RESEARCH ARTICLE

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The cosmological constant and dark energy in braneworlds

Abstract We review recent attempts to address the cosmological constant problem and the late-time acceleration of the Universe based on braneworld models. In braneworld models, the way in which the vacuum energy gravitates in the 4D spacetime is radically different from conventional 4D physics. It is possible that the vacuum energy on a brane does not curve the 4D spacetime and only affects the geometry of the extra-dimensions, offering a solution to the cosmological constant problem. We review the idea of supersymmetric large extra dimensions that could achieve this and also provide a natural candidate for a quintessence field. We also review the attempts to explain the late-time accelerated expansion of the universe from the large-distance modification of gravity based on the braneworld. We use the Dvali–Gabadadze–Porrati model to demonstrate how one can distinguish this model from dark energy models in 4D general relativity. Theoretical difficulties in this approach are also addressed.

1 Introduction

The cosmological constant problem is a long-standing problem in physics [1]. Particle physics predicts the existence of the vacuum energy density which is related to the fundamental scale of the theory, like the electroweak scale, $\rho_{\text{vac}} \sim (\text{TeV})^4$. This is typically more than 50 orders of magnitude larger than the observed value, $\rho_\Lambda \sim (10^{-3}\,\text{eV})^4$. Before the discovery of the accelerated expansion of the Universe, physicists tried to answer this question by seeking a theory that predicts the cosmological constant *should be* zero. However, the discovery of the accelerated expansion of the Universe makes this answer insufficient [2; 3; 4]. Now, we should explain why it is non-zero and yet it is so small. Moreover, there is a coincidence problem. The cosmological constant dominates the energy density of the Universe

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only recently. If the cosmological constant is really a constant, we should explain *why now*, does it become dominant.

One direction to answer these questions is to appeal to the anthropic principle [1]. If the cosmological constant is too large, the accelerated expansion started too early and it prevents structure from growing and we cannot exist. On the other hand, a universe with negative comsological constant re-collapses. Then observers will only exist within a tiny anthropic range of cosmological constant (see for example [5]). This idea is strengthened by the discovery in string theory that there are millions of low-energy vacua in the theory (the string theory landscape) [6]. It is argued that we might need the anthropic principle to select the low-energy vacuum. However, many theorists still hope to explain the problem without invoking the existence of ourselves in the Universe. Although significant efforts have been devoted to this attempt, we still have not succeeded yet to provide convincing models. However, the rapid progress of string theory has provided a new perspective for solving this problem. In this review, we focus on the attempts of using higher-dimensional gravity and branes to address the problem.

String theory is formulated in a 10D spacetime. On the other hand, our observed Universe is a 4D spacetime. Thus there should be a mechanism to hide the extra dimensions. The conventional idea is to compactify the extra dimensions by the Kaluza–Klein (KK) mechanism. The size of the extra dimensions L should be small, $L < \text{TeV}^{-1}$, in order not to spoil the success of the standard model of particle physics that is formulated in a 4D spacetime. Below the energy scale determined by the size of the extra dimensions, L^{-1}, the universe looks completely 4D if the radius of the extra dimensions is stabilized. Recently, a completely new way of hiding the extra dimensions has been proposed. This is the brane world mechanism where matter fields are confined to a 4D membrane in a higher dimensional spacetime (see [7] for a review). Only gravity and non-standard model particles can propagate into the whole higher-dimensional bulk spacetime. In this picture, the size of the extra dimensions can be much larger than that in the conventional KK compactification. In fact, the size of the extra dimensions could even be infinite. If the bulk is a spacetime with a negative cosmological constant, that is, an Anti-de Sitter (AdS) spacetime, it is shown that gravity behaves like 4D on scales larger than the AdS curvature length, even if the size of the extra dimensions is infinite [8]. Another way is to introduce induced gravity on a brane [9] (see [10] for an early attempt). If we assume there is an Einstein-Hilbert term on a brane, 4D gravity is recovered, in this case, on small scales even if the bulk is an infinite Minkowski spacetime. In these braneworld models, the behaviour of gravity can be dramatically different from the 4D theory, providing a new perspective to solve the cosmological constant and the dark energy problem.

This article will review several approaches to address the cosmological constant and the late-time acceleration problem based on braneworld gravity. Firstly, we explain the attempts to address the "old" cosmological constant problem—why the vacuum energy is incredibly small compared with the prediction of particle physics. These attempts exploit the modification of 4D gravity in the braneworld and change the way in which the vacuum energy gravitates in a 4D spacetime. Secondly, we introduce an idea to explain the late-time acceleration without introducing the cosmological constant. This idea also relies on the modification of gravity on large scales based on the braneworld idea.

In Sect. 2, we give a brief introduction to braneworlds. In Sect. 3, the attempts to solve the old cosmological constant problem are discussed. In Sect. 4, the idea to realize late-time acceleration without introducing a cosmological constant is explained. Section 5 is devoted to conclusions.

2 Braneworld models

The idea that ordinary matter fields are confined to a lower-dimensional domain wall was proposed in the 1980's [11; 12]. It was shown that fermion fields can be confined to a field theoretic domain wall. The progress in string theory, especially the discovery of D-branes, has revived these attempts [13]. The D-brane is defined by a membrane on which end-points of open strings lie. At the end-points of open strings, gauge fields can be attached. Then gauge fields are confined to the D-brane. On the other hand, closed strings that contain the graviton can propagate into the whole bulk. Then there arises a braneworld picture where usual matter fields are confined to a brane while gravity propagates into the whole bulk spacetime. A schematic picture of the braneworld is shown in Fig. 1. Based on this idea, several simplified braneworld models have been proposed that capture the basic features of the braneworld, yet in which we can address many important problems from a new perspective.

2.1 Arkani–Hamed-Dimopoulos–Dvali model

An interesting possibility in braneworld models is that some of the extra dimensions can be *large* [14; 15; 16]. In a conventional picture, extra dimensions are rolled up small so that we never observe them. More precisely, in order not to spoil the success of the standard model of particle physics that is formulated in a 4D spacetime, the size of the extra dimensions should be smaller than $TeV^{-1} \sim 10^{-19}$ m. However, in the braneworld, the standard model particles are confined to the 4D brane. Thus we do not need to worry about this constraint. The gravitational interactions are very weak and the 4D behaviour of the Newtonian force is only verified down to $44\,\mu m$ [17]. Thus the size of the extra-dimensions is allowed to be as large as $44\,\mu m$ (Fig. 2).

This opens up a new perspective to solve another serious problem in particle physics, namely the hierarchy problem: why the gravitational interaction is so weak compared with the other interactions. The answer could be that the gravitational field of an object on a brane leaks out into the large extra-dimensions and this leakage weakens the gravitational interactions on a brane. The gravitational

Fig. 1 A schematic picture of the braneworld. From [7]

Fig. 2 Constraints on Yukawa violations of the gravitational $1/r$ potential, $V(r) \propto (1/r)(1 + \alpha \exp(-r/\lambda))$. The shaded region is excluded at the 95% confidence level. From [17]

potential generated by an object with mass M is given by

$$\Psi(r) = -\frac{G_4 M}{r^2} \quad (r > L), \tag{1}$$

$$\Psi(r) = -\frac{G_D M}{r^{D-2}} \quad (r < L), \tag{2}$$

where L is the size of the $(D-4)$ dimensional extra-dimensions. Then the 4D gravitational constant is given in terms of the higher-dimensional gravitational constant as

$$G_4 = \frac{G_D}{L^{D-4}}, \quad M_4 = M_D^{(D-2)/2} L^{(D-4)/2}, \tag{3}$$

where $8\pi G_D = M_D^{-(D-2)}$. Then even if the fundamental scale of gravity M_D is TeV, the 4D gravitational constant can be 10^{19} GeV as long as L is appropriately large. For example, for $D = 6$, the current constraint on the deviation from the gravitational inverse-square law $L < 44\,\mu\text{m}$ implies $M_6 > 3.2$ TeV.

2.2 Randall–Sundrum model

The most difficult problem in the braneworld in terms of gravity is the inclusion of the self-gravity of the branes. In the ADD model, the self-gravity of the branes is implicitly neglected. The model proposed by Randall and Sundrum (RS) offers a consistent framework to deal with higher-dimensional gravity including the self-gravity of the branes [8]. They consider a 5D spacetime described by the action

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g}(^{(5)}R - 2\Lambda) - \sigma \int d^4x \sqrt{-\gamma} + \int d^4x \mathscr{L}_m, \tag{4}$$

where $\kappa_5^2 = 8\pi G_5$ and $\mathscr{L}_m$ represents the matter lagrangian confined to a brane. The introduction of the singular objects enforces the junction condition (Israel junction condition) at the location of the brane. The junction condition relates the extrinsic curvature at the brane to the energy momentum tensor localized on a brane. By solving the 5D bulk spacetime and imposing the junction condition at the brane, the solution for the gravitational field on the brane is obtained. The simplest solution is a solution with a Minkowski brane. The 5D metric is given by

$$ds^2 = dy^2 + \exp(-2|y|/\ell)\eta_{\mu\nu}dx^\mu dx^\nu. \tag{5}$$

A brane is located at $y = 0$ and the reflection symmetry (Z_2 symmetry) across the brane is imposed. The exponential "warp factor" is an essential ingredient of the model. Even if the physical size of the fifth dimension is infinite, low-momentum gravity is confined near the brane due to the curvature of the bulk spacetime and 4D gravity is recovered. It is shown that the solutions for weak gravity at large distances $r \gg \ell$ are given by [18]

$$ds^2 = -(1+2\Psi)dt^2 + (1+\Phi)\delta_{ij}dx^i dx^j, \tag{6}$$

$$\Psi = \frac{2G_4M}{r}\left(1 + \frac{2\ell^2}{3r^2}\right), \quad \Phi = \frac{2G_4M}{r}\left(1 + \frac{\ell^2}{3r^2}\right), \tag{7}$$

where $\kappa_4^2 = 8\pi G_4$ and is determined by

$$G_4 = G_5\ell. \tag{8}$$

Comparing this with Eq. (3), we notice that ℓ acts as the effective size of the extra dimension. Thus the RS model provides an "alternative to compactification".

Despite the remarkably simple setup of the model, gravity in this model is incredibly complicated. Fortunately, for a homogeneous and isotropic brane, the

generalized Birkoff theorem ensures that the bulk spacetime is AdS spacetime or AdS-Schwarzchild spacetime. Then the Friedmann equation on the brane is easily derived as [19; 20; 21]

$$H^2 = \frac{\Lambda_4}{3} + \frac{\kappa_4^2}{3}\rho + \frac{\kappa_5^4}{36}\rho^2 + \frac{C}{a^4}, \tag{9}$$

where

$$\Lambda_4 = \frac{\Lambda_5}{2} + \frac{\kappa_5^4 \sigma^2}{12}, \quad \kappa_4^2 = \frac{\kappa_5^2 \sigma}{6}. \tag{10}$$

The constant C is proportional to the black hole mass in the bulk. In accord with weak gravity, cosmology also shows the transition from 4D to 5D. At high energies $H\ell > 1$ where the horizon size H^{-1} is smaller than the effective size of the extra-dimension ℓ, the Friedmann equation is significantly modified and $H \propto \rho$. At low energies $H\ell < 1$, we recover the 4D Friedmann equation.

3 Cosmological constant problem in the braneworld

3.1 Self-tuning 5D braneworld

The relation between the vacuum energy and the effective cosmological constant on a brane is different from that in the usual 4D theory. In the RS braneworld, the vacuum energy in the brane σ is not directly related to the cosmological constant Λ_4 on the brane in the effective Einstein equation as in Eq. (10). In the RS braneworld, there should be a cancellation between the 4D and 5D contribution of the vacuum energy in order to have a vanishing cosmological constant on the brane. This requires a fine-tuning for the parameters in the action. Instead of having the cosmological constant in the bulk and tension on the brane, let us consider a scalar field with potentials [22; 23]. The action is given by

$$S = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g} \left(R - \frac{4}{3}(\partial_\mu \phi)^2 - V(\phi) \right) - \int d^4 x \sqrt{-\gamma} f(\phi). \tag{11}$$

The potentials can be taken as

$$V(\phi) = \Lambda_0 \exp(a\phi), \quad f(\phi) = V_0 \exp(b\phi). \tag{12}$$

with this choice, the action describes a family of theories parametrized by V_0, Λ_0, a and b. For simplicity, we take $\Lambda_0 = 0$. We look for a solution with a Minkowski spacetime on a brane. The 5D metric is given by

$$ds^2 = dy^2 + e^{2A(y)} \eta_{\mu\nu} dx^\nu dx^\mu. \tag{13}$$

The 5D Einstein equation gives the relation between the warp factor $A(y)$ and the scalar field $\phi(y)$

$$\phi'(y) = \pm \frac{1}{3} A'(y). \tag{14}$$

The solution for ϕ in the bulk is then obtained as

$$\phi(y) = \frac{3}{4}\log\left(\frac{4}{3}M_5 y + c_1\right) + d_1, \quad y < 0, \tag{15}$$

$$\phi(y) = -\frac{3}{4}\log\left(\frac{4}{3}M_5 y + c_2\right) + d_2, \quad y > 0, \tag{16}$$

where c_1, c_2, d_1 and d_2 are integration constants. The continuity of ϕ determines d_2. Then the junction conditions for the scalar field and the warp factor determine c_1 and c_2 in terms of b, V_0 and d_1 if $b \neq \pm 4/3$. This means that for a scalar coupling given by b, there is a Minkowski solution on a 4D brane for *any* value of the brane tension V_0. This is the idea of the "self-tuning". The vacuum energy in a 4D brane is cancelled by the integration constants in the solutions, not by the parameters in the original action. Thus this is not a fine-tuning. The hope is that the solution in the bulk adjusts itself so that the contribution from the vacuum energy on the brane is exactly cancelled.

Although the idea of self-tuning is very attractive, there are several problems in the original proposal [24; 25; 26; 27]. Firstly, there is a naked singularity in the above model with a scalar field. Any procedure that regularizes the singularity in the solutions would cause the re-introduction of the fine tuning. There is also a problem of stability. In the case of vanishing potential in the bulk, the static solution is unstable, leading to a singularity. A modified version of the model using the bulk black hole to hide the singularity inside the horizon was proposed [28], but it was argued that this model also cannot avoid the fine tuning [29].

3.2 6D braneworld

Another approach to realize the self-tuning is to consider a 6D bulk spacetime [30]. The action is given by

$$S = \int d^6 x \sqrt{-g}\left(\frac{1}{2\kappa_6^2}R - \Lambda_6 - \frac{1}{4}F_{ab}F^{ab}\right), \tag{17}$$

where the gauge field F_{ab} is required to stabilize the size of the extra dimensions. We decompose the coordinates into four macroscopic dimensions and the two extra dimensions. The metric is taken as

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \gamma_{ij}dx^i dx^j. \tag{18}$$

The gauge field is taken to consist of magnetic flux threading the extra dimensional space so that the field strength takes the form

$$F_{ij} = \sqrt{\gamma}B_0\varepsilon_{ij}, \tag{19}$$

where B_0 is a constant, γ is the determinant of γ_{ij} and ε_{ij} is the antisymmetric tensor normalized as $\varepsilon_{12} = 1$. All other components of F_{ab} vanish. A static and stable solution is obtained by choosing the extra-dimensional space to be a two-sphere

$$\gamma_{ij}dx^i dx^j = a_0^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{20}$$

Fig. 3 Removing a wedge from a sphere and identifying opposite sides to obtain a football geometry. Two equal-tension branes with conical deficit angles are located at either pole; outside the branes there is constant spherical curvature. From [30]

The magnetic field strength B_0 and the radius a_0 are fixed by the cosmological constant

$$B_0^2 = 2\Lambda_6, \quad a_0^2 = \frac{M_6^4}{2\Lambda_6}. \tag{21}$$

It should be noted that B_0 has to be tuned so that a Minkowski spacetime is induced in 4D. Now we add branes to this solution. The brane action is given by

$$S_4 = -\sigma \int d^4x \sqrt{-\gamma}. \tag{22}$$

The solution for the extra dimensions is now given by

$$\gamma_{ij} dx^i dx^j = a_0^2 (d\theta^2 + \alpha^2 \sin^2\theta d\varphi^2), \tag{23}$$

where

$$\alpha = 1 - \frac{\sigma}{2\pi M_6^2}, \quad a_0^2 = \frac{M_6^4}{2\Lambda_6}. \tag{24}$$

The coordinate φ ranges from 0 to 2π. Thus the effect of the brane makes a deficit angle $\delta = 2\pi(1 - \alpha)$ in the bulk. This is a 6D realization of the ADD model including the self-gravity of branes (Fig. 3).

The most interesting feature of this solution is that the 4D geometry is independent of the brane tension σ. The tension enters only in the deficit angle and not the radius a_0 and the magnetic field B_0 that need to be tuned to obtain a flat 4D spacetime. Thus the vacuum energy on the brane does not gravitate in the 4D spacetime but merely changes the geometry of the extra dimensions. Thus the outcome is similar to the self-tuning solutions discussed in Sect. 3.1. It should be noted that the cosmological constant problem is not fully solved even if this idea works. In order to obtain a flat spacetime, we need to tune the magnetic field and the bulk cosmological constant as in Eq. (21). However one could hope that some kinds of symmetry like supersymmetry in the bulk can ensure this tuning.

However, there have been objections to the self-tuning in this model [31]. Consider that a phase transition occurs and the tension of the brane changes from σ_1 to σ_2. Accordingly, α changes from $\alpha_1 = 1 - \sigma_1/(2\pi M^4)$ to $\alpha_2 = 1 - \sigma_2/(2\pi M^4)$. The magnetic flux is conserved as the gauge field strength is a closed form, $dF = 0$. Then the magnetic flux which is obtained by integrating the field strength over the extra dimensions should be conserved

$$\Phi_B = 4\pi\alpha_1 B_{0,1} = 4\pi\alpha_2 B_{0,2}. \tag{25}$$

The fine-tuning of Λ_6 and B_0, Eq. (21), that ensures the existence of Minkowski branes cannot be imposed both for $B_0 = B_{0,1}$ and $B_0 = B_{0,2}$ when $\alpha_1 \neq \alpha_2$. This becomes clear if we rewrite the conditions Eq. (21) as

$$\alpha^2 = \left(\frac{\Phi_B}{4\pi}\right)^2 \frac{\Lambda_6}{M_6^4}. \tag{26}$$

The left-hand side changes by the phase transition but the right-hand side cannot change. Moreover, the quantization condition must be imposed on the flux Φ_B. Then if the condition Eq. (21) is satisfied for some value of α, it will not be satisfied by neighbouring values. Thus after the phase transition, the 4D spacetime cannot be static [32].

3.3 Supersymmetric large extra dimensions

In the Einstein–Maxwell theory discussed in Sect. 3.2, the tuning between the magnetic flux and the cosmological constant in the 6D spacetime, Eq. (21), was necessary to obtain the flat 4D spacetime. This was the origin of the difficulty in realizing the self-tuning. To evade this problem, the Supersymmetric Large Extra Dimensions (SLED) model was proposed (see [33; 34] for a review). This is a supersymmetric version of the 6D model and the action is given by [35]

$$S = \int d^6x \sqrt{-g} \left[\frac{1}{2\kappa_6^2} \left(R - \partial_M \phi \partial^M \phi \right) - \frac{1}{4} e^{-\phi} F_{MN} F^{MN} - e^{\phi} \Lambda_6 \right]. \tag{27}$$

There exists a solution where the dilaton ϕ is constant, $\phi = \phi_0$, and the solution in the Maxwell–Einstein theory is a solution just by replacing $\Lambda_6 \to \Lambda_6 e^{\phi_0}$ and $B_0^2 \to B_0^2 e^{-\phi_0}$. The constant value ϕ_0 is determined by the condition that the potential for ϕ has minimum [36]

$$V'(\phi_0) = -\frac{1}{2} B_0^2 e^{-\phi_0} + \Lambda_6 e^{\phi_0} = 0. \tag{28}$$

This is exactly the condition to have a flat geometry on the brane (see Eq. (21))

$$B_0^2 e^{-\phi_0} = 2\Lambda_6 e^{\phi_0}. \tag{29}$$

Thus unlike the Einstein–Maxwell system, one might not need a tuning condition in the bulk. In fact, The known solutions in this model which have maximally symmetric 4D metric all have vanishing vacuum energy.

Again there were several objections to this version of the self-tuning [31; 36]. It is possible to derive the 4D effective theory by putting the metric in the form

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + M_6^{-2} e^{-2\psi(x)} (dr^2 + \sin^2 r d\theta^2), \tag{30}$$

and assuming $\phi = \phi(x)$. The potential which results from the two scalar fields is [31]

$$V(\psi, \phi) = M_6^{-4} e^{\sigma_2} U(\sigma_1), \quad U(\sigma_1) = \frac{B_0^2}{2\alpha^2} e^{-2\sigma_1} - 2M_6^2 e^{-\sigma_1} + 2\Lambda_6, \tag{31}$$

where $\sigma_1 = 2\psi + \phi$ and $\sigma_2 = 2\psi - \phi$. Unlike the Einstein–Maxwell theory, σ_2 ensures that $U(\sigma_1)$ vanishes at the minimum of the potential. We should note that σ_2 is related to the classical scaling property of the model. The 6D equation of motion is invariant under the constant rescaling $g_{MN} \to e^\omega g_{MN}$ and $e^\phi \to e^{\phi-\omega}$ and the lagrangian is scaled as $\mathscr{L} \to e^{2\omega} \mathscr{L}$. The modulus σ_2 can be identified as the one associated with this scaling property. Thus the flatness of the 4D spacetime

is ensured by the scaling property of the theory. However, this eventually leads to the same tuning condition (29) as in the Einstein–Maxwell theory. Then we can apply the same arguments as in the previous section. Suppose that the tension of the brane changes. Flux conservation (and flux quantization) means that the tuning condition cannot be maintained and $U(\sigma_1) \neq 0$. What happens would be that σ_2 acquires a runaway potential and the 4D spacetime becomes non-static.

A caveat in this argument is that the metric ansatz (30) is restrictive. In fact, there is a class of static solutions where there is a warping in the bulk. The solution has the form [37; 38]

$$ds_6^2 = W(\eta)^2 \eta_{\mu\nu} dx^\mu dx^\nu + a_0^2(W(\eta)^8 d\eta^2 + d\theta^2), \quad \phi = \phi_0 + 4\ln W(\eta) + 2\lambda_3 \eta, \tag{32}$$

where $W(\eta)$ is the warp factor. If both branes, at the north pole and the south pole, have the same tension, the warp factor becomes trivial. However, if the tensions are not equal, there is a warping. For $\lambda_3 \neq 0$, the metric near the branes no longer corresponds to that of a simple conical singularity. These solutions cannot be described by the ansatz (30). Thus one can still hope that the solutions will go to these solutions after a change of tension. An unambiguous way to investigate this problem is to study the dynamical solutions directly in the 6D spacetime. However, once we consider the case where the tension becomes time dependent, we encounter a difficulty to deal with the branes. This is because for co-dimension 2 branes, we encounter a divergence of metric near the brane if we put matter other than tension on a brane. Hence, without specifying how we regularize the branes, we cannot address the question what will happen if we change the tension. Is the self-tuning mechanism at work and does it lead to a 4D static solution? Or do we get a dynamical solution driven by the runway behaviour of the moduli field? There was a negative conclusion on the self-tuning in this supersymmetric model for a particular kind of regularization [36]. However, the answer could depend on the regularization of branes and the jury remains out. It is important to study the time-dependent dynamics in the 6D spacetime and the regularization of the branes [39; 40; 41; 42].

If the self-tuning mechanism works, then we should seek an explanation for the accelerated expansion today. The supersymmetry in the bulk would also provide a very interesting mechanism (see [33; 34] and references therein for detailed discussions). Supersymmetry is supposed to be broken at least at the electroweak scale M_w. Then in the 4D spacetime, this gives a vacuum energy of the order $\rho \sim M_w^4$ as the cancellation between the contribution to the vacuum energy from boson fields and fermion fields ceases to exist at M_w. However, if the self-tuning mechanism is at work, this vacuum energy does not give any contribution to the cosmological constant on the brane. However, the breakdown of supersymmetry is mediated to the bulk at least gravitationally. Then there arises a supersymmetry breaking scale in the bulk given by

$$M_{sb} = \frac{M_w^2}{M_4}. \tag{33}$$

Interestingly, this scale is related to the size of the extra dimensions. If we want to solve the hierarchy problem between the Planck scale and the electroweak scale, M_6 should be of the order M_w. Then from the relation between M_6, M_4 and the size

of the extra-dimension L, Eq. (3), the supersymmetry breaking scale in the bulk is given by

$$M_{sb} = \frac{1}{L}. \tag{34}$$

If L is $10\,\mu$m, we get the correct order of magnitude for the cosmological constant if $\rho_\Lambda \sim M_{sb}^4$. In order to confirm this expectation, we should compute the effective potential for the radion which describes the size of the extra dimensions generated by supersymmetry breaking. The potential for the radion obtained by integrating out the bulk loops is given by

$$V(L) = \frac{c_2 M_6^2}{L^2} + \frac{c_3}{L^4} (\log(M_6^2 L^2) + C). \tag{35}$$

The calculation of c_2 depends on the details of the spectrum of the theory at M_w and $c_2 = 0$ is critical for this model to work. If $c_2 = 0$, the potential leads to a natural realization of the quintessence model where the radion L acts as a quintessence field.

Thus SLED gives a consistent framework to address the cosmological constant problem and the dark energy model *provided that* the self-tuning mechanism works and the supersymmetry breaking on the brane generates the desired potential for the radion, $V(L) \sim L^{-4}$. In SLED, the 6D Planck scale is supposed to be M_w and the size of the extra dimensions today are $L \sim 10\,\mu$m. This leads to a lot of interesting phenomenology in local tests of gravity, collider physics and so on [33; 34].

4 Late-time acceleration in the braneworld

A new twist to the cosmological constant problem is the late time acceleration of the Universe. The simplest way to realize this is to assume that a tiny amount of the cosmological constant is left after cancelling the vacuum energies. But the vacuum energy is typically more than 50 orders of magnitude larger than the observed value of the cosmological constant. Thus this is an incredible fine-tuning. Moreover, if the self-tuning idea works and the vacuum energy does not gravitate, it is in general difficult to realize the accelerated expansion of universe (see however the SLED proposal discussed in Sect. 3.3). Alternatively, it is possible that there is no cosmological constant but that large-distance modification of GR accounts for the late-time acceleration. The braneword gravity provides a natural framework for the study of this possibility. For example in the model proposed by Dvali, Gabadadze and Porrtati (DGP), 4D GR is modified on large scales [9]. It is in fact possible to realize the accelerated expansion of the universe without a cosmological constant [43; 44]. This solution is known as the self-accelerating universe. We should note that in these attempts, we do not solve the old cosmological constant problem. In addition, in the DGP model, the coincidence problem is not solved and we should introduce a fine-tuned dimensional parameter related to the scale of the cosmological constant, $\rho_\Lambda \sim 10^{-3}$ eV. However this is a novel alternative to dark energy models in GR and it gives a new perspective to approach the problem.

4.1 Dvali–Gabadadze–Porrati model

In the DGP model, gravity leaks off the 4D Minkowski brane into the 5D bulk Minkowski spacetime at large scales. The 5D action describing the DGP model is given by

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g}R + \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-\gamma}\,^{(4)}R - \int d^4x \sqrt{-\gamma}\mathscr{L}_m. \qquad (36)$$

Instead of having the bulk cosmological constant and the tension on a brane as in the RS model, there is an induced Einstein–Hilbert term on the brane.

On small scales, gravity is effectively bound to the brane and 4D Newtonian dynamics is recovered to a good approximation. The transition from 4D to 5D behaviour is governed by a crossover scale

$$r_c = \frac{\kappa_5^2}{2\kappa_4^2}. \tag{37}$$

The weak-field gravitational potential behaves as

$$\Psi \sim \begin{cases} r^{-1} & \text{for } r < r_c, \\ r^{-2} & \text{for } r > r_c. \end{cases} \tag{38}$$

Unlike the RS model, gravity becomes 5D at large distances. The DGP model was generalized by Deffayet to a Friedman–Robertson–Walker brane in a Minkowski bulk [44]. The energy conservation equation remains the same as in general relativity, but the Friedman equation is modified:

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{39}$$

$$\frac{H}{r_c} = H^2 - \frac{8\pi G_4}{3}\rho. \tag{40}$$

The modified Friedmann equation shows that at late times in a CDM universe with $\rho \propto a^{-3} \to 0$, we have

$$H \to H_\infty = \frac{1}{r_c}. \tag{41}$$

Since $H_0 > H_\infty$, in order to achieve acceleration at late times, we require $r_c \gtrsim H_0^{-1}$, and this is confirmed by fitting SN observations [50]. Like the LCDM model, the DGP model has simple background dynamics, with a single parameter r_c to control the late-time acceleration.

On small scales, the Newtonian potential behaves as 4D. The Friedmann equation also shows that the universe behaves as 4D at early times, $Hr_c \gg 1$. However, the recovery of GR is very subtle in this model [45]. In fact, although the weak-field gravitational potential behaves as 4D on scales smaller than r_c, the linearized gravity is *not* described by GR. This is because there is no normalized zero-mode in this model and 4D gravity is recovered as a resonance of the massive KK gravitons. The massive graviton contains $5°$ of freedom compared with $2°$ of freedom in a massless graviton. One of them is a helicity-0 polarization. Due to this scalar degree of freedom, linearized gravity is described by Brans-Dicke (BD) gravity with vanishing BD parameter in the case of Minkowski spacetime. Thus this model would be excluded by solar system experiments. However, the non-linear interactions of the scalar mode becomes important on larger scales than expected [45; 46; 47; 48]. Let us consider a static source with mass M. Gravity becomes non-linear near the Schwarzshild radius $r_g = 2GM$. However, the scalar mode becomes non-linear at $r_* = (r_g r_c^2)^{1/3}$ (the Vainstein radius) which is much larger than r_g if $r_c \sim H_0^{-1}$. In fact, for the Sun r_* is much larger than the size of the solar system. A remarkable finding is at once the scalar mode becomes non-linear, GR is recovered. This non-linear shielding of the scalar mode is crucial to escape from the tight solar system constraints. Fig. 4 summarizes the behaviour of gravity in the DGP model (see [49] for a review on the DGP model).

Fig. 4 Summary of the behaviour of gravity in the DGP model. At large scales $r > r_c$, the theory is 5D. On small scales $r < r_c$, gravity becomes 4D but the linearized theory is described by a Brans-Dicke theory. This affects the large scale structure (LSS) and the Integrated Sachs-Wolfe (ISW) effect and its cross-correlation to LSS. Below the Vainstein radius $r < r_*$, the theory approaches GR. This transition can be probed by weak lensing and cluster abundance as the non-linear dynamics is important for these measures. The solar system tests also provide constraints on the model in the 4D Einstein phase

Fig. 5 Joint constraints (*solid thick*) on DGP models from the SNe data (*solid thin*), the BO measure A (*dotted*) and the CMB shift parameter S (*dot-dashed*). The left plot uses SNe Gold data, the right plot uses SNLS data. The thick dashed line represents the flat models, $\Omega_K = 0$. From [52]

4.2 Observational constraints on the self-accelerating universe

The self-accelerating universe provides useful example where we can study how various observations can be combined to test the model. It also provides a possibility to find a failure of GR at cosmological scales. A key is the complicated behaviour of gravity. We have various cosmological observations that cover various scales. Then combining the various data sets, we can probe the complicated behaviour of gravity in this model. A central question is whether we can distinguish the DGP model from dark energy models in GR.

4.2.1 Expansion history

The first question is whether one can distinguish between the self-accelerating universe and the simple ΛCDM model in GR. Both models have the same number of parameters and phenomenologically both theories have the same simplicity. In terms of density parameters, the Friedmann equation in the ΛCDM model is given by

$$\Omega_M + \Omega_\Lambda + \Omega_K = 1. \tag{42}$$

On the other hand in the DGP, we have [50]

$$\Omega_M + 2\sqrt{\Omega_{r_c}}\sqrt{1 - \Omega_K} + \Omega_K = 1, \tag{43}$$

where we defined

$$\Omega_{r_c} = \frac{1}{4H_0^2 r_c^2}. \tag{44}$$

In order to constrain the density parameters, we can combine data from supernovae, the cosmic microwave background shift parameter, and possibly the baryon oscillation peak [51; 52; 53; 54; 55; 56]. Interestingly, the current observations already give us a hint how we can distinguish the models. While the ΛCDM model fits the three data sets comfortably, there is some tension between the data and DGP (Fig. 5) [52]. It is suggested that a slightly open universe can fit the data set better in the DGP (Fig. 6) [53].

Note that the baryon acoustic oscillation measure requires the knowledge of the power spectrum thus the knowledge of perturbations. Precisely speaking, the analysis must be redone for the DGP model. We expect that only small corrections

Fig. 6 The $\Delta\chi^2$ between the best fit flat and open DGP versus that of a flat ΛCDM model. The Gold supernova (SN) data set is used in the top panel and the SNLS SN data set is used in the bottom panel. The DGP model requires curvature and a high Hubble constant. With the addition of Key Project (KP) direct Hubble constant measurements, open DGP is a marginally poorer fit to the data than flat ΛCDM. From [53]

Table 1 Best-fit parameters from SNe-CMB shift-Baryon Oscillation constraints, and χ^2 values, for the DGP and LCDM models

	Best-fit acceleration parameter	Best-fit density parameter	Best-fit curvature parameter	χ^2 value
DGP	$\Omega_{rc} = 0.125$	$\Omega_m = 0.270$	$\Omega_K = +0.0278$	185.0
LCDM	$\Omega_\Lambda = 0.730$	$\Omega_m = 0.285$	$\Omega_K = -0.0150$	177.8

The

Gold data is used for the SNe. From [52]

Table 2 As in Table 1 for the Legacy SNe data

	Best-fit acceleration parameter	Best-fit density parameter	Best-fit curvature parameter	χ^2 value
DGP	$\Omega_{rc} = 0.130$	$\Omega_m = 0.255$	$\Omega_K = +0.0300$	128.8
LCDM	$\Omega_\Lambda = 0.740$	$\Omega_m = 0.270$	$\Omega_K = -0.0100$	113.6

From

[52]

Fig. 7 Comparison between the results of fitting DGP and ΛCDM to the SNLS and ESSENCE supernova data set (filled in 68%, 95% and 99% confidence regions) and the Riess 07 Gold set (*dotted lines*). The solid black line corresponds to spatially flat universes. From [55]

are involved, but this problem must be addressed. The conclusion also seems to depend on the data set for supernovae (Tables 1 and 2) [52]. This is also true using the latest results from the ESSENCE and SNLS supernova data set and the Riess 07 Gold set (Fig. 7) [55].

In the future, precision data will enable us to distinguish between the DGP and the ΛCDM more clearly. Figure 8 shows the prediction of the baryon acoustic peak oscillation observed by a future WFMOS survey which is assumed to contain 2.1×10^6 galaxies, over 200 deg^2, at $0.5 < z < 1.3$ [57]. Clearly the difference between the two is much larger than the error bars.

4.2.2 Linear growth of structure

Although the DGP model can be distinguished from the ΛCDM model, background tests will never distinguish the DGP model from dark energy models in GR. This is because there always exists a dark energy model in GR that has exactly the same expansion history as in DGP. In fact as far as the background evolu-

Fig. 8 Theoretical predictions for $d\ln P(k)/d\ln k$ assuming a sample WFMOS, where $P(k)$ is the power spectrum of galaxies. The squares with error bars are evaluated with the simple simulation of the power spectrum for the ΛCDM model. The asterisks are the DGP model, but the error bars, which are almost the same as that of the ΛCDM model, are omitted for simplicity. Theoretical curves are the DGP model (dashed red curve) and the ΛCDM model (solid black curve). The parameters are $n_s = 0.95$, $\Omega_b = 0.044$, $\Omega_m = 0.27$ and the linear bias is taken as $b_0 = 1.5$. From [57]

tion of the Universe is concerned, the DGP is equivalent to the dark energy model whose equation of state is given by Lue et al. [58]

$$w = -\frac{1}{1 + \Omega_m(a)}.$$ (45)

For small red-shift, this is well fitted by $w = w_0 + w_a(1 - a)$ where $w_0 = -0.78$ and $w_a = 0.32$ if $\Omega_m = 0.3$ today [59]. Then we cannot distinguish the DGP from the dark energy model in GR.

However, even if the background dynamics is the same, this does not mean that the dynamics of perturbations is the same. Koyama and Maartens obtained the solutions for metric perturbations on sub-horizon scales by consistently solving the 5D perturbations under quasi-static approximations [60]. Scalar metric perturbations are given in longitudinal gauge by

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)\delta_{ij}dx^i dx^j,$$ (46)

and the perturbed energy-momentum tensor for matter is given by

$$\delta T_\nu^\mu = \begin{pmatrix} -\delta\rho & a\delta q_{,i} \\ -a^{-1}\delta q^{,i} & \delta p\,\delta^i_j \end{pmatrix}.$$ (47)

The solutions for the brane metric perturbations are [60]

$$\frac{k^2}{a^2}\Phi = 4\pi G_4\left(1 - \frac{1}{3\beta}\right)\rho\delta,$$ (48)

Fig. 9 The comoving distance $r(z)$ is shonw for ΛCDM (long dashed), DGP (solid, thick) and the equivalent GR dark energy model on the left. On the right, the growth history $g(a) = \delta(a)/a$ is shown for LCDM (*long dashed*) and DGP (*solid, thick*). The growth history for a dark energy model (*short dashed*) is also shown, with the same expansion history as DGP. Due to the time variation of Newton's constant through β in Eq. (53), the growth factor $g(a)$ receives an additional suppression compared with the dark energy model. DGP-4D (*solid, thin*) shows the incorrect result in which the inconsistent assumption is adopted. We set the density parameter for matter today as $\Omega_{m0} = 0.3$. From [60]

$$\frac{k^2}{a^2}\Psi = -4\pi G_4 \left(1 + \frac{1}{3\beta}\right)\rho\delta, \tag{49}$$

where

$$\beta = 1 - 2r_c H\left(1 + \frac{\dot{H}}{3H^2}\right), \tag{50}$$

and

$$\delta = \delta\rho - 3H\delta q. \tag{51}$$

This agrees with the results obtained by Lue, Scoccimarro and Starkman. They find spherically symmetric solutions by closing the 4D equations using an anzatz for
the metric and checking in retrospect that the obtained solutions satisfy regularity in the bulk. It was shown that the solutions (48) and (49) are uniquely determined by the regularity condition in the bulk within our approximations.

The modified Poisson equation (48) shows the suppression of growth. The rate of growth is determined by δ, and for CDM,

$$\ddot{\delta} + 2H\dot{\delta} = -\frac{k^2}{a^2}\Psi, \tag{52}$$

which leads to

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G_4 \left(1 + \frac{1}{3\beta}\right)\rho\delta. \tag{53}$$

Thus the growth rate receives an additional modification from the time variation of Newton's constant through β.

In Fig. 9, we show the linear growth factor δ/a for the DGP model, and compare it with ΛCDM and with the GR dark energy model whose background evolution matches that of the DGP model. We also show the incorrect DGP result, in which the inconsistent assumption of neglecting 5D perturbations is effectively adopted [61]. This inconsistent assumption has been made in various treatments but it leads to unreliable results. The correct equations for subhorizon density perturbations are crucial for meaningful tests of DGP predictions against structure formation observations. This highlights the fact that the growth rate is very sensitive to the modification of gravity.

There are several observations that can probe the growth of structure. Weak lensing measures the deflection of light generated by matter fluctuations (see [62] for a review). The deflection potential is given by

$$\phi = \Phi + \Psi. \tag{54}$$

Fig. 10 The galaxy-ISW cross-correlation coefficient R_l^i in each galaxy bin from $z = 0$ to $z = 3$. Solid curves denote flat ΛCDM and dashed curves denote open DGP. Note the much larger correlation at high z in open DGP. From [63]

We can relate ϕ to the matter overdensity δ:

$$\phi = \frac{8\pi G_4 a^2}{k^2} \delta.$$

(55)

Interestingly, this formula in the DGP is the same as the one in GR. However, the change of the growth rate leads to a different prediction of weak lensing. We should note that current observations measure weak lensing sourced by matter fluctuations in the non-linear regime. The solutions (48) can be applied only to linear perturbations and there is no justification to use the linear growth rate and predict the non-linear power spectrum using the mapping formula developed in GR. We will come back to this issue in section IV.B.

Another probe is the Integrated Sachs–Wolfe (ISW) effect. This is determined by the time variation of the deflection potential $\dot{\phi}$. On large scales, we should deal with the truly 5D effects and the quasi-static solutions are not applicable. There is some progress to deal with fully dynamical perturbations by adopting a scaling ansatz to solve the 5D equations [63]. They find that the quasi-static solution is an attractor on subhorizon scales. The ISW effects are sub-dominant compared with the primordial anisotropies formed at the last scattering surface. In order to extract the ISW effects, it is proposed to take cross correlation between the matter distributions and the CMB [64; 65; 66]. It was shown that the quasi-static solution is valid to calculate the corss correlation for large ℓ where a signal is maximized [63]. The growth function $g(a)$ changes at earlier times in the self-accelerating universe than in the ΛCDM model. This gives a larger signal in the cross correlation at high redshift. Thus higher red-shift galaxies can test the predictions in the self-accelerating universe with high significance (Fig. 10).

Hence, structure formation tests are essential for breaking the degeneracy with dark energy models in GR [67; 68; 69; 70; 71]. The distance-based SN observations draw only upon the background 4D Friedman equation (40) in DGP models, and therefore there are quintessence models in GR that can produce precisely the same SN redshifts as DGP. By contrast, structure formation observations require the 5D perturbations in DGP, and one cannot find equivalent GR models. This leads to an exciting possibility to find a failure of GR [68]. Suppose that our Universe is described by the DGP model. However, astronomers still try to fit the data by dark energy models in GR. For example, they use the parametrization of the equation of state of dark energy

$$w = w_0 + w_1 z.$$

(56)

Combining SN observations, CMB shift parameter and weak lensing, there appears an inconsistency. This is because weak lensing probes the growth of structure and the growth rate in the DGP model cannot be fitted by the growth rate in GR models given the same expansion history. Figure 11 demonstrates this possibility.

Fig. 11 Equations of state found using two different combinations of data sets. Solid contours are for fits to SN Ia and CMB data, while dashed contours are for fits to weak lensing and CMB data. The significant difference (inconsistency) between the equations of state found using these two combinations is a signature of the DGP model. The inconsistency is an observational detection of the underlying modified gravity DGP model (assumed here to generate the data). From [68]

Fig. 12 Confidence regions at 68% for the benchmark survey $z_{\text{mean}} = 0.9, d = 35$ (*outer contour*) and for $d = 50, 75$ (*inner contours*) for DGP. The dotted line represents the ΛCDM value. From [74]

In order to quantify the difference in the growth rate, it is convenient to parametrize the growth rate as [59]

$$g(a) = \exp\left\{ \int\limits_0^a d\ln a(\Omega(a)^\gamma - 1) \right\}.$$ (57)

In a quintessence model, γ is well approximated by

$$\gamma(w) = 0.55 + 0.05(1 + w(z = 1)).$$ (58)

In the DGP, γ is well approximated as $\gamma = 0.68$ [59]. Recently, several authors tried to estimate how accurately we can constrain γ using weak lensing in future surveys [72; 73; 74]. These results suggest that in the future, we will be able to discriminate ΛCDM and the DGP model from the difference in the growth rate. Figure 12 shows the constraint on γ for the DGP model assuming the "bench mark" survey on weak lensing, where the mean redshift is $z_{\text{mean}} = 0.9$ and the number of sources per arcmin^2 is $d = 35, 50, 75$ [74].

However, as we mentioned before, the weak lensing measure requires knowledge of the non-linear power-spectrum. In the DGP, this is a subtle problem. The DGP approaches GR on small scales. This is essential to evade the tight constraints from the solar system experiments. The non-linear power spectrum would be sensitive to this transition from Brans–Dicke linear theory to GR non-linear theory. The analyses so far have used the simple mapping formula developed in GR to derive the non-linear power spectrum. This approach could be inconsistent. Nevertheless, the conclusion that we will be able to distinguish the difference in the growth of structure would be valid and this is a very exciting possibility that we can achieve in future observations.

4.2.3 Non-linear structure formation

For quasi-static perturbations, it is possible to extend the linear result to non-linear perturbations by taking into account partially the non-linear effects of gravity. A key is the so called brane bending mode [48; 76; 77]. This mode describes the perturbations of the location of the brane and mediates an additional scalar interaction. In the linear regime, this is the scalar mode that makes the theory of BD type. This scalar mode becomes non-linear on much larger scales than gravity.

In terms of the brane bending mode φ, the effective equations on the brane are given by [75]

$$\frac{2}{a^2}\nabla^2\Phi = -8\pi G_4\delta\rho + \frac{1}{a^2}\nabla^2\varphi, \tag{59}$$

$$\Psi + \Phi = \varphi, \tag{60}$$

where the equation of motion for φ is given by

$$3\beta(t)\frac{\nabla^2}{a^2}\varphi + \frac{r_c^2}{a^4}\left[(\nabla^2\varphi)^2 - (\nabla_i\nabla_j\varphi)^2\right] = 8\pi G_4\delta\rho. \tag{61}$$

Again these equations are derived by properly solving the 5D equations and imposing the regularity condition in the bulk. Here we assume gravity is linear, $\Psi, \Phi \ll 1$, but we take into account the second order effects of φ. Note that the coefficient of the second order terms is given by r_c^2. As we take $r_c \sim H_0^{-1}$, the second order terms can be comparable to the linear term even if gravity remains linear.

These non-linear equations are difficult to solve in general. If we assume spherical symmetry, the solution for φ is given by

$$\frac{d\varphi}{dr} = \frac{r_g}{r^2}\Delta(r), \quad \Delta(r) = \frac{2}{3\beta}\left(\frac{r}{r_*}\right)^3\left(\sqrt{1+\left(\frac{r_*}{r}\right)^3}-1\right), \qquad (62)$$

where

$$r_* = \left(\frac{8r_c^2 r_g}{9\beta^2}\right)^{1/3}, \qquad (63)$$

and r_g is the Schwarzschild radius $r_g = 2G_4M$. The solutions for Φ and Ψ are obtained as

$$\Phi = \frac{r_g}{2r} + \frac{\varphi}{2}, \qquad (64)$$

$$\Psi = -\frac{r_g}{2r} + \frac{\varphi}{2}. \qquad (65)$$

For $r > r_*$, we recover the solutions for linear perturbations (48). For $r < r_*$, the solutions for metric perturbations are given by Lue et al. [58]

$$\Phi = \frac{r_g}{2r} + \frac{1}{\beta}\sqrt{\frac{\beta^2 r_g r}{2r_c^2}}, \qquad (66)$$

$$\Psi = -\frac{r_g}{2r} + \frac{1}{\beta}\sqrt{\frac{\beta^2 r_g r}{2r_c^2}}. \qquad (67)$$

In this region, the corrections to the solution in 4D GR are suppressed, so that Einstein gravity is recovered. The radius r_* is the Vainstein radius in the cosmological background.

The conservation of the energy momentum tensor holds as in GR. Then the continuity equation and the Euler equation are the same as in GR:

$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla^i(1+\delta)v_i = 0, \qquad (68)$$

$$\frac{\partial v_i}{\partial t} + \frac{1}{a}(v^j\nabla_j)v_i + Hv_i = -\frac{1}{a}\nabla_i\Psi, \qquad (69)$$

where v_i is the velocity perturbation of dark matter. Eqs. (59), (60), (61) (68) and (69) form a closed set of equations that has to be solved to address the non-linear structure formation problem in the DGP model. In order to see how GR is recovered dynamically, let us consider the evolution of a spherical top-hat perturbation $\delta(t,r)$ of top-hat radius R_t, where $\rho(t,r) = \rho(t)(1+\delta)$ is the full density distribution and $\rho(t)$ is the background density [58]. The Newtonian potential Ψ dominates the geodesic evolution of overdensity. Then the evolution equation for the over-density δ is given by

Fig. 13 Numerical solution of spherical collapse. The left panel shows the evolution for a spherical perturbation with $\delta_i = 3 \times 10^{-3}$ at $z_i = 1{,}000$ for $\Omega_m = 0.3$ in DGP gravity and in ΛCDM. The right panel shows the ratio of the solutions once they are both expressed as a function of their linear density contrasts. From [58]

$$\ddot{\delta} - \frac{4}{3}\frac{\dot{\delta}^2}{1+\delta} + 2H\dot{\delta} = 4\pi G\rho\delta(1+\delta)\left[1 + \frac{2}{3\beta}\frac{1}{\varepsilon}\left(\sqrt{1+\varepsilon} - 1\right)\right], \quad (70)$$

$$\varepsilon \equiv \frac{8r_c^2 r_g}{9\beta^2 R_i^3} = \frac{8}{9}\frac{(1+\Omega_m)^2}{(1+\Omega_m^2)^2}\Omega_m\delta. \quad (71)$$

In the linear regime, $\delta \ll 1$, $\varepsilon \ll 1$, we recover the linear evolution of the overdensity, Eq. (53). On the other hand for $\varepsilon \gg 1$, the right hand side of Eq. (70) becomes the same as in GR and the dynamics of the non-linear collapse becomes the same as in GR. Figure 13 shows the behaviour of δ in the DGP compared with the ΛCDM model.

For the non-spherically symmetric case, we need to solve the equations numerically. We should emphasize again that the analysis of the non-linear transition of the theory to GR is essential for the prediction of weak lensing and this is an outstanding open problem.

4.3 Theoretical consistency of the DGP model

Although the DGP model offers a concrete example for a modified gravity alternative to dark energy, this model is not free from problems. In fact, this model demonstrates the difficulties of modifying GR at large distances. One of the problems is related to the non-linearity of the scalar mode. The non-linear interactions

of the scalar mode become important at the Vainstein length $r_* \sim (r_c^2 r_g)^{1/3}$. If we consider a Planck scale mass particle, this length is given by $\Lambda_c^{-1} = (r_c^2 M_{pl}^{-1})^{-1/3}$, which is $\sim 1,000$ km for $r_c \sim H_0^{-1}$. This defines the length below which quantum corrections for the scalar mode become important. Thus Λ_c plays the same role as the Planck scale in GR. Then below the length Λ_c^{-1}, the classical theory loses its predictability. This is known as the strong coupling problem [76; 77; 78]. There have been debates whether this is indeed a problem or not [79]. It is suggested that there exists a consistent choice of counter-terms for which the model remains calculable [77].

The most serious problem in this model is that there are ghost-like excitations around the self-accelerating universe [76; 77; 80; 81; 82; 83]. In fact the growth rate already manifests this problem. The solution for the linearized perturbations is described by a BD theory with BD parameter given by [46; 60]

$$\omega = \frac{3}{2}(\beta - 1). \tag{72}$$

For large Hr_c, β is always negative. In fact, if $\omega < -3/2$, the BD scalar field has the wrong sign for its kinetic term and it becomes a ghost. For de Sitter spacetime, the condition $\omega < -3/2$ implies $Hr_c > 1/2$. This is exactly the condition that there exists a ghost in the theory. We can understand the extra suppression of the growth rate as due to the repulsive force mediated by the ghost. If we avoid the negative norm state when quantizing the theory with ghosts, the ghosts have unboundedly negative energy density and lead to the absence of a stable vacuum state. In a Lorentz invariant theory this instability is instant as the decay rate of the vacuum is infinity. It is suggested that if there is a Lorentz non-invariant cut-off in the theory and the cut-off scale is enough low, it is possible to keep the instability at unobservable level [84]. In the DGP model, the strong coupling scale Λ_c may serve as the cut-off scale. It is needed to calculate the decay rate of the vacuum and to see whether the self-accelerating universe can survive beyond the age of our Universe. It is also necessary to check the validity of the linearized analysis [75; 85; 86; 87; 88]. Several non-perturbative solutions indicate that the self-accelerating universe would be unstable [87; 88]. Then we are naturally lead to ask what does the solution decay to [89]. This is still an open question. See [90] for a review on the issue of the ghost in the DGP model.

Finally, it was pointed out that time-dependent perturbations around the spherically symmetric spacetime have a sound speed greater than 1 [91]. Again there are debates whether this is a problem or not. One subtlety is that this argument is based on the effective theory for the scalar mode and it is not clear this effective theory captures the property of the full gravitational perturbations in the model [92]. In addition, causality should be defined in a 5D spacetime and it is not clear that the super-luminality in the 4D effective theory really means the breakdown of causality in the full 5D theory.

Although it is still not clear whether we should deny the DGP model as a consistent theory due to these problems, this certainly demonstrates the difficulty for the large distance modification of gravity to explain the late time accelerated expansion of the Universe. It is necessary to seek improved models that can avoid these problems.

5 Conclusion

In this article, we review the attempts to address the cosmological constant problem and the dark energy problem in braneworlds.

The cosmological constant problem has resisted solution for many years. The conventional approach relies on 4D low-energy physics. This was a natural way of attacking the problem as in a conventional KK compactification, the size of extra dimensions must be smaller than TeV^{-1} and, below the TeV scale, our Universe can be described by the 4D effective theory. However, the braneworld picture completely changes the notion of extra dimensions. The extra dimensions can be large. For 6D spacetime, the size of the extra dimensions can be $L \sim 10\,\mu\text{m}$, with the 6D planck scale 10 TeV. Then above the scale L^{-1}, the Universe is described by $6D$ and the 4D effective theory cannot be used. In fact the energy density for the cosmological constant necessary to explain the present accelerated expansion is roughly $\rho_\Lambda \sim L^{-4}$. Moreover, the way the vacuum energy gravitates in our 4D Universe is completely different in the braneworld. Again in a 6D spacetime, the vacuum energy on a 4D brane does not curve the 4D spacetime but just changes the geometry of the extra dimensions. This leads to the self-tuning idea where the change of the vacuum energy in 4D spacetime is compensated by the modification in the geometry of extra dimensions. Although it was shown that the simple non-supersymmetric model does not work, it is hoped that the supersymmetric version of the model can realize the self-tuning. A close inspection reveals many problems in this approach, but further studies are necessary to judge whether the self-tuning idea really works or not. The outstanding problem is to know whether the 4D spacetime settles down to a static solution due to the self-tuning mechanism if there is a phase transition in the 4D spacetime. This requires a regularization of the branes and the analysis of the time dependent dynamics in the 6D spacetime. Based on the hope for the existence of self-tuning, the Supersymmetric Large Extra Dimensions (SLED) model is proposed as a framework to address the cosmological constant problem and the dark energy simultaneously. The self-tuning mechanism is supposed to cure the problem of the large vacuum energy produced by the phase transition in the 4D spacetime. This mechanism relies on the supersymmetry in the 6D spacetime, but supersymmetry is inevitably broken on a brane at least at TeV scale. This breakdown is mediated to the bulk only gravitationally and creates a weak potential for the radion which is the size of the extra dimensions. The potential energy is determined by the supersymmetric breaking scale in the bulk and it is argued that if the size of the extra dimensions is $10\,\mu\text{m}$, the 6D Planck scale is 10 TeV and the potential energy for the radion has the right amplitude to explain the present accelerated expansion of the Universe. The potential depends on the details of the spectrum of theory and it remains to be seen whether this proposal can work or not in a concrete realization of the models in string theory.

The late-time accelerated expansion of the Universe is a new problem forced by the discovery made by astronomers in 1998. An interesting possibility to explain this is a large distance modification of gravity. Again, the braneworld picture plays an essential role. The braneworld model provides a concrete example where gravity leaks off the brane and modifies the 4D GR on the brane at large distances. The DGP model is the simplest model that realizes this idea. The action for the

model is very simple. The 5D spacetime is just a Minkowski spacetime described by Einstein gravity. We are living on a 4D brane where 4D gravity is assumed to be induced. Despite the simple set-up of the model, gravity in this model is remarkably complicated. In fact there exists a solution (the self-accelerating universe) where the accelerated expansion of the Universe is realized just by the modification of gravity. We focused on the possibility to distinguish this model from dark energy models in GR by combining various observations. This leads to an interesting possibility to find a failure of GR at cosmological scales. Although the DGP model is the simplest model where we can address many issues from a simple action, the model is not free from problems. In particular, it has been shown that there exists a ghost in a self-accelerating universe. It is crucial to study how we can avoid the decay of the self-accelerating universe in order for the observational tests of the model to make sense.

The attempts to use higher-dimensional gravity and branes to address the cosmological constant problem and dark energy are new but there has been much progress. In this article, we only covered several attempts among them. We see that these attempts bring us a completely new way of attacking long-standing and tough problems although none of the models is completely successful so far. We hope further developments of the models based on these attempts lead to solutions for the long-standing problems.

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