Eur. Phys. J. C (2024) 84:653
https://doi.org/10.1140/epjc/s10052-024-13001-3

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Nonmetric geometric flows and quasicrystalline topological phases for dark energy and dark matter in $f(Q)$ cosmology

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Received: 29 February 2024 / Accepted: 9 June 2024

Abstract We elaborate on nonmetric geometric flow theory and metric-affine gravity with applications in modern cosmology. Two main motivations for our research follow from the facts that (1) cosmological models for $f(Q)$ modified gravity theories, MGTs, are efficient for describing recent observational data provided by the James Webb Space Telescope; and (2) the statistical thermodynamic properties of such nonmetric locally anisotropic cosmological models can be studied using generalizations of the concept of G. Perelman entropy. We derive nonmetric distorted R. Hamilton and Ricci soliton equations in such canonical nonholonomic variables when corresponding systems of nonlinear PDEs can be decoupled and integrated in general off-diagonal forms. This is possible if we develop and apply the anholonomic frame and connection deformation method involving corresponding types of generating functions and generating sources encoding nonmetric distortions. Using such generic off-diagonal solutions (when the coefficients of metrics and connections may depend generically on all spacetime coordinates), we model accelerating cosmological scenarios with quasi-periodic gravitational and (effective) matter fields; and study topological and nonlinear geometric properties of respective dark energy and dark matter, DE and DM, models. As explicit examples, we analyze some classes of nonlinear symmetries defining topological quasicrystal, QC, phases which can modified to generate other types of quasi-periodic and locally anisotropic structures. The conditions when such nonlinear systems possess a behaviour which is similar to that of the Lambda cold dark matter (ΛCDM) scenario are stated. We conclude that nonmetric geometric and cosmological flows can be considered as an alternative to the ΛCDM concordance models and speculate on how such theories can be elaborated. This is a partner work with generalizations and applications of the results published by Bubuianu, Vacaru et all. EPJC 84 (2024) 211; 80 (2020) 639; 78 (2018) 393; 78 (2018) 969; and CQG 35 (2018) 245009.

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Published online: 28 June 2024

1 Introduction and objectives

Modified gravity theories, MGTs, with nonmetricity have been known since 1918 when Weyl [1] considered nonmetric spaces as an attempt to construct a unified geometric theory of gravity and electromagnetism. In the Weyl geometry, the nonmetricity tensor $Q_{\alpha\beta\gamma} := D_\lambda g_{\beta\gamma} \neq 0$ is treated as an additional fundamental geometric object. This is different from the Einstein gravity theory (i.e. general relativity, GR) constructed for symmetric pseudo-Riemannian metrics $g_{\beta\gamma}$ of Lorentz signature $(+++-)$ and when the Levi-Civita, LC, connection ∇_λ is uniquely defined from the conditions of zero torsion, $T^\alpha_{\beta\lambda}[\nabla] = 0$, and metric compatibility, $\nabla_\lambda g_{\beta\gamma} = 0$.[1] The monographs [2–5] contain necessary geometric methods and results on mathematical relativity and exact solutions (they involve generally accepted abstract, frame and coordinate indices, and geometric and physical objects conventions). Nonmetric geometric and gravity theories were mostly ignored for many decades because of criticisms by A. Einstein and W. Pauli. We cite [6] as the first comprehensive review on metric-affine gravity involving independent metric, $g_{\beta\gamma}$, and linear connection, $D_\lambda = \{\Gamma^\alpha_{\beta\lambda}\}$, structures. Such theories are characterized by nontrivial torsion, $T^\alpha_{\beta\lambda}[D]$, and nonmetricity, $Q_{\alpha\beta\gamma}$, tensors and other geometric objects like the non-Riemannian curvature, the Ricci tensors, different scalar curvatures etc. In monograph [7], the metric-affine geometry and gravity theories were elaborated on (co) tangent super- and/or noncommutative bundle spaces and for various generalized Finsler–Lagrange–Hamilton spaces.

However, in physical literature, the position of MGTs, including Weyl geometries and various gravitational and matter field theories determined by generalized Lagrangians constructed as nonlinear functionals $f(R, T, Q, T_m)$, changed drastically during the last 25 years. That was due to observational pieces of evidence on late-time acceleration cosmology and related dark energy, DE, and dark matter, DM, problems. A plethora of MGTs have been elaborated alternatively to GR and standard particle physics. We recommend [8–11] as some recent reviews of results and applications in modern cosmology.[2] In this work, we study MGTs constructed for a gravi-

[1] In next sections and appendices, we shall provide necessary definitions and explain when our notations will be different from some generally accepted ones.

[2] In a partner work [15], a different system of notations is used. For instance, $F(...)$ is written instead of $f(...)$ because the symbol f was considered for normalizing functions of nonmetric geometric flows. In this paper, we write $f(Q)$ for nonmetric gravity as in [10,11], see also references therein. A comprehensive list of references on nonmetricity

tational Lagrangian $f(Q)$ involving a nonmetricity scalar Q (all necessary definitions and notations are provided in the next section and Appendix A). Such gravity theories may challenge the standard ΛCDM scenario, lead to interesting cosmological phenomenology, and can be confronted with various recent observational data provided by James Webb space telescope JWST, [12–14].[3]

Motivated by a number of new geometric and exciting theoretical properties of nonmetric MGTs, in this and a partner work [15], we employ the powerful mathematical tools exploited in theories of relativistic geometric flows and nonholonomic Lorentz manifolds [16] and consider such constructions for Q–deformations on nonholonomic metric-affine spaces. We develop also the anholonomic frame and connection deformation method, AFCDM, for generating exact and parametric cosmological solutions in $f(Q)$ gravity, see [17–19] for recent reviews of methods and results concerning MGTs and GR and applications in modern cosmology. So, the general goal of this work is to show how nonholonomic geometric flow and gravitational equations in $f(Q)$ MGTs and cosmology can be solved in exact and parametric forms using nonholonomic dyadic variables and defining necessary classes of distortions of canonical (non) linear connection structures and nonmetricity scalar $\widehat{\mathbf{Q}}$.

The main **Hypothesis** in this and the partner work [15] is that: mathematically self-consistent nonmetric geometric flow and related physically viable nonmetric gravitational and cosmological theories can be constructed considering Q-deformations of some models on nonholonomic Lorentz manifolds and certain classes of metric compatible gravity theories. Using nonholonomic dyadic variables and canonically adapted (non) linear connection structures, the corresponding nonmetric geometric evolution flow equations, or dynamical gravitational equations, can be integrated in certain generic off-diagonal forms determined by generating functions and generating sources depending on all spacetime coordinates. Such new classes of nonmetric geometric flow cosmological solutions describe various quasi-periodic spacetime (quasicrystal, QC) and quasicrystalline topological phases [17,20,21] being important for determining new features of the DE and DM theories.

The formulated hypothesis is supported by an important class of nonmetric quasi-stationary solutions which are with Killing symmetry on a time like vector – the typical examples include Q-deformed black ellipsoid, black holes, wormhole, solitonic etc. configurations [15]. Using similar geometric methods in dual forms on time and space coordinates (with corresponding nonholonomic space like Killing symmetry), we can generate various nonmetric locally anisotropic cosmological solutions with generic dependence on a time-like coordinate. Such off-diagonal metrics and generalized connection structures determining nonmetricity fields are constructed and studied in this work. All classes of quasi-periodic and cosmological solutions are characterized by corresponding nonlinear symmetries and described as Q-modified models of Grigori Perelman's thermodynamics introduced in the theory of Ricci flows [22–25]. We cite also [26–28], for reviews of mathematical results and methods, and [16,29,30], for recent developments and applications in non-standard particle physics, MGTs, and geometric and quantum information flow models.

The objectives, Objs, of this work are structured for corresponding sections:

The **Obj 1** stated for Sect. 2 is to formulate the $f(Q)$ gravity in nonholonomic (2+2) variables with canonical distortions of the LC-connection. The fundamental geometric and physical objects on such metric-affine spaces are redefined with respect to nonlinear connection, N-connection, adapted frames. This allows us to prove general decoupling and integration properties of geometric and physically important nonlinear systems of partial differential equations, PDEs, encoding nonmetricity.

In Sect. 3, we provide a generalization of the $f(Q)$ gravity to a model of nonmetric geometric flow theory. The **Obj 2** is to define Q-distorted relativistic R. Hamilton and D. Friedan equations in canonical nonholonomic variables and to consider respective nonholonomic Ricci soliton equations. Then, the **Obj 3** is to formulate a statistical thermodynamic model for $f(Q)$ geometric flows derived as a nonholonomic and canonical Q-deformation of G. Perelman constructions [22] and non Riemannian generalizations in [15,16,29,30].

The **Obj 4** of Sect. 4 is to study possible applications of our methods in modern DE and DM physics by constructing new classes of generic off-diagonal cosmological and geometric flow solutions encoding nonmetricity and generating topological QC structures. We prove that quasi-periodic cosmological structures may arise in generic off-diagonal forms from QC like generating functions, or from effective generating sources; and, in a general context, with mixed and different phases of topological QCs. Gravitational polarizations and effective sources are introduced and computed in such

Footnote 2 continued
gravity theories and applications is not provided in this paper. We do not discuss details on increased interest in cosmological phenomenology of $f(Q)$-type and cite only the most important papers that are directly related to the purposes of this work.

[3] We do not provide an exhaustive list of references and do not discuss details on recent observational data provided by James Webb Space Telescope, JWST, which have sparked debates about the validity of many cosmological models. Such results related to the accelerating cosmology and various problems of dark energy, DE, and dark matter, DM, physics motivate the importance for studying in GR and MGTs new classes of generic off-diagonal, non-homogeneous and locally anisotropic cosmological solutions.

forms when nonmetric QCs are generated as off-diagonal deformations of the ΛCDM model.

In Sect. 5, as the **Obj 5**, we provide explicit examples of how to compute Perelman's thermodynamic variables for nonmetric geometric flows inducing topological QC structures [17,20,21] for modeling DE and DM effects.

Finally, Sect. 6 is devoted to conclusions and discussion of the main results of the paper. Appendix A contains a reformulation of the AFCDM for constructing exact and parametric solutions in $f(Q)$ gravity and nonmetric geometric flow theories. We provide a brief review of nonholonomic 2+2 spacetime splitting and corresponding topological QC structures in Appendix B.

2 Metric-affine spaces and $f(Q)$ gravity in nonholonomic (2+2) variables

In this section, we formulate the four-dimensional, 4-d, $f(Q)$ gravity [10,11] in nonholonomic dyadic variables with canonical distortion of linear connection structures considered other type nonmetric gravity theories in our partner work [15]. Such a nonholonomic geometric formalism was developed in [16–19] but it will be generalized and applied in Sect. 4 for constructing exact and parametric solutions in nonmetric geometric flow and MGTs with generic off-diagonal cosmological metrics and generalized (non) linear connection structures. Appendix A contains a summary of necessary notations and formulas for applications in nonmetric geometric flow and gravity theories of the anholonomic frame and connection deformation method, AFCDM.

2.1 Nonlinear connections and nonholonomic dyadic splitting

Let V be a 4-d Lorentz manifold of necessary smooth class defined by a metric tensor $g=\{g_{\alpha\beta}(u^{\gamma})\}$ of signature $(+,+,+,-)$ and corresponding LC-connection $\nabla=\{\breve{\Gamma}^{\alpha}_{\beta\gamma}(u)\}$.[4] We can endow such a manifold with an independent linear (affine) connection structure $D=\{\Gamma^{\alpha}_{\beta\gamma}(u)\}$ and elaborate on certain physically important metric-affine geometric flow and gravity models determined by geometric data (g,D).

A nonholonomic structure with 2+2 splitting can be defined by local bases, $\mathbf{e}_{\nu}$, and co-bases (dual), $\mathbf{e}^{\mu}$,

$$\mathbf{e}_{\nu}=(\mathbf{e}_i,e_a)=(\mathbf{e}_i=\partial/\partial x^i-\ N_i^a(u)\partial/\partial y^a,\quad e_a=\partial_a=\partial/\partial y^a),\ \text{and} \tag{1}$$

$$\mathbf{e}^{\mu}=(e^i,\mathbf{e}^a)=(e^i=dx^i,\ \mathbf{e}^a=dy^a+N_i^a(u)dx^i). \tag{2}$$

We suppose that in local coordinate form a set of coefficients $N_i^a(u)$ is determined by a nonlinear connection, N-connection structure, $\mathbf{N}=N_i^a(x,y)dx^i\otimes\partial/\partial y^a$, which in global form can be defined on tangent bundle TV as a Whitney sum:

$$\mathbf{N}:\ TV=hV\oplus vV. \tag{3}$$

This states a conventional horizontal and vertical splitting (h- and v–decomposition) into respective 2-d and 2-d subspaces, hV and vV.[5] Decomposing geometric objects (tensors, connections etc.) with respect to N-adapted bases (1) and (2), we formulate a nonholonomic dyadic formalism for metric-affine geometry.

For nonholonomic manifolds enabled with N-connection structure (1), we shall use boldface symbols and write, for instance, $\mathbf{V}$, and consider respective nonholonomic tangent, $T\mathbf{V}$, and cotangent, $T^*\mathbf{V}$, bundles; their tensor products, $T\mathbf{V}\otimes T^*\mathbf{V}$, etc. The geometric objects on such geometric spaces (generalized spacetimes and/or phase spaces) can be N-adapted and written in boldface form as distinguished geometric objects (in brief, d-objects, d-vectors, d-tensors etc). For instance, we can write a d–vector as $\mathbf{X}=(hX,vX)$ and a second rank d-tensor as $\mathbf{F}=(hhF,hvF,vhF,vvF)$. Such dyadic decompositions can be written in N-adapted coefficient forms with respect to N-adapted bases, see details in [7,17–19].

2.2 N-adapted metric-affine structures

Any metric tensor $g\in T^*V\otimes T^*V$ can be written as a d-tensor $\mathbf{g}\in T^*\mathbf{V}\otimes T^*\mathbf{V}$ and parameterized in three equivalent

[4] Local coordinates are labeled as $u^{\alpha}=(u^{\acute{\imath}},u^4=t)=(x^i,y^a)$, for $\acute{\imath}=1,2,3$. We follow the conventions for local frames/ coordinates/ indices used, for instance, in [7,15] that typical 3-d space indices run values of type $\acute{\imath}=1,2,3$; for $u^4=y^4=ct$; the light velocity constant c can be always fixed as $c=1$ for corresponding systems of unities and coordinates; and typical indices of type $i=1,2$ and $a=3,4$ are used for a conventional 2+2 splitting. In brief, we write correspondingly $u=(x,t)=(x,y)$. Arbitrary local frames $e_{\alpha}=e_{\alpha}^{\alpha'}(u)\partial_{\alpha'}$ and (dual) frames, or co-frames, $e^{\beta}=e^{\beta}_{\beta'}(u)d^{\beta'}$ for respective coordinate (co) frames $\partial_{\alpha'}=\partial/\partial u^{\alpha'}$ and $d^{\beta'}=du^{\beta'}$, when matrices $e_{\alpha}^{\alpha'}(u)$ and $e^{\beta}_{\beta'}(u)$ define some tetradic (equivalently, vierbein coefficients). For primed indices, we may use similar conventions with a necessary 3+1 and/or 2+2 splitting. In our constructions, we can underline necessary indices, or drop any priming/underlying if that will not result in ambiguities.

[5] We use the term nonholonomic (equivalently, anholonomic) because, for instance, a N-elongated basis (1) satisfies certain nonholonomy relations $[\mathbf{e}_{\alpha},\mathbf{e}_{\beta}]=\mathbf{e}_{\alpha}\mathbf{e}_{\beta}-\mathbf{e}_{\beta}\mathbf{e}_{\alpha}=W^{\gamma}_{\alpha\beta}\mathbf{e}_{\gamma}$, with nontrivial anholonomy coefficients $W^b_{ia}=\partial_a N_i^b$, $W^a_{ji}=\Omega^a_{ij}=\mathbf{e}_j\left(N_i^a\right)-\mathbf{e}_i(N_j^a)$, where Ω^a_{ij} define the coefficients of N-connection curvature. If all W^b_{ia} are zero for a $\mathbf{e}_{\alpha}$, such a N-adapted base is holonomic and we can write it as a partial derivative ∂_{α} with $N_i^a=0$. The coefficients N_j^a may be nontrivial even all $W^{\gamma}_{\alpha\beta}=0$ (this can be in general curve coordinate bases but we can always define a diagonal holonomic base for a corresponding coordinate transforms).

forms:

$$
\begin{aligned}
g &= g_{\alpha'\beta'}(u) e^{\alpha'} \otimes e^{\beta'}, \\
&\quad \text{with respect to an arbitrary coframe } e^{\alpha'}; \\
&= \mathbf{g} = (hg, vg) = \ g_{ij}(x,y)\ e^i \otimes e^j \\
&\quad + \ g_{ab}(x,y)\ \mathbf{e}^a \otimes \mathbf{e}^b, \\
&\quad \text{in N-adapted form with } hg = \{\ g_{ij}\}, vg = \{g_{ab}\}; \qquad (4)\\
&= \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta, \\
&\quad \text{with respect to a coordinate coframe } du^\beta, \\
&\quad \text{where } \underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix}. \qquad (5)
\end{aligned}
$$

A metric $\mathbf{g} = \{\underline{g}_{\alpha\beta}\}$ is generic off–diagonal (for 4-d spacetimes, such a matrix can't be diagonalized via coordinate transforms) if the anholonomy coefficients $W^\gamma_{\alpha\beta}$ are not identical to zero.

Additionally to the metric structure g, we can consider an independent linear connection structure $D = \{\Gamma^\alpha_{\beta\lambda}\}$ with coefficients defined with respect to arbitrary frames and coframes, e_α and e^β [6]. In general, such a D is not adapted to a N-connection structure and can be a introduced to be independent both from the metric and N-connection structures, see details in [7,18,19].

A distinguished connection **(d–connection), D** $= (hD, vD)$, is defined as a linear connection preserving under parallelism the N–connection splitting (3). With respect to frames (1) and (2), we can write decompositions of **D** using h- and v-indices,

$$
\begin{aligned}
&\mathbf{D} = \{\mathbf{\Gamma}^\gamma_{\alpha\beta} = (L^i_{jk}, \acute{L}^a_{bk}; \acute{C}^i_{jc}, C^a_{bc})\}, \text{ where} \\
&hD = (L^i_{jk}, \acute{L}^a_{bk}) \text{ and } vD = (\acute{C}^i_{jc}, C^a_{bc}). \qquad (6)
\end{aligned}
$$

If a general metric-affine space is defined by geometric structures (V, g, D), a nonholonomic metric-affine space can be defined N-adapted form by geometric data $(\mathbf{V}, \mathbf{N}, \mathbf{g}, \mathbf{D})$.

The fundamental geometric objects of nonholonomic metric-affine space are defined and computed in standard form as for any linear connection. For a d-connection **D,** we have

$$
\begin{aligned}
&\mathcal{T}(\mathbf{X},\mathbf{Y}) := \mathbf{D_X Y} - \mathbf{D_Y X} - [\mathbf{X},\mathbf{Y}], \\
&\qquad \text{the torsion d-tensor, d-torsion;} \\
&\mathcal{R}(\mathbf{X},\mathbf{Y}) := \mathbf{D_X D_Y} - \mathbf{D_Y D_X} - \mathbf{D}_{[\mathbf{X},\mathbf{Y}]}, \\
&\qquad \text{the curvature d-tensor, d-curvature;} \\
&\mathcal{Q}(\mathbf{X}) := \mathbf{D_X g}, \\
&\qquad \text{the nonmetricity d-field, d-nonmetricity.} \qquad (7)
\end{aligned}
$$

Such geometric d-objects can be written in nonholonomic dyadic form with respect to N-adapted frames (1) and (2) with such parameterizations:

$$\mathcal{R} = \{\mathbf{R}^\alpha_{\beta\gamma\delta}\}, \mathcal{T} = \{\mathbf{T}^\gamma_{\alpha\beta}\}, \mathcal{Q} = \{\mathbf{Q}_{\gamma\alpha\beta}\}. \qquad (8)$$

We note that in various metric-affine gravity theories [6,8–11] the definitions and coefficient formulas for respective $R^\alpha_{\beta\gamma\delta}, T^\alpha_{\beta\lambda}$, and $Q_{\alpha\beta\gamma}$ are provided/computed for arbitrary frame/ coordinate decompositions for an affine connection $\Gamma^\alpha_{\beta\lambda}$ not considering N-adapted geometric constructions.

In geometric flow and gravity theories (this work and in [7,15], there are involved nontrivial nonmetricity d-tensors), there are used also another important geometric d-objects:

$$
\begin{aligned}
&\mathbf{R}ic = \{\mathbf{R}_{\beta\gamma} := \mathbf{R}^\alpha_{\beta\gamma\alpha}\}, \text{ the Ricci d-tensor;} \qquad (9)\\
&\mathbf{R}sc = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta}, \text{ the scalar curvature,} \qquad (10)
\end{aligned}
$$

where $\mathbf{g}^{\alpha\beta}$ are the coefficients of the inverse d-tensor of a d-metric (4).

The N-adapted coefficient formulas involving the coefficients (4) and (6) are provided in [7,18,19].

2.3 Canonical metric-affine d-structures, LC-connection, and nonmetricity

For a nonholonomic metric-affine space (**V,N,g,D**), we can construct different geometric and physical models defined by the same metric structure but involving different linear connection structures. Additionally to **D** (not determined by the metric), there are two other important linear connection structures determined by a d-metric **g** (4) following such definitions:

$$
(\mathbf{g}, \mathbf{N}) \rightarrow \begin{cases} \nabla : & \nabla \mathbf{g} = 0;\ {}_\nabla\mathcal{T} = 0, \text{ LC} - \text{connection;} \\ \widehat{\mathbf{D}} : & \widehat{\mathbf{Q}} = 0;\ h\widehat{\mathcal{T}} = 0,\ v\widehat{\mathcal{T}} = 0,\ hv\widehat{\mathcal{T}} \neq 0, \\ & \text{the canonical d-connection.} \end{cases} \qquad (11)
$$

Such an auxiliary d-connection defines a canonical distortion relation,[6]

$$\widehat{\mathbf{D}}[\mathbf{g}] = \nabla[\mathbf{g}] + \widehat{\mathbf{Z}}[\mathbf{g}], \qquad (12)$$

when the canonical distortion d-tensor, $\widehat{\mathbf{Z}}[\mathbf{g}]$, and $\nabla[\mathbf{g}]$ are determined by the same metric structure **g**. In our works, we prefer to work with $\widehat{\mathbf{D}}$ which allow to decouple in some general off-diagonal forms (5) physically important systems of nonlinear PDEs. The LC-connection does not have such a

[6] In our works, "hat" labels are used typically for geometric d-objects defined by $\widehat{\mathbf{D}}$ (11), when N-adapted coefficients are computed [7,18,19]: $\widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^\gamma_{\alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})\}$, for $\widehat{L}^i_{jk} = \frac{1}{2}g^{ir}(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk})$,
$\widehat{L}^a_{bk} = e_b(N^a_k) + \frac{1}{2}g^{ac}(\mathbf{e}_k g_{bc} - g_{dc}\ e_b N^d_k - g_{db}\ e_c N^d_k), \widehat{C}^i_{jc} = \frac{1}{2}g^{ik}e_c g_{jk}, \widehat{C}^a_{bc} = \frac{1}{2}g^{ad}(e_c g_{bd} + e_b g_{cd} - e_d g_{bc})$.

property (excepting some special diagonalizable ansatz), see details in [7,18,19].

A general d-connection $\mathbf{D}$ (6) can be characterized by

$$\mathbf{D} = \nabla + \mathbf{L} = \widehat{\mathbf{D}} + \widehat{\mathbf{L}}, \text{ where } \widehat{\mathbf{L}} = \mathbf{L} - \widehat{\mathbf{Z}} \quad (13)$$

for the disformation d-tensor $\mathbf{L} = \{\mathbf{L}^{\alpha}_{\ \beta\lambda} = \frac{1}{2}(\mathbf{Q}^{\alpha}_{\ \beta\lambda} - \mathbf{Q}^{\alpha}_{\ \beta\lambda} - \mathbf{Q}^{\alpha}_{\lambda\ \beta})\}$ with $\mathbf{Q}_{\alpha\beta\lambda} := \mathbf{D}_{\alpha}\mathbf{g}_{\beta\lambda}$, for $\widehat{\mathbf{Q}}_{\alpha\beta\lambda} = \widehat{\mathbf{D}}_{\alpha}\mathbf{g}_{\beta\lambda} = 0$. Here, we use boldface symbols $\mathbf{Q}$ and $\mathbf{L}$ because any tensor can be transformed into a d-tensor if a N-connection structure is prescribed. This holds true even ∇ is not a d-connection (nevertheless, all constructions can be performed with respect to N-elongated frames).

The two linear connection structure (6) and distortion relations (12) and (13) allow us to define and compute respective distortion relations of such geometric d-objects and, respective, objects:

- Nonmetricity d-vectors and vectors:

$$\mathbf{Q}_{\alpha} = \mathbf{g}^{\beta\lambda}\mathbf{Q}_{\alpha\beta\lambda} = \mathbf{Q}_{\alpha\ \lambda}^{\ \lambda},$$
$$^{\intercal}\mathbf{Q}_{\beta} = \mathbf{g}^{\alpha\lambda}\mathbf{Q}_{\alpha\beta\lambda} = \mathbf{Q}_{\alpha\beta}^{\ \ \ \alpha}; \text{ and, correspondingly,}$$
$$Q_{\alpha} = g^{\beta\lambda}Q_{\alpha\beta\lambda} = Q_{\alpha\ \lambda}^{\ \lambda},\ ^{\intercal}Q_{\beta} = g^{\beta\lambda}Q_{\alpha\beta}^{\ \ \ \alpha} = Q_{\alpha\beta}^{\ \ \ \alpha}, \quad (14)$$

  where there are used N-adapted frames (1) and (2) and, correspondingly, e_{α} and e^{β}.
- Nonmetricity conjugate d-tensor and tensor:

$$\widehat{\mathbf{P}}^{\gamma}_{\ \alpha\beta} = \frac{1}{4}\Big(-2\widehat{\mathbf{L}}^{\gamma}_{\alpha\beta} + \mathbf{Q}^{\gamma}\mathbf{g}_{\alpha\beta} -^{\intercal}\mathbf{Q}^{\gamma}\mathbf{g}_{\alpha\beta} - \frac{1}{2}\delta^{\gamma}_{\alpha}\mathbf{Q}_{\beta} - \frac{1}{2}\delta^{\gamma}_{\beta}\mathbf{Q}_{\alpha}\Big)$$
$$\mathbf{P}^{\gamma}_{\ \alpha\beta} = \frac{1}{4}\Big(-2\mathbf{L}^{\gamma}_{\ \alpha\beta} + \mathbf{Q}^{\gamma}\mathbf{g}_{\alpha\beta} -^{\intercal}\mathbf{Q}^{\gamma}\mathbf{g}_{\alpha\beta} - \frac{1}{2}\delta^{\gamma}_{\alpha}\mathbf{Q}_{\beta} - \frac{1}{2}\delta^{\gamma}_{\beta}\mathbf{Q}_{\alpha}\Big) \text{ and}$$
$$P^{\gamma}_{\ \alpha\beta} = \frac{1}{4}\Big(-2L^{\gamma}_{\ \alpha\beta} + Q^{\gamma}g_{\alpha\beta} -^{\intercal}Q^{\gamma}g_{\alpha\beta} - \frac{1}{2}\delta^{\gamma}_{\alpha}Q_{\beta} - \frac{1}{2}\delta^{\gamma}_{\beta}Q_{\alpha}\Big), \quad (15)$$

  where the formulas from the 3-d line for a general $D = \{\Gamma^{\alpha}_{\ \beta\gamma}(u)\}$ (which can be not a d-connection).
- Nonmetricity scalar for respective d-connection and LC-connection:

$$\widehat{\mathbf{Q}} = -Q_{\alpha\beta\lambda}\widehat{\mathbf{P}}^{\alpha\beta\lambda},\ Q = -Q_{\alpha\beta\lambda}\mathbf{P}^{\alpha\beta\lambda} \text{ and}$$
$$Q = -Q_{\alpha\beta\lambda}P^{\alpha\beta\lambda}. \quad (16)$$

  Here we note that, in general, the scalar $\widehat{\mathbf{Q}} \neq 0$ even $\widehat{\mathbf{Q}}_{\alpha\beta\lambda} = 0$ by definition ($\widehat{\mathbf{Q}}$ is not constructing only by contracting components of $\widehat{\mathbf{Q}}_{\alpha\beta\lambda}$).

Above geometric d-objects and objects can be used for elaborating and study models of nonmetric geometric flows and modified gravity theories. The motivation and priority of "hat" variables is that we can decouple and integrate in certain general off-diagonal forme corresponding physically important systems of nonlinear PDEs as we shall prove in next section and Appendix A.

2.4 Curvature, torsion and nonmetricity tensors for nonholonomic metric-affine spaces

A nonholonomic metric-affine space $\mathbf{V}$ with a prescribed metric and d-metric structure, $\mathbf{g} = \{g_{\underline{\alpha}\underline{\beta}}\} = (hg, vg)$, see formulas (4) and (5), is characterized by a corresponding multi-connection structure of type $(\nabla, D, \widehat{\mathbf{D}}, \mathbf{D})$, when the first two linear connections and the next two d-connections are correspondingly non-adapted and adapted to a prescribed N-connection structure $\mathbf{N}$. In result, we can define different types of curvature, Ricci, torsion and nonmetricity tensors and related scalars. This is important for formulating and investigating geometric and physical properties of nonmetric geometric flow and nonmetric gravitational theories. The goal of this subsection is to introduce necessary conventions and define in necessary abstract and N-adapted coefficient forms the above mentioned geometric objects and d-objects.

The fundamental geometric d-objects (7) and, with respective N-adapted coefficients (8), the Ricci d-tensor and curvature scalars (9) and (10) can be defined and computed in standard forms for any nonholonomic metric-affine space $(\mathbf{V},\mathbf{N},\mathbf{g},\mathbf{D} = \nabla + \mathbf{L} = \widehat{\mathbf{D}} + \widehat{\mathbf{L}})$ as we considered in [7,15]. Such d-objects are important for formulating nonmetric geometric theories and nonmetric gravitational equations. The last ones can be defined as a subclass of nonmetric nonholonomic Ricci soliton equations and written in alternative forms with (14), (15) and (16) encoded into certain effective sources and nonholonomic constraints.

The N-adapted and, respective, general frame/coordinate coefficients of the Riemannian d-curvatures and curvature can be labeled and computed as

$$\mathcal{R} = \{\mathbf{R}^{\alpha}_{\ \beta\gamma\delta}\},\ \widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\delta}\} \text{ and } \check{\mathcal{R}} = \{\check{R}^{\alpha}_{\ \beta\gamma\delta}\}. \quad (17)$$

Such values involve the same d-metric/metric structure $\mathbf{g} = (hg, vg) = \{g_{\underline{\alpha}\underline{\beta}}\}$, see parameterizations (4) and (5), but different linear connections. We omit explicit cumbersome formulas with h- and v-indices presented in [7]. It should be noted the coefficients (17) are subjected to certain distortion relations

$$\mathbf{R}^{\alpha}_{\ \beta\gamma\delta} = \check{R}^{\alpha}_{\ \beta\gamma\delta} + \check{\mathbf{Z}}^{\alpha}_{\ \beta\gamma\delta} = \widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\delta} + \widehat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma\delta}, \quad (18)$$

where the coefficients of the distortion d-tensors $\breve{\mathbf{Z}}^{\alpha}_{\beta\gamma\delta}$ and $\widehat{\mathbf{Z}}^{\alpha}_{\beta\gamma\delta}$ can be computed using respective distortion relations (12) and (13).

Contracting indices in (17) and (18), we define and compute the coefficients of respective Ricci d-tensors and Ricci tensor, see formulas (9),

$$\begin{aligned}
&\mathbf{R}ic = \breve{R}ic + \breve{Z}ic = \widehat{\mathbf{R}}ic + \widehat{\mathbf{Z}}ic, \\
&\qquad \text{for respective coefficients} \\
&\mathbf{R}ic = \{\mathbf{R}_{\beta\gamma} = \mathbf{R}^{\alpha}_{\ \beta\gamma\alpha}\};\ \breve{R}ic = \{\breve{R}_{\beta\gamma} = \breve{R}^{\alpha}_{\ \beta\gamma\alpha}\}, \\
&\breve{Z}ic = \{\breve{Z}_{\beta\gamma} = \breve{Z}^{\alpha}_{\ \beta\gamma\alpha}\}; \\
&\widehat{\mathbf{R}}ic = \{\widehat{\mathbf{R}}_{\beta\gamma} = \widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\alpha}\}, \widehat{\mathbf{Z}}ic = \{\widehat{\mathbf{Z}}_{\beta\gamma} := \widehat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma\alpha}\}.
\end{aligned} \tag{19}$$

Then, contracting respectively these formulas with $\mathbf{g}^{\alpha\beta}$, we compute the corresponding scalar curvatures (10),

$$\begin{aligned}
&\mathbf{R}sc = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta} = \breve{R}sc + \breve{Z}sc = \widehat{\mathbf{R}}sc + \widehat{\mathbf{Z}}sc, \ \text{where} \\
&\breve{R}sc = g^{\beta\gamma}\breve{R}_{\beta\gamma}, \breve{Z}sc = g^{\beta\gamma}\breve{Z}_{\beta\gamma}; \\
&\widehat{\mathbf{R}}sc = \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta}, \widehat{\mathbf{Z}}sc = \mathbf{g}^{\alpha\beta}\widehat{\mathbf{Z}}_{\alpha\beta}.
\end{aligned}$$

The nonmetric fundamental d-objects and objects (14), (15), (16) and (17), (18) encode certain nontrivial d-torsion and torison fields. Let us begin with the formulas for a general affine connection $D = \{\Gamma^{\gamma}_{\alpha\beta}\}$ with coefficients stated with respect to arbitrary frame, or coordinate, bases, ∂_α or e_α, and their dual. The fundamental geometric objects (7) are defined by D with coefficients formulas (8)

$$\mathcal{R} = \{\mathbf{R}^{\alpha}_{\ \beta\gamma\delta}\}, \mathcal{T} = \{T^{\gamma}_{\ \alpha\beta}\}, \mathcal{Q} = \{\mathbf{Q}_{\alpha\beta\gamma} := D_\alpha g_{\beta\gamma}\}.$$

Such coefficient formulas are provided in [6,7].

The distortions formulas (12) with distinguishing of coefficients for $\nabla = \{\breve{\Gamma}^{\gamma}_{\alpha\beta}\}$ can be parameterized:

$$\begin{aligned}
\Gamma^{\gamma}_{\ \alpha\beta} &= \breve{\Gamma}^{\gamma}_{\ \alpha\beta} + K^{\gamma}_{\ \alpha\beta} + {}^{q}Z^{\gamma}_{\ \alpha\beta}, \ \text{where} \\
K_{\alpha\beta\gamma} &= \frac{1}{2}(T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} - T_{\gamma\alpha\beta}), \\
S^{\ \alpha\beta}_{\gamma} &= \frac{1}{2}(K^{\alpha\beta}_{\ \ \gamma} + \delta^{\alpha}_{\gamma}T^{\tau\beta}_{\ \ \tau} - \delta^{\beta}_{\gamma}T^{\tau\alpha}_{\ \ \tau}), \\
{}^{q}Z_{\alpha\beta\gamma} &= \frac{1}{2}(Q_{\alpha\beta\gamma} - Q_{\beta\gamma\alpha} - Q_{\gamma\beta\alpha}).
\end{aligned} \tag{20}$$

Such geometric objects can be used for defining (as in formulas (9) and (10)) the Ricci tenor and three scalar values for D considered in the Weyl–Cartan geometry:

$$\begin{aligned}
&Ric[D] = \{R_{\beta\gamma} := R^{\alpha}_{\ \beta\gamma\alpha}\}, \\
&Rsc[D] = R = g^{\beta\gamma}\ R_{\beta\gamma}, {}^{s}T = S^{\ \alpha\beta}_{\gamma}T^{\gamma}_{\ \alpha\beta}, \\
&\qquad Q = {}^{q}Z^{\alpha}_{\ \beta\alpha}\, {}^{q}Z^{\beta\mu}_{\ \ \mu} - {}^{q}Z^{\alpha}_{\ \beta\mu}\, {}^{q}Z^{\beta\mu}_{\ \ \alpha}.
\end{aligned}$$

Here we note that for the LC-connection as $\nabla = \{\breve{\Gamma}^{\gamma}_{\alpha\beta}\}$, when the corresponding tensor and scalar geometric objects are defined satisfy such properties:

$$\begin{aligned}
&Ric[\nabla] = \{\breve{R}_{\beta\gamma} := \breve{R}^{\alpha}_{\ \beta\gamma\alpha}\}, T^{\gamma}_{\ \alpha\beta} \equiv 0 \\
&Rsc[\nabla] = \breve{R} = g^{\beta\gamma}\breve{R}_{\beta\gamma}, {}^{s}T = S^{\ \alpha\beta}_{\gamma}T^{\gamma}_{\ \alpha\beta} \equiv 0, \\
&\qquad Q = {}^{q}Z^{\alpha}_{\ \beta\alpha}\, {}^{q}Z^{\beta\mu}_{\ \ \mu} - {}^{q}Z^{\alpha}_{\ \beta\mu}\, {}^{q}Z^{\beta\mu}_{\ \ \alpha} \equiv 0.
\end{aligned}$$

A d-connection $\mathbf{D} = \{\mathbf{\Gamma}^{\gamma}_{\alpha\beta}\}$ with nontrivial nonmetric d-tensor $\mathbf{Q}_{\gamma\alpha\beta}$ allows to compute the N-adapted coefficients of distortion d-tensors with respect to $\widehat{\mathbf{D}} = \{\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}\}$,

$$\begin{aligned}
\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} &= \widehat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta} + \mathbf{K}^{\gamma}_{\ \alpha\beta} + {}^{q}\widehat{\mathbf{Z}}^{\gamma}_{\ \alpha\beta}, \ \text{where} \\
\mathbf{K}_{\alpha\beta\gamma} &= \frac{1}{2}(\mathbf{T}_{\alpha\beta\gamma} + \mathbf{T}_{\beta\gamma\alpha} - \mathbf{T}_{\gamma\alpha\beta}), \\
\mathbf{S}^{\ \alpha\beta}_{\gamma} &= \frac{1}{2}(\mathbf{K}^{\alpha\beta}_{\ \ \gamma} + \delta^{\alpha}_{\gamma}\mathbf{T}^{\tau\beta}_{\ \ \tau} - \delta^{\beta}_{\gamma}\mathbf{T}^{\tau\alpha}_{\ \ \tau}), \\
{}^{q}\mathbf{Z}_{\alpha\beta\gamma} &= \frac{1}{2}(\mathbf{Q}_{\alpha\beta\gamma} - \mathbf{Q}_{\beta\gamma\alpha} - \mathbf{Q}_{\gamma\beta\alpha}).
\end{aligned} \tag{21}$$

In above formulas, we can compute the canonical distortion $\mathbf{T}^{\gamma}_{\alpha\beta} = \widehat{\mathbf{T}}^{\gamma}_{\alpha\beta} + \widehat{\mathbf{Z}}^{\gamma}_{\alpha\beta}$ and ${}^{q}\mathbf{Z}^{\gamma}_{\alpha\beta} = {}^{q}\widehat{\mathbf{Z}}^{\gamma}_{\alpha\beta}$ involving the disfunction (13) computed for the canonical distortion (12) from $\widehat{\mathbf{D}} = \{\widehat{\Gamma}^{\gamma}_{\alpha\beta}\}$. This defines a nonholonomic Weyl-Cartan geometry when corresponding Ricci scalar, d-torsion scalar and Q-scalars are

$$\begin{aligned}
&\mathbf{R}sc[\mathbf{D}] = {}^{s}R = \mathbf{g}^{\beta\gamma}\ \mathbf{R}_{\beta\gamma},\ {}^{s}\mathbf{T} = \mathbf{S}^{\ \alpha\beta}_{\gamma}\mathbf{T}^{\gamma}_{\ \alpha\beta}, \\
&{}^{q}\mathbf{Q} = {}^{q}\mathbf{Z}^{\alpha}_{\ \beta\alpha}\, {}^{q}\mathbf{Z}^{\beta\mu}_{\ \ \mu} - {}^{q}\mathbf{Z}^{\alpha}_{\ \beta\mu}\, {}^{q}\mathbf{Z}^{\beta\mu}_{\ \ \alpha}.
\end{aligned}$$

In such formulas, we can always separate certain canonical "hat" components, which can be determined by some classes of exact/parametric off-diagonal solutions for physically important nonlinear PDEs when the Q-distortions are encoded in effective sources.

2.5 $f(Q)$ gravity in non-adapted or canonical dyadic variables

In this subsection, we analyze two possibilities to formulate models of nonmetric gravity theory, when 1) the gravitational Lagrange density ${}^{g}\mathcal{L} \approx f(Q)$ is a functional of the nonmetricity scalar Q, or 2) it is a functional ${}^{g}\widehat{\mathcal{L}} \approx \widehat{f}(\widehat{\mathbf{Q}})$ of $\widehat{\mathbf{Q}}$, see formulas (16). The first formulation is considered in the bulk of nonmetricity theories [6,8–11], when physical solutions are constructed for certain diagonal metrics depending on a radial coordinate (quasi-stationary solutions) or on a time like coordinate (cosmological solutions). For the second formulation, it is possible to apply the AFCDM and construct generic off-diagonal solutions as we proved in [7,17–19]. Such a modified geometric method can be applied also for nonmetric geometric flow theories [7,15] (in this work, we extend it for cosmological $\widehat{\mathbf{Q}}$–deformed solutions).

It should be noted that we can formulate theories with gravitational Lagrange density ${}^{g}\mathcal{L} \approx f(\mathbf{Q})$, where $\mathbf{Q}$ is defined by a general d-connection $\mathbf{D}$. Using nonholonomic distributions and distortions of connections, we can re-define the geometric constructions for some effective theories with ${}^{g}\mathcal{L} \approx {}^{g}\widehat{\mathcal{L}} + {}^{e}\widehat{\mathcal{L}}$, where ${}^{e}\widehat{\mathcal{L}}(\widehat{\mathbf{Q}}, ...)$ is a functional of $\widehat{\mathbf{Q}}$ and certain canonical scalars of d-torsion, distortions and disfunctions (parameterized in different forms (12), (13),(20), or (21)). In this work, we study for simplicity models with ${}^{g}\widehat{\mathcal{L}} \approx \widehat{f}(\widehat{\mathbf{Q}})$ when physically important systems of nonlinear PDEs can be decoupled and integrated in certain general forms for generic off-diagonal cosmological metrics and nontrivial nonmetricity. Such classes of locally anisotropic cosmological solutions can be restricted and modified for $f(Q)$ theories by imposing additional nonholonomic constraints; or distorted for more general configurations for $f(\mathbf{Q})$.

2.5.1 Nonmetric $f(Q)$ gravity in non-adapted variables and distortions of LC-connections

Using the Lagrange densities ${}^{g}\mathcal{L} = \frac{1}{2\kappa} f(Q)$, for gravitational constant κ, and the matter Lagrangian ${}^{m}\mathcal{L}(g_{\alpha\beta}, \phi)$, with matter fields ϕ and pseudo-Riemannian measure $\sqrt{|g|}d^4 u = \sqrt{|g_{\alpha\beta}|}d^4 u$, we can construct a nonmetric gravity theory for the action

$$\mathcal{S} = \int \sqrt{|g|}d^4 u({}^{g}\mathcal{L} + {}^{m}\mathcal{L}). \quad (22)$$

By varying $\mathcal{S}$ on $g^{\alpha\beta}$ and $\Gamma^{\gamma}_{\alpha\beta}$ for $f_Q := \partial f/\partial Q$, we obtain respective nonmetric gravitational field equations (see [10, 11,31]):

$$\frac{2}{\sqrt{|g|}}\nabla_{\gamma}(\sqrt{|g|} f_Q P^{\gamma}_{\ \alpha\beta}) + \frac{1}{2} f g_{\alpha\beta} + f_Q(P_{\beta\mu\nu} Q_{\alpha}^{\ \mu\nu} - 2P_{\alpha\mu\nu} Q^{\mu\nu}_{\ \ \beta}) = \kappa\, {}^{m}T_{\alpha\beta} \quad (23)$$

$$\text{and } \nabla_{\alpha}\nabla_{\beta}(\sqrt{|g|} f_Q P^{\alpha\beta}_{\ \ \gamma}) = 0. \quad (24)$$

In these formulas, $P^{\gamma}_{\ \alpha\beta}$ and $Q^{\gamma}_{\ \alpha\beta}$ are computed as in formulas (14), (15) and (16). The matter energy-momentum in (23) is defined variationally as

$${}^{m}T_{\alpha\beta} := -\frac{2}{\sqrt{|g|}}\frac{\delta(\sqrt{|g|}\, {}^{m}\mathcal{L})}{\delta g^{\alpha\beta}} = {}^{m}\mathcal{L} g_{\alpha\beta} + 2\frac{\delta({}^{m}\mathcal{L})}{\delta g^{\alpha\beta}}, \quad (25)$$

where the second formula holds if ${}^{m}\mathcal{L}$ does not depend in explicit form on $\Gamma^{\gamma}_{\alpha\beta}$. In this work, we shall study only such models of gravitational and matter field interactions. For cosmological applications, we can use a barotropic perfect fluid approximation when

$${}^{m}T_{\alpha\beta} = (\rho + p)v_{\alpha}v_{\beta} + p g_{\alpha\beta}, \quad (26)$$

with isotropic pressure ρ, energy density ρ and the 4-velocity vector v_{α}. We can generalize such formulas for certain locally anisotropic models with (26) describing locally anisotropic fluid matter, when with respect to N-adapted frames the data (ρ, p, v_{α}) may depend on a temperature/ geometric flow parameter τ and on spacetime coordinates $u = \{u^{\alpha}\}$, parameterized as $(\rho(\tau, u), p(\tau, u), v_{\alpha}(\tau, u))$.

The covariant representation of nonmetric gravitational equations (23) was formulated in [32] in a form using explicitly the Einstein tensor, $\breve{E} := \breve{R}ic - \frac{1}{2} g \breve{R}sc$, for ∇. There were studied also possible physical implications of some equivalent effective Einstein equations with effective energy-momentum tensor which may describe dark energy [11,32].[7] Such effective gravitational field equations can be written in the form

$$\breve{E}_{\alpha\beta} = \frac{\kappa}{f_Q}\, {}^{m}T_{\alpha\beta} + {}^{DE}T_{\alpha\beta}, \text{ where} \quad (27)$$

$${}^{DE}T_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta}\left(\frac{f}{f_Q} - Q\right) + 2\frac{f_{QQ}}{f_Q}\nabla_{\gamma}\left(Q P^{\gamma}_{\ \alpha\beta}\right). \quad (28)$$

The nonmetric modifications of GR are encoded into the effective energy-momentum tensor ${}^{DE}T_{\alpha\beta}$ (28). We note that the left label DE is used because it is considered that nonmetricity can be used to describe dark energy effects on modern cosmology (there are hundred of such works, see discussion and references in [6,8–11,32]). In our approach, we argue that additionally various DE and DM effects can be modelled also by generic off-diagonal terms of metrics for exact/parametric solutions in GR or MGTs and this holds true for theories of nonmetric geometric flows and various metric-affine gravity theories [7,15]. Another important remark is that we have to assume $f_Q < 0$ if we consider modified gravitational field equations (23) or (27) as Q-deformations of the standard Einstein equations in GR.

Let us discuss the properties of the connection field equations (27) studied in [11,32]: Such systems of nonlinear PDEs are trivially satisfied in a model-independent manner for various classes of geometric constructions and solutions [11,33]. For instance, we can consider the conditions $R^{\alpha}_{\ \beta\gamma\delta} = 0$, but $\breve{R}^{\alpha}_{\ \beta\gamma\delta} \neq 0$. This consists a class of restrictions for developing the $f(Q)$-theory. The term "coincident gauge" refers to such circumstances when there are certain coordinate frames for which $\Gamma^{\gamma}_{\alpha\beta} = 0$, but $\breve{\Gamma}^{\gamma}_{\alpha\beta} \neq 0$, and we can concentrate our sole attention to find physically important solutions, for instance, of the Q-modified Einstein equations (27). To study more general classes of metric-affine gravity and nonmetric geometric flow theories [15] is important to "relax" the conditions $R^{\alpha}_{\ \beta\gamma\delta} = 0$, considering non-zero values. In next subsec-

[7] We note that in this work we follow a different system of notations and chose an opposite sign before ${}^{m}T_{\alpha\beta}$ in (23) and (25).

tion, we analyze similar conditions in terms of the canonical d-connection structure $\widehat{\mathbf{D}}$ with canonical distortions (21).

2.5.2 Nonmetric $f(\widehat{Q})$ gravity in canonical nonholonomic variables

The systems of nonlinear PDEs (23) and (27) can not decoupled and integrated in some general forms for generic off-diagonal metrics formulated in arbitrary frames and coordinates. To generate exact and parametric solutions for such nonlinear systems of equations is possible if the geometric constructions are performed of nonholonomic dyadic variables introduced in Sect. 2.1 and using distortions from $\widehat{\mathbf{D}}$ of necessary type (non) linear connection structures. In such a N-adapted approach, we write the Lagrange density as ${}^{g}\widehat{\mathcal{L}} = \frac{1}{2\kappa}\widehat{f}(Q)$ and use the measure $\sqrt{|\mathbf{g}_{\alpha\beta}|}\delta^4 u$, where $\delta^4 u = du^1 du^2 \delta u^3 \delta u^4$ for $\delta u^a = \mathbf{e}^a$ being N-elongated differentials of type (2). We construct nonmetric nonholonomic gravity theories in canonical N-adapted variables for the action

$$\widehat{\mathcal{S}} = \int \sqrt{|\mathbf{g}_{\alpha\beta}|}\delta^4 u({}^{g}\widehat{\mathcal{L}} + {}^{m}\widehat{\mathcal{L}}), \tag{29}$$

where hats state that all Lagrange densities and geometric objects are written in nonholonomic dyadic form, with boldface indices and using $\widehat{\mathbf{D}}$. We can perform for $\widehat{\mathcal{S}}$ the same variational procedure as in previous subsection but in N-adapted form, on $\mathbf{g}^{\alpha\beta}$, and on $\mathbf{D} = \{\mathbf{\Gamma}^{\gamma}_{\alpha\beta}\}$ with distortions from $\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}$ as in (21), for $\widehat{f_Q} := \partial \widehat{f}/\partial \widehat{Q}$. If we chose $\widehat{\mathcal{S}} = \mathcal{S}$ with a $\mathcal{S}$ of type (22), we obtain equivalent classes of nonmetric gravity theories but formulated for different types of nonholonomic or holonomic variables.

So, following a N-adapted variational procedure, or applying "pure" geometric methods [2] with distoritions of geometric objects [7,15,17–19], we obtain such nonmetric gravitational field equations defined in nonholonomic canonical diadic variables

$$\frac{2}{\sqrt{|\mathbf{g}|}}\widehat{\mathbf{D}}_{\gamma}(\sqrt{|g|}\widehat{f_Q}\widehat{\mathbf{P}}^{\gamma}{}_{\alpha\beta}) + \frac{1}{2}\widehat{f}\mathbf{g}_{\alpha\beta} + \widehat{f_Q}(\widehat{\mathbf{P}}_{\beta\mu\nu}\mathbf{Q}_{\alpha}{}^{\mu\nu} - 2\widehat{\mathbf{P}}_{\alpha\mu\nu}\mathbf{Q}^{\mu\nu}{}_{\beta}) = \kappa\widehat{\mathbf{T}}_{\alpha\beta} \tag{30}$$

$$\text{and } \widehat{\mathbf{D}}_{\alpha}\widehat{\mathbf{D}}_{\beta}(\sqrt{|\mathbf{g}|}\widehat{f_Q}\widehat{\mathbf{P}}^{\alpha\beta}{}_{\gamma}) = 0. \tag{31}$$

In these formulas, $\widehat{\mathbf{P}}^{\gamma}{}_{\alpha\beta}$ and $\mathbf{Q}^{\gamma}{}_{\alpha\beta}$ are defined as in formulas (14), (15) and (16), when $\widehat{\mathbf{T}}_{\alpha\beta}$ are defined by N-adapted variations as in (25), or in the form (26), but with respect to N-elongated frames (1) and (2).

The covariant representation of (30) can be formulated by analogy to the formulas $\check{E}$ as in previous subsection but using explicitly the Einstein tensor, $\widehat{\mathbf{E}} := \widehat{\mathbf{R}}ic - \frac{1}{2}\mathbf{g}\widehat{\mathbf{R}}sc$, for $\widehat{\mathbf{D}}$. Here we consider a form of nonmetric gravitational equations with $\widehat{\mathbf{R}}ic = \{\widehat{\mathbf{R}}_{\alpha\beta}\}$ in the left side and certain effective sources $\widehat{\Upsilon}ic = \{\widehat{\Upsilon}_{\alpha\beta}\}$ in the right side. Such a representation is convenient for applying the AFCDM for constructing off-diagonal solutions, see details in Appendix A. Distorting the equations (27), for $\nabla \rightarrow \widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$ (12) and parameterizations (21), or redefining in N-adapted form the variation calculus,[8] we formulate the effective source (encoding both nonmetric geometric distortions and matter fields) in the form

$$\widehat{\mathbf{R}}_{\alpha\beta} = \widehat{\Upsilon}_{\alpha\beta}, \text{ for} \tag{32}$$

$$\widehat{\Upsilon}_{\alpha\beta} = {}^{e}\widehat{\mathbf{Y}}_{\alpha\beta} + {}^{m}\widehat{\mathbf{Y}}_{\alpha\beta}. \tag{33}$$

The source $\widehat{\Upsilon}_{\alpha\beta}$ (33) is defined by two d-tenors: 1) ${}^{e}\widehat{\mathbf{Y}}_{\alpha\beta} = \check{Z}ic_{\alpha\beta} - \widehat{\mathbf{Z}}ic_{\alpha\beta}$ is of geometric distorting nature, which can be computed in explicit form following formulas (12), (20) and (21) for (19). 2) The energy-momentum d-tensor ${}^{m}\widehat{\mathbf{Y}}_{\alpha\beta}$ of the matter fields in (33) encodes the nonmetricity scalar $\widehat{\mathbf{Q}}$ and d-tensor $\widehat{\mathbf{P}}^{\gamma}{}_{\alpha\beta}$ defined respectively by formulas (14), (15) and (16).[9]

The properties of the d-connection field equations (31) with $\widehat{\mathbf{D}}$ are different from the connection field equations (27) with ∇. Our goal is to formulate some well-defined conditions when such systems of nonlinear PDEs are trivially satisfied (for certain classes of models and their off-diagonal solutions) nonmetric nonholonomic geometric flow nonmetric gravitational field equations and solutions generalizing the constructions from [11,33]; see also the end of previous subsection. In N-adapted form, we can consider the conditions 1] $\mathbf{R}^{\alpha}_{\beta\gamma\delta} = 0$, but when $\check{R}^{\alpha}_{\beta\gamma\delta} \neq 0$ and $\widehat{\mathbf{R}}^{\alpha}_{\beta\gamma\delta} \neq 0$. We can always impose such constraints and construct different classes of $f(Q)$ or $\widehat{f}(\widehat{Q})$ gravity theories. The term canonical nonholonomic "coincident gauge" can be used for certain circumstances when in N-adapted form there are certain coordinate frames for which $\mathbf{D} = \{\Gamma^{\gamma}_{\alpha\beta} = 0\}$, but $\check{\Gamma}^{\gamma}_{\alpha\beta} \neq 0$ and/or $\widehat{\Gamma}^{\gamma}_{\alpha\beta} \neq 0$. 2] In a more general context, we can con-

[8] Certain physical implications of some equivalent effective Einstein equations were studied [11,32]

[9] To keep certain compatibility with the Einstein equations in GR for zero distortions and zero nonmetricity, we consider distortions of Ricci d-tensors and tensors related by formulas (19) with $\mathbf{R}ic = \check{R}ic + \check{Z}ic = \widehat{\mathbf{R}}ic + \widehat{\mathbf{Z}}ic$. For such distortions, the nonmetric modified Einstein equations (27) can be written in the form

$$\check{E}_{\alpha\beta} = \frac{\kappa}{f_Q}{}^{m}T_{\alpha\beta} + {}^{DE}T_{\alpha\beta} = \kappa\check{T}_{\alpha\beta}, \text{ or } \check{R}_{\alpha\beta} = \check{\Upsilon}_{\alpha\beta}, \text{ for}$$

$$\check{\Upsilon}_{\alpha\beta} = \kappa(\check{T}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\check{T}), \text{ for } \check{T} = g^{\alpha\beta}\check{T}_{\alpha\beta}.$$

The corresponding distortions of the Ricci d-tensors and Ricci tensor can be computed following formulas

$$\check{R}ic_{\alpha\beta} + \check{Z}ic_{\alpha\beta} = \widehat{\mathbf{R}}ic_{\alpha\beta} + \widehat{\mathbf{Z}}ic_{\alpha\beta} = \check{\Upsilon}_{\alpha\beta} + \check{Z}ic_{\alpha\beta};$$
$$\widehat{\mathbf{R}}ic_{\alpha\beta} + \widehat{\mathbf{Z}}ic_{\alpha\beta} = \check{\Upsilon}_{\alpha\beta} + \check{Z}ic_{\alpha\beta},$$
$$\widehat{\mathbf{R}}ic_{\alpha\beta} = \check{Z}ic_{\alpha\beta} - \widehat{\mathbf{Z}}ic_{\alpha\beta} + \check{\Upsilon}_{\alpha\beta}.$$

In canonical nonholonomic dyadic variables and for $\nabla \rightarrow \widehat{\mathbf{D}}$, above formulas transform respectively into those for (32) and (33).

sider nonmetric phase space determined by conditions

$$\widehat{\mathbf{D}}_{\beta}(\sqrt{|\mathbf{g}|}\widehat{f_Q}\widehat{\mathbf{P}}^{\alpha\beta}_{\gamma}) = const. \quad (34)$$

This imposes some nonholonomic covariant and N-adapted constraints on the class of affine d-connections $\mathbf{D} = \{\Gamma^{\gamma}_{\alpha\beta}\}$ and respective nonmetric d-tensors $\mathbf{Q}_{\alpha\beta\gamma}$. This way (considering assumptions of type 1] or 2]), we can study various types of nonmetric gravity theories but constrained in some forms which allow to construct general classes of exact/ parametric solutions when concentrating on physically important solutions of nonmetric modified Einstein equations (32).

Finally, we note that in Sect. 4 and Appendix A, we construct and study off-diagonal solutions of the system of nonlinear PDEs (32) considering that certain restrictions on the classes of integration functions and generating functions and sources imposed on (33) allow also to satisfy the canonical d-connection constraints (34). In general, such solutions with $\widehat{\mathbf{D}}$ do not consist of examples of solutions of the system (27) with ∇. But imposing additional nonholonomic constraints of type $\widehat{\mathbf{D}}_{|\mathcal{T}=0} = \nabla$, when the canonical nonholonomic torsion induced by N-coefficients become zero, $\widehat{\mathbf{T}}^{\gamma}_{\alpha\beta}$, we can extract exact/parametric solutions for (27) or (30). Such conditions can be satisfied by restricting respectively the class of generating functions and generating effective sources as we proved in [15–19]. They modify also the constructions in Appendix A for nontrivial nonmetric cosmological configurations. In this work, we concentrate on locally anisotropic cosmological solutions of (32), and their nonmetric geometric flow evolution in terms of $\widehat{\mathbf{D}}$ considering that we can reduce them in terms of LC-configurations ∇ by imposing additional nonholonomic constraints.

3 Nonmetric geometric flows and $f(Q)$ gravity

One of the main purposes of this work (stated for **Obj 4** in the introduction section) is to construct generic off-diagonal cosmological solutions of nonmetric gravitational equations(32) and analyze possible implications of such solutions in modern cosmology and DM and DE physics. Various classes of physically important such solutions were considered for nonholonomic MGTs [16–19] and, in nonmetric form, for quasi-stationary configurations, [15]. Unfortunately, to investigating physical properties of off-diagonal metrics and nonmetric deformations of connection is not possible if we apply only the concept of Bekenstein–Hawking thermodynamics [34–37] which works effectively for certain classes of solutions involving some hypersurface, duality conditions, or holographic configurations. We can apply advanced nonholonomic geometric methods involving relativistic and nonmetric generalizations [15] of the theory of geometric flows [22,23] and applications in GR and MGTs [29]. In such cases, we are able to define and compute another type, and more general, thermodynamic variables using the concept of W-entropy [22] which can generalized for nonmetric theories and corresponding quasi-stationary solutions [15]. In this work, we elaborate on geometric and gravitational models that are related to the nonmetric effective sources ${}^{e}\widehat{\mathbf{Y}}_{\alpha\beta}$ (33) and the canonical Ricci d-tensor $\widehat{\mathbf{R}}ic$ (19) describing off-diagonal cosmological solutions with topological QC structure. In this section, a model of nonmetric flows is elaborated for canonical deformations of $f(Q)$-gravity.

3.1 Nonmetric geometric flow equations in canonical dyadic variables

We elaborate on the theory of nonmetric geometric flows in canonical dyadic variables and families of geometric and physical data

$$(\mathbf{V},\mathbf{N}(\tau), \mathbf{g}(\tau), \mathbf{D}(\tau) = \nabla(\tau) + \mathbf{L}(\tau) = \widehat{\mathbf{D}}(\tau) + \widehat{\mathbf{L}}(\tau)\, {}^{g}\widehat{\mathcal{L}}(\tau) + {}^{m}\widehat{\mathcal{L}}(\tau)) \quad (35)$$

parameterized by a real parameter $\tau, 0 \leq \tau \leq \tau_1$. The geometric and physical objects for such theories depend, in general, on spacetime coordinates. To simplify our notations, we shall write (for instance) $\mathbf{g}(\tau)$ instead of $\mathbf{g}(\tau, u^{\beta})$ if that will not result in ambiguities. For any fixed value $\tau = \tau_0$, such data define a nonmetric gravity theory when the geometric d-objects are subjected to the conditions to define solutions of the nonlinear system of PDEs (32) and (34) with effective sources (33) as we considered in previous section. According to G. Perelman, τ can be treated as a temperature like parameter [22]. Such an interpretation and corresponding statistical thermodynamical formulations of the geometric flow theories can be considered also for relativistic and modified background manifolds.

3.1.1 Q-distorted R. Hamilton and D. Friedan equations

The nonmetric geometric flow evolution equations can be postulated in canonical dyadic variables in the form:

$$\partial_{\tau} g_{ij}(\tau) = -2[\widehat{\mathbf{R}}_{ij}(\tau) - \widehat{\yen}_{ij}(\tau)];$$
$$\partial_{\tau} g_{ab}(\tau) = -2[\widehat{\mathbf{R}}_{ab}(\tau) - \widehat{\yen}_{ab}(\tau)];$$
$$\widehat{\mathbf{R}}_{ia}(\tau) = \widehat{\mathbf{R}}_{ai}(\tau) = 0; \widehat{\mathbf{R}}_{ij}(\tau) = \widehat{\mathbf{R}}_{ji}(\tau); \widehat{\mathbf{R}}_{ab}(\tau) = \widehat{\mathbf{R}}_{ba}(\tau). \quad (36)$$

The equations (36) are written in N-adapted frames, where $\widehat{\square}(\tau) = \widehat{\mathbf{D}}^{\alpha}(\tau)\widehat{\mathbf{D}}_{\alpha}(\tau)$ is the canonical d'Alambert operator. The conditions $\widehat{\mathbf{R}}_{ia}(\tau) = \widehat{\mathbf{R}}_{ai}(\tau) = 0$ are imposed for $\widehat{Ric}[\widehat{\mathbf{D}}] = \{\widehat{\mathbf{R}}_{\alpha\beta} = [\widehat{R}_{ij}, \widehat{R}_{ia}, \widehat{R}_{ai}, \widehat{R}_{ab}]\}$ if we elaborate on a theory with symmetric d-metrics evolving under nonmetric nonholonomic Ricci flows. Such constraints are not considered, for instance, in nonassociative geometric flow theories with nonsymmetric metrics [29]. Systems of nonlinear PDEs of similar type are studied in [15] for models of nonmetric geometric flows related to $f(R, T, Q, T_m)$ theories of gravity. In this paper, the Q-deformations, distorting relations (21) and sources $\widehat{\yen}_{\alpha\beta}$ (33) are different from the sources studied in that work.

The geometric flow equations (36) consist of certain generalizations of the R. Hamilton equations [23] postulated for ∇. Here we note that equivalent equations were considered a few years before the mentioned mathematical works (by D. Friedan who was inspired by research on string theory and condensed matter physics [24,25]). We can derive nonmetric variants of geometric flow equations considering an approach which is similar to the abstract geometric method from [2] when certain τ-running fundamental geometric objects Ricci tensors and generalized sources are distorted to canonical nonholonomic data (35). In Sect. 4 and Appendix A, we prove that such systems of nonlinear PDEs are well-defined because they can be decoupled and integrated in certain general forms and certain classes of solutions describe physically important processes. Here we note that it is not possible to formulate and prove some general forms of nonmetric Thurston–Poincaré conjectures, as it was considered in [22], because there are an infinite number of non-Riemannian geometries when various topological and geometric analysis constructions are possible, being more sophisticate and not unique. Nevertheless, we can study certain physically important implications of nonmetric geometric flow and nonmetric gravity theories if we are able to find exact/parametric solutions of systems of nonlinear PDEs of type (36). For the classes of solutions with topological QC structures as in Appendix B, the approach involve nontrivial topological configurations.

3.1.2 *τ-Running Einstein equations, nonmetric geometric flows, and f(Q) Ricci solitons*

We can consider the term $\partial_\tau \mathbf{g}_{\mu'\nu'}(\tau)$ as an additional effective source defining τ-running of geometric flows of Ricci d-tensors. Using τ-families of vierbein transforms $\mathbf{e}^{\mu}_{\mu'}(\tau) = \mathbf{e}^{\mu}_{\mu'}(\tau, u^{\gamma})$ and their dual transform $\mathbf{e}_{\nu}{}^{\nu'}(\tau, u^{\gamma})$ with $\mathbf{e}^{\mu}(\tau) = \mathbf{e}^{\mu}_{\mu'}(\tau) du^{\mu'}$, we can introduce N-adapted effective sources

$$
{}^{\intercal}\yen^{\mu}_{\nu}(\tau) = \mathbf{e}^{\mu'}_{\mu}(\tau)\mathbf{e}_{\nu}{}^{\nu'}(\tau)\left[\yen_{\mu'\nu'}(\tau) - \frac{1}{2}\,\partial_\tau \mathbf{g}_{\mu'\nu'}(\tau)\right]
= \left[{}_{h}\yen(\tau, x^k)\delta^i_j,\ \yen(\tau, x^k, y^a)\delta^a_b\right]. \quad (37)
$$

The data ${}^{\intercal}\yen = [{}_{h}\yen,\ \yen]$ can be fixed as some generating functions for effective matter sources encoding contributions both by geometric flows and Q-deformations. Prescribing explicit values of ${}_{h}\yen$ and $\yen$, we impose certain nonholonomic constraints on the noncommutative geometric flow scenarios. It is not possible to solve in general form systems of PDEs of type $\partial_\tau \mathbf{g}_{\mu'\nu'} = 2(\yen_{\mu'\nu'} - {}^{\intercal}\yen_{\mu'\nu'})$, for prescribed generating sources in (37) and general effective distortions and matter energy-momentum. We can search for solutions with small parameters resulting in recurrent formulas for powers on such parameters. In a different approach, certain approximations allow to model relativistic models of geometric locally anisotropic geometric diffusion or nonlinear wave evolution. For applications to modern cosmology, we can consider that generating sources of type $[{}_{h}\yen, \yen]$ impose certain restrictions on the geometric evolution and dynamics of sources modified by Q-deformations.

Using the generating sources ${}^{\intercal}\yen^{\mu}_{\nu}(\tau)$ (37), we can write the nonmetric geometric flow equations (36) as τ-running and Q-deformed Einstein equations for $\widehat{\mathbf{D}}^{\alpha}(\tau)$,

$$
\widehat{\mathbf{R}}^{\alpha}{}_{\beta}(\tau) = {}^{\intercal}\yen^{\alpha}{}_{\beta}(\tau). \quad (38)
$$

Constraining nonholonomically this system for zero canonical d-connections, we model nonmetric τ-evolution scenarios in terms of LC-data:

$$
\widehat{\mathbf{T}}^{\gamma}_{\alpha\beta}(\tau) = 0,\ \text{for}\ \widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0}(\tau) = \nabla(\tau),\ \text{when} \quad (39)
$$

$$
\check{R}_{\beta\gamma}(\tau) = {}^{\intercal}\yen_{\beta\gamma}(\tau). \quad (40)
$$

Such systems of nonlinear PDEs define τ-running generalizations of Q-modified Einstein equations (27). Here we note that the torsion of the related affine connections D subjected to conditions (24) may be nontrivial and that ${}^{\intercal}\yen_{\beta\gamma}(\tau)$ encodes Q-deformations.

In [15], we defined nonholonomic and nonmetric Ricci soliton configurations is a self-similar one for the corresponding nonmetric geometric flow equations. Fixing $\tau = \tau_0$ in (36), we obtain the equations for the $\widehat{f}(\widehat{Q})$ Ricci solitons:

$$
\begin{aligned}
&\widehat{\mathbf{R}}_{ij} = {}^{\intercal}\yen_{ij},\ \widehat{\mathbf{R}}_{ab} = {}^{\intercal}\yen,\\
&\widehat{\mathbf{R}}_{ia} = \widehat{\mathbf{R}}_{ai} = 0; \widehat{\mathbf{R}}_{ij} = \widehat{\mathbf{R}}_{ji}; \widehat{\mathbf{R}}_{ab} = \widehat{\mathbf{R}}_{ba}.
\end{aligned} \quad (41)
$$

The nonholonomic variables can be chosen in such forms that (41) are equivalent to nonmetric modified Einstein equations (32) for $\widehat{\mathbf{D}}^{\alpha}(\tau_0)$.[10] For additional nonholonomic constraints, such equations define solutions of the nonmetric Einstein equations (27) for $\nabla(\tau_0)$.

Let us discus the issue of formulating conservation laws for Q-deformed systems. Using the canonical d-connection, we can check that for systems of type (41) and related nonholonomic modified Einstein equations there are satisfied the

[10] In various MGTs and GR, as in the theory of Ricci flows of Riemannian metrics, the Einstein spaces and certain nonholonomic deformations consist examples of (nonholonomic) Ricci solitons.

conditions

$$\widehat{\mathbf{D}}^{\beta}\widehat{\mathbf{E}}^{\alpha}_{\ \beta}=\widehat{\mathbf{D}}\left(\widehat{\mathbf{R}}^{\alpha}_{\ \beta}-\frac{1}{2}\,{}^{s}\widehat{R}\delta^{\alpha}_{\ \beta}\right)\neq 0 \text{ and } \widehat{\mathbf{D}}^{\beta}\ {}^{\intercal}\yen^{\alpha}_{\ \beta}\neq 0.$$

This is typical for nonholonomic systems and the issue of formulating conservation laws becomes more sophisticate because of nonmetricities. Similar problems exist also in nonholonomic mechanics when the conservation laws are formulated by solving nonholonomic constraints or introducing some Lagrange multiples associated to certain classes of nonholonomic constraints. Solving the constraint equations (they may depend on local coordinates, velocities or momentum variables), we can re-define the variables and formulate conservation laws in some explicit or non-explicit forms. Such nonholonomic variables allow us to introduce new effective (mechanical) Lagrangians and Hamiltonians. This allow us to define conservation laws in certain standard form if $\mathbf{Q}_{\alpha\beta\gamma}=0$. In some general forms and for nonmetric gravity theories, such constructions can be performed only for some special classes of solutions and corresponding geometric flow/ nonholonomic Ricci soliton models. Using distortions of connections, we can rewrite systems of type (32), (27), (41) etc. in terms of ∇, when $\nabla^{\beta}E^{\alpha}_{\ \beta}=\nabla^{\beta}T^{\alpha}_{\ \beta}=0$ for $\mathbf{Q}_{\alpha\beta\gamma}\rightarrow 0$ as in GR. Here we note that we shall formulate in next subsection a statistical thermodynamic energy type variable which is related to G. Perelman's paradigm for Ricci flows, which may encode also nonmetric contributions.

In Appendix A, we show how using the AFCDM the equations (32) and, if necessary, (27) with nonholonomic constraints (40) can be decoupled and integrated in general locally anisotropic cosmological forms for certain prescribed nonmetric effective sources (37). Fixing the running parameter, such (in general, off-diagonal) solutions with nonmetricity describe certain nonmetric cosmological spacetimes described by systems of nonlinear PDEs (32) and (41).

3.2 Statistical geometric thermodynamics for $f(Q)$ geometric flows

Perelman [22] introduced the so-called F- and W-functionals as certain Lyapunov type functionals, $\breve{F}(\tau,g,\nabla,\breve{R}sc)$ and $\breve{W}(\tau,g,\nabla,\breve{R}sc)$ depending on a temperature like parameter τ and fundamental geometric objects when $\breve{W}$ have properties of "minus entropy". Using $\breve{F}$ or $\breve{W}$, he elaborated a variational proof for geometric flow equations of Riemannian metrics, which was applied to developing a strategy for proving the Poincaré–Thorston conjecture. Respective details are provided in mathematical monographs [26–28]. It is not possible to formulate and prove in some general forms such conjectures for non-Riemannian geometries.

G. Perelman's geometric and analytic methods have a number of perspectives in modern particle physics, gravity and cosmology, see reviews of results in [16,18,29]. The main point is that using relativistic generalizations and various modifications of $\breve{W}$ we can formulate statistical and geometric thermodynamic models which are more general and very different from the constructions performed in the framework of the Bekenstein–Hawking paradigm [34–37]. The geometric flow constructions and related statistical thermodynamic approach can be generalized for various supersymmetric/noncommutative/nonassociative geometries etc. In [15], such constructions were extended in nonholonomic form for nonmetric geometric flow theories related to $f(R,T,Q,T_m)$ gravity and applied for investigating important physical properties of quasi-stationary solutions in such theories. To develop such geometric methods for the case of $f(Q)$ gravity when the nonmetric gravitational equations of type (23) involve directly the geometric objects $P^{\gamma}_{\alpha\beta}$ and $Q^{\gamma}_{\alpha\beta}$ it is convenient because the geometric flows are usually defined as Ricci flows. It is more appropriate to study models when the $f(Q)$ nonmetric gravitational equations are of type (27) and then to distort the constructions to include nonholonomic Ricci soliton equations (41). So, the goal of the next subsections is to formulate a nonmetric geometric thermodynamic model for nonmetric geometric flows of canonical d-connections in nonholonomic dyadic variables for τ-running modified Einstein equations (32).

3.2.1 Nonholonomic F-/ W-functionals for Q-deformed geometric flows and gravity

We postulate the modified Perelman's functionals following the same principles as in [15,16,18,22,26–29] but for nonmetric geometric flows involving canonical nonholonomic geometric objects and $\widehat{f}(\widehat{Q})$ distortions:

$$\widehat{\mathcal{F}}(\tau)=\int_{t_1}^{t_2}\int_{\Xi_t}e^{-\zeta(\tau)}\sqrt{|\mathbf{g}(\tau)|}\delta^4u[\widehat{f}(\widehat{\mathbf{R}}sc(\tau)) + {}^{e}\mathcal{L}(\tau)+{}^{m}\widehat{\mathcal{L}}(\tau)+|\widehat{\mathbf{D}}(\tau)\,\zeta(\tau)|^2], \quad (42)$$

$$\widehat{\mathcal{W}}(\tau)=\int_{t_1}^{t_2}\int_{\Xi_t}(4\pi\tau)^{-2}\,e^{-\zeta(\tau)}\sqrt{|\mathbf{g}(\tau)|}\delta^4u[\tau(\widehat{f}(\widehat{\mathbf{R}}sc(\tau)) + {}^{e}\mathcal{L}(\tau)+{}^{m}\widehat{\mathcal{L}}(\tau)+|\widehat{\mathbf{D}}(\tau)\zeta(\tau)|^2+\zeta(\tau)-4], \quad (43)$$

In these formulas, the condition $\int_{t_1}^{t_2}\int_{\Xi_t}(4\pi\tau)^{-2}\,e^{-\zeta(\tau)}\sqrt{|\mathbf{g}|}d^4u=1$ is imposed on the normalizing function $\zeta(\tau)=\zeta(\tau,u)$; and τ-families of Lagrange densities ${}^{g}\widehat{\mathcal{L}}+{}^{m}\widehat{\mathcal{L}}$ are considered as in (29).

Performing a N-adapted variational calculus for $\widehat{\mathbf{D}}$ (which is similar to that presented in [22,26–28] but N-adapted to nonmetric geometric structures) and redefining respectively the normalizing functions and nonholonomic distributions, we prove the nonmetric geometric flow equations (36) with an additional constraint equation for $\zeta(\tau)$:

$$\partial_\tau \zeta(\tau) = -\widehat{\square}(\tau)[\zeta(\tau)] + \left|\widehat{\mathbf{D}}^{\,n}\zeta(\tau)\right|^2 - \widehat{f}(\widehat{\mathbf{R}}sc(\tau)) - {}^{e}\mathcal{L}(\tau) - {}^{m}\widehat{\mathcal{L}}(\tau). \quad (44)$$

This equation for $\zeta(\tau)$ involves nonlinear partial differential operators of first and second order and relates τ-families of canonical Ricci scalars $\widehat{\mathbf{R}}sc(\tau)$ and nonmetric sources $\widehat{\yen}_{\alpha\beta}$ (33). Such a nonlinear PDE can't be solved in a general form which do not allow us to study in general forms models of flow evolution of topological configurations determined by arbitrary nonmetric structures and distributions of effective and real matter fields. We can prescribe a topological structure for an off-diagonal metric constructed as an exact/parametric solution of nonholonomic system of nonlinear PDEs (36). In such a case, we can chose a convenient $\zeta(\tau)$ (it can be prescribed to be a constant, or zero) which states certain constrains on nonmetric geometric evolution. Alternatively, we can solve the Eq. (44) in certain parametric forms and then to re-define the constructions for arbitrary systems of reference.

It should be noted here that the functionals $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{W}}$ were postulated in canonical nonholonomic variables in a form which is similar to the original F- and W-functionals [22], which were introduced for 3-d Riemannian τ-flows $(g(\tau), \nabla(\tau))$. In this work, we extend the constructions in $\widehat{f}(\widehat{Q})$-deformed 4-d Lorentz manifolds endowed with additional nonholonomic distributions determined by effective Lagrange densities. We can compute the functionals (42) and (43) for any 3+1 splitting with 3-d closed hypersurface fibrations $\widehat{\Xi}_t$ and considering nonholonomic canonical d-connections and respective geometric variables. Similar computations are provided in [16,18,29] and references therein. The $\widehat{\mathcal{W}}$-functional possess the properties of "minus" entropy if we re-define the normalizing function to "absorb" the contributions from Q-distortions and matter fields. This can be also modelled by choosing corresponding nonholonomic configurations along some causal curves taking values $\widehat{\mathcal{W}}(\tau)$ on $\widehat{\Xi}_t$.

The $\widehat{f}(\widehat{Q})$ modified G. Perelman's functionals can be defined and computed for any solution of nonmetric geometric flow equations (for instance, of type (32)), or for nonmetric Ricci soliton equations (41). They can be used for elaborating thermodynamic models both for nonmetric quasi-periodic configurations as in [15] and for locally anisotropic cosmological solutions encoding nonmetricity data (we shall provide details and analyze explicit examples in next sections).

3.2.2 *Canonical thermodynamic variables for $f(Q)$ theories*

Let us redefine the normalization function $\zeta(\tau) \to \widehat{\zeta}(\tau)$, for

$$\partial_\tau \zeta(\tau) + \widehat{\square}(\tau)[\zeta(\tau)] - \left|\widehat{\mathbf{D}}\zeta(\tau)\right|^2 - {}^{e}\mathcal{L}(\tau) - {}^{m}\widehat{\mathcal{L}}(\tau) = \partial_\tau \widehat{\zeta}(\tau) + \widehat{\square}(\tau)[\widehat{\zeta}(\tau)] - \left|\widehat{\mathbf{D}}\widehat{\zeta}(\tau)\right|^2,$$

when (44) transforms into

$$\partial_\tau \widehat{\zeta}(\tau) = -\widehat{\square}(\tau)[\widehat{\zeta}(\tau)] + \left|\widehat{\mathbf{D}}\widehat{\zeta}(\tau)\right|^2 - \widehat{f}(\widehat{\mathbf{R}}sc(\tau)). \quad (45)$$

In terms of a corresponding integration measure, the W-functional (43) can be written "without" the effective matter source[11]

$$\widehat{\mathcal{W}}(\tau) = \int_{t_1}^{t_2} \int_{\Xi_t} (4\pi\tau)^{-2} e^{-\widehat{\zeta}(\tau)} \sqrt{|\mathbf{g}(\tau)|}\delta^4 u \times [\tau(\widehat{f}(\widehat{\mathbf{R}}sc(\tau)) + |\widehat{\mathbf{D}}(\tau)\widehat{\zeta}(\tau)|^2 + \widehat{\zeta}(\tau) - 4]. \quad (46)$$

On a metric-affine space $\mathcal{M}$ endowed with canonical geometric variables and a nonholonomic (3+1) splitting,[12] we introduce the statistical partition function

$${}^{q}\widehat{Z}(\tau) = \exp\left[\int_{\widehat{\Xi}} [-\widehat{\zeta} + 2]\ (4\pi\tau)^{-2} e^{-\widehat{\zeta}}\ \delta\widehat{\mathcal{V}}(\tau)\right], \quad (47)$$

where the volume element is defined and computed as

$$\delta\widehat{V}(\tau) = \sqrt{|\mathbf{g}(\tau)|}\ dx^1 dx^2 \delta y^3 \delta y^4. \quad (48)$$

A left label q for ${}^{q}\widehat{Z}(\tau)$ is used in order to emphasize that nonmetric Q-contributions are encoded in $\mathbf{g}(\tau)$ considered as a solution of nonassociative geometric flow/ gravitational equations.

Using ${}^{q}\widehat{Z}$ (47) and $\widehat{\mathcal{W}}(\tau)$ (46) and performing the variational procedure in canonical N-adapted variables, on a closed region of $\mathcal{M}$ (which is similar to that provided in section 5 of [22]), we can define and compute respective (canonical geometric and statistical) thermodynamic variables:

$${}^{q}\widehat{\mathcal{E}}(\tau) = -\tau^2 \int_{\widehat{\Xi}} (4\pi\tau)^{-2} \left(\widehat{f}(\widehat{\mathbf{R}}sc) + |\,\widehat{\mathbf{D}}\,\widehat{\zeta}|^2 - \frac{2}{\tau}\right) \times e^{-\widehat{\zeta}}\,\delta\widehat{V}(\tau),$$

$${}^{q}\widehat{S}(\tau) = -\int_{\widehat{\Xi}} (4\pi\tau)^{-2} \left(\tau((\widehat{\mathbf{R}}sc) + |\widehat{\mathbf{D}}\widehat{\zeta}|^2) + \widehat{\zeta} - 4\right) \times e^{-\widehat{\zeta}}\,\delta\widehat{V}(\tau),$$

[11] Nevertheless the effective nonmetric and matter sources are encoded into geometric data if we consider an explicit class of solutions of (38).

[12] Such a conventional splitting is necessary for defining thermodynamic variables; in another turn, a nonholonomic 2+2 decomposition is important for generating off-diagonal solutions.

$$ {}^{q}\widehat{\sigma}(\tau) = 2\,\tau^{4}\int_{\widehat{\Xi}} (4\pi\tau)^{-2}\,|\,\widehat{\mathbf{R}}_{\alpha\beta} + \widehat{\mathbf{D}}_{\alpha}\,\widehat{\mathbf{D}}_{\beta}\widehat{\zeta}_{[1]} - \frac{1}{2\tau}\mathbf{g}_{\alpha\beta}|^{2} e^{-\widehat{\zeta}}\,\delta\widehat{V}(\tau). \quad (49) $$

Such nonmetric geometric thermodynamic variables were defined in [15] but for a different class of metric-affine theories. The fluctuation variable ${}^{q}\widehat{\sigma}(\tau)$ can be written as a functional of $\widehat{\mathbf{R}}_{\alpha\beta}$ even ${}^{q}\widehat{\mathcal{E}}\,(\tau)$ and ${}^{q}\widehat{\mathcal{S}}(\tau)$ are functionals of $\widehat{f}(\widehat{\mathbf{R}}sc)$ if we correspondingly re-define the normalizing functions $\widehat{\zeta} \rightarrow \widehat{\zeta}_{[1]}$. We omit such details in this work because we shall not compute ${}^{q}\widehat{\sigma}(\tau)$ for certain classes of solutions. Fixing the temperature in (49), we can compute thermodynamic variables $\left[\,{}^{q}\widehat{\mathcal{E}}(\tau_0),\ {}^{q}\widehat{\mathcal{S}}(\tau_0),\ {}^{q}\widehat{\sigma}(\tau_0)\right]$ for nonmetric Ricci solitons (41). Certain classes of solutions can be not well-defined in general form if, for instance, ${}^{q}\widehat{\mathcal{S}}(\tau_0) < 0$. We have to restrict some classes of nonholonomic distributions/distortions in order to generate physically viable solutions. In some spacetime regions, the nonmetric Q-deformations may result in un-physical models, but be well-defined for another nonholonomic conditions because of different sign contributions. We have to investigate this in explicit form for corresponding classes of exact/parametric solutions of physically important systems of nonlinear PDEs.

4 Off-diagonal cosmological solutions encoding nonmetric geometric flows

Recent observational data provided by James Webb Space Telescope, JWST, indicate that the Standard Cosmological Model, SCM, could require correcting or even a substantial revision [12–14]. Such modifications can be modelled as different cosmological scenarios determined by (non) metric geometric flow models and MGTs, or GR, with generic off-diagonal cosmological metrics. To be able to compare our nonholonomic approach with predictions of ΛCDM cosmological models involving nonmetricity fields we provide some basic formulas for the SCM. Then, we generate and analyse more general classes of nonmetric geometric flow and cosmological solutions. The main goal of this section is to prove that by applying the AFCDM (see a summary of results and basic references in Appendix A) we can elaborate on more general classes of cosmological theories with nonmetric geometric flows and MGTs with off-diagonal interactions. This opens new possibilities for revising the theory of accelerating Universe and the DE and DM paradigms. As explicit examples, we construct and analyse basic properties of off-diagonal cosmological metrics with topological quasicrystal, QC, structure (main definitions are provided in Appendix B) determined in different cases by quasi-periodic nonmetric geometric flows, effective and real matter sources, and/or off-diagonal components of metrics.

4.1 Diagonal metrics for $f(Q)$ cosmology

We cite [10,31–33] for reviews on $f(Q)$ gravity and cosmology with modified Einstein equations (22), or (23), or (27). For such a nonmetric spacetime, a prime off-diagonal metric $\mathring{\mathbf{g}} = [\mathring{g}_{\alpha},\ \mathring{N}_{i}^{a}]$ (A.15) is written using general curved coordinates $u^{\alpha}(x,y,z,t)$ and considered as a conformal transform,

$$ \mathring{\mathbf{g}}_{\alpha\beta} \simeq \mathring{a}^{-2}(t)\ {}^{RW}\mathbf{g}_{\alpha\beta}, \quad (50) $$

of the Friedmann–Lemaître–Robertson–Walker, FLRW, diagonal metric

$$ d\mathring{s}^{2} = {}^{RW}\mathbf{g}_{\alpha\beta}(u)\mathbf{e}^{\alpha}\mathbf{e}^{\beta} \simeq \mathring{a}^{2}(t)(dx^{2}+dy^{2}+dz^{2}) - dt^{2}. \quad (51) $$

In this formula, t is the cosmic time, $\mathring{a}(t)$ is the scale factor; and $x^{\grave{i}} = (x,y,z)$ are the Cartesian coordinates. A metric (51) defines a homogeneous, isotropic, and spatially flat cosmological spacetime as a solution of the Einstein equations in GR. Q-modifications of the energy–momentum tensor for matter are parameterized in the form (26) and generalized to effective tensors for DE (determined as in the modified gravitational equations (27) and (28)). Such metrics define Q-deformations to diagonal $f(Q)$ cosmological models determined by modified Friedman equations

$$ \begin{aligned} 3\mathring{H}^{2} &= \mathring{\rho} + \frac{1}{2}[f(\mathring{Q}) - \mathring{Q}] - \mathring{Q}[f_{Q}(\mathring{Q}) - 1], \\ &\times [2\mathring{Q}(f_{QQ}(\mathring{Q}) + f_{Q}(\mathring{Q}))]\mathring{H}^{\diamond} \\ &= \frac{1}{4}[f(\mathring{Q}) - 2\mathring{Q} + 2\mathring{Q}(1 - f_{Q}(\mathring{Q}))] - \frac{1}{2}\mathring{p}, \end{aligned} \quad (52) $$

for the Hubble function $\mathring{H} := \mathring{a}^{\diamond}/\mathring{a}$ and $\mathring{Q} = 6\mathring{H}^{2}$ for in the above equations, see the conventions on partial derivatives for Q–modified coefficients of the Ricci d-tensor (A.1). The conservation law (i.e. the matter equation-of-state) involving the prime energy density and pressure of matter fluid (respectively, $\mathring{\rho}$ and $\mathring{p}$) is

$$ \mathring{\rho}^{\diamond} + 3\mathring{H}(1+\mathring{w})\mathring{\rho} = 0, $$

for the parameter $\mathring{w} := \mathring{p}/\mathring{\rho}$. In this work, we follow our system of notation which is different from those used in other cosmological papers with diagonalizable metrics.

For applications in DE and DM physics, the Q-modified Friedman equation (52) are usually written in effective form

$$ {}^{m}\mathring{\Omega} + {}^{Q}\mathring{\Omega} = 1, $$

where the energy density parameters are introduced as

$$ {}^{m}\mathring{\Omega} = \mathring{\rho}/3\mathring{H}^{2} \text{ and } $$
$$ {}^{Q}\mathring{\Omega} = \left[\frac{1}{2}(f(\mathring{Q}) - \mathring{Q}) - \mathring{Q}(f_{Q}(\mathring{Q}) - 1)\right]/3\mathring{H}^{2}. $$

Such parameters allow to introduce some (prime) effective energy density and pressure, respectively, as

$$ {}^{eff}\mathring{\rho} = \mathring{\rho} + \frac{1}{2}[f(\mathring{Q}) - \mathring{Q}] - \mathring{Q}[f_Q(\mathring{Q}) - 1] \text{ and} $$
$$ {}^{eff}\mathring{p} = \frac{\mathring{\rho}(1+\mathring{w})}{2\mathring{Q} f_{QQ}(\mathring{Q}) + f_Q(\mathring{Q})} - \frac{\mathring{Q}}{2}. $$

In such thermodynamic variables, the total equation of state is written

$$ {}^{eff}\mathring{w} := \frac{{}^{eff}\mathring{p}}{{}^{eff}\mathring{\rho}} = -1 + \frac{{}^{m}\mathring{\Omega}(1+\mathring{w})}{2\mathring{Q} f_{QQ}(\mathring{Q}) + f_Q(\mathring{Q})}, $$

when ${}^{eff}\mathring{w} < -1/3$ for an accelerated universe.

A dynamical system analysis using cosmological parameters $({}^{m}\mathring{\Omega},\ {}^{Q}\mathring{\Omega},\ {}^{eff}\mathring{w})$ for various types of $f(\mathring{Q})$ was performed in section III of [10]. In this work, we elaborate on more general classes of nonmetric off-diagonal cosmological solutions which only for very special assumptions can be characterized by effective $({}^{m}\Omega,\ {}^{Q}\Omega,\ {}^{eff}w)$ determined by nonmetric geometric flows. For generic off-diagonal cosmological models, the concepts of above type thermodynamical models and related DM and DE theories are not applicable. We shall provide in Sect. 5 new examples which involve nonmetric generalizations of G. Perelman thermodynamics as considered in Sect. 3.2.

4.2 Ansatz for τ-running nonmetric cosmological spacetimes

In this subsection, we study a class of locally anisotropic cosmological solutions encoding in general form nonmetric Q-deformations and geometric flows on a temperature like parameter τ. Such generic off-diagonal metrics can be used for explaining recent observational data provided by JWST, [12–14], when the cosmological evolution and structure formation of the Universe are modelled in very different forms comparing to constructions in the frameworks of the SCM.

Nonmetric geometric flows of locally anisotropic cosmological solutions of the system of nonlinear PDEs (38) can be constructed for generic an off-diagonal ansatz[13]

$$ \hat{\mathbf{g}}(\tau) = g_i(\tau, x^k) dx^i \otimes dx^i + h_3(\tau, x^k, t)\mathbf{e}^3(\tau) \otimes \mathbf{e}^3(\tau) + h_4(\tau, x^k, t)\mathbf{e}^4(\tau) \otimes \mathbf{e}^4(\tau), $$
$$ \mathbf{e}^3(\tau) = dy^3 + n_i(\tau, x^k, t)dx^i, $$
$$ \mathbf{e}^4(\tau) = dy^4 + w_i(\tau, x^k, t)dx^i. \quad (53) $$

This is a d-metric of type (4) when the d-metric and N-connection coefficients depend on h-coordinates x^i, do not depend on the space coordinate y^3, but respective v-components generically depend as a v-coordinate on $y^4 = t$ and being parameterized in N-adapted form as

$$ g_i(\tau) = e^{\psi(\tau, x^j)},\ g_a(\tau) = h_a(\tau, x^k, t), $$
$$ N_i^3 = n_i(\tau, x^k, t), \quad N_i^4 = w_i(\tau, x^k, t). \quad (54) $$

We can find the solutions in explicit parametric form if we consider effective sources (37) determined by generating sources

$$ {}^{\intercal}\yen^{\mu}_{\nu}(\tau) = {}^{c}\yen^{\mu}_{\nu}(\tau) = [{}_{h}\yen(\tau, x^k)\delta^i_j,\ {}_{v}\yen(\tau, x^k, t)\delta^a_b], \quad (55) $$

where the left label "c" is used for cosmological configurations with h-v-decomposition. We shall use such splitting of effective generating v-sources:

$$ {}_{v}\yen(\tau, x^k, t) = {}^{m}\yen(\tau, x^k, t) + {}^{e}\yen(\tau, x^k, t) + {}^{DE}\yen(\tau, x^k, t), \quad (56) $$

associated to respective τ-running matter terms ${}^{m}T_{\alpha\beta}$ (25), effective sources ${}^{e}\widehat{\mathbf{Y}}_{\alpha\beta}$ (33) determined by distortions of connections encoding additional terms $\partial_\tau \mathbf{g}_{\mu\nu}$ as in (37); and effective DE terms ${}^{DE}T_{\alpha\beta}$ (28) computed by respective Q-deformations of gravitational Lagrangians. We shall not use similar decompositions for ${}_{h}\yen(\tau, x^k)$ because they contribute only to the sources of some 2-d Poisson equations which are linear and can be integrated in general form (see details in Appendix A). In another turn, the terms (56) are determined in parametric form by different physical constants which allows to compute different type physical implications to nonlinear geometric evolution and dynamical interactions in conventional v-subspaces.

4.3 Parametrization of d-metrics and sources describing nonmetric QC evolution

There are four possibilities to generate topological QCs and elaborate models of their nonmetric geometric evolution by cosmological d-metrics (53):

1. We can consider that the v-components a d-metric and respective N-coefficients are generated as functionals of a quasi-periodic function

$$ {}^{1}\widehat{\theta}^a \simeq \theta^I(u) = \theta^I_{[0]} + \mathbf{K}^I \mathbf{x}, \quad (57) $$

---

[13] With respect to coordinate dual frames, such a d-metric can be represented as $\hat{\mathbf{g}} = \underline{\widehat{g}}_{\alpha\beta}(\tau, u)du^\alpha \otimes du^\beta$, when the off-diagonal metrics are parameterized in the form

$$ \underline{\widehat{g}}_{\alpha\beta}(\tau, u) = \begin{bmatrix} e^\psi + (n_1)^2 h_3 + (w_1)^2 h_4 & n_1 n_2 h_3 + w_1 w_2 h_4 & n_1 h_3 & w_1 h_4 \\ n_1 n_2 h_3 + w_1 w_2 h_4 & e^\psi + (n_2)^2 h_3 + (w_2)^2 h_4 & n_2 h_3 & w_2 h_4 \\ n_1 h_3 & n_2 h_3 & h_3 & 0 \\ w_1 h_4 & w_2 h_4 & 0 & h_4 \end{bmatrix}, $$

where respective dependencies of coefficients on (τ, x^k, y^3) are omitted. Such a metric is generic off-diagonal if the W-coefficients defined in footnote 5 are not trivial.

with $\theta^I_{[0]}$ and $I=b$, for $b=3,4$. Such values are taken for a crystal like structure as we explain for topological QCs in Appendix B (for formulas (B.1)). For instance, the d-metric coefficients (54) can be defined by a functional $\ ^1\Psi(\tau,x^k,t) = \Psi[\ ^1\theta^a[\tau,x^k,t]]$ as for (A.7), when the generating sources $_{\upsilon}\ \yen(\tau,x^k,t)$ (56) are arbitrary ones (i. e. the components $^m\ \yen$, $^e\ \yen$, and $^{DE}\yen$ do not involve topological QC configurations). Correspondingly, the formulas (54) are considered as functionals

$$h_a[\ ^{\widehat{1}}\theta^a] = h_a[\ ^{\widehat{1}}\theta^a[\tau,x^k,t]],$$
$$n_i[\ ^{\widehat{1}}\theta^a] = n_i[\ ^{\widehat{1}}\theta^a[\tau,x^k,t]],$$
$$w_i[\ ^{\widehat{1}}\theta^a] = w_i[^{\widehat{1}}\theta^a[\tau,x^k,t]], \quad (58)$$

where the left label $\widehat{1}$ will be used for emphasizing quasi-periodic functional dependencies for generating functions of the coefficients of d-metrics and N-coefficients.

2. Topological QC configurations can be generated also by effective sources $^e\widehat{\mathbf{\Upsilon}}_{\alpha\beta}$ (33) if we consider that such N-adapted coefficients are related by frame/coordinate frame transforms (and including in nonholonomic form effective terms of type $\partial_\tau \mathbf{g}_{\mu\nu}$, see explanations for formulas (37)) to some

$$^e\widehat{\mathbf{T}}_{\alpha\beta} := -\frac{2}{\sqrt{|\mathbf{g}|}}\frac{\delta(\sqrt{|\mathbf{g}|}\ ^e\widehat{\mathcal{L}})}{\delta\mathbf{g}^{\alpha\beta}} = \ ^e\widehat{\mathcal{L}}\mathbf{g}_{\alpha\beta} + 2\frac{\delta(\ ^e\widehat{\mathcal{L}})}{\delta\mathbf{g}^{\alpha\beta}}.$$

Such a d-tensor is computed in N-adapted variational form as (25) but for an effective quasi-periodic Lagrangian $^e\widehat{\mathcal{L}} = \ ^{\theta 2}\widehat{\mathcal{L}}$ (B.2), or $^e\widehat{\mathcal{L}} = \ ^{\Theta 2}\widehat{\mathcal{L}}$ (B.3). In functional form, we parameterize the corresponding topological QC generating sources as

$$^e\widehat{\mathbf{\Upsilon}}_{\alpha\beta}[\ ^{\widehat{2}}\theta^a] = \ ^e\widehat{\mathbf{\Upsilon}}_{\alpha\beta}[\ ^{\widehat{2}}\theta^a[\tau,x^k,t]] \leftrightarrow \ ^e\Lambda, \quad (59)$$

where the label $\widehat{2}$ emphasize that we consider quasi-periodic structures related to canonical nonholonomic distortions of d-connections.

3. Quasi-periodic structures can be generated by effective DE sources $^{DE}\yen(\tau,x^k,t)$ (28) determined by Q-deformations of sources. In N-adapted form, we write $^{DE}\widehat{\mathbf{T}}_{\alpha\beta}$ for generating source functionals in (37) and consider DE generating sources as functionals

$$^{DE}\widehat{\mathbf{\Upsilon}}_{\alpha\beta}[\ ^{\widehat{3}}\theta^a] = \ ^{DE}\widehat{\mathbf{\Upsilon}}_{\alpha\beta}[\ ^{\widehat{3}}\theta^a[\tau,x^k,t]] \leftrightarrow \ ^{DE}\Lambda. \quad (60)$$

We use the label $\widehat{3}$ for distinguishing quasi-periodic structures of nonmetric origin.

Applying nonlinear transforms (A.8) and (A.9), the quasi-periodic generating sources (59) and (60) can be transformed into respective effective cosmological constants $^e\Lambda$ and $^{DE}\Lambda$ as in (A.10). This allows us to compute and distinguish contributions of different type quasi-periodic structures, which can be of distortion of connections and (non) metric geometric flow origin and/or of Q-deformation type. Because of nonlinear off-diagonal geometric evolution and gravitational and (effective) matter field interactions, considering nonmetric geometric data for $(\ ^m\Lambda\ ^e\Lambda,\ ^{DE}\Lambda)$, the coefficients of d-metrics and N-connections (in general frames of reference) became functionals on all types $\ ^{\widehat{q}}\theta^a$, for $q=1,2,3,$; when (58) are re-parameterized as

$$h_a[\ ^{\widehat{q}}\theta^a] = h_a[\ ^{\widehat{q}}\theta^a[\tau,x^k,t]],\ n_i[\ ^{\widehat{q}}\theta^a] = n_i[\ ^{\widehat{q}}\theta^a[\tau,x^k,t]],$$
$$w_i[\ ^{\widehat{q}}\theta^a] = w_i[\ ^{\widehat{q}}\theta^a[\tau,x^k,t]]. \quad (61)$$

Even for such general assumptions on generating data, we can construct exact and parametric solutions of physically important systems of nonlinear PDEs (32)–(41) if we apply the AFCDM summarized in Appendix A. Such generic off-diagonal metrics and generalized (non) linear connections describe the generation and nonmetric flow evolution of cosmological topological QC structure, which may be applied in modern accelerating cosmology and DE and DM physics. In this section, we provide explicit examples and discuss the main physical properties which may be important for explaining recent observational data provided by JWST, [12–14].

4.4 Generic non ΛCDM cosmological solutions with τ-running QC generating functions

Such generic off-diagonal cosmological solutions of (32) are of type (58) and generated by a functional $\ ^{\widehat{1}}\Psi = \Psi[\ ^{\widehat{1}}\theta^a[\tau,x^k,t]]$ introduced in τ-family of quadratic elements (A.7), when

$$ds^2(\tau) = e^{\psi(\tau,x^k)}[(dx^1)^2+(dx^2)^2]$$
$$+\left\{g_3^{[0]} - \int dt\frac{[\ ^{\widehat{1}}\Psi^2]^{\diamond}}{4(_{\upsilon}\ \yen)}\right\}\left\{dy^3 + \left[\ _1n_k + \ _2n_k\right.\right.$$
$$\left.\left.\times\int dt\frac{[(\ ^{\widehat{1}}\Psi)^2]^{\diamond}}{4(_{\upsilon}\ \yen)^2|g_3^{[0]} - \frac{\int dt[\ ^{\widehat{1}}\Psi^2]^{\diamond}}{4(_{\upsilon}\ \yen)}|^{5/2}}\right]dx^k\right\}$$
$$+\frac{[\ ^{\widehat{1}}\Psi^{\diamond}]^2}{4(_{\upsilon}\ \yen)^2\{g_3^{[0]} - \frac{\int dt[\ ^{\widehat{1}}\Psi^2]^{\diamond}}{4(_{\upsilon}\ \yen)}\}}\left(dt + \frac{\partial_i\ ^{\widehat{1}}\Psi}{\ ^{\widehat{1}}\Psi^{\diamond}}dx^i\right)^2. \quad (62)$$

The nonmetric geometric cosmological evolution described by (62) posses a topological QC structure even the effective sources $_{\upsilon}\ \yen = \ ^m\ \yen + \ ^e\ \yen + \ ^{DE}\yen$ (56) can be some general ones not involving a quasi-periodic structure.

The nonlinear symmetries (A.8) allow us to transform the generating data

$$(\Psi[\,^{1}\widehat{\theta}^{a}[\tau,x^{k},t]],_{v}\,\yen(\tau)) \rightarrow (\,^{1}\widehat{h}_{3}=h_{3}[\,^{1}\widehat{\theta}^{a}[\tau,x^{k},t]],\ _{v}\,\yen(\tau,x^{k},t)),$$
$$\text{when } \,^{1}\widehat{h}_{3}^{\diamond}(\tau) = -[(\,^{1}\widehat{\Psi})^{2}]^{\diamond}/4_{v}\,\yen(\tau). \tag{63}$$

The solutions (62) can be written in the form (A.12), when $h_3 \rightarrow \,^{1}\widehat{h}_3$ and $\,^{1}\widehat{h}_3$ is a generating functional for topological QC structure $\,^{1}\widehat{\theta}^{a}[\tau,x^{k},t]$ determining the quasi-periodicity of the v-components of the d-metric and N-connection coefficients.

Using formula (63) and

$$(\,^{1}\widehat{\Phi}[\,^{1}\widehat{\theta}^{a}[\tau,x^{k},t]])^{2} = -4\widehat{\Lambda}(\tau)\,^{1}\widehat{h}_{3}[\,^{1}\widehat{\theta}^{a}[\tau,x^{k},t]],$$
$$\text{for } \widehat{\Lambda}(\tau) = \,^{m}\Lambda(\tau) + \,^{e}\Lambda(\tau) + \,^{DE}\Lambda(\tau)$$

(derived for nonlinear symmetries (A.8) and (A.9) and for splitting of the effective τ-running cosmological constant (A.10) for (A.12)), we re-write the d-metric (62) in the form (A.11)), when

$$\begin{aligned}
ds^{2}(\tau) &= g_{\alpha\beta}(\tau,x^{k},t,\,^{1}\widehat{\Phi}(\tau),\widehat{\Lambda}(\tau))du^{\alpha}du^{\beta}\\
&= e^{\psi(\tau,x^{k})}[(dx^{1})^{2}+(dx^{2})^{2}] - \left\{g_{3}^{[0]}(\tau) - \frac{(\,^{1}\widehat{\Phi})^{2}(\tau)}{4\,\widehat{\Lambda}(\tau)}\right\}\\
&\times \Bigg\{dy^{3} + \Bigg[\,_{1}n_{k}(\tau) + \,_{2}n_{k}(\tau)\int dt\\
&\times \frac{(\,^{1}\widehat{\Phi})^{2}(\tau)[(\,^{1}\widehat{\Phi})^{\diamond}(\tau)]^{2}}{|\,\widehat{\Lambda}(\tau)\int dt\ _{v}\,\yen(\tau)\,[(\,^{1}\widehat{\Phi})^{2}(\tau)]^{\diamond}|[g_{3}^{[0]}(\tau) - \frac{(\,^{1}\widehat{\Phi})^{2}(\tau)}{4\,\widehat{\Lambda}(\tau)}]^{5/2}}\Bigg]\Bigg\}\\
&- \frac{(\,^{1}\widehat{\Phi})^{2}(\tau)[(\,^{1}\widehat{\Phi})^{\diamond}(\tau)]^{2}}{|\,\widehat{\Lambda}(\tau)\int dt\ _{v}\,\yen(\tau)[(\,^{1}\widehat{\Phi})^{2}(\tau)]^{\diamond}\ |[g_{3}^{[0]}(\tau) - \frac{(\,^{1}\widehat{\Phi})^{2}(\tau)}{4\,\widehat{\Lambda}(\tau)}]}\\
&\times \left\{dt + \frac{\partial_{i}\ \int dt\ _{v}\,\yen(\tau)\,[(\,^{1}\widehat{\Phi})^{2}(\tau)]^{\diamond}}{_{v}\,\yen(\tau)\,[(\,(\,^{1}\widehat{\Phi})(\tau))^{2}]^{\diamond}}dx^{i}\right\}^{2}.
\end{aligned} \tag{64}$$

This τ-family of cosmological solutions is determined by generating data $(\,^{1}\widehat{\Phi}[\,^{1}\widehat{\theta}^{a}[\tau,x^{k},t]];\ \,^{m}\Lambda(\tau) + \,^{e}\Lambda(\tau) + \,^{DE}\Lambda(\tau),_{v}\ \yen(\tau))$ and allow to distinguish and compare models with different types of τ-running effective cosmological constants. Such constants approximate the contributions of standard matter fields, effective distortions and Q-deformations. In general, we are not able to eliminate the effective generating source $_{v}\,\yen(\tau)$ but we can always chose certain nonholonomic configurations when the terms with some $(\,^{m}\,\yen(\tau),\ \,^{e}\,\yen(\tau),\ \,^{DE}\yen(\tau))$ are smaller than the terms with corresponding $(\,^{m}\Lambda(\tau),\ \,^{e}\Lambda(\tau),\ \,^{DE}\Lambda(\tau))$. Fixing $\tau = \tau_0$, we define generic off-diagonal cosmological solutions in nonmetric MGTs.

The τ-running d-metrics (62) allow us to model cosmological scenarios with topological QC structure induced by generating functions when the effective matter sources$_{v}\,\yen(\tau)$ are not obligatory quasi-periodic. The priority of parametrization of solutions written in the form (64) involving effective cosmological constants is that we can compute corresponding thermodynamic variables for generating topological QC configurations (see Sect. 5). Using such variables, we can distinguish contributions of nonholonomic distortions and off-diagonal terms, nonmetricity fields and metric fields and search for observational data which allows to decide which theory is more realistic and viable. Corresponding nonmetric generic off-diagonal cosmological scenarios are very different from those elaborated in the framework of the ΛCDM cosmological paradigm.

4.5 τ-Running cosmology with topological QC generating sources

Generic off-diagonal cosmological metrics can be generated by nonmetric topological QC sources of type (59) and/or (60) when the generating functions $\Psi(\tau)$, $\Phi(\tau)$ and $h_3(\tau)$ may encode, or not, quasi-periodic structures. In this subsection, we provide a modification of (64) with v-components of d-metric and N-connection coefficients of type (61), when

$$\begin{aligned}
ds^{2}(\tau) &= g_{\alpha\beta}(\tau,x^{k},t,\,^{1}\widehat{\Phi}(\tau),\widehat{\Lambda}(\tau))du^{\alpha}du^{\beta} = e^{\psi(\tau,x^{k})}[(dx^{1})^{2}+(dx^{2})^{2}] - \left\{g_{3}^{[0]}(\tau) - \frac{(\,^{q}\widehat{\Phi})^{2}(\tau)}{4\,\widehat{\Lambda}(\tau)}\right\}\\
&\times \left\{dy^{3} + [\,_{1}n_{k}(\tau) + \,_{2}n_{k}(\tau)\int dt \frac{(\,^{q}\widehat{\Phi})^{2}(\tau)[(\,^{q}\widehat{\Phi})^{\diamond}(\tau)]^{2}}{|\,\widehat{\Lambda}(\tau)\int dt\ (\,^{m}\,\yen(\tau) + \,^{e}\,\yen[\,^{2}\widehat{\theta}^{a}[\tau]] + \,^{DE}\yen[\,^{3}\widehat{\theta}^{a}[\tau]])[(\,^{q}\widehat{\Phi})^{2}(\tau)]^{\diamond}|[g_{3}^{[0]}(\tau) - \frac{(\,^{q}\widehat{\Phi})^{2}(\tau)}{4\,\widehat{\Lambda}(\tau)}]^{5/2}}]\right\}\\
&- \frac{(\,^{q}\widehat{\Phi})^{2}(\tau)[(\,^{q}\widehat{\Phi})^{\diamond}(\tau)]^{2}}{|\,\widehat{\Lambda}(\tau)\int dt\ (\,^{m}\,\yen(\tau) + \,^{e}\,\yen[\,^{2}\widehat{\theta}^{a}[\tau]] + \,^{DE}\yen[\,^{3}\widehat{\theta}^{a}[\tau]])[(\,^{q}\widehat{\Phi})^{2}(\tau)]^{\diamond}\ |[g_{3}^{[0]}(\tau) - \frac{(\,^{q}\widehat{\Phi})^{2}(\tau)}{4\,\widehat{\Lambda}(\tau)}]}\\
&\times \left\{dt + \frac{\partial_{i}\ \int dt\ (\,^{m}\,\yen(\tau) + \,^{e}\,\yen[\,^{2}\widehat{\theta}^{a}[\tau]] + \,^{DE}\yen[\,^{3}\widehat{\theta}^{a}[\tau]])\,[(\,^{q}\widehat{\Phi})^{2}(\tau)]^{\diamond}}{(\,^{m}\,\yen(\tau) + \,^{e}\,\yen[\,^{2}\widehat{\theta}^{a}[\tau]] + \,^{DE}\yen[\,^{3}\widehat{\theta}^{a}[\tau]])[\,(\,^{q}\widehat{\Phi})^{2}(\tau)]^{\diamond}}dx^{i}\right\}^{2}.
\end{aligned} \tag{65}$$

The τ-family of cosmological solutions (65) is determined by generating data

$$\begin{aligned}&({}^{\widehat{q}}\Phi[{}^{\widehat{q}}\theta^a[\tau, x^k, t]];\\ &\widehat{\Lambda}(\tau) = {}^m\Lambda(\tau) + {}^e\Lambda(\tau) + {}^{DE}\Lambda(\tau), {}_v\yen(\tau)),\end{aligned} \quad (66)$$

which allows us to compute the corresponding G. Perelman's thermodynamic variables (49) (see examples in section 5). In general forms, such nonmetric cosmological models can't be embedded into the framework of the ΛCDM cosmological paradigm. Such a general behaviour holds true even we consider additional constraints for extracting LC-configurations as in Appendix A.2.2 and, for further restrictions to locally anisotropic cosmological models in GR (with vanishing ${}^{DE}\yen(\tau)$ and ${}^{DE}\Lambda(\tau)$). Nevertheless, even in such special cases, topological QCs cosmological configurations can be generated because of generic off-diagonal terms of d-metrics and related N-connection coefficients.

4.6 Topological QC deformations of FLRW metrics under nonmetric geometric flows

Using the AFCDM, we can study how nonmetric geometric flows may result in deformation of certain prime off-diagonal metric $\mathring{\mathbf{g}} = [\mathring{g}_\alpha, \ \mathring{N}_i^a]$ (A.15) together with the formation of topological QC structure. An important physical example is to consider that $\mathring{\mathbf{g}}_{\alpha\beta} \simeq \mathring{a}^{-2}(t)\ {}^{RW}\mathbf{g}_{\alpha\beta}$ as in (50) and then to generate parametric solutions of (38) defined by gravitational η-polarizations (A.15). Considering a generating function of type (A.18) $\eta_3\ (\tau) \ \simeq \ {}^{\widehat{q}}\eta_3(\tau) \ = \ \eta_3[\ {}^{\widehat{q}}\theta^a]$ as in (61) (in particular cases, we can consider a functional dependence of type (63)) and generating sources of topological QC type (59) and/or (60), we generate τ-families of cosmological d-metrics (A.21), when

$$\begin{aligned}&\Big({}^{\widehat{q}}h_3(\tau) = \eta_3[\ {}^{\widehat{q}}\theta^a]\mathring{g}_\alpha;\ {}^m\yen(\tau)\\ &+ {}^e\yen[\ {}^{\widehat{2}}\theta^a[\tau]] + {}^{DE}\yen[\ {}^{\widehat{3}}\theta^a[\tau]];\\ &\widehat{\Lambda}\ (\tau) = {}^m\Lambda(\tau) + {}^e\Lambda(\tau) + {}^{DE}\Lambda(\tau)\Big).\end{aligned} \quad (68)$$

The primary cosmological data define a ΛCDM model with Q -deformations as we considered in Sect. 4.1. More general classes of solutions (67) deform such metrics in generic off-diagonal and describe arising of topological QC structures encoding via polarization functions nonmetric geometric flows and corresponding effective QC-sources. The physical interpretation of target classes of solutions is possible in terms of statistical/ geometric thermodynamic variables (49). Such τ-evolving locally anisotropic cosmological models are very different from the theories of ΛCDM type. Choosing corresponding parameters for the generating quasi-periodic structures (in general, they are not just of QC type but subjected to additional deformations because of random off-diagonal interactions and nonmetric geometric evolution), we may try to explain observational data provided by JWST [12–14].

4.7 ΛCDM configurations and small parametric deformations inducing QC structures

We can construct generic off-diagonal solutions for nonmetric geometric flows defining topological QC structures which can be analyzed in the framework of the ΛCDM paradigm but with nontrivial G. Perelman thermodynamic variables. This is possible if we apply the AFCDM for a small ε-parameter and respective χ-polarizations as defined by formulas (A.16), (A.17) and (A.20) $\chi_3(\tau, x^k, t) \ \simeq \ {}^{\widehat{q}}\chi_3(\tau) =$

$$\begin{aligned}d\widehat{s}^2(\tau) =& \widehat{g}_{\alpha\beta}(\tau, x^k, t; \mathring{g}_\alpha; \psi(\tau),\ {}^{\widehat{q}}\eta_3(\tau); ({}^m\yen(\tau) + {}^e\yen[\ {}^{\widehat{2}}\theta^a[\tau]] + {}^{DE}\yen[\ {}^{\widehat{3}}\theta^a[\tau]]))du^\alpha du^\beta\\ =& e^{\psi(\tau)}[(dx^1)^2 + (dx^2)^2] + {}^{\widehat{q}}\eta_3(\tau)\mathring{g}_3 \Bigg\{ dy^3 + [{}_1n_k(\tau) + {}_2n_k(\tau)\int dt\\ &\times \frac{[({}^{\widehat{q}}\eta_3(\tau)\mathring{g}_3)^\diamond]^2}{|\int dt\ ({}^m\yen(\tau) + {}^e\yen[\ {}^{\widehat{2}}\theta^a[\tau]] + {}^{DE}\yen[\ {}^{\widehat{3}}\theta^a[\tau]])({}^{\widehat{q}}\eta_3(\tau)\mathring{g}_3)^\diamond|\ ({}^{\widehat{q}}\eta_3(\tau)\mathring{g}_3)^{5/2}}]dx^k \Bigg\}^2\\ &- \frac{[({}^{\widehat{q}}\eta_3(\tau)\ \mathring{g}_3)^\diamond]^2}{|\int dt\ {}_v\ ({}^m\yen(\tau) + {}^e\yen[\ {}^{\widehat{2}}\theta^a[\tau]] + {}^{DE}\yen[\ {}^{\widehat{3}}\theta^a[\tau]])({}^{\widehat{q}}\eta_3(\tau)\ \mathring{g}_3)^\diamond|\ {}^{\widehat{q}}\eta_3(\tau)\mathring{g}_3}\\ &\times \Bigg\{ dt + \frac{\partial_i[\int dt\ {}_v\ ({}^m\yen(\tau) + {}^e\yen[\ {}^{\widehat{2}}\theta^a[\tau]] + {}^{DE}\yen[\ {}^{\widehat{3}}\theta^a[\tau]])\ ({}^{\widehat{q}}\eta_3(\tau)\mathring{g}_3)^\diamond]}{({}^m\yen(\tau) + {}^e\yen[\ {}^{\widehat{2}}\theta^a[\tau]] + {}^{DE}\yen[\ {}^{\widehat{3}}\theta^a[\tau]])({}^{\widehat{q}}\eta_3(\tau)\mathring{g}_3)^\diamond} dx^i \Bigg\}^2.\end{aligned} \quad (67)$$

For $({}^{\widehat{q}}\Phi[{}^{\widehat{q}}\theta^a[\tau, x^k, t]])^2 = -4\ \widehat{\Lambda}\eta_3[{}^{\widehat{q}}\theta^a]\mathring{g}_\alpha$, we can transform (67) in a variant of (65), or (A.19), with η-polarizations determined by the generating data

$\chi_3[\ {}^{\widehat{q}}\theta^a]$ as a generating function with quasi-periodic structure (as in (61). For some examples, we can use a functional dependence of type (63)) and generating sources of topological QC type (59) and/or (60). For such ε-linear decompositions, we obtain a small parametric version of type (A.21) for the d-metrics (67),

$$
\begin{aligned}
& d\widehat{s}^2(\tau) = \widehat{g}_{\alpha\beta}(\tau, x^k, t; \mathring{g}_\alpha; \psi(\tau), {}^{\widehat{q}}\chi_3(\tau); ({}^{\widehat{q}}\yen(\tau) = {}^{m}\yen(\tau) + {}^{e}\yen[{}^{\widehat{2}}\theta^a[\tau]] + {}^{DE}\yen[{}^{\widehat{3}}\theta^a[\tau]]))du^\alpha du^\beta \\
& = e^{\psi_0}(1 + \varepsilon\, {}^{\psi}\chi(\tau))[(dx^1)^2 + (dx^2)^2] + \zeta_3(\tau)(1 + \varepsilon\, {}^{\widehat{q}}\chi_3(\tau))\, \mathring{g}_3 \times \Bigg\{ dy^3 + [(\mathring{N}_k^3)^{-1}[{}_1n_k(\tau) + 16\, {}_2n_k(\tau) \\
& \quad \times \Big[\int dt \frac{\left([(\zeta_3(\tau)\mathring{g}_3)^{-1/4}]^\diamond\right)^2}{|\int dt[{}^{\widehat{q}}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)]^\diamond|} \Big] \\
& \quad + \varepsilon \frac{16\, {}_2n_k(\tau) \int dt \frac{\left([(\zeta_3(\tau)\mathring{g}_3)^{-1/4}]^\diamond\right)^2}{|\int dt[{}^{\widehat{q}}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)]^\diamond|} \left(\frac{[(\zeta_3(\tau)\mathring{g}_3)^{-1/4}\chi_3)]^\diamond}{2[(\zeta_3(\tau)\mathring{g}_3)^{-1/4}]^\diamond} + \frac{\int dt[{}^{\widehat{q}}\yen(\tau)(\zeta_3(\tau)\, {}^{\widehat{q}}\chi_3(\tau)\mathring{g}_3)]^\diamond}{\int dt[{}^{\widehat{q}}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)]^\diamond} \right)}{{}_1n_k(\tau) + 16\, {}_2n_k(\tau)[\int dt \frac{\left([(\zeta_3(\tau)\mathring{g}_3)^{-1/4}]^\diamond\right)^2}{|\int dt[{}^{\widehat{q}}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)]^\diamond|}]}]\mathring{N}_k^3 dx^k \Bigg\}^2 . \\
& \quad - \left\{ \frac{4[(|\zeta_3(\tau)\mathring{g}_3|^{1/2})^\diamond]^2}{\mathring{g}_3|\int dt \left\{ {}^{\widehat{q}}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)^\diamond \right\}|} - \varepsilon \left[\frac{({}^{\widehat{q}}\chi_3(\tau)|\zeta_3(\tau)\mathring{g}_3|^{1/2})^\diamond}{4(|\zeta_3(\tau)\mathring{g}_3|^{1/2})^\diamond} - \frac{\int dt\{ {}^{\widehat{q}}\yen[(\zeta_3(\tau)\mathring{g}_3)\, {}^{\widehat{q}}\chi_3(\tau)]^\diamond\}}{\int dt\{ {}^{\widehat{q}}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)^\diamond\}} \right] \right\} \mathring{g}_4 \\
& \quad \times \left\{ dt + \left[\frac{\partial_i \int dt\, {}^{\widehat{q}}\yen(\tau)\zeta_3^\diamond(\tau)}{(\mathring{N}_i^3)\, {}^{\widehat{q}}\yen(\tau)\zeta_3^\diamond(\tau)} + \varepsilon\left(\frac{\partial_i[\int dt\, {}^{\widehat{q}}\yen(\tau)\, (\zeta_3(\tau)\, {}^{\widehat{q}}\chi_3(\tau))^\diamond]}{\partial_i\, [\int dt\, {}^{\widehat{q}}\yen(\tau)\zeta_3^\diamond(\tau)]} - \frac{(\zeta_3(\tau)\, {}^{\widehat{q}}\chi_3(\tau))^\diamond}{\zeta_3^\diamond(\tau)} \right) \right] \mathring{N}_i^4 dx^i \right\}^2 . \qquad (69)
\end{aligned}
$$

We can re-define such small parametric formulas in terms of generating data $({}^{\widehat{q}}\Phi(\tau), \widehat{\Lambda}(\tau))$, when

$$
\begin{aligned}
& ({}^{\widehat{q}}\Phi[{}^{\widehat{q}}\theta^a[\tau, x^k, t]])^2 = -4\, \widehat{\Lambda}\zeta_3(\tau, x^k, t) \\
& (1 + \varepsilon\, {}^{\widehat{q}}\chi_3(\tau, x^k, t))[{}^{\widehat{q}}\theta^a]\mathring{g}_\alpha, \qquad (70)
\end{aligned}
$$

for $\widehat{\Lambda}(\tau) = {}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau)$. This allows to compute the G. Perelman thermodynamic variables as in next section.

The off-diagonal parametric solutions (69) can be parameterized, for instance, to define ellipsoidal deformations of spherical symmetric cosmological metrics into similar ones with ellipsoid symmetry or in certain N-adapted effective diagonal forms. Quasi-stationary solutions with nonmetricity fields are provided in a partner work [15] for generating functions of type $\chi_3(\tau, x^k, \varphi)$, with x^k and φ being space like coordinates and ε considered as an eccentricity parameter for some rotoid type deformations. Similar techniques can be applied if $\chi_3(\tau, x^k, \varphi) \to {}^{\widehat{q}}\chi_3(\tau, x^k, t)$ in order to generate ellipsoidal cosmological solutions encoding Q-deformations and quasi-periodic structures (for non-topological QCs, see [17,20]. We do not consider examples of such incremental work re-defining the constructions for topological QC configurations in this paper.

4.8 τ-evolution of nonmetric effective ΛCDM models

New τ-families of generic off-diagonal cosmological solutions of type (62), (64), (65), (67) and (69) motivate a new paradigm for elaborating cosmological models which describe nonlinear quasi-periodic structure formation. This way, new models of DE and DM theories can be elaborated that may encode possible nonmetric geometric evolution and various types of MGTs. Nevertheless, such modified cosmological models my possess some properties which can be described in the framework of the ΛCDM paradigm. In this subsection, we analyze such conditions for a prime metric $\mathring{\mathbf{g}}_{\alpha\beta}$ (50) (being conformal to a FLRW metric (51), in nonlinear coordinates) and transformed via frame/ coordinate transforms into a target metric (67) or (69) and parameterized in the form

$$
\begin{aligned}
ds^2 = {}& {}^{\widehat{q}}\mathbf{g}_{\alpha\beta}(\tau, t)\mathbf{e}^\alpha\mathbf{e}^\beta \simeq {}^{\widehat{q}}a^2(\tau, t)[dx^2 + dy^2 \\
& + (dz + {}^{\widehat{q}}w_i(\tau, t)dx^i)^2] - [dt + {}^{\widehat{q}}n_i(\tau, t)dx^i]^2. \qquad (71)
\end{aligned}
$$

In (71), t is the cosmic time, $x^{\grave{i}} = (x, y, z)$ are the Cartesian coordinates; ${}^{\widehat{q}}a(\tau, t) \simeq a([{}^{\widehat{q}}\theta^a(\tau, x, y, t)])$ defines a topological type QC scale factor, when ${}^{\widehat{q}}w_i(\tau, t)$ and ${}^{\widehat{q}}n_i(\tau, t)$ are proportional to a ε-parameter and can be stated to zero by choosing certain integration functions ${}_1n_k(\tau)$ and ${}_2n_k(\tau)$ and the generating functions are parameterized in a form to get small ${}^{\widehat{q}}w_i(\tau, t)$.

We suppose that an effective diagonal metric (in the chosen system of reference and coordinated) defines a model of τ-evolution of effective homogeneous, isotropic, and spatially flat cosmological spacetimes, which for a fixed temperature like parameter $\tau = \tau_0$ are solutions of the Q-modified Einstein equations (27) and (28)). For such configurations, the topological QC structure arise and evolve in diagonal form with effective density ${}^{\widehat{q}}\rho = \rho(\tau, t) = \frac{1}{8\pi^2}C_{ab}\varepsilon^{ij}N_i^a(\tau, t)N_j^a(\tau, t)$ as we explain for (B.6). Effectively, such diagonal quasi-periodic effective matter is described by a τ-parameter conservation law (i.e.

a family matter equations-of-state parameters) involving and effective energy density and certain pressure of an effective matter fluid (respectively, $\rho(\tau, t)$ and p),

$$\,^{\widehat{q}}\rho^{\diamond} + 3\,^{\widehat{q}}H(1+w)p = 0,$$

for the parameters $w(\tau) := p/p$ and effective Hubble functions $\,^{\widehat{q}}H(\tau, t) := \,^{\widehat{q}}a^{\diamond}/\,^{\widehat{q}}a$ and $\,^{\widehat{q}}Q = 6\,^{\widehat{q}}H^2$ which can be comuted for the Q–modified coefficients of the Ricci d-tensor (A.1). We note that the nonmetric vacuum gravitational structure of such cosmological τ-running spacetime is not trivial because it encode ε-deformations of topological QC deformations characterised by sources and conservation laws of type (B.4)–(B.5).

Metrics of type (71) define $f(\,^{\widehat{q}}Q)$ cosmological models determined by modified Friedman equations

$$3\,^{\widehat{q}}H^2 = \,^{\widehat{q}}\rho + \frac{1}{2}[f(\,^{\widehat{q}}Q) - \,^{\widehat{q}}Q] - \,^{\widehat{q}}Q[f_Q(\,^{\widehat{q}}Q) - 1],$$
$$\times [2\,^{\widehat{q}}Q(f_{QQ}(\,^{\widehat{q}}Q) + f_Q(\,^{\widehat{q}}Q))]\,^{\widehat{q}}H^{\diamond}$$
$$= \frac{1}{4}[f(\,^{\widehat{q}}Q) - 2\,^{\widehat{q}}Q + 2\,^{\widehat{q}}Q(1 - f_Q(\,^{\widehat{q}}Q))] - \frac{1}{2}p. \quad (72)$$

We can use such equations for modeling topological QC of DE and DM, when the quasi-periodic Q-modified Friedman equation (72) are written $\,^{m}\Omega + \,^{Q}\Omega = 1$, where the energy density parameters are introduced as

$$\,^{m}\Omega = \,^{\widehat{q}}\rho/3\,^{\widehat{q}}H^2 \text{ and}$$
$$\,^{Q}\Omega = \left[\frac{1}{2}(f(\,^{\widehat{q}}Q) - \,^{\widehat{q}}Q) - \,^{\widehat{q}}Q(f_Q(\,^{\widehat{q}}Q) - 1)\right]/3\,^{\widehat{q}}H^2.$$

A dynamical systems analysis can be performed as in section III of [10] (see also Sect. 4.1) but for effective $(\,^{m}\Omega, \,^{Q}\Omega, \,^{eff}w)$ determined by nonmetric geometric flows inducing topological QC structures. In nonholonomic variables keeping the diagonal character of such configurations, we can consider the target effective equation of state

$$\,^{eff}w := \frac{\,^{eff}p}{\,^{eff}\rho} = -1 + \frac{\,^{m}\Omega(1+w)}{2\,^{\widehat{q}}Qf_{QQ}(\,^{\widehat{q}}Q) + f_Q(\,^{\widehat{q}}Q)}, \quad (73)$$

energy density and pressure, respectively, as

$$\,^{eff}\rho = \,^{\widehat{q}}\rho + \frac{1}{2}[f(\,^{\widehat{q}}Q) - \,^{\widehat{q}}Q] - \,^{\widehat{q}}Q[f_Q(\,^{\widehat{q}}Q) - 1] \text{ and}$$
$$\,^{eff}p = \frac{\,^{\widehat{q}}\rho(1+w)}{2\,^{\widehat{q}}Qf_{QQ}(\,^{\widehat{q}}Q) + f_Q(\,^{\widehat{q}}Q)} - \frac{\,^{\widehat{q}}Q}{2}.$$

The condition $\,^{eff}w < -1/3$ holds for an accelerated universe. The formulas (72) and (73) depend in parametric form on effective temperature τ. This can be used for modelling diagonal nonmetric evolution on a time like parameter.

The equation of state (73) can be completed with G. Perelman thermodynamic variables (49) computed for a corresponding solution (71).

5 G. Perelman thermodynamics for topological QCs and nonmetric DE and DM

Physical properties of general τ-families of off-diagonal cosmological solutions encoding nonmetric and topological QC structures for DE and DM configurations can't be studied in the framework of the ΛCDM paradigm. Nevertheless, we can always characterize such models by respective G. Perelman thermodynamic variables, which can be computed in explicit form for all classes of solutions in geometric flow and gravity theories. The computations simplify substantially if we consider nonlinear symmetries transforming the data for generating functions and generating sources into equivalent ones re-defined in terms of equivalent new generating functions and effective τ-running cosmological constants. We considered such transforms in previous section for $\widehat{\Lambda}(\tau) = \,^{m}\Lambda(\tau) + \,^{e}\Lambda(\tau) + \,^{DE}\Lambda(\tau)$, see formulas (A.10). The goal of this section is to compute in explicit form the variables $\widehat{Z}$ (47) and $\,^{q}\widehat{\mathcal{E}}(\tau)$, $\,^{q}\widehat{S}(\tau)$ from (49) for such off-diagonal classes of cosmological solutions.[14]

For simplicity, we consider the same behavior for horizontal and vertical τ-running cosmological constants when $\,^{h}\Lambda(\tau) = \,^{v}\Lambda(\tau) = \widehat{\Lambda}(\tau)$ and the canonical Ricci scalar (see explanations for formulas (19)) is computed as

$$\widehat{\mathbf{R}}sc = 2\left[\,^{m}\Lambda(\tau) + \,^{e}\Lambda(\tau) + \,^{DE}\Lambda(\tau)\right]$$

for any generating data for cosmological solutions taken in a form (66), (68) or (70). The nonholonomic conditions for normalizing functions $\widehat{\zeta}(\tau)$ are stated as $\widehat{\mathbf{D}}_{\alpha}\widehat{\zeta} = 0$ and approximate $\widehat{\zeta} \approx 0$. To simplify computations we can fix a frame/coordinate system when such conditions are satisfied and then redefine the constructions for arbitrary bases and normalizing functions. We obtain:

$$\,^{q}\widehat{Z}(\tau) = \exp[\int_{\widehat{\Xi}} \frac{1}{8(\pi\tau)^2}\,\delta\,^{q}\mathcal{V}(\tau)],$$
$$\,^{q}\widehat{\mathcal{E}}(\tau) = -\tau^2\int_{\widehat{\Xi}} \frac{1}{8(\pi\tau)^2}$$
$$\times\left[2\left[\,^{m}\Lambda(\tau) + \,^{e}\Lambda(\tau) + \,^{DE}\Lambda(\tau)\right] - \frac{1}{\tau}\right]\delta\,^{q}\mathcal{V}(\tau),$$

14 We omit more cumbersome calculations for $\,^{q}\widehat{\sigma}(\tau)$. Here we also note that for quasi-stationary configurations a similar nonmetric geometric calculus was provided in section 4 of [15] (for instance, for nonmetric wormhole solutions and solitonic hierarchies). The computations for cosmological configurations are certain sense dual to those for quasi-stationary ones, see details in [17,19]. In this work, we consider a different class of nonmetric geometric flow and gravity theories, which involve different types of effective sources, generating functions and nonlinear symmetries corresponding to topological QC configurations.

$$ {}^{q}\widehat{S}(\tau) = -{}^{q}\widehat{W}(\tau) = -\int_{\widehat{\Xi}} \frac{1}{4(\pi\tau)^2} \times \left[\tau\left[{}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau)\right] - 1\right]\delta\, {}^{q}\mathcal{V}(\tau), \quad (74) $$

where the topological QC cosmological data are encoded into $\delta\, {}^{q}\mathcal{V}(\tau)$ determined by the determinant of corresponding off-diagonal solutions.

The volume form $\delta\, {}^{q}\mathcal{V}(\tau)$ (48) can be computed for cosmological d-metrics (67), with η-polarization functions, or (69), for χ-polarization functions, and including data for corresponding nonmetric generating sources. Respectively, we obtain

$$ \begin{aligned} {}^{\widehat{q}}\Phi(\tau) &= 2\sqrt{|[\,{}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau)]\,{}^{\widehat{q}}h_3(\tau)|} \\ &= 2\sqrt{|[\,{}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau)]\,{}^{\widehat{q}}\eta_3(\tau)\,\mathring{g}_3(\tau)|} \\ &\simeq 2\sqrt{|[\,{}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau)]\,\zeta_3(\tau)\,\mathring{g}_3|} \\ &\quad \times \left[1 - \frac{\varepsilon}{2}\,{}^{\widehat{q}}\chi_3(\tau)\right]. \end{aligned} \quad (75) $$

Considering nonholonomic and nonmetric evolution models with trivial integration functions ${}_1n_k = 0$ and ${}_2n_k = 0$ and introducing formulas (75) in (48), then separating terms with shell τ-running cosmological constants, we compute:

$$ \begin{aligned} \delta\,{}^{q}\mathcal{V} &= \delta\mathcal{V}[\tau,\ {}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau);\ {}^{q}_{h}\Upsilon(\tau), \\ &\quad \times\ {}^{m}¥[\theta^a[\tau]], {}^{e}¥[\theta^a[\tau]],\ {}^{DE}¥[\theta^a[\tau]];\ \psi(\tau),\ {}^{\widehat{q}}h_3(\tau)] \\ &= \delta\mathcal{V}(\,{}^{q}_{h}\Upsilon(\tau),\ {}^{m}¥^{\widehat{2}}[\theta^a[\tau]],\ {}^{e}¥[\theta^a[\tau]],\ {}^{DE}¥[\theta^a[\tau]]; \\ &\quad \times\ {}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau),\ {}^{\widehat{q}}\eta_3(\tau)\mathring{g}_3) \\ &= \frac{1}{{}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau)}\,\delta_\eta\mathcal{V}, \\ &\quad \text{where } \delta_\eta\mathcal{V} = \delta\,{}^{1}_{\eta}\mathcal{V} \times \delta^2_\eta\mathcal{V}. \end{aligned} $$

Such volume forms encoding topological QC structures are parameterized by products of two functionals:

In these formulas, we distinguish effective sources and cosmological constants with labels m, e and DE because such functionals induce, or not, topological QC structures of different types. The functions $\widetilde{\psi}(\tau)$ are defined as a τ–family of solutions of 2-d Poisson equations with effective source ${}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau)$, or we can use $\psi(\tau)$ for a respective source ${}_h\Upsilon(\tau)$. Integrating on a closed hypersurface $\widehat{\Xi}$ the products of h- and v-forms from (76), we obtain a running cosmological phase space volume functional

$$ {}^{\shortmid}_{\eta}\mathring{\mathcal{V}}(\tau) = \int_{\widehat{\Xi}} \delta_\eta\mathcal{V}({}_h\Upsilon(\tau), [\,{}^{m}¥(\theta^a[\tau]) + {}^{e}¥(\theta^a[\tau]) + {}^{DE}¥(\theta^a[\tau])],\ \mathring{g}_\alpha) \quad (77) $$

determined by prescribed classes of generating η-functions, effective generating topological QC cosmological sources $\left[{}_h\Upsilon(\tau),\ {}^{m}¥(\theta^a[\tau]) + {}^{e}¥(\theta^a[\tau]) + {}^{DE}¥(\theta^a[\tau])\right]$, coefficients of a prime s-metric $\mathring{g}_\alpha$ and nonholonomic distributions defining the hyper-surface $\widehat{\Xi}$. The explicit values of volume forms ${}^{\shortmid}_{\eta}\mathring{\mathcal{V}}(\tau)$ depend on the data we prescribe for $\widehat{\Xi}$ the type of topological QC Q-deformations (encoded into η- or ζ-polarizations) which are used for deforming a prime cosmological d-metric into cosmological topological QC as we considered in Sect. 4. We consider that it is always possible to compute ${}^{\shortmid}_{\eta}\mathring{\mathcal{V}}(\tau)$ for certain nonlinear quasi-periodic generating data and general Q-deformations. The thermodynamic variables depend explicitly on the τ-running effective cosmological constants when such dependencies can be chosen in some form explaining observational cosmological data and prescribing the evolution of off-diagonal DE and DM configurations.

Introducing functional (76) into the formulas (74), we compute

$$ {}^{q}\widehat{Z}(\tau) = \exp\left[\frac{{}^{\shortmid}_{\eta}\mathring{\mathcal{V}}(\tau)}{8(\pi\tau)^2}\right], $$

$$ {}^{q}\widehat{\mathcal{E}}(\tau) = \left[\frac{1}{\tau} - 2(\,{}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau))\right]\frac{{}^{\shortmid}_{\eta}\mathring{\mathcal{V}}(\tau)}{8\pi^2}, $$

$$ \begin{aligned} \delta^1_\eta\mathcal{V} &= \delta^1_\eta\mathcal{V}\left[\,{}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau), \eta_1(\tau)\,\mathring{g}_1\right] = e^{\widetilde{\psi}(\tau)}dx^1dx^2 \\ &= \sqrt{|\,{}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau)|}e^{\psi(\tau)}dx^1dx^2, \text{ for } \psi(\tau) \text{ being a solution of } (A.3), \\ \delta^2_\eta\mathcal{V} &= \delta^2_\eta\mathcal{V}[\,{}^{m}¥[\theta^a[\tau]],\ {}^{e}¥[\theta^a[\tau]],\ {}^{DE}¥[\theta^a[\tau]],\ {}^{\widehat{q}}\eta_3(\tau)\,\mathring{g}_3] \\ &= \frac{\partial_t|\,{}^{\widehat{q}}\eta_3(\tau)\,\mathring{g}_3|^{3/2}}{\sqrt{|\int dt\,[\,{}^{m}¥(\theta^a[\tau]) + {}^{e}¥(\theta^a[\tau]) + {}^{DE}¥(\theta^a[\tau])]\{\partial_t|\,{}^{\widehat{q}}\eta_3(\tau)\,\mathring{g}_3|\}^2|}} \\ &\quad \times \left[dt + \frac{\partial_i\left(\int dt\,[\,{}^{m}¥(\theta^a[\tau]) + {}^{e}¥(\theta^a[\tau]) + {}^{DE}¥(\theta^a[\tau])]\partial_t|\,{}^{\widehat{q}}\eta_3(\tau)\,\mathring{g}_3|\right)dx^i}{[\,{}^{m}¥(\theta^a[\tau]) + {}^{e}¥(\theta^a[\tau]) + {}^{DE}¥(\theta^a[\tau])]\partial_t|\,{}^{\widehat{q}}\eta_3(\tau)\,\mathring{g}_3|}\right]dt. \end{aligned} \quad (76) $$

$$
{}^{q}\widehat{S}(\tau) = -\,{}^{q}\widehat{W}(\tau) = \left[1 - \tau({}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau))\right] \frac{{}^{\shortmid}_{\eta}\mathring{\mathcal{V}}(\tau)}{4\,(\pi\tau)^{2}}. \quad (78)
$$

We can define the effective volume functionals (76) and geometric thermodynamic variables (78) for further parametric decompositions with ε-linear approximations (75) and ε-polarizations and find parametric formulas for τ-flows and Q-deformations of prime metrics,

$$
\begin{aligned}
\delta\mathcal{V} = {} & \delta\mathcal{V}_0[\tau,\ {}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau); \\
& \times_h \Upsilon(\tau), [\,{}^{m}\,¥(\theta^a[\tau]) + {}^{e}\,¥(\theta^a[\tau]) \\
& + {}^{DE}¥(\theta^a[\tau])];\ \psi(\tau),\ \mathring{g}_\alpha;\ \zeta_3(\tau)] \\
& + \varepsilon\delta\mathcal{V}_1[\tau,\ {}^{m}\Lambda(\tau) + {}^{e}\Lambda(\tau) + {}^{DE}\Lambda(\tau);\ {}_h\Upsilon(\tau), \\
& \times [{}^{m}\,¥(\theta^a[\tau]) + {}^{e}\,¥(\theta^a[\tau]) + {}^{DE}¥(\theta^a[\tau])]; \\
& \times \psi(\tau),\ \mathring{g}_\alpha;\ \zeta_3(\tau),\ {}^{\widehat{q}}\chi_3(\tau)]. \quad (79)
\end{aligned}
$$

Computing such a (79) for a family of off-diagonal cosmological solutions constructed in previous section, we can define corresponding ε-decompositions of the thermodynamic variables (78), expressed as

$$
\begin{aligned}
& \widehat{\mathcal{W}}(\tau) = \widehat{\mathcal{W}}_0 + \varepsilon\widehat{\mathcal{W}}(\tau), \widehat{\mathcal{Z}}(\tau) = \widehat{\mathcal{Z}}_0\ \widehat{\mathcal{Z}}_1(\tau), \\
& \widehat{\mathcal{E}}(\tau) = \widehat{\mathcal{E}}_0 + \varepsilon\widehat{\mathcal{E}}_1(\tau), \widehat{\mathcal{S}}(\tau) = \widehat{\mathcal{S}}_0 + \varepsilon\widehat{\mathcal{S}}_1(\tau). \quad (80)
\end{aligned}
$$

In our works on nonmetric geometric flows and τ-running cosmological theories, we do not present details of such cumbersome and incremental computations with ε–linear decomposition for $\delta\mathcal{V} = \delta\mathcal{V}_0 + \varepsilon\delta\mathcal{V}_1$ (79) and resulting (80) determined by corresponding χ-polarization functions. The final conclusion of this section is that we can always define and compute in general forms the corresponding generalized Perelman thermodynamical variables for different sources with labels m, e and DE, and elaborate on structure formation and statistical thermodynamic description of cosmological models with τ-running topological QC structures.

6 Conclusions

This is the second paper in a series of partner works devoted to nonmetric deformations of the theory of geometric flows, modified gravity, and applications in modern astrophysics and cosmology. The first one [15] was devoted to constructing and analyzing essential physical properties of quasi-stationary generic off-diagonal solutions in such theories. As the next steps (for this work), our main goals were to generalize our geometric and statistical thermodynamic methods for generating cosmological solutions and elaborate on possible applications in dark energy, DE, and dark matter, DM, physics. Such a multi- and interdisciplinary research program is motivated by recent results on nonmetric $f(Q)$-modified gravity and cosmology [8–11,31–33], when new classes of generic off-diagonal exact and parametric solutions for modified gravity theories, MGTs, with non-Riemannian connections [17–19] seem to describe in more adequate forms various observational data for accelerating cosmology recently provided by James Webb space telescope, JWST, [12–14].

To describe generic off-diagonal exact and parametric solutions for nonmetric MGTs in the framework of the Bekenstein–Hawking entropy paradigm is not possible because, in general, nonmetric and off-diagonal cosmological scenarios (for instance, the solutions with topological QC-like structure studied in this work) do not involve certain horizon/duality/holographic configurations. We had to elaborate on a new statistical/geometric thermodynamic paradigm introduced for geometric flow theories due to G. Perelman [22]; and developed in nonholonomic and relativistic forms for various MGTs and nonholonomic/nonassociative/nonmetric and other types modifications [16,29,30]. In this work, in addition to gaining a more complete understanding of cosmological effects of $f(Q)$ MGTs, we also studied new classes of τ-parametric generic off-diagonal and locally anisotropic cosmological solutions [17–19], elaborated on models with topological quasi-crystal, QC structure (see also other models with space and time structure [17,20,21]), and provided examples of how to compute G. Perelman thermodynamic variables for generic off-diagonal ansatz of cosmological metrics with topological QC structure.

Let us summarize and discuss the main new results and methods of this paper following the key ideas for solutions of objectives, **Obj 1- Obj 5**, formulated in the Introduction section:

1. In Sect. 2, we formulated the $f(Q)$ gravity theories in nonholonomic canonical $(2+2)$ variables which allowed us to prove general decoupling and integration properties of corresponding nonmetric deformed Einstein equations. Necessary technical results were provided in Appendix A as a summary of the anholonomic frame and connection deformation method, AFCDM, for constructing exact and parametric solutions for physically important systems of nonlinear PDEs in $f(Q)$ deformed nonmetric geometric flow and gravity theories. Such methods are not incremental but involve new ideas give new results for time dual nonholonomic transforms and nonmetric deformations of constructions from [15,17,18]. They can be summarized as the solution of **Ob1** for elaborating a general geometric formalism for constructing off-diagonal cosmological solutions in nonmetric geometric flow and gravity theories.

2. **Obj 2** was solved in Sect. 3.1 by formulating the $f(Q)$-distorted nonmetric geometric flow equations, derived as τ-running nonmetric Einstein equations and written in canonical nonholonomic variables as nonmetric Ricci solitons. Such formulations allowed us to apply the AFCDM and find exact and parametric solutions of such systems of nonlinear PDEs in general off-diagonal forms.
3. Then the statistical thermodynamics for nonholonomic $f(Q)$-distorted nonmetric geometric flows (i.e. the solution of **Obj 3)** was formulated in Sect. 3.2. Such models are different from those studied in [15] for $f(R, T, Q, T_m)$ nonmetric theories and their quasi-stationary solutions. Here we note that the nonmetric geometric flow τ-parameter is a temperature one as in [22], which can be exploited for describing new observational JWST data [12–14] when the cosmological evolution scenarios depends on a temperature like parameter being formulated in some generic nonlinear forms encoding or generating quasi-periodic structures, for instance, of topological QC type.
4. In Sect. 4 (providing solutions of **Obj 4** to consider applications of nonmetric geometric flow methods in DE and DM physics), we constructed and analyzed the most important physical properties of τ-families of generic off-diagonal cosmological solutions encoding nonmetricity and generating topological QC structures. Such solutions and defining toy topological models were formulated in nonholonomic forms (choosing corresponding classes of polarization functions and quasi-periodic effective sources) which may describe the geometric evolution of DM structure and correspond to effective running cosmological constants modelling DE configurations. In general, such generic off-diagonal cosmological scenarios are different from those defined for ΛCDM models and modifications.
5. Physical properties of τ-running families of nonmetric cosmological solutions, even for a fixed value of the geometric evolution parameter, can't be described in general form using the Bekenstein-Hawking thermodynamic paradigm. This requests the formulation of a new geometric and statistical thermodynamic concept of G. Perelman entropy. In explicit form (solving the **Obj 5**), we have shown how to compute Perelman's thermodynamic variables for nonmetric geometric flows and cosmological configurations with topological QC structure encoding $f(Q)$ nonmetric effects. We speculate how such variables may characterize the DE and DM physics determined by such a nonmetric cosmological scenarios. Necessary results on topological QC structures are provided in Appendix B by generalizing the approach in canonical nonholonomic variables which are important for applying the AFCDM.

The results listed and discussed in paragraphs 1–5 support the statements of the main **Hypothesis** formulated in Sect. 1 in such senses: (1) Nonmetric $f(Q)$ modifications of geometric flow and MGTs can be formulated in canonical nonholonomic variables which allow us to decouple and integrate in general forms physically important systems of nonlinear PDEs. (2) Such way constructed new classes of exact and parametric solutions are defined by generic off-diagonal metrics, nonmetric compatible affine connections and effective sources encoding nonmetric geometric flow deformations and possible quasi-periodic spacetime structures. (3) The cosmological solutions encoding various topological QC phases characterized by nonmetric versions of G. Perelman thermodynamic variables determine new features of the DE and DM physics and provide new geometric and thermodynamic methods for describing new observational JWST data.

Nevertheless, there are a number of important open questions and unsolved problems for nonmetricity physics which were laid out in [7,18] and (in questions on nonmetric geometric and information flow theories and gravity) QNGIFG1-4 from [15]. Perhaps, the next in the order to be solved in future works is to formulate a mathematically self-consistent and physically viable version of $f(Q)$-modified Einstein–Dirac theory and generalizing the AFCDM for constructing exact and parametric solutions describing nonmetric gravitational and spinor systems. In a more general context, a more fundamental and difficult task is to investigate possibilities of how to connect theories of nonmetric geometric flows to modern string theory. Such constructions should involve certain models of nonmetric twist products and nonassociative star deformations related to the research program on nonassociative geometric and information flows [29,30]. In future works, we plan to report on progress in solving such problems.

Acknowledgements This research on nonmetric geometric flows and applications extends SV and EN visit programs (respectively supported by the Fulbright USA-Romania and the Ministry of Education and Science of the Republic of Kazakhstan) at the physics department at California State University at Fresno, USA. The program on AFCDM for theories with effective sources encoding nonmetricity consists also a background for some objectives of a SV visiting fellowship at CAS LMU in Munich, Germany. SV and EN are grateful to the host professors D. Lüst and D. Singleton for kind support and collaboration.

Funding No funding or publication fee were provided for this research. SV states that his volunteer research in Chernivtsi, Ukraine, is related to former Fulbright and scientist at-risk visits at CSU Fresno, the USA; and CAS LMU, Munich, Germany.

Data Availability Statement This manuscript has no associated data. [Authors' comment: Data sharing does not apply to this article as no datasets were generated or analysed during the current study.]

Code Availability Statement The manuscript has no associated code/software. [Author's comment: Code/Software sharing not applica-

ble to this article as no code/software was generated or analysed during the current study.]

Funded by SCOAP[3].

Appendix A: Decoupling and integrability of nonmetric Ricci flows and cosmological equations

We review the anholonomic frame and connection deformation method, AFCDM), extended for constructing generic off-diagonal cosmological solutions of nonmetric geometric flow equations (38). Such solutions are "time" dual to certain quasi-stationary solutions, which can be generated by similar methods for different geometric variables; see details in sections 3.1, 3.2 and 3.5 of [19] and, for nonmetric geometric flows, see the partner work [15]. We apply abstract and symbolic geometric methods (for GR, see [2]) and analytic methods when similar proofs are considered in [17–19] and references therein.

A.1 Decoupling of nonlinear PDEs for nonmetric geometric cosmological flows

We consider the off-diagonal ansatz (53) with N-adapted coefficients of the canonical Ricci d-tensor and effective sources (55) for the system of nonlinear PDs (38) written in the form:

$$\widehat{R}^1_1(\tau) = \widehat{R}^2_2(\tau) = \frac{1}{2g_1 g_2}\left[\frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} + \frac{(g_2^{\bullet})^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1''\right] = -_h \yen(\tau),$$

$$\widehat{R}^3_3(\tau) = \widehat{R}^4_4(\tau) = \frac{1}{2h_3 h_4}\left[\frac{(h_3^{\diamond})^2}{2h_3} + \frac{h_3^{\diamond} h_4^{\diamond}}{2h_4} - h_3^{\diamond\diamond}\right] = -_v \yen(\tau),$$

$$\widehat{R}_{3k}(\tau) = \frac{h_3}{2h_4} n_k^{\diamond\diamond} + \left(\frac{3}{2}h_3^{\diamond} - \frac{h_3}{h_4}h_4^{\diamond}\right)\frac{n_k^{\diamond}}{2h_4} = 0,$$

$$\widehat{R}_{4k}(\tau) = \frac{w_k}{2h_3}\left[h_3^{\diamond\diamond} - \frac{(h_3^{\diamond})^2}{2h_3} - \frac{(h_3^{\diamond})(h_4^{\diamond})}{2h_4} + \frac{h_3^{\diamond}}{4h_3}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k(h_4^{\diamond})}{2h_4}\right] = 0. \quad (A.1)$$

In our works, we also use brief notations of partial derivatives when $\partial_1 q(u^\alpha) := q^\bullet$, $\partial_2 q(u^\alpha) := q'$, $\partial_3 q(u^\alpha) := q^*$ and $\partial_4 q(u^\alpha) := q^\diamond$, for an arbitrary function $q(u^\alpha)$. The d-metric ansatz (53) and corresponding nonmetric geometric flow equations (A.1) posses a Killing symmetry on d-vector ∂_3, i.e. the corresponding coefficients of the d-metric, N-connection (54) etc. do not depend on coordinate $u^3 = y^3$. To have at least one Killing symmetry on a v-coordinate is important for generating exact/parametric solutions of such off-diagonal equations in explicit form. In principle, we can consider more general ansatz without Killing symmetries but such constructions are technically more cumbersome and involve proofs on hundreds of pages, see discussions and examples in [17–19].[15]

We can write the system of equations (A.1) in a more simpler and explicit decoupled form if we express the h-components of the d-metric as $g_i(\tau) = e^{\psi(\tau,x^k)}$; introduce the coefficients

$$\alpha_i(\tau) := h_3^{\diamond}\partial_i[\varpi(\tau)],\ \beta(\tau) := h_3^{\diamond}(\tau)[\varpi(\tau)]^{\diamond},$$
$$\gamma(\tau) := (\ln|h_3(\tau)|^{3/2}/|h_4(\tau)|)^{\diamond}, \quad (A.2)$$

and consider a generating function $\Psi(\tau) = \exp[\varpi(\tau)]$ for $\varpi(\tau) := \ln|h_3^{\diamond}(\tau)/\sqrt{|h_3(\tau)h_4(\tau)|}|$. For simplicity, we do not write an explicit dependence of such coefficients on (τ, x^k), or (τ, x^k, y^3), but express:

$$\psi^{\bullet\bullet} + \psi'' = 2\ {}_h\yen(\tau), \quad (A.3)$$
$$(\varpi)^{\diamond} h_3^{\diamond} = 2h_3 h_4\ {}_v\yen(\tau), \quad (A.4)$$
$$n_k^{\diamond\diamond} + \gamma n_k^{\diamond} = 0, \quad (A.5)$$
$$\beta w_j - \alpha_j = 0, \quad (A.6)$$

Let us explain the explicit decoupling property of this system of nonlinear PDEs. The coefficients $g_i(\tau)$ are determined as solutions of a τ-family of 2-d Poisson equations (A.3). The coefficients $h_3(\tau)$ and $h_4(\tau)$ are related as solutions of (A.4) for any nontrivial $\varpi(\tau)$ and ${}_v\ \yen(\tau)$. This allows us to prescribe a value of $h_3(\tau)$ and find $h_4(\tau)$, or inversely. Having defined $h_a(\tau)$, we can compute the coefficients $\beta(\tau)$ and $\alpha_i(\tau)$ (using formulas (A.2)) and solve respective linear equations for $w_j(\tau)$ from (A.6). To find solutions for $n_k(\tau)$

[15] Here we note that the "t-duality" of locally anisotropic cosmological τ-evolution systems (53) and (A.1) means that such equations and solutions can be transformed into similar ones for quasi-stationary τ-evolving configurations if $h_3(\tau, x^k, t) \to h_4(\tau, x^k, y^3), h_4(\tau, x^k, t) \to h_3(\tau, x^k, y^3)$; $N_i^3 = n_i(\tau, x^k, t) \to N_i^4 = w_i(\tau, x^k, y^3)$, $N_i^4 = w_i(\tau, x^k, t) \to N_i^3 = n_i(\tau, x^k, y^3)$and $\partial_4 q(u^\alpha) := q^\diamond \to \partial_3 q(u^\alpha) := q^*$ etc. Nonmetric quasi-stationary models and their solutions are studied in the partner work [15] for different metric-affine configurations and effective sources.

we have to integrate two times on $y^4 = t$ in (A.5), when $\gamma(\tau)$ is determined by $h_3(\tau)$ and $h_4(\tau)$ as in (A.2).

Finally, we note that (A.3) defines 2-d conformal flat h-components of a d-metric in a form that allows a linear principle of superposition of solutions. The Eq. (A.4) is generic nonlinear and involve additional nonlinearities determined by coefficients (A.2). For a linear decomposition of effective sources ${}_{v}\yen = {}^{m}\yen + {}^{e}\yen + {}^{DE}\yen$ (56), the parametric contributions of different types of sources encode nonlinear nonmetric effects. The cosmological scenarios are generic off-diagonal even for small parametric interactions, and result in nonlinear nonmetric geometric evolution for nontrivial solutions for N-coefficients determined by (A.5) and (A.6).

A.2 Off-diagonal solutions for nonmetric cosmological configurations

Applying the AFCDM [17–19], we can integrate recurrently the system of nonlinear PDEs (A.3)–(A.6). Any such τ-family of generic off-diagonal metrics (53) is determined by respective families of a generating function $\varpi(\tau)$ (equivalently, $\Psi(\tau)$) and generating sources ${}_{h}\yen(\tau)$ and ${}_{v}\yen(\tau)$. In this subsection, we provide some classes and different parameterizations of such solutions which are important for elaborating cosmological models encoding nonmetric geometric evolution and generic off-diagonal interactions. The explicit form of such solutions depends on the type of parameterizations of generating functions and generating sources and how such values are related to integration functions and physical constants.

A.2.1 Nonlinear symmetries of nonmetric cosmological solutions

Nonmetric evolution models of anisotropic cosmological solutions with Killing symmetry and on ∂_3 and for effective sources (56) are defined by such generic off-diagonal quasi-stationary τ-families of d-metrics:

$$
\begin{aligned}
ds^2(\tau) &= e^{\psi(\tau,x^k)}[(dx^1)^2+(dx^2)^2] \\
&+\left\{g_3^{[0]} - \int dt \frac{[\Psi^2]^\diamond}{4({}_{v}\yen)}\right\}\left\{dy^3+\left[{}_{1}n_k + {}_{2}n_k\int dt \right.\right.\\
&\times \left.\left.\frac{[(\Psi)^2]^\diamond}{4({}_{v}\yen)^2|g_3^{[0]} - \int dt[\Psi^2]^\diamond/4({}_{v}\yen)|^{5/2}}\right]dx^k\right\}\\
&+\frac{[\Psi^\diamond]^2}{4\,({}_{v}\yen)^2\,\{g_3^{[0]} - \int dt[\Psi^2]^\diamond/4\,({}_{v}\yen)\}}\\
&\times\left(dt+\frac{\partial_i\Psi}{\Psi^\diamond}dx^i\right)^2.
\end{aligned} \tag{A.7}
$$

Such an exact/parametric solutions is determined by a generating function $\Psi(\tau,x^k,t)$, two generating effective sources ${}_{h}\yen(\tau,x^k)$ (encoded as a solution $\psi(\tau,x^k)$ of 2-d Poisson equation (A.3)) and ${}_{v}\yen(\tau,x^k,y^3)$, and integration functions ${}_{1}n_i(\tau,x^k), {}_{2}n_i(\tau,x^k)$ and $g_3^{[0]}(\tau,x^k)$. If for such d-metrics there are considered parametric decompositions as in (56), we generate recurrently compute Q-deformations and other type parametric solutions.

By straightforward computations, we can check that a τ-family of solutions (A.7) posses important nonlinear symmetries which allow us to re-write such d-metric in terms of different types of generating functions ($\Psi(\tau)$ or $\Phi(\tau)$) and transform effective sources into τ-running effective cosmological constants $\widehat{\Lambda}(\tau)$. Such nonlinear transforms, $(\Psi(\tau), {}_{v}\yen(\tau)) \leftrightarrow (\Phi(\tau), \widehat{\Lambda}(\tau) = const \neq 0$ for any $\tau_0)$, are defined by formulas

$$
\frac{[\Psi^2]^\diamond}{{}_{v}\yen(\tau)} = \frac{[\Phi^2(\tau)]^\diamond}{\widehat{\Lambda}(\tau)}, \text{ which can be integrated as} \tag{A.8}
$$

$$
\begin{aligned}
\Phi^2(\tau) &= \widehat{\Lambda}(\tau)\int dt({}_{v}\yen)^{-1}[\Psi^2(\tau)]^\diamond \text{ and/or}\\
\Psi^2(\tau) &= (\widehat{\Lambda}(\tau))^{-1}\int dt({}_{v}\yen)[\Phi^2(\tau)]^\diamond.
\end{aligned} \tag{A.9}
$$

For linear decompositions of effective sources (56, we can consider such nonlinear transforms into respective τ-running effective cosmological constants

$$
\begin{aligned}
{}_{v}\yen &= {}^{m}\yen + {}^{e}\yen + {}^{DE}\yen \leftrightarrow\\
\widehat{\Lambda} &= {}^{m}\Lambda + {}^{e}\Lambda + {}^{DE}\Lambda.
\end{aligned} \tag{A.10}
$$

We can prescribe nonholonomic dyadic structures when ${}^{e}\yen = {}^{e}\Lambda = 0$ for metric compatible configurations and generating sources ${}^{e}\widehat{\mathbf{Y}}_{\alpha\beta}$ (33) or ${}^{DE}\yen = {}^{DE}\Lambda = 0$ for trivial Q-deformations ${}^{DE}T_{\alpha\beta}$ (28). We can analyze cosmological scenarios with nonmetric geometric flow evolution prescribing certain small values of ${}^{e}\Lambda$ and/ or ${}^{DE}\Lambda$ even the terms ${}^{m}\yen$, ${}^{e}\yen$, and ${}^{DE}\yen$ may be not small and subjected to nonlinear symmetries (A.8) and (A.9).

Considering above stated nonlinear symmetries, we can write the quadratic element (A.7) in a form including effective cosmological constants,

$$
\begin{aligned}
&ds^2(\tau)\\
&= g_{\alpha\beta}(\tau,x^k,t,\Phi(\tau),\widehat{\Lambda}(\tau))du^\alpha du^\beta\\
&= e^{\psi(\tau,x^k)}[(dx^1)^2+(dx^2)^2] - \left\{g_3^{[0]}(\tau) - \frac{\Phi^2(\tau)}{4\,\widehat{\Lambda}(\tau)}\right\}\\
&\times\left\{dy^3+\left[{}_{1}n_k(\tau) + {}_{2}n_k(\tau)\int dt\right.\right.\\
&\times\left.\left.\frac{\Phi^2(\tau)[\Phi^\diamond(\tau)]^2}{|\,\widehat{\Lambda}(\tau)\int dt_{v}\yen(\tau)[\Phi^2(\tau)]^\diamond|[g_3^{[0]}(\tau) - \Phi^2(\tau)/4\,\widehat{\Lambda}(\tau)]^{5/2}}\right]\right\}\\
&- \frac{\Phi^2(\tau)[\Phi^\diamond(\tau)]^2}{|\,\widehat{\Lambda}(\tau)\int dt_{v}\yen(\tau)[\Phi^2(\tau)]^\diamond\,|[g_3^{[0]}(\tau) - \Phi^2(\tau)/4\,\widehat{\Lambda}(\tau)]}\\
&\times\left\{dt + \frac{\partial_i\int dt\,{}_{v}\yen(\tau)\,[\Phi^2(\tau)]^\diamond}{{}_{v}\yen(\tau)\,[\Phi^2(\tau)]^\diamond}dx^i\right\}^2.
\end{aligned} \tag{A.11}
$$

In these formulas, the indices run respectively $i, j, k, ... = 1, 2; a, b, c, ... = 3, 4$; there are used: generating functions $\psi(\tau, x^k)$ and $\Phi(\tau, x^{k_1}, t)$; generating sources $_h ¥(\tau, x^k)$ and $_v ¥(\tau, x^k, t)$; effective cosmological constants $\widehat{\Lambda}(\tau)$; and integration functions $_1n_k(\tau, x^j)$, $_2n_k(\tau, x^j)$ and $g_3^{[0]}(\tau, x^k)$.

Using (A.8), we express $h_3^{\diamond}(\tau) = -[\Psi^2(\tau)]^{\diamond}/4_v ¥(\tau)$, which allows to compute (we can chose necessary types of integration functions) in a form computed with $\Psi(\tau)$ from (A.7).[16] This means that taking $(h_3(\tau),_v ¥(\tau))$ as generating data, we can write the τ-families of cosmological d-metrics (A.7) or (A.11) in another equivalent form:

$$\begin{aligned} d\widehat{s}^2(\tau) &= \widehat{g}_{\alpha\beta}(\tau, x^k, t; h_4(\tau),_v ¥(\tau))du^\alpha du^\beta \\ &= e^{\psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] + h_3(\tau)\Big\{dy^3 + \Big[\ _1n_k(\tau) \\ &\quad +_2 n_k(\tau)\int dt \frac{[h_3^{\diamond}(\tau)]^2}{|\int dt[_v ¥(\tau)h_3(\tau)]^{\diamond}|\ [h_3(\tau)]^{5/2}}\Big]dx^k\Big\} \\ &\quad - \frac{[h_3^{\diamond}(\tau)]^2}{|\int dt[_v ¥(\tau)h_3(\tau)]^{\diamond}|\ h_3(\tau)} \\ &\quad \times \left\{dt + \frac{\partial_i[\int dt(_v ¥(\tau))\ h_3^{\diamond}(\tau)]}{_v ¥(\tau)\ h_3^{\diamond}(\tau)}dx^i\right\}^2 \end{aligned} \quad (A.12)$$

We can express $\Phi^2(\tau) = -4\widehat{\Lambda}(\tau)h_3(\tau)$ (also using the nonlinear symmetries (A.8) and (A.9)) and eliminate $\Phi(\tau)$ from the nonlinear quadratic element in (A.11). In a form which is similar to (A.12), we obtain a τ-family of solutions determined by generating data $(h_3(\tau); \widehat{\Lambda}(\tau),_v ¥(\tau))$.

A.2.2 τ-evolution of Q-deformed Levi-Civita configurations

The nonmetric off-diagonal locally anisotropic cosmological solutions considered in previous subsection were constructed for τ-families of canonical d-connections $\widehat{\mathbf{D}}(\tau)$. They encode Q-deformations and posses nonholonomically induced d-torsion coefficients $\widehat{\mathbf{T}}^{\gamma}_{\alpha\beta}(\tau)$, which are completely defined by the N-connection and d-metric structures. We can extract zero torsion LC-configurations for q-distortions of $\nabla(\tau)$ if we impose the conditions (39). By straightforward computations for quasi-stationary configurations, we can verify that all canonical d-torsion coefficients $\widehat{\mathbf{T}}^{\gamma}_{\alpha\beta}(\tau)$ vanish for ansatz (53) if there are satisfied such conditions:

$$\begin{aligned} &\mathbf{e}_i(\tau)\ln\sqrt{|\ h_3(\tau)|} = 0,\ w_i^{\diamond}(\tau) = \mathbf{e}_i(\tau)\ln\sqrt{|\ h_4(\tau)|}, \\ &\partial_i w_j(\tau) = \partial_j w_i(\tau) \text{ and} \\ &n_i^{\diamond}(\tau) = 0, n_k(\tau, x^i) = 0 \text{ and} \\ &\partial_i n_j(\tau, x^k) = \partial_j n_i(\tau, x^k). \end{aligned} \quad (A.13)$$

The solutions for necessary type of w- and n-functions depend on the class of vacuum, non-vacuum, Q-deformed and other type cosmological metrics which we attempt to generate.

We consider such a possibility to extract solutions which satisfy also the conditions (A.13): If we prescribe a generating function $\Psi(\tau) = \check{\Psi}(\tau, x^i, t)$, for which $[\partial_i(\check{\Psi})]^{\diamond} = \partial_i(\check{\Psi})^{\diamond}$, we solve the equations for w_j from (A.13) in explicit form if $_v ¥ = const$, or if such an effective source can be expressed as a functional $_v ¥(\tau, x^i, t) =_v ¥[\check{\Psi}(\tau)]$. Then, the conditions $\partial_i w_j(\tau) = \partial_j w_i(\tau)$ can be solved by any generating function $\check{A} = \check{A}(\tau, x^k, t)$ for which

$$w_i(\tau) = \check{w}_i(\tau) = \partial_i\ \check{\Psi}(\tau)/(\check{\Psi}(\tau))^{\diamond} = \partial_i\check{A}(\tau).$$

The equations for n-functions in (A.13) are solved for any $n_i(\tau) = \partial_i[\ ^2n(\tau, x^k)]$, i.e. is such N-connection coefficients do not depend on time like coordinate.

Putting together all above formulas for respective classes of generating functions with "inverse hats" and generating sources, we construct a nonlinear quadratic elements for locally anisotropic cosmological solutions (A.7) with zero canonical d-torsion,

$$\begin{aligned} d\check{s}^2(\tau) &= \check{g}_{\alpha\beta}(\tau, x^k, t)du^\alpha du^\beta = e^{\psi(\tau,x^k)}[(dx^1)^2 + (dx^2)^2] \\ &\quad + \left\{h_3^{[0]}(\tau) - \int dt \frac{[\check{\Psi}^2(\tau)]^{\diamond}}{4(_v ¥(\tau)[\check{\Psi}(\tau)])}\right\} \\ &\quad \times \left\{dy^3 + \partial_i[\ ^2n(\tau, x^k)]dx^i\right\}^2 \\ &\quad + \frac{[\check{\Psi}^{\diamond}(\tau)]^2}{4(_v ¥(\tau)[\check{\Psi}(\tau)])^2\{h_3^{[0]}(\tau) - \int dt[\check{\Psi}(\tau)]^{\diamond}/4_v ¥(\tau)[\check{\Psi}(\tau)]\}} \\ &\quad \times \left\{dt + [\partial_i(\check{A}(\tau))]dx^i\right\}^2. \end{aligned} \quad (A.14)$$

Under nonmetric geometric flows, the d-metrics (A.14) involve LC-configurations for $\nabla(\tau)$ but encode also Q-deformations included into $_v ¥(\tau)$. Such solutions can be generated for $_v¥ = {}^m ¥ + {}^e ¥ + {}^{DE}¥$ (56), or for ${}^e ¥ = 0$, or ${}^{DE}¥ = 0$. We can compare and decide if ceratin observational cosmological data are explained by any such solutions.

A.2.3 Nonmetric geometric cosmological evolution with small parametric polarizations

We can re-define the solutions constructed in previous subsection in such a form which allow to study τ-families of Q-deformations of a **prime** d-metric $\mathring{\mathbf{g}}$ (it can be an arbitrary one; or a solution of some equations in GR or a MGT) into a τ-family of **target** d-metrics $\mathbf{g}(\tau)$ of type (53),

$$\begin{aligned} \mathring{\mathbf{g}} &= [\mathring{g}_\alpha,\ \mathring{N}_i^a] \rightarrow \mathbf{g}(\tau) = [g_\alpha(\tau) = \eta_\alpha(\tau)\mathring{g}_\alpha, \\ & N_i^a(\tau) = \eta_i^a\ (\tau)\mathring{N}_i^a]. \end{aligned} \quad (A.15)$$

[16] We have to integrate on t the formula $[\Psi^2(\tau)]^{\diamond} = -4\int dt_v ¥(\tau)h_3^{\diamond}(\tau)$ for any prescribed $h_3(\tau)$ and $_v ¥(\tau)$.

Such nonholonomic transforms encode nonmetric deformations which in our approach are defined by some values $\eta_\alpha(\tau, x^k, t)$ and $\eta_i^a(\tau, x^k, t)$ called τ-running gravitational polarization (η-polarization) functions. The gravitational polarization functions are determined by respective τ-families of generating functions, generating sources and effective cosmological constants,

$$
\begin{aligned}
(\Psi(\tau),_v \ \yen(\tau)) &\leftrightarrow (\mathbf{g}(\tau),_v \ \yen(\tau)) \leftrightarrow (\eta_\alpha(\tau)\ \mathring{g}_\alpha \\
&\sim (\zeta_\alpha(\tau)(1+\varepsilon\chi_\alpha(\tau))\mathring{g}_\alpha,_v \ \yen(\tau)) \leftrightarrow \\
(\Phi(\tau),\ \widehat{\Lambda}(\tau)) &\leftrightarrow (\mathbf{g}(\tau),\ \widehat{\Lambda}(\tau)) \leftrightarrow (\eta_\alpha(\tau)\ \mathring{g}_\alpha \\
&\sim (\zeta_\alpha(\tau)(1+\varepsilon\chi_\alpha(\tau))\mathring{g}_\alpha,\ \widehat{\Lambda}(\tau)), \quad \text{(A.16)}
\end{aligned}
$$

where ε is a small parameter $0 \leq \varepsilon < 1$, with some $\zeta_\alpha(\tau, x^k, t)$ and $\chi_\alpha(\tau, x^k, t)$ (in brief, we shall use the term χ-polarization functions).

Using families of η- and/or χ-polarizations, the nonlinear symmetries (A.9) can be written in the form:

$$
\begin{aligned}
[\Psi^2(\tau)]^\diamond &= -\int dt_v \yen(\tau) h_3^\diamond(\tau) \simeq -\int dt_v \yen(\tau)(\eta_3(\tau)\ \mathring{g}_3)^\diamond \\
&\simeq -\int dt_v\ \yen(\tau)[\zeta_3(\tau)(1+\varepsilon\ \chi_3(\tau))\ \mathring{g}_3]^\diamond, \\
\Phi^2(\tau) &= -4\ \widehat{\Lambda}(\tau)h_3(\tau) \simeq -4\ \widehat{\Lambda}(\tau)\eta_3(\tau)\mathring{g}_3 \\
&\simeq -4\ \widehat{\Lambda}(\tau)\ \zeta_3(\tau)(1+\varepsilon\chi_3(\tau))\ \mathring{g}_3. \quad \text{(A.17)}
\end{aligned}
$$

So, the nonlinear symmetries of τ- and Q-deformations and generating functions for families of off-diagonal η-transforms of type (A.15) can be parameterized for η-polarizations,

$$\psi(\tau) \simeq \psi(\tau, x^k),\ \eta_3\ (\tau) \simeq \eta_3(\tau, x^k, t). \quad \text{(A.18)}$$

The η-polarization functions can be used for generating τ-families of locally anisotropic cosmological solutions of type (A.7),

$$
\begin{aligned}
&d\widehat{s}^2(\tau) = \widehat{g}_{\alpha\beta}(\tau, x^k, t; \mathring{g}_\alpha; \psi(\tau), \eta_3(\tau); \\
&{}_v \yen(\tau))du^\alpha du^\beta = e^{\psi(\tau)}[(dx^1)^2 + (dx^2)^2] \\
&\quad + \eta_3(\tau)\mathring{g}_3\Big\{dy^3 + \Big[{}_1 n_k(\tau) + {}_2 n_k(\tau) \\
&\quad \times \int dt \frac{[(\eta_3(\tau)\mathring{g}_3)^\diamond]^2}{|\int dt\ {}_v\ \yen(\tau)(\eta_3(\tau)\mathring{g}_3)^\diamond|\ (\eta_3(\tau)\mathring{g}_3)^{5/2}}\Big]dx^k\Big\}^2 \\
&\quad - \frac{[(\eta_3(\tau)\ \mathring{g}_3)^\diamond]^2}{|\int dt_v\ \yen(\tau)(\eta_3(\tau)\ \mathring{g}_3)^\diamond|\ \eta_3(\tau)\mathring{g}_3} \\
&\quad \times \left\{dt + \frac{\partial_i[\int dt\ {}_v\ \yen(\tau)\ (\eta_3(\tau)\mathring{g}_3)^\diamond]}{{}_v\ \yen(\tau)(\eta_3(\tau)\mathring{g}_3)^\diamond}dx^i\right\}^2. \quad \text{(A.19)}
\end{aligned}
$$

For $\Phi^2(\tau) = -4\ \widehat{\Lambda}h_3(\tau)$, we can transform (A.11) in a variant of (A.19) with η-polarizations determined by the generating data $(h_3(\tau);\ \widehat{\Lambda}\ (\tau))$. Here we note that it is difficult to understand if such off-diagonal metrics with general η- and Q-deformations may have, or not, certain physical importance even the primary data possess certain important physical interpretation.

We can study τ-flows and Q-deformations on a small parameter ε using ε-linear functions for generating data (A.16) and nonlinear symmetries (A.17). For such approximations, the η-polarizations in (A.19) describe nonholonomic deformations of a prime d-metric $\mathring{\mathbf{g}}$ into so-called ε-parametric τ-families of solutions with ζ-and χ-coefficients, when

$$
\begin{aligned}
&\psi(\tau) \simeq \psi(\tau, x^k) \simeq \psi_0(\tau, x^k)(1+\varepsilon_\psi \chi(\tau, x^k)), \text{ for} \\
&\eta_2(\tau) \simeq \eta_2(\tau, x^k) \simeq \zeta_2(\tau, x^k)(1+\varepsilon\chi_2(\tau, x^k)), \\
&\qquad \text{we can consider } \eta_2(\tau) = \eta_1(\tau); \\
&\eta_3(\tau) \simeq \eta_3(\tau, x^k, t) \simeq \zeta_3(\tau, x^k, t)(1+\varepsilon\chi_3(\tau, x^k, t)). \\
&\qquad \text{(A.20)}
\end{aligned}
$$

In such formulas, $\psi(\tau)$ and $\eta_2(\tau) = \eta_1(\tau)$ are such way chosen to be determined by solutions of the 2-d Poisson equation $\partial_{11}^2\psi(\tau) + \partial_{22}^2\psi(\tau) = 2\ {}_h\ \yen(\tau, x^k)$, see (A.3).

ε-parametric deformations (A.20) define such τ-families of locally anisotropic cosmological d-metrics with χ-generating functions,

$$
\begin{aligned}
& d\widehat{s}^2(\tau) = \widehat{g}_{\alpha\beta}(\tau, x^k, t; \psi(\tau), g_3(\tau); {}_{v}\yen(\tau)) du^\alpha du^\beta \\
& = e^{\psi_0}(1+\varepsilon\, {}^{\psi}\chi(\tau))[(dx^1)^2+(dx^2)^2] + \zeta_3(\tau)(1+\varepsilon\, \chi_3(\tau))\, \mathring{g}_3 \Bigg\{ dy^3 + \Bigg[(\mathring{N}^3_k)^{-1}[{}_1n_k(\tau) + 16\, {}_2n_k(\tau) \\
& \times \left[\int dt \frac{\left(\left[(\zeta_3(\tau)\mathring{g}_3)^{-1/4}\right]^\diamond\right)^2}{|\int dt[{}_{v}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)]^\diamond|} \right] + \varepsilon \frac{16{}_2n_k(\tau)\int dt \frac{([(\zeta_3(\tau)\mathring{g}_3)^{-1/4}]^\diamond)^2}{|\int dt[{}_{v}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)]^\diamond|}\left(\frac{[(\zeta_3(\tau)\mathring{g}_3)^{-1/4}\chi_3)]^\diamond}{2[(\zeta_3(\tau)\mathring{g}_3)^{-1/4}]^\diamond} + \frac{\int dt[{}_{v}\yen(\tau)(\zeta_3(\tau)\chi_3(\tau)\mathring{g}_3)]^\diamond}{\int dt[{}_{v}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)]^\diamond}\right)}{{}_1n_k(\tau) + 16{}_2n_k(\tau)\left[\int dt \frac{([(\zeta_3(\tau)\mathring{g}_3)^{-1/4}]^\diamond)^2}{|\int dt[{}_{v}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)]^\diamond|}\right]} \Bigg] \\
& \mathring{N}^3_k dx^k \Bigg\}^2 . - \left\{ \frac{4[(|\zeta_3(\tau)\mathring{g}_3|^{1/2})^\diamond]^2}{\mathring{g}_3|\int dt\{{}_{v}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)^\diamond\}|} - \varepsilon \left[\frac{(\chi_3(\tau)|\zeta_3(\tau)\mathring{g}_3|^{1/2})^\diamond}{4(|\zeta_3(\tau)\mathring{g}_3|^{1/2})^\diamond} - \frac{\int dt\{{}_{v}\yen\left[(\zeta_3(\tau)\mathring{g}_3)\chi_3(\tau)\right]^\diamond\}}{\int dt\{{}_{v}\yen(\tau)(\zeta_3(\tau)\mathring{g}_3)^\diamond\}} \right] \right\} \mathring{g}_4 \\
& \left\{ dt + \left[\frac{\partial_i \int dt_v\, \yen(\tau)\zeta_3^\diamond(\tau)}{(\mathring{N}^3_i)_v\, \yen(\tau)\zeta_3^\diamond(\tau)} + \varepsilon \left(\frac{\partial_i[\int dt_v\, \yen(\tau)\, (\zeta_3(\tau)\chi_3(\tau))^\diamond]}{\partial_i\, [\int dt_v\, \yen(\tau)\zeta_3^\diamond(\tau)]} - \frac{(\zeta_3(\tau)\chi_3(\tau))^\diamond}{\zeta_3^\diamond(\tau)} \right) \right] \mathring{N}^4_i dx^i \right\}^2
\end{aligned}
\tag{A.21}
$$

Such off-diagonal parametric solutions allow us to define, for instance, ellipsoidal deformations of spherical symmetric cosmological metrics into similar ones with ellipsoid symmetry. Locally anisotropic cosmological d-metrics of type (A.21) can be generated for by certain small parametric deformations and generating data $(\Phi(\tau), \widehat{\Lambda}(\tau))$.

Appendix B: A brief review on 2+2 spacetime topological quasicrystal structures

We outline necessary results on nonholonomic quasicrystalline structures which are necessary to elaborate on models of interacting topological phase of (effective) matter and nonmetric gravitational vacuum under geometric evolution, i.e. DE and DM phases. Considered elasticity models develop the constructions with cosmological quasicristals, QCs, [17,20] and do not consider nontrivial quantized topological terms with far richer structure than their crystalline counterparts studied in condensed matter physics [21].

B.1 Definition of spacetime QC and quasicrystalline topological phases

Let us state the definition of QC spacetime structure that we adopt in this work. Similarly to [21], we consider a countable set Vec^* with vectors $\mathbf{k} = \{k^{\alpha'}\} \in Vec^* \subset T^*\mathbf{V}$ for a point $\mathbf{x}(u) = \{x_{\alpha'}(u)\}$, for $u \in \mathbf{V}$ of a Lorentz manifold with Q-deformations. For a local (relativistic) quantum mechanical (QM) model, the expectation value $\langle\widehat{O}\rangle$ of a local observable $\widehat{O}$ can be expressed as a Fourier series

$$\langle\widehat{O}(\mathbf{x}(u))\rangle = \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{kx}} \text{ or } \langle\widehat{\mathbf{O}}(\mathbf{x}(u))\rangle = \sum_{\mathbf{k}} \mathbf{a}_{\mathbf{k}} e^{i\mathbf{kx}},$$

where $\mathbf{kx}$ denotes the scalar product in u and the coefficients $a_{\mathbf{k}} = a_{\mathbf{k}}(\widehat{O})$ depend on the choice of operator $\widehat{O}$. We can consider boldface symbols for d-operators adapted to a N-connection splitting (3). A crystal structure can be generated by a finite set Vec^* considered as a reciprocal lattice of a crystal defined for d' spacial dimensions (called as primitive reciprocal lattice vectors; for modelling spacetime and time like crystals with nonholonomic structure, we can consider pseudo-Euclidean signatures [17,20]). We say that such a system is a QC if Vec^* is defined in such a way but the smallest such set is of size greater than d'.

A quasicrystalline topological phase is defined by a family of Hamiltonians such that the ground state is always gapped and when the above QC conditions are satisfied. We can consider a QC with elasticity when $\langle\widehat{O}(\mathbf{x}(u))\rangle' = \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i(\phi_{\mathbf{k}}+\mathbf{kx})}$ for an independent choice on $\widehat{O}$ of phases subjected to the conditions $\phi_{\mathbf{k_1}} + \mathbf{k_2} = \phi_{\mathbf{k_1}} + \phi_{\mathbf{k_2}}[mod 2\pi]$ for any $\mathbf{k_1}, \mathbf{k_2} \in Vec^*$. The ϕ-filed can be thought as a "order parameter" for some spontaneous symmetry-breaking ground states, when the low-energy elastic deformation are considered as long-wavelength fluctuations (for such an order parameter). For such models, we replace $\phi_{\mathbf{k}}$ with a slow varying function $\phi_{\mathbf{k}} = \phi_{\mathbf{k}}(\mathbf{x}, t) = \phi_{\mathbf{k}}(u)$. Introducing phase fields

$$
\begin{aligned}
\theta_{\mathbf{k}}(u) &= \phi_{\mathbf{k}}(u) + \mathbf{kx}, \text{ for } \theta_{\mathbf{k}_1+\mathbf{k}_2}(u) \\
&= \theta_{\mathbf{k}_1}(u) + \theta_{\mathbf{k}_2}(u)\ [mod 2\pi], \\
&\text{when } \langle\widehat{O}(\mathbf{x}(u))\rangle' = \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\theta_{\mathbf{k}}(u)}.
\end{aligned}
\tag{B.1}
$$

We approximate for a low-energy deformation $\nabla\theta_{\mathbf{k}}(u) \approx \mathbf{k}$, when for some vacuum state $\nabla\theta_{\mathbf{k}}(u) = \mathbf{k}$. For canonical N-adapted deformations, such formulas transform into similar ones with $\nabla\theta_{\mathbf{k}}(u) \rightarrow \widehat{\mathbf{D}}\theta_{\mathbf{k}}(u)$.

For QC structures, the solutions of (B.1) can be parameterized by a set of reciprocal d-vectors $\mathbf{K}^1, ..., \mathbf{K}^{\widehat{d}}$ that generate $Vec^*(\widehat{d} > d'$, for a crystal structure, $\widehat{d} = d')$. There are imposed such properties:

1. Every d-vector can be expressed as an integer linear combination of type $\mathbf{k} = n_I \mathbf{K}^I \in Vec^*$ (for $I = 1, 2, ..., \widehat{d}$).
2. The reciprocal d-vectors are linearly independent over integers, i.e. if $n_I \mathbf{K}^I = 0$ for some integers n_I, then all $n_I = 0$ (we do no discuss variants of definition of QCs when this condition is dropped, for instance, when we have an over complete set of d-vectors).

Using properties 1 and 2, we can introduce phase angle fields $\theta^I(u)$ and write the solutions of (B.1) in the form $\theta_{\mathbf{k}}(u) = \theta^I(u) n_I(\mathbf{k})$, where $n_I(\mathbf{k})$ is a unique integer vector and $\nabla \theta^I(u) \approx K^I$ (or $\widehat{\mathbf{D}} \theta^I(u) \approx \mathbf{K}^I$ for N-adapted canonical constructions). Thus, we parameterize the elastic deformations in terms of $\widehat{d}$ fields $\theta^I(u)$. For a crystal type structure with $\widehat{d} = d'$, this defines phonon modes; for a QC structure with $\widehat{d} > d'$ there are more elastic modes than in a crystal. In this paper, we work directly with $\theta^I(u)$ even in condensed matter physics one consider decompositions into certain phonon and phason modes, see [21] and references therein.

Finally, we note that in the ground spacetime vacuum space, $\theta^I(u) = \theta^I_{[0]} + \mathbf{K}^I \mathbf{x}$, with $\theta^I_{[0]}$ taken for a crystal like structure. In such cases, the expectation values of observable in a QC deformation are given by linear mappings of a d' dimensional physical space (for modeling the crystal structure) into a hyperplane slice through a $\widehat{d}$-dimensional crystal "superspace".

B.2 Topological terms for nonholonomic spacetime QCs and elasticity

Let us consider a $\widehat{d}$-dimensional vector C_I, an antisymmetric (matrix $\widehat{d} \times \widehat{d}$) C_{IJ}, an antisymmetric tensor C_{IJK}, and a gauge field $\mathbf{A}_\mu(u)$ with local $U(1)$ symmetry. Using such algebraic data, vector A_μ, and angle space fields $\theta^I(u)$, we can introduce such effective Lagrange densities for topological QCs (see details in sections III and V of [21]):

$$
\begin{aligned}
{}^{\theta 1}\widehat{\mathcal{L}} &= {}^{\theta 1}\ \widehat{\mathcal{L}}_{[0]} + \frac{1}{2\pi} C_I \varepsilon^{\mu\nu} A_\mu \partial_\nu(\theta^I), \text{ for } d' = 1; \\
{}^{\theta 2}\widehat{\mathcal{L}} &= {}^{\theta 2}\ \widehat{\mathcal{L}}_{[0]} + \frac{1}{2\pi} C_{IJ} \varepsilon^{\mu\nu\gamma} A_\mu \partial_\nu(\theta^I) \partial_\gamma(\theta^J), \text{ for } d' = 2; \\
{}^{\theta 3}\widehat{\mathcal{L}} &= {}^{\theta 3}\ \widehat{\mathcal{L}}_{[0]} + \frac{1}{2\pi} C_{IJK} \varepsilon^{\mu\nu\gamma\sigma} A_\mu \partial_\nu(\theta^I) \partial_\gamma(\theta^J) \partial_\sigma(\theta^K), \\
&\quad \text{for } d' = 3, \qquad \text{(B.2)}
\end{aligned}
$$

where $\varepsilon^{\cdots}$ are absolute antisymmetric tensors and terms with label [0] can be defined for a specific crystal like structure.

In this work, we shall elaborate on models of dimensions $d' = 2, 3$. Such configurations are invariant (modulo 2π) under large gauge transformations of A on nonholonomic spacetime manifold $\mathbf{V}$ and for linearized equations each entry of C can be quantized to be integer. This defines a large family of symmetry-protected topological, SPT, phases which can be partially classified as QCs with $U(1)$ symmetry by integral antisymmetric rank-d' tensors of dimension $\widehat{d}$. In not N-adapted form, we can consider crystalline structures with $\widehat{d} = d'$, when all such tensors are some integer multiples of the Levi-Civita antisymmetric tensors. For a given QC SPT phase with underlying reciprocal lattice $\mathcal{L}$, the C-values depend on some choice of the generating reciprocal vectors K^I. Such phases are classified by the internal cohomology $H^{d'}(\mathcal{L}^*, \mathbb{Z})$, with integer $\mathbb{Z}$, where $\mathcal{L}$ is the space of homomorphisms from $\mathcal{L}^*$ to $U(1)$.

For QC spacetime configurations, there are possible additional topological terms which do not depend on $U(1)$ symmetry. If $\widehat{d} \geq d' + 1$, we can consider terms:

$$
\begin{aligned}
{}^{\Theta 1}\widehat{\mathcal{L}} &= \frac{1}{2\pi} \Theta_{IJ} \varepsilon^{\nu\gamma} \partial_\nu(\theta^I) \partial_\gamma(\theta^J), \text{ for } d' = 1; \\
{}^{\Theta 2}\widehat{\mathcal{L}} &= \frac{1}{2\pi} \Theta_{IJK} \varepsilon^{\nu\gamma\sigma} \partial_\nu(\theta^I) \partial_\gamma(\theta^J) \partial_\sigma(\theta^K) \text{ for } d' = 2; \qquad \text{(B.3)}
\end{aligned}
$$

and so on (for higher dimensions). Such terms are classified by $H^{d'+1}(\mathcal{L}^*, \mathbb{Z})$.

In this work, we study QC v-structures adapted to a N-splitting when $\theta^I \simeq \theta^a(x^i, t)$, for $I \simeq a$, and $\partial_j \theta^a(x^i, t) = K^a_j(x^i, t)$. For generating off-diagonal solutions, we can elaborate on models with $N^a_j(x^i, t) \simeq K^a_j(x^i, t)$ and define topological QC-configurations determined by certain θ^I terms.

Finally, we note that the angle space fields $\theta^I(u)$ can be related to another class of topological terms of Wess-Zumino type (see [21] and references therein) for $\widehat{d} \geq d' + 2$. There are not canonically expressible local Lagrangians in $d' + 1$ spacetime dimensions of an extension of the θ-fields. Such terms are characterized by $H^{d'+1}(\mathcal{L}^*, \mathbb{Z})$. For simplicity, in this work we work with QC structures of type ${}^{\theta 2}\widehat{\mathcal{L}}$ (B.2) or ${}^{\Theta 2}\widehat{\mathcal{L}}$ (B.3).

B.3 Mobility of dislocations and currents and charge density for spacetime QCs

In QCs, there are possible defects with dislocations which can move in N-adapted form only in the directions of their Burgers d-vector. Here, we provide necessary formulas for 2-d nonholonomic cosmological QC structures when

$$
\frac{1}{2\pi} \oint_C dx^i\ \mathbf{e}_i \theta^a(\tau, x^k, t) = \mathbf{b}^a \in \mathbb{Z}^{d'}, d' = 2,
$$

for a N-connection 2+2 splitting. In this formula, C is a loop surrounding the dislocations and $\mathbf{b}^a$ is the Burgers d-vectors. The nonholonomic and topological mobility constraints of a dislocations can be derived for a respective charge conservation law. We can consider a d-vector A_μ in (B.2), or introduce certain effective values determined by $\varepsilon^{\nu\gamma\sigma}\mathbf{e}_\nu(\theta^a)$ in (B.3), define certain "charged" mobility of $\theta^a(\tau, x^k, t)$ for temperature τ. The corresponding current is defined as the d-vector

$$\mathbf{J}^\nu = \frac{1}{8\pi^2} C_{ab}\varepsilon^{\nu\gamma\sigma}\mathbf{e}_\gamma(\theta^a)\mathbf{e}_\sigma(\theta^b).$$

The holonomic conservation law

$$\partial_\mu J^\mu = C_{ab}\varepsilon^{\mu\gamma\sigma}\left[\partial_\mu\partial_\gamma(\theta^a) - \partial_\gamma\partial_\mu(\theta^a)\right]\partial_\sigma(\theta^b) = 0$$

transforms into a nonholonomic relation

$$\mathbf{e}_\mu\mathbf{J}^\mu = C_{ab}\varepsilon^{\mu\gamma\sigma} W^\nu_{\mu\gamma}\mathbf{e}_\nu(\theta^a)\mathbf{e}_\sigma(\theta^b) = {}^J\mathbf{Z} \neq 0, \qquad \text{(B.4)}$$

where the anholonomy coefficients are defined in footnote 5 and ${}^J\mathbf{Z}$ is determined by N-coefficients. This reflects the fact that for nonholonomic and/or nonmetric systems the conservation laws became more sophisticate and may encode integration constants for corresponding Lagrange densities and distortion terms.

We can introduce double nonholonomic 2+2 and 3+1 splitting for space indices $i, j, ... = 1, 2$ and $\acute{i}, \acute{j}, ... = 1, 2, 3$, when $\partial_4 = \partial_t$. For off-diagonal vacuum models close to the equilibrium configurations, one approximates $\partial_t(\theta^a) \approx 0$, $\partial_i(\theta^a) \approx N^a_i(x^k)$ and $\partial_{j'}(\theta^a) \approx K^a_{j'}(x^i)$. In vicinity of such configurations, the nonholonmic distortions of the current conservation law (B.4) is computed

$$\mathbf{e}_\mu\mathbf{J}^\mu = C_{ab}\varepsilon^{ij} W^\nu_{4i}\mathbf{e}_\nu(\theta^a)N^b_j = {}^J\mathbf{Z},$$

which in holonomic frames is equivalent to

$$\partial_\mu J^\mu = C_{ab}\varepsilon^{\acute{i}\acute{j}}\left[\partial_4\partial_{\acute{j}}(\theta^a) - \partial_{\acute{j}}\partial_4(\theta^a)\right] K^a_{\acute{i}}(x^i) = 0.$$

Such formulas allow us to describe dislocations for a Burgers vector b^a at a position $x^{\acute{i}}(\zeta)$, with parameter ζ, moving at velocity $v^{\acute{i}}$, when

$$\left[\partial_4\partial_{\acute{j}} - \partial_{\acute{j}}\partial_4\right]\theta^a = \varepsilon_{\acute{j}\acute{k}} v^{\acute{k}}\delta[x^{\acute{i}} - x^{\acute{i}}(\zeta)], \text{ or}$$
$$W^\nu_{4i}\mathbf{e}_\nu(\theta^a) = \varepsilon_{jk} v^k\delta[x^i - x^i(\zeta)].$$

These local conservation law for (non) holonomic dislocations require that

$$C_{ab}b^a K^b_{j'} v^{j'} = 0, \text{ or } C_{ae}b^a N^e_j v^j = 0. \qquad \text{(B.5)}$$

The last condition is not satisfied if we consider general nonholonomic deformations. Equations (B.5) define the topological and nonholonomic mobility constraints of dislocations in a 2-d QC determined by the SPT invariant C_{ab} and the burgers vector b^a. Thus, dislocations in a spacetime QC are never completely immobilized and can move in some directions because there is at least one direction b^a which satisfies (non) holonomic conditions of type (B.5).

The effective gravitational vacuum fields A_μ in the action (B.2) can be evaluated by an effective charge density $\rho = \delta S/\delta A_4$ which for nonholonomic 2-d QC structures with $\partial_j(\theta^a) \approx K^a_j(x^i) \rightarrow N^a_j(x^i, t)$ and spacetime depending average density

$$\rho(x^i, t) = \frac{1}{8\pi^2} C_{ab}\varepsilon^{ij} N^a_i N^a_j. \qquad \text{(B.6)}$$

We assume that nearly gravitational vacuum equilibrium the non-topological part of the action does not contribute to average of such effective charge density but may result in nonholonomic modifications. Such an assumption is reasonable for effective phonon and phason fields which for (non) metric/ holonomic gravitational interactions does not give any contribution to the ground state charge density.

Finally, we note two simple tilling interpretations for QC structures with are similar to those presented in Figures 1 and 2 of section VIII A. of [21]. In the first case (called $k = 2$, for $\widehat{d} = 4 = d' + 2$ and $d' = 2$), there are generated the so-called QC topological phases with conventional 6 colours if no rotational symmetry is imposed. Imposing eight-fold rotational symmetry, we obtain two independent variants (for corresponding square and rhombus tiles). In the second case (conventional $k = 1$), four conventional colors can be generated and a 1-d SPT invariant is assigned for no rotational symmetry imposed. One independent SPT invariant is defined also if eightfold rotational symmetry is imposed. For equilibrium gravitational vacuum configurations, such nonholonomic configurations with nontrivial topology can be defined as quasi-stationary ones as in [15], Off-diagonal (non) metric gravitational interactions with average density $\rho(x^i, t)$ (B.6) result in nonholonomic deformations of QC structures which can be used for modeling DM and DE time depending configurations.

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