PATH INTEGRALS IN RIEMANNIAN MANIFOLDS

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I. Introduction

The configuration space $\mathbb{M}$ of a physical system is rarely $\mathbb{R}^n$. Indeed the configuration spaces of such common systems as a particle with spin, the harmonic oscillator, several hard spheres, a system of indistinguishable particles, etc. ... are Riemannian multiply connected manifolds [1]. The global properties of $\mathbb{M}$ contain much information on the generic properties of the physical system; nevertheless, they have received little attention. Why? Possibly because global problems are difficult, but also because physical laws have, since Newton, been largely stated as differential equations and investigated locally.

The Feynman formalism is the only global formalism of physics, but its global aspect is somewhat blurred in the original definition of path integrals. In this definition a path q mapping an interval $\mathbb{T} = [t_a, t_b]$ into $\mathbb{M}$ is replaced by p of its values $q^i = q(t_i)$ for a time subdivision of the interval $\mathbb{T}$

$$t_a = t_0 < t_1 < \ldots t_p < t_{p+1} = t_b \text{ with } q(t_a) = a,\ q(t_b) = b$$

The path integral over the space $\mathbb{P}(a,b)$ of all possible paths of the system between its states $A = (a,t_a)$ and $B = (b,t_b)$ is replaced by an integral over the space $\mathbb{R}^{pn}$ of the pn tuples $\{q^{i\alpha};\ i = 1 \ldots p,\ \alpha = 1 \ldots n\}$ where n is the dimension of $\mathbb{M}$, i.e., the number of degrees of freedom of the system. If it exists, the limit of the integral over $\mathbb{R}^{pn}$ when $p \to \infty$ is called the Feynman path integral. No unique prescription has been given for the $\mathbb{R}^{pn}$ integral beyond the requirement that the results agree with those obtained with the Schrödinger differential equation; the mathematical difficulties involved in defining the $\mathbb{R}^{pn}$ integral and its limit, and the computational difficulties in finding their values, whether unique or not! ..., has plagued both mathematicians and physicists. Thus most of the effort has been spent in showing that, locally, the Feynman formalism is equivalent to the Schrödinger formalism and in presenting the Feynman path integral as the solution of the Schrödinger equation satisfying some boundary conditions and other extraneous conditions as required by the given problem, such as the symmetry or antisymmetry property of the wave function. In this approach, the global properties of the space of all possible paths on $\mathbb{M}$ hardly enter the picture and much of the original beauty and power of the Feynman formalism is lost.

Recently [2] a new definition of the Feynman path integral which does not rest on the above limiting procedure has been proposed. It focuses on the space $\mathbb{P}(a,b)$ of all possible paths of the system from A to B. Besides a different emphasis, this definition leads to new techniques for computing path integrals. These techniques give readily results obtained laboriously by other methods; they are particularly well suited to the study of systems whose configuration space is not flat where other techniques are often ambiguous and encounter a great deal of difficulties.

In this paper we shall consider configuration spaces $\mathbb{M}$ which are Riemannian multiply connected manifolds with metric g and fundamental group π. We shall illustrate the propositions with the example of a free particle whose state at time t is $q(t) \in \mathbb{M}$. The action of this system is:

$$S(q) = \frac{m}{2}\int_{\mathbb{T}} \| \dot{q}(t) \|^2 \, dt = \frac{m}{2}\int_{\mathbb{T}} \left(\frac{Dq(t)}{dt}, \frac{Dq(t)}{dt}\right)_g dt$$

II. Pseudomeasures

The theory of promeasures (cylindrical measures) provides the framework for integration on function spaces; more precisely it is the basis for integration on Hausdorff topological vector spaces $\mathbb{X}$, locally convex. A promeasure is a family of bounded measures defined on a family of finite dimensional spaces suitably related to $\mathbb{X}$, satisfying some coherence conditions. The restriction to bounded measures makes it impossible to use the theory of promeasures for Feynman integration. However there is a one to one correspondence between the set of promeasures on $\mathbb{X}$ and their Fourier transforms on its dual $\mathbb{X}'$. One can thus define a promeasure by its Fourier transform and states the coherence conditions as conditions satisfied by the Fourier transforms. At this point it is possible to remove the restriction to bounded measures and to generalize the concept of promeasure: Indeed, the Fourier transforms of measures, considered as distributions of order zero, are defined for all measures, bounded or not. This new concept, given for convenience a name "pseudomeasure" and a symbol "w", enters our work only by its Fourier transform $\mathcal{F}w$. It is not known whether or not the mapping $w \to \mathcal{F}w$ of the set of pseudomeasures on $\mathbb{X}$ into the set of functions on $\mathbb{X}'$ is injective; i.e., whether or not a pseudomeasure w is uniquely defined by its Fourier transform $\mathcal{F}w$.

In the present study we consider only complex gaussian pseudomeasures; a gaussian pseudomeasure is a pseudomeasure whose Fourier transform is:

$$\mathcal{F}w = \exp(-iW/2)$$

where W is a quadratic form on X'. When $\mathbb{X}$ is the space of continuous paths x on $\mathbb{T}$, $\mathbb{X}'$ is the space of measures μ on $\mathbb{T}$ and

$$W(\mu) = \underline{W}(\mu,\mu) = \int_{\mathbb{T}} d\mu_\alpha(r) \int_{\mathbb{T}} d\nu_\beta(s) \; G^{\alpha\beta}(r,s)$$

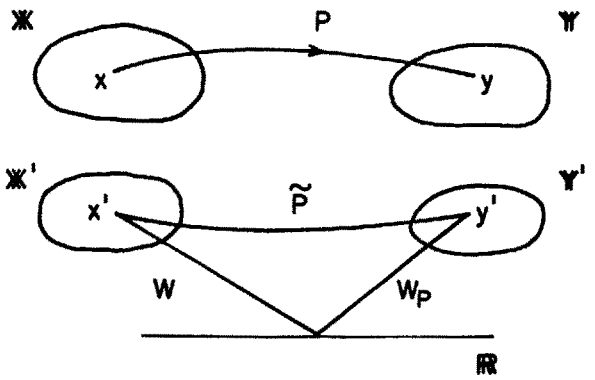
W is the variance of the gaussian, G its covariance. A normalized gaussian pseudomeasure is uniquely defined by its covariance.

Proposition 1. (Transformation of a Gaussian pseudomeasure under a linear mapping) [2,3] Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hausdorff, topological vector spaces, locally convex, let $\mathcal{X}'$ and $\mathcal{Y}'$ be their topological duals; let P be a linear continuous mapping from $\mathcal{X}$ into $\mathcal{Y}$, let $\tilde{P}$ be the transposed mapping from $\mathcal{X}'$ to $\mathcal{Y}'$ defined by

$$< \tilde{P}y', x > = < y', Px >$$

Let w be a Gaussian pseudomeasure on $\mathcal{X}$ of variance W. The image of w under P is a Gaussian pseudomeasure w_p on $\mathcal{Y}$ whose Fourier transform is

$$\mathcal{F}w_p = \exp(-iW_p/2) \qquad \text{with} \qquad W_p = W \cdot \tilde{P}$$



This proposition, together with the equation

$$\int_{\mathcal{Y}} F(y)dw_p(y) = \int_{\mathcal{X}} F \circ P(x)dw(x)$$

makes it possible to compute many Feynman integrals.

Example: Let $P : \mathcal{X} \to \mathbb{R}^{pn}$ by $x \mapsto y$ where y is the pn tuple $\{y^{i\alpha} = <\mu_{i\alpha}, x>\}$, then:

$$W_p(y') = y'_{i\alpha} W^{i\alpha j\beta} y'_{j\beta}$$

where $W^{i\alpha j\beta} = \underline{W}(\mu_{i\alpha}, \mu_{j\beta})$

and $$dw_p(y) = (2\pi i)^{-pn/2} (\det W^{-1})^{1/2} \exp(\tfrac{i}{2} y^{i\alpha} (W^{-1})_{i\alpha j\beta} y^{j\beta}) \, dy$$

The integrand of a Feynman path integral is often a function F of a set $\{<\mu_{i\alpha}, x>\}$, i.e., a function $F \circ P$ of x. The path integral over $\mathcal{X}$ is then equal to the integral of F(y) over $\mathbb{R}^{pn}$ with respect to the measure w_p computed in this example.
When $\mu_{i\alpha}$ is equal to the vector valued Dirac measure at t_i having only an α component $W^{i\alpha j\beta} = G^{\alpha\beta}(t_i, t_j)$ and one can use the mapping $P : \mathcal{X} \to \mathcal{Y}$ by $x \mapsto <\delta^{\alpha}_{t_i}, x> = x^{\alpha}(t_i)$ to compare the two definitions of Feynman integral.
For mappings P other than $\mathcal{X} \to \mathbb{R}^{pn}$ see [3].

III. The Feynman Green Function

Classical Physics is dominated by the Euler-Lagrange equation and Quantum Physics by the small disturbance equation, i.e., respectively by the first and the second variation of the action S. In this section we shall state the properties of the second variation, old and new, necessary to compute the propagation kernel.

The expansion of S around the classical path $\bar{q}$ can be written

$$S(q) = S(\bar{q}) + \frac{1}{2} S''(\bar{q})xx + \Sigma(x)$$

In the example considered $\bar{q}$ is the geodesic from a to b; the second variation $S''(\bar{q})$, also called the hessian of S at $\bar{q}$, is:

$$S''(\bar{q})xy = - \sum_t (y(t), \Delta\dot{x}(t)_g - \int_T (y(t), \ddot{x}(t) + R(\dot{\bar{q}}(t),x(t))\dot{\bar{q}}(t))_g dt$$

where Δ means "difference at a discontinuity": $\Delta \dot{x}(t) = \dot{x}(t^+) - \dot{x}(t^-)$ and where R is the Riemann tensor; x is a vector field along $\bar{q}$, it is an element of the tangent space at $\bar{q}$ of the space $\mathbb{F}$ of paths $q : \mathbb{T} \to \mathbb{M}$; i.e., $x \in T\mathbb{F}_{\bar{q}}$ and $x(t) \in T\mathbb{M}_{\bar{q}(t)}$
x is called a Jacobi field if and only if

$$S''(\bar{q})xy = 0 \qquad \text{for every} \quad y \in T\mathbb{F}_{\bar{q}}$$

A Jacobi field is a C^∞ - differentiable solution of the small disturbance equation, it will be denoted $\bar{x}$. The small disturbance equation, also called the Jacobi differential equation has 2n linearly independent solutions.
The Jacobi fields can be obtained by an m-parameter variation through geodesics [5]. An m-parameter variation $\bar{\alpha}$ of a path q is a mapping

$$\bar{\alpha} : \mathbb{U} \subset \mathbb{R}^m \to \mathbb{F} \quad \text{such that} \quad \bar{\alpha}(0) = \bar{q}$$

It is convenient to introduce the mapping α, also called m-parameter variation of q:

$$\alpha : \mathbb{U} \times \mathbb{T} \to \mathbb{M} \quad \text{by} \quad \alpha(u,t) = \bar{\alpha}(u)(t)$$

An m-parameter variation defines m "variation vector fields" $\{x_i\}$ by

$$x_i(t) = \frac{\partial \alpha}{\partial u^i}(0,t)$$

conversely

$$\alpha(u,t) = \exp_{\bar{q}(t)} (\Sigma u^i x_i(t))$$

When the family $\{\bar{\alpha}(u)\}$ is a family of geodesics, an m-parameter variation $\bar{\alpha}$ is called a variation through geodesics.

Two points $\bar{q}(t_a) = a$ and $\bar{q}(t_b) = b$ are said to be conjugate along $\bar{q}$ if there exists a non zero Jacobi field $\bar{x}$ along $\bar{q}$ vanishing at t_a and t_b. The dimension of the vector space of all such Jacobi fields is called the multiplicity of the conjugate points.
Lemma: Two points a and b are conjugate along $\bar{q}$ if and only if the mapping $\exp_a$ is critical at $(t_b-t_a)\ \dot{\bar{q}}(t_a)$; i.e., if its derivative mapping at the critical point is not one-one. Indeed, let $\dot{\bar{x}}(t_a)$ be the covariant derivative of the non-zero Jacobi field vanishing at t_a and t_b.

$$\exp'_a((t_b-t_a)\dot{\bar{q}}(t_a))\ (t_b-t_a)\ \dot{\bar{x}}_a = 0$$

A Jacobi field is determined by its values at any two non-conjugate points. Thus the Jacobi field equal to x_a at t_a and x_b at t_b is

$$\bar{x}(t) = -J(t,t_b)\ M(t_b,t_a)x_a\ -J(t,t_a)\ M(t_a,t_b)x_b$$

where $M(t_b,t_a)$ is the inverse of $J(t_b,t_a)$ and where $J(t,t_b)$ is the antisymmetric Jacobi two point vector along $\bar{q}$ such that

$$\begin{cases} J(t_b,t_a) = 0 \\ \dfrac{DJ}{dt} = (t = t_b,t_b) = g^{-1}(\bar{q}(t_b)) \text{ which will be abbreviated to } g^{-1}(t_b) \end{cases}$$

J is also known as the commutator function.
It has been shown [6] that M is the Van Vleck matrix:

$$M(t_b,t_a) = \partial^2\bar{S}(a,b)/\partial b\partial a \quad \text{where} \quad \bar{S}(a,b) = S(\bar{q}) \quad .$$

When the end points are conjugate, we determine the Jacobi field by its Cauchy data

$$\bar{x}(t) = J(t,t_a)\ g(t_a)\dot{\bar{x}}(t_a) + K(t,t_a)g(t_a)x_a$$

where $K(\cdot,t_a)$ is a Jacobi field along $\bar{q}$ defined by

$$K(t_a,t_a) = g^{-1}(t_a)$$

$$\frac{DK}{dt}\ (t = t_a,t_a) = 0 \quad .$$

Proposition 2. The end points a and b are conjugate along $\bar{q}$ if and only if $\det J(t_b,t_a) = 0$. The conjugate points are degenerate if and only if $\det K(t_b,t_a) = 0$.
This proposition provides a convenient criterion for the onset of catastrophes in path integrals, namely:
For a fixed point $a \in \mathbb{M}$ the catastrophe set of points $b \in \mathbb{M}$ satisfies:

$$\begin{cases} \det J(t_b,t_a) = 0 \\ \det K(t_b,t_a) = 0 \end{cases}$$

The proof of proposition 2 and the study of catastrophes is the subject of another paper.

The Feynman Green function is the Green function of the small disturbance operator that vanishes on the boundary.

Proposition 3: The Feynman Green function is equal to

$$G(r,s) = \frac{\hbar}{m} Y(r-s)J(r,t_b)M(t_b,t_a)J(t_a,s) - \frac{\hbar}{m} Y(s-r)J(r,t_a)M(t_a,t_b)J(t_b,s)$$

where Y is the Heaviside step function equal to unity for positive argument and zero otherwise.

Proof: see [6]

When a and b are conjugate along $\bar{q}$, the Feynman Green function is not defined, and we shall need another Green function G_- of the small disturbance operator to define the propagation kernel:

$$G_-(r,s) = \frac{\hbar}{m} Y(r-s)K(r,t_b)N(t_b,t_a)J(t_a,s) - Y(s-r)J(r,t_a)N(t_a,t_b)K(t_b,s)$$

where $N(t_b,t_a)$ is the inverse of $K(t_b,t_a)$ defined in the previous paragraph.

IV. The Propagation Kernel $K(B;A) = K(b,t_b;a,t_a)$

The propagation kernel of a system from a state A to a state B is the probability amplitude for the transition $A \to B$. It gives the wave function Ψ_{t_b} at t_a in terms of the wave function Ψ_{t_a} at t_a:

$$\Psi_{t_b}(b) = \int_{\mathbb{M}} K(b,t_b;a,t_a)\ \Psi_{t_a}(a)(\det g(a))^{1/2} da$$

According to Feynman's original definition

$$K(B;A) = \lim_{p\to\infty} \int_{\mathbb{R}^{pn}} K(B;p)K(p;p-1)\ldots K(1;A)dq^1\ldots dq^p \quad \text{where [4]}$$

$$K(k+1;k) = \left(\frac{1}{2\pi i\hbar}\right)^{n/2} \left(\det \frac{-1}{\hbar} \frac{\partial^2 \bar{S}(k+1;k)}{\partial q^{k+1}\partial q^k}\right)^{1/2} \exp\left(\frac{i}{\hbar} \bar{S}(k+1;k)\right)$$

$\bar{S}(k+1;k)$ is a function of q^{k+1}, t_{k+1}, q^k, t_k such that

$$\lim_{p\to\infty} \sum_{k=1}^{p} \bar{S}(k+1;k) = S(q) \overset{\text{def}}{=} \int_{\mathbb{T}} L(q(t), \dot{q}(t), t)\ dt$$

where L is the lagrangian of the system and S its action. We can write $\bar{S}(k+1;k) = \int_{t_k}^{t_{k+1}} L(q(t), \dot{q}(t), t)dt$ provided we give some prescription for the path $q: [t_k,t_{k+1}] \to M$. A natural prescription for $q \mid [t_k,t_{k+1}]$ is the classical path from q^k to q^{k+1}.

We shall give the new definition of the propagation kernel in three steps:

a) The paths $q \in F(a,b)$ map $\mathbb{T}$ in a geodesically convex neighborhood $\mathbb{N}$ of $\mathbb{M}$.

b) The end points a and b may be conjugate points along $\bar{q}$, but there is no other conjugate point along $\bar{q}$; the conjugate points are non-degenerate.

c) There are several conjugate points along $\bar{q}$; they are non-degenerate; the end points are not conjugate.

Proposition 4a. When $q : \mathbb{T} \to \mathbb{N}$, the propagation kernel is

$$K(B;A) = \exp \frac{i}{\hbar} S(\bar{q}) \int_{\mathcal{X}} \exp \frac{i}{\hbar} \Sigma(x) dw(x)$$

where w is the gaussian pseudomeasure on $\mathcal{X}$ defined by its Fourier transform

$$\mathcal{F}w = w(X)\exp(-iW/2)$$

$$W(\mu,\nu) = \int_{\mathbb{T}} d\mu_\alpha(r) \int_{\mathbb{T}} d\nu_\beta(s)\ G^{\alpha\beta}(r,s)$$

G is the Feynman Green function; $w(\mathcal{X}) = (\det M(t_b,t_a))^{1/2}/(2\pi i)^{n/2}$

Proof and applications: see [6] and [3]

This definition is equivalent to the original one when the original one is unambiguous, it does not require an ad hoc prescription for the $\mathbb{R}^{pn}$ integral, it is defined for a large class of physical systems, and leads to simple and powerful computational techniques.

Proposition 4b. Let $\mathcal{Y}$ be the space of vector fields along $\bar{q}$ which vanishes at t_a but take arbitrary values at t_b. Let w_- be the gaussian pseudomeasure of covariance G_- normalized to

$$w_-(\mathcal{Y}) = (\det g^{-1}(t_b)\ N(t_b,t_a)^{1/2}$$

Then w_- on $\mathcal{Y}$ induces w on $\mathcal{X} \subset \mathcal{Y}$:

$$\int_{\mathcal{Y}} x(y) \langle\delta_r,y\rangle \langle\delta_s,y\rangle\, dw_-(y) / \int_{\mathcal{Y}} x(y) dw_-(y) = iG(r,s)$$

$$\int_{\mathcal{Y}} x(y) dw_-(y) = \begin{cases} \int_{\mathcal{X}} dw(x) & \text{for non conjugate end points} \\ w_-(\mathcal{Y})\ \delta & \text{for conjugate non-degenerate end points} \end{cases}$$

where δ is the Dirac distribution at the origin of $\mathbb{R}^n$.

The propagation kernel is equal to

$$K(B;A) = \exp \frac{i}{\hbar} S(\bar{q}) \int_{\mathcal{Y}} \chi(y)\ \exp \frac{i}{\hbar} \Sigma(y)\ dw_-(y)$$

where $\chi(y)$ is the characteristic of function of $\mathcal{X} \subset \mathcal{Y}$.

The proof is given in [6]. This proposition serves two purposes:

It provides the formalism for integrating over paths with one fixed end point and one arbitrary point.

It makes it possible to approach a point b conjugate to a along $\bar{q}$ from a point b_u not conjugate to a along the geodesic $\bar{q}_u$ from a to b_u; the parameter $u \in \mathbb{R}^m$, where m is the multiplicity of the conjugate points, defines an m-parameter variation of $\bar{q}$ through geodesics such that $b_u = \exp_b uy_b$. One obtains in particular:

$$\int_{\mathcal{Y}} \chi(y;y(t_b) = y_b) dw_-(y)$$

$$= \frac{1}{(2\pi i)^{n/2}} \det{}^{1/2}(M(t_b,t_a)) \exp(-\frac{i}{2}\, y_b g(t_b) K(t_b,t_a) M(t_a,t_b) y_b)$$

giving the value of $\int_{\mathcal{Y}} x(y) dw_-(y)$ stated in the proposition

Schulman [7] has examined propagation kernels between conjugate points in terms of the eigenvalues of the small disturbance operator.

Proposition 4c. There may be several non degenerate conjugate points along $\bar{q}$. Let the end points a,b be two points in $\mathbb{M}$ which are not conjugate along any geodesic. Let π be the fundamental group of a countable CW-complex which contains one cell of dimension λ for each geodesic from a to b of Morse index λ. Let $K_\alpha(B;A)$ be the partial propagation kernel for all the paths from a to b in the same homotopy class α, computed according to proposition 1. Then the absolute value of the propagation kernel is

$$| K(B;A) | = | \sum_{\alpha\in\pi} \chi(\alpha) K_\alpha (B;A) |$$

where $\{\chi(\alpha);\ \alpha\in\pi\}$ is the set of characters of the fundamental group.

Proof: Because there is no unique way to label the homotopy classes by the elements α of the fundamental group, K(B;A) is determined only modulo an overall unobservable phase factor [1]. The proof rests on the fundamental theorem of Morse theory [5] which states the homotopy type of $\mathbb{F}(a,b)$ and on the theorem giving the propagation kernel on a multiply connected space [1]. The kernel K has been computed by Gutzwiller [8]. Proposition 4c and its proof give a rigorous derivation for his expression.

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