

Davide Gaiotto, Gregory W. Moore, Andrew Neitzke

Four-Dimensional Wall-Crossing via Three-Dimensional Field Theory

Received: 2 September 2009 / Accepted: 20 December 2009
© Springer-Verlag 2010

Abstract We give a physical explanation of the Kontsevich-Soibelman wall-crossing formula for the BPS spectrum in Seiberg-Witten theories. In the process we give an exact description of the BPS instanton corrections to the hyperkähler metric of the moduli space of the theory on $\mathbb{R}^3 \times S^1$. The wall-crossing formula reduces to the statement that this metric is continuous. Our construction of the metric uses a four-dimensional analogue of the two-dimensional tt^* equations.

Contents

1 Introduction and Summary

The main subject of this paper is a wall-crossing formula (WCF) for the degeneracies of BPS states in quantum field theories with $d = 4$, $\mathcal{N} = 2$ supersymmetry. Our conventions and a summary of relevant definitions can be found in Sect. 2. The space $\mathcal{H}_{\gamma, BPS}$ of BPS states of charge γ is the space of states in the one-particle Hilbert space, of electromagnetic charge γ , saturating the BPS bound $M \geq |Z_\gamma(u)|$. Here u denotes a point in the vector multiplet moduli space \mathcal{B} , that is, in the Coulomb branch of the moduli space of vacua.

The only available index for $d = 4$, $\mathcal{N} = 2$ supersymmetry is the second helicity supertrace:

$$\Omega(\gamma; u) := -\frac{1}{2} \text{Tr}_{\mathcal{H}_{BPS, \gamma}} (-1)^{2J_3} (2J_3)^2, \quad (1.1)$$

School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA. dgaiotto@ias.edu; neitzke@math.utexas.edu · NHETC and Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855–0849, USA. gmoore@physics.rutgers.edu

where J_3 is any generator of the rotation subgroup of the massive little group. It has been known for a long time that such indices are generally not independent of u but are only piecewise constant (1). Indeed, $\Omega(\gamma; u)$ can jump across walls of marginal stability, where $\gamma = \gamma_1 + \gamma_2$ and $\arg Z_{\gamma_1}(u) = \arg Z_{\gamma_2}(u)$. This fact played an important role in the development of Seiberg-Witten theory (2; 3).

In recent years a more systematic understanding of the u -dependence of the index has begun to emerge. Formulae for the change $\Delta\Omega$ across walls of marginal stability were given in (4) when at least one of the constituents in the decay $\gamma \rightarrow \gamma_1 + \gamma_2$ is primitive. These primitive and semiprimitive wall-crossing formulae were derived from physical pictures based on multicentered solutions of supergravity (5; 6). However, when both constituents have nonprimitive charges, the methods of (4) are difficult to employ.

Kontsevich and Soibelman (7) have proposed a remarkable wall-crossing formula for the $\Delta\Omega$ which applies to all possible decays. We review their formula, which we sometimes refer to as the KS formula, in Sect. 2.2.

On the one hand, Kontsevich and Soibelman's "Donaldson-Thomas invariants"

$\hat{\Omega}(\gamma; u)$ are not obviously the same as the $\Omega(\gamma; u)$ of interest in physics, and the techniques they use to arrive at their formula seem somewhat removed from standard physical considerations. On the other hand, their WCF involves striking new concepts compared to the formulation of the semiprimitive wall-crossing formulae of (4). In particular, the WCF is expressed in terms of a certain product of symplectomorphisms of a torus (see (2.18) below) which depends on the $\hat{\Omega}(\gamma; u)$, and hence *a priori* depends on u . The statement of the WCF is that this product is, in fact, independent of u . That in turn determines the u -dependence of $\hat{\Omega}(\gamma; u)$. This development raises the question of the physical derivation and interpretation of the KS formula and holds out the promise that some essential new physical ideas are involved. This will indeed prove to be the case.

In this paper we give a physical interpretation and proof of the KS formula in the case of $d = 4$, $\mathcal{N} = 2$ field theories. The generalization to supergravity is an interesting and important problem for future work.

Here is a sketch of the main ideas and the basic strategy. We consider the gauge theory on the space $\mathbb{R}^3 \times S^1$, where S^1 has radius R . At low energies this theory is described by a $d = 3$ sigma model with hyperkähler target space (\mathcal{M}, g) . This sigma model receives corrections from BPS instantons, in which the worldline of a BPS particle of the $d = 4$ theory is wrapped around S^1 . Expanding the metric g at large R , one can therefore read off the degeneracies $\Omega(\gamma; u)$ of the BPS particles. This immediately raises a puzzle: we know that the $\Omega(\gamma; u)$ are discontinuous, but g should be continuous! The continuity of the metric is based on the physical principle (which was crucial in (2; 3)) that the only singularities in the low energy effective field theory Lagrangian arise from the appearance at special moduli of massless particles (which should not have been integrated out in the effective theory).

Physically the resolution of this puzzle is similar to one recently discussed in (8). The exact metric g is indeed smooth, but it receives corrections from multiparticle as well as single-particle states. The disappearance of a 1-instanton contribution when a particle decays is compensated by a discontinuity in the multi-instanton contribution from its decay products. Similarly, disappearing n -instanton

contributions are compensated by discontinuities in the m -instanton contributions for $m > n$. To put this more precisely, the n -instanton corrections have the form $\sum \prod_{i=1}^n \Omega(\gamma_i; u) F_{(\gamma_i)}^{(n)}(u)$, where the sum runs over all n -tuples $\{\gamma_i\}$ of charges, and $F^{(n)}$ are essentially universal functions of R and the Z_{γ_i} . Upon crossing the wall each $\Omega(\gamma; u)$ has a discontinuity proportional to a sum of products of $\Omega(\gamma_j; u)$ with $\sum \gamma_j = \gamma$. At the same time, the functions $F^{(n)}$ have discontinuities proportional to the functions $F^{(n')}$ with $n' < n$. We will see that the Kontsevich-Soibelman wall-crossing formula expresses the consistency of this tower of cancellations.

The main technical hurdle in understanding the WCF is thus to give an efficient description of the corrections to g coming from the BPS instantons. A hyperkähler metric is a complicated object and it is hard to make progress by studying, say, the corrections to its components; nor is there generally a simple additive object like the Kähler potential available. To overcome this problem we borrow some ideas from twistor theory. Recall that a hyperkähler manifold is complex-symplectic with respect to a whole $\mathbb{C}\mathbb{P}^1$ worth of complex structures. The basic idea is that studying g is equivalent to studying the holomorphic Darboux coordinates on \mathcal{M} , provided that we consider *all* of these complex structures at once.

In the main body of this paper, we assume that the Kontsevich-Soibelman wall-crossing formula holds for $\Omega(\gamma; u)$. Under this assumption we construct the metric on \mathcal{M} , by giving a canonical set of functions $\mathcal{X}_\gamma(u, \theta; \zeta)$ on $\mathcal{M} \times \mathbb{C}^\times$, indexed by an electromagnetic charge γ . Here (u, θ) specifies a point of \mathcal{M} and the parameter ζ labels the complex structures on \mathcal{M} . Each \mathcal{X}_γ is *piecewise* holomorphic in ζ ; the effect of the BPS instantons is to create discontinuities in the $\mathcal{X}_\gamma(u, \theta; \zeta)$, along rays in the ζ -plane. These discontinuities are identified with the symplectomorphisms introduced by Kontsevich and Soibelman. In this approach the continuity of the metric is a consequence of the WCF. In the final section, we run the argument in reverse: using general principles of supersymmetric gauge theory, we deduce properties of the metric g which are sufficient to prove the WCF.

Summary

We begin in Sect. 2 with a review of the Seiberg-Witten solution of $d = 4$, $\mathcal{N} = 2$ gauge theories and the Kontsevich-Soibelman wall-crossing formula. We then discuss the formulation of the theory on $\mathbb{R}^3 \times S^1$. It is a sigma model into a manifold \mathcal{M} , which is topologically the Seiberg-Witten torus fibration over the $d = 4$ moduli space \mathcal{B} , equipped with a hyperkähler metric g . This metric depends on the radius R of S^1 . As $R \rightarrow \infty$ it approaches a simple form, which can be obtained by naive dimensional reduction of the $d = 4$ theory; we call this simple metric g^{sf} (for “semiflat”).

In Sect. 3 we explain our “twistorial” construction of hyperkähler metrics: given a collection of functions $\mathcal{X}_\gamma(u, \theta; \zeta)$ on \mathcal{M} , varying holomorphically with $\zeta \in \mathbb{C}^\times$ and obeying certain additional conditions, there is a hyperkähler metric for which $\mathcal{X}_\gamma(u, \theta; \zeta)$ are holomorphic Darboux coordinates. In particular, we give the functions $\mathcal{X}_\gamma^{\text{sf}}(u, \theta; \zeta)$ corresponding to the semiflat metric g^{sf} .

With this background in place we are ready to consider the instanton corrections. We begin this study in Sect. 4 with the simple case of a $U(1)$ gauge theory coupled to a single matter hypermultiplet of electric charge q . In this theory the corrected metric g is known exactly (9; 10). We explain how to obtain this corrected metric by including instanton corrections which modify the functions $\mathcal{X}_\gamma^{\text{sf}}(u, \theta; \zeta)$ to new ones $\mathcal{X}_\gamma(u, \theta; \zeta)$. In this construction we already see the building blocks of the Kontsevich-Soibelman formula appear: our $\mathcal{X}_\gamma(u, \theta; \zeta)$ naturally come out with discontinuities in the ζ -plane, which are precisely the elementary Kontsevich-Soibelman symplectomorphisms corresponding to the electric charges $\pm q$.

We then turn in Sect. 5 to the more interesting case where we have multiple kinds of BPS instanton corrections, coming from mutually non-local BPS particles in $d = 4$. In this case we find a natural ansatz for the $\mathcal{X}_\gamma(u, \theta; \zeta)$: essentially we just require that each BPS particle independently contributes a discontinuity like the one we found for a single particle. This discontinuity is most naturally located along a ray in the ζ -plane determined by the phase of the central charge of the BPS particle. The Kontsevich-Soibelman factors for mutually non-local particles do not commute; but this generically presents no problem since these particles have non-aligned central charges, and hence their discontinuities appear on distinct rays in the ζ -plane. The separation between rays disappears exactly at the walls of marginal stability; here the discontinuities pile up into products of Kontsevich-Soibelman factors. The WCF is the statement that this product is the same as we approach the wall from either side. This requirement is essential for us: it implies that the metric we construct from the $\mathcal{X}_\gamma(\zeta)$ is continuous.

More precisely, to determine the $\mathcal{X}_\gamma(u, \theta; \zeta)$ we specify both their discontinuities in the ζ -plane and also their asymptotics as $\zeta \rightarrow 0, \infty$. In other words, we formulate an infinite-dimensional ‘‘Riemann-Hilbert problem’’ whose solution is the $\mathcal{X}_\gamma(u, \theta; \zeta)$. We do not construct its solution exactly; rather we follow a strategy closely analogous to that employed by Cecotti and Vafa, who encountered a similar (but finite-dimensional) Riemann-Hilbert problem in the study of $d = 2$ theories with $\mathcal{N} = (2, 2)$ supersymmetry (11). A variation of their arguments allows us to show that the solution to our problem exists, at least for sufficiently large R . Indeed, in the large R limit the desired \mathcal{X}_γ can be obtained by successive approximations, where the zeroth approximation is just $\mathcal{X}_\gamma^{\text{sf}}$, and the n^{th} approximation incorporates multi-instanton effects up to n instantons.

Having constructed the functions $\mathcal{X}_\gamma(u, \theta; \zeta)$ and hence the metric g , we check that g has various properties which are expected on general field theory grounds; it passes all of these tests and we therefore argue that it should be the correct physical metric on \mathcal{M} , generalizing a similar argument in (12).

As we mentioned above, our construction of the $\mathcal{X}_\gamma(u, \theta; \zeta)$ bears a striking similarity to constructions which appeared in the $d = 2$ case (11). In that case a wall-crossing formula for the degeneracies of BPS domain walls was proven using the flat ‘‘ tt^* connection’’ in the bundle of vacua of the $d = 2$ theory. Two components of this connection give differential equations expressing the R-symmetry and scale invariance of the $d = 2$ theory. Our construction can similarly be phrased in terms of a flat connection \mathcal{A} over $\mathcal{B} \times \mathbb{C}\mathbb{P}^1 \times \mathbb{R}_+$, in the infinite-dimensional bundle of real-analytic functions on the torus fibers \mathcal{M}_u of \mathcal{M} . The Riemann-Hilbert

construction guarantees the existence of this \mathcal{A} . Each \mathcal{X}_γ defines a flat section. In particular, this flatness gives a pair of differential equations for the ζ and R dependence of \mathcal{X}_γ , which have their physical origin in the anomalous R-symmetry and scale invariance of the $d = 4$ theory.

As we describe in Sect. 7, this tt^* -like flat connection can be discovered using only general principles of supersymmetric gauge theory. Moreover, its mere *existence* is strong enough to justify our ansatz for the metric *a priori*. In particular, the wall-crossing formula, which appeared as a consistency requirement working within that ansatz, can be understood as the existence of an “isomonodromic deformation” constructed from \mathcal{A} . This gives a physical proof of the wall-crossing formula.

For convenience, in most of this paper we use a simple form of the wall-crossing formula which does not include information about flavor symmetries, and correspondingly we set all flavor masses to zero. In Sect. 6 we explain how to restore the flavor charge and mass information.

We include several appendices with additional details. In Appendix A we explain a direct verification that the wall-crossing formula gives the correct BPS degeneracies in the case of the pure $SU(2)$ theory. In Appendix B we describe the Cauchy-Riemann equations on \mathcal{M} , in a way that makes contact with our construction of the hyperkähler metric and with the tt^* equations of (13). In Appendix C we give the asymptotic analysis necessary for extracting the large- R corrections to the metric from our Riemann-Hilbert problem. In Appendix D we discuss some details of how to extract the differential equations from the solution of the Riemann-Hilbert problem. Finally, Appendix E explains a curious relation of one of our main results, Eq. (5.13), with the Thermodynamic Bethe Ansatz. There is much more to be said about this connection, but we leave that for another occasion.

Several subsections of the paper are devoted to global issues which are related to a subtle but important sign in the KS formula. On a first reading it would be reasonable to skip this discussion. Readers who choose this course should allow themselves to confuse T and \tilde{T} , as well as \mathcal{M} and $\tilde{\mathcal{M}}$, in the main text.

Discussion

Let us remark on a few particularly interesting points.

- Physically, our construction of the metric on \mathcal{M} amounts to a rule for “integrating out” mutually non-local particles in $d = 4$. This problem *a priori* appears to be difficult because one cannot find any duality frame in which all of the particles are electrically charged, so it is difficult to write a Lagrangian which includes all of the relevant fields. Here we have circumvented that difficulty.
- Our construction of the metric uses its twistorial description. The most natural physical context in which the twistor space occurs is projective superspace (14; 15; 16), in which the parameter ζ is a bosonic superspace coordinate. The fact that the corrections to g come only from BPS instantons, and that they are localized at specific rays in the ζ -plane, should have a natural explanation in the projective superspace language.

- One of the inspirations for the Kontsevich-Soibelman WCF was their earlier work (17), in which they gave a construction of the sheaf of holomorphic functions on a K3 surface, by “correcting” the sheaf of functions on the semiflat K3. The corrections were formulated in terms of products of symplectomorphisms similar to those which appear in the wall-crossing formula. This construction is closely related to ours, with K3 replaced by \mathcal{M} . The key new ingredient in our work is to consider all complex structures at once, thus introducing the parameter $\zeta \in \mathbb{C}\mathbb{P}^1$; having done so, we can formulate the crucial Riemann-Hilbert problem. This idea might also be useful in the original K3 context.
- The multi-instanton expansion of g is given as a sum of basic building blocks weighted by products of the BPS degeneracies $\Omega(\gamma; u)$. These basic building blocks have intricate discontinuities at the walls of marginal stability, which conspire with the jumps of $\Omega(\gamma; u)$ to make g continuous in u . All this is reminiscent of recent work of Joyce on wall-crossing (18). Moreover, Joyce’s work was interpreted by Bridgeland and Toledano Laredo in (19) in terms of isomonodromic deformation of a connection on $\mathbb{C}\mathbb{P}^1$, which somewhat resembles the one we consider here, but has a slightly different form: it has an irregular singularity only at $t = 0$, while ours has them both at $\zeta = 0$ and $\zeta = \infty$. There is an interesting scaling limit of our connection, $R \rightarrow 0$ and $\zeta \rightarrow 0$ with $\zeta/R = t$ fixed, which brings it into the form of the one in (19) (albeit with a different structure group). This limit retains the information about the BPS degeneracies and their wall-crossing. It would be interesting to see whether there is any sense in which it relates our connection to the one in (19).
- In our discussion we studied structures defined over the vector multiplet moduli space \mathcal{B} . However, both Kontsevich-Soibelman and Joyce formulate their invariants over a larger space, the space of “Bridgeland stability conditions” (20). We do not understand the meaning of our constructions when extended to this larger space.
- The wall-crossing formula as formulated by Kontsevich-Soibelman makes sense not only for $\mathcal{N} = 2$ field theories but also for supergravity, and indeed this was the main focus of (4). The moduli space \mathcal{M} of the theory on $\mathbb{R}^3 \times S^1$ is then a quaternionic-Kähler manifold rather than hyperkähler. Nevertheless, most of our considerations seem to make sense in that context, with appropriate modifications. For example, Hitchin’s theorem is replaced by LeBrun’s theorem characterizing the twistor space of a quaternionic-Kähler manifold in terms of holomorphic contact structures. In particular, there is still a natural notion of a “holomorphic” function $\mathcal{X}_\gamma(x, \zeta)$ (namely, a holomorphic function on the twistor space of \mathcal{M}), and the quaternionic-Kähler analogue of $\mathcal{X}_\gamma^{\text{sf}}$ has been worked out in (21). We expect that the instanton-corrected metric g on \mathcal{M} can be obtained by a method parallel to the one employed in this paper: formulate a Riemann-Hilbert problem for \mathcal{X}_γ , using $\mathcal{X}_\gamma^{\text{sf}}$ to fix the asymptotics, and the Kontsevich-Soibelman factors to fix the discontinuities. One important difficulty to overcome is that in gravity the degeneracies $\Omega(\gamma; u)$ grow very quickly with γ ; this makes the convergence of the iterative solution for \mathcal{X}_γ less obvious in this case. As in the hyperkähler case, the WCF should arise as a consistency condition ensuring that g is smooth.

- The analogy between the hyperkähler geometry of the fibration $\mathcal{M} \rightarrow \mathcal{B}$ and the tt^* geometry of (13; 1; 11) is striking: the two structures are very similar although one has to do with field theories in $d = 4$, the other in $d = 2$. Is there a direct relation between the two? One possibility is to relate them just by compactification, e.g. on S^2 . Different values of the $U(1)$ fluxes on S^2 would then correspond to different vacua of the $d = 2$ theory, and BPS states of the $d = 4$ theory could be identified with domain walls interpolating between these vacua in $d = 2$.¹ Related ideas have appeared in the literature before — in particular see (22; 23; 24). See also (25; 26) for a slightly different link between BPS spectra in $d = 2$ and $d = 4$.
- Infinitesimal deformations of a class of hyperkähler manifolds which include the semiflat geometry have been recently studied in (27). It would be interesting to describe the leading correction to the semiflat geometry in their language. Our Eq. (4.33) resembles their Eq. (3.38), with an appropriate choice of H and contours of integration.

2 Preliminaries

2.1 $d = 4$, $\mathcal{N} = 2$ gauge theory

We consider a gauge theory in $d = 4$ with $\mathcal{N} = 2$ supersymmetry, gauge group G of rank r , and a characteristic (complex) mass scale Λ . Seiberg-Witten theory (initiated in (2; 3), and reviewed more generally in e.g. (28; 29; 30)) gives a rather complete description of the behavior of such a gauge theory on its Coulomb branch at energies $\mu \ll \Lambda$, as follows.

The Coulomb branch is a complex manifold \mathcal{B} of complex dimension r , parameterized by the vacuum expectation values of the vector multiplet scalars. We denote a generic coordinate system on \mathcal{B} as (u^1, \dots, u^r) . At each point $u \in \mathcal{B}$ the gauge group is broken to a maximal torus $U(1)^r$. There is a lattice $\Gamma_u \simeq \mathbb{Z}^{2r}$ of electric and magnetic charges, equipped with an integral-valued symplectic pairing $\langle \cdot, \cdot \rangle$. This lattice is the fiber of a local system Γ over \mathcal{B} . That is, there is a fibration of lattices with fiber Γ_u over $u \in \mathcal{B}$, with nontrivial monodromy around the singular loci in \mathcal{B} , of complex codimension 1, where some BPS particles become massless. We sometimes write “ $\gamma \in \Gamma$ ” informally, meaning that γ is a local section of Γ .

There is a vector $Z(u) \in \Gamma_u^* \otimes_{\mathbb{Z}} \mathbb{C}$ of “periods,” which varies holomorphically with u . For any $\gamma \in \Gamma$ we define the central charge $Z_\gamma(u)$ by

$$Z_\gamma(u) = Z(u) \cdot \gamma. \quad (2.1)$$

$Z(u)$ plays a fundamental role in the description both of the massless and the massive sectors.

We begin with the massless part. Locally on \mathcal{B} one can choose a splitting $\Gamma = \Gamma^m \oplus \Gamma^e$ into Lagrangian sublattices of “magnetic” and “electric” charges respectively. Γ^m and Γ^e are then dual to one another using the pairing on Γ . Such

¹ This picture has been advocated to us by Cumrun Vafa.

a splitting is called an electric-magnetic duality frame. Concretely we may choose a basis $\{\alpha_1, \dots, \alpha_r\}$ for Γ^m and $\{\beta^1, \dots, \beta^r\}$ for Γ^e such that

$$\langle \alpha_I, \alpha_J \rangle = 0, \quad \langle \beta^I, \beta^J \rangle = 0, \quad \langle \alpha_I, \beta^J \rangle = \delta_I^J \quad (2.2)$$

with $I, J = 1, \dots, r$. After choosing such a frame, we obtain a system of ‘‘special coordinates’’ a^I on \mathcal{B} , which are nothing but the electric central charges, i.e.

$$Z_{\beta^I} = a^I. \quad (2.3)$$

The magnetic central charges are then holomorphic functions of the a^I . They are determined in terms of a single function $\mathcal{F}(a^I)$ (depending on the chosen frame), the $\mathcal{N} = 2$ prepotential:

$$Z_{\alpha_I} = \frac{\partial \mathcal{F}}{\partial a^I}. \quad (2.4)$$

This implies in particular that Z is not arbitrary: from the symmetry of mixed partial derivatives one obtains

$$\langle dZ, dZ \rangle = 0. \quad (2.5)$$

On the left side of (2.5) we are using the antisymmetric pairing $\langle \cdot, \cdot \rangle$ and also the antisymmetric wedge product of 1-forms on \mathcal{B} ; the combined pairing is symmetric, so this condition is not vacuous. Indeed, (2.5) says that around u , \mathcal{B} can be locally identified with a complex Lagrangian submanifold of $\Gamma_u^* \otimes_{\mathbb{Z}} \mathbb{C}$.

The prepotential completely determines the two-derivative effective Lagrangian, written in terms of the electric vector multiplets. To write this Lagrangian we introduce the symmetric matrix τ defined by

$$\tau_{IJ}(a) = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J}, \quad (2.6)$$

and then adopt a notation that suppresses the gauge index, e.g. $\tau|da|^2$ for $\tau_{IJ} da^I \wedge \star d\tilde{a}^J$. Then the bosonic part of the Lagrangian is

$$\mathcal{L}^{(4)} = \frac{\text{Im } \tau}{4\pi} (-|da|^2 - F^2) + \frac{\text{Re } \tau}{4\pi} F \wedge F. \quad (2.7)$$

The central charges Z_γ are also of fundamental importance for the massive spectrum. Indeed, the mass of any 1-particle state with charge γ obeys

$$M \geq |Z_\gamma| \quad (2.8)$$

with equality if and only if the state is BPS. BPS states belong to massive short multiplets of the super Poincare symmetry; under the little group $SU(2)$ the states at rest in such a multiplet transform as

$$[j] \otimes ([1/2] + 2[0]). \quad (2.9)$$

Choosing $j = 0$ gives the massive hypermultiplet, while $j = \frac{1}{2}$ is the massive vector multiplet.

There is a standard index which “counts” the short multiplets, namely the second helicity supertrace $\Omega(\gamma; u)$. This supertrace receives the contribution $+1$ for each massive hypermultiplet of charge γ in the spectrum of the theory at $u \in \mathcal{B}$, and similarly -2 for each massive vector multiplet. $\Omega(\gamma; u)$ is invariant under any deformation of the theory in which the 1-particle states do not mix with the continuum of multiparticle states. From (2.8) it follows that such mixing is very restricted; a BPS particle can decay only into other BPS particles, and then only if their central charges all have the same phase. Hence $\Omega(\gamma; u)$ is locally constant in u , away from the *walls of marginal stability* in \mathcal{B} . These walls of marginal stability are of real codimension 1 and are defined, for a pair of linearly independent charges γ, γ' , to be the locus of $u \in \mathcal{B}$, where Z_γ and $Z_{\gamma'}$ are nonzero and have the same phase.

Understanding the jumping behavior of $\Omega(\gamma; u)$ as u crosses a wall of marginal stability is one of the main motivations of this paper. We turn to it next.

2.2 The Kontsevich-Soibelman wall-crossing formula

In this section we review the Kontsevich-Soibelman wall-crossing formula. As originally proposed in (7) this formula determines the jumping behavior of “generalized Donaldson-Thomas invariants” $\hat{\Omega}(\gamma; u)$. As we will see below, if we identify the Donaldson-Thomas invariants with the helicity supertraces, $\hat{\Omega}(\gamma; u) = \Omega(\gamma; u)$, then the wall-crossing formula gives the physically expected answer in several nontrivial examples: in particular, it reproduces the “primitive wall-crossing formula” of (4), as well as the wall-crossing behavior of the BPS spectrum of Seiberg-Witten theory with gauge group $SU(2)$.

A technical point: for the KS formula to make sense, the $\Omega(\gamma; u)$ are not allowed to be completely arbitrary. Introducing a positive definite norm on Γ , one must require that there exists some $K > 0$ such that

$$\frac{|Z_\gamma|}{\|\gamma\|} > K \quad (2.10)$$

for all γ such that $\hat{\Omega}(\gamma; u) \neq 0$. Throughout this paper we will assume that this property, called the “Support Property,” holds.

The Kontsevich-Soibelman algebra.

The wall-crossing formula is given in terms of a Lie algebra defined by generators e_γ , with $\gamma \in \Gamma$, and a basic commutation relation

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}. \quad (2.11)$$

In this paper it will be important to realize this abstract Lie algebra as an algebra of complex symplectomorphisms of a complexified torus. Modulo a subtlety which will appear at the end of this section, this torus is the fiber \tilde{T}_u of the local system $\tilde{T} := \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^\times$.

Any $\gamma \in \Gamma$ gives a corresponding function X_γ on \tilde{T}_u , with $X_\gamma X_{\gamma'} = X_{\gamma + \gamma'}$. Upon choosing a basis $\{\gamma^1, \dots, \gamma^{2r}\}$ for Γ , we can choose $X^i := X_{\gamma^i}$ as coordinates for

\tilde{T}_u . The symplectic pairing on Γ^* gives a holomorphic symplectic form $\tilde{\omega}^T$ on \tilde{T}_u : if $\varepsilon^{ij} = \langle \gamma^i, \gamma^j \rangle$, and ε_{ij} is its inverse,

$$\tilde{\omega}^T = \frac{1}{2} \varepsilon_{ij} \frac{dX^i}{X^i} \wedge \frac{dX^j}{X^j}. \quad (2.12)$$

We would like to identify e_γ with the infinitesimal symplectomorphism of \tilde{T}_u generated by the Hamiltonian X_γ . This almost gives the algebra (2.11), but misses the extra sign $(-1)^{\langle \gamma_1, \gamma_2 \rangle}$. This sign will be important below in comparing to wall-crossing formulas known from physics; in that context it is related to the fact that the fermion number of a bound state of two particles of charges γ_1, γ_2 is shifted by $\langle \gamma_1, \gamma_2 \rangle$.

Over a local patch of \mathcal{B} , we can absorb this sign by introducing a ‘‘quadratic refinement’’ of the \mathbb{Z}_2 -valued quadratic form $(-1)^{\langle \gamma_1, \gamma_2 \rangle}$: this means a $\sigma : \Gamma \rightarrow \mathbb{Z}_2$ obeying

$$\sigma(\gamma_1)\sigma(\gamma_2) = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \sigma(\gamma_1 + \gamma_2). \quad (2.13)$$

One way to get such a σ is to choose a local electric-magnetic duality frame $\Gamma \cong \Gamma^e \oplus \Gamma^m$, write $\gamma = \gamma^e + \gamma^m$, and set $\sigma(\gamma) = (-1)^{\langle \gamma^e, \gamma^m \rangle}$. Notice that

$$\sigma(\gamma_1)\sigma(\gamma_2) = \sigma(\gamma_1 + \gamma_2)(-1)^{\langle \gamma_1^e, \gamma_2^m \rangle + \langle \gamma_2^e, \gamma_1^m \rangle} = \sigma(\gamma_1 + \gamma_2)(-1)^{\langle \gamma_1, \gamma_2 \rangle} \quad (2.14)$$

as needed. At any rate, having chosen any $\sigma(\gamma)$, we could identify e_γ with the symplectomorphism generated by the Hamiltonian $\sigma(\gamma)X_\gamma$.

Any two refinements σ, σ' obey $\sigma(\gamma)\sigma'(\gamma) = (-1)^{c(\sigma, \sigma') \cdot \gamma}$ for some fixed $c(\sigma, \sigma') \in \Gamma^*/2\Gamma^*$. The Hamiltonians $\sigma(\gamma)X_\gamma$ and $\sigma'(\gamma)X_\gamma$ associated to these two refinements are related by the automorphism of \tilde{T}_u which sends $X_\gamma \rightarrow (-1)^{c(\sigma, \sigma') \cdot \gamma} X_\gamma$.

The wall-crossing formula.

Now we are ready to formulate the wall-crossing formula. Its basic building block is the group element

$$\mathcal{H}_\gamma := \exp \sum_{n=1}^{\infty} \frac{1}{n^2} e_{n\gamma}. \quad (2.15)$$

Under our identification, \mathcal{H}_γ becomes a symplectomorphism acting on \tilde{T}_u , given by

$$\mathcal{H}_\gamma : X_{\gamma'} \rightarrow X_{\gamma'} (1 - \sigma(\gamma)X_\gamma)^{\langle \gamma', \gamma \rangle}. \quad (2.16)$$

Associate to each BPS particle of charge γ a ray in the complex ζ -plane, determined by the central charge,

$$\ell_\gamma := \{ \zeta : Z_\gamma(u) / \zeta \in \mathbb{R}_- \}. \quad (2.17)$$

As we vary $u \in \mathcal{B}$ these rays rotate in the ζ -plane. The cyclic ordering of the rays changes only when u reaches a wall of marginal stability. At such a wall a set of BPS rays ℓ_γ come together, corresponding to a set of charges γ for which

Z_γ become aligned. At a generic point on the wall of marginal stability, this set of charges can be parameterized as² $\{n\gamma_1 + m\gamma_2 : m, n > 0\}$, for some primitive vectors γ_1, γ_2 with $Z_{\gamma_1}/Z_{\gamma_2} \in \mathbb{R}_+$.

Now associate the group element \mathcal{K}_γ to each BPS particle of charge γ , and form the product over states which become aligned at the wall:

$$A := \prod_{\substack{\gamma=n\gamma_1+m\gamma_2 \\ m>0, n>0}} \mathcal{K}_\gamma^{\Omega(\gamma;u)}, \quad (2.18)$$

where the ordering of the factors corresponds to clockwise ordering of the rays ℓ_γ . We can consider this product for u on either side of the wall. As u crosses the wall, the order of the factors is reversed, and the $\Omega(\gamma;u)$ jump. The statement of the wall-crossing formula is that the whole product A is unchanged.

This condition is strong enough to determine the $\Omega(\gamma;u_+)$ from the $\Omega(\gamma;u_-)$, where u_\pm are points infinitesimally displaced on opposite sides of the wall. To understand how to do this in practice we first have to deal with an important subtlety: since the spectrum of BPS states is typically infinite, the product (2.18) generally involves infinitely many factors. Following (7), we can understand it as follows. The product only involves the generators $e_{n\gamma_1+m\gamma_2}$, where $m, n > 0$. The Lie algebra they generate can be consistently truncated by fixing some integer L and then setting $e_{n\gamma_1+m\gamma_2} = 0$ whenever $n+m > L$. (That is, the Lie algebra is filtered by Lie subalgebras with $n+m > L$, and we can take quotients by subalgebras with successively larger values of L .) After such a truncation (2.18) involves only finitely many nontrivial terms; the infinite product can be understood as the limit of these truncated products as $L \rightarrow \infty$.

In a similar spirit consider the power expansion of the symplectomorphism A ,

$$A : X_\gamma \rightarrow \left(1 + \sum_{m>0, n>0} c_\gamma^{m,n} X_{n\gamma_1+m\gamma_2}\right) X_\gamma, \quad (2.19)$$

and truncate it to $n+m \leq L$. We can compute this expansion on one side of the wall of marginal stability, and then recursively identify the $\Omega(\gamma;u)$ on the other side of the wall. Concretely, first set $L = 1$; then $\Omega(\gamma_1;u)$ and $\Omega(\gamma_2;u)$ are fixed by the requirement that they correctly reproduce $c_\gamma^{1,0}$ and $c_\gamma^{0,1}$. Next set $L = 2$ and consider the expansion of $A \mathcal{K}_{\gamma_1}^{-\Omega(\gamma_1;u)} \mathcal{K}_{\gamma_2}^{-\Omega(\gamma_2;u)}$ to extract the next set of degeneracies. This iteration can be continued in a straightforward way to determine all of the $\Omega(n\gamma_1 + m\gamma_2;u)$. What is far from obvious — but conjectured in (7) — is that the $\Omega(\gamma;u)$ computed in this way are integers!

Some examples.

In the above interpretation of the Kontsevich-Soibelman formula we identified their generalized Donaldson-Thomas invariants with the physically defined $\Omega(\gamma;u)$. To motivate this identification, we now describe a few examples.

² To establish the existence of these γ_1, γ_2 we need to use the Support Property: otherwise one can easily imagine situations in which the aligned Z_γ accumulate near the origin.

As explained above, at a generic point on a wall of marginal stability the symplectomorphisms which enter the WCF are generated by a two-dimensional lattice of charges, $\gamma = (p, q) \in \mathbb{Z}^2$ with canonical symplectic form $\langle (p, q), (p', q') \rangle = pq' - qp'$. We write correspondingly $X_{1,0} = x$, $X_{0,1} = y$. The symplectomorphisms $\mathcal{H}_{p,q}$ are then determined by their action on x and y , which is explicitly

$$\mathcal{H}_{p,q} : (x, y) \rightarrow ((1 - (-1)^{pq} x^p y^q)^q x, (1 - (-1)^{pq} x^p y^q)^{-p} y). \quad (2.20)$$

Consider a wall of marginal stability where the central charges for a single BPS particle of primitive charge $(1, 0)$ and a single particle of primitive charge $(0, 1)$ come together. Kontsevich and Soibelman notice a beautiful ‘‘pentagon identity’’:

$$\mathcal{H}_{1,0} \mathcal{H}_{0,1} = \mathcal{H}_{0,1} \mathcal{H}_{1,1} \mathcal{H}_{1,0}. \quad (2.21)$$

Hence the WCF predicts that crossing the wall, only one extra particle will be created, a dyonic bound state of one electrically charged particle and one magnetically charged particle. Indeed the ‘‘primitive wall-crossing formula’’ from supergravity (which is also valid in field theory) (4) predicts that this pair of particles will form a single bound state in a hypermultiplet representation. It also predicts that a single particle of charge $(1, 0)$ cannot be bound to more than one particle of charge $(0, 1)$. It is quite hard to count more general bound states of several particles of different type. Their absence is already a non-trivial prediction of the KS wall-crossing formula.

A further comparison with the primitive wall-crossing formula helps us understand the role of the sign in the commutation relation (2.11) of the e_γ . Consider the product $\mathcal{H}_{\gamma_1} \mathcal{H}_{\gamma_2}$ and try to rewrite it as a product in the opposite direction, (i.e. with the slopes of Z_{γ_i} increasing instead of decreasing) of the form $\mathcal{H}_{\gamma_2} \cdots \mathcal{H}_{\gamma_1}$. Suppose γ_1, γ_2 are primitive and consider the subalgebra generated by $e_{n\gamma_1 + m\gamma_2}$ quotiented by that with $n \geq 2, m \geq 2$. The result is a Heisenberg algebra. The KS formula in the truncated Heisenberg group reads

$$\mathcal{H}_{\gamma_1}^{\Omega(\gamma_1; u_+)} \mathcal{H}_{\gamma_1 + \gamma_2}^{\Omega(\gamma_1 + \gamma_2; u_+)} \mathcal{H}_{\gamma_2}^{\Omega(\gamma_2; u_+)} = \mathcal{H}_{\gamma_2}^{\Omega(\gamma_2; u_-)} \mathcal{H}_{\gamma_1 + \gamma_2}^{\Omega(\gamma_1 + \gamma_2; u_-)} \mathcal{H}_{\gamma_1}^{\Omega(\gamma_1; u_-)} \quad (2.22)$$

where u_\pm are points infinitesimally displaced on either side of the wall. Now, at a generic point on the wall of marginal stability we have $\Omega(\gamma_i; u_+) = \Omega(\gamma_i; u_-)$ for $i = 1, 2$. Moreover, $\mathcal{H}_{\gamma_1 + \gamma_2}$ is central in the Heisenberg group, and therefore, computing the group commutator we reproduce the corollary of the primitive wall-crossing formula:

$$\Delta \Omega = (-1)^{\langle \gamma_1, \gamma_2 \rangle - 1} \langle \gamma_1, \gamma_2 \rangle \Omega(\gamma_1; u) \Omega(\gamma_2; u). \quad (2.23)$$

A more elaborate version of this argument allows one to extract the semiprimitive wall-crossing formula of (4) from the KS formula.³

The example in (2.21) is exceptional in that both sides involve a finite number of terms. More typically one encounters infinite products. A second beautiful example presented by Kontsevich and Soibelman is the following:

$$\mathcal{H}_{1,0}^2 \mathcal{H}_{0,1}^2 = (\mathcal{H}_{0,1}^2 \mathcal{H}_{1,2}^2 \mathcal{H}_{2,3}^2 \cdots) \mathcal{H}_{1,1}^4 \mathcal{H}_{2,2}^{-2} (\cdots \mathcal{H}_{3,2}^2 \mathcal{H}_{2,1}^2 \mathcal{H}_{1,0}^2). \quad (2.24)$$

³ This was shown in unpublished work with Wu-yen Chuang.

We give an instructive proof of this identity in Appendix A. By a change of basis we obtain a physically very interesting formula,

$$\mathcal{H}_{1,-1}^2 \mathcal{H}_{0,1}^2 = (\mathcal{H}_{0,1}^2 \mathcal{H}_{1,1}^2 \mathcal{H}_{2,1}^2 \cdots) \mathcal{H}_{1,0}^4 \mathcal{H}_{2,0}^{-2} (\cdots \mathcal{H}_{3,-1}^2 \mathcal{H}_{2,-1}^2 \mathcal{H}_{1,-1}^2), \quad (2.25)$$

which captures the spectrum of an $SU(2)$ Seiberg-Witten theory with two massless flavors (more precisely, hypermultiplets transforming in the vector representation of an $SO(4) = SU(2)_A \times SU(2)_B$ flavor symmetry) as described in (31).⁴ On the right side we see the full weak coupling spectrum: one W boson of charge $(2, 0)$ (which contributes -2 to the helicity supertrace), the four hypermultiplets of charge $(1, 0)$, and a set of dyons of charge $(n, \pm 1)$, with multiplicity 2. (In fact these dyons are in doublets of $SU(2)_A$ or $SU(2)_B$, depending on the parity of n .) On the left side we see the strong coupling spectrum: a single monopole with multiplicity 2 (a doublet of $SU(2)_A$) and a single dyon with multiplicity 2 (a doublet of $SU(2)_B$).

The small change of variables $y \rightarrow -y^2$ converts the product formula (2.25) into

$$\mathcal{H}_{2,-1} \mathcal{H}_{0,1} = (\mathcal{H}_{0,1} \mathcal{H}_{2,1} \mathcal{H}_{4,1} \cdots) \mathcal{H}_{2,0}^{-2} (\cdots \mathcal{H}_{6,-1} \mathcal{H}_{4,-1} \mathcal{H}_{2,-1}). \quad (2.26)$$

This formula captures the wall-crossing behavior of the pure $SU(2)$ Seiberg-Witten theory.⁵ The left side includes the two particles present at strong coupling (32): a monopole of charge $(0, 1)$ and a dyon of charge $(2, -1)$. The right side includes the infinite spectrum of dyons at weak coupling, together with the W boson contribution $\mathcal{H}_{2,0}^{-2}$.

Adding flavor information.

The product (2.25) describes the BPS spectrum of $SU(2)$ Seiberg-Witten theory with $N_f = 2$, but does not carry information about the flavor charges of the BPS particles. We now describe a conjectural variant of the KS formula which includes the information about flavor charges. (We will see the physical motivation for this formula in Sect. 6.) Introduce a new lattice of flavor charges Γ^f , and a new parameter $\log \mu \in (\Gamma^f)^* \otimes_{\mathbb{Z}} \mathbb{C}^\times$. Then generalize the X_γ to new functions labeled by $(\gamma, \gamma^f) \in \Gamma \oplus \Gamma^f$:⁶ letting a run over a basis for Γ^f ,

$$X_{\gamma, \gamma^f} := \prod_i (X^i)^{\gamma_i} \prod_a (\mu^a)^{\gamma_a^f} = X_\gamma \prod_a (\mu^a)^{\gamma_a^f}. \quad (2.27)$$

Define refined symplectomorphisms carrying flavor information:

$$\mathcal{H}_{\gamma, \gamma^f} : X_\gamma \rightarrow X_\gamma (1 - \sigma(\gamma) X_{\gamma, \gamma^f})^{(\gamma', \gamma)}. \quad (2.28)$$

⁴ The relation of the identity (2.25) to Seiberg-Witten theory was first suggested by Frederik Denef. The precise relation of (2.25) to the $N_f = 2$ theory was worked out in collaboration with Wu-yen Chuang.

⁵ The close resemblance between (2.25) and (2.26) arises because the Seiberg-Witten curve for the $N_f = 2$ theory with zero masses is a double cover of that for the $N_f = 0$ theory.

⁶ Strictly speaking, the full local system $\hat{\Gamma}$ of charges does not split into $\Gamma \oplus \Gamma^f$ globally; we really have an extension $0 \rightarrow \Gamma^f \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 0$. However, we can always split this extension locally, and this is sufficient for our purposes.

The central charge now depends on the masses m^a , $Z_{\gamma, \gamma^f}(u) = Z_\gamma(u) + \gamma_a^f m^a$, and determines new walls of marginal stability. (μ^a are functions of the m^a . See Sect. 6 below.) We introduce a product analogous to (2.18),

$$A := \prod_{\substack{(\gamma, \gamma^f) = n\gamma_1 + m\gamma_2 \\ m > 0, n > 0}} \mathcal{H}_{\gamma, \gamma^f}^{\Omega(\gamma, \gamma^f; u)}. \quad (2.29)$$

The extended WCF states the continuity of A across the walls. We derive a refined version of the infinite product (2.25), including the flavor charges, in Appendix A; combining this with the extended WCF we obtain the correct wall-crossing for the $SU(2)$ theory with $N_f = 2$.

Global issues.

So far in this section we have worked over a local patch in \mathcal{B} , and chosen a fixed quadratic refinement σ in order to identify the Kontsevich-Soibelman algebra with an algebra of symplectomorphisms of the complexified torus \tilde{T}_u , a fiber of the local system \tilde{T} . It is impossible in general to choose such a refinement globally over \mathcal{B} , because of the monodromies of the local system Γ . Hence it is not true globally that the Kontsevich-Soibelman algebra is the algebra of symplectomorphisms acting on \tilde{T} .

However, by an appropriate twisting of \tilde{T} we can define a closely related complexified torus fibration T , on which the Kontsevich-Soibelman algebra does act. T is defined so that a local choice of quadratic refinement gives an identification $T \simeq \tilde{T}$, and given two different refinements σ, σ' , the corresponding identifications differ by the map $X_\gamma \rightarrow (-1)^{c(\sigma, \sigma') \cdot \gamma} X_\gamma$ on \tilde{T} . The fiberwise symplectic form $\omega^{\tilde{T}}$ induces a corresponding fiberwise symplectic form ω^T on T .

We can construct a twisted fibration T with the above properties as follows. Let R denote the local system over \mathcal{B} whose local sections are refinements σ . R is a torsor for $\Gamma^*/2\Gamma^*$, and T is the associated fibration,

$$T := (\tilde{T} \times R) / \left((X_\gamma, \sigma) \sim ((-1)^{c(\sigma, \sigma') \cdot \gamma} X_\gamma, \sigma') \right). \quad (2.30)$$

2.3 The low energy effective theory on $\mathbb{R}^3 \times S^1$

Our goal is to explain the Kontsevich-Soibelman WCF as a statement about the gauge theory on $\mathbb{R}^3 \times S^1$, with S^1 of radius R . We study the theory at an energy scale μ which is low compared to all other scales, i.e., $\mu \ll \Lambda$ and also $\mu \ll 1/R$. At this energy the theory looks effectively three-dimensional. In this section we describe some of its basic properties.

In the limit of large radius, $R \gg 1/\Lambda$, we can determine the three-dimensional dynamics using the infrared Lagrangian (2.7). The dynamical degrees of freedom are just the x^4 -independent modes of the four-dimensional fields. These include of course the scalars a^I . In addition, from the gauge sector we get the ‘‘electric’’ Wilson lines

$$\theta_e^I := \oint_{S^1} A_4^I dx^4, \quad (2.31)$$

as well as another set of periodic scalars $\theta_{m,I}$ obtained by dualizing the $d = 3$ gauge fields $A_\alpha^I dx^\alpha$. We will often think of these as “magnetic” Wilson lines,

$$\theta_{m,I} := \oint_{S^1} (A_{D,4})_I dx^4. \quad (2.32)$$

We can define $\theta_{m,I}$ either by working in a formulation treating the gauge fields as self-dual, or by working at fixed magnetic quantum numbers P^I and introducing $\theta_{m,I}$ as their Fourier duals.

All these periodic scalars coordinatize a $2r$ -torus \mathcal{M}_u at any fixed $u \in \mathcal{B}$. Letting u vary we obtain a torus fibration \mathcal{M} . The fiber \mathcal{M}_u degenerates over the singular loci in \mathcal{B} . The low energy theory on \mathbb{R}^3 is a sigma model with target space \mathcal{M} .

More precisely, $\theta = (\theta_\epsilon^I, \theta_{m,I})$ is an element in the fiber of a local system of $2r$ -tori $\widetilde{\mathcal{M}} := \Gamma^* \otimes_{\mathbb{Z}} (\mathbb{R}/2\pi\mathbb{Z})$. For any $\gamma \in \Gamma$, we get an angular coordinate on $\widetilde{\mathcal{M}}$ denoted $\theta_\gamma := \gamma \cdot \theta$. $\widetilde{\mathcal{M}}$ is not exactly the same as \mathcal{M} ; there is a global twisting which we glossed over above, and which we discuss at the end of this section.

In sum, the three-dimensional theory is a sigma model into a Riemannian manifold \mathcal{M} of real dimension $4r$, which is topologically a $2r$ -torus fibration over \mathcal{B} . The theory enjoys $\mathcal{N} = 4$ supersymmetry (8 real supercharges), which implies that the metric on \mathcal{M} is hyperkähler. This metric is the main object of study in this paper. It was studied previously in (12), where in particular the $R \rightarrow 0$ limit for pure $SU(2)$ gauge theory was identified as the Atiyah-Hitchin manifold. In this paper we are more interested in the opposite limit $R \rightarrow \infty$, because in this limit one can read off the imprint of the full BPS spectrum of the theory in $d = 4$. In the next section we begin by considering the leading behavior in this limit.

Global issues.

In the description above we were slightly naive about the precise definition of the Wilson lines. Our description is adequate over a local patch in \mathcal{B} , but as we will see in Sect. 4, it cannot be quite correct globally. Indeed, in order for the metric on \mathcal{M} to be smooth, we will see that the monodromies around paths in \mathcal{B} must generally be accompanied by shifts of the Wilson lines by π . This contradicts our naive description, since the torus fibration $\widetilde{\mathcal{M}}$ comes with a distinguished zero section.

We propose that the correct global picture is as follows: at any fixed $u \in \mathcal{B}$, the Wilson lines live in a torus \mathcal{M}_u which is isomorphic to $\widetilde{\mathcal{M}}_u$, but not canonically isomorphic. One obtains an isomorphism $\mathcal{M}_u \simeq \widetilde{\mathcal{M}}_u$ upon choosing a refinement σ of the quadratic form $(-1)^{\langle \gamma_1, \gamma_2 \rangle}$ on Γ_u .⁷ Such a refinement generally exists only locally, so the fibrations \mathcal{M} and $\widetilde{\mathcal{M}}$ are globally different. Given two local refinements σ, σ' the corresponding two local isomorphisms $\mathcal{M} \simeq \widetilde{\mathcal{M}}$ differ by the shift $\theta \rightarrow \theta + \pi c(\sigma, \sigma')$ acting on $\widetilde{\mathcal{M}}$. Of course, this discussion is closely parallel to the relation between the torus fibrations T and \widetilde{T} which we described at the end of Sect. 2.2.

⁷ Such quadratic refinements frequently appear in the precise formulations of self-dual gauge theories (33; 34; 35; 36). It seems likely that the origin of σ here can be explained in this way.

2.4 The semiflat geometry

The leading behavior of the metric on \mathcal{M} in the $R \rightarrow \infty$ limit is governed by the $d = 3$ effective action obtained by simply truncating (2.7) to its x^4 -independent sector. This gives

$$\begin{aligned} \mathcal{L}^{(3)} = & (\text{Im } \tau) \left(-\frac{R}{2} |da|^2 - \frac{R}{2} F^{(3)} \wedge \star F^{(3)} - \frac{1}{8\pi^2 R} d\theta_e^2 \right) \\ & + (\text{Re } \tau) \left(\frac{1}{2\pi} d\theta_e \wedge F^{(3)} \right). \end{aligned} \quad (2.33)$$

Then dualizing the $d = 3$ gauge field A^I to a scalar $\theta_{m,I}$ gives after a little rearranging

$$\mathcal{L}_{dual}^{(3)} = -\frac{R}{2} (\text{Im } \tau) |da|^2 - \frac{1}{8\pi^2 R} (\text{Im } \tau)^{-1} |d\theta_m - \tau d\theta_e|^2. \quad (2.34)$$

This is the Lagrangian for a sigma model into \mathcal{M} , with metric locally given by

$$g^{\text{sf}} = R(\text{Im } \tau) |da|^2 + \frac{1}{4\pi^2 R} (\text{Im } \tau)^{-1} |dz|^2, \quad (2.35)$$

where we introduced

$$dz_I = d\theta_{m,I} - \tau_{IJ} d\theta_e^J. \quad (2.36)$$

(While this notation is very convenient, we should emphasize that the form “ dz_I ” is not closed on the whole \mathcal{M} : it is only closed when restricted to each torus fiber \mathcal{M}_u .)

We call g^{sf} the “semiflat” metric on \mathcal{M} , because in this metric the torus fibers are flat. The expression (2.35) reflects the fact that g^{sf} is Kähler, with respect to a complex structure on \mathcal{M} for which da^I and dz_I are a basis for $\Omega^{1,0}$. In this complex structure \mathcal{M} is the Seiberg-Witten fibration by compact *complex* tori over \mathcal{B} . (We contrast this with other complex structures on \mathcal{M} which we will meet momentarily, in which the tori \mathcal{M}_u are *not* complex submanifolds.) The fibers \mathcal{M}_u all have volume

$$\text{vol}(\mathcal{M}_u) = \left(\frac{1}{R} \right)^r. \quad (2.37)$$

The expression (2.35) is valid only locally, since it uses a choice of duality frame. Nevertheless the expressions in different frames glue together into a smooth metric, everywhere except over the singular loci of \mathcal{B} , where g^{sf} has a singularity. Such a singularity would be unexpected from the point of view of effective field theory; we will see that it is resolved by BPS instanton corrections in the exact quantum-corrected metric g .

3 A Twistorial Construction of Hyperkähler Metrics

In this section we review some general facts about hyperkähler geometry, and then explain the basic idea underlying our description of g .

3.1 Holomorphic data from hyperkähler manifolds

We first recall some holomorphic data attached to any hyperkähler manifold. By definition, a hyperkähler manifold (\mathcal{M}, g) is Kähler with respect to a triplet of complex structures \mathcal{J} , obeying the relations

$$J_1 J_2 = J_3, \quad J_2 J_3 = J_1, \quad J_3 J_1 = J_2, \quad J_\alpha^2 = -1. \quad (3.1)$$

Let ω_α denote the three corresponding Kähler forms.

In fact, any hyperkähler (\mathcal{M}, g) is Kähler with respect to a more general complex structure, namely $a^\alpha J_\alpha$ with $\sum_{\alpha=1}^3 a_\alpha^2 = 1$, with corresponding Kähler form $a^\alpha \omega_\alpha$. So we have a whole S^2 worth of complex structures. One of the key insights of the twistor approach is that it is useful to consider this S^2 as the Riemann sphere, labeled by a complex parameter ζ . So we write the general complex structure and corresponding Kähler form as

$$J^{(\zeta)} = \frac{i(-\zeta + \bar{\zeta})J_1 - (\zeta + \bar{\zeta})J_2 + (1 - |\zeta|^2)J_3}{1 + |\zeta|^2}, \quad (3.2)$$

$$\omega^{(\zeta)} = \frac{i(-\zeta + \bar{\zeta})\omega_1 - (\zeta + \bar{\zeta})\omega_2 + (1 - |\zeta|^2)\omega_3}{1 + |\zeta|^2}. \quad (3.3)$$

We also organize the Kähler forms into a second combination,

$$\varpi(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-, \quad (3.4)$$

where we introduced the notation

$$\omega_\pm = \omega_1 \pm i\omega_2. \quad (3.5)$$

The essential property of $\varpi(\zeta)$ is that for any fixed $\zeta \in \mathbb{C}\mathbb{P}^1$ it is a holomorphic symplectic form on \mathcal{M} in complex structure $J^{(\zeta)}$. (To make sense of this statement for $\zeta = 0, \infty$ we have to rescale $\varpi(\zeta)$ by $\zeta, 1/\zeta$ respectively. Globally one could say that $\varpi(\zeta)$ is twisted by the line bundle $\mathcal{O}(2)$ over $\mathbb{C}\mathbb{P}^1$.)

3.2 Twistorial construction of g

Now we describe the method of determining g from holomorphic data on \mathcal{M} , which will be used in the rest of this paper.

First we specify our assumptions. Recall that \mathcal{M} is topologically a torus fibration over \mathcal{B} . For any choice of local patch in \mathcal{B} , quadratic refinement, and local section γ of the charge lattice Γ , we assume given a locally defined \mathbb{C}^\times -valued function $\mathcal{X}_\gamma(u, \theta; \zeta)$ of $(u, \theta) \in \mathcal{M}$ and $\zeta \in \mathbb{C}^\times$, with the following properties:

- The \mathcal{X}_γ are multiplicative,

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}. \quad (3.6)$$

- The \mathcal{X}_γ obey a reality condition,

$$\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}. \quad (3.7)$$

- All \mathcal{X}_γ are solutions to a single set of differential equations, of the form

$$\frac{\partial}{\partial u^i} \mathcal{X} = \left(\frac{1}{\zeta} \mathcal{A}_{u^i}^{(-1)} + \mathcal{A}_{u^i}^{(0)} \right) \mathcal{X}, \quad (3.8)$$

$$\frac{\partial}{\partial \bar{u}^i} \mathcal{X} = \left(\mathcal{A}_{\bar{u}^i}^{(0)} + \zeta \mathcal{A}_{\bar{u}^i}^{(1)} \right) \mathcal{X}, \quad (3.9)$$

where the operators $\mathcal{A}_{u^i}^{(n)}, \mathcal{A}_{\bar{u}^i}^{(n)}$ are complex vertical vector fields on the torus fiber \mathcal{M}_u , with the $\mathcal{A}_{u^i}^{(-1)}$ linearly independent at every point, and similarly $\mathcal{A}_{\bar{u}^i}^{(1)}$. (To motivate these equations, note that in Appendix B we show that the Cauchy-Riemann equations on (\mathcal{M}, g) have this form.)

- For each fixed $x \in \mathcal{M}$, $\mathcal{X}_\gamma(x; \zeta)$ is holomorphic in ζ on a dense subset of \mathbb{C}^\times . (In our application below, $\mathcal{X}_\gamma(x; \zeta)$ will be holomorphic away from a countable union of lines.)

To state our last three assumptions on the functions \mathcal{X}_γ we first define

$$\varpi(\zeta) := \frac{1}{8\pi^2 R} \varepsilon_{ij} \frac{d\mathcal{X}_{\gamma^i}}{\mathcal{X}_{\gamma^i}} \wedge \frac{d\mathcal{X}_{\gamma^j}}{\mathcal{X}_{\gamma^j}}, \quad (3.10)$$

where by d we mean the fiberwise differential, i.e. we treat ζ as a fixed parameter. We assume:

- $\varpi(\zeta)$ is globally defined (in particular the $\varpi(\zeta)$ defined over different local patches of \mathcal{B} agree with one another) and holomorphic in $\zeta \in \mathbb{C}^\times$. (Note that this does not imply that the \mathcal{X}_γ are holomorphic in ζ ; in our application they will be only piecewise holomorphic.)
- $\varpi(\zeta)$ is nondegenerate in the appropriate sense for a holomorphic symplectic form, i.e. $\ker \varpi(\zeta)$ is a $2r$ -dimensional subspace of the $4r$ -dimensional $T_{\mathbb{C}}\mathcal{M}$.
- $\varpi(\zeta)$ has only a simple pole as $\zeta \rightarrow 0$ or $\zeta \rightarrow \infty$.

In the rest of this section we explain how to define a hyperkähler metric g on \mathcal{M} , such that $\mathcal{X}_\gamma(\zeta)$ are holomorphic functions in complex structure $J^{(\zeta)}$, and $\varpi(\zeta)$ is the holomorphic symplectic form as in (3.4).

We consider the manifold $\mathcal{Z} := \mathcal{M} \times \mathbb{CP}^1$. It has the following properties:

1. \mathcal{Z} is a complex manifold. At any (x, ζ) the $2r$ equations (3.8), (3.9) define a half-dimensional subspace of $T_{\mathbb{C}}\mathcal{M}$ (if $\zeta = 0$ or $\zeta = \infty$ this is still true after rescaling one of the equations by a factor ζ). The direct sum of this subspace and the one generated by $\partial/\partial \bar{\zeta}$ is a half-dimensional subspace of $T_{\mathbb{C}}\mathcal{Z}$. We define $T^{0,1}\mathcal{Z}$ to be this subspace. This *a priori* defines only an almost complex structure on \mathcal{Z} . However, the existence of the functions \mathcal{X}_γ guarantees that this almost complex structure is actually integrable. (Of course, the \mathcal{X}_γ are not everywhere holomorphic in ζ ; but they are holomorphic on a dense set, which is enough to guarantee the vanishing of the Nijenhuis tensor. It follows in particular that there exist complex coordinates on \mathcal{Z} even around $\zeta = 0$ or $\zeta = \infty$.)

2. \mathcal{Z} is a holomorphic fibration over $\mathbb{C}\mathbb{P}^1$. The projection is simply $p(x, \zeta) = \zeta$.
3. There is a holomorphic section of $\Omega_{\mathcal{Z}/\mathbb{C}\mathbb{P}^1}^2 \otimes \mathcal{O}(2)$, giving a holomorphic symplectic form on each fiber $p^{-1}(\zeta)$. This is the globally defined $\varpi(\zeta)$.
4. There is a family of holomorphic sections $s : \mathbb{C}\mathbb{P}^1 \rightarrow \mathcal{Z}$, each with normal bundle $N \simeq \mathcal{O}(1)^{\oplus 2r}$. Indeed, for each $x \in \mathcal{M}$, we can define a section $s_x : \mathbb{C}\mathbb{P}^1 \rightarrow \mathcal{Z}$ by $s_x(\zeta) = (x, \zeta)$. To see that this is a holomorphic section, note first that it is holomorphic at least away from $\zeta = 0, \infty$, just because the local complex coordinates $\mathcal{X}_{\gamma^i}(x, \zeta)$ of \mathcal{Z} are holomorphic in ζ at fixed x ; but it extends continuously to $\zeta = 0, \infty$, so it must be holomorphic there as well by the Riemann removable singularity theorem. To show that the normal bundle $N(s_x) \simeq \mathcal{O}(1)^{\oplus 2r}$, first note that there is a 1-1 correspondence between holomorphic sections of $N^*(s_x)$ and holomorphic functions on the first infinitesimal neighborhood of s_x which vanish on s_x . But such functions are determined by their first-order Taylor expansion around x , i.e. they correspond to holomorphic sections of the trivial bundle $p^*((T_{\mathbb{C}})_x \mathcal{M})$ which annihilate the subbundle $B \subset p^*((T_{\mathbb{C}})_x \mathcal{M})$ defined by Eqs. (3.8), (3.9). Dualizing, we have $N(s_x) \simeq p^*((T_{\mathbb{C}})_x \mathcal{M})/B$. On the other hand (3.8), (3.9) give $2r$ trivializing sections of $B \otimes \mathcal{O}(1)$. So we conclude that $N(s_x) \otimes \mathcal{O}(-1)$ is trivial.
5. There is an antiholomorphic involution $\sigma : \mathcal{Z} \rightarrow \mathcal{Z}$, which covers the antipodal map on $\mathbb{C}\mathbb{P}^1$, and preserves ϖ in the sense that $\sigma^* \varpi = \overline{\varpi}$. This involution is just $\sigma(x, \zeta) = (x, -1/\bar{\zeta})$. Using the reality condition (3.7) we can check that it is antiholomorphic and preserves ϖ .

These are the characteristic properties of the *twistor space* of a hyperkähler manifold as described in (37; 38). In particular, using the recipe of (37; 38), one can reconstruct a hyperkähler metric g on \mathcal{M} from \mathcal{Z} . We can describe g concretely: note that from $\varpi(\zeta)^{r+1} = 0$ it follows that $\omega_+^r \wedge \omega_3 = 0$, which implies that the real 2-form ω_3 is of type $(1, 1)$ in complex structure J_3 . Therefore we can use J_3 and ω_3 to build a Kähler metric g on \mathcal{M} . This g coincides with the hyperkähler metric guaranteed by the twistor construction. In the following sections we will use this approach.

3.3 Twistorial construction of the semiflat geometry

The foregoing description of hyperkähler metrics is particularly convenient in the case of the semiflat metric g^{sf} which we introduced in Sect. 2.4. As above, we work over a local patch in \mathcal{B} , and make a local choice of quadratic refinement. Then for any $\gamma \in \Gamma$ we write the locally defined function⁸

$$\mathcal{X}_{\gamma}^{\text{sf}}(\zeta) := \exp[\pi R \zeta^{-1} Z_{\gamma} + i\theta_{\gamma} + \pi R \zeta \bar{Z}_{\gamma}]. \quad (3.11)$$

These functions obey ‘‘Cauchy-Riemann equations’’ of the form (3.8), (3.9), where

$$\mathcal{A}_{u^i}^{(-1)} = -i\pi R \frac{\partial Z}{\partial u^i} \cdot \frac{\partial}{\partial \theta}, \quad \mathcal{A}_{\bar{u}^i}^{(1)} = -i\pi R \frac{\partial \bar{Z}}{\partial \bar{u}^i} \cdot \frac{\partial}{\partial \theta}, \quad (3.12)$$

$$\mathcal{A}_{u^i}^{(0)} = 0, \quad \mathcal{A}_{\bar{u}^i}^{(0)} = 0, \quad (3.13)$$

⁸ This formula was first obtained in joint work with Boris Pioline, and is essentially the rigid limit of a formula in (21) for the quaternionic-Kähler case. It provided an important clue to discovering the constructions described in this paper.

and Z stands for the vector of periods. Then $\omega^{\text{sf}}(\zeta)$ is

$$\omega^{\text{sf}}(\zeta) := \frac{1}{8\pi^2 R} \varepsilon_{ij} \frac{d\mathcal{X}_{\gamma^i}^{\text{sf}}}{\mathcal{X}_{\gamma^i}^{\text{sf}}} \wedge \frac{d\mathcal{X}_{\gamma^j}^{\text{sf}}}{\mathcal{X}_{\gamma^j}^{\text{sf}}} \quad (3.14)$$

$$= \frac{1}{4\pi} \left[\frac{i}{\zeta} \langle dZ, d\theta \rangle + \left(\pi R \langle dZ, d\bar{Z} \rangle - \frac{1}{2\pi R} \langle d\theta, d\theta \rangle \right) + i\zeta \langle d\bar{Z}, d\theta \rangle \right]. \quad (3.15)$$

(Note that the vanishing condition (2.5) ensures that $\omega^{\text{sf}}(\zeta)$ has no terms of order ζ^{-2} or ζ^2 .) $\omega^{\text{sf}}(\zeta)$ and $\mathcal{X}_{\gamma}^{\text{sf}}(\zeta)$ obey the necessary conditions for the construction we described in Sect. 3.2, so they are the holomorphic symplectic form and complex coordinates for some hyperkähler metric on \mathcal{M} . As we now check, this metric is simply g^{sf} as desired.

First note that comparing the leading terms in (3.4) and (3.15) gives

$$\omega_+^{\text{sf}} = -\frac{1}{2\pi} \langle dZ, d\theta \rangle. \quad (3.16)$$

From ω_+^{sf} we can determine complex structure J_3^{sf} : indeed, after choosing an electric-magnetic duality frame, we can rewrite (3.16) as

$$\omega_+^{\text{sf}} = \frac{1}{2\pi} da^I \wedge dz_I. \quad (3.17)$$

This makes manifest that \mathcal{M} in complex structure J_3^{sf} is just the Seiberg-Witten fibration by complex tori. This is the complex structure we already described in Sect. 2.4.

Similarly, comparing the ζ -independent terms in (3.4) and (3.15) gives

$$\omega_3^{\text{sf}} = \frac{R}{4} \langle dZ, d\bar{Z} \rangle - \frac{1}{8\pi^2 R} \langle d\theta, d\theta \rangle, \quad (3.18)$$

which we can rewrite as

$$\omega_3^{\text{sf}} = \frac{i}{2} \left(R (\text{Im } \tau)_{IJ} da^I \wedge d\bar{a}^J + \frac{1}{4\pi^2 R} ((\text{Im } \tau)^{-1})^{IJ} dz_I \wedge d\bar{z}_J \right). \quad (3.19)$$

Comparing this with (2.35) we see that g^{sf} is indeed Kähler for complex structure J_3^{sf} and Kähler form ω_3^{sf} , and hence it is the hyperkähler metric guaranteed by the twistor construction starting from $\omega^{\text{sf}}(\zeta)$.

In this section we have seen that the semiflat metric on \mathcal{M} and its hyperkähler structure can be constructed from the functions $\mathcal{X}_{\gamma}^{\text{sf}}$ defined in (3.11). These functions are of fundamental importance for what follows.

4 Mutually Local Corrections

If we considered only the naive dimensional reduction of the massless sector, then the semiflat metric g^{sf} would be the end of the story. However, the theory in $d = 4$ also contains massive BPS particles. The metric receives corrections from “instanton” configurations in which one or more of these massive particles go around S^1 . These corrections will be weighted by a factor of at least $e^{-2\pi R|Z|}$, because of the bound $M \geq |Z|$ on the energy of states in the $d = 4$ theory.

In this section we study these corrections in examples in which all of the BPS particles are mutually local. This is much more tractable than the general situation, because we can choose a duality frame in which these particles are all electrically charged, and hence we can work completely within an effective Lagrangian description.

For most of the section we specialize further to the free $U(1)$ gauge theory coupled to a single charged hypermultiplet. In addition to being the simplest example, this theory is physically relevant because it describes the physics near a generic singularity in \mathcal{B} , where one BPS particle becomes much lighter than the others.

4.1 The exact single-particle metric

We consider a $U(1)$ gauge theory on $\mathbb{R}^3 \times S^1$, coupled to a single hypermultiplet of charge $q > 0$ (along with its CPT conjugate of charge $-q$). The metric we will describe has been considered previously in (9; 10).

The moduli space \mathcal{B} of the $d = 4$ theory is coordinatized by the vector multiplet scalar $a \in \mathbb{C}$. More precisely, \mathcal{B} is only an open patch in \mathbb{C} , because the $d = 4$ theory is not asymptotically free: there is a cutoff at $|a| \sim |\Lambda|$.

As we explained in Sect. 2.3, the moduli space \mathcal{M} of the $d = 3$ theory is a 2-torus fibration over \mathcal{B} . The torus fibers \mathcal{M}_a are coordinatized (temporarily ignoring the subtlety about quadratic refinements) by the electric Wilson line θ_e and the magnetic Wilson line θ_m , both with periodicity 2π .

The semiflat metric g^{sf} has an action of $U(1)^2$ by isometries, because shifts of θ_e and θ_m are exact symmetries. The electrically charged hypermultiplet couples to θ_e , and hence breaks the isometry which shifts it. However, there are no magnetically charged BPS states in the theory, so shifts of θ_m are still exact isometries. The corrected metric g is therefore of Gibbons-Hawking form.

For comparison with the Gibbons-Hawking ansatz we introduce a vector \mathcal{E} by

$$a = x^1 + ix^2, \quad \theta_e = 2\pi R x^3. \quad (4.1)$$

θ_m is a local coordinate on a $U(1)$ bundle over the open subset of $\mathbb{R}^2 \times S^1$ parameterized by \mathcal{E} . The metric is

$$g = V(\mathcal{E})^{-1} \left(\frac{d\theta_m}{2\pi} + A(\mathcal{E}) \right)^2 + V(\mathcal{E}) d\mathcal{E}^2, \quad (4.2)$$

where V is a positive harmonic function, to be calculated below, and A is a $U(1)$ connection with curvature

$$F = \star dV. \quad (4.3)$$

This is a slight generalization of the standard Gibbons-Hawking ansatz, in which one takes \mathcal{K} to lie in (an open subset of) \mathbb{R}^3 . (We can first work over a suitable subset of \mathbb{R}^3 and then divide by a \mathbb{Z} -action on the total space which shifts x^3 .) In the standard ansatz all A obeying (4.3) are gauge equivalent and so define the same metric. In our case this is not quite true: there is one additional gauge invariant degree of freedom associated to the holonomy around S^1 . This choice is related to the choice of a θ angle in $d = 4$.

$V(\mathcal{K})$ in our case can be calculated by integrating out the charged hypermultiplet at one loop. Reference (10) asserts a nonrenormalization theorem which implies that the computation is exact. The resulting V is a harmonic function with q singularities in $\mathbb{R}^2 \times S^1$. The periodicity in θ_e arises because one sums over the Kaluza-Klein momenta of the charged hypermultiplet on S^1 :

$$V = \frac{q^2 R}{4\pi} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{q^2 R^2 |a|^2 + (q \frac{\theta_e}{2\pi} + n)^2}} - \kappa_n \right). \quad (4.4)$$

Here κ_n is a regularization constant introduced to make the sum converge. Poisson resummation of (4.4) shows that

$$V = V^{\text{sf}} + V^{\text{inst}}, \quad (4.5)$$

with

$$V^{\text{sf}} = -\frac{q^2 R}{4\pi} \left(\log \frac{a}{\Lambda} + \log \frac{\bar{a}}{\Lambda} \right), \quad (4.6)$$

$$V^{\text{inst}} = \frac{q^2 R}{2\pi} \sum_{n \neq 0} e^{inq\theta_e} K_0(2\pi R |nqa|). \quad (4.7)$$

Here Λ is an ultraviolet cutoff related to the choice of κ_n .⁹

To specify the metric fully we must also give $A(\mathcal{K})$ obeying (4.3):

$$A = A^{\text{sf}} + A^{\text{inst}}, \quad (4.8)$$

where

$$A^{\text{sf}} = \frac{iq^2}{8\pi^2} \left(\log \frac{a}{\Lambda} - \log \frac{\bar{a}}{\Lambda} \right) d\theta_e, \quad (4.9)$$

$$A^{\text{inst}} = -\frac{q^2 R}{4\pi} \left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}} \right) \sum_{n \neq 0} (\text{sgn } n) e^{inq\theta_e} |a| K_1(2\pi R |nqa|). \quad (4.10)$$

At large R the leading terms in V and A are V^{sf} and A^{sf} . Keeping only these terms, g becomes the semiflat metric with

$$\tau = \frac{q^2}{2\pi i} \log \frac{a}{\Lambda}. \quad (4.11)$$

⁹ For example, if we choose $\kappa_n = (|\tilde{\Lambda}|^2 + n^2)^{-1/2}$, then we can choose $\Lambda = (qR)^{-1} \tilde{\Lambda} \exp[-2 \sum_{m=1}^{\infty} K_0(2\pi m |\tilde{\Lambda}|)]$.

This is the running coupling which comes from integrating out the hypermultiplet in $d = 4$.

The subleading terms $V^{\text{inst}}, A^{\text{inst}}$ yield corrections to the semiflat metric. They have the form of an instanton expansion as we expected, because of the asymptotic behavior $K_V(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ for $x \rightarrow +\infty$. They also break the translation invariance along θ_e as expected. Finally, they improve the singular behavior. Recall that in g^{sf} there is a singularity at $a = 0$. From (4.4) we see that the only possible singularities of g occur at $a = 0$, $q\theta_e = 2\pi n$. Studying the metric near these points we find that there is an A_{q-1} conical singularity at each one. So the singularity in g^{sf} is replaced by q higher-codimension singularities in g . In the simplest case $q = 1$, the singularity is completely smoothed.

Global issues.

There is a subtle issue regarding the global definition of the coordinate θ_m . We have chosen a gauge which is convenient for discussing the periodicity in θ_e . However, the presence of the logarithm in A^{sf} signals that this gauge is singular at $a = 0$. Moreover A^{sf} shifts by $-\frac{q^2}{2\pi} d\theta_e$ upon continuation around the origin $a \rightarrow e^{2\pi i} a$. This shift must be compensated by a gauge transformation

$$\theta_m \rightarrow \theta_m + q^2 \theta_e + C. \quad (4.12)$$

To fix C we make a gauge transformation to a new coordinate θ'_m :

$$\theta'_m = \theta_m + \frac{i}{4\pi} \left(\log \frac{a}{\Lambda} - \log \frac{\bar{a}}{\bar{\Lambda}} \right) (q^2 \theta_e + C). \quad (4.13)$$

The transformed θ'_m is single-valued as a goes around the origin. The gauge transformed A^{sf} is

$$(A')^{\text{sf}} = -\frac{i}{4\pi} \left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}} \right) (q^2 \theta_e + C). \quad (4.14)$$

Now we focus on the behavior at $q\theta_e = \pi$. Here we have $A^{\text{inst}} = 0$, so the exact gauge field is just given by $(A')^{\text{sf}}$. On the other hand, once the instanton corrections are included, there is no singularity either of the metric or of the $U(1)$ bundle at this point (recall that the only singularities occur at $a = 0$, $q\theta_e = 2\pi n$.) Since moreover θ'_m is single-valued, it follows that $(A')^{\text{sf}}$ cannot have a singularity here even if we go to $a = 0$ (or more precisely the only allowed singularity is a quantized Dirac string), which implies

$$C = -q\pi + 2\pi k \quad (4.15)$$

for some integer k . So we conclude that as we go around $a = 0$ the angular coordinates shift by

$$\theta_e \rightarrow \theta_e, \quad (4.16a)$$

$$\theta_m \rightarrow \theta_m + q^2 \theta_e - q\pi. \quad (4.16b)$$

The shift by $q^2\theta_e$ is as expected from the monodromy of the torus fibration. The shift by $-q\pi$ is more surprising, but fits into our discussion in the end of Sect. 2.3, where we proposed that the Wilson lines are well defined only after choosing a local quadratic refinement σ . So far in this section we have chosen the “standard” refinement $\sigma(\gamma_e, \gamma_m) = (-1)^{\gamma_e \gamma_m}$. The monodromy shifts $\gamma_e \rightarrow \gamma_e + q^2 \gamma_m$, and hence replaces σ by $\sigma'(\gamma_e, \gamma_m) = (-1)^{q^2 \gamma_m^2} \sigma(\gamma_e, \gamma_m)$. This change of refinement is compensated by the shift of θ_m by $-q\pi$.

4.2 Hyperkähler structure

Next we want to describe \mathcal{M} as a hyperkähler manifold. The hyperkähler structure of any Gibbons-Hawking metric is determined by the triplet of symplectic forms

$$\omega^\alpha = dx^\alpha \wedge \left(\frac{d\theta_m}{2\pi} + A(\mathcal{K}) \right) + \frac{1}{2} \varepsilon^{\alpha\beta\gamma} V dx^\beta \wedge dx^\gamma. \quad (4.17)$$

The holomorphic symplectic form (3.4) is then

$$\varpi(\zeta) = \frac{1}{4\pi^2 R} \xi_m \wedge \xi_e, \quad (4.18)$$

where

$$\xi_m = id\theta_m + 2\pi i A(\mathcal{K}) + \pi i V \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right), \quad (4.19)$$

$$\xi_e = id\theta_e + \pi R \left(\frac{1}{\zeta} da + \zeta d\bar{a} \right). \quad (4.20)$$

In particular it follows that ξ_e and ξ_m are of type $(1, 0)$.

Moreover, ξ_e can be written as

$$\xi_e = \frac{d\mathcal{X}_e}{\mathcal{X}_e}, \quad (4.21)$$

where

$$\mathcal{X}_e = \exp \left[\pi R \frac{a}{\zeta} + i\theta_e + \pi R \zeta \bar{a} \right]. \quad (4.22)$$

So \mathcal{X}_e is a holomorphic function on \mathcal{M} in complex structure $J^{(\zeta)}$. Notice that it coincides with the semiflat coordinate $\mathcal{X}_\gamma^{\text{sf}}$ given in (3.11), if we choose γ to be the unit electric charge, since in that case $Z_\gamma = a$ and $\theta_\gamma = \theta_e$. In other words, the “electric” complex coordinate is unaffected by the instanton corrections due to the electrically charged particle,

$$\mathcal{X}_e = \mathcal{X}_e^{\text{sf}}. \quad (4.23)$$

To finish describing the complex geometry of \mathcal{M} one should construct a second “magnetic” complex coordinate \mathcal{X}_m , such that

$$\varpi(\zeta) = -\frac{1}{4\pi^2 R} \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \frac{d\mathcal{X}_m}{\mathcal{X}_m}. \quad (4.24)$$

Such a \mathcal{X}_m is necessarily of the form

$$\mathcal{X}_m = e^{i\theta_m + \Phi(a, \bar{a}, \theta_e, \zeta)}. \quad (4.25)$$

The most obvious way of constructing \mathcal{X}_m would be to write out the Cauchy-Riemann equations on \mathcal{M} and look for a particular solution of the form (4.25). In the next section we follow a different approach: we give a particular solution for \mathcal{X}_m directly, in a form which will be especially convenient for what follows, and then rather than checking the Cauchy-Riemann equations we check (4.24) directly.

4.3 The solution for \mathcal{X}_m

Now we specialize to our \mathcal{M} . In this case we have

$$\varpi(\zeta) = \varpi^{\text{sf}}(\zeta) + \varpi^{\text{inst}}(\zeta), \quad (4.26)$$

where

$$\varpi^{\text{sf}}(\zeta) = -\frac{1}{4\pi^2 R} \xi_e \wedge \left[id\theta_m + 2\pi i A^{\text{sf}} + \pi i V^{\text{sf}} \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right) \right], \quad (4.27)$$

$$\varpi^{\text{inst}}(\zeta) = -\frac{1}{4\pi^2 R} \xi_e \wedge \left[2\pi i A^{\text{inst}} + \pi i V^{\text{inst}} \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right) \right]. \quad (4.28)$$

If we neglect the instanton corrections, the desired magnetic coordinate is

$$\mathcal{X}_m^{\text{sf}}(\zeta) = \exp \left[-i \frac{Rq^2}{2\zeta} \left(a \log \frac{a}{\Lambda} - a \right) + i\theta_m + i \frac{\zeta Rq^2}{2} \left(\bar{a} \log \frac{\bar{a}}{\Lambda} - \bar{a} \right) \right]. \quad (4.29)$$

This coincides with the expression (3.11) for the holomorphic coordinate $\mathcal{X}_\gamma^{\text{sf}}$ in the semiflat geometry, if we choose γ to be the unit magnetic charge, with $Z_\gamma = \frac{q^2}{2\pi i} (a \log \frac{a}{\Lambda} - a)$ and $\theta_\gamma = \theta_m$. A direct computation verifies that

$$\begin{aligned} \frac{d\mathcal{X}_m^{\text{sf}}}{\mathcal{X}_m^{\text{sf}}} &= \left[id\theta_m + 2\pi i A^{\text{sf}} + \pi i V^{\text{sf}} \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right) \right] \\ &\quad - \frac{iq^2}{4\pi} \left(\log \frac{a}{\Lambda} - \log \frac{\bar{a}}{\Lambda} \right) \frac{d\mathcal{X}_e}{\mathcal{X}_e}, \end{aligned} \quad (4.30)$$

and hence in particular

$$\varpi^{\text{sf}}(\zeta) = -\frac{1}{4\pi^2 R} \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \frac{d\mathcal{X}_m^{\text{sf}}}{\mathcal{X}_m^{\text{sf}}}, \quad (4.31)$$

as expected.

Notice that $\mathcal{X}_m^{\text{sf}}$ has a nontrivial monodromy around $a = 0$: the monodromies of $\log a$ and $\log \bar{a}$ combine with the monodromy of $e^{i\theta_m}$ given in (4.16b) to give

$$\mathcal{X}_m^{\text{sf}} \rightarrow (-1)^q \mathcal{X}_e^{q^2} \mathcal{X}_m^{\text{sf}}. \quad (4.32)$$

Next we include the instanton corrections. As we will demonstrate below, we can give the desired \mathcal{X}_m obeying (4.24) by the integral formula

$$\begin{aligned} \mathcal{X}_m &= \mathcal{X}_m^{\text{sf}} \exp \left[\frac{iq}{4\pi} \int_{\ell_+} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log[1 - \mathcal{X}_e(\zeta')^q] \right. \\ &\quad \left. - \frac{iq}{4\pi} \int_{\ell_-} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log[1 - \mathcal{X}_e(\zeta')^{-q}] \right], \end{aligned} \quad (4.33)$$

where we choose the contours ℓ_{\pm} to be any paths connecting 0 to ∞ which lie in the two half-planes

$$\mathcal{U}_{\pm} = \left\{ \zeta : \pm \text{Re} \frac{a}{\zeta} < 0 \right\}. \quad (4.34)$$

The two integral contributions in (4.33) come respectively from instanton corrections of positive and negative winding around S^1 .

In the rest of this section we verify that (4.33) is indeed correct. This amounts to verifying

$$-\frac{1}{4\pi^2 R} \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \frac{d\mathcal{X}_m}{\mathcal{X}_m} = \mathfrak{w}(\zeta). \quad (4.35)$$

From (4.33) we have

$$\frac{d\mathcal{X}_m}{\mathcal{X}_m} = \frac{d\mathcal{X}_m^{\text{sf}}}{\mathcal{X}_m^{\text{sf}}} + \mathcal{I}_+ + \mathcal{I}_-, \quad (4.36)$$

where

$$\mathcal{I}_{\pm} = -\frac{iq^2}{4\pi} \int_{\ell_{\pm}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \left[\frac{\mathcal{X}_e(\zeta')^{\pm q}}{1 - \mathcal{X}_e(\zeta')^{\pm q}} \frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')} \right]. \quad (4.37)$$

(Here we used the fact that the integrals in (4.33) depend on $a, \bar{a}, \theta_m, \theta_e$ only through $\mathcal{X}_e(\zeta')$, and are absolutely convergent, so we are free to bring the differential d inside.) Combining (4.35), (4.36), and (4.31), we see that the integrals \mathcal{I}_{\pm} need to give the instanton part $\mathfrak{w}^{\text{inst}}(\zeta)$ on the right side of (4.35), i.e. we need

$$\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge (\mathcal{I}_+ + \mathcal{I}_-) = \frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \left[2\pi i A^{\text{inst}} + \pi i V^{\text{inst}} \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right) \right]. \quad (4.38)$$

To check this we first note that

$$\frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \mathcal{I}_{\pm} = -\frac{iq^2}{4\pi} \int_{\ell_{\pm}} \frac{d\zeta'}{\zeta'} \left(\frac{\zeta' + \zeta}{\zeta' - \zeta} \frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')} \right) \left[\frac{\mathcal{X}_e(\zeta')^{\pm q}}{1 - \mathcal{X}_e(\zeta')^{\pm q}} \right] \quad (4.39)$$

and the two-form which appears here can be rewritten,

$$\begin{aligned} \frac{\zeta' + \zeta}{\zeta' - \zeta} \frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')} &= \frac{\zeta' + \zeta}{\zeta' - \zeta} \frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \left[\frac{d\mathcal{X}_e(\zeta')}{\mathcal{X}_e(\zeta')} - \frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \right] \\ &= -\pi R \frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \left[\left(\frac{1}{\zeta'} + \frac{1}{\zeta} \right) da - (\zeta' + \zeta) d\bar{a} \right], \end{aligned} \quad (4.41)$$

using the explicit form (4.22) of \mathcal{X}_e . Hence the left side of (4.38) becomes

$$\begin{aligned} & \frac{iq^2 R}{4} \frac{d\mathcal{X}_e(\zeta)}{\mathcal{X}_e(\zeta)} \wedge \left(\int_{\ell_+} \frac{d\zeta'}{\zeta'} \left[\left(\frac{1}{\zeta'} + \frac{1}{\zeta} \right) da - (\zeta' + \zeta) d\bar{a} \right] \frac{\mathcal{X}_e(\zeta')^q}{1 - \mathcal{X}_e(\zeta')^q} \right. \\ & \left. + \int_{\ell_-} \frac{d\zeta'}{\zeta'} \left[\left(\frac{1}{\zeta'} + \frac{1}{\zeta} \right) da - (\zeta' + \zeta) d\bar{a} \right] \frac{\mathcal{X}_e(\zeta')^{-q}}{1 - \mathcal{X}_e(\zeta')^{-q}} \right). \end{aligned} \quad (4.42)$$

Now we are ready to evaluate the integrals. It is convenient first to deform each of the contours ℓ_{\pm} to a canonical choice lying exactly in the middle of \mathcal{U}_{\pm} , i.e. to choose

$$\ell_{\pm} = \left\{ \zeta : \pm \frac{a}{\zeta} \in \mathbb{R}_- \right\}. \quad (4.43)$$

We first consider the terms which multiply ζ or $\frac{1}{\zeta}$. Expanding the geometric series we obtain:

$$\int_{\ell_+} \frac{d\zeta'}{\zeta'} \frac{\mathcal{X}_e(\zeta')^q}{1 - \mathcal{X}_e(\zeta')^q} = \sum_{n>0} \int_{\ell_+} \frac{d\zeta'}{\zeta'} \exp \left[\pi Rqn \frac{a}{\zeta'} + iqn\theta_e + \pi Rqn\zeta' \bar{a} \right] \quad (4.44)$$

$$= \sum_{n>0} 2e^{iqn\theta_e} K_0(2\pi Rq|na|). \quad (4.45)$$

The integral over ℓ_- in (4.42) gives a similar sum over $n < 0$. Altogether we find that the terms which multiply ζ or $\frac{1}{\zeta}$ in (4.42) equal

$$\begin{aligned} & \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \left(\frac{iq^2 R}{2} \sum_{n \neq 0} e^{iqn\theta_e} K_0(2\pi Rq|na|) \right) \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right) \\ & = \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge i\pi V^{\text{inst}} \left(\frac{1}{\zeta} da - \zeta d\bar{a} \right). \end{aligned} \quad (4.46)$$

For the remaining terms in (4.42) we use similarly

$$\int_{\ell_+} \frac{d\zeta'}{\zeta'} \zeta' \frac{\mathcal{X}_e(\zeta')^q}{1 - \mathcal{X}_e(\zeta')^q} = \sum_{n>0} \int_{\ell_+} \frac{d\zeta'}{\zeta'} \zeta' \exp \left[\pi Rqn \frac{a}{\zeta'} + iqn\theta_e + \pi Rqn\zeta' \bar{a} \right] \quad (4.47)$$

$$= - \sum_{n>0} 2 \frac{|a|}{\bar{a}} e^{iqn\theta_e} K_1(2\pi Rq|na|) \quad (4.48)$$

and

$$\int_{\ell_+} \frac{d\zeta'}{\zeta'} \frac{1}{\zeta'} \frac{\mathcal{X}_e(\zeta')^q}{1 - \mathcal{X}_e(\zeta')^q} = \sum_{n>0} \int_{\ell_+} \frac{d\zeta'}{\zeta'} \frac{1}{\zeta'} \exp \left[\pi Rqn \frac{a}{\zeta'} + iqn\theta_e + \pi Rqn\zeta' \bar{a} \right] \quad (4.49)$$

$$= - \sum_{n>0} 2 \frac{|a|}{a} e^{iqn\theta_e} K_1(2\pi Rq|na|). \quad (4.50)$$

Combining these with their counterparts from the integral over ℓ_- (which come with an extra minus sign), we see that these terms in (4.42) equal

$$\begin{aligned} \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge \left(-\frac{iq^2R}{2} \sum_{n \neq 0} e^{iqn\theta_e} (\text{sgn } n) |a| K_1(2\pi Rq|na|) \right) \left(\frac{da}{a} - \frac{d\bar{a}}{\bar{a}} \right) \\ = \frac{d\mathcal{X}_e}{\mathcal{X}_e} \wedge 2\pi i A^{\text{inst}}. \end{aligned} \quad (4.51)$$

So finally, summing (4.46) and (4.51), we obtain (4.38) as desired: differentiating the contour integrals in \mathcal{X}_m has correctly produced the instanton corrections V^{inst} and A^{inst} . This finishes the check that \mathcal{X}_m is the desired ‘‘magnetic’’ complex coordinate on \mathcal{M} .

Remark \mathcal{X}_m is closely related to the so-called ‘‘ Q function’’ in the theory of quantum integrable systems.¹⁰ We feel this is not a coincidence and points to some deep relation to integrable field theories. This feeling is reinforced by the fact that the crucial Eq. (5.13) below is a form of the Thermodynamic Bethe Ansatz, as explained in Appendix E.

4.4 Analytic properties

We now consider the analytic behavior of the pair $(\mathcal{X}_m, \mathcal{X}_e)$ on the ζ -plane.

For \mathcal{X}_e the story is simple: it is analytic for $\zeta \in \mathbb{C}^\times$, with essential singularities at $\zeta = 0, \infty$. For $\mathcal{X}_m^{\text{sf}}$ the same is true, but for the full \mathcal{X}_m the story is more intricate: the integrals in (4.33) are analytic in ζ only away from the contours ℓ_\pm . As ζ crosses either of these contours, the pole in the integrand crosses the path of integration. Therefore our expression for \mathcal{X}_m defines a *piecewise* analytic function, with the discontinuity determined by the residue of the pole. Introduce the notation $(\mathcal{X}_m)_{\ell_+}^+, (\mathcal{X}_m)_{\ell_+}^-$ for the limit of \mathcal{X}_m as ζ approaches ℓ_+ in the clockwise or counterclockwise direction respectively, and similar notation for ℓ_- . The discontinuity is then given by

$$(\mathcal{X}_m)_{\ell_+}^+ = (\mathcal{X}_m)_{\ell_+}^- (1 - \mathcal{X}_e^q)^{-q}, \quad (4.52a)$$

$$(\mathcal{X}_m)_{\ell_-}^+ = (\mathcal{X}_m)_{\ell_-}^- (1 - \mathcal{X}_e^{-q})^q. \quad (4.52b)$$

These discontinuities will play a crucial role for us below: indeed we will identify them with Kontsevich-Soibelman symplectomorphisms, as follows. We consider the pair of complex functions $(\mathcal{X}_m, \mathcal{X}_e)$ as giving a map

$$\mathcal{X} : \mathcal{M}_a \rightarrow T_a \quad (4.53)$$

from the real 2-torus \mathcal{M}_a coordinatized by (θ_m, θ_e) to a complexified 2-torus T_a coordinatized by (X_m, X_e) . The map \mathcal{X} varies as a function of ζ (and a, \bar{a}, R). In Sect. 2.2 we introduced the Kontsevich-Soibelman factors \mathcal{K}_γ as symplectomorphisms of T_a . Our discontinuities (4.52) say that at the ray ℓ_\pm , \mathcal{X}^+ and \mathcal{X}^- differ by composition with $\mathcal{K}_{0, \pm q}$.

¹⁰ We thank S. Lukyanov for sharing his notes on these functions with us.

An interesting phenomenon has occurred here. Consider the monodromy of \mathcal{X}_m in the a -plane around $a = 0$. This monodromy receives two contributions: the monodromy of $\mathcal{X}_m^{\text{sf}}$ given in (4.32) and the contributions from (4.52). These two contributions actually cancel one another! This fact is essentially related to the fact that the singularity of the semiflat metric at $a = 0$ has been smoothed out. On the other hand, if we analytically continue \mathcal{X}_m around $\zeta = 0$ it does not come back to itself. This monodromy does not create any problems. In particular, $\mathfrak{O}(\zeta)$ does behave well near $\zeta = 0$: it just has a simple pole, as one expects from the discussion in Sect. 3.1.

Now let us consider the asymptotics of $\mathcal{X}_e, \mathcal{X}_m$ as $\zeta \rightarrow 0, \infty$. The asymptotics of \mathcal{X}_e can be trivially read off from (4.22),

$$\mathcal{X}_e \sim \begin{cases} \exp\left[\pi R \frac{a}{\zeta} + i\theta_e\right] & \text{as } \zeta \rightarrow 0, \\ \exp\left[\pi R \zeta \bar{a} + i\theta_e\right] & \text{as } \zeta \rightarrow \infty. \end{cases} \quad (4.54)$$

For \mathcal{X}_m the asymptotics are more interesting. As $\zeta \rightarrow 0, \infty$ the integrand of (4.33) simplifies: the rational function just reduces to ± 1 . Then expanding the logarithm and evaluating the integral gives

$$\mathcal{X}_m \sim \begin{cases} \exp\left[-i \frac{Rq^2}{2\zeta} (a \log(a/\Lambda) - a) + i\theta_m + \frac{q}{2\pi i} \sum_{s \neq 0} \frac{1}{s} e^{isq\theta_e} K_0(2\pi Rq|sa|)\right] & \text{as } \zeta \rightarrow 0, \\ \exp\left[i \frac{\zeta Rq^2}{2} (\bar{a} \log(\bar{a}/\bar{\Lambda}) - \bar{a}) + i\theta_m - \frac{q}{2\pi i} \sum_{s \neq 0} \frac{1}{s} e^{isq\theta_e} K_0(2\pi Rq|sa|)\right] & \text{as } \zeta \rightarrow \infty. \end{cases} \quad (4.55)$$

These asymptotics hold for all phases of ζ . The discontinuities (4.52) along ℓ_{\pm} do not lead to discontinuities in the asymptotics, because the jump is exponentially close to 1 as $\zeta \rightarrow 0, \infty$ along ℓ_{\pm} : along ℓ_+ we have $\mathcal{X}_e \rightarrow 0$ exponentially fast, and along ℓ_- , $\mathcal{X}_e^{-1} \rightarrow 0$ exponentially fast.

On the other hand, we could also have defined a different function \mathcal{X}'_m , by analytically continuing \mathcal{X}_m across ℓ_+ clockwise. It follows from (4.52a) that on the clockwise side of ℓ_+ we have

$$\mathcal{X}'_m = \mathcal{X}_m (1 - \mathcal{X}_e^q)^q. \quad (4.56)$$

Suppose now that we analytically continue \mathcal{X}'_m further, clockwise to the boundary of \mathcal{U}_+ and then across into \mathcal{U}_- . In \mathcal{U}_- , \mathcal{X}_e is exponentially large as $\zeta \rightarrow 0$. So from (4.56) it follows that the $\zeta \rightarrow 0$ asymptotics of \mathcal{X}'_m and \mathcal{X}_m are different; in particular, \mathcal{X}'_m does not obey (4.55). Thus, the asymptotics of the analytic continuation of the function \mathcal{X}_m is not the analytic continuation of the asymptotics. This is the hallmark of Stokes' phenomenon.

Altogether, we have been led to consider a map $\mathcal{X} : \mathcal{M}_a \rightarrow T_a$, which depends holomorphically on ζ , and exhibits Stokes phenomena at $\zeta \rightarrow 0, \infty$, with Stokes factors given by composition with the Kontsevich-Soibelman symplectomorphisms acting on T_a . The crucial idea of this paper is that this picture is valid for general gauge theories, not just the abelian theory we considered here; indeed, as we will see in Sect. 5, it automatically incorporates multi-instanton effects from mutually non-local particles, and gives the exact metric on \mathcal{M} .

4.5 Differential equations

Above we saw that the hyperkähler geometry of \mathcal{M} is naturally described in terms of a map \mathcal{X} which exhibits Stokes phenomena. Stokes phenomena typically arise in the theory of linear ordinary differential equations with irregular singular points. Indeed, in our case there is such a differential equation

$$\zeta \partial_\zeta \mathcal{X} = \mathcal{A}_\zeta \mathcal{X}, \quad (4.57)$$

with an irregular singularity. In this section we identify this equation. In fact, at the same time we will find a companion equation, governing the dependence on the radius of S^1 ,

$$R \partial_R \mathcal{X} = \mathcal{A}_R \mathcal{X}. \quad (4.58)$$

Equations of the form (4.57), (4.58) are commonly encountered for finite-dimensional matrices $\mathcal{A}_\zeta, \mathcal{A}_R, \mathcal{X}$. Then \mathcal{A}_ζ and \mathcal{A}_R act on \mathcal{X} by matrix multiplication from the left, and the Stokes factors act from the right, so in particular the two actions commute. In our case the solution \mathcal{X} is a map $\mathcal{M}_a \rightarrow T_a$. The Stokes factors act as diffeomorphisms of T_a . \mathcal{A}_ζ and \mathcal{A}_R act as infinitesimal diffeomorphisms of \mathcal{M}_a , i.e. as differential operators in (θ_m, θ_e) . These two actions commute with one another because they act on different spaces.

Now what is the origin of the desired equations? They should be related to some symmetries of (\mathcal{M}, g) . At first glance (\mathcal{M}, g) would appear to have a $U(1)$ symmetry which just maps $a \mapsto e^{i\theta} a$. Such a symmetry would have an obvious physical origin: it would come from a $U(1)_R$ symmetry of the theory in $d = 4$. However, we know that this symmetry is actually anomalous once we include the matter hypermultiplet. Indeed, A^{sf} from (4.9) contains the factor $\log(a/\Lambda)$, which is invariant only under a *simultaneous* rotation of a and Λ . This simultaneous rotation hence leaves the metric invariant. It does not preserve the hyperkähler forms \mathcal{O} , but rather rotates ω_1 and ω_2 into one another; hence it leaves $\mathfrak{w}(\zeta)$ invariant if combined with the action $\zeta \mapsto e^{i\theta} \zeta$. By inspection, both \mathcal{X}_e and \mathcal{X}_m are invariant under this combined rotation of a, Λ and ζ , which leads to a differential equation:

$$\zeta \partial_\zeta \mathcal{X} = (-\Lambda \partial_\Lambda + \bar{\Lambda} \partial_{\bar{\Lambda}} - a \partial_a + \bar{a} \partial_{\bar{a}}) \mathcal{X}. \quad (4.59)$$

Similarly the anomalous scale invariance of the $d = 4$ theory leads to a symmetry which rescales R, a and Λ :

$$R \partial_R \mathcal{X} = (\Lambda \partial_\Lambda + \bar{\Lambda} \partial_{\bar{\Lambda}} + a \partial_a + \bar{a} \partial_{\bar{a}}) \mathcal{X}. \quad (4.60)$$

These equations are not yet of the desired form (4.57), (4.58) since they still involve derivatives with respect to the parameters $(\Lambda, \bar{\Lambda}, a, \bar{a})$. So let us consider the dependence on these parameters.

The dependence of \mathcal{X} on (a, \bar{a}) is completely determined in terms of the dependence on (θ_e, θ_m) , by the requirement that $(\mathcal{X}_e, \mathcal{X}_m)$ are holomorphic in complex structure $J^{(\zeta)}$. Indeed, using the basis (4.19), (4.20) for $(T^*)^{1,0} \mathcal{M}$, we see that the Cauchy-Riemann equations on \mathcal{M} are simply

$$\partial_a \mathcal{X} = \mathcal{A}_a \mathcal{X}, \quad (4.61)$$

$$\partial_{\bar{a}} \mathcal{X} = \mathcal{A}_{\bar{a}} \mathcal{X}, \quad (4.62)$$

where the connection form \mathcal{A} is defined by

$$\mathcal{A}_a = \frac{1}{\zeta} [-i\pi R \partial_{\theta_e} + \pi(V + 2\pi i R A_{\theta_e}) \partial_{\theta_m}] + 2\pi A_a \partial_{\theta_m}, \quad (4.63)$$

$$\mathcal{A}_{\bar{a}} = 2\pi A_{\bar{a}} \partial_{\theta_m} - \zeta [i\pi R \partial_{\theta_e} + \pi(V - 2\pi i R A_{\theta_e}) \partial_{\theta_m}]. \quad (4.64)$$

We can similarly dispose of the $(\Lambda, \bar{\Lambda})$ dependence. First note that \mathcal{X}_e is simply independent of $(\Lambda, \bar{\Lambda})$. For \mathcal{X}_m we have $\Lambda \frac{\partial \mathcal{X}_m}{\partial \Lambda} = \frac{iRq^2 a}{2\zeta} \mathcal{X}_m$, and similarly for $\bar{\Lambda}$. So writing

$$\mathcal{A}_\Lambda = \frac{q^2 R a}{2\zeta} \partial_{\theta_m}, \quad \mathcal{A}_{\bar{\Lambda}} = \frac{\zeta q^2 R \bar{a}}{2} \partial_{\theta_m}, \quad (4.65)$$

we have

$$\Lambda \partial_\Lambda \mathcal{X} = \mathcal{A}_\Lambda \mathcal{X}, \quad (4.66)$$

$$\bar{\Lambda} \partial_{\bar{\Lambda}} \mathcal{X} = \mathcal{A}_{\bar{\Lambda}} \mathcal{X}. \quad (4.67)$$

We can now recast Eqs. (4.59), (4.60) in the desired form (4.57), (4.58), with

$$\mathcal{A}_\zeta = -a \mathcal{A}_a + \bar{a} \mathcal{A}_{\bar{a}} - \Lambda \mathcal{A}_\Lambda + \bar{\Lambda} \mathcal{A}_{\bar{\Lambda}}, \quad (4.68)$$

$$\mathcal{A}_R = a \mathcal{A}_a + \bar{a} \mathcal{A}_{\bar{a}} + \Lambda \mathcal{A}_\Lambda + \bar{\Lambda} \mathcal{A}_{\bar{\Lambda}}. \quad (4.69)$$

Now we come to the crucial point: \mathcal{A}_ζ as given in (4.68) depends on ζ in a very simple way — it has only simple poles at $\zeta = 0, \infty$:

$$\mathcal{A}_\zeta = \frac{1}{\zeta} \mathcal{A}_\zeta^{(-1)} + \mathcal{A}_\zeta^{(0)} + \zeta \mathcal{A}_\zeta^{(1)}. \quad (4.70)$$

Equation (4.57) thus defines a meromorphic connection on \mathbb{CP}^1 , with two irregular singularities of rank 1. This motivates the appearance of Stokes phenomena, which we saw explicitly in the previous section.

A family of differential equations very similar to (4.57), (4.58), (4.61), (4.62), defining the “ tt^* connection,” appeared in (11; 39) in the context of the analysis and classification of $\mathcal{N} = (2, 2)$ field theories in $d = 2$. The similarity is not just formal. In particular, the interpretation of their equations for the ζ and R dependence was also in terms of $U(1)_R$ symmetry and scale transformations of the underlying field theory. A crucial point of their analysis is a direct relation between the large R asymptotics of the connection \mathcal{A} , the explicit form of the Stokes factors, and the degeneracies of BPS states in the $d = 2$ theory. There is a similar relation in our problem as well. Indeed this relation is the key to understanding the Kontsevich-Soibelman wall-crossing formula.

A look at (11; 39) also suggests a very useful technical tool for making further progress: we should convert the differential equations into a Riemann-Hilbert problem for \mathcal{X} , defined directly in terms of the Stokes data and asymptotics as $\zeta \rightarrow 0, \infty$. Using this tool we can immediately write down the generalization to multiple mutually non-local BPS instantons. We move to that problem in Sect. 5.

4.6 Higher spin multiplets

So far we have considered in some detail the corrections to g which come from integrating out a single electrically charged hypermultiplet. One can ask similarly about the corrections due to a single electrically charged higher spin multiplet — for example the vector multiplet containing the massive W boson.

In principle these corrections could be determined by a careful one-loop computation in three dimensions. Instead we exploit a trick: we consider the massive vector multiplet of an $\mathcal{N} = 4$ supersymmetric theory. Decomposing under $\mathcal{N} = 2$ supersymmetry this multiplet contains two hypermultiplets and one vector multiplet. On the other hand, because of the higher supersymmetry in the $\mathcal{N} = 4$ theory, one expects that the metric on \mathcal{M} will not receive any instanton corrections. The reason is that according to standard nonrenormalization theorems the Higgs branch is uncorrected (40), but the nonanomalous R -symmetry mixes the Higgs and Coulomb branches. It follows that the corrections from the $\mathcal{N} = 2$ vector multiplet must precisely cancel those from the two $\mathcal{N} = 2$ hypermultiplets. In other words, at least as far as these two $\mathcal{N} = 2$ multiplets are concerned, the corrections are weighted by the helicity supertrace $\Omega(\gamma; u)$.

More generally we may consider integrating out $\mathcal{N} = 2$ multiplets with arbitrary spin. Let a_j denote the weight multiplying the instanton correction from the j^{th} $\mathcal{N} = 2$ multiplet ($j = 0$ for the hypermultiplet, $j = 1$ for the vector, \dots), normalized to $a_0 = 1$. We saw above that $a_1 = -2$. Moreover, from the fact that the contribution from *any* multiplet of $\mathcal{N} = 4$ supersymmetry should vanish, we get $a_{j+2} + 2a_{j+1} + a_j = 0$. This determines $a_j = (-1)^j(j+1)$, so indeed the instanton corrections are weighted by the second helicity supertrace.

4.7 Higher rank generalization

All of our discussion can be easily generalized to the case of a rank r abelian gauge theory coupled to a set of electrically charged hypermultiplets. Let the charges be $q_I^{(s)}$, where $I = 1, \dots, r$ runs over the electric gauge fields, and s labels the set of hypermultiplets.

There is a $4r$ -dimensional generalization of the Gibbons-Hawking ansatz, with base $(\mathbb{R}^3)^r$ and a fiber $(S^1)^r$. We use coordinates $(x^{\alpha I}) = \mathcal{E}^I$ for the base, $\theta_{m,I}$ for the fiber, and write

$$g = [V(\mathcal{E})^{-1}]^{IJ} \left(\frac{d\theta_{m,I}}{2\pi} + A_I(\mathcal{E}) \right) \left(\frac{d\theta_{m,J}}{2\pi} + A_J(\mathcal{E}) \right) + V(\mathcal{E})_{IJ} d\mathcal{E}^I d\mathcal{E}^J, \quad (4.71)$$

where A and V are related by differential equations stating that

$$\omega^\alpha = dx^{\alpha I} \wedge \left(\frac{d\theta_{m,I}}{2\pi} + A_I \right) + \frac{1}{2} V_{IJ} \epsilon^{\alpha\beta\gamma} dx^{\beta I} \wedge dx^{\gamma J} \quad (4.72)$$

is a closed 2-form for $\alpha = 1, 2, 3$.

The 1-loop integral gives the natural result

$$V_{IJ} = \text{Im} \tau_{IJ}^0 + \sum_s \frac{q_I^{(s)} q_J^{(s)} R}{4\pi} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{|q_K^{(s)} R a^K|^2 + (q_K^{(s)} \frac{\theta_K}{2\pi} + n)^2}} - \kappa_n \right). \quad (4.73)$$

Poisson resummation of (4.4) shows that

$$V_{IJ} = V_{IJ}^{\text{sf}} + V_{IJ}^{\text{inst}}, \quad (4.74)$$

with

$$V_{IJ}^{\text{sf}} = \text{Im} \tau_{IJ}^0 - \sum_s \frac{q_I^{(s)} q_J^{(s)} R}{4\pi} \left(\log \frac{q_K^{(s)} a^K}{\Lambda} + \log \frac{q_K^{(s)} \bar{a}^K}{\Lambda} \right), \quad (4.75)$$

$$V_{IJ}^{\text{inst}} = \sum_s \frac{q_I^{(s)} q_J^{(s)} R}{2\pi} \sum_{n \neq 0} e^{inq_K^{(s)} \theta_e^K} K_0(2\pi R |nq_I^{(s)} a^I|). \quad (4.76)$$

Also

$$A_I = A_I^{\text{sf}} + A_I^{\text{inst}}, \quad (4.77)$$

where

$$A_I^{\text{sf}} = \text{Re} \tau_{IJ}^0 \frac{d\theta_e^J}{2\pi} + \sum_s \frac{iq_I^{(s)} q_J^{(s)}}{8\pi^2} \left(\log \frac{q_K^{(s)} a^K}{\Lambda} - \log \frac{q_K^{(s)} \bar{a}^K}{\Lambda} \right) d\theta_e^J, \quad (4.78)$$

$$A_I^{\text{inst}} = -\frac{q_I^{(s)} q_J^{(s)} R}{4\pi} \left(\frac{d\alpha^J}{q_K^{(s)} a^K} - \frac{d\bar{\alpha}^J}{q_K^{(s)} \bar{a}^K} \right) \sum_{n \neq 0} (\text{sgn} n) e^{inq_I^{(s)} \theta_e^I} |q_K^{(s)} a^K| K_1(2\pi R |nq_K^{(s)} a^K|). \quad (4.79)$$

At large R the leading terms in V and A are V^{sf} and A^{sf} . Keeping only these terms, g becomes the semiflat metric with

$$\tau_{IJ} = \tau_{IJ}^0 + \sum_s \frac{q_I^{(s)} q_J^{(s)}}{2\pi i} \log \frac{q_K^{(s)} a^K}{\Lambda}. \quad (4.80)$$

This is the coupling which comes from integrating out the hypermultiplets in $d = 4$.

The holomorphic symplectic form is

$$\omega(\zeta) = -\frac{1}{4\pi^2 R} \xi_e^I \wedge \xi_{m,I}, \quad (4.81)$$

where

$$\xi_e^I = id\theta_e^I + \pi R \left(\frac{d\alpha^I}{\zeta} + \zeta d\bar{\alpha}^I \right), \quad (4.82)$$

$$\xi_{m,I} = id\theta_{m,I} + 2\pi i A_I + i\pi V_{IJ} \left(\frac{d\alpha^J}{\zeta} - \zeta d\bar{\alpha}^J \right). \quad (4.83)$$

As before, the electric coordinates agree with their semiflat approximation,

$$\mathcal{X}_e^I = \exp \left[\pi R \frac{\alpha^I}{\zeta} + i\theta_e^I + \pi R \zeta \bar{\alpha}^I \right]. \quad (4.84)$$

The semiflat approximation to the magnetic ones is

$$\begin{aligned} \mathcal{X}_{m,I}^{\text{sf}} = \exp & \left[\frac{\pi R}{\zeta} \left(\tau_{IJ}^0 a^J + \sum_s \frac{q_I^{(s)}}{2\pi i} q_K^{(s)} a^K \log \frac{q_K^{(s)} a^K}{e\Lambda} \right) + i\theta_{m,I} \right. \\ & \left. + \pi R \zeta \left(\tau_{IJ}^0 \bar{a}^J + \sum_s \frac{q_I^{(s)}}{2\pi i} q_K^{(s)} \bar{a}^K \log \frac{q_K^{(s)} \bar{a}^K}{e\bar{\Lambda}} \right) \right]. \end{aligned} \quad (4.85)$$

The full magnetic coordinates are given by the integral formula

$$\begin{aligned} \mathcal{X}_{m,I} = \mathcal{X}_{m,I}^{\text{sf}} \exp & \left[\sum_s \frac{i q_I^{(s)}}{4\pi} \int_{\ell_+^s} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log [1 - \prod_J \mathcal{X}_e^J (\zeta')^{q_J^{(s)}}] \right. \\ & \left. - \frac{i q_I^{(s)}}{4\pi} \int_{\ell_-^s} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log [1 - \prod_J \mathcal{X}_e^J (\zeta')^{-q_J^{(s)}}] \right], \end{aligned} \quad (4.86)$$

where ℓ_{\pm}^s are any paths connecting 0 to ∞ which lie in the two half-planes

$$\mathcal{U}_{\pm}^s = \left\{ \zeta : \pm \text{Re} \frac{a^K q_K^{(s)}}{\zeta} < 0 \right\}. \quad (4.87)$$

Notice that $\mathcal{X}_{m,I}$ have discontinuities for each hypermultiplet, which as before are given by the Kontsevich-Soibelman symplectomorphisms $\mathcal{K}_{0,q_I^{(s)}}$.

5 Mutually Non-local Corrections

As we have just seen in the simplest nontrivial case, the exact hyperkähler metric g is not equal to the semiflat metric g^{sf} , because of the quantum corrections from instantons corresponding to $d = 4$ BPS states.

In general we expect such a quantum correction for each charge γ supporting a BPS state. These corrections should be weighted by the BPS multiplicities $\Omega(\gamma; u)$. However, we know that $\Omega(\gamma; u)$ can jump as u crosses a wall of marginal stability! So there seems to be a puzzle: will not the quantum corrections to g also jump discontinuously? How is this consistent with the field theory expectation that g should be smooth?

In this section we will give an explicit construction of the exact hyperkähler metric g for large R . We will see that it is indeed smooth, provided that the Kontsevich-Soibelman wall-crossing formula is satisfied. This is our physical interpretation of the WCF.

Expanding g around $R \rightarrow \infty$, we find the resolution of our puzzle: in addition to the contributions from single BPS particles, there are also multi-particle contributions. The discontinuity in the 1-particle contributions is compensated by a discontinuity in the multi-particle sector. See (8) for a related discussion in the $\mathcal{N} = 1$ context.

5.1 Defining the Riemann-Hilbert problem

We take our inspiration from the abelian theory studied in Sect. 4 and construct the metric by solving a Riemann-Hilbert problem in the ζ -plane. We work initially at fixed $u \in \mathcal{B}$, away from the walls of marginal stability. We also choose a fixed quadratic refinement σ at u .

The Riemann-Hilbert problem is formulated in terms of maps \mathcal{X} from the real torus $\widetilde{\mathcal{M}}_u$ to the complexified symplectic $2r$ -torus \widetilde{T}_u which we introduced in our review of the Kontsevich-Soibelman formula (Sect. 2.2). Given any such \mathcal{X} , we pull back the coordinate functions X_γ on \widetilde{T}_u to give functions \mathcal{X}_γ on $\widetilde{\mathcal{M}}_u$, defined by $\mathcal{X}_\gamma(\theta) = X_\gamma(\mathcal{X}(\theta))$. In particular, the $\mathcal{X}_\gamma^{\text{sf}}$ given in Sect. 3.3 come from a reference map \mathcal{X}^{sf} ; it is the zeroth approximation to the \mathcal{X} we construct below.

To formulate the Riemann-Hilbert problem we need to fix the *asymptotic behavior* of \mathcal{X} as $\zeta \rightarrow 0, \infty$ and its *discontinuities* in the ζ -plane.

We begin with the asymptotics. Introduce

$$\Upsilon := \mathcal{X}(\mathcal{X}^{\text{sf}})^{-1}. \quad (5.1)$$

In this section we are using an unconventional notation for composition of maps: $(fg)(x)$ means $g(f(x))$.¹¹ Thus Υ is a map from $\widetilde{\mathcal{M}}_u$ to itself (or more precisely to its complexification). Concretely Υ maps

$$e^{i\theta_i} \mapsto \mathcal{X}^i(\theta) \exp \left[-\pi R \frac{Z_i}{\zeta} - \pi R \zeta \bar{Z}_i \right]. \quad (5.2)$$

We require that the limit of Υ as $\zeta \rightarrow 0$ and $\zeta \rightarrow \infty$ exists,

$$\lim_{\zeta \rightarrow 0} \Upsilon = \Upsilon_0, \quad \lim_{\zeta \rightarrow \infty} \Upsilon = \Upsilon_\infty, \quad (5.3)$$

and moreover obeys

$$\Upsilon_0 = \widetilde{\Upsilon}_\infty. \quad (5.4)$$

Next we need to specify the discontinuities of \mathcal{X} , considered as a piecewise-analytic function of ζ . Assume temporarily that u does not lie on any wall of marginal stability. The discontinuities will be given in terms of the Kontsevich-Soibelman symplectomorphisms $\mathcal{K}_\gamma : \widetilde{T}_u \rightarrow \widetilde{T}_u$ associated to the BPS states. To each ray ℓ through the origin in the ζ -plane, we associate a subset of Γ_u ,

$$(\Gamma_u)_\ell := \{\gamma : Z_\gamma(u)/\zeta \in \mathbb{R}_- \text{ for } \zeta \in \ell\}, \quad (5.5)$$

and a corresponding product over BPS states:

$$S_\ell := \prod_{\gamma \in (\Gamma_u)_\ell} \mathcal{K}_\gamma^{\Omega(\gamma; u)}. \quad (5.6)$$

¹¹ One virtue of this notation can be seen by observing that the diagram $A \xrightarrow{f} B \xrightarrow{g} C$ composes to $A \xrightarrow{fg} C$. A second virtue will become apparent in Sect. 5.5.

(Since u does not lie on a wall, $(\Gamma_u)_\ell$ is at most one-dimensional, and the \mathcal{K}_γ for $\gamma \in (\Gamma_u)_\ell$ all commute; hence we do not have to specify the ordering in this product.) Since the charge lattice Γ_u is countable, for all but a countable set of rays ℓ we have $(\Gamma_u)_\ell = \emptyset$ and thus $S_\ell = 1$. We refer to rays for which $(\Gamma_u)_\ell \neq \emptyset$ as ‘‘BPS rays’’.

The most canonical choice of discontinuities is to require that

$$\mathcal{X}^+ = \mathcal{X}^- S_\ell, \quad (5.7)$$

where \mathcal{X}^+ , \mathcal{X}^- are the limit of \mathcal{X} as ζ approaches ℓ clockwise, counterclockwise respectively. This is the most straightforward generalization of what we found in Sect. 4: there we found a map $\mathcal{X} = (\mathcal{X}_m, \mathcal{X}_e)$ which was sectionally analytic in ζ , with two BPS rays ℓ_\pm across which \mathcal{X}_e was continuous and \mathcal{X}_m jumped according to (4.52). These two BPS rays corresponded to the single hypermultiplets of charge $(0, \pm q)$, and the discontinuity was exactly (5.7), with $S_{\pm\ell} = \mathcal{K}_{0, \pm q}$. We are now generalizing to include many BPS particles, just by requiring jumps along many BPS rays. In this more general situation there will be no \mathcal{X}_γ that is continuous everywhere.

We have now formulated our Riemann-Hilbert problem. Its solution is not unique: rather it is determined only up to a transformation $\mathcal{X} \rightarrow b\mathcal{X}$, with b an arbitrary diffeomorphism of $\widetilde{\mathcal{M}}_u$. We will fix this ambiguity in a convenient way when we solve the problem in the next section.

This Riemann-Hilbert problem might appear a bit unconventional since it is formulated in terms of \mathcal{X} and S_ℓ , which are not linear maps, but more general maps of manifolds. The concerned reader should feel free to ‘‘linearize’’ the problem by considering, instead of \mathcal{X} , the operation \mathcal{X}^* of pullback $C^\infty(\widetilde{T}_u) \rightarrow C^\infty(\widetilde{\mathcal{M}}_u)$. The price of doing so is that not every map $Q : C^\infty(\widetilde{T}_u) \rightarrow C^\infty(\widetilde{\mathcal{M}}_u)$ can be obtained as \mathcal{X}^* for some map \mathcal{X} ; so if we find a solution Q to the linear version of the Riemann-Hilbert problem, we face the extra difficulty of checking that $Q = \mathcal{X}^*$ for some \mathcal{X} . Fortunately the ‘‘functoriality’’ of the Riemann-Hilbert problem comes to the rescue. Q will be \mathcal{X}^* for some \mathcal{X} if and only if it preserves multiplication, $Q(fg) = Q(f)Q(g)$. Since all the data defining the linear problem is compatible with this structure, the solution is as well.

Finally let us discuss a reality property of our problem, which will be crucial for our construction of the hyperkähler metric. Thanks to the relations $\Omega(\gamma; u) = \Omega(-\gamma; u)$ our discontinuity conditions enjoy a discrete symmetry: given any solution \mathcal{X} , we can obtain another solution which we call $\bar{\mathcal{X}}$ by

$$\bar{\mathcal{X}}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}. \quad (5.8)$$

We claim that in fact our solution is invariant under this transformation,

$$\mathcal{X} = \bar{\mathcal{X}}. \quad (5.9)$$

To see this, consider the map $Y = \bar{\mathcal{X}} \mathcal{X}^{-1}$. Because both $\bar{\mathcal{X}}$ and \mathcal{X} have the same discontinuities, Y is actually *analytic* in ζ . On the other hand, because of our asymptotic condition (5.4), $Y \rightarrow 1$ as $\zeta \rightarrow 0, \infty$. Therefore by Liouville’s theorem we get $Y = 1$.

5.2 The role of the KS formula

Now we come to an important point, which was the main reason for writing this paper. Suppose we find a \mathcal{X} which solves our Riemann-Hilbert problem for any fixed u away from the walls of marginal stability. Then its behavior as a function of ζ is completely determined: it is continuous except at the BPS rays, where it jumps according to (5.7). But what can we say about its behavior as a function of u ?

The u dependence in our Riemann-Hilbert problem comes from two places. One is in the asymptotic boundary conditions (5.3); this dependence is certainly continuous. The other is in the discontinuity prescription (5.7). Here too the dependence is continuous as long as u stays away from the walls of marginal stability. But what happens as we cross the wall? Let u_w denote a generic point on a wall. As $u \rightarrow u_w$ from one side of the wall, BPS rays corresponding to charges $\gamma = n\gamma_1 + m\gamma_2$ collide with one another, coalescing into a single ray ℓ . Let A denote the total discontinuity of \mathcal{X} across this group of rays,

$$A = \prod_{\substack{\gamma = n\gamma_1 + m\gamma_2 \\ m > 0, n > 0}} \overset{\curvearrowright}{\mathcal{K}}_{\gamma}^{\Omega(\gamma, u)}. \quad (5.10)$$

Assuming that $\lim_{u \rightarrow u_w} \mathcal{X}$ from this side exists, it is the solution to a Riemann-Hilbert problem in which the discontinuity across ℓ is A (while the discontinuities along all other BPS rays are specified as before).

On the other hand, we could also consider $\lim_{u \rightarrow u_w} \mathcal{X}$ from the other side of the wall. For the two limits to agree, it is necessary and sufficient that they are solutions of the *same* Riemann-Hilbert problem: so this requires that A computed by (5.10) is the same on both sides of the wall. As we reviewed in Sect. 2.2, this is precisely the content of the KS wall-crossing formula!¹²

We conclude that, *assuming the BPS degeneracies obey the KS formula*, a solution \mathcal{X} of the Riemann-Hilbert problem is continuous as a function of u and ζ , except at the BPS rays. Moreover, the discontinuity across the BPS ray is given by a symplectomorphism.

5.3 Solving the Riemann-Hilbert problem

Having formulated the Riemann-Hilbert problem, we would like to see that it has a solution, and understand its large- R behavior. Unlike the simple cases we considered in Sect. 4, for which all of the S_{ℓ} commute with one another, here we cannot write an explicit integral formula for the desired \mathcal{X} ; we have to proceed more indirectly. We exploit the fact that the problem has a structure very similar to one considered in (39; 11). Indeed our problem is an infinite-dimensional version of the one considered there.

In (11) the Riemann-Hilbert problem is re-expressed as an integral equation for an analog Φ of $\Upsilon(\zeta)$. For large enough R , this equation describes Φ as a small correction of the identity matrix. It can therefore be solved iteratively, which proves

¹² The fact that the product in (5.10) is counterclockwise, while it was clockwise in Sect. 2.2, comes from our unusual convention on composition of maps in Sect. 5.

the existence of a solution for large enough R , and also gives an explicit formula for the leading corrections to the zeroth-order approximation $\Phi = 1$. These leading corrections are expressed directly in terms of the discontinuity factors.

This is exactly the sort of information we would like to find about our map \mathcal{X} . One direct approach would be to write down an infinite dimensional analogue of the integral equation in (11). This approach is directly applicable only to a linear Riemann-Hilbert problem, so one would have to pass to the linear problem mentioned at the end of the previous subsection. The solution of the integral equation would then give a linear map between the function spaces; as we have described, this linear map would be \mathcal{X}^* for some map $\mathcal{X} : \widetilde{\mathcal{M}}_u \rightarrow \widetilde{\mathcal{T}}_u$.

One minor issue is that if we follow precisely the prescription of (11) we will get a solution obeying the boundary condition $\mathcal{I}_0 = 1$. For our construction we need a different choice of boundary condition, namely (5.4), which has the advantage of being compatible with the reality condition $\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}$.¹³ Fortunately, it is straightforward to write a variant of the integral equation which takes into account this different choice of boundary condition, by a slight modification of the integral kernel.

This strategy seems good enough to prove the existence of a solution, but it has an important drawback: the intermediate steps of the iterative solution need not be of the form \mathcal{X}^* for any \mathcal{X} . It is useful to have a realization of the problem where each step in the approximation scheme is itself a map $\widetilde{\mathcal{M}}_u \rightarrow \widetilde{\mathcal{T}}_u$. This is possible if we write the following integral equation, using the abelian group structure on $\widetilde{\mathcal{T}}_u$:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \left[\frac{1}{4\pi i} \sum_{\ell} \int_{\ell} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log \frac{\mathcal{X}_\gamma(\zeta')}{(\mathcal{X}S_\ell)_\gamma(\zeta')} \right]. \quad (5.11)$$

Here the sum runs over BPS rays ℓ . Any solution of (5.11) obeys the discontinuity conditions (5.7). Moreover, our choice of integral kernel ensures that the solution will also obey the reality condition (5.4). Hence a solution of (5.11) is a solution of the Riemann-Hilbert problem.¹⁴

Using the explicit form of the Kontsevich-Soibelman factors from (2.16), we have

$$(\mathcal{X}S_\ell)_\gamma = \mathcal{X}_\gamma \prod_{\gamma' \in (\Gamma_u)_\ell} (1 - \sigma(\gamma') \mathcal{X}_{\gamma'})^{\Omega(\gamma'; u) \langle \gamma, \gamma' \rangle} \quad (5.12)$$

(with $(\Gamma_u)_\ell$ defined in (5.5)). Plug this into (5.11) to get the final integral equation for \mathcal{X} :

$$\begin{aligned} \mathcal{X}_\gamma(\zeta) = & \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \\ & \times \left[-\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta')) \right]. \end{aligned} \quad (5.13)$$

¹³ In this section we write $\mathcal{X} = \mathcal{X}(\zeta)$ explicitly, thinking of \mathcal{X} as a map which varies with ζ , and hence suppress the dependence on the coordinates θ of $\widetilde{\mathcal{M}}_u$.

¹⁴ Note that although the Riemann-Hilbert problem is invariant under diffeomorphisms of $\widetilde{\mathcal{M}}_u$, Eq. (5.11) is not; its solution is unique, not unique up to diffeomorphism.

As we have mentioned, Eq. (5.13) is a form of the Thermodynamic Bethe Ansatz. See Appendix E.

In Appendix C we argue that (5.13) has a solution for sufficiently large R , and describe its expansion as $R \rightarrow \infty$ for u away from the walls. The first nontrivial approximation is

$$\begin{aligned} \mathcal{X}_\gamma(\zeta) &\sim \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \\ &\times \left[-\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}^{\text{sf}}(\zeta')) \right], \end{aligned} \quad (5.14)$$

and is essentially a linear superposition of the 1-instanton corrections that we found in the abelian theory. Higher-order corrections involve multilinear in the $\Omega(\gamma'; u)$, and have an R dependence which identifies them as multi-instanton contributions.

Our arguments in Appendix C are closely related to ones given in (11) in the finite-dimensional tt^* context. In fact, our approach leads to a simplification of the asymptotic analysis even in the finite-dimensional case; hence in Appendix C we re-analyze that case as well.

Global issues.

By solving the Riemann-Hilbert problem, we have obtained a map $\mathcal{X} : \widetilde{\mathcal{M}}_u \rightarrow \widetilde{T}_u$ depending on the choice of the local quadratic refinement $\sigma(\gamma)$. This choice affects the Riemann-Hilbert problem through the definition of the discontinuities \mathcal{K}_γ . However, the solution \mathcal{X} depends on σ in a simple way. Recall that for any two refinements σ, σ' there is some $c(\sigma, \sigma') \in \Gamma_u^*/2\Gamma_u^*$ such that $\sigma(\gamma)\sigma'(\gamma) = (-1)^{\gamma \cdot c(\sigma, \sigma')}$. Given a solution $\mathcal{X}^{[\sigma]}$ of (5.11) with refinement σ , there is a corresponding solution $\mathcal{X}^{[\sigma']}$ with refinement σ' ,

$$\mathcal{X}_\gamma^{[\sigma']}(u, \theta; \zeta) = (-1)^{\gamma \cdot c(\sigma, \sigma')} \mathcal{X}_\gamma^{[\sigma]}(u, \theta + c\pi; \zeta). \quad (5.15)$$

It follows that if we use the refinement to identify $\widetilde{\mathcal{M}}_u \simeq \mathcal{M}_u$ and also $\widetilde{T}_u \simeq T_u$, we obtain $\mathcal{X} : \mathcal{M}_u \rightarrow T_u$ which is *independent* of the choice of refinement.

5.4 Constructing the symplectic form

So far, we have solved the Riemann-Hilbert problem to give a map $\mathcal{X} : \mathcal{M}_u \rightarrow T_u$, obeying the asymptotic conditions (5.3), the jump conditions (5.7), and the reality condition (5.9). Now letting u vary we obtain a map $\mathcal{X} : \mathcal{M} \rightarrow T$. We then construct a complex 2-form $\varpi(\zeta)$ on \mathcal{M} by pullback of the canonical fiberwise symplectic form on T ,

$$\varpi(\zeta) = \frac{1}{4\pi^2 R} \mathcal{X}^* \varpi^T = \frac{1}{8\pi^2 R} \varepsilon_{ij} \frac{d\mathcal{X}_{\gamma^i}}{\mathcal{X}_{\gamma^i}} \wedge \frac{d\mathcal{X}_{\gamma^j}}{\mathcal{X}_{\gamma^j}}. \quad (5.16)$$

A few properties of $\varpi(\zeta)$ follow directly from (5.16):

- Although \mathcal{X} is only piecewise analytic in ζ , $\varpi(\zeta)$ is honestly analytic (because the discontinuities S_γ are symplectomorphisms, i.e. they preserve ϖ^T).
- Using (5.9), we have $\varpi(-1/\bar{\zeta}) = \overline{\varpi(\zeta)}$.
- As $\zeta \rightarrow 0, \infty$ we can determine the behavior of $\varpi(\zeta)$ using the asymptotics (5.3) of \mathcal{X} and the explicit form (3.15) of $\varpi^{\text{sf}}(\zeta)$. We find that $\varpi(\zeta)$ has a simple pole in each case, with residue

$$\text{Res}_{\zeta=0}\varpi(\zeta) = \frac{i}{8\pi}\Upsilon_0^*\langle dZ, d\theta \rangle, \quad \text{Res}_{\zeta=\infty}\varpi(\zeta) = -\frac{i}{8\pi}\Upsilon_\infty^*\langle d\bar{Z}, d\theta \rangle. \quad (5.17)$$

- Using $\lim_{R \rightarrow \infty} \mathcal{X} = \mathcal{X}^{\text{sf}}$, it follows that $\varpi(\zeta)$ is nondegenerate (in the holomorphic sense) for large enough R .

These properties will be important in our construction of the hyperkähler metric.

5.5 Differential equations

Our Riemann-Hilbert problem has been formulated in terms of discontinuity factors which are universal (locally independent of all parameters of the gauge theory), together with asymptotics given by the functions $\mathcal{X}_\gamma^{\text{sf}}$, which depend on the parameters only in a very simple way. In this section, following a standard recipe, we show that this implies that the solution \mathcal{X} obeys a family of differential equations.

As we will see, the physical meaning of these equations is rather transparent. One group expresses the fact that the functions $\mathcal{X}(\zeta)$ which solve the Riemann-Hilbert problem are holomorphic on \mathcal{M} in complex structure $J(\zeta)$. These equations are essential for the construction of the hyperkähler metric. Another pair describe the renormalization group flow and a $U(1)_R$ -symmetry action. These are important for relating the metric to the KS wall-crossing formula.

A very similar family of equations were crucial in the story of “ tt^* geometry” which appeared in the context of massive $\mathcal{N} = (2, 2)$ -dimensional theories (11; 1; 13; 41).

We begin by recalling that the solution \mathcal{X} of our Riemann-Hilbert problem over \mathbb{CP}^1 is only sectionally analytic; it has jumps of the form $\mathcal{X} \rightarrow \mathcal{X} S_\ell$ along various rays $\ell \subset \mathbb{CP}^1$. So consider instead¹⁵

$$\mathcal{A}_\zeta := \zeta \partial_\zeta \mathcal{X} \mathcal{X}^{-1}. \quad (5.19)$$

¹⁵ This is the standard notation, but in our context it is somewhat mnemonic, so here is a longer description. The infinitesimal variation of the map \mathcal{X} by applying $\zeta \partial_\zeta$ gives a vector field on T , which we call $\zeta \partial_\zeta \mathcal{X}$. We then pull this back using \mathcal{X} to get the vector field \mathcal{A}_ζ on \mathcal{M}_u . We write this pullback operation as \mathcal{X}^{-1} , and because of our non-standard convention for composition, this \mathcal{X}^{-1} appears on the right rather than the left; this makes our equation agree with the usual form for Riemann-Hilbert problems, and in fact this agreement is the reason we use the non-standard convention in Sect. 5. In local coordinates one would write

$$\mathcal{A}_\zeta = \frac{\partial \mathcal{X}^i}{\partial \zeta} \left[\left(\frac{\partial \mathcal{X}}{\partial \theta} \right)^{-1} \right]_j^i \frac{\partial}{\partial \theta^j}. \quad (5.18)$$

The discontinuities of \mathcal{X} along the BPS rays cancel out in \mathcal{A}_ζ , which is therefore honestly analytic in ζ , except possibly for $\zeta = 0, \infty$ where \mathcal{X} becomes singular. So we can think of \mathcal{X} as a solution of an ordinary differential equation in ζ ,

$$\zeta \partial_\zeta \mathcal{X} = \mathcal{A}_\zeta \mathcal{X}. \quad (5.20)$$

We can describe this equation rather concretely, using our asymptotic information about \mathcal{X} . Note first that \mathcal{X}^{sf} obeys an equation of the same form. To write it we first introduce two vector fields on \mathcal{M}_u ,

$$\mathcal{A}_\zeta^{(-1),\text{sf}} := i\pi Z \cdot \partial_\theta, \quad \mathcal{A}_\zeta^{(1),\text{sf}} := i\pi \bar{Z} \cdot \partial_\theta. \quad (5.21)$$

Then we have

$$\zeta \partial_\zeta \mathcal{X}^{\text{sf}} = \mathcal{A}_\zeta^{\text{sf}} \mathcal{X}^{\text{sf}}, \quad (5.22)$$

where

$$\mathcal{A}_\zeta^{\text{sf}} = \frac{1}{\zeta} \mathcal{A}_\zeta^{(-1),\text{sf}} + \zeta \mathcal{A}_\zeta^{(1),\text{sf}}. \quad (5.23)$$

The important point is that the ζ dependence of $\mathcal{A}_\zeta^{\text{sf}}$ is very simple: just a simple pole at each of $\zeta = 0, \infty$. We can convert this information to information about \mathcal{A}_ζ , since we know from (5.3) that $\Upsilon = \mathcal{X} (\mathcal{X}^{\text{sf}})^{-1}$ remains finite at both $\zeta = 0, \infty$. This shows that \mathcal{A}_ζ also has only a simple pole at $\zeta = 0, \infty$, and even determines the residue,

$$\mathcal{A}_\zeta = \frac{1}{\zeta} \mathcal{A}_\zeta^{(-1)} + \mathcal{A}_\zeta^{(0)} + \zeta \mathcal{A}_\zeta^{(1)}, \quad (5.24)$$

where

$$\mathcal{A}_\zeta^{(-1)} = \Upsilon_0 \mathcal{A}_\zeta^{(-1),\text{sf}} \Upsilon_0^{-1}, \quad \mathcal{A}_\zeta^{(1)} = \Upsilon_\infty \mathcal{A}_\zeta^{(1),\text{sf}} \Upsilon_\infty^{-1}. \quad (5.25)$$

So we see that (5.20) defines a flat connection $\zeta \partial_\zeta - \mathcal{A}_\zeta$ over \mathbb{CP}^1 , valued in the infinite-dimensional algebra of vector fields on \mathcal{M}_u , with rank-1 irregular singularities at $\zeta = 0, \infty$. \mathcal{X} is a flat section for this connection.

So our solution to the Riemann-Hilbert problem leads directly to the construction of a flat connection over \mathbb{CP}^1 . In fact, this is a standard maneuver in the theory of ordinary differential equations. The connection we obtained has irregular singularities at $\zeta = 0$ and $\zeta = \infty$, and hence it exhibits Stokes' phenomenon. One of the virtues of the Riemann-Hilbert construction is that it is easy to determine the Stokes factors: they are simply the discontinuities S_ℓ which entered the Riemann-Hilbert problem.

The above discussion has an important extension. We have not just a single Riemann-Hilbert problem but a whole family of them, varying with additional parameters. These parameters include the coordinates u^i on \mathcal{B} , as well as the scale Λ , the radius R of S^1 , and perhaps some bare gauge couplings τ^0 . (For the moment we do *not* introduce mass parameters; but see Sect. 6 below.) We introduce the generic notation t^n to encompass all of these parameters.

Importantly, the discontinuities S_ℓ which define the Riemann-Hilbert problem do not depend on any of the t^n . Hence just as we did above for the ζ dependence, we consider

$$\mathcal{A}_n := \partial_{t^n} \mathcal{X} \mathcal{X}^{-1}. \quad (5.26)$$

As before, the discontinuities of \mathcal{X} cancel out, so \mathcal{A}_n is analytic in ζ away from $\zeta = 0, \infty$. Also as before, we can control the behavior near these singularities by first checking the behavior of $\mathcal{A}_n^{\text{sf}} := \partial_{t^n} \mathcal{X}^{\text{sf}} (\mathcal{X}^{\text{sf}})^{-1}$. For all of our t^n we have

$$\mathcal{A}_n^{\text{sf}} = \frac{1}{\zeta} \mathcal{A}_n^{(-1), \text{sf}} + \zeta \mathcal{A}_n^{(1), \text{sf}} \quad (5.27)$$

for some simple vector fields $\mathcal{A}_n^{(\pm 1), \text{sf}}$; then using the fact that Υ is finite as $\zeta \rightarrow 0, \infty$ as before, we obtain

$$\mathcal{A}_n = \frac{1}{\zeta} \mathcal{A}_n^{(-1)} + \mathcal{A}_n^{(0)} + \zeta \mathcal{A}_n^{(1)}, \quad (5.28)$$

where

$$\mathcal{A}_n^{(-1)} = \Upsilon_0 \mathcal{A}_n^{(-1), \text{sf}} \Upsilon_0^{-1}, \quad \mathcal{A}_n^{(1)} = \Upsilon_\infty \mathcal{A}_n^{(1), \text{sf}} \Upsilon_\infty^{-1}. \quad (5.29)$$

Also including (5.20), the full set of equations we obtain is

$$\partial_{u^j} \mathcal{X} = \left(\frac{1}{\zeta} \mathcal{A}_{u^j}^{(-1)} + \mathcal{A}_{u^j}^{(0)} \right) \mathcal{X}, \quad (5.30)$$

$$\partial_{\bar{u}^j} \mathcal{X} = \left(\mathcal{A}_{\bar{u}^j}^{(0)} + \zeta \mathcal{A}_{\bar{u}^j}^{(1)} \right) \mathcal{X}, \quad (5.31)$$

$$\Lambda \partial_\Lambda \mathcal{X} = \left(\frac{1}{\zeta} \mathcal{A}_\Lambda^{(-1)} + \mathcal{A}_\Lambda^{(0)} \right) \mathcal{X}, \quad (5.32)$$

$$\bar{\Lambda} \partial_{\bar{\Lambda}} \mathcal{X} = \left(\mathcal{A}_{\bar{\Lambda}}^{(0)} + \zeta \mathcal{A}_{\bar{\Lambda}}^{(1)} \right) \mathcal{X}, \quad (5.33)$$

$$R \partial_R \mathcal{X} = \left(\frac{1}{\zeta} \mathcal{A}_R^{(-1)} + \mathcal{A}_R^{(0)} + \zeta \mathcal{A}_R^{(1)} \right) \mathcal{X}, \quad (5.34)$$

$$\zeta \partial_\zeta \mathcal{X} = \left(\frac{1}{\zeta} \mathcal{A}_\zeta^{(-1)} + \mathcal{A}_\zeta^{(0)} + \zeta \mathcal{A}_\zeta^{(1)} \right) \mathcal{X}. \quad (5.35)$$

One also gets the extra relations

$$\mathcal{A}_R^{(-1)} = -\mathcal{A}_\zeta^{(-1)}, \quad \mathcal{A}_R^{(1)} = \mathcal{A}_\zeta^{(1)}, \quad (5.36)$$

from the fact that \mathcal{X}^{sf} is annihilated by $\zeta \partial_\zeta + R \partial_R$ as $\zeta \rightarrow 0$, and by $\zeta \partial_\zeta - R \partial_R$ as $\zeta \rightarrow \infty$.

We have finished constructing our equations. In Appendix D we discuss how to write them more concretely given the asymptotic expansion of \mathcal{X} around $\zeta = 0$. We conclude this section with a few remarks:

- Since the symplectic form $\varpi(\zeta)$ was constructed from \mathcal{X} , (5.35), (5.34) trivially imply equations for the ζ and R dependence of $\varpi(\zeta)$, of the form

$$\zeta \partial_\zeta \varpi = \left(\frac{1}{\zeta} \mathcal{L}_{\mathcal{A}_\zeta^{(-1)}} + \mathcal{L}_{\mathcal{A}_\zeta^{(0)}} + \zeta \mathcal{L}_{\mathcal{A}_\zeta^{(1)}} \right) \varpi, \quad (5.37)$$

$$R \partial_R (R \varpi) = \left(\frac{1}{\zeta} \mathcal{L}_{\mathcal{A}_R^{(-1)}} + \mathcal{L}_{\mathcal{A}_R^{(0)}} + \zeta \mathcal{L}_{\mathcal{A}_R^{(1)}} \right) (R \varpi). \quad (5.38)$$

Recalling from (3.4) that $\varpi(\zeta) = -\frac{i}{2\zeta} \omega^+ + \omega_3 - \frac{i}{2} \zeta \omega_-$, these equations can be expanded in powers of ζ to derive some interesting differential equations for the hyperkahler forms \mathfrak{G} .

- Recall that the solution \mathcal{X} of the Riemann-Hilbert problem was ambiguous up to a transformation $\mathcal{X} \rightarrow b \mathcal{X}$, with b any diffeomorphism of \mathcal{M}_u . This ambiguity leads to ζ -independent gauge transformations of the connection \mathcal{A} . There are several particularly convenient gauges. One is a gauge in which $\mathcal{A}_R^{(0)} = 0$. It follows from (5.38) that in this gauge the restriction of $R \omega_3$ to each \mathcal{M}_u is independent of R (and hence equals its $R \rightarrow \infty$ limit, namely $-\frac{1}{8\pi^2} \langle d\theta, d\theta \rangle$.) It would be interesting to know whether this gauge is the one chosen by our integral equation (5.11). If we allow b to be a *complexified* diffeomorphism, then at least formally we can also pick a gauge in which $\Upsilon_0 = 1$, so $\mathcal{A}^{(-1)} = \mathcal{A}^{(-1),\text{sf}}$; this is an analogue of the ‘‘topological gauge’’ of (11) (dually $\Upsilon_\infty = 1$ would be an ‘‘antitopological gauge’’).
- Two linear combinations of our equations have a simple physical meaning: they express the invariance under overall changes of scale and R-symmetry transformations. To see this first note that

$$(a^I \partial_{a^I} + \Lambda \partial_\Lambda) Z_\gamma = Z_\gamma \quad (5.39)$$

for all $\gamma \in \Gamma$. It follows that

$$(R \partial_R - a^I \partial_{a^I} - \bar{a}^I \partial_{\bar{a}^I} - \Lambda \partial_\Lambda - \bar{\Lambda} \partial_{\bar{\Lambda}}) \mathcal{X}^{\text{sf}} = 0, \quad (5.40)$$

$$(\zeta \partial_\zeta + a^I \partial_{a^I} - \bar{a}^I \partial_{\bar{a}^I} + \Lambda \partial_\Lambda - \bar{\Lambda} \partial_{\bar{\Lambda}}) \mathcal{X}^{\text{sf}} = 0. \quad (5.41)$$

These equations can be interpreted as the (anomalous) scale and R-symmetry invariance of the semiflat geometry. They imply relations among the $\mathcal{A}_n^{\text{sf}}$ (just by replacing $\partial \rightarrow \mathcal{A}$) which in turn give relations among $\mathcal{A}_n^{(\pm 1)}$: we find that

$$(R \partial_R - a^I \partial_{a^I} - \bar{a}^I \partial_{\bar{a}^I} - \Lambda \partial_\Lambda - \bar{\Lambda} \partial_{\bar{\Lambda}}) \mathcal{X} = \Delta \mathcal{X}, \quad (5.42)$$

$$(\zeta \partial_\zeta + a^I \partial_{a^I} - \bar{a}^I \partial_{\bar{a}^I} + \Lambda \partial_\Lambda - \bar{\Lambda} \partial_{\bar{\Lambda}}) \mathcal{X} = \Delta' \mathcal{X}, \quad (5.43)$$

where Δ, Δ' are ζ -independent vector fields on \mathcal{M}_u . We can set $\Delta = 0, \Delta' = 0$ by a gauge transformation. Indeed, our integral equation automatically picks the appropriate gauge: the recursive solution we give in Appendix C for large R satisfies (5.42), (5.43) term-by-term with $\Delta = \Delta' = 0$. So there is a sense in which the scale and R-symmetry invariance survive the instanton corrections.

- The compatibility between (5.34) and (5.35), together with the relations (5.36), implies a set of nonlinear differential equations for the R dependence of the quadruple $(\mathcal{A}_\zeta^{(\pm 1)}, \mathcal{A}_\zeta^{(0)}, \mathcal{A}_R^{(0)})$. These equations are a deformation of the Nahm equations, as we explain in Appendix D; the large R expansion of this quadruple can be produced directly by solving them iteratively. They are a possible tool for studying the behavior of our construction at *small* R . A generic solution of the Nahm equations would become singular at a finite value of R , and we expect that the same is true for our problem. Nevertheless, we expect that the particular solutions which we have described here, determined by the BPS degeneracies in $\mathcal{N} = 2$, $d = 4$ field theories, actually *are* regular for all values of R . It is possible that this gives an interesting constraint on the possible BPS spectra and IR prepotentials of $\mathcal{N} = 2$ theories. A very similar strategy was employed in (11) to constrain the properties of $d = 2$ theories.
- Our discussion in this section gives a new perspective on the role of the wall-crossing formula. The collection of Eqs. (5.30)–(5.35) describe a flat connection over $\mathbb{CP}^1 \times \mathcal{P}$, where \mathcal{P} is the parameter space coordinatized by the t^i . This flat connection can be viewed equivalently as an *isomonodromic family* of connections over \mathbb{CP}^1 , with irregular singularities of rank 1 at $\zeta = 0, \infty$. At each $t \in \mathcal{P}$ the Stokes data of the connection on \mathbb{CP}^1 are given by the Kontsevich-Soibelman factors. Using the parallel transport along \mathcal{P} , one shows that the Stokes data at the irregular singularities are “invariant” in an appropriate sense. To be precise: choosing any convex sector \mathcal{V} in the ζ -plane, the product

$$A_{\mathcal{V}} = \prod_{\ell \subset \mathcal{V}}^{\curvearrowright} S_\ell \quad (5.44)$$

is invariant, under any variation of $t \in \mathcal{P}$ for which no Stokes line ℓ enters or leaves \mathcal{V} . Applying this statement to variations of u , we recover the wall-crossing formula.

5.6 Constructing the metric and its large R asymptotics

So far we have constructed a family of functions $\mathcal{X}_\gamma(\zeta)$ on \mathcal{M} , and the corresponding holomorphic symplectic form $\varpi(\zeta)$. As we have discussed in Sect. 3.2, given $\varpi(\zeta)$ with the properties listed in Sect. 5.4, and \mathcal{X}_γ obeying “Cauchy-Riemann” equations of the form (5.30), (5.31), there exists a corresponding hyperkähler metric g on \mathcal{M} . This is our construction of g .

Given the exact functions $\mathcal{X}_\gamma(\zeta)$ solving (5.13), we can write g in closed form as follows. Use the expansion of the kernel in (5.13) for $|\zeta'/\zeta| < 1$ to obtain an asymptotic expansion for $\zeta \rightarrow 0$,

$$\log \mathcal{X}_\gamma = \frac{1}{\zeta} F_{-1}^\gamma + F_0^\gamma + \zeta F_1^\gamma + \mathcal{O}(\zeta^2), \quad (5.45)$$

where

$$\begin{aligned} F_{-1}^\gamma &= \pi R Z_\gamma, \\ F_0^\gamma &= i\theta_\gamma - \frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \log(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta')), \\ F_1^\gamma &= \pi R \bar{Z}_\gamma - \frac{1}{2\pi i} \sum_{\gamma'} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'^2} \log(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta')). \end{aligned} \quad (5.46)$$

Then, substituting into ω we extract

$$\begin{aligned} \omega_+ &= \frac{i}{2\pi^2 R} \varepsilon_{ij} dF_{-1}^{\gamma^i} \wedge dF_0^{\gamma^j}, \\ \omega_3 &= \frac{1}{8\pi^2 R} \varepsilon_{ij} \left(2dF_1^{\gamma^i} \wedge dF_{-1}^{\gamma^j} + dF_0^{\gamma^i} \wedge dF_0^{\gamma^j} \right). \end{aligned} \quad (5.47)$$

From these symplectic forms it is straightforward to obtain g .

Now let us consider the behavior of g for large R . In Sect. 5.3 we have discussed the large R asymptotics of the \mathcal{X}_γ , including the first BPS instanton correction, given in (5.14). Now we translate this into the correction to $\omega(\zeta)$. We begin by computing the correction to $\frac{d\mathcal{X}_\gamma}{\mathcal{X}_\gamma}$:

$$\frac{d\mathcal{X}_\gamma}{\mathcal{X}_\gamma} = \frac{d\mathcal{X}_\gamma^{\text{sf}}}{\mathcal{X}_\gamma^{\text{sf}}} + \mathcal{I}_\gamma + \dots, \quad (5.48)$$

where

$$\mathcal{I}_\gamma = \frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \frac{d\mathcal{X}_{\gamma'}^{\text{sf}}(\zeta')}{\mathcal{X}_{\gamma'}^{\text{sf}}(\zeta')} \frac{\sigma(\gamma') \mathcal{X}_{\gamma'}^{\text{sf}}(\zeta')}{1 - \sigma(\gamma') \mathcal{X}_{\gamma'}^{\text{sf}}(\zeta')}. \quad (5.49)$$

Note that \mathcal{I}_γ is exponentially suppressed as $R \rightarrow \infty$ as promised, since on ℓ we have $\mathcal{X}_{\gamma'}^{\text{sf}} \rightarrow 0$ exponentially as $R \rightarrow \infty$. The ellipsis in (5.48) indicates the multi-instanton corrections, which are even more suppressed. The leading correction to $\omega(\zeta)$ therefore arises from the wedge product between $\frac{d\mathcal{X}_\gamma^{\text{sf}}}{\mathcal{X}_\gamma^{\text{sf}}}$ and \mathcal{I}_γ .

To describe the correction more explicitly, it is convenient to consider each γ' separately, and adopt a symplectic basis $\{\gamma^1, \dots, \gamma^{2r}\}$ in which $\gamma' = q_\gamma \gamma^1$. Then the integral in (5.49) becomes essentially identical to the integral (4.37), which gave the instanton corrections to \mathcal{X}_m in Sect. 4.3. Evaluating the corresponding correction to $\omega(\zeta)$ just as we did there, we obtain

$$\omega(\zeta) = \omega^{\text{sf}}(\zeta) + \sum_{\gamma \in \Gamma} \omega_\gamma^{\text{inst}}(\zeta) + \dots, \quad (5.50)$$

where

$$\omega_\gamma^{\text{inst}}(\zeta) = -\Omega(\gamma'; u) \frac{1}{4\pi^2 R} \frac{d\mathcal{X}_{\gamma'}^{\text{sf}}(\zeta)}{\mathcal{X}_{\gamma'}^{\text{sf}}(\zeta)} \left[A_\gamma^{\text{inst}} + \frac{1}{2} V_\gamma^{\text{inst}} \left(\frac{1}{\zeta} da_\gamma - \zeta d\bar{a}_\gamma \right) \right], \quad (5.51)$$

with (cf. (4.7), (4.10))

$$V_{\gamma'}^{\text{inst}} = \frac{Rq_{\gamma'}^2}{2\pi} \sum_{n>0} \sigma(n\gamma') e^{in\theta_{\gamma'}} K_0(2\pi R|nZ_{\gamma'}|), \quad (5.52)$$

$$A_{\gamma'}^{\text{inst}} = -\frac{Rq_{\gamma'}^2}{4\pi} \left(\frac{dZ_{\gamma'}}{Z_{\gamma'}} - \frac{d\bar{Z}_{\gamma'}}{\bar{Z}_{\gamma'}} \right) \sum_{n>0} \sigma(n\gamma') e^{in\theta_{\gamma'}} |Z_{\gamma'}| K_1(2\pi R|nZ_{\gamma'}|). \quad (5.53)$$

From here one may expand in ζ to extract the leading corrections to ω_+ , ω_3 and hence obtain the leading correction to g .

5.7 Comparison to the physical metric

Having constructed a hyperkähler metric g on \mathcal{M} for large enough R , we now summarize some of its properties:

1. g is continuous,
2. g approaches the semiflat metric g^{sf} if all BPS particles have $|Z| \rightarrow \infty$,
3. g is smooth except for specific physically expected singularities, located over the singular loci in \mathcal{B} ,
4. g has $\text{vol}(\mathcal{M}_u) = \left(\frac{1}{R}\right)^r$,
5. (\mathcal{M}, g) in complex structure J_3 can be identified with the Seiberg-Witten torus fibration in its standard complex structure, and after this identification, the holomorphic symplectic form is $\omega_+ = -\frac{1}{4\pi} \langle dZ, d\theta \rangle$.

All of these properties agree with what is expected for the physical metric on $\mathbb{R}^3 \times S^1$ as described in (12). The simplest consistent picture is therefore that the metric we have constructed is indeed the physical one. (In the rank 1 case it was suggested in (12) that these properties indeed *determine* the metric, by a non-compact analogue of Yau's theorem. It is plausible that there could be a similar theorem more generally.)

In the rest of this section we establish these properties from our construction:

1. The continuity of g follows from the wall-crossing formula, as we have explained.
2. We need only look at the form of the corrections (5.51): they are all exponentially suppressed in $R|Z_{\gamma'}|$, and hence vanish exponentially fast if all $|Z_{\gamma'}| \rightarrow \infty$.
3. In any limit where $R|Z_{\gamma'}| \rightarrow \infty$ for all γ , the instanton contributions are exponentially suppressed and g approaches g^{sf} . This is enough to establish the smoothness of g at large R , *except* near a singular locus where some BPS particles with charges γ_i become massless ($Z_{\gamma_i} = 0$ and $\Omega(\gamma_i; u) \neq 0$). To understand the behavior near these points, we consider a scaling limit where $R \rightarrow \infty$ holding RZ_{γ_i} finite. One can approximate the Riemann-Hilbert problem in this limit by one in which we keep only the BPS rays ℓ_{γ_i} , dropping all the others. Indeed all other discontinuities involve factors of the form $(1 - \sigma(\gamma') \mathcal{X}_{\gamma'})$, which become exponentially close to 1 in this scaling limit. In the simplest case where only a single $Z_{\gamma} = 0$, we can always choose a duality frame such that γ is an electric charge. By shifting some of the angles θ by π , we can also arrange that the refinement σ is of the standard form $\sigma = (-1)^{\gamma_e \cdot \gamma_m}$ for this frame. Then we are in the situation we studied in Sect. 4,

where we found a hyperkähler metric which is smooth except for a periodic array of qA_{q-1} singularities. This agrees with the expectation from effective field theory in $d = 3$: a singularity occurs at the point where one of the Kaluza-Klein tower of charge- q hypermultiplets becomes massless.

In addition to the physical singularities we have examined, where a set of mutually local BPS particles become massless, there can also be superconformal points, where mutually nonlocal particles simultaneously become massless (42; 43). We have not analyzed these singularities, although we expect them to be interesting, and we expect the quadratic refinement to play an important role in their analysis.

4. Since \mathcal{M}_u is a complex torus with respect to J_3 , its volume is just $\frac{1}{r!} \int_{\mathcal{M}_u} \omega_3^r$. On the other hand, using (3.4) and the fact that ω_{\pm} restrict to zero on \mathcal{M}_u by (5.17), this is

$$\text{vol}(\mathcal{M}_u) = \frac{1}{r!} \int_{\mathcal{M}_u} \overline{\omega}^r(\zeta) = \frac{1}{(4\pi^2 R)^r r!} \int_{\mathcal{X}(\mathcal{M}_u)} (\overline{\omega}^T)^r = \left(\frac{1}{R}\right)^r, \quad (5.54)$$

as desired.

5. Complex structure J_3 can be determined from ω_1 and ω_2 , just by $J_3 = \omega_1^{-1} \omega_2$. But this information in turn is given by the residue of $\overline{\omega}(\zeta)$ at $\zeta = 0$; recall from (3.4) that $\omega_+ = \omega_1 + i\omega_2$ is given by

$$\omega_+ = 2i \text{Res}_{\zeta=0} \overline{\omega}(\zeta). \quad (5.55)$$

Our asymptotic condition (5.3) on $\overline{\omega}(\zeta)$ precisely ensures that this is related to the residue of $\overline{\omega}^{\text{sf}}(\zeta)$: indeed we just have

$$\omega_+ = \mathcal{Y}_0^* \omega_+^{\text{sf}}. \quad (5.56)$$

It follows that (\mathcal{M}, J_3) can be identified with $(\mathcal{M}, J_3^{\text{sf}})$ just by acting with the fiberwise diffeomorphism \mathcal{Y}_0 . As we explained in Sect. 3.3, the complex structure J_3^{sf} on \mathcal{M} is just that of the Seiberg-Witten torus fibration. Moreover, under this identification ω_+ is identified with ω_+^{sf} given in (3.16).

6 Adding Masses

In this section we briefly indicate how the results of the previous sections should be modified to include nontrivial mass parameters.

6.1 Single-particle corrections with masses

There is a simple variant of the $U(1)$ theory considered in Sect. 4: we can consider the $U(1)$ theory with several electrically charged hypermultiplets, of charges q_i . A theory with more than one species of particle will involve flavor charges, and depend non-trivially on mass parameters. The mass parameters in four dimensions are complex numbers m_i . Upon compactification to three dimensions an extra real periodic mass parameter m_i^3 appears, which is essentially a Wilson line for the flavor symmetry. We write $\psi_i := 2\pi R m_i^3$, with period 2π .

The mass parameters enter the corrected metric in a very simple fashion: each particle gives an additive contribution to V and A similar to the one we met before,

$$V = \sum_i \frac{q_i^2 R}{4\pi} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{R^2 |q_i a + m|^2 + (q_i \frac{\theta_e}{2\pi} + \frac{\psi_i}{2\pi} + n)^2}} - \kappa_n \right). \quad (6.1)$$

The coordinate \mathcal{X}_e is unchanged:

$$\mathcal{X}_e = \exp \left[\pi R \frac{a}{\zeta} + i\theta_e + \pi R \zeta \bar{a} \right]. \quad (6.2)$$

It is also useful to introduce a similar combination of the mass parameters:

$$\mu_i := \exp \left[\pi R \frac{m_i}{\zeta} + i\psi_i + \pi R \zeta \bar{m}_i \right]. \quad (6.3)$$

The semiflat \mathcal{X}_m receives contributions from integrating out all of the particles in $d = 4$:

$$\begin{aligned} \mathcal{X}_m^{\text{sf}}(\zeta) &= e^{i\theta_m} \prod_i \exp \left[-i \frac{R q_i}{2\zeta} \left((q_i a + m_i) \log \frac{q_i a + m_i}{e\Lambda} \right) \right. \\ &\quad \left. + i \frac{\zeta R q_i}{2} \left((q_i \bar{a} + \bar{m}_i) \log \frac{q_i \bar{a} + \bar{m}_i}{e\bar{\Lambda}} \right) \right]. \end{aligned} \quad (6.4)$$

The monodromy of $\mathcal{X}_m^{\text{sf}}(\zeta)$ around $q_i a + m_i = 0$ is

$$\mathcal{X}_m^{\text{sf}} \rightarrow (-\mu_i)^{q_i} \mathcal{X}_e^{q_i} \mathcal{X}_m^{\text{sf}}. \quad (6.5)$$

The full coordinate similarly receives instanton contributions from all of the particles,

$$\begin{aligned} \mathcal{X}_m &= \mathcal{X}_m^{\text{sf}} \prod_i \exp \left[\frac{i q_i}{4\pi} \int_{\ell_+^i} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log [1 - \mu_i \mathcal{X}_e(\zeta')^{q_i}] \right. \\ &\quad \left. - \frac{i q_i}{4\pi} \int_{\ell_-^i} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log [1 - \mu_i^{-1} \mathcal{X}_e(\zeta')^{-q_i}] \right], \end{aligned} \quad (6.6)$$

where we choose the contours ℓ_{\pm}^i to be any paths in the ζ -plane connecting 0 to ∞ which lie in the two half-planes

$$\mathcal{U}_{\pm}^i = \left\{ \zeta : \pm \text{Re} \frac{q_i a + m_i}{\zeta} < 0 \right\}. \quad (6.7)$$

The discontinuities depend now on the masses:

$$(\mathcal{X}_m)_{\ell_+^i}^+ = (\mathcal{X}_m)_{\ell_+^i}^- (1 - \mu_i \mathcal{X}_e^q)^{-q}, \quad (6.8a)$$

$$(\mathcal{X}_m)_{\ell_-^i}^+ = (\mathcal{X}_m)_{\ell_-^i}^- (1 - \mu_i^{-1} \mathcal{X}_e^{-q})^q. \quad (6.8b)$$

All of the formulas of this section can also be extended to higher rank along the lines of Sect. 4.7.

6.2 Multiple-particle corrections with masses

Now we are ready to understand the role of the mass parameters in the general Riemann-Hilbert and differential problems. Consider a gauge theory with n_f flavor symmetries. Denote the flavor charges as γ^f , and build μ_{γ^f} in the obvious way from the masses and flavor Wilson lines,

$$\mu_{\gamma^f} := \exp \left[\pi R \frac{m_{\gamma^f}}{\zeta} + i\psi_{\gamma^f} + \pi R \zeta \bar{m}_{\gamma^f} \right]. \quad (6.9)$$

The discontinuities of the abelian problem suggest generalized Kontsevich-Soibelman factors, of the form

$$\mathcal{K}_{\gamma, \gamma^f} := \mathcal{X}_{\gamma'} \rightarrow \mathcal{X}_{\gamma'} (1 - \sigma(\gamma) \mu_{\gamma^f} \mathcal{X}_{\gamma'})^{(\gamma, \gamma)}. \quad (6.10)$$

We can then define a Riemann-Hilbert problem similar to that of Sect. 5.1, which associates the discontinuity $\mathcal{K}_{\gamma, \gamma^f}$ to each particle of charge γ and flavor charge γ^f . Assuming that the wall-crossing formula still gives the correct BPS degeneracies when generalized to use these modified symplectomorphisms, we can use this Riemann-Hilbert problem to construct a hyperkähler metric on \mathcal{M} , which we propose is the correct one.

A standard trick in supersymmetric field theory is to regard the mass parameters as vacuum expectation values of vector multiplet scalars of an enhanced theory in which the flavor symmetry has been weakly gauged (44). Because of the weak gauging, the particles with magnetic flavor charge are very heavy and can be neglected in the limit in which the flavor gauge coupling goes to zero; the only flavor gauge charges that remain are electric. Using this trick, the generalized wall-crossing formula with masses can be interpreted as a zero-coupling limit of the standard wall-crossing formula.

Finally we would like to extend the differential formulation of Sect. 5.5 to deal with the mass parameters. In that section we relied on the fact that the factors S_ℓ were independent of the parameters. In our modified problem the S_ℓ depend explicitly on μ_{γ^f} , hence on m_i, R, ζ . However, it is true that all S_ℓ are annihilated by $\partial_{m_i} + i\frac{\pi R}{\zeta} \partial_{\psi_i}$ and by $\partial_{\bar{m}_i} + i\pi R \zeta \partial_{\psi_i}$. Then a slight modification of the arguments of that section shows that the solutions \mathcal{X} of the Riemann-Hilbert problem obey differential equations of the form

$$\partial_{m_i} \mathcal{X} = \left(\frac{1}{\zeta} \mathcal{A}_{m_i}^{(-1)} + \mathcal{A}_{m_i}^{(0)} \right) \mathcal{X}, \quad (6.11)$$

$$\partial_{\bar{m}_i} \mathcal{X} = \left(\mathcal{A}_{\bar{m}_i}^{(0)} + \zeta \mathcal{A}_{\bar{m}_i}^{(1)} \right) \mathcal{X}, \quad (6.12)$$

where $\mathcal{A}_{m_i}^{(-1)}$ is not just a vector field on \mathcal{M}_u but also includes the operator $-i\frac{\pi R}{\zeta} \partial_{\psi_i}$, and similarly for $\mathcal{A}_{\bar{m}_i}^{(1)}$.

The S_ℓ are also annihilated by the R-symmetry and scale invariance operators, so we obtain analogues of (6.13), (6.14) (after passing to an appropriate gauge),

$$(R\partial_R - a^l \partial_{a^l} - \bar{a}^l \partial_{\bar{a}^l} - \Lambda \partial_\Lambda - \bar{\Lambda} \partial_{\bar{\Lambda}} - m^i \partial_{m^i} - \bar{m}^i \partial_{\bar{m}^i}) \mathcal{X} = 0, \quad (6.13)$$

$$(\zeta \partial_\zeta + a^l \partial_{a^l} - \bar{a}^l \partial_{\bar{a}^l} + \Lambda \partial_\Lambda - \bar{\Lambda} \partial_{\bar{\Lambda}} + m^i \partial_{m^i} - \bar{m}^i \partial_{\bar{m}^i}) \mathcal{X} = 0. \quad (6.14)$$

Using these equations we can obtain our standard form (5.34), (5.35) for the R and ζ dependence of \mathcal{X} , again with the modification that \mathcal{A}_R and \mathcal{A}_ζ now involve derivatives with respect to the ψ_i .

7 A Proof of the Wall-Crossing Formula

In this paper we have given a construction of a hyperkähler metric g on \mathcal{M} and argued that it matches the physical metric on the moduli space of the gauge theory on $\mathbb{R}^3 \times S^1$. The Kontsevich-Soibelman wall-crossing formula arose as a consistency condition: without it our construction would not have given a smooth metric. We view this as strong circumstantial evidence that the wall-crossing formula is indeed correct.

However, these constructions do not quite give a *proof* of the wall-crossing formula. To give a proof we need to work directly from the physics of the gauge theory, rather than making any assumptions about what form the metric should take. In this approach we do not have the power of the Riemann-Hilbert construction available to us (at least initially). We use instead the alternative perspective which we described in Sect. 5.5. Let \mathcal{M} be the moduli space of the gauge theory on $\mathbb{R}^3 \times S^1$, and consider maps $\mathcal{X}(\zeta) : \mathcal{M} \rightarrow T$ (for $\zeta \neq 0, \infty$). We aim to construct an integrable set of equations for such $\mathcal{X}(\zeta)$, of the form (5.30)–(5.35), such that the connection (5.35) over $\mathbb{C}\mathbb{P}^1$ has Stokes rays ℓ carrying Stokes factors

$$S_\ell = \prod_{\gamma \in (T_u)_\ell} \mathcal{X}_\gamma^{\Omega(\gamma; u)}. \quad (7.1)$$

Having constructed such differential equations, the WCF would be the statement of isomonodromic deformation for the connection (5.35).

We now describe how to derive these differential equations directly from gauge theory. As we show in Appendix B, (5.30), (5.31) have a simple geometric meaning: they are just the Cauchy-Riemann equations, expressing the holomorphy of $\mathcal{X}(\zeta)$ in the complex structure $J^{(\zeta)}$. In particular, these equations can be understood purely in terms of the $\mathcal{N} = 4$ supersymmetry of the reduced theory. Next note that (5.32), (5.33) are of exactly the same form as (5.30), (5.31). Indeed they would become identical if we consider Λ as the scalar component of a “background” vector multiplet. This is a standard technique for proving non-renormalization theorems, see e.g. (44); applying it here should lead to the desired (5.32), (5.33). If the theory involves mass parameters we can prove (6.11), (6.12) similarly, by weakly gauging the flavor symmetry.

Finally we need to establish the key equations (5.34), (5.35) giving the R and ζ dependence of \mathcal{X} . These follow from the anomalous $U(1)_R$ symmetry and scale invariance of the $d = 4$ theory, as expressed by (5.42), (5.43) or (6.13), (6.14), together with the equations we have already established above. The functions \mathcal{X}_γ have a physical interpretation which we hope to describe elsewhere. They can be viewed as elements of a chiral ring of a three-dimensional topological field theory, or as certain line operator expectation values in the four-dimensional theory. Viewed in these terms Eqs. (5.34), (5.35) are anomalous Ward identities.

To finish the proof we have to show that the Stokes factors for the connection on $\mathbb{C}\mathbb{P}^1$ are indeed given by (7.1). For this we use the fact that the Stokes factors

are invariant under variation of R , thanks to (5.34). We can therefore go to very large R , where (away from the walls) the corrections to the metric should be well approximated by a linear superposition of the 1-instanton corrections we know from the abelian theory. Passing from the connection on \mathbb{CP}^1 to the corresponding Riemann-Hilbert problem, and running the same arguments we used in Sect. 5, we can show that these corrections correspond directly to the Stokes factors. This completes the proof of the wall-crossing formula, at least at a physical level of rigor.

Acknowledgements We would like to give special thanks to F. Denef, N. Nekrasov, and X. Yin for collaboration in the early stages of this project, and to B. Pioline for important discussions on the semiflat metric. We also thank Wu-yen Chuang, E. Diaconescu, D. Jafferis, D. Joyce, M. Kontsevich, S. Lukyanov, N. Seiberg, Y. Soibelman, V. Toledano Laredo, C. Vafa, E. Witten, and E. Zaslow for valuable discussions. The work of GM is supported by the DOE under grant DE-FG02-96ER40949. GM also thanks the Aspen Center for Physics and the KITP at UCSB (supported in part by the National Science Foundation under Grant No. PHY05-51164) for hospitality during the completion of this work. The work of AN is supported in part by the Martin A. and Helen Chooljian Membership at the Institute for Advanced Study, and by the NSF under grant number PHY-0503584. The work of DG is supported in part by DOE Grant DE-FG02-90ER40542.

A Verifying the KS Identity for Some $SU(2)$ Gauge Theories

There is an instructive way to prove the simple formula involving $\mathcal{H}_{1,0}$ and $\mathcal{H}_{0,1}$. Consider a sequence of numbers x_n satisfying the recursion

$$x_{n+1}x_{n-1} = 1 - x_n. \quad (\text{A.1})$$

Surprisingly, the recursion is periodic with period five:

$$x_2 = \frac{1-x_1}{x_0}, \quad x_3 = \frac{x_0+x_1-1}{x_0x_1}, \quad x_4 = \frac{1-x_0}{x_1}, \quad x_5 = x_0, \quad x_6 = x_1. \quad (\text{A.2})$$

Now, set $X_{1,0} = X_{1,0}^{(1)} = x_1^{-1}$ and $X_{0,1} = X_{0,1}^{(1)} = x_0$. Our strategy will be to define successive transformations $(X_{1,0}^{(n+1)}, X_{0,1}^{(n+1)}) = \mathcal{H}_n(X_{1,0}^{(n)}, X_{0,1}^{(n)})$ for an appropriate sequence of KS transformations \mathcal{H}_n until we obtain the identity transformation as $(X_{1,0}^{(1)}, X_{0,1}^{(1)}) \rightarrow (X_{1,0}^{(N)}, X_{0,1}^{(N)})$ (where $N = 6$ in our first example but will be infinite in the remaining examples). In order to avoid cluttering the notation we do not indicate the superscript (n) in what follows.

If we apply $\mathcal{H}_{1,0}$ it does not change the value of $X_{1,0} = x_1^{-1}$, but modifies $X_{0,1}$ to $X_{0,1} = x_0(1-x_1^{-1})^{-1} = -x_1x_2^{-1}$. Notice that $X_{1,1} = X_{1,0}X_{0,1} = -x_2^{-1}$.

We can then apply $\mathcal{H}_{1,1}$: this leaves $X_{1,1} = -x_2^{-1}$ and changes $X_{1,0}$. As a result, now $X_{0,1} = -x_1x_2^{-1}(1-x_2^{-1})^{-1} = x_3^{-1}$.

If we apply $\mathcal{H}_{0,1}$, $X_{0,1} = x_3^{-1}$ and $X_{1,0} = -x_3x_2^{-1}(1-x_3^{-1}) = x_4$. If we apply $\mathcal{H}_{1,0}^{-1}$ $X_{1,0} = x_4$, $X_{0,1} = x_3^{-1}(1-x_4) = x_5$. Finally if we apply $\mathcal{H}_{0,1}^{-1}$ we get $X_{0,1} = x_5 = x_0$, $X_{1,0} = x_4(1-x_5)^{-1} = x_6^{-1} = x_1^{-1}$. Hence we derive the desired

$$\mathcal{H}_{0,1}^{-1} \mathcal{H}_{1,0}^{-1} \mathcal{H}_{0,1} \mathcal{H}_{1,1} \mathcal{H}_{1,0} = 1. \quad (\text{A.3})$$

This was a useful warm-up exercise for more interesting formulae. Consider now a different recursion relation:

$$x_{n+1}x_{n-1} = (1-x_n)^2. \quad (\text{A.4})$$

This recursion is not in general periodic: it has the general solution

$$x_n = -\frac{\cosh^2(an+b)}{\sinh^2 a}. \quad (\text{A.5})$$

We can again relate the recursion to a product of \mathcal{K} factors.

We start again with $X_{0,1} = x_0$ and $X_{1,0} = x_1^{-1}$. If we apply $\mathcal{K}_{1,0}^2$ the result is $X_{1,0} = x_1^{-1}$, $X_{2,1} = x_1^{-2}x_0(1-x_1^{-1})^{-2} = x_2^{-1}$. If we apply $\mathcal{K}_{2,1}^2$ the result is $X_{2,1} = x_2^{-1}$, $X_{3,2} = x_2^{-2}x_1(1-x_2^{-1})^{-2} = x_3^{-1}$. We can keep acting with $\mathcal{K}_{n+1,n}^2$ for all n , following the recursion to arbitrarily large n . We can compute the infinite product by the infinite n limit of the relations $X_{n+1,n} = x_{n+1}^{-1}$ and $X_{n,n-1} = x_n^{-1}$. If we pick the real part of a, b positive, $X_{1,1} = e^{-2a}$ and $X_{1,0} = -e^{-2b}(1-e^{-2a})^2$.

On the other hand we can follow the recursion in the opposite direction: $X_{0,1} = x_0$ and $X_{1,0} = x_1^{-1}$ under $\mathcal{K}_{0,1}^2$ goes to $X_{0,1} = x_0$ and $X_{1,0} = x_1^{-1}(1-x_0)^2 = x_{-1}$. $\mathcal{K}_{1,0}^2$ sends this to $X_{1,0} = x_{-1}$ and $X_{0,1} = x_0(1-x_{-1})^{-2} = x_{-2}^{-1}$.

The latter relation is the image under $\mathcal{K}_{0,1}^2$ of $X_{0,1} = x_{-2}^{-1}$ and $X_{1,2} = x_{-2}^{-2}x_{-1}(1-x_{-2}^{-1})^{-2} = x_{-3}^{-1}$. We can now keep acting with the inverse of $\mathcal{K}_{n,n+1}$ for all n , computing again an infinite product. The large n limit of $X_{n,n+1} = x_{-n-2}^{-1}$ and $X_{n-1,n} = x_{-n-1}^{-1}$ is $X_{1,1} = e^{-2a}$ and $X_{1,0} = -e^{-2b}(1-e^{-2a})^{-2}$.

Hence by following the whole recursion from $n = -\infty$ to $n = \infty$ we can derive an expression for the infinite product

$$\cdots \mathcal{K}_{4,3}^2 \mathcal{K}_{3,2}^2 \mathcal{K}_{2,1}^2 \mathcal{K}_{1,0}^2 \mathcal{K}_{0,1}^{-2} \mathcal{K}_{1,0}^{-2} \mathcal{K}_{0,1}^2 \mathcal{K}_{1,2}^2 \mathcal{K}_{2,3}^2 \cdots \quad (\text{A.6})$$

The map between the limiting values of the recursion is $(X_{1,1}, X_{1,0}) \rightarrow (X_{1,1}, X_{1,0}(1-e^{-2a})^4 = X_{1,0}(1-e^{-4a})^4(1+e^{-2a})^{-4})$, which is the expected $\mathcal{K}_{2,2}^2 \mathcal{K}_{1,1}^{-4}$!

In the main text we related this formula to the wall-crossing behavior of a $SU(2)$ Seiberg-Witten theory with two flavors ($SO(4) = SU(2)_A \times SU(2)_B$ flavor symmetry). We argued that a similar relation should hold, which carries information about flavor charges. The relation should give the reordering of a product $\mathcal{K}_{1,0;1,0} \mathcal{K}_{1,0;-1,0} \mathcal{K}_{0,1;0,1} \mathcal{K}_{0,1;0,-1}$, i.e. the wall-crossing formula for a theory with a $SU(2)_A$ doublet of particles of charge $(1, 0)$ (flavor charge $(\pm 1, 0)$ under the Cartan generators of $SU(2)_A$ and $SU(2)_B$) and a $SU(2)_B$ doublet of particles of charge $(0, 1)$ (flavor charge $(0, \pm 1)$ under the Cartan generators of $SU(2)_A$ and $SU(2)_B$).

The basic transformations are

$$\mathcal{K}_{1,0;1,0} \mathcal{K}_{1,0;-1,0} : (X_{1,0}, X_{0,1}) \rightarrow (X_{1,0}, X_{0,1}(1-\mu_A X_{1,0})^{-1}(1-\mu_A^{-1} X_{1,0})^{-1}) \quad (\text{A.7})$$

and

$$\mathcal{K}_{0,1;0,1} \mathcal{K}_{0,1;0,-1} : (X_{1,0}, X_{0,1}) \rightarrow (X_{1,0}(1-\mu_B X_{0,1})(1-\mu_B^{-1} X_{0,1}), X_{0,1}). \quad (\text{A.8})$$

For this problem we need to alternate the factors from particles in doublets of $SU(2)_A$ or $SU(2)_B$. Let's take

$$x_n = -\frac{1}{2} \frac{\cosh u \cosh v}{\sinh^2 a} + (-1)^n \frac{1}{2} \frac{\sinh u \sinh v}{\sinh^2 a} - \frac{\sqrt{(\cosh 2a + \cosh 2u)(\cosh 2a + \cosh 2v)}}{\sinh^2 2a} \cosh(2an + 2b) \quad (\text{A.9})$$

with u, v to be determined in terms of μ_A, μ_B below.

This satisfies the recursion

$$x_{n+1} x_{n-1} = (1 - e^{u+(-1)^n v} x_n)(1 - e^{-u-(-1)^n v} x_n). \quad (\text{A.10})$$

We can again initialize the recursion as $X_{0,1} = x_0$ and $X_{1,0} = x_1^{-1}$. If we apply $\mathcal{K}_{1,0;1,0}\mathcal{K}_{1,0;-1,0}$ the result is again $X_{1,0} = x_1^{-1}$, $X_{2,1} = x_2^{-1}$, as long as we identify $\mu_A = e^{u-v}$. If we apply then $\mathcal{K}_{2,1;0,1}\mathcal{K}_{2,1;0,-1}$ the result is $X_{2,1} = x_2^{-1}$, $X_{3,2} = x_3^{-1}$, as long as we identify $\mu_B = e^{u+v}$. We can keep acting alternately with the $\mathcal{K}_{2n+1,2n;1,0}$, $\mathcal{K}_{2n+1,2n;-1,0}$ and the $\mathcal{K}_{2n+2,2n+1;0,1}$, $\mathcal{K}_{2n+2,2n+1;0,-1}$ for all n , following the recursion to arbitrarily large n . We can compute the infinite product by the infinite n limit of the relations $X_{n+1,n} = x_{n+1}^{-1}$ and $X_{n,n-1} = x_n^{-1}$. If we pick the real part of a, b positive, $X_{1,1} = e^{-2a}$ and

$$X_{1,0} = -e^{-2b}(1 - e^{-4a})^2(1 + e^{-2a-2u})^{-1/2} \\ \times (1 + e^{-2a+2u})^{-1/2}(1 + e^{-2a-2v})^{-1/2}(1 + e^{-2a+2v})^{-1/2}. \quad (\text{A.11})$$

On the other hand we can follow the recursion in the opposite direction. The large n limit of $X_{n,n+1} = x_{-n-2}^{-1}$ and $X_{n-1,n} = x_{-n-1}^{-1}$ is $X_{1,1} = e^{-2a}$ and

$$X_{1,0} = -e^{-2b}(1 - e^{-4a})^{-2}(1 + e^{-2a-2u})^{1/2}(1 + e^{-2a+2u})^{1/2} \\ \times (1 + e^{-2a-2v})^{1/2}(1 + e^{-2a+2v})^{1/2}. \quad (\text{A.12})$$

The total map is

$$(X_{1,1}, X_{1,0}) \\ \rightarrow \left(X_{1,1}, X_{1,0} \frac{(1 - X_{1,1}^2)^4}{(1 + \mu_A \mu_B X_{1,1})(1 + \mu_A^{-1} \mu_B X_{1,1})(1 + \mu_A \mu_B^{-1} X_{1,1})(1 + \mu_A^{-1} \mu_B^{-1} X_{1,1})} \right). \quad (\text{A.13})$$

We recognize the expected answer: a vector multiplet of charge $(2, 2)$ and no flavor charges, and a hypermultiplet of charge $(1, 1)$ in the $(2_A) \otimes (2_B)$ representation of the flavor symmetry (the vector of $SO(4)$).

B Cauchy-Riemann Equations on \mathcal{M}

In this appendix we explain how the Cauchy-Riemann equations on (\mathcal{M}, g) in complex structure $J^{(\zeta)}$ may be recast as flatness equations for a connection over \mathcal{B} , with a very simple ζ dependence. We do not assume that g arises from the construction we described in Sect. 5; rather, we use only general facts that follow from identifying (\mathcal{M}, g) as the moduli space of the gauge theory on $\mathbb{R}^3 \times S^1$.

For each $\zeta \in \mathbb{C}^\times$ we now consider the Cauchy-Riemann equations

$$\bar{\partial} f = 0 \quad (\text{B.1})$$

with respect to complex structure $J^{(\zeta)}$ on \mathcal{M} . We will rewrite these equations in the form

$$\partial_{u^i} f = \mathcal{A}_{u^i} f, \quad (\text{B.2})$$

$$\partial_{\bar{u}^{\bar{i}}} f = \mathcal{A}_{\bar{u}^{\bar{i}}} f, \quad (\text{B.3})$$

where \mathcal{A}_{u^i} and $\mathcal{A}_{\bar{u}^{\bar{i}}}$ are first-order differential operators acting along the torus fibers (so in coordinates (u, \bar{u}, θ) for \mathcal{M} they just involve derivatives with respect to θ), and moreover they depend on ζ in a simple way,

$$\mathcal{A}_{u^i} = \frac{1}{\zeta} \mathcal{A}_{u^i}^{(-1)} + \mathcal{A}_{u^i}^{(0)}, \quad (\text{B.4})$$

$$\mathcal{A}_{\bar{u}^{\bar{i}}} = \mathcal{A}_{\bar{u}^{\bar{i}}}^{(0)} + \zeta \mathcal{A}_{\bar{u}^{\bar{i}}}^{(1)}, \quad (\text{B.5})$$

with the $\mathcal{A}_{u^i}^{(-1)}$ linearly independent at every point, and similarly $\mathcal{A}_{\bar{u}^{\bar{i}}}^{(1)}$.

We begin by rewriting (B.1) as

$$(1 - iJ^{(\zeta, \bar{\zeta})})df = 0. \quad (\text{B.6})$$

If we treat ζ and $\bar{\zeta}$ as independent complex variables, then this equation is actually independent of $\bar{\zeta}$. To see this, it is enough to work at a single fixed ζ , say $\zeta = 0$. Specialize the general complex structure (3.2) to $\zeta = 0$,

$$J^{(\zeta=0, \bar{\zeta})} = J_3 + i\bar{\zeta}J_+, \quad (\text{B.7})$$

where we introduced $J_+ = J_1 + iJ_2$. Next note that $J_+J_3 = iJ_+$, so $J_+(J_3 - i) = 0$, so J_+ annihilates the $-i$ eigenspace of J_3 . So we have shown that $J^{(\zeta=0, \bar{\zeta})}$ and J_3 share an n -dimensional eigenspace with eigenvalue $-i$. To finish the argument we would like to know that $J^{(\zeta=0, \bar{\zeta})}$ does not have any *other* eigenvectors with eigenvalue $-i$. To see this we run a similar argument where we fix $\bar{\zeta}$ and let ζ vary; this produces n eigenvectors of $J^{(\zeta=0, \bar{\zeta})}$ with eigenvalue $+i$. Then by dimension counting there is no room for any more. So finally we see that the $-i$ eigenspace of $J^{(\zeta, \bar{\zeta})}$ is independent of $\bar{\zeta}$ as desired.

Thus we are free to choose any convenient $\bar{\zeta}$ in studying the Cauchy-Riemann equations (B.6). Since we want to understand how (B.6) looks in terms of the Seiberg-Witten fibration over \mathcal{B} , it is natural to choose $\bar{\zeta} = 0$; substituting this in (B.6) gives

$$(1 - iJ_3 - \zeta J_-)df = 0. \quad (\text{B.8})$$

We assume given an identification of the complex symplectic manifold $(\mathcal{M}, J_3, \omega_+)$ with the Seiberg-Witten torus fibration $(\mathcal{M}, J_3^{\text{sf}}, \omega_+^{\text{sf}} = -\frac{1}{4\pi} da^I \wedge dz_I)$. (It was argued in (12) that such an identification should exist at least for J_3 , using a weak coupling of the gauge theory to gravity; a similar argument shows the identification also for ω_+ .) Then contracting (B.8) with a vector field tangent to the torus fiber, $\frac{\partial}{\partial \bar{z}_I}$, gives

$$2\partial_{\bar{z}_I} f - \zeta(\partial_{\bar{z}_I} \cdot J_- df) = 0. \quad (\text{B.9})$$

To deal with the second term, we use

$$J_- = g^{-1}\omega_- = \frac{1}{4\pi}g^{-1}(d\bar{a}^I \wedge d\bar{z}_I) \quad (\text{B.10})$$

and multiply by $4\pi/\zeta$ to get

$$g^{-1}(df, d\bar{a}^I) = \frac{8\pi}{\zeta}\partial_{\bar{z}_I} f. \quad (\text{B.11})$$

This is almost of the form (B.2) which we want, but not quite: $g^{-1}(df, d\bar{a}^I)$ is a mixture of derivative operators acting on f . We want to make a change of basis to extract an equation for $\frac{\partial f}{\partial \bar{a}^I}$. To do this we consider the restriction of g to a horizontal subspace orthogonal to \mathcal{M}_u ; write this as $g = h_{IJ}d\bar{a}^I da^J$. Then multiplying by h_{IJ} we get

$$h_{IJ}g^{-1}(df, d\bar{a}^I) = \frac{8\pi}{\zeta}h_{IJ}\partial_{\bar{z}_I} f. \quad (\text{B.12})$$

Now consider the special case where f depends only on the base coordinates (a, \bar{a}) . Erecting an orthonormal basis at a point we see that $g^{-1}(df, d\bar{a}^I) = (h^{-1})^{IJ}\partial_{\bar{a}^J} f$. This implies that for general f the left side can be written as

$$(\partial_{a_J} - \mathcal{A}^J) f, \quad (\text{B.13})$$

where \mathcal{A}^J is a differential operator acting only in the fiber direction. (More intrinsically the full connection operator $\partial_{a_J} - \mathcal{A}^J$ is the derivative of f along the horizontal lift of the vector field ∂_{a_J} from \mathcal{B} to \mathcal{M} .)

Altogether then we have obtained

$$\frac{\partial}{\partial a_J} f = \frac{8\pi}{\zeta} h_{IJ} \frac{\partial f}{\partial \bar{z}_I} + \mathcal{A}^J f, \quad (\text{B.14})$$

which is of the desired form (B.2). An identical argument (starting with $\bar{\zeta} = \infty$ instead of $\bar{\zeta} = 0$) shows the conjugate Eq. (B.3).

The structure we have discovered here is very similar to the ‘‘improved connection’’ introduced in (13). To see the similarity most clearly, introduce an infinite-dimensional bundle V over \mathcal{B} , such that the fiber of V over $u \in \mathcal{B}$ is simply the space of real-analytic functions on the torus \mathcal{M}_u ,

$$V_u = C^\omega(\mathcal{M}_u). \quad (\text{B.15})$$

So a real-analytic complex-valued function f on the whole \mathcal{M} is equivalently a real-analytic section of V over \mathcal{B} . Then what we have found above is that the Cauchy-Riemann equations can be thought of as flatness equations for a 1-parameter family of connections in V , of a specific form. The flatness of these connections is a consequence of the integrability of the complex structures on \mathcal{M} . In (13) one also has a moduli space \mathcal{B} (parameterizing $\mathcal{N} = (2, 2)$ supersymmetric field theories in $d = 2$) and a vector bundle V over \mathcal{B} (the bundle of Ramond ground states.) One finds a family of flat connections in V parameterized by $\zeta \in \mathbb{C}^\times$, of the form

$$\nabla_i = \frac{1}{\zeta} C_i + D_i, \quad (\text{B.16})$$

$$\nabla_{\bar{i}} = \zeta \bar{C}_{\bar{i}} + \bar{D}_{\bar{i}}, \quad (\text{B.17})$$

where D_i is the standard connection provided by adiabatic variation of the couplings, and C_i are the ‘‘chiral ring’’ operators. The flatness of these connections is a consequence of the famous tt^* equations.

Throughout this paper, particularly in Sect. 5, many of the constructions — as well as their physical interpretations — are parallel to those which appeared in the tt^* story.

C Asymptotics of Integral Equations

In this appendix we will first show how to modify the asymptotic analysis of (11) in a situation with several BPS rays, and then adapt this analysis to our problem.

Finite-dimensional case

In (11) one studies a Riemann-Hilbert problem on the complex x -plane for an $m \times m$ matrix $\Psi(x)$, with a discontinuity along the real axis:

$$\begin{aligned} \Psi(ye^{-i\varepsilon}) &= \Psi(ye^{i\varepsilon})S & \text{for } y \in \mathbb{R}^+, \\ \Psi(ye^{-i\varepsilon}) &= \Psi(ye^{i\varepsilon})S^t & \text{for } y \in \mathbb{R}^-. \end{aligned} \quad (\text{C.1})$$

(We have now returned to the standard conventions for compositions of operators.) The analogues of the central charges Z_Y here are complex numbers Δ_{ij} , $i, j = 1, \dots, m$, which obey $\Delta_{ij} = w_i - w_j$ for some w_i . The matrix S is triangular, with 1 on the diagonal and $S_{ij} = 0$ if $\text{Re } \Delta_{ij} < 0$.

The asymptotic behavior of Ψ is determined by the constants w_i . If one defines

$$\Phi_{ij}(x) = \Psi_{ij}(x) e^{-\beta x w_j - \frac{\beta}{x} \bar{w}_j}, \quad (\text{C.2})$$

then $\Phi(x)$ tends to the identity matrix at $x \rightarrow \infty$ and to a certain ‘‘metric’’ g_{ij} at $x \rightarrow 0$.

The matrix S is the ‘‘Stokes multiplier’’ of the problem. It is convenient to re-express it as a product of more elementary ‘‘Stokes factors.’’ Indeed, assuming no three w_i are collinear in the complex plane, there are unique decompositions

$$S = \prod_{(ij): \operatorname{Re} \Delta_{ij} > 0} \overset{\curvearrowright}{s_{(ij)}}, \quad S' = \prod_{(ij): \operatorname{Re} \Delta_{ij} < 0} \overset{\curvearrowleft}{s_{(ji)}}, \quad (\text{C.3})$$

where the products are taken in the order of increasing $\arg \Delta_{ij}$, and each $s_{(ij)}$ has ones on the diagonal, and a single non-zero off-diagonal element at the location (ji) , with value $-\mu_{ij} = -\mu_{ji}$.

Using this decomposition, we can introduce our ‘‘multi-ray’’ version of the Riemann-Hilbert problem. Namely, introduce a set of rays through the origin in the x -plane,

$$\ell_{(ij)} = \{x : x\Delta_{ij} \in \mathbb{R}_+\}, \quad (\text{C.4})$$

and require

$$\Psi(ye^{-i\epsilon}) = \Psi(ye^{i\epsilon})s_{(ij)} \quad \text{for } y \in \ell_{(ij)}. \quad (\text{C.5})$$

Importantly, one can show that g_{ij} — which was the main object of interest in (11) — is the same whether we use the single-ray or multi-ray problem.

The integral equation (4.17) of (11) for $\Phi(x)$ reads (with some slight modifications to the ϵ conventions, for later convenience):

$$\begin{aligned} \Phi_{ij}(x) &= \delta_{ij} + \frac{1}{2\pi i} \int_0^\infty \frac{dy}{y-x} \sum_k \Phi_{ik}(e^{i\epsilon}y)(1-S)_{kj} e^{-\beta y \Delta_{kj} - \beta/y \bar{\Delta}_{kj}} \\ &\quad + \frac{1}{2\pi i} \int_{-\infty}^0 \frac{dy}{y-x} \sum_k \Phi_{ik}(e^{-i\epsilon}y)(1-S')_{kj} e^{-\beta y \Delta_{kj} - \beta/y \bar{\Delta}_{kj}}. \end{aligned} \quad (\text{C.6})$$

A solution to this equation gives a solution to the single-ray Riemann-Hilbert problem. Now we formulate a new integral equation which is equivalent to the multi-ray problem:

$$\Phi(x)_{ij} = \delta_{ij} + \sum_k \frac{1}{2\pi i} \int_{\ell_{(kj)}} \frac{dy}{y-x} \Phi_{ik}(y) \mu_{kj} e^{-\beta y \Delta_{kj} - \frac{\beta}{y} \bar{\Delta}_{kj}}. \quad (\text{C.7})$$

The integration here is understood to be the principal part integration. Note that when no three w_i are collinear the matrix elements Φ_{ik} are continuous across all the rays $\ell_{(kj)}$.

The recursive solution of (C.7) takes a simple form:

$$\Phi(x)_{ij} = \delta_{ij} + \Phi(x)_{ij}^{(1)} + \Phi(x)_{ij}^{(2)} + \dots, \quad (\text{C.8})$$

where

$$\begin{aligned} \Phi(x)_{ij}^{(1)} &= \frac{1}{2\pi i} \mu_{ij} \int_{\ell_{(ij)}} \frac{dy}{y-x} e^{-\beta y \Delta_{ij} - \frac{\beta}{y} \bar{\Delta}_{ij}}, \\ \Phi(x)_{ij}^{(2)} &= \frac{1}{(2\pi i)^2} \sum_{i_2} \mu_{i_2 i} \mu_{i_2 j} \int_{\ell_{(i_2 i)}} \int_{\ell_{(i_2 j)}} \frac{dy_1}{y_1 - y_2} \\ &\quad \times \frac{dy_2}{y_2 - x} e^{-\beta y_1 \Delta_{i_2 i} - \frac{\beta}{y_1} \bar{\Delta}_{i_2 i}} e^{-\beta y_2 \Delta_{i_2 j} - \frac{\beta}{y_2} \bar{\Delta}_{i_2 j}}, \end{aligned} \quad (\text{C.9})$$

and in general $\Phi(x)^{(n)}$ involves integrals over all chains of n rays $\ell_{(i_k i_{k+1})}$, where $i_1 = i$ and $i_{n+1} = j$. Each integral along $\ell_{(i_k i_{k+1})}$ contains the factor $e^{-\beta y \Delta_{i_k i_{k+1}} - \frac{\beta}{y} \bar{\Delta}_{i_k i_{k+1}}}$. On the ray this exponent is real and negative, with a single peak at $y = \exp(-i \arg \Delta_{i_k i_{k+1}})$. As β is taken to be

large, the integral is thus well approximated by the saddle point method, replacing the rest of the integrand by its value at the peak. As a result, the large β asymptotics at fixed x are simply:¹⁶

$$\Phi(x)_{ij}^{(1)} \sim \frac{1}{2\sqrt{\pi\beta|\Delta_{ij}|}i} \mu_{ij} \frac{1}{\exp[-i \arg \Delta_{ij}] - x} e^{-2\beta|\Delta_{ij}|}, \quad (\text{C.10})$$

and

$$\begin{aligned} \Phi(x)_{ij}^{(2)} &\sim \sum_{i_2} \mu_{ii_2} \mu_{i_2 j} \frac{1}{2\sqrt{\pi\beta|\Delta_{ii_2}|}i} \frac{1}{2\sqrt{\pi\beta|\Delta_{i_2 j}|}i} \times \frac{1}{\exp[-i \arg \Delta_{i_2 j}] - x} \\ &\quad \times \frac{1}{\exp[-i \arg \Delta_{ii_2}] - \exp[-i \arg \Delta_{i_2 j}]} e^{-2\beta|\Delta_{ii_2}|} e^{-2\beta|\Delta_{i_2 j}|}, \end{aligned} \quad (\text{C.11})$$

with similar estimates for the higher $\Phi^{(n)}$. By the triangle inequality we see that in the large β limit $\Phi^{(2)}$ is exponentially suppressed relative to $\Phi^{(1)}$, and similarly $\Phi^{(n+1)}$ is suppressed relative to $\Phi^{(n)}$. $\Phi^{(n)}$ has exactly the exponential suppression expected for an n -instanton correction.

We note that this asymptotic analysis is much simpler than the corresponding analysis of (C.6); in that case one has to deform the integration contour to pass through the appropriate saddle, and one encounters cuts and poles along the way, whose contributions have to be carefully tracked.

To finish this section we briefly discuss the analytic properties of this expansion. As we saw above, the n^{th} correction to Φ can be expressed in terms of certain iterated integrals:

$$\begin{aligned} &\mathcal{F}^{(n)}[x, \beta; \Delta_{i_1 i_2}, \dots, \Delta_{i_n i_{n+1}}] \\ &= \prod_{k=1}^n \left[\int_{\ell_{(i_k i_{k+1})}} \frac{dy_k}{2\pi i} e^{-\beta y_k \Delta_{i_k i_{k+1}} - \frac{\beta}{y_k} \bar{\Delta}_{i_k i_{k+1}}} \right] \prod_{k=1}^n \frac{1}{y_k - y_{k+1}} \Bigg|_{y_{n+1}=x}. \end{aligned} \quad (\text{C.12})$$

This $\mathcal{F}^{(n)}$ has an obvious discontinuity on the ray $\arg x = -\arg \Delta_{i_n i_{n+1}}$, which equals

$$\mathcal{F}^{(n-1)}[x, \beta; \Delta_{i_1 i_2}, \dots, \Delta_{i_{n-1} i_n}] e^{-\beta x \Delta_{i_n i_{n+1}} - \frac{\beta}{x} \bar{\Delta}_{i_n i_{n+1}}}. \quad (\text{C.13})$$

Note that $\mathcal{F}^{(n)}$ makes sense for generic values of the arguments $d_k = \Delta_{i_k i_{k+1}}$, without the restriction $\Delta_{ij} = w_i - w_j$. (In fact, one could even make Δ_{ij} and $\bar{\Delta}_{ij}$ into independent complex parameters, and put the rays at $x^2 \Delta_{ij} / \bar{\Delta}_{ij} \in \mathbb{R}^+$.) It has cuts whenever the phases of two consecutive arguments d_k, d_{k+1} align. The discontinuity is $\mathcal{F}^{(n-1)}$ with the same arguments, except for the substitution $d_k, d_{k+1} \rightarrow d_k + d_{k+1}$.

Essentially the same functions $\mathcal{F}^{(n)}$ appeared in the asymptotic analysis of (11); ours differ from those only in the placement of branch cuts.

Infinite-dimensional case

Now we give a similar analysis for the multiplicative integral Eq. (5.13). First, for any vector $\gamma \in \Gamma$, we define a vector $f^\gamma \in \Gamma_{\mathbb{Q}}$, by the power series expansion

$$-\sum_{\gamma' \in \Gamma} (\Omega(\gamma'; u) \log(1 - \sigma(\gamma') \mathcal{X}_{\gamma'})) \gamma' = \sum_{\gamma' \in \Gamma} f^{\gamma'} \mathcal{X}_{\gamma'}, \quad (\text{C.14})$$

¹⁶ These asymptotics are valid except when x lies exactly on the saddle point for the integral over y_n , i.e. $x = \exp[-i \arg \Delta_{i_n j}]$. At this point we find similar large- β asymptotics except that one of the $\sqrt{\beta}$ suppression factors is absent.

or more explicitly

$$f^\gamma = \sum_{n \geq 1 \text{ s.t. } \gamma = n\gamma'} \frac{\sigma(\gamma')^n}{n} \Omega(\gamma'; u) \gamma'. \quad (\text{C.15})$$

The point of this definition is that then (5.13) takes the form

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \left\langle \gamma, \frac{1}{4\pi i} \sum_{\gamma'} f^{\gamma'} \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \mathcal{X}_{\gamma'}(\zeta') \right\rangle. \quad (\text{C.16})$$

We aim to construct a solution \mathcal{X} to (C.16) as a limit of successive approximations $\mathcal{X}^{(n)}$, or the corresponding approximations $\Upsilon^{(n)}$ to Υ defined in (5.1). Υ is a map from \mathcal{M}_u to its complexification; we write its components as functions, $\Upsilon_\gamma(\theta) := \gamma \cdot \Upsilon(\theta)$. We begin by choosing $\Upsilon_\gamma^{(0)} = \theta_\gamma$. Recalling that Υ is defined so that

$$\mathcal{X}^{\text{sf}}(\Upsilon^{(n)}) = \mathcal{X}^{(n)}(\theta), \quad (\text{C.17})$$

we can write the iteration step as

$$e^{i\Upsilon_\gamma^{(n+1)}} = e^{i\theta_\gamma} \exp \left\langle \gamma, \frac{1}{4\pi i} \sum_{\gamma'} f^{\gamma'} \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \mathcal{X}_{\gamma'}^{\text{sf}}(\Upsilon^{(n)}, \zeta') \right\rangle. \quad (\text{C.18})$$

A fixed point of this iteration, $\Upsilon^{(n+1)} = \Upsilon^{(n)}$, would give a solution of (5.11). So to see that a solution exists we should verify that the iteration is a contraction, i.e. that

$$\max_{\zeta, \theta} \|\Upsilon^{(n+1)} - \Upsilon^{(n)}\| < C \max_{\zeta, \theta} \|\Upsilon^{(n)} - \Upsilon^{(n-1)}\| \quad (\text{C.19})$$

for some constant $C < 1$. More precisely, we will verify that this iteration is a contraction when acting on $\Upsilon^{(n)}$ which have $\max_{\zeta, \theta} \|\Upsilon^{(n)} - \theta\| < \infty$, and which obey a side condition expressing the fact that they are not too far from the real torus: we require $\max_{\zeta, \theta} |e^{i\Upsilon_\gamma^{(n)}}| < e^{\varepsilon \|\gamma\|}$, for a constant $\varepsilon > 0$ to be determined shortly.

First we need to see that the iteration preserves our side condition. Taking the absolute value of (C.18) and making the saddle point analysis, for large enough R we get the estimate¹⁷

$$|e^{i\Upsilon_\gamma^{(n+1)}}| < \exp \left[\sum_{\gamma'} |(f^{\gamma'}, \gamma)| e^{-2\pi R |Z_{\gamma'}| + \varepsilon \|\gamma\|} \right]. \quad (\text{C.20})$$

Now we assume that $\sum_{\gamma'} f^{\gamma'} \|\gamma'\| e^{-2\pi R |Z_{\gamma'}|}$ converges for large enough R . (This amounts to a requirement that the $\Omega(\gamma'; u)$ do not grow too quickly with γ' ; it appears very reasonable for field theory but would almost certainly be violated in the gravitational case.) We also use the Support Property recalled in Sect. 2.2, to bound $\|\gamma'\|$ by $K |Z_{\gamma'}|$. Then for large enough R we can pick ε so that the right side is smaller than $e^{\varepsilon \|\gamma\|}$. Doing this for γ running over a basis of Γ , we obtain our desired ε . (Indeed, we can take $\varepsilon \rightarrow 0$ exponentially fast for large R .)

¹⁷ As in the previous section, this analysis has to be supplemented by a separate discussion when ζ hits the saddle point, but that only reduces the suppression by a factor \sqrt{R} , and still allows us to establish (C.20).

Now we want to estimate

$$\|\Upsilon^{(n+1)} - \Upsilon^{(n)}\| = \frac{1}{4\pi} \left\| \sum_{\gamma \in \Gamma} \int_{\ell_\gamma} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} f^\gamma \left(\mathcal{X}_\gamma^{\text{sf}}(\Upsilon^{(n)}, \zeta') - \mathcal{X}_\gamma^{\text{sf}}(\Upsilon^{(n-1)}, \zeta') \right) \right\| \quad (\text{C.21})$$

$$\leq \frac{1}{4\pi} \left\| \sum_{\gamma \in \Gamma} \int_{\ell_\gamma} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} f^\gamma | \mathcal{X}_\gamma^{\text{sf}}(\theta, \zeta') | \left(e^{i\Upsilon^{(n)}(\zeta')} - e^{i\Upsilon^{(n-1)}(\zeta')} \right) \right\| \quad (\text{C.22})$$

$$\leq \frac{1}{4\pi} \left\| \sum_{\gamma \in \Gamma} \int_{\ell_\gamma} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} f^\gamma | \mathcal{X}_\gamma^{\text{sf}}(\theta, \zeta') | \left\| e^{\varepsilon \|\gamma\|} \|\gamma\| \max_{\zeta} \|\Upsilon^{(n)} - \Upsilon^{(n-1)}\| \right\|. \quad (\text{C.23})$$

The large- R saddle point analysis then gives

$$\begin{aligned} \|\Upsilon^{(n+1)} - \Upsilon^{(n)}\| &\leq \frac{1}{4\pi} \max_{\zeta} \|\Upsilon^{(n)}(\zeta) - \Upsilon^{(n-1)}(\zeta)\| \\ &\quad \times \left\| \sum_{\gamma \in \Gamma} f^\gamma \|\gamma\| e^{-2\pi R |Z_\gamma| + \varepsilon \|\gamma\|} \right\|. \end{aligned} \quad (\text{C.24})$$

For large enough R , with our convergence assumptions, this establishes the contraction property; indeed the iteration converges very quickly, with a speed determined by the largest $e^{-2\pi R |Z_\gamma|}$.

One can give an explicit expression for $\mathcal{X}_\gamma(\zeta)$ in terms of functions like the \mathcal{F} of the finite-dimensional case. The presence of the exponential in the recursion relation makes things a bit more intricate: instead of summing over chains one now gets a sum over decorated rooted trees. Let \mathcal{T} denote a rooted tree, with edges labeled by pairs (i, j) (where i is the node closer to the root), and each node decorated by a choice of $\gamma_i \in \Gamma$. Also call the decoration at the root node $\gamma_{\mathcal{T}}$. Then define the weight of the tree to be an element of $I_{\mathbb{Q}}$, determined by the $\Omega(\gamma; u)$,

$$\mathcal{W}_{\mathcal{T}} = \frac{f^{\gamma_{\mathcal{T}}}}{|\text{Aut}(\mathcal{T})|} \prod_{(i,j) \in \text{Edges}(\mathcal{T})} \langle \gamma_i, f^{j_i} \rangle. \quad (\text{C.25})$$

The iterative solution for $\mathcal{X}_\gamma(\zeta)$ then takes the form

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp \left\langle \gamma, \sum_{\mathcal{T}} \mathcal{W}_{\mathcal{T}} \mathcal{G}_{\mathcal{T}}(\zeta) \right\rangle, \quad (\text{C.26})$$

for some functions $\mathcal{G}_{\mathcal{T}}(\zeta)$. The integral Eq. (C.16) for \mathcal{X} becomes a formula expressing each $\mathcal{G}_{\mathcal{T}}(\zeta)$ in terms of the ones for smaller trees. Namely, deleting the root from \mathcal{T} leaves behind a set of rooted trees \mathcal{T}_a , and (C.16) will be satisfied if

$$\mathcal{G}_{\mathcal{T}}(\zeta) = \frac{1}{4\pi i} \int_{\ell_{\gamma_{\mathcal{T}}}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \mathcal{X}_{\gamma_{\mathcal{T}}}^{\text{sf}}(\zeta') \prod_a \mathcal{G}_{\mathcal{T}_a}(\zeta'). \quad (\text{C.27})$$

It follows that, as for the $\mathcal{F}^{(n)}$ of the finite-dimensional case, the discontinuity of $\mathcal{G}_{\mathcal{T}}$ along $\ell_{\gamma_{\mathcal{T}}}$ is determined by the product of the lower $\mathcal{G}_{\mathcal{T}_a}$.

From (C.26) we can also obtain Υ directly:

$$e^{i\Upsilon} = \exp \left\langle \gamma, \sum_{\mathcal{T}} \mathcal{W}_{\mathcal{T}} \mathcal{G}_{\mathcal{T}}(\zeta) \right\rangle. \quad (\text{C.28})$$

In particular this allows us to evaluate $\Upsilon(\zeta = 0)$. The expansion of the symplectic form $\varpi(\zeta)$ can similarly be analyzed in this fashion, and organized as a sum over trees.

D Asymptotics of Differential Equations

In this appendix we would like to understand how to compute the coefficients of the differential equations satisfied by \mathcal{X} . Consider the asymptotic expansion of \mathcal{X} around $\zeta = 0$:

$$\mathcal{X}(\theta, \zeta)_\gamma \sim \mathcal{X}_\gamma^{\text{sf}}(\Upsilon_0(\theta), \zeta) \exp \sum_{n>0} \zeta^n \gamma \cdot g_n(\theta). \quad (\text{D.1})$$

We consider the differential operators defined in Sect. 5.5,

$$\frac{1}{\mathcal{X}_\gamma} \frac{\partial \mathcal{X}_\gamma}{\partial t^n} = \frac{1}{\mathcal{X}_\gamma} \mathcal{A}_n \mathcal{X}_\gamma. \quad (\text{D.2})$$

Plug in the expansion (D.1) and keep only the first few terms:

$$\frac{\partial \log \mathcal{X}_\gamma^{\text{sf}}(\Upsilon_0(\theta))}{\partial t^n} = \left(\frac{1}{\zeta} \mathcal{A}_n^{(-1)} + \mathcal{A}_n^{(0)} \right) \Upsilon_0(\theta)_\gamma + \mathcal{A}_n^{(-1)} \gamma \cdot g_1(\theta). \quad (\text{D.3})$$

The leading part in ζ is a statement we already understood:

$$\pi R \frac{\partial Z_\gamma}{\partial t^n} = \mathcal{A}_n^{(-1)} \Upsilon_0(\theta)_\gamma. \quad (\text{D.4})$$

This means that $\mathcal{A}_n^{(-1)}$ is the pull-back by Υ_0 of $\mathcal{A}_n^{\text{sf}(-1)}$.

The next term in the expansion is

$$i \frac{\partial \Upsilon_0(\theta)_\gamma}{\partial t^n} = \mathcal{A}_n^{(0)} \Upsilon_0(\theta)_\gamma + \mathcal{A}_n^{(-1)} \gamma \cdot g_1(\theta), \quad (\text{D.5})$$

which determines $\mathcal{A}_n^{(0)}$, given a knowledge of $\Upsilon_0(\theta)$.

There is an alternative point of view, which is quite useful: consider the compatibility conditions between the various differential equations. For example, consider the equation $[R\partial_R - \mathcal{A}_R, \zeta \partial_\zeta - \mathcal{A}_\zeta] = 0$, i.e.

$$\left[R\partial_R + \frac{1}{\zeta} \mathcal{A}_\zeta^{(-1)} - \mathcal{A}_R^{(0)} - \zeta \mathcal{A}_\zeta^{(1)}, \zeta \partial_\zeta - \frac{1}{\zeta} \mathcal{A}_\zeta^{(-1)} - \mathcal{A}_\zeta^{(0)} - \zeta \mathcal{A}_\zeta^{(1)} \right] = 0, \quad (\text{D.6})$$

and expand it in powers of ζ . This gives three equations:

$$R\partial_R \mathcal{A}_\zeta^{(-1)} - [\mathcal{A}_R^{(0)}, \mathcal{A}_\zeta^{(-1)}] = [\mathcal{A}_\zeta^{(0)}, \mathcal{A}_\zeta^{(-1)}] + \mathcal{A}_\zeta^{(-1)}, \quad (\text{D.7})$$

$$R\partial_R \mathcal{A}_\zeta^{(0)} - [\mathcal{A}_R^{(0)}, \mathcal{A}_\zeta^{(0)}] = 2[\mathcal{A}_\zeta^{(1)}, \mathcal{A}_\zeta^{(-1)}], \quad (\text{D.8})$$

$$R\partial_R \mathcal{A}_\zeta^{(1)} - [\mathcal{A}_R^{(0)}, \mathcal{A}_\zeta^{(1)}] = [\mathcal{A}_\zeta^{(1)}, \mathcal{A}_\zeta^{(0)}] + \mathcal{A}_\zeta^{(1)}. \quad (\text{D.9})$$

These equations are strongly reminiscent of the Nahm equations, differing from them only by the two extra linear pieces on the right-hand side. These extra pieces are dominant at large radius. An alternative strategy to derive the large R asymptotics is again an iterative solution of these three equations around the semiflat solution.

Another interesting set of ‘‘isomonodromic’’ equations can be derived by similarly expanding $[\partial_u - \mathcal{A}_u, \zeta \partial_\zeta - \mathcal{A}_\zeta] = 0$:

$$0 = [\mathcal{A}_u^{(-1)}, \mathcal{A}_\zeta^{(-1)}], \quad (\text{D.10})$$

$$\frac{\partial}{\partial u} \mathcal{A}_\zeta^{(-1)} - [\mathcal{A}_u^{(0)}, \mathcal{A}_\zeta^{(-1)}] = [\mathcal{A}_u^{(-1)}, \mathcal{A}_\zeta^{(0)}] - \mathcal{A}_u^{(-1)}, \quad (\text{D.11})$$

$$\frac{\partial}{\partial u} \mathcal{A}_\zeta^{(0)} - [\mathcal{A}_u^{(0)}, \mathcal{A}_\zeta^{(0)}] = [\mathcal{A}_u^{(-1)}, \mathcal{A}_\zeta^{(1)}], \quad (\text{D.12})$$

$$\frac{\partial}{\partial u} \mathcal{A}_\zeta^{(1)} - [\mathcal{A}_u^{(0)}, \mathcal{A}_\zeta^{(1)}] = [\mathcal{A}_u^{(1)}, \mathcal{A}_\zeta^{(0)}] + \mathcal{A}_u^{(-1)}, \quad (\text{D.13})$$

$$0 = [\mathcal{A}_u^{(1)}, \mathcal{A}_\zeta^{(1)}]. \quad (\text{D.14})$$

E A Relation to the Thermodynamic Bethe Ansatz

Note added Nov. 20, 2009. It was pointed out to us some time ago by A. Zamolodchikov that one of the central results of this paper, Eq. (5.13), is in fact a version of the Thermodynamic Bethe Ansatz (TBA) (45). In this appendix we explain that remark. Another relation between four-dimensional super Yang-Mills theory and the TBA has recently been discussed by Nekrasov and Shatashvili (46).

The TBA equations for an integrable system of particles a with masses m_a , at inverse temperature β , with integrable scattering matrix $S_{ab}(\theta - \theta')$, where θ is the rapidity, are

$$\varepsilon_a(\theta) = m_a \beta \cosh \theta - \sum_b \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \phi_{ab}(\theta - \theta') \log(1 + e^{\beta \mu_b - \varepsilon_b(\theta')}), \quad (\text{E.1})$$

where $\phi_{ab}(\theta) = -i \frac{\partial}{\partial \theta} \log S_{ab}(\theta)$. Here the scattering matrix is diagonal, that is, the soliton creation operators obey $\Phi_a(\theta) \Phi_b(\theta') = S_{ab}(\theta - \theta') \Phi_b(\theta') \Phi_a(\theta)$.

We can put the logarithm of (5.13) in the form of (E.1) as follows. Clearly the particle labels a, b, \dots correspond to γ, γ', \dots . Now let $Z_\gamma = e^{i\alpha_\gamma} |Z_\gamma|$, where α_γ is real and only defined modulo 2π . For any γ we can make the change of variables $\zeta = -e^{i\alpha_\gamma + \theta}$, so that the BPS ray ℓ_γ is mapped out by $-\infty < \theta < \infty$. Under this change of variables the semiflat coordinate (3.11) becomes

$$\log \mathcal{X}_\gamma^{\text{sf}} = -2\pi R |Z_\gamma| \cosh \theta + i\varphi_\gamma. \quad (\text{E.2})$$

(Note that to avoid confusion with the rapidity θ we have changed the notation for the angular coordinate along the torus from θ_γ , used in the rest of this paper, to φ_γ .) Now we set

$$\beta \mu_\gamma := i\varphi_\gamma + \log(-\sigma(\gamma)) \pmod{2\pi i}. \quad (\text{E.3})$$

Note that $\beta \mu_\gamma$ is $i\varphi_\gamma$ or differs by $\pm i\pi$. In particular, it is pure imaginary. Define ‘‘quasiparticle energies’’ $\varepsilon_\gamma(\theta)$ by

$$\mathcal{X}_\gamma(\zeta = -e^{i\alpha_\gamma + \theta}) := -\sigma(\gamma) e^{\beta \mu_\gamma - \varepsilon_\gamma(\theta)} = e^{i\varphi_\gamma - \varepsilon_\gamma(\theta)}. \quad (\text{E.4})$$

More precisely, this defines ε_γ on the BPS ray ℓ_γ , where θ is real. For other θ we define ε_γ by analytic continuation — in contrast to \mathcal{X}_γ , which has discontinuities along certain lines of constant $\text{Im } \theta$ (the BPS rays).

We have chosen (E.3) so that the logarithm of (5.13) reads as

$$\varepsilon_\gamma(\theta) = 2\pi R |Z_\gamma| \cosh \theta + \sum_{\gamma'} \Omega(\gamma') \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} K_{\gamma, \gamma'}(\theta - \theta') \log(1 + e^{\beta \mu_{\gamma'} - \varepsilon_{\gamma'}(\theta')}), \quad (\text{E.5})$$

with

$$K_{\gamma, \gamma'}(\theta - \theta') = \frac{i}{2} \langle \gamma, \gamma' \rangle \frac{e^{\theta - \theta' + i\alpha_\gamma - i\alpha_{\gamma'}} + 1}{e^{\theta - \theta' + i\alpha_\gamma - i\alpha_{\gamma'}} - 1}. \quad (\text{E.6})$$

This kernel can also be written as

$$\begin{aligned} K_{\gamma, \gamma'}(\theta - \theta') &= i \langle \gamma, \gamma' \rangle \frac{\partial}{\partial \theta} \log \left[\sinh \left(\frac{1}{2} (\theta - \theta' + i\alpha_\gamma - i\alpha_{\gamma'}) \right) \right] \\ &= \frac{i}{2} \langle \gamma, \gamma' \rangle \coth \left(\frac{\theta - \theta' + i\alpha_\gamma - i\alpha_{\gamma'}}{2} \right). \end{aligned} \quad (\text{E.7})$$

The argument of the logarithm is not a pure phase, so $K_{\gamma, \gamma'}(\theta - \theta')$ does not correspond to a unitary scattering matrix, in general.

Let us comment briefly on the reality properties of the “quasiparticle energies.” The twistor coordinates satisfy the reality property

$$\overline{\mathcal{X}_\gamma(\zeta)} = \mathcal{X}_{-\gamma}(-1/\bar{\zeta}). \quad (\text{E.8})$$

Since $Z_{-\gamma} = -Z_\gamma$, we have

$$e^{i\alpha_{-\gamma}} = -e^{i\alpha_\gamma}. \quad (\text{E.9})$$

Hence, if $\zeta = -e^{i\alpha_\gamma + \theta}$ and θ is real, then $-1/\bar{\zeta} = -e^{i\alpha_{-\gamma} - \theta}$. Now, using $\sigma(-\gamma) = \sigma(\gamma)$ and $\varphi_{-\gamma} = -\varphi_\gamma$, we get the reality condition on the “quasiparticle energies”

$$\overline{\varepsilon_\gamma(\theta)} = \varepsilon_{-\gamma}(-\theta). \quad (\text{E.10})$$

The integral Eq. (5.13) is consistent with the reality condition (E.10) since for θ, θ' both real

$$\overline{K_{\gamma, \gamma'}(\theta - \theta')} = K_{-\gamma, -\gamma'}(-\theta + \theta'). \quad (\text{E.11})$$

References

1. S. Cecotti P. Fendley K.A. Intriligator C. Vafa (1992) A new supersymmetric index *Nucl. Phys.* **B386** 405 – 452
2. N. Seiberg E. Witten (1994) Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory *Nucl. Phys.* **B426** 19 – 52
3. N. Seiberg E. Witten (1994) Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD *Nucl. Phys.* **B431** 484 – 550
4. Denef, F., Moore, G.W.: “Split states, entropy enigmas, holes and halos.” <http://arXiv.org/abs/hep-th/0702146v2>, 2007
5. F. Denef (2000) Supergravity flows and D-brane stability *JHEP* **08** 050
6. Denef, F.: “On the correspondence between D-branes and stationary supergravity solutions of type II Calabi-Yau compactifications.” <http://arXiv.org/abs/hep-th/0010222v1>, 2000
7. Kontsevich, M., Soibelman, Y.: “Stability structures, motivic Donaldson-Thomas invariants and cluster transformations.” <http://arXiv.org/abs/0811.2435v1> [math.AG], 2008
8. I. Garcia-Etxebarria A.M. Uranga (2008) Non-perturbative superpotentials across lines of marginal stability *JHEP* **0801** 033
9. H. Ooguri C. Vafa (1996) Summing up D-instantons *Phys. Rev. Lett.* **77** 3296 – 3298
10. N. Seiberg S.H. Shenker (1996) Hypermultiplet moduli space and string compactification to three dimensions *Phys. Lett.* **B388** 521 – 523
11. S. Cecotti C. Vafa (1993) On classification of $\mathcal{N} = 2$ supersymmetric theories *Commun. Math. Phys.* **158** 569 – 644
12. Seiberg, N., Witten, E.: “Gauge dynamics and compactification to three dimensions.” <http://arXiv.org/abs/hep-th/9607163v1>, 1996
13. S. Cecotti C. Vafa (1991) Topological-antitopological fusion *Nucl. Phys.* **B367** 359 – 461
14. A. Karlhede U. Lindstrom M. Roček (1984) Selfinteracting tensor multiplets in $\mathcal{N} = 2$ superspace *Phys. Lett.* **B147** 297
15. U. Lindstrom M. Roček (1988) New hyperkahler metrics and new supermultiplets *Commun. Math. Phys.* **115** 21
16. U. Lindstrom M. Roček (2010) Properties of hyperkahler manifolds and their twistor spaces *Commun. Math. Phys.* **293** 257 – 278
17. Kontsevich, M., Soibelman, Y.: “Affine structures and non-Archimedean analytic spaces.” In: *The unity of mathematics*, Vol. **244** of Progr. Math., Boston, MA: Birkhäuser Boston, 2006, pp. 321–385

18. Joyce, D.: “*Holomorphic generating functions for invariants counting coherent sheaves on Calabi-Yau 3-folds.*” <http://arXiv.org/abs/hep-th/0607039v1>, 2006
19. Bridgeland, T., Toledano Laredo, V.: “*Stability conditions and Stokes factors.*” <http://arXiv.org/abs/0801.3974v1> [math.AG], 2008
20. T. Bridgeland (2007) Stability conditions on triangulated categories *Ann. Math. (2)* **166** 2 317 – 345
21. A. Neitzke B. Pioline S. Vandoren (2007) Twistors and black holes *JHEP* **04** 038
22. A. Losev N.A. Nekrasov S.L. Shatashvili (1998) Issues in topological gauge theory *Nucl. Phys.* **B534** 549 – 611
23. Losev, A., Nekrasov, N., Shatashvili, S.L.: “*The freckled instantons.*” <http://arXiv.org/abs/hep-th/9908204v2>, 1999
24. H. Ooguri C. Vafa E.P. Verlinde (2005) Hartle-Hawking wave-function for flux compactifications *Lett. Math. Phys.* **74** 311 – 342
25. N. Dorey T.J. Hollowood D. Tong (1999) The BPS spectra of gauge theories in two and four dimensions *JHEP* **05** 006
26. A. Ritz (2007) Superconformal R-charges and dyon multiplicities in $N = 2$ gauge theories *Phys. Rev.* **D75** 085008
27. S. Alexandrov B. Pioline F. Saueressig S. Vandoren (2009) Linear perturbations of quaternionic metrics - I. The Hyperkahler case *Lett. Math. Phys.* **87** 225 – 265
28. Lerche, W.: “*Introduction to Seiberg-Witten theory and its stringy origin.*” Prepared for CERN-Santiago de Compostela-La Plata Meeting on Trends in Theoretical Physics, CERN-Santiago de Compostela-La Plata, Argentina, 28 Apr - 6 May 1997
29. D.S. Freed (1999) Special Kaehler manifolds *Commun. Math. Phys.* **203** 31 – 52
30. Witten, E.: “Dynamics of quantum field theory.” In: *Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997)*, Providence, RI: Amer. Math. Soc., 1999, pp. 1119–1424
31. A. Bilal F. Ferrari (1996) Curves of Marginal Stability and Weak and Strong-Coupling BPS Spectra in $N = 2$ Supersymmetric QCD *Nucl. Phys.* **B480** 589 – 622
32. A. Bilal F. Ferrari (1996) The Strong-Coupling Spectrum of the Seiberg-Witten Theory *Nucl. Phys.* **B469** 387 – 402
33. E. Witten (1997) Five-brane effective action in M-theory *J. Geom. Phys.* **22** 103 – 133
34. Freed, D.S.: “*Dirac charge quantization and generalized differential cohomology.*” <http://arXiv.org/abs/hep-th/0011220v2>, 2001
35. M.J. Hopkins I.M. Singer (2005) Quadratic functions in geometry, topology, and M-theory *J. Diff. Geom.* **70** 329 – 452
36. Belov, D., Moore, G.W.: “*Holographic action for the self-dual field.*,” <http://arXiv.org/abs/hep-th/0605038v1>, 2006
37. N.J. Hitchin A. Karlhede U. Lindstrom M. Roček (1987) Hyperkahler metrics and supersymmetry *Commun. Math. Phys.* **108** 535
38. Hitchin, N.: “Hyper-Kähler manifolds.” *Astérisque* (1992), no. 206, Exp. No. 748, 3, 137–166. Séminaire Bourbaki, Vol. 1991/92
39. B. Dubrovin (1993) Geometry and integrability of topological-antitopological fusion *Commun. Math. Phys.* **152** 3 539 – 564
40. P.C. Argyres M.R. Plesser N. Seiberg (1996) The Moduli Space of $N = 2$ SUSY QCD and Duality in $N = 1$ SUSY QCD *Nucl. Phys.* **B471** 159 – 194
41. S. Cecotti C. Vafa (1993) Ising model and $\mathcal{N} = 2$ supersymmetric theories *Commun. Math. Phys.* **157** 139 – 178
42. P.C. Argyres M.R. Douglas (1995) New phenomena in $SU(3)$ supersymmetric gauge theory *Nucl. Phys.* **B448** 93 – 126
43. P.C. Argyres M.R. Plesser N. Seiberg E. Witten (1996) New $\mathcal{N} = 2$ Superconformal

- Field Theories in Four Dimensions *Nucl. Phys.* **B461** 71 – 84
44. N. Seiberg (1993) Naturalness versus supersymmetric nonrenormalization theorems *Phys. Lett.* **B318** 469 – 475
 45. A.B. Zamolodchikov (1990) Thermodynamic Bethe ansatz in relativistic models. Scaling three state Potts and Lee-Yang models *Nucl. Phys.* **B342** 695 – 720
 46. Nekrasov, N.A., Shatashvili, S.L.: “*Quantum integrability and supersymmetric vacua.*” <http://arXiv.org/abs/0901.4748v2> [hep-th], 2009