrak{B}^{\mathfrak{ab}}$ and $\mathfrak{C}^{\mathfrak{ab}}$ can be denoted as $\mathfrak{su}(2) \ltimes \mathfrak{psu}(1,1|2) \ltimes \mathbb{R}^3$.

We should point out that this $\mathfrak{su}(2)$ symmetry is only an accidental symmetry of this sector of $\mathcal{N} = 4$ SYM: The generators $\mathfrak{B}^{<<}$ and $\mathfrak{B}^{>>}$ transform between fermions Ψ and conjugate fermions $\dot{\Psi}$ in gauge theory, cf. (1.59). However, none of the $\mathfrak{psu}(2,2|4)$ generators of the full theory acts in such a way. Only the Cartan generator of the outer $\mathfrak{su}(2)$ is equivalent to a combination of the Lorentz generators, $\mathfrak{B}^{<>} = \mathfrak{L}^1{}_1 - \dot{\mathfrak{L}}^1{}_1$.

1.7 Hidden $\mathfrak{psu}(1|1)^2$ symmetry

Recall that we restrict to zero-momentum (shift-invariant) states for the gauge theory because of the origin of the spin chain from a trace. The $\mathfrak{psu}(1,1|2)$ spin chain is consistent for general momentum states, and we will study this more general case as well. However, for zero-momentum states there is a symmetry enhancement of a hidden $\mathfrak{psu}(1|1)^2$ symmetry and one central charge [41]. These symmetry generators descend from the full $\mathfrak{psu}(2,2|4)$

algebra, but vanish in the classical limit. This additional symmetry will enable us to write two-loop corrections to the symmetry generators in a simple way in Chapter 3. We now present the symmetry generators and their algebra, and show that these hidden supercharges act through length-changing interactions.

The $\mathfrak{su}(2)$ automorphism also rotates the $\mathfrak{psu}(1|1)^2$ supercharges, so we again introduce appropriate notation. We denote the fermionic generators (there are two for each $\mathfrak{psu}(1|1)$) by $\hat{\mathfrak{Q}}^{\mathfrak{a}}$ and $\hat{\mathfrak{S}}^{\mathfrak{a}}$. There is also one central charge, which we label $\hat{\mathfrak{D}}$. In terms of $\mathcal{N}=4$ SYM, they represent the supercharges

$$\begin{aligned} \hat{\mathfrak{Q}}^{<} &= \dot{\mathfrak{Q}}_{23}, & \hat{\mathfrak{Q}}^{>} &= -\mathfrak{Q}^{4}_{2}, \\ \hat{\mathfrak{S}}^{<} &= \mathfrak{S}^{2}_{4}, & \hat{\mathfrak{S}}^{>} &= \dot{\mathfrak{S}}^{32}. \end{aligned}$$

and the generator of anomalous dimensions

$$\hat{\mathfrak{D}} = \tfrac{1}{2}\mathfrak{D} + \mathfrak{L}^2{}_2 + \mathfrak{R}^4{}_4 = \tfrac{1}{2}\mathfrak{D} + \dot{\mathfrak{L}}^2{}_2 - \mathfrak{R}^3{}_3 = \tfrac{1}{2}\delta\mathfrak{D}. \tag{1.66}$$

The last two equalities are satisfied within the $\mathfrak{psu}(1,1|2)$ sector.

Using Appendix A.1, one can verify that these generators commute with the restrictions (1.33) and that they annihilate all the states of this sector at $\mathcal{O}(g^0)$. Furthermore, since the length generator $\mathcal{L}$ is written in terms of conserved charges in this sector (1.39), the algebra guarantees that

$$[\mathcal{L}, \hat{\mathfrak{Q}}] = \hat{\mathfrak{Q}} \quad \text{and} \quad [\mathcal{L}, \hat{\mathfrak{S}}] = -\hat{\mathfrak{S}}. \tag{1.67}$$

In other words, these fermionic generators change the length of the spin chain by one unit. It follows that the $\mathfrak{psu}(1|1)^2$ generators have an expansion in odd powers of g (consistent with their vanishing in the classical limit). These dynamic supercharges are a residue of the dynamic nature of the full $\mathfrak{psu}(2,2|4)$ spin chain.

It is now clear why this symmetry is only present in the zero-momentum sector of the $\mathfrak{psu}(1,1|2)$ spin chain. A state of length L has momentum $p = 2\pi n/L$ for $n = 0, \dots L-1$, and zero is the only momentum shared by both length-L and length-$(L+1)$ states. Since symmetry generators should commute with the shift operator $\mathfrak{U}$ (eigenvalues e^{ip}), dynamic generators are only possible in the zero-momentum sector.

The only nontrivial commutator of the $\mathfrak{psu}(1|1)^2$ algebra reads

$$\{\hat{\mathfrak{Q}}^{\mathfrak{a}}, \hat{\mathfrak{S}}^{\mathfrak{b}}\} = \hat{\mathfrak{C}}^{\mathfrak{ab}} + \varepsilon^{ab}\hat{\mathfrak{D}}. \tag{1.68}$$

For completeness we have introduced a triplet $\hat{\mathfrak{C}}^{\mathfrak{ab}}$ of central charges to accompany the singlet $\hat{\mathfrak{D}}$. In our spin chain model the triplet acts trivially, $\hat{\mathfrak{C}}^{\mathfrak{ab}} = 0$.

The $\mathfrak{B}^{\mathfrak{ab}}$ $\mathfrak{su}(2)$ outer automorphism acts canonically,

$$\begin{aligned}
[\mathfrak{B}^{\mathfrak{ab}}, \hat{\mathfrak{Q}}^{\mathfrak{c}}] &= \tfrac{1}{2}\varepsilon^{\mathfrak{ca}}\hat{\mathfrak{Q}}^{\mathfrak{b}} + \tfrac{1}{2}\varepsilon^{\mathfrak{cb}}\hat{\mathfrak{Q}}^{\mathfrak{a}},\\
[\mathfrak{B}^{\mathfrak{ab}}, \hat{\mathfrak{S}}^{\mathfrak{c}}] &= \tfrac{1}{2}\varepsilon^{\mathfrak{ca}}\hat{\mathfrak{S}}^{\mathfrak{b}} + \tfrac{1}{2}\varepsilon^{\mathfrak{cb}}\hat{\mathfrak{S}}^{\mathfrak{a}},\\
[\mathfrak{B}^{\mathfrak{ab}}, \hat{\mathfrak{C}}^{\mathfrak{cd}}] &= \varepsilon^{\mathfrak{cb}}\hat{\mathfrak{C}}^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\hat{\mathfrak{C}}^{\mathfrak{cb}},
\end{aligned} \tag{1.69}$$

The $\mathfrak{u}(1)$ $\mathcal{L}$ grading combined with the $\mathfrak{su}(2)$ $\mathfrak{B}^{\mathfrak{ab}}$ leads to a $\mathfrak{u}(2) \ltimes \mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}^4$ algebra, where the $\hat{\mathfrak{C}}^{\mathfrak{ab}}$ and $\hat{\mathfrak{D}}$ give the $\mathbb{R}^4$.

Finally, the generators of $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ commute with each other. This product structure of the full symmetry algebra for zero-momentum states will be very important for obtaining quantum corrections to the symmetry algebra.

1.8 Subsectors of the $\mathfrak{psu}(1,1|2)$ sector

To further simplify analysis, it is useful to restrict to even smaller subsectors of $\mathcal{N} = 4$ SYM. So we now introduce the subsectors that we will encounter. The $\mathfrak{su}(2|1)$ sector's

Yangian symmetry will be studied in Chapter 4. To restrict to the $\mathfrak{su}(2|1)$ sector, in addition to the restrictions (1.33) one requires

$$\dot{\mathfrak{L}}^1_1 = 0. \tag{1.70}$$

Now, generators consistent with this restriction include

$$\mathfrak{Q}^1_1,\ \mathfrak{Q}^2_1,\ \mathfrak{S}^1_1,\ \mathfrak{S}^1_2, \tag{1.71}$$

as well as the $\mathfrak{su}(2)$ generators

$$\mathfrak{R}^1_2,\ \mathfrak{R}^2_1,\ \mathfrak{R}^0 = \mathfrak{R}^1_1 - \mathfrak{R}^2_2, \tag{1.72}$$

and the dilatation generator, $\mathfrak{D}$. This sector is compact: There are only three types of fields,

$$\phi_1 = \varPhi_{13}, \quad \phi_2 = \varPhi_{23}, \quad \text{and} \quad \psi = \varPsi_{13}. \tag{1.73}$$

One can obtain the algebra relations and representation from those of the $\mathfrak{psu}(1,1|2)$ sector which contains this smaller sector. $\mathfrak{Q}^4_2$ and $\mathfrak{S}^2_4$ generate a hidden $\mathfrak{u}(1|1)$ symmetry for this sector, though we will not use it in this work.

Also, we will also encounter four rank-one subsectors. There are compact $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors, and two noncompact $\mathfrak{sl}(2)$ sectors. The $\mathfrak{su}(2)$ sector results from (1.33) combined with

$$\dot{\mathfrak{L}}^1_1 = 0, \quad \text{and} \quad \mathfrak{L}^1_1 = 0. \tag{1.74}$$

The only symmetry generators are the $\mathfrak{su}(2)$ R-symmetry generators and $\mathfrak{D}$, which act on the doublet of bosonic fields $\phi_1 = \varPhi_{13}$ and $\phi_2 = \varPhi_{23}$. Note that in this sector the symmetry group simply commutes with the dilatation generator.

The supersymmetric $\mathfrak{su}(1|1)$ sector satisfies (1.33),

$$\dot{\mathfrak{L}}^1_1 = 0, \quad \text{and} \quad \mathfrak{L}^1_1 + \mathfrak{R}^2_2 - \mathfrak{R}^1_1 + 2\,\mathfrak{R}^4_4 = 0. \tag{1.75}$$

Now the generators are $\mathfrak{Q}^2_1$, $\mathfrak{S}^1_2$, and $\mathfrak{D}$, and they act on $\phi = \Phi_{23}$ and $\psi = \Psi_{13}$[3].

To reach the bosonic noncompact $\mathfrak{sl}(2)$ sector we need (1.33) and

$$\mathfrak{R}^2_2 - \mathfrak{R}^1_1 = -2\,\mathfrak{R}^4_4. \tag{1.76}$$

The $\mathfrak{sl}(2)$ generators are $\mathfrak{P}_{11}$, $\mathfrak{K}^{11}$ and $\mathfrak{D}$, which act on fields $\phi^{(n)} = \mathcal{D}^n_{11}\Phi_{23}$.

For the fermionic $\mathfrak{sl}(2)$ sector we have (1.33) and

$$\mathfrak{L}^1_1 - \dot{\mathfrak{L}}^1_1 + 2\,\mathfrak{R}^4_4 = 0, \tag{1.77}$$

again with generators $\mathfrak{P}_{11}$, $\mathfrak{K}^{11}$ and $\mathfrak{D}$. However, now the fields are $\psi^{(n)} = \mathcal{D}^n_{11}\Psi_{13}$. Once again, $\mathfrak{Q}^4_2$ and $\mathfrak{S}^2_4$ generate a hidden $\mathfrak{u}(1|1)$ symmetry.

Occasionally we will find it useful to restrict to a different type of sector that only includes states with fewer than M spin chain excitations. An excitation corresponds to an impurity with respect to the vacuum state composed of only $\phi^{(0)}_1$. Every ϕ_2, fermion, and derivative counts as one excitation. Truncating by excitation number yields a sector because we can write the excitation number n_e of a state of length L within the $\mathfrak{psu}(1,1|2)$ sector in terms of conserved quantities

$$n_e = D_0 + R^{12} - \tfrac{1}{2}L. \tag{1.78}$$

Note that excitation sectors exist for the full theory as well (when L is no longer conserved). These excitation subsectors are especially useful for studying generators that conserve n_e. In addition to the dilatation generator, these include the $\mathfrak{psu}(1,1|2)$ sector's hidden $\mathfrak{psu}(1|1)^2$ supercharges.

[3] There is an isomorphic sector with generators $\dot{\mathfrak{Q}}_{11}$, $\dot{\mathfrak{S}}^{11}$, and $\mathfrak{D}$, which act on $\phi = \Phi_{23}$ and $\psi = \dot{\Psi}^4_1$.

Chapter 2

Symmetry enhancement at one loop

In this chapter we will discuss multiple facets of the symmetry structure of the $\mathfrak{psu}(1,1|2)$ sector at one loop. The most constraining symmetry is that due to integrability, which allows the use of Bethe equations to obtain the spectrum. These Bethe equations reflect the $\mathfrak{psu}(1,1|2)$ symmetry, as well as the hidden $\mathfrak{psu}(1|1)^2$ symmetry, which was introduced in Section 1.7. The central focus of this chapter, however, will be a curious observation made in [23]: The Bethe equations for the sector lead to a huge degeneracy of 2^M multiplets that does not follow from conventional integrable structures. The degeneracy is partially explained by the $\mathfrak{B}$ $\mathfrak{su}(2)$ automorphism. The degenerate $\mathfrak{psu}(1,1|2)$ multiplets essentially transform in a tensor product of $\mathfrak{su}(2)$ doublets. However, this tensor product is reducible, and therefore the $\mathfrak{su}(2)$ automorphism cannot explain the full degeneracy. To explain the remaining degeneracy at one loop, we will construct an infinite set of nonlocal spin chain generators. These generators carry one unit of $\mathfrak{B}$ charge and are built from the $\mathfrak{psu}(1|1)^2$ generators. We conjecture that these nonlocal symmetry generators form a parabolic subalgebra of the loop group of SU(2).

In Section 2.1, we introduce the Bethe equations and use them to observe this degeneracy. We give the $\mathcal{O}(g)$ $\mathfrak{psu}(1|1)^2$ symmetry generators in Section 2.2. To gain further intuition about the degeneracy, we study some degenerate spin chain states in Section 2.3. Finally, in Section 2.4 we present the nonlocal symmetry generators and discuss how they map between states and the algebra that they form. In Appendix B we present relevant multilinear operators for the $\mathfrak{psu}(1,1|2)$ sector, including a cubic operator that is a $\mathfrak{B}$-triplet and $\mathfrak{psu}(1,1|2)$ invariant. The proof that the nonlocal symmetry generators commute with the classical $\mathfrak{psu}(1,1|2)$ generators and the one-loop dilatation generator is given in Appendix C. This chapter and Appendices B and C are based on, and contain excerpts from, [40].

2.1 Integrability

As for many integrable systems, Bethe equations are a very useful tool for studying the spectrum of planar $\mathcal{N}=4$ SYM, see for example the reviews [41,12]. In this section, after an introduction to Bethe equations we review the one-loop $\mathfrak{psu}(1,1|2)$ Bethe equations and the degeneracies of the corresponding spectrum. Furthermore, we show that these degeneracies are also present for the transfer matrix, which gives the full integrable structure.

2.1.1 Introduction to Bethe equations

We begin by reviewing how the one-loop Bethe equations arise in the bosonic $\mathfrak{sl}(2)$ subsector. First we will need the Hamiltonian (dilatation generator) for this sector, which was computed in [14], truncated at two excitations[1].

$$
\begin{aligned}
\delta\mathfrak{D}_{(2)}\big|\phi_1^{(1)}\phi_1^{(0)}\big\rangle &= 2\big|\phi_1^{(1)}\phi_1^{(0)}\big\rangle - 2\big|\phi_1^{(0)}\phi_1^{(1)}\big\rangle,\\
\delta\mathfrak{D}_{(2)}\big|\phi_1^{(0)}\phi_1^{(1)}\big\rangle &= 2\big|\phi_1^{(0)}\phi_1^{(1)}\big\rangle - 2\big|\phi_1^{(1)}\phi_1^{(0)}\big\rangle,\\
\delta\mathfrak{D}_{(2)}\big|\phi_1^{(2)}\phi_1^{(0)}\big\rangle &= 3\big|\phi_1^{(2)}\phi_1^{(0)}\big\rangle - 2\big|\phi_1^{(1)}\phi_1^{(1)}\big\rangle - \big|\phi_1^{(0)}\phi_1^{(2)}\big\rangle,\\
\delta\mathfrak{D}_{(2)}\big|\phi_1^{(1)}\phi_1^{(1)}\big\rangle &= -2\big|\phi_1^{(2)}\phi_1^{(0)}\big\rangle + 4\big|\phi_1^{(1)}\phi_1^{(1)}\big\rangle - 2\big|\phi_1^{(0)}\phi_1^{(2)}\big\rangle,\\
\delta\mathfrak{D}_{(2)}\big|\phi_1^{(0)}\phi_1^{(2)}\big\rangle &= -\big|\phi_1^{(2)}\phi_1^{(0)}\big\rangle - 2\big|\phi_1^{(1)}\phi_1^{(1)}\big\rangle + 3\big|\phi_1^{(0)}\phi_1^{(2)}\big\rangle.
\end{aligned}
\tag{2.2}
$$

These two-site interactions act homogeneously across the length of the spin chain. The vacuum for a spin chain of length L is simply the state with L ϕ_1 and no derivatives.

$$\Psi_0 = \big|\phi_1^{(0)}\dots\phi_1^{(0)}\big\rangle. \tag{2.3}$$

Since $\delta\mathfrak{D}_{(2)}$ gives zero when acting on adjacent ϕ_1's with no derivatives, the vacuum has one-loop anomalous dimension zero. In fact the anomalous dimension vanishes to all orders, as it is a half-BPS state, see for example the reviews [44]. Next, we construct the one-excitation eigenstates. By translation invariance, the eigenstates are

$$\Psi_p = \sum_{x=1}^{L} e^{ipx}|x\rangle, \quad |x\rangle = \big|\phi_1^{(0)}\dots \overset{\overset{x}{\downarrow}}{\phi_1^{(1)}} \dots\phi_1^{(0)}\big\rangle, \tag{2.4}$$

where $|x\rangle$ has a derivative on site x. Acting with $\delta\mathfrak{D}_{(2)}$ we find

$$
\begin{aligned}
\delta\mathfrak{D}_{(2)}|x\rangle &= 4|x\rangle - 2|x-1\rangle - 2|x+1\rangle \Rightarrow\\
\delta\mathfrak{D}_{(2)}\Psi_p &= 4\sum_{x=1}^{L} e^{ipx}|x\rangle - 2\sum_{x=1}^{L} e^{ipx}|x-1\rangle - 2\sum_{x=1}^{L} e^{ipx}|x+1\rangle\\
&= 4\,\Psi_p - 2\sum_{x=1}^{L} e^{ip(x+1)}|x\rangle - 2\sum_{x=1}^{L} e^{ip(x-1)}|x\rangle\\
&= (4 - 2e^{ip} - 2e^{-ip})\Psi_p = 8\sin^2\frac{p}{2}\,\Psi_p.
\end{aligned}
\tag{2.5}
$$

[1] Note that the number of derivatives n_d is conserved in this subsector since it can be written in terms of conserved quantities,

$$n_d = \mathfrak{D}_0 - L. \tag{2.1}$$

To reach the second-to-last line we just shifted the summation variable x and used the periodicity of the chain to identify sites 0 and L, and sites 1 and L+1. Of course, periodicity also implies that

$$p = 2\pi n/L, \quad n \in \mathbb{Z}. \tag{2.6}$$

Next we look for two-excitation eigenstates. By translation invariance and homogeneity, since $\delta\mathfrak{D}_{(2)}$ acts locally we expect eigenstates that look like products of the one-excitation states, as long as the excitations are well separated. However, since only the complete two-excitation wave function needs to be periodic, we do not need to require (2.6) for the momenta (we will obtain a new periodicity condition). A general ansatz for this behavior is

$$\begin{aligned} \Psi_{p_1p_2} &= \sum_{1\le x_1\le x_2\le L} e^{ip_1x_1+ip_2x_2}|x_1x_2\rangle + (S(p_2,p_1)+c_0\delta_{x_1x_2})e^{ip_2x_1+ip_1x_2}|x_1x_2\rangle, \\ |x_1x_2\rangle &= \big|\phi_1^{(0)}\ldots \underset{\substack{x_1\\ \downarrow}}{\phi_1^{(1)}} \ldots \underset{\substack{x_2\\ \downarrow}}{\phi_1^{(1)}} \ldots \phi_1^{(0)}\big\rangle. \end{aligned} \tag{2.7}$$

Here $|x_1x_2\rangle$ has derivatives on sites x_1 and x_2.

Now, working out the action of $\delta\mathfrak{D}_{(2)}$ from (2.2), we find after taking into account the various possible cases with $x_1 < x_2$,

$$\begin{aligned} \delta\mathfrak{D}_{(2)}|x_1x_2\rangle = (8-2\delta_{x_1x_2})|x_1x_2\rangle - 2|(x_1-1)x_2\rangle - 2(1-\delta_{x_1x_2})|(x_1-1)x_2\rangle \\ - 2(1-\delta_{x_1x_2})|x_1(x_2-1)\rangle - 2|x_1(x_2+1)\rangle \\ - \delta_{x_1x_2}|(x_1-1)(x_2-1)\rangle - \delta_{x_1x_2}|(x_1+1)(x_2+1)\rangle. \end{aligned} \tag{2.8}$$

The Schrödinger equation,

$$\delta\mathfrak{D}_{(2)}\Psi_{p_1p_2} = E\Psi_{p_1p_2}, \tag{2.9}$$

then leads to three independent equations:

$$\begin{aligned}
x_2 - x_1 > 1: \quad & 8(1 + Se^{i(p_1-p_2)(x_2-x_1)}) - 2(e^{ip_1} + Se^{i(p_1-p_2)(x_2-x_1)+ip_2}) \\
& - 2(e^{-ip_1} + Se^{i(p_1-p_2)(x_2-x_1)-ip_2}) - 2(e^{ip_2} + Se^{i(p_1-p_2)(x_2-x_1)+ip_1}) \\
& - 2(e^{-ip_2} + Se^{i(p_1-p_2)(x_2-x_1)-ip_1}) = E(1 + Se^{i(p_1-p_2)(x_2-x_1)}), \\
x_2 - x_1 = 1: \quad & 8(e^{ip_2} + Se^{ip_1}) - 2((1 + S + c_0)e^{ip_1+ip_2}) - 2(e^{2ip_2} + Se^{2ip_1}) - 2(1 + S + c_0) \\
& - 2e^{ip_2-ip_1} - 2Se^{ip_1-ip_2} = E(e^{ip_2} + Se^{ip_1}), \\
x_2 - x_1 = 0: \quad & 6(1 + S + c_0) - (1 + S + c_0)e^{ip_1+ip_2} - (1 + S + c_0)e^{-ip_1-ip_2} \\
& - 2(e^{ip_2} + Se^{ip_1}) - 2(e^{-ip_1} + Se^{-ip_2}) = E(1 + S + c_0).
\end{aligned} \tag{2.10}$$

The first equation simplifies to

$$\begin{aligned}
(E - (4 - 4\cos p_1) - (4 - 4\cos p_2))(1 + e^{-i(p_1-p_2)(x_1-x_2)}S) = 0 \quad \Rightarrow \\
E = (4 - 4\cos p_1) + (4 - 4\cos p_2)) = 8\sin^2\frac{p_1}{2} + 8\sin^2\frac{p_2}{2}.
\end{aligned} \tag{2.11}$$

So the energy is just the sum of the energies of the one-excitation states with momentum p_1 and p_2: The excitations scatter elastically,

$$E(p_1, p_2) = E(p_1) + E(p_2). \tag{2.12}$$

The second equation of (2.10) simplifies as

$$\begin{aligned}
c_0\left(1 + e^{ip_1+ip_2} + (e^{ip_2} + Se^{ip_1})\right)(E - (4 - 4\cos p_1) - (4 - 4\cos p_2)) = 0 \quad \Rightarrow \\
c_0 = 0.
\end{aligned} \tag{2.13}$$

The vanishing of c_0 is a consequence of the fact that the $\delta\mathfrak{D}_{(2)}$ acts the same way on the position basis two-excitation states for excitations that are far apart or for those that are adjacent. The only special case of (2.8) is for excitations on the same site. Substituting for

E and setting c_0 to zero, the third equation of (2.10) reduces to

$$e^{ip_1} + Se^{ip_2} - \frac{1}{2}(1+S)(1+e^{ip_1+ip_2}) = 0, \quad \Rightarrow$$

$$S = S(p_2, p_1) = -\frac{1-2e^{ip_1}+e^{ip_1+ip_2}}{1-2e^{ip_2}+e^{ip_1+ip_2}}, \quad S(p_1, p_2) = -\frac{1-2e^{ip_2}+e^{ip_1+ip_2}}{1-2e^{ip_1}+e^{ip_1+ip_2}}. \quad (2.14)$$

We interpret $S(p_1, p_2)$ as the S-matrix for two-particle scattering: the phase that the wave function acquires after one particle of momentum p_1 scatters past another of momentum p_2. At this point, we are ready to apply the essential property due to integrability: factorized scattering. For arbitrarily many particles in the incoming scattering state, the set of initial and final momenta are the same, and the only possible outgoing state is a permutation of the particles with a factor of the two-particle S-matrix for every pair of particles that cross. Furthermore, the energy of these states is simply given as the sum of the $E(p_i)$. However, for periodic systems of finite length L, we have also the consistency requirement that scattering one excitation past all the others must be the same as just applying the shift operator to that excitation L times, which gives a factor of e^{-ipL}. The resulting system of equations, named Bethe equations in honor of Bethe's initial study of the integrable $\mathfrak{su}(2)$ Heisenberg spin chain [45], takes the form

$$1 = e^{ip_k L} \prod_{\substack{j=1 \\ j\neq k}}^{K} S(p_k, p_j). \quad (2.15)$$

These equations need to be solved for the K momenta p_i. Once that is done, the energy is the sum of the contributions from the individual momenta,

$$E = \sum_{j=1}^{K} 8\sin^2(\frac{p_j}{2}). \quad (2.16)$$

In fact, as proved in [15], the $\mathfrak{sl}(2)$ sector is integrable at one-loop, and (2.15) and (2.16) give the entire spectrum of one-loop anomalous dimensions of this sector. The procedure completed above could be repeated to obtain the spectrum for the other rank-one subsectors

($\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$) which are also integrable. Before moving on to the generalization we will need for higher rank sectors, we introduce a convenient change of variables

$$\exp(ip_k) = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}, \qquad u_k = \frac{1}{2}\cot\frac{p_k}{2}. \tag{2.17}$$

The advantage of this is that it makes the S-matrix and dispersion relation rational functions of these "Bethe roots" u_k

$$S(u_k,\, u_j) = \frac{u_j - u_k + i}{u_j - u_k - i}, \qquad E(u_k) = \frac{2i}{u_k + \frac{i}{2}} - \frac{2i}{u_k - \frac{i}{2}}. \tag{2.18}$$

2.1.2 One-loop Bethe equations for the $\mathfrak{psu}(1,1|2)$ sector

The Bethe equations for rank-one algebras have a generalization for higher rank algebras [46,47]. This was used to find the complete one-loop Bethe equations for $\mathcal{N} = 4$ SYM in [15]. For a rank r algebra, there are r types of Bethe roots, corresponding to the different flavors of excitations or simple roots of the algebra. Then a generic eigenstate will have excitations $u_{k,\,i}$ where $k = 1, \ldots r$ gives the flavor of the root and i labels the different roots of the same flavor. The Bethe equation become

$$\left(\frac{u_{k,\,i} - \frac{i}{2}V_k}{u_{k,\,i} + \frac{i}{2}V_k}\right)^L = \prod_{l=1}^{r} \prod_{\substack{j=1 \\ j\neq i \text{ if } l=k}}^{n_l} \frac{u_{k,\,i} - u_{l,\,j} - \frac{i}{2}M_{k,\,l}}{u_{k\,i} - u_{l,\,j} + \frac{i}{2}M_{k,\,l}}. \tag{2.19}$$

Here the V_k are the Dynkin labels of the representation[2] and M is the Cartan matrix of the algebra[3]. Also, the different excitations contribute to the total spin chain momentum and energy depending on the representation as

$$e^{ip_{k,\,i}} = \frac{u_{k,\,i} + \frac{i}{2}V_k}{u_{k,\,i} - \frac{i}{2}V_k}, \qquad E(u_{k,\,i}) = \frac{ic}{u_{k,\,i} + \frac{i}{2}V_k} - \frac{ic}{u_{k,\,i} - \frac{i}{2}V_k}. \tag{2.20}$$

[2]Equivalently, these are the Cartan generator eigenvalues of the highest weight states.

[3]This matrix gives the "eigenvalues" of the simple roots with respect to their commutators with the Cartan generators.

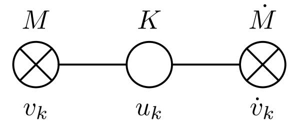

Figure 2.1: Dynkin diagram for $\mathfrak{psu}(1,1|2)$. The different flavors of Bethe roots and their overall numbers are indicated below/above the nodes, respectively.

In particular, Bethe roots for which the corresponding Dynkin label of the representation is zero carry no spin chain momentum or energy. c is a scale factor for the energy which is not fixed by the Bethe equations.

The symmetric Cartan matrix of $\mathfrak{psu}(1,1|2)$, which can be read from the Dynkin diagram in Fig. 2.1, is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.21}$$

while the spin representation with highest state $\phi_1^{(0)}$ has Dynkin labels [0, 1, 0]. The Dynkin diagram and Cartan matrix follows from the choice of simple roots

$$\{\mathfrak{Q}^{2+>},\, \mathfrak{R}^{11},\, \mathfrak{Q}^{2+<}\} \tag{2.22}$$

and basis of Cartan generators

$$\{\mathfrak{J}^{+-} + \mathfrak{R}^{12} + \mathfrak{C}^{<>},\, -2\,\mathfrak{R}^{12},\, \mathfrak{J}^{+-} + \mathfrak{R}^{12} - \mathfrak{C}^{<>}\}. \tag{2.23}$$

Applying the general result (2.19), we find that the Bethe equations for the planar $\mathfrak{psu}(1,1|2)$ sector of $\mathcal{N}=4$ SYM at leading order take the form [48]

$$\begin{aligned} 1 &= \prod_{j=1}^{K} \frac{v_k - u_j - \frac{i}{2}}{v_k - u_j + \frac{i}{2}}\,, \\ 1 &= \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K} \frac{u_k - u_j + i}{u_k - u_j - i} \prod_{j=1}^{M} \frac{u_k - v_j - \frac{i}{2}}{u_k - v_j + \frac{i}{2}} \prod_{j=1}^{\dot{M}} \frac{u_k - \dot{v}_j - \frac{i}{2}}{u_k - \dot{v}_j + \frac{i}{2}}\,, \\ 1 &= \prod_{j=1}^{K} \frac{\dot{v}_k - u_j - \frac{i}{2}}{\dot{v}_k - u_j + \frac{i}{2}}\,. \end{aligned} \tag{2.24}$$

Here we denote the bosonic excitations as $u_{1,\dots,K}$ and the two types of fermionic excitations as $v_{1,\dots,M}$, and $\dot{v}_{1,\dots,\dot{M}}$.

Because the representation has only one nonvanishing Dynkin label, the momentum and energy eigenvalues for eigenstates of this system are determined through the main Bethe roots $u_{1,\dots,K}$ alone

$$\exp(iP) = \prod_{j=1}^{K} \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}, \qquad E = \sum_{j=1}^{K} \left(\frac{2i}{u_j + \frac{i}{2}} - \frac{2i}{u_j - \frac{i}{2}} \right). \tag{2.25}$$

2.1.3 Symmetries of the Bethe equations

The $\mathfrak{psu}(1,1|2)$ symmetry is realized in the standard way for Bethe equations. One can add Bethe roots $v, u, \dot{v} = \infty$ to the set of Bethe roots for any eigenstate. It is easy to convince oneself that the Bethe equations (2.24) for the original roots as well as for the new root are satisfied. Moreover, the momentum and energy (2.25) are not changed by the introduction of the additional root. This means that the eigenstates come in multiplets with degenerate momentum and energy eigenvalues. These multiplets are modules of the symmetry algebra $\mathfrak{psu}(1,1|2)$. Note that the Bethe roots $v, u, \dot{v} = \infty$ are allowed to appear in eigenstates more than one time, and thus even very large or infinite multiplets can be swept out with this symmetry.

Another type of symmetry that is very important to $\mathcal{N} = 4$ SYM exists only in the zero-momentum sector. Here one adds a single root $v = 0$ or $\dot{v} = 0$ to an eigenstate configuration while decreasing the length L by one unit [23]. The original Bethe equations are preserved and the Bethe equation for $v = 0$ and $\dot{v} = 0$ is equal to the zero-momentum condition, cf. (2.25). As the momentum and energy eigenvalues depend explicitly on the main Bethe roots u_k only, they are not affected by this transformation. This symmetry leads to an additional four-fold degeneracy of states because each of the Bethe roots $v = 0$ and $\dot{v} = 0$

can only appear once at maximum. The associated algebra is two copies of $\mathfrak{su}(1|1)$ whose modules are typically two-dimensional. As we discussed in Section 1.7, these two additional algebras are required for a consistent embedding of the spin chain into the full $\mathfrak{psu}(2,2|4)$ spin chain [41]. Their generators were constructed in [41,38] at the leading order, and they transform one site of the spin chain into two or vice versa. We will present these generators in detail in Section 2.2.

The third and most obscure type of symmetry was observed in [23]. The way in which the auxiliary Bethe roots v_k and $\dot{v}_k$ appear in the Bethe equations (2.15) is completely symmetric: The Bethe equation for v_k is exactly the same as the one for $\dot{v}_k$. Furthermore, the product in the Bethe equation for u_k involves a product over all v_j and $\dot{v}_j$ with the same form of factor. Therefore we can freely interchange them

$$v_j \leftrightarrow \dot{v}_{j'} \tag{2.26}$$

without violating the Bethe equations. As for the previous type of symmetry, modifying only the auxiliary Bethe roots does not change the momentum nor the energy. It is straightforward to convince oneself that this leads to a degeneracy of 2^{M_0} states where M_0 is the number of v_j roots which are distinct from $\dot{v}_j$ (in order to avoid coincident Bethe roots of the same type).

This symmetry is partially explained by the $\mathfrak{su}(2)$ $\mathfrak{B}$ automorphism. Indeed, in terms of the Cartan charges, the transformation of Bethe roots (2.26) has the same effect as the generators $\mathfrak{B}^{<<}$ and $\mathfrak{B}^{>>}$. Effectively the two flavors of auxiliary Bethe roots v and $\dot{v}$ form a doublet of the outer $\mathfrak{su}(2)$. Now, if there are M_0 auxiliary Bethe roots in total, the degeneracy is realized as the M_0-fold tensor product of $\mathfrak{su}(2)$ doublets. This tensor product is reducible and $\mathfrak{su}(2)$ symmetry can only account for degeneracy within the irreducible

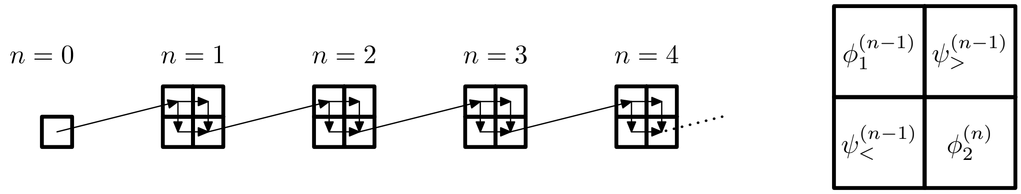

Figure 2.2: Structure of the spin representation (left). Each box represents one component of the module with the assignments shown on the right. Arrows represent simple roots of the algebra. The long diagonal arrows correspond to the middle node of the Dynkin diagram Fig. 2.1 while the short horizontal and vertical arrows correspond to the outer nodes.

components. Nevertheless, even the irreducible components turn out to be fully degenerate. Therefore the outer $\mathfrak{su}(2)$ explains only part of the extended degeneracy and there should be an even larger symmetry. This symmetry should have the full tensor product as one irreducible multiplet. It is somewhat reminiscent of the Yangian symmetry in the Haldane-Shastry model [49] which also displays fully degenerate tensor products. The closer investigation of this symmetry will be the main subject of the present chapter.

2.1.4 Commuting charges

A first question is whether the symmetry is merely constitutes an accidental degeneracy of the momentum and energy spectrum or whether it is a symmetry of the full integrable structure. This integrable structure includes an infinite family of generators or charges $\mathcal{Q}_r$ that commute with the Hamiltonian. Therefore, it is natural to look at the their eigenvalues. For the local $\mathcal{Q}_r$, these take the form[4]

$$Q_r = \frac{1}{r-1}\sum_{j=1}^{K}\left(\frac{i}{(u_j+\frac{i}{2})^{r-1}} - \frac{i}{(u_j-\frac{i}{2})^{r-1}}\right). \tag{2.27}$$

They depend on the main Bethe roots u_j only, just like the momentum and energy (2.25). Therefore, their spectrum displays this additional degeneracy.

However, this is not all there is to show; there are also nonlocal commuting charges whose

[4] With this normalization the energy eigenvalues are twice Q_2.

invariance properties might lead to some clues. For instance, the local charge eigenvalues Q_r are accurate only for $r \leq L$. For $r > L$ these charges wrap the spin chain state fully and they receive contributions from the auxiliary Bethe roots v_j and $\dot{v}_j$. This is best seen by considering the transfer matrix in the spin representation which serves as a generating function for the local charges as

$$T_{\text{spin}}(x) = \exp i \sum_{r=1}^{\infty} x^{r-1} Q_r. \tag{2.28}$$

A transfer matrix is a trace over a particular representation of the symmetry algebra. Therefore its eigenvalues in a particular representation are typically written as a sum with as many terms as there are components in the representation. The eigenvalues of a transfer matrix can often be reverse-engineered by a sort of analytic Bethe ansatz [46]. This requires some knowledge of the structure of the representations for which the transfer matrix is to be constructed. In particular, it is important to know what are the components and how are they connected by the simple roots of the algebra. The structure of the spin representation is depicted in Fig. 2.2. Now it is generally true that the transfer matrix has no dynamical poles, i.e. poles whose position depends on the Bethe roots. Conversely, the terms in the expression for the transfer matrix eigenvalue typically have many dynamical poles. These will have to cancel between the various terms once the Bethe equations are imposed. In particular, the Bethe equation for a particular type of Bethe root will have to ensure that the singularities between all terms cancel which are related by the simple root associated to that Bethe root, cf. Fig. 2.1. We are then led to the following expression for the transfer matrix eigenvalue in the spin representation

$$T_{\rm spin}(x) = \sum_{n=0}^{\infty} \left(\frac{x}{x-in}\right)^L \prod_{j=1}^{M} \frac{x-v_j}{x-v_j-in} \prod_{j=1}^{\dot{M}} \frac{x-\dot{v}_j}{x-\dot{v}_j-in}$$
$$\times \left(\delta_{n\neq 0} \prod_{j=1}^{K} \frac{x-u_j-i(n+\frac{1}{2})}{x-u_j-i(n-\frac{1}{2})} - 2\delta_{n\neq 0} + \prod_{j=1}^{K} \frac{x-u_j-i(n-\frac{1}{2})}{x-u_j-i(n+\frac{1}{2})} \right). \tag{2.29}$$

It is not difficult to see that the Bethe equations insure the cancellation of poles. For the pole at $x = u_k + i(m - \frac{1}{2})$, the contribution from the first term inside the parenthesis with $n = m$ cancels the contribution from the last term with $n = m - 1$, provided the second Bethe equation of (2.24) is satisfied. For the pole at $x = v_k + im$ $(m > 0)$, we find a coefficient proportional to

$$\prod_{j=1}^{K} \frac{v_k-u_j-\frac{i}{2}}{v_k-u_j+\frac{i}{2}} - 2 + \prod_{j=1}^{K} \frac{v_k-u_j+\frac{i}{2}}{v_k-u_j-\frac{i}{2}}$$
$$= \prod_{j=1}^{K} \frac{v_k-u_j+\frac{i}{2}}{v_k-u_j-\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{v_k-u_j-\frac{i}{2}}{v_k-u_j+\frac{i}{2}} - 1 \right)^2. \tag{2.30}$$

This is of the form $y^{-1}(y-1)^2$, where $y = 1$ by the first Bethe equation of (2.24). That is, the pole at $x = v_k + im$ is cancelled by a double zero. This double zero is essential as it cancels the double pole that arises for two coincident auxiliary Bethe roots $v_j = \dot{v}_{j'}$. Furthermore, it is straightforward to show that the local charge eigenvalues (2.27) (for $r \leq L$) follow from (2.28,2.29) and that only the one term with $n = 0$ contributes for $r \leq L$.

This expression for the transfer matrix is clearly invariant under the degeneracy transformation (2.26). Therefore the full transfer matrix obeys the enhanced symmetry, which is a clear hint that the integrable structure is compatible with the symmetry. It is however not fully invariant under it as the eigenvalues of transfer matrices in different representations show. For instance, for the fundamental and conjugate-fundamental representations it is easy to construct the transfer matrices

$$T_{\text{fund}}(x) = +\left(\frac{x+\frac{i}{2}}{x}\right)^L \prod_{j=1}^{M} \frac{x-v_j-\frac{i}{2}}{x-v_j+\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{x-u_j+i}{x-u_j} - 1\right) + \left(\frac{x-\frac{i}{2}}{x}\right)^L \prod_{j=1}^{\dot{M}} \frac{x-\dot{v}_j+\frac{i}{2}}{x-\dot{v}_j-\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{x-u_j-i}{x-u_j} - 1\right) \tag{2.31}$$

and

$$T_{\overline{\text{fund}}}(x) = +\left(\frac{x-\frac{i}{2}}{x}\right)^L \prod_{j=1}^{M} \frac{x-v_j+\frac{i}{2}}{x-v_j-\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{x-u_j-i}{x-u_j} - 1\right) + \left(\frac{x+\frac{i}{2}}{x}\right)^L \prod_{j=1}^{\dot{M}} \frac{x-\dot{v}_j-\frac{i}{2}}{x-\dot{v}_j+\frac{i}{2}} \left(\prod_{j=1}^{K} \frac{x-u_j+i}{x-u_j} - 1\right). \tag{2.32}$$

These expressions are clearly not invariant under the shuffling (2.26) of auxiliary Bethe roots. The violation of the symmetry may be related to the fact that the fundamental representations are centrally charged under $\mathfrak{su}(1,1|2)$ while the spin representation has zero central charge and thus belongs to $\mathfrak{psu}(1,1|2)$.[5]

Finally, we note that the transfer matrix in the spin representation (2.29) also has the degeneracy due to the $\mathfrak{psu}(1|1)$ symmetries (as do all of the $\mathcal{Q}_r$). Adding a v or $\dot{v}$ root at zero gives a factor of $x/(x-in)$ in each term of the sum. This is cancelled by decreasing L by one. However, again the degeneracy is not present for transfer matrix in the fundamental or conjugate-fundamental representations[6].

2.2 $\mathfrak{psu}(1|1)^2$ symmetry enhancement in the Lie algebra

As we have seen from examining the Bethe ansatz, and as discussed in Chapter 1, the $\mathfrak{psu}(1,1|2)$ symmetry of this sector is supplemented by a $\mathfrak{psu}(1|1)^2$ symmetry. Here we give

[5]It may be noted that the product $T_{\text{fund}}(x)\,T_{\overline{\text{fund}}}(x)$ is again invariant under switching the v and the $\dot{v}$. This is in agreement with the fact that the overall central charge for the two representations is zero.

[6]Their product does not have this degeneracy either.

its leading order representation ($\mathcal{O}(g)$) and again see that the restriction to cyclic states is required for this enhancement.

2.2.1 The representation at $\mathcal{O}(g)$

The action of the $\mathfrak{psu}(1|1)^2$ generators at leading order $\mathcal{O}(g)$ where initially obtained for the fermionic $\mathfrak{sl}(2)$ subsector (which has one hidden $\mathfrak{u}(1|1)$ symmetry) in [41]. From that result it is straightforward to obtain the action of these generators on the full set of fields by requiring the generators of $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ to commute at leading order. Alternatively, one can use the complete expression for the $\mathcal{O}(g)$ $\mathfrak{psu}(2,2|4)$ supersymmetry generators given in Appendix A.2. The generators $\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}$ act on two adjacent sites and turn them into a single site. Explicitly, the action takes the form

$$
\begin{aligned}
\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}|\phi^{(m)}_b\psi^{(n)}_{\mathfrak{c}}\rangle &= -\frac{1}{\sqrt{n+1}}\,\delta^{\mathfrak{a}}_{\mathfrak{c}}|\phi^{(n+m+1)}_b\rangle,\\
\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}|\psi^{(m)}_{\mathfrak{b}}\phi^{(n)}_c\rangle &= \frac{1}{\sqrt{m+1}}\,\delta^{\mathfrak{a}}_{\mathfrak{b}}|\phi^{(n+m+1)}_c\rangle,\\
\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}|\psi^{(m)}_{\mathfrak{b}}\psi^{(n)}_{\mathfrak{c}}\rangle &= \frac{\sqrt{n+1}}{\sqrt{(m+1)(m+n+2)}}\,\delta^{\mathfrak{a}}_{\mathfrak{b}}|\psi^{(n+m+1)}_{\mathfrak{c}}\rangle\\
&\quad+\frac{\sqrt{m+1}}{\sqrt{(n+1)(m+n+2)}}\,\delta^{\mathfrak{a}}_{\mathfrak{c}}|\psi^{(n+m+1)}_{\mathfrak{b}}\rangle,\\
\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}|\phi^{(m)}_b\phi^{(n)}_c\rangle &= \frac{1}{\sqrt{n+m+1}}\,\varepsilon_{bc}\varepsilon^{\mathfrak{a}\mathfrak{d}}|\psi^{(n+m)}_{\mathfrak{d}}\rangle.
\end{aligned}
\tag{2.33}
$$

Conversely, the generators $\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}$ act on a single site and turn it into two

$$
\begin{aligned}
\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}|\phi^{(n)}_b\rangle &= \sum_{k=0}^{n-1}\frac{1}{\sqrt{k+1}}\,\varepsilon^{\mathfrak{a}\mathfrak{c}}|\psi^{(k)}_{\mathfrak{c}}\phi^{(n-1-k)}_b\rangle-\sum_{k=0}^{n-1}\frac{1}{\sqrt{n-k}}\,\varepsilon^{\mathfrak{a}\mathfrak{c}}|\phi^{(k)}_b\psi^{(n-1-k)}_{\mathfrak{c}}\rangle,\\
\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}|\psi^{(n)}_{\mathfrak{b}}\rangle &= \sum_{k=0}^{n-1}\frac{\sqrt{n-k}}{\sqrt{(k+1)(n+1)}}\,\varepsilon^{\mathfrak{a}\mathfrak{c}}|\psi^{(k)}_{\mathfrak{c}}\psi^{(n-1-k)}_{\mathfrak{b}}\rangle\\
&\quad+\sum_{k=0}^{n-1}\frac{\sqrt{k+1}}{\sqrt{(n-k)(n+1)}}\,\varepsilon^{\mathfrak{a}\mathfrak{c}}|\psi^{(k)}_{\mathfrak{b}}\psi^{(n-1-k)}_{\mathfrak{c}}\rangle\\
&\quad-\sum_{k=0}^{n}\frac{1}{\sqrt{n+1}}\,\delta^{\mathfrak{a}}_{\mathfrak{b}}\varepsilon^{cd}|\phi^{(k)}_c\phi^{(n-k)}_d\rangle.
\end{aligned}
\tag{2.34}
$$

By inspection we see that the representations of $\mathfrak{psu}(1|1)^2$ has the manifest $\mathfrak{su}(2)$ $\mathfrak{B}^{\mathfrak{ab}}$ automorphism. The $\mathfrak{psu}(1|1)^2$ generators transform under conjugation as

$$(\hat{\mathfrak{Q}}^{\mathfrak{a}})^\dagger = \varepsilon_{\mathfrak{ab}}\hat{\mathfrak{S}}^{\mathfrak{b}}, \quad \hat{\mathfrak{D}}^\dagger = \hat{\mathfrak{D}}, \quad (\mathfrak{B}^{\mathfrak{ab}})^\dagger = -\varepsilon_{\mathfrak{ac}}\varepsilon_{\mathfrak{bd}}\mathfrak{B}^{\mathfrak{cd}}. \tag{2.35}$$

Importantly, note that from the leading order $\mathfrak{psu}(1|1)^2$ supercharges we can immediately obtain the one-loop $\mathcal{O}(g^2)$ dilatation generator $\delta\mathfrak{D}_{(2)}$, using (1.66) and (1.68),

$$\delta\mathfrak{D}_{(2)} = 2\{\hat{\mathfrak{Q}}^<_{(1)}, \hat{\mathfrak{S}}^>_{(1)}\} = -2\{\hat{\mathfrak{Q}}^>_{(1)}, \hat{\mathfrak{S}}^<_{(1)}\}. \tag{2.36}$$

This equation implies that $\delta\mathfrak{D}_{(2)}$ is the sum of interactions that remove two fields and insert two new fields (acting on two adjacent sites for the planar case). As first shown in [14], which generalized earlier results of [50], $\delta\mathfrak{D}_{(2)}$ acts by projecting two sites onto modules of definite $\mathfrak{psu}(2,2|4)$ "spin" j with coefficient $h(j)$. The quadratic Casimir of $\mathfrak{psu}(2,2|4)$ has eigenvalues $j(j+1)$ on a module of spin j, just as for $\mathfrak{su}(2)$. h gives the harmonic numbers

$$h(k) = \sum_{k'=1}^{k} \frac{1}{k'} = \psi(k+1) - \psi(1). \tag{2.37}$$

Here ψ is the digamma function. The harmonic numbers also play an essential role for $\delta\mathfrak{D}_{(4)}$, as we will see in Chapter 3.

We will not pursue it here, but it is also possible to work out the explicit interactions of $\delta\mathfrak{D}_{(2)}$. In terms of these two-site to two-site interactions $\delta\mathfrak{D}_{(2)}(1,2)$, we have (in the planar limit)

$$\delta\mathfrak{D}_{(2)}|X\rangle = \sum_{i=0}^{L-1} \mathfrak{U}^{-i}\delta\mathfrak{D}_{(2)}(1,2)\mathfrak{U}^i|X\rangle, \tag{2.38}$$

where $\delta\mathfrak{D}_{(2)}(1,2)$ acts on sites 1 and 2. From here it is also clear how to generalize to higher order corrections to symmetry generators that remove and insert equal numbers of fields. Note that the form (2.38) applies even to noncyclic states, giving the $\mathfrak{psu}(1,1|2)$ generator correction $2\mathfrak{J}^{+-}_{(2)}$.

For length-changing generators, which act on cyclic states only, we simply need to apply the interactions to the first site(s) of the chain (with a factor of L), and to project onto cyclic states. Cyclicity then insures that these generators act homogeneously. Explicitly, for the $\hat{\mathfrak{Q}}$ and $\hat{\mathfrak{S}}$ at $\mathcal{O}(g)$ acting on a cyclic state $|X\rangle$ we have

$$\hat{\mathfrak{Q}}_{(1)}|X\rangle = \frac{L}{L+1}\sum_{i=0}^{L}\mathfrak{U}^{-i}\hat{\mathfrak{Q}}_{(1)}(1)|X\rangle,$$
$$\hat{\mathfrak{S}}_{(1)}|X\rangle = \frac{L}{L-1}\sum_{i=0}^{L-2}\mathfrak{U}^{-i}\hat{\mathfrak{S}}_{(1)}(1,2)|X\rangle, \tag{2.39}$$

$\hat{\mathfrak{Q}}_{(1)}(1)$ acts on each component of $|X\rangle$ by replacing the field at the first site with (a sum of) two new fields (2.34) and $\hat{\mathfrak{S}}_{(1)}(1,2)$ replaces the fields of the first two sites with (a sum of) one new field (2.33).

2.2.2 Satisfying the algebra with gauge transformations

We now check that the action of the $\mathfrak{psu}(1|1)^2$ generators presented in the previous section satisfy the symmetry algebra at $\mathcal{O}(g)$. We need to check that the $\mathfrak{psu}(1|1)^2$ generators commute with the $\mathfrak{psu}(1,1|2)$ generators. Since the commutators vanish except when the generators act on common sites, it is sufficient to examine the commutators involving the

$\hat{\mathfrak{Q}}$ acting on one initial site (and two initial sites for $\hat{\mathfrak{S}}$). For example, consider

$$\begin{aligned}
\{\mathfrak{Q}^{1+<},\hat{\mathfrak{Q}}^{>}\}_{(1)}|\phi_a^{(n)}\rangle &= \{\mathfrak{Q}^{1+<}_{(0)},\hat{\mathfrak{Q}}^{>}_{(1)}\}|\phi_a^{(n)}\rangle \\
&= \sqrt{n+1}\,\delta_a^1\hat{\mathfrak{Q}}^{>}_{(1)}|\psi_>^{(n)}\rangle \\
&\quad -\sum_{k=0}^{n-1}\mathfrak{Q}^{1+<}_{(0)}\left(\frac{1}{\sqrt{k+1}}\big|\psi_<^{(k)}\phi_a^{(n-1-k)}\big\rangle-\frac{1}{\sqrt{n-k}}\big|\phi_a^{(k)}\psi_<^{(n-1-k)}\big\rangle\right) \\
&= -\sum_{k=0}^{n-1}\delta_a^1\left(\frac{\sqrt{n-k}}{\sqrt{k+1}}\big|\psi_<^{(k)}\psi_>^{(n-1-k)}\big\rangle+\frac{\sqrt{k+1}}{\sqrt{n-k}}\big|\psi_>^{(k)}\psi_<^{(n-1-k)}\big\rangle\right) \\
&\quad -\sum_{k=0}^{n}\delta_a^1\varepsilon^{de}\big|\phi_d^{(k)}\phi_e^{(n-k)}\big\rangle-\sum_{k=0}^{n-1}\big|\phi_2^{(k+1)}\phi_a^{(n-k-1)}\big\rangle \\
&\quad +\sum_{k=0}^{n-1}\delta_a^1\left(\frac{\sqrt{n-k}}{\sqrt{k+1}}\big|\psi_<^{(k)}\psi_>^{(n-1-k)}\big\rangle+\frac{\sqrt{k+1}}{\sqrt{n-k}}\big|\psi_>^{(k)}\psi_<^{(n-1-k)}\big\rangle\right) \\
&\quad +\sum_{k=0}^{n-1}\big|\phi_a^{(k)}\phi_2^{(n-k)}\big\rangle. \qquad (2.40)
\end{aligned}$$

We have substituted (1.61) and (2.34) and made straightforward simplifications. Note that we have included an extra minus sign due to fermions for the first term in the second-to-last line. The first and third lines of the last expression cancel, leaving three terms involving two bosonic fields. One can easily check that for $a = 1, 2$ the remaining terms almost cancel. What remains is

$$\{\mathfrak{Q}^{1+>},\hat{\mathfrak{Q}}^{<}\}_{(1)}|\phi_a^{(n)}\rangle = \big|\phi_2^{(0)}\phi_a^{(n)}\big\rangle-\big|\phi_a^{(n)}\phi_2^{(0)}\big\rangle. \qquad (2.41)$$

Repeating the same calculation for a fermionic initial site we find

$$\{\mathfrak{Q}^{1+<},\hat{\mathfrak{Q}}^{>}\}_{(1)}|\psi_{\mathfrak{a}}^{(n)}\rangle = \big|\phi_2^{(0)}\psi_{\mathfrak{a}}^{(n)}\big\rangle-\big|\psi_{\mathfrak{a}}^{(n)}\phi_2^{(0)}\big\rangle. \qquad (2.42)$$

Since this commutator acts the same way on all fields, it will vanish on cyclic states. The insertion of $\phi_2^{(0)}$ before one field will cancel with its insertion after the neighboring field to the left. For this cancellation to be complete, the cyclicity condition is essential. This type of cancellation can occur more generally for any generator $\mathfrak{M}$ that acts homogeneously as

$$\mathfrak{M}\big|Y_1 \dots Y_m X_1 \dots X_n\big\rangle = \sum C_{Y_1 \dots Y_m Z_1 \dots Z_{m'}} \big|Z_1 \dots Z_{m'} X_1 \dots X_n\big\rangle,$$
$$\mathfrak{M}\big|X_1 \dots X_n Y_1 \dots Y_m\big\rangle = \sum (-1)^{(\mathfrak{M}(X_1 \dots X_n))} C_{Y_1 \dots Y_m Z_1 \dots Z_{m'}} \big|X_1 \dots X_n Z_1 \dots Z_{m'}\big\rangle, \tag{2.43}$$

for all fields X_i, Y_i and Z_i, with some coefficients $C_{Y_1 \dots Y_m Z_1 \dots Z_{m'}}$. In the case above we have $\mathfrak{M} = \{\mathfrak{Q}^{1+<}, \hat{\mathfrak{Q}}^{>}\}_{(1)}$, $n = 1$, $m = 0$ (no fields Y), $m' = 1$, and the only nonvanishing C is $C_{\phi_2^{(0)}} = 1$. If $\mathfrak{M}$ is length-preserving, that is, $m{=}m'$, it vanishes on any periodic state, and we will call this term a chain derivative. However, for dynamic $\mathfrak{M}$, since this only vanishes on cyclic states as arise from gauge theory, we will call this a gauge transformation. In fact, this is very similar to terms that appear when checking that the supersymmetry variations of the fields satisfy the $\mathfrak{psu}(2,2|4)$ algebra. For instance, the supercharge variations $\mathfrak{Q}$ commute up to a commutator with the scalar fields,

$$\{\mathfrak{Q}^a_\alpha, \mathfrak{Q}^b_\beta\} X \sim g\, \varepsilon_{\alpha\beta} \varepsilon^{abcd} [\Phi_{cd}, X], \tag{2.44}$$

which vanishes for gauge-invariant product of fields X.

The full set of gauge transformations that appear at $\mathcal{O}(g)$ are

$$\begin{aligned} \{\mathfrak{Q}^{a+\mathfrak{b}}, \hat{\mathfrak{Q}}^{\mathfrak{c}}\}_{(1)} |X\rangle &= \varepsilon^{\mathfrak{bc}} \varepsilon^{ad} \left(\big|\phi^{(0)}_d X\big\rangle - \big|X \phi^{(0)}_d\big\rangle \right), \\ [\mathfrak{J}^{++}, \hat{\mathfrak{Q}}^{\mathfrak{a}}]_{(1)} |X\rangle &= \varepsilon^{\mathfrak{ab}} \left(\big|\psi^{(0)}_{\mathfrak{b}} X\big\rangle - (-1)^X \big|X \psi^{(0)}_{\mathfrak{b}}\big\rangle \right), \end{aligned} \tag{2.45}$$

$$\begin{aligned} \{\mathfrak{Q}^{a-\mathfrak{b}}, \hat{\mathfrak{S}}^{\mathfrak{c}}\}_{(1)} \big|\phi^{(0)}_d X\big\rangle &= -\delta^a_d \varepsilon^{\mathfrak{bc}} |X\rangle, & \{\mathfrak{Q}^{a-\mathfrak{b}}, \hat{\mathfrak{S}}^{\mathfrak{c}}\}_{(1)} \big|X \phi^{(0)}_d\big\rangle &= \delta^a_d \varepsilon^{\mathfrak{bc}} |X\rangle, \\ [\mathfrak{J}^{--}, \hat{\mathfrak{Q}}^{\mathfrak{a}}]_{(1)} \big|\psi^{(0)}_{\mathfrak{b}} X\big\rangle &= \delta^{\mathfrak{a}}_{\mathfrak{b}} |X\rangle, & [\mathfrak{J}^{--}, \hat{\mathfrak{Q}}^{\mathfrak{a}}]_{(1)} \big|X \psi^{(0)}_{\mathfrak{b}}\big\rangle &= -(-1)^X \delta^{\mathfrak{a}}_{\mathfrak{b}} |X\rangle. \end{aligned}$$

To complete the verification that the $\mathcal{O}(g)$ generators satisfy the symmetry constraints, we need to check all of their $\mathcal{O}(g^2)$ commutators. However, by hermiticity and $\mathfrak{B}$ symmetry it suffices to compute, for instance,

$$(\hat{\mathfrak{Q}}^{<}_{(1)})^2 = 0 \quad \text{and} \quad \{\hat{\mathfrak{Q}}^{<}_{(1)}, \hat{\mathfrak{S}}^{<}_{(1)}\} = 0. \tag{2.46}$$

We leave it as an exercise for the reader to confirm that these equations are indeed satisfied. This is only a consistency check: As shown in Appendix A, the $\mathcal{O}(g)$ algebra of the full $\mathcal{N} = 4$ SYM spin chain fixes uniquely all of the supercharges, including $\hat{\mathfrak{Q}}_{(1)}$ and $\hat{\mathfrak{S}}_{(1)}$.

2.3 Example degenerate states

Let us now return to the degeneracy of the Bethe equations. We will try to get acquainted with it by constructing explicitly some degenerate states. Here and in the following sections we will work only at leading order in the coupling constant g. In other words, the $\mathfrak{psu}(1,1|2)$ generators $\mathfrak{Q}, \mathfrak{J}$ are truncated at $\mathcal{O}(g^0)$, and for the $\mathfrak{psu}(1|1)^2$ generators $\hat{\mathfrak{Q}}, \hat{\mathfrak{S}}$ we take only the $\mathcal{O}(g^1)$ contributions $\hat{\mathfrak{Q}}_{(1)}, \hat{\mathfrak{S}}_{(1)}$.

2.3.1 Vacuum

The simplest state which is part of a nontrivial multiplet is

$$|0_L\rangle = |\psi^{(0)}_{<}\psi^{(0)}_{<}\psi^{(0)}_{<} \ldots \psi^{(0)}_{<}\rangle. \tag{2.47}$$

We shall call it the vacuum state of length L. Note that it is not the ground state, but it is a homogeneous eigenstate of the Hamiltonian and we can place excitations on it by flipping some of the spins. In the Bethe ansatz it is represented by a $K = L$ main Bethe roots and

$M = L$ auxiliary Bethe roots, so the Bethe equations become

$$1 = \prod_{j=1}^{L} \frac{v_k - u_j - \frac{i}{2}}{v_k - u_j + \frac{i}{2}},$$
$$1 = \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{L} \frac{u_k - u_j + i}{u_k - u_j - i} \prod_{j=1}^{L} \frac{u_k - v_j - \frac{i}{2}}{u_k - v_j + \frac{i}{2}}. \tag{2.48}$$

This system can be solved via introducing certain polynomials whose roots are the Bethe roots [51, 48]. First we define

$$P_1(x) = \prod_{j=1}^{L}(x - u_j + \tfrac{i}{2}) - \prod_{j=1}^{L}(x - u_j - \tfrac{i}{2}) \propto \prod_{j=1}^{L}(x - v_j). \tag{2.49}$$

$P_1(x)$ is proportional to the last product because its roots satisfy the first Bethe equation of (2.48) for v. Then using the two equivalent expressions for P_1 in $P_1(u_k + \frac{i}{2})/P_1(u_k - \frac{i}{2})$, and comparing to the second Bethe equation, we infer that the Bethe roots u satisfy

$$(u + \tfrac{i}{2})^L = (u - \tfrac{i}{2})^L. \tag{2.50}$$

Next we define another polynomial P_2,

$$P_2(x) = (x + \tfrac{i}{2})^L - (x - \tfrac{i}{2})^L. \tag{2.51}$$

Since the roots of P_2 are the u, we can write P_1 in terms of P_2,

$$P_1(x) = P_2(x + \tfrac{i}{2}) - P_2(x - \tfrac{i}{2}) = (x + i)^L + (x - i)^L - 2x^L. \tag{2.52}$$

However, the roots of P_1 are the Bethe roots v, so it follows that the v are solutions to

$$(v + i)^L + (v - i)^L = 2v^L. \tag{2.53}$$

The equation for the main Bethe roots (2.50) can be solved explicitly as $u_k = \frac{1}{2}\cot(\pi k/L)$. (Note that $u = \infty$ is a root, and similarly $v = \infty$ is a double root. This is not a highest weight state). From (2.25) we find that he momentum and energy of this state are given by

$$P = \pi(L - 1), \qquad E = 4L. \tag{2.54}$$

Using P_1 and P_2, one can also find the eigenvalue of the transfer matrix in the spin representation,

$$T_{\text{spin}}(x) = 1 + \sum_{n=0}^{\infty} \frac{x^L\big(2x^L - (x+i)^L - (x-i)^L\big)}{(x-in)^L(x-in-i)^L}\,. \tag{2.55}$$

Note that for even L the overall momentum is maximal, $P \equiv \pi$, while for odd L the overall momentum is zero, $P \equiv 0$. Therefore only the states with odd L are physical states of the gauge theory, and only for those the symmetry algebra enlarges by $\mathfrak{psu}(1|1)^2$.

The vacuum state is part of a $\mathfrak{su}(2)$ multiplet of $L+1$ states. The $L+1$ components are $(\mathfrak{B}^{<<})^{0,1,\ldots,L}|0_L\rangle$. Note also that it is part of a multiplet of $L-1$ multiplets of $\mathfrak{psu}(1,1|2)$.[7] The $L-1$ highest weight components are obtained by acting with the cubic operator given in Appendix B; they read $((\mathfrak{J}^3)^{<<})^{0,1,\ldots,L-2}|0_L\rangle$.

2.3.2 Degenerate eigenstates

Let us now consider the set of states where the flavor of one auxiliary Bethe root is flipped. One can convince oneself that a state is composed from basis states of the typical form

$$\mathfrak{Q}^{2-<}(k)\,\mathfrak{Q}^{1-<}(l)\,\mathfrak{J}^{++}(m)\,|0_L\rangle \sim |\ldots \overset{\overset{k}{\downarrow}}{\phi_1^{(0)}} \ldots \overset{\overset{l}{\downarrow}}{\phi_2^{(0)}} \ldots \overset{\overset{m}{\downarrow}}{\psi_<^{(1)}} \ldots\rangle. \tag{2.56}$$

The arguments of the generators correspond to the sites of the spin chain on which they should act. Here we have only displayed the excitations while the vacuum sites $\psi_<^{(0)}$ have been suppressed. The operators $\mathfrak{J}(k)$ act as the leading order generators in (1.60-1.61) on site k of the chain[8]. Some of the three excitations coincide on a single site giving rise to $\phi_1^{(1)}$, $\phi_2^{(1)}$ or $\psi_>^{(0)}$. Explicitly, we find precisely $L+1$ states of this form completely degenerate

[7] Due to the $\mathfrak{su}(2)$ grading of the $\mathfrak{psu}(1,1|2)$ algebra these two numbers differ by two.

[8] The statistics of the fermionic generators $\mathfrak{Q}(k)$ is taken into account by first permuting it to its place of action. This may cause a sign flip.

with the vacuum $|0_L\rangle$. Three of these states are descendants of $\mathfrak{psu}(1,1|2)$

$$\varepsilon_{ab}\mathfrak{Q}^{a-<}\mathfrak{Q}^{b-<}\mathfrak{J}^{++}|0_L\rangle, \quad \varepsilon_{ab}\mathfrak{Q}^{a-<}\mathfrak{Q}^{b+<}|0_L\rangle, \quad \varepsilon_{ab}\mathfrak{Q}^{a+<}\mathfrak{Q}^{b-<}|0_L\rangle, \tag{2.57}$$

and one is the $\mathfrak{su}(2)$ descendant

$$\mathfrak{B}^{<<}|0_L\rangle. \tag{2.58}$$

However, since $\mathfrak{B}^{\mathfrak{ab}}$ does not commute with $\mathfrak{psu}(1,1|2)$, it is more convenient to use instead the cubic operator $(\mathfrak{J}^3)^{\mathfrak{ab}}$ presented in Appendix B, (built from cubic combinations of ordinary $\mathfrak{psu}(1,1|2)$ and $\mathfrak{su}(2)$ generators),

$$(\mathfrak{J}^3)^{<<}|0_L\rangle. \tag{2.59}$$

The generator $(\mathfrak{J}^3)^{\mathfrak{ab}}$ commutes with $\mathfrak{psu}(1,1|2)$ and therefore moves between $\mathfrak{psu}(1,1|2)$ highest weight states.

For even L (and nonzero momentum) this exhausts the set of trivial descendants. There remain $L-3$ unexplained degenerate states. For odd L the vacuum is a zero-momentum state and therefore the additional $\mathfrak{psu}(1|1)^2$ symmetry applies. It yields one further descendant

$$\hat{\mathfrak{S}}^{<}\hat{\mathfrak{Q}}^{<}|0_L\rangle. \tag{2.60}$$

Consequently there are only $L-4$ unexplained degenerate states.

We find one degenerate state with the very simple form

$$\begin{aligned}|1_L\rangle = &\sum_{n,k=1}^{L}(-1)^k\mathfrak{J}^{++}(k+n)\,\varepsilon_{ab}\mathfrak{Q}^{a-<}(1+n)\,\mathfrak{Q}^{b-<}(L+n)\,|0_L\rangle \\ &-(1-(-1)^L)\mathfrak{B}^{<<}\,|0_L\rangle.\end{aligned} \tag{2.61}$$

One can easily confirm that it is a highest weight state of $\mathfrak{psu}(1,1|2)$. For even length this state is indeed linearly independent of the above descendants. For odd length, however, the state is proportional to a $\mathfrak{psu}(1|1)^2$ descendant, $|1_L\rangle \sim \hat{\mathfrak{S}}^{<}\hat{\mathfrak{Q}}^{<}|0_L\rangle$. This is a special case because of zero overall momentum. We will return to this issue in the next section.

We have also found a second degenerate state with a slightly more complicated form

$$\begin{aligned}|2_L\rangle = &\sum_{n,k=1}^{L}(-1)^k(2k-L-1+\delta_{k1}-\delta_{kL})\mathfrak{J}^{++}(k+n)\,\varepsilon_{ab}\mathfrak{Q}^{a-<}(1+n)\,\mathfrak{Q}^{b-<}(L+n)\,|0_L\rangle\\ &+\sum_{k=2}^{L}\sum_{n=1}^{L}(-1)^k\mathfrak{J}^{++}(k+n)\,\varepsilon_{ab}\mathfrak{Q}^{a-<}(2+n)\,\mathfrak{Q}^{b-<}(L+n)\,|0_L\rangle\\ &+(1+(-1)^L)(L-1)\mathfrak{B}^{<<}\,|0_L\rangle \end{aligned} \tag{2.62}$$

This state is also a highest weight state of $\mathfrak{psu}(1,1|2)$, and for odd length is not a $\mathfrak{psu}(1|1)^2$ descendant of $|0_L\rangle$.

2.3.3 Parity

The degenerate states do not all have the same parity. For L even or odd we find $\frac{1}{2}(L-2)$ or $\frac{1}{2}(L-3)$ states, respectively, which have opposite parity than the vacuum[9]. Recalling the above results, this means that after removing the trivial descendants there is always one more degenerate state with opposite parity than with equal parity. More explicitly, we can say that $|1_L\rangle$ has the opposite parity as $|0_L\rangle$ for even L and the same parity as $|0_L\rangle$ for odd L. Conversely, the state $|2_L\rangle$ has the same parity as $|0_L\rangle$ for even L and the opposite parity as $|0_L\rangle$ for odd L.

2.4 Nonlocal symmetry

To account for the additional degeneracy it is natural to seek new symmetry generators. We will take into account the finding regarding the Bethe ansatz and the form of the

[9]The definition of parity may include shifts $\mathcal{U}^k$ of the chain which act nontrivially on states with overall momentum. It is therefore more convenient to only specify the parity w.r.t. a reference state.

degenerate states found in the previous section to construct some nonlocal generators $\mathcal{Y}$. We will then investigate their algebra.

2.4.1 Bilocal generators

First of all, an elementary step between two degenerate Bethe states consists in changing the flavor of one auxiliary Bethe root as discussed in Section 2.1.3. The $\mathfrak{su}(2)$ generators $\mathfrak{B}^{\mathfrak{ab}}$ qualitatively act in the same way. This indicates that the new generators will be in the same representation, i.e. in the adjoint/spin-1/triplet representation of $\mathfrak{su}(2)$. We will thus denote them by $\mathcal{Y}^{\mathfrak{ab}} = \mathcal{Y}^{\mathfrak{ba}}$.

As the example degenerate states given in the previous section have multiple nonadjacent excitations, we should look for nonlocal generators. The simplest degenerate state $|1_L\rangle$ has a pair of adjacent excitations and a single excitation which is not near the pair. A generator which creates such a state from the vacuum $|0_L\rangle$ consequently has to be bilocal (at least). More complicated states with multilocal excitations could be generated by repeated application of these bilocal generators.

Furthermore, we know that the form of the example degenerate state $|1_L\rangle$ is qualitatively identical to the second order $\mathfrak{psu}(1|1)^2$ descendant $\hat{\mathfrak{S}}^{<}\hat{\mathfrak{Q}}^{<}|0_L\rangle$. Thus we expect $\mathcal{Y}^{\mathfrak{ab}}$ to act similarly as $\hat{\mathfrak{S}}^{\{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}\}}$.

Here we have to make a distinction between states with zero and states with nonzero momentum. For zero momentum the combination $\hat{\mathfrak{S}}^{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}}$ explains the degenerate state $|1_L\rangle$. However, due to the $\mathfrak{psu}(1|1)^2$ algebra, it cannot explain any of the other degenerate states. Conversely, in the case of nonzero momentum the individual generators $\hat{\mathfrak{S}}^{\mathfrak{a}}$ and $\hat{\mathfrak{Q}}^{\mathfrak{b}}$ cannot be defined independently because it is not possible to change the length of the spin chain preserving the momentum. It is nevertheless possible to consistently define the product

$\hat{\mathfrak{S}}^{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}}$ for nonzero momentum states. This is the bilocal length-preserving operator

$$\mathcal{Y}^{\mathfrak{ab}} = \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \tfrac{1}{2}\delta_{i,0} - \tfrac{1}{2}\delta_{i,L+1})\,\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{\{\mathfrak{a}}(1,2)\,\mathcal{U}^{i}\,\hat{\mathfrak{Q}}^{\mathfrak{b}\}}(1)\,\mathcal{U}^{-j}. \tag{2.63}$$

Recall that $\mathcal{U}$ is the operator that shifts the chain by one site to the right; it commutes with all of the local symmetry generators. The summation over j ensures that $\mathcal{Y}^{\mathfrak{ab}}$ acts homogeneously on the chain, and the symmetrization in the indices gives it one unit of $\mathfrak{B}$ charge, as needed to explain the degeneracy. The generator $\hat{\mathfrak{Q}}(1)$ removes the first site of the chain and replaces it with two sites, and $\hat{\mathfrak{S}}(1,2)$ replaces the first two sites of the spin chain with one. So, the generator $\mathcal{Y}^{\mathfrak{ab}}$ consists of products of the $\hat{\mathfrak{Q}}$ and $\hat{\mathfrak{S}}$ interactions acting all possible distances apart, with equal weight except for a symmetric regularization when the $\hat{\mathfrak{S}}$ interaction acts on both sites created by the $\hat{\mathfrak{Q}}$ interaction. The regularization resolves the one-site ambiguity in where to place newly created sites.

For zero-momentum states the action of $\mathcal{Y}^{\mathfrak{ab}}$ is equivalent to the action of $\hat{\mathfrak{S}}^{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}}$. Therefore, it cannot be used to immediately explain the additional degeneracy beyond the established $\mathfrak{psu}(1|1)^2$ symmetry in the zero-momentum sector. We will discuss this further in the Conclusions. However, $\mathcal{Y}^{\mathfrak{ab}}$ does commute exactly with $\mathfrak{psu}(1,1|2)$ and with the Hamiltonian, even if the momentum is nonzero; a proof is given in Appendix C. Therefore the existence of $\mathcal{Y}^{\mathfrak{ab}}$ proves the additional degeneracies for all states with nonzero momentum.

The generators $\mathcal{Y}^{\mathfrak{ab}}$ immediately explain the form of the simplest degenerate state (2.61) found in the last section; it is related to the vacuum by applying $\mathcal{Y}^{<<}$ once

$$|1_L\rangle \sim \mathcal{Y}^{<<}|0_L\rangle. \tag{2.64}$$

For even length $L \leq 10$ we have checked directly that the remaining descendants are given by

$$\mathcal{Y}^{<>}|1_L\rangle, \quad \ldots, \quad (\mathcal{Y}^{<>})^{(L-4)}|1_L\rangle. \tag{2.65}$$

For the odd-length states, which have vanishing momentum, one can easily convince oneself using $\mathcal{Y}^{ab} \simeq \hat{\mathfrak{S}}^{\{\mathfrak{a}}\hat{\mathfrak{Q}}^{\mathfrak{b}\}}$ that the states (2.65) are all proportional to $|1_L\rangle$. Furthermore it might be useful to know the eigenvalue of $\mathcal{Y}^{<>}$ on the vacuum state; we find

$$\mathcal{Y}^{<>}|0_L\rangle = \begin{cases} 2L|0_L\rangle & \text{for odd } L, \\ 0 & \text{for even } L. \end{cases} \tag{2.66}$$

It is curious to note that there exist a very similar bilocal generator $\mathcal{X}$ which is a $\mathfrak{B}$-singlet

$$\mathcal{X} = \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \tfrac{1}{2}\delta_{i,0} - \tfrac{1}{2}\delta_{i,L+1})\tfrac{1}{2}\varepsilon_{\mathfrak{ab}}\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\,\mathcal{U}^{i}\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1)\,\mathcal{U}^{-j}. \tag{2.67}$$

Like $\mathcal{Y}^{\mathfrak{ab}}$ it commutes with the $\mathfrak{psu}(1,1|2)$ algebra and the one-loop Hamiltonian as discussed in Appendix C. This generator apparently does not map between different $\mathfrak{psu}(1,1|2)$ multiplets. For instance, the eigenvalue of $\mathcal{X}$ on the states $|0_L\rangle$ as well as $|1_L\rangle$ is the same as in (2.66).

2.4.2 An infinite-dimensional algebra

Let us first understand the algebra of $\mathcal{Y}^{\mathfrak{ab}}$ in the zero-momentum sector where we have a representation in terms of $\mathfrak{psu}(1|1)^2$ generators. It is not difficult to convince oneself of the following relations

$$\begin{aligned} \big[\mathfrak{B}^{\mathfrak{ab}}, \hat{\mathfrak{D}}^m\hat{\mathfrak{Q}}^{\mathfrak{c}}\hat{\mathfrak{S}}^{\mathfrak{d}}\big] &= \hat{\mathfrak{D}}^m\varepsilon^{\mathfrak{c}\{\mathfrak{b}}\hat{\mathfrak{Q}}^{\mathfrak{a}\}}\hat{\mathfrak{S}}^{\mathfrak{d}} - \hat{\mathfrak{D}}^m\hat{\mathfrak{Q}}^{\mathfrak{c}}\hat{\mathfrak{S}}^{\{\mathfrak{b}}\varepsilon^{\mathfrak{a}\}\mathfrak{d}}, \\ \big[\hat{\mathfrak{D}}^m\hat{\mathfrak{Q}}^{\mathfrak{a}}\hat{\mathfrak{S}}^{\mathfrak{b}}, \hat{\mathfrak{D}}^n\hat{\mathfrak{Q}}^{\mathfrak{c}}\hat{\mathfrak{S}}^{\mathfrak{d}}\big] &= \hat{\mathfrak{D}}^{m+n+1}\varepsilon^{\mathfrak{cb}}\hat{\mathfrak{Q}}^{\mathfrak{a}}\hat{\mathfrak{S}}^{\mathfrak{d}} - \hat{\mathfrak{D}}^{m+n+1}\varepsilon^{\mathfrak{ad}}\hat{\mathfrak{Q}}^{\mathfrak{c}}\hat{\mathfrak{S}}^{\mathfrak{b}}. \end{aligned} \tag{2.68}$$

Denoting these combinations by $\mathcal{Y}_k^{\mathfrak{ab}}$, $k = 0, 1, 2, \ldots$, such that $\mathcal{Y}_0^{\mathfrak{ab}} = \mathfrak{B}^{\mathfrak{ab}}$ and $\mathcal{Y}_n^{\mathfrak{ab}} \simeq \hat{\mathfrak{D}}^{n-1}\mathcal{Y}^{\mathfrak{ab}}$, we obtain the infinite-dimensional algebra

$$[\mathcal{Y}_m^{\mathfrak{ab}}, \mathcal{Y}_n^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}\mathcal{Y}_{m+n}^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\mathcal{Y}_{m+n}^{\mathfrak{cb}}. \tag{2.69}$$

This algebra is a parabolic subalgebra of the loop algebra of $\mathfrak{su}(2)$.

We could also introduce the $\mathfrak{B}$-invariant generators $\mathcal{X}_0 = \hat{\mathfrak{A}}$ and $\mathcal{X}_n = \hat{\mathfrak{D}}^{n-1}\mathcal{X}$ which form an abelian algebra and which also commute with $\mathcal{Y}_n^{\mathfrak{ab}}$.

We conjecture that the same algebra (2.69) holds not only for the zero-momentum sector, but for all states if we identify

$$\mathcal{Y}_0^{\mathfrak{ab}} = \mathfrak{B}^{\mathfrak{ab}}, \qquad \mathcal{Y}_1^{\mathfrak{ab}} = \mathcal{Y}^{\mathfrak{ab}}, \qquad \mathcal{Y}_{n+1}^{\mathfrak{ab}} = -\tfrac{1}{2}\varepsilon_{\mathfrak{cd}}[\mathcal{Y}^{\mathfrak{c}\{\mathfrak{a}}, \mathcal{Y}_n^{\mathfrak{b}\}\mathfrak{d}}]. \tag{2.70}$$

It is quite clear that the relations with $m = 0$ or $n = 0$ hold by $\mathfrak{su}(2)$ symmetry. Furthermore, the relation with $m = n = 1$ merely defines $\mathcal{Y}_2^{\mathfrak{ab}}$. The relations with $m+n \geq 3$ are nontrivial and have to be verified.

In fact, the relations with $m + n = 3$ are the Serre relations for the algebra and they imply all the relations with $m + n > 3$. In the following we shall prove this statement by induction. For convenience, we switch to a basis for $\mathcal{Y}_n^{\mathfrak{i}}$ where the $\mathfrak{su}(2)$ structure constants are just the totally antisymmetric tensor $\varepsilon^{\mathfrak{ijk}}$. The commutations relations can now be written for all nonnegative integer levels N as

$$[\mathcal{Y}_m^{\mathfrak{i}}, \mathcal{Y}_{N-m}^{\mathfrak{j}}] = \varepsilon^{\mathfrak{ijk}}\mathcal{Y}_N^{\mathfrak{k}}, \quad m = 0, \dots N. \tag{2.71}$$

Assume (2.71) is satisfied at some level $N \geq 3$. Then we use five main steps to show that it is satisfied at level $N + 1$.

Step 1. Using our inductive assumption, consider the equations for $m = 1, \dots N-2$ and their cyclic permutations,

$$\begin{aligned} 0 &= [\mathcal{Y}_1^2, \mathcal{Y}_{N-m}^1] + [\mathcal{Y}_1^1, \mathcal{Y}_{N-m}^2] \\ &= [\mathcal{Y}_m^3, [\mathcal{Y}_1^2, \mathcal{Y}_{N-m}^1]] + [\mathcal{Y}_m^3, [\mathcal{Y}_1^1, \mathcal{Y}_{N-m}^2]] \\ &= [\mathcal{Y}_1^2, \mathcal{Y}_N^2] - [\mathcal{Y}_{m+1}^1, \mathcal{Y}_{N-m}^1] + [\mathcal{Y}_{m+1}^2, \mathcal{Y}_{N-m}^2] - [\mathcal{Y}_1^1, \mathcal{Y}_N^1]. \end{aligned} \tag{2.72}$$

Comparing the $m = M$ and $m = N - M - 1$ equations, we find that

$$[\mathcal{Y}^1_m, \mathcal{Y}^1_{N+1-m}] = [\mathcal{Y}^2_m, \mathcal{Y}^2_{N+1-m}] = [\mathcal{Y}^3_m, \mathcal{Y}^3_{N+1-m}], \quad m = 1, \dots N. \tag{2.73}$$

Step 2. We also have, for $m = 1, \dots N$

$$\begin{aligned} [\mathcal{Y}^1_m, \mathcal{Y}^1_{N+1-m}] &= [\mathcal{Y}^1_m, [\mathcal{Y}^2_1, \mathcal{Y}^3_{N-m}]] \\ &= [\mathcal{Y}^3_{m+1}, \mathcal{Y}^3_{N-m}] - [\mathcal{Y}^2_1, \mathcal{Y}^2_N], \end{aligned} \tag{2.74}$$

and cyclic permutations. Using the result from Step 1, we find

$$[\mathcal{Y}^1_m, \mathcal{Y}^1_{N+1-m}] = m[\mathcal{Y}^1_1, \mathcal{Y}^1_N], \quad m = 1, \dots N, \tag{2.75}$$

and similarly for cyclic permutations. However, since

$$[\mathcal{Y}^1_1, \mathcal{Y}^1_N] = -[\mathcal{Y}^1_N, \mathcal{Y}^1_1]. \tag{2.76}$$

we must have

$$0 = [\mathcal{Y}^1_m, \mathcal{Y}^1_{N+1-m}] = [\mathcal{Y}^2_m, \mathcal{Y}^2_{N+1-m}] = [\mathcal{Y}^3_m, \mathcal{Y}^3_{N+1-m}], \quad m = 1, \dots N. \tag{2.77}$$

Step 3. Commuting $\mathcal{Y}_0$ with $[\mathcal{Y}^1_m, \mathcal{Y}^1_{N+1-m}]$ (and cyclic permutations) yields

$$[\mathcal{Y}^{\mathfrak{i}}_m, \mathcal{Y}^{\mathfrak{j}}_{N+1-m}] = -[\mathcal{Y}^{\mathfrak{j}}_m, \mathcal{Y}^{\mathfrak{i}}_{N+1-m}], \quad m = 1, \dots N. \tag{2.78}$$

Step 4. We can now show that there is a unique consistent way to define $\mathcal{Y}^{\mathfrak{k}}_{N+1}$. For instance, consider the following equations for $m = 1, \dots N - 1$,

$$\begin{aligned} [\mathcal{Y}^1_m, \mathcal{Y}^2_{N+1-m}] &= [\mathcal{Y}^1_m, [\mathcal{Y}^3_1, \mathcal{Y}^1_{N-m}]] = -[\mathcal{Y}^2_{m+1}, \mathcal{Y}^1_{N-m}] \\ &= [\mathcal{Y}^1_{m+1}, \mathcal{Y}^2_{N-m}] \\ &\dots \\ &= [\mathcal{Y}^1_N, \mathcal{Y}^2_1] \\ &= \mathcal{Y}^3_{N+1}. \end{aligned} \tag{2.79}$$

Step 5. It is now straightforward to use any of the equivalent expressions for $\mathcal{Y}^{\mathfrak{k}}_{N+1}$ to check that

$$[\mathcal{Y}^{\mathfrak{i}}_0, \mathcal{Y}^{\mathfrak{j}}_{N+1}] = \varepsilon^{\mathfrak{ijk}} \mathcal{Y}^{\mathfrak{k}}_{N+1}. \tag{2.80}$$

This completes the set of equations at level $N+1$. Therefore, assuming the level-3 equations are satisfied, (2.71) is satisfied for all N.

At this time, a direct proof of the level-3 relations is beyond our technical capabilities. Note that to prove the level-3 relations, it is sufficient to check that (switching back to the previous $\mathfrak{su}(2)$ notation) $[\mathcal{Y}^{<>}_1, \mathcal{Y}^{<>}_2] = 0$, since commutators with the $\mathfrak{B}$ yield the remaining relations. This relation can also be written using only bilocal generators as

$$[\mathcal{Y}^{<>}, [\mathcal{Y}^{<<}, \mathcal{Y}^{>>}]] = 0. \tag{2.81}$$

Still, it would be best to confirm the level-3 relations. As a start, using `Mathematica` we have checked that they are satisfied on many states of small excitation number, including all states of length 4 with 4 or fewer excitations (above the half-BPS vacuum) and all state of Length 5 or 6 with 3 or fewer excitations. Also checked were states with larger lengths and excitation numbers, including a length-7, 7-excitation state. Checking much longer or higher excitation states rapidly becomes impractical because of the increasing number of interactions that need to be summed. However, the evidence described above is persuasive. Hopefully, a complete proof will become possible in the future.

As explained in Appendix B, assuming that the $\mathcal{Y}^{\mathfrak{ab}}$ really satisfy this algebra, there is a one-parameter generalization of these generators using the $(\mathfrak{J}^3)^{\mathfrak{ab}}$. The same subalgebra of the loop algebra of $\mathfrak{su}(2)$ is generated by the $\hat{\mathcal{Y}}^{\mathfrak{ab}}_0 = \mathfrak{B}^{\mathfrak{ab}}$ and the $\hat{\mathcal{Y}}^{\mathfrak{ab}}_1$, with

$$\hat{\mathcal{Y}}^{\mathfrak{ab}}_1 = \mathcal{Y}^{\mathfrak{ab}}_1 + \alpha\, (\mathfrak{J}^3)^{\mathfrak{ab}} \tag{2.82}$$

for any constant α.

Similar to the above reasoning, we can use the zero-momentum reduction for $\mathcal{X}$ to conjecture that it commutes with the $\mathcal{Y}_n^{\mathfrak{ab}}$ for all n. Again, we have obtained very strong evidence using `Mathematica`. We have checked that these commutators vanish for the same set of states described two paragraphs above. It is however presently not clear how to generalize $\mathcal{X}$ to an infinite dimensional algebra of $\mathcal{X}_m$.

2.4.3 Discussion

In this chapter we have investigated a curious 2^M-fold degeneracy of the integrable $\mathfrak{psu}(1,1|2)$ spin chain. This degeneracy was observed at the level of Bethe equations in [23]. Here we have considered the symmetry which explains the degeneracy. We have constructed $\mathfrak{B}$-triplet symmetry generators $\mathcal{Y}$ at the level of operators acting on spin chain states. These bilocal generators $\mathcal{Y}$ commute with $\mathfrak{psu}(1,1|2)$. Combined with $\mathfrak{B}$, they apparently generate a subalgebra of the loop algebra of $\mathfrak{su}(2)$. This extended symmetry algebra commutes with the Hamiltonian and thus explains the degeneracy.

It remains an open problem to identify the spin chain operators that generate the 2^M degeneracy for zero-momentum states. It is possible that these operators do not take a simple form. However, this still deserves further study especially because it is also possible that these operators for cyclic states would give new insight into the origin of the simple next-to-leading order corrections to the local symmetry generators obtained in [38], which will be the subject of the next chapter.

While we have restricted our study to the one-loop Hamiltonian, it is clear that the symmetry enhancement persists at higher loops. The 2^M degeneracy of the Bethe ansatz is preserved by the higher loop corrections [23]. Therefore, we expect the $\mathcal{Y}^{\mathfrak{ab}}$ symmetry generators to receive loop corrections so that they commute with the loop-corrected Hamiltonian. Note that the leading terms for the bilocal symmetry generators $\mathcal{Y}^{\mathfrak{ab}}_{(2)}$ discussed in this chapter appear at $\mathcal{O}(g^2)$. We will discuss the higher loop extension further in the Conclusions, following studies of loop corrections to other symmetry generators.

Chapter 3

The spin representation to two loops

3.1 Introduction

In this chapter we compute the two-loop dilatation generator of the $\mathfrak{psu}(1,1|2)$ sector, only using constraints from basic properties of Feynman diagrams and from superconformal symmetry. We introduce an auxiliary generator that satisfies special commutation relations with the leading order $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ generators. This enables us to find and verify solutions of the symmetry constraints up to two loops only using zero- and one-half-loop commutation relations of the algebra.

For our computation the extra restrictions from $\mathfrak{psu}(1|1)^2$ symmetry are essential. Furthermore, the odd-power expansion of these hidden supercharges enables us to find the two-loop dilatation generator from the next-to-leading order $\mathcal{O}(g^3)$ corrections. In fact, the two-loop dilatation generator is written simply as nested commutators of $\mathcal{O}(g^1)$ $\mathfrak{psu}(1|1)^2$ supercharges and the auxiliary generator. Our solution lifts consistently and naturally to nonplanar $\mathcal{N}=4$ gauge theory as well, that is for any choice of the gauge group. In particular, it includes wrapping interactions. These are interactions that are nonplanar in general, but become planar for short states that they wrap around. This result is especially interesting because properly accounting for wrapping interactions at higher loops is a significant remaining challenge for the Bethe ansatz approach.

Section 3.2 discusses the $\mathcal{O}(g^2)$ solution, and Section 3.3 discusses the $\mathcal{O}(g^3)$ solution and presents the two-loop dilatation generator. For simplicity, we assume planarity until Section 3.3.4, and in that section we present the lift to the finite-N solution. In Section 3.4, after verifying that our solution predicts the same anomalous dimensions as those of the field theory calculations of [18, 52–54], we confirm that our solution is consistent with integrability by computing the bosonic $\mathfrak{sl}(2)$ subsector two-loop S-matrix and some anomalous dimensions. Finally, Section 3.5 presents a solution for the $\mathcal{O}(g^4)$ $\mathfrak{psu}(1,1|2)$ symmetry generators and discusses directions for further research.

The $\mathcal{O}(g^3)$ symmetry algebra and the two-loop dilatation generator was presented in [38]. Here we give an improved proof that the solution satisfies the symmetry constraints, taking advantage of the $\mathfrak{su}(2)$ automorphism. Also, the $\mathcal{O}(g^4)$ solution has not been published previously.

3.2 Order g^2

At $\mathcal{O}(g^2)$, $\delta\mathfrak{D}$ and the $\mathfrak{psu}(1,1|2)$ generators receive quantum corrections. As we noted first in (2.36), we have

$$\delta\mathfrak{D}_{(2)} = 2\{\hat{\mathfrak{Q}}^{<}_{(1)}, \hat{\mathfrak{S}}^{>}_{(1)}\} = -2\{\hat{\mathfrak{Q}}^{>}_{(1)}, \hat{\mathfrak{S}}^{<}_{(1)}\}. \tag{3.1}$$

$\delta\mathfrak{D}$ is the only generator we need to compute because, as we now explain, once we know $\delta\mathfrak{D}$ the full $\mathfrak{su}(2)_{\mathfrak{B}}\times\mathfrak{psu}(1,1|2)\times\mathfrak{psu}(1|1)^2$ algebra's action is fixed by group theory. Knowing $\delta\mathfrak{D}$ means knowing its eigenstates and eigenvalues. Highest weight multiplets are then formed by states of equal eigenvalues, and the generators of $\mathfrak{su}(2)_{\mathfrak{B}} \times \mathfrak{psu}(1,1|2) \times \mathfrak{psu}(1|1)^2$ must connect states within the same multiplet with factors determined by group theory. Furthermore, since $\delta\mathfrak{D}$ is a central charge and its expansion starts at $\mathcal{O}(g^2)$, $\delta\mathfrak{D}_{(2n)}$ fixes the $\mathcal{O}(g^{2n-2})$ $\mathfrak{psu}(1,1|2)$ generators and the $\mathcal{O}(g^{2n-1})$ $\mathfrak{psu}(1|1)^2$ generators. Also, the $\mathfrak{psu}(1|1)^2$ algebra gives $\delta\mathfrak{D}_{(2n)}$ in terms of the $\mathcal{O}(g^{2n-1})$ $\mathfrak{psu}(1|1)^2$ generators.

At this point we should note that there are important degeneracies beyond those given by group theory. The local charges of (2.27), which commute with the dilatation generator, lead to further degeneracies for the planar theory. However, since these higher charges Q_r are $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ singlets, they are consistent with the group theory constraints on the spectrum[1]. Also, the nonlocal $\mathcal{Y}$ symmetries of the last chapter give the much larger degeneracy observed in the Bethe equations, but these degeneracies also are consistent with group theory since the $\mathcal{Y}$ commute with $\mathfrak{psu}(1,1|2)$ and are $\mathfrak{su}(2)$ triplets.

To obtain the perturbative corrections of $\delta\mathfrak{D}$ we will compute the corrections to the $\mathfrak{psu}(1|1)^2$ generators. For the latter we will find tight constraints by computing the corrections to the full set of $\mathfrak{psu}(1,1|2)$ generators. We begin by presenting the solution for the $\mathcal{O}(g^2)$ $\mathfrak{psu}(1,1|2)$ generators, and then we discuss its possible modifications and its proof.

3.2.1 The solution

We define two auxiliary generators that play central parts in our solution. $\mathfrak{h}$ is a one-site generator of harmonic numbers. Its action is

$$\mathfrak{h}|\phi_a^{(k)}\rangle = \tfrac{1}{2}h(k)|\phi_a^{(k)}\rangle, \quad \mathfrak{h}|\psi_{\mathfrak{a}}^{(k)}\rangle = \tfrac{1}{2}h(k+1)|\psi_{\mathfrak{a}}^{(k)}\rangle. \tag{3.2}$$

[1]Degeneracies due to the Q_r also have the distinctive feature of relating states of opposite parity, as the Q_r are parity odd for odd r.

Recall that the harmonic numbers appeared in the one-loop dilatation generator. It is natural that the harmonic numbers appear again; they are given by a difference of digamma functions, which are often associated with integrability. Since $\mathfrak{h}$ does not distinguish between the flavors of bosons or between flavors of fermions it commutes with $\mathfrak{R}^{ab}$ and $\mathfrak{B}^{\mathfrak{ab}}$. $\mathfrak{h}$ also commutes with $\mathfrak{J}^{+-}_{(0)}$ since both of these generators simply multiply individual sites by a spin-dependent numerical factor.

We now build a second auxiliary generator, labeled $\mathfrak{x}$, from the $\mathfrak{psu}(1|1)^2$ supercharges and $\mathfrak{h}$. $\mathfrak{x}$ is a two-site to two-site generator that we can write in two equivalent ways,

$$\begin{aligned}\mathfrak{x} &= \{\hat{\mathfrak{S}}^{>}_{(1)}, [\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{h}]\} + \{\hat{\mathfrak{Q}}^{>}_{(1)}, [\hat{\mathfrak{S}}^{<}_{(1)}, \mathfrak{h}]\} \\ &= -\{\hat{\mathfrak{S}}^{<}_{(1)}, [\hat{\mathfrak{Q}}^{>}_{(1)}, \mathfrak{h}]\} - \{\hat{\mathfrak{Q}}^{<}_{(1)}, [\hat{\mathfrak{S}}^{>}_{(1)}, \mathfrak{h}]\}.\end{aligned} \tag{3.3}$$

To show the equality in (3.3), subtract the second expression from the first and combine in pairs to obtain

$$\begin{aligned}&(\{\hat{\mathfrak{S}}^{>}_{(1)}, [\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{h}]\} + \{\hat{\mathfrak{Q}}^{<}_{(1)}, [\hat{\mathfrak{S}}^{>}_{(1)}, \mathfrak{h}]\}) + (\{\hat{\mathfrak{Q}}^{>}_{(1)}, [\hat{\mathfrak{S}}^{<}_{(1)}, \mathfrak{h}]\} + \{\hat{\mathfrak{S}}^{<}_{(1)}, [\hat{\mathfrak{Q}}^{>}_{(1)}, \mathfrak{h}]\}) \\ &= [\{\hat{\mathfrak{Q}}^{<}_{(1)}, \hat{\mathfrak{S}}^{>}_{(1)}\}, \mathfrak{h}] + [\{\hat{\mathfrak{Q}}^{>}_{(1)}, \hat{\mathfrak{S}}^{<}_{(1)}\}, \mathfrak{h}] \\ &= \frac{1}{2}[\delta\mathfrak{D}_{(2)}, \mathfrak{h}] - \frac{1}{2}[\delta\mathfrak{D}_{(2)}, \mathfrak{h}] \\ &= 0.\end{aligned} \tag{3.4}$$

We have used (3.1) to reach the second to last line. Averaging the two equivalent expression of (3.3), we have

$$\mathfrak{x} = \frac{1}{2}\varepsilon_{\mathfrak{ab}}\{\hat{\mathfrak{Q}}^{b}_{(1)}, [\hat{\mathfrak{S}}^{a}_{(1)}, \mathfrak{h}]\} + \frac{1}{2}\varepsilon_{\mathfrak{ab}}\{\hat{\mathfrak{S}}^{b}_{(1)}, [\hat{\mathfrak{Q}}^{a}_{(1)}, \mathfrak{h}]\}. \tag{3.5}$$

This form, makes it clear that $\mathfrak{x}$ commutes with $\mathfrak{B}^{ab}$ (since $\mathfrak{h}$ does). Furthermore, since the $\mathfrak{psu}(1|1)^2$ generators commute with $\mathfrak{R}^{ab}$ and $\mathfrak{J}^{+-}_{(0)}$, $\mathfrak{x}$ also commutes with them. Finally, because the two equal expressions in (3.3) are hermitian conjugates by (2.35), $\mathfrak{x}$ is hermitian.

We are now ready to present the $\mathcal{O}(g^2)$ solution. Let $A^{\pm}$ represent $\mathfrak{J}^{++}$, $\mathfrak{J}^{--}$, or the eight $\mathfrak{Q}$'s, where we retain only the $\mathfrak{su}(1,1)$ charge (the charge with respect to $\mathfrak{J}^{\alpha\beta}$). Then the following solves the symmetry and field theoretic constraints:

$$A^{\pm}_{(2)} = \pm[A^{\pm}_{(0)}, \mathfrak{x}] + [A^{\pm}_{(0)}, \mathfrak{y}]. \tag{3.6}$$

$\mathfrak{y}$ is any two-site to two-site generator that commutes with $\mathfrak{D}_{(0)}$, $\mathfrak{B}^{\mathfrak{ab}}$, and $\mathfrak{R}^{ab}$. Commuting all the generators of the $\mathfrak{psu}(1,1|2) \times \mathfrak{psu}(1|1)^2$ algebras with a generator such as $\mathfrak{y}$ maps one solution of the commutation relations to another. This is because it corresponds to the first term in the expansion of the similarity transformation

$$\mathfrak{J} \mapsto U\mathfrak{J}U^{-1}, \quad U = 1 - g^2\mathfrak{y} + \cdots, \quad \text{i.e.} \quad \mathfrak{J}_{(2)} \mapsto \mathfrak{J}_{(2)} + [\mathfrak{J}_{(0)}, \mathfrak{y}]. \tag{3.7}$$

Of course, the algebra relations

$$[\mathfrak{J}^A, \mathfrak{J}^B\} = f^{AB}{}_C\,\mathfrak{J}^C \tag{3.8}$$

are still satisfied after the application of such a similarity transformation. We require $\mathfrak{y}$ to commute with $\mathfrak{R}^{ab}$, $\mathfrak{B}^{\mathfrak{ab}}$, and $\mathfrak{D}_{(0)}$ to preserve R symmetry, the $\mathfrak{su}(2)$ automorphism, and $\delta\mathfrak{D}$'s eigenstates' classical dimensions. To maintain manifest consistency with the Feynman diagram rules, U's expansion must be in even powers of g, consisting of $(\frac{n}{2}+1)$-site to $(\frac{n}{2}+1)$-site interactions at $\mathcal{O}(g^n)$. For antihermitian (or vanishing) $\mathfrak{y}$, the hermitian conjugation relations (1.54, 1.58) are satisfied at $\mathcal{O}(g^2)$.

The simple structure of this solution enables us to quickly verify that it satisfies the field theoretic constraints. The commutator structure insures that the $\mathcal{O}(g^2)$ generators are built out of interactions acting on two adjacent sites, as required for the planar theory at this order. Furthermore, they are parity even since all the generators that appear within the commutator are parity even. We prove that the solution also satisfies the algebra relations in Section 3.2.3.

3.2.2 On the uniqueness of the solution

There are two possible sources of freedom for the solution at this order: interactions that vanish on periodic states (chain derivatives), and homogeneous solutions. We now exclude the former and discuss the latter.

The requirement of even parity rules out the possibility of applying chain derivatives to the solution at this order, since generators are sums of two-site to two-site interactions[2]. Any two-site gauge transformation for generator A, which acts on fields X and Y, is the sum of interactions of the form

$$A|X_0 X_i\rangle = |Y X_i\rangle, \quad A|X_i X_0\rangle = -(-1)^{X_i(X_0 Y)}|X_i Y\rangle, \quad \forall X_i. \tag{3.11}$$

The second interaction is precisely -1 times the parity-reflected first interaction, so cannot appear as long as we require even-parity local symmetry generators. This observation also implies that the algebra can only be satisfied exactly (instead of only modulo chain derivatives).

However, at this point, we cannot rule out modification by a homogeneous solution. Under this modification,

$$\mathfrak{J}^{++}_{(2)} \mapsto \mathfrak{J}^{++}_{(2)} + \delta\mathfrak{J}^{++}_{(2)}, \quad \mathfrak{J}^{--}_{(2)} \mapsto \mathfrak{J}^{--}_{(2)} + \delta\mathfrak{J}^{--}_{(2)}, \quad \mathfrak{Q}^{a\beta\mathfrak{c}}_{(2)} \mapsto \mathfrak{Q}^{a\beta\mathfrak{c}}_{(2)} + \delta\mathfrak{Q}^{a\beta\mathfrak{c}}_{(2)}. \tag{3.12}$$

In order for the symmetry constraints to remain satisfied, the $\delta\mathfrak{J}$'s and $\delta\mathfrak{Q}$'s must not make

[2] More precisely, we can choose to write all interactions as two-site to two-site. Expanding the commutators of (3.6) one would encounter one-site interactions of the form

$$A|X_0\rangle = |Y\rangle, \tag{3.9}$$

but this is equivalent to the sum of two-site interactions

$$A|X_0 X_i\rangle = \frac{1}{2}|Y X_i\rangle, \quad A|X_i X_0\rangle = \frac{1}{2}(-1)^{X_i(X_0 Y)}|X_i Y\rangle, \quad \forall X_i. \tag{3.10}$$

any net contributions to commutators of the algebra. For example[3]

$$\{\delta\mathfrak{Q}^{a\beta\mathfrak{c}}_{(2)},\mathfrak{Q}^{d\epsilon\mathfrak{f}}_{(0)}\}+\{\mathfrak{Q}^{a\beta\mathfrak{c}}_{(2)},\delta\mathfrak{Q}^{d\epsilon\mathfrak{f}}_{(2)}\}=-\varepsilon^{ad}\varepsilon^{\mathfrak{cf}}\delta\mathfrak{J}^{\beta\epsilon}_{(2)}. \tag{3.13}$$

We have not found any nontrivial homogeneous solutions, or ruled them out. However, from the above discussion regarding $\delta\mathfrak{D}$, we conclude that once $\delta\mathfrak{D}_{(4)}$ is found, this freedom is fixed. We will find the solution for $\delta\mathfrak{D}_{(4)}$ below. Since the $\mathcal{O}(g^2)$ solution presented in this section is consistent with it, this is the field theory $\mathcal{O}(g^2)$ solution.

3.2.3 The $\mathcal{O}(g^2)$ proof of the algebra

By taking full advantage of the symmetry algebra, it is possible to prove that the solution (3.6) satisfies the symmetry algebra at $\mathcal{O}(g^2)$ with minimal computation. Since the similarity transformation generated by $\mathfrak{y}$ preserves solutions of the algebra, we set $\mathfrak{y}$ to zero throughout this section without loss of generality. As there are multiple steps to the proof, we first give its outline.

- The $\mathfrak{Q}^{a\beta\mathfrak{c}}$ transform properly under commutation with $\mathfrak{R}^{ab}$ (1.56).

- The $\mathfrak{Q}^{a\beta\mathfrak{c}}$ transform properly under commutation with $\mathfrak{B}^{\mathfrak{ab}}$ (1.64).

- The commutator relations between supercharges with the same $\mathfrak{su}(1,1)$ index are satisfied

$$\{\mathfrak{Q}^{a+\mathfrak{b}},\mathfrak{Q}^{c+\mathfrak{d}}\}_{(2)}=-\varepsilon^{ac}\varepsilon^{\mathfrak{bd}}\mathfrak{J}^{++}_{(2)},\quad\{\mathfrak{Q}^{a-\mathfrak{b}},\mathfrak{Q}^{c-\mathfrak{d}}\}_{(2)}=-\varepsilon^{ac}\varepsilon^{\mathfrak{bd}}\mathfrak{J}^{--}_{(2)}. \tag{3.14}$$

- The following algebra relation is satisfied:

$$\{\mathfrak{Q}^{1+<},\mathfrak{Q}^{2->}\}_{(2)}=-\mathfrak{J}^{+-}_{(2)}. \tag{3.15}$$

[3]Of course, $\delta\mathfrak{J}^{+-}_{(2)}$, the modification to the one-loop dilatation generator, must vanish since $\delta\mathfrak{D}_{(2)}$ is fixed by the $\mathcal{O}(g)$ $\mathfrak{psu}(1|1)^2$ supercharges.

- By $\mathfrak{R}$ symmetry and $\mathfrak{B}$ symmetry, the previous step implies that all of the remaining 16 relations of (1.57) are satisfied at this order,

$$\{\mathfrak{Q}^{a+\mathfrak{b}}, \mathfrak{Q}^{c-\mathfrak{d}}\}_{(2)} = -\varepsilon^{ac}\varepsilon^{\mathfrak{b}\mathfrak{d}}\mathfrak{J}^{+-}_{(2)}. \tag{3.16}$$

- The above results for the supercharges guarantee that all commutator relations involving any $\mathfrak{J}^{\alpha\beta}$ inside the commutator are satisfied.

These steps include all of the commutators of the $\mathfrak{psu}(1,1|2)$ algebra that are nontrivially satisfied at $\mathcal{O}(g^2)$. Interestingly, we only need the explicit form of $\mathfrak{h}$ for the second-to-last step of the proof. There we use the identities

$$\begin{aligned} \{\mathfrak{Q}^{a+\mathfrak{b}}_{(0)}, [\mathfrak{Q}^{c-\mathfrak{d}}_{(0)}, \mathfrak{h}]\} &= -\frac{1}{2}\varepsilon^{ac}\mathfrak{B}^{\mathfrak{b}\mathfrak{d}} + \frac{1}{2}\varepsilon^{\mathfrak{b}\mathfrak{d}}\mathfrak{R}^{ac} - \frac{1}{4}\varepsilon^{ac}\varepsilon^{\mathfrak{b}\mathfrak{d}}\mathfrak{L}, \\ \{\mathfrak{Q}^{a-\mathfrak{b}}_{(0)}, [\mathfrak{Q}^{c+\mathfrak{d}}_{(0)}, \mathfrak{h}]\} &= -\frac{1}{2}\varepsilon^{ac}\mathfrak{B}^{\mathfrak{b}\mathfrak{d}} + \frac{1}{2}\varepsilon^{\mathfrak{b}\mathfrak{d}}\mathfrak{R}^{ac} + \frac{1}{4}\varepsilon^{ac}\varepsilon^{\mathfrak{b}\mathfrak{d}}\mathfrak{L}. \end{aligned} \tag{3.17}$$

These identities are shown in in Appendix D.

Commutators with $\mathfrak{R}$ and $\mathfrak{B}$. First, we note that the supercharges commute canonically with respect to $\mathfrak{R}^{ab}$ (1.56), because $\mathfrak{x}$ commutes with $\mathfrak{R}^{ab}$, and because the $\mathfrak{R}^{ab}$ are exact generators that simply transform indices canonically. Applying the Jacobi identity, we obtain

$$\begin{aligned} [\mathfrak{R}^{ab}, \mathfrak{Q}^{c\delta\mathfrak{e}}_{(2)}] &= \pm[\mathfrak{R}^{ab}, [\mathfrak{Q}^{c\delta\mathfrak{e}}_{(0)}, \mathfrak{x}]] \\ &= \pm\tfrac{1}{2}\varepsilon^{ca}[\mathfrak{Q}^{b\delta\mathfrak{e}}_{(0)}, \mathfrak{x}] + \pm\tfrac{1}{2}\varepsilon^{cb}[\mathfrak{Q}^{a\delta\mathfrak{e}}_{(0)}, \mathfrak{x}] \\ &= \tfrac{1}{2}\varepsilon^{ca}\mathfrak{Q}^{b\delta\mathfrak{e}}_{(2)} + \tfrac{1}{2}\varepsilon^{cb}\mathfrak{Q}^{a\delta\mathfrak{e}}_{(2)}. \end{aligned} \tag{3.18}$$

The last line is precisely what is required. Because $\mathfrak{x}$ commutes with $\mathfrak{B}^{\mathfrak{ab}}$ and because the $\mathfrak{B}^{\mathfrak{ab}}$ are also exact generators, a similar proof shows that the supercharges commute canonically with respect to $\mathfrak{B}^{\mathfrak{ab}}$ (1.64), as needed.

Commutators between $\mathfrak{Q}$ with same $\mathfrak{su}$(1,1) index. The simple commutator form of the solution (3.6) guarantees that all commutation relations between supercharges with the same $\mathfrak{su}(1|1)$ index are satisfied. We show this for a plus index, but the same steps apply for the minus-index proof (up to signs). The required commutation relation (1.57) simplifies in this case at $\mathcal{O}(g^2)$ to

$$\{\mathfrak{Q}^{a+\mathfrak{b}}, \mathfrak{Q}^{c+\mathfrak{d}}\}_{(2)} = -\varepsilon^{ac}\varepsilon^{\mathfrak{b}\mathfrak{d}}\,\mathfrak{J}^{++}_{(2)} \tag{3.19}$$

Now, substituting (3.6) for the left side, we obtain

$$\begin{aligned}
\{\mathfrak{Q}^{a+\mathfrak{b}}, \mathfrak{Q}^{c+\mathfrak{d}}\}_{(2)} &= \{\mathfrak{Q}^{a+\mathfrak{b}}_{(2)}, \mathfrak{Q}^{c+\mathfrak{d}}_{(0)}\} + \{\mathfrak{Q}^{a+\mathfrak{b}}_{(0)}, \mathfrak{Q}^{c+\mathfrak{d}}_{(2)}\} \\
&= \{[\mathfrak{Q}^{a+\mathfrak{b}}_{(0)}, \mathfrak{x}], \mathfrak{Q}^{c+\mathfrak{d}}_{(0)}\} + \{\mathfrak{Q}^{a+\mathfrak{b}}_{(0)}, [\mathfrak{Q}^{c+\mathfrak{d}}_{(0)}, \mathfrak{x}]\} \\
&= [\{\mathfrak{Q}^{a+\mathfrak{b}}_{(0)}, \mathfrak{Q}^{c+\mathfrak{d}}_{(0)}\}, \mathfrak{x}] \\
&= -\varepsilon^{ac}\varepsilon^{\mathfrak{b}\mathfrak{d}}\,[\mathfrak{J}^{++}_{(0)}, \mathfrak{x}] \\
&= -\varepsilon^{ac}\varepsilon^{\mathfrak{b}\mathfrak{d}}\,\mathfrak{J}^{++}_{(2)},
\end{aligned} \tag{3.20}$$

as required. The Jacobi identity gives the third equality, the leading-order algebra implies the second-to-last equality, and we have substituted (3.6) again for the last line.

(3.15) is satisfied. Recall that this relation is

$$\{\mathfrak{Q}^{1+<}, \mathfrak{Q}^{2->}\}_{(2)} = -\mathfrak{J}^{+-}_{(2)} = -\frac{1}{2}\,\delta\mathfrak{D}_{(2)}, \tag{3.21}$$

where the last equality follows from (1.41). First we use (3.6) and the Jacobi identity to simplify the left side of (3.15),

$$
\begin{aligned}
\{\mathfrak{Q}^{1+<}, \mathfrak{Q}^{2->}\}_{(2)} &= \{\mathfrak{Q}^{1+<}_{(2)}, \mathfrak{Q}^{2->}_{(0)}\} + \{\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{Q}^{2->}_{(2)}\} \\
&= \{[\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{x}], \mathfrak{Q}^{2->}_{(0)}\} - \{\mathfrak{Q}^{1+<}_{(0)}, [\mathfrak{Q}^{2->}_{(0)}, \mathfrak{x}]\} \\
&= 2\{[\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{x}], \mathfrak{Q}^{2->}_{(0)}\} - [\{\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{Q}^{2->}_{(0)}\}, \mathfrak{x}] \\
&= 2\{[\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{x}], \mathfrak{Q}^{2->}_{(0)}\}.
\end{aligned} \tag{3.22}
$$

For the last equality we used the leading order commutator

$$
\{\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{Q}^{2->}_{(0)}\} = \mathfrak{R}^{12} - \mathfrak{J}^{+-}, \tag{3.23}
$$

which commutes with $\mathfrak{x}$. For later reference observe the identity

$$
\{\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{Q}^{2->}_{(2)}\} = \{\mathfrak{Q}^{1+<}_{(2)}, \mathfrak{Q}^{2->}_{(0)}\}. \tag{3.24}
$$

Next we substitute (3.5), use the vanishing commutators between $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ generators, and (3.17),

$$
\begin{aligned}
& 2\{[\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{x}], \mathfrak{Q}^{2->}_{(0)}\} \\
&= \varepsilon_{\mathfrak{ab}}\{[\mathfrak{Q}^{1+<}_{(0)}, \{\hat{\mathfrak{Q}}^{b}_{(1)}, [\hat{\mathfrak{S}}^{a}_{(1)}, \mathfrak{h}]\}], \mathfrak{Q}^{2->}_{(0)}\} + \varepsilon_{\mathfrak{ab}}\{[\mathfrak{Q}^{1+<}_{(0)}, \{\hat{\mathfrak{S}}^{b}_{(1)}, [\hat{\mathfrak{Q}}^{a}_{(1)}, \mathfrak{h}]\}], \mathfrak{Q}^{2->}_{(0)}\} \\
&= \varepsilon_{\mathfrak{ab}}\{[\hat{\mathfrak{S}}^{a}_{(1)}, \{\mathfrak{Q}^{2->}_{(0)}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]\}], \hat{\mathfrak{Q}}^{b}_{(1)}\} + \varepsilon_{\mathfrak{ab}}\{[\hat{\mathfrak{Q}}^{a}_{(1)}, \{\mathfrak{Q}^{2->}_{(0)}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]\}], \hat{\mathfrak{S}}^{b}_{(1)}\} \\
&= -\frac{\varepsilon_{\mathfrak{ab}}}{2}\{[\hat{\mathfrak{S}}^{a}_{(1)}, \mathfrak{B}^{<>} - \mathfrak{R}^{12} - \frac{\mathfrak{L}}{2}], \hat{\mathfrak{Q}}^{b}_{(1)}\} - \frac{\varepsilon_{\mathfrak{ab}}}{2}\{[\hat{\mathfrak{Q}}^{a}_{(1)}, \mathfrak{B}^{<>} - \mathfrak{R}^{12} - \frac{\mathfrak{L}}{2}], \hat{\mathfrak{S}}^{b}_{(1)}\} \\
&= \ (\frac{1}{4} + \frac{1}{4})\{\hat{\mathfrak{S}}^{1}_{(1)}, \hat{\mathfrak{Q}}^{2}_{(1)}\} + (\frac{1}{4} - \frac{1}{4})\{\hat{\mathfrak{S}}^{2}_{(1)}, \hat{\mathfrak{Q}}^{1}_{(1)}\} \\
& \quad + (\frac{1}{4} - \frac{1}{4})\{\hat{\mathfrak{Q}}^{1}_{(1)}, \hat{\mathfrak{S}}^{2}_{(1)}\} + (\frac{1}{4} + \frac{1}{4})\{\hat{\mathfrak{Q}}^{2}_{(1)}, \hat{\mathfrak{S}}^{1}_{(1)}\} \\
&= \{\hat{\mathfrak{S}}^{1}_{(1)}, \hat{\mathfrak{Q}}^{2}_{(1)}\} \\
&= -\frac{1}{2}\,\delta\mathfrak{D}_{(2)}.
\end{aligned} \tag{3.25}
$$

The third-to-last equality follows from the commutators between $\mathfrak{psu}(1|1)^2$ generators and $\mathfrak{B}^{\mathfrak{ab}}$ and $\mathfrak{L}$, (1.69) and (1.67). The last line equals the right side of (3.15), as needed.

Taking full advantage of $\mathfrak{R}$ and $\mathfrak{B}$. Because the supercharges transform properly with respect to the $\mathfrak{R}^{ab}$ and the $\mathfrak{B}^{\mathfrak{ab}}$, the 16 commutators between supercharges with opposite $\mathfrak{su}(1|1)$ index are satisfied if one of the relations with all opposite indices is satisfied. One can see this by acting with various combinations of $\mathfrak{R}^{ab}$ and $\mathfrak{B}^{\mathfrak{ab}}$ on (3.15). However, there is also a simple group theory argument. Under $\mathfrak{su}(2)_{\mathfrak{B}} \times \mathfrak{su}(2)_{\mathfrak{R}}$, the sixteen commutators between supercharges with opposite $\mathfrak{su}(1,1)$ index transform as

$$(\mathbf{2},\mathbf{2}) \otimes (\mathbf{2},\mathbf{2}) = (\mathbf{3},\mathbf{3}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}). \tag{3.26}$$

Since (3.15) has nonvanishing components in all four irreducible representations, it follows immediately that the following 16 algebra relations are satisfied,

$$\{\mathfrak{Q}^{a+\mathfrak{b}}, \mathfrak{Q}^{c-\mathfrak{d}}\}_{(2)} = -\varepsilon^{ac}\varepsilon^{\mathfrak{bd}}\mathfrak{J}^{+-}_{(2)}. \tag{3.27}$$

Commutators involving $\mathfrak{J}^{\alpha\beta}$. For any commutator relation including a $\mathfrak{J}^{\alpha\beta}$ we can simply substitute a commutator of supercharges, and then the already proven properties of the supercharges guarantee that the relation is satisfied. For example,

$$\begin{aligned}
[\mathfrak{J}^{++}, \mathfrak{Q}^{1-<}]_{(2)} &= [\{\mathfrak{Q}^{1+>}, \mathfrak{Q}^{2+<}\}, \mathfrak{Q}^{1-<}]_{(2)} \\
&= -[\{\mathfrak{Q}^{1+>}, \mathfrak{Q}^{1-<}\}, \mathfrak{Q}^{2+<}]_{(2)} + [\mathfrak{Q}^{1+>}, \{\mathfrak{Q}^{2+<}, \mathfrak{Q}^{1-<}\}]_{(2)} \\
&= [\mathfrak{R}^{11}, \mathfrak{Q}^{2+<}_{(2)}] \\
&= -\frac{1}{2}\,\mathfrak{Q}^{1+<}_{(2)},
\end{aligned} \tag{3.28}$$

as required by (1.56). We use the Jacobi identity again to reach the second line, (1.57) and the fact the $\mathfrak{R}^{ab}$ receive no corrections for the third line, and (1.56) for the last line. In this way we can check all of the commutators between the $\mathfrak{J}^{\alpha\beta}$ and the $\mathfrak{Q}^{\alpha\beta\mathfrak{c}}$. Then to check the commutators between two $\mathfrak{J}^{\alpha\beta}$ we can repeat this substitution of supercharges for one of the $\mathfrak{J}^{\alpha\beta}$. This completes the proof that the (3.6) is a one-loop solution of the symmetry algebra.

3.3 Order g^3

With the $\mathcal{O}(g^2)$ solution, we can use the constraints to find the $\mathcal{O}(g^3)$ solution, which consists of corrections to the $\mathfrak{psu}(1|1)^2$ generators. We now present and discuss this $\mathcal{O}(g^3)$ solution and its proof, the two-loop dilatation generator that follows, and the lift to the finite-N dilatation generator.

3.3.1 The solution

Only the $\mathfrak{psu}(1|1)^2$ generators $\hat{\mathfrak{Q}}^{\mathfrak{a}}$ and $\hat{\mathfrak{S}}^{\mathfrak{a}}$ receive corrections at this order. Once again the form of the correction varies with the generator only through an overall sign,

$$\begin{aligned} \hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)} &= [\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, \mathfrak{x}] + [\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, \mathfrak{y}] + \alpha\, \hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, \\ \hat{\mathfrak{S}}^{\mathfrak{a}}_{(3)} &= -[\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}, \mathfrak{x}] + [\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}, \mathfrak{y}] + \alpha\, \hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}. \end{aligned} \tag{3.29}$$

As at the previous order, the $\mathfrak{y}$ commutator originates from a similarity transformation, and $\mathfrak{y}$ must be the same similarity transformation generator as for the $\mathcal{O}(g^2)$ solution. The α term results from the coupling constant transformation

$$g \mapsto g + \alpha g^3. \tag{3.30}$$

Such a transformation clearly preserves the algebra relations (1.10). As at $\mathcal{O}(g^2)$, the solution is hermitian, provided $\mathfrak{y}$ is antihermitian. It is difficult to imagine a simpler solution.

Beside the coupling constant transformation and the similarity transformation, the solution at this order is just a commutator with $\mathfrak{x}$, as was the case for $\mathcal{O}(g^2)$.

As at $\mathcal{O}(g^2)$, the commutator with $\mathfrak{x}$ insures that the generators act on the correct number of adjacent sites (2 to 3 or 3 to 2 in this case) and are parity even. We prove that this solution satisfies the symmetry algebra constraints in the next section.

At this order, we could add gauge transformations (which vanish on cyclic states) to the generators, and this is compatible with even-parity generators. Furthermore, as at $\mathcal{O}(g^1)$ the solution satisfies the commutation relations only up to gauge transformations. In the proof given in the next section, since this $\mathfrak{psu}(1|1)^2$ symmetry only holds for cyclic states, we will simply set gauge transformations (such as those of (2.45)) to zero. In fact, this was also implicit in the proof at $\mathcal{O}(g^2)$ leading up to (3.25).

The case for homogeneous solutions at this order exactly parallels that of the previous order. Under a homogeneous modification,

$$\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)} \mapsto \hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)} + \delta\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)}, \quad \hat{\mathfrak{S}}^{\mathfrak{a}}_{(3)} \mapsto \hat{\mathfrak{S}}^{\mathfrak{a}}_{(3)} + \delta\hat{\mathfrak{S}}^{\mathfrak{a}}_{(3)}. \tag{3.31}$$

In order for the symmetry constraints to remain satisfied, the $\delta\hat{\mathfrak{Q}}$'s and $\delta\hat{\mathfrak{S}}$'s must not make a net contribution to any commutator of the algebra, both for commutators among the $\mathfrak{psu}(1|1)^2$ generators and for those with $\mathfrak{psu}(1,1|2)$ generators. Again we have not found any nontrivial homogeneous solutions, or ruled them out. However, as for the $\mathcal{O}(g^2)$ solution, the successful checks of our solution with field theory computations implies that such a homogeneous contribution is not part of the field theory solution.

3.3.2 The $\mathcal{O}(g^3)$ proof of the algebra

Many of the same ingredients used at $\mathcal{O}(g^2)$ reappear in this proof. Also, we use repeatedly the vanishing $\mathcal{O}(g)$ commutators between $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ generators.

Without loss of generality, we set the similarity transformation generator $\mathfrak{y}$ and coupling constant transformation parameter α to zero. The main steps of the proof are as follows.

- The $\mathfrak{psu}(1|1)^2$ supercharges transform properly under commutation with $\mathfrak{B}^{\mathfrak{ab}}$ (1.69) and commute with $\mathfrak{R}^{ab}$.
- Commutators between the $\hat{\mathfrak{Q}}$ and the $\mathfrak{Q}^{a+\mathfrak{c}}$ vanish, as do those between $\hat{\mathfrak{S}}$ and the $\mathfrak{Q}^{a-\mathfrak{c}}$.
- The remaining commutators between $\mathfrak{psu}(1,1|2)$ supercharges and $\mathfrak{psu}(1|1)^2$ supercharges vanish. This guarantees that the $\mathfrak{psu}(1|1)^2$ supercharges commute with the $\mathfrak{J}^{\alpha\beta}$ too.
- Commutators between two $\hat{\mathfrak{Q}}$ vanish, as do those between two $\hat{\mathfrak{S}}$.
- Using the $\mathfrak{B}$ symmetry combined with verifying that

$$\{\hat{\mathfrak{Q}}^{>}, \hat{\mathfrak{S}}^{>}\}_{(4)} = 0 \tag{3.32}$$

  shows that the $\mathfrak{psu}(1|1)^2$ algebra (1.68) is satisfied.

One key identity for the proof is proved in Appendix D.

Before beginning the proof, we derive an alternate form for the solution that will be useful. First, observe the identity for any nilpotent generator Q ($Q^2 = 0$) and any other generator T,

$$[Q, \{Q, T\}] = 0. \tag{3.33}$$

To see this, expand the left side and use $Q^2 = 0$,

$$\begin{aligned}[Q, \{Q, T\}] &= Q^2T + QTQ - QTQ - TQ^2 \\ &= QTQ - QTQ \\ &= 0.\end{aligned} \tag{3.34}$$

For later use, also note that with the same assumption we have the opposite statistics version,

$$\{Q,[Q,T]\} = 0. \tag{3.35}$$

Consider (3.29) for $\hat{\mathfrak{Q}}^{<}_{(3)}$, with α and $\mathfrak{y}$ set to zero. Using the second expression of (3.3) for $\mathfrak{x}$, we find

$$\begin{aligned}\hat{\mathfrak{Q}}^{<}_{(3)} &= [\hat{\mathfrak{Q}}^{<}_{(1)},\mathfrak{x}]\\ &= -[\hat{\mathfrak{Q}}^{<}_{(1)},\{\hat{\mathfrak{S}}^{<}_{(1)},[\hat{\mathfrak{Q}}^{>}_{(1)},\mathfrak{h}]\}] - [\hat{\mathfrak{Q}}^{<}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(1)},[\hat{\mathfrak{S}}^{>}_{(1)},\mathfrak{h}]\}]\\ &= -[\hat{\mathfrak{Q}}^{<}_{(1)},\{\hat{\mathfrak{S}}^{<}_{(1)},[\hat{\mathfrak{Q}}^{>}_{(1)},\mathfrak{h}]\}].\end{aligned} \tag{3.36}$$

(3.33) was used to reach the last line. Repeating this for the other three $\mathfrak{psu}(1|1)^2$ supercharges, we obtain the alternative expressions equivalent to (3.29) (with α and $\mathfrak{y}$ set to zero)

$$\begin{aligned}\hat{\mathfrak{Q}}^{>}_{(3)} &= [\hat{\mathfrak{Q}}^{>}_{(1)},\{\hat{\mathfrak{S}}^{>}_{(1)},[\hat{\mathfrak{Q}}^{<}_{(1)},\mathfrak{h}]\}], \quad \hat{\mathfrak{Q}}^{<}_{(3)} = -[\hat{\mathfrak{Q}}^{<}_{(1)},\{\hat{\mathfrak{S}}^{<}_{(1)},[\hat{\mathfrak{Q}}^{>}_{(1)},\mathfrak{h}]\}],\\ \hat{\mathfrak{S}}^{>}_{(3)} &= [\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{>}_{(1)},[\hat{\mathfrak{S}}^{<}_{(1)},\mathfrak{h}]\}], \quad \hat{\mathfrak{S}}^{<}_{(3)} = -[\hat{\mathfrak{S}}^{<}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(1)},[\hat{\mathfrak{S}}^{>}_{(1)},\mathfrak{h}]\}].\end{aligned} \tag{3.37}$$

Commutators with $\mathfrak{B}$ and $\mathfrak{R}$. Since $\mathfrak{x}$ commutes with $\mathfrak{B}^{\mathfrak{ab}}$, the $\mathcal{O}(g^3)$ correction to the $\mathfrak{psu}(1|1)^2$ generators transforms properly with respect to $\mathfrak{B}^{\mathfrak{ab}}$. Similarly, since $\mathfrak{x}$ commutes with $\mathfrak{R}^{ab}$, the $\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)}$ and $\hat{\mathfrak{S}}^{\mathfrak{a}}_{(3)}$ commute with $\mathfrak{R}^{ab}$ too.

$\{\hat{\mathfrak{Q}},\mathfrak{Q}^{+}\}$ and $\{\hat{\mathfrak{S}},\mathfrak{Q}^{-}\}$. The same argument that showed that the $\mathfrak{Q}^{a+\mathfrak{b}}$ anticommute with each other properly at $\mathcal{O}(g^2)$ (the steps leading to (3.20)) can be applied to show

$$\{\hat{\mathfrak{Q}}^{\mathfrak{a}},\mathfrak{Q}^{b+\mathfrak{c}}\}_{(3)} = 0 \quad \text{and} \quad \{\hat{\mathfrak{S}}^{\mathfrak{a}},\mathfrak{Q}^{b-\mathfrak{c}}\}_{(3)} = 0. \tag{3.38}$$

For instance, we have

$$\begin{aligned}\{\hat{\mathfrak{Q}}^{\mathfrak{a}}, \mathfrak{Q}^{b+\mathfrak{c}}\}_{(3)} &= \{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)}, \mathfrak{Q}^{b+\mathfrak{c}}_{(0)}\} + \{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, \mathfrak{Q}^{b+\mathfrak{c}}_{(2)}\} \\ &= \{[\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, \mathfrak{x}], \mathfrak{Q}^{b+\mathfrak{c}}_{(0)}\} + \{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, [\mathfrak{Q}^{b+\mathfrak{c}}_{(0)}, \mathfrak{x}]\} \\ &= [\{\hat{\mathfrak{Q}}^{\mathfrak{a}}, \mathfrak{Q}^{b+\mathfrak{c}}\}_{(1)}, \mathfrak{x}] \\ &= 0, \end{aligned} \tag{3.39}$$

and similarly for $\{\hat{\mathfrak{S}}^{\mathfrak{a}}, \mathfrak{Q}^{b-\mathfrak{c}}\}_{(3)}$. To reach the last line, we used the vanishing $\mathcal{O}(g)$ commutators between $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ generators.

$\{\hat{\mathfrak{Q}}, \mathfrak{Q}^-\}$ and $\{\hat{\mathfrak{S}}, \mathfrak{Q}^+\}$. For this step, we require an identity that depends on the explicit form of $\mathfrak{h}$. For all $\mathfrak{a}$ and without summation we have

$$\{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, [\mathfrak{Q}^{b-\mathfrak{a}}, \mathfrak{h}]\} = 0 \quad \text{and} \quad \{\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}, [\mathfrak{Q}^{b+\mathfrak{a}}, \mathfrak{h}]\} = 0. \tag{3.40}$$

This is proved in Appendix D. Next, we verify that

$$\{\hat{\mathfrak{Q}}^{<}, \mathfrak{Q}^{1-\>}\}_{(3)} = 0. \tag{3.41}$$

We simplify using the Jacobi identity and the vanishing $\mathcal{O}(g)$ commutators,

$$\begin{aligned}\{\hat{\mathfrak{Q}}^{<}, \mathfrak{Q}^{1->}\}_{(3)} &= \{\hat{\mathfrak{Q}}^{<}_{(3)}, \mathfrak{Q}^{1->}_{(0)}\} + \{\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{Q}^{1->}_{(2)}\} \\ &= \{[\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{x}], \mathfrak{Q}^{1->}_{(0)}\} - \{\hat{\mathfrak{Q}}^{<}_{(1)}, [\mathfrak{Q}^{1->}_{(0)}, \mathfrak{x}]\} \\ &= 2\{[\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{x}], \mathfrak{Q}^{1->}_{(0)}\} - [\{\hat{\mathfrak{Q}}^{<}, \mathfrak{Q}^{1->}\}_{(1)}, \mathfrak{x}] \\ &= 2\{[\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{x}], \mathfrak{Q}^{1->}_{(0)}\}. \end{aligned} \tag{3.42}$$

To finish, we apply (3.37) and the new identity (3.40),

$$\begin{aligned}2\{[\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{x}], \mathfrak{Q}^{1->}_{(0)}\} &= -2\{[\hat{\mathfrak{Q}}^{<}_{(1)}, \{\hat{\mathfrak{S}}^{<}_{(1)}, [\hat{\mathfrak{Q}}^{>}_{(1)}, \mathfrak{h}]\}], \mathfrak{Q}^{1->}_{(0)}\} \\ &= -2\{\hat{\mathfrak{Q}}^{<}_{(1)}, [\hat{\mathfrak{S}}^{<}_{(1)}, \{\hat{\mathfrak{Q}}^{>}_{(1)}, [\mathfrak{Q}^{1->}_{(0)}, \mathfrak{h}]\}]\} \\ &= 0. \end{aligned} \tag{3.43}$$

As a corollary of this proof we have

$$\{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, \mathfrak{Q}^{b-\mathfrak{c}}_{(2)}\} = 0,\ \{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)}, \mathfrak{Q}^{b-\mathfrak{c}}_{(0)}\} = 0,\ \{\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)}, \mathfrak{Q}^{b+\mathfrak{c}}_{(2)}\} = 0,\ \text{and}\ \{\hat{\mathfrak{S}}^{\mathfrak{a}}_{(3)}, \mathfrak{Q}^{b+\mathfrak{c}}_{(0)}\} = 0. \tag{3.44}$$

Now, by $\mathfrak{R}$ symmetry and $\mathfrak{B}$ symmetry (3.41) implies that all commutators of the type $\{\hat{\mathfrak{Q}}, \mathfrak{Q}^{a-\mathfrak{b}}\}$ vanish. The argument follows the one used after (3.25). The eight possibilities transform under $\mathfrak{su}(2)_{\mathfrak{B}} \times \mathfrak{su}(2)_{\mathfrak{R}}$ as

$$(\mathbf{1}, \mathbf{2}) \otimes (\mathbf{2}, \mathbf{2}) = (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{3}). \tag{3.45}$$

Since (3.41) has components in both irreducible representations, it follows that the algebra relation is satisfied for all of the index combinations. Similarly, one can prove

$$\{\hat{\mathfrak{S}}^{<}, \mathfrak{Q}^{1+>}\}_{(3)} = 0, \tag{3.46}$$

which implies that all commutators of the form $\{\hat{\mathfrak{S}}, \mathfrak{Q}^{a+\mathfrak{b}}\}$ vanish. By substituting commutators of supercharges for $\mathfrak{J}^{\alpha\beta}$, as we did in (3.28), one can see that the $\mathfrak{psu}(1|1)^2$ generators commute with $\mathfrak{J}^{\alpha\beta}$ at this order. This completes the $\mathcal{O}(g^3)$ portion of the proof.

$\{\hat{\mathfrak{Q}}, \hat{\mathfrak{Q}}\}$ and $\{\hat{\mathfrak{S}}, \hat{\mathfrak{S}}\}$. This proof works similarly to the previous proofs for (3.20) and (3.38), which also involved commutators between generators that received corrections of the same form (same-sign commutators with $\mathfrak{x}$),

$$\begin{aligned}
\{\hat{\mathfrak{Q}}^{\mathfrak{a}}, \hat{\mathfrak{Q}}^{\mathfrak{b}}\}_{(4)} &= \{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)}, \hat{\mathfrak{Q}}^{\mathfrak{b}}_{(1)}\} + \{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, \hat{\mathfrak{Q}}^{\mathfrak{b}}_{(3)}\} \\
&= \{[\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, \mathfrak{x}], \hat{\mathfrak{Q}}^{\mathfrak{b}}_{(1)}\} + \{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, [\hat{\mathfrak{Q}}^{\mathfrak{b}}_{(1)}, \mathfrak{x}]\} \\
&= [\{\hat{\mathfrak{Q}}^{\mathfrak{a}}, \hat{\mathfrak{Q}}^{\mathfrak{b}}\}_{(2)}, \mathfrak{x}] \\
&= 0,
\end{aligned} \tag{3.47}$$

and similarly for $\{\hat{\mathfrak{S}}^{\mathfrak{a}}, \hat{\mathfrak{S}}^{\mathfrak{b}}\}_{(4)}$. For the last line, we used the leading order $\mathfrak{psu}(1|1)^2$ symmetry algebra.

$\{\hat{\mathfrak{Q}}, \hat{\mathfrak{S}}\}$. We begin by checking that

$$\{\hat{\mathfrak{Q}}^{>}, \hat{\mathfrak{S}}^{>}\}_{(4)} = 0 \quad \text{or} \quad 2\{[\hat{\mathfrak{Q}}^{>}_{(1)}, \mathfrak{x}], \hat{\mathfrak{S}}^{>}_{(1)}\} = 0, \tag{3.48}$$

where the second expression follows from the Jacobi identity and the leading order $\mathfrak{psu}(1|1)^2$ symmetry algebra. Using the alternative expression for the $\mathcal{O}(g^3)$ correction (3.37) and (3.33) we obtain

$$\begin{aligned} 2\{[\hat{\mathfrak{Q}}^{>}_{(1)}, \mathfrak{x}], \hat{\mathfrak{S}}^{>}_{(1)}\} &= 2\{[\hat{\mathfrak{Q}}^{>}_{(1)}, \{\hat{\mathfrak{S}}^{>}_{(1)}, [\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{h}]\}], \hat{\mathfrak{S}}^{>}_{(1)}\} \\ &= -2\{[\hat{\mathfrak{S}}^{>}_{(1)}, \{\hat{\mathfrak{S}}^{>}_{(1)}, [\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{h}]\}], \hat{\mathfrak{Q}}^{>}_{(1)}\} \\ &= 0. \end{aligned} \tag{3.49}$$

For the second-to-last line we again used the leading order $\mathfrak{psu}(1|1)^2$ symmetry algebra. By $\mathfrak{B}$ symmetry, this guarantees that all four algebra relations for commutators between $\hat{\mathfrak{Q}}$ and $\hat{\mathfrak{S}}$ are satisfied (they split as $\mathbf{1} \oplus \mathbf{3}$ and we checked a relation with components in both representations). This completes the proof that the solution (3.29) provides a one- and one-half-loop solution to the symmetry algebra.

3.3.3 The two-loop dilatation generator

From the $\mathfrak{psu}(1|1)^2$ algebra (1.68) we can now compute $\delta\mathfrak{D}_{(4)}$ directly,

$$\delta\mathfrak{D}_{(4)} = 2\{\hat{\mathfrak{Q}}^{<}, \hat{\mathfrak{S}}^{>}\}_{(4)} = -2\{\hat{\mathfrak{Q}}^{>}, \hat{\mathfrak{S}}^{<}\}_{(4)}. \tag{3.50}$$

It follows that $\delta\mathfrak{D}_{(4)}$ is composed only of the leading order $\mathfrak{psu}(1|1)^2$ supercharges and $\mathfrak{h}$, the one-site harmonic number generator. After setting the similarity transformation $\mathfrak{y}$ to zero, without loss of generality, and using the form of the $\mathcal{O}(g^3)$ solution (3.37), we find

$$\begin{aligned} \delta\mathfrak{D}_{(4)} &= 2\{\hat{\mathfrak{Q}}^{<}_{(1)}, [\hat{\mathfrak{S}}^{>}_{(1)}, \{\hat{\mathfrak{Q}}^{>}_{(1)}, [\hat{\mathfrak{S}}^{<}_{(1)}, \mathfrak{h}]\}]\} - 2\{\hat{\mathfrak{S}}^{>}_{(1)}, [\hat{\mathfrak{Q}}^{<}_{(1)}, \{\hat{\mathfrak{S}}^{<}_{(1)}, [\hat{\mathfrak{Q}}^{>}_{(1)}, \mathfrak{h}]\}]\} \\ &= 2\{\hat{\mathfrak{Q}}^{>}_{(1)}, [\hat{\mathfrak{S}}^{<}_{(1)}, \{\hat{\mathfrak{Q}}^{<}_{(1)}, [\hat{\mathfrak{S}}^{>}_{(1)}, \mathfrak{h}]\}]\} - 2\{\hat{\mathfrak{S}}^{<}_{(1)}, [\hat{\mathfrak{Q}}^{>}_{(1)}, \{\hat{\mathfrak{S}}^{>}_{(1)}, [\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{h}]\}]\}. \end{aligned} \tag{3.51}$$

In this expression we have left out the coupling constant transformation parameterized by α in (3.29), which leads to

$$\delta\mathfrak{D}_{(4)} \mapsto \delta\mathfrak{D}_{(4)} + 2\alpha\,\delta\mathfrak{D}_{(2)}. \tag{3.52}$$

However, to match field theory results α must be zero.

3.3.4 Nonplanar lift and wrapping interactions

By lifting our expressions for the building blocks of $\delta\mathfrak{D}_{(4)}$ to their nonplanar generalization, we will construct a candidate for the finite-N $\delta\mathfrak{D}_{(4)}$. To support our conjecture that this is the correct solution, we will observe that it accurately includes wrapping interactions. The two-loop nonplanar solution for the $\mathfrak{su}(2)$ sector (a subsector of the $\mathfrak{psu}(1,1|2)$ sector) was found in [18]. In that case, there is a unique lift from the planar to the nonplanar theory.

Let the gauge group of the theory (this generalization does not require $\mathrm{U}(N)$) have generators $\mathfrak{t}_\mathfrak{m}$ and metric $\mathfrak{g}^{\mathfrak{m}\mathfrak{n}}$. Recall that for the nonplanar case we write interactions as traces of fields and variations. So we need the gauge group expansions, for X_i a field $\phi_a^{(k)}$ or $\psi_\mathfrak{a}^{(k)}$,

$$X_i = X_i^\mathfrak{m}\mathfrak{t}_\mathfrak{m},$$
$$\check{X}^i = \mathfrak{t}_\mathfrak{m}\mathfrak{g}^{\mathfrak{m}\mathfrak{n}}\frac{\delta}{\delta X_i^\mathfrak{n}}, \qquad \frac{\delta}{\delta X_i^\mathfrak{m}}X_j^\mathfrak{n} = \delta_j^i\delta_\mathfrak{m}^\mathfrak{n}. \tag{3.53}$$

Now the nonplanar action for the one-site generators, including the $\mathcal{O}(g^0)$ terms and $\mathfrak{h}$ are straightforward to obtain. For instance,

$$\mathfrak{h} = \sum_{k=0}^{\infty} h(k)\,\mathrm{Tr}\,\phi_a^{(k)}\,\check{\phi}^{(k)a} + \sum_{k=0}^{\infty} h(k+1)\,\mathrm{Tr}\,\psi_\mathfrak{a}^{(k)}\,\check{\psi}^{(k)\mathfrak{a}}. \tag{3.54}$$

The $\mathfrak{psu}(1|1)^2$ supercharges also have a natural generalization for the nonplanar theory,

$$\begin{aligned}\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)} = &\sum_{\substack{0\le n\\0\le k<n}} \frac{\varepsilon^{\mathfrak{a}\mathfrak{c}}}{\sqrt{k+1}} \operatorname{Tr}[\psi_{\mathfrak{c}}^{(k)}, \phi_b^{(n-1-k)}]\check{\phi}^{(n)b} \\ &+ \sum_{\substack{0\le n\\0\le k<n}} \frac{\sqrt{n-k}}{\sqrt{(k+1)(n+1)}}\varepsilon^{\mathfrak{a}\mathfrak{c}} \operatorname{Tr}\{\psi_{\mathfrak{c}}^{(k)}, \psi_{\mathfrak{b}}^{(n-1-k)}\}\check{\psi}^{(n)\mathfrak{b}} \\ &- \sum_{\substack{0\le n\\0\le k\le n}} \frac{1}{2\sqrt{n+1}}\delta^{\mathfrak{a}}_{\mathfrak{b}}\varepsilon^{cd} \operatorname{Tr}[\phi_c^{(k)}, \phi_d^{(n-k)}]\check{\psi}^{(n)\mathfrak{b}}. \end{aligned} \tag{3.55}$$

With minor switching of summation variables, it is straightforward to check that this expression reduces to (2.34) in the planar limit. For the hermitian conjugates, $\varepsilon_{\mathfrak{a}\mathfrak{b}}\hat{\mathfrak{S}}^{\mathfrak{b}}_{(1)}$, simply perform the switch

$$X_i \leftrightarrow \check{X}^i, \quad \forall X_i. \tag{3.56}$$

Substituting these expressions into the expressions for $\mathfrak{x}$ gives its nonplanar version. Then the expressions given for $\hat{\mathfrak{Q}}_{(3)}$, $\hat{\mathfrak{S}}_{(3)}$ and $\delta\mathfrak{D}_{(4)}$ (3.29) and (3.51) become nonplanar. Because the proof that the planar solution satisfies the symmetry constraints is independent of planarity, the nonplanar generalization still satisfies the symmetry constraints.

While we do not have a proof that this is the correct nonplanar solution, our solution accurately includes wrapping interactions, which can be thought of as special cases of nonplanar interactions. A two-site wrapping interaction, for example, could be written as

$$\operatorname{Tr}(X_i X_j)\operatorname{Tr}(\check{X}^k \check{X}^m), \tag{3.57}$$

which is nonplanar in general, but on a two-site state its action becomes planar. Similarly, the planar solution and the nonplanar generalization are equivalent for two-site states. Since the leading-order $\mathfrak{psu}(1|1)^2$ supercharges map one site to two sites or vice-versa, the action of $\delta\mathfrak{D}_{(4)}$ (3.51) is well defined even on two-site states. Furthermore, since acting with the $\hat{\mathfrak{Q}}$'s on two-site states yields three-site states, two-site states are in the same $\mathfrak{psu}(2,2|4)$

multiplets as three-site states. Therefore, adding special wrapping interactions that only change the anomalous dimensions of two-site states would be inconsistent with the symmetry constraints.

3.4 Verifications of the two-loop solution and of integrability

Using the solution for the two-loop dilatation operator, we first provide strong evidence that it is correct via direct diagonalization and comparison to rigorous field theory computations. We then use our solution to present strong evidence in favor of integrability by computing the internal two-loop S-matrix in the bosonic $\mathfrak{sl}(2)$ sector and by comparing anomalous dimension predictions of the higher-loop Bethe ansatz, which was first proposed in [21], with the results of direct diagonalization.

3.4.1 Comparison with field theory calculations

Expanding the expression for $\delta\mathfrak{D}_{(4)}$ (3.64) in terms of interactions, we find the planar anomalous dimensions by direct diagonalization. We first identify the spin chain states of the subspaces of certain (small) values of classical dimension, $\mathfrak{R}$ and $\mathfrak{B}$ charge, and length. Then we apply $g^2\delta\mathfrak{D}_{(2)} + g^4\delta\mathfrak{D}_{(4)}$ to these subspaces and compute its eigenvalues (the anomalous dimensions) and eigenstates, as in the simple one-loop example (1.21). For these two-loop computations, we have used `Mathematica`. We check states with rigorously known anomalous dimensions. These include twist-two operators [52], a pair of states of length three and bare dimension six [53], two-excitation states (BMN operators) [55, 18], and length-three states built from one type of fermion and from derivatives (in the fermionic $\mathfrak{sl}(2)$ subsector) [54]. The states we check, given in Table 3.1, are in complete agreement with these previous computations. Therefore, we conclude that we have found almost certainly

$\mathfrak{D}_0$	(R,L,B)	$(\delta\mathfrak{D}_2,\delta\mathfrak{D}_4)^P$
4	$(1,2,0)$	$(12,-48)^+$
5	$(\frac{3}{2},3,0)$	$(8,-24)^-$
6	$(1,2,0)$	$(\frac{50}{3},-\frac{1850}{27})^+$
6	$(\frac{3}{2},3,0)$	$(15,-\frac{225}{4})^\pm$
6	$(2,4,0)$	$(5.5279,-11.6393)^+$ $(14.4721,-56.361)^+$
7	$(\frac{5}{2},5,0)$	$(4,-6)^-$ $(12,-42)^-$
7.5	$(0,3,\frac{3}{2})$	$(20,-\frac{245}{3})^\pm$
8	$(1,2,0)$	$(\frac{98}{5},-\frac{91238}{1125})^+$
8	$(3,6,0)$	$(3.0121,-3.32025)^+$ $(9.7802,-29.2249)^+$ $(15.2078,-59.455)^+$
9	$(3,8,1)$	$(2.3431,-1.95838)^-$ $(8,-20)^-$ $(13.6569,-50.042)^-$
9.5	$(0,3,\frac{3}{2})$	$(\frac{133}{6},-\frac{131117}{1440})^\pm$
10	$(\frac{7}{2},9,1)$	$(1.87164,-1.21946)^+$ $(6.6108,-13.7752)^+$ $(12,-40)^+$ $(15.5175,-61.005)^+$
10.5	$(0,3,\frac{3}{2})$	$(\frac{761}{35},-\frac{138989861}{1543500})^+$ $(\frac{761}{30},-\frac{419501}{4000})^\pm$

Table 3.1: Two-loop spectrum for states with rigorously known planar anomalous dimensions. We use the normalization $g^2 = \frac{g_{\mathrm{YM}}^2 N}{16\pi^2}$, which needs to be taken into account when comparing with previous results. The P exponent of the anomalous dimensions gives the states' eigenvalues under parity. The $\pm$ pairs for P are a consequence of integrability. The twist-two operators are those with length two, the two-excitation states satisfy $\mathfrak{D}_0 - L \leq 2$, and the three-fermion states have $R = 0$.

the correct solution for $\delta\mathfrak{D}_{(4)}$. Since our comparison includes length-two states, we find confirmation that no additional wrapping terms are needed.

3.4.2 Two-loop spin chain S-matrix

We now perform a two-loop check of the bosonic $\mathfrak{sl}(2)$ sector Bethe ansatz of [21]. Instead of only checking anomalous dimension predictions, we also verify a key part of its derivation, the S-matrix. This was also verified by a direct field theory calculation in [56]. As at one-loop (Section 2.1.1), we restrict to the two-excitation $\mathfrak{sl}(2)$ sector consisting of states composed only of ϕ_1 and two or fewer derivatives[4]. We have computed the two-loop internal S-matrix as in [21], which used ideas introduced in [57]. At two loops, the Schrödinger equation becomes

$$H|\Psi\rangle = E|\Psi\rangle, \qquad H = g^2\,\delta\mathfrak{D}_{(2)} + g^4\,\delta\mathfrak{D}_{(4)}. \tag{3.58}$$

It is solved to this order by the ansatz

$$\begin{aligned} \Psi_{p_1p_2} = &\sum_{1\le x_1\le x_2\le L} (1 + c_1\,\delta_{x_1(x_2-1)}) e^{ip_1x_1+ip_2x_2}|x_1x_2\rangle \\ &+ \sum_{1\le x_1\le x_2\le L} (S(p_2,p_1) + c_0\,\delta_{x_1x_2}) e^{ip_2x_1+ip_1x_2}|x_1x_2\rangle, \\ S = S_{(0)} &+ g^2 S_{(2)}, \quad c_0 = g^2 c_{0,(2)}, \quad c_1 = g^2 c_{1,(2)}. \end{aligned} \tag{3.59}$$

$|x_1x_2\rangle$ was defined in (2.7), and as before p_i are the momenta of the excitations which scatter off each other. As at one loop, since the Hamiltonian is still short ranged and translationally invariant, for large separation (large $(x_2 - x_1)$) the solutions of the Schrödinger equation reduce to superpositions of one excitation eigenstates, proportional to e^{ipx}. The S-matrix, now including a two-loop correction, gives the phase that one excitation's wave function acquires when passing the other excitation. Because the Hamiltonian has interactions involving at most three adjacent sites, we need to introduce the new coefficient, c_1[5].

Using `Mathematica`, we have solved the Schrödinger equation using this ansatz and our expression for the dilatation generator. Including the one-loop solution presented earlier,

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[4]Of course, by $\mathfrak{R}$ symmetry, the sector composed of ϕ_2 instead has the same S-matrix and anomalous dimensions.

[5]For simplicity we have not included the argument of p_i, but all functions still depend on them. Note that c_1 and c_0 are unphysical. They will transform nontrivially under a similarity transformation for $\delta\mathfrak{D}$, unlike S and E.

the solution for the energy and the S-matrix is

$$E = E(p_1) + E(p_2), \qquad E(p) = 8\sin^2(\frac{p}{2}) - 32g^2\sin^4(\frac{p}{2}) \tag{3.60}$$

$$S(p_1, p_2) = S_{(0)} + g^2 S_{(2)}$$

$$S_{(0)} = -\frac{1 - 2e^{ip_2} + e^{ip_1+ip_2}}{1 - 2e^{ip_1} + e^{ip_1+ip_2}} \tag{3.61}$$

$$S_{(2)} = \frac{16ie^{ip_1+ip_2}\sin(\frac{p_1}{2})(\sin(\frac{p_1-3p_2}{2}) - 4\sin(\frac{p_1-p_2}{2}) + \sin(\frac{3p_1-p_2}{2}))\sin(\frac{p_2}{2})}{(1 - 2e^{ip_1} + e^{ip_1+ip_2})^2}. \tag{3.62}$$

To two-loop order, after accounting for a factor of $\sqrt{2}$ difference in normalization of g, this agrees with the solution given by equations (3.3) and (6.4), (4.27), and (3.7) of [21].

3.4.3 Comparisons with Bethe equation calculations

Having computed the two-particle S-matrix to two-loop order, assuming diffractionless scattering and requiring periodicity yields the two-loop Bethe equation for this sector, which can be used to compute anomalous dimensions for states with arbitrary numbers of excitations, as in [21]. As shown in Table 3.2, we find perfect agreement between the predictions of the Bethe ansatz and direct diagonalization of the two-loop dilatation generator. This provides compelling evidence for two-loop integrability in the bosonic $\mathfrak{sl}(2)$ subsector. Further support comes from the proof that scattering is diffractionless for three excitation states [56].

Finally, we provide evidence of integrability including fermions as well. We compute anomalous dimensions for the $\mathfrak{su}(1|1)$ sector(s), again via direct diagonalization. As described in Section 1.8, this sector includes states made of only one type of ϕ and only one type of ψ and no derivatives. Again, our findings are in complete agreement with those found assuming integrability in [21]. These anomalous dimensions were also found by direct diagonalization of the compact $\mathfrak{su}(2|3)$ dilatation operator in [19].

D_0	(R,L,B)	$\left(\delta\mathfrak{D}_2,\delta\mathfrak{D}_4\right)^P$
7	$(\frac{3}{2},3,0)$	$(12,-39)^-$
7	$(2,4,0)$	$(12,-42)^\pm$
8	$(\frac{3}{2},3,0)$	$(\frac{35}{2},-\frac{18865}{4608})^\pm$
8	$(2,4,0)$	$(8.7655,-21.001)^+$ $(16.7185,-64.272)^+$ $(23.1826,-92.4124)^+$ $(\frac{46}{3},-\frac{1331}{27})^\pm$
8	$(\frac{5}{2},5,0)$	$(9.4586,-28.0586)^\pm$ $(15.5414,-57.692)^\pm$

Table 3.2: Two-loop spectrum of highest weight states in the bosonic $\mathfrak{sl}(2)$ sector(s) found by direct diagonalization. Again note that we use $g^2 = \frac{g_{\text{YM}}^2 N}{16\pi^2}$.

D_0	$(R,L,B))$	$\left(\delta\mathfrak{D}_2,\delta\mathfrak{D}_4\right)^P$
7	$(\frac{1}{2},5,2)$	$(20,-80)^-$
7.5	$(\frac{3}{2},6,1.5)$	$(16,-56)^\pm$
8	$(1,6,2)$	$(16,-56)^+$
8.5	$(2,7,1.5)$	$(14,-48)^\pm$
9	$(\frac{3}{2},7,2)$	$(12.7922,-37.597)^-$ $(18.2198,-68.411)^-$ $(24.99,-97.992)^-$
9.5	$(\frac{5}{2},8,1.5)$	$(12,-38)^\pm$ $(16,-58)^\pm$

Table 3.3: Two-loop spectrum of states in the $\mathfrak{su}(1|1)$ sector found by direct diagonalization.

3.5 Order g^4

Given the simple structure to the quantum corrections through $\mathcal{O}(g^3)$ and for the two-loop dilatation generator, only involving $\mathfrak{h}$ and the leading order symmetry generators, one might hope that this continues to hold at higher orders. As we now show, it is possible to find a $\mathcal{O}(g^4)$ solution built only from these ingredients. However, an exhaustive search suggests that new ingredients are needed beyond this order. We discuss the implications of this in Section 3.5.3.

3.5.1 A solution

Before presenting the solution, we given an alternative form for the $\mathcal{O}(g^2)$ solution (3.6). To obtain these equalities we use (3.40) and the two equivalent expression for $\mathfrak{x}$ (3.3).

$$\begin{aligned}
\mathfrak{Q}^{a+<}_{(2)} &= [\hat{\mathfrak{S}}^{>}_{(1)}, \{\hat{\mathfrak{Q}}^{<}_{(1)}, [\mathfrak{Q}^{a+<}_{(0)}, \mathfrak{h}]\}], \quad \mathfrak{Q}^{a+>}_{(2)} = -[\hat{\mathfrak{S}}^{<}_{(1)}, \{\hat{\mathfrak{Q}}^{>}_{(1)}, [\mathfrak{Q}^{a+>}_{(0)}, \mathfrak{h}]\}], \\
\mathfrak{Q}^{a-<}_{(2)} &= -[\hat{\mathfrak{Q}}^{>}_{(1)}, \{\hat{\mathfrak{S}}^{<}_{(1)}, [\mathfrak{Q}^{a-<}_{(0)}, \mathfrak{h}]\}], \quad \mathfrak{Q}^{a->}_{(2)} = [\hat{\mathfrak{Q}}^{<}_{(1)}, \{\hat{\mathfrak{S}}^{>}_{(1)}, [\mathfrak{Q}^{a->}_{(0)}, \mathfrak{h}]\}].
\end{aligned} \tag{3.63}$$

Given the iterative structure appearing at lower orders, it is natural to suspect that the next corrections take a similar form, for instance having a leading order generator replaced by its first correction. In fact, we find a solution that comes close to this intuition:

$$\begin{aligned}
\mathfrak{Q}^{a+<}_{(4)} &= 2[\hat{\mathfrak{S}}^{>}_{(1)}, \{\hat{\mathfrak{Q}}^{<}_{(3)}, [\mathfrak{Q}^{a+<}_{(0)}, \mathfrak{h}]\}] - \frac{1}{2}[\mathfrak{Q}^{a+<}_{(2)}, \mathfrak{x}], \\
\mathfrak{Q}^{a+>}_{(4)} &= -2[\hat{\mathfrak{S}}^{<}_{(1)}, \{\hat{\mathfrak{Q}}^{>}_{(3)}, [\mathfrak{Q}^{a+>}_{(0)}, \mathfrak{h}]\}] - \frac{1}{2}[\mathfrak{Q}^{a+>}_{(2)}, \mathfrak{x}], \\
\mathfrak{Q}^{a-<}_{(4)} &= -2[\hat{\mathfrak{Q}}^{>}_{(1)}, \{\hat{\mathfrak{S}}^{<}_{(3)}, [\mathfrak{Q}^{a-<}_{(0)}, \mathfrak{h}]\}] + \frac{1}{2}[\mathfrak{Q}^{a-<}_{(2)}, \mathfrak{x}], \\
\mathfrak{Q}^{a->}_{(4)} &= 2[\hat{\mathfrak{Q}}^{<}_{(1)}, \{\hat{\mathfrak{S}}^{>}_{(3)}, [\mathfrak{Q}^{a->}_{(0)}, \mathfrak{h}]\}] + \frac{1}{2}[\mathfrak{Q}^{a->}_{(2)}, \mathfrak{x}].
\end{aligned} \tag{3.64}$$

For each $\mathfrak{Q}^{a\beta\mathfrak{c}}_{(4)}$, the first term replaces one $\mathcal{O}(g^1)$ generator in (3.63) with twice its $\mathcal{O}(g^3)$ correction, and the second term replaces the $\mathcal{O}(g^0)$ generator in the solution (3.6) with

$-1/2$ of its one-loop correction.

As at previous orders, there are similarity and coupling constant transformations that map one solution to another. At this order we can expand a similarity transformation as

$$U = 1 - g^2\,\mathfrak{y}_{(2)} + \tfrac{1}{2}g^4\,\mathfrak{y}_{(2)}^2 - g^4\,\mathfrak{y}_{(4)} + \dots \tag{3.65}$$

which corresponds to the mapping

$$\mathfrak{Q}_{(4)}^{a\beta\mathfrak{c}} \mapsto \mathfrak{Q}_{(4)}^{a\beta\mathfrak{c}} + [\mathfrak{Q}_{(2)}^{a\beta\mathfrak{c}}, \mathfrak{y}_{(2)}] + [\mathfrak{Q}_{(0)}^{a\beta\mathfrak{c}}, \mathfrak{y}_{(4)}] + \tfrac{1}{2}[[\mathfrak{Q}_{(0)}^{a\beta\mathfrak{c}}, \mathfrak{y}_{(2)}], \mathfrak{y}_{(2)}]. \tag{3.66}$$

The $\mathfrak{y}_{(2)}^2$ term is needed for the similarity transformation to preserve the locality of the generators. The coupling constant transformation (3.30) acts at this order as

$$\mathfrak{Q}_{(4)}^{a\beta\mathfrak{c}} \mapsto \mathfrak{Q}_{(4)}^{a\beta\mathfrak{c}} + 2\alpha\,\mathfrak{Q}_{(2)}^{a\beta\mathfrak{c}}. \tag{3.67}$$

Note that at this order, the generators act on three (adjacent) sites, and that this solution (for antihermitian $\mathfrak{y}$) is consistent with the hermitian conjugation transformation (1.58).

We have not given expressions for $\mathfrak{J}_{(4)}^{++}$ and $\mathfrak{J}_{(4)}^{--}$, but these can be obtained using the algebra relation (1.57). For instance,

$$\begin{aligned}
\mathfrak{J}_{(4)}^{++} &= \{\mathfrak{Q}^{1+>}, \mathfrak{Q}^{2+<}\}_{(4)} = -\{\mathfrak{Q}^{1+<}, \mathfrak{Q}^{2+>}\}_{(4)} \\
&= \{\mathfrak{Q}_{(4)}^{1+>}, \mathfrak{Q}_{(0)}^{2+<}\} + \{\mathfrak{Q}_{(2)}^{1+>}, \mathfrak{Q}_{(2)}^{2+<}\} + \{\mathfrak{Q}_{(0)}^{1+>}, \mathfrak{Q}_{(4)}^{2+<}\} \\
&= -\{\mathfrak{Q}_{(4)}^{1+<}, \mathfrak{Q}_{(0)}^{2+>}\} - \{\mathfrak{Q}_{(2)}^{1+<}, \mathfrak{Q}_{(2)}^{2+>}\} - \{\mathfrak{Q}_{(0)}^{1+<}, \mathfrak{Q}_{(4)}^{2+>}\}.
\end{aligned} \tag{3.68}$$

As part of our proof we will confirm that these two different ways to generate $\mathfrak{J}_{(4)}^{++}$ are equal.

3.5.2 The $\mathcal{O}(g^4)$ proof of the algebra

As at previous orders we will begin by checking the commutators with $\mathfrak{R}^{ab}$ and $\mathfrak{B}^{\mathfrak{ab}}$, and the latter check is nontrivial for this solution. After proving these relations are satisfied, we will need only to verify directly two commutators of supercharges with equal $\mathfrak{su}(1,1)$ charge and one commutator between supercharges with unequal $\mathfrak{su}(1,1)$ charge. The commutators involving equal $\mathfrak{su}(1,1)$ charge depend on one new identity (and its hermitian conjugate) that depends on the explicit form of $\mathfrak{h}$,

$$[\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{1+<}_{(0)},\mathfrak{h}]\},\{\hat{\mathfrak{Q}}^{>}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\}]=[\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{1+<}_{(0)},\mathfrak{h}]\},\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]\}]. \tag{3.69}$$

The difference between the two sides appears in the $\mathfrak{B}$ indices of the last two supercharges on each side. The identity can be confirmed via computing the actions of these nested commutators of generators on the fields. Note that both sides are sums of interactions that map one initial site to three final sites. Furthermore, both sides have $\mathfrak{B}$ charge and $\mathfrak{R}$ charge equal to 1. Therefore, they automatically annihilate

$$|\psi^{(k)}_{>}\rangle \quad \text{and} \quad |\phi^{(k)}_{2}\rangle \tag{3.70}$$

since there are no interactions with these charges that map these states to length-3 final states. One only needs to check (3.69) on the other two possible initial states (for which there are possible interactions).

By inspection of (3.64), we see that the solution transforms properly under commutation with the $\mathfrak{R}^{ab}$ and with $\mathfrak{B}^{<>}$. However, checking the transformation with respect to $\mathfrak{B}^{<<}$ and $\mathfrak{B}^{>>}$ is less simple.

First we observe three identities that follow from previous results,

$$\{\hat{\mathfrak{S}}^{\mathfrak{a}}_{(1)},\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(3)}\}=0 \quad \text{(no summation)}, \tag{3.71}$$

$$\{\hat{\mathfrak{S}}^{>}_{(1)},\hat{\mathfrak{Q}}^{<}_{(3)}\}=-\{\hat{\mathfrak{S}}^{<}_{(1)},\hat{\mathfrak{Q}}^{>}_{(3)}\}, \tag{3.72}$$

$$\{\hat{\mathfrak{S}}^{>}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\}=-\{\hat{\mathfrak{S}}^{<}_{(1)},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]\}. \tag{3.73}$$

The first identity is implied by the proof given for the vanishing of $\{\hat{\mathfrak{Q}}^{>},\hat{\mathfrak{S}}^{>}\}_{(4)}$, the second follows from the first by acting with $\mathfrak{B}^{\mathfrak{ab}}$, and the third follows via $\mathfrak{B}^{\mathfrak{ab}}$ from (3.40).

The terms of (3.64) involving the commutator with $\mathfrak{x}$ clearly transform properly with respect to $\mathfrak{B}^{\mathfrak{ab}}$, so we need to check only the transformation of the first terms. Then we have, for example,

$$\begin{aligned}[\mathfrak{B}^{<<},[\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(3)},[\mathfrak{Q}^{1+<}_{(0)},\mathfrak{h}]\}]] &= -[\hat{\mathfrak{S}}^{<}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(3)},[\mathfrak{Q}^{1+<}_{(0)},\mathfrak{h}]\}] \\ &= 0, \end{aligned} \tag{3.74}$$

where we used (3.40) and (3.71). This (and the three parallel computations) shows that half of the commutator relations with the $\mathfrak{B}^{<<}$ and $\mathfrak{B}^{>>}$ are satisfied. For the remainder, as an example we compute the commutator with 1/2 of the first term of $\mathfrak{Q}^{a+<}_{(4)}$,

$$\begin{aligned}[\mathfrak{B}^{>>},[\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(3)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\}]] &= [\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{>}_{(3)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\}] + [\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(3)},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]\}] \\ &= -[\hat{\mathfrak{Q}}^{>}_{(3)},\{\hat{\mathfrak{S}}^{>}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\}] + [\{\hat{\mathfrak{S}}^{>}_{(1)},\hat{\mathfrak{Q}}^{<}_{(3)}\},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]] \\ &= [\hat{\mathfrak{Q}}^{>}_{(3)},\{\hat{\mathfrak{S}}^{<}_{(1)},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]\}] - [\{\hat{\mathfrak{S}}^{<}_{(1)},\hat{\mathfrak{Q}}^{>}_{(3)}\},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]] \\ &= -[\hat{\mathfrak{S}}^{<}_{(1)},\{\hat{\mathfrak{Q}}^{>}_{(3)},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]\}]. \end{aligned} \tag{3.75}$$

The second equality depends on (3.71) and (3.40), and the third equality follows from (3.73) and (3.72). The last expression is 1/2 of the first term of $\mathfrak{Q}^{a+>}_{(4)}$, as needed. The other three nonvanishing commutators involving $\mathfrak{B}^{<<}$ and $\mathfrak{B}^{>>}$ can be shown similarly, completing the proof that the solution (3.64) transforms properly with respect to $\mathfrak{B}$ symmetry.

For the commutators between supercharges with equal $\mathfrak{su}(1,1)$ charge we prove

$$\{\mathfrak{Q}^{1+<},\mathfrak{Q}^{1+<}\}_{(4)} = 0. \tag{3.76}$$

This lengthy proof appears in Appendix E and combined with $\mathfrak{B}$ symmetry, $\mathfrak{R}$ symmetry, and hermitian conjugation shows that the algebra relations for commutators between supercharges with like $\mathfrak{su}(1,1)$ charge are satisfied at two loops. Also, $\mathfrak{B}$ symmetry and $\mathfrak{R}$ symmetry guarantee that the two different expressions for $\mathfrak{J}^{++}_{(4)}$ (or $\mathfrak{J}^{--}_{(4)}$) are equal.

For the commutators between supercharges with unequal $\mathfrak{su}(1,1)$ charge, in Appendix E we show that

$$\{\mathfrak{Q}^{1+<}, \mathfrak{Q}^{2->}\}_{(4)} = -\frac{1}{2}\delta\mathfrak{D}_4. \tag{3.77}$$

As at one loop, $\mathfrak{R}$ symmetry and $\mathfrak{B}$ symmetry then imply that all 16 relations for commutators between supercharges with opposite $\mathfrak{su}(1,1)$ charge are satisfied. This completes the proof that the $\mathcal{O}(g^4)$ solution (3.64) satisfies the symmetry algebra constraints.

3.5.3 Discussion

We have performed a thorough search at $\mathcal{O}(g^5)$ for a solution only built from leading order generators and $\mathfrak{h}$. We truncated to two excitations and used `Mathematica` to check the algebra relations for hundreds of possible expressions for the $\mathcal{O}(g^5)$ generators. Since we did not find a solution, it appears very likely that at least starting at $\mathcal{O}(g^5)$ a new ingredient is needed. In fact, at two excitations we have confirmed that introducing one new generator $\mathfrak{h}_{(2)}$ that acts on two sites is sufficient to satisfy the symmetry algebra and match three-loop anomalous dimension predictions of the Bethe ansatz. However, at this point $\mathfrak{h}_{(2)}$ is simply a large, finite-sized numerical matrix, and further work is required to find its all-excitation form. In retrospect it is not surprising that $\mathfrak{h}_{(2)}$ is necessary. Like the symmetry generators, $\mathfrak{h}$ should receive quantum corrections. On the other hand, we have not yet identified what algebra is generated by the combination of $\mathfrak{h}$ (and its corrections) and the $\mathfrak{psu}(1,1|2)$ generators. Understanding this would certainly make much more progress possible.

We should mention also that the $\mathcal{O}(g^5)$ two-excitation solution is not consistent with the

$\mathcal{O}(g^4)$ solution we presented. This implies two possibilities: either there is a homogeneous solution to the symmetry constrains starting from $\mathcal{O}(g^4)$, or else the "solution" we found is not compatible with any solution for the three-loop dilatation generator. In fact, studies of the phase function for the Bethe ansatz suggest that there is one homogeneous solution at this order consistent with integrability [27, 32].

There are numerous promising directions for extension to higher loops. Further study of the new generator $\mathfrak{h}_{(2)}$ and the (currently unidentified) enlarged algebra including $\mathfrak{h}$ is the most direct. Also, studying the next-to-leading order corrections to the nonlocal symmetry enhancement of Chapter 2 may provide key insights. Finally, it makes sense to look for a method to require integrability directly for the quantum corrections to the symmetry generators. We investigate this possibility by studying Yangian symmetry in the next chapter.

Chapter 4

Yangian symmetry at two loops in the $\mathfrak{su}(2|1)$ subsector

In this chapter we construct the next-to-leading order Yangian symmetry of the $\mathfrak{su}(2|1)$ subsector. As we discussed in the Introduction, in typical integrable systems the existence of Yangian charges follows from a nearest neighbor R-matrix, which also generates an infinite family of local commuting charges. Since it is unclear if it is possible to apply the R-matrix formalism to this system beyond leading order, it is reasonable to use the existence of Yangian symmetry as a substitute definition of integrability. Therefore, the results of this chapter prove that the $\mathfrak{su}(2|1)$ sector is integrable at two loops[1].

The dual string theory's classical Yangian charges were introduced in [25], and [58] argues that an infinite family of nonlocal charges persists quantum mechanically. For the gauge theory, the Yangian charges are known at leading order for the full theory [36, 37]. For the $\mathfrak{su}(2)$ sector, they are know up to second [59] and fourth order [60], and [61] gave the next-to-leading order Yangian symmetry of the $\mathfrak{su}(1|1)$ sector. By considering the $\mathfrak{su}(2|1)$ sector at next-to-leading order, we encounter important features that have not appeared in these previous studies of the gauge theory Yangian. The Hamiltonian (dilatation generator)

[1]Note that the leading order Hamiltonian is $\mathcal{O}(g^2)$ or one loop, so that showing it has nth order Yangian symmetry proves $(n+1)$-loop integrability.

is part of the local symmetry algebra, and other local symmetry generators have two-site interactions.

Following a brief introduction, in Section 4.2 we review the $\mathfrak{su}(2|1)$ algebra and its Yangian generalization, and in Section 4.3 we review the leading order representation. The next-to-leading order corrected Yangian charges and the proofs that they satisfy the Yangian algebra and commute with the Hamiltonian are given in Section 4.4. We discuss our results in Section 4.5. This chapter is based on [39].

4.1 Introduction

At leading order, the infinite tower of Yangian charges are generated by repeated commutators of the bilocal generators:

$$\mathfrak{Y}^a = \sum_{i<j} f^A{}_{CB} \mathfrak{J}^B(i) \mathfrak{J}^C(j), \tag{4.1}$$

where the $\mathfrak{J}^A$ are local symmetry generators and $f^A{}_{BC}$ are the structure constants. As in previous chapters, $\mathfrak{J}(i)$ represents the generator acting on site i of the spin chain.

We propose that the $\mathcal{O}(g^n)$ perturbative corrections to $\mathfrak{Y}^a$ are of the schematic form

$$\mathfrak{Y}^a_{(n)} = \sum_{\substack{p+q=n \\ i<<j}} f^A{}_{CB} \mathfrak{J}^B_{(p)}(i) \mathfrak{J}^C_{(q)}(j) + \text{local terms} \tag{4.2}$$

The $<<$ symbol in the summation excludes terms for which the two (generically multisite) symmetry generators act on common sites, which are instead included in the local terms. Via explicit computation we confirm that there are Yangian charges of this form for the $\mathfrak{su}(2|1)$ sector at next-to-leading order. Furthermore, in this case the local terms can be expressed simply as sums over products of two overlapping local symmetry generators.

As is well known, for typical integrable systems the Yangian charges only commute with the spin chain Hamiltonian on infinite-length chains. The reason is that it is not possible to consistently define periodic boundary conditions for the Yangian charges. At next-to-leading order, we will see that an infinite chain is required for the Yangian charges to transform properly under local symmetry transformations, to satisfy the Serre relations, and to commute with the Hamiltonian.

4.2 The algebra

4.2.1 The $\mathfrak{su}(2|1)$ algebra

As we discussed in Section 1.8, there are 4 supersymmetry generators for the $\mathfrak{su}(2|1)$ algebra (1.71), as well as the $\mathfrak{su}(2)$ $\mathfrak{R}$ symmetry generators (1.72) and the dilatation generator. For the Yangian algebra it will be convenient to use a single index for the generators. We introduce $\mathfrak{J}^A$, $A = 1, \ldots 8$, related to the original generators as

$$\mathfrak{J}^1 = \mathfrak{S}^1_1 + \mathfrak{Q}^1_1, \quad \mathfrak{J}^2 = i(\mathfrak{S}^1_1 - \mathfrak{Q}^1_1), \quad \mathfrak{J}^3 = \mathfrak{S}^1_2 + \mathfrak{Q}^2_1, \quad \mathfrak{J}^4 = i(\mathfrak{S}^1_2 - \mathfrak{Q}^2_1),$$
$$\mathfrak{J}^5 = \mathfrak{R}^1_2 + \mathfrak{R}^2_1, \quad \mathfrak{J}^6 = i(\mathfrak{R}^1_2 - \mathfrak{R}^2_1), \quad \mathfrak{J}^7 = \mathfrak{R}^1_1 - \mathfrak{R}^2_2, \quad \mathfrak{J}^8 = 2\mathfrak{D}_0 - \mathfrak{L} + \delta\mathfrak{D}. \qquad (4.3)$$

Recall that $\mathfrak{L}$ is the length generator, which commutes with all of these length-preserving symmetry generators. In this basis, the supersymmetry generators have indices between 1 and 4, the bosonic $\mathfrak{su}(2)$ generators have indices between 5 and 7, and $\mathfrak{J}^8$ is a linear combination of the dilatation generator and the length generator.

The algebra is

$$[\mathfrak{J}^A, \mathfrak{J}^B\} = f^{ABC} g_{CD} \mathfrak{J}^D = f^{AB}{}_D \mathfrak{J}^D, \qquad (4.4)$$

where g is the Cartan-Killing form or metric for the $\mathfrak{su}(2|1)$ algebra, with nonvanishing components,

$$g_{12} = g_{34} = \frac{i}{2}, \quad g_{21} = g_{43} = -\frac{i}{2}, \quad g_{55} = g_{66} = g_{77} = -\frac{1}{2} \quad g_{88} = \frac{1}{2}. \qquad (4.5)$$

Note that the metric is symmetric, up to an extra minus sign for switching fermionic indices. The structure constants f^{ABC} are totally antisymmetric, with extra minus signs for every interchange of fermionic indices. Up to permutations of the indices, the only nonvanishing components are:

$$\begin{aligned} f^{117} &= f^{135} = f^{227} = f^{236} = f^{245} = -2, \\ f^{118} &= f^{146} = f^{228} = f^{337} = f^{338} = f^{447} = f^{448} = 2, \\ f^{567} &= 4i. \end{aligned} \tag{4.6}$$

Because of Fermi statistics, we must be careful about the order of the indices of the (inverse) metric when (raising) lowering indices. In our convention, the first index of the (inverse) metric is summed over, as in (4.4).

We will also use the symmetric invariant tensor, defined by

$$d^{ABC} = \mathrm{sTr}\{\mathfrak{J}_0^A, \mathfrak{J}_0^B]\mathfrak{J}_0^C. \tag{4.7}$$

For this definition, all the generators act on the same site, and the mixed brackets with the curly bracket first means that we use the anticommutator for commuting generators and vice-versa. Up to permutations of the indices, the only nonvanishing components are:

$$\begin{aligned} &d^{127} = d^{136} = d^{145} = d^{246} = -2i, && d^{235} = d^{347} = 2i, \\ &d^{128} = d^{348} = -6i, && d^{558} = d^{668} = d^{788} = 4, \\ &d^{888} = -12. \end{aligned} \tag{4.8}$$

4.2.2 The $\mathfrak{su}(2|1)$ Yangian algebra

The $\mathfrak{su}(2|1)$ Yangian Algebra is an infinite-dimensional algebra generated by $\mathfrak{J}^A$ and $\mathfrak{Y}^A$. Commutators of these generators yield an infinite sequence of generators,

$\mathfrak{Y}_i^A$, $(i = 0, 1, 2, \ldots)$. $i = 0, 1$ correspond to $\mathfrak{J}^A$ and $\mathfrak{Y}^A$. The algebra is defined by the following commutation relations.

$$[\mathfrak{J}^A, \mathfrak{J}^B\} = f^{AB}{}_C \mathfrak{J}^C, \tag{4.9}$$

$$[\mathfrak{J}^A, \mathfrak{Y}^B\} = f^{AB}{}_C \mathfrak{Y}^C, \tag{4.10}$$

$$f^{[BC}{}_E[\mathfrak{Y}^{A\}}, \mathfrak{Y}^E\} = h^2(-1)^{(EM)} f^{AK}{}_D f^B{}_E{}^L f^C{}_F{}^M f_{KLM}\{\mathfrak{J}^D, \mathfrak{J}^E, \mathfrak{J}^F\}. \tag{4.11}$$

The notation $\{\mathfrak{J}^D, \mathfrak{J}^E, \mathfrak{J}^F\}$ denotes the totally symmetric product (with extra minus signs for every interchange of fermionic generators). $(-1)^{(EM)}$ gives -1 when both indices are fermionic, and 1 otherwise. Our conventions lead to $h^2 = -\frac{64}{3}$. The mixed brackets around the raised indices on the left side of the last equation mean that it is summed anti-symmetrically over all permutations of A, B and C. We obtained (4.11) (the Serre relation) by substituting (4.10) into the standard form for this Serre relation. Usually this would have no effect, but in our case this simplifies the proof of that the Serre relation is satisfied.

4.3 The leading order representation

4.3.1 The $\mathfrak{su}(2|1)$ algebra

To obtain the leading order representation of the generators, we first relate the $\mathfrak{J}^A$ to the $\mathfrak{psu}(1,1|2)$ sector generators. We find, using (1.55) for the supercharges,

$$\mathfrak{J}^1 = \mathfrak{Q}^{1+>} - \mathfrak{Q}^{2-<}, \qquad \mathfrak{J}^2 = -i(\mathfrak{Q}^{1+>} + \mathfrak{Q}^{2-<}),$$
$$\mathfrak{J}^3 = \mathfrak{Q}^{1-<} + \mathfrak{Q}^{2+>}, \qquad \mathfrak{J}^4 = i(\mathfrak{Q}^{1-<} - \mathfrak{Q}^{2+>}), \tag{4.12}$$

and, using (1.40) for the $\mathfrak{su}(2)$ generators,

$$\mathfrak{J}^5 = \mathfrak{R}^{11} - \mathfrak{R}^{22}, \quad \mathfrak{J}^6 = i(\mathfrak{R}^{11} + \mathfrak{R}^{22}), \quad \mathfrak{J}^7 = -2\,\mathfrak{R}^{12}. \tag{4.13}$$

Also, using the conditions satisfied by the $\mathfrak{su}(2|1)$ sector (1.33) and (1.70), and the expressions for $\mathfrak{L}$ (1.39) and $\mathfrak{J}^{+-}$ (1.41), we have

$$\mathfrak{J}^8 = 2\,\mathfrak{J}^{+-}. \tag{4.14}$$

Since there are no derivatives in this sector the only bosonic fields are $\phi_1 = \phi_1^{(0)}$ and $\phi_2 = \phi_2^{(0)}$, while there is only one fermionic field $\psi = \psi_<^{(0)}$. Now we can read the leading order representation from the $\mathfrak{psu}(1,1|2)$ expressions (1.48-1.49, 1.60-1.61). The nonvanishing contributions are

$$\begin{array}{llll}
\mathfrak{J}^1_{(0)}|\phi_1\rangle = -|\psi\rangle, & \mathfrak{J}^1_{(0)}|\psi\rangle = |\phi_1\rangle, & \mathfrak{J}^2_{(0)}|\phi_1\rangle = i|\psi\rangle, & \mathfrak{J}^2_{(0)}|\psi\rangle = i|\phi_1\rangle, \\
\mathfrak{J}^3_{(0)}|\phi_2\rangle = -|\psi\rangle, & \mathfrak{J}^3_{(0)}|\psi\rangle = |\phi_2\rangle, & \mathfrak{J}^4_{(0)}|\phi_2\rangle = i|\psi\rangle, & \mathfrak{J}^4_{(0)}|\psi\rangle = i|\phi_2\rangle, \\
\mathfrak{J}^5|\phi_1\rangle = |\phi_2\rangle, & \mathfrak{J}^5|\phi_2\rangle = |\phi_1\rangle, & \mathfrak{J}^6|\phi_1\rangle = i|\phi_2\rangle, & \mathfrak{J}^6|\phi_2\rangle = -i|\phi_1\rangle, \\
\mathfrak{J}^7|\phi_1\rangle = |\phi_1\rangle, & \mathfrak{J}^7|\phi_2\rangle = -|\phi_2\rangle, & \mathfrak{J}^8_{(0)}|\phi_a\rangle = |\phi_a\rangle, & \mathfrak{J}^8_{(0)}|\psi\rangle = 2|\psi\rangle.
\end{array} \tag{4.15}$$

We have neglected the subscript (0) for $\mathfrak{J}^5$, $\mathfrak{J}^6$ and $\mathfrak{J}^7$ since these are the $\mathfrak{R}$ symmetry generators, which receive no quantum corrections. For later convenience, note that switching $\{\mathfrak{J}^1, \mathfrak{J}^2\}$ and $\{\mathfrak{J}^3, \mathfrak{J}^4\}$ is equivalent to switching ϕ_1 and ϕ_2.

The leading term of $\delta\mathfrak{D}$, $\mathfrak{J}^8_{(2)}$, acts on adjacent sites of the spin chain as [19]

$$\delta\mathfrak{D}_{(2)} = 2(1 - \mathit{\Pi}), \tag{4.16}$$

where 1 is the identity operator, and $\mathit{\Pi}$ is the graded permutation operator. $\delta\mathfrak{D}_{(2)}$ also equals the quadratic Casimir operator:

$$\delta\mathfrak{D}_{(2)}(1,2) = 2\,g_{ab}\mathfrak{J}^a_{(0)}(1)\mathfrak{J}^b_{0)}(2). \tag{4.17}$$

For larger sectors (where the Casimir has more than two distinct eigenvalues on two-site chains), this simple relation between the Hamiltonian and the quadratic Casimir is replaced by a relation involving the digamma function, as we discussed earlier (2.37).

4.3.2 The $\mathfrak{su}(2|1)$ Yangian algebra

The Yangian algebra includes the local symmetry generators $\mathfrak{J}^A_{(0)}$ given in the previous section. The Yangian generators $\mathfrak{Y}^A_{(0)}$ then act as

$$\mathfrak{Y}^A_{(0)} = f^A{}_{CB} \sum_{i<j} \mathfrak{J}^B_{(0)}(i)\mathfrak{J}^C_{(0)}(j). \tag{4.18}$$

The adjoint transformation rule (4.10) is satisfied because of the Jacobi identity. The Serre relation follows from a straightforward generalization of the proof given for $\mathfrak{su}(n)$ Yangians in [37].

To show that the $\mathfrak{Y}^A_{(0)}$ commute with the one-loop dilatation generator, one can modify the arguments used in [36] for the full $\mathfrak{psu}(2,2|4)$ spin chain, or check by explicit computation that on a chain of length 2

$$[\delta\mathfrak{D}_{(2)}, \mathfrak{Y}^A_{(0)}](1,2) = 2(\mathfrak{J}^A_{(0)}(1) - \mathfrak{J}^A_{(0)}(2)). \tag{4.19}$$

Since $\delta\mathfrak{D}_{(2)}$ commutes with the $\mathfrak{J}^A_{(0)}$, the commutator on longer chains is just the sum of this adjacent 2-site commutator over the length of the chain. However, this yields a total chain derivative that vanishes on an infinite chain.

4.4 The next-to-leading order corrections

4.4.1 The $\mathfrak{su}(2|1)$ Algebra

Using the transformation (4.12) between the single-index notation and the $\mathfrak{psu}(1,1|2)$ supercharges, we can compute the one-loop corrections from the solution we found in Chapter 3. For $\mathfrak{J}^1$ and $\mathfrak{J}^2$ we find

$$\begin{aligned}
\mathfrak{J}^1_{(2)}|\phi_a\phi_b\rangle &= \delta^1_{[a}|\psi\phi_{b]}\rangle - \delta^1_{[a}|\phi_{b]}\psi\rangle, & \mathfrak{J}^1_{(2)}|\psi\psi\rangle &= \tfrac{1}{2}(|\phi_1\psi\rangle - |\psi\phi_1\rangle),\\
\mathfrak{J}^1_{(2)}|\phi_a\psi\rangle &= \tfrac{1}{2}(|\phi_a\phi_1\rangle - |\phi_1\phi_a\rangle) + \tfrac{1}{2}\delta^1_a|\psi\psi\rangle, & \mathfrak{J}^1_{(2)}|\psi\phi_a\rangle &= \tfrac{1}{2}(|\phi_1\phi_a\rangle - |\phi_a\phi_1\rangle) - \tfrac{1}{2}\delta^1_a|\psi\psi\rangle,\\
\mathfrak{J}^2_{(2)}|\phi_a\phi_b\rangle &= -i\,\delta^1_{[a}|\psi\phi_{b]}\rangle + i\,\delta^1_{[a}|\phi_{b]}\psi\rangle, & \mathfrak{J}^2_{(2)}|\psi\psi\rangle &= \tfrac{i}{2}(|\phi_1\psi\rangle - |\psi\phi_1\rangle),\\
\mathfrak{J}^2_{(2)}|\phi_a\psi\rangle &= \tfrac{i}{2}(|\phi_a\phi_1\rangle - |\phi_1\phi_a\rangle) - \tfrac{i}{2}\delta^1_a|\psi\psi\rangle, & \mathfrak{J}^2_{(2)}|\psi\phi_a\rangle &= \tfrac{i}{2}(|\phi_1\phi_a\rangle - |\phi_a\phi_1\rangle) + \tfrac{i}{2}\delta^1_a|\psi\psi\rangle.
\end{aligned} \tag{4.20}$$

For $\mathfrak{J}^3$ and $\mathfrak{J}^4$ we can simply switch ϕ_1 and ϕ_2 in the above expressions, obtaining

$$\begin{aligned}
\mathfrak{J}^3_{(2)}|\phi_a\phi_b\rangle &= \delta^2_{[a}|\psi\phi_{b]}\rangle - \delta^2_{[a}|\phi_{b]}\psi\rangle, & \mathfrak{J}^3_{(2)}|\psi\psi\rangle &= \tfrac{1}{2}(|\phi_2\psi\rangle - |\psi\phi_2\rangle),\\
\mathfrak{J}^3_{(2)}|\phi_a\psi\rangle &= \tfrac{1}{2}(|\phi_a\phi_2\rangle - |\phi_2\phi_a\rangle) + \tfrac{1}{2}\delta^2_a|\psi\psi\rangle, & \mathfrak{J}^3_{(2)}|\psi\phi_a\rangle &= \tfrac{1}{2}(|\phi_2\phi_a\rangle - |\phi_a\phi_2\rangle) - \tfrac{1}{2}\delta^2_a|\psi\psi\rangle,\\
\mathfrak{J}^4_{(2)}|\phi_a\phi_b\rangle &= -i\,\delta^2_{[a}|\psi\phi_{b]}\rangle + i\,\delta^2_{[a}|\phi_{b]}\psi\rangle, & \mathfrak{J}^4_{(2)}|\psi\psi\rangle &= \tfrac{i}{2}(|\phi_2\psi\rangle - |\psi\phi_2\rangle),\\
\mathfrak{J}^4_{(2)}|\phi_a\psi\rangle &= \tfrac{i}{2}(|\phi_a\phi_2\rangle - |\phi_2\phi_a\rangle) - \tfrac{i}{2}\delta^2_a|\psi\psi\rangle, & \mathfrak{J}^4_{(2)}|\psi\phi_a\rangle &= \tfrac{i}{2}(|\phi_2\phi_a\rangle - |\phi_a\phi_2\rangle) + \tfrac{i}{2}\delta^2_a|\psi\psi\rangle.
\end{aligned} \tag{4.21}$$

Alternatively, we can generalize the quadratic product for the one-loop dilatation generator to a compact expression for all of the one-loop generators,

$$\begin{aligned}
\mathfrak{J}^A_{(2)}(1,2) =& \frac{1}{4}\left(d_{BC}{}^A - d^A{}_{BC}\right)\mathfrak{J}^B_{(0)}(1)\mathfrak{J}^C_{(0)}(2)\\
&+\frac{1}{4}((-1)^{(AA)}-1)\left(\mathfrak{J}^A_{(0)}(1)\mathfrak{J}^8_{(0)}(2) + \mathfrak{J}^8_{(0)}(1)\mathfrak{J}^A_{(0)}(2)\right) + g^{A8}g_{BC}\mathfrak{J}^B_{(0)}(1)\mathfrak{J}^C_{(0)}(2).
\end{aligned} \tag{4.22}$$

Importantly, note that this expression is basis dependent. Also, the first two terms vanish for all of the bosonic generators, and the last term gives the dilatation generator.

We will not write the lengthy expression for the two-loop dilatation generator that can be computed from the solution given in Chapter 3. Instead, we present the more compact expression,

$$\begin{aligned}\delta\mathfrak{D}_{(4)}(1,2,3)=&\left(\frac{1}{2}d_{CBA}-\frac{1}{2}(-1)^{AB}d_{BCA}\right)\mathfrak{J}^A_{(0)}(1)\mathfrak{J}^B_{(0)}(2)\mathfrak{J}^C_{(0)}(3)\\&+\frac{1}{2}(g_{AB}-g_{BA})\left(\mathfrak{J}^8_{(0)}(1)\mathfrak{J}^A_{(0)}(2)\mathfrak{J}^B_{(0)}(3)+\mathfrak{J}^A_{(0)}(1)\mathfrak{J}^B_{(0)}(2)\mathfrak{J}^8_{(0)}(3)\right)\\&-\mathfrak{J}^8_{(0)}(2)\delta\mathfrak{D}_{(2)}(1,3)+2\delta\mathfrak{D}_{(2)}(1,3)-2\,\delta\mathfrak{D}_{(2)}(1,2)-2\,\delta\mathfrak{D}_{(2)}(2,3).\end{aligned} \tag{4.23}$$

As above, this expression is basis dependent. The expression in Chapter 3 is more general, as it includes possible similarity transformations, but they have no effect on the spectrum. Similarly, it is also possible to add chain derivatives to $\delta\mathfrak{D}_{(4)}$, with no effect on the spectrum.

4.4.2 The $\mathfrak{su}(2|1)$ Yangian algebra

We find that the next-to-leading order corrections to the Yangian generators can still be written in terms of the $\mathfrak{J}^A$.

$$\mathfrak{Y}^A_{(2)}=\mathfrak{Y}^A_{(2)\ \mathrm{nonlocal}}+\mathfrak{Y}^A_{(2)\ \mathrm{local}}, \tag{4.24}$$

$$\mathfrak{Y}^A_{(2)\ \mathrm{nonlocal}}=f^A{}_{CB}\left(\sum_{i<j-1}\mathfrak{J}^B_{(2)}(i,i+1)\mathfrak{J}^C_{(0)}(j)+\sum_{i<j}\mathfrak{J}^B_{(0)}(i)\mathfrak{J}^C_{(2)}(j,j+1)\right), \tag{4.25}$$

$$\begin{aligned}\mathfrak{Y}^A_{(2)\ \mathrm{local}}=&\sum_i f^A{}_{CB}\mathfrak{J}^B_{(2)}(i,i+1)(\mathfrak{J}^C_{(0)}(i+1)-\mathfrak{J}^C_{(0)}(i))\\&+\sum_i f^A{}_{CB}(\mathfrak{J}^B_{(0)}(i)-\mathfrak{J}^B_{(0)}(i+1))\mathfrak{J}^C_{(2)}(i,i+1)\\&+\sum_i\left(\alpha\,\mathfrak{Y}^A_{(0)}(i,i+1)+\gamma\sum_{B=1}^{8}(g^{BA}-g^{AB})\left(\mathfrak{J}^B_{(0)}(i)-\mathfrak{J}^B_{(0)}(i+1)\right)\right).\end{aligned} \tag{4.26}$$

It is interesting to note that if the minus signs in the first two lines of (4.26) were replaced by plus signs, these lines would simplify to $\mathfrak{J}^A_{(2)}$ acting on the entire chain. Also, for $A=8$ the first two lines of (4.26) vanish. Note that the term proportional to γ vanishes on infinite chains since it is a chain derivative. We include it because the choice $\gamma=2$ simplifies the proof that the Serre relation is satisfied. However, when checking commutators of the

$\mathfrak{Y}^A$ with the $\mathfrak{J}^B$ we set $\gamma = 0$, since commutators of chain derivatives and local symmetry generators are chain derivatives[2]. For vanishing α and γ, (4.26) is the sum over the chain of the two-site interaction

$$\mathfrak{Y}^A_{(2)\,\mathrm{local}}(1,2) = \left(4\, f_{BC}{}^A + 6\, f^A{}_{CB} + 6\, f_{CB}{}^A\right) \mathfrak{J}^B_{(0)}(1)\mathfrak{J}^C_{(0)}(2)$$
$$+ \sum_{B=1}^{8} (g^{BA} - g^{AB}) \left(\mathfrak{J}^B_{(0)}(1)\mathfrak{J}^8_{(0)}(2) - \mathfrak{J}^8_{(0)}(1)\mathfrak{J}^B_{(0)}(2)\right) - 2g^A{}_8 \mathfrak{Y}^8_{(0)}(1,2). \qquad (4.27)$$

Again this is a basis dependent expression. We have verified that the Yangian $\mathfrak{su}(2)$ generators restricted to the $\mathfrak{su}(2)$ subsector agree with the expression found in [59] for $\alpha = 1$. We cannot check with the $\mathfrak{su}(1|1)$ Yangian presented in [61] since there the corrections to the Yangian generators are local, and bilocal corrections are essential when the $\mathfrak{su}(1|1)$ sector's Yangian is embedded in a larger sector's Yangian[3].

Similarity transformations. As explained in Chapter 3, the local symmetry algebra is preserved by similarity transformations, which act as

$$\mathfrak{J}^A \mapsto U\mathfrak{J}^A U^{-1}, \quad U = 1 - g^2\mathfrak{y} + \cdots, \quad \text{i.e.} \quad \mathfrak{J}^A_{(2)} \mapsto \mathfrak{J}^A_{(2)} + [\mathfrak{J}^A_{(0)}, \mathfrak{y}], \qquad (4.28)$$

and the Yangian symmetry is preserved also if we apply the same transformation to the $\mathfrak{Y}^A$. In general, U could be any generator acting on the chain, but we require similarity transformations to preserve the coupling dependence that arises from Feynman diagrams, the manifest $\mathfrak{R}$ symmetry, and the even parity of the local symmetry generators. As a result, the $\mathcal{O}(g^2)$ contributions from U can be linear combination of only five possible two-

[2] To see this, note that a chain derivative acts with a factor of $(1 - \mathfrak{U})$, and any (length-preserving) generator that acts homogeneously with such a factor is a chain derivative. Then the statement follows since local symmetry generators commute with $(1 - \mathfrak{U})$.

[3] The construction of [61] maintained manifest $\mathfrak{su}(1|1)$ symmetry, i.e. the supercharges received no quantum corrections, and the Hamiltonian (including the next-to-leading order correction) simply commuted with this $\mathfrak{su}(1|1)$. Of course, to match the gauge theory dilatation generator for larger sectors at next-to-leading order, the Hamiltonian must become a nonseparable part of the algebra.

site interactions. Two linear combinations of these, the identity and the one-loop dilatation generator, commute with all of the $\mathfrak{J}^A_{(0)}$ and $\mathfrak{Y}^A_{(0)}$. So there are three nontrivial independent similarity transformations. One linear combination preserves the form (4.24), while the remaining two are not consistent with this form. It is intriguing that the solution we found in Chapter 3 selected a basis in which the Yangian corrections can be written simply in terms of the local generators.

Adjoint transformation. We now verify that the adjoint transformation rule (4.10) is satisfied at next-to-leading order. First we consider only the commutator with $\mathfrak{J}^8$, which beyond leading order is the anomalous part of the dilatation generator. Since corrections to each of the $\mathfrak{Y}^A$ have the same classical dimension as at leading order, the vanishing commutator of the leading order $\mathfrak{Y}^A$ with the one-loop dilatation generator implies that at next-to-leading order the Yangian charges transform properly under commutators with $\mathfrak{J}^8$ [62].

The same proof as at leading order shows that the term of $\mathfrak{Y}^B$ proportional to α transforms properly with respect to the $\mathfrak{J}^A$. This term is needed (again with $\alpha = 1$) for the commutator with the dilatation generator at $\mathcal{O}(g^4)$, but is not necessary for the $\mathfrak{J}^A$ and $\mathfrak{Y}^A$ to form a Yangian algebra at next-to-leading order. Of the remaining terms, those that involve products of generators far apart transform properly under commutation with the $\mathfrak{J}^A$, as a straightforward generalization of the leading order proof would still work. However, we need to check terms in the commutator that involve one generator that intersects the other two generators. All of these terms involve only two adjacent sites of the chain, so it will be sufficient to examine this commutator on a chain with just two sites. Explicitly, we need to check the second of the following equalities:

$$\begin{aligned}
[\mathfrak{J}^A, \mathfrak{Y}^B\}_{(2)\,\mathrm{local}}(1,2) &= f^B{}_{DC}\Big([\mathfrak{J}^A_{(2)}(1,2), \mathfrak{J}^C_{(0)}(1)\mathfrak{J}^D_{(0)}(2)\} \\
&\quad +[\mathfrak{J}^A_{(0)}(1)+\mathfrak{J}^A_{(0)}(2), \mathfrak{J}^C_{(2)}(1,2)(\mathfrak{J}^D_{(0)}(2) - (\mathfrak{J}^D_{(0)}(1))\} \\
&\quad + [\mathfrak{J}^A_{(0)}(1)+\mathfrak{J}^A_{(0)}(2), (\mathfrak{J}^C_{(0)}(1) - \mathfrak{J}^C_{(0)}(2))(\mathfrak{J}^D_{(2)}(1,2)\}\Big) \\
&= f^{AB}{}_C\mathfrak{Y}^C_{(2)\,\mathrm{local}}(1,2) + \text{terms that vanish on an infinite chain.}
\end{aligned} \tag{4.29}$$

Canceling terms, we find that this is equivalent to

$$\begin{aligned}
0 =& f^B{}_{DC}\Big([\mathfrak{J}^A_{(2)}(1,2), \mathfrak{J}^C_{(0)}(1)\}\mathfrak{J}^D_{(0)}(1) - (-1)^{AC}\mathfrak{J}^C_{(0)}(1)[\mathfrak{J}^A_{(2)}(1,2), \mathfrak{J}^D_{(0)}(1)\}\Big) - \text{parity} \\
&-\frac{1}{8}[\mathfrak{J}^A_{(2)}(1,2), \mathfrak{Y}^B_{(0)}(1,2)\} + \text{terms that vanish on an infinite chain.}
\end{aligned} \tag{4.30}$$

Here 'parity' just means interchange sites 1 and 2. Since $\mathfrak{J}^A$ receives no quantum corrections for $A = 5,\ 6,$ or 7, clearly Yangian generators transform properly with respect to these local symmetry generators. More generally, (4.30) simplifies to

$$m^{AB}{}_C\left(\mathfrak{J}^C_{(0)}(1) - \mathfrak{J}^C_{(0)}(2)\right), \tag{4.31}$$

where, in our basis

$$m^{AB}{}_C = \frac{1}{4}((-1)^{AA} - 1)\left(\sum_{D=1}^{8} g^{DA} f^{BD}{}_C + 2\,g^{B8}g^A{}_C + 2\,g^8{}_C g^{AB}\right) - 2g^{A8}g^B{}_C. \tag{4.32}$$

The expression (4.31) is just a chain derivative. Therefore, the Yangian charges transform properly under the local symmetry algebra.

Commutator with the Hamiltonian. We have also verified that the dilatation generator commutes with this Yangian representation up to $\mathcal{O}(g^4)$. Again the commutator splits into local and bilocal parts. The local part includes terms where one of the generators intersects both of the others, as well as the commutator involving the last term in the expression for $\mathfrak{Y}^B_{(2)}$. The rest of the commutator is bilocal. The bilocal part vanishes because the dilatation generator commutes with the $\mathfrak{J}^A$. The local piece involves three adjacent sites. Using `Mathematica`, we have checked this commutator on a chain of three sites:

$$\begin{aligned}
[\delta\mathfrak{D},\mathfrak{Y}^A]_{(4)\ \mathrm{local}}(1,2,3) &= [\delta\mathfrak{D}_{(2)},\mathfrak{Y}^A_{(2)}]_{\mathrm{local}}(1,2,3)+[\delta\mathfrak{D}_{(4)},\mathfrak{Y}^A_{(0)}]_{\ \mathrm{local}}(1,2,3)\\
&= \mathfrak{W}^A(2,3)-\mathfrak{W}^A(1,2),\\
\mathfrak{W}^A(1,2) &= d^A{}_{CB}\mathfrak{J}^B_{(0)}(1)\mathfrak{J}^C_{(0)}(2)+\mathfrak{J}^A_{(0)}(1)+\mathfrak{J}^A_{(0)}(2)+4\,\mathfrak{J}^A_{(2)}(1,2). \qquad (4.33)
\end{aligned}$$

Since this difference of $\mathfrak{W}^A$ is a chain derivative, the $\mathfrak{Y}^A$ commute with $\delta\mathfrak{D}$ at $\mathcal{O}(g^4)$.

4.4.3 The Serre relation

The Serre relation is satisfied up to a new type of term that vanishes on an infinite chain ($L=\infty$),

$$\begin{aligned}
f^{[BC}{}_E[\mathfrak{Y}^{A\}},\mathfrak{Y}^E\}_{(2)} = &-\frac{64}{3}(-1)^{(EM)}f^{AK}{}_Df^B{}_E{}^Lf^C{}_F{}^Mf_{KLM}\{\mathfrak{J}^D,\mathfrak{J}^E,\mathfrak{J}^F\}_{(2)}\\
&+a^{ABC}{}_{DE}\left(\mathfrak{J}^D_{(0)}(1)\sum_{i=2}^{L}\mathfrak{J}^E_{(0)}(i)+\mathfrak{J}^D_{(0)}(L)\sum_{i=1}^{L-1}\mathfrak{J}^E_{(0)}(i)\right).
\end{aligned} \qquad (4.34)$$

We have not found simple expressions for the coefficients $a^{ABC}{}_{DE}$ (we have found their values in our basis), but this will not be necessary. To show that this last term vanishes on an infinite chain, we introduce the set of parity-odd two-site operators

$$\tilde{\mathfrak{Y}}^{AB}(i,j)=\mathfrak{J}^A_{(0)}(i)\mathfrak{J}^B_{(0)}(j)-\mathfrak{J}^A_{(0)}(j)\mathfrak{J}^B_{(0)}(i). \qquad (4.35)$$

It is always possible to find coefficients $c_{ABC}{}^{DE}$ such that

$$c_{ABC}{}^{DE}[\mathfrak{J}^A_{(0)}(i),\tilde{\mathfrak{Y}}^{BC}(i,j)\}=\mathfrak{J}^D_{(0)}(i)\mathfrak{J}^E_{(0)}(j). \qquad (4.36)$$

Using this, we write the extra term in (4.34) as

$$\sum_{i<j} \tilde{a}^{ABC}{}_{DEF}[\mathfrak{J}^D_{(0)}(1) - \mathfrak{J}^D_{(0)}(L), \tilde{\mathfrak{Y}}^{EF}(i,j)\}, \tag{4.37}$$

for some new coefficients $\tilde{a}$. The first term in the commutator vanishes on infinite chains. Therefore, satisfying (4.34) is equivalent to satisfying the Serre relations on infinite chains. (4.34) is a simpler form for checking the relation.

We will now prove that (4.34) is satisfied. We have checked this equation on two- and three-site chains for $\gamma = 2$. In fact, this is sufficient to guarantee that this relation holds for any length chain. Define[4]

$$\mathfrak{Z}^{ABC}(1,L) = f^{[BC}{}_E[\mathfrak{Y}^{A\}}, \mathfrak{Y}^E\}_{(2)} + \frac{64}{3}(-1)^{(EM)} f^{AK}{}_D f^B{}_E{}^L f^C{}_F{}^M f_{KLM}\{\mathfrak{J}^D, \mathfrak{J}^E, \mathfrak{J}^F\}_{(2)}, \tag{4.38}$$

evaluated on a chain of length L, which we need to show equals the extra term in (4.34) for any L. Also, define $\mathfrak{B}^{ABC}(1,L)$ to be the terms of $\mathfrak{Z}^{ABC}(1,L)$ simultaneously including generators acting on the first site and generators acting on the last site.

The terms entering the Serre relation are local, bilocal, or trilocal. Local terms are those that would appear on a chain of length two. The contribution from the cubic $\mathfrak{J}$ term actually vanishes on two sites, but the $\mathfrak{Y}$ commutators yield

$$\mathfrak{Z}^{ABC}(1,2) = a^{ABC}{}_{DE}(\mathfrak{J}^D_{(0)}(1)\mathfrak{J}^E_{(0)}(2) + \mathfrak{J}^D_{(0)}(2)\mathfrak{J}^E_{(0)}(1)), \tag{4.39}$$

which agrees with (4.34) for $L = 2$.

Bilocal terms first appear for chains of length three. The bilocal terms from the cubic $\mathfrak{J}$ expression are canceled by commutators of the nonlocal part of $\mathfrak{Y}_{(2)}$ with a twice-intersecting $\mathfrak{Y}_{(0)}$ (not including terms where a nearest neighbor $\mathfrak{Y}_{(0)}$ acts on the same sites as a $\mathfrak{J}_{(2)}$).

[4]Note that $\mathfrak{Z}^{ABC}(1,L)$ acts on all sites between and including sites 1 and L.

The remaining bilocal pieces either include a local $\mathfrak{Y}_{(2)}$ or a $\mathfrak{J}_{(2)}$ intersecting a $\mathfrak{Y}_{(0)}$ on two sites. For these terms, all that matters is that two sites are adjacent, so that a $\mathfrak{J}_{(2)}$ or $\mathfrak{Y}_{(2)\ \text{local}}$ act on them, and the location of the third site does not matter. Using this, we can determine the contribution from the bilocal terms just from a three-site `Mathematica` computation. For the bilocal part involving the boundaries of a chain of length 3 we find

$$\mathfrak{B}^{ABC}(1,3) = a^{ABC}{}_{DE}\left((\mathfrak{J}^D_{(0)}(1) - \mathfrak{J}^D_{(0)}(2))\mathfrak{J}^E_{(0)}(3) + (\mathfrak{J}^D_{(0)}(3) - \mathfrak{J}^D_{(0)}(2))\mathfrak{J}^E_{(0)}(1)\right). \tag{4.40}$$

The remaining terms are trilocal. The trilocal piece gives zero total contribution to the $\mathfrak{Z}$ since the $\mathfrak{J}$ satisfy the $\mathfrak{su}(2|1)$ algebra at one loop exactly without chain derivatives. Therefore, the bilocal boundary contributions (4.40) for chains of length three are the only contributions to $\mathfrak{B}^{ABC}$ for any chain of length greater than two,

$$\mathfrak{B}^{ABC}(1,L) = a^{ABC}{}_{DE}\left((\mathfrak{J}^D_{(0)}(1) - \mathfrak{J}^D_{(0)}(2))\mathfrak{J}^E_{(0)}(L) + (\mathfrak{J}^D_{(0)}(L) - \mathfrak{J}^D_{(0)}(L-1))\mathfrak{J}^E_{(0)}(1)\right). \tag{4.41}$$

Now we can finish the proof using induction. Assume that (4.34) is satisfied for chains of length L. Then split terms of $\mathfrak{Z}^{ABC}(1, L+1)$ into those that only act on the first L sites, those that only act on the last L sites, and those that act on both boundaries. However, we also need to subtract the terms that only act on the intersection of the first and last L

sites. Then using our assumption and substituting (4.41) is sufficient:

$$\begin{aligned}
\mathfrak{Z}(1,L+1) =& \mathfrak{Z}^{ABC}(1,L)+\mathfrak{Z}^{ABC}(2,L+1)-\mathfrak{Z}^{ABC}(2,L)+\mathfrak{B}^{ABC}(1,L+1)\\
=& a^{ABC}{}_{DE}\left(\mathfrak{J}^D_{(0)}(1)\sum_{i=2}^{L}\mathfrak{J}^E_{(0)}(i)+\mathfrak{J}^D_{(0)}(L)\sum_{i=1}^{L-1}\mathfrak{J}^E_{(0)}(i)+\mathfrak{J}^D_{(0)}(2)\sum_{i=3}^{L+1}\mathfrak{J}^E_{(0)}(i)\right.\\
&\left.+\,\mathfrak{J}^D_{(0)}(L+1)\sum_{i=2}^{L}\mathfrak{J}^E_{(0)}(i)-\mathfrak{J}^D_{(0)}(2)\sum_{i=3}^{L}\mathfrak{J}^E_{(0)}(i)-\mathfrak{J}^D_{(0)}(L)\sum_{i=2}^{L-1}\mathfrak{J}^E_{(0)}(i)\right)\\
&+\mathfrak{B}^{ABC}(1,L+1)\\
=& a^{ABC}{}_{DE}\left(\mathfrak{J}^D_{(0)}(1)\sum_{i=2}^{L}\mathfrak{J}^E_{(0)}(i)+\mathfrak{J}^D_{(0)}(2)\mathfrak{J}^E_{(0)}(L+1)+\mathfrak{J}^D_{(0)}(L)\mathfrak{J}^E_{(0)}(1)\right.\\
&\left.+\,\mathfrak{J}^D_{(0)}(L+1)\sum_{i=2}^{L}\mathfrak{J}^E_{(0)}(i)\right)+\mathfrak{B}^{ABC}(1,L+1)\\
=& a^{ABC}{}_{DE}\left(\mathfrak{J}^D_{(0)}(1)\sum_{i=2}^{L+1}\mathfrak{J}^E_{(0)}(i)+\mathfrak{J}^D_{(0)}(L+1)\sum_{i=1}^{L}\mathfrak{J}^E_{(0)}(i)\right),
\end{aligned} \tag{4.42}$$

in agreement with (4.34). The Serre relation is satisfied regardless of the value of α (the coefficient of $\mathfrak{Y}^A_{(0)}(i,i+1)$ in the expression for $\mathfrak{Y}^A_{(2)}$ (4.26)), but we do not have a simple explanation for this.

4.5 Discussion

We have constructed the next-to-leading order corrections to the $\mathfrak{su}(2|1)$ sector Yangian, proving integrability at two loops. Furthermore, these corrections are built in a simple way from the local symmetry generators. Perhaps the most important result is the generalization of the standard definition of Yangian symmetry to include this system. The local symmetry generators still transform as usual,

$$[\mathfrak{J}^A,\mathfrak{J}^B\} = f^{AB}{}_C\mathfrak{J}^C. \tag{4.43}$$

However, the adjoint transformation rule of the Yangian charges is generalized to allow for chain derivatives. This had to be the case since at leading order the Hamiltonian only commutes with the Yangian generators up to chain derivatives, and the Hamiltonian becomes part of the symmetry algebra starting at next-to-leading order. On a chain of length L the adjoint transformation rule is

$$[\mathfrak{J}^A, \mathfrak{Y}^B\} = f^{AB}{}_C \mathfrak{Y}^C + m^{AB}{}_C(\hat{\mathfrak{J}}^C(\text{left}) - \hat{\mathfrak{J}}^C(\text{right})). \tag{4.44}$$

The only essential part of the last term is that it is a boundary term. At next-to-leading order in this sector, $\hat{\mathfrak{J}}^A(\text{left})$ is $\mathfrak{J}^A_{(0)}(1)$ and $\hat{\mathfrak{J}}^A(\text{right})$ is $\mathfrak{J}^A_{(0)}(L)$. At higher orders we expect that the $\hat{\mathfrak{J}}^A$ will at least involve higher order corrections to the $\mathfrak{J}^A$ acting on the boundary sites and their immediate neighbors, and in principle they do not even need to be simply related to the $\mathfrak{J}^A$. Finally, the Serre relation is generalized to include commutators of boundary terms,

$$\begin{aligned} f^{[BC}{}_E[\mathfrak{Y}^{A\}}, \mathfrak{Y}^E\} =& h^2(-1)^{(EM)} f^{AK}{}_D f^B{}_E{}^L f^C{}_F{}^M f_{KLM}\{\mathfrak{J}^D, \mathfrak{J}^E, \mathfrak{J}^F\} \\ &+ \tilde{a}^{ABC}{}_{DEF}[\hat{\mathfrak{J}}^D(\text{left}) - \hat{\mathfrak{J}}^D(\text{right}), \tilde{\mathfrak{Y}}^{EF}\}. \end{aligned} \tag{4.45}$$

Again the only essential part of the new term is that $\hat{\mathfrak{J}}^D(\text{left}) - \hat{\mathfrak{J}}^D(\text{right})$ is a boundary term.

Finally, the results of this chapter give a foundation for generalizing to the noncompact $\mathfrak{psu}(1,1|2)$ sector. Since many computations in this chapter depended on compactness, new insights will be needed. However, the simple structure of the corrections for the local $\mathfrak{psu}(1,1|2)$ generators found in Chapter 3 should make this generalization tractable.

Conclusions

The integrability of planar AdS/CFT has been central to recent progress on the duality. In this dissertation, we have studied the $\mathcal{N} = 4$ SYM spin chain realization of multiple types of symmetry related to integrability. We have seen how multisite and length-changing spin chain interactions and supersymmetry combine to produce novel symmetries. Due to its embedding within $\mathcal{N} = 4$ SYM, the $\mathfrak{psu}(1,1|2)$ sector has a hidden $\mathfrak{psu}(1|1)^2$ symmetry, which acts on the spin chain through length-changing interactions. This hidden symmetry enabled us to find simple iterative expressions for the two-loop dilatation generator and for other local symmetry generator corrections. Furthermore, constructing a bilocal product of the $\mathfrak{psu}(1|1)^2$ generators (at leading order) led to a new understanding of a large degeneracy of the spin chain's spectrum. The bilocal products combined with a $\mathfrak{su}(2)$ automorphism to generate a previously unknown infinite-dimensional symmetry of this sector. We presented very strong evidence that this symmetry forms a subgroup of the loop group of $SU(2)$. Finally, by constructing the next-to-leading order Yangian symmetry for the $\mathfrak{su}(2|1)$ subsector we showed how Yangian symmetry can be realized with multisite local symmetry generators.

The iterative structure of the two-loop corrections to the local symmetry generators naturally motivates extensions to higher loops. New features should appear at three and four loops. The general form for the overall phase function of the Bethe equations [27,32] implies that at three loops there is at least one homogeneous solution for the dilatation generator. Furthermore, to match the spectrum that follows from the conjectured S-matrix phase [31], the dilatation generator must have interactions with transcendental coefficients starting at four loops. Still, even the transcendental coefficients must have a special structure since they should preserve the transcendentality principle [63]. This principle describes a property of the anomalous dimensions of twist-2 operators with large Lorentz spin S. The coefficient of $\log S$, labeled $f(g)$, has degree of transcendentality $2l-2$ at l loops, where $\zeta(n)$ and π^n have degree of transcendentality n. It is possible that finding the corresponding corrections to the dilatation generator will lead to new insights into this transcendentality pattern.

Wrapping interactions may appear first at four loops also. These interactions do not contribute at lower orders because supersymmetry relates length-two states to length-four states. Optimistically, it is possible that the iterative structure of the dilatation generator corrections would suggest a natural solution for the wrapping interactions. This would be very useful since the wrapping interactions are less constrained by superficial properties of Feynman diagrams, and the AdS/CFT Bethe ansatz does not take corrections from wrapping interactions into account.

More generally, we should note that iterative structures have appeared in other $\mathcal{N}=4$ SYM calculations. Planar scattering amplitudes[5] have iterative structure at two and three loops [64]. In fact, the gluon amplitudes found in these works depend on the same function $f(g)$ that was mentioned above in the context of anomalous dimensions[6]. Also,

[5] While a (spacetime) S-matrix does not make sense for a conformal field theory like $\mathcal{N}=4$ SYM, one can still compute matrix elements between on-shell states.

[6] This made possible the impressive four-loop test [33] of the proposal for the phase function of the spin chain S-matrix.

following Witten's work relating gauge theory to a string theory in twistor space [65], recurrence relations between amplitudes involving different numbers of particles have been found [66]. The two-loop dilatation generator has some qualitative resemblance to this recursive structure. The $\hat{\mathfrak{Q}}_{(1)}$ and $\hat{\mathfrak{S}}_{(1)}$ are analogous to the three gluon on-shell amplitudes, and $\mathfrak{h}$ is similar to a Feynman propagator. Adding to this analogy, just like a (Lorentz scalar or vector) propagator is a Green's function satisfying an equation quadratic in momentum, $\mathfrak{h}$ satisfies the key equations (3.17) and (3.40), which are quadratic in local spin chain symmetry generators. From this perspective, since a propagator of an interacting field theory receives quantum corrections, it would not be surprising if an extension to higher loops require corrections to $\mathfrak{h}$ too. It would be wonderful to make a closer link between the iterative structures of the dilatation generator and scattering amplitudes.

Alternatively, the iterative structure of the two-loop dilatation generator could originate from a generalization of the result of Rej, Serban, and Staudacher [67]. Up to three loops, the $\mathfrak{su}(2)$ sector spin chain matches the strong coupling expansion of a twisted Hubbard model. This is exciting because the Hubbard model has a simple nearest neighbor Hamiltonian. While it is now clear that the agreement breaks down beyond three loops, it appears that it still gives the correct rational part of the anomalous dimension matrix [31] (not including wrapping contributions). Perhaps it is possible to generalize the Hubbard model proposal to apply to the $\mathfrak{psu}(1,1|2)$ sector, to three loops or higher.

As we noted at the end of Chapter 2, the nonlocal symmetry of that chapter should also continue at higher loops. Similar to the quantum corrections to the Yangian generators of Chapter 4, we conjecture that the corrections for the bilocal generators involve substituting the appropriate loop corrections for the $\mathfrak{psu}(1|1)^2$ generators in the expression for the $\mathcal{Y}^{\mathfrak{ab}}$. Because the $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{Q}}$ act on more sites at once at higher orders, explicit calculation would be required to find the regularization of their overlap. However, as at leading order, for cyclic states the $\mathcal{Y}^{\mathfrak{ab}}$ would reduce to an ordinary product of the (loop-corrected) $\mathfrak{psu}(1|1)^2$ generators. Therefore, we conjecture that the loop corrections will also preserve the algebra of the $\mathcal{Y}_n^{\mathfrak{ab}}$. Since we already have the next-to-leading order corrections for the $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{Q}}$, it will be straightforward to verify this at that order. Furthermore, the study of these corrections would add yet another constraint for the higher loop contributions to the local symmetry generators.

In addition to higher loop corrections, there are multiple possible extensions of our results beyond the $\mathfrak{psu}(1,1|2)$ sector. In any larger sector of $\mathcal{N} = 4$ SYM containing the $\mathfrak{psu}(1,1|2)$ sector, the dilatation generator has length-changing interactions. It would be interesting to see if recursive structure still appears. These length-changing interactions would also appear in the Yangian generators, and it would make sense to study this first by extending the $\mathfrak{su}(2|1)$ Yangian to the dynamic (but still compact) $\mathfrak{su}(2|3)$ sector. In that sector, the local symmetry generators only commute up to spin chain gauge transformation. Therefore, it is likely that the defining Yangian algebra equations (4.44-4.45) will require even further generalization. We anticipate that the adjoint transformation rule will need to include commutators with boundary terms, and the Serre relation will need to include commutators with commutators of boundary terms. In fact, the study of the spin chain gauge transformations themselves is worth further study, as suggested in [68]. The gauge transformations should combine with the local symmetry generators to form a very large algebra. Understanding this algebra would be of great use, since gauge transformations contribute to the commutators involving the nonlocal charges of Chapters 2 and also would appear in commutators involving the Yangian charges of dynamic sectors. An understanding of the gauge transformations might give new clues about corrections needed for $\mathfrak{h}$ at higher orders.

Finally, the infinite-dimensional symmetry of Chapter 2 may have applications even beyond $\mathcal{N} = 4$ SYM. It might be relevant for certain superstring models on $AdS_3 \times S^3$ or $AdS_2 \times S^2$ which also possess $\mathfrak{psu}(1,1|2)$ symmetry. Further suitable models include the principal chiral/WZW model on the group manifold $\widetilde{\mathrm{PSU}}(1,1|2)$ or some of it cosets. For instance, in some of these cases an additional su(2) and some even larger unexplained degeneracies were noticed in [69]. It is conceivable that they are of a similar origin as the ones discussed in Chapter 2.

To conclude, we have developed multiple techniques for studying the diverse symmetries of $\mathcal{N} = 4$ SYM. It is clear that there are many intriguing possibilities for research motivated by the results of this dissertation. We hope that such research will lead to new insights into gauge/string duality, and even into QCD.

Appendix A

The oscillator representation and the $\mathcal{O}(g)$ supercharges

In this appendix we take advantage of a very useful oscillator representation for $\mathfrak{psu}(2,2|4)$ and the fields of the $\mathcal{N}=4$ SYM spin chain [14, 70]. We first review this representation at leading order. Then, in Section A.2 we present oscillator expressions for the full set of supersymmetry generators at $\mathcal{O}(g)$. We prove that these expressions satisfy the $\mathfrak{psu}(2,2|4)$ algebra at $\mathcal{O}(g)$ and that they give the unique solution of the symmetry algebra.

A.1 The oscillator representation at leading order

Following [14], we represent the $\mathfrak{psu}(2,2|4)$ algebra in terms of two sets of bosonic oscillators $(\mathbf{a}^\alpha, \mathbf{a}^\dagger_\alpha)$, $(\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\alpha})$ with $\alpha, \dot\alpha = 1,2$ and one set of fermionic oscillator $(\mathbf{c}^a, \mathbf{c}^\dagger_a)$ with $a = 1,2,3,4$. The nonvanishing commutators take the standard form

$$[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] = \delta^\alpha_\beta, \qquad [\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}, \qquad \{\mathbf{c}^a, \mathbf{c}^\dagger_b\} = \delta^a_b. \tag{A.1}$$

Then the $\mathfrak{psu}(2,2|4)$ generators (at leading order in g) can be written as products of two oscillators. The Lorentz and $\mathfrak{R}$ symmetry generators (to all orders in g) are

$$\begin{aligned}
\mathfrak{L}^{\alpha}_{\beta} &= \mathbf{a}^{\dagger}_{\beta}\mathbf{a}^{\alpha} - \tfrac{1}{2}\delta^{\alpha}_{\beta}\mathbf{a}^{\dagger}_{\gamma}\mathbf{a}^{\gamma}, \\
\dot{\mathfrak{L}}^{\dot\alpha}_{\dot\beta} &= \mathbf{b}^{\dagger}_{\dot\beta}\mathbf{b}^{\dot\alpha} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\mathbf{b}^{\dagger}_{\dot\gamma}\mathbf{b}^{\dot\gamma}, \\
\mathfrak{R}^{a}_{b} &= \mathbf{c}^{\dagger}_{b}\mathbf{c}^{a} - \tfrac{1}{4}\delta^{a}_{b}\mathbf{c}^{\dagger}_{c}\mathbf{c}^{c}.
\end{aligned} \tag{A.2}$$

The (classical) dilatation generator counts $\mathbf{a}$ and $\mathbf{b}$ oscillators,

$$\mathfrak{D} = 1 + \tfrac{1}{2}\mathbf{a}^{\dagger}_{\gamma}\mathbf{a}^{\gamma} + \tfrac{1}{2}\mathbf{b}^{\dagger}_{\dot\gamma}\mathbf{b}^{\dot\gamma}, \tag{A.3}$$

and the supercharges, translations and boosts are

$$\begin{aligned}
\mathfrak{Q}^{b}_{\alpha} &= \mathbf{a}^{\dagger}_{\alpha}\mathbf{c}^{b}, & \mathfrak{S}^{\alpha}_{b} &= \mathbf{c}^{\dagger}_{b}\mathbf{a}^{\alpha}, \\
\dot{\mathfrak{Q}}_{\dot\alpha b} &= \mathbf{b}^{\dagger}_{\dot\alpha}\mathbf{c}^{\dagger}_{b}, & \dot{\mathfrak{S}}^{\dot\alpha b} &= \mathbf{b}^{\dot\alpha}\mathbf{c}^{b}, \\
\mathfrak{P}_{\alpha\dot\beta} &= \mathbf{a}^{\dagger}_{\alpha}\mathbf{b}^{\dagger}_{\dot\beta}, & \mathfrak{K}^{\alpha\dot\beta} &= \mathbf{a}^{\alpha}\mathbf{b}^{\dot\beta}.
\end{aligned} \tag{A.4}$$

Finally, there are two additional $\mathfrak{u}(1)$ generators that are not part of $\mathfrak{psu}(2,2|4)$,

$$\begin{aligned}
\mathfrak{B} &= \phantom{1 - {}} \tfrac{1}{2}\mathbf{a}^{\dagger}_{\gamma}\mathbf{a}^{\gamma} - \tfrac{1}{2}\mathbf{b}^{\dagger}_{\dot\gamma}\mathbf{b}^{\dot\gamma}, \\
\mathfrak{C} &= 1 - \tfrac{1}{2}\mathbf{a}^{\dagger}_{\gamma}\mathbf{a}^{\gamma} + \tfrac{1}{2}\mathbf{b}^{\dagger}_{\dot\gamma}\mathbf{b}^{\dot\gamma} - \tfrac{1}{2}\mathbf{c}^{\dagger}_{c}\mathbf{c}^{c}.
\end{aligned} \tag{A.5}$$

The external automorphism $\mathfrak{B}$ measures hypercharge, which equals $\mathfrak{B}^{<>}$ for the $\mathfrak{psu}(1,1|2)$ sector. $\mathfrak{C}$ is the central charge which must vanish for $\mathfrak{psu}(2,2|4)$. Therefore, the condition $\mathfrak{C} = 0$ will serve as a physicality constraint for states.

Using the commutation relations for the oscillators (A.1), it is straightforward to obtain the commutation relations for $\mathfrak{psu}(2,2|4)$, which are satisfied at all orders in g (quantum corrections deform the generators but not the algebra). The commutators with the Lorentz and $\mathfrak{R}$ symmetry generators take the canonical form for $\mathfrak{su}(n)$, with $n = 2,\, 4$. Let J^{A}_{B} be one of these $\mathfrak{su}(n)$ generators, and let K^C and K_C be any $\mathfrak{psu}(2,2|4)$ generator (with one index suppressed). Then we have,

$$[J^A_B, K^C] = -\delta^C_B K^A + \frac{1}{n}\delta^A_b K^C, \quad [J^A_B, K_C] = \delta^A_C K_B - \frac{1}{n}\delta^A_B K_C. \tag{A.6}$$

Since $\mathfrak{D}$ counts $\mathbf{a}$ and $\mathbf{b}$ oscillators (with a factor of $\frac{1}{2}$), we see that the supercharges have dimension $\pm\frac{1}{2}$, translations and boosts have dimension ± 1, and the remaining generators commute with $\mathfrak{D}$. Similarly, only the supercharges have nonvanishing values ($\pm\frac{1}{2}$) of hypercharge. Of course, all of the $\mathfrak{psu}(2,2|4)$ generators commute with $\mathfrak{C}$. The remaining nonvanishing commutators are a little more involved. Commutators with net nonzero dimension are

$$\begin{aligned}
&[\mathfrak{S}^\alpha{}_b, \mathfrak{P}_{\dot\gamma\delta}] = \delta^\alpha_\delta \dot{\mathfrak{Q}}_{\dot\gamma b}, \qquad [\mathfrak{K}^{\alpha\dot\beta}, \dot{\mathfrak{Q}}_{\dot\gamma d}] = \delta^{\dot\beta}_{\dot\gamma}\mathfrak{S}^\alpha_d, \\
&[\dot{\mathfrak{S}}^{a\dot\beta}, \mathfrak{P}_{\dot\gamma\delta}] = \delta^{\dot\beta}_{\dot\gamma}\mathfrak{Q}^a_\delta, \qquad [\mathfrak{K}^{\alpha\dot\beta}, \mathfrak{Q}^c_\delta] = \delta^\alpha_\delta \dot{\mathfrak{S}}^{c\dot\beta}, \\
&\{\dot{\mathfrak{Q}}_{\dot\alpha b}, \mathfrak{Q}^c_\delta\} = \delta^c_b \mathfrak{P}_{\dot\alpha\delta}, \qquad \{\dot{\mathfrak{S}}^{a\dot\beta}, \mathfrak{S}^\gamma_d\} = \delta^a_d \mathfrak{K}^{\gamma\dot\beta}.
\end{aligned} \tag{A.7}$$

The other commutators have $\mathfrak{D}$ and Lorentz and $\mathfrak{R}$ symmetry generators on the right side,

$$\begin{aligned}
[\mathfrak{K}^{\alpha\dot\beta}, \mathfrak{P}_{\dot\gamma\delta}] &= \delta^{\dot\beta}_{\dot\gamma}\mathfrak{L}^\alpha_\delta + \delta^\alpha_\delta \dot{\mathfrak{L}}^{\dot\beta}_{\dot\gamma} + \delta^\alpha_\delta\delta^{\dot\beta}_{\dot\gamma}\mathfrak{D}, \\
\{\mathfrak{S}^\alpha_b, \mathfrak{Q}^c_\delta\} &= \delta^c_b\mathfrak{L}^\alpha_\delta + \delta^\alpha_\delta\mathfrak{R}^c_b + \tfrac{1}{2}\delta^c_b\delta^\alpha_\delta\mathfrak{D}, \\
\{\dot{\mathfrak{S}}^{a\dot\beta}, \dot{\mathfrak{Q}}_{\dot\gamma d}\} &= \delta^a_d\dot{\mathfrak{L}}^{\dot\beta}_{\dot\gamma} - \delta^{\dot\beta}_{\dot\gamma}\mathfrak{R}^a_d + \tfrac{1}{2}\delta^a_d\delta^{\dot\beta}_{\dot\gamma}\mathfrak{D}.
\end{aligned} \tag{A.8}$$

The algebra is symmetric under hermitian conjugation, so it is consistent (but not essential) to require the solution for the quantum-corrected generators to be hermitian. That is,

$$(\mathfrak{Q}^b_\alpha)^\dagger = \mathfrak{S}^\alpha_b, \quad (\dot{\mathfrak{Q}}_{\dot\alpha b})^\dagger = (\dot{\mathfrak{S}}^{\dot\alpha b}), \quad (\mathfrak{P}_{\dot\alpha\beta})^\dagger = \mathfrak{K}^{\beta\dot\alpha}, \quad \mathfrak{D}^\dagger = \mathfrak{D}. \tag{A.9}$$

Next, still following [14], we represent the fields of $\mathcal{N} = 4$ SYM (and covariant derivatives) in terms of the oscillators. We introduce the nonphysical vacuum $|0\rangle$, which is annihilated by **a**, **b**, **c**. Then, fields or states without any covariant derivatives are,

$$
\begin{aligned}
|\mathcal{F}_{\alpha\beta}\rangle &\sim \mathbf{a}^\dagger_\alpha \mathbf{a}^\dagger_\beta|0\rangle, \\
|\varPsi_{\alpha b}\rangle &\sim \mathbf{a}^\dagger_\alpha \mathbf{c}^\dagger_b|0\rangle, \\
|\varPhi_{ab}\rangle &\sim \mathbf{c}^\dagger_a \mathbf{c}^\dagger_b|0\rangle, \\
|\dot{\varPsi}^b_{\dot\alpha}\rangle &\sim \frac{1}{3!}\varepsilon^{bcde}\mathbf{b}^\dagger_{\dot\alpha}\mathbf{c}^\dagger_c\mathbf{c}^\dagger_d\mathbf{c}^\dagger_e|0\rangle, \\
|\dot{\mathcal{F}}_{\dot\alpha\dot\beta}\rangle &\sim \frac{1}{4!}\varepsilon^{cdef}\mathbf{b}^\dagger_{\dot\alpha}\mathbf{b}^\dagger_{\dot\beta}\mathbf{c}^\dagger_c\mathbf{c}^\dagger_d\mathbf{c}^\dagger_e\mathbf{c}^\dagger_f|0\rangle.
\end{aligned} \tag{A.10}
$$

We have used tildes since there is freedom in for normalization of these states. To obtain a generic field composed of arbitrarily many covariant derivatives (in various directions), we can proceed inductively. For any field X (with or without covariant derivatives), we have

$$
|\mathcal{D}_{\alpha\dot\beta}X\rangle \sim \mathbf{a}^\dagger_\alpha \mathbf{b}^\dagger_{\dot\beta}|X\rangle. \tag{A.11}
$$

Note that the oscillator representation automatically symmetrizes indices and removes the trace for products of covariant derivatives, as required for an irreducible representation Throughout this work we normalize states so that hermitian conjugation simply reverses initial and final states. To accomplish this, for any n identical oscillators (acting on the same site) include a factor of $1/\sqrt{n!}$.

As claimed in Section 1.1, it is now clear that the full $\mathfrak{psu}(2,2|4)$ single-site highest weight module is given by acting with the $\mathfrak{psu}(2,2|4)$ (lowering) generators on the highest weight state $\varPhi_{34} = \mathbf{c}^\dagger_3\mathbf{c}^\dagger_4|0\rangle$. The $\mathfrak{R}$ map the highest weight state to the other $\varPhi$, acting once or twice with the $\mathfrak{Q}$ or $\dot{\mathfrak{Q}}$ yield the fermionic fields and field strengths, while repeated applications of $\mathfrak{P}$ give all possible (symmetrized and traceless) combinations of covariant derivatives.

Finally, the fields of the $\mathfrak{psu}(1,1|2)$ sector are easy to identify using this oscillator formalism. The restrictions (1.33) are equivalent to

$$L + \tfrac{1}{2}n_{\mathbf{a}} + \tfrac{1}{2}n_{\mathbf{b}} = n_{\mathbf{a}_1} - n_{\mathbf{a}_2} - \frac{3}{2}n_{\mathbf{c}_4} + \tfrac{1}{2}(n_{\mathbf{c}_1} + n_{\mathbf{c}_2} + n_{\mathbf{c}_3}),$$
$$L + \tfrac{1}{2}n_{\mathbf{a}} + \tfrac{1}{2}n_{\mathbf{b}} = n_{\mathbf{b}_1} - n_{\mathbf{b}_2} + \frac{3}{2}n_{\mathbf{c}_3} - \tfrac{1}{2}(n_{\mathbf{c}_1} + n_{\mathbf{c}_2} + n_{\mathbf{c}_4}). \tag{A.12}$$

Adding the two equations and simplifying we obtain

$$2L + 2n_{\mathbf{a}_2} + 2n_{\mathbf{b}_2} = 2n_{\mathbf{c}_3} - 2n_{\mathbf{c}_4}, \tag{A.13}$$

where $n_{\mathbf{x}}$ is the number of $\mathbf{x}$ oscillators. Since the number of any type of oscillator is nonnegative and because there can be at most one of each type of fermionic oscillator on a single site (L on a length-L chain), this equation can only be satisfied if the following four conditions are satisfied,

$$n_{\mathbf{a}_2} = n_{\mathbf{b}_2} = n_{\mathbf{c}_4} = 0, \quad n_{\mathbf{c}_3} = L. \tag{A.14}$$

When these conditions are imposed, the remaining independent condition reduces to $\mathfrak{C} = 0$, which is required anyway for physical states. It is now straightforward to apply the projection of (A.14) to the fields and generators of the full $\mathfrak{psu}(2,2|4)$ system, and this yields precisely the fields and generators of the $\mathfrak{psu}(1,1|2)$ sector described in Section 1.4.

A.2 The supersymmetry generators at $\mathcal{O}(g)$

A.2.1 The solution

At $\mathcal{O}(g)$, $\mathfrak{Q}_{(1)}$ and $\dot{\mathfrak{Q}}_{(1)}$ have one-site to two-site interactions given by

$$\mathfrak{Q}^b_{\alpha(1)}(1) = \mathcal{H}'\epsilon^{bdef}\epsilon_{\alpha\gamma}\mathbf{a}^{\gamma}(1)\mathbf{c}^{\dagger}_d(2)\mathbf{c}^{\dagger}_e(2)\mathbf{c}^{\dagger}_f(2)\check{Z}(2), \tag{A.15}$$

$$\dot{\mathfrak{Q}}_{\dot{\alpha}b(1)}(1) = \mathcal{H}'\epsilon_{\dot{\alpha}\dot{\gamma}}\mathbf{b}^{\dot{\gamma}}(1)\mathbf{c}^{\dagger}_b(2)\check{Z}(2), \tag{A.16}$$

where $\check{Z}$ inserts a new site without any oscillator excitations,

$$\check{Z}(2)|X\rangle = |X0\rangle. \tag{A.17}$$

As usual, the full action of these supercharges is given by summing these interactions homogeneously over the length of the spin chain. $\mathcal{H}'$ acts very similarly to the one-loop dilatation generator, or the harmonic action $\mathcal{H}$, given in [14]. Following the presentation given there, we introduce a collective oscillator $\mathbf{A}^\dagger_A = (\mathbf{a}^\dagger_\alpha, \mathbf{b}^\dagger_{\dot\alpha}, \mathbf{c}^\dagger_a)$. A general two-site state is written as

$$|s_1, \ldots, s_n; A\rangle = \mathbf{A}^\dagger_{A_1}(s_1) \ldots \mathbf{A}^\dagger_{A_n}(s_n)|00\rangle, \tag{A.18}$$

and the label $s_k = 1, 2$ specifies the site on which the k-th oscillator acts. Both $\mathcal{H}$ and $\mathcal{H}'$ act on two fields by moving indices between them, and each of them acts in the same way on all types of oscillators:

$$\begin{aligned} \mathcal{H}|s_1, \ldots, s_n; A\rangle &= \sum_{s'_1,\ldots s'_n} c_{n,n_{12},n_{21}}\, \delta_{C_1,0}\delta_{C_2,0}\, |s'_1, \ldots, s'_n; A\rangle, \\ \mathcal{H}'|s_1, \ldots, s_n; A\rangle &= \sum_{s'_1,\ldots s'_n} a_{n,n_{12},n_{21}}\, \delta_{C_1,0}\delta_{C_2,0}\, |s'_1, \ldots, s'_n; A\rangle. \end{aligned} \tag{A.19}$$

The sums go over the sites $1, 2$ and $\delta_{C_1,0}$, $\delta_{C_2,0}$ project to states where the central charge at each site is zero, that is states $|XY\rangle$ which satisfy $\mathfrak{C}(1)|XY\rangle = \mathfrak{C}(2)|XY\rangle = 0$. n_{12} and n_{21} are the number of oscillators that shift from site 1 to 2 or vice versa. The coefficients $a_{n,n_{12},n_{21}}$ are slightly different than the corresponding $c_{n,n_{12},n_{21}}$ of the harmonic action of [14][1]

$$\begin{aligned} c_{n,n_{12},n_{21}} &= (-1)^{1+n_{21}} B(\tfrac{1}{2}(n_{12}+n_{21}), 1 + \tfrac{1}{2}(n - n_{12} - n_{21})), \\ a_{n,n_{12},n_{21}} &= (-1)^{n_{21}} B(\tfrac{1}{2}(n_{12}+n_{21}+1), \tfrac{1}{2}(n - n_{12} - n_{21} + 1)). \end{aligned} \tag{A.20}$$

B is the beta function

$$B(p,\, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{A.21}$$

[1]The $(-1)^{1+n_{21}}$ factor is slightly different than the corresponding factor given in [14], but it leads to the same action because of the central charge projection.

Also, the $\mathcal{H}$ coefficients need to be regularized for no hopping as $c_{n,0,0} = h(\frac{1}{2}n)$. No regularization is needed for $\mathcal{H}'$.

As at leading order, Hermitian conjugation gives the remaining supercharges, $\mathfrak{S}_{(1)}$ and $\dot{\mathfrak{S}}_{(1)}$. They have two-site to one-site interactions given by

$$\mathfrak{S}^{\alpha}_{b(1)}(1,2) = -\delta_{C_1,0}\,\hat{Z}(1,2)\epsilon_{bdef}\epsilon^{\alpha\gamma}\mathbf{a}^{\dagger}_{\gamma}(1)\mathbf{c}^{d}(2)\mathbf{c}^{e}(2)\mathbf{c}^{f}(2)\mathcal{H}'^{\dagger}, \tag{A.22}$$

$$\dot{\mathfrak{S}}^{\dot{\alpha}b}_{(1)}(1,2) = \delta_{C_1,0}\,\hat{Z}(1,2)\epsilon^{\dot{\alpha}\dot{\gamma}}\mathbf{b}^{\dagger}_{\dot{\gamma}}(1)\mathbf{c}^{b}(2)\mathcal{H}'^{\dagger}. \tag{A.23}$$

$\hat{Z}(1,2)$ removes a (second) site without any oscillator excitations,

$$\hat{Z}(1,2)|X0\rangle = |X\rangle, \tag{A.24}$$

and annihilates two-site state with any oscillators on the second site. Importantly, $\mathcal{H}'^{\dagger}$ first projects onto $\mathfrak{C} = 0$ physical states and then shifts oscillators, so generically it maps physical states to nonphysical states. The other difference between $\mathcal{H}'$ and its hermitian conjugate is that n_{12} and n_{21} are interchanged, which only affects the exponent of the minus sign in the expression for the coefficients, $a_{n,n_{12},n_{21}}$ (A.20).

Substituting the solution for the supercharges in the algebra relations (A.7), one can compute the $\mathcal{O}(g)$ corrections to the $\mathfrak{K}$ and $\mathfrak{P}$. However, there are four alternative commutators for any single $\mathfrak{P}$ or $\mathfrak{K}$ component. Below, we will see that $\mathfrak{R}$ symmetry guarantees that these four expressions are equal. Since the dilatation generator receives no corrections at this order, we have now given the full set of $\mathcal{O}(g)$ corrections.

Parity. Since local symmetry generators are parity even, we must check that the expressions for the $\mathfrak{Q}_{(1)}$ and $\dot{\mathfrak{Q}}_{(1)}$ satisfy this requirement. Consider the parity-reflected version of $\dot{\mathfrak{Q}}_{(1)}$ (A.16),

$$P\dot{\mathfrak{Q}}_{\dot{\alpha}a(1)}(1)P^{-1} = -\mathcal{H}'_{P}\epsilon_{\dot{\alpha}\dot{\beta}}\mathbf{b}^{\dot{\beta}}(2)\mathbf{c}^{\dagger}_{a}(1)\check{Z}(1). \tag{A.25}$$

Importantly, there is an extra overall minus sign from the parity reversal of $\check{Z}$ since there is a factor of $(-1)^L$ in the definition of parity (1.32) ($\check{Z}(1) - \check{Z}(2)$ is parity even). Examining the expressions defining $\mathcal{H}'$, we find that its parity-reflected version $\mathcal{H}'_P$ only differs by replacing $(-1)^{(n_{21})}$ with $(-1)^{(n_{12})}$.

We will prove that this parity reflected supercharge equals the initial supercharge by showing that the amplitudes for any fixed initial to final state transition are equal. Observe that the argument of the beta function in the definition of $\mathcal{H}'$ (A.20) is symmetric between the number of oscillators that switch sites and the number that remain at the same site. Therefore, $\dot{\mathfrak{Q}}_{(1)}$ and its parity reflection have the same magnitude for any given initial to final state transition. Furthermore, for $\dot{\mathfrak{Q}}_{(1)}$ or its parity reflection, the only oscillator shifts that can contribute a minus sign are those of the inserted $\mathbf{c}^\dagger$. Also for any given initial to final state transition, the inserted $\mathbf{c}^\dagger$ must shift for $\dot{\mathfrak{Q}}_{(1)}$ or for its parity reflection, but not for both. It follows that $\dot{\mathfrak{Q}}_{(1)}$ and its parity reflection receive opposite sign contributions from the (-1) exponent factor. This cancels the overall minus of (A.25). We conclude that the transition amplitudes are the same, completing the proof that the $\dot{\mathfrak{Q}}_{(1)}$ are parity even. A similar proof works for the $\mathfrak{Q}_{(1)}$ since they also have an odd number of $\mathbf{c}^\dagger$ insertions.

Discrete symmetry. For later use we observe that there is a discrete symmetry R that switches $\mathbf{a}$ and $\mathbf{b}$ and applies a particle-hole transformation to the $\mathbf{c}$. Under R (not to be confused with $\mathfrak{R}^a_b$), the generators transform as

$$\begin{aligned} &\mathfrak{Q}^b_\alpha \leftrightarrow \dot{\mathfrak{Q}}_{\alpha b}, && \mathfrak{S}^\alpha_b \leftrightarrow \dot{\mathfrak{S}}^{\alpha b}, && \mathfrak{P}_{\alpha\beta} \leftrightarrow \mathfrak{P}_{\beta\alpha}, && \mathfrak{K}^{\alpha\beta} \leftrightarrow \mathfrak{K}^{\beta\alpha}, \\ &\mathfrak{L}^\alpha_\beta \leftrightarrow \dot{\mathfrak{L}}^\alpha_\beta, && \mathfrak{R}^a_b \leftrightarrow -\mathfrak{R}^b_a, && \mathfrak{D} \rightarrow \mathfrak{D}, && \mathfrak{B} \rightarrow -\mathfrak{B}. \end{aligned} \tag{A.26}$$

Since R maps $\mathfrak{C}$ to $-\mathfrak{C}$, it maps physical states to physical states. The fields transform as

$$\mathcal{F}_{\alpha\beta} \leftrightarrow \dot{\mathcal{F}}_{\alpha\beta}, \quad \Psi_{\alpha b} \leftrightarrow \dot{\Psi}^b_\alpha, \quad \Phi_{ab} \rightarrow \tfrac{1}{2}\varepsilon^{abcd}\Phi_{cd}. \tag{A.27}$$

It is straightforward to check that this symmetry is respected both by the leading order generators and by the $\mathcal{O}(g)$ supersymmetry generators. This will enable a more efficient check that the supercharges satisfy the algebra.

Alternative representation. We finish this section with an equivalent representation of $\mathcal{H}'$, which is conceptually simpler. To write this version we order the oscillators by site,

$$\mathcal{H}'\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\mathbf{A}^\dagger_{A_i}(1)\mathbf{A}^\dagger_{B_j}(2)|00\rangle = \\ \delta_{C_1,0}\delta_{C_2,0}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\, 2\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\left(\mathbf{A}^\dagger_{A_i}(1)\cos\theta+\mathbf{A}^\dagger_{A_i}(2)\sin\theta\right)\left(\mathbf{A}^\dagger_{B_j}(2)\cos\theta-\mathbf{A}^\dagger_{B_j}(1)\sin\theta\right)|00\rangle. \tag{A.28}$$

$\mathcal{H}'$ integrates over the possible ways to rotate the oscillators collectively between the sites, and projects onto physical states. Interestingly, there is a similar representation for the harmonic action $\mathcal{H}$ of [14], which gives the complete one-loop dilatation generator. In that case an integration weight of $\cot\theta$ is required, and an additional contribution proportional to the identity operator is needed for regularization,

$$\mathcal{H}\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\mathbf{A}^\dagger_{A_i}(1)\mathbf{A}^\dagger_{B_j}(2)|00\rangle = \delta_{C_1,0}\delta_{C_2,0}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\, 2\cot\theta\bigg\{\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\mathbf{A}^\dagger_{A_i}(1)\mathbf{A}^\dagger_{B_j}(2)|00\rangle \\ -\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\left(\mathbf{A}^\dagger_{A_i}(1)\cos\theta+\mathbf{A}^\dagger_{A_i}(2)\sin\theta\right)\left(\mathbf{A}^\dagger_{B_j}(2)\cos\theta-\mathbf{A}^\dagger_{B_j}(1)\sin\theta\right)|00\rangle\bigg\}. \tag{A.29}$$

The equivalence between these expressions and the previous definitions of $\mathcal{H}'$ and $\mathcal{H}$ follows from the identity

$$\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\,\cos^{2p-1}\theta\sin^{2q-1}\theta=\tfrac{1}{2}B(p,\,q). \tag{A.30}$$

A.2.2 A $\mathfrak{psu}(1,1|2)$ sector example

Here we use the general solution for the supercharges to compute part of the expressions for the $\mathcal{O}(g)$ supercharges of the $\mathfrak{psu}(1,1|2)$ sector, given in Section 2.2.1. We compute

$$\hat{\mathfrak{Q}}^{<}_{(1)}|\psi^{(n)}_{>}\rangle = \dot{\mathfrak{Q}}_{23(1)}|\mathcal{D}^{n}_{11}\Psi_{13}\rangle = \dot{\mathfrak{Q}}_{23(1)}\frac{1}{n!\sqrt{n+1}}(\mathbf{a}^{\dagger}_{1})^{n+1}(\mathbf{b}^{\dagger}_{1})^{n}\mathbf{c}^{\dagger}_{3}|0\rangle. \tag{A.31}$$

We used (1.66) and (1.46) to transform between the $\mathfrak{psu}(1,1|2)$ sector notation and the notation for the full theory, and (A.10-A.11) for the oscillator representation. Applying the definition of (A.16), and using the commutation relation for the $\mathbf{b}$ (A.1), we obtain

$$\begin{aligned}\hat{\mathfrak{Q}}^{<}_{(1)}|\psi^{(n)}_{>}\rangle &= \mathcal{H}'(-\mathbf{b}^{1}(1)\mathbf{c}^{\dagger}_{3}(2))\frac{1}{n!\sqrt{n+1}}\left(\mathbf{a}^{\dagger}_{1}(1)\right)^{n+1}\left(\mathbf{b}^{\dagger}_{1}(1)\right)^{n}\mathbf{c}^{\dagger}_{3}(1)|00\rangle \\ &= \mathcal{H}'\frac{1}{(n-1)!\sqrt{n+1}}\left(\mathbf{a}^{\dagger}_{1}(1)\right)^{n+1}\left(\mathbf{b}^{\dagger}_{1}(1)\right)^{n-1}\mathbf{c}^{\dagger}_{3}(1)\mathbf{c}^{\dagger}_{3}(2)|00\rangle. \end{aligned} \tag{A.32}$$

The minus sign from the first line cancels against the minus sign for switching the order of the $\mathbf{c}^{\dagger}_{3}$. Because of the projection onto physical states included in the definition of $\mathcal{H}'$ (A.19), the only possible final states are

$$\left(\mathbf{a}^{\dagger}_{1}(1)\right)^{m+1}\left(\mathbf{b}^{\dagger}_{1}(1)\right)^{m}\mathbf{c}^{\dagger}_{3}(1)\left(\mathbf{a}^{\dagger}_{1}(2)\right)^{n-m}\left(\mathbf{b}^{\dagger}_{1}(2)\right)^{n-m-1}\mathbf{c}^{\dagger}_{3}(2)|00\rangle, \quad m=0,\dots n-1. \tag{A.33}$$

There are two cases of oscillator shifts that we need to consider. If the $\mathbf{c}^{\dagger}$ are not shifted, $2(n-m)-1$ oscillators are moved from site 1 to site 2, or $n_{12}=2(n-m)-1$. In this case, $n_{21}=0$. The other possibility is that the $\mathbf{c}^{\dagger}$ are flipped, and $n_{12}=2(n-m)$, $n_{21}=1$. Since there are a total of $2n+2$ oscillators, (A.20) gives $B(n-m,\, m+2)$ and $-B(n-m+1,\, m+1)$ for the two cases. However, the minus sign of the second coefficient is cancelled if we keep the same ordering of the $\mathbf{c}^{\dagger}$ as in (A.33) for the final state. Since $\mathcal{H}'$ sums over all possible shifts of oscillators, we need to include combinatoric factors for choosing which oscillators to move. In either case this gives a factor of

$$\binom{n+1}{m+1}\binom{n-1}{m}. \tag{A.34}$$

Putting together all the pieces, we find the states of (A.33) come with a coefficient of

$$\begin{aligned}\frac{1}{(n-1)!\sqrt{n+1}}\binom{n+1}{m+1}\binom{n-1}{m}\left(B(n-m,\, m+2)+B(n-m+1,\, m+1)\right) =\\ \frac{1}{(n-1)!\sqrt{n+1}}\frac{(n+1)!(n-1)!}{(m+1)!(n-m)!m!(n-m-1)!}\left(\frac{(n-m-1)!(m+1)!}{(n+1)!}+\frac{(n-m)!m!}{(n+1)!}\right)\\ =\frac{\sqrt{n+1}}{(m+1)!(n-m)!}.\end{aligned} \tag{A.35}$$

Finally, again using (1.46) and (A.10-A.11), we write the final states in terms of the $\mathfrak{psu}(1,1|2)$ sector notation for the field $\psi_>^{(n)}$,

$$\begin{aligned}\hat{\mathfrak{Q}}^{<}_{(1)}|\psi_>^{(n)}\rangle &= \sum_{m=0}^{n-1}\frac{\sqrt{n+1}}{(m+1)!(n-m)!}(m!\sqrt{m+1})\left((n-m-1)!\sqrt{n-m}\right)|\psi_>^{(m)}\psi_>^{(n-m-1)}\rangle\\ &= \sum_{m=0}^{n-1}\frac{\sqrt{n+1}}{\sqrt{(m+1)(n-m)}}|\psi_>^{(m)}\psi_>^{(n-m-1)}\rangle,\end{aligned} \tag{A.36}$$

in agreement with (2.34). The remaining calculations to obtain the full action of the $\hat{\mathfrak{Q}}_{(1)}$ proceed similarly, and then hermitian conjugation can be used to obtain the $\hat{\mathfrak{S}}_{(1)}$.

A.2.3 Checking the algebra and uniqueness at $\mathcal{O}(g)$

We will now prove that the solution for the $\mathcal{O}(g)$ generators indeed satisfies the $\mathfrak{psu}(2,2|4)$ algebra. At the end of the section we will give a concise proof that it is the unique solution, up to normalization.

We begin with some properties that are essential for the proof. First, $\mathcal{H}'$ has the key property

$$[\mathfrak{J}_{(0)}, \mathcal{H}'] = 0 \tag{A.37}$$

for any classical $\mathfrak{psu}(2,2|4)$ generator $\mathfrak{J}_{(0)}$. The proof of this follows the same proof as given in [14] for the unprimed $\mathcal{H}$. Although the coefficients are slightly different, the same arguments work because the coefficients still satisfy the essential identities

$$a_{n+2,n_{12},n_{21}} + a_{n+2,n_{12},n_{21}+2} = a_{n,n_{12},n_{21}}, \quad a_{n+2,n_{12},n_{21}} + a_{n+2,n_{12}+2,n_{21}} = a_{n,n_{12},n_{21}}. \tag{A.38}$$

Furthermore, with the sign $(-1)^{n_{21}}$ of the $a_{n,n_{12},n_{21}}$ (A.19), we find that the $\mathfrak{J}_{(0)}$ commute even when (A.37) is evaluated on arbitrary nonphysical states, such as appear at intermediate stages of the supercharges' actions. Therefore, in checking commutators between $\mathcal{O}(g^1)$ supercharges and classical generators, we can ignore $\mathcal{H}'$.

Also, $\check{Z}$ commutes with all of the classical generators besides the $\mathfrak{P}$ and $\dot{\mathfrak{Q}}$. This is because any $\mathbf{a}$, $\mathbf{b}$, or $\mathbf{c}$ annihilates the inserted (nonphysical) vacuum site.

We now check all of the $\mathfrak{psu}(2,2|4)$ algebra relations involving a commutator with a $\dot{\mathfrak{Q}}$.

Commutators with Lorentz and R symmetry generators. Since the $\dot{\mathfrak{Q}}_{\dot{\alpha}b(1)}$ do not involve any $\mathbf{a}$ (besides through $\mathcal{H}'$), clearly they commute with the $\mathfrak{L}$ as required. The commutators with $\dot{\mathfrak{L}}$ and $\mathfrak{R}$ are only slightly more involved. The only contribution to the commutators with the $\dot{\mathfrak{L}}$ comes from the $\mathbf{b}^{\dot{\gamma}}$ in the expression for $\dot{\mathfrak{Q}}_{\dot{\alpha}b(1)}$. But $\epsilon_{\dot{\alpha}\dot{\gamma}}\mathbf{b}^{\dot{\gamma}}$ transforms with a lower Lorentz $\mathfrak{su}(2)$ index, so these canonical commutators are satisfied. Similarly, the canonical commutators with $\mathfrak{R}$ are satisfied because the only contributions come from the $\mathbf{c}^{\dagger}_{b}$ in $\dot{\mathfrak{Q}}_{\dot{\alpha}b(1)}$.

Commutators with $\mathfrak{S}^{\delta}_{e}$. In this case, we have

$$\{\dot{\mathfrak{Q}}_{\dot{\alpha}b(1)}, \mathfrak{S}^{\delta}_{e(0)}\} = 0, \tag{A.39}$$

because $\mathfrak{S}_{(0)}$ is a product of $\mathbf{a}$ and $\mathbf{c}^{\dagger}$ which clearly commute with the $\mathbf{b}$ and $\mathbf{c}^{\dagger}$ of $\dot{\mathfrak{Q}}_{\dot{\alpha}b(1)}$. The discrete R transformation combined with hermitian conjugation then imply

$$\{\dot{\mathfrak{Q}}_{\dot{\alpha}b(0)}, \mathfrak{S}^{\delta}_{e(1)}\} = 0, \tag{A.40}$$

so the total commutator at $\mathcal{O}(g)$ vanishes as required.

Commutators with $\dot{\mathfrak{S}}^{e}_{\dot{\delta}}$. Since there is no contribution to $\delta\mathfrak{D}$ at $\mathcal{O}(g)$, from (A.8) we see that these commutators are required to vanish also. Again, we first observe that

$$\{\dot{\mathfrak{Q}}_{\dot{\alpha}b(1)}, \dot{\mathfrak{S}}^{e}_{\dot{\delta}(0)}\} = 0, \tag{A.41}$$

because the only possible contribution could come from the $\dot{\mathfrak{S}}_{(0)} \sim \mathbf{bc}$ acting on the second site commuted with the $\mathbf{c}^{\dagger}_{b}(2)$ of $\dot{\mathfrak{Q}}_{\dot{\alpha}b(1)}$. However, this vanishes because the $\mathbf{b}$ of $\dot{\mathfrak{S}}_{(0)}$ annihilates the second site even after $\mathbf{c}^{\dagger}_{b}(2)$ acts. Combined with the hermitian conjugate vanishing commutators, we find

$$\{\dot{\mathfrak{Q}}_{\dot{\alpha}b}, \dot{\mathfrak{S}}^{e}_{\dot{\delta}}\}_{(1)} = 0, \tag{A.42}$$

as needed.

Commutators with $\mathfrak{Q}^{e}_{\delta}$. At $\mathcal{O}(g)$, these commutation relations (A.7) reduce to

$$\{\dot{\mathfrak{Q}}_{\dot{\alpha}b}, \mathfrak{Q}^{e}_{\delta}\}_{(1)} = \delta^{e}_{b}\mathfrak{P}_{\dot{\alpha}\delta(1)}. \tag{A.43}$$

For $e \neq b$, the terms in the commutator either include a $\mathbf{c}$ that annihilates the second site or two identical $\mathbf{c}^{\dagger}$ that annihilate the second site. Since there is no contribution from commutators between oscillators acting on the first site, the algebra relations are satisfied for $e \neq b$. Then $\mathfrak{R}$ symmetry implies that $\mathfrak{P}_{\delta\dot{\alpha}(1)}$ is uniquely and consistently defined by (A.43) for any $b = e$. Therefore, these commutators are satisfied as well.

Commutators between the $\dot{\mathfrak{Q}}_{\dot{\alpha}b}$. These are the most involved relations to check because the commutator only vanishes up to gauge transformations. The exact relation is

$$\{\dot{\mathfrak{Q}}_{\dot{\alpha}b}, \dot{\mathfrak{Q}}_{\dot{\delta}e}\}_{(1)} = \varepsilon_{\dot{\alpha}\dot{\delta}}(\check{Z}_{be}(1) - \check{Z}_{be}(2)), \tag{A.44}$$

where $\check{Z}_{be}(i)$ inserts a Φ_{be} on site i[2],

$$\check{Z}_{be}(1)|X\rangle = |\Phi_{be}X\rangle, \quad \check{Z}_{be}(2)|X\rangle = |X\Phi_{be}\rangle. \tag{A.45}$$

To show this it will be sufficient to check one commutator with $b \neq e$ and $\dot{\alpha} \neq \dot{\delta}$, because $\dot{\mathfrak{L}}$ and $\mathfrak{R}$ symmetry then imply all of the remaining commutators. So we will verify that

$$\{\dot{\mathfrak{Q}}_{11}, \dot{\mathfrak{Q}}_{22}\}_{(1)} = \check{Z}_{12}(1) - \check{Z}_{12}(2). \tag{A.46}$$

Furthermore, since $\{\dot{\mathfrak{Q}}_{21}, \dot{\mathfrak{Q}}_{21}\}_{(1)}$ vanishes by inspection, $\mathfrak{R}$ symmetry implies that $\{\dot{\mathfrak{Q}}_{21}, \dot{\mathfrak{Q}}_{22}\}_{(1)}$ vanishes. Therefore, $\dot{\mathfrak{L}}^1_2$ commutes with the left side (and right side) of (A.46). It follows that we can check (A.46) on (physical) states without $\mathbf{b}^\dagger_2$ (any state with $\mathbf{b}^\dagger_2$ can be obtained by acting with $\dot{\mathfrak{L}}^1_2$ which preserves (A.46)). Similarly, since $\mathfrak{L}$ clearly commutes with both sides of (A.46), we only need to check states without any $\mathbf{a}^\dagger_2$. Applying this type of argument a third time, now using $\mathfrak{S}_{(0)}$, we conclude that it is sufficient to check (A.46) on states of the form

$$\frac{(\mathbf{a}^\dagger_1)^{n+2}(\mathbf{b}^\dagger_1)^n}{n!\sqrt{(n+1)(n+2)}}|0\rangle = |\mathcal{F}^{(n)}\rangle, \tag{A.47}$$

where we use the same notation as for the $\mathfrak{psu}(1,1|2)$ sector, with the superscript (n) corresponding to $\mathcal{D}^n_{11}$. Before proceeding with the calculation we introduce notation for the other fields we will encounter,

$$\begin{aligned}
|\psi^{(n)}_{11}\rangle &= \frac{(\mathbf{a}^\dagger_1)^{n+1}(\mathbf{b}^\dagger_1)^n\mathbf{c}^\dagger_1}{n!\sqrt{n+1}}|0\rangle, \qquad & |\psi^{(n)}_{12}\rangle &= \frac{(\mathbf{a}^\dagger_1)^{n+1}(\mathbf{b}^\dagger_1)^n\mathbf{c}^\dagger_2}{n!\sqrt{n+1}}|0\rangle, \\
|\phi^{(n)}_{12}\rangle &= \frac{(\mathbf{a}^\dagger_1)^{n}(\mathbf{b}^\dagger_1)^n\mathbf{c}^\dagger_1\mathbf{c}^\dagger_2}{n!}|0\rangle, \qquad & |\psi^{(n,1)}_{12}\rangle &= \frac{(\mathbf{a}^\dagger_1)^{n+1}(\mathbf{b}^\dagger_1)^{n-1}\mathbf{b}^\dagger_2\mathbf{c}^\dagger_2}{(n-1)!\sqrt{n(n+1)}}|0\rangle.
\end{aligned} \tag{A.48}$$

Next we give the relevant leading order supercharge actions,

$$\begin{aligned}
\dot{\mathfrak{Q}}_{11(0)}|\mathcal{F}^{(n)}_{11}\rangle &= \sqrt{n+1}|\psi^{(n+1)}_{11}\rangle, \qquad & \dot{\mathfrak{Q}}_{11(0)}|\psi^{(n)}_{12}\rangle &= \sqrt{n+1}|\phi^{(n+1)}_{12}\rangle, \\
\dot{\mathfrak{Q}}_{22(0)}|\mathcal{F}^{(n)}_{11}\rangle &= |\psi^{(n+1,1)}_{12}\rangle, & &
\end{aligned} \tag{A.49}$$

[2]Of course, $\check{Z}_{be}$ vanishes for $b = e$.

and relevant ones at $\mathcal{O}(g)$,

$$\begin{aligned}
\dot{\mathfrak{Q}}_{22(1)}|\mathcal{F}_{11}^{(n)}\rangle &= \sum_{m=0}^{n-1} \frac{\sqrt{(n+2)(m+1)}}{\sqrt{(n+1)(m+2)(n-m)}}(|\mathcal{F}_{11}^{(m)}\psi_{12}^{(n-m-1)}\rangle - |\psi_{12}^{(n-m-1)}\mathcal{F}_{11}^{(m)}\rangle), \\
\dot{\mathfrak{Q}}_{22(1)}|\psi_{11}^{(n)}\rangle &= \sum_{m=0}^{n-1} \frac{\sqrt{n-m}}{\sqrt{(n+1)(n-m+1)}}(|\phi_{12}^{(m)}\mathcal{F}_{11}^{(n-1-m)}\rangle - |\mathcal{F}_{11}^{(n-1-m)}\phi_{12}^{(m)}\rangle) \\
&\quad - \sum_{m=0}^{n-1} \frac{\sqrt{m+1}}{\sqrt{(n+1)(n-m)}}(|\psi_{11}^{(m)}\psi_{12}^{(n-m-1)}\rangle + |\psi_{12}^{(n-m-1)}\psi_{11}^{(m)}\rangle), \\
\dot{\mathfrak{Q}}_{11(1)}|\psi_{12}^{(n,1)}\rangle &= \sum_{m=0}^{n-1} \frac{\sqrt{n-m}}{\sqrt{n(n+1)(n-m+1)}}\left(|\phi_{12}^{(m)}\mathcal{F}_{11}^{(n-1-m)}\rangle - |\mathcal{F}_{11}^{(n-1-m)}\phi_{12}^{(m)}\rangle\right) \\
&\quad + \sum_{m=0}^{n-1} \frac{\sqrt{n-m}}{\sqrt{n(n+1)(m+1)}}\left(|\psi_{11}^{(m)}\psi_{12}^{(n-m-1)}\rangle + |\psi_{12}^{(n-m-1)}\psi_{11}^{(m)}\rangle\right). \quad \text{(A.50)}
\end{aligned}$$

Now we are ready to check the commutator of $\dot{\mathfrak{Q}}$ explicitly, using (A.49-A.50). We compute the two $\mathcal{O}(g)$ contributions separately. First, since $\dot{\mathfrak{Q}}_{11(1)}$ annihilates $|\mathcal{F}_{11}^{(n)}\rangle$ we find

$$\begin{aligned}
\{\dot{\mathfrak{Q}}_{11(1)}, \dot{\mathfrak{Q}}_{22(0)}\}|\mathcal{F}_{11}^{(n)}\rangle &= \dot{\mathfrak{Q}}_{11(1)}|\psi_{12}^{(n+1,1)}\rangle \\
&= \sum_{m=0}^{n} \frac{\sqrt{n-m+1}}{\sqrt{(n+1)(n+2)(n-m+2)}}\left(|\phi_{12}^{(m)}\mathcal{F}_{11}^{(n-m)}\rangle - |\mathcal{F}_{11}^{(n-m)}\phi_{12}^{(m)}\rangle\right) \\
&\quad + \frac{\sqrt{n-m+1}}{\sqrt{(n+1)(n+2)(m+1)}}\left(|\psi_{11}^{(m)}\psi_{12}^{(n-m)}\rangle + |\psi_{12}^{(n-m)}\psi_{11}^{(m)}\rangle\right).
\end{aligned}$$

The second contribution is

$$
\begin{aligned}
\{\dot{\mathfrak{Q}}_{11(0)}, \dot{\mathfrak{Q}}_{22(1)}\}|\mathcal{F}_{11}^{(n)}\rangle &= \sqrt{n+1}\dot{\mathfrak{Q}}_{22(1)}|\psi_{11}^{(n+1)}\rangle \\
&+ \sum_{m=0}^{n-1} \frac{\sqrt{(n+2)(m+1)}}{\sqrt{(n+1)(m+2)(n-m)}} \dot{\mathfrak{Q}}_{11(0)}|\mathcal{F}_{11}^{(m)}\psi_{12}^{(n-m-1)}\rangle \\
&- \sum_{m=0}^{n-1} \frac{\sqrt{(n+2)(m+1)}}{\sqrt{(n+1)(m+2)(n-m)}} \dot{\mathfrak{Q}}_{11(0)}|\psi_{12}^{(n-m-1)}\mathcal{F}_{11}^{(m)}\rangle \\
&= \sum_{m=0}^{n} \frac{\sqrt{(n+1)(n-m+1)}}{\sqrt{(n+2)(n-m+2)}} (|\phi_{12}^{(m)}\mathcal{F}_{11}^{(n-m)}\rangle - |\mathcal{F}_{11}^{(n-m)}\phi_{12}^{(m)}\rangle) \\
&- \sum_{m=0}^{n} \frac{\sqrt{(n+1)(m+1)}}{\sqrt{(n+2)(n-m+1)}} (|\psi_{11}^{(m)}\psi_{12}^{(n-m)}\rangle + |\psi_{12}^{(n-m)}\psi_{11}^{(m)}\rangle) \\
&+ \sum_{m=0}^{n} \frac{m\sqrt{(n+2)}}{\sqrt{(n+1)(m+1)(n-m+1)}} (|\psi_{11}^{(m)}\psi_{12}^{(n-m)}\rangle + |\psi_{12}^{(n-m)}\psi_{11}^{(m)}\rangle) \\
&+ \sum_{m=1}^{n} \frac{\sqrt{(n+2)(n-m+1)}}{\sqrt{(n+1)(n-m+2)}} \left(|\mathcal{F}_{11}^{(n-m)}\phi_{12}^{(m)}\rangle - |\phi_{12}^{(m)}\mathcal{F}_{11}^{(n-m)}\rangle\right).
\end{aligned}
$$

We have changed the summation variables for the last two lines to make it easier to collect terms. We find that the two contributions almost completely cancel, with the final result

$$
\{\dot{\mathfrak{Q}}_{11}, \dot{\mathfrak{Q}}_{22}\}_{(1)}|\mathcal{F}_{11}^{(n)}\rangle = |\phi_{12}^{(0)}\mathcal{F}_{11}^{(n)}\rangle - |\mathcal{F}_{11}^{(n)}\phi_{12}^{(0)}\rangle, \tag{A.51}
$$

in agreement with (A.46). It follows that the commutators between $\dot{\mathfrak{Q}}$ also satisfy the algebra, though the restriction to cyclic spin chain states is necessary.

Using hermitian conjugation and the discrete R symmetry, we conclude that all algebra relations except those involving commutators with the $\mathfrak{P}$ or $\mathfrak{K}$ are satisfied. However, these last relations are also satisfied because we can write $\mathfrak{P}$ and $\mathfrak{K}$ as commutators of supercharges. This completes the proof that the solution satisfies the $\mathfrak{psu}(2,2|4)$ algebra at $\mathcal{O}(g)$.

Uniqueness We will prove that the action of $\dot{\mathfrak{Q}}_{14(1)}$ is fixed by symmetry and constraints from the basic structure of Feynman diagrams. Then hermitian conjugation, discrete R symmetry, and closure of the algebra imply that the full $\mathcal{O}(g)$ solution is unique.

First we show that at $\mathcal{O}(g)$ $\mathfrak{Q}$ and $\dot{\mathfrak{Q}}$ must have only one-site to two-site interactions, and $\mathfrak{S}$ and $\dot{\mathfrak{S}}$ must have two-site to one-site interactions. Since a total of three sites are involved in $\mathcal{O}(g)$ interactions, and using symmetry under hermitian conjugation, it suffices to show that $\mathfrak{Q}_{(1)}$ and $\dot{\mathfrak{Q}}_{(1)}$ cannot have two-site to one-site interactions. For these interactions, Lorentz and $\mathfrak{R}$ symmetry, the conservation of classical dimension of generators, and the central charge constraint would require the supercharges to insert or remove at least one $\mathfrak{su}(2)$-singlet combination of $\mathbf{a}$ or $\mathbf{b}$ (i.e. $\varepsilon^{\alpha\beta}\mathbf{a}^\dagger_\alpha\mathbf{a}^\dagger_\beta$). However, such antisymmetric combinations of bosonic oscillators necessarily vanish because there is only one final site ($\varepsilon^{\alpha\beta}\mathbf{a}^\dagger_\alpha(1)\mathbf{a}^\dagger_\beta(1) = 0$).

It follows immediately that the supercharges satisfy

$$\{\mathfrak{Q}_{(1)}, \mathfrak{S}_{(0)}\} = \{\mathfrak{Q}_{(0)}, \mathfrak{S}_{(1)}\} = \{\dot{\mathfrak{Q}}_{(1)}, \dot{\mathfrak{S}}_{(0)}\} = \{\dot{\mathfrak{Q}}_{(0)}, \dot{\mathfrak{S}}_{(1)}\} = 0, \tag{A.52}$$

since the one-site to two-site and two-site to one-site contributions must vanish separately.

Next, we need the following elegant result of Beisert. In [41], he used the fermionic $\mathfrak{sl}(2)$ sector to show that the action of $\dot{\mathfrak{Q}}_{14(1)}$ on the $|\mathcal{D}^n_{22}\Psi_{24}\rangle$ for all nonnegative n is uniquely fixed by symmetry, up to normalization. Now, $\mathfrak{L}$ and $\dot{\mathfrak{L}}^2_1$ must commute with $\dot{\mathfrak{Q}}_{14(1)}$. Also, from the previous discussion we know that $\mathfrak{S}_{(0)}$ and $\dot{\mathfrak{S}}_{(0)}$ commute with $\dot{\mathfrak{Q}}_{14(1)}$. Using this set of generators that commute with $\dot{\mathfrak{Q}}_{14(1)}$, we can map the set of $|\mathcal{D}^n_{22}\Psi_{24}\rangle$ to the full set of single-site states. Therefore, the action of $\dot{\mathfrak{Q}}_{14(1)}$ on all fields is fixed uniquely. As we noted at the beginning of this section, it then follows that the full $\mathcal{O}(g)$ solution for the $\mathfrak{psu}(2,2|4)$ generators is unique up to normalization[3].

[3] If we do not require a hermitian solution, there is freedom for two normalization constants: one for the positive dimension supercharges and one for the negative dimension supercharges.

Appendix B

Multilinear operators

In this appendix we list some relevant multilinear operators for the $\mathfrak{psu}(1,1|2)$ symmetry algebra. These include the quadratic Casimir invariant, but also an interesting triplet of cubic operators. We then show that the cubic operators satisfy the same algebra as the $\mathcal{Y}^{\mathfrak{ab}}$ and can be used to deform the $\mathcal{Y}^{\mathfrak{ab}}$ while preserving this algebra.

B.1 Quadratic invariants

It is straightforward to construct the quadratic Casimir for the maximally extended $\mathfrak{psu}(1,1|2)$ algebra

$$\mathfrak{J}^2 = 2\varepsilon_{\mathfrak{bc}}\varepsilon_{\mathfrak{da}}\mathfrak{B}^{\mathfrak{ab}}\mathfrak{C}^{\mathfrak{cd}} + \varepsilon_{bc}\varepsilon_{da}\mathfrak{R}^{ab}\mathfrak{R}^{cd} - \varepsilon_{\beta\gamma}\varepsilon_{\delta\alpha}\mathfrak{J}^{\alpha\beta}\mathfrak{J}^{\gamma\delta} - \varepsilon_{ad}\varepsilon_{\beta\epsilon}\varepsilon_{\mathfrak{cf}}\mathfrak{Q}^{a\beta\mathfrak{c}}\mathfrak{Q}^{d\epsilon\mathfrak{f}}. \tag{B.1}$$

For the algebra without central extensions, $\mathfrak{C}^{\mathfrak{ab}} = 0$, the first term simply drops out. The centrally extended algebra without outer automorphism, on the other hand, does not have a quadratic invariant because the first term is important, but it requires $\mathfrak{B}^{\mathfrak{ab}}$.

For the maximally extended $\mathfrak{psu}(1|1)^2$ the quadratic Casimir operator reads

$$\hat{\mathfrak{J}}^2 = \varepsilon_{\mathfrak{bc}}\varepsilon_{\mathfrak{da}}\{\hat{\mathfrak{B}}^{\mathfrak{ab}}, \hat{\mathfrak{C}}^{\mathfrak{cd}}\} + \{\hat{\mathfrak{A}}, \hat{\mathfrak{D}}\} - \varepsilon_{\mathfrak{ab}}[\hat{\mathfrak{Q}}^{\mathfrak{a}}, \hat{\mathfrak{S}}^{\mathfrak{b}}]. \tag{B.2}$$

In the combined algebra of $\mathfrak{psu}(1,1|2)$ and $\mathfrak{psu}(1|1)^2$ with identified outer automorphisms $\mathfrak{B}^{\mathfrak{ab}} = \hat{\mathfrak{B}}^{\mathfrak{ab}}$ also the central charges have to be identified, $\mathfrak{C}^{\mathfrak{ab}} = \hat{\mathfrak{C}}^{\mathfrak{ab}}$, in order for a quadratic invariant to exist. This invariant is the sum of (B.1) and (B.2) but with the first term in both expressions appearing only once.

Of course, other invariant quadratic generators include the quadratic combinations of the central charges

$$\mathfrak{C}^2 = \varepsilon_{\mathfrak{bc}}\varepsilon_{\mathfrak{da}}\mathfrak{C}^{\mathfrak{ab}}\mathfrak{C}^{\mathfrak{cd}}, \qquad \hat{\mathfrak{C}}^2 = \varepsilon_{\mathfrak{bc}}\varepsilon_{\mathfrak{da}}\hat{\mathfrak{C}}^{\mathfrak{ab}}\hat{\mathfrak{C}}^{\mathfrak{cd}}, \qquad \hat{\mathfrak{D}}^2. \tag{B.3}$$

B.2 A triplet of cubic $\mathfrak{psu}(1,1|2)$ invariants

Curiously, there exist three cubic $\mathfrak{psu}(1,1|2)$ invariants $(\mathfrak{J}^3)^{\mathfrak{ab}} = (\mathfrak{J}^3)^{\mathfrak{ba}}$ for the algebra without central extensions, $\mathfrak{C}^{\mathfrak{ab}} = 0$,

$$\begin{aligned}(\mathfrak{J}^3)^{\mathfrak{ab}} = {} & \varepsilon_{ce}\varepsilon_{dh}\varepsilon_{\zeta\iota}\mathfrak{R}^{cd}\mathfrak{Q}^{e\zeta\mathfrak{a}}\mathfrak{Q}^{h\iota\mathfrak{b}} + \varepsilon_{eh}\varepsilon_{\gamma\zeta}\varepsilon_{\delta\iota}\mathfrak{J}^{\gamma\delta}\mathfrak{Q}^{e\zeta\mathfrak{a}}\mathfrak{Q}^{h\iota\mathfrak{b}} \\ & + \varepsilon_{de}\varepsilon_{fc}\mathfrak{B}^{\mathfrak{ab}}\mathfrak{R}^{cd}\mathfrak{R}^{ef} - \varepsilon_{\delta\epsilon}\varepsilon_{\zeta\gamma}\mathfrak{B}^{\mathfrak{ab}}\mathfrak{J}^{\gamma\delta}\mathfrak{J}^{\epsilon\zeta} - \varepsilon_{cf}\varepsilon_{\delta\eta}\varepsilon_{\mathfrak{eh}}\mathfrak{B}^{\mathfrak{ab}}\mathfrak{Q}^{c\delta\mathfrak{e}}\mathfrak{Q}^{f\eta\mathfrak{h}}.\end{aligned} \tag{B.4}$$

They transform as a triplet under $\mathfrak{B}$ and commute with the $\mathfrak{psu}(1,1|2)$ algebra. These cubic generators are important for the multiplet structure in the algebra with the outer automorphism. For a multiplet of the extended algebra, the highest weight states of $\mathfrak{psu}(1,1|2)$ form a multiplet of $\mathfrak{su}(2)$. To move about in this multiplet one cannot simply use the $\mathfrak{su}(2)$ generators $\mathfrak{B}^{\mathfrak{ab}}$ because they do not commute with $\mathfrak{psu}(1,1|2)$. Instead, the cubic generators map between highest weight states of $\mathfrak{psu}(1,1|2)$, i.e. they can be understood as $\mathfrak{su}(2)$ ladder generators.

B.3 A one-parameter deformation of the $\mathcal{Y}$

First we show that the $(\mathfrak{J}^3)^{\mathfrak{ab}}$ generate the same algebra as the $\mathcal{Y}^{\mathfrak{ab}}$. We define,

$$(\tilde{\mathfrak{J}}^3)^{\mathfrak{ab}} = \varepsilon_{ce}\varepsilon_{dh}\varepsilon_{\zeta\iota}\mathfrak{R}^{cd}\mathfrak{Q}^{e\zeta\mathfrak{a}}\mathfrak{Q}^{h\iota\mathfrak{b}} + \varepsilon_{eh}\varepsilon_{\gamma\zeta}\varepsilon_{\delta\iota}\mathfrak{J}^{\gamma\delta}\mathfrak{Q}^{e\zeta\mathfrak{a}}\mathfrak{Q}^{h\iota\mathfrak{b}}. \tag{B.5}$$

Then $(\tilde{\mathfrak{J}}^3)^{\mathfrak{ab}}$ gives the difference between $(\mathfrak{J}^3)^{\mathfrak{ab}}$ and the product of $\mathfrak{B}^{\mathfrak{ab}}$ and the quadratic Casimir (with $\mathfrak{C}^{\mathfrak{ab}} = 0$),

$$(\mathfrak{J}^3)^{\mathfrak{ab}} = (\tilde{\mathfrak{J}}^3)^{\mathfrak{ab}} + \mathfrak{J}^2\mathfrak{B}^{\mathfrak{ab}}. \tag{B.6}$$

Now, $(\mathfrak{J}^3)^{\mathfrak{ab}}$ commutes with ordinary $\mathfrak{psu}(1,1|2)$ generators, and the $\tilde{\mathfrak{J}}^3$ are products of ordinary $\mathfrak{psu}(1,1|2)$ generators only. Therefore, the commutator of two nonidentical $\mathfrak{J}^3$ generators yields simply a product of the quadratic Casimir and a $\mathfrak{J}^3$,

$$[(\mathfrak{J}^3)^{\mathfrak{ab}}, (\mathfrak{J}^3)^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}\mathfrak{J}^2(\mathfrak{J}^3)^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}\mathfrak{J}^2(\mathfrak{J}^3)^{\mathfrak{cb}}. \tag{B.7}$$

From this, it is straightforward to obtain the entire algebra generated by the cubic invariants. Define $(\mathfrak{J}^3_0)^{\mathfrak{ab}} = \mathfrak{B}^{\mathfrak{ab}}$ and

$$(\mathfrak{J}^3_n)^{\mathfrak{ab}} = \left(\mathfrak{J}^2\right)^{n-1}(\mathfrak{J}^3)^{\mathfrak{ab}}, \quad n \geq 1. \tag{B.8}$$

It only takes a short computation to show that these $\mathfrak{J}^3_n$ satisfy the same algebra as the $\mathcal{Y}_n$,

$$[(\mathfrak{J}^3_m)^{\mathfrak{ab}}, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] = \varepsilon^{\mathfrak{cb}}(\mathfrak{J}^3_{m+n})^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}}(\mathfrak{J}^3_{m+n})^{\mathfrak{cb}}. \tag{B.9}$$

For n or m equal to 0, this algebra is satisfied since the quadratic Casimir commutes even with $\mathfrak{B}^{\mathfrak{ab}}$. Assuming n and m are greater than 0, we substitute the definition (B.8) to obtain

$$\begin{aligned}
[(\mathfrak{J}^3_m)^{\mathfrak{ab}}, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] &= [\left(\mathfrak{J}^2\right)^{m-1} (\mathfrak{J}^3)^{\mathfrak{ab}}, \left(\mathfrak{J}^2\right)^{n-1} (\mathfrak{J}^3)^{\mathfrak{cd}}] \\
&= \left(\mathfrak{J}^2\right)^{n+m-2} [(\mathfrak{J}^3)^{\mathfrak{ab}}, (\mathfrak{J}^3)^{\mathfrak{cd}}] \\
&= \varepsilon^{\mathfrak{cb}} \left(\mathfrak{J}^2\right)^{n+m-1} (\mathfrak{J}^3)^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}} \left(\mathfrak{J}^2\right)^{n+m-1} (\mathfrak{J}^3)^{\mathfrak{cb}} \\
&= \varepsilon^{\mathfrak{cb}} (\mathfrak{J}^3_{n+m})^{\mathfrak{ad}} - \varepsilon^{\mathfrak{ad}} (\mathfrak{J}^3_{n+m})^{\mathfrak{cb}}, \qquad \text{(B.10)}
\end{aligned}$$

as required. We used the vanishing commutator between $\mathfrak{J}^2$ and $(\mathfrak{J}^3)^{\mathfrak{ab}}$, and (B.7). It is interesting that the role of the quadratic Casimir operator here resembles that of $\hat{\mathfrak{D}}$ in the $\mathcal{Y}$ algebra for cyclic states above (2.69).

We are now ready to present the deformation of the $\mathcal{Y}^{\mathfrak{ab}}_n$, which we label $\hat{\mathcal{Y}}^{\mathfrak{ab}}_n$. We parameterize the deformation with α. $\hat{\mathcal{Y}}^{\mathfrak{ab}}_0$ is still given by $\mathfrak{B}^{\mathfrak{ab}}$, but all other generators become

$$\hat{\mathcal{Y}}^{\mathfrak{ab}}_n = \alpha^n (\mathfrak{J}^3_n)^{\mathfrak{ab}} + \sum_{m=0}^{n-1} \alpha^m \binom{n}{m} \left(\mathfrak{J}^2\right)^m \mathcal{Y}^{\mathfrak{ab}}_{n-m}, \quad n \geq 1. \qquad \text{(B.11)}$$

Again, the algebra relations take the same form,

$$[\hat{\mathcal{Y}}^{\mathfrak{ab}}_m, \hat{\mathcal{Y}}^{\mathfrak{cd}}_n] = \varepsilon^{\mathfrak{cb}} \hat{\mathcal{Y}}^{\mathfrak{ad}}_{m+n} - \varepsilon^{\mathfrak{ad}} \hat{\mathcal{Y}}^{\mathfrak{cb}}_{m+n}. \qquad \text{(B.12)}$$

The commutators with n or m equal to zero are again satisfied because $\mathfrak{J}^2$ commutes with $\mathfrak{B}^{\mathfrak{ab}}$. In order to check the relations with $m = 1$, we need the commutators between the $(\mathfrak{J}^3_n)^{\mathfrak{ab}}$ and the $\mathcal{Y}^{\mathfrak{ab}}_1$. The vanishing commutator between the $\mathcal{Y}$ and the ordinary $\mathfrak{psu}(1,1|2)$ generators implies

$$\begin{aligned}
&[\mathcal{Y}^{\mathfrak{ab}}_1, (\mathfrak{J}^3)^{\mathfrak{cd}}] = \mathfrak{J}^2 \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_1 - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_1\right), \qquad [(\mathfrak{J}^3)^{\mathfrak{ab}}, \mathcal{Y}^{\mathfrak{cd}}_1] = \mathfrak{J}^2 \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_1 - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_1\right), \\
&[\mathcal{Y}^{\mathfrak{ab}}_1, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] = \left(\mathfrak{J}^2\right)^n \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_1 - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_1\right), \quad [(\mathfrak{J}^3_n)^{\mathfrak{ab}}, \mathcal{Y}^{\mathfrak{cd}}_1] = \left(\mathfrak{J}^2\right)^n \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_1 - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_1\right). \quad \text{(B.13)}
\end{aligned}$$

Now we expand the left side of the relations (B.12) with $m = 1$ using (B.11). We simplify using (B.13) and the algebras of the $(\mathfrak{J}^3_n)^{\mathfrak{ab}}$ and of the $\mathcal{Y}^{\mathfrak{ab}}_n$,

$$\begin{aligned}
[\hat{\mathcal{Y}}^{\mathfrak{ab}}_1, \hat{\mathcal{Y}}^{\mathfrak{cd}}_n] &= \alpha^n [\mathcal{Y}^{\mathfrak{ab}}_1, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] + \alpha^{n+1}[(\mathfrak{J}^3)^{\mathfrak{ab}}, (\mathfrak{J}^3_n)^{\mathfrak{cd}}] \\
&\quad + \sum_{m=0}^{n-1} \alpha^m \binom{n}{m} \left([\mathcal{Y}^{\mathfrak{ab}}_1, \left(\mathfrak{J}^2\right)^m \mathcal{Y}^{\mathfrak{cd}}_{n-m}] + \alpha [(\mathfrak{J}^3)^{\mathfrak{ab}}, \left(\mathfrak{J}^2\right)^m \mathcal{Y}^{\mathfrak{cd}}_{n-m}] \right) \\
&= \alpha^n \varepsilon^{\mathfrak{cb}} \left(\mathfrak{J}^2\right)^n \mathcal{Y}^{\mathfrak{ad}}_1 - \alpha^n \varepsilon^{\mathfrak{ad}} \left(\mathfrak{J}^2\right)^n \mathcal{Y}^{\mathfrak{cb}}_1 + \alpha^{n+1} \varepsilon^{\mathfrak{cb}} (\mathfrak{J}^3_{n+1})^{\mathfrak{ad}} - \alpha^{n+1} \varepsilon^{\mathfrak{ad}} (\mathfrak{J}^3_{n+1})^{\mathfrak{cb}} \\
&\quad + \sum_{m=0}^{n-1} \alpha^m \binom{n}{m} \left(\mathfrak{J}^2\right)^m \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_{n+1-m} - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_{n+1-m} \right) \\
&\quad + \sum_{m=0}^{n-1} \alpha^{m+1} \binom{n}{m} \left(\mathfrak{J}^2\right)^{m+1} \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_{n-m} - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_{n-m} \right) \\
&= \alpha^{n+1} \varepsilon^{\mathfrak{cb}} (\mathfrak{J}^3_{n+1})^{\mathfrak{ad}} - \alpha^{n+1} \varepsilon^{\mathfrak{ad}} (\mathfrak{J}^3_{n+1})^{\mathfrak{cb}} \\
&\quad + \sum_{m=0}^{n} \alpha^m \binom{n+1}{m} \left(\mathfrak{J}^2\right)^m \left(\varepsilon^{\mathfrak{cb}} \mathcal{Y}^{\mathfrak{ad}}_{n+1-m} - \varepsilon^{\mathfrak{ad}} \mathcal{Y}^{\mathfrak{cb}}_{n+1-m} \right) \\
&= \varepsilon^{\mathfrak{cb}} \hat{\mathcal{Y}}^{\mathfrak{ad}}_{n+1} - \varepsilon^{\mathfrak{ad}} \hat{\mathcal{Y}}^{\mathfrak{cb}}_{n+1}.
\end{aligned} \tag{B.14}$$

We combined terms to reach the second-to-last expression, and substituted the definition (B.11) for the last line. The calculation proceeds in parallel for (B.12) with $n = 1$. Since this algebra's Serre relations are the level three equations, which have n or m equal to one, it follows that (B.12) is satisfied.

Appendix C

Symmetries of the bilocal generators $\mathcal{Y}$

In this appendix, we prove that the $\mathcal{Y}^{\mathfrak{ab}}$ commute with the $\mathfrak{psu}(1,1|2)$ generators, including the one-loop dilatation generator. The proofs can be modified straightforwardly to show the same for $\mathcal{X}$.

As in Chapter 2, since we work only at leading order the $\mathfrak{psu}(1,1|2)$ generators $\mathfrak{Q}, \mathfrak{J}$ are truncated at $\mathcal{O}(g^0)$ and the $\mathfrak{psu}(1|1)^2$ generators $\hat{\mathfrak{Q}}, \hat{\mathfrak{S}}$ only act with $\hat{\mathfrak{Q}}_{(1)}, \hat{\mathfrak{S}}_{(1)}$.

C.1 $\mathfrak{psu}(1,1|2)$ invariance

We now prove that $\mathcal{Y}^{\mathfrak{ab}}$ commutes with the classical $\mathfrak{psu}(1,1|2)$ generators. It is sufficient to prove that the commutator with the $\mathfrak{Q}$ vanish since the $\mathfrak{Q}$ generate the complete algebra. Furthermore, using $\mathfrak{B}$ symmetry, it is sufficient to prove this for $\mathcal{Y}^{<<}$. Now, $\mathfrak{Q}^{a\beta<}$ commute exactly with the $\hat{\mathfrak{Q}}^{<}$ and $\hat{\mathfrak{S}}^{<}$, so it is clear that these commutators vanish. However, it is nontrivial to show that the $\mathfrak{Q}^{a+>}$ commute with $\mathcal{Y}^{<<}$, since they only commute with $\hat{\mathfrak{Q}}^{<}$

up to a gauge transformation (2.45)

$$\{\mathfrak{Q}^{a+>}, \hat{\mathfrak{Q}}^{<}\}|X\rangle = \frac{\varepsilon^{ab}}{\sqrt{2}}|X\phi_b^{(0)}\rangle - \frac{\varepsilon^{ab}}{\sqrt{2}}|\phi_b^{(0)}X\rangle = \check{Z}^a(2) - \check{Z}^a(1). \tag{C.1}$$

Again we use the notation $\check{Z}^a(i)$ for the insertion of a bosonic field at a new site between the original sites i and $i+1$. It will be useful to note that we can use $\mathcal{U}$ to change the site indices of any generator that acts on site i and any number of following sites,

$$X(i+1\ldots) = \mathcal{U}X(i\ldots)\mathcal{U}^{-1}. \tag{C.2}$$

We are now ready to check the commutator directly. We use that the $\mathfrak{Q}^{a+>}$ still commute exactly with $\hat{\mathfrak{S}}^{<}$ and apply (C.1) and (C.2),

$$\begin{aligned}
[\mathfrak{Q}^{a+>}, \mathcal{Y}^{<<}] &= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \frac{\delta_{i,0}}{2} - \frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,(\check{Z}^a(1) - \check{Z}^a(2))\mathcal{U}^{-j} \\
&= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \frac{\delta_{i,0}}{2} - \frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,\check{Z}^a(1)\mathcal{U}^{-j} \\
&\quad - \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \frac{\delta_{i,0}}{2} - \frac{\delta_{i,L+1}}{2})\mathcal{U}^{(j+1)-(i+1)}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i+1}\,\check{Z}^a(1)\mathcal{U}^{-(j+1)} \\
&= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \frac{\delta_{i,0}}{2} - \frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,\check{Z}^a(1)\mathcal{U}^{-j} \\
&\quad - \sum_{j=1}^{L}\sum_{i=1}^{L+2}(1 - \frac{\delta_{i,1}}{2} - \frac{\delta_{i,L+2}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,\check{Z}^a(1)\mathcal{U}^{-j}
\end{aligned} \tag{C.3}$$

We shifted summation variables to obtain the last line, $i \to (i+1)$ and $j \to (j+1)$. Since the chain is of length L initially and after the application of the commutator, $j = L$ is equivalent to $j = 0$. Now we can combine the two lines (being careful with the different ranges for i) and simplify,

$$
\begin{aligned}
[\mathfrak{Q}^{a+>}, \mathcal{Y}^{<<}] &= \sum_{j=0}^{L-1}\sum_{i=0}^{L+2}\Big[((1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2}-\delta_{i,L+2})-(1-\frac{\delta_{i,1}}{2}-\frac{\delta_{i,L+2}}{2}-\delta_{i,0}))\\
&\quad\times\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}^{i}\,\check{Z}^{a}(1)\mathcal{U}^{-j}\Big]\\
&= \frac{1}{2}\sum_{j=0}^{L-1}(\mathcal{U}^{j-1}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}\,\check{Z}^{a}(1)\mathcal{U}^{-j}+\mathcal{U}^{j}\,\hat{\mathfrak{S}}^{<}(1,2)\,\check{Z}^{a}(1)\mathcal{U}^{-j}\\
&-\mathcal{U}^{j-2}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}\,\check{Z}^{a}(1)\mathcal{U}^{-j}-\mathcal{U}^{j-1}\,\hat{\mathfrak{S}}^{<}(1,2)\,\check{Z}^{a}(1)\mathcal{U}^{-j})\\
&= \frac{1}{2}(1-\mathcal{U}^{-1})\sum_{j=0}^{L-1}\mathcal{U}^{j}(\mathcal{U}^{-1}\,\hat{\mathfrak{S}}^{1}(1,2)\,\mathcal{U}\,\check{Z}^{a}(1)+\hat{\mathfrak{S}}^{<}(1,2)\,\check{Z}^{a}(1))\mathcal{U}^{-j}. \qquad \text{(C.4)}
\end{aligned}
$$

To reach the middle expressions, we used that the length of the chain is $L+1$ after $\check{Z}^{a}$ acts. The expression in parenthesis inside the sum in the last line gives a chain derivative by parity. To see this, we write the chain with site $0 = L$ first:

$$
\begin{aligned}
&(\mathcal{U}^{-1}\,\hat{\mathfrak{S}}^{<}(1,2)\,\mathcal{U}\,\check{Z}^{a}(1)+\hat{\mathfrak{S}}^{<}(1,2)\,\check{Z}^{a}(1))|Y_0\,Y_1\,Y_2\ldots\rangle =\\
&\frac{\varepsilon^{ab}}{2}(\mathfrak{S}^{<}(0,1)|Y_0\phi_b^{(0)}\,Y_1\,Y_2\ldots\rangle+\mathfrak{S}^{<}(1,2)|Y_0\,\phi_b^{(0)}\,Y_1\,Y_2\ldots\rangle) =\\
&\frac{\varepsilon^{ab}}{2}(-\mathfrak{S}^{<}(0,1)|\phi_b^{(0)}Y_0\,Y_1\,Y_2\ldots\rangle+\mathfrak{S}^{<}(1,2)|Y_0\,\phi_b^{(0)}\,Y_1\,Y_2\ldots\rangle). \qquad \text{(C.5)}
\end{aligned}
$$

We used parity to reach the last line. Since this term acts homogeneously on the chain, the first and second terms cancel.

The proof for the $\mathfrak{Q}^{a-2}$ is similar. They only commute with $\hat{\mathfrak{S}}^{<}$ up to the gauge transformations (2.45)

$$
\begin{aligned}
\{\mathfrak{Q}^{a->}, \hat{\mathfrak{S}}^{<}\}|X\phi_b^{(0)}\rangle &= -\frac{\delta_b^a}{\sqrt{2}}|X\rangle = -\hat{Z}^{a}(2),\\
\{\mathfrak{Q}^{a->}, \hat{\mathfrak{S}}^{<}\}|\phi_b^{(0)}X\rangle &= \frac{\delta_b^a}{\sqrt{2}}|X\rangle = \hat{Z}^{a}(1). \qquad \text{(C.6)}
\end{aligned}
$$

Here we have defined $\hat{Z}^{a}(i)$. Since the $\mathfrak{Q}^{a->}$ commute exactly with the $\hat{\mathfrak{Q}}^{<}$, again using

(C.2) to shift site indices we find

$$[\mathfrak{Q}^{a-\>}, \mathcal{Y}^{<<}] = \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1 - \frac{\delta_{i,0}}{2} - \frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,(\hat{Z}^a(1) - \hat{Z}^a(2))\,\mathcal{U}^i\,\hat{\mathfrak{Q}}^{<}(1)\mathcal{U}^{-j}$$
$$= \frac{1}{2}(\mathcal{U}^{-1} - 1)\sum_{j=0}^{L-1}\mathcal{U}^j(\mathcal{U}\,\hat{Z}^a(1)\,\mathcal{U}^{-1}\,\hat{\mathfrak{Q}}^{<}(1) + \hat{Z}^a(1)\,\hat{\mathfrak{Q}}^{<}(1))\mathcal{U}^{-j}. \quad \text{(C.7)}$$

Again, the term in parenthesis is a chain derivative by parity. This completes the proof that the $\mathfrak{Q}$ commute with $\mathcal{Y}^{<<}$. It follows by $\mathfrak{B}$ and $\mathfrak{psu}(1,1|2)$ symmetry that the $\mathcal{Y}^{\mathfrak{ab}}$ commute with all of the classical $\mathfrak{psu}(1,1|2)$ generators.

It is clear from the above proof that $\mathcal{X}$ (2.67) also commutes with the classical $\mathfrak{psu}(1,1|2)$ generators, since the bilocal product of $\hat{\mathfrak{S}}^{<}$ and $\hat{\mathfrak{Q}}^{>}$ (or $\hat{\mathfrak{S}}^{>}$ and $\hat{\mathfrak{Q}}^{<}$) by itself commutes.

C.2 Conservation

To prove that the $\mathcal{Y}$ commute with the Hamiltonian, $\hat{\mathfrak{D}}$, we first need to consider how the $\mathfrak{psu}(1|1)^2$ generators commute with the Hamiltonian. Locally, we have

$$\hat{\mathfrak{D}} = \{\hat{\mathfrak{Q}}^{<}, \hat{\mathfrak{S}}^{>}\} + \text{chain derivative},$$
$$= -\{\hat{\mathfrak{Q}}^{>}, \hat{\mathfrak{S}}^{<}\} + \text{chain derivative},$$
$$= \tfrac{1}{2}\delta\mathfrak{D}_2 + \text{chain derivative}. \quad \text{(C.8)}$$

Here "locally" refers to the interactions that are summed over the length of the chain. For instance, the local expression for the one-loop commutators expand as one-site to one-site and two-site to two-site interactions,

$$\{\hat{\mathfrak{Q}}^{\mathfrak{a}}, \hat{\mathfrak{S}}^{\mathfrak{b}}\} = (\hat{\mathfrak{S}}^{\mathfrak{b}}(1,2)\hat{\mathfrak{Q}}^{\mathfrak{a}}(1)) + (\hat{\mathfrak{Q}}^{\mathfrak{a}}(1)\hat{\mathfrak{S}}^{\mathfrak{b}}(1,2) + \hat{\mathfrak{S}}^{\mathfrak{b}}(2,3)\hat{\mathfrak{Q}}^{\mathfrak{a}}(1) + \hat{\mathfrak{S}}^{\mathfrak{b}}(1,2)\hat{\mathfrak{Q}}^{\mathfrak{a}}(2)). \quad \text{(C.9)}$$

The term inside the first parenthesis is one-site to one-site, and the remaining terms are two-site to two-site. As explained in Section 2.2.2, a chain derivative summed over the length of a periodic chain gives zero, so when we commute $\mathcal{Y}^{\mathfrak{ab}}$ with the Hamiltonian, we can use any of the equivalent forms in (C.8) as long as each one acts homogeneously on the chain. We will use this freedom to always commute any $\mathfrak{psu}(1,1)^2$ generator with the commutator in (C.8) that involving the same generator. Therefore, it will be convenient to define

$$\begin{aligned} \mathfrak{D}_L &= \{\hat{\mathfrak{Q}}^<, \hat{\mathfrak{S}}^>\} \\ \mathfrak{D}_R &= -\{\hat{\mathfrak{Q}}^>, \hat{\mathfrak{S}}^<\}. \end{aligned} \tag{C.10}$$

Furthermore, the $\mathfrak{D}_L$ and $\mathfrak{D}_R$ split into local one-site to one-site and two-site to two-site interactions (C.9). Then we have the exact local equalities only involving the two-site to two-site interactions of $\mathfrak{D}_L$ and $\mathfrak{D}_R$,

$$\begin{aligned} [\hat{\mathfrak{Q}}^<(i), \mathfrak{D}_L] &= \hat{\mathfrak{q}}^<(i-1,i) - \hat{\mathfrak{q}}^<(i,i+1), \\ [\hat{\mathfrak{Q}}^>(i), \mathfrak{D}_R] &= \hat{\mathfrak{q}}^>(i-1,i) - \hat{\mathfrak{q}}^>(i,i+1), \\ \hat{\mathfrak{q}}^<(i-1,i) &= \hat{\mathfrak{Q}}^<(i)\mathfrak{D}_L(i-1,i) - \mathfrak{D}_L(i-1,i)\hat{\mathfrak{Q}}^<(i) \\ \hat{\mathfrak{q}}^>(i-1,i) &= \hat{\mathfrak{Q}}^>(i)\mathfrak{D}_R(i-1,i) - \mathfrak{D}_R(i-1,i)\hat{\mathfrak{Q}}^>(i). \end{aligned} \tag{C.11}$$

Note that $\hat{\mathfrak{q}}(i,i+1)$ is a two-site to three-site interaction, with final sites $(i,i+1,i+2)$. These equalities can be shown easily by expanding $\mathfrak{D}_L$ and $\mathfrak{D}_R$ and using the fact that $(\hat{\mathfrak{Q}}^{\mathfrak{a}})^2 = 0$ is even satisfied on a one-site chain:

$$(\hat{\mathfrak{Q}}^{\mathfrak{a}}(1) + \hat{\mathfrak{Q}}^{\mathfrak{a}}(2))\hat{\mathfrak{Q}}^{\mathfrak{a}}(1) = 0 \quad \text{(no sum)}. \tag{C.12}$$

Similarly, we have

$$[\hat{\mathfrak{S}}^{<}(i,i+1),\mathfrak{D}_R] = \hat{\mathfrak{s}}^{<}(i-1,i,i+1) - \hat{\mathfrak{s}}^{<}(i,i+1,i+2),$$
$$[\hat{\mathfrak{S}}^{>}(i,i+1),\mathfrak{D}_L] = \hat{\mathfrak{s}}^{>}(i-1,i,i+1) - \hat{\mathfrak{s}}^{>}(i,i+1,i+2),$$
$$\hat{\mathfrak{S}}^{<}(i-1,i,i+1) = \hat{\mathfrak{S}}^{<}(i,i+1)\mathfrak{D}_R(i-1,i) - \mathfrak{D}_R(i-1,i)\hat{\mathfrak{S}}^{<}(i,i+1)$$
$$\hat{\mathfrak{s}}^{>}(i-1,i,i+1) = \hat{\mathfrak{S}}^{>}(i,i+1)\mathfrak{D}_L(i-1,i) - \mathfrak{D}_L(i-1,i)\hat{\mathfrak{S}}^{>}(i,i+1). \tag{C.13}$$

$\hat{\mathfrak{s}}(i,i+1,i+2)$ is a three-site to two-site interaction, with final sites $(i,i+1)$. Now, using these commutation relations, and the identities that follow from (C.2)

$$\hat{\mathfrak{q}}^{\mathfrak{a}}(i-1,i) = \mathcal{U}^{-1}\,\hat{\mathfrak{q}}^{\mathfrak{a}}(i,i+1)\,\mathcal{U},$$
$$\hat{\mathfrak{s}}^{\mathfrak{a}}(i-1,i,i+1) = \mathcal{U}^{-1}\,\hat{\mathfrak{s}}^{\mathfrak{a}}(i,i+1,i+2)\,\mathcal{U}, \tag{C.14}$$

we find

$$\begin{aligned}
[\hat{\mathfrak{D}},\mathcal{Y}^{\mathfrak{ab}}] &= \sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,(\hat{\mathfrak{s}}^{\mathfrak{a}}(0,1,2)-\hat{\mathfrak{s}}^{\mathfrak{a}}(1,2,3))\,\mathcal{U}^{i}\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1)\mathcal{U}^{-j}\\
&+\sum_{j=0}^{L-1}\sum_{i=0}^{L+1}(1-\frac{\delta_{i,0}}{2}-\frac{\delta_{i,L+1}}{2})\mathcal{U}^{j-i}\,\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\mathcal{U}^{i}\,(\hat{\mathfrak{q}}^{\mathfrak{b}}(0,1)-\hat{\mathfrak{q}}^{\mathfrak{b}}(1,2))\,\mathcal{U}^{-j}\\
&= -\frac{1}{2}(1-\mathcal{U}^{-1})\sum_{j=0}^{L-1}\mathcal{U}^{j}\,(\hat{\mathfrak{s}}^{\mathfrak{a}}(1,2,3)\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1)+\mathcal{U}^{-1}\,\hat{\mathfrak{s}}^{\mathfrak{a}}(1,2,3)\,\mathcal{U}\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1))\mathcal{U}^{-j}\\
&+\frac{1}{2}(1-\mathcal{U}^{-1})\sum_{j=0}^{L-1}\mathcal{U}^{j}\,(\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\,\hat{\mathfrak{q}}^{\mathfrak{b}}(1,2)+\mathcal{U}\,\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\,\mathcal{U}^{-1}\,\hat{\mathfrak{q}}^{\mathfrak{b}}(1,2))\mathcal{U}^{-j}.
\end{aligned} \tag{C.15}$$

To complete the proof, we will now show that this vanishes since it is a homogeneous sum of a chain derivative. Equivalently,

$$\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\,\hat{\mathfrak{q}}^{\mathfrak{b}}(1,2)+\mathcal{U}\,\hat{\mathfrak{S}}^{\mathfrak{a}}(1,2)\,\mathcal{U}^{-1}\,\hat{\mathfrak{q}}^{\mathfrak{b}}(1,2)-\hat{\mathfrak{s}}^{\mathfrak{a}}(1,2,3)\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1)-\mathcal{U}^{-1}\,\hat{\mathfrak{s}}^{\mathfrak{a}}(1,2,3)\,\mathcal{U}\,\hat{\mathfrak{Q}}^{\mathfrak{b}}(1), \tag{C.16}$$

acts as a chain derivative on sites 1 and 2.

First we simplify the first term. For simplicity, we consider the $(<<)$ component. By definition and using the two-site to two-site interactions of the defining commutator of $\mathfrak{D}_L$

(C.10), we have

$$\hat{\mathfrak{q}}^{<}(1,2) = \hat{\mathfrak{Q}}^{<}(2)\mathfrak{D}_L(1,2) - \mathfrak{D}_L(1,2)\hat{\mathfrak{Q}}^{<}(2)$$
$$= \hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2) + \hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2) + \hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)$$
$$- \hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2) - \hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{Q}}^{<}(2) - \hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{Q}}^{<}(2). \tag{C.17}$$

Now, in the second term of the last expression (on the second-to-last line), we can switch the order of $\hat{\mathfrak{Q}}^{<}(2)$ and $\hat{\mathfrak{S}}^{<}(1,2)$ (with a minus sign) since these two operators do not act on any shared sites, but being careful with site indices, we must use $\hat{\mathfrak{Q}}^{<}(3)$ instead. Then, by the identity (C.12) that $\hat{\mathfrak{Q}}^2 = 0$ even on one site, we find that the second term and the fifth term cancel, and we are left with the simpler expression

$$\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)+\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)-\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2)-\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{Q}}^{<}(2). \tag{C.18}$$

Now the first two terms of (C.16) can be written as

$$\hat{\mathfrak{S}}^{<}(1,2)\,\hat{\mathfrak{q}}^{<}(1,2) + \hat{\mathfrak{S}}^{<}(2,3)\,\hat{\mathfrak{q}}^{<}(1,2). \tag{C.19}$$

The contributions from the first term of (C.18) cancel using (C.12) and the identity[1]

$$(\hat{\mathfrak{S}}^{<}(1,2)\,\hat{\mathfrak{Q}}^{<}(1) - \hat{\mathfrak{S}}^{<}(2,3)\,\hat{\mathfrak{Q}}^{<}(2))\hat{\mathfrak{Q}}^{<}(1) = 0 \tag{C.20}$$

So we are left with the following six terms for (C.19) (the first two terms of (C.16))

$$\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1) - \hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2) -$$
$$\hat{\mathfrak{S}}^{<}(1,2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{Q}}^{<}(2) + \hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{Q}}^{<}(2)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1) -$$
$$\hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{S}}^{>}(1,2)\hat{\mathfrak{Q}}^{<}(2) - \hat{\mathfrak{S}}^{<}(2,3)\hat{\mathfrak{S}}^{>}(2,3)\hat{\mathfrak{Q}}^{<}(1)\hat{\mathfrak{Q}}^{<}(2). \tag{C.21}$$

[1] This identity can be proved without too much difficulty. The second term is minus the parity image of the first term, so one just needs to check that the first term is parity even. This can be done with a short computation because, by $\mathfrak{B}$ charge conservation, the only possible interactions are $\sim |\psi_{<}\rangle \to |\psi_{>}\psi_{>}\rangle$.

Similar steps can be used for the last two terms of (C.16). We find

$$\hat{\mathfrak{S}}^<(2,3)\hat{\mathfrak{Q}}^>(1)\hat{\mathfrak{S}}^<(1,2)\hat{\mathfrak{Q}}^<(1)+\hat{\mathfrak{S}}^<(2,3)\hat{\mathfrak{S}}^<(1,2)\hat{\mathfrak{Q}}^>(2)\hat{\mathfrak{Q}}^<(1)-$$
$$\hat{\mathfrak{S}}^<(1,2)\hat{\mathfrak{Q}}^>(2)\hat{\mathfrak{S}}^<(2,3)\hat{\mathfrak{Q}}^<(1)+\hat{\mathfrak{S}}^<(2,3)\hat{\mathfrak{Q}}^>(1)\hat{\mathfrak{S}}^<(1,2)\hat{\mathfrak{Q}}^<(2)+$$
$$\hat{\mathfrak{S}}^<(2,3)\hat{\mathfrak{S}}^<(1,2)\hat{\mathfrak{Q}}^>(2)\hat{\mathfrak{Q}}^<(2)-\hat{\mathfrak{S}}^<(1,2)\hat{\mathfrak{Q}}^>(2)\hat{\mathfrak{S}}^<(2,3)\hat{\mathfrak{Q}}^<(2). \tag{C.22}$$

Recall that we need to show that (C.16) is a gauge transformation. (C.16) is the sum of (C.21) and (C.22). At this point, it is necessary to explicitly expand these terms as a sum of interactions. However, we can use discrete symmetries to greatly reduce the amount of computation. Under the discrete transformation R that acts as[2]

$$R|\psi_<^{(n)}\rangle=|\psi_>^{(n)}\rangle,\quad R|\psi_>^{(n)}\rangle=|\psi_<^{(n)}\rangle,\quad R|\phi_1^{(n)}\rangle=|\phi_1^{(n)}\rangle,\quad R|\phi_2^{(n)}\rangle=-|\phi_2^{(n)}\rangle, \tag{C.23}$$

the supercharges transform as

$$R\hat{\mathfrak{Q}}^<R^{-1}=-\hat{\mathfrak{Q}}^>,\quad R\hat{\mathfrak{S}}^<R^{-1}=\hat{\mathfrak{S}}^>, \tag{C.24}$$

as can be confirmed by examining the expressions for the $\mathfrak{psu}(1|1)^2$ generators (2.34). Then under the combined operation

$$X\to RX^\dagger R^{-1}, \tag{C.25}$$

(C.21) transforms into minus (C.22) (term by term). Also, (C.21) and (C.22) are both parity odd. Using these discrete symmetries, as well as $\mathfrak{R}$ symmetry and conservation of $\mathfrak{B}$ charge, one can infer the complete action of (C.16) by computing the following four types

[2]This transformation is similar but different from the discrete R transformation presented in Appendix A.

of interactions:

$$
\begin{aligned}
&|\phi_1^{(n)}\phi_2^{(m)}\rangle \to \sum_{k=0}^{n+m-1} f_1(n,m,k)|\psi_>^{(k)}\psi_>^{(n+m-k-1)}\rangle, \\
&|\phi_1^{(n)}\psi_<^{(m)}\rangle \to \sum_{k=0}^{n+m} f_2(n,m,k)|\phi_1^{(k)}\psi_>^{(n+m-k)}\rangle + f_3(n,m,k)\sum_{k=0}^{n+m}|\psi_>^{(k)}\phi_1^{(n+m-k)}\rangle, \\
&|\psi_<^{(n)}\psi_<^{(m)}\rangle \to \sum_{k=0}^{n+m} f_4(n,m,k)|\psi_<^{(k)}\psi_>^{(n+m-k)}\rangle.
\end{aligned}
\tag{C.26}
$$

Completing this still lengthy computation, and applying the known symmetries, we find that the $<<$ component of (C.16) is given by the chain derivative $X^{<<}(1) - X^{<<}(2)$, where the only nonvanishing action of $X^{<<}$ is

$$
X^{<<}|\psi_<^{(n)}\rangle = \frac{2}{(n+1)^2}|\psi_>^{(n)}\rangle. \tag{C.27}
$$

Therefore, the $<<$ component of the commutator with the Hamiltonian vanishes on periodic states, and by $\mathfrak{B}$ symmetry the $\mathcal{Y}^{\mathfrak{ab}}$ commute with $\hat{\mathfrak{D}}$.

Analogous steps to those above can be used to show that $\mathcal{X}$ also commutes with the Hamiltonian. However, we have only computed (via `Mathematica`) the two-site to two-site interactions in this case up to five excitations. That computation was consistent with the commutator being a chain derivative, but another lengthy computation is needed to complete the proof in this case (the five-excitation computation is extremely strong evidence).

Appendix D

Identities involving $\mathfrak{h}$

In this appendix we prove the essential identities (3.17) and (3.40) for the proof that the symmetry algebra is satisfied at $\mathcal{O}(g^2)$ and $\mathcal{O}(g^3)$.

Recall that (3.17) was

$$\begin{aligned}\{\mathfrak{Q}_{(0)}^{a+\mathfrak{b}},[\mathfrak{Q}_{(0)}^{c-\mathfrak{d}},\mathfrak{h}]\} &= -\frac{1}{2}\varepsilon^{ac}\mathfrak{B}^{\mathfrak{bd}}+\frac{1}{2}\varepsilon^{\mathfrak{bd}}\mathfrak{R}^{ac}-\frac{1}{4}\varepsilon^{ac}\varepsilon^{\mathfrak{bd}}\mathfrak{L},\\ \{\mathfrak{Q}_{(0)}^{a-\mathfrak{b}},[\mathfrak{Q}_{(0)}^{c+\mathfrak{d}},\mathfrak{h}]\} &= -\frac{1}{2}\varepsilon^{ac}\mathfrak{B}^{\mathfrak{bd}}+\frac{1}{2}\varepsilon^{\mathfrak{bd}}\mathfrak{R}^{ac}+\frac{1}{4}\varepsilon^{ac}\varepsilon^{\mathfrak{bd}}\mathfrak{L}.\end{aligned} \tag{D.1}$$

By $\mathfrak{B}$ and $\mathfrak{R}$ symmetry it is sufficient to check each equation for one case where all the indices of one supercharge are opposite those of the others. We will show that

$$\{\mathfrak{Q}_{(0)}^{1+<},[\mathfrak{Q}_{(0)}^{2->},\mathfrak{h}]\} = -\frac{1}{2}\mathfrak{B}^{<>}+\frac{1}{2}\mathfrak{R}^{12}-\frac{1}{4}\mathfrak{L} \tag{D.2}$$

The proof for a case of the second identity in (D.1) follows the same steps, or hermitian conjugation can be used. We first compute using (1.61) and (3.2)

$$\begin{aligned}[][\mathfrak{Q}_{(0)}^{2->},\mathfrak{h}]|\psi_{\mathfrak{a}}^{(n)}\rangle &= \frac{1}{2}h(n+1)\mathfrak{Q}_{(0)}^{2->}|\psi_{\mathfrak{a}}^{(n)}\rangle+\sqrt{n+1}\,\delta_{\mathfrak{a}}^{>}\mathfrak{h}|\phi_1^{(n)}\rangle\\ &= -\frac{1}{2}h(n+1)\sqrt{n+1}\,\delta_{\mathfrak{a}}^{>}|\phi_1^{(n)}\rangle+\frac{1}{2}h(n)\sqrt{n+1}\,\delta_{\mathfrak{a}}^{>}|\phi_1^{(n)}\rangle\\ &= -\frac{1}{2\sqrt{n+1}}\,\delta_{\mathfrak{a}}^{>}|\phi_1^{(n)}\rangle.\end{aligned} \tag{D.3}$$

The commutator vanishes when acting on bosonic fields,

$$[\mathfrak{Q}^{2-<}_{(0)}, \mathfrak{h}]|\phi^{(n)}_a\rangle = 0 \tag{D.4}$$

since in this case $\mathfrak{h}$ gives $h(n)$ for both terms of the commutator.

It follows that the left side of (D.2) is only nonzero when acting on $|\psi^{(n)}_>\rangle$ (since $[\mathfrak{Q}^{2->}_{(0)}, \mathfrak{h}]$ acts nontrivially) or $|\phi^{(n)}_1\rangle$ (since $\mathfrak{Q}^{1+<}_{(0)}$ maps it to $|\psi^{(n)}_>\rangle$). So we compute

$$\begin{aligned}\{\mathfrak{Q}^{1+<}_{(0)}, [\mathfrak{Q}^{2->}_{(0)}, \mathfrak{h}]\}|\psi^{(n)}_>\rangle &= -\frac{1}{2\sqrt{n+1}}\mathfrak{Q}^{1+<}_{(0)}|\phi^{(n)}_1\rangle + [\mathfrak{Q}^{2->}_{(0)}, \mathfrak{h}]\mathfrak{Q}^{1+<}_{(0)}|\psi^{(n)}_>\rangle \\ &= -\frac{1}{2}|\psi^{(n)}_>\rangle.\end{aligned} \tag{D.5}$$

Since the eigenvalue $B^{<>}$ of $|\psi^{(n)}_>\rangle$ is $1/2$ and $B^{<>}$ of $|\psi^{(n)}_<\rangle$ is $-1/2$ by (1.62), it follows that the identity (D.2) is satisfied on fermionic fields (recall that $\mathfrak{L}$ counts the length of a spin chain, so it gives one on a one-site state).

For $|\phi_1\rangle$ we find

$$\begin{aligned}\{\mathfrak{Q}^{1+<}_{(0)}, [\mathfrak{Q}^{2->}_{(0)}, \mathfrak{h}]\}|\phi^{(n)}_1\rangle &= \sqrt{n+1}[\mathfrak{Q}^{2->}_{(0)}, \mathfrak{h}]|\psi^{(n)}_>\rangle \\ &= -\frac{1}{2}|\phi^{(n)}_1\rangle.\end{aligned} \tag{D.6}$$

From (1.48) we see that the eigenvalue R^{12} of $|\phi^{(n)}_2\rangle$ is $1/2$ and R^{12} of $|\phi^{(n)}_1\rangle$ is $-1/2$, so (D.2) is also satisfied on bosonic fields, completing the proof.

We now prove (3.40), which states that the following one-site to two-site commutator vanishes,

$$\{\hat{\mathfrak{Q}}^{\mathfrak{a}}_{(1)}, [\mathfrak{Q}^{b-\mathfrak{a}}_{(0)}, \mathfrak{h}]\}. \tag{D.7}$$

as does its hermitian conjugate. By $\mathfrak{B}$ and $\mathfrak{R}$ symmetry, it is sufficient to check the case $\mathfrak{a} = >$ and $b = 2$. Then, the left side could be nonzero only on $|\psi^{(n)}_>\rangle$, since that is the only flavor on which $[\mathfrak{Q}^{2->}_{(0)}, \mathfrak{h}]$ acts nontrivially, and $\hat{\mathfrak{Q}}^{>}_{(1)}$ never inserts a $|\psi^{(n)}_>\rangle$. To complete the proof, we use (2.34) and (D.3) to check that the action on $|\psi_>^{(n)}\rangle$ vanishes,

$$
\begin{aligned}
\{\hat{\mathfrak{Q}}^>_{(1)},[\mathfrak{Q}^{2-\gt}_{(0)},\mathfrak{h}]\}|\psi_>^{(n)}\rangle &= -\frac{1}{2\sqrt{n+1}}\,\hat{\mathfrak{Q}}^>_{(1)}|\phi_1^{(n)}\rangle \\
&\quad -\sum_{k=0}^{n-1}\frac{\sqrt{n-k}}{\sqrt{(k+1)(n+1)}}\,[\mathfrak{Q}^{2-\gt}_{(0)},\mathfrak{h}]\big|\psi_<^{(k)}\psi_>^{(n-1-k)}\big\rangle \\
&\quad -\sum_{k=0}^{n-1}\frac{\sqrt{k+1}}{\sqrt{(n-k)(n+1)}}\,[\mathfrak{Q}^{2-\gt}_{(0)},\mathfrak{h}]\big|\psi_>^{(k)}\psi_<^{(n-1-k)}\big\rangle \\
&\quad -\sum_{k=0}^{n}\frac{1}{\sqrt{n+1}}\,\varepsilon^{cd}[\mathfrak{Q}^{2-\gt}_{(0)},\mathfrak{h}]\big|\phi_c^{(k)}\phi_d^{(n-k)}\big\rangle \\
&= \sum_{k=0}^{n-1}\frac{1}{2\sqrt{k+1}\sqrt{n+1}}\,\big|\psi_<^{(k)}\phi_1^{(n-1-k)}\big\rangle \\
&\quad -\sum_{k=0}^{n-1}\frac{1}{2\sqrt{n-k}\sqrt{n+1}}\,\big|\phi_1^{(k)}\psi_<^{(n-1-k)}\big\rangle \\
&\quad -\sum_{k=0}^{n-1}\frac{1}{2\sqrt{(k+1)(n+1)}}\,\big|\psi_<^{(k)}\phi_1^{(n-1-k)}\big\rangle \\
&\quad +\sum_{k=0}^{n-1}\frac{1}{2\sqrt{(n-k)(n+1)}}\,\big|\phi_1^{(k)}\psi_<^{(n-1-k)}\big\rangle \\
&= 0. \qquad \text{(D.8)}
\end{aligned}
$$

Appendix E

$\mathcal{O}(g^4)$ supercharge commutators

Here we prove two essential relations satisfied by the $\mathcal{O}(g^4)$ solution (3.64), beginning with

$$\{\mathfrak{Q}^{1+<}, \mathfrak{Q}^{1+<}\}_{(4)} = 0. \tag{E.1}$$

Expanding and substituting the solution (3.64) we obtain

$$\begin{aligned}
\{\mathfrak{Q}^{1+<}, \mathfrak{Q}^{1+<}\}_{(4)} &= 2\{\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{Q}^{1+<}_{(4)}\} + \{\mathfrak{Q}^{1+<}_{(2)}, \mathfrak{Q}^{1+<}_{(2)}\} \\
&= 4\{\mathfrak{Q}^{1+<}_{(0)}, [\hat{\mathfrak{S}}^{>}_{(1)}, \{\hat{\mathfrak{Q}}^{<}_{(3)}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]\}]\} - \{\mathfrak{Q}^{1+<}_{(0)}, [\mathfrak{Q}^{1+<}_{(2)}, \mathfrak{r}]\} \\
&\quad + \{\mathfrak{Q}^{1+<}_{(2)}, \mathfrak{Q}^{1+<}_{(2)}\} \\
&= -4\{\hat{\mathfrak{S}}^{>}_{(1)}, [\mathfrak{Q}^{1+<}_{(0)}, \{\hat{\mathfrak{Q}}^{<}_{(3)}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]\}]\} + 2\{\mathfrak{Q}^{1+<}_{(2)}, \mathfrak{Q}^{1+<}_{(2)}\} \\
&= 4\{\hat{\mathfrak{S}}^{>}_{(1)}, [\{\mathfrak{Q}^{1+<}_{(2)}, \hat{\mathfrak{Q}}^{<}_{(1)}\}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]]\} + 2\{\mathfrak{Q}^{1+<}_{(2)}, \mathfrak{Q}^{1+<}_{(2)}\}.
\end{aligned} \tag{E.2}$$

To reach the third equality we used the $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$ algebra relations, and the last equality follows from the $\mathcal{O}(g^3)$ algebra and the nilpotent identity (3.35). Using the alternative expression for the $\mathcal{O}(g^2)$ supercharge correction (3.63) and the observation (3.44) we find

$$\{\mathfrak{Q}^{1+<}_{(2)}, \mathfrak{Q}^{1+<}_{(2)}\} = -\{\hat{\mathfrak{S}}^{>}_{(1)}, [\{\mathfrak{Q}^{1+<}_{(2)}, \hat{\mathfrak{Q}}^{<}_{(1)}\}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]]\} + \{\hat{\mathfrak{S}}^{>}_{(1)}, [\hat{\mathfrak{Q}}^{<}_{(1)}, \{\mathfrak{Q}^{1+<}_{(2)}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]\}]\}. \tag{E.3}$$

With the previous result, this implies that we need to show

$$[\{\mathfrak{Q}^{1+<}_{(2)}, \hat{\mathfrak{Q}}^{<}_{(1)}\}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]] = -[\hat{\mathfrak{Q}}^{<}_{(1)}, \{\mathfrak{Q}^{1+<}_{(2)}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]\}]. \tag{E.4}$$

We first simplify the left side. Using the second expression for $\mathfrak{x}$ in (3.3) we have yet another expression for $\mathfrak{Q}_{(2)}$,

$$\mathfrak{Q}^{a+<}_{(2)} = -[\hat{\mathfrak{Q}}^{<}_{(1)}, \{\hat{\mathfrak{S}}^{>}_{(1)}, [\mathfrak{Q}^{a+<}_{(0)}, \mathfrak{h}]\}] - [\hat{\mathfrak{S}}^{<}_{(1)}, \{\hat{\mathfrak{Q}}^{>}_{(1)}, [\mathfrak{Q}^{a+<}_{(0)}, \mathfrak{h}]\}]. \tag{E.5}$$

This implies

$$\begin{aligned} \{\hat{\mathfrak{Q}}^{<}_{(1)}, \mathfrak{Q}^{1+<}_{(2)}\} &= -\{\hat{\mathfrak{Q}}^{<}_{(1)}, [\hat{\mathfrak{S}}^{<}_{(1)}, \{\hat{\mathfrak{Q}}^{>}_{(1)}, [\mathfrak{Q}^{a+<}_{(0)}, \mathfrak{h}]\}]\} \\ &= \{\hat{\mathfrak{S}}^{<}_{(1)}, [\hat{\mathfrak{Q}}^{<}_{(1)}, \{\hat{\mathfrak{Q}}^{>}_{(1)}, [\mathfrak{Q}^{a+<}_{(0)}, \mathfrak{h}]\}]\}, \end{aligned} \tag{E.6}$$

where we have also used (3.35). Now, we must have

$$\{[\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}], [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]\} = 0, \tag{E.7}$$

since there is no one-site interaction which has both $\mathfrak{R}$ and $\mathfrak{B}$ charge equal to one. Using this and (3.35) again, as well as the special identity we needed for the $\mathcal{O}(g^3)$ proof (3.40), we obtain

$$[\{\mathfrak{Q}^{1+<}_{(2)}, \hat{\mathfrak{Q}}^{<}_{(1)}\}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]] = [\hat{\mathfrak{S}}^{<}_{(1)}, [\{\hat{\mathfrak{Q}}^{<}_{(1)}, [\mathfrak{Q}^{1+<}_{(0)}, \mathfrak{h}]\}, \{\hat{\mathfrak{Q}}^{>}_{(1)}, [\mathfrak{Q}^{a+<}_{(0)}, \mathfrak{h}]\}]]. \tag{E.8}$$

Next we simplify the right side of (E.4) using the alternative expression for the $\mathcal{O}(g^2)$ solution (3.63), (E.7), the nilpotent identity (3.33), and (3.40) and its $\mathfrak{B}$ "descendant" (3.73).

$$\begin{aligned}\{\mathfrak{Q}^{1+<}_{(2)},[\mathfrak{Q}^{1+<}_{(0)},\mathfrak{h}]\} &= -[\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\},\{\hat{\mathfrak{S}}^{>}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\}] \\ &= [\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\},\{\hat{\mathfrak{S}}^{<}_{(1)},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]\}] \\ &= \{\hat{\mathfrak{S}}^{<}_{(1)},[\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]]\}. \qquad \text{(E.9)}\end{aligned}$$

It follows, after again using (3.33), that

$$\begin{aligned}-[\hat{\mathfrak{Q}}^{<}_{(1)},\{\mathfrak{Q}^{1+<}_{(2)},[\mathfrak{Q}^{1+<}_{(0)},\mathfrak{h}]\}] &= -[\hat{\mathfrak{Q}}^{<}_{(1)},\{\hat{\mathfrak{S}}^{<}_{(1)},[\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]]\}] \\ &= [\hat{\mathfrak{S}}^{<}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(1)},[\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]]\}] \\ &= [\hat{\mathfrak{S}}^{<}_{(1)},[\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{1+<}_{(0)},\mathfrak{h}]\},\{\hat{\mathfrak{Q}}^{<}_{(1)},[\mathfrak{Q}^{a+>}_{(0)},\mathfrak{h}]\}]]. \qquad \text{(E.10)}\end{aligned}$$

Comparing this final expression for the right side of (E.4) with (E.8) for the left side, we see that they are equal by the special identity (3.69), which completes the proof that

$$\{\mathfrak{Q}^{1+<},\mathfrak{Q}^{1+<}\}_{(4)} = 0. \qquad \text{(E.11)}$$

We now prove that

$$\{\mathfrak{Q}^{1+<},\mathfrak{Q}^{2->}\}_{(4)} = -\frac{1}{2}\delta\mathfrak{D}_{(4)}. \qquad \text{(E.12)}$$

Expanding the left side we have,

$$\begin{aligned}\{\mathfrak{Q}^{1+<},\mathfrak{Q}^{2->}\}_{(4)} &= \{\mathfrak{Q}^{1+<}_{(4)},\mathfrak{Q}^{2->}_{(0)}\} + \{\mathfrak{Q}^{1+<}_{(2)},\mathfrak{Q}^{2->}_{(2)}\} + \{\mathfrak{Q}^{1+<}_{(0)},\mathfrak{Q}^{2->}_{(4)}\} \\ &= 2\{[\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(3)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\}],\mathfrak{Q}^{2->}_{(0)}\} - \frac{1}{2}\{[\mathfrak{Q}^{1+<}_{(2)},\mathfrak{x}],\mathfrak{Q}^{2->}_{(0)}\} \\ &\quad + \{\mathfrak{Q}^{1+<}_{(2)},\mathfrak{Q}^{2->}_{(2)}\} \\ &\quad + 2\{\mathfrak{Q}^{1+<}_{(0)},[\hat{\mathfrak{Q}}^{<}_{(1)},\{\hat{\mathfrak{S}}^{>}_{(3)},[\mathfrak{Q}^{2->}_{(0)},\mathfrak{h}]\}]\} + \frac{1}{2}\{\mathfrak{Q}^{1+<}_{(0)},[\mathfrak{Q}^{2->}_{(2)},\mathfrak{x}]\}.\end{aligned}$$

(E.13)

We first simplify the second term of the second line.

$$
\begin{aligned}
-\frac{1}{2}\{[\mathfrak{Q}^{1+<}_{(2)},\mathfrak{x}],\mathfrak{Q}^{2->}_{(0)}\} &= -\frac{1}{2}[\{\mathfrak{Q}^{1+<}_{(2)},\mathfrak{Q}^{2->}_{(0)}\},\mathfrak{x}] + \frac{1}{2}\{\mathfrak{Q}^{1+<}_{(2)},[\mathfrak{Q}^{2->}_{(0)},\mathfrak{x}]\} \\
&= -\frac{1}{2}[\{\mathfrak{Q}^{1+<}_{(2)},\mathfrak{Q}^{2->}_{(0)}\},\mathfrak{x}] - \frac{1}{2}\{\mathfrak{Q}^{1+<}_{(2)},\mathfrak{Q}^{2->}_{(2)}\}. \qquad \text{(E.14)}
\end{aligned}
$$

We repeat for the last term of (E.13) and then apply our observation from $\mathcal{O}(g^2)$ (3.24),

$$
\begin{aligned}
\frac{1}{2}\{\mathfrak{Q}^{1+<}_{(0)},[\mathfrak{Q}^{2->}_{(2)},\mathfrak{x}]\} &= \frac{1}{2}[\{\mathfrak{Q}^{1+<}_{(0)},\mathfrak{Q}^{2->}_{(2)}\},\mathfrak{x}] - \frac{1}{2}\{\mathfrak{Q}^{1+<}_{(2)},\mathfrak{Q}^{2->}_{(2)}\} \\
&= \frac{1}{2}[\{\mathfrak{Q}^{1+<}_{(2)},\mathfrak{Q}^{2->}_{(0)}\},\mathfrak{x}] - \frac{1}{2}\{\mathfrak{Q}^{1+<}_{(2)},\mathfrak{Q}^{2->}_{(2)}\} \qquad \text{(E.15)}
\end{aligned}
$$

Combing these simplified expressions, we find that the second, third and fifth terms of the last expression of (E.13) cancel. The first term contributes

$$
\begin{aligned}
2\{[\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(3)},[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}]\}],\mathfrak{Q}^{2->}_{(0)}\} &= 2[\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(3)},\{[\mathfrak{Q}^{a+<}_{(0)},\mathfrak{h}],\mathfrak{Q}^{2->}_{(0)}\}\}] \\
&= [\hat{\mathfrak{S}}^{>}_{(1)},\{\hat{\mathfrak{Q}}^{<}_{(3)},\mathfrak{B}^{<>} - \mathfrak{R}^{12} + \frac{\mathfrak{L}}{2}\}] \\
&= -\{\hat{\mathfrak{S}}^{>}_{(1)},\hat{\mathfrak{Q}}^{<}_{(3)}\}, \qquad \text{(E.16)}
\end{aligned}
$$

with the second equality following from (3.17). Similarly, the second-to-last term gives

$$
2\{\mathfrak{Q}^{1+<}_{(0)},[\hat{\mathfrak{Q}}^{<}_{(1)},\{\hat{\mathfrak{S}}^{>}_{(3)},[\mathfrak{Q}^{2->}_{(0)},\mathfrak{h}]\}]\} = -\{\hat{\mathfrak{Q}}^{<}_{(1)},\hat{\mathfrak{S}}^{>}_{(3)}\}. \qquad \text{(E.17)}
$$

Combining these last two expressions, we find

$$
\{\mathfrak{Q}^{1+<},\mathfrak{Q}^{2->}\}_{(4)} = -\{\hat{\mathfrak{Q}}^{<},\hat{\mathfrak{S}}^{>}\}_{(4)} = -\frac{1}{2}\delta\mathfrak{D}_{(4)}, \qquad \text{(E.18)}
$$

as needed.

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