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Non-Relativistic Holography from Hořava Gravity

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Abstract

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Holography is a powerful theoretical duality that relates quantum gravitational theories to non-gravitational theories in one less dimension. The most explored example of this tool is the correspondence between general relativity on five dimensional Anti-de Sitter space and a four dimensional supersymmetric Yang-Mills theory. This case is extremely useful as the strong coupling regime of the Yang-Mills theory is solved by weakly coupled gravity.

Another interesting class of strongly coupled field theories are those of a non-relativistic nature that arise in condensed matter systems. Herein, motivated by the generic symmetry structure of these theories, a non-relativistic version of holography is proposed using an alternate theory of gravity, Hořava gravity. Justifications of this proposal are thoroughly discussed. Various checks of the duality, such as correlation functions and black hole thermodynamics are presented. This new holographic correspondence provides a crucial tool to tackle strongly coupled problems in condensed matter systems. On the other hand, Hořava gravity, thought to be a UV complete theory of quantum gravity, will allow holography to move away from the strong coupling, large number of colors limit that has restricted traditional AdS/CFT.

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INTRODUCTION

1.1 Duality

1.1.1 Electromagnetic Duality

The notion of duality is a powerful concept available to modern physicists. Dualities arise when it is discovered that one and the same physical system can have multiple descriptions in the mathematical language of theoretical physics. The simplest and longest know example is that of classical electromagnetism. In vacuum, Maxwell's equations describing the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, respectively, are remarkably symmetric:

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t}. \tag{1.1}$$

These equations are invariant under the transformation $\vec{E} \rightarrow \vec{B}$ and $\vec{B} \rightarrow -\vec{E}$. This means that a solution with a various field configuration is equivalent to the configuration related by the above duality transformation, in the sense that they are both solutions to Maxwell's equations. Therefore, if we are interested in solving Maxwell's equations we have two options: we can either solve the question posed to us involving various boundary conditions on the electric and magnetic fields; or we can make the duality transformation, solve that problem, and then transform the solution back to the original situation of concern. Although the simplicity of equations 1.1 make this approach somewhat unnecessary, we will shortly discuss much more useful dualities.

1.1.2 T-Duality

A more interesting example is that of T-duality. It illustrates an equivalence of some theories having a small compact spatial direction with other theories having a large compact direction. This can be simply illustrated in terms of a classical string. For any object propagating in a compact direction Fourier analysis tells us that its spatial momentum $\vec{p}$ is quantized in Kaluza-Klein modes. For a circle of radius R this gives a kinetic contribution to the energy: $E_{kin}^2 \equiv \vec{p}^{\,2} = (n/R)^2$, where n is an integer classifying the quantized mode of the momentum. An object also has a contribution to its energy due to its rest mass. For an ideal string this is just given by its tension times its length. In a compact dimension a string has the interesting option of wrapping around the direction multiple times, quantified by the integer winding number m. The physical length of the string in that case is simply the circumference of the compact direction times the number of times the string wraps around: $L = 2\pi Rm$. This gives a rest mass of $M \equiv LT = (2\pi Rm)(l_s^2/2\pi) = mR/l_s^2$, where the string tension T has been expressed in terms of a constant l_s, called the string length[1]: $T \equiv l_s^2/2\pi$. The total energy of the string in this compact background is therefore given by

$$E^2 \equiv \vec{p}^{\,2} + M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{l_s^2}\right)^2, \tag{1.2}$$

which is invariant under the switching of the momentum mode and winding number, $n \leftrightarrow m$, if the radius of the circle is also inverted, $R \to l_s^2/R$. Therefore the physical spectrum given by possible energies of the string is equally described by propagation around a large circle or a small one (with respect to l_s.) This is sensible: a large circle has relatively small Kaluza-Klein modes, but winding will contribute a large mass due to the long length of the stretched string; for a small circle, the winding costs relatively little energy, while the Kaluza-Klein modes have a small wavelength, and

[1] Not to be confused with the actual length of the string, L.

hence large energy. T-duality shows that these two mathematical descriptions have the same physical energy spectrum.

Although this example of T-duality for a classical string is illustrative, it may appear trivial. Fortunately, at the level of quantum superstring theory T-duality is still present and has a shocking consequence: of the five consistent superstring theories, Type IIA and Type IIB are related by T-duality, as are the Heterotic-O and Heterotic-E theories. This means that IIA and IIB are two possible descriptions of the same underlying physical situation, analogues to the small and large compact directions of the previous discussion. It is now believed that all five superstring theories are related by some form of duality, and therefore they form a unique description.

1.1.3 Seiberg Duality

One final example duality shows another powerful aspect of the concept. As will be taken advantage of, dualities are often at their most useful when the strong coupling regime of one theory is dual to the weak coupling behavior of the other. This allows difficult strongly coupled physics to be understood in the formalism of perturbation theory. Glossing over details, Seiberg duality [77] is an example of such a duality: it states the equivalence of the low energy behavior of two a priori different gauge theories, and relates the strong coupling regime of one to the weak coupling regime of the other.

1.2 Holography

1.2.1 The Holographic Principle

Gauge-gravity duality [69, 88, 41], or holography for short, is a powerful duality displaying many of the above features. In its most general form, the holographic principle is the statement that a quantum theory of gravity is equivalently described by a theory without gravity and in one less dimension [86, 84]. This conclusion is reached by applying basic concepts of quantum mechanics and thermodynamics to gravitational black hole solutions.

The heart of the argument is a measure of the amount of information needed to describe a physical theory. This count of the degrees of freedom is quantified by examining the size of the Hilbert space. Consider the simple quantum mechanical system consisting of n spins. Each one can be in one of two states, spin-up or spin-down, so the total number of possible states of the system is $\mathcal{N} = 2^n$. This is the dimension of the Hilbert space, which importantly is related to the maximal entropy possible for the system, S_{max},

$$\mathcal{N} \equiv e^{S_{max}}. \tag{1.3}$$

For the case of n spins the maximal entropy is $S_{max} = n \log 2$. This is a measure of the amount of information the system can store, in this case each of n spins can store one bit of information, which has the numerical entropy $S_{bit} = \log 2$.

For the more complex example of a quantum field theory (QFT) the maximal number of degrees of freedom possible in a volume of space is infinite due to infinitesimal wavelength fluctuations of the field. An understanding of how this entropy scales with the volume can be obtained by introducing a cutoff and treating the theory on a lattice. When considering gravity this is somewhat natural: near the Planck length l_p simple QFT can no longer be trusted and effects of quantum gravity must be included. The number of sites in the lattice is V/a^3, where V is the volume of space under consideration, and a is the cutoff spacing of the lattice. On each site the field can be quantified by a discrete set of ν possible values, which above was 2 for spin-up or spin-down. This size of the Hilbert space of the lattice theory is therefore $\mathcal{N} = \nu^{V/a^3}$. Invoking 1.3 the maximal entropy of this system is

$$S_{max} = \frac{V}{a^3} \log \nu, \tag{1.4}$$

which is proportional to the volume of the system, given in units of the cutoff. This scaling is a generic conclusion for field theory configurations with a finite energy density [84]. It implies that the number of bits needed to describe the theory is proportional to the volume under consideration.

When the effects of gravity are incorporated the scaling of the size of the Hilbert space is radically different. This is concluded from examining the entropy of black holes. A black hole is a solution to a gravitational theory that contains regions that are causally disconnected from an asymptotic region. This means that there are observers inside the black hole region who can never communicate with observers who are very far away from the black hole. The dividing boundary between locations that can contact distant observers and those that cannot is called the horizon of the black hole. Such objects are fascinating for various reasons, and, as to be discussed, are of fundamental interest in holography.

The first interesting property of black holes is that they have an inherent entropy [12]. This is a necessity if the second law of thermodynamics is to be maintained, otherwise when entropy is thrown into a black hole the total entropy of the universe would decrease. Various perpetual motion machines could then be constructed by using black holes to transform thermal energy into work. The requirement that this entropy is discernible from the exterior implies it is some function of the mass of the black hole[2]. The proper quantity can be argued to be the area of the horizon of the black hole. Primarily, the area of the horizon is a non-decreasing quantity in any dynamic process [42], in analogy to entropy. By considering the process of dropping a single bit of information into a black hole one can determine the minimal amount of increase in entropy of the black hole and relate it to the change in area. This determines the entropy to be proportional to the horizon area, while further thermodynamical considerations [43] fix the proportionality constant,

$$S_{BH} = \frac{k_B c^3 A}{4\hbar G_N}, \tag{1.5}$$

[2]More precisely, it can only be a function of the externally visible properties of a black hole, its "hair": for a black hole in general relativity these are the mass, the angular momentum, and any conserved charges.

where A is the area of the event horizon, k_B is the Boltzmann constant, c is the speed of light, $\hbar$ is Planck's constant, and G_N is Newton's gravitational constant.

Even more interesting than the existence of black hole entropy is that it is the maximal amount of entropy a system in a given volume can obtain [13]. Consider a system such as a QFT as described above. Suppose in a given volume V this system has an initial entropy in excess of S_{BH}, equation 1.5. One can then consider adding additional energy to this system until it is dense enough that it collapses to a black hole filling up the volume, with horizon area A. This process has decreased the entropy to the final value of 1.5, and therefore violates the second law of thermodynamics[3]. It is concluded that the system could not have had the supposed large initial entropy and that the maximal entropy contained in a region of space with surface area A is given by[4] $A/4$ is the Planck length.

A remarkable result has emerged: when considering the influence of gravity the maximal entropy contained in a system does not scale like its volume V, as true for a lattice QFT, but like its surface area A, as given by S_{BH}. The information theoretical consequences of this is that the number of bits needed to describe the system is not equal to the volume (in cutoff units) but is equal to the surface area (in Planck units). This "holographic principle" refers to the reduction in the information contained in a system by a whole dimension.

The arguments that lead to this conclusion require that the entropy of a black hole is proportional to its horizon area. Although [12, 42, 43] show that this is indeed the case in general relativity (GR), and derive the exact value 1.5, it can be expected to hold in other geometric theories of gravity. As discussed in later sections, the gravitational theories known as Einstein-Aether theory [54] and Hořava gravity [49] can be shown to obey forms of the first law, that is, their black holes obey a relation

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[3]Importantly, this process also decreases the entropy of the region exterior to V.

[4]Natural units will often be used where k_B, c, $\hbar$, and G_N are all equal to 1. Note that $l_p \equiv \sqrt{G_N \hbar / c^3}$

between their energy and a measure of their horizon area that can be interpreted thermodynamically. This gives evidence that their black holes also have an entropy proportional to their area. It is therefore generically expected that these gravity theories obey the holographic principle as well.

The holographic principle is a form of a duality, the concept introduced in Section 1.1. It states that any system involving gravity has a dual description in terms of a theory in one less dimension. Although in this generality it does not explain how, holography guarantees that there are two ways to answer a physical inquiry: it can be posed as a question concerning a gravitational system, or it can be asked in the dual holographic theory. A precise example of this duality will be discussed in Section 1.2.3.

1.2.2 A Gedanken Allegory

This discussion of black holes, entropy, bits and information can feel removed from our everyday physical intuition. The following story concretely illustrates the radical departure from everyday experience that the holographic principle entails.

Suppose you are a successful graduate student, having recently obtained your doctorate you are preparing to move across the country for your first post-doc. This requires packing up your Jackson's "Classical Electrodynamics," your Wald's "General Relativity," your Srednicki's "Quantum Field Theory," and the rest of your trusty textbooks that have gotten you this far. Being still in the grad student mindset you wish to save as much money and space as possible, and are attempting to fit them all in one box. Yet, now officially being a physicist, your scatterbrainedness is setting in, and you think it best to label the box with every book you put in, to keep track.

It quickly becomes obvious that all the books will not fit, so you cleverly decide to instead take copies of them: after scanning them you can shrink the page size down to save space. You first cut them down to half the size, then a tenth, a millionth. You keep shrinking down the page size as small as you possibly can: eventually you are left with pages that have one word[5] per Planck area, any smaller and the quantum nature of spacetime becomes apparent, fluctuations scrambling the text and making the sentences illegible.

As you are putting these tiny copies in, and writing the name of each book on the box, you realize another problem: during the coming drive across the country you are likely going to need to look up the LSZ reduction formula, the Penrose-Hawking singularity theorem, or some other physical factoid. Although the name of each book is on the exterior of the box, it will take far too long to dig through and pull out the one you need, so you come up with another clever solution: you will make two minimally sized copies of each book; one is put in the box; while the pages of the other are pasted to the surface of the box, so their total contents are quickly located, as needed.

You happily carry out this process for a while: scanning your books, shrinking down their pages, making two copies, pasting the pages of one to the exterior of the box, while putting the other inside. But eventually you run into a final problem: despite shrinking down the pages to a minimal size while maintaining legibility, grad school has left you with so many texts that you are running out of space to paste the pages on the exterior of the box. Finally, the surface is completely covered in text, you dare not paste over any pages on the exterior, and yet there must surely be more room inside the volume of the box to hold additional books. Figuring that the worse that can happen is that you forget what the last few unlabeled books are, you decide to throw them in the box anyway. But the worse does happen: as this final book enters the box, it and all it's contents disappear to you, and in it's place is a black hole!

This tragic swallowing of your prized collection of texts is a consequence of the holographic principle. Although it seems obvious that one can put more pages into the

[5] One bit really.

volume of a box than one can put on its exterior, this is not so at a fundamental level. A fully covered box surface was the signal that the maximal amount of information that the box could store was reached, any more information added and your system transformed to the only object that can store that much entropy, a black hole.

1.2.3 AdS/CFT

The Anti-de Sitter/Conformal Field Theory (AdS/CFT) [69, 88, 41] correspondence is a precise example of a holographic duality. It states that the gravitational theory of Type IIB string theory on the five dimensional, uniformly negatively curved spacetime, known as Anti-de Sitter space, is equivalent to the non-gravitational, four dimensional theory known as $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory.

This duality arises from considering the behavior of Dp-branes in ten dimensional Type IIB string theory. These are objects with p spatial dimensions on which the end points of open strings of the theory can end. Considering a stack of N coincident $D3$-branes leads to two possible descriptions of their low energy behavior. Figure 1.1 shows the situation. One approach uses the perturbative description of the branes given by the strings that can attach to them. The dynamics of these strings describe the low energy behavior of the stack: only their massless excitations are important, and it is seen that the strings attached to the branes decouple from any strings in the ambient ten dimensional background that are not attached, in particular they no longer interact with the closed strings that describe the graviton. Therefore the low energy behavior of the $D3$-branes is described by a non-gravitational effective theory, which the massless excitations of the strings determine to be $3+1$ dimensional $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory. The gauge group of the Yang-Mills theory is $SU(N)$ because the string endpoints can have “labels” describing which of the N $D3$-branes they attach to. It has the $\mathcal{N} = 4$ supersymmetry as as a consequence of being derived from the excitations of superstrings.

The other description of the low energy behavior of the stack of $D3$-branes follows

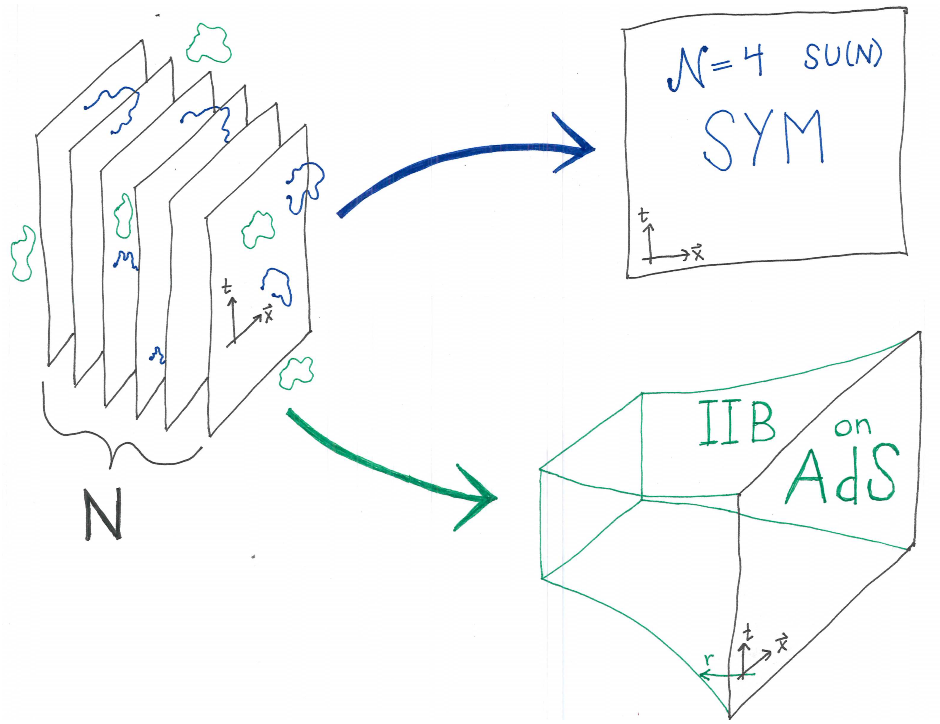

Figure 1.1: Two descriptions of N $D3$ branes. One results in four dimensional $\mathcal{N} = 4$ $SU(N)$ SYM, while the other is IIB superstring theory on AdS_5.

from letting them backreact on the geometry. The branes have inherent tension and hence a mass density. Therefore when placed in the flat ten dimensional background they will gravitationally interact and cause the spacetime geometry to warp. Near the stack this backreacted geometry can be seen to be described by five dimensional Anti-de Sitter (AdS) spacetime. Therefore, the low energy behavior of the $D3$-branes is described by the gravitational Type IIB string theory on an AdS_5 background. The AdS/CFT correspondence captures this duality: IIB on AdS_5 is dual to $\mathcal{N} = 4$ super Yang-Mills (SYM).

This example of holography was found as dual ways to describe a stack of branes, but some general lessons are learned about what properties such a duality must exhibit. One property that is given explicitly in this example is a "holographic dictionary," that is, how are questions and formulas from one theory translated to the other. For example, the simple electromagnetic duality of Section 1.1 had the translation of $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$, while the dictionary of T-duality contains $n \leftrightarrow m$ and $R \to l_s^2/R$. Table 1.2 gives a partial dictionary for the AdS/CFT correspondence.

IIB on AdS$_5$	$\mathcal{N} = 4$ SYM
string coupling: g_s	Yang-Mills coupling: $g_{YM}^2 \equiv g_s$
radius of curvature: $L^4 \equiv 4\pi g_s N l_s^4$	rank of gauge group: N
boundary values of fields: $\phi_i\|_{boundary}$	sources for operators: $\mathcal{L} = \mathcal{L}_{SYM} + \phi_i\|_{boundary}\mathcal{O}_i$
spin of field ϕ_i	spin of operator $\mathcal{O}_i$
mass of field ϕ_i	scaling dimension of operator $\mathcal{O}_i$
partition function with boundary values: $Z_{IIB\ on\ AdS_5}(\phi_i\|_{boundary})$	partition function with sources: $Z_{SYM}(\phi_i\|_{boundary}\mathcal{O}_i)$

Figure 1.2: Entries in the holographic dictionary for AdS/CFT showing how quantities of the gravitational theory on the left are related to those of the non-gravitational one on the right.

Examining Table 1.2 allows determination of how the different regimes of the dual theories align. If the coupling of the strings is small, $g_s \ll 1$, they are very unlikely to interact, quantum loop corrections will be unimportant, and the theory can be treated as a classical theory of strings. The second row of the dictionary says that this corresponds to a weak Yang-Mills coupling, $g_{YM} \ll 1$ for the SYM theory. Classical string theory on a curved background is still a very difficult theory, it is often necessary to consider the geometry being weakly curved. In this regime the geometry does not probe the extended nature of the strings and in first approximation they behave as point particles. For AdS, this requires a large radius of curvature, as compared to the string length. The dictionary, Table 1.2, determines that this requirement, $L/l_s \gg 1$, implies that $g_s N \gg 1$. Being in the classical regime with $g_s \ll 1$, this in turn requires that N, the rank of the gauge group of SYM is much larger than one. In gauge theories with large gauge group it is understood that the perturbative expansion parameter is more properly captured by the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$, than the Yang-Mills coupling g_{YM} itself [85].

This relation of parameters show that AdS/CFT has a particularly useful limit: when the gravitational theory is classical and the string size can be neglected, the gauge theory is at large N with a large 't Hooft coupling; that is, classical supergravity on AdS_5 is dual to strongly coupled large N SYM. This is incredibly powerful as it makes queries concerning the non-perturbative strongly coupled regime of SYM much more approachable by translating them into questions concerning a weakly coupled gravitational theory, general relativity[6]. Thus, in this regime, AdS/CFT is a weak/strong duality, as was the Seiberg duality of Section 1.1.3.

An example of an explicit property that can be calculated in AdS/CFT is the correlation function of the operator $\mathcal{O}_i$. The last row of the dictionary 1.2 shows the heart of the duality: the partition function of the two theories correspond. To calculate the correlation function of $\mathcal{O}_i$, two variations of the SYM partition function with respect to the source $\phi_i|_{boundary}$ are made. By the duality, this can instead be calculated by varying the IIB partition function with respect to the boundary values of the field ϕ_i.

Another very basic property that this explicit holographic duality illustrates is that the symmetries of the dual theories must match. In particular are the symmetries which the fields and operators of the theories form representations. $\mathcal{N} = 4$ SYM, as a $3+1$ dimensional relativistic QFT, has fields that transform as representations of the Lorentz group, $SO(1,3)$. The conformal nature of this field theory (the "C" in

[6]The supersymmetry can be thought of as simply dictating which types of matter fields exist.

Conformal Field Theory) enhances this symmetry to the conformal group $SO(2,4)$. On the gravitational side of the duality this symmetry group manifests itself as the geometric isometries of AdS, which is a maximally symmetric solution to GR.

Another type of symmetry many field theories exhibit is global "spurionic" symmetry. The theory is not symmetric under such a transformation acting on just the fields, the coupling parameters need to transform as well. The case of the theory of a massive Dirac fermion is an example. The mass term explicitly breaks the axial symmetry. However, this symmetry can be restored if the mass is assigned an axial charge and is transformed as well. Although such spurionic symmetries do not in general generate conserved charges, they are useful as they constrain how couplings can appear in the low energy effective theory, or correlation functions.

While diffeomorphism invariance, the statement that physics does not depend on the coordinate system used for spacetime, is often seen as a hallmark of the theory of general relativity, it is already a property of any relativistic QFT formulated on a fixed, potentially curved, background spacetime metric. One is always free to change coordinate systems if it facilitates analysis, the only requirement is that the metric needs to transform as well in order to maintain the proper geometric quantities, such as covariant derivatives or volume measures. Therefore, diffeomorphisms act as a spurionic symmetry of a QFT: in addition to transforming the dynamic fields of the theory, the non-dynamic parameters captured by the metric need to be transformed as well. In a QFT on a fixed spacetime background the metric acts analogously to the Dirac fermion mass. One should think of the metric as a set of coupling constants specified at every point in spacetime. Position dependent diffeomorphisms are now a global symmetry under which these coupling constants transform.

This general situation holds true for SYM, and so must manifest itself in the holographic AdS/CFT duality. On the gravitational bulk side the diffeomorphisms that are spurionic symmetries of the field theory are part of the higher dimensional diffeomorphism invariance. Diffeomorphisms are a gauge invariance of GR. This means they are not a symmetry at all, but a redundancy in the description. Gauge variant quantities are simply not physical. The gauge variant description introduced non-physical degrees of freedom to simplify the Lagrangian; the gauge invariance of observables removes those extra degrees of freedom again. Bulk diffeomorphisms that vanish near the boundary of AdS space correspond to such a gauge invariance, therefore they should not be interpreted as a global symmetry, instead they correspond to a redundancy in the description of the bulk theory. On the other hand, changes of coordinates that do not vanish at the boundary are not a redundancy: they are global symmetries of the gravitational theory because they act on the boundary data. They correspond to the spurionic diffeomorphisms acting on the metric of the field theory. This matching of symmetries between the gravitational theory and the field theory is a basic requirement for a holographic duality.

1.3 A Problem in Want of a Tool

The power of the AdS/CFT correspondence is that it allows the hard problem of strongly coupled SYM to be solved in terms of weakly coupled GR. This is an amazing tool that has been used toward understanding the viscosity of the quark-gluon plasma [74], jet quenching [67], thermalization [10], chaotic fluid dynamics [2], and other physically applicable problems.

Another class of strongly interacting field theories are those arising in condensed matter systems, such as the quantum Hall effect [65] or Fermions at unitarity [28]. These systems are often describable in terms of an effective quantum field theory that is inherently non-relativistic (NR). Lacking perturbative tools to tackle these problems, there has recently been much interest in formulating holography for NR QFTs, starting with the work of [80, 8]. In addition to the metric of GR, these works included a massive vector field in the gravitational theory. This has the effect of changing the solution from being that of AdS to a geometry that captures the symmetries of the non-relativistic conformal group known as Schrödinger symmetry.

Importantly, the dual NR CFTs thus obtained are seen to be a light-like reductions of a relativistic CFT, from which they inherit most of their properties. They hardly constitute generic NR QFTs.

The driving goal of this thesis is to find a holographic description of NR QFTs, not by breaking Lorentz invariance by adding fields to a fundamentally relativistic duality such as AdS/CFT, but by proposing a new holographic duality for a gravitational theory that is inherently non-relativistic. A guiding principle is that the symmetries of both dual theories, including the spurionic symmetries, must match, as discussed in the previous section. In Chapter 2 the formal development of non-relativistic quantum field theories is explored, with an emphasis on symmetries. This leads to Chapter 3 where a variant of Hořava gravity [49] is shown to have the same symmetry structure as a generic NR QFT. Chapter 4 contains an explicit holographic dictionary relating quantities of Hořava gravity and those of NR QFTs. Important example calculations, including correlation functions and the thermodynamics of some black hole solutions is presented in Chapter 5. Lastly, Chapter 6 summarizes the key lessons learned within, and poses possible future developments for this subject.

Chapter 2
NON-RELATIVISTIC FIELD THEORIES

Much of this chapter is taken from [56, 57] which was coauthored with Andreas Karch.

2.1 Spurionic Symmetries of NR QFTs

As illustrated in the introduction, the spurionic symmetries of the field theory are a crucial property that needs to be captured by a possible holographic duality. For the case of relativistic QFTs this requirement was met by generic diffeomorphisms being the gauge symmetries of general relativity. The nature and symmetries of non-relativistic QFTs deserves further elucidation, being far from the modern theorists' everyday thought.

In a non-relativistic quantum field theory, time plays a special role: there is a preferred notion of spatial slices consisting of events happening simultaneously. This can be implemented by considering the spacetime manifold to be equipped with a co-dimension one foliation consisting of the spatial leaves. A global time defines the invariant notion of whether one event occurs before or after another, and is hence required in order to have a well defined causality. Non-relativistic theories can have instantaneous interactions that, when turned on, have immediate influence at arbitrarily large spatial distances, but they cannot influence events that occurred at an earlier global time. In this way causality is preserved in the absence of light cones.

Usually one wants to insist on translation invariance in time, $t \to \tilde{t} = t - f$ where f is a constant, so that the system allows for a conserved energy. Sometimes this symmetry can be extended to include the case where f is linear in t, or even to the case where f is an arbitrary function of t. As will be reviewed, these two special cases correspond to NR QFTs which are scale and conformally invariant, respectively. Any such time coordinate has the right to be called a global time: the leaves of the foliation remain at constant time even after the transformation $f(t)$. On the other hand, the Lorentzian diffeomorphism where f has spatial dependence, violates the preferred foliation as it changes the time ordering of events. Such a redefined temporal coordinate cannot be considered a global time because it would alter the notion of which events occur before or after another, and hence violate causality. Although in a NR QFT one can always work in a global time, and restrict f to be a function of time only, insight and information can be gained by considering a "non-physical" time and allowing spatial dependence of f. This is analogous to using an arbitrary metric in a relativistic field's action so that one can calculate the stress-energy tensor, even if only interested in flat Minkowski space. As will be discussed, from the non-relativistic viewpoint one can still consider these non-physical temporal transformations by having them act on a background source coupling to energy current.

For a NR QFT defined in d spatial dimensions invariance under purely spatial diffeomorphisms is still expected. Furthermore, for many NR QFTs a different change of spatial coordinates at different times can be performed, that is $x_i \rightarrow \tilde{x}_i(x_i, t)$. In particular, these time dependent spatial diffeomorphisms include Galilean boosts. Together with translations and rotations these boosts play a special role as they leave a flat space background with no electromagnetic field invariant. Correspondingly, they do not just constrain the low energy effective action but are true (non-spurionic) symmetries, and give rise to conserved charges.

Additionally, most NR QFTs allow for a conserved particle number current. In this case the theory in the presence of background electric and magnetic fields coupling to particle number can be formulated, and the theory possesses a position dependent $U(1)$ global spurionic symmetry acting on the associated background potential. These symmetries — time-dependent spatial diffeomorphisms, a $U(1)$ rotation acting on the background gauge field coupled to particle number, and time translation invariance — can be taken as the defining symmetries of a large class of NR QFTs. This class includes most interacting electron systems and in particular the quantum Hall states. As will be shown in Section 2.4, if the theory allows for arbitrary reparametrizations of time it describes a conformal NR QFT, of which the unitary Fermi gas is an example. For these conformal theories there again exists a subgroup of transformations that leaves the trivial field theory metric and gauge potential invariant. This subgroup is often referred to as the Schrödinger group. The mathematical structure of these symmetries will be worked out in the following sections.

Spurionic symmetries put strong constraints on possible terms in the low energy effective action of an interacting NR QFT. This was exploited for the unitary Fermi gas in [81], and for quantum Hall states in [52]. For the quantum Hall states these symmetries allow one to relate the Hall viscosity and the change in filling fraction when the theory is put on a sphere to a single coefficient in the low energy effective action. Furthermore, the leading correction to the Hall conductivity in the presence of a background electric field with slow spatial variation is completely determined by the symmetry in terms of thermodynamic quantities. As the Hall states describe gapped states, the only fields appearing in the low energy effective action are the background metric and background electric fields, making symmetries very powerful. In the unitary Fermi gas the interplay between NR conformal invariance and NR diffeomorphisms constrains several transport coefficients in the hydrodynamic description of this system [81].

As in the relativistic case, a guiding principle for constructing a gravitational dual should be the defining spurionic symmetries of a generic NR QFT. A holographic gravity dual should have the same set of symmetry transformations as the field theory of interest: time dependent spatial diffeomorphisms, spatially dependent temporal diffeomorphisms, and the $U(1)$ symmetry acting on the background gauge field coupled to particle number. This set of transformations will be referred to as "NR electro-diffeomorphisms". If the temporal diffeomorphisms are restricted, excluding the non-physical spatially dependent ones that violate the preferred foliation, one has the "NR general covariance" of [50]. Furthermore, by excluding the $U(1)$ gauge symmetry one has the "foliation preserving diffeomorphisms" of [49]. It should be emphasized that any NR QFT that has NR electro-diffeomorphisms as its symmetry group must still have a notion of global time in order to have a well defined causality. This means the spacetime manifold comes equipped with a foliation by spatial leaves parametrized by a global time. Such a theory can therefore be restricted to have only NR general covariance by working in coordinates adapted to the foliation. Although the symmetry group of NR electro-diffeomorphisms can give more information about a theory, it can only describe the same causal theories that NR general covariance can.

2.2 Non-relativistic Electro-diffeomorphism Symmetry

In the previous section's discussion of NR QFTs it was seen that coordinate changes and global $U(1)$ rotations are spurionic symmetries of many systems of interest. These transformations are now formalized.

As first introduced in [81], and extended in [80], many NR QFTs with conserved particle number are invariant under diffeomorphism and $U(1)$ transformations if the background fields transform as:

$$\begin{aligned}
\delta A_t &= \xi^\mu \partial_\mu A_t + \dot{f} A_t + A_k \dot{\xi}^k - \dot{\lambda}, \\
\delta A_i &= \xi^\mu \partial_\mu A_i + A_k \partial_i \xi^k + A_t \partial_i f + m e^{\Phi} g_{ik} \dot{\xi}^k - \partial_i \lambda, \\
\delta \Phi &= \xi^\mu \partial_\mu \Phi + B_k \dot{\xi}^k - \dot{f}, \\
\delta B_i &= \xi^\mu \partial_\mu B_i + B_k \partial_i \xi^k + B_i (B_k \dot{\xi}^k - \dot{f}) - \partial_i f, \\
\delta g_{ij} &= \xi^\mu \partial_\mu g_{ij} + g_{ik} \partial_j \xi^k + g_{kj} \partial_i \xi^k + (B_i g_{jk} + B_j g_{ik}) \dot{\xi}^k.
\end{aligned} \tag{2.1}$$

The diffeomorphism parameters, $\xi^t \equiv f$ and ξ^i, and the gauge parameter λ can be arbitrary functions of space and time. This is the symmetry group of "NR electro-diffeomorphisms." Interpretation can be given to these background fields by examining an action with this symmetry. Consider free NR particles described by the action

$$S = \int dt d^d x \sqrt{g} e^{-\Phi} \left[\frac{\imath}{2} e^{\Phi} (\psi^\dagger D_t \psi - D_t \psi^\dagger \psi) - \frac{g^{ij}}{2m} D_i \psi^\dagger D_j \psi \right.$$
$$\left. - \frac{g^{ij} B_j}{2m} (D_t \psi^\dagger D_i \psi + D_i \psi^\dagger D_t \psi) - \frac{g^{ij} B_i B_j}{2m} D_t \psi^\dagger D_t \psi \right], \qquad (2.2)$$

where $D_\mu \psi \equiv \partial_\mu \psi - \imath A_\mu \psi$ is the gauge covariant derivative. This action is invariant under the transformations 2.1 if the field ψ transforms as

$$\delta\psi = \xi^\mu \partial_\mu \psi - \imath \lambda \psi. \qquad (2.3)$$

By varying the action 2.2 with respect to the background fields they are given physical meaning [80]: g_{ij} is the spatial metric and couples to the stress tensor T^{ij}; A_μ is the gauge field and couples to the particle number density and current $(n, \vec{j})$; and $(\Phi, \vec{B})$ are the sources that couple to the energy density and current $(\epsilon, \vec{E})$.

Among the general transformations described by equations 2.1 is a subgroup that leaves the trivial background, $g_{ij} = \delta_{ij}$ and $A_\mu = \Phi = B_i = 0$, invariant. These are determined to be translations, spatial rotations, and Galilean boosts. The latter are given by

$$\vec{\xi}(t, \vec{x}) = \vec{v} t, \qquad \lambda(t, \vec{x}) = \vec{v} \cdot \vec{x}. \qquad (2.4)$$

In this sense the only true non-trivial symmetries that are a consequence of NR electro-diffeomorphism invariance are Galilean boosts. More general transformations are only a symmetry if we treat the background fields as spurions, transforming according to 2.1.

2.3 Conservation Laws

The spurionic symmetry transformations of the background fields, as captured in 2.1, leads to expressions for the conservation of particle number, momentum, and energy

[81, 80]. In general backgrounds the latter two are only conserved if one takes into account the momenta and energy stored in the external fields. The connected part of the generating functional, W, is defined as $e^{\imath W} \equiv \int D\psi^\dagger D\psi e^{\imath S}$. Assuming that W can be written as an integral of a local density,

$$W[\Phi, B_i, g_{ij}, A_t, A_i] = \int dt d^d x \mathcal{W}, \tag{2.5}$$

the invariance of the action S under the field transformations 2.1 implies, upon integrating by parts, the conservation laws:

$$\partial_t n + \partial_k j^k = 0, \quad \partial_t \pi_i + \partial_k T_i^k = 0, \quad \partial_t \epsilon + \partial_k E^k = 0, \tag{2.6}$$

which are the conservation of particle number, momentum, and energy, respectively. The conserved densities and currents are given by:

$$\begin{aligned}
n &\equiv -\frac{\delta\mathcal{W}}{\delta A_t}, \qquad j^k \equiv -\frac{\delta\mathcal{W}}{\delta A_k},\\
\pi_i &\equiv -B_i\left(\frac{\delta\mathcal{W}}{\delta\Phi} + B_j\frac{\delta\mathcal{W}}{\delta B_j}\right) - (B_k g_{ij} + B_j g_{ik})\frac{\delta\mathcal{W}}{\delta g_{kj}} - A_i\frac{\delta\mathcal{W}}{\delta A_t} - me^{\Phi} g_{ij}\frac{\delta\mathcal{W}}{\delta A_j},\\
T_i^k &\equiv \delta_i^k\mathcal{W} - B_i\frac{\delta\mathcal{W}}{\delta B_k} - 2g_{ij}\frac{\delta\mathcal{W}}{\delta gkj} + A_i\frac{\delta\mathcal{W}}{\delta A_k},\\
\epsilon &\equiv \frac{\delta\mathcal{W}}{\delta\Phi} + B_i\frac{\delta\mathcal{W}}{\delta B_i} - A_t\frac{\delta\mathcal{W}}{\delta A_t}, \qquad E^k \equiv \frac{\delta\mathcal{W}}{\delta B_k} - A_t\frac{\delta\mathcal{W}}{\delta A_k}.
\end{aligned} \tag{2.7}$$

In what follows the case of $B_i = 0$ is of main interest. For such backgrounds the momentum density and particle number current are linked [81, 37],

$$\pi_i = nA_i + me^{\Phi} j_i. \tag{2.8}$$

Even when $\Phi = B_i = 0$ the variation of $\mathcal{W}$ with respect to these fields is needed to to calculate the energy density ϵ and current $\vec{E}$.

2.4 NR Scale and Conformal Invariance

In addition to the above diffeomorphism and $U(1)$ transformations the spurionic symmetry of some NR QFTs can be extended to include a type of conformal invariance

[80]. The additional generator $\omega(t, \vec{x})$ acts on the background fields via:

$$\delta_\omega \Phi = -2\omega, \qquad \delta_\omega g_{ij} = 2\omega g_{ij}, \tag{2.9}$$

with the rest invariant.

Although the action 2.2 is not invariant under this transformation, it can be made so by exchanging the "minimal coupling" used here for "conformal coupling" [80]. Alternatively, the restricted case of $\Phi = B_i = 0$ can be considered, as in [81]. To maintain $B_i = 0$, the transformations 2.1 require that $\partial_i f = 0$, which is simply the statement that temporal diffeomorphisms that violate the preferred foliation by a global time are not allowed in NR general covariance. Conversely, the existence of a global time allows[1] $B_i = 0$, by working in adapted coordinates. From the transformations 2.1 and 2.9, it is seen that $\Phi = 0$ is maintained for $\omega = -\dot{f}(t)/2$. In this way it is apparent how in the restricted case of [81] time reparametrization contains the information of the conformal structure of the theory. They are intimately linked by demanding that Φ remains zero.

It is useful to define the notion of the conformal dimension of an operator [81]. By the argument above, for NR general covariance this can be determined by the operator's behavior under infinitesimal time reparametrization. In general, an operator/field transforming as

$$\delta\mathcal{O} \supset f\dot{\mathcal{O}} + \Delta_{\mathcal{O}}\dot{f}\mathcal{O} \tag{2.10}$$

is said to have the conformal dimension $\Delta_{\mathcal{O}}$. In the restricted case of $\Phi = B_i = 0$, and using using $\omega = -\dot{f}/2$, equations 2.1 and 2.9 imply the remaining background fields transform as:

$$\delta A_t \supset f\dot{A}_0 + \dot{f}A_t, \quad \delta A_i \supset f\dot{A}_i, \quad \delta g_{ij} \supset f\dot{g}_{ij} + 2\omega g_{ij} = f\dot{g}_{ij} - \dot{f}g_{ij}. \tag{2.11}$$

[1]On the spatial leaves defined by a global time the above action 2.2 is required to reproduce the Schrödinger equation. This in turn gives $B_i = 0$ in such a coordinate frame. Using the observation that $n_\mu \equiv (e^{-\Phi}, -e^{-\Phi}B_i)$ transforms as a spacetime one-form, one can interpret the action 2.2 as giving time evolution in the n_μ direction.

Therefore, A_t, A_i, and g_{ij} are conformal operators with dimensions 1, 0, and -1, respectively. With these transformations of the background fields the action 2.2 (with $\Phi = B_i = 0$) is invariant under arbitrary $f(t)$; the free action is "conformally invariant" if the scalar field is assigned the conformal transformation

$$\delta\psi \supset -\frac{d}{2}\omega\psi = \frac{d}{4}\dot{f}\psi. \tag{2.12}$$

If a theory is formulated with this conformal invariance, there is a subgroup of the spurionic symmetry transformations that leave the trivial background $g_{ij} = \delta_{ij}$ and $A_\mu = 0$ invariant. It has already been seen that translations, rotations, and Galilean boosts maintain this background. A second special case is the scale transformation. This corresponds to a constant conformal transformation, $\omega = -\kappa/2$, which, by above, requires the time reparametrization $f = \kappa t$. In order to leave the trivial background metric invariant these transformations need to be combined with a spatial diffeomorphism that corresponds to rescaling the spatial coordinates

$$\xi^i = \frac{\dot{f}}{2}x^i = \frac{\kappa}{2}x^i. \tag{2.13}$$

The relative weight of 1/2 between the rescaling of time and space corresponds to a dynamical critical exponent of $z = 2$, as expected for a Schrödinger system.

In later sections examples of gravity backgrounds that have scaling symmetries for $z \neq 2$ will be discussed. In order for this more general scale transformation to be a symmetry of a Galilean invariant QFT the conformal transformation must be modified to

$$\delta_\omega \Phi = -z\omega, \qquad \delta_\omega g_{ij} = 2\omega g_{ij}, \qquad \delta_\omega m = (z-2)\omega m, \tag{2.14}$$

that is, the mass m must now be treated as a spurionic field. Preserving the trivial background under the temporal rescaling $f = \kappa t$ then requires the conformal transformation $\omega = -\kappa/z$ and the spatial rescaling $\xi^i = \kappa x^i/z$, as expected for dynamical critical exponent z. It has been argued in [8], based on a holographic construction, that such scale and Galilean invariant fixed points should exist in interacting NR QFTs. In the action 2.2, as in Schrödinger's equation, m is a parameter, not a field. In this case, the $z = 2$ scale transformation gets singled out as the only true scale symmetry that leaves the mass invariant. All other values of z can formally be realized as spurionic symmetries under which m transforms. This is also the case in the $z \neq 2$ backgrounds of [8], where the compact light-like direction scales non-trivially for $z \neq 2$, and hence so does the compactification radius which sets the mass of the Kaluza-Klein particles. In principle a system with $z \neq 2$ scaling can be constructed by promoting m to a dynamical field in all the above, and adding a hidden sector action S_m which sets the scaling of m to be given by the transformation 2.14, but this will not be attempted here.

For the $z = 2$ case realized by the free field theory above, there is one more symmetry generator that leaves the trivial background invariant. It is usually referred to as the "special conformal" transformation of the Schrödinger group, and corresponds to the combination

$$\omega = -Ct, \quad f = Ct^2, \quad \xi^i = Ctx^i, \quad \lambda = \frac{1}{2}C\vec{x}^2. \tag{2.15}$$

Interactions can be added to the free theory that preserve the spatial diffeomorphism and the global $U(1)$ symmetries. In particular, the physically important case of a Coulomb interaction (as relevant for electron systems) has the full NR general covariance, while a short range interaction in the limit of infinite scattering length (as relevant for the unitary Fermi gas) additionally has NR conformal invariance. Hence their low energy physics are constrained by these symmetries. General interactions need not preserve the full conformal symmetry of the free theory. If the theory remains invariant under transformations of the form $f(t) = f_0 + f_1 t$ then it has time translation and scale invariance. For the case of $f_1 = 0$ the theory only has time translation invariance.

2.5 Relativistic Parent Theory

One can obtain a NR QFT by taking the speed of light $c \to \infty$ limit of a relativistic field theory. In order to yield non-trivial results, a chemical potential μ must be turned on to provide the rest mass m of particles. This causes the free energy associated with a particle to remain finite in the large c limit, while the free energy associated with an antiparticle goes to infinity as twice its rest energy and they therefore completely decouple. The absence of antiparticles in a NR QFT means that virtual pairs cannot lead to particle creation. Instead the existence of particles requires a chemical potential to pay their rest mass. The non-relativistic theory then describes fluctuations around this energy. Extensive use of this concept and the $c \to \infty$ limit will be made throughout.

An illustrative example of this process, which will be paralleled later in Section 3.2.1, is to consider the Kaluza-Klein compactification of a free complex relativistic scalar. It is widely appreciated that momentum around the compact direction gives a tower of lower dimensional modes of mass $m_n \equiv |n| m_{kk}$, where the compactification radius is related to the Kaluza-Klein mass as $R_{kk} \equiv 1/(m_{kk}c)$.[2] Additionally, the compact momentum also gives the lower dimensional modes a $U(1)$ charge: the symmetry of translation around the circle acts as a phase rotation on the n-th Kaluza-Klein mode, in accordance to it having charge $q_n \equiv n$. Turning on a chemical potential $\mu = m_{kk}c^2$, the modes have the energy

$$E_n = \sqrt{\vec{p}^{\,2} c^2 + \left(m_n c^2\right)^2} - q_n \mu. \tag{2.16}$$

Taking the $c \to \infty$ limit, while keeping the Kaluza-Klein mass m_{kk} fixed, corresponds to taking the null $R_{kk} \to 0$ limit and reduces the mode's energy to

$$E_n \approx \frac{\vec{p}^{\,2}}{2n m_{kk}} + m_{kk}c^2(|n| - n). \tag{2.17}$$

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[2] Non-relativistically one still has $\hbar = 1$, so energy is measured in inverse time. $E = mc^2$ (or more precisely $KE = mv^2/2$) implies that $mc = E/c$ has units of inverse length.

The modes with positive charge n have the expected non-relativistic dispersion relation, while the "anti-particles" with negative charge n have an energy that grows like c^2, and hence decouple from the low energy theory.

In the same spirit, the transformations 2.1 can easily be derived by taking a non-relativistic limit of the relativistic theory of a charged massive field. Of course this procedure does not give the most general NR QFT, but it does give a simple way to derive the transformation properties of the free field theory. This is easiest to illustrate in the case of a scalar [81]. The relativistic action

$$S = -\int d^d x dt \sqrt{-g} \frac{1}{2} \left(g^{\mu\nu} \mathcal{D}_\mu \phi^\dagger \mathcal{D}_\nu \phi + c^2 m^2 e^{2\sigma} \phi^\dagger \phi \right), \tag{2.18}$$

where $\mathcal{D}_\mu \phi \equiv \partial_\mu \phi - \imath C_\mu \phi$ is the gauge covariant derivative of the gauge field C_μ, is invariant under the infinitesimal general relativistic coordinate and $U(1)$ gauge transformations:

$$\begin{aligned}
\delta\phi &= \xi^\rho \partial_\rho \phi - \imath \Lambda \phi, \\
\delta C_\mu &= \xi^\rho \partial_\rho C_\mu + C_\rho \partial_\mu \xi^\rho - \partial_\mu \Lambda, \\
\delta g_{\mu\nu} &= \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\rho\nu} \partial_\mu \xi^\rho.
\end{aligned} \tag{2.19}$$

Powers of the speed of light have been explicitly displayed throughout so the non-relativistic $c \to \infty$ limit can be taken. Note that the relativistic mass is defined as me^σ, this is crucial as it has a different scaling dimension than the non-relativistic mass m, as discussed above. Additionally, m and σ will be treated as spurionic fields and be allowed to have spacetime dependence. Following [81] one could now define the non-relativistic field by factoring out the fast phase rotation due to the scalar field's rest mass: $\phi \equiv e^{-\imath mc^2 t} \phi_{NR}/\sqrt{c}$. For charged scalars this phase can instead be gauged away via the gauge transformation $\Lambda = -c^2 mt$. Therefore $\sqrt{c}\phi$ itself can be treated as a non-relativistic charged scalar by working with the background gauge potential $C_\mu = -\partial_\mu \Lambda = \delta_{\mu t} mc^2$. Although relativistically such a gauge field would be considered highly trivial as it has zero field strength, here it plays an important role due to the fact that c dependent gauge transformations, such as the above Λ, and the $c \to \infty$ limit do not commute. Also note that, unlike constant spatial vector potentials, a constant C_t can, in general, not be completely gauged away. The term $\int_M j^\mu C_\mu$ (where M is the space-time manifold) is usually taken to be gauge invariant as long as j^μ is a conserved current. Under a gauge transformation $\delta C = -d\Lambda$, the change in action is

$$\delta S = -\int_M j^\mu \partial_\mu \Lambda = -\int_{\partial M} (\Lambda j_\mu) dS^\mu + \int_M \Lambda \partial_\mu j^\mu. \tag{2.20}$$

The second term vanishes by current conservation. The contributions to the boundary term from spatial boundaries vanish for any localized current. However, for the boundaries of the integral at the final and initial times, $t = t_f$ and $t = t_i$, j^0 generically will not vanish. The total charge Q is conserved, and if it is non-zero at one time, it is non-zero at all times. In particular, for $\Lambda = mc^2 t$ (which would be needed to set the above constant C_t to zero) one has

$$\delta S = -mc^2 Qt\Big|_{t_i}^{t_f} = mc^2 Q(t_i - t_f) \tag{2.21}$$

which clearly is non-zero as long as Q is non-zero. The action is only invariant under the restricted class of gauge transformations which vanish at t_f and t_i. However, to remove a constant C_t would require a gauge transformation which is non-vanishing on initial and final surfaces.

As a warm-up, consider, as in [81], the metric expansion[3]:

$$g_{\mu\nu} = \begin{pmatrix} -c^2 + 2\frac{A_t}{m} & \frac{A_i}{m} \\ \frac{A_j}{m} & g_{ij} \end{pmatrix}. \tag{2.22}$$

For a constant m and $\sigma = 0$, substituting this form of the metric, the gauge field background $C_t = mc^2$, and the rescaled field $\psi = \sqrt{mc}\phi$ into the relativistic action

[3]The leading piece of g_{tt} goes as c^2 because the non-relativistic time t is used as the temporal coordinate, not $x^0 \equiv ct$. Likewise for the behavior of g_{ti}.

2.18, and after discarding negative powers of c, one obtains

$$S = \int dtd^d x\sqrt{g}\left[\frac{\imath}{2}(\psi^\dagger\partial_t\psi - \partial_t\psi^\dagger\psi) + A_t\psi^\dagger\psi - \frac{g^{ij}}{2m}(D_i\psi^\dagger D_j\psi)\right], \tag{2.23}$$

which is the non-relativistic action 2.2 with $\Phi = B_i = 0$. The action of spatial diffeomorphisms and the global $U(1)$ on the remaining background fields can be determined from the transformation 2.19 for the generators $\xi^\mu = (\lambda/mc^2, \xi^i)$.

There are two important points in the details of this calculation, both concerning the gauge field C_μ. First, the mass term in the relativistic action 2.18, for $\sigma = 0$, would contribute the term $-1/2c^2m^2\psi^\dagger\psi$ to the non-relativistic action, forcing $\psi = 0$ in the $c \to \infty$ limit. For the background $C_t = mc^2$ this mass term is canceled by the $-C_\mu g^{\mu\nu} C_\nu \psi^\dagger\psi$ term coming from the covariant derivative. This is understood as tuning a chemical potential that provides the rest mass of the particles, so that the non-relativistic action only describes fluctuations around this energy. Thus the magnitude of the gauge field acts as a chemical potential and needs to be fixed to cancel the mass term and allow a non-trivial non-relativistic limit.

Secondly, it needs to be assured that a consistent expansion in powers of c has been done, both of the metric and the gauge field. C_μ naturally has a piece of order c^2 from performing the gauge transformation removing the fast phase of the scalar field. A consistent expansion can be made so that the next term comes in at zeroth order: $C_\mu = c^2 b_\mu + v_\mu + \mathcal{O}(c^{-2})$. As will be discussed in more detail in Sections 3.2.2 and 4.2, for the expansion of the temporal diffeomorphism generator $\xi^t = f - \alpha/c^2$ the gauge field transforms as

$$\delta v_\mu \supset -C_t\partial_\mu\alpha/c^2 = -b_t\partial_\mu\alpha, \tag{2.24}$$

that is, the $\mathcal{O}(c^0)$ piece of C_μ is generated by the subleading temporal diffeomorphism α. As the warm-up example above had $v_\mu = 0$ throughout, to maintain this restriction an $\mathcal{O}(c^0)$ gauge transformation was implicitly performed. Therefore, the appearance of the gauge transformation λ as the subleading term of the temporal diffeomorphism is only an artifact of demanding $\alpha = -\lambda/m$, so that v_μ stayed zero.

The warm-up example can be extended to include general backgrounds. As discussed above, the gauge field can be consistently expanded as $C_\mu \equiv c^2 b_\mu + v_\mu$. The consistent expansion of the metric can be determined by first considering the case of $C_\mu = c^2 b_t \delta^t_\mu$, similar to the warm-up example. In this frame the metric can be decomposed in the ADM form[4]

$$g_{\mu\nu} = \begin{pmatrix} -c^2 N^2 + N^k N_k & N_i \\ N_j & G_{ij} \end{pmatrix}, \tag{2.25}$$

where $N^k = G^{ki} N_i$. The general leading gauge field $C_\mu = c^2 b_\mu$ can be obtained from $C_t = c^2 b_t$ by performing a coordinate transformation, $C_\mu \to J^\nu_\mu C_\nu$. Under this transformation the metric change by two Jacobian factors, $g_{\mu\nu} \to J^\rho_\mu g_{\rho\sigma} J^\sigma_\nu$, and it can be see that all components generically gain $\mathcal{O}(c^2)$ pieces. One is therefore lead to expand the metric as[5]

$$g_{\mu\nu} = \begin{pmatrix} -c^2 N^2 + N^k N_k & N_i + c^2 P_i \\ N_j + c^2 P_j & G_{ij} - c^2 \frac{P_i P_j}{N^2} \end{pmatrix}. \tag{2.26}$$

The non-relativistic action 2.2 and transformations 2.1 can now be derived by taking the formal $c \to \infty$ limit of this relativistic theory. First, the chemical potential must provide the rest energy of the particles. As discussed above, this is achieved by a cancellation between the mass term and the magnitude of the gauge field. In general there are also $\mathcal{O}(c^4)$ pieces of $C_\mu C^\mu$. For the action to have a non-trivial non-relativistic limit this piece must vanish, requiring

$$\frac{P_i}{N^2} = -\frac{b_i}{b_t}. \tag{2.27}$$

This can be understood as the requirement that the theory has a global time, needed for a causal non-relativistic theory. As discussed above in Section 2.4, in adapted

[4]The lack of A_t in this expansion will be discussed further in Section 4.4.

[5]This can be seen to be a consistent expansion, meaning no other positive power of c pieces get turned on by coordinate transformations.

coordinates the vector B_i vanishes. The relation of this NR field to b_i and P_i, given by 2.30 to follow, justifies the identification in 2.27: in general these fields only arise due to coordinate changes to a non-adapted frame. The $\mathcal{O}(c^2)$ piece of $C_\mu C^\mu$ will play the role of a chemical potential and cancel the mass term. Explicitly this requires

$$\frac{b_t}{N} = me^{\sigma}. \tag{2.28}$$

Plugging the rescaled field $\phi \to \phi/\sqrt{c}$ and the expansions for the gauge field and metric into the relativistic action 2.18 (with the generalized spacetime dependent mass), and discarding negative powers of c, one obtains

$$S = \int dtd^dxme^{\sigma}\mathcal{L}, \tag{2.29}$$

where $\mathcal{L}$ is the Lagrangian density of the action 2.2 if the following identifications are made:

$$\begin{aligned}
e^{-\Phi} &\equiv \frac{mN^2}{b_t}, \\
B_i &\equiv -\frac{b_i}{b_t} = \frac{P_i}{N^2}, \\
A_t &\equiv v_t + \frac{b_t N^k N_k}{2N^2}, \\
A_i &\equiv v_i + \frac{b_t N_i}{N^2} - \frac{b_i N^k N_k}{2N^2}, \\
g_{ij} &\equiv G_{ij} - \frac{b_i N_j}{b_t} - \frac{b_j N_i}{b_t} + \frac{b_i b_j N^k N_k}{b_t^2}.
\end{aligned} \tag{2.30}$$

These field combinations transform as 2.1 if the relativistic generators of the transformations 2.19 are expanded as $\xi^\mu = (f, \xi^i)$ and $\Lambda = \lambda$. Gauge transformations of order $\mathcal{O}(c^2)$ can no longer be performed as they would not leave the $\mathcal{O}(c^2)$ piece of $C_\mu C^\mu$ invariant, which was needed to have a well defined $c \to \infty$ limit. The role of the subleading temporal diffeomorphism α, introduced previously, and its relation to the specific combinations of relativistic fields in the dictionary 2.30 will be discussed in Section 4.2.

For constant m and σ the field can be rescaled as $\psi \equiv \phi\sqrt{m}e^{\sigma/2}$ and the above exactly reproduces the Lagrangian density of the action 2.2. This rescaling only changes the dimension of the field and, in fact, can be done even for time dependent m and σ. The role of the field σ can most easily be understood by enforcing NR conformal invariance on the above action. For the general $z \neq 2$, the transformation 2.14 will be a spurionic symmetry of the action if

$$\delta_\omega e^\sigma = -(d+z-2)\omega e^\sigma. \tag{2.31}$$

It is now clear why the NR action 2.23 of [81] has NR conformal invariance. The transformation ω can be used to set me^σ to a constant. Recall from Section 2.4 that the restricted case of [81] with $B_i = \Phi = 0$ is maintained by performing $\omega = -\dot{f}/z$ whenever the temporal redefinition $f(t)$ is performed. In turn, transformations 2.14 and 2.31 imply that me^σ will generically become a function of time. But such a factor can be absorbed into the fields by a redefinition, due to the anti-symmetric nature of the time derivative in the action 2.2. This means the restricted case [81] will have NR conformal invariance and the field has the transformation

$$\delta_\omega \psi \equiv \delta_\omega \left(\phi\sqrt{m}e^{\sigma/2}\right) = \frac{1}{2\sqrt{m}}\phi e^{\sigma/2}\delta_\omega m + \frac{1}{2}\phi\sqrt{m}e^{-\sigma/2}\delta_\omega e^\sigma = -\frac{d}{2}\omega\psi, \tag{2.32}$$

in essence deriving 2.12.

Chapter 3

A NON-RELATIVISTIC GRAVITATIONAL THEORY

Much of this chapter is taken from [56, 57] which was coauthored with Andreas Karch, as well as [55].

3.1 Hořava Gravity

A gravitational theory centered around foliation preserving diffeomorphisms was introduced by Hořava in [49]. In its most simple form, Hořava-Lifshitz theory describes the dynamics of a lapse field N, a shift vector $N_I(t, x_I)$, and a spatial metric $G_{IJ}(t, x_I)$[1]. In the language of [49] the theory is "projectable" if N is a function of t only, and non-projectable when N is allowed to have spatial dependence as well. As shown in the next section, the most general low energy action consistent with symmetries and containing up to two derivatives is almost completely fixed to be that of Einstein gravity written in terms of these fields. In addition to the two free dimensionful parameters of Einstein's gravity, the Newton's constant G_N and the cosmological constant Λ, the low energy limit of projectable Hořava gravity has two additional free parameters: $\tilde{\lambda}$, which determines the relative coefficient of the two allowed kinetic terms for the spatial metric; and $\tilde{\beta}$, which determines the speed of the spin two graviton[2]. In the non-projectable case, which will be the main interest in

[1]Indices have the following meaning: i, j, ... run over the d spatial dimensions of the field theory; μ, ν, ... run over the $d+1$ field theory directions including time; I, J, ... run over the $D = d+1$ spatial dimensions of the bulk including the radial coordinate r; and, last but not least, M, N, ... run over all $d+2$ bulk directions including time and r. Section 3.2.1 will require discussion of a $d+3$ dimensional bulk, there indices X, Y, ... will cover the $d+2$ directions of M, N, ... plus one additional direction ζ.

[2]Without Lorentz invariance the speeds of light and gravity can be different.

this work, there is another two derivative term involving spatial derivatives of N that can be included in the low energy action. The corresponding coupling constant is referred to as $\tilde{\alpha}$. These parameters are one of the issues that makes it difficult to find a version of Hořava gravity that is a consistent theory of our world. In order to agree with the observed Lorentz invariance one needs a mechanism to set $\tilde{\lambda} \approx \tilde{\beta} \approx \tilde{\alpha} \approx 0$, the values they take in general relativity. For applications to NR holography, this is of no concern. In fact, one could hope that by adjusting these couplings Hořava gravity could holographically describe a wide class of NR QFTs.

The projectable version of Hořava-Lifshitz theory was extended in [50, 25] to include NR general covariance (that is, the $U(1)$ symmetry corresponding to particle number conservation in addition to foliation preserving diffeomorphisms). In this case, the theory contains two additional non-dynamical fields, the "potential" $A(t, x_I)$ (which arises as the subleading term of N in a non-relativistic expansion and in that sense it restores spatial dependence to N) and at least one of the following: a field A_{IJ}, which can be thought of as the subleading term of the spatial metric, or the so called "prepotential" field $\nu(t, x_I)$. The one exception is the case of $D = 2$ spatial dimensions, for which no extra field beyond A is required. These versions do not allow a straightforward holographic interpretation. In $D = 2$ dimensions the equations of motion for A immediately force spatial slices to be flat, whereas for holographic interpretations following the standard recipe an asymptotically hyperbolic spatial slice is expected[3]. Similarly, the theory with A_{IJ} requires a flat spatial slice[4]. The scenario with the prepotential ν has a different problem. Under the $U(1)$ symmetry ν shifts. Therefore, as discussed more in Section 3.1.2, the $U(1)$ gauge invariance in the

[3] AdS in flat slicing has been found as a solution to projectable Hořava gravity [36], but given in the Gullstrand-Painleve coordinates, which do not extend to the boundary. These are related to the traditional Fefferman-Graham coordinates [32] by a "non-physical" temporal transformation, and so correspond to gauge inequivalent configurations of Hořava gravity.

[4] While it is possible to introduce a "spatial cosmological constant" Ω in the theory with A_{IJ}, the constraints that arise as the equations of motion of A and A_{IJ} are only satisfied if $\Omega = 0$.

bulk is completely fixed by choosing $\nu = 0$ gauge; there are no residual transformations left that could be interpreted as global symmetries acting on the background data of the dual field theory. One could instead adopt the $N_r = 0$ gauge, which leaves r-independent gauge transformations as a residual symmetry. In this case the asymptotic value of ν would have to be interpreted as the source of a boundary operator. Like the background electric and magnetic fields, this background spurionic field would not be invariant under the $U(1)$ global transformation. Unlike the former, which do transform exactly like background fields should under a $U(1)$ transformation, ν shifts also in the boundary theory. The only example of an operator that transforms like this would be the phase of a $U(1)$ charged operator; if either added to the Lagrangian or having acquired an expectation value, the presence of this operator would signal that in the boundary theory the $U(1)$ symmetry is broken (explicitly or spontaneously, respectively). Thus the theory with ν can at best capture the dual to a NR QFT with a broken $U(1)$.

The following sections will derive a different field content that obeys the symmetries of NR electro-diffeomorphisms by taking a particular Kaluza-Klein compactification of GR, as well as by taking the infinite speed of light limit of Einstein-Maxwell theory. The main thrust of this thesis is that by working in coordinates adapted to the global time, and restricting the symmetry transformations to exclude the non-physical temporal diffeomorphisms, non-projectable Hořava gravity coupled to electric and magnetic fields captures NR general covariance, and therefore should be holographically dual to a generic NR QFT with these same symmetries.

3.1.1 Foliation Preserving Diffeomorphisms

Hořava gravity is an alternate metric theory of gravity [49]. Like general relativity, its low energy behavior can be expressed in the context of geometry. In GR a dynamic spacetime manifold encodes the influence of gravity: the manifold responds to the presence of mass and energy, while simultaneously the geometry dictates the evolution of said matter. A hallmark of GR, capturing its geometric nature, is the coordinate invariance of observables. This is captured in the physical description by demanding that diffeomorphisms, that is, local changes of coordinates on the manifold, are a gauge symmetry of the theory.

Hořava gravity shares much of this geometric flavor, but, unlike GR, in addition to a dynamic spacetime there is a fundamentally special foliation of the manifold, Σ_t. Hořava gravity has a preferred notion of time which is geometrically captured by the leaves of a co-dimension one foliation: these slices consist of simultaneous events. The foliation structure breaks the general covariance enjoyed by just the manifold. The preferred global time of Hořava gravity means that Lorentzian coordinate changes that mix spatial directions into a new time coordinate are no longer allowed. These would alter the notion of which events are simultaneous and violate the preferred foliation of the manifold. The temporal coordinate can still be reparametrized: sending $t \to \tilde{t}(t)$ for an arbitrary monotonic function $\tilde{t}$ preserves the simultaneity of events, and hence the foliation structure. Spatial diffeomorphisms that change coordinates on a leaf are still allowed, and indeed can even be time dependent. These transformations, spatial diffeomorphisms $x_I \to \tilde{x}_I(t, x_I)$ and time reparametrizations $t \to \tilde{t}(t)$, are collectively called foliation preserving diffeomorphisms, and are a gauge symmetry of Hořava gravity.

The degrees of freedom of Hořava gravity can be understood in this low energy geometric picture. A spacetime manifold in coordinates adapted to the foliation will have a metric of the ADM form

$$\tilde{G}_{MN} dx^M dx^N = -N^2 dt^2 + G_{IJ} \left(dx^I + N^I dt\right) \left(dx^J + N^J dt\right) . \tag{3.1}$$

Here G_{IJ} is the spatial metric on the leaves of the foliation at a constant t, N is the lapse function, and N^I is the shift, a spatial vector. The low energy action of Hořava gravity is expressed in terms of these fields as [39][5]

$$S = \int dtd^dxdr\left(\mathcal{L}_{kin} - \mathcal{L}_V\right), \tag{3.2}$$

where the kinetic term is given in terms of the extrinsic curvature of the leaves,

$$K_{IJ} \equiv \frac{1}{2N}\left(\dot{G}_{IJ} - \nabla_I N_J - \nabla_J N_I\right), \tag{3.3}$$

and its trace, $K = G^{IJ}K_{IJ}$, by

$$\mathcal{L}_{kin} = \frac{1}{16\pi G_H}\sqrt{G}N\left[K_{IJ}K^{IJ} - \left(1+\tilde{\lambda}\right)K^2\right]. \tag{3.4}$$

Here G is the determinant of the spatial metric G_{IJ}, and ∇_I is its Levi-Civita connection. The gravitational constant G_H is a dimensionful parameter that sets the scale where quantum effects become important, that is, it determines the Planck length. The simplest potential term involving up to two derivatives, as appropriate for the low energy limit, is given by [18, 20]

$$-\mathcal{L}_V = \frac{1}{16\pi G_H}\sqrt{G}N\left[(1+\tilde{\beta})(R-2\Lambda) + \tilde{\alpha}\frac{(\nabla_I N)(\nabla^I N)}{N^2}\right], \tag{3.5}$$

where R is the Ricci scalar of G_{IJ} and Λ is the cosmological constant.

The constants $(\tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$ are free dimensionless coupling constants that are allowed by demanding only foliation preserving diffeomorphisms and not the full relativistic diffeomorphism invariance of GR[6]. For $\tilde{\lambda} = \tilde{\alpha} = \tilde{\beta} = 0$ this action becomes, up to a total derivative, the standard Einstein-Hilbert one, written in terms of the ADM decomposition of the full $d+2$ dimensional bulk metric $\tilde{G}_{MN}$. Even in the $\tilde{\lambda} = \tilde{\alpha} = \tilde{\beta} = 0$ limit this is not the theory of standard GR. Despite identical actions, the gauge invariances of Hořava gravity lack the general temporal diffeomorphism $t \to \tilde{t}(t, x_I)$. As a consequence Hořava gravity contains an extra scalar degree of freedom as compared to GR.

[5] Note that the coupling constants are not the same as [39], despite similar names.

[6] With no further field content $\tilde{\beta}$ can be set to zero by performing a field and parameter redefinition [33].

Hořava gravity departs much more drastically from GR in its high energy behavior. Although this regime will not be further explored, a brief discussion is pertinent. Because of the fundamental foliation of spacetime, a preferred notion of time exists in Hořava gravity, and Lorentz symmetry is broken. This allows temporal and spatial coordinates to have different mass dimensions and is captured by the dynamical critical exponent z_H: $[x_I] = -1$ while $[t] = -z_H$, implying $[G_H] = z_H - D$. An interesting case is near a UV fixed point with $z_H = D$. The UV action will be dominated by terms with $2D$ spatial derivatives, while there will be the same kinetic term 3.4, involving K_{IJ} and which contains only two time derivatives. This appears to lead to a unitary, power counting renormalizable theory [49, 5, 3], as evidenced by the fact that now $[G_H] = 0$. The low energy action 3.5 with an effective $z_{IR} = 1$ would then be the flow from this UV fixed point due to relevant deformations.

Under spatial diffeomorphisms ξ^I and time reparametrizations $f(t)$ the fields transform as

$$\begin{aligned}
\delta G_{IJ} &= \xi^K \partial_K G_{IJ} + f \dot{G}_{IJ} + G_{IK} \partial_J \xi^K + G_{KJ} \partial_I \xi^K, \\
\delta N_I &= \xi^K \partial_K N_I + f \dot{N}_I + N_K \partial_I \xi^K + G_{IK} \dot{\xi}^K + \dot{f} N_I, \\
\delta N &= \xi^K \partial_K N + f \dot{N} + \dot{f} N.
\end{aligned} \tag{3.6}$$

These can be derived by taking the $c \to \infty$ limit of the transformation of the relativistic metric $\tilde{G}_{MN}$ with the diffeomorphism parameters $\xi^M = \left(f, \xi^I\right)$ (after explicitly restoring the speed of light to the metric: $N \to cN$) [48].

3.1.2 NR General Covariance and the Scalar Khronon

Hořava gravity can be usefully embedded into standard GR via a Stückelberg-like mechanism [34, 18, 20]. This formalism makes the extra degree of freedom explicit by coupling Einstein gravity to an additional scalar field ϕ. When the scalar field acquires an expectation value the gauge symmetry of GR is broken down to only spatial diffeomorphisms along the level sets of ϕ. In this way ϕ can be used to define the preferred foliation by a global time, and is referred to as the khronon [18, 20]. This view of Hořava gravity as GR with diffeomorphism invariance broken by a background field has also been recently emphasized in [63].

To have the symmetries of foliation preserving diffeomorphisms, ϕ needs to have the reparametrization symmetry in field space $\phi \to \tilde{\phi}(\phi)$, which becomes the time reparametrization symmetry of Hořava gravity[7]. This reparametrization invariance can be made explicit by working with the time-like unit vector normal to the leaves of constant ϕ,

$$u_M \equiv \frac{-\partial_M \phi}{\sqrt{-\tilde{G}^{NP}\partial_N\phi\partial_P\phi}}. \tag{3.7}$$

In the "unitary gauge" the time coordinate is chosen to be the expectation value of the khronon, $t = \phi$, and this vector has $u_0 = -N$ and all its spatial components vanish. The geometric quantities of the foliation appearing in Hořava gravity can all be expressed in terms of the khronon field. In particular, in unitary gauge the spatial components of

$$\mathcal{K}_{MN} \equiv \left(\tilde{G}_{MP} + u_M u_P\right)\tilde{\nabla}^P u_N \tag{3.8}$$

become the extrinsic curvature K_{IJ}. The most general covariant action involving two derivatives of the normal vector u_M is related to Einstein-Aether theory [54], and given by

$$\begin{aligned} S_K =& \frac{1}{16\pi G_K}\int dt d^D x\sqrt{-\tilde{G}}\Big[\tilde{R} - 2\Lambda \\ &+ c_4 u^M\tilde{\nabla}_M u^N u^P \tilde{\nabla}_P u_N - c_2\left(\tilde{\nabla}_M u^M\right)^2 - c_3\tilde{\nabla}_M u^N\tilde{\nabla}_N u^M\Big]. \end{aligned} \tag{3.9}$$

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[7] As explained in [20], a similar construction also underlies other modified theories of gravity. A time dependent condensate of a scalar with a shift symmetry (giving rise to a theory with time dependent spatial diffeomorphisms together with time translation symmetry) underlies the "ghost condensation" model [6] as well as shift-symmetric k-essence [7]. When even time translation symmetry is absent and only time dependent spatial diffeomorphisms are preserved, the symmetry group governs the effective theory of standard inflation [23, 87]. If time translation invariance is combined with time independent diffeomorphisms one has the symmetry of Einstein-aether theory [54] or gauged ghost condensation [22].

The reader should recall that the tilded quantities refer to those derived from the full $d+2$ dimensional Lorentzian metric.

To make contact with Hořava gravity one chooses the time coordinate t to be the scalar field ϕ, breaking the general covariance of the khronon-metric theory. Under this gauge fixing the khronon action 3.9 then becomes to the low energy Hořava action 3.2 upon making the identification of constants [11]:

$$\frac{G_H}{G_K} = 1+\tilde{\beta} = \frac{1}{1-c_3}, \quad 1+\tilde{\lambda} = \frac{1+c_2}{1-c_3}, \quad \tilde{\alpha} = \frac{c_4}{1-c_3}. \tag{3.10}$$

By examining the weak-field, slow-motion limit of the action 3.9 it is seen that the effective Newton's constant is [21, 11]

$$G_N \equiv \frac{1-c_3}{1-\frac{c_4}{2}} G_H. \tag{3.11}$$

Lastly, the equations of motion following from the action 3.9 can be linearized around the flat background solution to determine the speed of the modes. The low momentum dispersion relations determine the wave speeds squared to be [53, 39]:

$$s_2^2 = \frac{1}{1-c_3}, \quad s_0^2 = \frac{(c_2+c_3)(D-1-c_4)}{c_4(1-c_3)(D-1+Dc_2+c_3)}, \tag{3.12}$$

for the spin two modes of the metric and the spin zero mode of the foliation, respectively[8]

A powerful use of the khronon formalism is in the probe regime where the c_i are parametrically small. In this case the backreaction of the khronon on the geometry can be ignored and any solution of vacuum GR descends to a solution of Hořava gravity. In the probe limit, the action 3.9 is just that of GR, so the metric equations of motion are clearly satisfied by vacuum GR solutions. The equation of motion for the

[8] In terms of the khronon language, the field redefinition that sets $\tilde{\beta}=0$ (equivalently $c_3=0$) involves the redefinition $\tilde{G}^{eff}_{MN} \equiv \tilde{G}_{MN} - \sigma u_M u_N$. This is equivalent to determining which effective metric $\tilde{G}^{eff}_{MN}$ matter couples minimally to. For example, in an electromagnetic action, contracting the field strength F_{MN} via $\tilde{G}^{eff}_{MN}$ for $\sigma=0$ or $\sigma=\tilde{\beta}$ is equivalent to working in units with the speed of the photon or that of the spin two graviton equal to one, respectively.

khronon ϕ can then be solved on the fixed background given by the metric solution. By making the Lorentzian coordinate change to unitary gauge $t \to \hat{t} \equiv \phi(t, x_I)$ the resulting lapse N, shift N_I, and spatial metric G_{IJ} of the ADM decomposition 3.1 are now a solution to Hořava gravity. In the probe limit the khronon does not influence the geometry of the manifold, it merely imprints the preferred notion of time via its level sets.

From the identification of parameters 3.10, the probe limit of the khronon formalism gives the Hořava coupling constants:

$$\tilde{\beta} \approx c_3 \ll 1, \quad \tilde{\lambda} \approx c_2 + c_3 \ll 1, \quad \tilde{\alpha} \approx c_4 \ll 1, \tag{3.13}$$

while the speeds of the modes 3.12 reduce to

$$s_2^2 \approx 1, \quad s_0^2 \approx \frac{c_2 + c_3}{c_4} = \frac{\tilde{\lambda}}{\tilde{\alpha}}. \tag{3.14}$$

Interestingly, even in the probe limit the scalar mode can have arbitrary sound speed. In taking the limit this physical speed should be held fixed; that is, the ratio $\tilde{\lambda}/\tilde{\alpha}$ is held fixed while both constants are taken to zero. This technique will be used in Section 5.6 to compute numerical profiles for the khronon in the background of an Anti-de Sitter-Schwarzschild black hole.

One can also use the scalar khronon to formulate the generally covariant version of Hořava gravity [50, 25]. As initially introduced in [48] the transformations of the Hořava fields 3.6 can be extended to include a $U(1)$ transformation by expanding to the next order in the speed of light. For $N \to N + A(t, x_I)/c^2$ and $\xi^t = f - \alpha(t, x_I)/c^2$ the action of the $U(1)$ transformation α is:

$$\begin{aligned}
\delta_\alpha N &= 0, \quad \delta_\alpha G_{IJ} = 0, \\
\delta_\alpha N_I &= N^2 \partial_I \alpha, \\
\delta_\alpha A &= -(\alpha \dot{N}) + N N^I \partial_I \alpha,
\end{aligned} \tag{3.15}$$

while under foliation preserving diffeomorphisms A transforms as N does. As it stands the action 3.2 is not invariant under this transformation. First developed in [50], and later generalized in [25], this can be fixed by postulating the "prepotential" field ν that shifts under the α transformation. In fact, this field can be associated with the scalar khronon, as follows.

For a consistent interpretation of α as a gauge transformation how it acts on the khronon needs to be understood. Restoring factors of the speed of light the expansion of the khronon around unitary gauge is

$$\phi = c^2 t + \chi(t, x_I). \tag{3.16}$$

From this the transformation $t \to t + \alpha/c^2$ is expected to be reinterpreted as the shift $\chi \to \chi - \alpha$, that is, the subleading relativistic temporal diffeomorphism α can be interpreted in a non-relativistic foliation preserving way as instead shifting the khronon fluctuation χ. Therefore, the prepotential ν is naturally identified with χ, the subleading piece of the khronon in the $c \to \infty$ expansion. The transformation of χ can also be found by considering the khronon to be the phase of a complex scalar. Expanding the relativistic transformation of a scalar, and demanding the reparametrization invariance of the khronon field, one finds

$$\delta\chi = \xi^K \partial_K \chi + f\dot{\chi} - \dot{f}\chi - \alpha. \tag{3.17}$$

It is easy to check that the following combinations are invariant under the $U(1)$ transformation α

$$\hat{N}_I \equiv N_I + N^2 \partial_I \chi, \quad \hat{A} \equiv A - (\dot{\chi N}) + N N^I \partial_I \chi + \frac{N^3}{2} G^{IJ} \partial_I \chi \partial_J \chi. \tag{3.18}$$

In the projectable case, this reproduces the "minimal substitution" of [25] if the identification $\nu \equiv -N\chi$ of the prepotential with the khronon fluctuation is made[9]. In particular $\hat{A}$ is equivalent to [25]'s $A - a$.

Using this form of covariant Hořava gravity an obstruction in creating a holographic duality is apparent. The khronon must be added to the bulk action to yield

[9]The factor of the lapse N is due to differing definitions of α as the subleading piece of the temporal diffeomorphism when compared with [50, 25].

invariance under α. From the statement of holography, this action can give the correlation function of the operator dual to the khronon by examining its on-shell boundary value. This operator is not gauge invariant though, and will shift under α as χ does. The only operator that shifts under a gauge transformation is the phase of a charged field; it acquiring a nontrivial correlation function indicates that the $U(1)$ symmetry is in fact broken in the field theory. This is apparent by considering how the bulk transformation generated by α manifests itself as the global $U(1)$ rotation of the field theory. If one uses the freedom 3.15 to gauge fix $N_r = 0$, it is seen that r independent α maintains this bulk gauge choice and would be expected to correspond to boundary $U(1)$ transformations, leading to the above issue by shifting χ. Alternatively, α could be used to gauge away the khronon in the bulk. This does not solve the issue as now there are no residual α transformations that could be interpreted as acting on the boundary data. In this case the boundary $U(1)$ appears broken too.

There are two additional issues with the scalar khronon formulation leading to its abandonment as a holographic gravitational theory that captures the full NR general covariance symmetry. First is a purely classical gravitational consideration. By its nature the khronon field needs a uniform spatial distribution to define the leaves of the foliation. Such a configuration should generically be gravitationally unstable to clumping, and therefore may not even define a consistent theory[10]. The second issue is quantum in nature. In order to recover the time reparametrization invariance of Hořava gravity the khronon ϕ needs to have a global field redefinition symmetry. In quantum gravity there is expected to be no global symmetries so this construction seems problematic beyond the classical level.

These shortcomings hint at a solution; as the khronon is seen to transform as the phase of a complex scalar, this scalar should be considered charged, and the accompanying gauge fields included in the bulk. Being a gauged phase, this field would have

[10]Problems along these lines are known in the related ghost condensation theories [26, 76].

no stress tensor and therefore avoid the issue of clumping. Time reparametrization can be implemented without the need of postulating global symmetries, and therefore can be consistent with tenets of quantum gravity. As this construction requires the inclusion of a bulk vector field to set a preferred time slicing it will be referred to as a vector khronon. The hope of [50, 25], that the shift N_I could play a dual role as a gauge field for both spatial diffeomorphisms ξ^I and the $U(1)$ generator α seems to not be borne out, at least for holographic purposes. The role of bulk gauge fields will be pursued shortly, but first an alternate motivation for their necessity is discussed.

3.2 Vector Khronons

3.2.1 Kaluza-Klein Vector Khronon

The first attempts [80, 8, 35] at a gravitational dual to a non-relativistic field theory shared an unexpected feature: they had two extra dimensions compared to the field theory they described. This can be understood by realizing that these NR QFTs are basically light-like compactifications of relativistic field theories in one higher dimension. The simplest example is given by a light-like compactification with periodic boundary conditions for all fields. More interesting examples can be obtained by imposing twisted boundary conditions for R-charged fields along the light-like circle [47, 1, 68]. This twisting removes some of the zero modes on the circle and makes the field theory more tractable. With compactification on a light-like circle, the lower dimensional field theory preserves a non-relativistic subgroup of the higher dimensional relativistic Lorentz symmetry, the Schrödinger group. The holographic dual description correspondingly is also a light-like compactification, of general relativity on AdS spacetime. Momentum modes along the light-like direction, ζ, appear as separate conserved particle number sectors in the NR QFT, not as spatial momentum modes. This direction and the traditional holographic radial coordinate gives two extra dimensions to the bulk geometry. For an interesting non-relativistic interpretation of this geometry see [27].

Near the boundary $r \to 0$, the metric of these bulk geometries can be parametrized as [80][11]

$$d\hat{s}^2 = -\frac{2e^{-\Phi}}{mr^2}\left(dt - B_i dx^i\right)\left(d\zeta - A_t dt - A_i dx^i\right) + \frac{g_{ij}dx^i dx^j + dr^2}{r^2}. \tag{3.19}$$

The gauge $g_{\mu r} = g_{\zeta r} = 0$ has been chosen, but this does not completely fix the diffeomorphisms of the theory. Under the residual transformations these fields parametrizing the metric transform exactly like the NR QFT fields 2.1, for $\xi^\zeta \equiv \lambda$. This justifies this example as a non-relativistic holographic duality.

The NR QFT holographically described by GR on this background is highly constrained: most of its properties are inherited from the relativistic theory upon the light-like compactification, even with twisted boundary conditions. For $d = 2$ it is known that the field theory dual to the spacetime 3.19 is simply the discrete light cone quantization of $\mathcal{N} = 4$ SYM theory in four spacetime dimensions [47, 68]. Field theory properties, such as hydrodynamics and thermodynamics, follow from this relativistic reduction [47, 75]. Here this known non-relativistic duality is used as motivation: it has long been understood that a light-like compactification can be equivalent to a spatial compactification on a circle of vanishing radius, plus an appropriate boost [78, 79]. In what follows a $c \to \infty$ scaling limit will be shown to make a spatial compactification light-like and recover the metric 3.19.

This construction is equivalent to considering a chemical potential that provides the rest mass of the charged Kaluza-Klein momentum modes for a purely spatial circle and then taking the $c \to \infty$ limit[12], exactly as executed in the field theory

[11] The r^{-4} “Lifshitz” term in [80, 8] is unimportant for this discussion. It is separately invariant under the symmetry transformations. It encodes the effect of R-twisted boundary conditions in the field theory [47, 1, 68]. To get a non-trivial field theory with the desired Schrödinger invariance twisting is not needed and the light-like circle compactification with periodic boundary conditions suffices.

[12] The same is also true when considering twisted boundary conditions, even though in that case the construction is a little more complicated. It was shown in [16] that $\mathcal{N} = 4$ Super Yang-Mills

construction of Section 2.5. This allows direct identification of the correct bulk fields that map to the field theory sources of Section 2.2, as well as the bulk version of the constraint relating the chemical potential to the rest mass.

Consider a $d+3$ dimensional spacetime with metric $\hat{G}_{XY}$, and compactify along the last direction ζ. The Kaluza-Klein decomposition of the metric is

$$\hat{G}_{XY} \equiv L^2 \begin{pmatrix} \tilde{G}_{MN} + G_{\zeta\zeta} C_M C_N & -G_{\zeta\zeta} C_N \\ -G_{\zeta\zeta} C_M & G_{\zeta\zeta} \end{pmatrix}, \tag{3.20}$$

where L is a characteristic length scale of the geometry, such that the displayed metric components, as well as chosen coordinates, are unitless. The proper size of the compactified direction is $d\hat{s}^2 \equiv L^2 R_{kk}^2 e^{-2\Sigma} d\zeta^2$, where the dimensionless Kaluza-Klein radius R_{kk} is introduced. To recover a light-like compactification the limit $R_{kk} \to 0$ needs to be taken. The formal dimensionless expansion parameter is this radius, but defining $R_{kk} \equiv (L m_{kk} c)^{-1}$ in terms of a Kaluza-Klein mass the formal $c \to \infty$ limit can be taken instead. The Kaluza-Klein mass m_{kk} can be identified with the non-relativistic field theory mass m. The bulk proper Kaluza-Klein mass on the other hand is me^{Σ}, in units with $L = 1$. It should be emphasized that this limit is simply a coordinate scaling limit: the proper size of the compact direction is taken to zero, while time is rescaled such that the Kaluza-Klein mass remains finite.

Expanding the Kaluza-Klein gauge field as $C_M = c^2 b_M + v_M$, and the asymptotic $d+2$ dimensional metric as

$$\tilde{G}_{MN} = \begin{pmatrix} -c^2 N^2 + N^K N_K & c^2 P_I + N_I \\ c^2 P_J + N_J & -c^2 \frac{P_I P_J}{N^2} + G_{IJ} \end{pmatrix}, \tag{3.21}$$

yields a line element, $d\hat{s}^2 = \hat{G}_{XY} dx^X dx^Y$, with pieces of $\mathcal{O}(c^2)$ and $\mathcal{O}(c^0)$, as well as vanishing negative powers of c. To be a non-singular consistent scaling limit of the

compactified on a spatial circle with twisted boundary conditions can be obtained by a combination of T-dualities and shifts from the usual black D3-brane metric. The Null-Melvin Twist procedure of [47, 1, 68] used to generate the metric 3.19, including the additional r^{-4} Lifshitz term, is an infinite boost limit of this compactification, which once again can be interpreted as setting the chemical potential equal to the rest energy followed by a $c \to \infty$ limit.

$d+3$ dimensional geometry the $\mathcal{O}(c^2)$ pieces must vanish. Additionally, matching the $\mathcal{O}(c^0)$ components to those of the asymptotic metric 3.19 yields restrictions and identifications. Examining the $\mathcal{O}(c^2)$ term of the dt^2 piece, the asymptotic restriction

$$me^{\Sigma} = \frac{b_t}{N}, \tag{3.22}$$

is required of the fields. This is the bulk implementation of the field theory constraint 2.28, which is the requirement that the chemical potential compensates the rest energy and allows a NR limit. Combining this with the "null" $dtd\zeta$ and $dx^i d\zeta$ pieces, and matching to the metric 3.19, the identifications:

$$\frac{e^{-\Phi}}{m} \equiv \frac{r^2 N^2}{b_t}, \tag{3.23}$$

$$B_i \equiv -\frac{b_i}{b_t}, \tag{3.24}$$

are obtained, where it is understood that this is a matching of the asymptotic $r \to 0$ fields. The vanishing of the $\mathcal{O}(c^2)$ term of the $dtdx^i$ piece yields the restriction

$$P_I = -N^2 \frac{b_I}{b_t}, \tag{3.25}$$

which, recalling Section 2.5, encodes the requirement of the existence of a global time. Matching the remaining metric components to equation 3.19, the following identifications are obtained:

$$A_t \equiv v_t + \frac{b_t N^I N_I}{2N^2}, \tag{3.26}$$

$$A_i \equiv v_i + \frac{b_t N_i}{N^2} - \frac{b_i N^I N_I}{2N^2}, \tag{3.27}$$

$$g_{ij} \equiv r^2 \left(G_{ij} - \frac{b_i N_j}{b_t} - \frac{b_j N_i}{b_t} + \frac{b_i b_j N^I N_I}{b_t^2} \right). \tag{3.28}$$

It should be noted that the same partial gauge fixing which yielded the $d+3$ dimensional metric 3.19 has been used to set $\hat{G}_{rr} = 1/r^2$ and $\hat{G}_{r\zeta} = \hat{G}_{r\mu} = 0$. In terms of the Kaluza-Klein fields this can be seen to yield:

$$G_{rr} = \frac{1}{r^2}, \quad b_r = P_r = 0, \quad N_r + \frac{e^{-2\Sigma}}{m^2} b_t v_r = 0, \quad G_{ri} + \frac{e^{-2\Sigma}}{m^2} v_r b_i = 0. \tag{3.29}$$

Therefore, extending the above definitions 3.24, 3.27, and 3.28 to hold when an index is r this partial gauge fixing gives $B_r = A_r = g_{ri} = 0$.

Compared to the field theory non-relativistic limit 2.30 the above identifications are equivalent, up to powers of r. While the fields $(\Phi, B_i, A_t, A_i, g_{ij})$ of metric 3.19 are functions of only the field theory coordinates t and x_i, the Kaluza-Klein fields $(\Sigma, P_I, N, N_I, b_M, v_M, G_{IJ})$ generically depend on the holographic radial direction as well[13]. The above identifications can be taken to determine the asymptotic r behavior of these fields. From the identification 3.24, b_t and b_i must have the same asymptotic radial dependence, which, combined with 3.28, gives the leading asymptotic behavior of G_{ij} and N_i as r^{-2}. From the definition 3.23, N^2/b_t goes as r^{-2}, while the relations 3.26 and 3.27 determine v_M to be asymptotically independent of r.

Further determination requires assumptions on the behavior of Σ. For the asymptotic form $e^{-\Sigma} \equiv e^{-\sigma(t,\vec{x})}/r^{\delta}$ and using 3.22, the asymptotic behaviors $N \sim r^{\delta-2}$ and $b_M \sim r^{2\delta-2}$ are determined. Note that for $\delta = 1$ the lapse goes as $N \sim r^{-1}$ and the metric is asymptotically AdS. One can extend the symmetries to include the non-relativistic conformal transformations of 2.9 by considering radial diffeomorphisms, as in [80], which in fact fix $\delta = 1$. These transformations will be more fully explored in the next section.

The Kaluza-Klein viewpoint illuminates the factor of me^{σ} arising in the non-relativistic Lagrangian density derived by the $c \to \infty$ limit of the relativistic field theory in Section 2.5. Upon dimensional reduction the volume density of the higher dimensional theory yields the lower dimensional volume density, as well as a factor related to the proper Kaluza-Klein mass. In this case this gives an overall factor of $\sqrt{G_{\zeta\zeta}} = e^{-\Sigma}/mc$ causing the non-relativistic Lagrangian to be exactly that of 2.2, even for spacetime dependent m and Σ.

---

[13] In each case the ζ independent modes are being considered, corresponding to unbroken $U(1)$ invariance.

3.2.2 Einstein-Maxwell Vector Khronon

To the point of excess, a more general derivation of a holographic map relating bulk and NR QFT fields will now be presented. The motivation follows from the previous sections: it was seen that GR on a $d+3$ dimensional manifold can capture the generic symmetries of a $d+1$ dimensional NR QFT by taking a particular compactification and scaling limit. This specific duality is overly restrictive; despite containing fields that obey NR electro-diffeomorphism invariance most of the properties are simply inherited from the relativistic derivation.

Instead one can start with the Kaluza-Klein reduced field content of Section 3.2.1, a graviton and a Maxwell field (the scalar will not play a role here), and show that the NR limit can be taken directly in Einstein-Maxwell theory. Previously, spatial compactification and a scaling limit gave a light-like compactification of $d+3$ dimensional general relativity. Now, starting with the $d+2$ dimensional field content of the Kaluza-Klein theory, that is the Einstein-Maxwell system, a true[14] $d+2$ dimensional non-relativistic $c \to \infty$ limit will be taken.

The relativistic diffeomorphism generators are expanded as $\xi^M = (f - \alpha/c^2, \xi^I)$, under which the $d+2$ dimensional metric transforms as

$$\delta\tilde{G}_{MN} = \xi^P \partial_P \tilde{G}_{MN} + \tilde{G}_{MP}\partial_N \xi^P + \tilde{G}_{NP}\partial_M \xi^P. \tag{3.30}$$

For the consistent expansion

$$\tilde{G}_{MN} \equiv \begin{pmatrix} -c^2N^2 - 2N^2A + N^K N_K & c^2 P_I + N_I \\ c^2 P_J + N_J & -c^2 \frac{P_I P_J}{N^2} + G_{IJ} \end{pmatrix}, \tag{3.31}$$

under the diffeomorphism transformations in the $c \to \infty$ limit, the metric fields

[14] This is to contrast with the scaling limit of the previous section. There, after the $c \to \infty$ limit, the $d+3$ dimensional spacetime metric was finite. In this section if the Einstein-Maxwell fields were recombined back into a higher dimensional spacetime metric it would contain nonsensical $\mathcal{O}(c^2)$ pieces.

transform as:

$$
\begin{aligned}
\delta N =& \xi^K \partial_K N + f\dot{N} + \dot{f}N - \frac{P_K}{N}\dot{\xi}^K, \\
\delta A =& \xi^K \partial_K A + f\dot{A} - (\dot{\alpha} - N^I \partial_I \alpha)\left(1 + \frac{N^K P_K}{N^2}\right) + 2\frac{AP_K}{N^2}\dot{\xi}^K - 2AN^K \partial_K f, \\
\delta N_I =& \xi^K \partial_K N_I + N_K \partial_I \xi^K + f\dot{N}_I + \dot{f}N_I + G_{IK}\dot{\xi}^K \\
&+ \partial_I f\left(N^K N_K - 2N^2 A\right) + N^2 \partial_I \alpha - \dot{\alpha}P_I, \\
\delta G_{IJ} =& \xi^K \partial_K G_{IJ} + G_{IK}\partial_J \xi^K + G_{JK}\partial_I \xi^K + f\dot{G}_{IJ} \\
&+ N_I \partial_J f + N_J \partial_I f - P_I \partial_J \alpha - P_J \partial_I \alpha, \\
\delta P_I =& \xi^K \partial_K P_I + P_K \partial_I \xi^K + f\dot{P}_I + \dot{f}P_I - \frac{P_I P_K}{N^2}\dot{\xi}^K - N^2 \partial_I f.
\end{aligned}
\tag{3.32}
$$

The relativistic Maxwell gauge field can be expanded as $C_M \equiv c^2 b_M + v_M$. It transforms under the action of the gauge generator $\Lambda \equiv c^2\beta + \lambda$ and the relativistic diffeomorphisms ξ^M as

$$
\delta C_M = \xi^N \partial_N C_M + C_N \partial_M \xi^N - \partial_M \Lambda. \tag{3.33}
$$

Taking the $c \to \infty$ limit gives the transformations for the gauge fields:

$$
\begin{aligned}
\delta b_t =& \xi^K \partial_K b_t + f\dot{b}_t + \dot{f}b_t + b_K \dot{\xi}^K - \dot{\beta}, \\
\delta b_I =& \xi^K \partial_K b_I + b_K \partial_I \xi^K + f\dot{b}_I + b_t \partial_I f - \partial_I \beta, \\
\delta v_t =& \xi^K \partial_K v_t + f\dot{v}_t + \dot{f}v_t + v_K \dot{\xi}^K - \dot{\lambda} - b_t \dot{\alpha}, \\
\delta v_I =& \xi^K \partial_K v_I + v_K \partial_I \xi^K + f\dot{v}_I + v_t \partial_I f - \partial_I \lambda - b_t \partial_I \alpha.
\end{aligned}
\tag{3.34}
$$

Lastly, consider a complex scalar Ψ charged under the gauge field. It has the relativistic transformation

$$
\delta \Psi = \xi^M \partial_M \Psi - \imath \Lambda \Psi. \tag{3.35}
$$

Expanding the field as $\Psi \equiv \rho e^{-\imath\eta}$ for $\eta \equiv c^2\phi + \chi$, in the $c \to \infty$ limit, the real magnitude and phases transform as:

$$\begin{aligned}
\delta\rho =&\xi^K \partial_K \rho + f\dot{\rho}, \\
\delta\phi =&\xi^K \partial_K \phi + f\dot{\phi} + \beta, \\
\delta\chi =&\xi^K \partial_K \chi + f\dot{\chi} + \lambda - \dot{\phi}\alpha.
\end{aligned} \tag{3.36}$$

In the backgrounds to be considered $\Psi = 0$, so the particular form of the matter fields is not essential. What is needed is that some charged matter exists in the bulk, so that a constant A_t can not be simply gauged away. In the Kaluza-Klein example of the previous subsection the role of the charged matter was played by the massive Kaluza-Klein gravitons.

This procedure has given a consistent set of fields that transform sensibly in the $c \to \infty$ non-relativistic limit. To go further, for example to construct an action and determine which fields have non-trivial dynamics, some simplifying restrictions will be made. Most importantly, the theory is required to have a global time. As discussed in Section 2.1 this is necessary to have a causal non-relativistic theory. It can be implemented by constructing a spacetime foliation whose leaves contain events that happen at the same global time. Parallel to the previous discussion, this could be achieved by considering a scalar field whose level sets define the foliation leaves. The shortcomings of this scalar khronon formalism, enumerated in Section 3.1.2, requires a different approach in the pursuit of a bulk theory.

These problems will be circumvented by considering ϕ to be the gauged phase of a charged field, but global time will not be defined via its level sets. Instead, given the expectation value $\phi = t$, this phase is set to zero by performing the gauge transformation $\beta = -t$, which will turn on a constant time component of the gauge field, b_t. Thus the vector b_M acts as a "khronon" and determines the foliation by a global time: when in adapted coordinates it has only a temporal component. Once the expectation value of ϕ has been gauged away, in order to preserve $\phi = 0$, the "large" gauge transformations β can no longer be performed. Time reparametrizations are present in the theory as performing a spatially independent $f(t)$ maintains $b_I = 0$, that is, it stays within a physical global time.

Chapter 4

A NON-RELATIVISTIC HOLOGRAPHIC DUALITY

Much of this chapter is taken from [56, 57] which was coauthored with Andreas Karch, as well as [55].

4.1 Holographic Map

As discussed in the introduction, a holographic duality must capture the spurionic symmetries exhibited by the theories. For the NR QFTs of Section 2.2 the background sources transform as 2.1 under the non-relativistic electro-diffeomorphism transformations. From the previous Chapter a set of possible bulk fields that transform in a non-relativistic way was derived. What remains is to determine a map between bulk gravitational quantities and boundary field theory ones, using the spurionic symmetries as a guide.

By examining the above transformations 3.32-3.34 of bulk fields, combinations which asymptotically transform as the NR QFT fields 2.1 can be determined. Firstly, for $\beta = 0$, the two combinations

$$-\frac{b_i}{b_t}, \quad \frac{P_i}{N^2}, \tag{4.1}$$

both transform as the non-relativistic field B_i, with which they will be identified. This relation between the metric field P_I and the gauge field b_I, as discussed in Section 2.5, is required for the existence of a global time.

It is then seen that both N and b_t transform like $e^{-\Phi}$, and in generality the asymptotic identification

$$e^{-\Phi} \equiv r^{\gamma(\delta_\Phi+1)} N \left(\frac{N}{b_t}\right)^{\delta_\Phi}, \tag{4.2}$$

can be made, where the factor $r^{\gamma(\delta_\Phi+1)}$ is required to strip off the asymptotic radial behavior of the bulk fields, and δ_Φ is an arbitrary power. This parametrization assumes that asymptotically $b_t \sim r^0$, which is natural for the vector khronon, and that therefore $N \sim 1/r^\gamma$. Additional restrictions on δ_Φ and γ due to the conformal dimensions of the NR fields will be discussed shortly. Lastly, it can be seen that the combinations:

$$\begin{aligned}
A_t \equiv & v_t + \left(\frac{b_t}{N}\right)^{\frac{2}{\gamma}-1}\left(\frac{N^I N_I}{2N} - NA\right), \\
A_i \equiv & v_i + \left(\frac{b_t}{N}\right)^{\frac{2}{\gamma}-1}\left[\frac{N_i}{N} - \frac{b_i}{b_t}\left(\frac{N^I N_I}{2N} - NA\right)\right], \\
g_{ij} \equiv & r^2 \hat{g}_{ij} = r^2\left[G_{ij} - \frac{b_i N_j}{b_t} - \frac{b_j N_i}{b_t} + 2\frac{b_i b_j N}{b_t^2}\left(\frac{N^I N_I}{2N} - NA\right)\right],
\end{aligned} \tag{4.3}$$

asymptotically transform under f, ξ^i, and λ as the field theory gauge fields and metric if the mass is identified as

$$m \equiv r^{\gamma(\delta_\Phi+1)-2}\left(\frac{b_t}{N}\right)^{\frac{2}{\gamma}-\delta_\Phi-1}. \tag{4.4}$$

This requirement comes from examining the transformation of A_i, and equating the coefficient of $\hat{g}_{ij}\dot{\xi}^j$ with the bulk fields corresponding to me^Φ, to reproduce 2.1.

4.2 Subleading Temporal Diffeomorphisms

The role of the subleading temporal diffeomorphism α is overdue for a discussion. The field theory quantities are not affected by this transformation, as seen in 2.1. There are two different scenarios for the role of α in the bulk; both of them have an interesting holographic interpretation and lead to physically distinct pictures. One option is that the bulk action is not invariant under α transformations, it is simply not a symmetry of the gravity dual either, and should never be performed. This is a consistent truncation of the $c \to \infty$ expansion, and also allows A to be set to zero, that is, the subleading expansion of the lapse N need not be considered. The above then gives a well defined dictionary between bulk gravitational and field theory quantities, parametrized by the two constants γ and δ_Φ. The fields defined in 4.3 are then just a part of the boundary sources; there are additional gauge invariant bulk fields, such as e.g. N_r, and hence also additional field theory sources.

Alternatively, the subleading temporal diffeomorphism α can be a gauge invariance of the bulk theory. That is, it can be interpreted as a redundancy of the bulk description, and that is why it does not effect the field theory data. The fields defined in 4.3 are only invariant under α for $\gamma = 1$, or equivalently $N \sim 1/r$. Appearing mysterious in the Kaluza-Klein derivation of Section 3.2.1, this justifies the combinations of bulk fields that give the field theory ones. As that bulk theory contains the full diffeomorphism invariance of GR, the only physical boundary fields are those that are invariant under the bulk redundancy α, and therefore the ones appearing in 4.3 with $\gamma = 1$. This also elucidates the appearance of the subleading temporal diffeomorphism in the NR QFT work of [81] and the generally covariant Hořava-Lifshitz theory of [50, 25]. As they inherently consider uncharged fields they do not have the explicit gauge field v_μ. From equation 3.34, to consistently consider the transformation α, but to maintain $v_\mu = 0$, one must implicitly perform a gauge transformation λ. The α variant piece v_I of the invariant A_I, defined above, was held fixed. Thus the redundancy α was made physical by linking it to the transformation λ, which is a global symmetry of the field theory.

4.3 NR Scale and Conformal Invariance

Additionally, it is desirable to be able to describe NR QFTs that have the NR conformal symmetry of 2.14. As with traditional holography, this transformation is captured by symmetries of the bulk theory. Unlike the usual AdS/CFT correspondence, these symmetries are not strict isometries of the spacetime geometry, but instead manifest as transformations acting on the above combinations identified as field theory quantities. As in traditional holography and [80], the conformal structure of the field theory is captured by radial diffeomorphisms in the gravitational bulk. Under $\xi^r = -\omega(t, \vec{x})r$

the field theory data transform as:

$$
\begin{aligned}
\delta e^{-\Phi} \supset & r^{\gamma(\delta_\Phi+1)}\xi^r\partial_r\left(N\left(\frac{N}{b_t}\right)^{\delta_\Phi}\right) = \gamma(\delta_\Phi+1)\omega e^{-\Phi}, \\
\delta g_{ij} \supset & r^2\xi^r\partial_r\left(\hat{g}_{ij}\right) = 2\omega\hat{g}_{ij}r^2 = 2\omega g_{ij}, \\
\delta m \supset & r^{\gamma(\delta_\Phi+1)-2}\xi^r\partial_r\left(\frac{b_t}{N}\right)^{\frac{2}{\gamma}-\delta_\Phi-1} = \left(\gamma(\delta_\Phi+1)-2\right)\omega m,
\end{aligned}
\tag{4.5}
$$

which agrees with the field theory conformal transformation 2.14 for $z \equiv \gamma(\delta_\Phi + 1)$. The interesting case when the bulk is AdS (that is, $N \sim 1/r$) and the mass is invariant under scale transformations, corresponds to

$$
\gamma = 1, \quad z = 2, \quad \delta_\Phi = 1. \tag{4.6}
$$

Another interesting possibility of a case with an invariant mass (that is, $z = 2$) is the Lifshitz background with $\gamma = 2$, requiring $\delta = 0$.

For NR general covariance the bulk transformations that preserve the trivial asymptotic background $\Phi = B_I = A_t = A_I = 0$ and $G_{IJ} = \delta_{IJ}/r^2$, should agree with the field theory symmetries. The first case of a scale transformation starts with the temporal rescaling $f = \kappa t$. To maintain $\Phi = 0$, from above, the radial rescaling $\xi^r = \kappa r/z$ is required. To maintain G_{ij} the spatial rescaling $\xi^i = \kappa x^i/z$ must be performed, in agreement with a dynamical critical exponent of z. Lastly, B_I, G_{rI}, and $A_M = 0$ are automatically maintained under these scale transformations. In complete parallel to the field theory discussion in Section 2.4, m changes for $z \neq 2$, in which case the symmetry is only spurionic. Although these bulk combinations have the same isometries as the field theory quantities with which they are identified, the bulk fields themselves may not be invariant. Under the scale transformations generically

$$
\delta b_t = \kappa b_t, \quad \delta N = \kappa\left(1 - \frac{\gamma}{z}\right)N. \tag{4.7}
$$

As discussed further in Section 5.3, this non-invariance of N can be interpreted as evidence for hyperscaling violation of the theory. On the other hand, in the bulk action of probe fields, bulk fields are expected to enter only in the invariant combinations identified above. There factors of b_t/N act like the σ field of previous sections, adding it to the action can change the dimension of the probe fields.

For the special case of $z = 2$ there is an additional transformation of the bulk fields preserving the trivial background. This "special conformal" transformation involves the time reparametrization $f = Ct^2$. To preserve $\Phi = 0$, from above, the radial redefinition $\xi^r = Ctr$ must also be performed. Preservation of the trivial metric then requires $\xi^i = Ctx^i$. Lastly, maintaining the form of the bulk fields that correspond to the trivial gauge configuration requires the gauge transformation $\lambda = C(\vec{x}^2 + r^2)/2$. As with the scale transformation, not all bulk fields are invariant under this special conformal transformation. In addition to N and b_t, and the issues discussed above, the shift vector is not invariant under the time dependent ξ^I, but transforms as

$$\delta N_I = \frac{Cx^I}{r^2}. \tag{4.8}$$

These fields should correspond to gauge invariant operators in the field theory, and thus it appears that NR conformal invariance is generically untenable. It can be recovered for the special case of bulk invariance under the subleading temporal diffeomorphism α. This transformation allows the shift N_I to be held to zero, as well as the maintenance of $A = 0$ for the subleading term of the lapse. As shown above, α invariance restricts $N \sim 1/r$, that is, the bulk background is that of AdS. The NR scale and conformal isometries of Section 2.4 are therefore realized by this bulk theory.

4.4 Bulk Action

Consider, initially, bulk theories without the α transformation. This also allows the consistent setting of $A = 0$; the subleading piece of the lapse N does not need to be considered in the c expansion. As previously discussed, by working in a global time $b_I = P_I = 0$ can be maintained. This gives the following consistent field content: the metric is decomposed in the ADM variables N, N_I, and G_{IJ} adapted to the preferred foliation; the gauge vector behaves as the non-relativistic decomposition v_t and v_I with respect to the global time. The background "large" gauge field b_t determines the foliation by a global time, and should be considered a parameter that must be tuned to yield a NR holographic duality, much like the cosmological constant in traditional holography. The gauge transformations are spatial diffeomorphisms ξ^I, temporal reparametrization $f(t)$, and the $U(1)$ transformation λ. This is exactly the field content and symmetries of Hořava gravity coupled to non-relativistic electromagnetic fields: our proposal for a holographic dual to a generic NR QFT obeying the symmetries 2.1 is this non-relativistic gravity theory, on a background spacetime with a non-zero b_t. The bulk action will therefore be determined by the couplings $(\tilde{\lambda}, \tilde{\beta}, \tilde{\alpha})$ of Hořava gravity, as well as those introduced with non-relativistic electromagnetic fields. To go further and consider bulk theories with the α transformation, it is noted that the covariant Hořava-Lifshitz theory of [50, 25], coupled to electromagnetic fields, is a bulk theory with α invariance and the same fields and symmetries as above. It therefore is capable of holographically describing Schrödinger invariant NR CFTs.

4.5 Horizons and Thermodynamics

As illustrated in the introduction, holography and AdS/CFT grew out of understanding black hole thermodynamics in GR, and from combining principles of quantum mechanics and black hole entropy. If these same arguments can be applied to black holes of Hořava gravity, the notion of holography naturally extends to a much larger class of theories, most notably those of an intrinsic non-relativistic nature [49, 14, 15, 56, 57, 39]. Beyond this, such solutions are extremely important in holography: black holes geometries at a Hawking temperature T_H are dual to thermal field theory states at the same temperature, as was know very early on in AdS/CFT [40, 69, 88]. Black holes can be understood geometrically as spacetimes that have regions that are causally disconnected from asymptotic observers at temporal infinity.

The lack of Lorentz invariance makes causality a subtler notion in Hořava gravity than it is in GR. In Einstein gravity the causal structure of a solution is most apparent when brought into Penrose form. Light cones form forty-five degree diagonal lines and define the invariant notion of whether one event is space-like, null, or time-like separated from another. This nature of separation between two events and the domain of influence for a given region are easily deduced from the Penrose diagram. From this construction event horizons are identified as the null boundary that separates the domain of dependence of future infinity from the rest of the manifold.

In Hořava gravity a light cone is not a limiting object, and the notion of causality is maintained by the existence of a preferred global time instead. Indeed, as seen in the previous sections, fields in Hořava gravity can have arbitrarily fast propagation speed, there is no limiting role of the speed of light that is fundamental to GR. Despite this, a mode traveling with any speed in the preferred frame can only propagate forward in global time. The leaves of the foliation labeled by the preferred time define the invariant notion of whether one event is before, simultaneous with, or after another.

Likewise, the foliation by a preferred time can define the notion of causal boundaries and horizons in Hořava gravity. Simplistically, the leaf of the foliation labeled by $t = \infty$ forms a causal boundary of the manifold. Only events on leaves labeled by earlier times can possibly influence this boundary. Naively, this leaf may be expected to simply form the boundary of the manifold defined as future infinity, much as $t = \infty$ is future null infinity in Minkowski space. More interestingly, this leaf can "bend down" as it foliates the manifold and may never penetrate some region. This is analogous to the asymptotically flat Schwarzschild black hole in Schwarzschild time: the line $t = \infty$ is partially asymptotic null infinity, but partially the event horizon at $r = r_h$, see Figure 4.1.

It is important to note that in GR the fact that leaves of Schwarzschild time converge at the event horizon is not an invariant statement about the geometry of the spacetime: different coordinates can have a foliation that does nothing special at

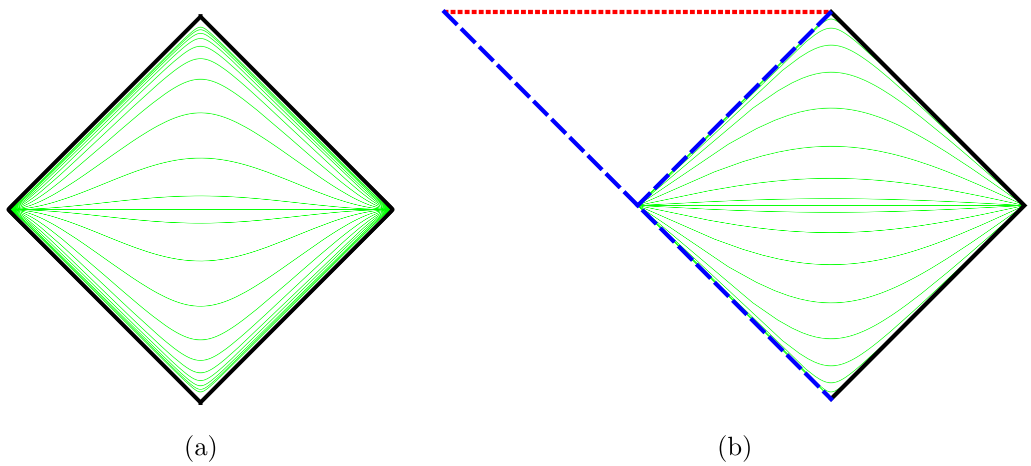

Figure 4.1: a) The Penrose diagram for Minkowski space. The thin green lines are slices of constant time, while $t = \pm\infty$ is null infinity in solid black. b) The Penrose diagram for a Schwarzschild black hole. The thin green lines are of constant Schwarzschild time, $t = \pm\infty$ is partially null infinity in solid black, and partially the event horizon in dashed blue. The singularity at $r = 0$ is in dotted red.

r_h. To identify the leaf $t = \infty$ as a causal boundary of the black hole spacetime in GR one also need to use the fact that it is a null surface. Only then is it concluded that nothing inside the radius r_h can influence events outside: that is, the leaf $t = \infty$ does form a causal boundary of the foliation.

The case in Hořava gravity is somewhat simpler in this respect. Causality is encoded in the foliation itself, not any light cone structure. The convergence of its leaves is an invariant statement, and such a region is a causal boundary. Such a slice (that isn't asymptotic infinity) is called a "universal horizon" [29, 11, 19]: an event behind it cannot influence events outside as they are all at "earlier" times as measured by the preferred foliation. Solutions to Hořava gravity with universal horizons will be called black holes, and explored in the next chapter.

The existence of causal horizons in Hořava gravity begs the question of whether they obey a thermodynamic description as do their counterparts in GR [19, 14, 15]. Motivation for such a description follows from arguments analogous to Bekenstein's original proposals [12]: a causal horizon must have intrinsic entropy if the second law is not to be violated when exterior entropy falls in. Consistency with a first law then implies a temperature of the horizon. For Hořava gravity, additional motivation comes from its claim as a UV complete quantum theory of gravity [19]. From a microscopic description, the entropy of a macroscopic system is a measure of the number of fundamental degrees of freedom contained. The macroscopic second law is then a reflection of the unitarity of the microscopic theory. If Hořava gravity is truly a sensible quantum theory of gravity macroscopic systems such as black holes defined by their universal horizons must obey a second law of thermodynamics.

Chapter 5

EVIDENCE AND SOLUTIONS

Much of this chapter is taken from [55], as well as [56, 57] which were coauthored with Andreas Karch. The work of Section 5.4 is from [58], to which the author contributed.

5.1 String Theory Embeddings

The previous chapter has argued that, based on its symmetry structure, Hořava gravity is the natural holographic dual of a generic NR QFT. To check that this is correct, it would be nice to confirm that the construction can be consistently embedded into string theory. This embedding is facilitated by the observation that NR systems can be derived as a $c \to \infty$ limit of a relativistic theory by setting the chemical potential equal to the rest energy of the lightest charged particle. In order to give string theory embeddings of this scenario, examples of relativistic holographically dual pairs where the field theory side has a global $U(1)$ symmetry with massive charged particles need to be found.

One such example was in fact already presented in Section 3.2.1. Start with the known duality between AdS$_5$ in type IIB string theory and $\mathcal{N} = 4$ SYM gauge theory and compactify the latter on a space-like circle with periodic boundary conditions. In this case the resulting $2+1$ dimensional relativistic field theory has a new global $U(1)$ symmetry associated with shifts around the compact circle. The charged particles are the momentum modes in the compact direction and they naturally have a mass equal to the inverse circle radius. The non-relativistic limit in this theory introduces a chemical potential for this $U(1)$ particle number equal to the rest mass of the Kaluza-

Klein particles and then takes the $c \to \infty$ limit. This is exactly what was done in Section 3.2.1 so that in this limit the circle becomes light-like. In this Kaluza-Klein example, a massless scalar in the relativistic geometry (for example the IIB dilaton) is described exactly in terms of the non-relativistic action 2.2 translated into Hořava fields via the map 4.3. The at first sight unnatural N/b_t prefactor is introduced in the action from the higher dimensional origin; it is exactly the $\sqrt{G_{\zeta\zeta}}$ prefactor in the bulk action due to the line element in the compact direction. This can be interpreted as needed order to avoid hyperscaling violation, as alluded to at the end of Section 3.2.1.

While many examples of holographic dualities in the presence of finite chemical potential are understood by now, the task of finding additional examples where the charge carriers are massive so that the NR limit can be implemented is somewhat more non-trivial. One example is ABJM theory which allows a supersymmetric preserving mass term and a non-relativistic limit [71, 66]. Gravitational solutions of M-theory matching the global symmetries of this NR CFT were studied in detail in [59]. There it was found that the prospective gravity dual did not have the same amount of supersymmetry as the NR ABJM field theory. This leads one to question the role of supersymmetry in non-relativistic holography. Although it is crucial in traditional AdS/CFT, often providing stability to the best known examples, it may not be as important for NR physics.

Another large class of examples of holographically dual pairs with a finite density of massive charge carriers is based on probe branes [61] which were first studied at finite chemical potential in [64]. In this situation the thermodynamics and the spectrum of hydrodynamic modes was recently analyzed in the NR limit advertised here [62, 4]. While in those papers the results were not phrased in the language of Hořava gravity, the findings, especially of the latter, are completely consistent with the picture developed here. For two physically distinct probe systems (with $d = 3$ and $d = 2$ spatial dimensions respectively), in the scaling limit the probe brane system is found to be governed by a NR CFT with dynamical critical exponent $z = 2$ and hyperscaling violating exponent $\Theta = 1$.

All of these examples provide realization of the non-relativistic holographic duality proposed here, by embedding it into a known relativistic dual. Although this is expected to not always be possible, it does provide evidence that Hořava gravity has a holographic interpretation.

5.2 *Asymptotically Hyperbolic Solutions*

Understanding the asymptotic structure of gravitational theories has proven to be a fruitful endeavor. Various "No Hair" theorems have used these techniques to determine that black holes in GR are labeled by a small number of parameters. In the context of Einstein-Aether theory and Hořava gravity this has been done for asymptotically flat solutions [29, 11, 14]. Generically, static spherically-symmetric solutions have three defining parameters. The requirement of asymptotic flatness reduces this number to two. The solutions are reduced to a one parameter family, the mass, upon requiring regularity at the spin-zero sound horizon, that is, the trapped surface for waves of the speed s_0 given in 3.12 [29, 11].

The focus of this section is on understanding the space of solutions to Hořava gravity that have an asymptotically Lifshitz metric:

$$\lim_{r\to 0} ds^2 \approx -\left(\frac{L}{r}\right)^{2z} dt^2 + \left(\frac{L}{r}\right)^2 dr^2 + \left(\frac{L}{r}\right)^2 d\vec{x}^2, \tag{5.1}$$

where z is the dynamical critical exponent that controls the anisotropy of time versus space, and L is a length giving the scale of the curvature. As the spatial metric given by slices of constant t asymptotically has uniform negative curvature, spacetimes 5.1 will be referred to as hyperbolic. Metrics of this form are important in the arena of holography. Allowing $z = 1$, one obtains the metric of Anti-de Sitter space in Poincaré coordinates. Such boundary conditions are crucial from the viewpoint of standard AdS/CFT [69, 88, 41]. For $z \neq 1$ the metric 5.1 exhibits an anisotropic scaling symmetry of time relative to space. General relativity on such backgrounds has been argued to be dual to non-relativistic field theories [60, 9]. More pertinently, although GR requires additional matter to support this geometry, such a metric is a vacuum solution to Hořava gravity [39, 56]. This leads to speculation that the Lifshitz metric may play the fundamental role in a holographic duality involving Hořava gravity that AdS space plays in traditional holography.

The Poincaré-like coordinates of 5.1 are a natural choice to work in as they most simply lead to the definition of the boundary at $r = 0$ via an anisotropic conformal transformation [51]. However, in order to fully explore the interior geometry of an asymptotically hyperbolic spacetime it is important to use coordinates that are nowhere singular. From experience with black holes in Minkowski and AdS spaces, the partial-null Eddington-Finkelstein-like (EF) coordinates are a better choice: they are not singular at the metric horizon, unlike Schwarzschild and Poincaré coordinates. To obtain the Lifshitz metric 5.1 in EF coordinates one first defines the radial tortoise coordinate $r_* \equiv 1/z(r/L)^z$. From this the EF time is defined as the null coordinate $v \equiv t - r_*$, and the metric becomes

$$ds^2 = -\frac{dv^2}{r^{2z}} - 2\frac{dvdr}{r^{z+1}} + \frac{d\vec{x}^2}{r^2}, \tag{5.2}$$

where units have been chosen such that $L = 1$.

In Hořava gravity, in addition to the asymptotic behavior of the metric, the boundary conditions of the foliation must be specified. For holography it is natural to use a time coordinate that is asymptotically the Poincaré time of 5.1: this time appears as the conformal boundary time and would correspond to a global time coordinate of the dual field theory on a flat background. In the khronon language of Section 3.1.2 this requires $\phi|_{r\to 0} = t = v + r_*$, or in terms of the foliation normal vector in EF coordinates,

$$u_M = \left(-\frac{1}{r^z}, -\frac{1}{r}, \vec{0}\right). \tag{5.3}$$

Respecting the stationarity and planar symmetry properties of the Lifshitz metric

5.1, the general ansatz of the full metric in EF coordinates is

$$ds^2 = -e(r)dv^2 - 2f(r)dvdr + \frac{d\vec{x}^2}{r^2}. \tag{5.4}$$

The general foliation normal vector respecting these symmetries is

$$u_M = \left(-\frac{a(r)^2 e(r) + f(r)^2}{2a(r)f(r)}, -a(r), \vec{0}\right), \tag{5.5}$$

where the unit norm constraint has been imposed. The required asymptotic behavior to reproduce the hyperbolic spacetime of 5.2 and 5.3 gives the leading behavior as $r \to 0$ of the free functions:

$$e(r) \sim \frac{1}{r^{2z}}, \quad f(r) \sim \frac{1}{r^{z+1}}, \quad a(r) \sim \frac{1}{r}. \tag{5.6}$$

The remainder of this section will be restricted to $D = 3$, that is, a four dimensional spacetime. This allows explicit expressions to be written down simply, although generalization is straightforward. A useful trick to simplify the khronon action 3.9 is to note that a hypersurface orthogonal vector such as u_M has vanishing curl

$$u_{[N}\nabla_P u_{Q]} = 0. \tag{5.7}$$

Therefore, in four dimensions, for $\omega_M \equiv \epsilon_{MNPQ} u^N \nabla^P u^Q = 0$, one can write

$$(u^M \nabla_M u_N)^2 = (u^M \nabla_M u_N)^2 - \omega_M \omega^M = -\frac{1}{2} F_{MN} F^{MN}, \tag{5.8}$$

for the "field strength" $F_{MN} \equiv \partial_M u_N - \partial_N u_M$. This simplifies the c_4 term in the khronon action 3.9, as it removes any factors of the connection in the derivative. The task is to now examine the equations of motion coming from the action 3.9 with the ansatzes 5.4 for the metric and 5.5 for the foliation normal vector. Plugging in a series solution for the functions (e, f, a), with leading behaviors given by 5.6, the equations of motion can be solved order by order as $r \to 0$. This gives evidence to the number of free parameters that classify a general solution by determining the number of free coefficients in the series expansions of (e, f, a). The full equations of motion are cumbersome and not very enlightening, and will not be shown here. To zeroth order the solution is simply the Lifshitz background 5.2, requiring [39]

$$c_4 = \frac{z-1}{z}, \quad \Lambda = -\frac{(D-2+z)(D-1+z)}{2}, \tag{5.9}$$

that is, the dynamical critical exponent z is determined by the coupling c_4, and the cosmological constant is in turn fixed. To higher order there is a drastic difference whether $z = 1$ or not.

5.2.1 $z = 1$

For $z = 1$ ($c_4 = \tilde{\alpha} = 0$) the metric is asymptotically Anti-de Sitter space. The general asymptotic series expansion has coefficients proportional to $(z-1)$, therefore this case must be treated separately. Examining the order by order expansion of the equations of motion as $r \to 0$ it appears the series for e and f truncates. To order r^{30} the functions are determined to be:

$$a(r) = \frac{1}{r} + C_a r^2 + \frac{1}{2}\left((c_3+1)C_a^2 + (c_3-1)C_a C_e + \frac{c_3}{4}C_e^2\right) r^5 + \cdots, \tag{5.10}$$

$$e(r) = \frac{1}{r^2} + C_e r - \frac{1}{4} c_3 (C_e + 2C_a)^2 r^4, \tag{5.11}$$

$$f(r) = \frac{1}{r^2}, \tag{5.12}$$

where C_e and C_a are constants, and only the first three terms of $a(r)$ are shown, although importantly no more free parameters appear. The truncation of an asymptotic series solution is often the signature of an analytic solution, which will indeed be constructed in Section 5.5.

5.2.2 $z \neq 1$

For $z \neq 1$ the asymptotic series analysis is more subtle. Stripping off the leading boundary behavior given by 5.6, and expanding the free functions in a series solution, gives:

$$
\begin{aligned}
a(r) =& \frac{1}{r}\left(1 - \frac{1}{2}C_e r^{z+2} + \frac{(2z^2+2z-1)C_e^2}{4z(z+1)} r^{2(z+2)} - \frac{(9z^2+9z-8)C_e^3}{16z(z+1)} r^{3(z+2)} + \cdots\right) \\
e(r) =& \frac{1}{r^{2z}}\left(1 + C_e r^{z+2} + \frac{(z-1)(z+2)C_e^3}{24z(z+1)} r^{3(z+2)} - \frac{(z-1)(z+2)C_e^4}{24z(z+1)} r^{4(z+2)} + \cdots\right) \\
f(r) =& \frac{1}{r^{z+1}}\left(1 + \frac{(z-1)(z+2)C_e^2}{8z(z+1)} r^{2(z+2)} - \frac{(z-1)(z+2)C_e^3}{6z(z+1)} r^{3(2+z)} + \cdots\right). \qquad (5.13)
\end{aligned}
$$

Importantly, up to order $r^{5(z+2)}$, the expansion is seen to contain only one free coefficient, C_a, contrary to general expectations [29]. Also evident is the fact that, apart from the leading singular factor, the functions (a, e, f) are only functions of r^{z+2}.

To find the missing constant it helps to recognize when the above procedure fails. It has been assumed that the free functions can be expanded as their leading singular behavior times an analytic function. Whether this analyticity is justified depends on what the subleading characteristic exponents are. These can be determined by making the ansatz

$$
e(r) = \frac{1}{r^{2z}}, \quad f(r) = \frac{1}{r^{z+1}}, \quad a(r) = \frac{1}{r}\left(1 + a_\Delta r^\Delta\right), \qquad (5.14)
$$

and using the equations of motion for $r \to 0$ to determine the allowed powers Δ. This procedure yields

$$
\Delta_\pm = \frac{1}{2}\left(z + 2 \pm \sqrt{(z+2)^2 - \frac{8(1-c_3)(z-1)}{c_2+c_3}}\right). \qquad (5.15)
$$

The only requirement on Δ is that it is non-negative in order to maintain the desired boundary behavior of $a(r)$. Generically, $\Delta_\pm$ is non-integer and therefore the ansatz of analyticity is not justified. Indeed, $z = 1$ is a special case for which $\Delta_\pm = (0, 3)$.

In general for Δ to be an integer requires a specific choice of couplings. For concreteness take the case of $z = 2$. Then for the choice $c_2 = (1 - 3c_3)/2$ it is seen that $\Delta_\pm = 2$. In this case the asymptotic series expansion of the functions is:

$$\begin{aligned}
a(r) =& \frac{1}{r}\left(1 + C_a r^2 + \frac{1}{2}(C_a^2 - C_e)r^4 + (C_a(C_a^2 - C_e) - C_a^3)r^6 + \cdots\right)\\
e(r) =& \frac{1}{r^4}\left(1 + C_e r^4 - \frac{1}{72}\left(C_e + C_a^2\right)^2\left(2(C_a^2 - C_e) + C_a^2(3c_3 - 4)\right)r^{12} + \cdots\right)\\
f(r) =& \frac{1}{r^3}\left(1 + \frac{1}{12}\left(C_e + C_a^2\right)^2 r^8 + \frac{1}{32}\left(C_e + C_a^2\right)^2\left(C_a^2(c_3 - 5) - 4C_e\right)r^{12} + \cdots\right).
\end{aligned} \tag{5.16}$$

Up to $\mathcal{O}(r^{12})$ the expansion is seen to have two free parameters, C_a and C_e, and reduces to 5.13 for $C_a = 0$. Importantly, as evident in the general case from equation 5.14, only the subleading behavior of the function $a(r)$ is modified; generically the subleading piece of $e(r)$ goes as r^{2-z}, while that of $f(r)$ goes as r^{z+3}.

Thus for any z it is found that planar symmetric stationary solutions of Hořava gravity are determined by two constants: C_e, the coefficient of r^{2-z} in the expansion of $e(r)$, and C_a, the coefficient of $r^{\Delta-1}$ in the expansion of $a(r)$. The demand for an asymptotically hyperbolic spacetime has reduced the three dimensional parameter space of solutions to these two. As in [29, 11], this parameter space can be argued to reduce to one constant for physically acceptable solutions. The key point is that although desired boundary conditions have been imposed at asymptotic infinity, there is no guarantee that the solutions are non-singular in the interior (disregarding singularities hidden by horizons). An important place to demand regularity of the solution is the spin-zero sound horizon, that is, the trapped surface for waves of the speed s_0 in 3.12. This is physically reasonable as it is expected to be true for solutions that describe the late stages of gravitational collapse, as is argued in general relativity for the regularity of the metric horizon. This requirement of regularity reduces the two parameters describing an asymptotically hyperbolic Hořava solution to one, the mass.

5.2.3 Spacetime Mass

An important use of the near boundary asymptotic expansion of solutions is in the determination of the mass of a spacetime. Hořava gravity has a preferred notion of time due to its foliation Σ_t. This leads to the definition of the mass of the spacetime as the on-shell Hamiltonian with respect to the preferred slicing, following [44]. The first step in this process is to include the Gibbons-Hawking term in the action in order to make the variational problem well-posed. This is accomplished by defining the total action

$$S_T \equiv \frac{1}{16\pi G_K} \int_{\mathcal{M}} \sqrt{-\tilde{G}} \mathcal{L}_K + \frac{1}{8\pi G_K} \int_{\partial\mathcal{M}} \sqrt{|h|} \mathcal{K}, \tag{5.17}$$

where: $\mathcal{L}_K$ is determined from the khronon action 3.9; the first integral is over the manifold $\mathcal{M}$, while the second is the Gibbons-Hawking term over its boundary $\partial\mathcal{M}$; and $\mathcal{K}$ is the trace of the extrinsic curvature of the boundary, while h is the determinant of its induced metric.

When the Ricci scalar in $\mathcal{L}_K$ is decomposed with respect to the ADM variables of the foliation there arise two total derivatives:

$$-\nabla_N \left(u^N \nabla_M u^M\right) + \nabla_N \left(u^M \nabla_M u^N\right). \tag{5.18}$$

These lead to additional boundary terms, and the structure of $\partial\mathcal{M}$ deserves elucidation. For the asymptotically hyperbolic spacetimes of concern there is a time-like boundary at $r = 0$. This has the outward space-like normal s^N, and it will be assumed that $s^N u_N = 0$ at $r = 0$. Therefore the first boundary term arising from equation 5.18 does not contribute here. The second term combines with the Gibbons-Hawking term to give the extrinsic curvature of the two-surface of constant t at $r = 0$ as embedded in the preferred foliation and denoted ${}^2\mathcal{K}$.

The next components of $\partial\mathcal{M}$ are the past and future space-like boundaries of the foliation, given schematically by $t = \pm\infty$. These surfaces have normal u^N, and therefore the boundary contribution from the second term of 5.18 vanishes due to the unit norm of u^N. The first term can then be seen to completely cancel the Gibbons-Hawking term, leading to no net boundary contributions from past and future infinity.

The final boundary possible is that of the universal horizon for black hole spacetimes. As discussed in Section 4.5, and explored further later, this is a surface that the asymptotic foliation Σ_t does not penetrate, and is the causal boundary of the spacetime. To understand its boundary contributions to the Hamiltonian it pays to be more precise. Consider a spacetime with a Killing vector χ^N that is asymptotically time-like. As one travels inwards along the leaves of Σ_t the product $\chi^N u_N$, which is initially negative, can approach 0 at some value of $r = r_h$ [14]. The Killing vector is therefore tangent to this surface. Causal evolution in the direction of increasing global time t is therefore necessarily toward the center of the spacetime, and can never reach the asymptotic boundary at $r = 0$. Thus this surface at a constant radius r_h is a universal horizon, and a boundary of the leaves of the foliation. Unlike GR, the surface $r = r_h$ is space-like, with time-like normal u^N. Indeed, it is none other than the surface at $t = \pm\infty$, as schematically argued in the introduction. Therefore, the universal horizon contributes no additional boundary terms from 5.18.

The next step in transforming to a Hamiltonian is to write the Lagrangian in terms of P_{MN}, the momentum conjugate to the spatial metric of the ADM decomposition, G_{MN}[1]. Recalling the definition of the extrinsic curvature in terms of the ADM fields, it is seen that only spatial derivatives of the shift N_I appear in the action, highlighting its nature as a constraint. To transform to a Hamiltonian these spatial derivatives are integrated by parts to give another boundary term that contributes on the boundaries of the leaves of the foliation, $\partial\Sigma_t$:

$$-\frac{1}{8\pi G_K}\int_{\partial\Sigma_t}\sqrt{H}d\Omega\frac{N^M P_{MN} n^N}{\sqrt{G}}, \tag{5.19}$$

where: H is the determinant of the induced metric on surfaces of constant t and r;

[1]The purely spatial indices $I, J\ldots$ can be extended to spacetime indices $M, N\ldots$ through the definition of the spatial metric as a projector: $G_{MN} \equiv \tilde{G}_{MN} + u_M u_N$.

Ω are the coordinates along these surfaces; the momentum is $P_{MN} \equiv \sqrt{G}(K_{MN} - \frac{1+c_2}{1-c_3} K G_{MN})$; and n^N is the outward normal to the boundaries of the leaves of the foliation.

The first contribution comes from the component of $\partial\Sigma_t$ at the asymptotic boundary $r = 0$. Here the outward normal n^N is the space-like vector s^N, and this term generically contributes. The other possible contribution comes from the internal boundary of the universal horizon at $r = r_h$. Here the normal is u^N, the time-like vector that is orthogonal to the leaves of Σ_t, and therefore $P_{MN} u^N = 0$ and there is no contribution.

Putting this all together gives the value of the on-shell Hamiltonian for a solution to Hořava gravity:

$$\mathcal{H} \equiv -\frac{1}{8\pi G_K} \int_{S_t^0} \sqrt{H} d\Omega \left({}^2\mathcal{K} N - \frac{N^I P_{IJ} s^J}{\sqrt{G}} \right), \tag{5.20}$$

where S_t^0 is the surface at the boundary $r = 0$ and constant t. For the asymptotically hyperbolic solutions at hand this quantity generically diverges. The physical mass of a spacetime will therefore be defined to be the difference between its on-shell Hamiltonian and that of a reference background. Importantly, as the Hamiltonians are regulated by a cut-off near the $r = 0$ boundary, these surfaces need to be chosen for each background such that the value of the fields agrees upon them. Therefore the lapse on the cut off is equal for each spacetime: $N(\epsilon) = N_0(\epsilon_0)$, where N is the lapse for the solution under examination, evaluated on the surface $r = \epsilon$, while N_0 is the lapse of the reference background, evaluated on the surface $r = \epsilon_0$.

For $z = 1$ the solution asymptotically approaches Anti-de Sitter space, which will be used as the reference background. Converting to the ADM coordinates of Σ_t, the expansion of the functions 5.10 can be used to determine the behavior of the integrand of the Hamiltonian 5.20 near $r = \epsilon$. For these solutions the first term concerning ${}^2\mathcal{K}$ contributes

$$-\frac{1}{8\pi G_K} \int_{S_t^\epsilon} d^2x \left(\frac{2}{\epsilon^3} + 2C_e \right). \tag{5.21}$$

The second term in the integrand involving the momentum P_{IJ} contributes a term of order ϵ^3, and therefore vanishes as the cut off is removed. Requiring that the lapse on the cut off surfaces of the solution and AdS space agree determines the relation

$$\frac{1}{\epsilon} + \frac{C_e \epsilon^2}{2} = \frac{1}{\epsilon_0}. \tag{5.22}$$

Using equation 5.20 this determines the mass of the solution to be

$$M_{z=1} \equiv \mathcal{H}_{z=1} - \mathcal{H}_{AdS} = \frac{-C_e A}{8\pi G_K}, \tag{5.23}$$

where $A \equiv \int d^2x$ is the volume of the transverse space.

For $z \neq 1$ one again finds that the on-shell Hamiltonian is divergent. In this case it can be regulated by performing a background subtraction by Lifshitz space, with appropriate z. Repeating the above arguments using the naive expansion 5.13 leads to the identical value of 5.23 for the mass of the solution. On the other hand, the correct expansion for $z \neq 1$ has the subleading power Δ given in 5.15, which generically contributes to the mass. For the example given by the series 5.16 the above procedure yields the mass

$$M_{z=2,\Delta=2} = \frac{(2C_a^2 - C_e)A}{8\pi G_K}. \tag{5.24}$$

The generic behavior of the mass of a $z \neq 1$ spacetime remains to be understood. An analytic solution going beyond an asymptotic series expansion would shed light on this point. Regardless, the definition of mass as the on-shell Hamiltonian 5.20 for a solution remains well defined, up to regularization as discussed.

A "first law" could now be derived relating the variation of mass of two solutions that have a small variation of the dimensionful constants C_e or C_a. As it stands this is not a very useful statement: from the asymptotic expansion alone there is no explicit relation between the parameters C_e and C_a and the radius of the universal horizon, r_h. A more rigorous first law is generally derived by making use of the identity $\nabla^M(\nabla_N \chi_M) = R_{NM}\chi^M$, the equations of motion, and Gauss's law. Such a derivation can indeed be done in the case of Hořava gravity and has been derived for Einstein-aether theory in [17]. See also [70] for a derivation following Wald's Noether charge method applicable to asymptotically flat solutions. An explicit example of the first law, following from an analytic solution, will be given in Section 5.5.

5.3 Lifshitz Backgrounds and Galilean Correlation Functions

An important use of a holographic duality is in the calculation of correlation functions of field theory operators. As discussed in the Introduction concerning the holographic dictionary for AdS/CFT, the duality relates the gravitational partition function, as a functional of the boundary value of fields, to the QFT partition function, as a functional of the sources of operators. Therefore, by studying the behavior of a bulk field as a function of its boundary value, the correlation function of the dual field theory operator can be calculated.

As discussed in Section 4.1 the asymptotic behavior of the lapse function N is captured by the exponent γ and sets the form of the holographic map. Interestingly, the radial behavior of the lapse N for metric 5.1 with $\tilde{\alpha} < 1$ is capable of reproducing an arbitrary γ. The Lifshitz spacetime undoubtedly deserves further study in a holographic context.

To calculate correlation functions one needs to examine the on-shell action of the dual bulk fields. For a bulk probe scalar with $z = 2$ the non-relativistic action 2.2 will be used, written in terms of the Hořava fields. This gives the following bulk action for a charged probe scalar:

$$\begin{aligned} S_\Psi = \int dt dr d^d x \sqrt{G} \frac{N^2}{b_t} \Bigg[\left(\frac{\imath b_t}{2N^2} \Psi^\dagger \left(\mathcal{D}_t - N^J \mathcal{D}_J \right) \Psi + h.c. \right) \\ - \frac{G^{IJ}}{2m} \mathcal{D}_I \Psi^\dagger \mathcal{D}_J \Psi - \frac{M^2}{2m} \Psi^\dagger \Psi \Bigg], \end{aligned} \tag{5.25}$$

where the metric and gauge covariant derivatives are given by $\mathcal{D}_t = \partial_t - \imath v_0$, $\mathcal{D}_I = \nabla_I - \imath v_I$, and M is the non-relativistic bulk mass. The combination of temporal and spatial derivatives in the kinetic term is expected for invariance under foliation preserving diffeomorphisms. For $\tilde{\alpha} = 0$, by the metric 5.1 above AdS is a solution, on this background the probe scalar action becomes:

$$S_\Psi = \int dt dr d^d x \frac{1}{b_t r^{d+3}} \left[\imath b_t r^2 \Psi^\dagger \partial_t \Psi - \frac{r^2}{2m} \partial_I \Psi^\dagger \partial_I \Psi - \frac{M^2}{2m} \Psi^\dagger \Psi \right], \tag{5.26}$$

where it is assumed that m and b_t are constant. This agrees with the action of [80] up to overall normalization. Therefore, their calculation of the correlation function of the field theory operator dual to this scalar follows through, and in momentum space gives[2]

$$\langle \mathcal{O}\mathcal{O} \rangle \sim \left(\vec{k}^2 - 2 m b_t \omega \right)^\nu, \tag{5.27}$$

where

$$\nu = \sqrt{\frac{(d+2)^2}{4} + M^2}.$$

Upon Fourier transforming to real space this gives the restrictive form dictated by Galilean and scale symmetry [46, 45, 72], providing a quantitative check of the duality. Comparison to [80, 8] shows that the constant $m b_t$ plays the role of the charge or particle number of the operator $\mathcal{O}$.

This form of the probe scalar action can also be motivated as a derivative expansion. At zero derivatives is simply the mass term. At one derivative, using the NR fields, the following terms can be constructed:

$$\Psi^\dagger b^t \partial_t \Psi, \quad \Psi^\dagger N^I \partial_I \Psi, \quad \Psi^\dagger b^I \partial_I \Psi, \quad \Psi^\dagger P^I \partial_I \Psi. \tag{5.28}$$

For Hořava gravity the last two terms are absent, while the first two are taken in the combination that is invariant under foliation preserving diffeomorphisms. At two derivatives the leading term is simply the canonical spatial gradient squared term. Other bulk probe actions are possible, given only the symmetry restrictions of Hořava gravity. In particular, the Lagrangian can be multiplied by the overall factor $(b_t/N)^\Theta$.

[2] [80] contains a typo involving the incorrect exponent in this expression, compare to [8].

The effect of this factor is to shift the dimension of the operator coupled to the bulk field, mimicking the σ field of Section 2.5. As discussed in Section 5.1 it can also be understood to represent hyperscaling violation, as the dimension of the operator is changed by replacing $d \to d-\Theta$, modifying the effective number of spatial dimensions of the theory. The scalar action with $\Theta = 1$ in many ways appears to be the most natural. In that case no inverse powers of b_t appear in the action and the potential simply has an overall prefactor of N as part of the usual measure. This is exactly the scalar action one would have written down in Hořava gravity without the extra Maxwell field.

5.4 AdS_2 Background Correlation Functions

An interesting class of NR QFTs are those that have background $U(1)$ gauge fields. This motivates examining Hořava gravity coupled to an electromagnetic potential V_M, to capture this symmetry. In the khronon formalism of Section 3.1.2, this is accomplished by adding to the action the Lagrange density

$$\mathcal{L}_{EM} = \frac{1}{16\pi G_K}\sqrt{-\tilde{G}}\left[-\mu\mathcal{F}_{MN}\mathcal{F}^{MN} + \frac{\kappa}{4}u^M\mathcal{F}_{MN}u^P\mathcal{F}^N_P\right], \tag{5.29}$$

where $\mathcal{F}_{MN} \equiv \partial_M V_N - \partial_N V_M$ is the electromagnetic field strength, and the κ term is a novel additional Lorentz scalar allowed by the existence of the aether vector u_M.

There is undoubtedly a rich structure of solutions to this theory, but a relatively simple one is readily available [58]. Consider a metric ansatz, in EF coordinates, for a spacetime with an asymptotically flat region at $r \to 0$:

$$ds^2 = -e(r)dv^2 - 2f(r)dvdr + \frac{1}{r^2}\left(d\theta^2 + \sin(\theta)d\phi^2\right), \tag{5.30}$$

where θ and ϕ are the usual angular coordinates on the two-sphere $\mathbb{S}^2$. The normal vector u_M has the same form as the ansatz 5.5, while the gauge potential can be taken to have only a time-like component V_t. To be asymptotically flat requires $\Lambda = 0$ and that the free functions of this ansatz behave like

$$e(r) \sim 1, \quad f(r) \sim \frac{1}{r^2}, \quad a(r) \sim \frac{1}{r^2}, \quad V_t \sim Qr, \tag{5.31}$$

as $r \to 0$. Inserting this ansatz into the equations of motion following from the khronon action 3.9 with the electromagnetic term 5.29 included, one sees that the near boundary expansion leads to an obvious truncation of all of functions except $a(r)$:

$$\begin{aligned} e(r) &= (1 - C_a r)^2 \,, \\ f(r) &= \frac{1}{r^2}, \\ a(r) &= \frac{1}{r^2} - \frac{C_a}{r} + C_a^2 - C_a^3 r + \cdots \,, \\ V_t(r) &= \sqrt{\frac{4(2-c_4)C_a^2}{\kappa + 8\mu}} r. \end{aligned} \tag{5.32}$$

Closer inspection shows that $a(r)$ is in fact a geometric series and can be resummed to give

$$a(r) = \frac{1}{r^2(1 - C_a r)} = \frac{1}{r^2\sqrt{e(r)}}. \tag{5.33}$$

Transforming to a preferred time coordinate of Hořava gravity shows that this solution is in fact the extremal Reissner-Nordstrom black hole, with the global time given by the canonical time.

Although it is interesting to see that the flat extremal Reissner-Nordstrom black hole is a solution of Hořava gravity coupled to electromagnetism, it is not directly an easily interpretable holographic spacetime. On the other hand, it is well known that the near horizon geometry of this solution is $\mathrm{AdS}_2 \times \mathbb{S}^2$, and does have a clear holographic interpretation. This motivates looking for solutions of the form $\mathrm{AdS}_2 \times \mathcal{M}^2$, where $\mathcal{M}^2$ is a maximally symmetric two-manifold. Plugging this metric ansatz into 3.9 and 5.29 it is indeed found that $\mathrm{AdS}_2 \times \mathcal{M}^2$ is a solution for $V_t = QL/r$ and $\Lambda = (-2 + c_4 + (8\mu + \kappa)Q^2)/2L^2$, where L is the curvature radius of the AdS_2 and Q

is a charge. The radius of curvature of the manifold $\mathcal{M}^2$ is given by

$$L^2_{\mathcal{M}^2} = \frac{L^2}{(8\mu+\kappa)Q^2 + c_4 - 1}, \tag{5.34}$$

that is: it is a flat plane for $|Q| = \sqrt{(1-c_4)/(8\mu+\kappa)}$; a sphere for $|Q| > \sqrt{(1-c_4)/(8\mu+\kappa)}$; and a hyperplane for $|Q| < \sqrt{(1-c_4)/(8\mu+\kappa)}$

Given these solutions to Hořava gravity with a straightforward holographic interpretation, the correlation function for operators dual to probe scalars can be calculated, as in Section 5.3. Specifically, using the action 5.25 the correlation function in the background of $\mathrm{AdS}_2 \times \mathbb{R}^2$ can be calculated. The results of this calculation can be compared with the fully relativistic situation in [31], in order to probe the differences between employing holography for fields on the same background that respect different boundary symmetries.

Given the translational symmetry of the background $\mathrm{AdS}_2 \times \mathbb{R}^2$ the ansatz $\Psi = \psi(r)e^{-\imath(\omega t - \vec{k}\cdot\vec{x})}$ can be made. Plugging this into the action 5.25, the equation of motion of this probe scalar is determined to be

$$\psi''(r) - \frac{\psi'(r)}{r} + \left(2q\left(\omega + \frac{Q}{r}\right) - \frac{\vec{k}^2 + M^2}{r^2}\right)\psi(r) = 0, \tag{5.35}$$

where $q \equiv mb_t$ is the charge of the operator dual to Ψ. This can be solved exactly in terms of Whittaker functions of the first kind $\mathcal{M}[a;b;z]$:

$$\psi(r) = C_1\sqrt{r}\mathcal{M}\left[-\imath Q\sqrt{\frac{q}{2\omega}};\nu;\imath r\sqrt{8q\omega}\right] + C_2\sqrt{r}\mathcal{M}\left[-\imath Q\sqrt{\frac{q}{2\omega}};-\nu;\imath r\sqrt{8q\omega}\right], \tag{5.36}$$

where C_1 and C_2 are constants, and $\nu \equiv \sqrt{1 + \vec{k}^2 + M^2}$.

Deep in the interior of this geometry, as $r \to \infty$, the Whittaker functions are seen to have both infalling and outgoing contributions. Boundary conditions are chosen to eliminate the outgoing modes, in order to reconstruct a retarded correlation function from these perturbations. This fixes the ratio of C_1/C_2. The near boundary behavior of the solution is of the form

$$\psi(r \to 0) \sim A(\omega,\vec{k})r^{\Delta_-} + B(\omega,\vec{k})r^{\Delta_+}, \tag{5.37}$$

where $\Delta_\pm = 1 \pm \nu$ and determines the scaling dimension of the dual field theory operator to be Δ_+. Following [31, 30], the retarded correlation function of the dual operator is given by the ratio of its expectation value, coming from the normalizable term r^{Δ_+}, to its source value, coming from the non-normalizable term r^{Δ_-}:

$$G_R(\omega,\vec{k}) \equiv \frac{B(\omega,\vec{k})}{A(\omega,\vec{k})} = e^{-2\pi\imath\nu}\frac{\Gamma(-2\nu)\Gamma\left(\frac{1}{2}+\nu-\imath Q\sqrt{\frac{q}{2\omega}}\right)}{\Gamma(2\nu)\Gamma\left(\frac{1}{2}-\nu-\imath Q\sqrt{\frac{q}{2\omega}}\right)}\left(\imath\sqrt{8q\omega}\right)^{2\nu}. \tag{5.38}$$

While there are similarities between this correlation function and those calculated in [31, 30], there are fundamental differences that reflect the non-relativistic nature of the current duality. From above, ν is strictly real and positive, while for relativistic charged scalars ν can become imaginary as the gauge coupling increases. In the relativistic case this is understood to be due to the pair production of charged particles by a strong electric field [31]. This non-relativistic result is in concordance with that interpretation: due to the absence of anti-particles, a non-relativistic field theory will not undergo pair production, regardless of the strength of the electric field.

It is reassuring to note that as the electric field is turned off in the $Q \to 0$ limit, the behavior of the correlation function 5.38 with frequency ω is

$$\lim_{Q\to 0} G_R(\omega,\vec{k}) = e^{-2\pi\imath\nu}\frac{\Gamma(-2\nu)}{\Gamma(2\nu)}\left(\imath\sqrt{q\omega/2}\right)^{2\nu}, \tag{5.39}$$

which recovers the power law behavior $G_R(\omega,\vec{k}) \sim \omega^\nu$ for the correlation function of the non-relativistic scalar from Section 5.3 and [80, 8]. For generic Q the solution 5.38 has a nontrivial dependence on the frequency ω due to the Gamma functions. In particular, the presence of the combination $Q\sqrt{q/\omega}$ indicates the emergence of a new scale in the dual NR field theory. This is partially expected: the gauge coupling Q scales as a velocity, which is a dimensionful quantity non-relativistically. This velocity may be analogous to a Fermi-like velocity.

Another interesting limit possible for NR QFTs, for the reasons mentioned above, is the strong field limit with $Q \to \infty$:

$$\lim_{Q\to\infty} G_R(\omega,\vec{k}) = e^{-3\pi\imath\nu}\frac{\Gamma(-2\nu)}{\Gamma(2\nu)}\left(\imath 2Qq\right)^{2\nu}. \tag{5.40}$$

Of note, is the independence of this correlation function of frequency ω, implying instantaneous response. This is sensible in light of the previous paragraph: the velocity set by the gauge coupling Q has been sent to infinity. Whether the analogy between this velocity scale and the presence of a Fermi-like surface can be made more precise is an open question.

5.5 An Analytic Solution

For $z = 1$, by examining the asymptotic expansion of the equations of motion 5.10 it is seen that the power series solutions for $e(r)$ and $f(r)$ appear to terminate. Therefore, making the ansatz:

$$e(r) = \frac{1}{r^2} + C_e r - \frac{1}{4} c_3 (C_e + 2C_a)^2 r^4, \qquad f(r) = \frac{1}{r^2}, \tag{5.41}$$

the equations of motion are seen to be solved by

$$a(r) = \frac{2\left(\sqrt{4 + 4C_e r^3 + (1 - c_3)(C_e + 2C_a)^2 r^6} + (C_e + 2C_a) r^3\right)}{4(1 + C_e r^3) r - c_3 (C_e + 2C_a)^2 r^7}. \tag{5.42}$$

This solution still depends on two parameters, C_e and C_a, and it needs to be checked whether it is non-singular in the interior. As mentioned above, due to the nature of the equations of motion, a possible singular point of solutions is the sound horizon for the scalar mode with speed $s_0^2 = ((c_2+c_3)(2-c_4))/(c_4(1-c_3)(2+3c_2+c_3))$. For asymptotically AdS solutions, from equation 5.9, $z = 1$ implies that $c_4 = 0$, and therefore the scalar sound speed is $s_0 \to \infty$. Intuitively, for an infinite speed scalar mode, its sound horizon should be at the same position as the universal horizon which traps modes of any speed. In Section 5.2.3 it was determined that the condition for the location of the universal horizon is $\chi^M u_M = 0$, for χ_M the asymptotically time-like Killing vector.

For the above analytic solution to be physical it must be non-singular at the universal horizon. One quantity to examine is $\left(\chi^M u_M\right)^2$: being a square it must be non-negative for a physical spacetime, while by above it must vanish at the universal horizon. Therefore its first derivative must also vanish there in order to satisfy these two properties. For the above solution

$$\left(\chi^M u_M\right)^2 = \frac{1}{r^2} + C_e r + (1 - c_3)(C_e + 2C_a)^2 r^4, \tag{5.43}$$

and requiring that this and its first derivative vanishes at the universal horizon, $r = r_h$, implies

$$C_e = -\frac{2}{r_h^3} \qquad C_a = \frac{1 - 1/\sqrt{1 - c_3}}{r_h^3}. \tag{5.44}$$

This simplifies the solution to be:

$$e(r) = \frac{1}{r^2} - \frac{2r}{r_h^3} - \frac{c_3 r^4}{(1 - c_3) r_h^6}, \qquad f(r) = \frac{1}{r^2}, \qquad a(r) = \frac{r_h^3}{r_h^3 r + \left(\frac{1}{\sqrt{1-c_3}} - 1\right) r^4}. \tag{5.45}$$

5.5.1 Adapted Coordinates and the Universal Horizon

It is important to note that the metric and foliation normal vector following from the solution 5.45 are not written in the preferred global time of Hořava gravity. In order to more fully understand the causal structure of this solution it is useful to change to the ADM coordinates adapted to the foliation. This is done by choosing the time coordinate to be the khronon ϕ, so that u_M has only a time component. The transformation from the above EF coordinates $x^M = (v, r, \vec{x})$ to the ADM coordinates t and $y^I = (r, \vec{x})$ has Jacobian's:

$$\begin{aligned} t^M &\equiv \frac{\partial x^M}{\partial t}, \qquad & e_I^M &\equiv \frac{\partial x^M}{\partial y^I}, \\ \tilde{t}_M &\equiv \frac{\partial t}{\partial x^M}, \qquad & \tilde{e}_M^I &\equiv \frac{\partial y^I}{\partial x^M}. \end{aligned} \tag{5.46}$$

The global time of the ADM variables, t, and the null time of the EF coordinates v are related by the ansatz $t = v + h(r)$, for h a function of the radial coordinate r. Under this coordinate change the hypersurface orthogonal vector u_M becomes:

$$\tilde{u}_t = \frac{\partial x^M}{\partial t} u_M = u_v, \qquad \tilde{u}_I = e_I^M u_M = (-h'(r) u_v + u_r, \vec{0}). \tag{5.47}$$

By definition of adapted coordinates $\tilde{u}_I = 0$ is required, and therefore determines $h'(r) = u_r/u_v$, which can be evaluated for the above solution. This then gives the ADM spatial metric G_{IJ}

$$G_{IJ} \equiv e_I^M g_{MN} e_J^N = \begin{pmatrix} h'(r)\left(g_{vv}h'(r) - 2g_{vr}\right) & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix} = \begin{pmatrix} \frac{r_h^6}{r^2(r^3-r_h^3)^2} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix}. \tag{5.48}$$

The ADM shift vector N_I is given by the time-space component of the transformed spacetime metric:

$$N_I \equiv t^M g_{MN} e_I^N = \left(-h'(r)g_{vv} + g_{vr}, \vec{0}\right) = \left(\frac{r}{\sqrt{1-c_3}(r^3 - r_h^3)}, \vec{0}\right). \tag{5.49}$$

Lastly, the ADM lapse function N is determined by the time-time component of the transformed spacetime metric:

$$N^2 \equiv N^I N_I - t^M g_{MN} t^N = \frac{(r^3 - r_h^3)^2}{r^2 r_h^6}, \tag{5.50}$$

giving the final ADM form of the spacetime metric:

$$ds^2 = -\frac{(r^3 - r_h^3)^2}{r^2 r_h^6} dt^2 + \frac{r_h^6}{r^2(r^3 - r_h^3)^2}\left(dr + \frac{r^3(r^3 - r_h^3)}{\sqrt{1-c_3} r_h^6} dt\right)^2 + \frac{d\vec{x}^2}{r^2}. \tag{5.51}$$

This allows one to interpret the meaning of the constant r_h in the context of adapted coordinates. On the two-sphere at this radial coordinate the lapse function N vanishes, but the lapse function is what determines the normal distance from one spatial leaf of the foliation at global time t_0 to the next at $t_0 + \Delta t$. Therefore the distance between the leaves is ever decreasing and they bunch up as $r = r_h$ is approached on any of them. As the preferred asymptotic global time runs to infinity, the leaves of Σ_t never penetrate this two-sphere. This shows how causality and causal boundaries can arise in a theory like Hořava gravity that have no intrinsic limiting speed. Disturbances can propagate arbitrarily fast, but they can only propagate forward in the preferred global time t. As the entire region exterior to $r = r_h$ is to the past of the interior region (always at an "earlier" global time) nothing can escape the interior. The location of the universal horizon is therefore a radius where the lapse function vanishes.

5.5.2 Near Horizon Geometry and Temperature

A classic method to derive the temperature of a black hole in GR is to examine the near horizon geometry. Generically the Euclidean manifold has a conical singularity at the Killing horizon unless the Euclidean time has a specific periodicity. The formalism of finite temperature QFT then dictates the temperature to be the inverse of this period. It therefore is expected it to be beneficial to examine the geometry near the universal horizon, although the findings will be radically different then the generic case in GR.

Examining the near horizon $r \to r_h$ limit of the solution 5.51 can be tricky due to the off-diagonal $dtdr$ term coming from the shift vector 5.49. In GR this term would be removed by a temporal diffeomorphism, but this is not allowed in the foliation preserving diffeomorphisms of Hořava gravity. On the other hand, a time-dependent radial diffeomorphism to eliminate this cross term is still allowed. To this end it is first useful to redefine the radial coordinate,

$$\rho \equiv \sqrt{1-c_3} r_h^6 \left[\frac{1}{2r^2 r_h^3} - ArcTan\left(\frac{2r+r_h}{\sqrt{3}r_h}\right) + \frac{Log(r_h - r)}{3r_h^5} - \frac{Log(r^2 + r_h r + r_h^2)}{6r_h^5} \right], \tag{5.52}$$

which goes to $+\infty$ at the boundary $r \to 0$, and $-\infty$ at the universal horizon $r \to r_h$. The reason for this definition is that it makes the $dr + N^r dt$ term of the metric 5.51 conformally flat. A new time dependent radial coordinate which diagonalizes the metric can be defined: $\xi = \rho + t$. For a fixed t, ξ has the same behavior as ρ: $\xi \to +\infty$ at the boundary, and $\xi \to -\infty$ at the horizon. This has the price of making the metric non-static, giving the near-horizon behavior

$$r_h - r \approx exp(3\rho/(\sqrt{1-c_3} r_h)) = exp(3(\xi - t)/(\sqrt{1-c_3} r_h)), \tag{5.53}$$

which gives the near-horizon form of the metric:

$$ds^2 \approx -\frac{9e^{\frac{6(\xi-t)}{\sqrt{1-c_3}r_h}}dt^2}{r_h^4} + \frac{d\xi^2}{r_h^2(1-c_3)} + \frac{dx^2}{r_h^2}. \tag{5.54}$$

The non-static term can be removed by performing an allowed time reparametrization,

$$\tau = -\frac{\sqrt{1-c_3}r_h exp(-3t/(\sqrt{1-c_3}r_h))}{3}, \tag{5.55}$$

which goes to $-\infty$ as $t \to -\infty$, and $\tau \to 0$ as $t \to +\infty$. One last radial coordinate R can be defined to put the ξ-τ terms of the metric into a conformal form:

$$\frac{d\xi^2}{r_h^2(1-c_3)} \equiv \frac{9exp(6\xi/(\sqrt{1-c_3}r_h)dR^2}{r_h^4}. \tag{5.56}$$

This gives the final form of the near-horizon metric,

$$ds^2 = \frac{1}{(3R)^2}\left(-d\tau^2 + dR^2\right) + \frac{dx^2}{r_h^2}, \tag{5.57}$$

which is recognized to be AdS_2 with curvature radius 1/3 (relative to the asymptotic AdS_4 geometry of the solution) crossed with $\mathbb{R}^2$.

Unlike generic black holes in GR, the near horizon geometry is not Rindler space. Therefore an approach following Euclideanization and the subsequent periodicity of imaginary time is not available to define a temperature. Fortunately, despite having no extrinsic scale, it is understood how AdS_2 can have a notion of temperature [83]. This arises because different choices of time coordinate in AdS_2 can lead to inequivalent vacua of a QFT upon quantization. For the case at hand, there is an inherited time t, which is the Poincaré time of the decoupled asymptotic AdS_4 geometry. This is the time that would correspond to the Minkowski time of the dual NR QFT in flat space. Of interest is the behavior of objects like correlation functions with respect to the time t. Due to the relation between the AdS_2 time τ and the AdS_4 time t from equation 5.55, it is seen that the former is periodic in the imaginary part of the latter. This implies that any calculation performed in the vacuum of the near horizon geometry will be periodic in the imaginary part of t, and therefore exhibits thermal behavior from the view of the boundary observer. The inverse of this period gives the temperature of the spacetime

$$T_H = \frac{3}{2\pi r_h \sqrt{1-c_3}}. \tag{5.58}$$

5.5.3 Entropy and the First Law

The formula 5.23 for the mass of an asymptotically hyperbolic spacetime allows the first law to be displayed for this black hole. The solution 5.51 has $C_e = -2/r_h^3$ and therefore determines the mass to be:

$$M = \frac{A}{4\pi G_K r_h^3}. \tag{5.59}$$

Taking the on-shell action to be the Helmholtz free energy divided by temperature allows the calculation of the entropy. The on-shell action needs to be regulated by a background subtraction, and the extent of temporal integration of the two spacetimes is related to maintain the same geometries on the cut off surfaces [89]. This process yields the regulated on-shell action I for this solution:

$$I = \frac{-\beta A}{8\pi G_K r_h^3}, \tag{5.60}$$

where β is the inverse of the black hole temperature T_H given by 5.58. Thermodynamic identities now give the entropy:

$$S \equiv \beta M - I = \frac{3\beta A}{8\pi G_K r_h^3} = \frac{\sqrt{1-c_3}A_h}{4G_N}, \tag{5.61}$$

where A_h is the transverse area of the horizon and, recalling equation 3.11, for this solution $G_N = (1-c_3)G_H = G_K$ is the Newton constant. The first law is thus checked and this Hořava black hole is seen to obey sensible thermodynamics.

5.5.4 Tunneling Method

An alternate and intuitively pleasing method to calculate a black hole's temperature is the so-called tunneling method [73]. The foundational idea is that near a horizon the virtual pairs of particles in the quantum vacuum can be disassociated with the "negative mass" partner ending up in the black hole, in order to maintain energy conservation. The positive energy partner is then free to travel to the asymptotic region and be interpreted as Hawking radiation.

Calculationally, this method makes use of the fact that a given quanta of this radiation was extremely blueshifted near the horizon, and therefore a semiclassical approach can be used. Considering a scalar field, this allows a WKB approximation of the wavefunction Φ of an excitation as:

$$\Phi(x) \approx \phi_0 exp\left(\imath \mathcal{S}[\phi(x)]\right), \tag{5.62}$$

where $\mathcal{S}$ is the action of the scalar field, ϕ is its classical solution, and ϕ_0 is a constant amplitude. This wavefunction determines the rate that the scalar field can tunnel through the horizon. With the WKB approximation the quantum probability of emission is:

$$\Gamma \equiv \Phi^* \Phi \propto exp(-2\ Im[\mathcal{S}]). \tag{5.63}$$

If this emission distribution is Boltzmann than a temperature can be meaningfully associated to the process.

Although the WKB approximation requires knowledge of the equation of motion of the scalar field, the simpler eikonal/Hamilton-Jacobi approximation only requires a dispersion relation for the field. From the wavefunction 5.62, the Hamilton-Jacobi equations can be derived by acting the momentum operator on Φ:

$$k_M = \nabla_M \mathcal{S}[\phi(x)]. \tag{5.64}$$

Combining this with the dispersion relation for ϕ allows the determination of $Im[\mathcal{S}]$, and therefore the tunneling probability.

In GR it is argued that due to the extreme blueshift near the horizon the appropriate dispersion relation to use is that of a massless particle, $k_t^2 = \vec{k}^2$, regardless of the actual equation of motion of the field. While this dispersion relation is fixed in

GR due to Lorentz invariance, in Hořava gravity it could take on the very general form:

$$k_t^2 = \frac{\vec{k}^{2z_\phi}}{k_0^{2(z_\phi-1)}} + \cdots \,, \tag{5.65}$$

where z_ϕ is an integer determining the nature of the dispersion, k_0 is a constant of dimension one, and the dots schematically imply that all powers of $\vec{k}^2$ less than z_ϕ, as well as derivatives of $\vec{k}$ can be included, see [17] for a more precise discussion.

A tractable, and seemingly natural, choice is $z_\phi = 2$. This gives a dispersion that is similar to the Schrödinger equation. Proceeding with the traditional methods of calculating $Im\mathcal{S}$, one obtains the temperature of a static spherically symmetric black hole in Hořava gravity [17, 15]:

$$T_{UH} = \left.\frac{a^M s_M |\chi|}{4\pi}\right|_{r=r_h} \,, \tag{5.66}$$

where s_M is the outward pointing space-like vector orthogonal to u_M, $a_M \equiv u^N \nabla_N u_M$, and $\chi^M = (1,0,0,0)$ is the asymptotically time-like Killing vector. For the above analytic solution 5.51 this gives the temperature:

$$T_{UH} = \frac{1}{2} T_H, \tag{5.67}$$

that is, one half of the value as determined by the geometric method of Section 5.5.2. Interestingly, another tractable value is the $z_\phi \to \infty$ limit [17], for which the temperature is twice that of the $z_\phi = 2$ case, and therefore agrees with the geometric method.

The fact that the tunneling method calculation has the ambiguity of the free parameter z_ϕ makes it a somewhat unappealing technique to calculate the Hawking temperature, which is expected to be universal. In GR, the possible ambiguity arising for fields of different mass or spin has been shown to be irrelevant in the calculation of Hawking radiation because the extreme blueshift of the near-horizon region dictates that only the linear high energy dispersion relation, constrained by Lorentz invariance, plays a role. It can be hoped that similarly the high energy dispersion relation of fields near the universal horizon in Hořava gravity is unique. Some evidence, presented in Section 6, points towards the $z_\phi = \infty$ limit being this unique relation. In this case the temperature as calculated via the tunneling method agrees with that extracted from the near horizon geometry.

The temperature derived via the tunneling method for $z_\phi = 2$ has recently been reproduced in [24] by arguments concerning ray tracing near the universal horizon and their relation to the surface gravity. It is claimed this calculation is universal regardless of the exact dispersion, but that would contradict the tunneling method in the $z_\phi \to \infty$ limit. Whether the ray tracing method can be used in this regime would provide crucial insight into the temperature of universal horizons.

5.5.5 An Asymptotically Flat Solution

Another analytic black hole which provides a testing ground of the ideas developed above has been presented in [17]. The four dimensional metric is given as:

$$ds^2 = -e(r)^2 dt^2 + \frac{1}{e(r)^2}\left(dr - f(r)e(r)dt\right)^2 + r^2 d\Omega^2, \tag{5.68}$$

where:

$$e(r) = -1 + \frac{r_h}{r}, \qquad f(r) = \sqrt{\frac{\mu r_h}{r} + \frac{(2-c_4)r_h^2}{2(1-c_3)r^2}}. \tag{5.69}$$

Asymptotic infinity is the region $r \to \infty$, which is seen to be Minkowski space in spherical coordinates. Importantly, the metric is written in a preferred global time, and therefore $e(r)$ is the lapse function of the ADM decomposition and, as above, it vanishing at $r = r_h$ signals the location of a universal horizon. The constant μ further parametrizes the solution.

Similar manipulations as applied to the asymptotically AdS black hole of Section 5.5.2 can be brought to bear. After diagonalizing the metric by performing a time-dependent radial diffeomorphism, followed by a time reparametrization, the near horizon geometry of 5.68 becomes:

$$ds^2 \approx \frac{r_h^2}{R^2}\left(-d\tau^2 + dR^2\right) + r_h^2 d\Omega^2, \tag{5.70}$$

where: $R \to \infty$ at the universal horizon; and the geometry is again recognized to be AdS_2, now crossed with a sphere, both of radius r_h.

The temperature of this solution follows from the relation between the near horizon AdS_2 time τ and the asymptotic Minkowskian time t:

$$\tau \equiv -\frac{r_h}{\sqrt{\mu + \frac{2-c_4}{2(1-c_3)}}} exp\left(-\frac{r_h t}{\sqrt{\mu + \frac{2-c_4}{2(1-c_3)}}}\right). \tag{5.71}$$

Periodicity in the imaginary part of t determines the temperature to be:

$$T_H = \frac{1}{2\pi r_h}\sqrt{\mu + \frac{2-c_4}{2(1-c_3)}}. \tag{5.72}$$

Comparing to the calculation of temperature presented in [17], it is again seen that the tunneling method agrees with this geometric method for $z_\phi = \infty$, and not for $z_\phi = 2$.

5.6 Numerical Solutions

5.6.1 Probe Limit

As discussed in Section 3.1.2, a powerful use of the khronon formalism is the probe limit regime. When the khronon does not backreact, any solution to GR is a solution to Hořava gravity; the khronon simply determines what time coordinate needs to be used to be a preferred global time.

An interesting class of manifolds to examine with this technique are those which are black holes of GR [19]. These are defined by having event horizons: null causal boundaries of the asymptotic region. Whether the event horizon[3] maintains an im-

[3]From hereon the event horizon will be referred to as the metric horizon to avoid confusion with the universal horizon, which is the true causal boundary of Hořava gravity.

portant status can be explored with the probe limit technique. A particularly interesting black hole geometry, from the viewpoint of holography, is the Anti-de Sitter-Schwarzschild (AdS-S) solution. In four dimensions the metric in Poincaré coordinates is:

$$ds^2_{AdS-S} = -\frac{1-r^3}{r^2}dt^2 + \frac{1}{r^2(1-r^3)}dr^2 + \frac{1}{r^2}d\vec{x}^2, \tag{5.73}$$

where the boundary is at $r = 0$, and the radius of the metric horizon has been set to one. It is beneficial to work in coordinates that are non-singular at the metric horizon. Using the Eddington-Finkelstein time $v \equiv t - r_*$, where the tortoise coordinate is given by $r_* = \int^r dr'/(1 - r'^3)$, the metric becomes:

$$ds^2_{AdS-S} = -\frac{1-r^3}{r^2}dv^2 - \frac{2}{r^2}dvdr + \frac{1}{r^2}d\vec{x}^2. \tag{5.74}$$

On this background the khronon equation of motion can be derived from the probe action[4]:

$$S_\phi = -\frac{c_4}{16\pi G_K}\int dvdrdx^2\sqrt{-g}\left[\frac{1}{2}F_{MN}F^{MN} + s_0^2(\nabla_M u^M)^2\right], \tag{5.75}$$

recalling that $F_{MN} \equiv \partial_M u_N - \partial_N u_M$, and s_0 is the sound speed of the scalar mode of Hořava gravity, given by $s_0^2 = (c_2 + c_3)/c_4$ in the probe limit, whereas the sound speed of the spin two graviton is $s_2 = 1$. From the action 5.75, it is seen that in the probe limit the only effective coupling constant is the speed s_0. A useful parametrization of the khronon orthogonal vector for a static, transversely constant ansatz is:

$$u_M = \left(-\frac{1+f(r)}{2r}\sqrt{\frac{1-r^3}{f(r)}}, -\frac{1}{r}\sqrt{\frac{f(r)}{1-r^3}}, 0, 0\right). \tag{5.76}$$

As such it is explicitly normalized to be unit time-like in the AdS-S background, $u_M u^M = -1$. The boundary condition that the global time is asymptotically Poincaré time requires $\phi \to t = v + r_*$ at the boundary. In EF coordinates this is equivalent to $u_M \to (-1/r, -1/r, 0, 0)$, or $f(0) = 1$ for the above parametrization.

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[4]This is because $R_{MN}u^M u^N = const$ for AdS-S and a normalized u_M.

Varying the action 5.75 with respect to $f(r)$ gives the equation of motion for the khronon. This second order non-linear ODE is seen to have the expected singular points at the boundary, $r = 0$, and the metric horizon, $r = 1$. Additionally, there is a singular point whenever the magnitude of the khronon vanishes, $f(r_{crit}) = 0$, but, from examining the expression 5.76 for the khronon normal vector, u_M is non-singular for $f(r_{crit}) = 0$ only if $r_{crit} = 1$, that is, this is not a new singular point, but just the metric horizon again. This is understood by recognizing that in the probe limit $s_2 = 1$ and therefore the metric horizon is more properly understood as the sound horizon for the spin two degrees of freedom of the metric. Lastly, there is a singular point at the sound horizon for the scalar mode, r_s, where the magnitude of the khronon function satisfies

$$f(r_s)^2 - 2f(r_s)\frac{1+s_0^2}{1-s_0^2} + 1 = 0. \tag{5.77}$$

This gives $f_s \equiv f(r_s) = (1 \pm s_0)^2/((1-s_0)(1+s_0))$. The bottom sign is chosen as the physically acceptable condition for two reasons: there should be a non-singular solution for the limit $s_0 \to 1$ corresponding to the scalar and spin two sound horizons coinciding, which requires $f \to 0$ as above; additionally, in the following numerical construction, solutions using the top sign either do not match the desired boundary condition $f(0) = 1$ or are singular at the metric horizon.

A final important radial coordinate can be seen from the parametrization for u_M in 5.76. Recall from Section 3.1.2 that when written in the preferred global time the hypersurface normal vector u_M has only a temporal component which is given by the ADM lapse function N. Additionally, in transforming from the EF time v to the preferred global time the temporal component of u_M is unaltered, see equations 5.47. This implies that at a coordinate where u_v vanishes, so does the lapse function N of the preferred foliation and therefore this is the location of the universal horizon, r_h. From the parametrization 5.76 this determines the universal horizon to be the radial coordinate such that $f(r_h) = -1$. This further implies, from examining u_r, that physical solutions must have $r_h > 1$, that is the universal horizon is inside of the metric horizon.

5.6.2 Solutions

The second order non-linear ODE for $f(r)$ can be numerically solved via a shooting method. At the scalar sound horizon the function is given by $f(r_s) = f_s$, see equation 5.77 above. Requiring a regular solution at this singular point in turn determines the value of $f'(r_s)$ from two possible choices. Using this data as boundary conditions, numerical integration can be performed in either direction, increasing or decreasing r, to find a full solution. In practice, boundary conditions are imposed a small distance ϵ from the scalar sound horizon, and a Taylor series is used that matches the desired behavior at r_s. This is the typical trick to improve numerical stability while integrating near a singular point. The technique of matching a numerical solution to a local Taylor series must also be done to jump over the singular point at the metric horizon $r = 1$.

For a given scalar speed s_0, taking the location of the sound horizon r_s as the shooting parameter, and using the previously mentioned boundary conditions there, the equation of motion can be numerically integrated out to the boundary $r = 0$. By varying r_s one can obtain the value that is needed to meet the boundary condition $f(0) = 1$, which is the requirement that the foliation asymptotically tends to that of Poincaré time. For every s_0 this gives a unique regular solution that is asymptotically AdS.

Case I: $s_0 < 1$

For the scalar speed $s_0 < 1$ the sound horizon is outside of the metric horizon, that is $r_s < 1$. To implement the above numerical procedure a Taylor series expansion about r_s which solves the equation of motion to order $(r - r_s)^4$ is used, and the required boundary conditions are implemented at $r = r_s - \epsilon$ for $\epsilon = 10^{-3}$. Figure 5.1 shows two plotted solutions for r_s differing by 10^{-9}, the accuracy used throughout. The

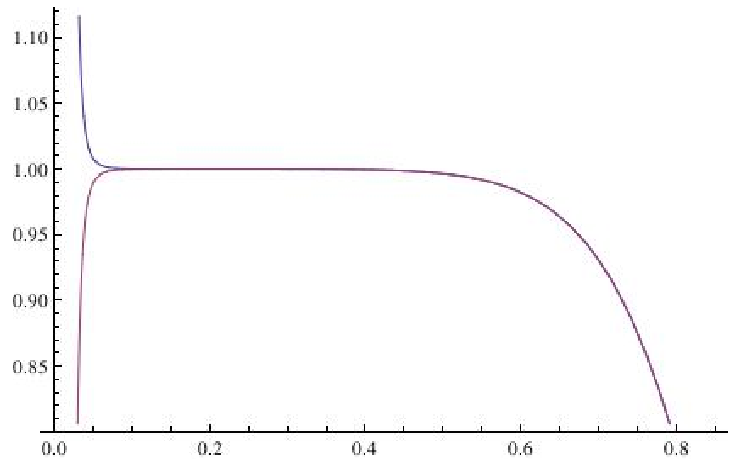

Figure 5.1: The profile of the khronon function $f(r)$ for $s_0 = 2/10$ is plotted from r_s to the boundary at $r = 0$. The lower red branch has $r_s = 212192121/250000000 \sim 0.848768$ while the upper blue branch has r_s larger by 10^{-9}.

desired solution with $f(0) = 1$ lies between the plotted two. Figure 5.2 presents crucial results for speeds $s_0 < 1$: the r_s giving the correct boundary conditions is shown, along with r_h, the radial coordinate at which $f(r)$ first becomes equal to -1, which is the location of the universal horizon, as argued above.

The three other possible combinations of boundary conditions for $f(r_s)$ and $f'(r_s)$ do not give physically acceptable solutions. Two of them always have $f(0) > 1$ or $f(0) < 1$ for all r_s, never switching as in Figure 5.1. This means they do not realize the desired condition of Poincaré time at the asymptotic boundary. The final possible class of solutions, those with $f_s = (1+s_0)/(1-s_0)$ and $f'(r_s) > 0$, do exhibit the correct asymptotic $f(0) \to 1$ behavior, but are divergent at the metric horizon. Examining u_M in 5.76 it is seen that this leads to a non-regular solution for the hypersurface orthogonal vector. An example of this class is shown in Figure 5.3.

s_0	r_s	r_h
0.1	0.823037721	1.12386952
0.2	0.848768484	1.13782822
0.3	0.872378252	1.15121426
0.4	0.894577562	1.16436206
0.5	0.915465270	1.17473414
0.6	0.935021618	1.18484700
0.7	0.953227190	1.19334154
0.8	0.970092978	1.20092572
0.9	0.985661688	1.20752533

Figure 5.2: The location of the scalar sound horizon r_s, and the radius of the universal horizon r_h, for speeds $s_0 < 1$.

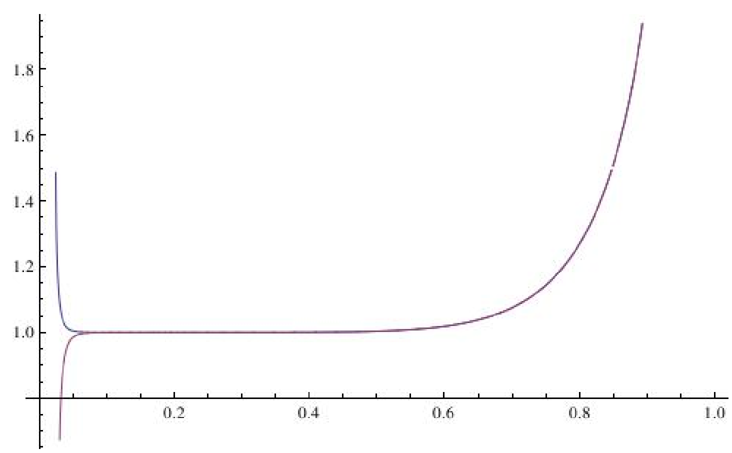

Figure 5.3: The profile of the khronon function $f(r)$ for $s_0 = 2/10$ with incorrect boundary conditions at r_s. Although the behavior is as desired at $r = 0$, $f(r)$ is divergent at the metric horizon $r = 1$.

Case II: $s_0 > 1$

For the khronon speed $s_0 > 1$ the scalar sound horizon is inside the metric horizon, that is $r_s > 1$. The numerics are implemented by the shooting method as above. Figure 5.4 gives the data of the solutions: r_s and r_h for each s_0. Of importance it

s_0	r_s	r_h
1.1	1.013189612	1.218154746
1.2	1.025420211	1.223152116
1.3	1.036500000	1.226276947
1.4	1.046758982	1.229434208
1.5	1.056222636	1.232232389
1.6	1.064967826	1.234712679
1.7	1.073108780	1.237040172
1.8	1.080589512	1.238919367
1.9	1.087514072	1.240540799
2	1.094025032	1.242147735
4	1.166242711	1.254600129
8	1.210285515	1.258505374
16	1.234393424	1.259559808
32	1.246977962	1.259830158
64	1.253404370	1.259898282
128	1.256651377	1.259915354

Figure 5.4: The location of the scalar sound horizon r_s, and the radius of the universal horizon r_h, for speeds $s_0 > 1$.

is seen that the universal horizon r_h is always behind the metric horizon at $r = 1$, and always behind the scalar sound horizon r_s. From the data it appears that r_h is bound between $(1.11, 1.26)$ as $s_0 \to (0, \infty)$, respectively. In fact, the analytic solution of Section 5.5 can be used to determine the $s_0 \to \infty$ behavior. Since that solution is asymptotically Anti-de Sitter, if the probe limit is taken it will correspond to one of the numerical solutions above. Recalling that the solution has $c_4 = 0$, in the probe limit this solution has a scalar speed of $s_0^2 = (c_2 + c_3)/c_4 \to \infty$. From the metric 5.45, in the probe limit, the analytic solution has the metric component:

$$g_{vv} = -\frac{1}{r^2} + \frac{2r}{r_h^3}. \tag{5.78}$$

The metric horizon is where this component vanishes, giving $g_{vv} = -1 + 2/r_h^3 = 0$, that is $r_h = 2^{1/3} \approx 1.2599$, agreeing with the numerical bound above.

5.6.3 Universal Horizon

These numerical solutions all have the behavior that $u_v \to 0$ as $r \to r_h$. By writing the khronon as $\phi = v + h(r)$ it is easy to see that $h(r) = \int^r dr' u_r/u_v + const$, as in Section 5.5.1. Therefore the vanishing of u_v implies that the khronon diverges[5] at this radius, consequently the spatial slices of the foliation defined by the level sets of ϕ pile up at this radius and do not penetrate to smaller r. This is shown, for $s_0 = 7/10$, in Figure 5.5. Despite foliating the entire exterior of the black hole the leaves coming from the asymptotic boundary pile up at $r_h \approx 1.19$ and do not reach further into the interior. It is important to note that the interior region $r > r_h$ still has a foliation by a preferred global time: Figure 5.5 only shows the foliation that is connected to the asymptotic boundary for clarity, the numerical solution for the khronon, and hence the foliation, can be constructed arbitrarily far into the interior, as in [11]. This shows that the foliations of the interior are disconnected from the foliation that reaches the asymptotic boundary.

[5] This divergence is physically acceptable as it can be removed via an allowed temporal reparametrization. The invariant field u_M is non singular.

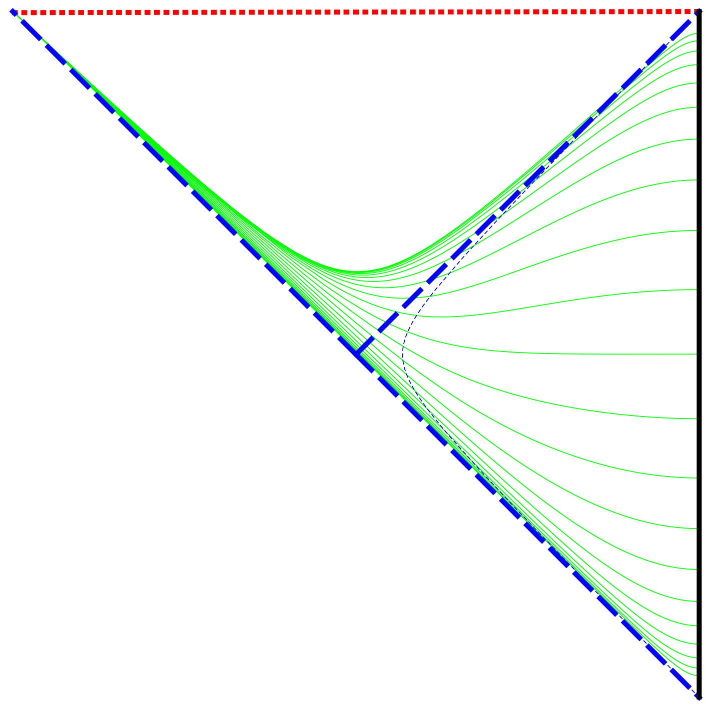

Figure 5.5: The Penrose diagram of the AdS black hole. The singularity at $r \to \infty$ is in dotted red at the top, the boundary $r = 0$ is in thick black on the right, the metric horizon at $r = 1$ is in thick dashed blue, while the thin dashed blue is the scalar sound horizon at r_s. The level sets of the khronon with $s_0 = 7/10$ are shown in thin green.

The khronon defines the preferred global time coordinate of Hořava gravity, while causality means that influences can only propagate forward in global time, although arbitrary speed is allowed. In particular any disturbance at $r > r_h$ can only propagate towards larger r: everywhere exterior to r_h is at an "earlier" global time. In this sense the surface r_h, which is the boundary of the exterior foliation, defines a causal boundary for the asymptotic observer and is therefore justly a universal horizon.

The status of the metric horizon at $r = 1$ can now be made clear. In Hořava gravity this surface is properly understood as the sound horizon for the spin 2 graviton. Like the scalar sound horizon at r_s, these spheres are trapped surfaces for the low energy modes of their respective gravitons. On the other hand, higher energy corrections to Hořava gravity will modify the dispersion relations of the gravitons to allow arbitrary speed[6]. This allows the gravitons, and any other fields considered, to escape their respective sound horizons, but they will still be inevitably trapped by the universal horizon as this is a trapped surface for modes of infinite speed. Therefore the relative unimportance of the metric horizon as a sound horizon, as compared to the universal horizon as a causal boundary, is understood in Hořava gravity.

[6]These corrections are not expected to modify the low energy picture of the universal horizon as the curvature is generically small there.

Chapter 6

DISCUSSION

6.1 Lessons Learned

In the large landscape of internally consistent quantum field theories, the highly constrained class of relativistic quantum field theories occupies only a small corner. This work suggests that something similar should be true on the dual holographic side. While the well studied case of gravitational theories with the full relativistic diffeomorphism invariance of Einstein gravity seems to require string theory for its high energy completion, the holographic dual to a generic NR QFT seems to simply be a high energy fixed point of Hořava gravity, with non-trivial dynamical scaling exponent z_{HL}, coupled to an almost arbitrary matter sector. This basic picture has been one of the motivations behind the original work of [48, 49] and was also recently emphasized in [38, 15].

The duality proposed here was largely motivated by the similar symmetry structures of this alternate theory of gravity [49] and a generic non-relativistic quantum field theory [81]. The work of [80, 8, 35] involving light-like compactifications of relativistic holographic duals also gave crucial clues hinting at the correct field content of the bulk theory, as discussed in Sections 3.2.1 and 3.2.2.

An important check of this proposal is the calculation of correlation functions. In Section 5.3 this was performed for a probe scalar on the background of AdS_4 and was shown to obey the rigid structure required by Galilean and scale symmetries. In Section 5.4 a correlation function was calculated for a probe scalar on the background of $\mathrm{AdS}_2 \times \mathbb{R}^2$. The results show distinct non-relativistic characteristics, as compared to the relativistic calculations of [31, 30] on this same background.

Section 5.2 explores the class of solutions to Hořava gravity that asymptote to Lifshitz spacetimes. These are an important class of solutions from the viewpoint of holographic application. Lifshitz spacetimes in general relativity have been argued to be dual to non-relativistic quantum field theories. Given that these spacetimes are vacuum solutions of Hořava gravity lends evidence to the proposal that this alternate theory of gravity is dual to generic NR QFTs.

The crucial object at the heart of holography is the black hole. This gravitational object is fascinating classically for its causal and geometric structure. Quantum mechanically black holes present the remarkable fact that they are thermodynamic objects, with a temperature and entropy. Simple counting then shows that the amount of information needed to describe any volume of space is equivalent to that of a field theory on the bounding surface area, and holography is born.

If Hořava gravity is to follow this paradigm in the same way that general relativity does, understanding of its black holes is a requisite. A primary feature, discussed throughout, is that despite lacking Lorentz invariance and the geometric structure of null light cones, Hořava gravity does have a well defined causal structure due to signals only propagating forward in the preferred time. This leads to the possibility of a causally disconnected region of the spacetime, when the foliation by global time that covers asymptotic infinity does not penetrate the entire manifold, as discussed in Section 5.6.3. The boundary of this region behaves like an event horizon of general relativity: it is a surface from which no causal signals can escape and influence infinity. Black holes exist in Hořava gravity and are the region behind this boundary or "universal horizon".

Numerical Hořava black holes were constructed in Section 5.6, proving existence of universal horizons. These solutions exhibit interesting properties, such as the dependence of the location of the universal horizon on the speed of the scalar graviton, but, being numeric, they would be cumbersome to explore the important issue of thermodynamics. Luckily, Section 5.5 provides an analytic spacetime that is seen to have a universal horizon. Calculations of its mass 5.59, entropy 5.61, and temperature 5.58 obey a first law of thermodynamics, giving crucial evidence that the holographic principle applies to Hořava gravity.

6.2 Open Questions

There is much to be explored in the subject of non-relativistic holography. Initial directions to be explored is to simply follow routes that that traditional AdS/CFT has opened. One arena is referred to as the "fluid/gravity" correspondence, this allows the exploration of hydrodynamics in terms of the gravitational behavior of black holes. Interesting properties of non-relativistic fluids such as viscosity parameters can be calculated with this holography.

To get truly novel results along these lines black hole solutions of Hořava gravity that are asymptotically Lifshitz spaces would be desirable. Such solutions would also clear up the uncertainty in the definition of mass in $z \neq 1$ backgrounds, as discussed in Section 5.2.3.

To make contact with possible real world systems further developments are required. To make contact with the quantum Hall effect, solutions of Hořava gravity with magnetic fields present are needed. In order to capture the NR conformal symmetry of Fermions at unitarity the additional $U(1)$ invariance of Section 4.2 needs to be incorporated. This is particularly tempting to explore in three dimensions as it can be accomplished with the inclusion of only one additional field.

The prospect that Hořava gravity may be a complete quantum theory would allow holography to move away from the weakly curved gravity regime. This would explore dual field theories outside of the strongly coupled, large number of colors regime that have restricted traditional AdS/CFT. Hořava gravity comes with an intrinsic scale, the Planck mass M_{pl}. For energies far below the Planck mass the action should be limited to two derivative terms and is uniquely fixed to be 3.2. This is the appropriate action to use when L, the typical curvature radius of spacetime, is large in Planck units. From experience with relativistic holography, this limit corresponds to a large N_c limit in the dual QFT, which allows one to study a classical bulk theory. One of the big selling points of Hořava gravity is that it is a candidate for a UV finite quantum theory of gravity. At energies far above the Planck scale the theory is argued to flow to a UV fixed point with a different dynamical critical exponent z_{HL}. As a consequence, at this putative UV fixed point the counting of derivatives needs to distinguish between spatial derivatives, which have dimension 1, and temporal derivatives, which have dimension z_{HL}. All marginal and relevant terms (that is terms with dimension less than or equal to $D+z_{HL}$, which compensates the dimension $-D-z_{HL}$ of the integration measure $d^Dx\,dt$) need to be included in the action. In particular, the potential energy, which depends on the curvature of the spatial metric G_{IJ} and its spatial derivatives, should include terms with up to $D+z_{HL}$ derivatives of the metric. For the special case of $D=3$, $z_{HL}=3$ a full list of the possible terms in the potential, subject to certain discrete symmetry assumptions, has been worked out[1] in [82].

Section 5.5 confirmed the thermodynamics of an analytic black hole. This is reassuring, but it also raised some interesting questions. The first of these is that the near horizon geometry is not Rindler space, as it generically is in GR. This also arises in the asymptotically flat black hole of [17]. Instead the geometry is AdS_2 crossed with the transverse space, $\mathbb{R}^2$ or S^2, respectively. This indeed appears to be the generic situation in Hořava gravity: the universal horizon occurs where the lapse N vanishes, if it does so linearly, then g_{tt} vanishes quadratically, once the metric is diagonalized as in Section 5.5.2. Despite this, a notion of temperature can still be defined via the methods of [83].

The second interesting, and likely related issue, arises in the calculation of the

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[1]In the original work of [49] a simpler potential has been used by imposing the additional constraint of detailed balance. It seems to still be under debate whether this constraint can be imposed at the full quantum level. This question is not relevant for the $M_{pl}L \gg 1$ case.

temperature via the tunneling method. This approach has the weakness that the dispersion relation Eq. 5.65 is not unique once Lorentz invariance is abandoned. When using the tunneling method in GR the extreme blueshift of the horizon is used to justify the linear dispersion $k_t = |\vec{k}|$ regardless of the mass of the particle. In Hořava gravity the same logic can be used. For modes of extremely high momenta, the leading power z_ϕ in Eq. 5.65 dominates all lower powers in the dispersion. As z_ϕ is not uniquely fixed by symmetry, as it is in GR, this argument favors as large a z_ϕ as possible. It therefore seems natural to use $z_\phi \to \infty$ for the dispersion in the near universal horizon calculation of the tunneling method. It is also reassuring that this form of the dispersion gives a temperature that agrees with the geometric method of Section 5.5.2, while the "minimal" choice $z_\phi = 2$ disagrees by a factor of two. Why this may be related to the near universal horizon geometry is the fact that AdS_2 can be seen as the $z \to \infty$ limit of the Lifshitz spacetime Eq. 5.1. A better understanding of the natural action for a non-relativistic scalar in these background geometries will prove crucial in justifying these arguments.

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