

GRAVITATIONAL DYNAMICS OF NEAR
EXTREMAL BLACK HOLES

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Abstract

This thesis is devoted to trying to understand the quantum mechanics of gravity. We consider the low energy dynamics of near extremal black holes. We find that the gravitational dynamics of near extremal black holes are controlled by a one dimensional effective action and the quantum mechanics of the effective action is solvable. We give a simple formula for gravitational backreacted correlation functions and the Wheeler de Witt wavefunctional to all orders in perturbation theory.

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To my family.

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Chapter 1

Introduction

A black hole is a geometry in general relativity where a certain spacetime singularity occurs [1]. It exhibits strong gravitational acceleration such that classically nothing can escape its horizon. Moreover, the classical gravity description is expected to break down around the spacetime singularity, due to the high curvature scale. It was shown that under general circumstances, the Einstein equation predicts spacetime singularity from gravitational collapse [2] and therefore one can not prevent the creation of black holes. If the singularity is always behind the event horizon such that no exterior observers can detect it then one can in principle avoid discussing the singularity. Nevertheless, such a classical statement is false once we consider the quantum fluctuations of matter field around the horizon. Hawking's calculation [3] showed that the black hole horizon does emit particle like a thermal body and therefore black holes can evaporate. It is natural to ask then what is the final stage of the evaporation and what happens to the singularity. Since the thermal radiation does not carry information, this will lead to a non-unitary evolution if one trusts this picture till the end of the evaporation. This lack of unitary causes the black hole information paradox since in principle the probability of the universe should be conserved [4]. Among various versions of information paradox, the AMPS paradox [5, 6, 7] draws direct tension

between the smoothness condition of black hole horizon and the entanglement entropy monogamy. Monogamy is one of the most fundamental properties of entanglement which states that if one system is maximumly entangled with another system, then it cannot be correlated with any other systems. The basic argument of AMPS is coming from this: in order to preserve the unitarity, the hawking radiation should entangle with the early radiations after some evaporation time (the Page time [8]). This is inconsistent with the smoothness condition of the black hole horizon which requires the hawking radiation to be maximally entangled with its pair at the black hole interior. This smoothness condition roots from the equivalence principle which says the spacetime at the horizon can be locally approximated as a Minkowski space and the matter state should be close to the vacuum defined with respect to the Minkowski time. How delicate is the vacuum structure of the quantum field theory at the horizon then? Moreover, will small perturbation destroy the vacuum from the gravitational blueshift? In other words, will small change of initial state invalid the picture we just described by scrambling the black hole system? Shenker and Stanford [9, 10, 11] studied such problem and connected the scrambling phenomenon with gravitational shockwave scattering [12]. They were considering a special system of two black holes that are entangled in a thermofield double state [13]. The thermofield double state is prepared by a euclidean evolution of the gravitational system between two sides such at each side it appears as a thermal system. Although an initial perturbation destroys the entanglement between modes in the two black holes, what they found is that, because of the shockwave backreaction on the geometry, there is a new spacetime region generated which still makes at least one of the horizon smooth. In a holographic setup, such shockwave geometries correspond to out of time ordered correlators (OTOC) in a strongly coupled quantum mechanical system and are closely related with thermalization. For systems with a large degree of freedom (including black holes), a Lyapunov exponent [14] can be defined from OTOC to characterize the rate of growth of chaos. From

causality and unitarity, one can prove that the Lyapunov exponent should be universally bounded by $2\pi k_B T/\hbar$ [15] which is saturated by black holes. On that account, the maximum Lyapunov exponent is another unique feature of black hole horizon, in addition to its thermodynamics property such as Bekenstein-Hawking entropy [16, 17]. The maximum Lyapunov exponent characterizes the dynamical behavior of black holes out of thermal equilibrium.

This thesis considers the chaos behavior of near extremal black holes. In general relativity, a black hole can both have charge Q and mass M [18]. The requirement of no naked singularity implies the relation $Q \leq M$, when the equality holds it is called an extremal black hole and a near extremal black hole is when the charge is close to its maximum value. Near the extremality, the Hawking temperature is linear with respect to its near extremal entropy and vanishes at the extremal limit. This behavior is due to the universal $\text{AdS}_2 \times \text{S}_2$ throat of near extremal black hole near its horizon. Anti de-Sitter Space (AdS) is a highly symmetric spacetime with constant negative curvature. The gravitational dynamics on $\text{AdS}_2 \times \text{S}_2$ is controlled by the Jackiw-Teitelboim gravity [19, 20] which comes from the spherical reduction of general relativity on this background. The Jackiw-Teitelboim gravity is a 2d dilaton gravity model consists of the 2d metric and a scalar field called the dilaton field. From the four dimensional picture, the 2d metric describes the AdS_2 geometry and the dilaton is the size of the two sphere. An AdS_2 geometry has an infinity set of asymptotic symmetries generated by large diffeomorphism transformations. In pure AdS_2 limit, the asymptotic symmetry is spontaneously broken to $\text{SL}(2, \mathbb{R})$ and the infinity goldstone modes make the path integral divergent. However, because we are considering the near extremal case, the symmetry is also explicitly broken by the boundary location and the soft modes are lifted by a Schwarzian effective action that is uniquely determined by the $\text{SL}(2, \mathbb{R})$ symmetry. The Schwarzian action and JT gravity are equivalent to each other when we put the boundary location close to infinity

and the propagating degree of freedom are the boundary gravitons. In quantum mechanics theories such as SYK like models [21, 22, 23, 24, 25, 26] where the Schwarzian action is the low energy effective action, one can think of the propagating modes are the collective motions of states close to the ground state. The coupling constant of the Schwarzian action becomes small at low energies or low temperatures. For this reason, gravitational corrections, though formally suppressed by the Newton constant, lead to important effects in the IR. When we consider matters in this near extremal geometry, the matter field is located with respect to the boundary and the fluctuations of the boundary gravitons will change the relative location between matters. In this sense, the Schwarzian action is a bit like a realization of the Mach Principle, where the boundary is the “distant star”. The boundary itself can be understood as a massive particle with a large electric charge carrying momenta and the shockwave geometries mentioned above have a simple explanation in terms of the Newton’s law of the boundary particle [27].

Using this point of view one can think of the full quantum gravity problem as the combination of two problems. First, we consider quantum fields propagating in AdS_2 (or H_2 in the Euclidean case) and then we add the “gravitational particle” which couples to the quantum fields by changing their boundary location in AdS_2 . The discussion of quantum fields will be standard and depends on the particular model one interested in, therefore we will mainly focus on solving the second problem. Generically, solving the gravitational problem is challenging and is not exactly equivalent to a quantum mechanical particle. One needs to worry about what functional space one will integrate over. For example, in path integrals, one usually integrates over all trajectories including those with self-intersections. However self-intersecting boundaries in gravitational system have no obvious meaning. On that account, more precisely the gravitational problem is equal to a self-avoiding particle. Nevertheless, it turns out that one can take a particular limit of this model, namely when the boundary is close to infinity, to avoid this issue and the treatment of the boundary

theory as an ordinary particle is justified. It is also true that the JT gravity can be rewritten as a Schwarzian action only in this limit. We call this the Schwarzian limit and will only focus on solving the JT action in the Schwarzian limit. Solving the model away from Schwarzian limit was considered recently by Kitaev and Suh [28].

The main portion of the dissertation consists of two papers, the first paper build the connection between JT gravity and Schwarzian action, the second paper devotes the quantization of JT gravity.

Chapter 2

Conformal Symmetry and its Breaking in Two dimensional Nearly Anti-de-Sitter space

This chapter consists of a paper [29] written in collaboration with Juan Maldacena and Douglas Stanford. It is published as “Conformal Symmetry and its Breaking in Two dimensional Nearly Anti-de-Sitter space”, PTEP 2016 (2016) no.12, 12C104 [arXiv:1606.01857]. The original abstract is as follows:

We study a two dimensional dilaton gravity system, recently examined by Almheiri and Polchinski, which describes near extremal black holes, or more generally, nearly AdS_2 spacetimes. The asymptotic symmetries of AdS_2 are all the time reparametrizations of the boundary. These symmetries are spontaneously broken by the AdS_2 geometry and they are explicitly broken by the small deformation away from AdS_2 . This pattern of spontaneous plus explicit symmetry breaking governs the gravitational backreaction of the system. It determines several gravitational properties such as the linear in temperature dependence of

the near extremal entropy as well as the gravitational corrections to correlation functions. These corrections include the ones determining the growth of out of time order correlators that is indicative of chaos. These gravitational aspects can be described in terms of a Schwarzian derivative effective action for a reparametrization.

2.1 Pure AdS_2

2.1.1 Coordinate systems

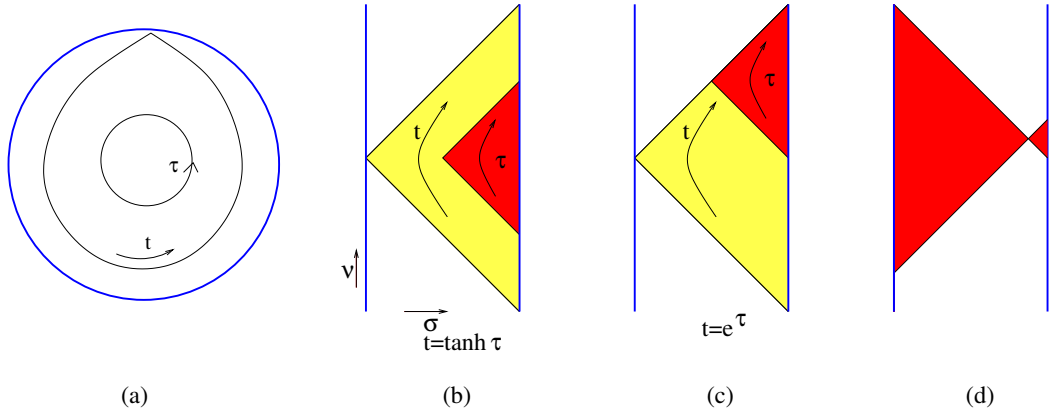


Figure 2.1: (a) Hyperbolic space or Euclidean AdS_2 . The orbits of τ translations look like circles. Orbits of t are curves that touch the boundary at $t = \pm\infty$. (b) Lorentzian AdS_2 . The ν, σ coordinates cover the whole strip. The \hat{t}, z coordinates describe the Poincare patch denoted here in yellow. The red region is covered by the $\hat{\tau}, \rho$ coordinates. There are different choices for how to place the $\hat{\tau}, \rho$ region that are generated by $SL(2)$ isometries. In (b) and (c) we show two choices and give the relation between the Poincare time \hat{t} and the $\hat{\tau}$ at the boundary of the space. In (d) we show a generic pair of Rindler wedges.

On AdS_2 (with unit radius) it is convenient to use the following coordinate systems

$$\text{Euclidean :} \quad ds^2 = \frac{dt^2 + dz^2}{z^2}, \quad ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2 \quad (2.1.1)$$

$$\text{Lorentzian :} \quad ds^2 = \frac{-dt^2 + dz^2}{z^2}, \quad ds^2 = d\rho^2 - \sinh^2 \rho d\hat{\tau}^2, \quad ds^2 = \frac{-d\nu^2 + d\sigma^2}{\sin^2 \sigma}$$

$$\text{Embedding :} \quad -Y_{-1}^2 - Y_0^2 + Y_1^2 = -1, \quad ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2 \quad (2.1.2)$$

With Euclidean signature both coordinate choices cover all of hyperbolic space. In Lorentzian signature they cover different regions of the global space. The hatted versions of the times are Lorentzian e.g. $t = i\hat{t}$. The causal structure of the global space is displayed clearly in the ν, σ coordinates. The \hat{t}, z and $\hat{\tau}, \rho$ coordinates cover different patches as seen in figure 2.1. The $\hat{\tau}, \rho$ coordinates can be viewed as describing the exterior of a finite temperature black hole. We can also view them as Rindler coordinates of AdS_2 . Note that the finite temperature and zero temperature solutions are just different coordinate patches of the same space.

2.1.2 Symmetries and a family of solutions

Let us imagine that we have a spacetime that is exactly AdS_2 , with a finite Newton constant. Then the gravitational action is

$$I = -\frac{\phi_0}{2} \left[\int d^2x \sqrt{g} R + 2 \int K \right] + I_m[g, \chi] \quad (2.1.3)$$

where I_m is the matter action and χ are the matter fields. Here ϕ_0 is a constant, which sets the entropy $S_0 = 2\pi\phi_0$.

We now want to imagine a situation where this spacetime arises as a low energy limit of a well defined UV theory. For this purpose we imagine that we cut off the spacetime. The UV theory has some time coordinate u . Throughout the paper, we denote the time in the boundary theory by u . Let us say that the (Euclidean) AdS_2 spacetime has the metric in (2.1.1). We want to cut off the space along a trajectory given by $(t(u), z(u))$. We expect to fix the proper length of the boundary curve

$$g|_{\text{bdy}} = \frac{1}{\epsilon^2}, \quad \frac{1}{\epsilon^2} = g_{uu} = \frac{t'^2 + z'^2}{z^2} \longrightarrow z = \epsilon t' + O(\epsilon^3) \quad (2.1.4)$$

where primes are u derivatives. Note that, given an arbitrary $t(u)$, we can choose $z(u) = \epsilon t'(u)$ in order to obey the above equations. Since all other fields are constant on the AdS_2 vacuum, when we set the boundary conditions for all the fields to be such constants, we will obey all other boundary conditions. Therefore we find that we have a family of solutions to the problem, given by $t(u)$.

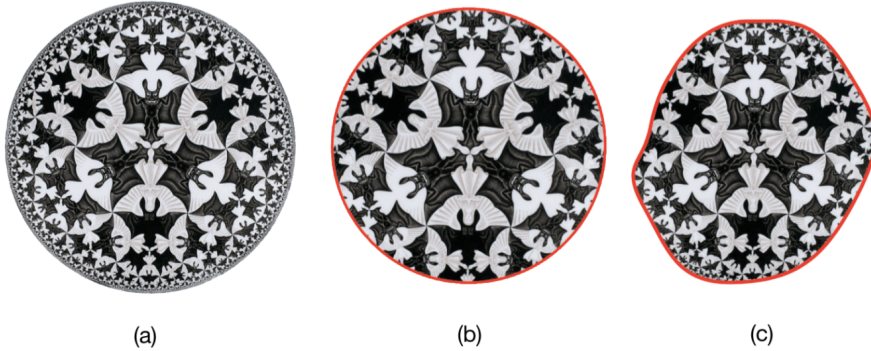


Figure 2.2: In (a) we see the full AdS_2 space. In (b) we cut it off at the location of a boundary curve. In (c) we choose a more general boundary curve. The full geometry of the cutout space does depend on the choice of the boundary curve. On the other hand, the geometry of this cutout region remains the same if we displace it or rotate it by an $SL(2)$ transformation of the original AdS_2 space.

Let us clarify in what sense these are different solutions. The main point is that we are cutting out a region of AdS_2 , with different shapes that depend on the function $t(u)$, see figure 2.2. Though the interior AdS_2 space is locally the same, the full cutout shape does depend on $t(u)$. For example, correlation functions of matter fields will depend on the shape chosen by the function $t(u)$. Note however, that overall translations or rotations of the whole shape in hyperbolic space do not change the physics. These are described by the action of an $SL(2)$ symmetry group on AdS_2 . It acts by sending

$$t(u) \rightarrow \tilde{t}(u) = \frac{at(u) + b}{ct(u) + d}, \quad \text{with } ad - cb = 1 \quad (2.1.5)$$

We see that $t(u)$ or $\tilde{t}(u)$ produce exactly the same cutout shape. Therefore the full set of different interior geometries is given by the set of all functions $t(u)$ up to the above $SL(2)$ transformations. (Or modded out by these $SL(2)$ transformations (2.1.5)).

It is worth noting that we can also look at the asymptotic symmetries of AdS_2 . They are generated by reparametrizations of the asymptotic form

$$\zeta^t = \varepsilon(t), \quad \zeta^z = z\varepsilon'(t) \tag{2.1.6}$$

These will map one boundary curve into another. In fact, (2.1.6) sends the curve $t(u) = u$ to $t(u) = u + \varepsilon(u)$.

If we insert these geometries into the action (2.1.3) the Gauss-Bonnet theorem implies that we always get the same action, namely the extremal entropy. Thus we have a set of exact zero modes parametrized by $t(u)$ (up to the $SL(2)$ identification (2.1.5)).

Notice that, near the boundary, the geometries are indistinguishable, we need to go through the bulk in order to distinguish them. In fact, this is the realization of the full reparametrization symmetry that we expect in this problem. In other words, we expect that $SL(2)$ is enhanced to a full Virasoro like symmetry, which in this case, are just the reparametrization symmetries. However, the reparametrization symmetry is spontaneously broken by AdS_2 . It is broken to $SL(2, R)$. The zero modes are characterized by the functions $t(u)$. These can be viewed as Goldstone bosons. Except that here we consider them in the Euclidean problem. We can call these zero modes “boundary gravitons”. They are similar to the ones that appear in three dimensions. An important difference with the three dimensional case is that, here, these modes have precisely zero action in the conformal limit, there is no local conformal invariant action we can write down for them.

2.2 $NAdS_2$, or nearly AdS_2 spacetimes

The pure AdS_2 gravity theory discussed above is not consistent with any configuration with non-zero energy, since the variation of the metric imposes that the stress tensor of matter is identically zero. The Einstein term is topological and does not contribute to the equation of motion for the metric. If one is only interested in understanding the ground state entropy this can be enough [30, 31].

In order to obtain a reasonable gravity theory it is important to consider a nearly AdS_2 geometry. In other words, we need to keep track of the leading effects that break the conformal symmetry. This is a configuration that still remembers that the conformal symmetry is slightly broken. A model that correctly captures a large number of situations where AdS_2 arises from a higher dimensional system (or from some otherwise well defined UV theory) is the following [32]

$$I = -\frac{\phi_0}{2} \left[\int \sqrt{g} R + 2 \int_{bdy} K \right] - \frac{1}{2} \left[\int d^2x \phi \sqrt{g} (R + 2) + 2 \int_{bdy} \phi_b K \right] + I_M[g, \chi] + \dots \quad (2.2.7)$$

Here we imagine that $\phi_0 \gg \phi$ and the dots denote higher order terms in ϕ . We will neglect all such higher order terms here. ϕ_b is the boundary value of ϕ . If AdS_2 is arising from the near horizon geometry of a near extremal black hole, then $\phi_0 + \phi$ is the area of the two sphere, and ϕ_0 is the area of the extremal black hole, with ϕ denoting the deviations from this extremal value. The middle term in the action is the Jackiw Teitelboim two dimensional gravity theory [33, 34]. The first term is purely topological and its only role is to give the extremal entropy. We have included the extrinsic curvature terms at the boundary to make the metric variational problem well defined. From now on, we will ignore the dots in (2.2.7). Since the first term in the action is topological we will also ignore it.

A thorough analysis of this model was presented in an article by Almheiri and Polchinski [32]. Here we simply emphasize how the pattern of breaking of the reparametrization symmetry determines many aspects of the theory. Now, let us analyze the equations of motion of the Jackiw Teitelboim theory

$$I_{JT} = -\frac{1}{2} \left[\int d^2x \phi \sqrt{g} (R + 2) + 2 \int_{bdy} \phi_b K \right]. \quad (2.2.8)$$

The equations of motion for ϕ imply that the metric has constant negative curvature or is AdS_2 . This is also the case if we include the matter term in (2.2.7) since it is independent of the dilaton ϕ . The equations of motion for the metric are

$$T_{\mu\nu}^\phi \equiv \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi + g_{\mu\nu} \phi = 0 \quad (2.2.9)$$

Due to the Bianchi identity, this implies that $T_{\mu\nu}^\phi$ is automatically conserved. It turns out that the general solution is

$$\phi = \frac{\alpha + \gamma t + \delta(t^2 + z^2)}{z} = Z.Y \quad (2.2.10)$$

where we also rewrote the expression in embedding coordinates (2.1.2), where Z is an arbitrary vector.¹

The solution breaks the $SL(2)$ isometries to $U(1)$. In fact, the vector $\zeta^\mu = \epsilon^{\mu\nu} \partial_\nu \phi$ is always a Killing vector of the metric thanks to the equations (2.2.9) [35]. Thus, the combined dilaton gravity theory always preserves this isometry.

Since ϕ is diverging near the boundary, we now have a new dimensionful coupling constant which is the strength of that divergence. In other words, beyond the condition

¹More precisely, in (2.2.10) we use the Euclidean version of the embedding coordinates.

(2.1.4) we also need to impose the condition

$$\phi_b = \phi|_{\text{bdy}} = \frac{\phi_r(u)}{\epsilon} \quad (2.2.11)$$

where $\phi_r(u)$ is an arbitrary dimension -1 coupling. The r stands for “renormalized,” in the sense that it remains finite in the $\epsilon \rightarrow 0$ limit. For generality we have chosen it to depend on u , but we could also choose it to be independent of u . When we choose it to be constant we will denote it as $\bar{\phi}_r$.

When we embed this into a full higher dimensional picture, we have in mind situations where $\phi_b \propto 1/\epsilon$ is large, but $\phi_b \ll \phi_0$ so that we are still in the near extremal region.² In other words, we cut off the space before ϕ becomes too large. Note that the notion of “too large” is really external to the theory (2.2.8).

Now, once we impose both (2.1.4) and (2.2.11) we determine completely the shape of the curve or reparametrization $t(u)$. It is simply given by computing $z(u)$ from (2.1.4), substituting in (2.2.10) and then using (2.2.11) to obtain

$$\frac{\alpha + \gamma t(u) + \delta t(u)^2}{t'(u)} = \phi_r(u). \quad (2.2.12)$$

It is interesting that this condition can also be obtained from an effective action for $t(u)$. A simple way to obtain the effective action is the following. Starting from (2.2.8) we impose the equation of motion for ϕ which implies that we have an AdS_2 space. Inserting that into the action (2.2.8) we find that the first term vanishes and we only get the boundary term, which involves the boundary value of ϕ (2.2.11),

$$I_{TJ} \rightarrow - \int \frac{du}{\epsilon} \frac{\phi_r(u)}{\epsilon} K \quad (2.2.13)$$

² This type of expansion is somewhat analogous to the slow roll expansion for inflationary universes.

where we also used that the induced metric is given by du/ϵ , (2.1.4). The extrinsic curvature is given by

$$\begin{aligned} K &= \frac{t'(t'^2 + z'^2 + zz'') - zz't''}{(t'^2 + z'^2)^{\frac{3}{2}}} = 1 + \epsilon^2 \text{Sch}(t, u) , \\ \text{Sch}(t, u) &\equiv -\frac{1}{2} \frac{t''^2}{t'^2} + \left(\frac{t''}{t'} \right)' \end{aligned} \quad (2.2.14)$$

Inserting this into I_{TJ} we get

$$I = - \int du \phi_r(u) \text{Sch}(t, u) \quad (2.2.15)$$

We see that the zero modes get an action determined by the Schwarzian. Here $\phi_r(u)$ is an external coupling and $t(u)$ is the field variable.

It is interesting to contemplate why we obtained this. We expect that the breaking of conformal symmetry should be local along the boundary, and proportional to $\phi_r(u)$. In addition, we expect to obtain a local action which involves the Pseudo-Nambu Goldstone modes. Since these are specified by $t(u)$ up to global $SL(2)$ transformations, we conclude that the simplest term is the Schwarzian action, which is indeed $SL(2)$ invariant; $\text{Sch}(t, u) = \text{Sch}\left(\frac{at+b}{ct+d}, u\right)$.

Finally, it is easy to check that by varying (2.2.15) with respect to $t(u)$ we obtain the equation

$$\left[\frac{1}{t'} \left(\frac{(t' \phi_r)'}{t'} \right) \right]' = 0 \quad (2.2.16)$$

which can be easily integrated to (2.2.12), where α, γ, δ are integration constants.³ Thus we see that the action (2.2.15), which is defined purely on the boundary, captures the same information as the bulk expression for the dilaton ϕ . Notice that this also implies that

³A fourth integration constant arises by integrating (2.2.12).

the equations of motion of the action (2.2.15) are equivalent to imposing the equations of motion that result from varying the metric, which were not imposed in deriving (2.2.15). The time dependence of $\phi_r(u)$ allows us to pick an arbitrary $t(u)$ as the saddle point geometry. On the other hand, we can also remove it by picking a new time coordinate via $d\tilde{u} = \bar{\phi}_r du / \phi_r(u)$. When $\phi_r(u)$ is constant (2.2.16) becomes $\bar{\phi}_r \frac{[\text{Sch}(t,u)]'}{t'} = 0$.

The Schwarzian action summarizes many gravitational effects of the model. As we have explained, it follows from the symmetries of the problem and its applicability can go beyond systems that are described by a local gravity theory. In fact, this Schwarzian action was introduced, for these reasons, by Kitaev in his analysis of certain interacting fermion models [36] (see [22] for a description).

2.2.1 The near extremal entropy

It is convenient to make a change of field variable in the Schwarzian action from t to τ of the form

$$t = \tan \frac{\tau}{2}. \quad (2.2.17)$$

We can then use the general transformation rule for the Schwarzian to find

$$\begin{aligned} I &= -C \int du \text{Sch}(t, u) = -C \int du \left[\text{Sch}(\tau, u) + \tau'^2 \text{Sch}(t, \tau) \right] \\ &= -C \int du \left[\text{Sch}(\tau, u) + \frac{1}{2} \tau'^2 \right], \quad C \equiv \bar{\phi}_r. \end{aligned} \quad (2.2.18)$$

We could have derived this form of the action by starting with AdS_2 in terms of the coordinates $ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2$ and viewing the boundary as parametrized by $\tau(u)$, with $\rho(u)$ determined by the analog of (2.1.4).

This is an interesting action, whose solutions are $\tau = \frac{2\pi}{\beta} u$ (up to $SL(2)$ transformations). Note that $\tau \sim \tau + 2\pi$. For these solutions, only the term involving τ'^2 in (2.2.18) is

important. On such solutions the action gives

$$\log Z = -I = 2\pi^2 \frac{C}{\beta} = 2\pi^2 CT \tag{2.2.19}$$

which leads to a near extremal entropy $S = S_0 + 4\pi^2 CT$ which is linear in the temperature. Note that $4\pi^2 CT$ is also the specific heat. This linear in T behavior is a simple consequence of the reparametrization symmetry and its breaking.

This gives us only the near extremal entropy. The extremal entropy, S_0 , can be obtained by adding a purely topological term to the above action of the form

$$-I_{\text{top}} = \phi_0 \int du \tau'. \tag{2.2.20}$$

It might seem unusual that we reproduce the entropy from a classical action. This is familiar from the bulk point of view, but it seems unusual to reproduce it from a boundary-looking action. However, this is common in discussions of hydrodynamics. In that case, the free energy is reproduced from a classical action. Here the crucial feature is that the solution depends on the temperature through the condition that τ winds once as we go from $u = 0$ to $u = \beta$.

We could wonder whether we should consider solutions where τ winds n times, $\tau = n \frac{2\pi}{\beta} u$. It appears that this effective action makes sense only for the case with winding number one.⁴

Note that this is not a microscopic derivation of the entropy. This is simply phrasing the computation of the entropy as a consequence of a symmetry. We have not given an explicit description of the black hole microstates. If one had a microscopic system which

⁴ For $n = 0$ the τ' terms in the numerator are a problem. For $n > 1$ the small fluctuations around the solution have negative modes.

displayed this symmetry breaking pattern, then we would microscopically explain the form of the entropy.

It is also possible to compute the ADM energy of the system. This is given in terms of the boundary values of the fields. In this case, we get [32]

$$M = \frac{1}{\epsilon}[\phi_b - \partial_n \phi] = \bar{\phi}_r \text{Sch}(t, u) = C \text{Sch}(t, u) = -C \text{Sch}(\hat{t}, \hat{u}) \quad (2.2.21)$$

The second expression is giving the mass terms of the Schwarzian action. This can be obtained by either solving the equations for ϕ or by deriving the conserved quantity associated u translations for (2.2.15). It is valid in the absence of boundary sources for massive fields. The general formula is given in (A.1.11).

2.3 Adding matter

We can now add matter as in (2.2.7). Since ϕ does not appear in the matter action, the metric is still fixed to the AdS_2 metric by the ϕ equation of motion, and the matter fields move on this fixed AdS_2 geometry. The gravitational backreaction is completely contained in the equations obeyed by the dilaton, which are simply (2.2.9) but with the matter stress tensor in the right hand side⁵. These are three equations for a single variable ϕ , but the conservation of the matter stress tensor implies that the equations are consistent. The boundary is located by finding the curve where $\phi = \phi_b$. This can be done by first solving for ϕ in the bulk as described above and then finding the trajectory of the boundary curve. Alternatively, one can show that the final equation for the trajectory is given by an equation where we add a new term in the right hand side of (2.2.16). For the case of

⁵This structure is similar to other models of dilaton gravity where the metric is forced to be flat, instead of AdS_2 , see [37] for a review.

massless matter fields we obtain

$$C \frac{(\text{Sch}(t, u))'}{t'} = -t' T_{tz} \quad (2.3.22)$$

A simple derivation is obtained by equating the change in energy (2.2.21) to the flux of energy, $-t'^2 T_{tz}$, into the space. A factor of t' comes from redshifting the energy from t to u time and another factor from going from energy per unit t to energy per unit u ⁶. Solving (2.3.22) we find $t'(u)$ and solve directly for the trajectory of the boundary curve. The correction to (2.3.22) when we have sources for massive fields is given in appendix A.1.

For correlation function computations it is useful to calculate the effective action as a function of the boundary conditions for the matter fields, $\chi_r(u)$, which can be functions of the boundary time.

It is convenient to solve first an auxiliary problem, which consists of finding the effective action for the matter fields in AdS , with boundary conditions $\tilde{\chi}_r(t)$ which are functions of the AdS_2 boundary time. For a free field in AdS_2 this is simple to compute and we obtain

$$-I_{eff} = D \int dt dt' \frac{\tilde{\chi}_r(t) \tilde{\chi}_r(t')}{|t - t'|^{2\Delta}} \quad (2.3.23)$$

where D is a constant.⁷ Once we specify the trajectory of the actual boundary curve via $t(u)$ we can transform this to the desired boundary conditions

$$\chi \sim z^{1-\Delta} \tilde{\chi}_r(t) = \epsilon^{1-\Delta} t'^{1-\Delta} \tilde{\chi}_r(t) = \epsilon^{1-\Delta} \chi_r(u) \longrightarrow \chi_r(u) = [t'(u)]^{1-\Delta} \tilde{\chi}_r(t(u)). \quad (2.3.24)$$

⁶In Lorentzian signature we get a minus sign in (2.3.22) from the minus in (2.2.21).

⁷ $D = \frac{(\Delta - \frac{1}{2})\Gamma(\Delta)}{\sqrt{\pi}\Gamma(\Delta - \frac{1}{2})}$, or $D = 1/2\pi$ for $\Delta = 1$.

The first expression defines $\tilde{\chi}_r(t)$, then we used the expression for z from (2.1.4), and finally we compared it to the expression for χ that defines $\chi_r(u)$. Using (2.3.24), we rewrite (2.3.23) as

$$-I_{eff} = D \int dud u' \left[\frac{t'(u)t'(u')}{[t(u) - t(u')]^2} \right]^\Delta \chi_r(u)\chi_r(u'). \quad (2.3.25)$$

Though we did this for the two point function, the same is true for any n point function. If we had a self interacting matter theory in AdS_2 and we computed the AdS_2 n point correlation function, then the correct physical one in $NAdS_2$, after coupling to gravity, would be obtained by writing them using $t(u)$, and rescaling by a factor of $t'(u_i)^{\Delta_i}$ at the insertion of each operator.⁸ Even if we had free fields in AdS_2 this coupling to gravity makes them interact with each other. What is remarkable is the simplicity of this coupling.

In this way we have found a coupling between $t(u)$ and the matter action. We see that the coupling proceeds by a reparametrization of the original two point function. The full correlation functions are obtained by integrating over $t(u)$, after we add the Schwarzian action (2.2.15). These are the same formulas derived in [32]. The classical equations for $t(u)$ that follow from the variation of the Schwarzian action, (2.2.15), plus (2.3.25) are the same as the ones in [32].

2.3.1 Perturbative expansion of the Schwarzian action

Since the Schwarzian action is of order C we can evaluate its effects using perturbation theory around a solution. To avoid carrying unnecessary factors of the temperature we set $\beta = 2\pi$. The factors of temperature can be reinstated by dimensional analysis. We then

⁸ We can say that if $Z_M[\tilde{\chi}_r(t), z(t)]$ is the partition function of the pure matter theory with boundary conditions at $z = z(t)$, then the one in the theory coupled to dilaton gravity is $Z_{Dressed}[\chi_r(u), \epsilon] = Z_M[t'^{\Delta-1}\chi(u), z(t(u))]$, where $z = \epsilon t' + \dots$.

set

$$\tau = u + \varepsilon(u) \tag{2.3.26}$$

in (2.2.18) and expand to second order in ε to obtain

$$I_\epsilon = \frac{C}{2} \int du [\varepsilon''^2 - \varepsilon'^2], \quad \text{for } \beta = 2\pi \tag{2.3.27}$$

We would like to compute the propagator for this action. A problem is that the action has three zero modes, going like $\epsilon = 1, e^{iu}, e^{-iu}$. These zero modes arise from $SL(2)$ transformations of the background solution $\tau = u$. Recall that these $SL(2)$ transformations did not generate new geometries. Therefore we should not be integrating over them in the first place, since the integral over ε is only over distinct geometries. This is equivalent to viewing the $SL(2)$ symmetry as a gauge symmetry, so that we can gauge fix those three zero modes to zero and invert the propagator. The answer is

$$\langle \epsilon(u)\epsilon(0) \rangle = \frac{1}{2\pi C} \left[-\frac{(|u| - \pi)^2}{2} + (|u| - \pi) \sin |u| + a + b \cos u \right] \tag{2.3.28}$$

The last two terms are proportional to $SL(2)$ zero modes and cancel in any gauge invariant computation.⁹ We will now use this propagator for some computations. The effective coupling is β/C which is the same as the inverse of the near extremal entropy.¹⁰

2.3.2 Gravitational contributions to the four point function

Suppose that we have operators V, W , which are dual to two different fields which are free in AdS_2 before coupling to gravity. The gravitational contribution to the four point function can be computed as follows. (Some four point functions were also considered

⁹ A direct inversion of the operator gives $a = 1 + \pi^2/6$ and $b = 5/2$ [22].

¹⁰This is the reason that there is trouble with naive black hole thermodynamics at $C/\beta \sim 1$ [38].

in [32]. These steps are identical to the ones discussed in [22], since the effective action is the same.) We start from the factorized expression for the four point function, $\langle V(t_1)V(t_2)W(t_3)W(t_4) \rangle = \frac{1}{t_{12}^{2\Delta}} \frac{1}{t_{34}^{2\Delta}}$. We then insert the reparametrizations (2.2.17) and (2.3.26) into (2.3.25) and expand to linear order in ε to obtain

$$\frac{1}{t_{12}^{2\Delta}} \longrightarrow \mathcal{B}(u_1, u_2) \frac{\Delta}{[2 \sin \frac{u_{12}}{2}]^{2\Delta}}, \quad \mathcal{B}(u_1, u_2) \equiv \left[\varepsilon'(u_1) + \varepsilon'(u_2) - \frac{\varepsilon(u_1) - \varepsilon(u_2)}{\tan \frac{u_{12}}{2}} \right]. \quad (2.3.29)$$

We make a similar replacement for $t_{34}^{-2\Delta}$, and then contract the factors of ε using the propagator (2.3.28). This gives the $O(1/C)$ contribution to the four point function. Note that the bilocal operator \mathcal{B} is $SL(2)$ invariant.¹¹ The final expression depends on the relative ordering of the four points. When $u_4 < u_3 < u_2 < u_1$ we obtain the factorized expression

$$\frac{\langle V_1 V_2 W_3 W_4 \rangle_{\text{grav}}}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \Delta^2 \langle \mathcal{B}(u_1, u_2) \mathcal{B}(u_3, u_4) \rangle = \frac{\Delta^2}{2\pi C} \left(-2 + \frac{u_{12}}{\tan \frac{u_{12}}{2}} \right) \left(-2 + \frac{u_{34}}{\tan \frac{u_{34}}{2}} \right) \quad (2.3.30)$$

As discussed in [22], this expression can be viewed as arising from energy fluctuations. Each two point function generates an energy fluctuation which then affects the other. Since energy is conserved, the result does not depend on the relative distance between the pair of points. In other words, we can think of it as

$$\langle V_1 V_2 W_3 W_4 \rangle_{\text{grav}} = \partial_M \langle V_1 V_2 \rangle \partial_M \langle W_3 W_4 \rangle \frac{1}{-\partial_M^2 S(M)} = \partial_\beta \langle V_1 V_2 \rangle \partial_\beta \langle W_3 W_4 \rangle \frac{1}{\partial_\beta^2 \log Z(\beta)} \quad (2.3.31)$$

where M is the mass of the black hole background, or β its temperature, and $S(M)$ or $\log Z$ are its entropy or partition function.¹² Both expressions give the same answer, thanks to

¹¹ Indeed a and b in (2.3.28) disappear if we consider $\langle \varepsilon(u) \mathcal{B}(u_1, u_2) \rangle$.

¹² The correlator at finite β is $\langle VV \rangle = \left[\frac{\beta}{\pi} \sin \frac{\pi u_{12}}{\beta} \right]^{-2\Delta}$.

thermodynamic identities between entropy and mass.¹³ If one expands as $u_{12} \rightarrow 0$ we get a leading term going like u_{12}^2 which one would identify with an operator of dimension two. In this case this is the Schwarzian itself, which is also the energy and it is conserved (2.2.21). Its two point functions are constant.¹⁴

It is also interesting to evaluate the correlator in the other ordering $u_4 < u_2 < u_3 < u_1$. We get

$$\begin{aligned} \frac{\langle V_1 W_3 V_2 W_4 \rangle_{\text{grav}}}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} &= \frac{\Delta^2}{2\pi C} \left[\left(-2 + \frac{u_{12}}{\tan \frac{u_{12}}{2}} \right) \left(-2 + \frac{u_{34}}{\tan \frac{u_{34}}{2}} \right) + \right. \\ &+ \left. \frac{2\pi [\sin(\frac{u_1 - u_2 + u_3 - u_4}{2}) - \sin(\frac{u_1 + u_2 - u_3 - u_4}{2})]}{\sin \frac{u_{12}}{2} \sin \frac{u_{34}}{2}} + \frac{2\pi u_{23}}{\tan \frac{u_{12}}{2} \tan \frac{u_{34}}{2}} \right] \end{aligned} \quad (2.3.32)$$

This expression interpolates between (2.3.30) when $u_3 = u_2$ and an expression like (2.3.30), but with $u_{34} \rightarrow -2\pi + u_{34}$, when $u_3 = u_1$. Note that now the answer depends on the overall separation of the two pairs. This dependence, which involves the second sine term in the numerator as well as the u_{23} factor, looks like we are exciting the various zero modes of the Schwarzian action, including the exponential ones. It is interesting to continue (2.3.32) to Lorentzian time and into the chaos region which involves the correlator in the out of time order form

$$\langle V(a)W_3(b + \hat{u})V(0)W(\hat{u}) \rangle \sim \frac{\beta \Delta^2}{C} e^{\frac{2\pi \hat{u}}{\beta}}, \quad \frac{\beta}{2\pi} \ll \hat{u} \ll \frac{\beta}{2\pi} \log \frac{C}{\beta} \quad (2.3.33)$$

where $a, b \sim \beta$. Here we restored the temperature dependence in (2.3.32) by multiplying by an overall a factor of $\frac{\beta}{2\pi}$ and sending $u_i \rightarrow \frac{2\pi}{\beta} u_i$.

We can also connect (2.3.33) to a scattering process. It is peculiar that in this setup the two particles do not scatter since they behave like free fields on a fixed AdS_2 background.

¹³(2.3.31) is valid for a general spherically symmetric reduction of general relativity to two dimensions.

¹⁴ Note that this is different than in 1+1 dimensions, where the stress tensor correlators go like $1/z^4$.

On the other hand, they create a dilaton profile which gives rise to a non-trivial interaction once we relate the AdS_2 time to the boundary time. The net result is the same as what is usually produced by the scattering of shock waves, see appendix A.2. Here we see that the gravitational effects are very delocalized, we can remove them from the bulk and take them into account in terms of the boundary degree of freedom $t(u)$.

2.3.3 Loop corrections

We can use the Schwarzian action as a full quantum theory and we can compute loop corrections. The simplest example corresponds to the one loop correction to the free energy [39]. This arises from computing the functional determinant of the quadratic operator in (2.4.38). This was done in detail in [22] and we will not repeat the details. The important point is simply that it gives a temperature dependent correction to the free energy going like

$$\log Z|_{\text{one loop}} = -\frac{3}{2} \log \left(\frac{\beta}{C} \right) \quad (2.3.34)$$

This is a correction to the leading classical expression (2.2.19). The determinants of all matter fields in AdS_2 are conformally invariant and should not give rise to a temperature dependent contribution, but they can and do contribute to the extremal entropy [40, 41]. The correction (2.3.34) is such that there is no logarithmic correction to the entropy as a function of energy. This is good, since there are cases, such as BPS black holes where do not expect corrections that diverge in the IR.

As a second example we can consider a loop correction to the two point function. We expand the reparameterization

$$\frac{1}{t_{12}^{2\Delta}} \rightarrow \frac{(1 + \varepsilon'_1)^\Delta (1 + \varepsilon'_2)^\Delta}{\left(\sin \frac{u_{12} + \varepsilon_1 - \varepsilon_2}{2} \right)^{2\Delta}} \quad (2.3.35)$$

to quadratic order in the ε , and then contract using the propagator (2.3.28)

$$\begin{aligned}
\frac{\langle V_1 V_2 \rangle_{\text{one loop}}}{\langle V_1 V_2 \rangle_{\text{tree}}} &= \Delta \left\langle \frac{(\varepsilon_1 - \varepsilon_2)^2}{4 \sin^2 \frac{u}{2}} - \frac{1}{2} (\varepsilon_1'^2 + \varepsilon_2'^2) \right\rangle + \frac{\Delta^2}{2} \left\langle \left(\varepsilon_1' + \varepsilon_2' - \frac{(\varepsilon_1 - \varepsilon_2)}{\tan \frac{u}{2}} \right)^2 \right\rangle \\
&= \frac{1}{2\pi C} \left[\Delta \frac{(u^2 - 2\pi u + 2 - 2 \cos u + 2(\pi - u) \sin u)}{4 \sin^2 \frac{u}{2}} + \right. \\
&\quad \left. + \frac{\Delta^2}{2} \left(-2 + \frac{u}{\tan \frac{u}{2}} \right) \left(-2 + \frac{(u - 2\pi)}{\tan \frac{u}{2}} \right) \right], \quad u = u_1 - u_2 > 0 \quad (2.3.36)
\end{aligned}$$

It is interesting to continue these formulas to Lorentzian signature $u \rightarrow i\hat{u}$ and then expand them for large Lorentzian times. The largest term, which goes at \hat{u}^2 for large Lorentzian time can arise from energy fluctuations in a manner analogous to (2.3.31). The tree level correlator includes a quasinormal decay as $e^{-\Delta\hat{\tau}} \sim e^{-\Delta 2\pi\hat{u}/\beta}$. But the energy fluctuations cause a temperature fluctuation which would then lead to a correction for the ratio of the one loop to tree level as

$$\frac{\Delta^2}{2} \frac{4\pi^2 \hat{u}^2}{\beta^4} \frac{1}{\partial_\beta^2 \log Z} = \frac{\Delta^2 \hat{u}^2}{2\beta C} \quad (2.3.37)$$

which agrees with the \hat{u}^2 piece from (2.3.36).¹⁵

2.4 Lorentzian picture and the SL(2) symmetry

2.4.1 SL(2) symmetry of the Schwarzian action

We have seen that gravitational effects are summarized by the Schwarzian action (2.2.15). This action seems problematic when viewed as an action in Lorentzian signature since it involves higher derivative terms. These usually lead to ghosts. We can see this more

¹⁵ In fact, (2.3.36) can be written exactly as $\langle V_1 V_2 \rangle_{\text{one loop}} = \frac{1}{2\partial_\beta^2 \log Z} \partial_\beta \left\{ [\partial_\beta \langle V_1 V_2 \rangle_{\text{tree}}]_{u \rightarrow \beta - u} \right\}$. The derivatives seem to be a way of varying the temperature in a way that maintains periodicity.

explicitly by starting with the Lorentzian action for small fluctuations

$$iI_L = -iC \int d\hat{u} \text{Sch}(\hat{t}, \hat{u}) = i\frac{C}{2} \int d\hat{u} (\varepsilon''^2 + \varepsilon'^2), \quad \text{for } \beta = 2\pi, \quad C \equiv \frac{\bar{\phi}_r}{8\pi G} \quad (2.4.38)$$

It is possible to rewrite this higher derivative action in terms of a two derivative action for two fields by introducing a new field η

$$\int d\hat{u} [\varepsilon''^2 + \varepsilon'^2] \rightarrow \int d\hat{u} \left[\eta (\varepsilon'' - \frac{\varepsilon}{2}) - \frac{\eta^2}{4} - \frac{\varepsilon^2}{4} \right] = \int d\hat{u} [-r'^2 - r^2 + q'^2]$$

$$\varepsilon = r + q, \quad \eta = r - q \quad (2.4.39)$$

Integrating out η we get $\eta = 2\varepsilon'' - \varepsilon$ and recover the original action, (2.4.38). We can also use this expression for η to express r and q in terms of ε which gives

$$r = \varepsilon'', \quad q = -\varepsilon'' + \varepsilon \quad (2.4.40)$$

The full set of solutions of the original Lorentzian action (2.4.38) is given by

$$\varepsilon = (\alpha e^{\hat{u}} + \beta e^{-\hat{u}}) + (\gamma \hat{u} + \delta) \quad (2.4.41)$$

We see that the first parenthesis corresponds to the ghost like mode r and the last one to the mode q . Note that in Euclidean space we started out with a full function worth of nearly zero modes, but in the Lorentzian theory these only give rise to the two degrees of freedom r and q .

Should we be worried by the appearance of the ghost like mode that has a negative sign in its kinetic term in (2.4.39)? Should we view the exponentially growing solutions in (2.4.41) as an instability? To answer these questions we need to recall that the original metric had an unbroken $SL(2)$ symmetry. Such $SL(2)$ diffeomorphisms do not generate a

new cutout geometry. Thus, we should not include them in our integration over Pseudo-Goldstone modes. One way to remove them is to treat such diffeomorphisms as a gauge symmetry. More precisely, the Schwarzian action has an $SL(2)$ global symmetry. This global symmetry has its associated conserved charges

$$\begin{aligned}
Q^- &= C \left[\frac{\hat{t}'''}{\hat{t}'^2} - \frac{\hat{t}''^2}{\hat{t}'^3} \right] = C e^{-\hat{\tau}} \left[\frac{\hat{\tau}'''}{\hat{\tau}'^2} - \frac{\hat{\tau}''^2}{\hat{\tau}'^3} + \frac{\hat{\tau}''}{\hat{\tau}'} \right], \\
Q^0 &= C \left[\frac{\hat{t}'''\hat{t}}{\hat{t}'^2} - \frac{\hat{t}\hat{t}''^2}{\hat{t}'^3} - \frac{\hat{t}''}{\hat{t}'} \right] = C \left[\frac{\hat{\tau}'''}{\hat{\tau}'^2} - \frac{\hat{\tau}''^2}{\hat{\tau}'^3} - \hat{\tau}' \right] \\
Q^+ &= C \left[\frac{\hat{t}'''\hat{t}^2}{\hat{t}'^2} - \frac{\hat{t}''^2\hat{t}^2}{\hat{t}'^3} - \frac{2\hat{t}\hat{t}''}{\hat{t}'} + 2\hat{t}' \right] = C e^{\hat{\tau}} \left[\frac{\hat{\tau}'''}{\hat{\tau}'^2} - \frac{\hat{\tau}''^2}{\hat{\tau}'^3} - \frac{\hat{\tau}''}{\hat{\tau}'} \right]
\end{aligned} \tag{2.4.42}$$

where we also wrote it after setting $\hat{t} = e^{\hat{\tau}}$, which is appropriate for Lorentzian finite temperature computations. Treating them as gauge symmetry amounts to saying that the full state should be invariant under these symmetries. However, we see that a solution with nonzero $\hat{\tau}'$ cannot have zero charges! Recall, though, that in the bulk this $SL(2)$ symmetry acts on the full AdS_2 spacetime. This means that it is a symmetry of the thermofield double. In the quantum mechanical description of the thermofield double we have two sides and the charges are equal and opposite on the two sides. $Q_L^a = -Q_R^a$, so that the total charge can be zero. Therefore, purely on one side the charges can be anything. We can view the charges Q^a as proportional to the vector Z^a in (2.2.10) that determines the location of the bifurcation point. The $SL(2)$ transformations move this point in AdS_2 . This motion has no physical consequence because the location of the boundary is determined by the value of the dilaton and thus the boundary curve moves together with the bifurcation point as we perform an $SL(2)$ transformation.

For the simplest solution $\hat{\tau} = \frac{2\pi}{\beta}u$, the charges are

$$Q^\pm = 0, \quad Q^0 = -C \frac{2\pi}{\beta} \tag{2.4.43}$$

This value of Q^0 (when $Q^\pm = 0$) can be viewed as (minus) the near extremal entropy of the black hole. More precisely $S = -2\pi Q$. As Wald has pointed out [42], we can view black hole entropy as a Noether charge associated to the translation generated by the horizon generating Killing vector.

There is an additional conserved quantity of the Schwarzian action, which is simply associated to u time translations. This is the Hamiltonian discussed in (2.2.21). It is interesting to note the relation

$$H = \frac{1}{2C} [-Q^+Q^- + (Q^0)^2] \quad (2.4.44)$$

between the energy and the charges. Here the Q are the charges of only the t field on one side, as in (2.4.42).

It is also interesting to evaluate the charges and the Hamiltonian for a first order perturbation around the thermal solution, $\hat{\tau} = u + \varepsilon(u)$

$$\begin{aligned} Q^\pm &\sim C e^{\pm u} [\varepsilon''' \mp \varepsilon''] \quad , \quad Q^0 \sim C [-1 + \varepsilon''' - \varepsilon'] \\ H &\sim C \left[\frac{1}{2} - (\varepsilon''' - \varepsilon') \right] \end{aligned} \quad (2.4.45)$$

With these expressions we see that the zero mode $\varepsilon = e^u$ only contributes to Q^- and $\varepsilon = e^{-u}$ only to Q^+ . Two point functions of H are constant, as expected for a conserved quantity.¹⁶ Saying that we treat the $SL(2)$ symmetry as a gauge symmetry implies that we are not free to excite these modes. These modes are excited in an amount that is set by the value of the charges.

¹⁶Two point functions of the Q^a , such as $\langle Q^a(u)Q^b(0) \rangle$ are *not* constant, despite their classical conservation law. This is due to the fact that we needed to break the $SL(2)$ symmetry to compute the propagator for ε (2.3.28). This is not a problem because these are not gauge invariant quantities.

At this linear order in ε , we can also show that the Hamiltonian has the expected commutation relation with operators. From (2.4.45) and (2.4.40) we have $H \sim C(\frac{1}{2} + q')$. Assuming a canonical quantization of the non-ghost mode q , we conclude that $[H, \varepsilon] = C[q', q] = -i$ and $[H, \varepsilon'] = 0$. To evaluate the commutator with V , we include the reparameterization dressing $V \rightarrow (1 + \varepsilon')^\Delta V(u + \varepsilon)$ and then expand to linear order in ε . This immediately gives $[H, V] = -iV'$.

2.4.2 Adding matter

If we have matter in AdS_2 then the matter can also carry $SL(2)$ charges. The total $SL(2)$ charge is the sum of the matter one plus the one carried by the field $\hat{\tau}(\hat{u})$ that appears in the Schwarzian action. For massless matter we simply have

$$Q_T^a = Q^a(\hat{\tau}) + q_M^a \tag{2.4.46}$$

where q_M^a are the standard charges associated to the AdS_2 isometries for the matter fields. The $SL(2)$ gauge symmetry is saying that Q_T will remain constant as we add matter. This is compatible with the equations of motion (2.3.22), and the fact that the $SL(2)$ charges change by a flux of energy; for massless matter we have simply $\partial_{\hat{u}}(Q^-, Q^0, Q^+) = T_{\hat{t}\hat{z}}\hat{t}'(1, \hat{t}, \hat{t}^2)$. When sources for massive fields are turned on, one has to add an extra stress tensor term to q_M to define matter charges that satisfy the correct conservation conditions, see (A.1.13). This $SL(2)$ gauge symmetry implies that we cannot purely excite one of the ghost modes, we have to excite them together with some matter fields.

The total energy is still given by the ADM expression (2.2.21) and it is written purely in terms of the $\hat{\tau}$ variable. In particular, (2.4.44) continues to be true where Q^a in (2.4.44) are the $SL(2)$ charges of the τ system only, they are *not* the total $SL(2)$ charge appearing

in (2.4.46). So we see that the matter inside AdS_2 only carries $SL(2)$ charge and their contribution to the mass only appears through the $SL(2)$ constraints that relate these charges to the $SL(2)$ charge of the $\hat{\tau}$ variable.

Suppose that we start from the thermofield double state and then we add matter on the right part. Then from the right part point of view we can view the charges and masses of the left part of the thermofield double as being carried at the horizon. Then the condition that the total charge vanishes becomes simply the condition that

$$Q_h^a = Q^a + q_M^a = Q_R^a \quad (2.4.47)$$

where Q_R^a are the total charges of the right system, and q_a^M are the $SL(2)$ charge of possible matter falling into the black hole. And $Q_h^a = -Q_L^a$ is the value of the charge at the bifurcation point, and also equal to minus the charges of the left side. Here we assumed that there is no matter on the left side of the spacetime. We have seen that for the simple solution $\tau = u$ the charge $Q^0 < 0$, and it is related to the energy (2.4.44). On the other hand, with the same conventions, the matter charge q_M^0 of a matter particle would be *positive*. That makes (2.4.47) compatible with the energy conservation condition for small fluctuations which says that the mass of the black hole plus the energy of matter should be the same as the energy measured at the boundary.

When we throw matter into the black hole, the values of these charges change, but always in agreement with the conservation law. As we send in matter from the boundary, its additional mass is immediately recorded in the new value for the on shell Schwarzschild. The $SL(2)$ charges of the matter, together with the boundary charges, are constrained to add to the same value that the boundary system had before we threw in the matter. For an initial configuration with $Q^\pm = 0$, the changes of the Q^0 charges demanded by (2.4.47)

can be viewed as a consequence of the first law, once we remember that Q^0 is related to the entropy.

Let us add a classical massive particle following a geodesic in the background AdS_2 spacetime. The equations for this geodesic are given by $A.Y = 0$ in embedding coordinates. If we choose $A^2 = \pm m^2$, then \vec{A} is also proportional to the $SL(2)$ charges. Then the dilaton on the other side of the geodesic is given by $\Phi = (Z + A).Y$, where $Z + A$ reflect the $SL(2)$ charges on the other side of the geodesic.

An important point to note is that the Schwarzian action breaks conformal symmetry for the u time, so that general $SL(2)$ transformation of u , such as $u \rightarrow (au + b)/(cu + d)$ is *not* a symmetry. (Only $u \rightarrow u + \text{constant}$ is a symmetry.) Nevertheless the $SL(2)$ charges acting on t are still conserved, since they are gauge symmetries. In other words, we should not confuse the $SL(2)$ symmetry acting on t , which is gauged and thus unbroken, with the $SL(2)$ symmetry acting on u which is not gauged, and it is broken by the Schwarzian action.

These charges are analogous to the edge modes of the electromagnetic field discussed in [43, 44], or the “center” in [45], or horizon symmetries in [46].

It is likely that there is a more elegant way to think about these $SL(2)$ charges using the $SL(2)$ gauge theory formulation of (2.2.8) [47, 48].

2.5 Higher orders in the chaos region

The order $G \sim 1/C$ term in the four point function (2.3.33) is exponentially growing in the time separation \hat{u} of the V and W operators. However, we actually expect the full correlator to become small at large \hat{u} . This is due to higher order effects in powers of $1/C$. As an application of the Schwarzian action, we will show how to sum powers of $(e^{\hat{u}}/C)$,

which will be enough to capture the late-time decay. More precisely, we work in a limit $C \rightarrow \infty$, $\hat{u} \rightarrow \infty$ with $e^{\hat{u}}/C$ fixed.

2.5.1 Bulk inspiration

The procedure is equivalent to a bulk analysis where the tree-level amplitude is upgraded to an eikonal \mathcal{S} matrix. We will briefly review this analysis (see [11], using [49, 50, 51]) which is very simple in AdS_2 . The V and W operators are represented by bulk quanta with momenta p_- and q_+ , respectively. To capture all powers of $e^{\hat{u}}/C$, one can replace the bulk metric by two effective “shock wave” modes. These are parameterized by shifts X^+ and X^- on the future and past horizons of the black hole. The bulk metric perturbations and stress tensor are

$$h_{++} = 4\delta(x^+)X^-, \quad h_{--} = 4\delta(x^-)X^+, \quad T_{--} = -p_- \delta(x^-), \quad T_{++} = -q_+ \delta(x^+). \quad (2.5.48)$$

Here we are using Kruskal coordinates x^+, x^- to describe the region near the horizon, see (A.2.16). The bulk action $I_{JT} + S_M$ (2.2.7) for these quantities to quadratic order is

$$I_L = -2CX^+X^- - X^+q_+ - X^-p_-. \quad (2.5.49)$$

The integral over X^+, X^- of e^{iI_L} gives the scattering matrix $\mathcal{S} = e^{ip_-q_+/2C}$. The four point function is an in-out overlap with this \mathcal{S} matrix. The essential feature is that at late time the product p_-q_+ is large, and the \mathcal{S} matrix implements a large translation of the wave packets, making the overlap small.

2.5.2 Full resummation from the Schwarzian action

We will now do the calculation for real, using the boundary formulation of the theory. The metric is always exactly AdS_2 ; the only variable is the reparameterization $t(u)$. We have to identify X^+, X^- with certain modes of $t(u)$ and then evaluate the Schwarzian action and the coupling to V, W . The main subtlety is that for the out-of-time-order correlator, we have to think about the function $t(u)$ on a folded time contour.

We start with a single shock, so just X^+ is nonzero. The bulk solution consists of two black holes glued together with a shift along the horizon. In terms of $t(u)$, we glue two solutions together at $\hat{u} = \infty$ with an $SL(2)$ transformation. In an $SL(2)$ frame where the original black hole solution is $\hat{t} = e^{\hat{u}}$, the transformation is a simple translation, and the shock wave solution is

$$\hat{t} = e^{\hat{u}} \quad (\text{sheet 1}), \quad \hat{t} = e^{\hat{u}} + X^+ \quad (\text{sheet 2}). \quad (2.5.50)$$

Notice that these can be glued at $\hat{u} = \infty$. We will find it more convenient to work in a different $SL(2)$ frame, where the original solution is $\hat{t} = \tanh(\frac{\hat{u}}{2})$, see figure 2.1. Then the one-shock solution is

$$\hat{t} = x \quad (\text{sheet 1}), \quad \hat{t} = x + \frac{(1-x)^2 X^+}{2 + (1-x)X^+} \quad (\text{sheet 2}), \quad x \equiv \tanh(\frac{\hat{u}}{2}). \quad (2.5.51)$$

This new $SL(2)$ frame makes it clear that if X^+ is small then we have a small perturbation to \hat{t} for all values of \hat{u} .

To compute the $\langle V_1 W_3 V_2 W_4 \rangle$ correlator, we need a contour with four folds, as in figure 2.3. We would like to superpose the (2.5.51) solution and a similar expression for X^- in the right way to capture the important part of the functional integral over $t(u)$. We can guess the answer based on the bulk picture, or from the discussion of $SL(2)$ charges in

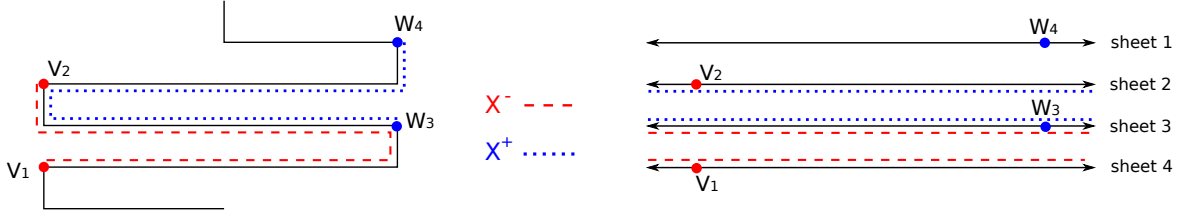


Figure 2.3: The black contour at left is the minimal time contour needed to compute $\langle V_1 W_3 V_2 W_4 \rangle$. Horizontal is real time, vertical is imaginary time, the ends are identified. It is convenient to extend the folds to infinity, as shown at right. The blue and red indicate which sheets have the X^- and X^+ perturbations turned on.

the previous section, which suggests that each pair of operators is associated to a relative $SL(2)$ transformation between the portion of the contour inside and outside the pair. So we consider the configuration

$$\hat{t} = x + \frac{(1-x)^2 X^+}{2 + (1-x)X^+} \theta(2, 3) - \frac{(1+x)^2 X^-}{2 + (1+x)X^-} \theta(3, 4), \quad x \equiv \tanh\left(\frac{\hat{u}}{2}\right). \quad (2.5.52)$$

The θ symbols are defined to be equal to one on the sheets indicated and zero elsewhere, see figure 2.3. This is not a solution to the equations of motion, it is an off-shell configuration of $t(u)$. The idea is that by integrating over X^+, X^- , we are capturing the part of the integral over $t(u)$ that gives powers of $e^{\hat{u}}/C$. The entire dependence of the action (2.2.15) on X^+, X^- comes from the fold where both terms are nonzero. The product $X^+ X^-$ is small, of order $1/C$ in the limit we are taking, so it is enough to compute the action to quadratic order:

$$iI_L \supset -iC \int_{\text{sheet 3}} d\hat{u} \text{Sch}(\hat{t}, \hat{u}) = -2iC X^- X^+ + (X\text{-independent}). \quad (2.5.53)$$

To compute the four point function, we also have to consider the reparameterized two-point functions. If the V operators are at early time, then only the X^+ part of the reparameterization is important. It acts on sheets two and three, and therefore affects

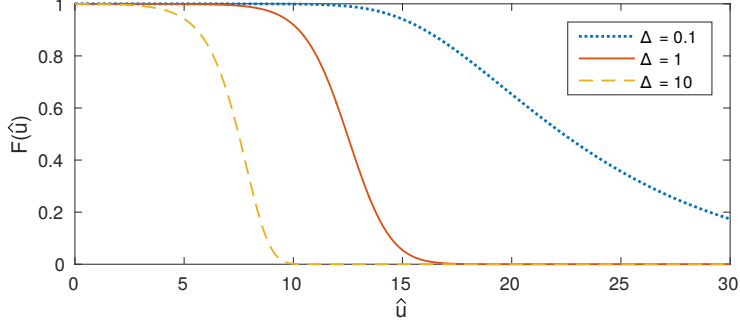


Figure 2.4: The correlation function (2.5.57) in the configuration (2.5.58), with $8C = 10^6$. Scaling up C simply translates all of the curves to the right without changing their shape. The initial descent from the plateau is characterized by the $e^{\frac{2\pi}{\beta}\hat{u}}$ behavior. The final approach to zero is determined by quasinormal decay.

only V_2 (see figure 2.3). Similarly, for the W operators we only have to consider the X^- part acting on W_3 :

$$G_V(\hat{u}_1, \hat{u}_2) = \left[\frac{-\hat{t}'(\hat{u}_1)\hat{t}'(\hat{u}_2)}{(\hat{t}(\hat{u}_1) - \hat{t}(\hat{u}_2))^2} \right]^\Delta \approx \left[\frac{-i}{2 \sinh \frac{\hat{u}_{12}}{2} - X^+ e^{-(\hat{u}_1 + \hat{u}_2)/2}} \right]^{2\Delta} \quad (2.5.54)$$

$$G_W(\hat{u}_3, \hat{u}_4) = \left[\frac{-\hat{t}'(\hat{u}_3)\hat{t}'(\hat{u}_4)}{(\hat{t}(\hat{u}_3) - \hat{t}(\hat{u}_4))^2} \right]^\Delta \approx \left[\frac{-i}{2 \sinh \frac{\hat{u}_{34}}{2} - X^- e^{-(\hat{u}_3 + \hat{u}_4)/2}} \right]^{2\Delta} \quad (2.5.55)$$

Now, to compute the four point function, we simply integrate these expressions over X^\pm with the weighting given by (2.5.53). Up to measure factors, we have

$$\langle V_1 W_3 V_2 W_4 \rangle \propto \int dX^+ dX^- e^{-2iCX^+ X^-} G_V(\hat{u}_1, \hat{u}_2) G_W(\hat{u}_3, \hat{u}_4). \quad (2.5.56)$$

One of the integrals can be done simply, and the second can be expressed using the confluent hypergeometric function $U(a, 1, x) = \Gamma(a)^{-1} \int_0^\infty ds e^{-sx} \frac{s^{a-1}}{(1+s)^a}$. The answer is

$$\frac{\langle V_1 W_3 V_2 W_4 \rangle}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \frac{U(2\Delta, 1, \frac{1}{z})}{z^{2\Delta}}, \quad z = \frac{i}{8C} \frac{e^{(\hat{u}_3 + \hat{u}_4 - \hat{u}_1 - \hat{u}_2)/2}}{\sinh \frac{\hat{u}_{12}}{2} \sinh \frac{\hat{u}_{34}}{2}}. \quad (2.5.57)$$

The real part of z is positive for the ordering of operators we have assumed. A simple configuration to keep in mind is the one where the V, W operators are equally spaced around the Euclidean circle, for example

$$\hat{u}_1 = -\frac{\hat{u}}{2} - i\pi, \quad \hat{u}_2 = -\frac{\hat{u}}{2}, \quad \hat{u}_3 = \frac{\hat{u}}{2} - i\frac{\pi}{2}, \quad \hat{u}_4 = \frac{\hat{u}}{2} + i\frac{\pi}{2} \quad \Longrightarrow \quad z = \frac{e^{\hat{u}}}{8C}. \quad (2.5.58)$$

Here \hat{u} is the separation of the early V operators and the late W operators. Notice that the z variable is real and positive in this type of configuration. We give a plot of (2.5.57) in figure 2.4. For early and late times, we have the limiting behaviors

$$\frac{U(2\Delta, 1, \frac{1}{z})}{z^{2\Delta}} \approx 1 - 4\Delta^2 z \quad (z \ll 1), \quad \frac{U(2\Delta, 1, \frac{1}{z})}{z^{2\Delta}} \approx \frac{\log z}{\Gamma(2\Delta)z^{2\Delta}} \quad (z \gg 1). \quad (2.5.59)$$

The small z expression reproduces the initial exponential growth of the connected correlator (2.3.33). The large z behavior gives exponential decay of the full correlator at late time, where it is dominated by the decay of the quasinormal modes.

2.5.3 Role of the $SL(2)$ charges

To understand the charges of the matter, it is helpful represent (2.5.54) and (2.5.55) in a basis that diagonalizes certain $SL(2)$ generators. It is convenient to return to the $SL(2)$ frame where the background solution is $\hat{t} = e^{\hat{u}}$. One can write

$$G_V(\hat{u}_1, \hat{u}_2) = \frac{1}{\Gamma(2\Delta)} \int_{-\infty}^0 \frac{dq_+}{-q_+} \Psi_1(q_+) \Psi_2(q_+) e^{-iX^+ q_+} \quad (2.5.60)$$

$$G_W(\hat{u}_3, \hat{u}_4) = \frac{1}{\Gamma(2\Delta)} \int_{-\infty}^0 \frac{dp_-}{-p_-} \Psi_3(p_-) \Psi_4(p_-) e^{-iX^- p_-} \quad (2.5.61)$$

where the wave functions are given by

$$\Psi_j = x_j^\Delta e^{-x_j}, \quad x_1 = -iq_+ e^{\hat{u}_1}, \quad x_2 = iq_+ e^{\hat{u}_2}, \quad x_3 = ip_- e^{-\hat{u}_3}, \quad x_4 = -ip_- e^{-\hat{u}_4}. \quad (2.5.62)$$

Equations (2.5.60) and (2.5.61) decompose the bilocals into pieces with definite charge, e.g. $q_M^- = -q_+$. Based on the discussion in section 2.4.2, we expect these charges to be related to the charges of the shocks in $t(u)$. For example, when we pass the V operator, the X^- shock turns on, and one can check from (2.4.42) that the charge changes by $Q_{after}^- - Q_{before}^- = -2CX^-$. This gets related to q_+ as follows: when we insert (2.5.60) and (2.5.61) into (2.5.56) and integrate over X^+ , we will get a delta function setting $-2CX^- - q_+ = 0$, so that

$$Q_{after}^- - Q_{before}^- = -q_M^-. \quad (2.5.63)$$

This means that the total charge Q_T indeed remains constant.

The charges give a new perspective on the exponential growth of the four point function. Acting with V at an early time changes Q^- of the $\hat{\tau}$ sector. Because the charge is conserved, and because of the explicit factor of $e^{-\hat{\tau}}$ in (2.4.42), this has an exponentially growing effect on $\hat{\tau}(u)$ as we move toward the future. We can always make an $SL(2)$ gauge transformation to remove this effect on either the portion of the contour before V or the portion after, but not both. In the out-of-time-order four point function we have W operators probing both sides, so the exponential effect is physical.

Chapter 3

The Quantum Gravity Dynamics of Near Extremal Black Holes

We consider the quantization of JT gravity in this chapter, to simplify the expression we will rescale the boundary time to absorb the C dependence in last chapter. This corresponding to measure the boundary time in units of ϕ_b . This chapter consists of a paper [52], and published as “The Quantum Gravity Dynamics of Near Extremal Black Holes”, [arXiv:1809.08647]. The original abstract is as follows:

We study the quantum effects of Near-Extremal black holes near their horizons. The gravitational dynamics in such backgrounds are closely connected to a particle in AdS_2 with constant electric field. We use this picture to solve the theory exactly. We will give a formula to calculate all correlation functions with quantum gravity backreactions as well as the exact Wheeler-DeWitt wavefunction. Using the WdW wavefunction, we investigate the complexity growth in quantum gravity.

3.1 Charged Particle in AdS_2

There are various ways to describe the boundary motion (2.2.15). Here we will think of it as arising from the motion of the physical boundary of AdS_2 inside a rigid AdS_2 space. This picture is most clear for finite ϵ in (2.1.4), but it is true even as $\epsilon \rightarrow 0$. The dynamics of the boundary is $SL(2)$ invariant. This $SL(2)$ invariance is a gauge symmetry since it simply reflects the freedom we have for cutting out a piece of AdS_2 space that we will call the “inside”. It is important that the dilaton field we discussed above is produced after we put in the boundary and it moves together with the boundary under this $SL(2)$ gauge transformation. It is a bit like the Mach principle, the location in AdS_2 is only defined after we fix the boundary (or distant “stars”).

We can make this picture of a dynamical boundary more manifest as follows. Since the bulk Jackiw-Teitelboim action (2.2.7) is linear in ϕ , we can integrate out the dilaton field which sets the metric to that of AdS_2 and removes the bulk term in the action, leaving only the term involving the extrinsic curvature

$$I = -\frac{\phi_r}{\epsilon} \int du \sqrt{g} K \tag{3.1.1}$$

This action, however, is divergent as we take ϵ to zero. This divergence is simply proportional to the length of the boundary and can be interpreted as a contribution to the ground state energy of the system. So we introduce a counterterm proportional to the length of the boundary to cancel it. This is just a common shift of the energies of all states. It is also convenient to use the Gauss-Bonnet theorem to relate the extrinsic curvature to an integral over the bulk

$$\int_{\partial M} du \sqrt{g} K = 2\pi \chi(M) - \frac{1}{2} \int_M R \tag{3.1.2}$$

Since the curvature is a constant, the bulk integral is actually proportional to the total area A of our space. That is we have the regularized action:

$$\begin{aligned}
I &= -\frac{\phi_r}{\epsilon} \int_{\partial M} du \sqrt{g} (K - \underbrace{1}_{\text{counterterm}}) = -\frac{\phi_r}{\epsilon} \left(2\pi\chi(M) - \frac{1}{2} \int_M R - \int_{\partial M} du \sqrt{g} \right) \\
&= -2\pi q \chi(M) - qA + qL, \qquad q \equiv \frac{\phi_r}{\epsilon}, \quad L = \frac{\beta\phi_r}{\epsilon} \qquad (3.1.3)
\end{aligned}$$

We now define an external gauge field a_μ as

$$a_\varphi = \cosh \rho - 1, \quad a_\rho = 0, \quad f_{\rho\varphi} = \sinh \rho = \sqrt{g}, \qquad (3.1.4)$$

and write the action as follows

$$I = -2\pi q + qL - q \int a \qquad (3.1.5)$$

where we used that $\chi(M)$ is a topological invariant equal to one, in our case, where the topology is that of a disk. The term qL is just the length of the boundary. So this action has a form somewhat similar to the action of a relativistic charged particle moving in AdS_2 in the presence of a constant electric field. There are a couple of important differences. First we are summing only over trajectories of fixed proper length set by the inverse temperature β . Second, in the JT theory we are treating the $SL(2)$ symmetry as a gauge symmetry. And finally, in the JT theory we identify the proper length with the boundary time, viewing configurations which differ only by a shift in proper time as inequivalent. In fact, all these changes simplify the problem: we can actually think of the problem as a non-relativistic particle moving on H_2 in an electric field. In appendix B.4 we discuss in more detail the connection to the relativistic particle.

In fact, precisely the problem we are interested in has been discussed by Polyakov in [53], Chapter 9, as an intermediate step for the sum over paths. Now we would also like to point out that we can directly get to the final formula by using the discussion there, where he explicitly shows that for a particle in flat space the sum over paths of fixed proper length that stretch between two points \vec{x} and \vec{x}' gives

$$\int \mathcal{D}\vec{x} e^{-m_0\tilde{\tau}} \delta(\dot{\vec{x}}^2 - 1) = e^{-\frac{1}{2}\mu^2\tau} \langle x' | e^{-\tau H} | x \rangle = e^{-\frac{1}{2}\mu^2\tau} \int \mathcal{D}\mathbf{x} \exp\left(-\int_0^\tau d\tau' \frac{1}{2}\dot{x}^2\right) \quad (3.1.6)$$

μ^2 is the regularized mass and $\tilde{\tau}$ is related to τ by a multiplicative renormalization. The JT model consists precisely of a functional integral of this form, where we fix the proper length along the boundary. There are two simple modifications, first the particle is in a curved H_2 space and second we have the coupling to the electric field. These are minor modifications, but the arguments leading to (3.1.6) continue to be valid so that the partition function of the JT model can be written directly:

$$\int \mathcal{D}\vec{x} e^{2\pi q - m_0\tilde{\tau} + q \int a} \delta\left(\frac{\dot{x}^2 + \dot{y}^2}{y^2} - q^2\right) = e^{2\pi q - \frac{1}{2}\mu^2\tau} \text{Tr} e^{-\tau H} = e^{2\pi q - \frac{1}{2}\mu^2\tau} \int \mathcal{D}\mathbf{x} \exp\left(-\int_0^\tau d\tau' \frac{1}{2} \frac{\dot{x}^2 + \dot{y}^2}{y^2} - q \frac{\dot{x}}{y}\right) \quad (3.1.7)$$

The delta function implements the first condition in (2.1.4) at each point along the path (Notice that we have absorbed the C dependence here as mentioned at the beginning of the chapter). The last path integral can be done exactly by doing canonical quantization of the action (section 3.2) and by comparing the result with the one from the Schwarzian action [54] we can determine that τ is the inverse temperature β .

In the above discussion we have been fixing the time along the boundary. Instead we can fix the energy at the boundary, where the energy is the variable conjugate to time. This can be done by simply integrating (3.1.7) times $e^{\beta E}$ over β along the imaginary axis. This fixes the energy of the non-relativistic problem by generating a $\delta(H - E)$. More precisely,

we will argue that after doing a spectral decomposition we can write the propagator at coincident points as

$$Z_{JT}(\beta) = \int_0^\infty \rho(E) e^{-\beta E} dE \longrightarrow \rho(E) = \int_{-i\infty}^{i\infty} \frac{d\beta}{i} e^{E\beta} Z_{JT}(\beta) \quad (3.1.8)$$

where the function $\rho(E)$ can then be interpreted as a “density of states” in the microcanonical ensemble. We will give its explicit form in section (4.2). For now, we only want to contrast this integral with a superficially similar one that appears when we compute the relativistic propagator

$$e^{-2\pi q} \int_0^\infty e^{E\beta} Z_{JT}(\beta) = \langle \phi(x) \phi(x) \rangle \quad (3.1.9)$$

which gives the relativistic propagator of a massive particle in an electric field at coincident points (we can also compute this at non-coincident points to get a finite answer). The total mass of the particle is

$$m = q - \frac{E}{q} \quad (3.1.10)$$

For large q this is above threshold for pair creation¹. The pair creation interpretation is appropriate for the problem in (3.1.9), but not for (3.1.8). In both problems we have a classical approximation to the dynamics that corresponds to a particle describing a big circular trajectory in hyperbolic space at radius ρ_c :

$$\tanh \rho_c = \frac{m}{q} \quad (3.1.11)$$

For the problem in (3.1.9), fluctuations around this circle lead to an instability, with a single negative mode and an imaginary part in the partition function (3.1.9). This single negative mode corresponds to small fluctuations of the overall size of the circular trajectory around

¹See Appendix B.4

(3.1.11). On the other hand in (3.1.8) we are integrating the same mode along a different contour, along the imaginary axis, where we get a real and finite answer. Furthermore, the imaginary part in the partition function (3.1.9) comes precisely from the trajectory describing pair creation, which is also the type of contribution captured in (3.1.8).

Finally, in the relativistic particle problem, we expect that the pair creation amplitude should be exponentially suppressed for large q , while the partition function for the JT model is not. In fact, for large q the exponential suppression factor for pair creation goes as $e^{-2\pi q}$, which is precisely cancelled by a similar factor in (B.4.19), to obtain something finite in the large q limit.

3.2 Solving the Quantum Mechanical Problem

As we explained above the solution of the JT theory is equivalent to considering a non-relativistic particle in AdS_2 or H_2 . We first consider the Euclidean problem, of a particle moving in H_2 . An ordinary magnetic field in H_2 leads to an Euclidean action of the form

$$S = \int du \frac{1}{2} \frac{\dot{x}^2 + \dot{y}^2}{y^2} + ib \int du \frac{\dot{x}}{y} - \frac{1}{2} (b^2 + \frac{1}{4}) \int du, \quad b = iq \quad (3.2.12)$$

If b is real we will call it a magnetic field, when q is real we will call it an “electric” field. The last term is a constant we introduced for convenience. Its only effect will be to shift the ground state energy. It is interesting to compute the classical solutions and the corresponding action for (3.2.12). These solutions are simplest in the ρ and φ coordinates, using the $SL(2)$ symmetry we find that the trajectories are given by ($t = -iu$):

$$\frac{1}{2} \sinh^2 \rho \left(\frac{d\varphi}{dt} \right)^2 + \frac{q^2}{2} - \frac{1}{8} = E, \quad \cosh \rho = \frac{q\beta}{2\pi}, \quad \frac{d\varphi}{du} = \frac{2\pi}{\beta}. \quad (3.2.13)$$

In this classical limit we get the following relations for the action and the temperature:

$$\begin{aligned}\frac{\beta}{2\pi} &= \frac{1}{\sqrt{2E + \frac{1}{4}}} \\ -S &= \frac{2\pi^2}{\beta} + \frac{\beta}{8} - 2\pi q\end{aligned}\tag{3.2.14}$$

When b is real, this system is fairly conventional and it was solved in [55]. Its detailed spectrum depends on b . For very large b we have a series of Landau levels and also a continuous spectrum. In fact, already the classical problem contains closed circular orbits, related to the discrete Landau levels, as well as orbits that go all the way to infinity.² The number of discrete Landau levels decreases as we decrease the magnetic field and for $0 < b < \frac{1}{2}$ we only get a continuous spectrum. The system has a $SL(2)$ symmetry and the spectrum organizes into $SL(2)$ representations, which are all in the continuous series for $0 < b < 1/2$. For real q we also find a continuous spectrum which we can view as the analytic continuation of the one for this last range of b .

The canonical momenta of the action (3.2.12) are:

$$p_x = \frac{\dot{x}}{y^2} + \frac{iq}{y}; \quad p_y = \frac{\dot{y}}{y^2}.\tag{3.2.15}$$

And the Hamiltonian conjugate to τ_L is thus:

$$H = \frac{\dot{x}^2 + \dot{y}^2}{2y^2} + \frac{q^2}{2} = \frac{y^2}{2}[(p_x - i\frac{q}{y})^2 + p_y^2] + \frac{q^2}{2} - \frac{1}{8}\tag{3.2.16}$$

Note that the Hamiltonian is not Hermitian. However, it is PT-symmetric (here parity reflects x and p_x) and for that reason the spectrum is still real, see [56]. The action is

²See Appendix B.6

invariant under $SL(2, R)$ transformations generated by

$$L_0 = xp_x + yp_y; \quad L_{-1} = p_x; \quad L_1 = (y^2 - x^2)p_x - 2xyp_y - 2iqy \quad (3.2.17)$$

Notice the extra q dependent term in L_1 that arises due to the presence of a magnetic field. Up to a simple additive constant, the Hamiltonian is proportional to the Casimir operator

$$H = \frac{1}{2} \left(L_0^2 + \frac{1}{2} L_{-1} L_1 + \frac{1}{2} L_1 L_{-1} \right) + \frac{q^2}{2} - \frac{1}{8} \quad (3.2.18)$$

As is common practice, let us label the states by quantum numbers $j = \frac{1}{2} + is$ and k , so that $H|j, k\rangle = j(1 - j)|j, k\rangle$ and $L_{-1}|j, k\rangle = k|j, k\rangle$. We can find the eigenfunctions by solving the *Schrödinger* equation with boundary condition that the wavefunction should vanish at the horizon $y \rightarrow \infty$ [57, 55, 58]:

$$\omega_{s,k} = \frac{s^2}{2}, \quad f_{s,k}(x, y) = \begin{cases} \left(\frac{s \sinh 2\pi s}{4\pi^3 k} \right)^{\frac{1}{2}} |\Gamma(is - b + \frac{1}{2})| e^{-ikx} W_{b, is}(2ky), & k > 0; \\ \left(\frac{s \sinh 2\pi s}{4\pi^3 |k|} \right)^{\frac{1}{2}} |\Gamma(is + b + \frac{1}{2})| e^{-ikx} W_{-b, is}(2|k|y), & k < 0. \end{cases} \quad (3.2.19)$$

where ω_{ks} is giving the energy of the states labelled by s and k , and W is the Whittaker function. The additive constant in (3.2.12) was introduced to simplify this equation. We can think of s as the quantum number of the continuous series representation of $SL(2)$ with spin $j = \frac{1}{2} + is$.

After continuing $b \rightarrow iq$ we find that the gravitational system has a continuous spectrum

$$E(s) = \frac{s^2}{2}. \quad (3.2.20)$$

3.2.1 The Propagator

It is useful to compute the propagator for the non-relativistic particle in a magnetic field, $K(u, \mathbf{x}_1, \mathbf{x}_2) = \langle \mathbf{x}_1 | e^{-uH} | \mathbf{x}_2 \rangle$. Here, \mathbf{x} stands for x, y . The propagator for a real magnetic particle was obtained in [55]:

$$\begin{aligned}
 G(u, \mathbf{x}_1, \mathbf{x}_2) &= e^{i\varphi(\mathbf{x}_1, \mathbf{x}_2)} \int_0^\infty ds s e^{-u\frac{s^2}{2}} \frac{\sinh 2\pi s}{2\pi(\cosh 2\pi s + \cos 2\pi b)} \frac{1}{d^{1+2is}} \times \\
 &\quad \times {}_2F_1\left(\frac{1}{2} - b + is, \frac{1}{2} + b + is, 1, 1 - \frac{1}{d^2}\right). \\
 d &= \sqrt{\frac{(x_1 - x_2)^2 + (y_1 + y_2)^2}{4y_1 y_2}} \\
 e^{i\varphi(\mathbf{x}_1, \mathbf{x}_2)} &= e^{-2ib \arctan \frac{x_1 - x_2}{y_1 + y_2}}
 \end{aligned} \tag{3.2.21}$$

In the case that we have a real magnetic field the prefactor is a phase and it is gauge dependent. It is equal to the value of Wilson Line $e^{i\int a}$ stretched along the geodesic between x_2 and x_1 . Here we quoted the value in the gauge where the action is (3.2.12). The second equation defines the parameter d , which is a function of the geodesic distance between the two points. Note that $d = 1$ corresponds to coincident points. We can get the answer we want by making the analytic continuation $b \rightarrow iq$ of this formula. We can check that this is the right answer for our problem by noticing the following. First one can check that this expression is invariant under the $SL(2)$ symmetry $L_a = L_a^1 + L_a^2$ where L_a are the generators (3.2.17) acting on \mathbf{x}_1 and L_a^2 are similar generators as in (3.2.17), but with $q \rightarrow -q$. It is possible to commute the phase $e^{i\varphi(\mathbf{x}_1, \mathbf{x}_2)}$, in (3.2.21) past these generators which would remove the q dependent terms. This implies that the rest should be a function of the proper distance, which is the case with (3.2.21). Then we can check the equation

$$0 = (\partial_u + H_1)G(u, \mathbf{x}_1, \mathbf{x}_2) \tag{3.2.22}$$

which is also indeed obeyed by this expression. The s dependent prefactor is fixed by the requirement that the propagator composes properly, or more precisely, by saying that for $u = 0$ we should get a δ function.

3.2.2 Partition Function

The gravitational partition function is related with the particle partition function with inverse temperature β . The canonical partition function of the quantum mechanical system

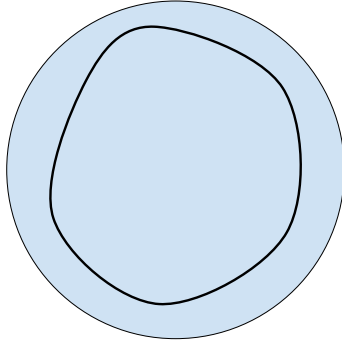


Figure 3.1: Free Energy diagram with inverse temperature β .

is

$$\begin{aligned}
 Z_{Particle} &= \text{Tr} e^{-\beta H} = \int_0^\infty ds \int_{-\infty}^\infty dk \int_M \frac{dxdy}{y^2} e^{-\beta \frac{s^2}{2}} f_{s,k}^*(x,y) f_{s,k}(x,y) \\
 &= V_{AdS} \int_0^\infty ds e^{-\beta \frac{s^2}{2}} \frac{s}{2\pi} \frac{\sinh(2\pi s)}{\cosh(2\pi q) + \cosh(2\pi s)}. \tag{3.2.23}
 \end{aligned}$$

The volume factor V_{AdS} arises because after momentum integration there is no position dependence. In a normal quantum mechanical system, the volume factor means that the particle can have independent configurations at different locations of our space, however for a gravitational system this should be thought as redundant and should be cancelled by

the volume of $SL(2, R)$ gauge group $2\pi V_{AdS}$ ³. In gravitational system, there can also other contributions to the entropy from pure topological action. These give a contribution to the ground state entropy S_0 . Including the topological action in (3.1.5), we find a gravitational “density of states” as

$$\rho(s) = \underbrace{e^{S_0} e^{2\pi q}}_{\text{extra terms}} \underbrace{\frac{1}{2\pi}}_{\text{residue gauge}} \underbrace{\frac{s}{2\pi \cosh(2\pi q) + \cosh(2\pi s)}}_{\text{particle in magnetic field}} = e^{S_0} e^{2\pi q} \frac{s}{2\pi^2} \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2\pi q k} \sinh(2\pi s k). \quad (3.2.24)$$

We have not given an explicit description of these states in the Lorentzian theory. More details were discussed in [28, 59].

This expression has some interesting features. Notice that the classical limit corresponds to large q and large s , where we reproduce (3.2.14). After approximating, the density of states are $\log \rho(s) \sim S_0 + 2\pi s$ for $s/q < 1$ and $S_0 + 2\pi q$ for $s/q > 1$.

We can also expand the partition function for very small and very large temperatures where we obtain

$$\begin{aligned} Z_{JT} &\sim e^{S_0} e^{2\pi q} \frac{1}{4\pi^2 \beta}, & \beta &\ll \frac{1}{q} \\ Z_{JT} &\sim e^{S_0} \frac{1}{\sqrt{2\pi} \beta^{3/2}}, & \beta &\gg 1 \end{aligned} \quad (3.2.25)$$

Notice that at leading order we get an almost constant entropy both at low and high temperatures, with the high temperature one being higher. In both cases there are power law corrections in temperature.

Before we try to further elucidate the interpretation of this result, let us emphasize a couple of important defects of our discussion. First, when we replaced the partition function of the JT theory by the action of a non-relativistic particle in an electric field, we

³There might be a multiplicative factor in the volume of gauge group, but we can always absorb that into S_0 .

were summing over paths in H_2 . This includes paths that self intersect see figure 3.2(b). Such paths do not have an obvious interpretation in the JT theory and it is not even clear that we should include them. For example, the sum over k in (3.2.24) can be understood in terms of classical solutions which wind k times around the circle. These make sense for the problem of the particle in the electric field but apparently not in the JT theory. Maybe

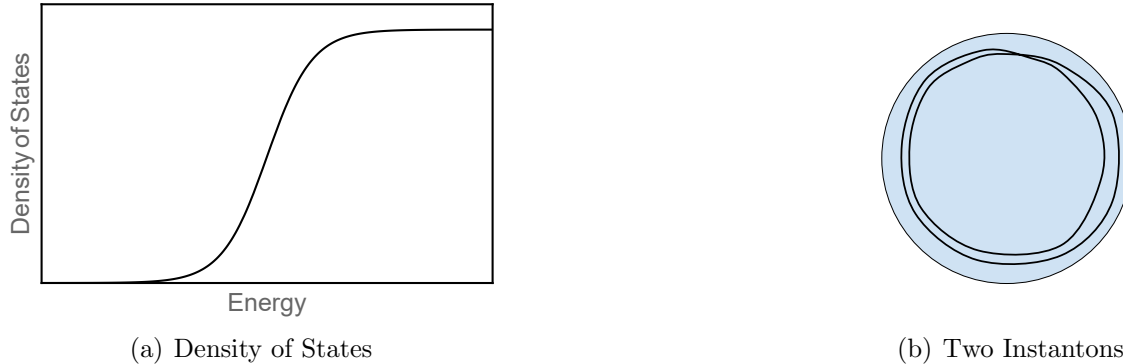


Figure 3.2: Density of States and the Two Instantons configuration

such paths could be given some interpretation in the gravity theory. Alternatively, we might want to sum over paths that do not self intersect. A second defect is that we would be eventually interested in adding some matter fields propagating in the bulk geometry. These matter fields have boundary conditions at the boundary of the region of hyperbolic space cut out by the boundary trajectory. The partition function of the fields with such an arbitrary boundary trajectory could also modify the results we described above. Of course, this issue does not arise if we have the pure JT theory. It is only important if we want to introduce bulk matter fields to define more complex observables.

Instead of attempting to address the above issues, we will take an easy route, which is to consider the system only in the large q (or small ϵ) limit. In this regime, we address the above issues, and we can still trust the description of the particle in the electric field. This large q or small ϵ limit is the same one that isolates the Schwarzian action from the JT theory [29, 60, 61]. It turns out that the limit can be taken already at the level of

the mechanical system, a simple rescaled version of the above system. This provides an alternative method for quantizing the Schwarzian theory. It has the advantage of being a straightforward second order action of a particle moving in a region near the boundary of hyperbolic. Of course, the Schwarzian theory was already quantized using a variety of methods in [62, 63, 54, 64, 65]. We will simply provide yet another perspective, recover the old results, and write a few new expressions.

3.3 Quantum Gravity at Schwarzian Limit

Before getting into the details notice that the large q limit of (3.2.24) gives

$$\rho(s) = e^{S_0} \frac{s}{2\pi^2} \sinh(2\pi s), \quad E = \frac{s^2}{2}, \quad Z_{JT} = \int_0^\infty ds \rho(s) e^{-\beta \frac{s^2}{2}} = e^{S_0} \frac{1}{\sqrt{2\pi} \beta^{\frac{3}{2}}} e^{\frac{2\pi^2}{\beta}}. \quad (3.3.26)$$

This reproduces what was found in [54, 64, 66, 67] by other methods. We see that we get a finite answer and also that the contributions from the $k > 1$ terms in (3.2.24) have disappeared. Because the S_0 part decouples with JT gravity, from now on, we will drop it and discuss S_0 only when it is necessary.

3.3.1 The Propagator

To get a limit directly at the level of the mechanical system it is useful to define a rescaled coordinate, z , via

$$y = z/q. \quad (3.3.27)$$

After taking the large q limit, the boundary particle propagator becomes ⁴:

$$G(u, \mathbf{x}_1, \mathbf{x}_2) = \frac{1}{q} e^{-2\pi q \theta(x_2 - x_1)} \tilde{K}(u, \mathbf{x}_1, \mathbf{x}_2); \quad q \gg 1. \quad (3.3.28)$$

$$\tilde{K}(u, \mathbf{x}_1, \mathbf{x}_2) = e^{-2\frac{z_1+z_2}{x_1-x_2}} \frac{2\sqrt{z_1 z_2}}{\pi^2 |x_1 - x_2|} \int_0^\infty ds s \sinh(2\pi s) e^{-\frac{s^2}{2} u} K_{2is}\left(\frac{4\sqrt{z_1 z_2}}{|x_1 - x_2|}\right); \quad (3.3.29)$$

$$= e^{-2\frac{z_1+z_2}{x_1-x_2}} \frac{\sqrt{2}}{\pi^{3/2} u^{3/2}} \frac{\sqrt{z_1 z_2}}{|x_1 - x_2|} \int_{-\infty}^\infty d\xi (\pi + i\xi) e^{-2\frac{(\xi - i\pi)^2}{u} - \frac{4\sqrt{z_1 z_2}}{|x_1 - x_2|} \cosh \xi} \quad (3.3.30)$$

The original phase factor $e^{i\varphi(\mathbf{x}_1, \mathbf{x}_1)}$ factorizes into a product of singular “phase” $e^{-2\pi q \theta(x_2 - x_1)}$, with θ the step function, and a regular “phase” $e^{-2\frac{z_1+z_2}{x_1-x_2}}$. The singular “phase” is the same order as the topological piece in (3.1.5). In order to have a finite result they should cancel between each other. This can only be satisfied if the x_i s are in cyclic order. As shown in figure (3.3(a)), the product of singular “phase” gives $-2\pi q$ for cyclic order x_i s and this would cancel with the topological action $2\pi q$. While for other ordering of the x_i s, this would have $-2\pi n q$ for $n = 2, 3, \dots$ and is highly suppressed in the limit q goes to infinity. This cyclic order is telling us where the interior of our space time is. The magnetic field produces a preferred orientation for the propagator. After fixing the order, all our formulas only depend on $\tilde{K}(u, \mathbf{x}_1, \mathbf{x}_2)$ which has no q dependence. The residual q factor in 3.3.28 cancels out the additional q from the measure of coordinate integral, $\frac{dx dy}{y^2} \rightarrow q \frac{dx dz}{z^2}$. In conclusion, after taking the limit we get a finite propagator equal to (3.3.29), which should be multiplied by a step function $\theta(x_1 - x_2)$ that imposes the right order.

The final function $\tilde{K}(u, \mathbf{x}_1, \mathbf{x}_2)$ has the structure of $e^{-2\frac{z_1+z_2}{x_1-x_2}} f(u, \frac{z_1 z_2}{(x_1-x_2)^2})$. This can be understood directly from the $SL(2)$ symmetry. After taking the large q limit, the $SL(2, R)$

⁴see the Appendix B.5 for details

charges become

$$L_0 = i(x\partial_x + z\partial_z); \quad L_{-1} = i\partial_x; \quad L_1 = -ix^2\partial_x - 2ixz\partial_z - 2iz. \quad (3.3.31)$$

We can check that they still satisfy the $SL(2)$ algebra. If we drop the last term in L_1 , the $SL(2, R)$ charges become the usual differential operators on $EAdS_2$. And the propagator will have only dependence on the geodesic distance. When L_1 operator is deformed, the condition of $SL(2, R)$ invariance fixes the structure of the propagator as follows. The L_0 and L_{-1} charges are not deformed and they imply that the only combinations that can appear are

$$v \equiv \frac{z_1 + z_2}{x_1 - x_2} \quad \text{and} \quad w \equiv \frac{z_1 z_2}{(x_1 - x_2)^2}. \quad (3.3.32)$$

Writing the propagator as $\tilde{K}(u, \mathbf{x}_1, \mathbf{x}_2) = k(v, w)$ and requiring it to be invariant under L_1 gives the following equation for α :

$$\partial_v k + 2k = 0 \longrightarrow k = e^{-2v} h(w) \quad (3.3.33)$$

$$\tilde{K}(u, \mathbf{x}_1, \mathbf{x}_2) = e^{-2\frac{z_1+z_2}{x_1-x_2}} f\left(u, \frac{z_1 z_2}{(x_1 - x_2)^2}\right), \quad (3.3.34)$$

The full function can also be determined directly as follows. Again we impose the propagator equation (or heat equation)

$$\begin{aligned} 0 &= \left[\partial_u + \frac{1}{2} \left(L_0^2 + \frac{1}{2} L_{-1} L_1 + \frac{1}{2} L_1 L_{-1} \right) - \frac{1}{8} \right] \tilde{K} \\ 0 &= \left[\frac{s^2}{2} + \frac{w^2}{2} \partial_w^2 + 2w + \frac{1}{8} \right] K_s(w) \end{aligned} \quad (3.3.35)$$

where L_a are given in (3.3.31) and are acting only on the first argument of \tilde{K} . The solution of the last equation which is regular at short distances ($w \rightarrow \infty$) is \sqrt{w} times the Bessel K function in (3.3.28).

We can also directly determine the measure of integration for s by demanding that the propagator at $u = 0$ is a δ function or by demanding the propagator compose properly. This indeed is the case with the $s \sinh 2\pi s$ function in (3.3.29). To explicitly show the above statement, it will be useful to use spectral decomposition of the propagator:

$$\tilde{K}(u, \mathbf{x}_1, \mathbf{x}_2) = \int ds \frac{2s \sinh(2\pi s)}{\pi^3} e^{-\frac{s^2 u}{2}} \int dk \sqrt{z_1 z_2} e^{ik(x_1 - x_2)} K_{2is}(2\sqrt{2ikz_1}) K_{2is}(2\sqrt{2ikz_2}). \quad (3.3.36)$$

It can be easily checked that the special functions $f_{k,s}(x, z) = \sqrt{z} e^{ikx} K_{2is}(2\sqrt{2ikz})$ are delta function normalizable eigenmodes of the large q Hamiltonian:

$$\int \frac{dx dz}{z^2} f_{k_1, s_1} f_{k_2, s_2} = \delta(k_1 - k_2) \delta(s_1 - s_2) \frac{\pi^3}{2s \sinh(2\pi s)} \quad (3.3.37)$$

Notice that the inner product fixes the integral measure completely in (3.3.36), and the composition relation is manifestly true:

$$\int \frac{dx dz}{z^2} \tilde{K}(u_1, \mathbf{x}_1, \mathbf{x}) \tilde{K}(u_2, \mathbf{x}, \mathbf{x}_2) = \tilde{K}(u_1 + u_2, \mathbf{x}_1, \mathbf{x}_2) \quad (3.3.38)$$

At short time the propagator has the classical behavior:

$$\tilde{K}(u, \mathbf{x}_1, \mathbf{x}_2) \sim \delta(x_1 - x_2 + uz_2) e^{-\frac{(z_1 - z_2)^2}{2uz_2}} \quad (3.3.39)$$

This form of singularity is expected since we are taking the large q limit first and thus the velocity in x direction is fixed to be z . In the original picture of finite q we are looking

at the time scale which is large compare to AdS length but relatively small such that the quantum fluctuations are not gathered yet.

The integral structure in the propagator (3.3.29) has an obvious meaning: integrating over s represents summing over all energy states with Boltzmann distribution e^{-Eu} , and the Bessel function stands for fixed energy propagator. We want to stress that the argument in the Bessel function is unusual, and at short distance it approaches a funny limit:

$$K_{2is}\left(\frac{4}{\ell}\right) \simeq \sqrt{\frac{\pi}{8\ell}} e^{-\frac{4}{\ell}}, \quad \ell = \frac{|x_1 - x_2|}{\sqrt{z_1 z_2}} \rightarrow 0. \quad (3.3.40)$$

One should contrast this exponential suppression with the short distance divergence in QFT which is power law. In our later discussion of exact correlation function with gravity backreaction, we will see that this effect kills UV divergence from matter fields.

To obtain the expression (3.3.30), we use the integral representation for the Bessel function and the final result has some interesting physical properties:

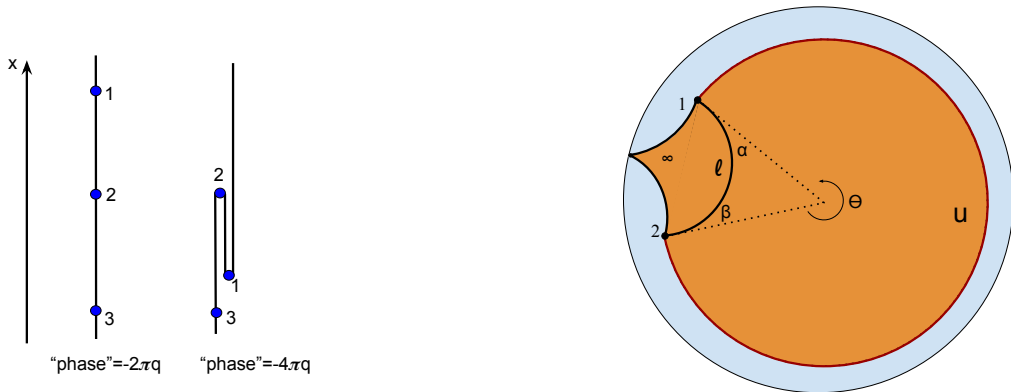
Firstly, we see that at large u the time dependence and coordinate dependence factorized. So, at large time we have a universal power law decay pointed out in [62].

Secondly, as we said before, the phase factor $e^{-2\frac{z_1+z_2}{x_1-x_2}}$ is equal to the Wilson line $e^{-q \int_1^2 a}$ stretched along the geodesic connection between location 1 and 2 (Figure 3.3(b)). The field a depends on our choice of gauge, our convention corresponds to fix the minimum value of a at infinity and then the Wilson line is equal to e^{-qA} , where A is the area of a hyperbolic triangle spanned by 1, 2 and ∞ .

Thirdly, defining $2\pi + 2i\xi$ as θ , then θ has the meaning of the spanned angle at the horizon (Figure 3.3(b)). Then the gaussian weight $e^{-2\frac{(\xi-i\pi)^2}{u}} = e^{\frac{\theta^2}{2u}}$ can be understood from the classical action along the boundary with fixed span angle θ . The boundary drawn in the figure represents a curve with fixed (regularized) proper length u in H_2 .

Lastly, the factor $e^{-\frac{4\sqrt{z_1 z_2}}{|x_1 - x_2|} \cosh \xi} = e^{-\frac{4 \cos \theta}{\ell}}$ is equal to $e^{q(\alpha + \beta)}$, which is a corner term that arise from JT gravity in geometry with jump angles. Here α and β are defined as the angle spanned by the geodesic with fixed length and the ray coming from horizon to the boundary.

In summary the propagator can be understood as an integral of JT gravity partition functions over geometries 3.3(b) with different θ s.



(a) singular “phase” factor for different ordering

(b) A geometric representation of the propagator. Here we fix the span angle θ , the propagator is a summation over such geometries.

Figure 3.3: The singular “phase” for different ordering and the geometric representation of the propagator

Finally, let us comment on the issues we raised in section 4.2. In the large q limit we are considering the propagator at relatively large distances and in a regime where locally in AdS the integration over paths that fluctuate wildly is suppressed. Alternatively we can say that in the integration over paths we put a UV cutoff which is large compared to $1/q$ but small compared to the AdS radius. This is the non-relativistic regime for the boundary particle. The quantum effects are still important at much longer distances due to the large size of AdS . In addition, if we have quantum fields in AdS , then their partition functions for these fluctuating contours that have fluctuations over distances larger than the AdS radius are expected to depend on this shape in a local way. Due to the symmetries of AdS_2 ,

this is simply expected to renormalize the action we already have without introducing extra terms. This can be checked explicitly for conformal field theories by using the conformal anomaly to compute the effective action of the CFT₂ on a portion of H_2 (Appendix B.3).

3.3.2 Wheeler-DeWitt Wavefunction

In the pure JT theory we can think about quantizing the bulk theory and obtaining the Wheeler-DeWitt wavefunction. This was discussed in the classical limit by Harlow and Jafferis [68].

The Wheeler-DeWitt wavefunction can be created by Euclidean evolution of the boundary and hence is closely related to the propagator we have discussed above. The wavefunction in Lorentzian signature could then be obtained by analytic continuation of the boundary time. The Euclidean evolution can be specified by either of the two parameters: the proper length u or energy E . Choosing a different parameter corresponds to imposing a different boundary condition in JT theory. In general there are four possible choices of boundary conditions in 2d dilaton gravity, there are two sets of conjugate variables: $\{\phi_b, K\}$, and $\{u, E\}$ ⁵. In preparing the wavefunction we fix the boundary value of dilaton and hence there are only two choices of the parameter (u or E). We denote the corresponding wavefunction as $|u\rangle_G$ and $|E\rangle_G$ respectively. In terms of holographic considerations, $|u\rangle_G$ represents a thermofield double state:

$$|u\rangle_G \sim \sum_n e^{-E_n u} |E_n\rangle_L |\bar{E}_n\rangle_R \tag{3.3.41}$$

⁵Energy E is proportional to the normal derivative of the dilaton field at the boundary.

and $|E\rangle_G$ is like an average of energy eigenstates in a window of energy E :

$$|E\rangle_G \sim \frac{1}{\delta E} \sum_{|E-E_n|<\delta E} |E_n\rangle_L |\bar{E}_n\rangle_R. \quad (3.3.42)$$

The width of the energy window is some coarse graining factor such that the summation contains e^{S_0} states and does not show up clearly in gravity.⁶

With the definition of the states, one can evaluate them in terms of different basis. There are three natural bases turn out to be useful, we call them S , η and ℓ bases. Basis S corresponding to fix the horizon value of dilaton field ϕ_h , or equivalently by Bekenstein-Hawking formula, the entropy of the system. The canonical conjugate variable of S will be called η and that characterizes the boost angle at the horizon. ℓ stands for fixing geodesic distance between two boundary points. To see that the horizon value of the dilaton field is a gauge invariant quantity, one can do canonical analysis of JT gravity. With ADM decomposition of the spacetime metric, one can get the canonical momenta and Hamiltonian constraints of the system [70]:

$$ds^2 = -N^2 dt^2 + \sigma^2(dx + N^x dt)^2; \quad (3.3.43)$$

$$\mathcal{H} = -\Pi_\phi \Pi_\sigma + \sigma^{-1} \phi'' - \sigma^{-2} \sigma' \phi' - \sigma \phi; \quad \mathcal{H}_x = \Pi_\phi \phi' - \sigma \Pi'_\sigma; \quad (3.3.44)$$

$$\Pi_\phi = N^{-1}(-\dot{\sigma} + (N^x \sigma)') = K\sigma; \quad \Pi_\sigma = N^{-1}(-\dot{\phi} + N^x \phi') = \partial_n \phi \quad (3.3.45)$$

That is the dilaton field is canonically conjugate to the extrinsic curvature and boundary metric is canonical conjugate to the normal derivative of the dilaton field (both are pointing inwards). By a linear combination of the Hamiltonian constraints (3.3.44), one can

⁶If one understand getting $|E\rangle$ state from integrating over thermofield double state in time direction, then a natural estimate on δE is $\frac{1}{T}$, where T is the total time one integrate over. The validity of JT description of boundary theory is $T < e^{S_0}$, and we get $\delta E > e^{-S_0}$. For $T > e^{S_0}$, there are other possible instanton contributions. The proper gravitational theory at this regime is studied in paper [69]

construct the following gauge invariant quantity C :

$$-\frac{1}{\sigma}(\phi'\mathcal{H} + \Pi_\sigma\mathcal{H}_x) = \frac{1}{2}(\Pi_\sigma^2 + \phi^2 - \frac{\phi'^2}{\sigma^2})' \equiv C[\Pi_\sigma, \phi, \sigma]' \sim 0 \quad (3.3.46)$$

The Dirac quantization scheme then tells us that the quantity C has a constant mode which is gauge invariant (commute with Hamiltonian constraint). Choosing the gauge that normal derivative of the dilaton is zero, we can solve the Hamiltonian constraint:

$$\phi^2 - (\partial_X\phi)^2 = 2C \equiv S^2 \quad \rightarrow \quad \phi(X) = S \cosh X, \quad (3.3.47)$$

where $dX = \sigma dx$ is the proper distance along the spatial slice. Because the normal derivative of dilaton field is zero, the minimum value of dilaton at this spatial slice is actually a local extremum in both directions. Therefore, the minimal value of dilaton field, namely S , is a global variable. The classical geometry in this gauge is a ‘‘Pac-Man’’ shape (right figure in figure 3.4). Focusing on the intersection region of the spatial slice and the boundary, we have the spatial slice is orthogonal to the boundary. This is because we are gauge fixing $\partial_n\phi = 0$ on the spatial slice, and $\phi = \phi_b$ on the boundary. The ADM mass of the system, after regularization, is then $M = \phi_b(\phi_b - \partial_X\phi)$ [29]. Substituting the behavior of $\phi(X)$ we get:

$$M = \frac{S^2}{2}. \quad (3.3.48)$$

This is the same relation in 3.2.20 and therefore we can interpret the s variable as entropy of our system S .

For the purpose of fixing geodesic distance, it is convenient to think of doing the path integral up to a slice L with zero extrinsic curvature. This picks out a particular slice (left figure in Figure 3.4) among the solutions obeying the Hamiltonian constraint. The WdW wavefunction can be evaluated as an Euclidean path integral with fixed (rescaled) geodesic

distance d between the two boundary points:

$$\begin{aligned}\Psi(u; d) &= \int \mathcal{D}g \mathcal{D}\phi e^{\frac{1}{2} \int \phi(R+2) + \int_L \phi K + \phi_b \int_{Bdy} (K-1)} = \int \mathcal{D}f e^{\phi_b \int_{Bdy} (K-1)} \\ &= \int \mathcal{D}\mathbf{x} e^{-m \int_{Bdy} \sqrt{g} + q \int_{Bdy} a + q \int_L a + (\pi - \alpha_1 - \alpha_2) q}\end{aligned}\quad (3.3.49)$$

Here we are fixing the total length of L to be d and the proper length of the boundary to be u . α_1 and α_2 in the last expression denotes the jump angle at the corner coming from the singular contribution of the extrinsic curvature and should be integrated over. Without the $e^{q \int_L a}$ factor in (3.3.49), the path integral corresponds to the propagator (3.3.29). Remember that the phase factor is equal to $e^{-q \int_L a}$, so the wavefunction in ℓ basis is actually the propagator (3.3.29), with the phase factor stripped out

$$\Psi(u; \ell) \equiv \langle \ell | u \rangle_G = \frac{2}{\pi^2 \ell} \int_0^\infty ds s \sinh(2\pi s) e^{-\frac{s^2}{2} u} K_{2is}\left(\frac{4}{\ell}\right), \quad \ell = \frac{|x_1 - x_2|}{\sqrt{z_1 z_2}}. \quad (3.3.50)$$

ℓ is a function of the regularized geodesic distance d between \mathbf{x}_1 and \mathbf{x}_2 : $\ell = e^{\frac{d}{2}}$. The semiclassical of $\Psi(u; \ell)$ can be obtained using formula (3.3.30), in the exponent we get saddle point result:

$$\Psi(u; \ell) \sim \exp\left[-\frac{2(\xi_* - i\pi)^2}{u} + \frac{4\xi_* - i\pi}{u \tanh \xi_*}\right]; \quad \frac{\xi_* - i\pi}{u} = -\frac{\sinh \xi_*}{\ell}. \quad (3.3.51)$$

The same saddle point equation and classical action was obtained in [68] by a direct evaluation in JT gravity.

The wavefunction with fixed energy boundary condition can be obtained by multiplying $\Psi(u; \ell)$ by e^{+Eu} and integrating over u along the imaginary axis. This sets $E = \frac{s^2}{2}$ in the

above integral over s . So this wavefunction has a very simple expression:

$$\Psi(E; \ell) \equiv \langle \ell | E \rangle_G = \rho(E) \frac{4}{\ell} K_{i\sqrt{8E}}\left(\frac{4}{\ell}\right). \quad (3.3.52)$$

The classical geometry for $\Psi(E; \ell)$ is the same as the left figure in Figure 3.4, with fixing energy on the boundary. We want to stress that it is important to have the $\rho(E)$ factor in (3.3.52) for a classical geometry description since we are averaging over the states. We can roughly think of $\frac{4}{\ell} K_{i\sqrt{8E}}\left(\frac{4}{\ell}\right)$ as a gravitational “microstate” $|\mathcal{E}\rangle$ with fixed energy E . Such a “microstate” will not have a classical geometry representation and therefore is just a formal definition. The inner product between wavefunctions is defined as $\langle \Psi_1 | \Psi_2 \rangle = \int_0^\infty d\ell \Psi_1^*(\ell) \Psi_2(\ell)$.

Going to the entropy basis S , it is easy to start with $\Psi(E)$. Because of the identity $E = \frac{S^2}{2}$, expanding $\Psi(E)$ in the S basis is diagonal:

$$\Psi(E; S) \equiv \langle S | E \rangle_G = \sqrt{\rho(S)} \delta\left(E - \frac{S^2}{2}\right) \quad (3.3.53)$$

We put this square root of $\rho(S)$ factor in the definition of S basis such that inner product between different S state is a delta function $\langle S | S' \rangle = \delta(S - S')$. This factor is also required such that the classical limit matches with gravity calculation. Integrating over energy with Boltzman distribution, we can get the expression of thermofield double state in the S basis:

$$\Psi(u; S) \equiv \langle S | u \rangle_G = \int dE e^{-uE} \langle S | E \rangle_G = \sqrt{\rho(S)} e^{-\frac{uS^2}{2}} \quad (3.3.54)$$

In the semiclassical limit, the wavefunction becomes gaussian and coincides with the on shell evaluation of the “Pac-Man” geometry (Figure 3.4):

$$\Psi(u, S) \sim \sqrt{S} e^{\pi S - \frac{uS^2}{2}} \quad (3.3.55)$$

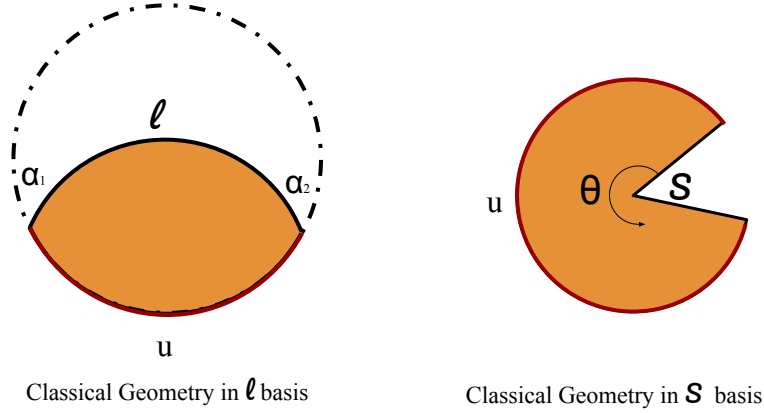


Figure 3.4: Classical Geometry in ℓ and S basis

The on shell calculation is straightforward: JT action in this geometry contains two parts: the Schwarzian action $\int(K - 1)$ on the boundary and a corner contribution at the center: $S(\pi - \theta)$, where θ is the span angle at the horizon (Figure 3.4). The Schwarzian action simply gives $Eu = \frac{S^2 u}{2}$ by direct evaluation. We can determine θ from u since they are related with redshift: $\theta = uS$. Therefore the corner term gives: $\pi S - S^2 u$. Adding them up then gives us the classical action. We can also expand S in terms of the ℓ basis, and relate $\Psi(\ell)$ with $\Psi(S)$ by a change of basis:

$$\langle \ell | S \rangle = \sqrt{\rho(S)} \frac{4}{\ell} K_{2iS} \left(\frac{4}{\ell} \right); \quad \Psi(E; \ell) = \int_0^\infty dS \langle \ell | S \rangle \langle S | E \rangle_G; \quad (3.3.56)$$

Before discussing our last basis, we want to stress the simplicity of the wavefunction in S basis (3.3.55) and the Gaussian factor resembles an ordinary particle wavefunction in momentum basis. We introduce our last basis η as canonical conjugate variable of S , with

an analog of going to position space of the particle picture in mind:

$$|\eta\rangle = \int_0^\infty dS \cos(\eta S) |S\rangle; \quad \langle \ell | \eta \rangle = \int_0^\infty dS \cos(\eta S) \sqrt{\rho(S)} K_{2iS} \left(\frac{4}{\ell} \right) \quad (3.3.57)$$

$$\Psi(E, \eta) = \sqrt{\frac{\sinh(2\pi\sqrt{2E})}{2\pi^2\sqrt{2E}}} \cos(\eta\sqrt{2E}); \quad \Psi(u, \eta) = \int_0^\infty dS \sqrt{\rho(S)} \cos(\eta S) e^{-\frac{uS^2}{2}}. \quad (3.3.58)$$

To understand the meaning of η better, we can look at the classical behavior of $\Psi(u; \eta)$:

$$\Psi(u; \eta) \sim \frac{1}{u} \exp\left[\frac{\pi^2}{2u} - \frac{\eta^2}{2u}\right] \left(e^{\frac{i\eta\pi}{u}} \sqrt{\pi + i\eta} + e^{-\frac{i\eta\pi}{u}} \sqrt{\pi - i\eta} \right) \quad (3.3.59)$$

When u is real, the wavefunction is concentrated at $\eta = 0$ and has classical action of a half disk in the exponent. When $u = \frac{\beta}{2} + it$ which corresponds to the case of analytically continuing into Lorentzian signature, the density of the wavefunction $|\Psi(u, \eta)|^2$ is dominated by:

$$|\Psi(\frac{\beta}{2} + it, \eta)|^2 \sim \frac{\sqrt{\pi^2 + \eta^2}}{\beta^2 + 4t^2} \exp\left[\frac{2\pi^2}{\beta}\right] \left(\exp\left[-\frac{2\beta(\eta - \frac{2\pi t}{\beta})^2}{\beta^2 + 4t^2}\right] + \exp\left[-\frac{2\beta(\eta + \frac{2\pi t}{\beta})^2}{\beta^2 + 4t^2}\right] \right) \quad (3.3.60)$$

showing the fact that η is peaked at the Rindler time $\frac{2\pi t}{\beta}$. We can therefore think of fixing η as fixing the IR time or the boost angle at the horizon. The classical intuition for the boost angle is most clear in Euclidean geometry, where for fixed boundary proper time there can be different cusps at the horizon (Figure 3.5).

One application of those wavefunctions is that we can take an inner product and get the partition function. However, there are also other ways to get the partition function. For example, we can concatenate three propagators and integrate over their locations. This also gives the partition function by the composition rule of propagator. By the relation between propagator and wavefunction, we can also view this as taking an inner product of three wavefunctions with an interior state as in figure 3.6, where the interior state

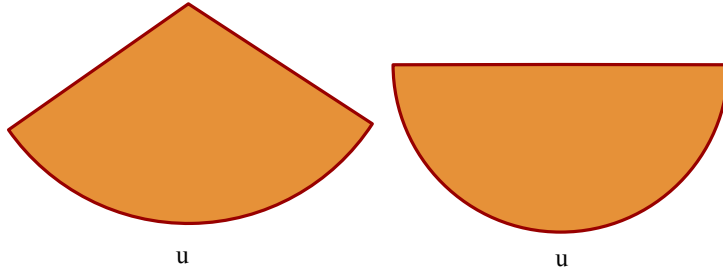


Figure 3.5: Euclidean geometries with different cusps.

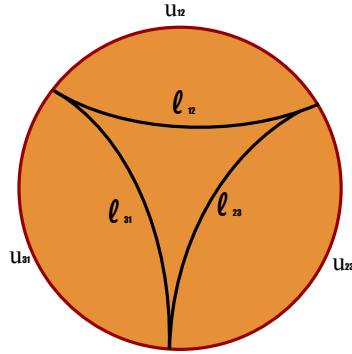


Figure 3.6: Partition function from inner product of three wavefunctions.

can be understood as an entangled state for three universes. To be more precise, we can view the wavefunction as the result of integrating the bulk up to the geodesics with zero extrinsic curvature. Then the interior state is given by the area of the hyperbolic triangle in figure 3.6. The path integral for the hyperbolic triangle (denoted as $I(\ell_{12}, \ell_{23}, \ell_{31})$, where $\ell_{ij} = \frac{|x_i - x_j|}{\sqrt{z_i z_j}}$), is a product of three phase factors, which satisfies a nontrivial equality (with

ordering $x_1 > x_2 > x_3$):

$$I(\ell_{12}, \ell_{23}, \ell_{31}) = e^{-2\left(\frac{z_1+z_2}{x_1-x_2} + \frac{z_2+z_3}{x_2-x_3} + \frac{z_3+z_1}{x_3-x_1}\right)} = \frac{16}{\pi^2} \int_0^\infty d\tau \tau \sinh(2\pi\tau) K_{2i\tau}\left(\frac{4}{\ell_{12}}\right) K_{2i\tau}\left(\frac{4}{\ell_{23}}\right) K_{2i\tau}\left(\frac{4}{\ell_{31}}\right). \quad (3.3.61)$$

Recalling that the Bessel function represents the fixed energy “microstate” $|\mathcal{E}\rangle$ (3.3.52) and $\frac{\tau}{2\pi^2} \sinh(2\pi\tau)$ is the density of state, this formula tells us that the interior state is a GHZ state for three universe:

$$I_{123} \sim \sum_n |\mathcal{E}_n\rangle_1 |\mathcal{E}_n\rangle_2 |\mathcal{E}_n\rangle_3. \quad (3.3.62)$$

I can also be viewed as a scattering amplitude from two universes into one universe. It constrains the $SL(2, \mathbb{R})$ representation of the three wavefunctions to be the same.⁷ We can write down the partition function as:

$$Z_{JT} = \int_0^\infty \prod_{\{ij\} \in \{12, 23, 31\}} dl_{ij} \Psi(u_{12}, \ell_{12}) \Psi(u_{23}, \ell_{23}) \Psi(u_{31}, \ell_{31}) I_{\ell_{12}, \ell_{23}, \ell_{31}}. \quad (3.3.63)$$

This same result also holds if we repeat the process n times. It is interesting that we can view the full disk amplitude in these various ways.

One can also extend our analysis to include matter field. One type of such wavefunction can be created by inserting operator during Euclidean evolution, and is analysed in appendix B. Note that because of the $SL(2, \mathbb{R})$ symmetry is a gauge symmetry, our final state has to be a gauge singlet including matter field.

⁷Some thing similar happens for 2d Yang-Mills theory [71, 72, 65].

3.4 Correlation Functions in Quantum Gravity

3.4.1 Gravitational Feynman Diagram

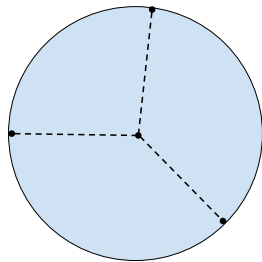
The propagator enables us to “dress” quantum field theory correlators to produce quantum gravity ones. Namely, we imagine that we have some quantum field theory in H_2 and we compute correlation functions of operators as we take the points close to the boundary where they take the form

$$\langle O_1(\mathbf{x}_1) \dots O_n(\mathbf{x}_n) \rangle_{\text{QFT}} = q^{-\sum \Delta_i} z_1^{\Delta_1} \dots z_n^{\Delta_n} \langle O_1(x_1) \dots O_n(x_n) \rangle_{\text{CFT}} \quad (3.4.64)$$

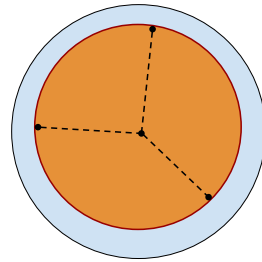
The factor of q arises from (3.3.27), and the last factor is simply defined as the function that results after extracting the z dependence. For example, for a two point function we get

$$\langle O_1(\mathbf{x}_1) O_2(\mathbf{x}_2) \rangle_{\text{QFT}} = q^{-2\Delta} z_1^\Delta z_2^\Delta \frac{1}{|x_1 - x_2|^{2\Delta}}. \quad (3.4.65)$$

We can now use the propagator (3.3.29) to couple the motion of the boundary and thus obtain the full quantum gravity expression for the correlator. The factors of q are absorbed as part of the renormalization procedure for defining the full quantum gravity correlators. In this way we obtain



(a) Witten Diagram



(b) Gravitational Feynman Diagram

Figure 3.7: Summation of $\frac{1}{N}$ effects fluctuates the boundary of Witten Diagram

$$\begin{aligned} \langle O_1(u_1) \dots O_n(u_n) \rangle_{\text{QG}} &= e^{2\pi q} \int \frac{\prod_{i=1}^n \frac{dx_i dy_i}{y_i^2}}{V(\text{SL}(2, \mathbb{R}))} G(u_{12}, \mathbf{x}_1, \mathbf{x}_2) G(u_{23}, \mathbf{x}_2, \mathbf{x}_3) \dots G(u_{n1}, \mathbf{x}_n, \mathbf{x}_1) \times \\ &\times \langle O_1(\mathbf{x}_1) \dots O_n(\mathbf{x}_n) \rangle_{\text{QFT}} q^{\sum_i \Delta_i} \end{aligned} \quad (3.4.66)$$

where the left hand side is the full quantum gravity correlator by definition. The last factor is the usual renormalization necessary to get something finite.

The factor of $e^{2\pi q}$ cancels with the q dependent ‘‘phase’’ factors in (3.3.28) to give one if we order the points cyclically ($x_1 > x_2 \dots > x_n$). This requires that we define more carefully the last propagator $G(u_{n1}, \mathbf{x}_n, \mathbf{x}_1)$ as:

$$e^{-2\pi q} \tilde{K}(u_{n1}, \mathbf{x}_n, \mathbf{x}_1) = e^{-2\pi q} e^{-2\frac{z_n+z_1}{x_n-x_1}} \frac{2\sqrt{z_n z_1}}{\pi^2 |x_n - x_1|} \int_0^\infty ds s \sinh(2\pi s) e^{-\frac{s^2}{2} u_{n1}} K_{2is} \left(\frac{4\sqrt{z_n z_1}}{|x_n - x_1|} \right). \quad (3.4.67)$$

The factor $\frac{1}{V(\text{SL}(2, \mathbb{R}))}$ in (3.4.66) means that we should fix the $SL(2, R)$ gauge symmetry (Appendix B.1).

In the end we can write down an expression where we have already taken the $q \rightarrow \infty$ limit

$$\boxed{\langle O_1(u_1) \dots O_n(u_n) \rangle_{\text{QG}} = \int_{x_1 > x_2 \dots > x_n} \frac{\prod_{i=1}^n dx_i dz_i}{V(\text{SL}(2, \mathbb{R}))} \tilde{K}(u_{12}, \mathbf{x}_1, \mathbf{x}_2) \dots \tilde{K}(u_{n1}, \mathbf{x}_n, \mathbf{x}_1) z_1^{\Delta_1-2} \dots z_n^{\Delta_n-2} \langle O_1(x_1) \dots O_n(x_n) \rangle_{\text{CFT}}.} \quad (3.4.68)$$

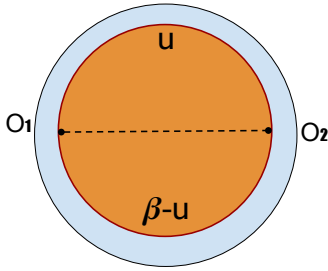
This is one of the main results of our paper and it gives a detailed expression for correlation function in 2 dimensional quantum gravity in terms of the correlation functions of the QFT in hyperbolic space, or AdS_2 .

Notice that in usual AdS/CFT the correlators $\langle O_1(x_1) \dots O_n(x_n) \rangle_{\text{CFT}}$ are an approximation to the full answer. This is sometimes computed by Witten diagrams. We get a better approximation by integrating over the metric fluctuations. In this case, the non-trivial gravitational mode is captured by the boundary propagator. The formula (3.4.68) includes all the effects of quantum gravity in the JT theory (in the Schwarzian limit). The

final diagrams consist of the Witten diagrams for the field theory in AdS plus the propagators for the boundary particle and we can call them “Gravitational Feynman Diagrams”, see figure 3.7.

3.4.2 Two Point Function

Using formula (3.4.68), we can study gravitational effects on bulk fields such as its two point function:

$$\langle O_1(u)O_2(0) \rangle_{QG} = \text{Diagram} \quad (3.4.69)$$


The explicit expression for $\langle O_1(u)O_2(0) \rangle_{QG}$ with dimension Δ at temperature $\frac{1}{\beta}$ is ⁸:

$$\frac{1}{V(\text{SL}(2,\mathbb{R}))} \int_{x_1 > x_2} \frac{dx_1 dx_2 dz_1 dz_2}{z_1^2 z_2^2} \int_0^\infty ds_1 ds_2 \rho(s_1) \rho(s_2) e^{-\frac{s_1^2}{2}u - \frac{s_2^2}{2}(\beta-u)} K_{2is_1} \left(\frac{4\sqrt{z_1 z_2}}{|x_1 - x_2|} \right) K_{2is_2} \left(\frac{4\sqrt{z_1 z_2}}{|x_1 - x_2|} \right) \left(\frac{\sqrt{z_1 z_2}}{|x_1 - x_2|} \right)^{2\Delta+2}. \quad (3.4.70)$$

To fix the $SL(2, R)$ gauge, we can choose $z_1 = z_2 = 1$ and $x_2 = 0$. Then the integral over H_2 space is reduced to a single integral over x_1 , with a Jacobian factor $2x_1$ (Appendix B.1):

$$\int_0^\infty ds_1 ds_2 \rho(s_1) \rho(s_2) e^{-\frac{s_1^2}{2}u - \frac{s_2^2}{2}(\beta-u)} \int_0^\infty dx_1 \left(\frac{1}{x_1} \right)^{2\Delta+1} K_{2is_1} \left(\frac{4}{x_1} \right) K_{2is_2} \left(\frac{4}{x_1} \right). \quad (3.4.71)$$

⁸We will just keep the Δ dependent constant since at last we will normalized with respect of partition function which corresponds to set $\Delta = 0$.

the last integral can be interpreted as a matrix element of two point operator $O_1 O_2$ between states $|E_1, \psi\rangle$ and $|E_2, \psi\rangle$, where $|\psi\rangle$ is the wavefunction of quantum field theory and $|E\rangle_G$ represents the fixed energy gravitational state. Integrating over x can be thought as integrating over a particular gravitational basis, and we can see that the gravity wavefunction suppress the UV contributions from quantum field theory ($K_{2is}(\frac{4}{x}) \sim \sqrt{\frac{\pi x}{8}} e^{-4/x}$ for $x \sim 0$). The final expression for the two point function is:

$$\langle O_1(u) O_2(0) \rangle_{QG} = \frac{1}{\mathcal{N}} \int ds_1 ds_2 \rho(s_1) \rho(s_2) e^{-\frac{s_1^2}{2}u - \frac{s_2^2}{2}(\beta-u)} \frac{|\Gamma(\Delta - i(s_1 + s_2)) \Gamma(\Delta + i(s_1 - s_2))|^2}{2^{2\Delta+1} \Gamma(2\Delta)} \quad (3.4.72)$$

$$= \frac{1}{\mathcal{N}} \frac{\Gamma(2\Delta)}{u^{3/2}(\beta-u)^{3/2} 2^{4\Delta+4} \pi^3} \int_{c-i\infty}^{c+i\infty} d\theta_1 d\theta_2 \theta_1 \theta_2 e^{\frac{\theta_1^2}{2u} + \frac{\theta_2^2}{2(\beta-u)}} \frac{1}{(\cos \frac{\theta_1}{2} + \cos \frac{\theta_2}{2})^{2\Delta}}. \quad (3.4.73)$$

In the second expression we write the integral in terms of variable θ using the second integral representation of the propagator (3.3.30). The normalization constant can be determined by taking the $\Delta = 0$ limit: $\mathcal{N} = Z_{JT}$.

If we contemplate the result (3.4.72) a little bit, then we find that the two integrals of s_1 and s_2 just represent the spectral decomposition of the two point function. Indeed, under spectral decomposition we have $\langle O(u) O(0) \rangle = \sum_{n,m} e^{-E_n u - E_m(\beta-u)} |\langle E_n | O | E_m \rangle|^2$. Compare with (3.4.72), we can read out the square of matrix element of operator O :

$${}_G \langle E_1 | O_L O_R | E_2 \rangle_G = \delta E^{-2} \sum_{\substack{|E_n - E_1| < \delta E \\ |E_m - E_2| < \delta E}} |\langle E_n | O | E_m \rangle|^2 = \rho(E_1) \rho(E_2) \frac{|\Gamma(\Delta - i(\sqrt{2E_1} + \sqrt{2E_2})) \Gamma(\Delta + i(\sqrt{2E_1} - \sqrt{2E_2}))|^2}{2^{2\Delta+1} \Gamma(2\Delta)}. \quad (3.4.74)$$

Remember the notation is that $|E\rangle_G$ stands for a gravitational state with energy E and $|E_n\rangle$ stands for one side microstate (3.3.42). We have put the measure $\rho(E) = \frac{1}{2\pi^2} \sinh(2\pi\sqrt{2E})$ in the definition of matrix element for the reason that in gravity it is more natural to consider an average of energy states as a bulk state. To understand this formula a little bit better, we can consider the classical limit, namely large E . In this limit the matrix

element squared can be approximated as a nonanalytic function:

$${}_G\langle E_1|O_L O_R|E_2\rangle_G \propto |E_1 - E_2|^{2\Delta-1} e^{2\pi\min(\sqrt{2E_1}, \sqrt{2E_2})}. \quad (3.4.75)$$

If we fix E_1 and varying E_2 from 0 to infinity, the matrix element changes from $|E_1 - E_2|^{2\Delta-1}\rho(E_2)$ to $|E_1 - E_2|^{2\Delta-1}\rho(E_1)$ after E_2 cross E_1 . We can understand this behavior qualitatively as a statistical effect: the mapping from energy subspace E_1 to E_2 by operator O is surjective when the Hilbert space dimension of E_2 is less than E_1 and is injective otherwise. Another understanding is the following: the two point function is finite in a fixed energy state $|E_n\rangle$, which means the following summation of intermediate states $|E_m\rangle$ is order one: $\sum_m |\langle E_n|O|E_m\rangle|^2$. Looking at the case $E_m > E_n$, because of the density of states grows rapidly, the matrix element squared has to be proportional to $\frac{1}{\rho(E_m)}$ to get a finite result. Multiplied by $\rho(E_n)\rho(E_m)$, we have $\rho(E_n)\rho(E_m)|\langle E_n|O|E_m\rangle|^2 \sim \rho(\min(E_n, E_m))$.

3.4.3 ETH and the KMS condition

The Eigenstate Thermalization Hypothesis (ETH) is a general expectation for chaotic system. It expresses that the operator expectation value in an energy eigenstate can be approximated by thermal expectation value with effective temperature determined from the energy. Such hypothesis can be tested with the knowledge of operator matrix elements.

The two point function in microcanonical ensemble is:

$$\begin{aligned} \frac{1}{\delta E} \sum_{|E_n - E| < \delta E} \langle E_n|O(u)O(0)|E_n\rangle &= \rho(E)e^{uE} \langle E|Oe^{-uH}O|E\rangle \\ \langle E|Oe^{-uH}O|E\rangle &= \int_0^\infty ds \rho(s) e^{-\frac{s^2}{2}u} \frac{|\Gamma(\Delta + i(s + \sqrt{2E}))|^2 |\Gamma(\Delta + i(s - \sqrt{2E}))|^2}{2^{2\Delta+1}\Gamma(2\Delta)} \end{aligned} \quad (3.4.76)$$

Notice that $|E\rangle$ is not $|E\rangle_G$, the former represents a one side microstate, while the later is a gravitational state. Accordingly $\langle E|O(u)O(0)|E\rangle$ stands for a two point function in a microstate. To study ETH, we will consider the case of a heavy black hole $E = \frac{S^2}{2} = \frac{2\pi^2}{\beta^2} \gg 1$. From the discussion in last section, we know that the matrix element tries to concentrate s around $\sqrt{2E}$ and thus we can approximate $\rho(E)\rho(s)|\Gamma(\Delta + i(s + S))|^2$ as proportional to $sS^{2\Delta-1}e^{\pi(s+S)}$. Using integral representation for $|\Gamma(\Delta + i(s - S))|^2$ we derive the two point function in microcanonical essemble with energy E is proportional to:

$$\frac{\rho(E)S^{2\Delta-1}}{u^{3/2}} \int d\xi (\pi + i\xi) e^{-\frac{2}{u}(\xi - i\frac{\pi}{2})^2 - (\pi + 2i\xi)S + u\frac{S^2}{2}} \frac{1}{(\cosh \xi)^{2\Delta}}. \quad (3.4.77)$$

The ξ variable can be understood as the measure of time separation in units of effective temperature between two operators and its fluctuation represents the fluctuation of the effective temperature. And the final integral can be understood as a statistical average of correlation functions with different temperatures. If we put back the Newton Constant $G_N = \frac{1}{N}$, we have $S \sim N$ and $u \sim N^{-1}$. As can be seen from the probability distribution, the fluctuation is of order $\frac{1}{\sqrt{N}}$, and hence for large N system we can use saddle point approximation:

$$\xi = i\left(\frac{\pi}{2} - \frac{S}{2}u\right) - \frac{u\Delta \tanh \xi}{2} = i\left(\frac{\pi}{2} - \frac{\pi u}{\beta}\right) - \frac{u\Delta \tanh \xi}{2}. \quad (3.4.78)$$

The first piece gives the typical temperature of the external state, while the last piece comes from the backreaction of operator on the geometry. If we first take the limit of large N, one simply get that the two point function in microcanonical essemble is the same as canonical essemble. However, the euclidean correlator in canonical essemble is divergent as euclidean time approach to inverse temperature β because of KMS condition. Such singular behavior plays no role in the microcanonical essemble so is called a “forbidden singularity” in ETH

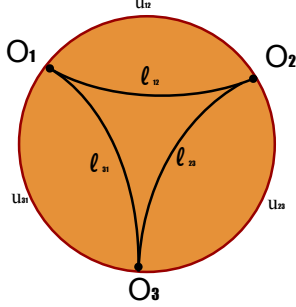


Figure 3.8: Bulk Diagram for Three Point Function

[73, 74]. In our situation we can see directly how the forbidden singularity disappears in the microcanonical ensemble. When ξ approach $-i\frac{\pi}{2}$ at the forbidden singularity the backreaction on the geometry becomes large and hence the effective temperature becomes lower:

$$\frac{2\pi}{\beta^*} \rightarrow \frac{2\pi}{\beta} - \frac{\Delta}{\pi - \frac{\pi u}{\beta}}. \quad (3.4.79)$$

At the time $\frac{\beta-u}{\beta} \sim \frac{\Delta}{N}$, the backreaction is important and we expect to see deviation from thermal correlators. Therefore the correlation function in microcanonical ensemble will never have singularity away from coincide point.

3.4.4 Three Point Function

The bulk diagram of the three point function will be like Figure 3.6 with additional operator inserting at the intersection points (See Figure 3.8). The QFT three point correlation function in AdS_2 is fixed by conformal symmetry and we can write it down as:

$$\langle O_1 O_2 O_3 \rangle = C_{123} \frac{z_1^{\Delta_1} z_2^{\Delta_2} z_3^{\Delta_3}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{13}|^{\Delta_1+\Delta_3-\Delta_2}} = \frac{C_{123}}{\ell_{12}^{\Delta_1+\Delta_2-\Delta_3} \ell_{23}^{\Delta_2+\Delta_3-\Delta_1} \ell_{13}^{\Delta_1+\Delta_3-\Delta_2}}. \quad (3.4.80)$$

Δ_i is the conformal dimension of O_i . Putting them in formula 3.4.68 and rewrite the propagator in terms of the wavefunction (3.3.50), we have the quantum gravitational three point function:

$$\langle O_1 O_2 O_3 \rangle_{QG} = \int_{x_1 > x_2 > x_3} \frac{\prod_{i=1}^3 dx_i dz_i}{V(\text{SL}(2, \mathbb{R}))} \Psi_{u_{12}, \ell_{12}} \Psi_{u_{23}, \ell_{23}} \Psi_{u_{13}, \ell_{13}} I_{\ell_{12}, \ell_{23}, \ell_{13}} \frac{C_{123}}{\ell_{12}^{\Delta_1 + \Delta_2 - \Delta_3} \ell_{23}^{\Delta_2 + \Delta_3 - \Delta_1} \ell_{13}^{\Delta_1 + \Delta_3 - \Delta_2}}. \quad (3.4.81)$$

We can view this expression as an inner product of three universe wavefunction with the interior, inserting three bilocal operators $\tilde{O}_{ij;k} \tilde{O}_{ij;k}$ with dimension $\tilde{\Delta}_{ij;k} = \frac{1}{2}(\Delta_i + \Delta_j - \Delta_k)$ between them. One can fix the $SL(2, R)$ symmetry and express the integral in terms of ℓ_{ij} , it is the same exercise as in open string calculation to find the Jacobian factor (Appendix B.1). Here we can just argue that in order to get the partition function at $\Delta = 0$, the measure has to be flat. Therefore the three point function factorizes into form:

$$\langle O_1(u_1) O_2(u_2) O_3(u_3) \rangle_{QG} \propto C_{123} \int_0^\infty d\tau \rho(\tau) \mathcal{I}_\tau(u_{12}, \tilde{\Delta}_{12;3}) \mathcal{I}_\tau(u_{23}, \tilde{\Delta}_{23;1}) \mathcal{I}_\tau(u_{31}, \tilde{\Delta}_{31;2}) \quad (3.4.82)$$

while $\mathcal{I}_\tau(u_{ij}, \tilde{\Delta}_{ij;k})$ is an integral of ℓ_{ij} which gives the two point function in microstate E_τ (3.4.76) with the e^{uE_τ} factor stripped off:

$$\mathcal{I}_\tau(u_{ij}, \tilde{\Delta}_{ij;k}) = \frac{1}{2} \int_0^\infty d\ell_{ij} \Psi_{u_{ij}, \ell_{ij}} \frac{1}{\ell_{ij}^{\Delta_i + \Delta_j - \Delta_k}} K_{2i\tau} \left(\frac{4}{\ell_{ij}} \right) = \langle E_\tau | \tilde{O}_{ij;k} e^{-u_{ij} H} \tilde{O}_{ij;k} | E_\tau \rangle; \quad E_\tau = \frac{\tau^2}{2}. \quad (3.4.83)$$

Again the normalization constant can be fixed by choosing O_i to be identity.

3.4.5 Einstein-Rosen Bridge

The Einstein-Rosen Bridge in a classical wormhole keeps growing linearly with time and this behavior was conjectured to related with the growth of computational complexity of the dual quantum state [75]. Based on the universal behavior of complexity growth, Susskind proposed a gravitational conjecture in a recent paper [76] about the limitation

of classical general relativity description of black hole interior. The conjecture was stated as follows:

Classical general relativity governs the behavior of an ERB for as long as possible.

In this section, we will test this conjecture using the exact quantum wavefunction of JT gravity (3.3.50). We will in particular focusing on the behavior of ERB at time bigger than 1. The size of Einstein-Rosen Bridge \mathcal{V} in two dimensions is the geodesic distance d between two boundaries, and can be calculated in thermofield double state $|u\rangle$ as:

$$\mathcal{V} = \langle u|d|u\rangle \tag{3.4.84}$$

We want to focus on the dependence of volume on Lorentzian time evolution. Therefore we do analytic continuation of u in Lorentzian time: $u = \frac{\beta}{2} + it$. Using the WdW wavefunction in ℓ basis (3.3.50) and the relation between d and ℓ , we can calculate the expectation value exactly. This can be done by taking the derivative of the two point function (3.4.72) with respect to Δ at $\Delta = 0$. Using the integral representation for $|\Gamma(\Delta + i(s_1 - s_2))|^2$, the only time dependence of volume is given by:

$$\mathcal{V}(t) = \frac{1}{\mathcal{N}} \int_{-\infty}^{\infty} d\xi \int_0^{\infty} ds_1 ds_2 \rho_1 \rho_2 e^{i(s_1 - s_2)\xi - i(s_1 - s_2)\frac{(s_1 + s_2)}{2}t - \frac{\beta}{4}(s_1^2 + s_2^2)} \log(2 \cosh \frac{\xi}{2}) |\Gamma(is_1 + is_2)|^2. \tag{3.4.85}$$

The limit we are interested in is $\beta \ll 1 \ll t$ ⁹, in which case the integral has a saddle point at¹⁰:

$$s_{1,2} = \frac{2\pi}{\beta}; \quad \xi = \frac{2\pi t}{\beta}. \tag{3.4.86}$$

⁹Remember that we are measuring time in units of ϕ_r , so time order 1 is a quantum gravity region.

¹⁰Actually this saddle point is valid for any range of t as long as $\beta \ll 1$.

Therefore the volume has linear dependence in time:

$$\mathcal{V}(t) \sim \frac{2\pi t}{\beta}. \quad (3.4.87)$$

Using the complexity equal to volume conjecture [75, 77], the complexity of thermofield double state is proportional to the maximum volume:

$$\mathcal{C}(t) = \#\mathcal{V}(t) = \#\frac{2\pi t}{\beta}. \quad (3.4.88)$$

The proportionality constant is suggested in [78] to be S_0 based on classical calculation of near extremal black hole. This, however, is not very clear in our model since S_0 is the coupling constant of the pure Einstein-Hilbert action and decouples with JT theory (2.2.7). Since the saddle point (3.4.86) is actually valid from early time to late time, the proportionality constant can be fixed at classical level and once we fix it we can conclude that the length of Einstein-Rosen Bridge (or complexity of the state) keeps linearly growing even considering quantum gravity effects in JT theory. We want to comment that this is not an obvious result that one can expect from classical observables. For example, one might argue that we can extract the information of the ERB from two sided correlators for the reason that semiclassically we can approximate the correlator as e^{-md} . Therefore one can conclude the ERB has linear growth from the quasinormal behavior of the correlator. However such observables can only give us information of ERB up to time order 1, which is the same time scale we can trust the classical general relativity calculation. After that the correlation function changes from exponential decay into universal power law decay $\frac{1}{t^3}$ as one can directly derive from analytic continuation of result (3.4.72). If we still use such correlator to extract information about ERB we would get the wrong conclusion that it stops its linear growth after time order 1. The reason why it is incorrect is that at this

time scale the operator disturbs the state and causes different energy states interfere each other strongly. It is simply that the correlator can no longer be described by the classical geometry, rather than the interior stops to behave classically. From our calculation, we see that if we probe of the state in a weaker and weaker way, we are still able to see the classical geometry. Lastly, we want to talk a little about when JT gravity needs to be modified. A naive estimate can be made from the partition function that when β approaches $e^{\frac{2}{3}S_0}$, the partition function becomes less than one and definitely at this time scale we need new physics. A recently study of gravitational physics at this time scale was discussed in [69, 79].

Chapter 4

Conclusion

4.1 Discussion

We have pointed out how the asymptotic symmetries of AdS_2 can be used to determine many aspects of the gravitational dynamics of nearly AdS_2 spacetimes, or $NAdS_2$. The essential feature is the emergence of a reparametrization symmetry which is both spontaneously and explicitly broken. The corresponding pseudo-Goldstone bosons are described by a reparametrization $t(u)$ that expresses AdS time t in terms of the physical boundary time u . The explicit breaking leads to a Schwarzian action for $t(u)$ (2.2.15). In addition, we also have a simple coupling to bulk fields (2.3.25). These together give rise to several features of $NAdS_2$ or near extremal black holes. These include the computation of the near extremal free energy as well as several gravitational effects involving correlation functions. These include gravitational corrections to the four point function (2.3.30) and (2.3.33) as well as corrections to the two point function (2.3.36).

Our result gives an explicit formula (3.4.68) to calculate all order corrections to correlation functions from quantum gravity in two dimensions. The formula can be understood diagrammatically and we call it Gravitational Feynman Diagram. We also give the exact

Wheeler-DeWitt wavefunction and discuss the growth of its complexity quantum mechanically.

For all these features it was important to assume that the Schwarzian action was the leading effect that breaks the reparametrization symmetry. This is the case in many interesting physical situations. However, one can imagine cases involving AdS_2 spaces with particularly light fields, dual to operators with dimensions $1 < \Delta < 3/2$. In these cases, if these fields are excited, then we have larger irrelevant perturbations and the infrared dynamics is different. See appendix A.4 for a detailed discussion.

The Schwarzian action involves higher derivative terms, which raise ghost fears. The ghosts are made invisible by treating the $SL(2)$ symmetry of the Schwarzian action as a gauge symmetry. This reflects the fact that the whole configuration, including the boundary, can be shifted around in AdS_2 space with no physical consequence. This is distinct from the physical $SL(2)$ symmetry acting on u which is broken by the Schwarzian action. The ghost-like degrees of freedom lead to exponentially growing corrections in the out of time ordered configuration.

Note that the Schwarzian action has the flavor of a hydrodynamical theory. Namely, it reproduces the thermodynamics of the system. The fact that the entropy is the conserved charge associated to the τ circle translations also resonates with recent discussions of a $U(1)_T$ symmetry in [80], see also [60]. It is also important to include both sides of the thermofield double to make sense of the $SL(2)$ constraints. Now, this Schwarzian action goes beyond ordinary long distance hydrodynamics, because it is including modes whose time variation rate is comparable to the temperature. Such modes are crucial for reproducing the out of time order correlator in the chaos regime.

Two dimensional black holes are a very useful testing ground for ideas for solving the information paradox. Any general idea should work in this simplest context. A important element seems to be a better understanding of the emergence of the charges Q^\pm . These

are the symmetries that allow us to move into the interior!. These charges are analogous to the edge modes of the electromagnetic field discussed in [43, 44], or the “center” in [45], or horizon symmetries in [46] (see also [81])¹.

There are several other questions remaining to be answered. In two dimensions similar holomorphic reparametrizations give rise to a Virasoro algebra with a central charge. Here we have mentioned neither the algebra nor the central charge. It would be nice to see whether and how it can be defined. Several papers have discussed a central charge for AdS_2 , including [82, 83, 84, 85], but we have not understood how they are connected to the present discussion.

Although we are focusing on theoretical description of two dimensional black holes, the near-extremal black holes in nature should contain these features. Both Reissner–Nordström black holes and Kerr black holes have an AdS_2 throat near their extremality. For those black holes, the gravitational effects are enhanced by their near extremal entropies (the coupling constant is ϕ_h rather than $\phi_0 + \phi_h$) and therefore are better backgrounds to test gravitational effects. We should however point out that the observational black holes all have large near extremal entropies and thus are very classical [86]. In addition, the Thorne limit of Kerr black hole sets a lower bound on the near extremal entropy in nature. But for the Primordial black holes in early universe, our story might play a role and it will be interesting to study the physical consequence in that situation.

¹ It was emphasized in [46] that the number of charges is infinite in more than two dimensions. In two dimensions we simply have a finite number of charges, the $SL(2)$ charges discussed here.

Appendix A

Appendix for Chapter 2

A.1 Massive fields in AdS_2 and their coupling to gravity

In this appendix we study in some detail the effect that sources for massive fields have on the time-dependence of the Hamiltonian and the $SL(2)$ charges. A subtlety is that the Schwarzian is not equal to the ADM Hamiltonian while such sources are turned on, it differs by a term involving T_{zz} . For the $SL(2)$ charges, one has to add a similar term to the naive matter charges to get exact conservation.

We will start by considering free fields. We imagine we add classical sources at the boundary by specifying the boundary conditions $\chi_r(u)$, see (2.3.24). As we explained in section (2.3), we can go from the effective action (2.3.23) to (2.3.25). We can add this to the Schwarzian action (2.2.15) and then vary the resulting effective action for $t(u)$ to

obtain a new classical equation

$$C \frac{[\text{Sch}(t, u)]'}{t'} = -\frac{1}{t'} \{ \chi_r'(u) \mathcal{O}(u) + \partial_u [(\Delta - 1) \chi_r(u) \mathcal{O}(u)] \} \quad (\text{A.1.1})$$

$$\text{with } \mathcal{O}(u) \equiv 2D t'(u)^\Delta \int du' \frac{t'(u')^\Delta \chi_r(u')}{(t(u) - t(u'))^{2\Delta}} \quad (\text{A.1.2})$$

where \mathcal{O} can also be interpreted as the classical expectation value of the operator dual to the source $\chi_r(u)$. \mathcal{O} and χ_r are related to the small z behavior of the field by

$$\chi(z, t) = \left[\chi_r(u) - \frac{\epsilon^{2\Delta-1} \mathcal{O}(u)}{2\Delta - 1} \right] \left(\frac{z}{t'(u)} \right)^{1-\Delta} + \dots + \frac{\mathcal{O}(u)}{2\Delta - 1} \left(\frac{z}{t'(u)} \right)^\Delta + \dots \quad (\text{A.1.3})$$

The explicit ϵ term is to ensure $\chi(\epsilon t'(u), t(u)) = \epsilon^{1-\Delta} \chi_r(u)$ so that (2.3.24) is satisfied.

We now want to relate (A.1.1) to the energy conservation condition. In the bulk, given any vector ζ^μ , we can construct a current $(*j_\zeta)_\mu = \epsilon_\mu^\nu T_{\nu\delta} \zeta^\delta$, which is conserved when ζ is a Killing vector. In general the ADM mass M is given by the first equality in (2.2.21) as a function of the dilaton. We expect that its first derivative should give us the flux of energy into the system

$$\partial_u M = \frac{1}{8\pi G} \partial_u (t' \partial_z \phi - z' \partial_t \phi) = (*j)_\mu(t', z')^\mu = \sqrt{\hbar} T_{un} \quad (\text{A.1.4})$$

where u is the coordinate along the boundary and n is the normal direction. Here we have equated the flux with the energy corresponding locally to a direction tangent to the boundary curve. We also assumed $\partial_u \phi_b = 0$. For a scalar field we see that

$$\partial_u M = \sqrt{\hbar} T_{un} = \sqrt{\hbar} \partial_u \chi \partial_n \chi = \partial_u \chi_r [\epsilon^{1-2\Delta} (\Delta - 1) \chi_r - \mathcal{O}] \quad (\text{A.1.5})$$

where we used (A.1.3). Note, that, as expected, energy is conserved as long as the sources are time independent. The first term diverges as $\epsilon \rightarrow 0$ and can be cancelled by a counter term. Here we assumed that $1 < \Delta < 3/2$ in order to avoid further divergent terms.

Comparing this with (A.1.1) we conclude that

$$M = C\text{Sch}(t, u) + \epsilon^{1-2\Delta}(\Delta - 1)\frac{\chi_r(u)^2}{2} + (\Delta - 1)\chi_r(u)\mathcal{O}(u) \quad (\text{A.1.6})$$

In fact, we can compute the relation between the mass M and the Schwarzian directly by using the definition of the ADM mass

$$(8\pi G)M \equiv -\partial_n\phi + \phi\sqrt{h} = t'\partial_z\phi - z'\partial_t\phi + \frac{\bar{\phi}_r}{\epsilon^2} = \quad (\text{A.1.7})$$

$$= t'\left(\frac{\phi}{z} + \partial_z\phi\right) - \frac{z'}{z}\partial_t(z\phi) - \left[\frac{t'}{z} - \frac{1}{\epsilon}\right]\frac{\bar{\phi}_r}{\epsilon} \quad (\text{A.1.8})$$

where $\sqrt{h} = 1/\epsilon$ is the boundary metric and $\phi = \bar{\phi}_r/\epsilon$ at the boundary (2.2.11). We have added and subtracted various terms. In the first term we use the T_{zz} equation

$$\partial_t^2\phi - \frac{1}{z^2}\phi - \frac{1}{z}\partial_z\phi = 8\pi G T_{zz}. \quad (\text{A.1.9})$$

In the third term in (A.1.8) we expand the constant proper length condition as $t'/z = \frac{1}{\epsilon} - \frac{1}{2}\epsilon\frac{t''^2}{t'^2}$. We also assume that

$$\phi = \frac{\phi_-}{z} + \phi_{sl} \quad (\text{A.1.10})$$

where ϕ_{sl} is less singular than $1/z$, so that $z\phi$ has a finite limit. Then we see that $\phi_- = t'\bar{\phi}_r$ to leading order. We can also convert ∂_t into $\frac{1}{t'}\partial_u$ for the terms that are finite. All these terms together then give

$$M = \frac{1}{8\pi G} \left[t'(\partial_t^2(z\phi) - 8\pi G z T_{zz}) - \frac{t''^2}{t'^2}\bar{\phi}_r + \frac{1}{2}\frac{t''^2}{t'^2}\bar{\phi}_r \right] = C\text{Sch}(t, u) - t'zT_{zz}. \quad (\text{A.1.11})$$

This derivation of the relation between the Schwarzian and the mass is valid also for a self interacting matter theory (that is not directly coupled to the dilaton in the lagrangian). Evaluating zT_{zz} for a free field we obtain the extra terms in (A.1.6).

We can similarly consider the expressions for the SL(2) charges. We expect that the total SL(2) charges should be preserved *even with time dependent boundary conditions*. We first define a naive matter SL(2) charge, $q_M^{(k)}$, as the integral of $*j_\zeta$ over a spatial slice with $\zeta^\mu = (k(t), \epsilon k'(t))$ near the boundary, with $k(t) = 1, t, t^2$, for each of the SL(2) generators. For a free field, the fluxes are then given as

$$\partial_u q_M^{(k)} = -\sqrt{h} T_{n\mu} \zeta^\mu = \frac{k}{t'} \chi_r'(u) \mathcal{O}(u) - (\Delta - 1) \left(\frac{k}{t'} \right)' \chi_r \mathcal{O} + \partial_u \left((1 - \Delta) \epsilon^{1-2\Delta} \frac{\chi_r^2 k}{2t'} \right) \quad (\text{A.1.12})$$

We now define a new matter charge that includes some extra terms of the form

$$\begin{aligned} Q_M^{(k)} &= q_M^{(k)} - k z T_{zz} \\ Q_M^{(k)} &= \int *j_\zeta + (\Delta - 1) \epsilon^{1-2\Delta} \frac{\chi_r^2 k}{2t'} + (\Delta - 1) \frac{k}{t'} \chi_r \mathcal{O} \end{aligned} \quad (\text{A.1.13})$$

The extra terms are boundary terms that we can add in the definition of the charge. Then we see that the total SL(2) charges defined as

$$Q_T^{(k)} = Q^{(k)} + Q_M^{(k)}, \quad \partial_u Q_T^{(k)} = 0 \quad (\text{A.1.14})$$

are conserved, once we use the equations of motion (A.1.1). Here $Q^{(k)}$ are the charges constructed purely out of the $t(u)$ variable as in (2.4.42), $(Q^-, Q^0, Q^+) = (Q^{(1)}, Q^{(t)}, Q^{(t^2)})$.

They obey

$$\partial_u Q^{(k)} = C \frac{k [\text{Sch}(t, u)]'}{t'}, \quad k = 1, t, t^2. \quad (\text{A.1.15})$$

A.2 Gravitational shock wave scattering

Here we consider the scattering of two pulses in two dimensional gravity. This is simplest to discuss in a frame where one pulse is highly boosted, created by $V(-\hat{u})$ where \hat{u} large, and the other is unboosted, created by $V(0)$. As in higher dimensions, the scattering can then be described by studying the propagation of the probe V particle on the background created by W [49].

In the theory (2.2.7), the metric is always exactly AdS_2 , so there must be a set of coordinates in which this background is trivial, and the particles simply pass through each other, without detecting any local gravitational effect. If this is the case, how can there be any scattering at all? The answer is that these coordinates are related to the physical boundary coordinate \hat{u} in a nontrivial way [50, 32]. This is simplest to explain with a drawing, which we attempt in figure A.1.

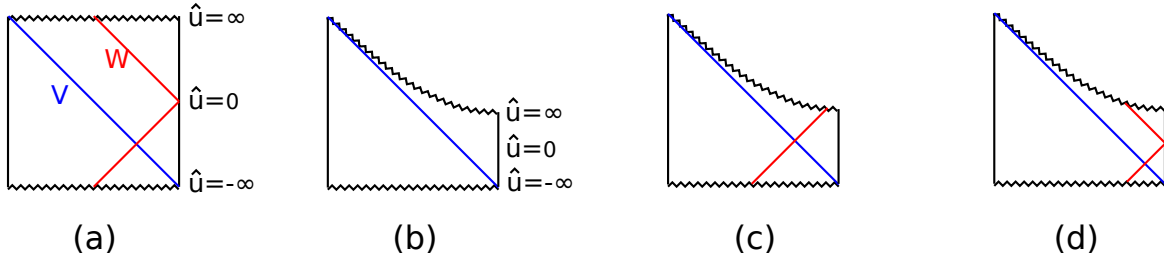


Figure A.1: In (a) we show the trajectories of the V, W quanta without backreaction. In (b) we show the backreaction of the V particle. This is still a piece of AdS_2 , but it is a smaller piece. In (c) we add back W in the arrangement appropriate for the operator ordering $V(-\hat{u})W(0)$. The trajectories are (almost) the same as (a) relative to the fixed AdS_2 coordinates of the diagram, but they change relative to the physical \hat{u} coordinate. In (d) we show the other ordering $W(0)V(-\hat{u})$. Now the red line touches the boundary at time $\hat{u} = 0$. Although it is difficult to see in this frame, the V line has moved down slightly, so that it no longer reaches the boundary.

We can also relate this discussion to the standard shock wave picture. The backreaction of V can be described by a metric

$$ds^2 = -\frac{4dx^+dx^-}{(1+x^+x^-)^2} + 4X^-\delta(x^+)(dx^+)^2 \quad (\text{A.2.16})$$

where X^- is proportional to the large p_+ momentum of V . In these coordinates, the dilaton is a function of x^+x^- only, and the physical time coordinate \hat{u} at the right boundary ($x^+x^- = -1, x^+ > 0$) is given by $e^{\hat{u}} = x^+$. However, it is not manifest that they describe a piece of AdS_2 . It is simple to check that we can rewrite (A.2.16) as

$$ds^2 = -\frac{4d\tilde{x}^+d\tilde{x}^-}{(1+\tilde{x}^+\tilde{x}^-)^2}, \quad \tilde{x}^+ = \begin{cases} x^+ & x^+ < 0 \\ \frac{x^+}{1+X^-x^+} & x^+ > 0, \end{cases} \quad \tilde{x}^- = x^- - X^-\theta(x^+). \quad (\text{A.2.17})$$

These coordinates make it clear that we have a piece of AdS_2 . In fact, these are the coordinates of the fixed AdS_2 space on which the drawings in figure A.1 are represented. In these coordinates, the dilaton profile does not simply depend on $\tilde{x}^+\tilde{x}^-$. Their relationship to the time \hat{u} at the right boundary depends on X^- . At the boundary we have

$$e^{\hat{u}} = x^+ = \frac{\tilde{x}^+}{1 - X^-\tilde{x}^+}. \quad (\text{A.2.18})$$

In summary, the particles simply pass through each other in the bulk, but nevertheless there is an effect on the boundary time. In this way, the gravity dual manages to encode chaos in non-interacting particles.

A.3 Corrections to the matter two point functions

It is natural to ask about the form of the leading correction to the matter two point functions due to further couplings to the dilaton field such as

$$\int d^2x \sqrt{g} [(\nabla\chi)^2 + m^2\chi^2 + \alpha\phi\chi^2] \quad (\text{A.3.19})$$

We can now consider the background value for $\phi = \phi_h \cosh \rho$ to find the leading correction to the thermal two point function. This can be found by noticing that the integral we need to do for the dilaton field has the same form as the one expected for the insertion of a bulk to boundary propagator for a $\Delta = -1$ boundary operator. Thus the correction to the correlator has the form $\int du \langle O_\Delta(u_1) O_\Delta(u_2) V_{-1}(u) \rangle$. Since the three point function is fixed by conformal symmetry we find that

$$\langle O(1)O(2) \rangle = \left(\frac{\pi}{\beta \sin \frac{\pi\tau_{12}}{\beta}} \right)^{2\Delta} \left[1 + c_0 \alpha \frac{\phi_r}{\beta} \left(2 + \pi \frac{(1 - 2\tau_{12}/\beta)}{\tan \frac{\pi\tau_{12}}{\beta}} \right) \right] \quad (\text{A.3.20})$$

where c_0 is a numerical constant. We see that this correction depends on a new parameter α that depends on the details of the theory. This has the same form as the corrections found in [22] for the Sachdev-Ye-Kitaev model.

A.4 A case where the Schwarzian is not dominating

Throughout this paper we have considered nearly AdS_2 situations where the Schwarzian is the leading irrelevant deformation. We now ask the question of whether this always happens or whether there are also situations where other corrections dominate.

For simplicity we will focus only on systems that have a large N expansion, or a weakly coupled gravity description. By assumption we have an IR fixed point, therefore

we assume that there are no relevant operators turned on. We can consider the effects of turning on irrelevant single trace operators which correspond to changing the boundary values of massive bosonic fields in AdS_2 .

We will see that if we turn on an operator with

$$1 < \Delta < 3/2 \tag{A.4.21}$$

then its effects dominate over the ones due to the Schwarzian action and the IR dynamics is different from the one described in this article.

As before, we still have the zero modes in the IR parametrized by the field $t(u)$. But due to the presence of the irrelevant operator with dimension Δ we get an effective action given by (2.3.25)

$$-I_{eff} = \lambda^2 \int dud u' \left[\frac{t'(u)t'(u')}{(t(u) - t(u'))^2} \right]^\Delta \tag{A.4.22}$$

here λ is the coefficient of the operator in the action $\int du \lambda O(u)$. As a simple check that we obtain some effect that dominates over the Schwarzian, we consider the finite temperature configuration with $t = \tan \frac{\pi u}{\beta}$ where we obtain a free energy of the form

$$\begin{aligned} \log Z &= \lambda^2 \beta \int_\epsilon^{\beta-\epsilon} du \left[\frac{\pi}{\beta \sin \frac{\pi u}{\beta}} \right]^{2\Delta} \\ &= \beta \lambda^2 \frac{1}{\Delta - \frac{1}{2}} \frac{1}{\epsilon^{2\Delta-1}} + \lambda^2 \beta^{2-2\Delta} \frac{\pi^{2\Delta-\frac{1}{2}} \Gamma(\frac{1}{2} - \Delta)}{\Gamma(1 - \Delta)} \end{aligned} \tag{A.4.23}$$

The first term is a UV divergence, but is proportional to β so that it is a correction to the ground state energy. The second term is finite. We see that if $\Delta < 3/2$, this second term dominates, for large β , over the Schwarzian answer (2.2.19), which goes as β^{-1} .

Thus in the range (A.4.21) this operator gives the leading IR correction. The effective action for the reparametrizations (A.4.22) is non-local. This can be checked more explicitly

by setting $t = u + \varepsilon(u)$ and expanding in fourier space. We end up with an action of the form $I_{eff} \propto \int dp |p|^{1+2\Delta} \varepsilon(p) \varepsilon(-p)$. which indeed has a non-local form for the range (A.4.21).

The Sachdev-Ye-Kitaev model [21, 36] has no operators in the range (A.4.21) [36], therefore the Schwarzian dominates in the IR.

A.5 Lack of reparametrization symmetry in “conformal” quantum mechanics

There are simple quantum mechanical theories that display an $SL(2)$ conformal symmetry. An example is a lagrangian of the form

$$S = \int dt \left[\left(\frac{dX}{dt} \right)^2 - \frac{\ell^2}{X^2} \right] \quad (\text{A.5.24})$$

Under a transformation of the form

$$t \rightarrow t(\tilde{t}) , \quad X(t(\tilde{t})) = (t')^{\frac{1}{2}} \tilde{X}(\tilde{t}) \quad (\text{A.5.25})$$

(A.5.24) changes to

$$S \rightarrow \int d\tilde{t} \left[\left(\frac{d\tilde{X}}{d\tilde{t}} \right)^2 - \frac{\ell^2}{\tilde{X}^2} - \frac{1}{2} \text{Sch}(t, \tilde{t}) \tilde{X}^2 \right] \quad (\text{A.5.26})$$

We see that if $t(\tilde{t})$ is an $SL(2)$ transformation, then the action is invariant. However, if it is a more general reparametrization the action is not invariant.

Appendix B

Appendix for Chapter 3

B.1 Gauge Fix $SL(2, \mathbb{R})$

This section reviews the procedure to fix $SL(2, \mathbb{R})$ gauge which is needed for calculating correlation functions in quantum gravity using formula (3.4.68). With the parametrization of group elements in $SL(2, \mathbb{R})$ by $g = e^{i\epsilon_\alpha L_\alpha}$ ($\alpha = \pm 1, 0$) near the identity, we have g acting on \mathbf{x} as following (3.3.31):

$$gx = x - \epsilon_{-1} - \epsilon_0 x + \epsilon_1 x^2; \quad gz = z - \epsilon_0 z + \epsilon_1 2xz. \quad (\text{B.1.1})$$

Choosing the gauge fixing condition as $f_\alpha(g\mathbf{x}) = 0$, we can fix the $SL(2, \mathbb{R})$ symmetry in (3.4.68) using Faddeev-Popov method. First we have the identity:

$$1 = M(\mathbf{x}) \int dg \delta(f_\alpha(g\mathbf{x})) \quad (\text{B.1.2})$$

Because the measure is invariant under group multiplication, $M(\mathbf{x})$ is equal to $M(g\mathbf{x})$ and we can calculate it at the solution \mathbf{x}_0 of the gauge constraints on its orbit: $f_\alpha(\mathbf{x}_0) = 0$,

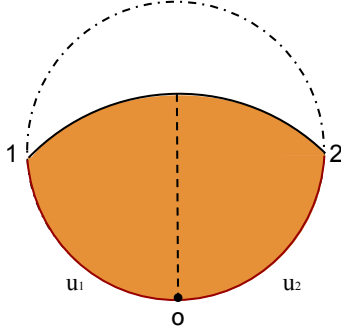


Figure B.1: WdW wavefunction with matter

$\mathbf{x}_0 \in G(\mathbf{x})$. Since the Haar measure is flat near the identity we have ¹:

$$M(\mathbf{x}) = \det \left(\frac{\partial f_\alpha(g\mathbf{x}_0)}{\partial \epsilon_\beta} \right) \Big|_{\epsilon_\beta=0}. \quad (\text{B.1.3})$$

Inserting 1 in integrals of $\text{SL}(2,\mathbb{R})$ invariant function $F(\mathbf{x})$ like the one in (3.4.68), we have:

$$\int d\mathbf{x} F(\mathbf{x}) = \int d\mathbf{x} M(\mathbf{x}) \int dg \delta(f_\alpha(g\mathbf{x})) F(\mathbf{x}) = \int dg \int d\mathbf{x} M(\mathbf{x}) \delta(f_\alpha(\mathbf{x})) F(\mathbf{x}). \quad (\text{B.1.4})$$

We see that the volume of $\text{SL}(2,\mathbb{R})$ factorizes out and we have the gauge fixed expression:

$$\int d\mathbf{x} \det \left(\frac{\partial f_\alpha(g\mathbf{x}_0)}{\partial \epsilon_\beta} \right) \Big|_{\epsilon_\beta=0} \delta(f_\alpha(\mathbf{x})) F(\mathbf{x}). \quad (\text{B.1.5})$$

B.2 Wavefunction with matter

Including matter sector in JT gravity (action 2.2.7), we can discuss the exact wavefunction including matter backreaction. Schematically, since the geometry on which the matter field

¹The normalization constant is arbitrary and we choose it to be one.

propagates is not changed, the WdW wavefunction Φ including matter sector will be :

$$|\Phi\rangle = \sum_n |\Psi_n\rangle \otimes |n\rangle, \quad (\text{B.2.6})$$

where $|n\rangle$ denotes the matter state in fixed AdS background, and $|\Psi_n\rangle$ means the gravitational wavefunction after backreaction from matter state $|n\rangle$. By specifying the boundary condition of the matter in Euclidean evolution, one can create different types of states such as vacuum state. The vacuum state in AdS is stable and will not backreact on the gravity sector and thus one will get the same story discussed in section 3.3.2. We will consider another type of state that is created by inserting one boundary operator O during the euclidean evolution (Figure B.1). The operator O creates a single $\text{SL}(2,\text{R})$ representation with conformal dimension Δ . For the reason that the boundary is fluctuating, O does not create only one asymptotic state, but a superposition of its descendants: $|\Delta, n\rangle$. The transition amplitude from $|O(\mathbf{x})\rangle$ to $|\Delta, n\rangle$ can be determined from the asymptotic behavior of two point function:

$$\lim_{x' \rightarrow \infty} \sum_n \langle O(x') | n \rangle \langle n | O(x) \rangle = \frac{z'^{\Delta} z^{\Delta}}{|x - x'|^{2\Delta}} = \frac{z'^{\Delta} z^{\Delta}}{x'^{2\Delta}} \left(1 + 2\Delta \frac{x}{x'} + \dots + \frac{\Gamma(2\Delta + n)}{\Gamma(2\Delta)\Gamma(n)} \frac{x^n}{x'^n} + \dots \right) \quad (\text{B.2.7})$$

Therefore we have:

$$|O(x)\rangle = \sum_n \sqrt{\frac{\Gamma(2\Delta + n)}{\Gamma(2\Delta)\Gamma(n)}} z^{\Delta} x^n |\Delta, n\rangle \quad (\text{B.2.8})$$

Notice that because the matter carries $\text{SL}(2,\text{R})$ charge, the gravitational part is not a singlet and in particular will depend on the location of two boundary points \mathbf{x}_1 and \mathbf{x}_2 . By choosing our time slice to be the one with zero extrinsic curvature, we can get the

backreacted gravitational wavefunction:

$$\Psi_n(\mathbf{x}_1, \mathbf{x}_2) = e^{-2\frac{z_2+z_2}{x_2-x_1}} \int d\mathbf{x} \tilde{K}(u_1, \mathbf{x}_1, \mathbf{x}) \tilde{K}(u_2, \mathbf{x}, \mathbf{x}_2) \sqrt{\frac{\Gamma(2\Delta+n)}{\Gamma(2\Delta)\Gamma(n)}} z^\Delta x^n. \quad (\text{B.2.9})$$

B.3 Boundary effective action from CFT

CFT partition function in two dimension has simple dependence on the shape of geometry by Liouville action. More precisely, the CFT partition function of central charge c on geometries related by $g = e^{2\rho} \hat{g}$ is related:

$$Z[g] = e^{\frac{c}{6} S_L} Z[\hat{g}]; \quad S_L = \frac{1}{4\pi} \left[\int (\hat{\nabla} \rho)^2 + \rho \hat{R} + 2 \int \rho \hat{K} \right]. \quad (\text{B.3.10})$$

Our strategy to get the effective action of the boundary shape is to first find a conformal map that maps the boundary into a circle, and then evaluate the Liouville action on that new metric. By Cauchy's Theorem, such a conformal map always exists and is uniquely determined up to $\text{SL}(2, \mathbb{R})$ transformation. The $\text{SL}(2, \mathbb{R})$ transformation does not change the weyl factor and therefore does not affect our final result. The original metric has constant negative curvature and we will parametrize it by a complex coordinate h as $\frac{4}{(1-|h|^2)^2} dh d\bar{h}$. If we denote the conformal map as $h(z)$, where z is the coordinate in which the boundary is a circle $|z| = 1$, then the new metric in coordinate z is:

$$ds^2 = \frac{4\partial h \bar{\partial} \bar{h}}{(1-|h|^2)^2} dz d\bar{z}. \quad (\text{B.3.11})$$

The holomorphic function $h(z)$ determines the boundary location in h coordinate (parametrized by u) at ²:

$$r(u)e^{i\tilde{\theta}(u)} = h(e^{i\theta(u)}), \quad (\text{B.3.12})$$

where $r(u)$ and $\tilde{\theta}(u)$ are related by the metric boundary condition:

$$\frac{4(r'^2 + r^2\tilde{\theta}'^2)}{(1-r^2)^2} = q^2; \quad r(u) = 1 - q^{-1}\tilde{\theta}'(u) + O(q^{-2}). \quad (\text{B.3.13})$$

Combine these two equations at large q we get a Riemann Hilbert type problem:

$$e^{i\tilde{\theta}(u)}(1 - q^{-1}\tilde{\theta}'(u)) = h(e^{i\theta(u)}). \quad (\text{B.3.14})$$

This equation can be solved by the holomorphic property of $h(z)$ and the solution is:

$$h(z) = z \left(1 - \frac{1}{2\pi q} \int d\alpha \frac{e^{i\alpha} + z}{e^{i\alpha} - z} \theta'(\alpha) \right) \quad (\text{B.3.15})$$

Choosing our reference metric \hat{g} to be flat, we have:

$$\rho = \frac{1}{2}(\log \partial h + \log \bar{\partial} \bar{h}) - \log(1 - h\bar{h}). \quad (\text{B.3.16})$$

Evaluation of the Liouville action (B.3.10) is then straightforward and gives us a Schwarzian action:

$$S_L = -\frac{1}{2} - \frac{1}{4\pi q} \int du Sch(\tan(\frac{\theta}{2})). \quad (\text{B.3.17})$$

We want to remark that the sign in front of the Schwarzian action is negative so a naive attempt to get induced gravity from large number of quantum fields does not work.³

² $z = e^{i\theta}$ and $h = re^{i\tilde{\theta}}$

³For an other interpretation of this result, see [87]. Matter quantization with JT theory was also considered in [88] in the context of non-critical string theory.

B.4 Connection with the relativistic particle and Pair Production

We will start from a formal expression for the relativistic particle with mass m and charge q in an electric field and gradually implement these changes to get the partition function of the JT theory. The partition function for the relativistic particle has the form

$$Z_{\text{rel}}(m_0, q) = \sum_{\text{Paths}} e^{-m_0 L} e^{-q \int a} = \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{1}{2}\tau\mu^2} \int \mathcal{D}x \mathcal{D}y \exp \left(- \int_0^\tau d\tau' \frac{1}{2} \frac{\dot{x}^2 + \dot{y}^2}{y^2} - q \int \frac{dx}{y} \right) \quad (\text{B.4.18})$$

where L is the length of the path. In the right hand side τ is Schwinger's proper time, which is related by a renormalization factor to the actual proper time of the path [53] (Chapter 9). Also, we have that $\mu^2 = \frac{(m_0 - m_{cr})}{\tilde{\epsilon}}$, where $\tilde{\epsilon}$ is a UV cutoff for the path integral (not to be confused with ϵ in (2.1.4)). If we are interested in the JT partition function at finite temperature, then we are interested in fixing the length of the paths. As we mentioned, this is the same as fixing the Schwinger time in (B.4.18). More explicitly, we can multiply $Z_{\text{rel}}(m_0, q)$ by $e^{\frac{m_0 \beta}{\tilde{\epsilon}}}$ and then integrate over m_0 along the imaginary axis (with a suitable) real part to fix the length of the path. This then gives $\beta = \tau$ in the above expression. The precise value of μ^2 can be absorbed by shifting the ground state energy. It will be convenient for further purposes to set $\mu^2 = q^2 - 1/4$. The path integral in the right hand side of (B.4.18) has an infinite volume factor. We drop this factor when we divide by the volume of $SL(2)$. In addition, the factor of $1/\tau$ should be dropped because we view configurations that differ by a shift in proper time as inequivalent. After all these modifications we find

$$Z_{JT}(\beta) = e^{S_0} e^{2\pi q} \frac{1}{2\pi} G(\beta, x, y; x, y) \quad (\text{B.4.19})$$

where G denotes the propagator of the non-relativistic problem of a particle in an electric field

$$G(\tau; x, y; x', y') = \langle x, y | e^{-\tau H} | x', y' \rangle = \int \mathcal{D}x \mathcal{D}y \exp \left(- \int_0^\tau d\tau' \frac{1}{2} \frac{\dot{x}^2 + \dot{y}^2}{y^2} - q \int \frac{dx}{y} \right) \quad (\text{B.4.20})$$

At first sight, the statement that a gravitational system is equivalent to a particle makes no sense, since we know that the entropy of a particle is very small. Usually the partition function is of the form $Z|_{particle} \sim (\hbar)^\#$ for a particle system, but black hole has entropy of order $\frac{1}{\hbar}$, that is $Z|_{BH} \sim e^{\frac{\#}{\hbar}}$. This is because in the particle case, the major contribution in functional integral is given by stationary solution, and the fluctuations near the stationary solution give the power of \hbar , while for the gravitational system, a stationary solution will corresponding to no geometry and we have the requirement of the boundary should have winding number one. A solution with winding number one in the particle system is an instanton contribution for particle pair production, which is usually very small and is in addition imaginary, so how can this matches with gravity system? The pair creation rate for a particle with charge q and mass m in AdS can be estimated from Euclidean solution which is a big circle with radius $\rho_b = \text{arctanh}(\frac{m}{q})$:

$$I = mL - qA \sim 2\pi m \sinh \rho_b - 2\pi q (\cosh \rho_b - 1) + \pi (m \sinh \rho_b - q \cosh \rho_b) \delta \rho^2 \sim 2\pi q. \quad (\text{B.4.21})$$

We see that the damping factor is exactly cancelled out by our gravitational topological piece. The negative norm mode is related with the rescaling of the circle. That is not allowed in canonical ensemble because of the temperature constraint.

B.5 Details on the Schwarzian limit of Propagator

The main technical difficulty in finding the large q limit of propagator (3.2.21) is the hypergeometric function. To properly treat it, we can first use transformation of variables:

$$\begin{aligned} & \frac{1}{d^{1+2is}} F\left(\frac{1}{2} - iq + is, \frac{1}{2} + iq + is, 1, 1 - \frac{1}{d^2}\right) = \\ & \frac{\Gamma(-2is)}{d^{1+2is} \Gamma(\frac{1}{2} - is + iq) \Gamma(\frac{1}{2} - is - iq)} F\left(\frac{1}{2} + is - iq, \frac{1}{2} + is + iq, 1 + 2is, \frac{1}{d^2}\right) + (s \rightarrow -s) \end{aligned} \quad (\text{B.5.22})$$

In the limit of large q (d scales with q), we have approximation of hypergeometric function:

$$F\left(\frac{1}{2} + is - iq, \frac{1}{2} + is + iq, 1 + 2is, \frac{1}{d^2}\right) \sim \Gamma(1 + 2is) \left(\frac{d}{q}\right)^{2is} I_{2is}\left(\frac{2q}{d}\right). \quad (\text{B.5.23})$$

Using reflection property of gamma function together with large q approximation of $\Gamma(\frac{1}{2} - is + iq) \Gamma(\frac{1}{2} - is - iq) \sim 2\pi e^{-\pi q} q^{-2is}$, we have:

$$(\text{B.5.22}) \sim -\frac{ie^{\pi q}}{2 \sinh(2\pi s) d} \left(I_{-2is}\left(\frac{2q}{d}\right) - I_{2is}\left(\frac{2q}{d}\right)\right) = \frac{e^{\pi q}}{\pi d} K_{2is}\left(\frac{2q}{d}\right). \quad (\text{B.5.24})$$

Putting everything together will give us (3.3.28).

B.6 Trajectories in Real Magnetic Field

The equation of motions for a particle in real magnetic field b are:

$$\frac{\dot{x}^2 + \dot{y}^2}{2y^2} = E; \quad (\text{B.6.25})$$

$$\frac{\dot{x}}{y^2} + \frac{b}{y} = k, \quad (\text{B.6.26})$$

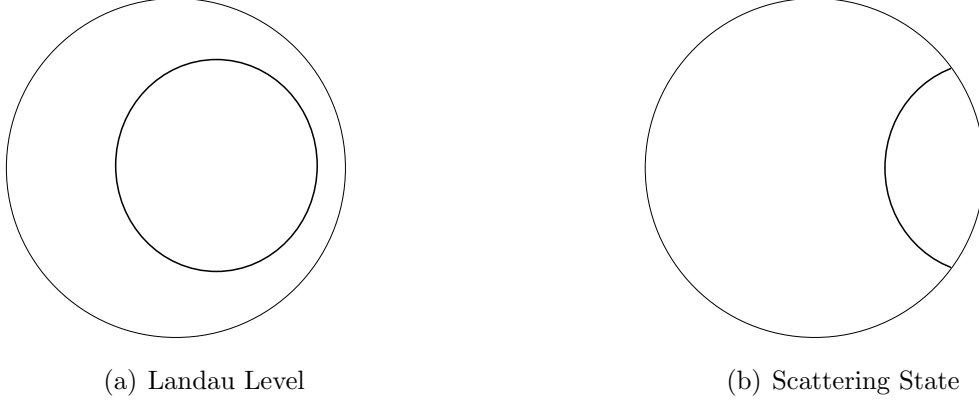


Figure B.2: Particle in real magnetic field

where E and k are the conserved energy and momentum respectively. Since we are only interested in the trajectories, we can introduce a time parametrization ξ such that $\frac{d\tau}{d\xi} = \frac{1}{2ky}$. Then in coordinate ξ , we have:

$$(\partial_\xi x)^2 + (\partial_\xi y)^2 = \frac{E}{2k^2}; \quad \partial_\xi x = \left(y - \frac{b}{2k}\right). \quad (\text{B.6.27})$$

This means that we have solutions:

$$x^2 + \left(y - \frac{b}{2k}\right)^2 = \frac{E}{2k^2}. \quad (\text{B.6.28})$$

Those are circles with radius $\sqrt{\frac{E}{2}} \frac{1}{k}$ and center at location $(0, \frac{b}{2k})$. So classically we have two types of states as shown in Figure B.2 : for $E < \frac{b^2}{2}$, the particle is confined by magnetic field and becomes Landau level in the hyperbolic plane; for $E > \frac{b^2}{2}$, the gravitational effect dominates and particle scatters out of the space.

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