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Andrea Fuster Pérez

geboren te Barakaldo, Spanje

promotor: prof.dr. J.W. van Holten

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1 Introduction

Gravity is one of the four fundamental forces in Nature. It is the one we are more familiar with in everyday life, as we cannot possibly neglect the effect of Earth's gravity on all objects and ourselves. This is also why gravity was the first fundamental force to be discovered and described quantitatively, by Sir Isaac Newton. Gravity is responsible for the Universe we observe, from driving its expansion to keeping the planets in orbit around the Sun. Newtonian gravity, however, can only describe our Universe in a limited way. It is in fact part of a more complex theory of gravity, Einstein's General Relativity. And that is what this thesis is all about.

1.1. General Relativity and more

General Relativity is generally accepted as the most successful theory of gravitation. In General Relativity gravitation is understood as the curvature of spacetime caused by energy and matter. General Relativity reduces to Newtonian gravity for weak gravitational fields (see equation (1.10)) and low velocities $v/c << 1$. General Relativity is mathematically described by the Einstein equations, a set of highly non-linear tensorial-differential equations. They were derived by Albert Einstein and published for the first time back in 1915 [1]. The main ingredient of these equations is the *metric tensor* $g_{\mu\nu}$, from which every other object is constructed. The *Christoffel symbols* $\Gamma^{\mu}_{\nu\rho}$ are defined by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\sigma\mu} (g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}) \tag{1.1}$$

where $g^{\sigma\mu}$ is the inverse metric and , indicates partial derivative. The *Riemann tensor* $R^{\lambda}{}_{\mu\rho\nu}$ can be calculated from the metric and the Christoffel symbols as follows

$$R^{\lambda}{}_{\mu\rho\nu} = \Gamma^{\lambda}_{\mu\nu,\rho} - \Gamma^{\lambda}_{\mu\rho,\nu} + \Gamma^{\lambda}_{\sigma\rho}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\mu\rho} \tag{1.2}$$

The Riemann tensor is related to the curvature of spacetime. In flat spacetime, described by the Minkowski metric $\eta_{\mu\nu} \equiv (-1, +1, +1, +1)$, the Riemann tensor vanishes. The *Ricci tensor* $R_{\mu\nu}$ is obtained by contracting certain indices in the Riemann tensor

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} \tag{1.3}$$

The *Ricci scalar* R is the following scalar object

$$R = g^{\mu\nu} R_{\mu\nu} \tag{1.4}$$

We are now in position to give Einstein's equations in vacuum[1]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{1.5}$$

The left-hand side is actually defined to be the *Einstein tensor* $G_{\mu\nu}$, so equivalently

$$G_{\mu\nu} = 0 \tag{1.6}$$

In four dimensions, the indices μ, ν run from 0 to 3. The Einstein's equations are symmetric, and so there are ten of them. The vacuum equations describe spacetime outside the source of curvature. These equations can be generalized to describe spacetime when matter is present. In that case there is an additional term corresponding to the matter *energy-momentum tensor* $T_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{1.7}$$

where G is Newton's gravitational constant and c is the velocity of light. The energy-momentum tensor represents both the flux and density of energy and momentum of the present matter. It is the source of the non-vacuum gravitational field. For consistency the energy-momentum tensor must satisfy

$$D^{\mu} T_{\mu\nu} = 0 \tag{1.8}$$

where D^{μ} is the spacetime covariant derivative. This condition usually follows from, or is equivalent to, the equations of motion of the matter fields. On

[1] In this case $R_{\mu\nu} = 0$ suffices.

the other hand, a *cosmological constant* Λ can be introduced in the above equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.9}$$

This was done for the first time by Einstein. At the time, it was believed that the universe was not only uniform but also essentially static. He introduced the cosmological constant to allow for a time-independent solution for a uniform universe. Later it was discovered that the universe is not static but expanding. He then called the cosmological constant "the biggest blunder of my life"[2]. Much later, supernovae observations suggested that the universe is expanding at an accelerated rate. Other independent observations seem to corroborate this. In practice this could mean that the cosmological constant has a very small, but non-zero value. So maybe Einstein was right after all when he introduced the cosmological term.

Clearly, to find solutions of the Einstein's equations is not an easy task. One can consider simplified versions of the equations to make them more tractable, like linearized gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.10}$$

where $h_{\mu\nu}$ represents the (small) deviation of the metric from flat spacetime. Another possibility is to look for approximate solutions, for example by numerical methods. On the other hand, a given metric is an *exact solution* if it solves the full Einstein's equations. Many results in the field of exact solutions have been collected in the classic book by Kramer et al [3]. But what is the use of such solutions? There is no doubt that there is always some loss of information in studying only solutions of approximate equations, specially if those are linear while the original equations are not. Exact solutions are sometimes seen as pure mathematical objects without physical applications. Some exact solutions are actually of great physical relevance, like the Schwarzschild and Kerr solutions for stars and black holes or the Friedmann solutions for cosmology.

Another important physical phenomenon predicted by General Relativity is the existence of gravitational waves. They are mathematically described by *wave-like* solutions of the Einstein's equations. In this context, wave-like just means that the solutions satisfy the wave equation for propagating radiation. So heuristically, if there is propagating radiation in vacuum it can only be gravitational radiation. It is well-known that wave-like solutions of the linearized Einstein's equations exist. The question arises whether there are wave-like exact solutions. The answer is yes, and in the second part of this chapter we will study an example of such solutions in detail, namely, the *pp-waves*. These solutions are not only important in four dimensions, but also in higher dimensions.

Why consider higher dimensions? It is a fact that General Relativity as it is cannot be unified with Quantum Mechanics. One of the possible scenarios to solve this puzzle is *Superstring Theory*. This is a theory of quantum gravity which describes particles as different vibration modes of strings. Such a theory turns out to be consistent only in ten spacetime dimensions. Although this seemed to be a difficulty for the theory to describe the real world at first, it is now generally accepted that it can be solved by curling up the extra dimensions to make them "invisible" in practice.

A necessary ingredient of Superstring Theory is *Supergravity*. This is a certain extension of General Relativity where a new underlying symmetry is present, the so-called *Supersymmetry*. Supersymmetry postulates the existence of a new particle for each of the known ones, in such a way that if the original particle is a boson the new particle is a fermion, and the other way around. The original and new particle are supposed to share all other properties, like the mass. This new symmetry, if it exists at all, must be somehow broken in Nature as we do not observe those new particles at the expected mass range. Supergravity can be defined in any number of dimensions up to eleven.

Coming back to pp-waves, their higher-dimensional counterpart turned out to be of relevance in the context of superstrings and supergravity for various reasons[2]. In fact, the subject of this thesis are exact solutions which generalize the pp-waves in some sense, in four and higher dimensions. The motivation to do so is the possible physical applications of such solutions, in the fashion of pp-waves. General Relativity is defined in higher dimensions in an analogous way to the four-dimensional case. The solutions we treat have the property that certain scalar quantities constructed from the Riemann tensor are zero (VSI solutions) or constant (CSI solutions). The pp-waves are in the first class. A well-known example of a CSI spacetime is (anti-)de Sitter space. The VSI spaces belong to a class of metrics called *Kundt class*; the CSI ones can

[2] We discuss this in more detail in section 4.2

be in this class (*Kundt CSI*) or not. We only treat the former. VSI solutions are not compatible with a cosmological constant while for the CSI ones it is, in general, needed.

We now summarize the structure of this thesis. In Chapter 2 we construct VSI solutions of the coupled Einstein-Yang-Mills system in four dimensions. In Chapter 3 we derive the explicit form of the VSI metrics in arbitary number of dimensions $N \geq 4$. In Chapter 4 we construct VSI solutions of Type IIB Supergravity. In Chapter 5 we study higher-dimensional Kundt CSI spacetimes. In Chapter 6 we summarize the presented results and comment on possible future work directions.

1.2. Four-dimensional pp-wave spacetimes

A covariantly constant null vector (CCNV) k is defined by

$$\nabla_\mu k_\nu = 0, \;\; k_\mu k^\mu = 0 \tag{1.11}$$

The spacetimes admitting such a vector are called *plane-fronted waves with parallel rays*, or *pp-waves* for short. A covariantly constant vector is automatically a *Killing vector*[3]

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \tag{1.12}$$

The most general pp-wave spacetime in four dimensions is in principle given by the Brinkmann metric [4]

$$\begin{aligned} ds^2 \;\; &= \;\; -2du\,[dv + H(u,x,y)\;du + W_x(u,x,y)\;dx + W_y(u,x,y)\;dy] \\ &\quad\; + g_{ij}(u,x,y)\;dx^i dx^j \end{aligned} \tag{1.13}$$

where $u = (t-z)/\sqrt{2}$, $v = (t+z)/\sqrt{2}$ are light-cone coordinates, $i, j = x, y$ and all of the metric functions are real. The functions W_x, W_y can be transformed away so long as the transverse metric depends on u [5]. The transformations

$$x' = h^x(u,x,y), \;\; y' = h^y(u,x,y) \tag{1.14}$$

leave the form of the metric (1.13) invariant. The transformed metric functions are related to the original ones by

$$H \;\; = \;\; H' + W'_i h^i_{,u} - \frac{1}{2} g'_{ij} h^i_{,u} h^j_{,u} \tag{1.15}$$

$$W_i \;\; = \;\; W'_j h^j_{,i} - g'_{jk} h^j_{,u} h^k_{,i} \tag{1.16}$$

$$g_{ij} \;\; = \;\; g'_{kl} h^k_{,i} h^l_{,j} \tag{1.17}$$

[3] Note that the converse is not true. The existence of a Killing vector is a necessary but not sufficient condition for that vector to be covariantly constant.

where $i,j = x,y$. From (1.16) it is clear that we can accomplish $W_i = 0$ by solving the differential equations

$$W'_j h^j_{,i} = g'_{jk} h^j_{,u} h^k_{,i}, \quad g'^{ik} g'_{kl} = \delta^i_l \tag{1.18}$$

for the functions h^x, h^y. Furthermore, since any two-dimensional metric is conformally equivalent to flat space, we can write

$$ds^2 = -2du\,[dv + H(u,x,y)\,du] + P(u,x,y)\,(dx^2 + dy^2) \tag{1.19}$$

This is a vacuum solution of the Einstein equations if $P = P(u)$ [4] but then it can be absorbed into the redefinition of x, y and v

$$x' = \sqrt{P}x, \quad y' = \sqrt{P}y, \quad v' = v + \frac{1}{4}P_{,u}(x^2 + y^2) \tag{1.20}$$

In this way we get the usual form of the pp-wave metric

$$ds^2 = -2du\,[dv + H(u,x,y)\,du] + dx^2 + dy^2 \tag{1.21}$$

Or, in complex notation $z = (x+iy)/\sqrt{2}$, $\bar{z} = (x-iy)/\sqrt{2}$,

$$ds^2 = -2du\,[dv + H(u,z,\bar{z})\,du] + 2dzd\bar{z} \tag{1.22}$$

Note that the pp-wave metric reduces to flat space when $H = 0$. The only non-vanishing Christoffel symbols are

$$\Gamma_{uu}{}^{v} = 2H_{,u}, \;\; \Gamma_{uu}{}^{x} = \Gamma_{xu}{}^{v} = H_{,x}, \;\; \Gamma_{uu}{}^{y} = \Gamma_{yu}{}^{v} = H_{,y} \tag{1.23}$$

By using these one can write down the geodesic equations (see, for example, [6]). On the other hand, since the pp-wave metric does not depend on the light-cone coordinate v, the (null) vector

$$k = k^\nu \partial_\nu = \partial_v, \quad k_u = 1 \tag{1.24}$$

is a Killing vector[5]. It is also CCNV, as can be seen from the absence of connection coefficients of the form $\Gamma_{\mu u}{}^{u}$. On the other hand, the non-zero Riemann curvature components read

$$R_{uxux} = -H_{,xx}\,, \;\; R_{uyuy} = -H_{,yy}\,, \;\; R_{uxuy} = -H_{,xy} \tag{1.25}$$

[4] Otherwise there are, in general, non-zero components of the Ricci tensor R_{xx}, R_{yy}, R_{ux} and R_{uy}.

[5] A metric which does not depend on a certain coordinate x^α is obviously invariant under transformations of the form $x^\alpha \to x^\alpha + a$, where a is a constant. This implies the existence of a Killing vector lying in the direction of the symmetry $k = \partial_\alpha$.

The metric (1.21) is an exact vacuum solution when the function H is harmonic, i.e.,

$$R_{uu} = -\triangle H = 0 \tag{1.26}$$

with $\triangle$ the Laplacian on the (x, y) plane. Or in complex coordinates

$$H,_{z\bar{z}} = 0 \tag{1.27}$$

The obvious general solution reads

$$H = f(u, z) + \bar{f}(u, \bar{z}) \tag{1.28}$$

with f an arbitrary holomorphic function. Note that for H linear in x, y we just have flat space as the Riemann tensor is zero. On the other hand, the Ricci scalar of pp-waves is zero[6].

1.3. Physical interpretation of pp-waves

A wave propagating in Minkowski space with $c \equiv 1$ in the z direction is described by the wave equation

$$\frac{\partial^2 H}{\partial t^2} - \frac{\partial^2 H}{\partial z^2} = 0 \tag{1.29}$$

This equation can also be written in the factor form

$$(\partial_t - \partial_z)(\partial_t + \partial_z)H = 0 \tag{1.30}$$

Or, in light-cone coordinates u, v,

$$\partial_u \partial_v H = 0 \tag{1.31}$$

Any function of the form $H = F(u) + G(v)$ satisfies this wave equation. Both terms represent travelling waves, in the positive (F) or negative (G) direction of the z axis. The function H in (1.21) does not depend on v and obviously satisfies equation (1.31). The pp-wave metric describes exact gravitational waves propagating in the positive z direction.

Now we can understand why the metric is called pp in the first place. The 2-surfaces $u, v =$ constant with metric $dx^2 + dy^2$ are called *wave surfaces*. The term "plane-fronted waves" means that the wave surfaces are flat, while "parallel rays" refers to the existence of a null vector whose spacelike component is parallel to the direction of propagation.

[6] $R = g^{\mu\nu} R_{\mu\nu}$, while R_{uu} is the only non-zero Ricci tensor component, $g^{uu} = 0$.

1.4. Subclasses of pp-wave spacetimes

Various subclasses of pp-wave spacetimes can be defined for specific choices of the metric function H in (1.21). For some special choices there are additional Killing vectors. All of these were given in [7]. Here we comment on the most interesting cases.

Impulsive pp-waves [8] are described by the function

$$H = \delta(u)F(x, y) \tag{1.32}$$

with F an arbitrary harmonic function. They can be seen as an idealization of sandwich waves $u_1 < H \neq 0 < u_2$ of infinitely short duration, $u_2 \to 0$. The Aichelburg-Sexl boost is in this class [9]. Another interesting class occurs for $H_{,u} = 0$. Such metrics obviously have ∂_u as additional Killing vector. On the other hand, *plane waves* are those pp-waves given by

$$H(u, x, y) = A_{ij}(u)x^i x^j \tag{1.33}$$

with $i, j = x, y$ and A_{ij} a real symmetric matrix. This is the simplest case for which the metric (1.21) is non-flat. Additionally, the Riemann curvature components (1.25) are finite everywhere. Plane waves were first considered in [10]. In this case the Einstein equation (1.26) simplifies to

$$\text{Tr } A_{ij} = 0 \tag{1.34}$$

We can write (1.33) explicitly

$$H(u, x, y) = A_{11}(u)(x^2 - y^2) + 2A_{12}(u)xy \tag{1.35}$$

The function A_{11} (A_{12}) describes the wave profile of the "+" ("×") polarization mode of the gravitational wave. Plane waves have four additional Killing vectors of the form

$$k = p(u)\partial_x + q(u)\partial_y + (p'x + q'y)\partial_v \tag{1.36}$$

where the functions $p(u)$, $q(u)$ satisfy

$$p'' + A_{11}p - A_{12}q = 0 \tag{1.37}$$
$$q'' - A_{11}q + A_{12}p = 0 \tag{1.38}$$

Homogeneous plane waves are plane waves with at least one additional Killing vector with a non-zero u-component. The simplest example are the plane waves with constant A_{ij}, which have the extra Killing vector ∂_u. These are known as Cahen-Wallach spaces [11, 12]. Another example is given by

$$A_{ij}(u) = \frac{C_{ij}}{u^2} \tag{1.39}$$

with C_{ij} a constant matrix. In this case we have the Killing vector

$$k = u\partial_u - v\partial_v \tag{1.40}$$

The homogeneous plane waves have been studied in detail in [13].

Plane waves have gained momentum in string theory thanks to an old result by Penrose, namely, the fact that any Lorentzian spacetime becomes a plane wave near a null geodesic (*Penrose limit*) [14]. The topic of colliding plane waves have also received much attention (see [15] for a thorough review).

1.5. Non-vacuum pp-wave spacetimes

So far we have only studied vacuum pp-wave spacetimes. However, for certain types of matter there exist pp-wave solutions of the coupled Einstein-matter system. In practice, this means that the metric is still of the form (1.21) but the function H has extra matter-related terms. Let us consider Einstein-Maxwell (EM), i.e., the theory of gravity coupled to the electromagnetic field A_μ. It is described by the equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = F_{\mu\rho}F_\nu^{\ \rho} - \frac{1}{4}g_{\mu\nu}\, F_{\alpha\beta}F^{\alpha\beta} \tag{1.41}$$

$$\nabla_\lambda F^{\lambda\mu} = 0 \tag{1.42}$$

where we use the convention $8\pi G \equiv 1$ and the electromagnetic field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.43}$$

The right-hand side of (1.41) is the energy-momentum tensor of the electromagnetic field. Before giving the pp-wave solution of (1.41), (1.42) we note the following curiosity. The Killing vector of a given vacuum spacetime can be seen as an electromagnetic potential. It can be shown that the corresponding field strength, called *Papapetrou field*, satisfies the (curved) Maxwell equation (1.42). This was noted for the first time in [16]. Unfortunately, equation (1.41) is not satisfied in general. Nevertheless, this fact has been used in several subjects (see, for example, [17] and references therein).

We now give pp-wave solutions with a null electromagnetic field [3], i.e., $F_{\mu\nu}F^{\mu\nu} = *F_{\mu\nu}F^{\mu\nu} = 0$. For convenience, we use complex notation. Consider the electromagnetic potential

$$A = \alpha(u, z, \bar{z})\, du \tag{1.44}$$

The only non-zero components of the field strength are

$$F_{zu} = \alpha_{,z}\,, \quad F_{\bar{z}u} = \alpha_{,\bar{z}} \tag{1.45}$$

Equation (1.42) reads[7]

$$\alpha_{,z\bar{z}} = 0 \tag{1.46}$$

and the obvious general solution is

$$\alpha(u, z, \bar{z}) = F(u, z) + \overline{F}(u, \bar{z}) \tag{1.47}$$

where F is an arbitrary holomorphic function. On the other hand, equation (1.41) is

$$H_{,z\bar{z}} = \alpha_{,z}\, \alpha_{,\bar{z}} \tag{1.48}$$

And so

$$H(u, z, \bar{z}) = f(u, z) + \bar{f}(u, \bar{z}) + F(u, z)\overline{F}(u, \bar{z}) \tag{1.49}$$

Note that this is the vacuum H function (1.28) plus a term corresponding to the electromagetic wave. In the case of plane waves we have[8]

$$H(u, z, \bar{z}) = A(u)z^2 + \overline{A}(u)\bar{z}^2 + B(u)z\bar{z} \tag{1.50}$$

with B a real function describing the wave profile of the electromagetic radiation. The electromagnetic term does not alter the form of the Killing vectors (1.36) [9]. Homogeneous plane waves occur, for example, for $A(u) = A_0$, $B(u) = B_0$, with A_0, B_0 (real) constants, and for $A = A_0/u^2$, $B = B_0/u^2$.

These solutions can be generalized to include scalar fields, such as the dilaton [6, 18]. The scalar fields are then usually constrained to depend only on the light-cone coordinate u. On the other hand, pp-waves also admit null radiation solutions. However, there are no non-null electromagnetic fields, perfect fluids or solutions with non-zero cosmological constant [3].

[7] With the considered ansatz, $\nabla_\lambda F^{\lambda\mu} \equiv \partial_\lambda F^{\lambda\mu}$.

[8] The relation between the coefficients in real and complex notation is $A_{11} = 2\mathrm{Re}A$, $A_{12} = -2\mathrm{Im}(A)$.

[9] But equations (1.37), (1.38) acquire an extra term $B(u)p$, $B(u)q$ on the left-hand side, respectively.

2 Einstein-Yang-Mills VSI solutions

In this chapter we first briefly review the Petrov and Plebański-Petrov classification of the Weyl and Ricci tensor in four dimensions. We then study some generalities about the four-dimensional Kundt class. We define VSI and CSI spacetimes, key concepts in this thesis. We explain the basics of the Einstein-Yang-Mills (EYM) model. The four-dimensional EYM has been extensively studied since particle-like solutions were found [19] and shortly thereafter, the first EYM hairy black hole solutions [20]. We consider EYM in the context of exact solutions of General Relativity, and construct VSI-Yang-Mills solutions. In a certain limit, the solutions can be reduced to Einstein-Maxwell solutions. This chapter is based on reference [21].

2.1. Classification of the Weyl and Ricci tensor in four dimensions

2.1.1. Preliminaries

Consider a four-dimensional manifold M endowed with a Lorentzian metric, i.e., of signature $(-,+,+,+)$. Tensor quantities defined on M can be expressed in *coordinate* bases $\{\partial_\mu\}$ and $\{\mathrm{d}x^\mu\}$ or *non-coordinate* bases $\{e_a = e_a^\mu \partial_\mu\}$ and $\{\theta^a = e^a_\mu \mathrm{d}x^\mu\}$. The coefficients e_a^μ (e^a_μ) are the so-called *frame* (*coframe*) components. The metric on a non-coordinate basis is constant, which obviously makes calculations easier. In a *orthonormal* coframe

$$g = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab}\, \theta^a \otimes \theta^b \tag{2.1}$$

where η_{ab} is the diagonal Minkowski metric. In a *non-orthonormal* coframe η_{ab} is not diagonal. Consider the following coframe

$$\{\theta^a\} = (n, \ell, m, \overline{m}) \tag{2.2}$$

where n and ℓ are real and m, $\overline{m}$ are each other's complex conjugate. In addition

$$\ell^a \ell_a = n^a n_a = 0, \;\; \ell^a n_a = -1, \;\; m^a \overline{m}_a = 1 \tag{2.3}$$

and so ℓ and n are null. We can identify $\theta^0 = n$, $\theta^1 = \ell$, $\theta^2 = m$, $\theta^3 = \overline{m}$. In terms of coordinate bases

$$\begin{aligned} &\theta^0 = -n_\mu \mathrm{d}x^\mu, \;\; \theta^1 = -l_\mu \mathrm{d}x^\mu, \;\; \theta^2 = m_\mu \mathrm{d}x^\mu, \;\; \theta^3 = \overline{m}_\mu \mathrm{d}x^\mu \\ &e_0 = l^\mu \partial_\mu, \qquad e_1 = n^\mu \partial_\mu, \qquad e_2 = \overline{m}^\mu \partial_\mu, \qquad e_3 = m^\mu \partial_\mu \end{aligned} \tag{2.4}$$

The basis $\{e_a\}$ is a *complex null tetrad*. The metric can be written

$$g = -2\,\theta^0 \otimes \theta^1 + 2\,\theta^2 \otimes \theta^3 \tag{2.5}$$

Or, equivalently,

$$\eta_{ab} = -2\ell_{(a} n_{b)} + 2m_{(a}\overline{m}_{b)} \tag{2.6}$$

where parentheses indicate symmetrization of indices with weight 1/2. We can express any other tensor T on M in this frame.

The complex null tetrad (2.4) can also be used to construct a basis of 2-forms or *bivectors*

$$Z^\alpha = (U, V, W), \quad \alpha = 1,2,3 \tag{2.7}$$

with components

$$U_{ab} = -l_{[a}\overline{m}_{b]}, \quad V_{ab} = n_{[a} m_{b]}, \quad W_{ab} = m_{[a}\overline{m}_{b]} - n_{[a} l_{b]} \tag{2.8}$$

Here, square brackets indicate antisymmetrization of indices with weight 1. The above bivectors are actually self-dual. For complex bivectors this means ${}^*Z^\alpha = -iZ^\alpha$, where $*$ is the Hodge operator[1]. A general self-dual bivector can be expanded in terms of the basis (2.7). Their complex conjugates

$$\overline{Z}^\alpha = (\overline{U}, \overline{V}, \overline{W}), \quad \alpha = 1,2,3 \tag{2.9}$$

are anti-self-dual, i.e., ${}^*\overline{Z}^\alpha = i\overline{Z}^\alpha$. They form a basis for anti-self-dual bivectors. A general bivector X_{ab} can then be expanded in terms of the basis $\{Z^\alpha,\ \overline{Z}^\alpha\}$

$$X_{ab} = c_\alpha Z^\alpha_{ab} + d_\alpha \overline{Z}^\alpha_{ab} \tag{2.10}$$

[1] This can be verified with the aid of $\epsilon_{abcd} m^a \overline{m}^b l^c n^d = i$, where ϵ is the Levi-Civita tensor.

2.1.2. The Petrov classification

In this section we briefly explain the classification of the Weyl tensor in four dimensions, the *Petrov classification*. A complete review can be found in [3]. The Weyl tensor can be defined in terms of the Riemann and Ricci tensor, and the Ricci scalar

$$C_{abcd} = R_{abcd} - \frac{1}{2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{1}{6}Rg_{a[c}g_{d]b} \tag{2.11}$$

These are all components with respect to the basis (2.4). In vacuum spacetimes the Weyl tensor is equal to the Riemann tensor. The Weyl tensor has the same symmetries as the Riemann tensor. In addition, it is completely traceless

$$C^a{}_{bad} = 0 \tag{2.12}$$

This reduces the number of linearly independent components from twenty (Riemann tensor) to ten. The Weyl tensor remains invariant under a conformal transformation of the metric

$$g'_{ab} = e^{2U(x^n)}g_{ab}, \quad C'_{abcd} = C_{abcd} \tag{2.13}$$

For this reason the Weyl tensor is also called the *conformal tensor*.

The Weyl tensor can be expanded in terms of the basis of bivectors (2.7)

$$\begin{aligned} C_{abcd} &= \Psi_0\, U_{ab}U_{cd} + \Psi_1\, (U_{ab}W_{cd} + W_{ab}U_{cd}) \\ &+ \Psi_2\, (V_{ab}U_{cd} + U_{ab}V_{cd} + W_{ab}W_{cd}) \\ &+ \Psi_3\, (V_{ab}W_{cd} + W_{ab}V_{cd}) + \Psi_4\, V_{ab}V_{cd} \end{aligned} \tag{2.14}$$

Here, the five complex coefficients $\Psi_0, \ldots, \Psi_4$ are defined by

$$\begin{aligned} \Psi_0 &= C_{abcd}n^a m^b n^c m^d \\ \Psi_1 &= C_{abcd}n^a l^b n^c m^d \\ \Psi_2 &= C_{abcd}n^a m^b \overline{m}^c l^d \\ \Psi_3 &= C_{abcd}n^a l^b \overline{m}^c l^d \\ \Psi_4 &= C_{abcd}\overline{m}^a l^b \overline{m}^c l^d \end{aligned} \tag{2.15}$$

We can classify spacetimes according to the *Petrov type* of the associated Weyl tensor, i.e., the vanishing (or not) of the above coefficients in the following way

$$
\begin{aligned}
&\text{Type } I: && \Psi_0 = 0 \\
&\text{Type } II: && \Psi_0 = \Psi_1 = 0 \\
&\text{Type } D: && \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \\
&\text{Type } III: && \Psi_0 = \Psi_1 = \Psi_2 = 0 \\
&\text{Type } N: && \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \\
&\text{Type } O: && \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \Leftrightarrow C_{abcd} = 0
\end{aligned}
\tag{2.16}
$$

A Weyl tensor of type I is called *algebraically general*, and a Weyl tensor of types II, D, III or N is called *algebraically special*.

An important remark is that vacuum spacetimes of Petrov type N are exact gravitational waves[2]. On the other hand, a space is called *conformally flat* if it can be related to flat space by a conformal transformation, equation (2.13). A necessary and sufficient condition for this to happen is that the Weyl tensor vanishes[3]. Therefore, a Weyl tensor of type O is called conformally flat.

2.1.3. The Plebański-Petrov classification

The algebraic types of the Ricci tensor can be specified in detail by means of the *Segre classification* (see, for example, Chapter 5 in [3]). However, it is also possible to define a (simpler) classification of the Ricci tensor in analogy with the Petrov classification for the Weyl tensor.

Tensors in General Relativity can also be written in spinor notation (see [22, 23]). For each tensor one can define an spinor equivalent. Relations involving null vectors and bivectors simplify considerably in spinor language. Also, it is a necessary ingredient of the Newman-Penrose (NP) formalism [24].

The so-called *Plebański spinor* [25] is constructed from the spinor equivalent of the traceless Ricci tensor

$$S_{ab} = R_{ab} - \frac{1}{4} R g_{ab} \tag{2.17}$$

[2] This is always true for non-twisting spacetimes like the Kundt spacetimes in this thesis. Twisting spacetimes of Petrov type N, the Hauser metric [3] for example, cannot be interpreted in this way.

[3] This is only true in dimensions $N \geq 4$.

The tensor equivalent of the Plebański spinor is [26]

$$S^{[a}{}_{[c}S^{b]}{}_{d]} + \delta^{[a}{}_{[c}S_{d]e}S^{b]e} - \frac{1}{6}\delta^{[a}{}_{[c}\delta^{b]}{}_{d]}S_{ef}S^{ef} \tag{2.18}$$

This tensor has exactly the same symmetries as the Weyl tensor. The coefficients (2.15) can also be defined for the Plebański tensor[4] (2.18). Therefore, it can be classified according to its Petrov type. This classification of the Ricci tensor is called the *Plebański-Petrov (PP) classification* [26]. It can be sketched (in tensorial language) as follows

$$R_{ab} \overset{\text{construct}}{\Longrightarrow} S_{ab} \overset{\text{construct}}{\Longrightarrow} \text{Plebański tensor} \underset{\text{translate back to}}{\overset{\text{classify}}{\Longrightarrow}} R_{ab} \tag{2.19}$$

In this thesis we will only encounter metrics of PP type N and O. A Ricci tensor of PP type N reads

$$R_{ab} = 2R_{12}\,\ell_{(a}m_{b)} + 2R_{13}\,\ell_{(a}\overline{m}_{b)} + R_{11}\,\ell_a\ell_b \tag{2.20}$$

The PP type is O if $R_{12} = R_{13} = 0$. If in addition $R_{11} = 0$, we have PP type O-vacuum.

2.2. Four-dimensional Kundt class

We now treat the so-called *Kundt class* of metrics, named after the German physicist W. Kundt. We closely follow the discussion in [3]. The Kundt class is characterized by having a shear-free, non-expanding and non-twisting geodesic null congruence $\mathbf{l} = \partial_v$. It is described by the following line element

$$ds^2 = -2du\,(dv + H\,du + W\,dz + \overline{W}\,d\bar{z}) + 2P^{-2}\,dzd\bar{z}, \quad P_{,v} = 0 \tag{2.21}$$

Here, P and H are real functions while W is complex; $u = (t-z)/\sqrt{2}$, $v = (t+z)/\sqrt{2}$ are light-cone coordinates and $z = (x+iy)/\sqrt{2}$, $\bar{z} = (x-iy)/\sqrt{2}$ are complex conjugate coordinates in the transverse plane. The coordinate transformations preserving the form of (2.21) are

- $(v', u', z', \bar{z}') = (v, u, f(u,z), \bar{f}(u,\bar{z}))$

$$\begin{aligned} H' &= H - (f_{,u}\,\bar{f}_{,u}/P^2 + W f_{,u}\,\bar{f}_{,\bar{z}} + \overline{W}\bar{f}_{,u}\,f_{,z})/(f_{,z}\,\bar{f}_{,\bar{z}}), \\ W' &= W/f_{,z} + \bar{f}_{,u}/(P^2 f_{,z}\,\bar{f}_{,\bar{z}}), \quad P'^2 = P^2 f_{,z}\,\bar{f}_{,\bar{z}} \end{aligned} \tag{2.22}$$

[4] Although they are much simpler when written in spinor notation, i.e., for the Plebański spinor.

- $(v', u', z', \bar{z}') = (v + h(u, z, \bar{z}), u, z, \bar{z})$

$$H' = H - h_{,u}\,, \quad W' = W - h_{,z}\,, \quad P' = P \tag{2.23}$$

- $(v', u', z', \bar{z}') = (v/h_{,u}(u), h(u), z, \bar{z})$

$$H' = \frac{1}{h_{,u}^2}\left(H + v\frac{h_{,uu}}{h_{,u}}\right), \quad W' = \frac{1}{h_{,u}}W, \quad P' = P \tag{2.24}$$

Now, if $(\ln P)_{,z\bar{z}} = 0$, then one can always transform P to $P = 1$ by means of the coordinate transformation (2.22). We can divide the Kundt class into two main subgroups according to this condition. This also determines the Gaussian curvature of the surfaces $(u, v) = \text{constant}$

$$K = 2P^2(\ln P)_{,z\bar{z}} \tag{2.25}$$

This is obviously zero for Kundt metrics for which the above condition holds, and non-zero otherwise. In this thesis we are interested in Kundt metrics of Petrov type III. In this case the function P can always be transformed to $P = 1$. However, without further requirements, Kundt metrics with $P = 1$ are of Petrov type II (and PP type III).

The pp-waves discussed in section 1.2 are obviously contained in Kundt class, for the choice of metric functions

$$H = H(u, z, \bar{z}), \quad W = \overline{W} = 0, \quad P = 1 \tag{2.26}$$

The pp-waves are of PP type O and Petrov type N or O. In higher dimensions this is not true anymore, as we will see in Chapter 3.

2.2.1. Curvature Invariants

A *curvature invariant of order k* is a scalar constructed from the metric, a polynomial in the Riemann tensor and its covariant derivatives up to the order k. The simplest example is the Ricci scalar, which is obviously of zeroth order. Another invariant of zeroth order is the Kretschmann curvature scalar

$$R_{abcd}R^{abcd} \tag{2.27}$$

An invariant of order 1 is, for example,

$$R_{abcd;e}R^{abcd;e} \tag{2.28}$$

where ; denotes covariant differentiation. In general, scalar invariants can be useful in proving whether metrics given in different coordinate systems are identical or not. Hence, they are often used to identify physical singularities. Curvature invariants of zeroth order have been extensively studied in the literature (see, for example, [27] and references therein). This is not the case for higher order invariants.

2.2.2. VSI and CSI spacetimes

We define *VSI_k spacetimes* as Lorentzian spacetimes for which all curvature invariants of order up to k are zero, and *VSI spacetimes* as those for which all invariants of all orders vanish. The acronym VSI stands for vanishing scalar invariants. It is clear that VSI spacetimes cannot be distinguished from each other or flat space by using curvature invariants. VSI_1 spacetimes have been investigated in [28]. On the other hand, it was proven in [29] that if a spacetime is VSI_2, then it is necessarily VSI. It is well-known that pp-waves have this property [30]. In this thesis we treat the complete set of VSI metrics, both in four and higher dimensions (Chapters 2 and 3, respectively).

In [29] it was shown that VSI spacetimes belong to the Kundt class and are of PP type N or O, and of Petrov type III, N or O. It can be seen that such spacetimes are a subclass of the Kundt metrics with $P = 1$. Little is known about the physical interpretation of VSI spacetimes more general than pp-waves[5]. The character of singularities can only be studied from the behaviour of free test particles. We do know, however, that VSI spacetimes do not have black hole regions as they admit no horizons[6] [31]. This was already proven for pp-waves in [32].

In the same way, we can also define *CSI spacetimes* as those Lorentzian spacetimes for which all invariants of all orders are constant [33]. Accordingly, CSI stands for constant scalar invariants. Obviously, VSI $\subset$ CSI. CSI spacetimes can be physically interesting, as it is guaranteed that curvature invariants do not blow up. Some of the Kundt metrics are CSI. It is likely that all of the four-dimensional CSI spacetimes have been constructed in [33]. Further in this thesis we study higher-dimensional CSI spacetimes (section 3.3.2 and the whole Chapter 5).

VSI spacetimes have vanishing cosmological constant. However, they can

[5] When possible, we will comment on this throughout the thesis.
[6] This result holds in arbitrary number of dimensions.

be generalized to include a cosmological constant [34, 35, 36]. In this case $P = P(\Lambda)$ and they become CSI. The only non-zero curvature invariants are then zeroth order invariants constructed from the cosmological constant [29]. On the other hand, CSI spacetimes generally have a non-zero cosmological constant.

2.3. Einstein-Yang-Mills

The EYM system is obtained by coupling gravity to the Yang-Mills field

$$A = A_\mu \mathrm{d}x^\mu = A^a_\mu T_a \mathrm{d}x^\mu \tag{2.29}$$

Here, latin indices are gauge indices running from 1 to dim G, the dimension of the considered Lie group. The generators T_a obey

$$[T_a,\, T_b] = i f_{abc} T_c, \quad \mathrm{tr}\ (T_a T_b) = K \delta_{ab} \tag{2.30}$$

The EYM equations read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{K g^2} \mathrm{tr}\ \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu}\, F_{\alpha\beta} F^{\alpha\beta} \right) \tag{2.31}$$

$$D_\lambda F^{\lambda\mu} = 0 \tag{2.32}$$

where $8\pi G \equiv 1$, g is the gauge coupling constant and the Yang-Mills field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu,\, A_\nu] = F^a_{\mu\nu} T_a \tag{2.33}$$

The gauge covariant derivative in (2.32) is given by

$$D_\lambda = \nabla_\lambda - i\, [A_\lambda,\,] \tag{2.34}$$

We distinguish between two types of EYM solutions. First of all, solutions which have an abelian character. They can be easily constructed from Einstein-Maxwell solutions [37]. On the other hand, one would like to find solutions which have a genuine non-abelian character, i.e., solutions which cannot be constructed as mentioned above. This was first achieved in pp-wave background [38, 39]. These results were based on a flat space solution found by Coleman [40] and known as *non-abelian plane waves*. The solutions presented in the next section are the VSI generalization of the one in [38].

2.4. VSI-Yang-Mills solutions

We construct VSI-Yang-Mills solutions of PP type O[7], i.e.,

$$R_{ab} = R_{11}\,\ell_a \ell_b \tag{2.35}$$

To do so, we rely on the vacuum Kundt solutions of Petrov type III or lower. They were given in [41] (but only shown to be VSI in [29]). They are described by the Kundt line element with $P = 1$

$$ds^2 = -2du\,(dv + H\,du + W\,dz + \overline{W}\,d\bar{z}) + 2dzd\bar{z} \tag{2.36}$$

and further divided into the following subclasses

$$1)\ W_{,v} = 0, \quad 2)\ W_{,v} = -\frac{2}{z+\bar{z}} \tag{2.37}$$

In our conventions[8]

$$R_{11} = R_{uu} \tag{2.38}$$

2.4.1. Case $\mathbf{W_{,v} = 0}$

The metric functions are given by

$$\begin{aligned} W &= W(u,\bar{z}) && (2.39)\\ H &= H^{(0)}(u,z,\bar{z}) + \frac{1}{2}\left(W_{,\bar{z}} + \overline{W}_{,z}\right) v && (2.40) \end{aligned}$$

Here, W is an arbitrary complex function and $H^{(0)}$ is a real function. A function with the superscript (0) does not depend on the light-cone coordinate v. This is a vacuum solution of the Einstein equations if the function $H^{(0)}$ satisfies

$$\frac{1}{2}R_{uu} = H^{(0)}_{,z\bar{z}} - \mathrm{Re}\left(W_{,\bar{z}}^{2} + WW_{,\bar{z}\bar{z}} + W_{,\bar{z}u}\right) = 0 \tag{2.41}$$

This class of solutions degenerates to Petrov type N when

$$\Psi_3 = \frac{1}{2}W_{,\bar{z}\bar{z}} = 0 \tag{2.42}$$

[7] Recall that VSI spaces can also be of PP type N. However, the Yang-Mills field we consider here can (by itself) only act as a source to the PP type O VSI spacetimes. Besides, no energy-momentum tensor corresponding to a PP type N spacetime can satisfy the weak energy condition $T_{ab}u^a u^b \geq 0$ (u is the 4-velocity of an observer). Hence, they are not regarded as physically significant.

[8] $e_0 = \partial_v,\ \ e_1 = \partial_u + \overline{W}\,\partial_z + W\,\partial_{\bar{z}} - (H + W\overline{W})\,\partial_v,\ \ e_2 = \bar{e}_3 = \partial_z.$

In this case one can use the remaining freedom in the coordinate transformations (2.22)-(2.24) to make $W = 0$. The metric reduces then to the pp-waves (1.22). Petrov type O is obtained for

$$\Psi_4 = H^{(0)}{}_{,\bar{z}\bar{z}} = 0 \tag{2.43}$$

In vacuum, this is just flat space[9]. The function W can have an arbitrary dependence on u but must be at least quadratic in $\bar{z}$ so that the solution is truly of type III. This holds for non-vacuum PP type O solutions as well[10].

We can generalize the solution to include a null Yang-Mills field ($F^a_{\mu\nu}F^{a\mu\nu} = *F^a_{\mu\nu}F^{a\mu\nu} = 0$) of the form

$$A = \alpha(u, z, \bar{z})\, du = \alpha^a(u, z, \bar{z}) T_a\, du \tag{2.44}$$

The only non-zero components of the field strength are

$$F_{zu} = \alpha_{,z}\,, \quad F_{\bar{z}u} = \alpha_{,\bar{z}} \tag{2.45}$$

The Yang-Mills equation in flat space is

$$\partial_\lambda F^{\lambda\mu} - i[A_u,\ F^{u\mu}] = 2\alpha_{,z\bar{z}} = 0 \tag{2.46}$$

and it has the general solution

$$\alpha^a(u, z, \bar{z}) = \chi^a(u, z) + \overline{\chi}^a(u, \bar{z}) \tag{2.47}$$

where χ^a are arbitrary complex functions. In the spacetime (2.39)-(2.40), the Yang-Mills equation reads

$$\partial_\lambda F^{\lambda\mu} + \Gamma_{\lambda\nu}{}^{\lambda} F^{\nu\mu} - i[A_u,\ F^{u\mu}] \tag{2.48}$$

The third term vanishes since the components $F^{u\mu}$ are still zero in the considered geometry. On the other hand, the only non-zero Christoffel symbols of the form $\Gamma_{\lambda\nu}{}^{\lambda}$ are precisely $\Gamma_{\lambda u}{}^{\lambda}$, and so the second term vanishes as well. The Yang-Mills equation has the same form and solution as in flat space.

[9] A vacuum solution of Petrov type O has zero Ricci and Weyl tensor. From equation (2.11) we see that the Riemann tensor is zero as well.

[10] Matter fields only enter the solution through the differential equation for $H^{(0)}$; the function W is as in vacuum.

Of course equation (2.41) has to be modified to account for the energy-momentum tensor of the Yang-Mills field

$$T_{uu} = \frac{2}{g^2}\delta_{ab}\ \alpha,^a_z \alpha,^b_{\bar{z}} \tag{2.49}$$

Therefore, equation (2.41) transforms to

$$H,^{(0)}_{z\bar{z}} - \mathrm{Re}\left(W,^2_{\bar{z}} + WW,_{\bar{z}\bar{z}} + W,_{\bar{z}u}\right) = \frac{1}{g^2}\delta_{ab}\alpha,^a_z \alpha,^b_{\bar{z}} \tag{2.50}$$

It is straightforward to solve for the function $H^{(0)}$

$$H^{(0)}(u,z,\bar{z}) = f(u,z) + \bar{f}(u,\bar{z}) + \frac{1}{g^2}\delta_{ab}\chi^a\overline{\chi}^b + \mathrm{Re}\left\{(W,_u + WW,_{\bar{z}})\, z\right\} \tag{2.51}$$

Here f is an arbitrary complex function. This is an exact solution of the EYM system. It is of Petrov type III, so long as the function W is (at least) quadratic in $\bar{z}$. In the type N reduction [38] the fourth term on the right-hand side of (2.51) vanishes. This type N solution has also been considered in the context of supergravity [42, 18]. The solution becomes of Petrov type O for

$$\bar{f}(u,\bar{z}),_{\bar{z}\bar{z}} = 0, \quad \delta_{ab}\chi^a\overline{\chi}^b,_{\bar{z}\bar{z}} = 0 \tag{2.52}$$

In this limit we recover Coleman's non-abelian plane waves

$$\chi^a(u,z) = \lambda^a(u) z \tag{2.53}$$

where $\lambda^a(u)$ are bounded complex functions. They are the non-abelian analogues of the electromagnetic plane waves (1.50). In fact, (2.53) is the only form of (2.47) which gives rise to a bounded energy-momentum tensor. Note that in this case the space is only curved by the presence of the Yang-Mills field. In other words, the Riemann tensor (1.25) is only non-zero because of the YM term $\sim z\bar{z}$ in (2.51).

2.4.2. Case $\mathbf{W,_v = -\frac{2}{z+\bar{z}}}$

We consider now the metric (2.36) with

$$W = W^{(0)}(u,z) - \frac{2v}{z+\bar{z}} \tag{2.54}$$

$$H = H^{(0)}(u,z,\bar{z}) + \frac{W^{(0)} + \overline{W}^{(0)}}{z+\bar{z}}v - \frac{v^2}{(z+\bar{z})^2} \tag{2.55}$$

Here, $W^{(0)}$ is an arbitrary complex function and $H^{(0)}$ is a real function. This is a vacuum solution of Einstein equations if the function $H^{(0)}$ satisfies

$$\frac{1}{2}R_{uu} = (z+\bar{z})\left(\frac{H^{(0)}+\overline{W}^{(0)}W^{(0)}}{z+\bar{z}}\right)_{,z\bar{z}} - W,_{z}^{(0)}\,\overline{W},_{\bar{z}}^{(0)} = 0 \tag{2.56}$$

This class of solutions become of Petrov type N when $\Psi_3 = \overline{W},_{\bar{z}}^{(0)}/(z+\bar{z}) = 0$. In this case one can transform the function $W^{(0)}(u)$ to zero. The resulting metric is known as *Kundt waves*. It represents plane-fronted[11] gravitational waves. Note that in this case

$$H = H^{(0)}(u,z,\bar{z}) - \frac{v^2}{(z+\bar{z})^2} \tag{2.57}$$

satisfies the wave equation (1.31). In [43] it was shown that the Kundt waves have a physical singularity at $x = 1/\sqrt{2}(z+\bar{z}) = 0$. Moreover, it was recently proved that chaotic geodesic motion arises for the Kundt waves [44], as for (non-plane waves) pp-waves [45]. Roughly speaking, this means that the trajectories of test particles are very much dependent on the initial conditions. They will actually follow substantially different paths for very similar initial conditions.

Petrov type O is obtained for

$$\Psi_4 = (z+\bar{z})\left(\frac{H^{(0)}}{z+\bar{z}}\right),_{\bar{z}\bar{z}} = 0 \tag{2.58}$$

In [46] it was shown that Einstein-Maxwell null fields, scalar and neutrino fields do not exist for this type of metrics.

We consider the field (2.44), (2.47) in this background. Features in this geometry are the same ones as in the previous case, except for extra terms in the Yang-Mills equation

$$(\partial_z + \Gamma_{uz}{}^{u} + \Gamma_{vz}{}^{v})F^{zv} + (\partial_{\bar{z}} + \Gamma_{u\bar{z}}{}^{u} + \Gamma_{v\bar{z}}{}^{v})F^{\bar{z}v} = 0 \tag{2.59}$$

However, these extra terms cancel each other since

$$\Gamma_{uz}{}^{u} = -\Gamma_{vz}{}^{v}, \quad \Gamma_{u\bar{z}}{}^{u} = -\Gamma_{v\bar{z}}{}^{v} \tag{2.60}$$

[11] Since $P \equiv 1$ in (2.21).

And so the Yang-Mills equation is still the same as in flat space, equation (2.46). Finally, equation (2.56) has to be modified to include the YM energy-momentum tensor

$$\frac{1}{2}R_{uu} = (z+\bar{z})\left(\frac{H^{(0)}+\overline{W}^{(0)}W^{(0)}}{z+\bar{z}}\right)_{,z\bar{z}} - W,_{z}^{(0)}\,\overline{W},_{\bar{z}}^{(0)} = \frac{1}{g^2}\delta_{ab}\alpha,_{z}^{a}\,\alpha,_{\bar{z}}^{b} \quad (2.61)$$

We refine our ansatz in order to solve explicitly for $H^{(0)}$. We choose the specific function $W^{(0)} = g(u)z$, where $g(u)$ is an arbitrary complex function. This is the simplest form of $W^{(0)}$ for which the solution is of Petrov type III[12]. On the other hand we consider the Yang-Mills field (2.53). Under these assumptions the equation to solve reads

$$(z+\bar{z})\left(\frac{H^{(0)}+g\bar{g}z\bar{z}}{z+\bar{z}}\right)_{,z\bar{z}} - g(u)\bar{g}(u) = \frac{1}{g^2}\delta_{ab}\lambda^a(u)\bar{\lambda}^b(u) \quad (2.62)$$

It is not difficult to see that

$$\begin{aligned} H^{(0)}(u,z,\bar{z}) &= \left(f(u,z)+\bar{f}(u,\bar{z})\right)(z+\bar{z}) - g\bar{g}z\bar{z} \\ &\quad +\sigma(u)(z+\bar{z})^2\left\{\ln(z+\bar{z})-1\right\} \end{aligned} \quad (2.63)$$

for $\mathrm{Re}(z) > 0$. Here f is again an arbitrary complex function while σ is the real function

$$\sigma(u) = \frac{1}{g^2}\delta_{ab}\lambda^a(u)\bar{\lambda}^b(u) + g(u)\bar{g}(u) \quad (2.64)$$

This is another exact EYM solution of Petrov type III. The corresponding vacuum solution ($\lambda^a \equiv 0$) has to our knowledge not been considered before. The case $g(u) \equiv 0$ yields the type N reduction. A related solution of Petrov type N was given in [47], for a Yang-Mills field with arbitrary higher polynomial dependence on z, $\bar{z}$. The solution is of Petrov type O if

$$\bar{f}(u,\bar{z}),_{\bar{z}\bar{z}} = 0, \quad \delta_{ab}\frac{\lambda^a(u)\bar{\lambda}^b(u)}{z+\bar{z}} = 0 \quad (2.65)$$

The second equation would imply that the YM field is absent. Therefore, such solutions do not exist[13]. This is consistent with the fact that null EM solutions

[12] In the case $g(u) \equiv 1$ the linear term in v of the metric function H can be transformed away by a coordinate transformation (2.24) with $h = -e^{-u}$. Regardless, the solution is of Petrov type III as $W^{(0)'} = e^u z'$, i.e., $\Psi_3 \neq 0$.

[13] At least for the non-abelian plane waves (2.53).

do not exist either.

We conclude the chapter with a few general remarks. The term $[A_\mu,\ A_\nu]$ in the Yang-Mills curvature (2.33) vanishes for the considered YM field. Still, the solutions have a full non-abelian character as long as the gauge fields $\alpha^a(u, z, \bar{z})$ are distinct. The case

$$\alpha^a(u, z, \bar{z}) = \beta^a \alpha(u, z, \bar{z}) \tag{2.66}$$

with β^a some parameters, corresponds to "abelian" YM solutions. These can be reduced to VSI-Maxwell solutions by considering $U(1)$ as gauge group. Kundt Einstein-Maxwell solutions of Petrov type III with $W_{,v} = 0$ were already considered in [48], and EM Kundt waves in [49].

The geodesic equations for certain choices of the metric function W have been given in [21]. The case $W_{,v} \neq 0$ has been further studied in [50], where the possibility of chaotic behaviour was pointed out.

We end up the chapter summarizing the solutions presented in Table 2.1.

$\mathbf{W_{,v}}$	**Petrov type**	**Metric functions**
0	III	$W = W(u, \bar{z})$ $H = H^{(0)}(u, z, \bar{z}) + \frac{1}{2}\left(W_{,\bar{z}} + \overline{W}_{,z}\right) v$ Eq. (2.50)
	N	*pp-waves:* $W = 0$ $H = H^{(0)}(u, z, \bar{z})$ Eq. (2.50) with $W = 0$
	O	$W = 0$ $H = H^{(0)}(u, z, \bar{z})$ Eq. (2.50) with $W = 0$ and YM field (2.53)
$-\frac{2}{z+\bar{z}}$	III	$W = W^{(0)}(u, z) - \frac{2v}{z+\bar{z}}$ $H = H^{(0)}(u, z, \bar{z}) + \frac{W^{(0)} + \overline{W}^{(0)}}{z+\bar{z}} v - \frac{v^2}{(z+\bar{z})^2}$ Eq. (2.61)
	N	*Kundt waves:* $W = -\frac{2v}{z+\bar{z}}$ $H = H^{(0)}(u, z, \bar{z}) - \frac{v^2}{(z+\bar{z})^2}$ Eq. (2.61) with $W = 0$
	O	do not exist

Table 2.1: Four-dimensional VSI-Yang-Mills solutions of PP type O. The YM field is given by (2.44), (2.47). The solutions are PP type O-vacuum when the YM field is absent.

3 Higher-dimensional VSI spacetimes

In this chapter we first introduce the framework we are working in: General Relativity in higher dimensions. Next, we briefly review the recently developed algebraic classification of higher-dimensional spacetimes. We study the higher-dimensional Kundt class and its most interesting subclasses: higher-dimensional pp-waves, CSI and VSI spacetimes, as well as their overlap. At this point we are ready to derive the explicit form of the higher-dimensional VSI spacetimes and classify them completely, which is the aim of this chapter. This chapter is based on reference [51].

3.1. Einstein gravity in higher dimensions

Gravity in higher dimensions is described by an N-dimensional version of the usual Einstein-Hilbert action plus a possible cosmological constant

$$\mathcal{S} \sim \int d^N x \sqrt{-g}\,(R + \Lambda) \tag{3.1}$$

Varying this action with respect to the N-dimensional metric leads to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3.2}$$

where the tensor indices run now from 0 to $N-1$. In the presence of matter there is also a term corresponding to the energy-momentum tensor

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{3.3}$$

Clearly, finding exact solutions in higher dimensions is not an easy task. Yet, many solutions have been found by now. Among the most interesting ones, the Myers-Perry solution [52] and the black ring [53].

As in four dimensions, more general theories of gravitation can be considered. Some of these alternative theories are specific to higher dimensions, like Gauss-Bonnet (GB) gravity [54, 55]. GB gravity often appears in the context of braneworld cosmology and string theory (see section 4.2). In fact, GB gravity is contained in the more general Lovelock theories [56]. These theories are characterized by having higher curvature terms in the Lagrangian, such as R^2.

3.2. Classification of the Weyl and Ricci tensor in higher dimensions

Four-dimensional spacetimes can be classified according to the Petrov and Plebanski-Petrov classification. An analogous classification of the Weyl and Ricci tensor for higher-dimensional spacetimes was recently proposed [57, 58]. Some of the known higher-dimensional spacetimes have been classified accordingly in [59]. We will review it briefly in this section, more details can be found in the mentioned literature.

3.2.1. Preliminaries

In this section we generalize the tetrad formalism described in 2.1.1 to arbitrary higher number of dimensions[1]. Consider an N-dimensional manifold M endowed with a Lorentzian metric, i.e., of signature $(-,+,\ldots,+)$. We will use the following coframe

$$(n, \ell, m^i), \quad i = 2, \ldots, N-1 \tag{3.4}$$

where all of the vectors are real[2] and

$$\ell^a \ell_a = n^a n_a = 0, \;\; \ell^a n_a = 1, \;\; m^{ia} m^j_a = \delta^{ij} \tag{3.5}$$

In other words ℓ and n are null while m^i are spacelike. Recall that the metric can be written in terms of a non-orthonormal coframe as follows

$$g = \eta_{ab}\, \theta^a \otimes \theta^b \tag{3.6}$$

[1] Although conventions here are slightly different.
[2] A complex coframe $(n, \ell, m^j, \overline{m}^j)$ can only be used in even number of dimensions.

where η_{ab} is not diagonal. We can identify $\theta^0 = n$, $\theta^1 = \ell$ and $\theta^i = m^i$. In terms of coordinate bases

$$\theta^0 = n_\mu \mathrm{d}x^\mu, \quad \theta^1 = l_\mu \mathrm{d}x^\mu, \quad \theta^i = m^i_\mu \mathrm{d}x^\mu \tag{3.7}$$

$$e_0 = l^\mu \partial_\mu, \qquad e_1 = n^\mu \partial_\mu, \quad e_i = m^\mu_i \partial_\mu \tag{3.8}$$

The metric reads

$$g = 2\,\theta^0 \otimes \theta^1 + \theta^i \otimes \theta^i \tag{3.9}$$

Or, equivalently,

$$\eta_{ab} = 2\ell_{(a} n_{b)} + \delta_{jk} m^j_a m^k_b \tag{3.10}$$

where parentheses indicate symmetrization of indices with weight 1/2. The frame transforms under a Lorentz boost in the (n, ℓ)-plane as follows

$$\hat{\ell} = \lambda^{-1}\ell, \quad \hat{n} = \lambda n, \quad \hat{m}_i = m_i \tag{3.11}$$

for some $\lambda \neq 0$ (and similarly for the coframe). A certain frame component $T_{a_1 \ldots a_p}$ of a tensor T of rank p transforms as

$$\hat{T}_{a_1 \ldots a_p} = \lambda^b T_{a_1 \ldots a_p} \tag{3.12}$$

where b is the *boost weight* of the considered component. Clearly, components with boost weight zero are those left invariant by a boost transformation. In this way, a tensor can be decomposed according to the boost weights of its components. The *boost order* of a tensor T is the highest boost weight occurring among the components.

The boost order of a tensor T depends on the rank and symmetry properties of the tensor. In principle, the boost order is equal to the rank of the tensor, but it is smaller when the tensor has some skew-symmetry. For example, a fully antisymmetric tensor of any rank can be decomposed as

$$T = (T)_1 + (T)_0 + (T)_{-1} \tag{3.13}$$

where 1, 0, -1 are the only possible boost weights. We will make use of this fact in the next chapter.

In particular, this machinery can be used to classify higher-dimensional spacetimes algebraically. Given a certain higher-dimensional spacetime, the Weyl and Ricci tensor can be calculated in the proposed frame as it is done for the metric in (3.10). Both tensors can then be classified according to their boost order. The classification of the Weyl tensor is the higher-dimensional analogue of the Petrov classification; the Ricci one is analogue to the Plebanski-Petrov (PP) classification (see Chapter 2). The Petrov and PP classification are recovered in the four-dimensional limit.

3.2.2. Weyl type

The Weyl tensor in higher dimensions is defined by

$$C_{abcd} = R_{abcd} - \frac{1}{N-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{1}{(N-1)(N-2)}Rg_{a[c}g_{d]b} \quad (3.14)$$

where R is the Ricci scalar. The Weyl tensor can be decomposed in the following way

$$C_{abcd} = \overbrace{4C_{0i0j}n_{\{a}m^i{}_b n_c m^j{}_{d\}}}^{2} + \overbrace{8C_{010i}n_{\{a}\ell_b n_c m^i{}_{d\}} + 4C_{0ijk}n_{\{a}m^i{}_b m^j{}_c m^k{}_{d\}}}^{1} + \left\{ \begin{array}{l} 4C_{0101}n_{\{a}\ell_b n_c \ell_{d\}} + 4C_{01ij}n_{\{a}\ell_b m^i{}_c m^j{}_{d\}} + \\ 8C_{0i1j}n_{\{a}m^i{}_b \ell_c m^j{}_{d\}} + C_{ijkl}m^i_{\{a}m^j{}_b m^k{}_c m^l{}_{d\}} \end{array} \right\}^0 + \overbrace{8C_{101i}\ell_{\{a}n_b \ell_c m^i{}_{d\}} + 4C_{1ijk}\ell_{\{a}m^i{}_b m^j{}_c m^k{}_{d\}}}^{-1} + \overbrace{4C_{1i1j}\ell_{\{a}m^i{}_b \ell_c m^j{}_{d\}}}^{-2}$$

where 2, 1, 0, −1, −2 indicate the boost weight of the different components. Note that, in general, the Weyl tensor is of boost order 2. The Weyl components are not completely independent from each other,

$$C_{0i0}{}^i = 0, C_{010j} = C_{0ij}{}^i, C_{0(ijk)} = 0, C_{0101} = C_{0i1}{}^i, C_{i(jkl)} = 0,$$
$$C_{0i1j} = -\frac{1}{2}C_{ikj}{}^k + \frac{1}{2}C_{01ij}, C_{1(ijk)} = 0, C_{1i1}{}^i = 0 \quad (3.15)$$

We are now in position to define the *principal type* of a Lorentzian spacetime (making use of the relations above):

$$\begin{array}{ll} \text{Type } G: & C_{abcd} \neq 0 \\ \text{Type } I: & C_{0i0j} = 0 \\ \text{Type } II: & C_{0i0j} = C_{0ijk} = 0 \\ \text{Type } III: & C_{0i0j} = C_{0ijk} = C_{ijkl} = C_{01ij} = 0 \\ \text{Type } N: & C_{0i0j} = C_{0ijk} = C_{ijkl} = C_{01ij} = C_{101i} = C_{1ijk} = 0 \\ \text{Type } O: & C_{abcd} = 0 \end{array} \quad (3.16)$$

In other words, we have Weyl type G if the boost order is 2, type I if the boost order is 1 and so forth.

It is also possible to define a more detailed classification, by specifying the *secondary type* as well. This means we not only take into account the boost order, but also the boost weight of individual components. For example, Type D, for which the Weyl tensor only has boost weight zero components. Also,

$$\text{Type } III(a): \quad C_{101i} = 0, \quad C_{1ijk} \neq 0 \tag{3.17}$$

We will study extensively certain metrics of this type in this chapter. Both types of boost weight -1 Weyl components are actually related by [60]

$$C_{101i} = C_{1jij} \tag{3.18}$$

where there is a sum over j. This relation holds for Ricci type N (see next section) and vacuum spacetimes [61]. Clearly, C_{101i} can be zero while C_{1jij} or C_{1ijk} are not. However, if $C_{1ijk} = 0$ then automatically $C_{101i} = 0$ as well. This will prove useful when calculating the Weyl type N reduction of type III metrics further in this chapter.

The Riemann tensor can be decomposed in the same way [60]. In general it is also of boost order 2 and it is subject to similar (but not equal) constraints (3.15). The Riemann tensor can also be classified according to its boost order. It is important to note that, in general, the Riemann and Weyl tensor can be of different boost orders. Both tensors are related through equation (3.14). It is then clear that the boost order of the Ricci tensor (see next section) has to be taken into account as well.

3.2.3. Ricci type

A second order symmetric tensor can be decomposed as follows

$$\begin{aligned} R_{ab} \quad = \quad & \overbrace{R_{00}\, n_a n_b}^{2} + \overbrace{2R_{0i}\, n_{(a} m^i{}_{b)}}^{1} + \overbrace{2R_{01}\, \ell_{(a} n_{b)} + R_{ij}\, m^i{}_a m^j{}_b}^{0} \\ & + \overbrace{2R_{1i}\, \ell_{(a} m^i{}_{b)}}^{-1} + \overbrace{R_{11}\, \ell_a \ell_b}^{-2} \end{aligned} \tag{3.19}$$

where 2, 1, 0, -1, -2 indicate the boost weight of the different components. In particular, this is true for the Ricci tensor. We are now in position to define the *Ricci type* of a Lorentzian spacetime:

$$\begin{aligned}
&\text{Type } I: \quad R_{0i} = 0 \\
&\text{Type } II: \quad R_{00} = R_{0i} = 0 \\
&\text{Type } III: \quad R_{00} = R_{0i} = R_{01} = R_{ij} = 0 \\
&\text{Type } N: \quad R_{00} = R_{0i} = R_{01} = R_{ij} = R_{1i} = 0 \\
&\text{Type } O: \quad R_{ab} = 0
\end{aligned} \tag{3.20}$$

Note that the metric tensor can also be decomposed in the above form. In fact, we have already done so in (3.10). Comparing both expressions we see that the metric only has boost weight zero components. This must necessarily be the case as any metric is invariant under boosts.

3.3. Higher-dimensional Kundt class

The Kundt class can be generalized to higher arbitrary number of dimensions. As in four dimensions, it is characterized by having a shear-free, non-expanding, non-twisting geodesic null congruence $\mathbf{l} = \partial_v$. The higher-dimensional Kundt class can be written in canonical form as follows [33]:

$$ds^2 = 2du\,[dv + H(u,v,x^k)\,du + W_i(u,v,x^k)\,dx^i] + g_{ij}(u,x^k)\,dx^i dx^j \tag{3.21}$$

In N dimensions, the light-cone coordinates[3] are $u = 1/\sqrt{2}\ (x^{N-1} - t)$, $v = 1/\sqrt{2}\ (x^{N-1} + t)$. The spatial coordinates are $(x^1, \ldots, x^{N-2})$; g_{ij} is the spatial metric. In contrast to the four-dimensional case, we choose *real* spatial coordinates to treat odd and even number of dimensions on the same footing. The metric functions W_i are real as well. Of course one can always go to complex notation in even number of dimensions[4]. There exist coordinate transformations $x^{i'} = f^i(u,x^k)$ which can be used to transform *either* the v-independent part of the metric functions W_i away *or* to eliminate the u-dependence in the transverse metric [62]. In the first case the transformed metric g_{ij} necessarily depends on u. Metrics of the form (3.21) are (at most) of Riemann type I, i.e., without further constraints the boost order of the Riemann tensor is 1. The higher-dimensional Kundt class contains a number of interesting subclasses, which we describe in what follows.

[3] Here we use a different convention than in Chapter 2.

[4] In Appendix *A* we give the relation between the metric functions in both notations for arbitrary even number of dimensions N (and flat transverse metric g_{ij}).

3.3.1. Higher-dimensional pp-wave spacetimes

Higher-dimensional pp-wave spacetimes are defined as in four dimensions, i.e., those spacetimes admitting a covariantly constant null vector (CCNV). The most general N-dimensional pp-wave spacetime is given by the Brinkmann metric [63]

$$ds^2 = 2du\,[dv + H(u,x^k)\,du + W_i(u,x^k)\,dx^i] + g_{ij}(u,x^k)\,dx^i dx^j \qquad (3.22)$$

A detailed discussion on these metrics can be found in [5]. From (3.22) it is clear that $k = k^\mu \partial_\mu = \partial_v$ is a Killing vector. Let us see it is also covariantly constant

$$\partial_\mu k_\nu - \Gamma^\lambda_{\mu\nu} k_\lambda = 0 \qquad (3.23)$$

In principle, the expression on the left-hand side is non-zero for $\nu = u$

$$\partial_\mu k_u - \Gamma^u_{\mu u} k_u \qquad (3.24)$$

Since k_u is constant and all Christoffel symbols $\Gamma^u_{\mu u}$ are zero, (3.24) vanishes. In particular this is true for flat transverse space.

The higher-dimensional pp-wave spacetimes belong to the higher-dimensional Kundt class. Note that in the literature people usually call higher-dimensional pp-waves to the following metric

$$ds^2 = 2du\,[dv + H(u,x^k)\,du] + \delta_{ij}(u,x^k)\,dx^i dx^j \qquad (3.25)$$

This is misleading, as the class of CCNV spacetimes is much broader.

3.3.2. Higher-dimensional CSI and VSI spacetimes

Higher-dimensional CSI and VSI spacetimes are defined as in four dimensions. In general, higher-dimensional CSI metrics are exact solutions of (3.3) with non-zero cosmological constant (although there is a few of them for which it vanishes [64]). It has been conjectured that higher-dimensional CSI spacetimes are either locally homogeneous spaces[5] or a subclass of the Kundt metrics [33]. In this thesis we will treat the latter, which we call Kundt CSI metrics.

The higher-dimensional Kundt CSI metrics are given by (3.21), where the spatial metric is locally homogeneous. Classical examples of homogeneous

[5] The distinction between globally and locally homogeneous spaces is a subtle one and we will not discuss it here. We refer the interested reader to [65, 66].

spaces are those with euclidean, spherical or hyperbolic geometry. This is a necessary but not sufficient condition. Kundt CSI spacetimes are conjectured to be (at most) of Riemann type II. Therefore, one additionally has to impose the vanishing of the boost weight 1 components of the Riemann tensor. We will treat Kundt CSI spacetimes in detail in Chapter 5.

The higher-dimensional VSI spacetimes are exact solutions of (3.3) with zero cosmological constant. It has been proven that any such metric can be written in the form [67, 33]

$$ds^2 = 2du\,[dv + H(u,v,x^k)\,du + W_i(u,v,x^k)\,dx^i] + \delta_{ij}\,dx^i dx^j \qquad (3.26)$$

The higher-dimensional VSI spacetimes are therefore a subclass of the higher-dimensional Kundt CSI spacetimes with *euclidean* spatial metric. Again, this condition is necessary but not sufficient. VSI spacetimes are (at most) of Riemann type III [68]. The vanishing of the boost weight 1 and 0 components of the Riemann tensor is thus also required. Note that, in general, the v-independent part of the metric functions W_i in (3.26) cannot be transformed away as there is no u-dependence left in the transverse metric. This chapter is devoted to the study of the explicit form and classification of such spacetimes.

We summarize the main subclasses of the higher-dimensional Kundt spacetimes in the table below.

	Line element	**CSI**	**VSI**
HD Kundt	Eq. (3.21)	g_{ij} (locally) homogeneous Riemann type II	$g_{ij} = \delta_{ij}$ Riemann type III
HD pp-waves	Eq. (3.22)		

Table 3.1: Higher-dimensional Kundt class and main subclasses.

Note that CSI spacetimes of the Kundt form are a subset of all CSI metrics, while VSI spacetimes necessarily belong to the Kundt class. It is also worth to notice the following. In four dimensions, pp-wave metrics are *always* VSI. A higher-dimensional pp-wave spacetime (3.22), however, can be VSI, CSI or none of them.

3.4. Higher-dimensional VSI spacetimes

Recall that any higher-dimensional VSI metric can be written in the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}v + H(v,u,x^n)\mathrm{d}u + W_i(v,u,x^n)\mathrm{d}x^i\right] + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{3.27}$$

with $i,j = 1,\ldots,N-2$. The purpose of this chapter is twofold. We classify VSI spacetimes in terms of their Ricci and Weyl type. We find out the explicit form of the metric functions H and W_i in (3.27). This is analogous to the results on four-dimensional VSI spacetimes by Coley et al, presented in Chapter 2.

All of the coordinate transformations preserving the form of metric (3.27) follow:

- $(v',u',x'^i) = (v,u,f^i(x^k))$ and $J^i{}_j \equiv \frac{\partial f^i}{\partial x^j}$

$$H' = H, \quad W'_i = W_j\left(J^{-1}\right)^j{}_i, \quad \delta'_{ij} = \delta_{kl}\left(J^{-1}\right)^k{}_i\left(J^{-1}\right)^l{}_j \tag{3.28}$$

- $(v',u',x'^i) = (v + h(u,x^k), u, x^i)$

$$H' = H - h_{,u}, \quad W'_i = W_i - h_{,i}, \quad \delta'_{ij} = \delta_{ij} \tag{3.29}$$

- $(v',u',x'^i) = (v/g_{,u}(u), g(u), x^i)$

$$H' = \frac{1}{g_{,u}^2}\left(H + v\frac{g_{,uu}}{g_{,u}}\right), \quad W'_i = \frac{1}{g_{,u}}W_i, \quad \delta'_{ij} = \delta_{ij} \tag{3.30}$$

- $(v',u',x'^i) = (v,u,f^i(u;x^k))$, where $f^i(u_0;x^k)$ is an isometry of flat space, and $J^i{}_j \equiv \frac{\partial f^i}{\partial x^j}$

$$\begin{aligned} H' &= H + \delta_{ij}f^i_{,u}f^j_{,u} - W_j\left(J^{-1}\right)^j{}_i f^i_{,u}, \quad W'_i = W_j\left(J^{-1}\right)^j{}_i - \delta_{ij}f^j_{,u}, \\ \delta'_{ij} &= \delta_{kl}\left(J^{-1}\right)^k{}_i\left(J^{-1}\right)^l{}_j \equiv \delta_{ij} \end{aligned} \tag{3.31}$$

The coordinate transformations (3.28) form the group of diffeomorphisms of the transverse space. The set of transformations (3.31) consists of translations, rotations and reflections of the coordinates x^i. For every constant $u = u_0$, they form the isometry group of $(N-2)$-dimensional euclidean space; for arbitrary

u it becomes an infinite-dimensional group[6].

As we have already said in the previous section VSI spacetimes are known to be of Riemann type III, at most. We learned from section 3.2.2 in this chapter what this means: the Riemann tensor can only have negative boost weight components. The next step is therefore to calculate the positive and zero boost weight components of the Riemann tensor for (3.27) in a certain null frame and require them to be zero. We introduce the null coframe

$$\begin{aligned} \ell &= \mathrm{d}u & (3.32) \\ n &= \mathrm{d}v + H\mathrm{d}u + W_i \mathrm{d}x^i & (3.33) \\ m^{i+1} &= \mathrm{d}x^i & (3.34) \end{aligned}$$

The line element (3.27) can be written as in (3.9), where we identify $\ell = \theta^1$, $n = \theta^0$, $m^{i+1} = \theta^{i+1}$. It is clear that tetrad indices run from 0 to $N-1$. Now we are in position to calculate the desired Riemann components. It turns out they have boost weight 1 and 0. The linearly independent such components of the Riemann tensor are[7]

$$\begin{aligned} R_{010(i+1)} &= -\frac{1}{2} W_{i,vv} & (3.35) \\ R_{0101} &= -H_{,vv} + \frac{1}{4}\left(W_{i,v}\right)\left(W^{i,v}\right) & (3.36) \\ R_{01(i+1)(j+1)} &= W_{[i}W_{j],vv} + W_{[i;j],v} & (3.37) \\ R_{0(i+1)1(j+1)} &= \frac{1}{2}\left[-W_j W_{i,vv} + W_{i;j,v} - \frac{1}{2}\left(W_{i,v}\right)\left(W_{j,v}\right)\right] & (3.38) \end{aligned}$$

where square brackets denote antisymmetrization and ; denotes covariant derivation. The components (3.35) have boost weight 1 while (3.36)-(3.38) have boost weight 0.

Hence, the negative boost order conditions of the Riemann tensor yield

$$\begin{aligned} W_{i,vv} &= 0 & (3.39) \\ H_{,vv} - \frac{1}{4}\left(W_{i,v}\right)\left(W^{i,v}\right) &= 0 & (3.40) \end{aligned}$$

[6] This also applies to the analogous set of coordinate transformations in the four-dimensional VSI case, i.e., transformations of the form (2.22) with $P \equiv 1$ (for general P the transverse space has no isometries).

[7] Riemann and Weyl tensor components throughout the chapter have been calculated with Maple; Ricci tensor components have been calculated with the GREAT [69] package for Mathematica.

from which it follows that

$$W_i(v,u,x^k) = vW_i^{(1)}(u,x^k) + W_i^{(0)}(u,x^k) \tag{3.41}$$

$$H(v,u,x^k) = \frac{v^2}{8}(W_i^{(1)})(W^{(1)i}) + vH^{(1)}(u,x^k) + H^{(0)}(u,x^k) \tag{3.42}$$

subject to

$$W_{[i;j],v} = 0 \tag{3.43}$$

and

$$W_{(i;j),v} - \frac{1}{2}\left(W_{i,v}\right)\left(W_{j,v}\right) = 0 \tag{3.44}$$

In equations (3.41), (3.42) the superscript (0) denotes functions without v-dependence; the superscript (1) denotes functions appearing as coefficients of terms linear in v.

Actually only one of the W_i functions can have v-dependence (if at all), namely W_1. Let us show this. From equation (3.43) it follows that $W_i^{(1)}$ can be written locally as a gradient:

$$W_i^{(1)} = [\phi(u,x^k)]_{,i} \tag{3.45}$$

Eq. (3.44) now simplifies to

$$\left(e^{-\frac{1}{2}\phi}\right)_{,ij} = 0 \tag{3.46}$$

which can be integrated to yield:

$$\phi = -2\ln\left[a_i(u)x^i + C(u)\right] \tag{3.47}$$

where $a_i(u)$ and $C(u)$ are arbitrary functions of u. Now, utilizing the rotations and translations of x^i we can simplify ϕ:

$$\phi = -2\ln[a(u)x^1], \quad \text{or} \quad \phi = -2\ln[C(u)] \tag{3.48}$$

Inserting these in equation (3.45) we obtain the two general cases:

$$\text{(i):} \qquad W_1^{(1)} = -\frac{2}{x^1}; \quad W_i^{(1)} = 0,\ i \neq 1$$

$$\text{(ii):} \qquad W_i^{(1)} = 0$$

Therefore the metric functions are

$$W_i(v,u,x^k) = -\delta_{i1}\frac{2\epsilon}{x^1}v + W_i^{(0)}(u,x^k) \tag{3.49}$$

$$H(v,u,x^k) = \frac{\epsilon v^2}{2(x^1)^2} + vH^{(1)}(u,x^k) + H^{(0)}(u,x^k) \tag{3.50}$$

where $H^{(1)}(u,x^k)$ and $H^{(0)}(u,x^k)$ are the redefined functions. The parameter ϵ can be 1 or 0, corresponding to case (i) or (ii) above. This is analogous to the four-dimensional case, where the spin coefficient τ can be different from zero or 0, making the distinction between metrics with $W_{,v} \neq 0$ or 0.

Recall that in four dimensions, case $\tau \neq 0$, we have

$$W_{,v} = -\frac{2}{z+\bar{z}} \tag{3.51}$$

In real spatial coordinates (x,y) this is just

$$W_{1,v} = -\frac{2}{x}, \qquad W_{2,v} = 0 \tag{3.52}$$

And so case (i) is the straightforward generalization of the four-dimensional situation. It is perhaps surprising that in higher dimensions one can get rid of the v-dependence in all functions W_i but one, just like in 4D.

The spacetimes (3.49), (3.50) above are in general of *Ricci and Weyl type III*. This can be seen from the definition of the Weyl tensor, equation (3.14). As the Riemann tensor only has negative boost weight components, so do the Ricci and Weyl tensor.

We compute next the Ricci tensor (in the coordinate basis) for (3.49), (3.50). The non-zero components are R_{uu} (which we do not give here) and R_{ui}, $i = 1,\dots,N-2$

$$R_{u1} = H^{(1)}{}_{,1} - \frac{\epsilon}{(x^1)^2}W_1^{(0)} - \frac{\epsilon}{x^1}W^{(0)m}{}_{,m} - \frac{1}{2}\Delta W_1^{(0)} + \frac{1}{2}W^{(0)i}{}_{,i1} \tag{3.53}$$

$$R_{un} = H^{(1)}{}_{,n} - \frac{1}{2}\Delta W_n^{(0)} + \frac{1}{2}W^{(0)i}{}_{,in} + \frac{\epsilon}{x^1}W_{1,n}^{(0)} \tag{3.54}$$

$m,n = 2,\dots,N-2$. The relation between the components in the coordinate and tetrad basis is

$$R_{ui} = R_{1i}, \quad R_{uu} = R_{11} \tag{3.55}$$

Indeed, comparing with (3.20) we see that the Ricci tensor is of type III. Without specific sources it is difficult to make further progress[8]. We will therefore only treat a subclass of all VSI Ricci type III metrics in the next section.
The fact that only one of the functions W_i is allowed to depend on v can also be seen at the level of the equations of motion. If we postulate an ansatz such that for example two of the W_i functions depend on v, say

$$W_{1,v} \neq 0, \quad W_{m,v} \neq 0 \tag{3.56}$$

where W_m can be any of the W_i functions other than W_1. The Ricci tensor has then a non-zero component of the form

$$R_{1m} = -W_{1,v}W_{m,v} \tag{3.57}$$

And so we see that such an spacetime cannot be VSI since the most general VSI metrics only have non-zero R_{uu}, R_{ui} Ricci components.

Further progress is most easily made by first performing a coordinate transformation such that we transform away $W_1^{(0)}$ in (3.49)[9]. This will make calculations a bit simpler. We make a coordinate transformation (3.29)

$$v' = v + h(u, x^k) \tag{3.58}$$

In the new coordinates

$$W_1' \;=\; -\frac{2\epsilon}{x^1}v' + \underbrace{\frac{2h\epsilon}{x^1} + W_1^{(0)} - h_{,1}}_{W_1'^{(0)}} \tag{3.59}$$

In order for $W_1'^{(0)}$ to vanish the function h must satisfy

$$h_{,1} - 2\frac{h\epsilon}{x^1} = W_1^{(0)} \tag{3.60}$$

This is an inhomogeneous linear differential equation of order one with solution

$$h = (x^1)^{2\epsilon}\left(\int \frac{W_1^{(0)}}{(x^1)^{2\epsilon}}dx^1 + C\right) \tag{3.61}$$

[8]In Chapter 4, section 4.4.2 we construct a specific Ricci type III solution where the source is a dilaton field.

[9]Actually, in the $\epsilon = 0$ case any one of the $W_i^{(0)}$ could be transformed away. We choose $W_1^{(0)}$ for consistency with the $\epsilon = 1$ case.

The remaining metric functions undergo the following redefinitions

$$W'_m = W_m^{(0)} - h_{,m} \quad (3.62)$$

$$H'^{(0)} = H^{(0)} - h_{,u} - hH^{(1)} + \frac{\epsilon h^2}{2(x^1)^2} \quad (3.63)$$

$$H'^{(1)} = H^{(1)} - \frac{\epsilon h}{(x^1)^2} \quad (3.64)$$

where m ranges over $2, \ldots, N-2$. The structure of (3.49), (3.50) is preserved and we can just reabsorb the extra factors in the definition of the functions

$$W_1 = -2\frac{v\epsilon}{x^1} \quad (3.65)$$

$$W_m = W_m^{(0)}(u, x^k) \quad (3.66)$$

$$H(v, u, x^k) = \frac{\epsilon v^2}{2(x^1)^2} + vH^{(1)}(u, x^k) + H^{(0)}(u, x^k) \quad (3.67)$$

As an aside, we remark that it is also possible to choose a gauge with $W_1^{(0)}$ non-zero. In general, both calculations and expressions become more difficult in this gauge but in some cases they are actually simpler (Weyl type O, $\epsilon = 0$). Another motivation to study this case is the fact that the metric functions for $\epsilon = 1$ are the higher-dimensional analogue of the 4-D ones. The corresponding results have been presented in [70]. Although we don't study this case here in full detail we do summarize the main results in Appendix F for completeness.

Equations (3.53), (3.54) simplify considerably when $W_1^{(0)} = 0$

$$R_{u1} = {H^{(1)}}_{,1} - \frac{\epsilon}{x^1}{W^{(0)m}}_{,m} + \frac{1}{2}{W^{(0)m}}_{,m1} \quad (3.68)$$

$$R_{un} = {H^{(1)}}_{,n} - \frac{1}{2}\Delta W_n^{(0)} + \frac{1}{2}{W^{(0)m}}_{,mn} \quad (3.69)$$

3.4.1. CCNV Ricci III VSI spacetimes

We treat Ricci and Weyl type III VSI spacetimes with no dependence on the light-cone coordinate v. Their metric functions are obtained from (3.65)-(3.67) by considering $\epsilon = H^{(1)} = 0$

$$W_1 = 0 \quad (3.70)$$

$$W_m = W_m^{(0)}(u, x^k) \quad (3.71)$$

$$H = H^{(0)}(u, x^i) \quad (3.72)$$

Clearly, they are of the form (3.22) with $g_{ij} = \delta ij$, i.e., they belong to the class of higher-dimensional pp-waves. In four dimensions, pp-waves are (at most) of PP type O and Petrov type N (Ricci and Weyl type N). However, in higher dimensions we have the following important result:

Higher-dimensional VSI pp-waves are (at most) of Ricci and Weyl type III

In this case we can give explicitly all of the Einstein equations

$$\triangle H^{(0)} - \frac{1}{4}W_{mn}W^{mn} - W^{(0)m}{}_{,mu} + T_{uu} = 0 \qquad (3.73)$$

$$\frac{1}{2}W^{(0)m}{}_{,m1} = T_{u1} \qquad (3.74)$$

$$\frac{1}{2}W^{(0)m}{}_{,mn} - \frac{1}{2}\Delta W^{(0)}_n = T_{un} \qquad (3.75)$$

where W_{mn} is defined by $W_{mn} = W^{(0)}_{m,n} - W^{(0)}_{n,m}$ [10] and we use the convention $8\pi G_N \equiv 1$. Note that equations (3.74), (3.75) come from (3.68), (3.69). These spacetimes only exist in the presence of matter giving rise to an energy-momentum tensor with non-zero T_{ui} components. In Chapter 4, section 4.4.2 we prove that this type of spacetimes do not exist in the context of supergravity. This means it is not possible to find supergravity sources with an energy-momentum tensor of the required form. At this point it is not known if sources of other type might do.

The spacetimes above reduce to Ricci type N when $T_{ui} = 0$, $T_{uu} \neq 0$. In that case one obtains the class discussed in section 3.4.2.

3.4.2. Ricci N VSI spacetimes

On the other hand, the discussion can be kept completely general qua sources if we restrict ourselves to VSI spacetimes of Ricci type N, for which

$$R_{ui} = 0 \qquad (3.76)$$

(see (3.20)). This condition allows us to determine the form of $H^{(1)}(u, x^k)$ in the metric function (3.50) in terms of the functions $W^{(0)}_i(u, x^k)$. With the condition above as a starting point we are now in position to classify the metrics

[10] Note that, in general, $W^{(0)}_1 = 0$ but $W_{1n} = -W_{n1} \neq 0$.

in terms of their Ricci type (N or O-vacuum) and their Weyl type (III, N or O).

When $\epsilon = 0, 1$ the Ricci type N conditions (3.76) reduce to

$$2H^{(1)}{}_{,1} = \frac{2\epsilon}{x^1} W^{(0)m}{}_{,m} - W^{(0)m}{}_{,m1} \tag{3.77}$$

$$2H^{(1)}{}_{,n} = \Delta W^{(0)}_n - W^{(0)m}{}_{,mn} \tag{3.78}$$

subject to

$$\Delta W^{(0)}_{n,1} = \frac{2\epsilon}{x^1} W^{(0)m}{}_{,mn} \tag{3.79}$$

$$\Delta W^{(0)}_{m,n} = \Delta W^{(0)}_{n,m} \tag{3.80}$$

where $\Delta = \partial^i \partial_i$ is the spatial Laplacian and $m, n \geq 2$. A partial integration of (3.77)-(3.80) reduces these constraints to a divergence and a Laplacian that must be satisfied by $W^{(0)}_m$, namely

$$W^{(0)m}{}_{,m} = \epsilon (x^1)^2 \left[F - \int \frac{4}{(x^1)^3} H^{(1)} dx^1 \right] + (1-\epsilon)F - 2H^{(1)} \tag{3.81}$$

$$\Delta W^{(0)}_n = \epsilon (x^1)^2 \left[F_{,n} - \int \frac{4}{(x^1)^3} H^{(1)}{}_{,n} dx^1 \right] + (1-\epsilon) F_{,n} \tag{3.82}$$

where $F = F(u, x^n)$ is an arbitrary function independent of x^1. Note when $\epsilon = 0$ (3.81) defines $H^{(1)}$ with F determined from (3.82). We note that, in general, F cannot be transformed away.

Therefore, the equations above describe all VSI metrics of Ricci type N, with $\epsilon = 0$ or $\epsilon = 1$. In general the Weyl tensor is of type III [11].

Weyl type III

As noted above, in general the Weyl tensor is of type III. We treat both the $\epsilon = 0$, $\epsilon = 1$ cases.

- $\boxed{\epsilon = 1}$

$$W_1 = -\frac{2}{x^1} v \tag{3.83}$$

$$W_m = W^{(0)}_m (u, x^k) \tag{3.84}$$

$$H = H^{(0)}(u, x^i) + \frac{1}{2}\left(\tilde{F} - W^{(0)m}{}_{,m} \right) v + \frac{v^2}{2(x^1)^2} \tag{3.85}$$

[11] In general, we will not display all non-zero components of the Weyl tensor. In some cases they are given in the text, a few others are listed in the Appendix at the end of the chapter. Full calculations are available upon request.

where $\tilde{F} = \tilde{F}(u, x^i)$ is a function satisfying:

$$\tilde{F}_{,1} = \frac{2}{x^1} W^{(0)m}{}_{,m}, \quad \tilde{F}_{,n} = \Delta W^{(0)}_n \tag{3.86}$$

In addition, we have the Einstein equation

$$\begin{aligned} x^1 \triangle \left(\frac{H^{(0)}}{x^1} \right) &+ \left(\frac{W^{(0)m} W^{(0)}_m}{x^1} \right)_{,1} - 2H^{(1)}{}_{,m} W^{(0)m} - H^{(1)} W^{(0)m}{}_{,m} \\ &- \frac{1}{4} W_{mn} W^{mn} - W^{(0)m}{}_{,mu} + T_{uu} = 0 \end{aligned} \tag{3.87}$$

where $W_{mn} = W^{(0)}_{m,n} - W^{(0)}_{n,m}$ [12] and T_{uu} is determined by the matter field; in the case of vacuum we have $T_{uu} = 0$.

- $\boxed{\epsilon = 0}$

$$W_1 = 0, \tag{3.88}$$

$$W_m = W^{(0)}_m(u, x^k) \tag{3.89}$$

$$H = H^{(0)}(u, x^i) + \frac{1}{2} \left(F - W^{(0)m}{}_{,m} \right) v \tag{3.90}$$

where $F = F(u, x^n)$ is a function satisfying:

$$F_{,1} = 0, \quad F_{,n} = \Delta W^{(0)}_n \tag{3.91}$$

Finally we have:

$$\begin{aligned} \triangle H^{(0)} &- \frac{1}{4} W_{mn} W^{mn} - 2H^{(1)}{}_{,m} W^{(0)m} - H^{(1)} W^{(0)m}{}_{,m} \\ &- W^{(0)m}{}_{,mu} + T_{uu} = 0 \end{aligned} \tag{3.92}$$

VSI spacetimes with (3.83)-(3.85) and (3.88)-(3.90) are the higher-dimensional analogues of the four-dimensional spacetimes of Petrov (Weyl) type III, PP-type O with $\tau \neq 0$ and $\tau = 0$, respectively. Further subclasses can be considered [58]. We discuss the subclass III(a) defined by (3.17).

[12] Note that, in general, $W^{(0)}_1 = 0$ but $W_{1n} = -W_{n1} \neq 0$.

Weyl Type III(a), $\epsilon = 1$

In this case the Weyl components C_{011i} $(= -C_{101i})$ read

$$\begin{aligned} C_{0112} &= H^{(1)}{}_{,1} \\ C_{011(m+1)} &= H^{(1)}{}_{,m} - \frac{1}{2}\frac{W^{(0)}_{m,1}}{x^1} \end{aligned} \tag{3.93}$$

where $H^{(1)}$ is the coefficient of v in (3.85) and $m = 2, \ldots, N-2$. Vanishing of the Weyl components (3.93) implies $H^{(1)}{}_{,m1} = 0$ for consistency; these constraints give already

$$W^{(0)}_m = \tilde{W}_m(u, x^n)\,\frac{(x^1)^2}{2} + \tilde{\tilde{W}}_m(u, x^n) \tag{3.94}$$

where the functions $\tilde{W}_m$, $\tilde{\tilde{W}}_m$ do not depend on x^1. The Weyl components (3.93) vanish themselves when the functions $W^{(0)}_m$ satisfy

$$W^{(0)m}{}_{,m1} = \frac{2}{x^1} W^{(0)m}{}_{,m} \tag{3.95}$$

$$W^{(0)m}{}_{,mn} = \Delta W^{(0)}_n - \frac{W^{(0)}_{n,1}}{x^1} \tag{3.96}$$

Inserting here (3.94) we get the following conditions for the functions $\tilde{W}_n(u, x^n)$, $\tilde{\tilde{W}}_n(u, x^n)$:

$$\tilde{W}^m{}_{,mn} = \Delta \tilde{W}_n \tag{3.97}$$

$$\tilde{\tilde{W}}^m{}_{,m} = \Delta \tilde{\tilde{W}}_n = 0 \tag{3.98}$$

The class is characterized by

$$W_1 = -\frac{2}{x^1} v \tag{3.99}$$

$$W_m = \tilde{W}_m(u, x^n)\,\frac{(x^1)^2}{2} + \tilde{\tilde{W}}_m(u, x^n) \tag{3.100}$$

$$H = H^{(0)}(u, x^i) + \frac{1}{2} f(u, x^n)\, v + \frac{v^2}{2(x^1)^2} \tag{3.101}$$

where the function $f(u, x^n)$ does not depend on x^1 and it is such that $f_{,m} = \tilde{W}_m(u, x^n)$. A necessary (but not sufficient) condition for a metric to be in this subclass is that the functions $\tilde{W}_m$, $\tilde{\tilde{W}}_m$ satisfy equations (3.97), (3.98). Furthermore, they have to be such that (some of) the left boost weight -1 C_{1ijk} Weyl components are non-vanishing[13] in order for the considered metric

[13] There is no compact way to write down the general C_{1ijk} in a similar fashion to that in the III(a) $\epsilon = 0$ case (equation (3.118)), so we shall not explicitly display them here.

to be truly of type III(a).

An example of a spacetime in this subclass is constructed as follows. For odd N, consider the functions

$$\tilde{\tilde{W}}_n = \frac{-p_n(u)x^{n+1}}{Q^{\frac{N-3}{2}}} \tag{3.102}$$

$$\tilde{\tilde{W}}_{n+1} = \frac{p_n(u)x^n}{Q^{\frac{N-3}{2}}} \tag{3.103}$$

where n only takes on even values of $m = 2, \ldots, N-2$ and $Q = \delta_{mm'}x^m x^{m'}$. The p_n are arbitrary functions of u. For even N, consider the same functions except for $n = N-2$

$$\tilde{\tilde{W}}_{N-2} = 0 \tag{3.104}$$

Note that none of these functions depend on x^1, as required. The spacetimes are then characterized by equations (3.99)-(3.101), with $\tilde{W}_m(u, x^n) = f(u, x^n) = 0$. The choice of functions W_m in N odd (even) dimensions coincides with those of the higher-dimensional gyraton solution [71] in $N-1$ even (odd) dimensions. However, the spacetimes presented here are more general, since the functions W_1 and H depend on v [14]. These solutions might be referred to as *Kundt gyratons*, in analogy with the pp-waves (no v-dependence) and Kundt waves (v-dependence) but which have the same (vanishing) W_m functions.

Another example of a spacetime in this subclass with $\tilde{W}_m = 0$ is given by the six-dimensional metric

$$W_1 = -\frac{2}{x^1}v \tag{3.105}$$

$$W_2 = f_2(u)x^3x^4 \tag{3.106}$$

$$W_3 = f_3(u)x^2x^4 \tag{3.107}$$

$$W_4 = f_4(u)x^2x^3 \tag{3.108}$$

$$H = H^{(0)}(u, x^i) + \frac{v^2}{2(x^1)^2} \tag{3.109}$$

We list the Weyl components C_{1ijk} in Appendix B.

[14] In the next section we treat the gyraton solution in detail.

In the last example, H contains a linear dependence on v (the two examples above can be generalized to include a linear dependence on v as well). From equations (3.99) - (3.101), we consider the symmetric and antisymmetric parts of $\tilde{\tilde{W}}_{m,n}$. We impose the condition

$$\left(\tilde{\tilde{W}}_{m,n} - \tilde{\tilde{W}}_{n,m}\right)_{,l} = 0 \tag{3.110}$$

along with the constraint (3.98) and recall that $f, \tilde{W}_m$ and $\tilde{\tilde{W}}_m$ contain no x^1 dependence. This results in the vanishing of all components, $C_{011(n+1)}$, of the Weyl tensor, whereas the remaining boost weight -1 components, C_{1ijk}, are related to the symmetric part of $\tilde{\tilde{W}}_{m,n}$. We note that, if instead of (3.110), we required that $\tilde{\tilde{W}}_{(m,n)} = 0$ (and also (3.98)), the resulting Weyl tensor is of reduced type N.

Weyl type III(a), $\epsilon = 0$

In this case we have

$$C_{011(n+1)} = H^{(1)}{}_{,n} \tag{3.111}$$

where $H^{(1)}$ is the coefficient of v in (3.90) and $n = 1, \ldots, N-2$. The Weyl components (3.111) vanish when the functions $W^{(0)}_m$ satisfy:

$$W^{(0)m}{}_{,m1} = 0 \tag{3.112}$$

$$W^{(0)m}{}_{,mn} = \Delta W^{(0)}_n \tag{3.113}$$

These conditions can be written more compactly

$$\partial_m W^{mn} = 0 \tag{3.114}$$

(and are consequently expressed in a form similar to the Maxwell equations in Euclidean space). The following class is obtained

$$W_1 = 0 \tag{3.115}$$

$$W_m = W^{(0)}_m(u, x^k) \tag{3.116}$$

$$H = H^{(0)}(u, x^i) \tag{3.117}$$

A necessary (but not sufficient) condition for a metric to be in this subclass is that the functions W_m satisfy equations (3.112), (3.113). Furthermore, the functions W_m have to be such that (some of) the left boost weight -1 Weyl components are non-vanishing:

$$C_{1(l+1)(n+1)(m+1)} = \frac{1}{2}\left(W_{m,n} - W_{n,m}\right)_{,l} \neq 0 \qquad (3.118)$$

This is equivalent to:

$$\partial_l W^{mn} \neq 0 \qquad (3.119)$$

From equations (3.115)-(3.117) it is clear that metrics in this subclass do *not* depend on v. In fact, they are obtained by making the Ricci type N reduction of metrics in section 3.4.1. As such, they belong to the class of higher-dimensional pp-waves. We have the following interesting result:

Higher-dimensional VSI pp-waves of Ricci type N are (at most) of Weyl type III(a)

Note that this result only concerns metrics of Weyl type III(a) in the case $\epsilon = 0$; Weyl type III(a) metrics in the case $\epsilon = 1$ *do* depend on v. We now give two explicit examples of metrics in this novel subclass. First of all, the higher-dimensional relativistic gyraton spacetime in [71]. Let us for example consider the ten-dimensional gyraton, for which the metric functions (3.116) read[15]

$$\begin{aligned} W_i &= -p_j(u)\frac{x^{i+1}}{Q^4} \\ W_{i+1} &= p_j(u)\frac{x^i}{Q^4} \end{aligned} \qquad (3.120)$$

where i only takes odd values 1, 3, 5, 7; p_j are arbitrary functions of u with $j = (i+1)/2$ and $Q = (x^1)^2 + \ldots + (x^8)^2$. Note that in this particular example $W_1^{(0)} \neq 0$; we can nevertheless use the equations in this section with transverse indices including 1. The above functions satisfy

$$\triangle W_n = 0, \quad W^{(0)m}{}_{,m} = 0 \qquad (3.121)$$

for each n and $n, m = 1, \ldots, 8$. Condition (3.113) is obviously satisfied. On the other hand, one can check that the components (3.118) are indeed non-zero (see Appendix C). The above solution describes the gravitational field of

[15] In [71] they use complex coordinates, we translate their results to real notation using Appendix A.

a spinning beam pulse of electromagnetic radiation ("gyraton").

A five-dimensional example of another spacetime in this subclass is given by

$$W_1 = 0 \quad (3.122)$$
$$W_2 = f_2(u)x^1x^3 \quad (3.123)$$
$$W_3 = f_3(u)x^1x^2 \quad (3.124)$$
$$H = H^{(0)}(u, x^i) \quad (3.125)$$

Here, f_2 and f_3 are arbitrary functions of u. These metric functions trivially satisfy (3.112), (3.113). The non-vanishing Weyl components (3.118) are

$$C_{1234} = \frac{1}{2}(f_3(u) - f_2(u))$$
$$C_{1324} = \frac{1}{2}f_3(u)$$
$$C_{1423} = \frac{1}{2}f_2(u)$$

In Chapter 4, section 4.5.2 we will come back to the previous two examples and calculate their supersymmetry properties.

Weyl type N

The spacetime is of Weyl type N if:

$$C_{1ijk} = C_{101i} = 0 \quad (3.126)$$

Further progress can then be made by requiring the vanishing of the C_{1ijk} components of the Weyl tensor; this guarantees that $C_{101i} = 0$ as well (see section 3.2.2). Using the Weyl type III results and the calculated expressions for C_{1ijk} we obtain the following metric functions.

- $\epsilon = 1$ - *Generalized Kundt waves*

In this case we have (the corresponding C_{1ijk} are listed in Appendix D):

$$W_1 = -2\frac{v}{x^1} \quad (3.127)$$
$$W_m = x^n B_{nm}(u) + C_m(u) \quad (3.128)$$
$$H = \frac{v^2}{2(x^1)^2} + H^{(0)}(u, x^i) \quad (3.129)$$

And (3.87) simplifies to

$$x^1 \triangle \left(\frac{H^{(0)}}{x^1}\right) - \frac{1}{(x^1)^2}\sum W_m^2 - 2\sum_{m<n} B_{mn}^2 + T_{uu} = 0, \tag{3.130}$$

where W_m is given by (3.128). In the case $B_{nm} = C_m \equiv 0$ one obtains the higher-dimensional Kundt waves in [67].

- $\epsilon = 0$ - *Generalized pp-waves*

In the case $\epsilon = 0$ we impose the vanishing of (3.118) and obtain

$$W_1 = 0 \tag{3.131}$$
$$W_m = x^1 C_m(u) + x^n B_{nm}(u) \tag{3.132}$$
$$H = H^{(0)}(u, x^i) \tag{3.133}$$

and (3.92) reduces to:

$$\triangle H^{(0)} - \frac{1}{2}\sum C_m^2 - 2\sum_{m<n} B_{mn}^2 + T_{uu} = 0 \tag{3.134}$$

$B_{nm} = B_{[nm]}$ in (3.128) and (3.132). In (3.132) a type (3.29) coordinate transformation has been used to remove a gradient term. Also, in (3.133) a type (3.30) transformation was used to eliminate a u-dependent term linear in v. This can *always* be done for higher-dimensional VSI spacetimes (see Appendix E). As in the four-dimensional case the term $x^1 C_m(u)$ can be transformed away (at the expense of introducing a non-vanishing W_1). However, unlike the four-dimensional case, terms linear in x^n in W_m cannot be transformed away (even if a non-zero W_1 is allowed).

Since both Weyl and Ricci tensors have only boost weight -2 components (i.e. both are of type N), VSI spacetimes with (3.127)-(3.129) and (3.131)-(3.133) are the higher-dimensional generalizations of Kundt and pp-waves (i.e., are the higher-dimensional analogues of the four-dimensional spacetimes of Petrov (Weyl) type N, PP-type O with $\tau \neq 0$ and $\tau = 0$, respectively).

Weyl type O

The spacetime is of type O if the Weyl tensor vanishes. We therefore require boost weight -2 Weyl components to be zero. These components are not only related to the metric functions W_m (the boost weight -1 components do), but also to the function $H^{(0)}$. This is similar to the four-dimensional case, where the scalar Ψ_4 is related to the function H^0 and Ψ_3 to W (see Chapter 2).

There are many boost weight -2 components, and they are lengthy. In requiring them to vanish one has to take into account that the function $H^{(0)}$ is given by the corresponding differential equation: (3.134) for $\epsilon = 0$, (3.130) for $\epsilon = 1$. So we need to find such a function $H^{(0)}$ which on top of that makes Weyl components C_{1i1j} zero. The functions W_m are just the same ones as for Weyl type N. The coordinate dependence of the energy-momentum tensor also plays a rol. We begin the discussion in each case by considering the case where $T_{uu} = T_{uu}(u)$ and build up from there.

- $\boxed{\epsilon = 1}$

  In this case the boost weight -2 components only vanish if the function $H^{(0)}$ satisfies

$$x^1 \left(\frac{H^{(0)}}{x^1} \right)_{,11} = \frac{1}{(x^1)^2} \sum W_m^2 - \frac{1}{8} T_{uu}, \quad H^{(0)}{}_{,mm} = \sum_n B_{mn}^2 - \frac{1}{8} T_{uu} \tag{3.135}$$

  It can be seen that this cannot be accomplished in the case $T_{uu} = T_{uu}(u)$. However, the Weyl tensor does vanish for $T_{uu} = \Phi(u) x^1$ and

$$\begin{aligned} H^{(0)} &= \frac{1}{2} \sum W_m^2 - \frac{1}{16} \Phi(u) x^1 \left[(x^1)^2 + (x^m)^2 \right] \\ &\quad + x^1 F_0(u) + x^1 x^i F_i(u) \end{aligned} \tag{3.136}$$

  where $\Phi(u)$, $F_0(u)$, $F_i(u)$ are arbitrary functions of u. The last two terms in $H^{(0)}$ correspond to the Weyl type O reduction of the higher-dimensional Kundt waves. This particular form of T_{uu} is likely to be the most general one compatible with a vanishing Weyl tensor. It remains to clarify the associated type of matter field for this conformally flat spacetime and the given Ricci tensor. Null Maxwell fields and massless scalar fields can already be excluded, as it is the case in four dimensions [46].

- $\boxed{\epsilon = 0}$

$$
\begin{aligned}
H^{(0)} &= \frac{1}{8}\left(\sum C_m^2 (x^1)^2 + \sum_{m\leq n} C_m C_n x^m x^n\right) + \frac{1}{2}x^1 x^m (C'_m + B_{mn}C_n) \\
&+\frac{1}{2}B_{ml}B_{nl}x^m x^n + x^i F_i(u) - \frac{1}{16}T_{uu}\ \left[(x^1)^2 + (x^m)^2\right] \qquad (3.137)
\end{aligned}
$$

Here $'$ means derivative with respect to u and $l \neq m, n$; $F_i(u)$ are arbitrary functions of u. The term $x^i F_i(u)$ corresponds to the Weyl type O reduction of generalized pp-waves with $W_m = 0$. In the case of matter depending on the spatial coordinates as well, $T_{uu}(u, x^i)$, the last term on the right-hand side of equation (3.137) has to be replaced by some $H^{(0)}_{(T_{uu})}$ such that:

$$
H^{(0)}_{(T_{uu}),ii} = -\frac{1}{8}T_{uu}, \quad H^{(0)}_{(T_{uu}),ij} = 0, \qquad i,j = 1,\ldots,8 \qquad (3.138)
$$

3.4.3. Ricci type O - vacuum

In the vacuum case and Weyl type III or N, equations are the same as for Ricci type N except T_{uu} is now zero. We note that vacuum spacetimes of Weyl type N can be interpreted as higher-dimensional exact gravitational waves, if at all. Finally, in the case of Weyl type O, we simply have N-dimensional Minkowski space.

We conclude this chapter by summarizing the results in Tables 3.2 and 3.3.

ϵ	**Weyl type**	**Metric functions**
0	III	$W_1 = 0$ $W_m = W_m^{(0)}(u, x^k)$ $H = H^{(0)}(u, x^i) + \frac{1}{2}\left(F - W^{(0)m}{}_{,m}\right) v;\ \ F(u, x^i)$ defined by (3.91) Eq. (3.92)
	N	*Generalized pp-waves:* $W_1 = 0$ $W_m = x^1 C_m(u) + x^n B_{nm}(u)$ $H = H^{(0)}(u, x^i)$ Eq. (3.134)
	O	W_1, W_m as in type N; $H^{(0)}$ given by (3.137), (3.138)
1	III	$W_1 = -\frac{2}{x^1} v$ $W_m = W_m^{(0)}(u, x^k)$ $H = H^{(0)}(u, x^i) + \frac{1}{2}\left(\tilde{F} - W^{(0)m}{}_{,m}\right) v + \frac{v^2}{2(x^1)^2};\ \ \tilde{F}(u, x^i)$ defined by (4.54) Eq. (3.87)
	N	*Generalized Kundt waves:* $W_1 = -2\frac{v}{x^1}$ $W_m = x^n B_{nm}(u) + C_m(u)$ $H = \frac{v^2}{2(x^1)^2} + H^{(0)}(u, x^i)$ Eq. (3.130)
	O	W_1, W_m as in type N; $H^{(0)}$ given by (3.136) and $T_{uu} = \Phi(u) x^1$

Table 3.2: All higher dimensional VSI spacetimes of Ricci type N. In the above, $m = 2, \ldots, N-2$. Ricci type O (vacuum) spacetimes occur for $T_{uu} = 0$ in (3.87), (3.92), (3.130), (3.134).

ϵ	**Ricci type**	**Weyl type**	**Metric functions**
0	III	III	$W_1 = 0$ $W_m = W_m^{(0)}(u, x^k)$ $H = H^{(0)}(u, x^i)$ $-\triangle H^{(0)} + \frac{1}{4} W_{mn} W^{mn} + W^{(0)m}{}_{,mu} = T_{uu}$ $W^{(0)m}{}_{,m1} = 2T_{u1}$ $-\Delta W_n^{(0)} + W^{(0)m}{}_{,mn} = 2T_{un}$
	N	III (a)	$W_1 = 0$ $W_m = W_m^{(0)}(u, x^k)$ $H = H^{(0)}(u, x^i)$ Eq. (3.92) $\left(H^{(1)} = 0\right)$ W_m satisfy (3.112), (3.113), (3.118)
		N	*Generalized pp-waves*; Eqns. (3.131) - (3.133)

Table 3.3: All higher-dimensional VSI spacetimes with a covariantly constant null vector.

4 VSI spacetimes in supergravity

In this chapter we study certain generalizations of pp-wave supergravity solutions, namely, VSI supergravity solutions. We explicitly consider the case of IIB supergravity, but similar solutions are expected in other supergravity theories. First of all, we introduce supergravity theories in general. We comment on the relevance of higher-dimensional pp-wave spacetimes in the context of supergravity and (super)strings. We briefly review the IIB supergravity theory. We explicitly construct VSI supergravity solutions of Ricci type N and III. We are not aware of any other Ricci type III supergravity solutions in the literature. We argue that the solutions supported by NS-NS fields are exact string solutions. Finally, we study the supersymmetry properties of the solutions. We find that the only supersymmetric solutions are those in the CCNV VSI subclass. We therefore show that supersymmetric solutions of more general algebraic type than the usual pp-waves can be found. This chapter is based on reference [62].

4.1. Supergravity theories

Supergravity is an extended theory of gravity with an extra (local) symmetry called *supersymmetry*, or *susy* for short. There is plenty of literature on supersymmetry and supergravity, see for example the classic reviews [72, 73]. In any supergravity theory the graviton coexists with a fermionic field called *gravitino*. The two fields differ in their spin: 2 for the graviton, 3/2 for the gravitino. The gravitino is actually a vector-spinor

$$\Psi_{\mu\,\alpha} \tag{4.1}$$

where μ is a vector index and α is a spinor index[1]. This is, in fact, a basic feature of supersymmetric theories in general. Each bosonic (fermionic) field in the non-supersymmetric version of the theory is postulated to have a fermionic (bosonic) partner with the same properties as the original field but spin differing in one half. In turn, the number of bosonic degrees of freedom must match the fermionic ones. Sometimes additional fields or constraints are needed to achieve this.

In a supergravity theory an action can be defined which includes both the graviton and gravitino, plus any other additional fields. The fields are related to each other by supersymmetry transformations which are parametrized by one or more local spinors

$$\epsilon(x^\mu) \tag{4.2}$$

Roughly speaking, they are such that bosons (fermions) are transformed into fermions (bosons), e.g.,

$$\delta_{\rm susy} B \sim \epsilon F \tag{4.3}$$

$$\delta_{\rm susy} F \sim \partial\epsilon + \epsilon\partial B \tag{4.4}$$

The action is left invariant under the supersymmetry transformations, i.e., it is *supersymmetric*

$$\delta_{\rm susy}\, \mathcal{S}(B,F) = 0 \tag{4.5}$$

The corresponding equations of motion can be derived from the action. As in General Relativity, the challenge is to find exact solutions to the equations of motion. In supergravity the challenge is actually double as one is often interested in finding solutions which are not only exact but also supersymmetric. This means that the solution itself is invariant under supersymmetry transformations

$$\delta_{\rm susy} B = 0 \tag{4.6}$$

$$\delta_{\rm susy} F = 0 \tag{4.7}$$

The above transformations are called *Killing equations*, and the spinor involved, (4.2), is called *Killing spinor* in this case. The number of linearly independent components of the Killing spinor determines the fraction of supersymmetry preserved by the solution. For example, if all (half) of the spinor

[1] A good introductory review on spinors in the context of supersymmetry can be found in [74].

components are linearly independent the solution is said to preserve all (one-half) supersymmetry and so forth.

Supergravity theories can be defined in various number of dimensions up to eleven [75]. In eleven dimensions there is only one such theory, D=11 supergravity [76]. This theory is conjectured to be the low-energy effective limit of M-theory. In ten dimensions there are three different supergravity theories: N=1 [77, 78], N=2A[2] [80, 81, 82] and N=2B [83, 84, 85] (aka IIA and IIB). N refers to the number of spinors needed to parametrize the supersymmetry transformations of the fields. IIA supergravity can be obtained as the ten-dimensional reduction of D=11 supergravity. IIA and IIB supergravity are the low-energy effective theories of IIA and IIB superstring theories, respectively. Moreover, the fermionic fields in IIA (IIB) supergravity have opposite (the same) chirality. In this thesis we will focus on IIB supergravity, perhaps the most studied one among the supergravity theories. In this way we can construct explicitly a certain type of VSI supergravity solutions. However, similar solutions are to be expected in other theories of supergravity.

4.2. Higher-dimensional pp-waves

We have studied the higher-dimensional pp-waves in Chapter 3, section 3.3.1. Metrics of the form (3.25) have been extensively considered in the context of supergravity and (super)strings (see [5] and references therein). One of the main reasons is the fact that such spacetimes are exact string solutions to all orders in the (inverse) string tension parameter α' [86, 87]. Let us briefly comment on this statement. It is clear that exact vacuum solutions of the Einstein equations are also exact supergravity solutions with all fields but the metric turned off. In string theory, the equation of motion for the metric receives corrections of all orders related to the parameter α' and the scalar invariants [88, 89]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha' H_{\mu\nu} + \ldots = 0 \tag{4.8}$$

Here H is the Lanczos tensor[3] which contains, among others, the Gauss-Bonnet term

$$H_{\mu\nu} \supset \frac{1}{2}g_{\mu\nu}(R_{\rho\sigma\lambda\eta}R^{\rho\sigma\lambda\eta} - 4R_{\rho\sigma}R^{\rho\sigma} + R^2) \tag{4.9}$$

[2] There is also a massive version of this theory [79].

[3] This should not be confused with another tensor called Lanczos, namely, a three index tensor in terms of which the Weyl tensor can be expressed [90].

The Lanczos tensor arises from the Gauss-Bonnet term in the action. This term only contributes to the equations of motion in five or higher dimensions. The dots denote higher order corrections constructed from derivatives and higher powers of the curvature.

A vacuum solution is an exact string solution when all α' corrections of all orders vanish. Such solutions are rare in string theory. This is precisely the case for higher-dimensional pp-waves due to the vanishing invariants property[4]. Of course, all vacuum VSI spacetimes presented in Chapter 3 have the same property; therefore they are exact string solutions as well. In the non-vacuum case there are also higher order corrections related to the matter fields. Using the arguments of [86], higher-dimensional pp-waves supported by appropiate fields are also known to be exact string solutions (for example, [87, 91, 92]).

There are other reasons to consider pp-waves in this context. Certain pp-wave solutions are *maximally supersymmetric* solutions of IIB and eleven-dimensional supergravities (see Chapter 5, section 5.4). This means that they preserve all of the supersymmetries. The maximally supersymmetric solutions of ten- and eleven-dimensional supergravities are the vacua of string and M theories. Moreover, type IIB superstring theory on the IIB maximally supersymmetric pp-wave solution is exactly solvable [93, 94]. In other words, the string spectrum can be calculated completely.

4.3. Type IIB supergravity

The field content of the IIB supergravity theory is

$$\{\ \underbrace{\phi, g_{\mu\nu},\ B_{\mu\nu},\ A_0,\ A_{\mu\nu},\ A_{\mu\nu\rho\sigma}}_{\text{bosons}},\ \underbrace{\lambda,\ \Psi_\mu}_{\text{fermions}}\ \} \tag{4.10}$$

Here ϕ is the scalar *dilaton* and λ is its supersymmetric partner, the *dilatino*; g is the metric tensor and Ψ_μ is the graviton supersymmetric partner, the gravitino (4.1). The field B is the so-called *Kalb-Ramond* field. B and A_n are the gauge potentials for the antisymmetric field H and *Ramond-Ramond* (RR) fields F_{n+1}, respectively

$$H = dB \tag{4.11}$$

$$F_{n+1} = dA_n \tag{4.12}$$

[4] This is a sufficient but not necessary condition. Some spacetimes with constant scalar invariants are exact string solutions as well (see Chapter 5).

The bosonic fields ϕ, g, B are called *Neveu-Schwarz/Neveu-Schwarz* (NS-NS) fields. They are also called the *common sector* because they are part of all three ten-dimensional supergravity theories.

The bosonic part of the type IIB supergravity action in the string frame[5] is

$$\begin{aligned}\mathcal{S}_{IIB} &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{|g|}\Big(e^{-2\phi}\left[R+4(\partial\phi)^2-\frac{1}{2\cdot 3!}H^2\right]-\frac{1}{2\cdot 3!}(\tilde{F}_3)^2 \\ &\qquad -\frac{1}{2}(F_1)^2-\frac{1}{4\cdot 5!}(\tilde{F}_5)^2\Big)-\frac{1}{4\kappa^2}\int d^{10}x A_4F_3H \qquad (4.13)\end{aligned}$$

where $\kappa=\sqrt{8\pi G_{10}}\, e^{\langle\phi\rangle}$, G_{10} is the ten-dimensional Newton constant and

$$\tilde{F}_3 = F_3 - HA_0 \qquad (4.14)$$

$$\tilde{F}_5 = F_5 - \frac{1}{2}(B\wedge F_3 - A_2\wedge H) \qquad (4.15)$$

The above action consists of kinetic terms for the fields and a (topological) Chern-Simons term. The complete set of supersymmetry transformation rules for the fields can be found in [5]. Varying the action with respect to all the potentials we have

$$\nabla_l\partial^l A_0 = -\frac{1}{3!}H_{ijk}(F_3-A_0H)^{ijk} \qquad (4.16)$$

$$\nabla^k(F_3-A_0H)_{kij} = \frac{1}{3!}(F_5)_{ijklm}H^{klm} \qquad (4.17)$$

$$\nabla_k[(e^{-2\phi}+(A_0)^2)H-A_0F_3]_{kij} = -\frac{1}{3!}(F_5)_{ijklm}(F_3)^{klm} \qquad (4.18)$$

$$\nabla_l\partial^l\phi = -\frac{1}{4}R+\frac{1}{4\cdot 2\cdot 3!}H_{ijk}H^{ijk}+\partial_k\phi\partial^k\phi \qquad (4.19)$$

$$\begin{aligned}R_{ij} &= -2D_i\partial_j\phi+\frac{3}{2\cdot 3!}H_{(i}{}^{lm}H_{j)lm} \\ &\quad +e^{2\phi}\frac{1}{2}\left((F_1)_i(F_1)_j-\frac{1}{2}g_{ij}(F_1)^k(F_1)_k\right) \\ &\quad +e^{2\phi}\frac{1}{4\cdot 4!}(\tilde{F}_5)_{(i}{}^{lmnp}(\tilde{F}_5)_{j)lmnp} \\ &\quad +e^{2\phi}\frac{1}{2\cdot 3!}\left[3(\tilde{F}_3)_{(i}{}^{lm}(\tilde{F}_3)_{j)lm}-\frac{1}{2}g_{ij}(\tilde{F}_3)^{jlm}(\tilde{F}_3)_{jlm}\right] \qquad (4.20)\end{aligned}$$

$$\partial_{[i}(\tilde{F}_5)_{jklmn]} = \frac{5!}{3!\cdot 3!}(F_3)_{[ijk}(H_3)_{lmn]} \qquad (4.21)$$

[5] The string-frame metric is the (target space) metric appearing in the Polyakov action [95, 96].

In addition, we have

$$(\tilde{F}_5)_{ijklm} = \frac{1}{5!}\varepsilon_{ijklmnopqr}(\tilde{F}_5)^{nopqr} \tag{4.22}$$

This is just the self-duality condition for $\tilde{F}_5$

$$\tilde{F}_5 = *\tilde{F}_5 \tag{4.23}$$

where $*$ denotes the Hodge operator. This condition does not follow from the action (4.13) [84, 97]. However, it has to be imposed at the level of the field equations so that the number of bosonic degrees of freedom matches the fermionic one.

Note we have given the bosonic part of the action and therefore we only obtain the bosonic equations of motion. In general, fermionic solutions are not considered at all due to breaking of Lorentz invariance. Nevertheless, fermionic solutions are easily generated by acting with (non-preserved) supersymmetry transformations on purely bosonic solutions[6]. Solutions with a non-zero gravitino have been constructed in four- and eleven-dimensional supergravity [98, 99, 100, 101].

4.4. VSI spacetimes in IIB supergravity

Our aim is to construct bosonic solutions of IIB supergravity for which the spacetime is VSI; they will generalize higher-dimensional pp-wave solutions. The motivation is clear: it is likely that VSI supergravity solutions have as nice properties as the pp-wave solutions. We consider solutions with non-zero dilaton, Kalb-Ramond field and RR 5-form. The corresponding field equations are

$$\begin{aligned}
R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} &= -2\nabla_\mu\partial_\nu\phi + \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} + e^{2\phi}\frac{1}{4\cdot 4!}F_{\mu\lambda\rho\kappa\sigma}F_\nu{}^{\lambda\rho\kappa\sigma} && (4.24)\\
0 &= F_{ijklm}H^{klm} && (4.25)\\
\nabla_\mu\partial^\mu\phi &= -\frac{1}{4}R + \frac{1}{4\cdot 2\cdot 3!}H^2 + \partial_k\phi\partial^k\phi && (4.26)\\
0 &= \nabla_\lambda H^{\lambda\mu\nu} - 2(\partial_\lambda\phi)H^{\lambda\mu\nu} && (4.27)\\
H &= dB && (4.28)\\
dF &= 0 && (4.29)\\
F &= *F && (4.30)
\end{aligned}$$

[6] Clearly, this can only be done for bosonic solutions which are not maximally supersymmetric.

In the next two sections we make use of boost weight arguments to motivate an ansatz for the fields, and also to sort out the Ricci type of the solutions.

4.4.1. Fields ansatz

In Chapter 3, section 3.4 we saw that the VSI requirement implies that all curvature tensors must be of negative boost order. It is therefore reasonable that we similarly require that the quadratic terms in H and F in equation (4.24) are also of negative boost order. Since H and F are antisymmetric we must have (see equation (3.13))

$$F = (F)_{-1} + (F)_0 + (F)_1, \quad H = (H)_{-1} + (H)_0 + (H)_1 \tag{4.31}$$

where $(\ )_b$ means projection onto the boost weight b components. As an example consider $(F)_1$ which can be written

$$(F_{\mu\lambda\rho\kappa\sigma})_1 = n_{[\mu}\xi_{\lambda\rho\kappa\sigma]} \tag{4.32}$$

where n is the basis vector (3.33) and ξ is a four-form satisfying

$$\xi_{\lambda\rho\kappa\sigma}n^{\lambda} = 0 \tag{4.33}$$

so that $(F)_1$ is fully antisymmetric. We then have the following term in equation (4.24) (and an analogous term for H)

$$\left(F_{\mu\lambda\rho\kappa\sigma}F_{\nu}{}^{\lambda\rho\kappa\sigma}\right)_2 \sim n_{\mu}n_{\nu}\xi_{\lambda\rho\kappa\sigma}\xi^{\lambda\rho\kappa\sigma}$$

This term has boost weight 2 [7]. Requiring this term to vanish implies $\xi = 0$ since the factor $\xi_{\lambda\rho\kappa\sigma}\xi^{\lambda\rho\kappa\sigma}$ is a sum of squares, and hence, $(F)_1 = 0$. A similar calculation can be done for the boost weight 0 components F_0. We conclude that if the quadratic terms of equation (4.24) only have negative boost weight terms, then F and H only possess negative boost weight terms[8]

$$H = (H)_{-1}, \quad F = (F)_{-1} \tag{4.34}$$

This means that H and F can be written as

$$H_{\mu\nu\rho} = \ell_{[\mu}\tilde{B}_{\nu\rho]}, \ F_{\mu\lambda\rho\kappa\sigma} = \ell_{[\mu}\varphi_{\lambda\rho\kappa\sigma]} \tag{4.35}$$

[7] For a general tensor product $(T \otimes S)_b = \sum_{b=b'+b''}(T)_{b'} \otimes (S)_{b''}$.

[8] One can, in principle, imagine a very special (and unnatural) situation where the derivatives of ϕ exactly cancel the non-negative boost weight terms of $H^2 + F^2$; however, we shall not consider this possibility further here.

where ℓ is the null vector field (3.32) and the forms $\tilde{B}$, φ are such that

$$\ell^\mu \tilde{B}_{\mu\nu} = \ell^\mu \varphi_{\mu\rho\kappa\sigma} = 0 \tag{4.36}$$

This means that $\tilde{B}$ and φ only have transverse components.

Last, we look at the dilaton kinetic term in (4.24). In principle we could have $\phi = \phi(u, v, x^i)$ so that

$$\mathrm{d}\phi = \partial_\mu\phi\,\mathrm{d}x^\mu = \overbrace{\phi_{,u}\,\mathrm{d}u}^{\text{b.w. }-1} + \overbrace{\phi_{,v}\,\mathrm{d}v}^{\text{b.o. }1} + \overbrace{\phi_{,i}\,\mathrm{d}x^i}^{\text{b.w. }0} \tag{4.37}$$

Using (3.33), the second term on the right-hand side can be written

$$\phi_{,v}\,\mathrm{d}v = \phi_{,v}\,(\overbrace{n}^{\text{b.w. }1} - \overbrace{H\,\mathrm{d}u}^{\text{b.w. }-1} - \overbrace{W_i\,\mathrm{d}x^i}^{\text{b.w. }0}) \tag{4.38}$$

If we require $\partial_\mu\phi$ to have negative boost weight as well, then

$$\phi_{,v} = \phi_{,i} = 0 \tag{4.39}$$

and therefore

$$\phi = \phi(u) \tag{4.40}$$

4.4.2. Ricci type of the solutions

We learned in the previous chapter that VSI spacetimes are of Ricci type N or III. In the first case, the Ricci tensor has boost order -2 and the only non-zero component is R_{uu}. In the second case, the Ricci tensor has boost order -1 and the non-zero components are R_{uu} (b.w. -2) and R_{ui} (b.w. -1).

Consider the form fields (4.34). The corresponding quadratic terms in (4.24) clearly have boost weight -2. Therefore, they can only be sources to VSI solutions for which the Ricci tensor is of boost order -2, i.e., Ricci type N solutions. Ricci type III solutions exist if the dilaton term in (4.24) has a non-zero boost weight -1 component

$$(\nabla_\mu\partial_\nu\phi)_{-1} \neq 0 \tag{4.41}$$

We write below the dilaton term explicitly

$$\nabla_\mu\partial_\nu\phi = \partial_\mu\partial_\nu\phi - \Gamma^u_{\mu\nu}\partial_u\phi \tag{4.42}$$

Clearly, the first term on the right-hand side only gives a (uu) contribution $\partial_u\partial_u\phi$ (of boost weight -2). The second term, however, gives not only a (uu) contribution

$$-\Gamma^u_{uu}\partial_u\phi = H_{,v}\,\partial_u\phi \tag{4.43}$$

but also a $(u1)$ contribution

$$\Gamma^u_{u1}\partial_u\phi = \frac{1}{2}W_{1,v}\ \partial_u\phi \tag{4.44}$$

The (uu) contribution has boost weight -2 and the $(u1)$ contribution boost weight -1 [9]. While (4.43) is non-zero for both $\varepsilon = 0, 1$ VSI spaces[10], (4.44) is only present in the $\varepsilon = 1$ case. We conclude that there are no Ricci type III solutions for $\varepsilon = 0$. This means there are no Ricci III CCNV (see section 3.4.1) supergravity solutions either[11]. This is perhaps unfortunate, as such solutions would have been good candidates to preserve supersymmetry (see section 4.5 later on).

4.4.3. Ricci type N solutions

We first construct solutions with Ricci type N VSI spacetimes (see Chapter 3, section 3.4.2). We postulate the following ansatz motivated by the argument in 4.4.1

$$g_{\mu\nu} = g^{\rm VSI}_{\mu\nu},\ \phi = \left\{ \begin{array}{cc} \phi(u) & (\varepsilon = 0) \\ \phi_0 & (\varepsilon = 1) \end{array} \right\},\ H_{\mu\nu\rho} = \frac{1}{4}\ell_{[\mu}\tilde{B}_{\nu\rho]},\ F_{\mu\lambda\rho\kappa\sigma} = \ell_{[\mu}\varphi_{\lambda\rho\kappa\sigma]} \tag{4.45}$$

where ϕ_0 is a constant, $\tilde{B}$ and φ are a two- and four-forms with no dependence on v. Equations (4.25), (4.26) are automatically satisfied. From equations (4.28), (4.29), $\tilde{B}_{\nu\rho} = \tilde{B}_{\nu,\rho} - \tilde{B}_{\rho,\nu}$ [12] and φ has to be a closed form. On the other hand, it is clear that the second term on the right-hand side of equation (4.27) vanishes. The covariant derivative in the first term reduces to

[9] It can be shown that the boost weights of Γ^u_{uu}, Γ^u_{u1} are -1 and 0, respectively.

[10] In this chapter we rename ε the parameter ϵ throughout Chapter 3, as the latter is widely used to denote the supersymmetry spinor 4.2.

[11] This is in the context of type IIB supergravity and so long as the situation described in footnote 7 does not apply.

[12] The relation between B in (4.28) and $\tilde{B}$ is $B_{\nu\rho} = \ell_{[\nu}\tilde{B}_{\rho]}$.

an ordinary one for both $\varepsilon = 0, 1$ [13]. Equations (4.24)-(4.30) become

$$x^1 \triangle \left(\frac{H^{(0)}}{x^1} \right) + \left(\frac{W^{(0)m} W^{(0)}_m}{x^1} \right)_{,1} - 2H^{(1)}{}_{,m} W^{(0)m} - H^{(1)} W^{(0)m}{}_{,m} - \frac{1}{4} W_{mn} W^{mn} - W^{(0)m}{}_{,mu} = -\frac{1}{4 \cdot 4} \tilde{B}_{ij} \tilde{B}^{ij} - 3!\, e^{2\phi_0} \varphi^2 \quad (4.46)$$

$$\triangle H^{(0)} - \frac{1}{4} W_{mn} W^{mn} - 2H^{(1)}{}_{,m} W^{(0)m} - H^{(1)} W^{(0)m}{}_{,m} - W^{(0)m}{}_{,mu} = 2\phi'' + 2H^{(1)} \phi' - \frac{1}{4 \cdot 4} \tilde{B}_{ij} \tilde{B}^{ij} - 3!\, e^{2\phi} \varphi^2 \quad (4.47)$$

$$\partial_i \tilde{B}^{ij} = 0 \quad (4.48)$$

$$\varphi = *_8 \varphi \quad (4.49)$$

In the above equations $m, n = 2, \ldots, 8$ and $i, j = 1, \ldots, 8$. Prime denotes derivative with respect to u and $*_8$ is the Hodge operator in the eight-dimensional transverse space. Note that equations (4.46), (4.47) are analogue to (3.87), (3.92).

The reason why the dilaton field has to be constant in the $\varepsilon = 1$ case is the following. In the $\varepsilon = 1$ case, the dilaton contribution (4.43) has v-dependence due to the v-quadratic term in (3.85). At the same time, nor the Ricci tensor not the terms quadratic in the form fields in (4.46) have any dependence on v. The only way to get rid of the v-dependence in (4.43) is by considering a constant dilaton. Not only that, a non-constant dilaton would also give a contribution (4.44) to the Einstein equations, which means the solution is not of Ricci type N but III.

The solutions above are of Weyl type III. They contain previously known solutions such as the string gyratons [102] and pp-wave supergravity solutions (for example, [91, 92]). The latter arise in the Weyl type N limit of the $\varepsilon = 0$ solutions (see [51]).

4.4.4. Ricci type III solutions

Motivated by the arguments in sections 4.4.1 and 4.4.2 we construct a solution with non-constant dilaton $\phi = \phi(u)$ in the $\varepsilon = 1$ case. Equations (4.24)

[13] This can be shown with a calculation similar to those in Chapter 2, sections 2.4.1, 2.4.2 for the Yang-Mills equation. The only difference being the Christoffel symbols $\Gamma^u_{u1} = -\Gamma^v_{v1}$ as we use real notation here.

read

$$x^1 \triangle \left(\frac{H^{(0)}}{x^1}\right) + \left(\frac{W^{(0)m} W^{(0)}_m}{x^1}\right)_{,1} - 2H^{(1)}{}_{,m} W^{(0)m} - H^{(1)} W^{(0)m}{}_{,m}$$

$$-\frac{1}{4} W_{mn} W^{mn} - W^{(0)m}{}_{,mu} + \frac{2v}{(x^1)^2}\phi' = 2\phi'' + 2\left(H^{(1)} + \frac{v}{(x^1)^2}\right)\phi' \quad (4.50)$$

$$H^{(1)}{}_{,1} = \frac{1}{x^1} W^{(0)m}{}_{,m} - \frac{1}{2} W^{(0)m}{}_{,m1} + \frac{2}{x^1}\phi' \quad (4.51)$$

$$2H^{(1)}{}_{,n} = \Delta W^{(0)}_n - W^{(0)m}{}_{,mn} \quad (4.52)$$

Note that the v-dependent terms in (4.50) cancel each other. On the other hand, $H^{(1)}$ is determined by (4.51), (4.52). The function $H^{(0)}$ can then be found from (4.50). The complete metric function H is

$$H = H^{(0)}(u, x^i) + \frac{1}{2}\left(\tilde{F} - W^{(0)m}{}_{,m}\right) v + \frac{v^2}{2(x^1)^2} \quad (4.53)$$

where $\tilde{F} = \tilde{F}(u, x^i)$ is a function satisfying

$$\tilde{F}_{,1} = \frac{2}{x^1}(W^{(0)m}{}_{,m} + 2\phi'), \quad \tilde{F}_{,n} = \Delta W^{(0)}_n \quad (4.54)$$

Vanishing $\mathbf{W^{(0)}_m}$ functions

In this case we can write down the metric explicitly. Equations (4.51), (4.52) read

$$H^{(1)}{}_{,1} = \tilde{F}_{,1} = \frac{2}{x^1}\phi', \quad H^{(1)}{}_{,n} = 0 \quad (4.55)$$

Therefore we have

$$H^{(1)} = 2\phi' \ln(x^1) + f(u) \quad (4.56)$$

where f is an arbitrary function of u. As we have pointed out before, such a function in $H^{(1)}$ can be transformed away (see Appendix F). Now we can insert the above expression in equation (4.50)

$$x^1 \triangle \left(\frac{H^{(0)}}{x^1}\right) = 2\left(\phi'' + 2(\phi')^2 \ln(x^1)\right) \quad (4.57)$$

A possible solution for $H^{(0)}$ is

$$H^{(0)} = \frac{1}{7}\phi'' x_m x^m + 2(x^1)^2(\phi')^2 \left([\ln(x^1)]^2 - 2\ln(x^1) + 2\right) \quad (4.58)$$

And so the complete metric function H in this simple case is

$$\begin{aligned} H &= \frac{1}{7}\phi'' x_m x^m + 2(x^1)^2(\phi')^2 \left([\ln(x^1)]^2 - 2\ln(x^1) + 2\right) \\ &\quad + 2\phi' \ln(x^1) v + \frac{v^2}{2(x^1)^2} \end{aligned} \tag{4.59}$$

The solution presented here is, to our knowledge, the first supergravity solution of Ricci type III. The dilaton dependence on u is crucial to construct the solution, and it reduces to Ricci type N when the dilaton is constant (or absent). VSI supergravity solutions of Ricci type III with form fields only do not exist. However, the solution in this section can be generalized in a straightforward way to include the form fields in (4.45).

The Ricci type N, Weyl type III solutions in the section 4.4.1 can be reduced to Weyl type N. On the contrary, the Ricci type III solution presented here can only have Weyl type III [14].

4.4.5. Solutions with all Ramond-Ramond fields

The above solutions can be generalized to include non-zero F_1, F_3 RR fields. It is well-known that $SL(2,\mathbb{R})$ is the classical S-duality symmetry group for IIB supergravity (see, for example, [5]). Such a transformation can be parametrized by [103]

$$S = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \tag{4.60}$$

with $ps - qr = 1$. The fields transform according to

$$\begin{pmatrix} A_2' \\ B' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} A_2 \\ B \end{pmatrix}, \quad \tau' = \frac{p\tau + q}{r\tau + s} \tag{4.61}$$

where A_2, B are the RR 2-potential and the Kalb-Ramond field; $\tau = A_0 + ie^{-\phi}$, with A_0 the RR scalar and ϕ the dilaton. The metric and RR 5-form remain invariant[15]. The VSI solutions presented have $A_0 = A_2 = 0$. Under a transformation (4.60)

$$A_2' = qB, \quad B' = sB \tag{4.62}$$

[14] Requiring the solution to have vanishing boost weight -1 Weyl components reduces it to the Ricci and Weyl type N solution.

[15] The S-duality symmetry becomes manifest when the metric is in the Einstein frame. The Einstein- and string-frame metric are related by $g_E = e^{-\phi/2} g_S$ [104]. Such a rescaling does not affect the character of the VSI solutions presented.

In this way one can generate a non-zero F'_3 which is proportional to H; the Kalb-Ramond field gets rescaled. The dilaton and RR scalar can be read from

$$sA'_0 - e^{-(\phi+\phi')}r = q \quad (4.63)$$

$$rA'_0 + se^{(\phi-\phi')}s = p \quad (4.64)$$

For solutions with $\phi = \phi(u)$ one obtains $(F'_1)_u = \partial_u A'_0$. For solutions with a constant dilaton the RR 1-form remains zero.

4.4.6. String corrections

In section 4.2 we argued that higher-dimensional vacuum VSI spacetimes are exact string solutions. In four dimensions VSI spacetimes are known to be exact string solutions even in the presence of dilaton and Kalb-Ramond field [105]. Using the same line of reasoning, it can be argued that the higher-dimensional VSI spacetimes supported by dilaton and Kalb-Ramond field are exact string solutions, and thus they might be relevant in string theory.

We sketch the argument here. In the presence of fields other than the metric there are extra terms in equation (4.8). They are second-rank tensors constructed from the fields, the Riemann tensor and their powers and derivatives. In our case, both $\nabla\phi$ and H are proportional to $\ell = \mathrm{d}u$. This means that terms containing contractions of at least one Riemann tensor and one or more $\nabla\phi$ or H vanish, since $\ell^\mu R_{\mu\rho\kappa\sigma} = 0$. Similarly, it can be argued that terms constructed from two or more fields and their derivatives vanish. At the moment it is not clear whether this is also the case for VSI solutions with non-zero RR fluxes [106].

We conclude section 4.4 by summarizing the found solutions in the table below.

Ricci type	$\varepsilon = 0$	$\varepsilon = 1$
III	none	$\phi = \phi(u)$, H, F
N	$\phi = \phi(u)$, H, F	ϕ constant, H, F

Table 4.1: VSI IIB supergravity solutions. H and F are as in (4.45).

4.5. Supersymmetry

Given a spinor ϵ on a Lorentzian manifold, the vector constructed from its Dirac current

$$k^a = \bar{\epsilon}\gamma^a\epsilon \tag{4.65}$$

is null or timelike. Moreover, if ϵ is a Killing spinor then k^a is a Killing vector[16]. This result has been proven for a number of supergravity theories (for example, $D = 11$ [107], type IIB [108]), and it is generally believed to hold in all theories of supergravity (although the details may vary in each particular theory depending on the specific field equations). Therefore, a necessary (but not sufficient) condition for a particular supergravity solution to preserve some supersymmetry is that the involved spacetime possesses a null or timelike Killing vector.

The existence of Killing vectors in VSI spacetimes in an arbitrary number of dimensions has been studied. It is known that there can exist no null or timelike Killing vector unless $\varepsilon = 0$ and H is independent of v (see Appendix in [62]). We conclude that there are no supersymmetric solutions with $H_{,v} \neq 0$ in any supergravity theory. We therefore only study the supersymmetry properties of VSI IIB supergravity solutions with a covariantly constant null vector. These are of Ricci type N and Weyl type III(a) or N [62]. We will focus on solutions of Weyl type III(a), as the Weyl type N ones have been discussed extensively in the literature. We will consider two different cases, namely, vacuum solutions and NS-NS solutions.

4.5.1. Vacuum case

The only Killing equation for pure gravitational solutions is given by the supersymmetry transformation of the gravitino

$$\delta_{\text{susy}}\Psi_\mu = (\partial_\mu - \frac{1}{4}w_{\mu\ ab}\ \gamma^{ab})\epsilon = 0 \tag{4.66}$$

where $w_{\mu\ ab}$ is the so-called *spin connection*. The spin connection is related to the connection coefficients with respect to the frame (see section 2.1)

$$w_{\mu\ a}^{\ \ b} = \Gamma^b_{ca} e^c_\mu \tag{4.67}$$

And

$$\gamma^{ab} = \frac{1}{2}[\gamma^a,\ \gamma^b] \tag{4.68}$$

[16] The details on how to construct the Dirac current (4.65) depend on the character of the Killing spinor, this is, on the specific theory of supergravity.

where the ten-dimensional gamma matrices satisfy

$$\{\gamma^a,\ \gamma^b\} = 2\eta^{ab} \tag{4.69}$$

In our conventions, η^{ab} is given by (3.10). In case of anticommuting gamma matrices (4.68) is just

$$\gamma^{ab} = \gamma^a\gamma^b \tag{4.70}$$

Greek indices are curved indices $u, v, 1, ..., 8$ and Latin indices are tangent space indices $0, ..., 9$. The supersymmetry parameter ϵ is a 32-component $SO(1,9)$ chiral spinor (so in turn 16 non-zero components). We make use of the coframe (3.32)-(3.34). The corresponding frame is

$$\begin{aligned} e_{i+1} &= \partial_i - W_i\,\partial_v \\ e_1 &= \partial_u - H\,\partial_v \\ e_0 &= \partial_v \end{aligned}$$

The components of the spin connection for Weyl III(a) VSI spacetimes are[17]

$$w_{u\ (i+1)(j+1)} = \frac{1}{2}W_{ij}, \qquad w_{u\ (i+1)1} = -H^{(0)}{}_{,i} + W^{(0)}_{i,u} \tag{4.71}$$

$$w_{i\ (j+1)1} = \frac{1}{2}W_{ji} \tag{4.72}$$

Recall from Chapter 3 that $W_{ij} = W^{(0)}_{i,j} - W^{(0)}_{j,i}$. It has been proved[18] that the only case where supersymmetry arises is when [110]

$$\partial_k W_{ij} = 0 \tag{4.73}$$

This condition is equivalent to the functions W_i being linear in the transverse coordinates. In that case the spacetime reduces to Weyl type N [51].

4.5.2. NS-NS case

We discuss next supersymmetry on NS-NS solutions. The fermion supersymmetry transformations are given by

$$\delta_{\text{susy}}\psi_\mu = (\partial_\mu - \frac{1}{4}\Omega_{\mu\ ab}\,\gamma^{ab})\epsilon \tag{4.74}$$

$$\delta_{\text{susy}}\lambda = (\gamma^\mu\partial_\mu\phi - \frac{1}{6}\gamma^{\mu\nu\rho}H_{\mu\nu\rho})\epsilon \tag{4.75}$$

[17] The spin connection has been calculated with the CARTAN package [109] for Mathematica.

[18] The analysis in [110] concerns five dimensions. However, the result holds in higher dimensions so long as the transverse space is flat [106].

where Ω is the torsionful spin connection

$$\Omega_{\mu\ ab} = w_{\mu\ ab} + H_{\mu\ ab} \tag{4.76}$$

and $\gamma^{\mu\nu\rho}$ is the antisymmetric gamma symbol

$$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]} \tag{4.77}$$

The curved gamma matrices are given by

$$\gamma^{\mu} = e^{\mu}_{a}\gamma^{a}, \quad \{\gamma^{\mu},\ \gamma^{\nu}\} = 2g^{\mu\nu} \tag{4.78}$$

The non-zero components of the torsionful spin connection are

$$\Omega_{u\ (i+1)(j+1)} = \frac{1}{2}(W_{ij} + \tilde{B}_{ij}), \qquad \Omega_{u\ (i+1)1} = -H^{(0)},_{i} + W^{(0)}_{i,u} \tag{4.79}$$

$$\Omega_{i\ (j+1)1} = \frac{1}{2}W_{ji} \tag{4.80}$$

We consider solutions with at most one half of the supersymmetries broken [19]. This is achieved by requiring $\gamma^{u}\epsilon = 0$ [20]. The dilatino variation vanishes automatically since

$$(\gamma^{u})^{2} = 0 \tag{4.81}$$

The gravitino Killing equation reduces to

$$\left(\partial_{u} - \frac{1}{4}(W_{ij} + \tilde{B}_{ij})\gamma^{(i+1)(j+1)}\right)\epsilon = 0, \quad \partial_{v}\epsilon = \partial_{i}\epsilon = 0 \tag{4.82}$$

We see that for consistency

$$W_{ij} = \tilde{B}_{ji} + f_{ij}(u) \tag{4.83}$$

where f_{ij} are arbitrary functions of u. However, for our purposes we can take $f_{ij} = 0$ since such functions are related to Weyl type N solutions. We therefore obtain eight (complex) constant Killing spinors and half of the supersymmetry is preserved.

From equations (4.83) and (4.48), the metric functions W_i satisfy

$$\partial_{i}\ (W_{i,j} - W_{j,i}) = 0 \tag{4.84}$$

[19] In some cases, different fractions of supersymmetry can be preserved [111].

[20] A $SO(1,9)$ chiral spinor can be written as $\epsilon = \frac{1}{2}\gamma^{u}\gamma^{v}\epsilon + \frac{1}{2}\gamma^{v}\gamma^{u}\epsilon = \epsilon_{+} + \epsilon_{-}$, where ϵ_{+} (ϵ_{-}) is a 16-component $SO(8)$ positive (negative) chiral spinor (so in turn 8 non-zero components). Requiring $\gamma^{u}\epsilon = 0$ amounts to $\epsilon = \epsilon_{+}$.

This is the necessary condition for the spacetime to be of Weyl type III(a), equation (3.114). Recall that in addition it is required that (equation (3.118))

$$\partial_k \left(W_{i,j} - W_{j,i}\right) \neq 0 \tag{4.85}$$

The supersymmetry analysis is similar to that of [112] and the resulting solutions are the IIB analogues to the supersymmetric string waves therein. However, we have shown that the solutions can have a more general algebraic type than pp-waves. We present a few examples of such solutions below.

Consider the VSI metric[21]

$$\begin{aligned} W_1 &= 0 && (4.86)\\ W_m &= f_{mn}(u)x^n x^1 && (4.87)\\ H &= H^{(0)}(u, x^i) && (4.88) \end{aligned}$$

where the antisymmetric f_{mn} are arbitrary functions of u and $m, n = 2, ..., 8$. This spacetime satisfies (3.112), (3.113) and it is therefore of Weyl type III(a). Supported by the dilaton and Kalb-Ramond field

$$\phi = \phi(u) \quad \tilde{B}_{1m} = f_{mn}(u)x^n, \ \tilde{B}_{mn} = 2f_{nm}(u)x^1, \tag{4.89}$$

it is a supersymmetric solution of the type discussed above. The function $H^{(0)}$ can be determined from equation (3.92).

Another example involves the gyraton metric (3.120) discussed in Chapter 3, section 3.4.2. Recall that

$$\begin{aligned} W_i &= -\frac{\tilde{p}_i(u)x^{i+1}}{Q^4} && (4.90)\\ W_{i+1} &= \frac{\tilde{p}_i(u)x^i}{Q^4} && (4.91)\\ H &= H^{(0)}(u, x^i), && (4.92) \end{aligned}$$

where i only takes odd values 1, 3, 5, 7; p_j are arbitrary functions of u with $j = (i+1)/2$ and $Q = (x^1)^2 + \ldots + (x^8)^2$. This metric was already shown to be of Weyl type III(a). In [102] this spacetime was considered in the context of supergravity, together with a constant dilaton and Kalb-Ramond field of the form presented in the ansatz. As we have seen such solution can be generalized

[21] This is the ten-dimensional generalization of the metric (3.122)-(3.125).

to include a dilaton depending on u. The (one-half) supersymmetric gyraton will be the one with

$$\phi = \phi(u), \quad \tilde{B}_{jk} = W_{kj} \tag{4.93}$$

Again, $H^{(0)}$ can be determined from equation (3.92). This solution belongs to the class of *saturated* (meaning the angular momentum is equal to the energy density) string gyratons in [102]. Another supersymmetric (AdS) gyraton solution is given in [113].

5 Higher-dimensional CSI spacetimes

In this chapter we study higher-dimensional Kundt CSI spacetimes [33, 114]. There are enough examples in the literature, specially in the supergravity/strings context. However, they are not recognized as members of this class of metrics and their mathematical properties are not known. First, we will give the general structure of Kundt CSI metrics. Then we consider a best-known example, Anti-de Sitter space, and its possible generalizations. Along the way we review Kundt CSI metrics in the literature and comment on their supersymmetry properties, where possible. Some new examples of Kundt CSI spacetimes which are not discussed in this chapter can be found in [114].

5.1. Kundt CSI spacetimes

Any higher-dimensional Kundt CSI spacetime can be written in the canonical form

$$ds^2 = 2du\ [dv + H(u,v,x^k)\ du + W_i(u,v,x^k)\ dx^i] + g^{\perp}_{ij}(x^k)\ dx^i dx^j \qquad (5.1)$$

where $i,j = 1,\ldots,n-2$ [1]; $\perp$ indicates thoroughout the chapter a quantity in the transverse space. The transverse metric can without loss of generality be assumed not to depend on the light-cone coordinate u. Note that the metric functions W_i in (5.1) cannot be transformed away as there is no u-dependence left in the transverse metric (see section 3.3). Moreover, $g^{\perp}_{ij}$ is locally homogeneous. The coordinate transformations preserving the above form of the metric are

[1] In this chapter we denote the number of dimensions by n rather than N.

1. $(v', u', x'^i) = (v, u, f^i(x^k))$ and ${J^i}_j \equiv \frac{\partial f^i}{\partial x^j}$
$$H' = H, \quad W'_i = W_j \left(J^{-1}\right)^j{}_i, \quad {g^{\perp}_{ij}}' = g^{\perp}_{kl} \left(J^{-1}\right)^k{}_i \left(J^{-1}\right)^l{}_j$$

2. $(v', u', x'^i) = (v + h(u, x^k), u, x^i)$
$$H' = H - h_{,u}, \quad W'_i = W_i - h_{,i}, \quad {g^{\perp}_{ij}}' = g^{\perp}_{ij}$$

3. $(v', u', x'^i) = (v/k_{,u}(u), k(u), x^i)$
$$H' = \frac{1}{k_{,u}^2}\left(H + v\frac{k_{,uu}}{k_{,u}}\right), \quad W'_i = \frac{1}{k_{,u}} W_i, \quad {g^{\perp}_{ij}}' = g^{\perp}_{ij}$$

Recall that Kundt CSI spaces are conjectured to be (at most) of Riemann type II (see section 3.3.2). Consequently, their Ricci and Weyl type is II or lower. The Riemann tensor of (5.1) is of boost order 1. Therefore we have to impose the vanishing of the boost weight 1 components of the Riemann tensor; the boost weight 0 components must be constant. The following form of the metric functions satisfy both requirements [33]

$$W_i(v, u, x^k) = vW_i^{(1)}(u, x^k) + W_i^{(0)}(u, x^k) \tag{5.2}$$

$$H(v, u, x^k) = \frac{v^2}{8}\left[4\sigma + (W_i^{(1)})(W^{(1)i})\right] + vH^{(1)}(u, x^k) + H^{(0)}(u, x^k) \tag{5.3}$$

where $W^{(1)i} = (g^{\perp})^{ij} W_j^{(1)}$. Note that, unlike the VSI case, all of the functions W_i can depend on v.

In general, the Ricci tensor has non-zero components

$$R_{uu}, \;\; R_{ui}, \;\; R_{uv}, \;\; R_{ij} = R^{\perp}_{ij} \tag{5.4}$$

and therefore is of Ricci type II. Note that spacetimes of the form (5.1) with flat (or Ricci flat[2], $R_{ij} = 0$) transverse metric are *not* automatically VSI since the vanishing of the 1 and 0 boost weight Riemann components still has to be imposed. This can be seen at the Ricci tensor level in a non-zero R_{uv} component. If both $R_{uv} = R_{ij} = 0$ we obtain VSI spaces of Ricci type III. Pure ($\equiv$ not VSI) CSI spacetimes are of Ricci type II and not lower.

[2] The reason being that if a Riemannian metric is Ricci flat and homogeneous, then it is flat [115, 116].

5.2. CCNV Kundt CSI spacetimes

An interesting case occurs for metric functions H, W_i in (5.1) with no dependence on v [117]. Then we have a subclass of the higher-dimensional pp-waves (3.22), namely, the subclass with (locally) homogeneous transverse space. As such, it is CCNV. We have the following result:

Higher-dimensional CSI pp-waves are (at most) of Ricci and Weyl type II

In this relatively simple case we can give the expressions for the Ricci tensor and Ricci scalar

$$R_{uu} = \triangle^{\perp} H^{(0)} - \frac{1}{4} W_{ij} W^{ij} - g^{\perp\, ij} \nabla_i W^{(0)}_{j,u} \tag{5.5}$$

$$R_{ui} = \frac{1}{2} \nabla^j W_{ji} \tag{5.6}$$

$$R_{ij} = R^{\perp}_{ij} \tag{5.7}$$

$$R = R^{\perp} \tag{5.8}$$

Here indices are raised by the transverse metric. The Laplacian on a curved space is the so-called Laplace-Beltrami operator. Its action on a scalar function f is defined by

$$\triangle^{\rm LB} f = g^{ij} \nabla_i \partial_j f \tag{5.9}$$

On the other hand[3]

$$\frac{1}{2} W_{ji} = W^{(0)}_{i,j} - W^{(0)}_{j,i} \tag{5.10}$$

These spacetimes *do* become VSI when the transverse metric is (Ricci) flat, namely, CCNV VSI of Ricci type III. In this case equations (5.5), (5.6) reduce to (3.73)-(3.75).

We now give an example in this class. Consider the following five-dimensional metric [118]

$$ds^2 = 2du \left[dv + H^{(0)}(u, x^k)\, du \right] + d\Omega_3^2 \tag{5.11}$$

where $d\Omega_3^2$ is the standard round metric on S^3. The unit sphere[4] S^n is an example of an Einstein space

$$R_{\mu\nu} = k g_{\mu\nu} \tag{5.12}$$

[3] Note the slightly different definition with respect to (3.87).

[4] We can multiply $d\Omega_n^2$ by l^2; then $r^2 = 1/l^2$ is the radius of S^n and $k = (n-1)/r^2$.

The Ricci scalar of an Einstein space

$$R = g^{\mu\nu} R_{\mu\nu} = k g^{\mu\nu} g_{\mu\nu} = nk \tag{5.13}$$

An Einstein space is a vacuum solution of the n-dimensional Einstein equations with cosmological constant

$$\Lambda = k \frac{n-2}{2} \tag{5.14}$$

In the case of S^n, $k = n-1$. More on the n-sphere can for example be found in [5].

Note (5.11) is already in the Kundt form (5.1), with $W_i^{(1)} = W_i^{(0)} = \sigma = H^{(1)} = 0$. The Ricci tensor has components

$$R_{uu} = \triangle^{(S^3)} H^{(0)} \tag{5.15}$$
$$R_{ij} = 2 g_{ij}(S^3) \tag{5.16}$$

where $\triangle^{(S^3)}$ is the Laplace-Beltrami operator on S^3.
The metric (5.11) together with a constant dilaton and appropiate antisymmetric field is an exact solution to bosonic string theory[5].

5.3. Anti-de Sitter space

The metric of Anti-de Sitter space, AdS_n, is[6]

$$ds^2 = \frac{l^2}{z^2}\left[2\mathrm{d}u\mathrm{d}\tilde{v} + (\mathrm{d}x^i)^2 + \mathrm{d}z^2\right] \tag{5.17}$$

l is the curvature radius of AdS, $i = 1, \ldots, n-3$. AdS_n is an Einstein space with

$$k = -\frac{n-1}{l^2}, \quad R = -\frac{n(n-1)}{l^2}, \quad \Lambda = -\frac{(n-1)(n-2)}{2l^2} \tag{5.18}$$

An important property of Anti-de Sitter space is that is *maximally symmetric* [119], i.e., it admits the maximum number of linearly independent Killing vectors: $n(n+1)/2$. It is interesting to point out that all maximally symmetric spaces are Einstein and have vanishing Weyl tensor. They are Minkowski, de Sitter and Anti-de Sitter space, and their Riemannian versions Euclidean,

[5] However, it is not a vacuum solution of five-dimensional gravity.
[6] Here (and in the rest of the chapter) we use $\tilde{v}$ and reserve v for the Kundt form of the metric.

n-sphere and hyperbolic space (see, for example, [120]).

Another important property is that all curvature invariants are constant. In fact, Anti-de Sitter is one of the simplest examples of Kundt CSI spacetimes. We can write (5.17) in the Kundt form by making a coordinate transformation

$$(v, u, x^i) = (\tilde{v}\frac{l^2}{z^2}, u, x^i) \tag{5.19}$$

In this way we have

$$\mathrm{d}s^2 = 2\mathrm{d}u(\mathrm{d}v + \frac{2v}{z}\mathrm{d}z) + \frac{l^2}{z^2}\left[(\mathrm{d}x^i)^2 + \mathrm{d}z^2\right] \tag{5.20}$$

which is of the form (5.2, 5.3) with $H^{(1)} = H^{(0)} = W_i^{(0)} = 0$, $W_z^{(1)} = 2/z$. In this case the coefficient of v^2 vanishes because $\sigma = -1$ and $W_z^{(1)}W^{(1)z} = 4$. The transverse space is the (homogeneous) hyperbolic space $\mathbb{H}^{n-2}$.
From (5.17), it is clear that AdS_n is conformally flat and therefore the Weyl tensor vanishes. The Weyl type is O.

5.4. Generalizations of Anti-de Sitter space

Anti-de Sitter space can be generalized in a number of ways while staying in the class of Kundt CSI metrics. We will discuss two (complementary) approaches.

1. Warping a n-dimensional (Ricci type N) VSI spacetime (see Chapter 3) with warp factor $(l/z)^2$:

$$ds^2 = \frac{l^2}{z^2}\left[2\mathrm{d}u\mathrm{d}\tilde{v} + 2H(\tilde{v}, u, x^k)\mathrm{d}u^2 + 2W_i(\tilde{v}, u, x^k)\mathrm{d}u\mathrm{d}x^i + (\mathrm{d}x^i)^2\right] \tag{5.21}$$

$i = 1, \dots, n-2$. Note that (5.21) reduces to AdS_n when the functions H, W_i are zero. If the VSI metric is Ricci flat (5.21) is an Einstein space with $k = -(n-1)/l^2$ as pure AdS_n; the functions H, W_i will satisfy new differential equations. By construction, all of these metrics have the same curvature invariants as AdS_n. On the other hand, (5.21) is obviously conformal to the VSI space and so the Weyl type is III or lower. The Weyl type of (5.21) is however not one-to-one to the Weyl type of the VSI metric. Note that the Weyl tensor is the same one but the metric functions satisfy different equations of motion. This can cause small differences in the Weyl type as well (we will give an example later).

In general, the symmetry properties of AdS_n are lost, as the metric functions of the VSI space can depend on all coordinates. An interesting subclass arises when the VSI metric is CCNV; in that case (5.21) has the null Killing vector

$$k^v = 1, \;\; k_u = \frac{l^2}{z^2} \tag{5.22}$$

This property makes this subclass attractive from a supersymmetry point of view. However, the metric constructed in this way is not CCNV itself

$$\partial_\mu k_u - \Gamma^u_{\mu u} k_u \overset{\Gamma^u_{zu}=-1/z}{=} -2\frac{l^2}{z^3} + \frac{l^2}{z^3} \neq 0 \tag{5.23}$$

This can also be seen from the Kundt form of the metric. By using (5.19), such metrics are easily written in Kundt form (5.2, 5.3)

$$\mathrm{d}s^2 = 2\mathrm{d}u\left(\mathrm{d}v + \frac{2l^2}{z^2}H(u,x^k)\mathrm{d}u + \frac{2l^2}{z^2}W_i(u,x^k)\mathrm{d}x^i + \frac{2v}{z}\mathrm{d}z\right) + \frac{l^2}{z^2}(\mathrm{d}x^i)^2 \tag{5.24}$$

where $H^{(1)} = 0$, $W_z^{(1)} = 2/z$, $W_i^{(0)} = \frac{l^2}{z^2}W_i$, $H^{(0)} = \frac{l^2}{z^2}H$. As for AdS_n, the coefficient of v^2 in the metric function H vanishes and the transverse space is the hyperbolic space $\mathbb{H}^{n-2}$. Note the dependence of the metric on v; clearly it is not CCNV. In the new coordinates[7] the null Killing vector is

$$k^v = \frac{l^2}{z^2}, \;\; k_u = \frac{l^2}{z^2} \tag{5.25}$$

Let us give two explicit examples in this class. The n-dimensional Siklos spacetime [121] arises when the VSI considered is (3.25)

$$ds^2 = \frac{l^2}{z^2}\left[2\mathrm{d}u\mathrm{d}\tilde{v} + 2H(u,x^k)\mathrm{d}u^2 + (\mathrm{d}x^i)^2\right] \tag{5.26}$$

$i = 1,\ldots,n-2$. The function H satisfies the Siklos equation

$$\triangle H - \frac{n-2}{z}H_{,z} = 0 \tag{5.27}$$

which is, in fact, the Laplace equation in hyperbolic space. A (rather) general solution of the Siklos equation can be found in [122]. Note the difference

[7] Under coordinate transformations, a vector in the tangent space transforms $k'^\mu = k^\nu \partial x'^\mu/\partial x^\nu$.

with respect to (3.25), where H satisfies the flat Laplace equation. The Siklos metric is of Weyl type N. It represents exact gravitational waves propagating in the anti-de Sitter universe [123].

A particular example of a Siklos metric is the Kaigorodov spacetime, K_n, for which

$$H = z^{n-1} \tag{5.28}$$

It is straightforward to see that this function satisfies (5.27). The four-dimensional metric was given in [124, 125] and generalized to higher dimensions in [126]. In general, the Siklos spacetime is inhomogeneous but the Kaigorodov metric is actually homogeneous. This is analogous to the pp-waves and homogeneous plane waves case. The Kaigorodov metric has $n-1$ obvious Killing vectors (but ∂_v is the only null one).
All of the Siklos metrics preserve 1/4 of the supersymmetries[8], regardless the form of the function H in (5.27) [127]. In particular, this is true for the Kaigorodov metric as was previously shown in [126].

The second example is the n-dimensional AdS gyraton [113], which describes a gyraton (see section 3.4.2) in anti-de Sitter space. The metric reads

$$ds^2 = \frac{l^2}{z^2}\left[2\mathrm{d}u\mathrm{d}\tilde{v} + 2H(u,x^k)\mathrm{d}u^2 + 2W_i(u,x^k)\mathrm{d}u\mathrm{d}x^i + (\mathrm{d}x^i)^2\right] \tag{5.29}$$

with $i = 1,\dots,n-2$. The functions H, W_i satisfy the following equations

$$\triangle H - \frac{n-2}{z}H_{,z} - \frac{1}{4}W_{ij}W^{ij} - W^i{}_{,iu} + \frac{n-2}{z}W_{z,u} = 0 \tag{5.30}$$

$$\partial_i W^{ij} = \frac{n-2}{z}W^{zj} \tag{5.31}$$

Note that the AdS gyraton is of Weyl type III, and not III(a) as the gyraton (see section). Since the AdS gyraton is conformal to the gyraton, the Weyl tensor is the same for both solutions. In the case of the gyraton, the Weyl components (3.111) vanish because the functions W_i satisfy equation (3.114)

$$\partial_i W^{ij} = 0 \tag{5.32}$$

From equation (5.31) it is clear that, in general, the Weyl components (3.111) do not vanish for the AdS gyraton; its Weyl type is III.

[8] In the context of a n-dimensional supergravity theory with cosmological constant.

The five-dimensional AdS gyraton has been considered in the context of gauged supergravity, and both gauged and ungauged supergravity coupled to an arbitrary number of vector supermultiplets [128]. Some of these solutions preserve 1/4 of the supersymmetry [128], [129], [130].

Spaces of the form (5.21) with a non-CCNV VSI can be constructed in a systematic way. We pointed out in Chapter 4 that non-CCNV VSI spaces do not have any null or timelike Killing vector. Most likely this is also the case for CSI spaces constructed from them, which makes them unattractive from a supersymmetry point of view.

2. We consider product manifolds $M \times K$, for which

$$\mathrm{d}s^2_{M\times K} = \mathrm{d}s^2_M + \mathrm{d}s^2_K, \quad \dim\,(M \times K) = \dim M + \dim K \tag{5.33}$$

The metric on M is of the form (5.21) while K is a locally homogeneous space, for example, $\mathbb{E}^d$, S^d or $\mathbb{H}^d$. We therefore have

$$\mathrm{d}s^2 = \frac{l^2}{z^2} ds^2_{VSI} + ds^2_{hom} \tag{5.34}$$

The functions H and W_i in the VSI metric can now be fibred, $H(v, u, x^k, y^a)$, $W_i(v, u, x^k, y^a)$, where y^a $a = 1, ..., \dim K$ are the coordinates on K. We already mentioned that M is Einstein if the VSI seed is Ricci flat; if K is Einstein then $M \times K$ is Einstein as well. The simplest example occurs for $M = AdS_n$ and Euclidean transverse space. Supersymmetric solutions of this type in $D = 5$ gauged supergravity were given in [131].

Best-known examples in this class are the spaces of the form $AdS \times S$. They are of Weyl type O [9]. To fix ideas let us discuss $AdS_5 \times S^5$

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[2\mathrm{d}u\mathrm{d}\tilde{v} + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] + \mathrm{d}\Omega_5^2 \tag{5.35}$$

where $\mathrm{d}\Omega_5^2$ is the standard round metric on S^5. Note (5.35) is of the form (5.34), with the simplest VSI (Minkowski space). In the Kundt form it is just (5.20), but now the transverse space is $\mathbb{H}^3 \times S^5$.

Let us comment on the relevance of these metrics in supergravity and string theory. Spaces of the form $AdS \times S$, together with appropriate five- or

[9] Provided their curvature radii match, otherwise they are Weyl type D.

four-form fields, are maximally supersymmetric solutions of IIB and eleven-dimensional supergravities [132, 133, 84]. Moreover, they are exact string and M-theory solutions [134]. A complete classification of the maximally supersymmetric solutions in ten- and eleven-dimensional supergravities can be found in [135]. We summarize the results here. In IIB supergravity there are three maximally supersymmetric solutions, namely, flat space (with zero fluxes), $AdS_5 \times S^5$ and the Blau Figueroa-O'Farrill Hull Papadopoulos (BFHP) solution [91]. The later has a metric of the plane wave type (see section 1.4) and constant five-form field. On the other hand, flat space (with constant dilaton) is the only maximally supersymmetric solution of both IIA and $N = 1$ ten-dimensional supergravities [135]. In eleven-dimensional supergravity there are four maximally supersymmetric solutions. These are $AdS_4 \times S^7$, $AdS_7 \times S^4$, flat space (with zero flux) and the Kowalski-Glikman (KG) solution [136]. The KG solution has a metric of the plane wave type and constant four-form field. The BFHP solution is in fact the ten-dimensional analogue of the KG solution.

Both the BFHP and KG solution are actually related to the $AdS \times S$ solutions. The Penrose limit was generalized to include supergravity fields other than the metric in [137]. This defined the so-called Penrose-Güven limit of supergravity solutions. The number of supersymmetries of the original solution is preserved (or increased) [138, 139, 137]. The BFHP and KG solution arise as the Penrose-Güven limit of $AdS_5 \times S^5$, and $AdS_4 \times S^7$ or $AdS_7 \times S^4$ respectively [140, 138].

Finally, $AdS_5 \times S^5$ plays a key role in the AdS/CFT correspondence [141]. The idea is that Type IIB superstrings on this background can be related to a four-dimensional conformal field theory without gravity, namely, $N = 4$ super-Yang-Mills (SYM). However, this is not an easy task. An smart approach is to consider Type IIB superstrings on the BFHP background instead. This is known as the Berenstein-Maldacena-Nastase (BMN) correspondence [140].

$AdS_5 \times S^5$ can be generalized in, at least, two ways. In the first place, by considering VSI metrics other than Minkowski in M. Such spacetimes have the same Weyl type as the VSI metric in question (III at most[10]). For example,

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[2\mathrm{d}u\mathrm{d}\tilde{v} + 2H(u,x,y,z,y^a)\mathrm{d}u^2 + (\mathrm{d}x^i)^2\right] + \mathrm{d}\Omega_5^2 \tag{5.36}$$

In the Kundt form we have now $H^{(0)} = 2H/z^2$. The Weyl type is N. Such spacetimes are supersymmetric solutions of IIB supergravity (and there are

[10] II if the curvature radii do not match.

analogous solutions in $D = 11$ supergravity) [142]. One could also consider solutions where AdS_5 is replaced by the (five-dimensional) AdS gyraton (5.29) or a space constructed from a non-CCNV VSI.

A second possibility to generalize $AdS_5 \times S^5$ is by swapping S^5 with another Einstein space. Such backgrounds are called Freund-Rubin. The idea of considering spaces of the form $AdS \times M$, with M an Einstein-Sasaki[11] manifold goes back to [144]. Roughly speaking it is an odd-dimensional Einstein space which admits Killing spinors, examples are the n-sphere or the squashed sphere. The latter is an n-sphere squashed in one direction, see for example [5]. In [144] the authors present supersymmetric solutions of $D = 11$ supergravity where M is, for example, the squashed sphere S^7. Another well-known Einstein-Sasaki space is $T^{1,1}$. It corresponds to $U(1)$ bundles over $SU(2) \times SU(2)$ in which the $U(1)$ fibres wind once over each of the $SU(2)$. In ten dimensions, solutions of the form $AdS_5 \times T^{1,1}$ have been extensively studied.

The last possibility we discuss here is the case where the metric on M is not of the form (5.21). An example follows

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}v + (H^{(0)}(u, x^k) + v^2)\,\mathrm{d}u\right] + \mathrm{d}\Omega_3^2 \tag{5.37}$$

with $\mathrm{d}\Omega_3^2$ the standard round metric on S^3. We can write it as

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\xi^2 + \sin\xi^2 \mathrm{d}\Omega_2^2 \tag{5.38}$$

This spacetime is a generalization of the metric (5.11), where the function H is allowed to depend on v (so clearly it is not CCNV). At the same time it is a special case of a solution presented in [114]. The latter's transverse space is the squashed S^3, (5.37) is obtained in the round sphere limit. The solution is in the Kundt form (5.1), with $\sigma = 1$, $W_i^{(1)} = W_i^{(0)} = H^{(1)} = 0$. The Ricci tensor is

$$\begin{aligned} R_{uu} &= \triangle^{(S^3)} H^{(0)} + 4(H^{(0)} + v^2) && (5.39)\\ R_{uv} &= 2 && (5.40)\\ R_{ij} &= 2 g_{ij}(S^3) && (5.41)\end{aligned}$$

where $\triangle^{(S^3)}$ is the Laplace-Beltrami operator on S^3. It is straightforward to see that (5.37) is an Einstein space with $k = 2$, when

$$\triangle^{(S^3)} H^{(0)} = 0 \tag{5.42}$$

[11] See [143] for a short review of Einstein-Sasaki spaces in this context.

Assuming $SO(3)$ symmetry, the general solution is [118]

$$H^{(0)} = C_1 + C_2 \cot \xi \tag{5.43}$$

with C_1, C_2 constants.

6 Conclusions and outlook

In this thesis we have considered Kundt spacetimes, both in four and higher dimensions. The original motivation to study this class of metrics was the fact that they generalize the well-known pp-waves. Perhaps the most interesting feature of pp-waves is that they are exact string solutions to all orders in α'. This is a consequence of the pp-waves vanishing invariants property. The VSI spacetimes are a subclass of the Kundt spacetimes defined by this property. Clearly, the vacuum VSI spacetimes are immediately exact string solutions. Some Kundt spacetimes with constant invariants, like $AdS_5 \times S_5$, are known to be exact string solutions as well. Therefore, Kundt CSI spacetimes are also worthy of study. The Kundt spacetimes could have other applications similar to pp-waves.

The four-dimensional Kundt class was almost completely characterized long ago. Kundt metrics are algebraically special, i.e., of Petrov type II, D, III or N. The Kundt class can be subdivided into two main subclasses, according to whether the transverse space is flat or conformally flat. Kundt metrics of Petrov type III and N are in the first class, the latter being the usual pp-waves. In the second case the metrics are of type II or D.
The vanishing invariant property of pp-waves was also known for a long time. However, it was only recently shown that all such metrics belong to the Kundt class and are of Petrov type III and N. They have been called vanishing scalar invariant (VSI) metrics. In this thesis we have constructed VSI solutions of the Einstein-Yang-Mills (EYM) equations (and Einstein-Maxwell (EM) in the abelian limit). It is worth to note that, in these VSI solutions, the ansatz for the source fields is exactly the same one as for plain pp-waves. This was the first hint that pp-wave solutions could be easily generalized to VSI solutions.

Higher-dimensional (HD) pp-wave solutions have been extensively considered in the context of supergravity and superstrings. This brought us to the idea of generalizing those to HD VSI solutions. In order to do this we first had to construct the HD VSI metrics themselves. A very important tool we used to do so was the recently developed algebraic classification of HD metrics. These can be classified according to their Ricci and Weyl type. These are analogous to the four-dimensional Plebanski-Petrov (PP) and Petrov classification, respectively. The VSI theorem had already been generalized to higher dimensions. It was postulated that HD VSI metrics belong to the HD Kundt class and are of Weyl type III and N ($\sim$ Petrov type III and N), and Ricci type III or N ($\sim$ PP type N or O). However, the explicit form of the HD VSI metrics was not known. In this thesis we have characterized completely the HD VSI spacetimes of Ricci type N, and those of Ricci type III partially. In practice, this knowledge can be used to construct new solutions in any theories for which pp-wave solutions already exist. For example, the four-dimensional VSI-YM solutions could be generalized to HD VSI EYM solutions. In this thesis we have restricted ourselves to supergravity solutions.

The structure of the HD VSI metrics is fairly similar to that of the four-dimensional ones. We found, however, the following crucial difference. In four dimensions pp-waves are always of Petrov type N. In higher dimensions, pp-wave metrics (of Ricci type N) can not only have Weyl type N but also III (III(a) to be precise). This means they can be of a more general algebraic type. We showed that the relativistic gyraton, which describes the gravitational field of a spinning beam pulse of electromagnetic radiation, is in this very special class. We also presented a (simpler) explicit example of another metric of the same type. Ricci and Weyl type III pp-wave metrics exist only if appropiate sources can be found.

Next, we considered the ten-dimensional VSI spacetimes in the context of type IIB supergravity (but similar results are expected in other supergravity theories). We were able to construct VSI solutions with non-zero dilaton and both NS-NS and RR fluxes. As in the four-dimensional EYM case, the field ansatz coincides with the pp-waves one. We found solutions of both Ricci type N and III. The first ones are generalizations of the usual HD pp-wave solutions. The Ricci type III solutions are, to our knowledge, the first of their kind in the literature. They are made possible by the dilaton dependence on the light-cone coordinate u. At the same time, we showed that there are no pp-wave supergravity solutions of Ricci type III.
The HD VSI supergravity solutions in this thesis are also exact string solu-

tions to all orders in α', at least in the presence of dilaton and NS-NS flux. Future research should determine whether this is also the case for RR fluxes. On the other hand, we studied the supersymmetry properties of the solutions. We found that HD VSI metrics with dependence on the light-cone coordinate v do not posses any null or timelike Killing vector. This implies that the only potentially supersymmetric HD VSI solutions are those of Ricci type N and Weyl type III(a) or N. The result concerning the Weyl type III(a) class is novel (although we had hoped for a bigger class). Subsequently, we analyzed the supersymmetries of this class in detail, both for vacuum and NS-NS solutions. We gave explicit examples of the latter. It would be interesting to study the supersymmetries of the solutions with RR fluxes as well.

The HD VSI metrics are actually a special case of HD constant scalar invariant (CSI) metrics. These have been postulated to be either HD homogeneous spacetimes or a subclass of the HD Kundt class being of Ricci and Weyl type II at most. In this thesis we have only considered the latter, which we refer to as HD Kundt CSI metrics. Although we did not study them in full detail we were able to present a general characterization. We found out that pure (not VSI) HD Kundt CSI metrics are of Ricci type II and not lower. They include HD CSI pp-wave spacetimes, which will be analyzed elsewhere.
We showed that many of the spacetimes commonly considered in supergravity and superstrings are actually in this class, beginning already with Anti-de Sitter space. We sketched how to construct generalizations of existing solutions while staying in this class. Some of them are specially interesting, namely, the generalizations of Anti-de Sitter by using a HD VSI seed. Those have exactly the same invariants as pure Anti-de Sitter. This means they can be used to generate new exact string or M-theory solutions, by generalizing the exact Anti-de Sitter solutions like the $AdS \times S$ spaces. Therefore, it would be very interesting to construct those metrics explicitly.

The HD pp-waves belong to the HD Kundt class, and most of them are CSI or VSI. They can have general algebraic Ricci and Weyl types. Therefore we conclude that they are far more rich than their four-dimensional counterpart. On the other hand we showed that HD Kundt spacetimes are of relevance in supergravity and string theory, which makes them worthy of further study.

Appendices

Appendix A

Consider the following metric in arbitrary even number of dimensions N

$$ds^2 = 2du\,[dv + H\,du + W_i\,dz^i + \overline{W}_i\,d\bar{z}^i] + 2dz^i d\bar{z}^i \tag{A.1}$$

where $i = 1, \ldots, (N-2)/2$. We define the complex coordinates z^i, $\bar{z}^i$ as follows

$$z^i = \frac{1}{\sqrt{2}}\,(x^j + ix^{j+1}), \;\; \bar{z}^i = \frac{1}{\sqrt{2}}\,(x^j - ix^{j+1}) \tag{A.2}$$

where j takes odd values between 1 and $N-3$ and $i = (j+1)/2$. We can rewrite (A.1) in terms of the real coordinates x^j

$$ds^2 = 2du\,[dv + H\,du + W_j\,dx^j + W_{j+1}\,dx^{j+1}] + (dx^j)^2 + (dx^{j+1})^2 \tag{A.3}$$

The metric functions are then related by

$$W_j = \sqrt{2}\,(W_i + \overline{W}_i), \;\; W_{j+1} = \sqrt{2}i\,(W_i - \overline{W}_i) \tag{A.4}$$

Appendix B

We list below the Weyl components C_{1ijk} for the six-dimensional metric (3.105)-(3.109).

$$
\begin{aligned}
C_{1324} &= \frac{1}{2}\left(f_3(u) - f_2(u)\right)\frac{x^4}{x^1} \\
C_{1325} &= \frac{1}{2}\left(f_4(u) - f_2(u)\right)\frac{x^3}{x^1} \\
C_{1345} &= \frac{1}{2}(f_4(u) - f_3(u)) \\
C_{1423} &= \frac{1}{2}\left(-f_3(u) - f_2(u)\right)\frac{x^4}{x^1} \\
C_{1425} &= \frac{1}{2}\left(-f_3(u) - f_4(u)\right)\frac{x^2}{x^1}
\end{aligned}
$$

$$
\begin{aligned}
C_{1435} &= \frac{1}{2}(f_4(u) - f_2(u)) \\
C_{1523} &= \frac{1}{2}\left(-f_4(u) - f_2(u)\right)\frac{x^3}{x^1} \\
C_{1524} &= \frac{1}{2}\left(-f_4(u) - f_3(u)\right)\frac{x^2}{x^1} \\
C_{1534} &= \frac{1}{2}(f_3(u) - f_2(u))
\end{aligned}
$$

Appendix C

We list below some of the Weyl components C_{1ijk} for the ten-dimensional gyraton (3.120), namely, those of the form C_{1j2j}.

$$
\begin{aligned}
C_{1323} &= \frac{-8x^2p_1(-3Q+5(x^3)^2+5(x^4)^2+5(x^5)^2+5(x^6)^2+5(x^7)^2+5(x^8)^2)}{Q^6} \\
C_{1424} &= \frac{4(-x^2p_1Q+10x^2p_1(x^3)^2-10p_2x^4x^1x^3)}{Q^6} \\
C_{1525} &= \frac{4(-x^2p_1Q+10x^2p_1(x^4)^2+10p_2x^4x^1x^3)}{Q^6} \\
C_{1626} &= \frac{4(-x^2p_1Q+10x^2p_1(x^5)^2-10p_3x^6x^1x^5)}{Q^6} \\
C_{1727} &= \frac{4(-x^2p_1Q+10x^2p_1(x^6)^2+10p_3x^6x^1x^5)}{Q^6} \\
C_{1828} &= \frac{4(-x^2p_1Q+10x^2p_1(x^7)^2-10p_4x^7x^1x^8)}{Q^6} \\
C_{1929} &= \frac{4(-x^2p_1Q+10x^2p_1(x^8)^2+10p_4x^7x^1x^8)}{Q^6}
\end{aligned}
$$

As an aside, we can check relation (3.18) for $i=2$

$$C_{1012} = C_{1j2j} = 0$$

Appendix D

We display here the Weyl components C_{1ijk} for VSI spacetimes of type III, in the $\epsilon = 1$ case and six dimensions. The generalization to higher number of dimensions is straightforward.

$$C_{1223} = \frac{1}{2}\left(W_{2,11} - \frac{W_{2,1}}{x^1}\right) \qquad C_{1423} = \frac{1}{2}\left(W_{2,13} - \frac{W_{2,3} + W_{3,2}}{x^1}\right)$$

$$C_{1224} = \frac{1}{2}\left(W_{3,11} - \frac{W_{3,1}}{x^1}\right) \qquad C_{1424} = \frac{1}{2}\left(W_{3,13} - \frac{W_{3,3}}{x^1}\right)$$

$$C_{1225} = \frac{1}{2}\left(W_{4,11} - \frac{W_{4,1}}{x^1}\right) \qquad C_{1425} = \frac{1}{2}\left(W_{4,13} - \frac{W_{4,3} + W_{3,4}}{x^1}\right)$$

$$C_{1234} = \frac{1}{2}(W_{3,12} - W_{2,13}) \qquad C_{1434} = \frac{1}{2}(W_{3,23} - W_{2,33})$$

$$C_{1235} = \frac{1}{2}(W_{4,12} - W_{2,14}) \qquad C_{1435} = \frac{1}{2}(W_{4,23} - W_{2,34})$$

$$C_{1245} = \frac{1}{2}(W_{4,13} - W_{3,14}) \qquad C_{1445} = \frac{1}{2}(W_{4,33} - W_{3,34})$$

$$C_{1323} = \frac{1}{2}\left(W_{2,12} - \frac{W_{2,2}}{x^1}\right) \qquad C_{1523} = \frac{1}{2}\left(W_{2,14} - \frac{W_{2,4} + W_{4,2}}{x^1}\right)$$

$$C_{1324} = \frac{1}{2}\left(W_{3,12} - \frac{W_{2,3} + W_{3,2}}{x^1}\right) \qquad C_{1524} = \frac{1}{2}\left(W_{3,14} - \frac{W_{3,4} + W_{4,3}}{x^1}\right)$$

$$C_{1325} = \frac{1}{2}\left(W_{4,12} - \frac{W_{2,4} + W_{4,2}}{x^1}\right) \qquad C_{1525} = \frac{1}{2}\left(W_{4,14} - \frac{W_{4,4}}{x^1}\right)$$

$$C_{1334} = \frac{1}{2}(W_{3,22} - W_{2,23}) \qquad C_{1534} = \frac{1}{2}(W_{3,24} - W_{2,34})$$

$$C_{1335} = \frac{1}{2}(W_{4,22} - W_{2,24}) \qquad C_{1535} = \frac{1}{2}(W_{4,24} - W_{2,44})$$

$$C_{1345} = \frac{1}{2}(W_{4,23} - W_{3,24}) \qquad C_{1545} = \frac{1}{2}(W_{4,34} - W_{3,44})$$

Appendix E

In the metric function H in (3.27), a u-dependent term only which is linear in v

$$H \supset H^{(1)}(u)v \tag{E.1}$$

can always be transformed away. Consider a coordinate transformation (3.30) parametrized by some function $g(u)$. The transformed function $H'^{(1)}$

$$H'^{(1)} = \frac{1}{g_{,u}^{2}} \left(H^{(1)}(u) + \frac{g_{,uu}}{g_{,u}} \right) \tag{E.2}$$

Therefore, the transformed function H' will not contain a term linear in v provided that

$$g_{,u} = e^{-\int H^{(1)}(u)\,\mathrm{d}u} \tag{E.3}$$

Appendix F

The metric functions in the gauge $W_1^{(0)} \neq 0$ are displayed in Table 1 at the end of this appendix. The corresponding Einstein equations are given below.

$$\triangle H^{(0)} - \frac{1}{4}W_{ik}W^{ik} - 2H^{(1)}{}_{,i}W^{(0)i} - H^{(1)}W^{(0)i}{}_{,i} - W^{(0)i}{}_{,iu} + T_{uu} = 0 \quad \text{(F.1)}$$

$$\triangle H^{(0)} - 2\sum_{i<j} B_{ij}^2 + T_{uu} = 0 \quad \text{(F.2)}$$

$$H^{(0)}_{T_{uu},ii} = -\frac{1}{8}T_{uu}, \quad H^{(0)}_{T_{uu},ik} = 0 \quad \text{(F.3)}$$

$$x^1\triangle\left(\frac{H^{(0)}}{x^1}\right) + \left(\frac{W^{(0)j}W_j^{(0)} - (W_1^{(0)})^2}{x^1}\right)_{,1} + \frac{2}{x^1}\left(W_1^{(0)}W_{1,1}^{(0)} - W^{(0)j}W_{1,j}^{(0)}\right)$$
$$-2H^{(1)}{}_{,i}W^{(0)i} - H^{(1)}W^{(0)i}{}_{,i} - \frac{1}{4}W_{ik}W^{ik} - W^{(0)i}{}_{,iu} + T_{uu} = 0 \quad \text{(F.4)}$$

$$x^1 \triangle\left(\frac{H^{(0)}}{x^1}\right) - \frac{1}{(x^1)^2}\left((W_1^{(0)})^2 + \sum_j (W_j - x^1 B_{1j})^2\right) - \sum_j B_{1j}^2$$
$$-2\sum_{i<j} B_{ij}^2 + T_{uu} = 0 \quad \text{(F.5)}$$

In equations (3.58)-(3.64) we give the rules to transform $W_1^{(0)}$ away, using the function $h(u, x^k)$ defined by (3.61). In the Weyl type III case we cannot be more precise as $W_1^{(0)}$ is undetermined. We know however, that for $\epsilon = 1$ the functions $\tilde{F}$ are related by

$$\tilde{F}_{W_1^{(0)}=0} = \tilde{F}_{W_1^{(0)}\neq 0} - \triangle h \quad \text{(F.6)}$$

In the Weyl N case we can give the explicit form of the function h. For $\epsilon = 1$ we have

$$h(u, x^k) = x^1(x^m B_{1m}(u) - C_1(u)) \quad \text{(F.7)}$$

Note that by transforming away $W_1^{(0)}$ we also transform away the coefficient $H^{(1)} = -W_1^{(0)}/x^1$ in (3.50). For $\epsilon = 0$

$$h(u, x^k) = x^1 x^m B_{m1}(u) \quad \text{(F.8)}$$

And $C_m(u) = 2B_{1m}(u)$ in (3.132). In the Weyl O case one can obtain the transformed functions $H^{(0)}$ by using (3.63) with h given by (F.7) or (F.8).

ϵ	**Weyl**	**Metric functions**	**Eq.**
	III	$W_i = W_i^{(0)}(u, x^k)$ $H = H^{(0)}(u, x^i) + \frac{1}{2}\left(F - W^{(0)i}{}_{,i}\right)v,\ \ F(u, x^i)$ defined by $F_{,i} = \Delta W_i^{(0)}$	(F.1)
0	N	$W_i = x^k B_{ki}(u),\ \ B_{ki}(u)$ antisymmetric $H = H^{(0)}(u, x^i)$	(F.2)
	O	W_i as in type N $H^{(0)} = \frac{1}{2}W^i W_i + x^i f_i(u) + H^{(0)}_{T_{uu}}$	$(F.2)$ $(F.3)$
	III	$W_1 = -\frac{2}{x^1}v + W_1^{(0)}(u, x^k)$ $W_j = W_j^{(0)}(u, x^k)$ $H = H^{(0)}(u, x^i) + \frac{1}{2}\left(\tilde{F} - W^{(0)i}{}_{,i} - \frac{2W_1^{(0)}}{x^1}\right)v + \frac{v^2}{2(x^1)^2},$ $\tilde{F}(u, x^i)$ defined by $\tilde{F}_{,1} = \frac{2}{x^1}W^{(0)i}{}_{,i} + \Delta W_1^{(0)},\ \tilde{F}_{,j} = \Delta W_j^{(0)}$	(F.4)
1	N	$W_1 = -2\frac{v}{x^1} + x^j B_{j1}(u) + C_1(u)$ $W_j = x^i B_{ij}(u) + C_j(u),\ \ B_{j1}(u), B_{ij}(u)$ antisymmetric $H = H^{(0)}(u, x^i) - \frac{W_1^{(0)}}{x^1}v + \frac{v^2}{2(x^1)^2}$	(F.5)
	O	W_i as in type N $H^{(0)} = \frac{1}{2}(W_1^{(0)})^2 + \frac{1}{2}\sum_j (W_j - x^1 B_{1j})^2 + x^1 g_0(u) + x^1 x^i g_i(u) - \frac{1}{16}\Phi(u) x^1 x^i x_i,$ $T_{uu} = \Phi(u) x^1$	(F.5)

Table 1: All higher-dimensional VSI spacetimes of Ricci type N in the gauge $W_1^{(0)} \neq 0$. Ricci type O (vacuum) for $T_{uu} = 0$ in (F.1)-(F.5).

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Resumen

Espacios-tiempo de Kundt en Relatividad General y Supergravedad

Todos sabemos que cuando hay gravedad los objetos caen. Es decir, si soltamos algo no flota en el aire sino que cae al suelo. O si nosotros mismos saltamos, volvemos a caer. Esto se debe a que en la Tierra hay gravedad. La gravedad no se da únicamente en la Tierra, sino que está presente allí en el espacio donde haya materia (la Tierra en este caso). Newton fue el primero en desarrollar una teoría de la gravedad. Una teoría es un modelo capaz de predecir y describir los fenómenos físicos que observamos. La teoría de Newton, sin embargo, no puede explicar muchos de los fenómenos relacionados con la gravedad. En 1916 Einstein publica su teoría de la relatividad, capaz de predecir con precisión fenómenos que sólo han sido observados mucho después. Por ejemplo, gracias a la relatividad disponemos del sistema de navegación GPS que permite determinar nuestra posición con un margen de error de tan sólo unos metros. Si intentaramos usar este sistema aplicando exclusivamente la teoría de Newton dejaría de funcionar correctamente en cuestión de minutos.

En la teoría de la relatividad el tiempo y el espacio no se tratan independientemente sino que forman parte de un todo: el espacio-tiempo. Por otra parte, la gravedad se entiende como la curvatura de este espacio-tiempo causada por la presencia de materia. Esta curvatura está relacionada con la masa de dicha materia: cuanto más masa mayor es la curvatura producida. Objetos cercanos "caen" debido a esa curvatura y siguen las trayectorias que ella determina. Para hacer una analogía, podemos imaginarnos el efecto que causa un balón en una red. El balón curva la red, y este efecto es claramente mayor cuanto más pesado es el balón. Si ponemos otro objeto más pequeño en la red "caerá" en dirección al balón debido a la curvatura que este último causa.

La teoría de la relatividad está descrita matemáticamente por un conjunto de ecuaciones llamadas ecuaciones de Einstein. El tipo de materia presente determina la forma de estas ecuaciones. En esta tesis presentamos soluciones nuevas de estas ecuaciones. Esencialmente esto significa que hemos encontrado una nueva manera en la que se puede curvar el espacio-tiempo, al menos teóricamente. Si este tipo de curvatura se puede observar o no es otra cuestión a la que por ahora no podemos responder. En realidad hemos estudiado la teoría de la relatividad asumiendo que el espacio-tiempo puede tener más de cuatro dimensiones. Es decir, que los objetos no sólo tienen largo, alto y ancho en un momento (=tiempo) dado sino también otras dimensiones adicionales, demasiado pequeñas para ser vistas. Esta revolucionaria idea aparece en el contexto de las teorías de unificación. Este tipo de teorías describen conjuntamente fenómenos físicos causados por diferentes tipos de fuerzas. Entre 1921 y 1926 Kaluza y Klein desarrollaron una teoría que asumía la existencia de una dimensión adicional. La teoría de Kaluza y Klein podía describir tanto la gravedad como el electromagnetismo, la fuerza responsable de la electricidad y las propiedades magnéticas de los materiales como por ejemplo los imanes.

Uno de los problemas por resolver más importantes de la física moderna es precisamente el encontrar una "teoría del todo" capaz de describir de una manera unificada las cuatro fuerzas presentes en la Naturaleza: la gravedad, el electromagnetismo, la fuerza débil y la fuerza fuerte. La fuerza fuerte hace que partículas como el protón sean estables y puedan por tanto formar parte de los átomos, y éstos a su vez formar toda la materia existente. La fuerza débil está relacionada con la radioactividad, el proceso por el cual un átomo inestable pierde energía emitiendo radiación y transformandose así en un átomo diferente. Ésta es la base de la energía nuclear.

Aunque aún no disponemos de una teoría del todo, hay teorías que consiguen dar una solución parcial a este problema. La más estudiada es probablemente la teoría de las supercuerdas, nacida en los años ochenta. Los objetos fundamentales en esta teoría son cuerdas de diferentes tipos (cerradas, abiertas) que se encuentran en el espacio-tiempo. Al vibrar estas cuerdas dan lugar a todas las partículas fundamentales, por ejemplo el electrón, y las diferentes fuerzas que actúan entre ellas incluida la gravedad. Curiosamente, la teoría de las supercuerdas solamente funciona bien matemáticamente cuando dichas cuerdas existen en diez dimensiones (nueve dimensiones espaciales y tiempo). Es decir, si esta teoría está en lo cierto existirían nada más y nada menos que seis dimensiones adicionales. La teoría de las supercuerdas está descrita matemáticamente por un conjunto de ecuaciones que determinan como vibran las diferentes cuerdas en este espacio-tiempo con dimensiones adicionales.

No es difícil imaginar que las ecuaciones de supercuerdas no son en absoluto fáciles de resolver. Sin embargo en algunos casos es posible, usando soluciones de las ecuaciones de Einstein en el espacio-tiempo con dimensiones adicionales. Se da el caso de que una de ellas es, gracias a una propiedad especial del espacio-tiempo, no sólo solución de las ecuaciones de Einstein sino también de las ecuaciones de supercuerdas. Hasta ahora sólo se conocía una solución de las ecuaciones de Einstein con esta propiedad tan importante. Dicha solución representa ondas gravitacionales propagándose en el espacio-tiempo y ha resultado ser crucial en la teoría de supercuerdas. En esta tesis presentamos nuevas soluciones de las ecuaciones de Einstein que generalizan la solución ya conocida, y que a la vez tienen esa misma propiedad del espacio-tiempo. Esto las convierte inmediatamente en nuevas soluciones de supercuerdas. Otro aspecto interesante es que algunas de las soluciones de las ecuaciones de Einstein encontradas representan ondas gravitacionales al igual que la solución original.

No es impensable que las ondas gravitacionales descritas por algunas de las soluciones presentadas pudieran llegar a ser observadas en el futuro. Por otra parte, las nuevas soluciones de las ecuaciones de Einstein en esta tesis podrían llegar a ser tan importantes en la teoría de supercuerdas como la solución de ondas gravitacionales conocida con anterioridad. Y de este modo contribuir al desarrollo de una teoría del todo.

Samenvatting

Kundt ruimtetijden in Algemene Relativiteitstheorie en Supergravitatie

Iedereen weet dat voorwerpen vallen als er zwaartekracht is. Met andere woorden, als wij een voorwerp loslaten zweeft het niet in de lucht maar valt naar de grond. Als wij in de lucht springen, vallen we ook weer terug. Dit komt door de zwaartekracht op Aarde. De zwaartekracht komt niet alleen voor op Aarde; overal waar materie aanwezig is is er ook zwaartekracht. Newton was de eerste die een theorie van de zwaartekracht ontwikkelde. Een theorie is een model die in staat is om de fysische verschijnselen die wij waarnemen te beschrijven en te voorspellen. De theorie van Newton kan echter vele verschijnselen uit de zwaartekracht niet uitleggen. In 1916 publiceerde Einstein zijn relativiteitstheorie. Deze voorspelde nauwkeurig verschijnselen die pas veel later zijn waargenomen. Dankzij de relativiteitstheorie hebben we bijvoorbeeld het GPS navigatiesysteem tot onze beschikking. Dit maakt mogelijk om onze positie te bepalen met een foutmarge van slechts enkele meters. Als wij alleen Newton's theorie zouden toepassen, zou het GPS navigatiesysteem binnen enkele minuten een foute positie weergeven.

In de relativiteitstheorie worden ruimte en tijd niet meer onafhankelijk van elkaar beschouwd. Ruimte en tijd worden verenigd in de zogeheten ruimte-tijd. Aan de andere kant, de zwaartekracht wordt begrepen als de kromming van deze ruimte-tijd veroorzaakt door de aanwezigheid van materie. Deze kromming is gerelateerd aan de massa van de materie; hoe meer massa hoe krommer de ruimte-tijd wordt. Nabije voorwerpen "vallen" door die kromming en bewegen in de banen die de kromming toelaat. Wij kunnen bijvoorbeeld denken aan het effect dat een bal veroorzaakt in een net. Het net kromt door de bal en dit effect is duidelijk groter hoe zwaarder de bal. Als wij een ander (kleiner) voorwerp in het net plaatsen, zal het "vallen" in de richting van de bal die de kromming veroorzaakt.

De relativiteitstheorie wordt wiskundig beschreven door een stel vergelijkingen die bekend zijn onder de naam Einstein-vergelijkingen. Hun precieze vorm wordt bepaald door de aanwezige materie. In dit proefschrift stellen we nieuwe oplossingen voor van de Einstein-vergelijkingen. Dit betekent, theoretisch gezien, dat wij een nieuwe manier hebben gevonden waarop de ruimte-tijd gekromd kan worden. Of dit type kromming waargenomen kan worden is een andere vraag waarop wij voorlopig geen antwoord hebben. In het bestuderen van de relativiteitstheorie zijn we er van uit gegaan dat de ruimte-tijd meer dan vier dimensies kan hebben. Met andere woorden, voorwerpen hebben niet alleen lengte, hoogte en dikte op een bepaalde moment (=tijd) maar misschien ook andere extra dimensies die te klein zouden zijn om gemerkt te worden. Dit revolutionaire idee komt voor in het kader van de theorieën van unificatie. Deze theorieën beschrijven tegelijkertijd fysische verschijnselen die veroorzaakt worden door verschillende krachten. Kaluza en Klein ontwikkelden tussen 1921 en 1926 een theorie die er vanuit ging dat er een extra dimensie bestond. Deze theorie was in staat om zowel de zwaartekracht als electromagnetisme, de kracht die elektrische en de magnetische verschijnselen veroorzaakt (zoals de magnetische eigenschappen van materialen) te beschrijven.

Een van de belangrijkste kwesties in de huidige fysica is zo een "theorie van alles" te kunnen ontwikkelen. Een dergelijke theorie zou in staat zijn om alle vier bestaande krachten in de natuur te beschrijven: de zwaartekracht, electromagnetisme, de zwakke kracht en de sterke kracht. De sterke kracht zorgt ervoor dat bepaalde deeltjes stabiel zijn, zoals het proton. Op deze manier kunnen die deeltjes deel uit maken van atomen, en deze dan van alle chemisch actieve materie. De zwakke kracht is onder andere gerelateerd aan de radioactiviteit. Dit is het proces waardoor een niet-stabiel atoom energie kwijtraakt die als warmte wordt vrijgegeven, waarbij het verandert in een ander atoom. De kernenergie is daarop gebaseerd.

Wij hebben voorlopig geen theorie van alles tot onze beschikking. Er zijn wel bepaalde theorieën die met een gedeeltelijke oplossing daarvan komen. De meest bestudeerde theorie is zonder twijfel supersnarentheorie, die in de jaren tachtig geboren werd. De fundamentele objecten daarbij zijn snaren van verschillende types (gesloten, open) die zich in de ruimte-tijd bevinden. De trillingen van deze snaren veroorzaken het verschijnen van alle fundamentele deeltjes in de natuur (zoals het elektron) en de verschillende krachten die de deeltjes voelen, de zwaartekracht inclusief. Vreemd genoeg doet supersnarentheorie het wiskundig alleen goed als de snaren in tien dimensies bestaan (negen ruimtelijke dimensies en tijd). Als supersnarentheorie juist blijkt te zijn bestaan er dus niet meer en niet minder dat zes extra dimensies. Supersnarentheorie wordt wiskundig beschreven door een stel vergelijkingen die bepalen hoe de snaren trillen in deze uitgebreide ruimte-tijd.

Men kan zich voorstellen dat het bepaald niet makkelijk is om met een oplossing van de supersnarentheorie vergelijkingen te komen. Heel soms is het wel mogelijk door gebruik te maken van oplossingen van de Einstein-vergelijkingen in de ruimte-tijd met extra dimensies. Een van deze oplossingen is, dankzij een speciale eigenschap van de ruimte-tijd, niet alleen een oplossing van de Einstein-vergelijkingen maar ook van die van snarentheorie. Tot nu toe was er slechts een oplossing van de Einstein-vergelijkingen met die belangrijke eigenschap bekend. Deze oplossing beschrijft (exacte) gravitatiegolven die propageren in de ruimte-tijd en is cruciaal gebleken voor de supersnarentheorie. In dit proefschrift stellen we nieuwe oplossingen voor van de Einstein-vergelijkingen die de reeds bekende oplossing generaliseren, en die tegelijkertijd ook dezelfde eigenschap van de ruimte-tijd hebben. Dit maakt ze meteen nieuwe oplossingen van de vergelijkingen voor supersnaren. Ook belangrijk is de feit dat een aantal van de gevonden oplossingen van de Einstein-vergelijkingen zijn ook gravitatiegolven zoals de originele oplossing.

Het is niet ondenkbaar dat de gravitatiegolven beschreven door sommige oplossingen in dit proefschrift waargenomen zouden kunnen worden in de toekomst. Aan de andere kant, de algemene nieuwe oplossingen van de Einstein-vergelijkingen zouden even belangrijk kunnen worden voor de supersnarentheorie als de reeds bekende gravitatiegolfoplossing. En onze oplossingen zouden op deze manier bijdragen aan het ontwikkeling van een theorie van alles.

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