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Generalized Lie Algebraic Geometry in $R^3 \times$ SO(3) Configuration Space for SU(3) of Elementary Particles and for Wave-packing of Atomic Structure

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Abstract. In this paper, we show, by extending Lie's original 1871 thesis "on philosophical reflections upon the nature of Cartesian geometry" based on "transformation by which surfaces that touch each other are turned into similar surfaces ... between the Plucker line geometry and a geometry whose elements are the space's spheres" to include toroidal deformation of the sphere, how an algebraic geometric principle of duality between points, lines and planes of $3-$dimesional space provides a sufficiently general framework for realizations of Lie-algebra and its Lie-isotopic and Lie-admissible generalizations in solid state configuration space $R^3 \times$SO(3) compatible with translational periodicity of 3-dimensional space lattice. The generalization provide not only representation of SU(3) symmetry of extended (string-like) elementary particles with complementary duality of leptons and baryons, but also dual wave-packet representation of atomic structure and the periodic table, highlighting the significance of the fact that Mendeleev originally moulded his two-dimensional rendering of the periodic system on the dual Sanskrit grammar/phonetics.

1. Introduction

The general relativity theory (GRT) based on curved space-time has continued to provide a wealth of knowledge on matter-energy gravity [1-4]. However, since its proposal more than a century ago, there have been a number of opposing groups [1, 5-7]. For example, after numerous discussions in the early years of the biennial meetings of the Physical Interpretations of Relativity Theories (PIRT) organized at the Imperial College in London, Santilli pointed out nine inconsistency theorems for the GRT that originate from the use of Riemannian curvature and the abandonment of universal invariance [5]. Another issue with the GRT is the long open problem on the possibility of unifying the Einstein GTR and quantum field theory (QFT) $[8-9]$ which through the concept of duality field(wave)-particle provides the best understanding of matter-energy exchange. It is expected that unifying these two theories of matter-energy gravity and matter-energy exchange will be a great boost to achieving the unified force field theory (UFFT) also known as the theory of everything (TOE) [8]. Now like the GRT, there is a school of thought that believes that the QFT is not a complete theory of matter-energy exchange

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[10-12]. One of the reasons is that QFT has emanated from a truncated Hamiltonian leaving out the external forces as envisaged in the original description of nature [10]. The truncated Hamiltonian origin of QFT has a mathematical unitary structure, namely, that its basic time evolution constitutes a unitary transformation on a Hilbert space. This mathematical structure is based on the well-known conventional Lie-algebra [13-14], $[A, B] = AB - BA$ between generic matrices or operators A, B for time-reversible systems. It has been pointed that for irreversible systems, we need to include the external terms [10-11]. This extension naturally results to an algebraic inconsistency [11] which can be resolved by generalization of the time-reversible systems leading to hadronic mechanics that is characterized by the Lie-Santilli isotopic product [10,11], $[A, B] = A\hat{T}B - B\hat{T}A$, where $\hat{T}$ is the inverse of an "isounit", $\hat{I}$. However, for time-irreversible processes, like deep-inelastic non-unitary scattering theory of deformable particles in hadronic mechanic, a further generalization to Lie-admissible product $[A, B] = A\hat{T}B - B\hat{T}^{+}A$, $\hat{T} \neq \hat{T}^{+} \neq \hat{I}$ is required, so that in summary:

$$\left.\begin{array}{c} APB - BP^{+}A,\ (P^{+} = P = I) \rightarrow\ \text{Lie algebraic product} \\ APB - BP^{+}A,\ (P^{+} = P \neq I) \rightarrow\ \text{Lie} - \text{Santilli isotopic algebraic product} \\ APB - BQA,\ (P \neq Q \neq I) \rightarrow\ \text{Lie} - \text{admissible algebraic product,} \end{array}\right\} \tag{1}$$

where P^{+} is the transpose of, in general, a non-unitary matrix P and I is the conventional "unit" matrix. In digital signal processing (DSP) [15], the progressive generalization in Eq.(1) is applicable with A as an "input" and B an "output" while P characterizes the cuboct Lie symmetries of the nanostructure of the "processor".

A typical non-trivial example of an algebraic geometric realization of the Lie-admissible product in Eq.(1) arises naturally in non-unitary scattering theory of hadronic mechanics as the left and right dichotomy of a 2×2 matrix (A) defined by the *left and right genotopy (transformation)*:

$$\begin{array}{c} APB \equiv \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \leftarrow A \equiv \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \equiv A \rightarrow \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \equiv BQA, \\ \Rightarrow APB - BQA = 0,\ (P \neq Q \neq I) \end{array} \tag{2}$$

where, $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \sigma_1$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv i\sigma_2$, $P \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \equiv -\sigma_3$ and $(\sigma_1, \sigma_2, \sigma_3)$ are Pauli spin matrices.

In algebraic geometric terms, the matrix (A) corresponds to Lax pairing of a symmetric matrix $(g^{0}_{\mu\nu})$ and antisymmetric matrix $(\widetilde{\beta})$ as follows

$$\begin{array}{c} A \equiv (\tilde{g}_{\mu\nu}) = (1 + \tilde{\beta})(g^{0}_{\mu\nu}) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}; \\ \text{where, } (g^{0}_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \tilde{\beta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix};\ \tilde{\beta}^2 = -1. \end{array} \tag{3}$$

Thus in Minkowski space-time geometry, it characterizes a transformation of the "distance":

$$s^2 = x^{\mu} g^{0}_{\mu\nu} x^{\nu} \equiv (ct)^2 - x^2 - y^2 - z^2 \tag{4}$$

into

$$s^2 = x^{\mu} \widetilde{g}_{\mu\nu} x^{\nu} \equiv (ct)^2 - x^2 - y^2 - z^2 - 2cty, \tag{5}$$

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The objective of our study (in Sec.2) of such progressive generalizations in solid state configuration space $R^3 \times SO(3)$ of the six-vector, $(\mathbf{r}, \mathbf{c}t)$ of 3-vector coordinates $\{\mathbf{r} = (x, y, z)\}$ and 3-vector velocity-time coordinates $\{\mathbf{c}t = (c_x t, c_y t, c_z t)\}$ is to achieve a representation of SU(3) symmetry of elementary particles with complementary duality of leptons and baryons, as well as the dual wave-packet representation of atomic structure and the periodic table. The results will be discussed and conclusions drawn in Sec.3.

2. Transformation of sphere into a torus

Owing to a growing interest in the torus as the fundamental pattern for all creation from elementary particles [16] to cosmology [17] and the recent demonstration of the generic torus as a possible geometric object of the grand unification theorem [8, 18], we will adopt the torus as our geometric object here. Therefore, in this section we start by extending Lie's original 1871 thesis "on philosophical reflections upon the nature of Cartesian geometry" [13] based on "transformation by which surfaces that touch each other are turned into similar surfaces ... between the Plucker line [projective] geometry and a geometry whose elements are the space's spheres" by considering toroidal deformation of a "point" sphere into a torus in 3-dimensional space

$$0 = c^2t^2 + r^2 \to \hat{c}^2t^2 - r^2 - 2 \in \hat{c}tr = 0 \tag{6}$$

where $\hat{c}^2 \equiv c^2/n^2$, n being the refractive index of the medium and the torus is characterized by the parametric equations:

$$x = (\in r + ct\cos\theta)\cos\varphi, \quad y = (\in r + ct\cos\theta)\sin\varphi, \quad z = ct\sin\theta, \tag{7}$$

t being the time, φ the meridian angle, and θ the latitude angle. By eliminating φ from the parametric equations, one obtains

$$(ct)^2 - x^2 - y^2 - z^2 + (\in r)^2 = 2(ct)(\in r)\sin\theta \tag{8}$$

from which the *quantization into a lattice* results in the form [8, 19],

$$2(ct)(\in r)\sin\theta \equiv \begin{cases} \pm 2(ct)(\in r), & \text{if } \theta = (n + \frac{1}{2})\pi, \\ 0, & \text{if } \theta = \pi n. \end{cases} \tag{9}$$

$$\text{i.e.} \begin{cases} (ct \pm \in r)^2 - x^2 - y^2 - z^2 = 0, & \text{if } \theta = (n + \frac{1}{2})\pi \\ (ct)^2 + (\in r)^2 - x^2 - y^2 - = 0 \ \& \ z^2 = 0, & \text{if } \theta = \pi n \end{cases} \tag{10}$$

where $n = 0, 1, 2, ...$ The only values of the angle θ compatibles with perfect translational symmetry of a crystal lattice in three-dimensional space are those for which $(2\cos - 1)$ =integer, and hence $\theta = 2\pi/n$ where $n = 1, 2, 3, 4, 6$ include cubic and hexagonal lattices, but not pentagonal lattices (as shown in Figure 1(a)). The visual images of the types of geometric objects represented by Eqs.(10) are easily constructed by rewriting the first of the two equations in the form

$$\left(x^2 + y^2 + z^2 - (ct - \in r)^2\right)\left(x^2 + y^2 + z^2 - (ct + \in r)^2\right) = 0, \tag{11}$$

so that, for $\theta = (n + \frac{1}{2})\pi$, a pair of concentric spheres in $\vec{r}$-space of radii $ct \pm \in r$ define a spherical shell of thickness, $2 \in r$, with one sphere circumscribing the cube and the other sphere circumscribing the hexagon as shown in Figure 1(b). This may be termed the "black hole" solution associated with the primeval "Big Bang" theory of the creation of the universe.

The second of the two equations in (10) is a ruled quadric surface with real (u, v) line-generators in 3-dimensional projective space with homogeneous coordinates $(ct, \in r, x, y)$ given by

$$\begin{pmatrix} ct + x & \in r - y \\ \in r + y & ct - x \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0, \begin{pmatrix} ct + y & \in r - x \\ \in r + x & ct - y \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0, \text{if } \theta = \pi n \tag{12}$$

which may be termed "string" or "worm hole" solution as shown in Figure 1(c). However, this needs more geometrical proof to fully establish that the generalization can yield the string" or "worm hole" solution.

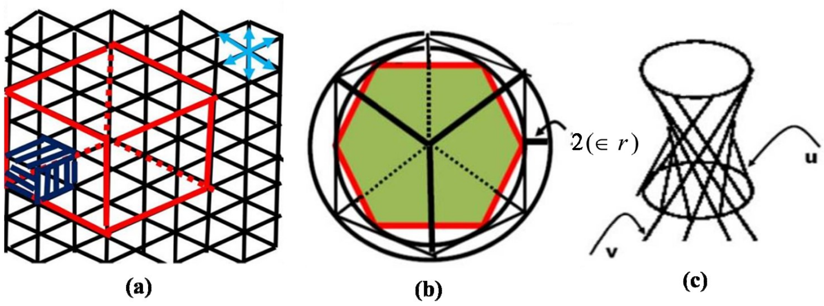

Figure 1. a) cubic and hexagonal lattices b) Cube hexagonal representation of 'black hole' c) Envelope of line generators ('strings').

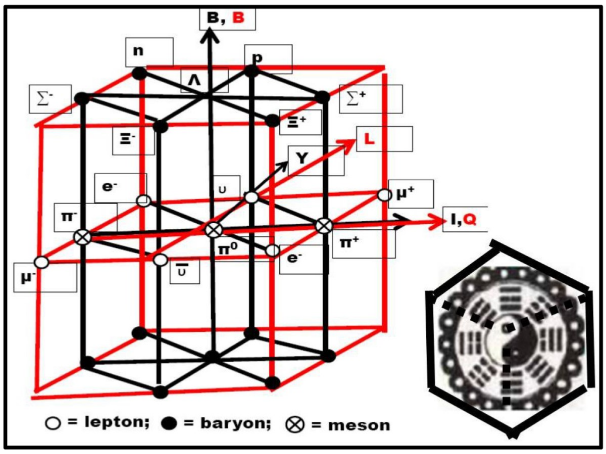

Figure 2. Representation of SU(3) for Baryons and Leptons and the Chinese I-Ching triagram well-spring of 8 fold way.

By recalling the well-known fact in solids state physics (see chapter 1 of [20]) that a face-centred cubic (fcc) packing of hard spheres has the same packing fraction ($f = 0.74$) as an ideal hexagonal close-packed (hcp) arrangement for which the ratio of the dimensions of axial to basal lattice cells, $c/a = \sqrt{8/3} = 1.633$ is the so-called φ-ratio that accounts for growth in the universe, we can construct the cube-hexagon hyperspace, in Figure 2 by using the fact [20] that the plane passing through the mid-points of six sides of the cube produces a hexagon which is the base of a hexagonal-based pyramid, and if the side of the cube is 1 unit, then the side of the hexagon is $\sqrt{1/2}$. As a result we obtain the cube-hexagon hyperspace shown in Figure 2, which is suitable for representations of hadrons (baryons and mesons) in hexagonal (I,B,Y)-space according to SU(3) symmetry and leptons (electron, muon and neutrino) in cubic (Q,L,B)-space, I being the isotopic spin, Y the hypercharge, B the baryon number, L the lepton number, and eQ the electric charge (in unit of proton charge, $e = 1$). Also indicated in Figure 2 is the SU(3) correlation with Chinese I-Ching [21] (8-fold triagram symbol of dynamic weather change) as a well-spring for the apparent *doubling of axes* which makes the cube-hexagon hyperspace a 6-dimensional realization of Lie-algebra and Lie-admissible generalizations in solid state configuration space $\mathrm{R}^3\times\mathrm{SO}(3)$ which is the result we are after. This has enabled us to correlate artistically in Figure 3 a synthesis of 3- and 4-prong VEDA wheel of motion [21] and screw pyramid clouds with 6-dimensional lattice $\mathrm{R}^3\times\mathrm{SO}(3)$, cube-hexagon scaling $(3^2+4^2=5^2)\rightarrow(6^2+8^2=10^2)\rightarrow(24^2+32^2=40^2)$ from representations of SU(3) as $[1^3+2^3+3^3=36=6\times6])\rightarrow[1^3+2^3+3^3+4^3=100=10\times10]$ (dual cube-hexagon lattice) defining two string lengths, $3+4+5=12$ and $24+32+40+32+24=152$ such that the proton-electron mass ratio is $12\times(1+152)=12\times153=1836[23-24]$.

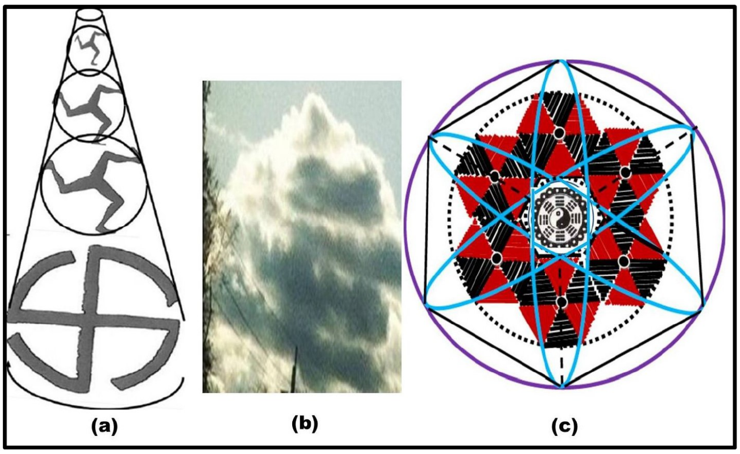

Figure 3. Correlation of a) Synthesis of 3 and 4-prong VEDA wheel of motion with b) Screw pyramid clouds (after AOL Today September 2, 2013) and c) 6-dimensional lattice $\mathrm{R}^3\times\mathrm{SO}(3)$, cube-hexagon scaling.

3. Discussion and Conclusion

The synthesis captured artistically and mathematically in Figure 3(c) as the result of our previous [14, 25] and current studies of the generalized Lie algebraic geometry in solid state configuration space $\mathrm{R}^3\times\mathrm{SO}(3)$ for matter at elementary particle and atomic levels (in as much as it accounts for the proton/electron mass ratio) has all the key features of a Theory of Everything. Apparently, Figure 3(c) also highlights the semiology (logic and physics of culture) [26] of the fact that Mendeleev originally moulded his two-dimensional rendering of the periodic system on the dual Sanskrit grammar/phonetics. The application in engineering to digital signal processing [15,25] in nanotechnology also points towards its promise for the future.

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