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The alternating presentation of $U_q(\widehat{gl_2})$ from Freidel-Maillet algebras

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This paper is dedicated to Paul Terwilliger for his 65th birthday

Abstract

An infinite dimensional algebra denoted $\bar{\mathcal{A}}_q$ that is isomorphic to a central extension of U_q^+ - the positive part of $U_q(\widehat{sl_2})$ - has been recently proposed by Paul Terwilliger. It provides an 'alternating' Poincaré-Birkhoff-Witt (PBW) basis besides the known Damiani's PBW basis built from positive root vectors. In this paper, a presentation of $\bar{\mathcal{A}}_q$ in terms of a Freidel-Maillet type algebra is obtained. Using this presentation: (a) finite dimensional tensor product representations for $\bar{\mathcal{A}}_q$ are constructed; (b) explicit isomorphisms from $\bar{\mathcal{A}}_q$ to certain Drinfeld type 'alternating' subalgebras of $U_q(\widehat{gl_2})$ are obtained; (c) the image in U_q^+ of all the generators of $\bar{\mathcal{A}}_q$ in terms of Damiani's root vectors is obtained. A new tensor product decomposition for $U_q(\widehat{sl_2})$ in terms of Drinfeld type 'alternating' subalgebras follows. The specialization $q \to 1$ of $\bar{\mathcal{A}}_q$ is also introduced and studied in details. In this case, a presentation is given as a non-standard Yang-Baxter algebra.

1. Introduction

Quantum affine algebras are known to admit at least three presentations. For $U_q(\widehat{sl_2})$, the first presentation originally introduced in [40,29] - referred as the Drinfeld-Jimbo presentation in the

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literature - is given in terms of generators $\{E_i, F_i, K_i^{\pm 1} | i = 0, 1\}$ and relations, see Appendix A. The so-called Drinfeld second presentation was found later on [30], given in terms of generators $\{\mathsf{x}_k^{\pm}, \mathsf{h}_\ell, \mathsf{K}^{\pm 1}, C^{\pm 1/2} | k \in \mathbb{Z}, \ell \in \mathbb{Z}\backslash\{0\}\}$ and relations. The third one, obtained in [54], takes the form of a Faddeev-Reshetikhin-Takhtajan (FRT) presentation [31,32]. In these definitions, note that the so-called derivation generator is omitted (see [22, Remark 2, p. 393]). In the following, we denote respectively U_q^{DJ}, U_q^{Dr} and U_q^{RS} these presentations of $U_q(\widehat{sl_2})$. In addition, for $U_q(\widehat{sl_2})$ note that a fourth presentation called 'equitable', denoted U_q^{IT}, has been introduced in [38]. It is generated by $\{y_i^{\pm}, k_i^{\pm} | i = 0, 1\}$. For the explicit isomorphism $U_q^{IT} \rightarrow U_q^{DJ}$, see [38, Theorem 2.1].

The construction of a Poincaré-Birkhoff-Witt (PBW) basis for $U_q(\widehat{sl_2})$ [25,18] on one hand, and the FRT presentation of Ding-Frenkel [28] on the other hand brought major contributions to the subject, by establishing the explicit isomorphisms between U_q^{DJ}, U_q^{Dr} and U_q^{RS} (see also [41,27]). To motivate the goal of the present paper, as a preliminary let us briefly review the main results of [25,18] and [28].

• To establish the isomorphism between U_q^{DJ} and U_q^{Dr}, the main ingredient is the construction of a PBW basis. In [25], it is shown that the so-called positive part of $U_q(\widehat{sl_2})$ denoted $U_q^{DJ,+}$ - cf. Notation 1.2 - is generated by positive (real and imaginary) root vectors [25, Section 3.1]. The root vectors are obtained using Lusztig's braid group action on U_q^{DJ} [49]. Based on the structure of the commutation relations among the root vectors, a PBW basis for $U_q^{DJ,+}$ is first obtained [25, Section 4]. Then, introduce the subalgebras $U_q^{DJ,-}$, $U_q^{DJ,0}$ of U_q^{DJ}. Thanks to the tensor product decomposition $U_q^{DJ} \cong U_q^{DJ,+} \otimes U_q^{DJ,0} \otimes U_q^{DJ,-}$ [49] and some automorphism of U_q^{DJ}, the PBW basis for $U_q^{DJ,+}$ induces a PBW basis for $U_q(\widehat{sl_2})$ [25, Section 5]. Then, the explicit isomorphism $U_q^{Dr} \rightarrow U_q^{DJ}$ [18] maps Drinfeld generators to root vectors. See [19, Lemma 1.5], [27].

• To establish the explicit isomorphism between U_q^{RS} and U_q^{Dr}, the main ingredient in [28] is the construction of a FRT presentation for $U_q(\widehat{gl_2})$, which can be interpreted as a central extension of $U_q(\widehat{sl_2})$ [35]. In this approach, the defining relations are written in the form of a Yang-Baxter algebra. Namely, two quantum Lax operators $L^{\pm}(z)$ which entries are generating functions with coefficients in two different subalgebras of U_q^{Dr} are introduced. They satisfy certain functional relations (the so-called 'RTT' relations) characterized by an R-matrix. The explicit isomomorphism $U_q^{RS} \rightarrow U_q^{Dr}$ is obtained as a corollary of the FRT presentation of $U_q(\widehat{gl_2})$.

In these works, Damiani's root vectors (or equivalently the Drinfeld generators), associated PBW bases and the Yang-Baxter algebra play a central role. Later on, these objects found several applications. For instance, the universal R-matrix is built from elements in PBW bases of $U_q(\widehat{sl_2})$ subalgebras [26]. Also, irreducible finite dimensional representations of $U_q(\widehat{sl_2})$ are classified using U_q^{Dr} [21]. A natural question is the following: for $U_q(\widehat{sl_2})$, is it possible to construct a different 'triplet' of mutually isomorphic algebras other than U_q^{DJ} (or U_q^{IT}), U_q^{Dr} and U_q^{RS}?

Recent works by Paul Terwilliger bring a new light on this subject, and give a starting point for a precise answer. Indeed, in [61,62] Terwilliger investigated the description of PBW bases of $U_q(\widehat{sl_2})$ from the perspective of combinatorics, using a q-shuffle algebra $\mathbb{V}$ introduced earlier by Rosso [53]. Remarkably, using an injective algebra homomorphism $U_q^{DJ,+} \rightarrow \mathbb{V}$ a closed form for the images in $\mathbb{V}$ of Damiani's root vectors of $U_q^{DJ,+}$ - the basic building elements of Damiani's PBW basis - was obtained in terms of Catalan words [61, Theorem 1.7]. Then, in [62], he introduced a set of elements $\{W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1} | k \in \mathbb{N}\}$ into the q-shuffle algebra named as 'alternating' words. It was shown that the alternating words generate an algebra denoted U [62, Section 5] for which a PBW basis was constructed [62, Theorem 10.1,10.2]. Considering the preimage in $U_q^{DJ,+}$ of the alternating words of U, a new PBW in basis - called alternating - for $U_q^{DJ,+}$ arises, besides Damiani's one [25, Theorem 2]. A comparison between the images in $\mathbb{V}$ of both PBW bases was done, see [62, Section 11]. More recently [63], a central extension of the preimage of the algebra U arising from the exchange relations between alternating words, denoted $\mathcal{U}_q^+$, has been introduced. Its generators are in bijection with 'alternating' generators recursively built in $U_q^{DJ,+}$ and form an 'alternating' PBW basis for the new algebra $\mathcal{U}_q^+$ [63, Section 10].

In this paper, we investigate further these new 'alternating' algebras motivated by the construction of a new triplet of presentations for $U_q(\widehat{sl_2})$. To this aim, following [63] we introduce the algebra $\bar{\mathcal{A}}_q$ with generators $\{\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1}|k \in \mathbb{N}\}$ - see Definition 2.1. Note that to enable a non-trivial specialization $q \to 1$, the definitions of $\bar{\mathcal{A}}_q$ and $\mathcal{U}_q^+$ slightly differ. However, for $q \neq 1$ $\bar{\mathcal{A}}_q$ and $\mathcal{U}_q^+$ are essentially the same object. Also, the center $\mathcal{Z}$ of $\bar{\mathcal{A}}_q$ is introduced. Adapting the results of [63], the 'alternating' PBW basis of $\bar{\mathcal{A}}_q$ is given, see Theorem 2.12. Following [62], similarly we introduce the algebra $\bar{A}_q$ with generators $\{W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1}|k \in \mathbb{N}\}$. One has:

$$\bar{\mathcal{A}}_q \cong \bar{A}_q \otimes \mathcal{Z} \, . \tag{1.1}$$

Let $\langle W_0, W_1\rangle$ denote the subalgebra of $\bar{A}_q$ generated by W_0, W_1. The simplest relations satisfied by W_0, W_1 are the q-Serre relations (2.44), (2.45), of $U_q^{DJ,+}$ - see (A.1). Actually, according to [62], $\bar{A}_q \cong U_q^{DJ,+} \cong U_q^{DJ,-}$. So, having in mind the isomorphic pair consisting of $U_q^{DJ,+}$ (or $U_q^{DJ,-}$) and certain subalgebras of U_q^{Dr} [18,19], an 'alternating' isomorphic pair is provided by $\langle \mathcal{W}_0, \mathcal{W}_1\rangle$ and $\bar{A}_q$. Furthermore, by analogy with [18], the explicit isomorphism $\bar{A}_q \to \langle W_0, W_1\rangle$ follows from Lemma 2.9 using a map $\gamma : \bar{\mathcal{A}}_q \to \bar{A}_q$. Details are reviewed in Section 2. For completeness, the specialization $q \to 1$ of $\bar{\mathcal{A}}_q$, denoted $\bar{\mathcal{A}}$, is also introduced.

The main result of this paper is a presentation for $\bar{\mathcal{A}}_q$ which sits into the family of Freidel-Maillet type algebras[1] [33] for generic q, see Theorem 3.1. For the specialization $\bar{\mathcal{A}}$, a FRT type presentation is obtained. It sits into the family of non-standard Yang-Baxter algebras, see Proposition 3.6. This is done in Section 3. This Freidel-Maillet type presentation of $\bar{\mathcal{A}}_q$ gives an efficient framework for studying in more details this algebra and clarifying its relation with U_q^{DJ} (or U_q^{IT}), U_q^{Dr} and U_q^{RS}. The following results are obtained:

(a) Tensor product realizations of $\bar{\mathcal{A}}_q$ in $U_q(sl_2)^{\otimes N}$ are explicitly constructed. They generate certains quotients of $\bar{\mathcal{A}}_q$, characterized by a set of linear relations satisfied by the fundamental generators. See Proposition 4.5. This is done in Section 4.

(b) Explicit isomorphisms between $\bar{\mathcal{A}}_q$ and certain 'alternating' subalgebras of $U_q(\widehat{gl_2})$, denoted $U_q(\widehat{gl_2})^{\triangleright,+}$ and $U_q(\widehat{gl_2})^{\triangleleft,-}$, are obtained. See Propositions 5.18, 5.20. The main ingredient in the analysis is the use of the Ding-Frenkel isomorphism [28]. As a corollary, similar results for $\bar{A}_q$ and the 'alternating' subalgebras of $U_q(\widehat{sl_2})$ follow. Also, it is shown that $\bar{\mathcal{A}}_q$ can be regarded as a left (or right) comodule of alternating subalgebras of $U_q(\widehat{gl_2})$. An example of coaction map is given in Lemma 5.25. See Example 5.26.

1 See also [51,2,48].

(c) The explicit isomorphism $\iota : \langle \mathsf{W}_0, \mathsf{W}_1\rangle \to U_q^{DJ,+}$ given by (2.46) is extended to the whole set of generators of $\bar{\mathcal{A}}_q$: a set of functional equations that determine the explicit relation between Damiani's root vectors $\{E_{n\delta+\alpha_i}, E_{n\delta}|i=0,1\} \in U_q^{DJ,+}$ (or $\{F_{n\delta+\alpha_i}, F_{n\delta}|i=0,1\} \in U_q^{DJ,-}$) and the generators of $\bar{A}_q$ is derived, see Proposition 5.27.

The results (b) and (c) are given in Section 5. All together, if we denote $\bar{A}_q^{FM}$ as the Freidel-Maillet type presentation of $\bar{A}_q$, we get the isomorphic 'triplet'

$$U_q^{DJ,+} \cong \bar{A}_q \cong \bar{A}_q^{FM} \ .$$

In the last section, we point out a straightforward application of [62,63] combined with the results of Section 5. One has the 'alternating' tensor product decomposition of $U_q(\widehat{sl_2})$:

$$U_q(\widehat{sl_2}) \cong \bar{A}_q^{\triangleright} \otimes U_q^{DJ,0} \otimes \bar{A}_q^{\triangleleft} \ , \tag{1.2}$$

where $\bar{A}_q^{\triangleright(\triangleleft)} (\cong U_q^{DJ,+(-)})$ are certain alternating subalgebras of U_q^{Dr}. The corresponding 'alternating' PBW basis is given in Theorem 6.1.

Let us conclude this introduction with some additional comments. In the literature, it is known that solutions of the Yang-Baxter equation find many applications in the theory of quantum integrable systems such as vertex models, spin chains,... They can be obtained by specializing solutions of the universal Yang-Baxter equation, the so-called universal R-matrices. As already mentioned, the construction of a universal R-matrix for $U_q(\widehat{sl_2})$ (and similarly for higher rank cases) essentially relies on the tensor product decomposition

$$U_q(\widehat{sl_2}) \cong U_q^{DJ,+} \otimes U_q^{DJ,0} \otimes U_q^{DJ,-}, \tag{1.3}$$

and the use of root vectors [46,45,26,35,42,43]. Now, the 'alternating' tensor product decomposition (1.2) rises the question of an 'alternating' universal K-matrix built from a product of solutions to a universal Freidel-Maillet type equation. See [24,52,16,55,1] for related problems. In view of the importance of the R-matrix in mathematical physics, it looks as an interesting problem that might be considered elsewhere.

It should be mentioned that the analysis here presented is also motivated by the subject of the q-Onsager algebra O_q [59,4] and its applications to quantum integrable systems. See e.g. [10,11,5,15,66,67,13]. The original presentation of O_q is given in terms of generators A, B satisfying a pair of q-Dolan-Grady relations. The algebra $\bar{\mathcal{A}}_q$ studied in this paper can be viewed as a limiting case of the algebra $\mathcal{A}_q$ introduced in [14,6]. For $\mathcal{A}_q$, the original presentation [8] takes the form of a reflection algebra introduced by Sklyanin [57], see [14]. Let us denote this presentation by $\mathcal{A}_q^S$. Using $\mathcal{A}_q^S$, it has been conjectured that $\mathcal{A}_q$ is a central extension of O_q. Initial supporting evidences were based on a comparison between the 'zig-zag' basis of O_q [39] and the one conjectured for $\mathcal{A}_q$ [6, Conjecture 1]. Other evidences are also given in [64]. More recently, the conjecture is finally proved [65]. So, using a surjective homomorphism $\mathcal{A}_q \to O_q$, one gets a triplet of isomorphic algebras $O_q \cong A_q \cong A_q^S$. Independently, more recently the analog of Lusztig's automorphism and Damiani's root vectors denoted $B_{n\delta+\alpha_0}, B_{n\delta+\alpha_1}, B_{n\delta}$ for the q-Onsager algebra have been obtained [12] (see also [60]). In terms of the root vectors, a PBW basis has been constructed. In addition, a Drinfeld type presentation is now identified [50]. However, at the moment the precise relation between the presentation of O_q given in [12] or its Drinfeld type presentation denoted O_q^{Dr} [50] and A_q is yet to be clarified. To prove $O_q \cong A_q \cong A_q^S$ provides an 'alternating' triplet of presentation for the q-Onsager algebra and $O_q^{Dr} \cong A_q$, the analysis here presented sketches the strategy that may be considered elsewhere.

Clearly, alternating subalgebras for higher rank affine Lie algebras and corresponding generalizations of (1.2) may be considered as well following a similar approach.

Notation 1.1. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \cdots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$. Let $\mathbb{K}$ denote an algebraically closed field of characteristic 0. $\mathbb{K}(q)$ denotes the field of rational functions in an indeterminate q. The q-commutator $\big[X, Y\big]_q = qXY - q^{-1}YX$ is introduced. We denote $[x] = (q^x - q^{-x})/(q - q^{-1})$.

Notation 1.2. U_q^{DJ} is the Drinfeld-Jimbo presentation of $U_q(\widehat{sl_2})$. $U_q^{DJ,+}, U_q^{DJ,0}, U_q^{DJ,-}$ are the subalgebras of U_q^{DJ} generated respectively by $\{E_0, E_1\}$, $\{K_0, K_1\}$, $\{F_0, F_1\}$. We also introduce the subalgebras $U_q^{DJ,+,0}$ (resp. $U_q^{DJ,-,0}$) generated by $\{E_0, E_1, K_0, K_1\}$ (resp. $\{F_0, F_1, K_0, K_1\}$).

2. The algebra $\bar{\mathcal{A}}_q$ and its specialization $q \to 1$

In this section, the algebra $\bar{\mathcal{A}}_q$ and its specialization $q \to 1$ denoted $\bar{\mathcal{A}}$ are introduced. The algebra $\bar{\mathcal{A}}_q$ is nothing but a slight modification of the algebra $\mathcal{U}_q^+$ introduced in [63, Section 3]. Compared with $\mathcal{U}_q^+$, the modification here considered aims to ensure that the specialization $q \to 1$ of $\bar{\mathcal{A}}_q$ is non-trivial. Also, the parameter $\bar{\rho}$ is introduced for normalization convenience. So, part of the material in this section is mainly adapted from [63]. Besides, Lemma 2.3 and Lemma 2.4 solve [62, Problem 13.1]. At the end of this section, we prepare the discussion for Sections 3 and 5.

2.1. Defining relations

We refer the reader to [63, Definition 3.1] for the definition of $\mathcal{U}_q^+$. We now introduce the algebra $\bar{\mathcal{A}}_q$.

Definition 2.1. Let $\bar{\rho} \in \mathbb{K}(q)$. $\bar{\mathcal{A}}_q$ is the associative algebra over $\mathbb{K}(q)$ generated by $\{\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1} | k \in \mathbb{N}\}$ subject to the following relations:

$$[\mathsf{W}_0, \mathsf{W}_{k+1}] = [\mathsf{W}_{-k}, \mathsf{W}_1] = \frac{(\tilde{\mathsf{G}}_{k+1} - \mathsf{G}_{k+1})}{q + q^{-1}}, \tag{2.1}$$

$$[\mathsf{W}_0, \mathsf{G}_{k+1}]_q = [\tilde{\mathsf{G}}_{k+1}, \mathsf{W}_0]_q = \bar{\rho}\mathsf{W}_{-k-1}, \tag{2.2}$$

$$[\mathsf{G}_{k+1}, \mathsf{W}_1]_q = [\mathsf{W}_1, \tilde{\mathsf{G}}_{k+1}]_q = \bar{\rho}\mathsf{W}_{k+2}, \tag{2.3}$$

$$[\mathsf{W}_{-k}, \mathsf{W}_{-\ell}] = 0, \qquad [\mathsf{W}_{k+1}, \mathsf{W}_{\ell+1}] = 0, \tag{2.4}$$

$$[\mathsf{W}_{-k}, \mathsf{W}_{\ell+1}] + [\mathsf{W}_{k+1}, \mathsf{W}_{-\ell}] = 0, \tag{2.5}$$

$$[\mathsf{W}_{-k}, \mathsf{G}_{\ell+1}] + [\mathsf{G}_{k+1}, \mathsf{W}_{-\ell}] = 0, \tag{2.6}$$

$$[\mathsf{W}_{-k}, \tilde{\mathsf{G}}_{\ell+1}] + [\tilde{\mathsf{G}}_{k+1}, \mathsf{W}_{-\ell}] = 0, \tag{2.7}$$

$$[\mathsf{W}_{k+1}, \mathsf{G}_{\ell+1}] + [\mathsf{G}_{k+1}, \mathsf{W}_{\ell+1}] = 0, \tag{2.8}$$

$$[\mathsf{W}_{k+1}, \tilde{\mathsf{G}}_{\ell+1}] + [\tilde{\mathsf{G}}_{k+1}, \mathsf{W}_{\ell+1}] = 0, \tag{2.9}$$

$$[\mathsf{G}_{k+1}, \mathsf{G}_{\ell+1}] = 0, \qquad [\tilde{\mathsf{G}}_{k+1}, \tilde{\mathsf{G}}_{\ell+1}] = 0, \tag{2.10}$$

$$[\tilde{\mathsf{G}}_{k+1}, \mathsf{G}_{\ell+1}] + [\mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{\ell+1}] = 0\,. \tag{2.11}$$

Remark 2.2. The defining relations of $\bar{\mathcal{A}}_q$ coincide with the defining relations (30)-(40) in [63] of the algebra $\mathcal{U}_q^+$ for the identification:

$$\mathsf{W}_{-k} \mapsto \mathcal{W}_{-k} , \quad \mathsf{W}_{k+1} \mapsto \mathcal{W}_{k+1} , \tag{2.12}$$

$$\mathsf{G}_{k+1} \mapsto q^{-1}(q^2-q^{-2})\mathcal{G}_{k+1} , \quad \tilde{\mathsf{G}}_{k+1} \mapsto q^{-1}(q^2-q^{-2})\tilde{\mathcal{G}}_{k+1} , \tag{2.13}$$

$$\bar{\rho} \mapsto q^{-1}(q^2-q^{-2})(q-q^{-1}) . \tag{2.14}$$

Note that there exists an automorphism σ and an antiautomorphism S (for $\mathcal{U}_q^+$, see [63, Lemma 3.9]) such that:

$$\sigma : \mathsf{W}_{-k} \mapsto \mathsf{W}_{k+1} , \quad \mathsf{W}_{k+1} \mapsto \mathsf{W}_{-k} , \quad \mathsf{G}_{k+1} \mapsto \tilde{\mathsf{G}}_{k+1} , \quad \tilde{\mathsf{G}}_{k+1} \mapsto \mathsf{G}_{k+1} , \tag{2.15}$$

$$S : \mathsf{W}_{-k} \mapsto \mathsf{W}_{-k} , \quad \mathsf{W}_{k+1} \mapsto \mathsf{W}_{k+1} , \quad \mathsf{G}_{k+1} \mapsto \tilde{\mathsf{G}}_{k+1} , \quad \tilde{\mathsf{G}}_{k+1} \mapsto \mathsf{G}_{k+1} . \tag{2.16}$$

For completeness (see [63, Note 2.6]) and the discussion in the next section, a set of additional relations can be derived from the defining relations (2.1)-(2.11), given in Lemmas 2.3, 2.4 below.

Lemma 2.3. *In $\bar{\mathcal{A}}_q$, the following relations hold:*

$$[\mathsf{W}_{-k}, \mathsf{G}_\ell]_q = [\mathsf{W}_{-\ell}, \mathsf{G}_k]_q , \qquad [\mathsf{G}_k, \mathsf{W}_{\ell+1}]_q = [\mathsf{G}_\ell, \mathsf{W}_{k+1}]_q , \tag{2.17}$$

$$[\tilde{\mathsf{G}}_k, \mathsf{W}_{-\ell}]_q = [\tilde{\mathsf{G}}_\ell, \mathsf{W}_{-k}]_q , \qquad [\mathsf{W}_{\ell+1}, \tilde{\mathsf{G}}_k]_q = [\mathsf{W}_{k+1}, \tilde{\mathsf{G}}_\ell]_q . \tag{2.18}$$

Proof. Consider the first equation in (2.17). For convenience, substitute $\ell \to \ell+1$ and multiply by $\bar{\rho}$ the equality. From the r.h.s. of the resulting equation, using (2.2) one has:

$$\begin{aligned}
[\underbrace{\bar{\rho}\mathsf{W}_{-\ell-1}}_{=[\mathsf{W}_0,\mathsf{G}_{\ell+1}]_q}, \mathsf{G}_k]_q &= q^2\mathsf{W}_0 \underbrace{\mathsf{G}_{\ell+1}\mathsf{G}_k}_{=\mathsf{G}_k\mathsf{G}_{\ell+1}} - \mathsf{G}_{\ell+1}\mathsf{W}_0\mathsf{G}_k - \mathsf{G}_k\mathsf{W}_0\mathsf{G}_{\ell+1} \\
&\quad + q^{-2}\underbrace{\mathsf{G}_k\mathsf{G}_{\ell+1}}_{=\mathsf{G}_{\ell+1}\mathsf{G}_k}\mathsf{W}_0 \qquad \text{by (2.10)} \\
&= \underbrace{q^2\mathsf{W}_0\mathsf{G}_k\mathsf{G}_{\ell+1}}_{=q[\mathsf{W}_0,\mathsf{G}_k]_q\mathsf{G}_{\ell+1}+\mathsf{G}_k\mathsf{W}_0\mathsf{G}_{\ell+1}} - \underbrace{\mathsf{G}_{\ell+1}\mathsf{W}_0\mathsf{G}_k}_{=q^{-1}\mathsf{G}_{\ell+1}[\mathsf{W}_0,\mathsf{G}_k]_q+q^{-2}\mathsf{G}_{\ell+1}\mathsf{G}_k\mathsf{W}_0} \\
&\quad - \mathsf{G}_k\mathsf{W}_0\mathsf{G}_{\ell+1} + q^{-2}\mathsf{G}_{\ell+1}\mathsf{G}_k\mathsf{W}_0 \\
&= [[\mathsf{W}_0,\mathsf{G}_k]_q, \mathsf{G}_{\ell+1}]_q \\
&= \bar{\rho}[\mathsf{W}_{-k}, \mathsf{G}_{\ell+1}]_q ,
\end{aligned}$$

which coincides with the l.h.s. The three other equations are shown similarly. □

Lemma 2.4. *In $\bar{\mathcal{A}}_q$, the following relations hold:*

$$[\mathsf{G}_k, \tilde{\mathsf{G}}_{\ell+1}] - [\mathsf{G}_\ell, \tilde{\mathsf{G}}_{k+1}] = \bar{\rho}(q+q^{-1})\left([\mathsf{W}_{-\ell}, \mathsf{W}_{k+1}]_q - [\mathsf{W}_{-k}, \mathsf{W}_{\ell+1}]_q\right), \tag{2.19}$$

$$[\tilde{\mathsf{G}}_k, \mathsf{G}_{\ell+1}] - [\tilde{\mathsf{G}}_\ell, \mathsf{G}_{k+1}] = \bar{\rho}(q+q^{-1})\left([\mathsf{W}_{\ell+1}, \mathsf{W}_{-k}]_q - [\mathsf{W}_{k+1}, \mathsf{W}_{-\ell}]_q\right), \tag{2.20}$$

$$[\mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{\ell+1}]_q - [\mathsf{G}_{\ell+1}, \tilde{\mathsf{G}}_{k+1}]_q = \bar{\rho}(q+q^{-1})\left([\mathsf{W}_{-\ell}, \mathsf{W}_{k+2}] - [\mathsf{W}_{-k}, \mathsf{W}_{\ell+2}]\right), \tag{2.21}$$

$$[\tilde{\mathsf{G}}_{k+1}, \mathsf{G}_{\ell+1}]_q - [\tilde{\mathsf{G}}_{\ell+1}, \mathsf{G}_{k+1}]_q = \bar{\rho}(q+q^{-1})\left([\mathsf{W}_{\ell+1}, \mathsf{W}_{-k-1}] - [\mathsf{W}_{k+1}, \mathsf{W}_{-\ell-1}]\right). \tag{2.22}$$

Proof. Consider (2.19). One has:

$$
\begin{aligned}
[\mathsf{G}_k,\tilde{\mathsf{G}}_{\ell+1}] &= [\mathsf{G}_k,\tilde{\mathsf{G}}_{\ell+1}-\mathsf{G}_{\ell+1}] = (q+q^{-1})[\mathsf{G}_k,[\mathsf{W}_0,\mathsf{W}_{\ell+1}]] \qquad \text{by (2.1)}\\
&= (q+q^{-1})\Big(\underbrace{\mathsf{G}_k\mathsf{W}_0}_{=q^2\mathsf{W}_0\mathsf{G}_k-\bar{\rho}q\mathsf{W}_{-k}}\mathsf{W}_{\ell+1}-\mathsf{G}_k\mathsf{W}_{\ell+1}\mathsf{W}_0-\mathsf{W}_0\mathsf{W}_{\ell+1}\mathsf{G}_k\\
&\quad +\mathsf{W}_{\ell+1}\underbrace{\mathsf{W}_0\mathsf{G}_k}_{=q^{-2}\mathsf{G}_k\mathsf{W}_0+\bar{\rho}q^{-1}\mathsf{W}_{-k}}\Big)\\
&= (q+q^{-1})\left([\mathsf{W}_0,[\mathsf{G}_k,\mathsf{W}_{\ell+1}]_q]_q-\bar{\rho}[\mathsf{W}_{-k},\mathsf{W}_{\ell+1}]_q\right) .
\end{aligned}
$$

It follows:

$$
\begin{aligned}
[\mathsf{G}_k,\tilde{\mathsf{G}}_{\ell+1}]-[\mathsf{G}_\ell,\tilde{\mathsf{G}}_{k+1}] &= \bar{\rho}(q+q^{-1})\left([\mathsf{W}_{-\ell},\mathsf{W}_{k+1}]_q-[\mathsf{W}_{-k},\mathsf{W}_{\ell+1}]_q\right)\\
&\quad +(q+q^{-1})[\mathsf{W}_0,\underbrace{[\mathsf{G}_k,\mathsf{W}_{\ell+1}]_q-[\mathsf{G}_\ell,\mathsf{W}_{k+1}]_q}_{=0\ \text{ by (2.17)}}]_q
\end{aligned}
$$

which reduces to (2.19). One shows (2.20) similarly.

Consider (2.22). One has:

$$
\begin{aligned}
\bar{\rho}[\mathsf{W}_{\ell+1},\mathsf{W}_{-k-1}] &= [\mathsf{W}_{\ell+1},[\mathsf{W}_0,\mathsf{G}_{k+1}]_q]\\
&= q\mathsf{W}_0[\mathsf{W}_{\ell+1},\mathsf{G}_{k+1}]+q^{-1}[\mathsf{G}_{k+1},\mathsf{W}_{\ell+1}]\mathsf{W}_0+\frac{q^{-1}}{(q+q^{-1})}\mathsf{G}_{k+1}\tilde{\mathsf{G}}_{\ell+1}\\
&\quad -\frac{q}{(q+q^{-1})}\tilde{\mathsf{G}}_{\ell+1}\mathsf{G}_{k+1}+\frac{(q-q^{-1})}{(q+q^{-1})}\mathsf{G}_{k+1}\mathsf{G}_{\ell+1} ,
\end{aligned}
$$

where (2.2), (2.1) and (2.10) have been used successively. Using (2.8) it follows:

$$
\begin{aligned}
\bar{\rho}\left([\mathsf{W}_{\ell+1},\mathsf{W}_{-k-1}]-[\mathsf{W}_{k+1},\mathsf{W}_{-\ell-1}]\right) &= \frac{q^{-1}}{(q+q^{-1})}\left(\mathsf{G}_{k+1}\tilde{\mathsf{G}}_{\ell+1}-\mathsf{G}_{\ell+1}\tilde{\mathsf{G}}_{k+1}\right)\\
&\quad -\frac{q}{(q+q^{-1})}\left(\tilde{\mathsf{G}}_{\ell+1}\mathsf{G}_{k+1}-\tilde{\mathsf{G}}_{k+1}\mathsf{G}_{\ell+1}\right) .
\end{aligned}
\tag{2.23}
$$

From (2.11), note that:

$$
\tilde{\mathsf{G}}_{\ell+1}\mathsf{G}_{k+1}-\tilde{\mathsf{G}}_{k+1}\mathsf{G}_{\ell+1}=\mathsf{G}_{k+1}\tilde{\mathsf{G}}_{\ell+1}-\mathsf{G}_{\ell+1}\tilde{\mathsf{G}}_{k+1}
$$

which implies:

$$
(q-q^{-1})\left(\mathsf{G}_{\ell+1}\tilde{\mathsf{G}}_{k+1}-\mathsf{G}_{k+1}\tilde{\mathsf{G}}_{\ell+1}\right)=[\tilde{\mathsf{G}}_{k+1},\mathsf{G}_{\ell+1}]_q-[\tilde{\mathsf{G}}_{\ell+1},\mathsf{G}_{k+1}]_q .
$$

Using this last equality in the r.h.s. of (2.23), eq. (2.22) follows. The other relation (2.21) is shown similarly. □

Remark 2.5. The relations (41)-(46) in [63] follow from Lemmas 2.3, 2.4, using the identification (2.12)-(2.14).

2.2. The center $\mathcal{Z}$

For the algebra $\mathcal{U}_q^+$, central elements denoted $Z_{n+1}^\vee$ are known [63, eq. (52) and Lemma 5.2] (see also equivalent expressions [63, Corollary 8.4]). With minor modifications using the correspondence (2.12)-(2.14), central elements in $\bar{\mathcal{A}}_q$ are obtained in a straightforward manner. Thus, we omit the proof of the following lemma and refer the reader to [63, Section 13] for details.

Lemma 2.6. *For $n \in \mathbb{N}$, the element*

$$Y_{n+1} = \mathsf{G}_{n+1} q^{-n-1} + \tilde{\mathsf{G}}_{n+1} q^{n+1} - (q^2 - q^{-2}) \sum_{k=0}^{n} q^{-n+2k} \mathsf{W}_{-k} \mathsf{W}_{n+1-k} + \frac{(q - q^{-1})}{\bar{\rho}} \sum_{k=0}^{n-1} q^{-n+1+2k} \tilde{\mathsf{G}}_{k+1} \mathsf{G}_{n-k} \quad (2.24)$$

is central in $\bar{\mathcal{A}}_q$.

Remark 2.7. Central elements for the algebra $\mathcal{U}_q^+$ [63, Lemma 5.2, Corollary 8.4] are obtained using the identification (2.12)-(2.14):

$$Y_{n+1} \mapsto q^{-1}(q^2 - q^{-2}) Z_{n+1}^\vee \ . \quad (2.25)$$

Note that the central elements are fixed under the action of (anti)automorphisms of $\bar{\mathcal{A}}_q$. Applying σ and S according to (2.15), (2.16), three other expressions for Y_{n+1} follow (for $\mathcal{U}_q^+$, see [63, Corollary 8.4]). In particular, for further convenience, define the combination:

$$\Delta_{n+1} = \frac{1}{q^{n+1} + q^{-n-1}} \left(Y_{n+1} + \sigma(Y_{n+1}) \right) \ . \quad (2.26)$$

Using (2.5), one has $S(\Delta_{n+1}) = \Delta_{n+1}$. Thus, Δ_{n+1} is invariant under the action of σ, S.

Example 2.8.

$$\Delta_1 = \mathsf{G}_1 + \tilde{\mathsf{G}}_1 - (q - q^{-1})\big(\mathsf{W}_0\mathsf{W}_1 + \mathsf{W}_1\mathsf{W}_0\big), \quad (2.27)$$

$$\Delta_2 = \mathsf{G}_2 + \tilde{\mathsf{G}}_2 - \frac{(q^2 - q^{-2})}{(q^2 + q^{-2})}(q^{-1}\mathsf{W}_0\mathsf{W}_2 + q\mathsf{W}_2\mathsf{W}_0 + q^{-1}\mathsf{W}_1\mathsf{W}_{-1} + q\mathsf{W}_{-1}\mathsf{W}_1) + \frac{(q - q^{-1})}{(q^2 + q^{-2})}\Big(\frac{\tilde{\mathsf{G}}_1\mathsf{G}_1 + \mathsf{G}_1\tilde{\mathsf{G}}_1}{\bar{\rho}}\Big), \quad (2.28)$$

$$\Delta_3 = \mathsf{G}_3 + \tilde{\mathsf{G}}_3 - \frac{(q - q^{-1})}{(q^2 + q^{-2} - 1)}(q^{-2}\mathsf{W}_0\mathsf{W}_3 + q^2\mathsf{W}_3\mathsf{W}_0 + q^{-2}\mathsf{W}_1\mathsf{W}_{-2} + q^2\mathsf{W}_{-2}\mathsf{W}_1) - \frac{(q - q^{-1})}{(q^2 + q^{-2} - 1)}(\mathsf{W}_2\mathsf{W}_{-1} + \mathsf{W}_{-1}\mathsf{W}_2) + \frac{(q - q^{-1})}{(q^2 + q^{-2} - 1)}\Big(\frac{\tilde{\mathsf{G}}_2\mathsf{G}_1 + \mathsf{G}_2\tilde{\mathsf{G}}_1}{\bar{\rho}}\Big) \ . \quad (2.29)$$

By construction, the elements Δ_{n+1} are central in $\bar{\mathcal{A}}_q$. Let $\mathcal{Z}$ denote the subalgebra of $\bar{\mathcal{A}}_q$ generated by $\{\Delta_{n+1}\}_{n \in \mathbb{N}}$. By [63, Proposition 6.2], $\mathcal{Z}$ is the center of $\bar{\mathcal{A}}_q$.

2.3. Generators and recursive relations

Following [63], combining the defining relations (2.1)-(2.3) together with (2.26) it follows:

Lemma 2.9. *In $\bar{\mathcal{A}}_q$, the following recursive relations hold:*

$$G_{n+1} = \frac{(q^2-q^{-2})}{2(q^{n+1}+q^{-n-1})}\sum_{k=0}^{n} q^{-n+2k}\left(W_{-k}W_{n+1-k}+W_{k+1}W_{k-n}\right) \tag{2.30}$$
$$-\frac{(q-q^{-1})}{2\bar{\rho}(q^{n+1}+q^{-n-1})}\sum_{k=0}^{n-1} q^{-n+1+2k}\left(G_{k+1}\tilde{G}_{n-k}+\tilde{G}_{k+1}G_{n-k}\right)$$
$$+\frac{(q+q^{-1})}{2}\big[W_{n+1},W_0\big]+\frac{1}{2}\Delta_{n+1}\ ,$$

$$\tilde{G}_{n+1} = G_{n+1}+(q+q^{-1})\big[W_0,W_{n+1}\big]\ , \tag{2.31}$$

$$W_{-n-1} = \frac{1}{\bar{\rho}}\big[W_0,G_{n+1}\big]_q\ , \tag{2.32}$$

$$W_{n+2} = \frac{1}{\bar{\rho}}\big[G_{n+1},W_1\big]_q\ . \tag{2.33}$$

Iterating the recursive formulae (2.30), (2.31), (2.32), (2.33), given n fixed, the corresponding generator is a polynomial in $\mathsf{W}_0, \mathsf{W}_1$ and $\{\Delta_{k+1}|k=0,...,n\}$.

Example 2.10. The first generators read:

$$\mathsf{G}_1 = q\mathsf{W}_1\mathsf{W}_0 - q^{-1}\mathsf{W}_0\mathsf{W}_1 + \frac{1}{2}\Delta_1\ , \tag{2.34}$$

$$\mathsf{W}_{-1} = \frac{1}{\bar{\rho}}\left((q^2+q^{-2})\mathsf{W}_0\mathsf{W}_1\mathsf{W}_0 - \mathsf{W}_0^2\mathsf{W}_1 - \mathsf{W}_1\mathsf{W}_0^2\right) + \frac{1}{2}\frac{\Delta_1(q-q^{-1})}{\bar{\rho}}\mathsf{W}_0\ , \tag{2.35}$$

$$\mathsf{G}_2 = \frac{1}{\bar{\rho}(q^2+q^{-2})}\Big((q^{-3}+q^{-1})\mathsf{W}_0^2{\mathsf{W}_1}^2 - (q^3+q){\mathsf{W}_1}^2\mathsf{W}_0^2 \tag{2.36}$$
$$+(q^{-3}-q^3)(\mathsf{W}_0{\mathsf{W}_1}^2\mathsf{W}_0+\mathsf{W}_1\mathsf{W}_0^2\mathsf{W}_1)$$
$$-(q^{-5}+q^{-3}+2q^{-1})\mathsf{W}_0\mathsf{W}_1\mathsf{W}_0\mathsf{W}_1 + (q^5+q^3+2q)\mathsf{W}_1\mathsf{W}_0\mathsf{W}_1\mathsf{W}_0\Big)$$
$$+\frac{1}{2}\frac{\Delta_1(q-q^{-1})}{\bar{\rho}}\big(q\mathsf{W}_1\mathsf{W}_0-q^{-1}\mathsf{W}_0\mathsf{W}_1\big) - \frac{1}{4}\frac{\Delta_1^2(q-q^{-1})}{\bar{\rho}(q^2+q^{-2})} + \frac{1}{2}\Delta_2\ .$$

Expressions of $\tilde{\mathsf{G}}_1, \mathsf{W}_2, \tilde{\mathsf{G}}_2$ are obtained using the automorphism σ.

Corollary 2.11. *The algebra $\bar{\mathcal{A}}_q$ is generated by W_0, W_1 and $\mathcal{Z}$.*

2.4. PBW basis

Following [63, Lemma 3.10], the algebra $\bar{\mathcal{A}}_q$ has an $\mathbb{N}^2$-grading. Define $\deg : \bar{\mathcal{A}}_q \to \mathbb{N}\times\mathbb{N}$. For the alternating generators one has:

$$\deg(\mathsf{W}_{-k}) = (k+1, k)\ , \quad \deg(\mathsf{W}_{k+1}) = (k, k+1)\ ,$$
$$\deg(\mathsf{G}_{k+1}) = \deg(\tilde{\mathsf{G}}_{k+1}) = (k+1, k+1)\ .$$

Note that the expressions in Lemma 2.9 are homogeneous with respect to the grading assignment. The dimension $d_{i,j}$ of the vector space spanned by linearly independent vectors of the same degree (i, j) is obtained from the formal power series in the indeterminates λ, μ:

$$\Phi(\lambda,\mu) = \mathcal{H}(\lambda,\mu)\mathcal{Z}(\lambda,\mu)\ ,$$
$$= \sum_{(i,j)\in\mathbb{N}} d_{i,j}\lambda^i\mu^j \quad \text{for } |\lambda|, |\mu| < 1$$

with

$$\mathcal{H}(\lambda,\mu) = \prod_{\ell=1}^{\infty} \frac{1}{1-\lambda^\ell\mu^{\ell-1}}\frac{1}{1-\lambda^{\ell-1}\mu^\ell}\frac{1}{1-\lambda^\ell\mu^\ell}\ , \quad \mathcal{Z}(\lambda,\mu) = \prod_{\ell=1}^{\infty}\frac{1}{1-\lambda^\ell\mu^\ell}\ .$$

In [63, Section 10], a PBW basis for $\mathcal{U}_q^+$ is obtained. The proof solely uses the defining relations corresponding to (2.1)-(2.11). The following theorem is a straightforward adaptation of [63, Theorem 10.2].

Theorem 2.12. *(see [63]) A PBW basis for $\bar{\mathcal{A}}_q$ is obtained by its alternating generators*

$$\{\mathsf{W}_{-k}\}_{k\in\mathbb{N}}\ , \quad \{\mathsf{G}_{\ell+1}\}_{\ell\in\mathbb{N}}\ , \quad \{\tilde{\mathsf{G}}_{m+1}\}_{m\in\mathbb{N}}\ , \quad \{\mathsf{W}_{n+1}\}_{n\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$\mathsf{W}_{-k} < \mathsf{G}_{\ell+1} < \tilde{\mathsf{G}}_{m+1} < \mathsf{W}_{n+1}\ , k, \ell, m, n \in \mathbb{N}\ .$$

Note that combining σ, S given by (2.15), (2.16), other PBW bases can be obtained.

2.5. *The algebra $\bar{A}_q$*

By construction [63], the algebra U_q^+ studied in [62] is a quotient of the algebra $\mathcal{U}_q^+$. This quotient is characterized by the fact that the images of all the central elements $Z_n^\vee$ of [63, Definition 5.1] in U_q^+ are vanishing, see [63, Lemma 2.8]. Recall (2.25), (2.26).

Definition 2.13. The algebra $\bar{A}_q$ is defined as the quotient of the algebra $\bar{\mathcal{A}}_q$ by the ideal generated from the relations $\{\Delta_{k+1} = 0 | \forall k \in \mathbb{N}\}$. The generators are $\{W_{-k}, W_{k+1}, G_{k+1}, \tilde{G}_{k+1} | k \in \mathbb{N}\}$.

Following [63, Lemma 3.3], let us denote by $\gamma : \bar{\mathcal{A}}_q \to \mathcal{A}_q$ the corresponding surjective homomorphism. It is such that:

$$\gamma: \quad \mathsf{W}_{-k} \mapsto W_{-k}\ , \quad \mathsf{W}_{k+1} \mapsto W_{k+1}\ , \quad \mathsf{G}_{k+1} \mapsto G_{k+1}\ , \quad \tilde{\mathsf{G}}_{k+1} \mapsto \tilde{G}_{n+1}\ . \tag{2.37}$$

So, they can be obtained as polynomials in W_0, W_1 applying γ to the expressions given in Lemma 2.9, where $\gamma(\Delta_{k+1}) = 0$ for all k.

In [62,63], the embedding of $U_q^{DJ,+}$ into a q-shuffle algebra leads to $\bar{A}_q$, providing an 'alternating' presentation for $U_q^{DJ,+}$. Adapting this result to our conventions, it follows:

Proposition 2.14. *(see ([62,63]) $\bar{A}_q \cong U_q^{DJ,+} \cong U_q^{DJ,-}$.*

In [62, Section 10], an alternating' PBW basis for $U_q^{DJ,+}$ is obtained. We refer to [62, Theorem 10.1].

Theorem 2.15. *(see [62]) A PBW basis for $\bar{A}_q$ is obtained by its alternating generators*

$$\{W_{-k}\}_{k\in\mathbb{N}}\ ,\quad \{G_{\ell+1}\}_{\ell\in\mathbb{N}}\ ,\quad \{W_{n+1}\}_{n\in\mathbb{N}}$$

in any linear order $<$ that satisfies

$$W_{-k} < G_{\ell+1} < W_{n+1}\ , k,\ell,n\in\mathbb{N}\ ;$$
$$W_{k+1} < G_{\ell+1} < W_{-n}\ , k,\ell,n\in\mathbb{N}\ .$$

Using automorphisms of $\bar{A}_q$, other PBW bases can be obtained.

2.6. The specialization $q\to 1$ and the algebra $\bar{\mathcal{A}}$

For the specialization $q\to 1$, according to the identification (2.13), (2.14), the defining relations [63, Definition 3.1] of the algebra $\mathcal{U}_q^+$ drastically simplify to those of a commutative algebra. Instead, the specialization $q\to 1$ of the defining relations of the algebra $\bar{\mathcal{A}}_q$ lead to an associative algebra called $\bar{\mathcal{A}}$, as explained below. To define properly the specialization, we follow the method described in e.g. [47, Section 10] (see also references therein).

Let $\mathsf{A}=\mathbb{K}[q]_{q-1}$ $(=\mathcal{S}^{-1}\mathbb{K}[q]$ where $\mathcal{S}=\mathbb{K}[q]\backslash(q-1))$. Let $\mathcal{U}_{\mathsf{A}}$ be the A-subalgebra of $\bar{\mathcal{A}}_q$ generated by $\{\mathsf{W}_{-k},\mathsf{W}_{k+1},\mathsf{G}_{k+1},\tilde{\mathsf{G}}_{k+1}|k\in\mathbb{N}\}$. Note that contrary to $U_q(\widehat{sl_2})$ [22, page 289], according to the structure of the defining relations (2.1)-(2.11) for the specialization $q\to 1$ of $\bar{\mathcal{A}}_q$ there is no need to introduce other generators. One has the natural isomorphism of A-algebras $\mathcal{U}_{\mathsf{A}}\otimes_{\mathsf{A}}\mathbb{K}(q)\to\bar{\mathcal{A}}_q$. Consider $\mathbb{K}$ as an A-module via evaluation at $q=1$. The algebra

$$\mathcal{U}_1=\mathcal{U}_{\mathsf{A}}\otimes_{\mathsf{A}}\mathbb{K}$$

is the specialization of $\bar{\mathcal{A}}_q$ at $q=1$. Similarly, one defines $\mathcal{Z}_{\mathsf{A}}$, and $\mathcal{Z}_1=\mathcal{Z}_{\mathsf{A}}\otimes_{\mathsf{A}}\mathbb{K}$.

Definition 2.16. $\bar{\mathcal{A}}$ is the associative algebra over $\mathbb{K}$ with unit and generators $\{\mathsf{w}_{-k},\mathsf{w}_{k+1},\mathsf{g}_{k+1},\tilde{\mathsf{g}}_{k+1}|k\in\mathbb{N}\}$ satisfying the following relations:

$$\big[\mathsf{w}_{-\ell},\mathsf{w}_{k+1}\big]=\frac{1}{2}(\tilde{\mathsf{g}}_{k+\ell+1}-\mathsf{g}_{k+\ell+1})\ , \tag{2.38}$$
$$\big[\tilde{\mathsf{g}}_{k+1},\mathsf{w}_{-l}\big]=\big[\mathsf{w}_{-l},\mathsf{g}_{k+1}\big]=16\mathsf{w}_{-k-\ell-1}\ , \tag{2.39}$$
$$\big[\mathsf{w}_{\ell+1},\tilde{\mathsf{g}}_{k+1}\big]=\big[\mathsf{g}_{k+1},\mathsf{w}_{\ell+1}\big]=16\mathsf{w}_{\ell+k+2}\ , \tag{2.40}$$
$$\big[\mathsf{w}_{-k},\mathsf{w}_{-\ell}\big]=0\ ,\quad \big[\mathsf{w}_{k+1},\mathsf{w}_{\ell+1}\big]=0\ ,\quad \big[\mathsf{g}_{k+1},\mathsf{g}_{\ell+1}\big]=0\ ,\quad \big[\tilde{\mathsf{g}}_{k+1},\tilde{\mathsf{g}}_{\ell+1}\big]=0\ . \tag{2.41}$$

Remark 2.17. An overall parameter $\bar{\rho}_c\in\mathbb{K}^*$ may be introduced in the r.h.s. of (2.39), (2.40).

Proposition 2.18. *There exists an algebra isomorphism $\mathcal{U}_1\to\bar{\mathcal{A}}$ such that:*

$$\begin{aligned}&W_{-k}\mapsto w_{-k}\ ,\quad W_{k+1}\mapsto w_{k+1}\ ,\quad G_{k+1}\mapsto g_{k+1}\ ,\\ &\tilde{G}_{k+1}\mapsto\tilde{g}_{k+1}\ ,\quad \bar{\rho}\mapsto 16\ ,\quad q\mapsto 1\ .\end{aligned} \tag{2.42}$$

Proof. First, we show how to obtain the defining relations for $\bar{\mathcal{A}}$ from those of $\bar{\mathcal{A}}_q$ at $q=1$ and $\bar{\rho}=16$. From eqs. (2.4), (2.10), one immediately obtains the four equations in (2.41). From (2.26), one gets

$$\delta_{k+1} = \mathsf{g}_{k+1} + \tilde{\mathsf{g}}_{k+1} \ , \tag{2.43}$$

where $\{\delta_k\}_{k\in\mathbb{N}}$ are central with respect to the algebra generated by $\{\mathsf{w}_{-k}, \mathsf{w}_{k+1}, \mathsf{g}_{k+1}, \tilde{\mathsf{g}}_{k+1} | k \in \mathbb{N}\}$. This implies the first equalities in (2.39), (2.40). The second equalities in (2.39), (2.40) are obtained from elementary computation using the Jacobi identity together with (2.5)-(2.10) and (2.1)-(2.4). For instance:

$$\begin{aligned}\big[\mathsf{w}_{-1}, \mathsf{w}_{k+1}\big] &= \frac{1}{16}\big[\big[\mathsf{w}_0, \mathsf{g}_1\big], \mathsf{w}_{k+1}\big] = -\frac{1}{16}\big[\underbrace{\big[\mathsf{g}_1, \mathsf{w}_{k+1}\big]}_{=[\mathsf{g}_{k+1},\mathsf{w}_0]=16\mathsf{w}_{k+2}}, \mathsf{w}_0\big] \\ &\quad - \frac{1}{16}\big[\underbrace{\big[\mathsf{w}_{k+1}, \mathsf{w}_0\big]}_{=-\frac{1}{2}(\tilde{\mathsf{g}}_{k+1}-\mathsf{g}_{k+1})}, \mathsf{g}_1\big] = \big[\mathsf{w}_0, \mathsf{w}_{k+2}\big] \\ &= \frac{1}{2}(\tilde{\mathsf{g}}_{k+2} - \mathsf{g}_{k+2}) \ .\end{aligned}$$

By induction, it follows:

$$\big[\mathsf{w}_{-\ell}, \mathsf{w}_{k+1}\big] = \big[\mathsf{w}_{-\ell+1}, \mathsf{w}_{k+2}\big] = \cdots = \big[\mathsf{w}_0, \mathsf{w}_{k+\ell+1}\big] = \frac{1}{2}(\tilde{\mathsf{g}}_{k+\ell+1} - \mathsf{g}_{k+\ell+1}) \ .$$

Similarly, by induction one easily finds:

$$\begin{aligned}\big[\tilde{\mathsf{g}}_{k+1}, \mathsf{w}_{-\ell}\big] &= \big[\tilde{\mathsf{g}}_{k+2}, \mathsf{w}_{-\ell+1}\big] = \cdots = \big[\tilde{\mathsf{g}}_{k+\ell+1}, \mathsf{w}_0\big] = 16\mathsf{w}_{-k-\ell-1} \ , \\ \big[\mathsf{w}_{\ell+1}, \tilde{\mathsf{g}}_{k+1}\big] &= \big[\mathsf{w}_{\ell}, \tilde{\mathsf{g}}_{k+2}\big] = \cdots = \big[\mathsf{w}_1, \tilde{\mathsf{g}}_{k+\ell+1}\big] = 16\mathsf{w}_{\ell+k+2} \ .\end{aligned}$$

Thus, the defining relations (2.38)-(2.41) of $\bar{\mathcal{A}}$ are recovered from the specialization $q \to 1$, $\bar{\rho} \to 16$ of the defining relations (2.1)-(2.11) of $\bar{\mathcal{A}}_q$. The converse statement is easily checked. □

In the following, we call $\bar{\mathcal{A}}$ the specialization $q \to 1$ of $\bar{\mathcal{A}}_q$.

2.7. *Relation with $U_q^{DJ,\pm}$ and specialization*

The following comments give some motivation for Sections 3 and 5. We first describe the relation between $\bar{\mathcal{A}}_q$ and $U_q(\widehat{sl_2})$ with respect to the Drinfeld-Jimbo presentation, adapting directly the results of [63]. On one hand, recall that the defining relations for $U_q^{DJ,+}$, $U_q^{DJ,-}$ are respectively given by (A.1), (A.2). On the other hand, inserting (2.35) in (2.4) for $k=0, \ell=1$ one finds that $\mathsf{W}_0, \mathsf{W}_1$ satisfy the q-Serre relations:

$$[\mathsf{W}_0, [\mathsf{W}_0, [\mathsf{W}_0, \mathsf{W}_1]_q]_{q^{-1}}] = 0 \ , \tag{2.44}$$

$$[\mathsf{W}_1, [\mathsf{W}_1, [\mathsf{W}_1, \mathsf{W}_0]_q]_{q^{-1}}] = 0 \ . \tag{2.45}$$

Let $\langle \mathsf{W}_0, \mathsf{W}_1 \rangle$ denote the subalgebra of $\bar{\mathcal{A}}_q$ generated by $\mathsf{W}_0, \mathsf{W}_1$. According to [63, Proposition 6.4] combined with Remark 2.2, it follows that the map $\langle \mathsf{W}_0, \mathsf{W}_1 \rangle \to U_q^{DJ,+}$:

$$\mathsf{W}_0 \mapsto E_1 \ , \qquad \mathsf{W}_1 \mapsto E_0 \tag{2.46}$$

is an algebra isomorphism. Obviously, a similar statement holds for $U_q^{DJ,-}$. Let $\mathcal{Z}^+$ denote the image of $\mathcal{Z}$ by the map (2.46), and similarly $\mathcal{Z}^-$ the image associated with the negative part. In both cases, it is a polynomial algebra [63, Section 4]. Adapting [63, Proposition 6.5] and using Remark 2.2, by Corollary 2.11 one concludes:

$$\bar{\mathcal{A}}_q \cong U_q^{DJ,+} \otimes \mathcal{Z}^+ \cong U_q^{DJ,-} \otimes \mathcal{Z}^- \ . \tag{2.47}$$

For this reason, $\bar{\mathcal{A}}_q$ is called the central extension of $U_q^{DJ,+}$ (or $U_q^{DJ,-}$).

Let us also add the following comment. In view of the isomorphism (2.46), $\bar{\mathcal{A}}_q$ can be equipped with a comodule structure [22]. For instance, examples of left (or right) coaction maps can be considered for the subalgebra $\langle \mathsf{W}_0, \mathsf{W}_1 \rangle$. Define the 'left' coaction such that

$$\bar{\mathcal{A}}_q \to U_q^{DJ,+,0} \otimes \bar{\mathcal{A}}_q \ . \tag{2.48}$$

Consider its restriction to $\langle \mathsf{W}_0, \mathsf{W}_1 \rangle \cong U_q^{DJ,+}$. As an example of coaction, we may consider:

$$\mathsf{W}_0 \to E_0 \otimes 1 + K_0 \otimes \mathsf{W}_0 \ , \tag{2.49}$$

$$\mathsf{W}_1 \to E_1 \otimes 1 + K_1 \otimes \mathsf{W}_1 \ . \tag{2.50}$$

A 'right' coaction could be introduced similarly, as well as a coaction $\bar{\mathcal{A}}_q \to U_q^{DJ,-,0} \otimes \bar{\mathcal{A}}_q$. In Section 5, a comodule algebra homomorphism δ is obtained, see Lemma 5.25.

The relation between $\bar{\mathcal{A}}$ and the Lie algebra $\widehat{sl_2}^{SC}$ can be considered through specialization. Recall the isomorphism $\mathcal{U}_\mathsf{A} \otimes_\mathsf{A} \mathbb{K}(q) \to \bar{\mathcal{A}}_q$ and similarly for $\langle \mathsf{W}_0, \mathsf{W}_1 \rangle$ and $\mathcal{Z}$. One has the injection $\langle \mathsf{W}_0, \mathsf{W}_1 \rangle_\mathsf{A} \otimes_\mathsf{A} \mathcal{Z}_\mathsf{A} \to \mathcal{U}_\mathsf{A}$ by [47, Lemma 10.6]. By Lemma 2.9, the latter map is also surjective. Using the fact that $\langle \mathsf{W}_0, \mathsf{W}_1 \rangle_\mathsf{A}$ and $\mathcal{Z}_\mathsf{A}$ are free A-modules, one calculates:

$$\begin{aligned}\mathcal{U}_1 = \mathcal{U}_\mathsf{A} \otimes_\mathsf{A} \mathbb{K} &= (\langle \mathsf{W}_0, \mathsf{W}_1 \rangle_\mathsf{A} \otimes_\mathsf{A} \mathcal{Z}_\mathsf{A}) \otimes_\mathsf{A} \mathbb{K} \\ &= (\langle \mathsf{W}_0, \mathsf{W}_1 \rangle_\mathsf{A} \otimes_\mathsf{A} \mathbb{K}) \otimes_\mathbb{K} (\mathcal{Z}_\mathsf{A} \otimes_\mathsf{A} \mathbb{K}) \ .\end{aligned}$$

Let $\langle \mathsf{w}_0, \mathsf{w}_1 \rangle$ denote the subalgebra of $\bar{\mathcal{A}}$. By Proposition 2.18 one has $\langle \mathsf{w}_0, \mathsf{w}_1 \rangle \cong \langle \mathsf{W}_0, \mathsf{W}_1 \rangle_\mathsf{A} \otimes_\mathsf{A} \mathbb{K}$. The generators $\mathsf{w}_0, \mathsf{w}_1$ satisfy the Serre relations (i.e. (2.44)-(2.45) for $q = 1$). Recall the Lie algebra $\widehat{sl_2}^{SC}$ in the Serre-Chevalley presentation of $\widehat{sl_2}$ with defining relations reported in Appendix A. Denote $\widehat{sl_2}^{SC,+}$ (resp. $\widehat{sl_2}^{SC,-}$) the subalgebra generated by $\{e_0, e_1\}$ (resp. $\{f_0, f_1\}$). Combining the isomorphism (2.46) and the well-known result about the specialization $q \to 1$ of $U_q^{DJ,+}$ given by $U(\widehat{sl_2}^{SC,+})$, it follows that the map $\langle \mathsf{w}_0, \mathsf{w}_1 \rangle \to U(\widehat{sl_2}^{SC,+})$ is an isomorphism. Also, $\mathcal{Z}$ is a polynomial ring in the $\{\Delta_{k+1}\}_{k\in\mathbb{N}}$. $\mathcal{Z}_1 = \mathcal{Z}_\mathsf{A} \otimes_\mathsf{A} \mathbb{K} = U(\mathsf{z})$ where z is the linear span of $\{\delta_{k+1}\}_{k\in\mathbb{N}}$, see (2.43). Denote $\mathsf{z}^\pm$ the images of z in $\widehat{sl_2}^{SC,\pm}$. It follows:

$$\bar{\mathcal{A}} \cong U\left(\widehat{sl_2}^{SC,+} \oplus \mathsf{z}^+\right) \cong U\left(\widehat{sl_2}^{SC,-} \oplus \mathsf{z}^-\right) \ . \tag{2.51}$$

The structure of the isomorphisms (2.47) and (2.51) suggests a close relationship between $\bar{\mathcal{A}}_q$ (resp. $\bar{\mathcal{A}}$) and certain subalgebras of the quantum universal enveloping algebra $U_q(\widehat{gl_2})$ (resp. its specialization $U(\widehat{gl_2})$). To clarify this relation in Section 5, a new presentation for $\bar{\mathcal{A}}_q$ (and $\bar{\mathcal{A}}$) is given in the next section.

3. A Freidel-Maillet type presentation for $\bar{\mathcal{A}}_q$ and its specialization $q \to 1$

In this section, it is shown that the algebra $\bar{\mathcal{A}}_q$ introduced in Definition 2.1 admits a presentation in the form of a K-matrix satisfying the defining relations of a quadratic algebra within the family introduced by Freidel and Maillet [33], see Theorem 3.1. In this framework, by Theorem 3.1 and Proposition 3.3, several results obtained in [63] for $\mathcal{U}_q^+$ are derived in a straightforward manner. For the specialization $q \to 1$, a presentation of the Lie algebra $\bar{\mathcal{A}}$ - see Definition 2.16 - is obtained in terms of a non-standard classical Yang-Baxter algebra, see Proposition 3.6.

3.1. A quadratic algebra of Freidel-Maillet type

Let $R(u)$ be the intertwining operator (called quantum R-matrix) between the tensor product of two fundamental representations $\mathcal{V}_1 \otimes \mathcal{V}_2$ for $\mathcal{V} = \mathbb{C}^2$ associated with the algebra $U_q(\widehat{sl_2})$. The element $R(u)$ depends on the deformation parameter q and is defined by [17]

$$R(u) = \begin{pmatrix} uq - u^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & u - u^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & u - u^{-1} & 0 \\ 0 & 0 & 0 & uq - u^{-1}q^{-1} \end{pmatrix}, \tag{3.1}$$

where u is an indeterminate, called 'spectral parameter' in the literature on integrable systems. It is known that $R(u)$ satisfies the quantum Yang-Baxter equation in the space $\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3$. Using the standard notation

$$R_{ij}(u) \in \mathrm{End}(\mathcal{V}_i \otimes \mathcal{V}_j), \tag{3.2}$$

the Yang-Baxter equation reads

$$R_{12}(u/v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u/v). \tag{3.3}$$

As usual, introduce the permutation operator $P = R(1)/(q - q^{-1})$. Here, note that $R_{12}(u) = PR_{12}(u)P = R_{21}(u)$.

We now show that the algebra $\bar{\mathcal{A}}_q$ is isomorphic to a quadratic algebra of Freidel-Maillet type [33], which can be viewed as a limiting case of the standard quantum reflection equation (also called the boundary quantum Yang-Baxter equation) introduced in the context of boundary quantum inverse scattering theory [23,57]. In addition to (3.1), define:

$$R^{(0)} = diag(1, q^{-1}, q^{-1}, 1)\ . \tag{3.4}$$

Define the generating functions:

$$\mathcal{W}_+(u) = \sum_{k\in\mathbb{N}} \mathsf{W}_{-k} U^{-k-1}\ , \quad \mathcal{W}_-(u) = \sum_{k\in\mathbb{N}} \mathsf{W}_{k+1} U^{-k-1}\ , \tag{3.5}$$

$$\mathcal{G}_+(u) = \sum_{k\in\mathbb{N}} \mathsf{G}_{k+1} U^{-k-1}\ , \quad \mathcal{G}_-(u) = \sum_{k\in\mathbb{N}} \tilde{\mathsf{G}}_{k+1} U^{-k-1}\ , \tag{3.6}$$

where the shorthand notation $U = qu^2/(q + q^{-1})$ is used. Let $k_\pm$ be non-zero scalars in $\mathbb{K}(q)$ such that

$$\bar{\rho} = k_+k_-(q + q^{-1})^2\ . \tag{3.7}$$

Theorem 3.1. *The algebra $\bar{\mathcal{A}}_q$ has a presentation of Freidel-Maillet type. Let $K(u)$ be a square matrix such that*

$$K(u)=\begin{pmatrix} uq\mathcal{W}_+(u) & \frac{1}{k_-(q+q^{-1})}\mathcal{G}_+(u)+\frac{k_+(q+q^{-1})}{(q-q^{-1})} \\ \frac{1}{k_+(q+q^{-1})}\mathcal{G}_-(u)+\frac{k_-(q+q^{-1})}{(q-q^{-1})} & uq\mathcal{W}_-(u) \end{pmatrix} \tag{3.8}$$

with (3.5)-(3.6). The defining relations are given by:

$$R(u/v)\ (K(u)\otimes I\!I)\ R^{(0)}\ (I\!I\otimes K(v))\ =\ (I\!I\otimes K(v))\ R^{(0)}\ (K(u)\otimes I\!I)\ R(u/v)\ . \tag{3.9}$$

Proof. Inserting (3.8) into (3.9), the system of (sixteen in total) independent equations for the entries $(K(u))_{ij}$ coming from the Freidel-Maillet type quadratic algebra (3.9) leads to a system of commutation relations between the generating functions $\mathcal{W}_\pm(u)$, $\mathcal{G}_\pm(u)$. Using the identification (3.7), after simplifications these commutation relations read:

$$\big[\mathcal{W}_\pm(u),\mathcal{W}_\pm(v)\big]=0\ , \tag{3.10}$$

$$\big[\mathcal{W}_+(u),\mathcal{W}_-(v)\big]+\big[\mathcal{W}_-(u),\mathcal{W}_+(v)\big]=0\ , \tag{3.11}$$

$$\begin{aligned}(U-V)\big[\mathcal{W}_\pm(u),\mathcal{W}_\mp(v)\big]&=\frac{(q-q^{-1})}{\bar\rho(q+q^{-1})}\left(\mathcal{G}_\pm(u)\mathcal{G}_\mp(v)-\mathcal{G}_\pm(v)\mathcal{G}_\mp(u)\right)\\&\quad+\frac{1}{(q+q^{-1})}\big(\mathcal{G}_\pm(u)-\mathcal{G}_\mp(u)+\mathcal{G}_\mp(v)-\mathcal{G}_\pm(v)\big)\ ,\end{aligned} \tag{3.12}$$

$$(U-V)\big[\mathcal{G}_\pm(u),\mathcal{G}_\mp(v)\big]=\bar\rho(q^2-q^{-2})UV\big(\mathcal{W}_\pm(u)\mathcal{W}_\mp(v)-\mathcal{W}_\pm(v)\mathcal{W}_\mp(u)\big)\ , \tag{3.13}$$

$$U\big[\mathcal{G}_\mp(v),\mathcal{W}_\pm(u)\big]_q-V\big[\mathcal{G}_\mp(u),\mathcal{W}_\pm(v)\big]_q+\ \bar\rho\big(U\mathcal{W}_\pm(u)-V\mathcal{W}_\pm(v)\big)=0\ , \tag{3.14}$$

$$U\big[\mathcal{W}_\mp(u),\mathcal{G}_\mp(v)\big]_q-V\big[\mathcal{W}_\mp(v),\mathcal{G}_\mp(u)\big]_q+\ \bar\rho\big(U\mathcal{W}_\mp(u)-V\mathcal{W}_\mp(v)\big)=0\ , \tag{3.15}$$

$$\big[\mathcal{G}_\epsilon(u),\mathcal{W}_\pm(v)\big]+\big[\mathcal{W}_\pm(u),\mathcal{G}_\epsilon(v)\big]=0\ ,\quad \forall\epsilon=\pm\ , \tag{3.16}$$

$$\big[\mathcal{G}_\pm(u),\mathcal{G}_\pm(v)\big]=0\ , \tag{3.17}$$

$$\big[\mathcal{G}_+(u),\mathcal{G}_-(v)\big]+\big[\mathcal{G}_-(u),\mathcal{G}_+(v)\big]=0\ . \tag{3.18}$$

The commutation relations among the generators of $\bar{\mathcal{A}}_q$ are now extracted. Inserting (3.5), (3.6) into (3.10)-(3.18), expanding and identifying terms of same order in $U^{-k}V^{-l}$ one finds equivalently the set of defining relations (2.1)-(2.11) together with the set of relations (2.17), (2.18) and (2.19)-(2.22) as we now show in details. Precisely, inserting (3.5) into (3.10), (3.11), one gets (2.4), (2.5), respectively. Inserting (3.5), (3.6) into (3.12), one gets (2.1), (2.21), (2.22). Inserting (3.5), (3.6) into (3.13), one gets (2.19), (2.20). Inserting (3.5), (3.6) into (3.14) and (3.15), one gets (2.2), (2.3) as well as (2.17), (2.18). Inserting (3.5), (3.6) into (3.16)-(3.18), one gets (2.5)-(2.11). As the relations (2.17), (2.18) and (2.19)-(2.22) follow from the defining relations (2.1)-(2.11) by Lemmas 2.3, 2.4, it follows that the Freidel-Maillet type algebra (3.9) is isomorphic to $\bar{\mathcal{A}}_q$. □

Remark 3.2. The relations (3.10)-(3.18) coincide with the relations [63, Lemmas 13.3,13.4] in the algebra $\mathcal{U}_q^+$ for the identification:

$$U\mapsto t^{-1}\ ,\quad V\mapsto s^{-1}\ , \tag{3.19}$$

$$\mathcal{W}_\pm(u)\mapsto t\mathcal{W}^\mp(t)\ ,\quad \mathcal{W}_\pm(v)\mapsto s\mathcal{W}^\mp(s)\ , \tag{3.20}$$

$$\mathcal{G}_+(u)\mapsto q^{-1}(q^2-q^{-2})(\mathcal{G}(t)-1)\ ,\quad \mathcal{G}_-(u)\mapsto q^{-1}(q^2-q^{-2})(\tilde{\mathcal{G}}(t)-1)\ , \tag{3.21}$$

$$\mathcal{G}_+(v)\mapsto q^{-1}(q^2-q^{-2})(\mathcal{G}(s)-1)\ ,\quad \mathcal{G}_-(v)\mapsto q^{-1}(q^2-q^{-2})(\tilde{\mathcal{G}}(s)-1)\ , \tag{3.22}$$

$$\bar\rho\mapsto q^{-1}(q^2-q^{-2})(q-q^{-1})\ . \tag{3.23}$$

For completeness, let us mention that an alternative presentation of $\bar{\mathcal{A}}_q$ can be considered instead, that involves power series in u in the opposite direction. Indeed, consider the system of relations (3.10)-(3.18) with (3.5)-(3.6). Applying the transformation:

$$\mathcal{W}_\pm(u) \mapsto -\mathcal{W}_\mp(u^{-1}q^{-1}) , \qquad \mathcal{G}_\pm(u) \mapsto -\mathcal{G}_\pm(u^{-1}q^{-1}) ,$$
$$u \mapsto u^{-1} , \quad q \mapsto q^{-1} ,$$

and similarly for $u \to v$, one finds that

$$K'(u) = \begin{pmatrix} u^{-1}q^{-1}\mathcal{W}_-(u^{-1}q^{-1}) & \frac{1}{k_-(q+q^{-1})}\mathcal{G}_+(u^{-1}q^{-1}) + \frac{k_+(q+q^{-1})}{(q-q^{-1})} \\ \frac{1}{k_+(q+q^{-1})}\mathcal{G}_-(u^{-1}q^{-1}) + \frac{k_-(q+q^{-1})}{(q-q^{-1})} & u^{-1}q^{-1}\mathcal{W}_+(u^{-1}q^{-1}) \end{pmatrix} \tag{3.24}$$

satisfies the Freidel-Maillet type equation:

$$\begin{aligned} & R(u/v)\ (K'(u) \otimes I\!I)\ (R^{(0)})^{-1}\ (I\!I \otimes K'(v)) \\ & \quad = (I\!I \otimes K'(v))\ (R^{(0)})^{-1}\ (K'(u) \otimes I\!I)\ R(u/v) . \end{aligned} \tag{3.25}$$

This second presentation of $\bar{\mathcal{A}}_q$ will be used in Section 5.

3.2. Central elements

For the Freidel-Maillet type algebra (3.9), central elements can be derived from the so-called Sklyanin determinant by analogy with [57, Proposition 5]. Define $P_{12}^- = (1-P)/2$. As usual, below 'tr$_{12}$' stands for the trace over $\mathcal{V}_1 \otimes \mathcal{V}_2$.

Proposition 3.3. *Let $K(u)$ be a solution of (3.9). The quantum determinant*

$$\Gamma(u) = \text{tr}_{12}\big(P_{12}^-(K(u) \otimes I\!I)\ R^{(0)}(I\!I \otimes K(uq))\big) , \tag{3.26}$$

is such that $\big[\Gamma(u), (K(v))_{ij}\big] = 0$.

Proof. Recall the notation (3.2). Introduce the vector space $\mathcal{V}_0$. With respect to the tensor product $\mathcal{V}_0 \otimes \mathcal{V}_1 \otimes \mathcal{V}_2$, we denote:

$$K_0(u) = K(u) \otimes I\!I \otimes I\!I , \qquad K_1(u) = I\!I \otimes K(u) \otimes I\!I , \qquad K_2(u) = I\!I \otimes I\!I \otimes K(u) . \tag{3.27}$$

Consider the product $(a) \equiv K_0(v)\Gamma(u)$:

$$\begin{aligned} (a) &= K_0(v)\text{tr}_{12}\big(P_{12}^- K_1(u)\ R_{12}^{(0)} K_2(uq)\big) , \\ &= q\, K_0(v)\text{tr}_{12}\big(P_{12}^- R_{01}^{(0)} R_{02}^{(0)} K_1(u)\ R_{12}^{(0)} K_2(uq)\big) \quad (\text{using} \quad P_{12}^- = q\, P_{12}^- R_{01}^{(0)} R_{02}^{(0)}) \\ &= q\text{tr}_{12}\big(P_{12}^- K_0(v) R_{01}^{(0)} R_{02}^{(0)} K_1(u)\ R_{12}^{(0)} K_2(uq)\big) \quad (\text{using} \quad [K_0(v), P_{12}^-] = 0) \\ &= q\text{tr}_{12}\big(P_{12}^- K_0(v) R_{01}^{(0)} K_1(u) R_{02}^{(0)}\ R_{12}^{(0)} K_2(uq)\big) \quad (\text{using} \quad [K_1(u), R_{02}^{(0)}] = 0) \\ &= q\text{tr}_{12}\big(P_{12}^- R_{01}^{-1}(v/u) K_1(u) R_{01}^{(0)} K_0(v) R_{01}(v/u) R_{02}^{(0)}\ R_{12}^{(0)} K_2(uq)\big) \quad (\text{using (3.9)}) . \end{aligned}$$

Then we use $[K_0(v), R_{12}^{(0)}] = 0$, $[K_2(uq), R_{01}(v/u)] = 0$ and

$$R_{01}(v/u)R_{02}^{(0)}\ R_{12}^{(0)} = \ R_{12}^{(0)}R_{02}^{(0)}R_{01}(v/u)$$

to show:

$$\begin{aligned}&K_0(v)\mathrm{tr}_{12}\big(P_{12}^{-}K_1(u)\ R_{12}^{(0)}K_2(uq)\big)\\&\quad= q\,\mathrm{tr}_{12}\big(P_{12}^{-}R_{01}^{-1}(v/u)K_1(u)R_{01}^{(0)}K_0(v)R_{12}^{(0)}R_{02}^{(0)}R_{01}(v/u)K_2(uq)\big)\\&\quad= q\,\mathrm{tr}_{12}\big(P_{12}^{-}R_{01}^{-1}(v/u)K_1(u)R_{01}^{(0)}R_{12}^{(0)}K_0(v)R_{02}^{(0)}K_2(uq)R_{01}(v/u)\big)\end{aligned}$$

Applying again (3.9) to the combination $K_0(v)R_{02}^{(0)}K_2(uq)$ and using $R_{02}(v/uq)R_{01}^{(0)}\ R_{12}^{(0)} = R_{12}^{(0)}R_{01}^{(0)}R_{02}(v/uq)$, it follows:

$$\begin{aligned}(a) &= q\,\mathrm{tr}_{12}\big(P_{12}^{-}R_{01}^{-1}(v/u)K_1(u)R_{01}^{(0)}R_{12}^{(0)}R_{02}^{-1}(v/uq)K_2(uq)R_{02}^{(0)}K_0(v)R_{02}(v/uq)R_{01}(v/u)\big)\\&= q\,\mathrm{tr}_{12}\big(P_{12}^{-}R_{01}^{-1}(v/u)K_1(u)R_{02}^{-1}(v/uq)R_{12}^{(0)}R_{01}^{(0)}K_2(uq)R_{02}^{(0)}K_0(v)R_{02}(v/uq)R_{01}(v/u)\big)\\&= q\,\mathrm{tr}_{12}\big(P_{12}^{-}R_{01}^{-1}(v/u)R_{02}^{-1}(v/uq)K_1(u)R_{12}^{(0)}K_2(uq)R_{01}^{(0)}R_{02}^{(0)}K_0(v)R_{02}(v/uq)R_{01}(v/u)\big).\end{aligned}$$

Then, using $P_{12}^{-}R_{02}(x/q)R_{01}(x) = P_{12}^{-}(x^2-q^2)(x^2-q^{-2})/x^2$, $q\,P_{12}^{-}R_{01}^{(0)}R_{02}^{(0)} = P_{12}^{-}$, eq. (3.9) and the cyclicity of the trace, the last expression simplifies to:

$$\begin{aligned}(a) &= q\,\mathrm{tr}_{12}\big(P_{12}^{-}K_1(u)R_{12}^{(0)}K_2(uq)R_{01}^{(0)}R_{02}^{(0)}K_0(v)P_{12}^{-}\big)\\&= \mathrm{tr}_{12}\big(P_{12}^{-}K_1(u)R_{12}^{(0)}K_2(uq)\big)K_0(v)\\&= \Gamma(u)K_0(v)\ . \quad \square\end{aligned}$$

Now, define:

$$\Gamma(u) = \frac{1}{2(q-q^{-1})}\left(\Delta(u) - \frac{2\bar{\rho}}{(q-q^{-1})}\right) .$$

Using the entries of (3.8), by Proposition 3.3 it implies $[\Delta(u), \mathcal{W}_\pm(v)] = [\Delta(u), \mathcal{G}_\pm(v)] = 0$. Using (3.5), (3.6), it follows:

Corollary 3.4.

$$\begin{aligned}\Delta(u) = &(q-q^{-1})u^2q^2\Big(\mathcal{W}_+(u)\mathcal{W}_-(uq) + \mathcal{W}_-(u)\mathcal{W}_+(uq)\Big)\\&- \frac{(q-q^{-1})}{\bar{\rho}}\Big(\mathcal{G}_+(u)\mathcal{G}_-(uq) + \mathcal{G}_-(u)\mathcal{G}_+(uq)\Big)\\&- \mathcal{G}_+(u) - \mathcal{G}_+(uq) - \mathcal{G}_-(u) - \mathcal{G}_-(uq)\end{aligned} \tag{3.28}$$

provides a generating function for central elements in $\bar{\mathcal{A}}_q$.

Expanding $\Delta(u)$ in power series of $U = qu^2/(q+q^{-1})$, the coefficients produce the central elements of $\bar{\mathcal{A}}_q$ given by (2.26). Namely, by straightforward calculations one gets:

$$\Delta(u) = -\sum_{n=0}^{\infty} U^{-n-1}q^{-n-1}(q^{n+1}+q^{-n-1})\Delta_{n+1}\ .$$

Remark 3.5. In [63, Lemma 13.8], a generating function for central elements is given. By [63, Corollary 8.4] and [63, Definition 13.1], alternatively three other generating functions may be considered. For instance:

$$Z^\vee(t) = \mathcal{G}(qt)\tilde{\mathcal{G}}(q^{-1}t) - qt\mathcal{W}^+(qt)\mathcal{W}^-(q^{-1}t) ,$$
$$\sigma(Z^\vee(t)) = \tilde{\mathcal{G}}(qt)\mathcal{G}(q^{-1}t) - qt\mathcal{W}^-(qt)\mathcal{W}^+(q^{-1}t) .$$

Using the identification (3.19)-(3.23), the image of the generating function $\Delta(u)$ in the algebra $\mathcal{U}_q^+$ follows:

$$\Delta(u) \mapsto -q^{-1}(q^2 - q^{-2}) \left(Z^\vee(q^{-1}t) + \sigma(Z^\vee(q^{-1}t)) \right) .$$

3.3. *Specialization* $q \to 1$

Due to the presence of poles at $q = 1$ in the off-diagonal entries of $K(u)$ in (3.8), the relations (3.9) are not suitable for the specialization $q \to 1$. However, it is possible to solve this problem within the framework of the non-standard classical Yang-Baxter algebra [20,56,3,58] in order to obtain an alternative presentation of $\bar{\mathcal{A}}$, besides Definition 2.16, viewed as a specialization $q \to 1$ of the Freidel-Maillet type algebra (3.9). Introduce the r-matrix[2]

$$\bar{r}(u, v) = \frac{1}{(u^2/v^2 - 1)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2u/v & 0 \\ 0 & 2u/v & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.29}$$

solution of the non-standard classical Yang-Baxter equation [3]:

$$[\, \bar{r}_{13}(u_1, u_3)\, ,\, \bar{r}_{23}(u_2, u_3)\,] = [\, \bar{r}_{21}(u_2, u_1)\, ,\, \bar{r}_{13}(u_1, u_3)\,] + [\, \bar{r}_{23}(u_2, u_3)\, ,\, \bar{r}_{12}(u_1, u_2)\,] , \tag{3.30}$$

where $\bar{r}_{21}(u, v) = P\bar{r}_{12}(u, v)P$ $(= \bar{r}_{12}(u, v)$ for (3.29)). Define the generating functions:

$$w_+(u) = \sum_{k=0}^{\infty} \mathsf{w}_{-k} U^{-k-1} , \quad w_-(u) = \sum_{k=0}^{\infty} \mathsf{w}_{k+1} U^{-k-1} , \tag{3.31}$$

$$g_+(u) = \sum_{k=0}^{\infty} \mathsf{g}_{k+1} U^{-k-1} \;, \quad g_-(u) = \sum_{k=0}^{\infty} \tilde{\mathsf{g}}_{k+1} U^{-k-1} \quad \text{with} \quad U = u^2/2 . \tag{3.32}$$

Proposition 3.6. *The algebra $\bar{\mathcal{A}}$ admits a FRT presentation given by*

$$B(u) = \frac{1}{2} \begin{pmatrix} \frac{1}{4}\, g_-(u) & u w_-(u) \\ u w_+(u) & \frac{1}{4}\, g_+(u) \end{pmatrix} \tag{3.33}$$

that satisfies the non-standard classical Yang-Baxter algebra

$$[\, B_1(u)\, ,\, B_2(v)\,] = [\, \bar{r}_{21}(v, u)\, ,\, B_1(u)\,] + [\, B_2(v)\, ,\, \bar{r}_{12}(u, v)\,] . \tag{3.34}$$

Proof. Insert (3.33) into (3.34) with (3.29). Define the formal variables $U = u^2/2$ and $V = v^2/2$. One obtains equivalently:

---

[2] Note that this r-matrix can be obtained from a limiting case of a r-matrix considered in [7].

$$(U-V)\big[w_\pm(u),w_\mp(v)\big]=\frac{1}{2}(g_\pm(u)-g_\mp(u)+g_\mp(v)-g_\pm(v))\ ,$$
$$(U-V)\big[g_\epsilon(u),w_\pm(v)\big]\mp\epsilon 16\big(Uw_\pm(u)-Vw_\pm(v)\big)=0\ ,\quad \epsilon=\pm 1\ ,$$
$$\big[g_\pm(u),g_\mp(v)\big]=0\ ,$$
$$\big[w_\pm(u),w_\pm(v)\big]=0\ ,\quad \big[g_\pm(u),g_\pm(v)\big]=0\ .$$

These relations are equivalent to the specialization $q\to 1$ of (3.10)-(3.18) ($\bar\rho\mapsto 16$). Using (3.31), the above equations are equivalent to (2.38)-(2.41). □

Remark 3.7. For the specialization $q\to 1$, the generating function (3.28) reduces to $\delta(u)=-2(g_+(u)+g_-(u))$.

4. Quotients of $\bar{\mathcal{A}}_q$ and tensor product representations

In this section, a class of solutions - so-called 'dressed' solutions - of the Freidel-Maillet type equation (3.9) are constructed and studied in details by adapting known techniques of the so-called reflection equation [57], see Proposition 4.1. By Lemma 4.3, it is shown that the entries of the dressed solutions can be written in terms of the 'truncated' generating functions (4.28)-(4.29), whose generators act on N-fold tensor product representations of $U_q(sl_2)$ according to (4.16)-(4.19). Realizations of $\bar{\mathcal{A}}_q$ in $U_q(sl_2)^{\otimes N}$ are obtained, see Proposition 4.5.

4.1. Dressed solutions of the Freidel-Maillet type equation

The starting point of the following analysis is an adaptation of [57, Proposition 2], [33], to the Freidel-Maillet type equation (3.9), thus we skip the proof of the proposition below. Let $K_0(u)$ be a solution of (3.9). Assume there exists a pair of quantum Lax operators satisfying the exchange relations:

$$R(u/v)\ (L(u)\otimes I\!I)\ (I\!I\otimes L(v))=(I\!I\otimes L(v))\ (L(u)\otimes I\!I)\ R(u/v)\ , \tag{4.1}$$
$$R(u/v)\ (L_0(u)\otimes I\!I)\ (I\!I\otimes L_0(v))=(I\!I\otimes L_0(v))\ (L_0(u)\otimes I\!I)\ R(u/v)\ , \tag{4.2}$$
$$R^{(0)}\ (L_0(u)\otimes I\!I)\ (I\!I\otimes L(v))=(I\!I\otimes L(v))\ (L_0(u)\otimes I\!I)\ R^{(0)}. \tag{4.3}$$

Using (4.1)-(4.3), it is easy to show that $L_0(uv_1)K_0(u)L(u/v_1)$ for any $v_1\in\mathbb{K}^*$ is also a solution of (3.9) (similar to [57, Proposition 2]). More generally it follows[3]

Proposition 4.1. *Let $K_0(u)$ be a solution of (3.9). Let N be a positive integer and $\{v_i\}_{i=1}^N\in\mathbb{K}^*$. Let $L(u),L_0(u)$ be such that (4.1)-(4.3) hold. Then*

$$K^{(N)}(u)=(L_0(uv_N))_{[N]}\cdots(L_0(uv_1))_{[1]}K_0(u)(L(u/v_1)))_{[1]}\cdots(L(u/v_N))_{[N]} \tag{4.5}$$

satisfies (3.9).

[3] Here the index $[j]$ characterizes the 'quantum space' $V_{[j]}$ on which the entries of $L(u),L_0(u)$ act. With respect to the ordering $V_{[2]}\otimes V_{[1]}$ used below for (4.16)-(4.19), one has:

$$((T)_{[2]}(T')_{[1]}(T'')_{[2]})_{ij}=\sum_{k,\ell=1}^{2}(T)_{ik}(T'')_{\ell j}\otimes(T')_{k\ell}\ . \tag{4.4}$$

This proposition provides a tool for the explicit construction of so-called ‘dressed’ solutions of (3.9). Below, we construct explicit examples of such solutions. To this end, we first introduce some known basic material. Recall the algebra $U_q(sl_2)$ consists of three generators denoted $S_\pm, s_3$. They satisfy

$$[s_3, S_\pm] = \pm S_\pm \qquad \text{and} \qquad [S_+, S_-] = \frac{q^{2s_3} - q^{-2s_3}}{q - q^{-1}} . \tag{4.6}$$

The central element of $U_q(sl_2)$ is the so-called Casimir operator:

$$\Omega = \frac{q^{-1}q^{2s_3} + qq^{-2s_3}}{(q-q^{-1})^2} + S_+S_- = \frac{qq^{2s_3} + q^{-1}q^{-2s_3}}{(q-q^{-1})^2} + S_-S_+. \tag{4.7}$$

Let V be the spin-j irreducible finite dimensional representation of $U_q(sl_2)$ of dimension $2j+1$. The eigenvalue ω_j of Ω is such that

$$\omega_j \equiv \frac{w_0^{(j)}}{(q-q^{-1})^2} \qquad \text{with} \qquad w_0^{(j)} = q^{2j+1} + q^{-2j-1}. \tag{4.8}$$

Define the so-called quantum Lax operators

$$L_0(u) = \begin{pmatrix} uq^{1/2}q^{s_3} & 0 \\ 0 & uq^{1/2}q^{-s_3} \end{pmatrix} \qquad \text{and}$$
$$L(u) = \begin{pmatrix} uq^{1/2}q^{s_3} - u^{-1}q^{-1/2}q^{-s_3} & (q-q^{-1})S_- \\ (q-q^{-1})S_+ & uq^{1/2}q^{-s_3} - u^{-1}q^{-1/2}q^{s_3} \end{pmatrix} . \tag{4.9}$$

Recall the R-matrices (3.1) and (3.4). One routinely checks that the relation (4.1) holds. The relations (4.2)-(4.3) follow as a limiting case of (4.1). Note that the overall factor $uq^{1/2}$ in the expression of $L_0(u)$ is kept for further convenience only. Let $k_\pm, \bar{\epsilon}_\pm \in \mathbb{K}$. Define:

$$K_0(u) = \begin{pmatrix} u^{-1}\bar{\epsilon}_+ & \frac{k_+}{(q-q^{-1})} \\ \frac{k_-}{(q-q^{-1})} & u^{-1}\bar{\epsilon}_- \end{pmatrix} . \tag{4.10}$$

It is checked that $K_0(u)$ satisfies (3.9). As a basic example of dressed solution, consider the case $N=1$ of (4.5). Define the four operators in $U_q(sl_2)$:

$$\mathcal{W}_0^{(1)} = k_+v_1q^{1/2}S_+q^{s_3} + \bar{\epsilon}_+q^{2s_3} , \tag{4.11}$$
$$\mathcal{W}_1^{(1)} = k_-v_1q^{1/2}S_-q^{-s_3} + \bar{\epsilon}_-q^{-2s_3} , \tag{4.12}$$
$$\mathcal{G}_1^{(1)} = k_+k_-v_1^2\frac{(w_0^{(j_1)} - (q+q^{-1})q^{2s_3})}{(q-q^{-1})} + (q^2-q^{-2})k_-\bar{\epsilon}_+v_1q^{-1/2}S_-q^{s_3} + (q-q^{-1})\bar{\epsilon}_+\bar{\epsilon}_- , \tag{4.13}$$
$$\tilde{\mathcal{G}}_1^{(1)} = k_+k_-v_1^2\frac{(w_0^{(j_1)} - (q+q^{-1})q^{-2s_3})}{(q-q^{-1})} + (q^2-q^{-2})k_+\bar{\epsilon}_-v_1q^{-1/2}S_+q^{-s_3} + (q-q^{-1})\bar{\epsilon}_+\bar{\epsilon}_- . \tag{4.14}$$

Computing explicitly the entries of (4.5) for $N=1$, one finds that the dressed solution can be written as:

$$K^{(1)}(u) = \begin{pmatrix} uq\mathcal{W}_0^{(1)} - u^{-1}v_1^2\bar{\epsilon}_+ & \frac{\mathcal{G}_1^{(1)}}{k_-(q+q^{-1})} + \frac{k_+qu^2}{(q-q^{-1})} - \frac{k_+v_1^2w_0^{(j_1)}}{(q^2-q^{-2})} - \frac{\bar{\epsilon}_+\bar{\epsilon}_-(q-q^{-1})}{k_-(q+q^{-1})} \\ \frac{\tilde{\mathcal{G}}_1^{(1)}}{k_+(q+q^{-1})} + \frac{k_-qu^2}{(q-q^{-1})} - \frac{k_-v_1^2w_0^{(j_1)}}{(q^2-q^{-2})} - \frac{\bar{\epsilon}_+\bar{\epsilon}_-(q-q^{-1})}{k_+(q+q^{-1})} & uq\mathcal{W}_1^{(1)} - u^{-1}v_1^2\bar{\epsilon}_- \end{pmatrix} . \tag{4.15}$$

4.2. General dressed solutions

The structure of the above solution (4.15) can be generalized to dressed solutions of arbitrary size as we now show. According to the ordering of the 'quantum' vector spaces $V^{(N)} = V_{[N]} \otimes \cdots \otimes V_{[2]} \otimes V_{[1]}$, let us first define recursively the four families of operators $\{\mathcal{W}_{-k}^{(N)}, \mathcal{W}_{k+1}^{(N)}, \mathcal{G}_{k+1}^{(N)}, \tilde{\mathcal{G}}_{k+1}^{(N)} | k = 0, 1, ..., N\}$, where N is a positive integer:

$$\mathcal{W}_{-k}^{(N)} = \frac{(q-q^{-1})}{k_-(q+q^{-1})^2}\left(v_N q^{1/2} S_+ q^{s_3} \otimes \mathcal{G}_k^{(N-1)}\right) + q^{2s_3} \otimes \mathcal{W}_{-k}^{(N-1)} - \frac{v_N^2}{(q+q^{-1})} I\!I \otimes \mathcal{W}_{-k+1}^{(N-1)} + \frac{v_N^2 w_0^{(j_N)}}{(q+q^{-1})^2}\mathcal{W}_{-k+1}^{(N)} , \tag{4.16}$$

$$\mathcal{W}_{k+1}^{(N)} = \frac{(q-q^{-1})}{k_+(q+q^{-1})^2}\left(k_- v_N q^{1/2} S_- q^{-s_3} \otimes \tilde{\mathcal{G}}_k^{(N-1)}\right) + q^{-2s_3} \otimes \mathcal{W}_{k+1}^{(N-1)} - \frac{v_N^2}{(q+q^{-1})} I\!I \otimes \mathcal{W}_k^{(N-1)} + \frac{v_N^2 w_0^{(j_N)}}{(q+q^{-1})^2}\mathcal{W}_k^{(N)} , \tag{4.17}$$

$$\mathcal{G}_{k+1}^{(N)} = (q^2-q^{-2})k_- v_N q^{-1/2} S_- q^{s_3} \otimes \mathcal{W}_{-k}^{(N-1)} - \frac{v_N^2}{(q+q^{-1})} q^{2s_3} \otimes \mathcal{G}_k^{(N-1)} + I\!I \otimes \mathcal{G}_{k+1}^{(N-1)} + \frac{v_N^2 w_0^{(j_N)}}{(q+q^{-1})^2}\mathcal{G}_k^{(N)} , \tag{4.18}$$

$$\tilde{\mathcal{G}}_{k+1}^{(N)} = (q^2-q^{-2})k_+ v_N q^{-1/2} S_+ q^{-s_3} \otimes \mathcal{W}_{k+1}^{(N-1)} - \frac{v_N^2}{(q+q^{-1})} q^{-2s_3} \otimes \tilde{\mathcal{G}}_k^{(N-1)} + I\!I \otimes \tilde{\mathcal{G}}_{k+1}^{(N-1)} + \frac{v_N^2 w_0^{(j_N)}}{(q+q^{-1})^2}\tilde{\mathcal{G}}_k^{(N)} . \tag{4.19}$$

Here for the special case $k=0$ we identify[4]

$$\mathcal{W}_k^{(N)}|_{k=0} \equiv 0 , \quad \mathcal{W}_{-k+1}^{(N)}|_{k=0} \equiv 0 , \quad \mathcal{G}_k^{(N)}|_{k=0} = \tilde{\mathcal{G}}_k^{(N)}|_{k=0} \equiv \frac{k_+k_-(q+q^{-1})^2}{(q-q^{-1})} I\!I^{(N)} \tag{4.20}$$

together with the 'initial' conditions for $k \geq 1$ (the notation (4.27) is used)

[4] Although the notation is ambiguous, one must keep in mind that $\mathcal{W}_k^{(N)}|_{k=0} \neq \mathcal{W}_{-k}^{(N)}|_{k=0}$, $\mathcal{W}_{-k+1}^{(N)}|_{k=0} \neq \mathcal{W}_{k+1}^{(N)}|_{k=0}$ for any N.

$$\mathcal{W}_{-k}^{(0)} \equiv \left(\frac{\alpha_1}{q+q^{-1}}\right)^{k-1} \left(\frac{\alpha_1}{q+q^{-1}}\right)_{|v_1=0} \mathcal{W}_0^{(0)} ,$$
$$\mathcal{W}_{k+1}^{(0)} \equiv \left(\frac{\alpha_1}{q+q^{-1}}\right)^{k-1} \left(\frac{\alpha_1}{q+q^{-1}}\right)_{|v_1=0} \mathcal{W}_1^{(0)} , \qquad (4.21)$$

$$\mathcal{G}_{k+1}^{(0)} = \tilde{\mathcal{G}}_{k+1}^{(0)} \equiv \left(\frac{\alpha_1}{q+q^{-1}}\right)^{k} \mathcal{G}_1^{(0)} , \qquad (4.22)$$

where

$$\mathcal{W}_0^{(0)} \equiv \bar{\epsilon}_+ , \quad \mathcal{W}_1^{(0)} \equiv \bar{\epsilon}_- \quad \text{and} \quad \mathcal{G}_1^{(0)} = \tilde{\mathcal{G}}_1^{(0)} \equiv \bar{\epsilon}_+\bar{\epsilon}_-(q-q^{-1}) . \qquad (4.23)$$

A crucial ingredient in the construction of dressed solutions by induction from (4.5) is the existence of a set of linear relations satisfied by the operators (4.16)-(4.19). We proceed by strict analogy with [8, Appendix B], thus we skip most of the details of the proof. For further convenience, introduce the notation:

$$\bar{\epsilon}_{\pm}^{(N)} = (-1)^N \left(\prod_{k=1}^{N} v_k^2\right) \bar{\epsilon}_{\pm} . \qquad (4.24)$$

Lemma 4.2. *The operators (4.16)-(4.19) satisfy the linear relations:*

$$\sum_{k=0}^{N} c_k^{(N)} \mathcal{W}_{-k}^{(N)} + \bar{\epsilon}_+^{(N)} = 0 , \qquad \sum_{k=0}^{N} c_k^{(N)} \mathcal{W}_{k+1}^{(N)} + \bar{\epsilon}_-^{(N)} = 0 , \qquad (4.25)$$

$$\sum_{k=0}^{N} c_k^{(N)} \mathcal{G}_{k+1}^{(N)} = 0 , \qquad \sum_{k=0}^{N} c_k^{(N)} \tilde{\mathcal{G}}_{k+1}^{(N)} = 0 \qquad (4.26)$$

with[5] $c_k^{(N)} = (-1)^{N-k-1}(q+q^{-1})^k \mathrm{e}_{N-k}(\alpha_1, \alpha_2, \cdots, \alpha_N)$,

$$\alpha_1 = \frac{v_1^2 w_0^{(j_1)}}{(q+q^{-1})} + \frac{\bar{\epsilon}_+\bar{\epsilon}_-(q-q^{-1})^2}{k_+k_-(q+q^{-1})} , \quad \alpha_k = \frac{v_k^2 w_0^{(j_k)}}{(q+q^{-1})} \quad \textit{for} \quad k = 2, ..., N . \qquad (4.27)$$

Proof. For $N = 1, 2$, the four relations (4.25)-(4.26) are explicitly checked. Then we proceed by induction. □

The result below is obtained after some straightforward calculations similar to those performed in [8,9], thus we just sketch the proof. Introduce the 'truncated' generating functions:

$$\mathcal{W}_+^{(N)}(u) = \sum_{k=0}^{N-1} f_{k+1}^{(N)}(u) \mathcal{W}_{-k}^{(N)} , \quad \mathcal{W}_-^{(N)}(u) = \sum_{k=0}^{N-1} f_{k+1}^{(N)}(u) \mathcal{W}_{k+1}^{(N)} \qquad (4.28)$$

[5] For the elementary symmetric polynomials in the variables $\{x_i | i = 1, ..., n\}$, we use the notation:

$$\mathrm{e}_k(x_1, x_2, ..., x_n) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} x_{j_1} x_{j_2} \cdots x_{j_k} .$$

$$\mathcal{G}_+^{(N)}(u) = \sum_{k=0}^{N-1} f_{k+1}^{(N)}(u)\mathcal{G}_{k+1}^{(N)} , \quad \mathcal{G}_-^{(N)}(u) = \sum_{k=0}^{N-1} f_{k+1}^{(N)}(u)\tilde{\mathcal{G}}_{k+1}^{(N)} \tag{4.29}$$

where

$$f_k^{(N)}(u) = \sum_{p=k}^{N}(-1)^{N-p}(q+q^{-1})^{p-1}\mathsf{e}_{N-p}(\alpha_1,\alpha_2,...,\alpha_N)U^{p-k} \quad \text{with}$$
$$U = qu^2/(q+q^{-1}) . \tag{4.30}$$

Lemma 4.3. *Dressed solutions of the form (4.5) can be written as:*

$$K^{(N)}(u) = \begin{pmatrix} uq\mathcal{W}_+^{(N)}(u) + u^{-1}\bar{\epsilon}_+^{(N)} & \frac{1}{k_-(q+q^{-1})}\mathcal{G}_+^{(N)}(u) + \frac{k_+(q+q^{-1})}{(q-q^{-1})} f_0^{(N)}(u) \\ \frac{1}{k_+(q+q^{-1})}\mathcal{G}_-^{(N)}(u) + \frac{k_-(q+q^{-1})}{(q-q^{-1})} f_0^{(N)}(u) & uq\mathcal{W}_-^{(N)}(u) + u^{-1}\bar{\epsilon}_-^{(N)} \end{pmatrix} \tag{4.31}$$

with (4.28)-(4.29) and (4.24).

Proof. For $N = 1$, one checks that (4.31) coincides with (4.15). Then, we proceed by induction. Assume $K^{(N)}(u)$ is of the form (4.31) for N fixed. We compute $((L_0(uv_{N+1}))_{[\mathsf{N+1}]}K^{(N)}(u)(L(u/v_{N+1}))_{[\mathsf{N+1}]})_{ij}$ for $i, j = 1, 2$. For instance, consider the entry $(11)_{N+1}$. Explicitly, it reads:

$$\begin{aligned}(11)_{N+1} = uq\Bigg(&(q-q^{-1})v_{N+1}q^{1/2}S_+q^{s_3}\\ &\otimes\left(\frac{1}{k_-(q+q^{-1})}\mathcal{G}_+^{(N)}(u) + \frac{k_+(q+q^{-1})}{(q-q^{-1})}f_0^{(N)}(u)\right) + q^{2s_3}\otimes\bar{\epsilon}_+^{(N)}\\ &+(u^2qq^{2s_3}-v_{N+1}^2)\otimes\mathcal{W}_+^{(N)}(u)\Bigg) - u^{-1}v_{N+1}^2\bar{\epsilon}_+^{(N)} .\end{aligned}$$

Inserting (4.28), (4.29) and using the definitions (4.16)-(4.19), (4.24) for $N \to N+1$, after some simple operations and reorganizing all terms one gets:

$$\begin{aligned}(11)_{N+1} = uq\Bigg(&\sum_{k=0}^{N-1}\left((q+q^{-1})f_k^{(N)}(u) - \alpha_{N+1}f_{k+1}^{(N)}(u)\right)\mathcal{W}_{-k}^{(N+1)}\\ &+(q+q^{-1})f_N^{(N)}(u)\mathcal{W}_{-N}^{(N+1)}\Bigg) + u^{-1}\bar{\epsilon}_+^{(N+1)} + q^{2s_3}\\ \otimes&\underbrace{\left(\sum_{k=0}^{N-1}\left(qu^2f_{k+1}^{(N)}(u) - (q+q^{-1})f_k^{(N)}(u)\right)\mathcal{W}_{-k}^{(N+1)} - (q+q^{-1})f_N^{(N)}(u)\mathcal{W}_{-N}^{(N+1)} + \epsilon_+^{(N)}\right)}_{\equiv\Gamma(u)} .\end{aligned}$$

Identifying $(11)_{N+1}$ with $(K^{(N+1)}(u))_{11}$ leads to a set of constraints. They read:

$$(q+q^{-1})f_k^{(N)}(u) - \alpha_{N+1}f_{k+1}^{(N)}(u) = f_{k+1}^{(N+1)}(u) \quad \text{for} \quad k = 0, ..., N-1 , \tag{4.32}$$
$$(q+q^{-1})f_N^{(N)}(u) = f_{N+1}^{(N+1)}(u) \tag{4.33}$$

and $\Gamma(u) = 0$. The solution of the constraints (4.32)-(4.33) is given by (4.30). Using this expression, one finds that $\Gamma(u)$ coincides with the l.h.s. of the first equation in (4.25). By Lemma 4.2, it follows $\Gamma(u) = 0$, so $(11)_{N+1} = (K^{(N+1)}(u))_{11}$. By similar arguments, one shows $(ij)_{N+1} = (K^{(N+1)}(u))_{ij}$ using (4.25), (4.26). □

4.3. Realizations of $\bar{\mathcal{A}}_q$ in $U_q(sl_2)^{\otimes N}$

According to previous results, dressed solutions of the form (4.31) automatically generate the finite set of operators (4.16)-(4.19). In this section, we show (4.16)-(4.19) extends to $k \in \mathbb{N}$ and provide realizations of $\bar{\mathcal{A}}_q$ in $U_q(sl_2)^{\otimes N}$. To this aim, we need a generalization of Lemma 4.2.

Lemma 4.4. *For any $p \in \mathbb{N}$, the operators (4.16)-(4.19) satisfy the linear relations:*

$$\sum_{k=0}^{N} c_k^{(N)} \mathcal{W}_{-k-p}^{(N)} + \delta_{p,0}\bar{\epsilon}_+^{(N)} = 0\,, \qquad \sum_{k=0}^{N} c_k^{(N)} \mathcal{W}_{k+1+p}^{(N)} + \delta_{p,0}\bar{\epsilon}_-^{(N)} = 0\,, \tag{4.34}$$

$$\sum_{k=0}^{N} c_k^{(N)} \mathcal{G}_{k+1+p}^{(N)} = 0\,, \qquad \sum_{k=0}^{N} c_k^{(N)} \tilde{\mathcal{G}}_{k+1+p}^{(N)} = 0\,. \tag{4.35}$$

Proof. For $p = 0$ the four relations hold by Lemma 4.2. For $N = 1$ and any $p \geq 1$, the four relations are checked using (4.21), (4.22). Then we proceed by induction on N. □

Define $\bar{\mathcal{A}}_q^{(N)}$ as the algebra generated by $\{\mathcal{W}_{-k}^{(N)}, \mathcal{W}_{k+1}^{(N)}, \mathcal{G}_{k+1}^{(N)}, \tilde{\mathcal{G}}_{k+1}^{(N)} | k \in \mathbb{N}\}$. We are now in position to give the main result of this section.

Proposition 4.5. *The map $\bar{\mathcal{A}}_q \to \bar{\mathcal{A}}_q^{(N)}$ given by:*

$$\mathsf{W}_{-k} \mapsto \mathcal{W}_{-k}^{(N)}\,, \quad \mathsf{W}_{k+1} \mapsto \mathcal{W}_{k+1}^{(N)}\,, \quad \mathsf{G}_{k+1} \mapsto \mathcal{G}_{k+1}^{(N)}\,, \quad \tilde{\mathsf{G}}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1}^{(N)}$$

with (4.16)-(4.19) for $k \in \mathbb{N}$ and (3.7) is a surjective homomorphism.

Proof. Consider the image of (3.8) such that the generators in (3.5), (3.6) map to (4.16)-(4.19). For instance, one has:

$$\mathcal{W}_+(u) \mapsto \sum_{k \in \mathbb{N}} \mathcal{W}_{-k}^{(N)} U^{-k-1} = \sum_{k=0}^{N-1} \mathcal{W}_{-k}^{(N)} U^{-k-1} + \sum_{k=N}^{\infty} \mathcal{W}_{-k}^{(N)} U^{-k-1}\,. \tag{4.36}$$

Using (4.34):

$$\begin{aligned}
\sum_{k=N}^{\infty} \mathcal{W}_{-k}^{(N)} U^{-k-1} &= \sum_{p=0}^{\infty} \mathcal{W}_{-N-p}^{(N)} U^{-N-p-1} \\
&= -\frac{1}{c_N^{(N)}} \sum_{p=0}^{\infty} \sum_{k=0}^{N-1} c_k^{(N)} \mathcal{W}_{-k-p}^{(N)} U^{-N-p-1} - \frac{\bar{\epsilon}_+^{(N)}}{c_N^{(N)}} U^{-N-1} \\
&= -\frac{1}{c_N^{(N)}} \left(\sum_{k=0}^{N-1} c_k^{(N)} U^{k-N} \right) \mathcal{W}_+(u)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{c_N^{(N)}}\sum_{k=1}^{N-1}\sum_{p=0}^{k-1}c_k^{(N)}\mathcal{W}_{-p}^{(N)}U^{-N-p-1+k}-\frac{\bar{\epsilon}_+^{(N)}}{c_N^{(N)}}U^{-N-1}\\
=&-\frac{1}{c_N^{(N)}}\left(\sum_{k=0}^{N-1}c_k^{(N)}U^{k-N}\right)\mathcal{W}_+(u)\\
&+\frac{1}{c_N^{(N)}}\sum_{k=0}^{N-1}U^{-k-1}\mathcal{W}_{-k}^{(N)}\left(\sum_{p=k+1}^{N-1}c_p^{(N)}U^{p-N}\right)-\frac{\bar{\epsilon}_+^{(N)}}{c_N^{(N)}}U^{-N-1}\ .
\end{aligned}
$$

Replacing the last expression into (4.36) and using (4.28), (4.29) and (4.30), one gets:

$$f_0^{(N)}(u)\mathcal{W}_+(u)\mapsto\mathcal{W}_+^{(N)}(u)+u^{-2}q^{-1}\bar{\epsilon}_+^{(N)}\ .$$

Similarly, using (4.34), (4.35) one finds:

$$f_0^{(N)}(u)\mathcal{W}_-(u)\mapsto\mathcal{W}_-^{(N)}(u)+u^{-2}q^{-1}\bar{\epsilon}_-^{(N)}\ ,\quad f_0^{(N)}(u)\mathcal{G}_\pm(u)\mapsto\mathcal{G}_\pm^{(N)}(u)\ .$$

It follows $f_0^{(N)}(u)K(u)\mapsto K^{(N)}(u)$. Thus, the operators (4.16)-(4.19) for $k\in\mathbb{N}$ generate a quotient of the algebra $\bar{\mathcal{A}}_q$ by the relations (4.34), (4.35). □

Remark 4.6. For the specialization $q\to 1$ in (4.16)-(4.19), realizations of $\bar{\mathcal{A}}$ in $U(sl_2)^{\otimes N}$ are obtained.

5. The algebra $\bar{\mathcal{A}}_q$, alternating subalgebras of $U_q(\widehat{gl_2})$ and root vectors

Recall that the quantum affine Kac-Moody algebra $U_q(\widehat{sl_2})$ admits a Drinfeld second presentation denoted U_q^{Dr} with generators $\{\mathsf{x}_k^\pm,\mathsf{h}_\ell,\mathsf{K}^{\pm1},C^{\pm1/2}|k\in\mathbb{Z},\ell\in\mathbb{Z}\backslash\{0\}\}$ [30,18,41]. For $q\to 1$, this presentation specializes to the universal enveloping algebra of $\widehat{sl_2}$ with generators $\{x_k^\pm,h_k,c|k\in\mathbb{Z}\}$ - called the Cartan-Weyl presentation - see e.g. [18, top of page 566]. According to (2.47) (similarly (2.51)), a natural question concerns the interpretation of $\bar{\mathcal{A}}_q$ in terms of subalgebras of U_q^{Dr} (and similarly for $\bar{\mathcal{A}}$ in terms of subalgebras of $\widehat{sl_2}$). Although this problem may look complicated at first sight for $q\neq 1$, it is solved using the framework of Freidel-Maillet algebras combined with the results of Ding-Frenkel [28], as shown in this section. In this section, we fix $\mathbb{K}=\mathbb{C}$.

We start with the simplified situation $q\to 1$, see Definition 5.3 and Proposition 5.4.

5.1. *The algebra $\bar{\mathcal{A}}$ and 'alternating' subalgebras of $\widehat{gl_2}$*

The affine general Lie algebra $\widehat{gl_2}$ admits a presentation of Serre-Chevalley type and Cartan-Weyl type, closely related with the presentations of the affine Lie algebra $\widehat{sl_2}$ [44,37]. Consider the presentation of Cartan-Weyl type for $\widehat{gl_2}$. In the definition below, [., .] denotes the Lie bracket.

Definition 5.1. (Cartan-Weyl presentation $\widehat{gl_2}^{CW}$) The affine general Lie algebra $\widehat{gl_2}$ over $\mathbb{C}$ is generated by $\{x_k^\pm,\epsilon_{1,k},\epsilon_{2,k},c|k\in\mathbb{Z}\}$ subject to the relations:

$$\left[\epsilon_{i,k},\epsilon_{j,\ell}\right]=kc\delta_{i,j}\delta_{k+\ell,0}\ ,\tag{5.1}$$

$$\left[\epsilon_{1,k},x_\ell^\pm\right]=\pm x_{k+\ell}^\pm\ ,\tag{5.2}$$

$$\big[\epsilon_{2,k}, x_\ell^{\pm}\big] = \mp x_{k+\ell}^{\pm} \;, \tag{5.3}$$
$$\big[x_k^{+}, x_\ell^{-}\big] = \epsilon_{1,k+\ell} - \epsilon_{2,k+\ell} + \delta_{k+\ell,0}kc \;, \tag{5.4}$$
$$\big[x_k^{\pm}, x_{k\pm1}^{\pm}\big] = 0 \tag{5.5}$$

and c is central.

Note the automorphism θ such that:

$$\theta:\; x_k^{\pm} \mapsto x_k^{\mp} \;, \quad \epsilon_{1,k} \mapsto \epsilon_{2,k} \;, \quad \epsilon_{2,k} \mapsto \epsilon_{1,k} \;, \quad c \mapsto c \;. \tag{5.6}$$

Let

$$h_k = \epsilon_{1,k} - \epsilon_{2,k} \;. \tag{5.7}$$

The subalgebra generated by $\{x_k^{\pm}, h_k, c | k \in \mathbb{Z}\}$, denoted $\widehat{sl_2}^{CW}$, is isomorphic to the affine Lie algebra $\widehat{sl_2}$. The commutation relations are given by (5.4), (5.5) with (5.7) and

$$\big[h_k, h_\ell\big] = \delta_{k+\ell,0}2kc \;, \tag{5.8}$$
$$\big[h_k, x_\ell^{\pm}\big] = \pm 2x_{k+\ell}^{\pm} \;. \tag{5.9}$$

Recall the Serre-Chevalley presentation $\widehat{sl_2}^{SC}$ in Appendix A.

Remark 5.2. An isomorphism $\widehat{sl_2}^{SC} \to \widehat{sl_2}^{CW}$ is given by:

$$k_0 \mapsto -h_0 - c \;, \quad k_1 \mapsto h_0 \;, \quad e_1 \mapsto x_0^{+} \;, \quad e_0 \mapsto x_1^{-} \;, \quad f_1 \mapsto x_0^{-} \;,$$
$$f_0 \mapsto x_{-1}^{+} \;, \quad c \mapsto -c \;.$$

In view of (2.51), we now study the relation between $\bar{\mathcal{A}}$ and $\widehat{gl_2}$. Isomorphisms between certain subalgebras of $\widehat{gl_2}$ and $\bar{\mathcal{A}}$ can be identified through a direct comparison of the defining relations (5.1)-(5.5) and (2.38)-(2.41). However, although not necessary for $q = 1$, to prepare the analysis for $q \neq 1$ in the next section it is instructive to exhibit these isomorphisms using the FRT presentation of $U(\widehat{gl_2})$, which follows from $U(\widehat{sl_2})$'s one.[6]

Introduce the following classical (traceless) r-matrix for an indeterminate $z \neq 1$ associated with $\widehat{sl_2}$:

$$r(z) = \frac{1}{z-1}\begin{pmatrix} -\frac{1}{2}(z+1) & 0 & 0 & 0 \\ 0 & \frac{1}{2}(z+1) & -2 & 0 \\ 0 & -2z & \frac{1}{2}(z+1) & 0 \\ 0 & 0 & 0 & -\frac{1}{2}(z+1) \end{pmatrix} . \tag{5.10}$$

Note that $r_{12}(z) = -r_{21}(1/z) = -r_{12}(z)^{t_1 t_2}$. It satisfies the classical Yang-Baxter equation

$$[\, r_{13}(z_1/z_3) \;,\; r_{23}(z_2/z_3) \,] = [\, r_{13}(z_1/z_3) + r_{23}(z_2/z_3) \;,\; r_{12}(z_1/z_2) \,] \,. \tag{5.11}$$

For simplicity, we keep the same notation for the generators of $U(\widehat{sl_2})$ and $\widehat{sl_2}$. Defining:

6 We expect this presentation appears in the literature, although we could not find a reference. Here it is taken from [7].

$$T^+(z) = \begin{pmatrix} h_0/2 & 2x_0^- \\ 0 & -h_0/2 \end{pmatrix} + \sum_{k\geq 1} z^k \begin{pmatrix} h_k & 2x_k^- \\ 2x_k^+ & -h_k \end{pmatrix}, \tag{5.12}$$

$$T^-(z) = \begin{pmatrix} -h_0/2 & 0 \\ -2x_0^+ & h_0/2 \end{pmatrix} + \sum_{k\geq 1} z^{-k} \begin{pmatrix} -h_{-k} & -2x_{-k}^- \\ -2x_{-k}^+ & h_{-k} \end{pmatrix}, \tag{5.13}$$

one checks that the relations[7]

$$[T^\pm(z), c] = 0\,, \tag{5.14}$$

$$[T_1^\pm(z), T_2^\pm(w)] = [T_1^\pm(z) + T_2^\pm(w), r_{12}(z/w)]\,, \tag{5.15}$$

$$[T_1^+(z), T_2^-(w)] = [T_1^+(z) + T_2^-(w), r_{12}(z/w)] - 2c\, r'_{12}(z/w)z/w\,, \tag{5.16}$$

are equivalent to the relations (5.4), (5.5), (5.8), (5.9), where $[.,.]$ now denotes the usual commutator $[.,.]_1$. The FRT presentation for $U(\widehat{gl_2})$ is obtained from (5.12), (5.13) as follows. Define the 2×2 matrix

$$H^\pm(z) = \pm\left(\frac{1}{2}(\epsilon_{1,0} + \epsilon_{2,0}) + \sum_{k\geq 1} z^{\pm k}(\epsilon_{1,\pm k} + \epsilon_{2,\pm k})\right) I\!I\ .$$

The corresponding pair of Lax operators for $U(\widehat{gl_2})$ is given by $T^\pm_{\widehat{gl_2}}(z) = T^\pm(z) + H^\pm(z)$, and satisfy classical Yang-Baxter relations that follow from (5.14)-(5.16).

We now relate $\bar{\mathcal{A}}$ to certain subalgebras of $\widehat{gl_2}$ using the FRT presentation. By straightforward computation, it is found that

$$B(u) \mapsto \tilde{B}^-(u) = -T^-_{\widehat{gl_2}}(u^2) - t_0 \quad \text{or} \quad B(u) \mapsto \tilde{B}^+(u) = T^+_{\widehat{gl_2}}(u^{-2}) - t_0 \tag{5.17}$$

with $t_0 = diag(\epsilon_{1,0}, \epsilon_{2,0})$, satisfy the non-standard classical Yang-Baxter equation (3.34) for the identification $\bar{r}(u, v) = -r(u^2/v^2) - r_0$, where $r_0 = diag(1/2, -1/2, -1/2, 1/2)$. In particular, let us consider the first map in (5.17). Applying a similarity transformation:

$$B^-(u) = -M(u)\tilde{B}^-(u)^t M(u)^{-1} \quad \text{with} \quad M(u) = \begin{pmatrix} 0 & -u \\ 1 & 0 \end{pmatrix}$$

one finds for instance that

$$B^-(u) = \begin{pmatrix} 0 & 0 \\ 2u^{-1}x_0^+ & 0 \end{pmatrix} + \sum_{k\geq 1} u^{-2k} \begin{pmatrix} 2\epsilon_{1,-k} & 2ux_{-k}^- \\ 2u^{-1}x_{-k}^+ & 2\epsilon_{2,-k} \end{pmatrix} \tag{5.18}$$

satisfies (3.34) for the symmetric r-matrix (3.29). Similarly, from the second map in (5.17) one gets a second solution of (3.34) with (3.29):

$$B^+(u) = \begin{pmatrix} 0 & 2u^{-1}x_0^- \\ 0 & 0 \end{pmatrix} + \sum_{k\geq 1} u^{-2k} \begin{pmatrix} 2\epsilon_{1,k} & 2u^{-1}x_k^- \\ 2ux_k^+ & 2\epsilon_{2,k} \end{pmatrix}\ . \tag{5.19}$$

According to the structure of the matrices (5.18), (5.19) and the automorphism (5.6), different subalgebras that combine half of the positive/negative root vectors, together with half of the imaginary root vectors are now introduced.

[7] We denote $r'(z) = \frac{d}{d}r(z)$.

Definition 5.3.

$$\widehat{gl_2}^{\triangleright,\pm} = \{x_k^{\pm}, x_{k+1}^{\mp}, \epsilon_{1,k+1}, \epsilon_{2,k+1} | k \in \mathbb{N}\}\ , \tag{5.20}$$

$$\widehat{gl_2}^{\triangleleft,\pm} = \{x_{-k}^{\pm}, x_{-k-1}^{\mp}, \epsilon_{1,-k-1}, \epsilon_{2,-k-1} | k \in \mathbb{N}\}\ . \tag{5.21}$$

We call $\widehat{gl_2}^{\triangleright,\pm}$ and $\widehat{gl_2}^{\triangleleft,\pm}$ the right and left alternating subalgebras of $\widehat{gl_2}$. The subalgebra generated by $\{\epsilon_{1,0}, \epsilon_{2,0}, c\}$ is denoted $\widehat{gl_2}^{\diamond}$.

Inserting (5.18) (resp. (5.19)) into (3.34), the relations satisfied by the generators $\{x_{\pm k}^{\pm}, \epsilon_{1,\pm\ell}, \epsilon_{2,\pm\ell}\}$ are extracted. They are identical to the defining relations of the subalgebra $\widehat{gl_2}^{\triangleleft,+}$ (resp. $\widehat{gl_2}^{\triangleright,-}$). Thus, FRT presentations for $\widehat{gl_2}^{\triangleright,-}$ and $\widehat{gl_2}^{\triangleleft,+}$ are given respectively by (5.19), (5.18) satisfying (3.34). Applying the automorphism (5.6) to (5.19), (5.18), one gets the FRT presentations of $\widehat{gl_2}^{\triangleright,+}$ and $\widehat{gl_2}^{\triangleleft,-}$, respectively.

In particular, combining above results with those of Section 3 it follows:

Proposition 5.4. *There exists an algebra isomorphism* $\bar{\mathcal{A}} \to U(\widehat{gl_2}^{\triangleright,+})$ *(resp.* $\bar{\mathcal{A}} \to U(\widehat{gl_2}^{\triangleleft,-})$*) such that:*

$$\begin{aligned} &\mathsf{w}_{-k} \mapsto 2^{1-k} x_{k+1}^{-}\ , \quad \mathsf{w}_{k+1} \mapsto 2^{1-k} x_k^{+}\ , \quad \mathsf{g}_{k+1} \mapsto 2^{3-k}\epsilon_{1,k+1}\ ,\\ &\tilde{\mathsf{g}}_{k+1} \mapsto 2^{3-k}\epsilon_{2,k+1}\\ (\textit{resp.}\ &\mathsf{w}_{-k} \mapsto 2^{1-k} x_{-k}^{-}\ , \quad \mathsf{w}_{k+1} \mapsto 2^{1-k} x_{-k-1}^{+}\ , \quad \mathsf{g}_{k+1} \mapsto 2^{3-k}\epsilon_{1,-k-1}\ ,\\ &\tilde{\mathsf{g}}_{k+1} \mapsto 2^{3-k}\epsilon_{2,-k-1}\ . \end{aligned}$$

Proof. Identify $\theta(B^{\pm}(u))$ for (5.19), (5.18), to (3.33). □

Observe that the elements $\delta_{k+1}^{\pm} = \epsilon_{1,\pm(k+1)} + \epsilon_{2,\pm(k+1)}$ are central. If we denote $\mathrm{Z}^{\pm} = \{\delta_{k+1}^{\pm}\}_{k\in\mathbb{N}}$ and introduce the alternating subalgebras $\widehat{sl_2}^{\triangleright,+} = \{x_k^{+}, x_{k+1}^{-}, h_{k+1} | k \in \mathbb{N}\}$ (resp. $\widehat{sl_2}^{\triangleleft,-} = \{x_{-k}^{-}, x_{-k-1}^{+}, h_{-k-1} | k \in \mathbb{N}\}$), in addition to (2.51) one has the decompositions $\widehat{gl_2}^{\triangleright,+} = \widehat{sl_2}^{\triangleright,+} \oplus \mathrm{Z}^{+}$ and $\widehat{gl_2}^{\triangleleft,-} = \widehat{sl_2}^{\triangleleft,-} \oplus \mathrm{Z}^{-}$. So, the images become:

$$\mathsf{g}_{k+1} \mapsto 2^{2-k}(h_{k+1} + \delta_{k+1}^{+})\ , \quad \tilde{\mathsf{g}}_{k+1} \mapsto 2^{2-k}(-h_{k+1} + \delta_{k+1}^{+}) \tag{5.22}$$

$$(\text{resp. } \mathsf{g}_{k+1} \mapsto 2^{2-k}(h_{-k-1} + \delta_{k+1}^{-})\ , \quad \tilde{\mathsf{g}}_{k+1} \mapsto 2^{2-k}(-h_{-k-1} + \delta_{k+1}^{-}))\ . \tag{5.23}$$

In the next section, by analogy we use the Freidel-Maillet type presentation given in Section 3 to derive q-analogs of the isomorphisms of Proposition 5.4.

5.2. The algebra $\bar{\mathcal{A}}_q$ and 'alternating' subalgebras of $U_q(\widehat{gl_2})$

The Drinfeld second presentation [36,35] and FRT presentation of $U_q(\widehat{gl_2})$ [54,28] are first reviewed, see Definition 5.5 and Theorem 5.7. Then, 'alternating' subalgebras of $U_q(\widehat{gl_2})$ that can be viewed as q-analogs of (5.20), (5.21) are identified, see Definition 5.12. Using the Ding-Frenkel isomorphism [28], K-matrices $K^{\pm}(u)$ (or $K'^{+}(u)$) that satisfy the Freidel-Maillet type equation (3.9) (or (3.25)) are constructed using a dressing procedure, see Lemmas 5.15, 5.16 or 5.17. By a direct comparison of the K-matrix (3.8) (resp. (3.24)) to the K-matrix $K^{-}(u)$ (resp. $K'^{+}(u)$), explicit isomorphisms from $\bar{\mathcal{A}}_q$ to alternating subalgebras of $U_q(\widehat{gl_2})$ are derived, see Propositions 5.18, 5.20. For the first generators, Examples 5.19, 5.21 are given.

5.2.1. Drinfeld second presentation and FRT presentation of $U_q(\widehat{gl_2})$

In this subsection, we review some necessary material. For the quantum affine Kac-Moody algebra $U_q(\widehat{sl_2})$, there are two standard presentations: the *Drinfeld-Jimbo* presentation denoted U_q^{DJ} and the *Drinfeld (second) presentation* denoted U_q^{Dr}, see e.g. [22, p. 392], [34,27]. For $U_q(\widehat{gl_2})$, an analog of Drinfeld second presentation is known [36,35].

Definition 5.5. The quantum affine algebra $U_q(\widehat{gl_2})$ is isomorphic to the associative algebra over $\mathbb{C}(q)$ with generators $\{\mathsf{x}_k^\pm, \mathcal{E}_{1,\ell}, \mathcal{E}_{2,\ell}, \mathsf{K}^{\pm 1} | k \in \mathbb{Z}, \ell \in \mathbb{Z}\backslash\{0\}\}$, central elements $C^{\pm 1/2}$ and the following relations:

$$C^{1/2}C^{-1/2} = 1 , \quad \mathsf{K}\mathsf{K}^{-1} = \mathsf{K}^{-1}\mathsf{K} = 1 , \tag{5.24}$$

$$\big[\mathcal{E}_{i,k}, \mathcal{E}_{j,\ell}\big] = \frac{[k]_q}{k}\frac{C^k - C^{-k}}{q-q^{-1}}\delta_{i,j}\delta_{k+\ell,0} , \quad \mathsf{K}\mathcal{E}_{i,k} = \mathcal{E}_{i,k}\mathsf{K} , \tag{5.25}$$

$$\big[\mathcal{E}_{1,k}, \mathsf{x}_\ell^\pm\big] = \pm\frac{[k]_q}{k}C^{\mp|k|/2}q^{|k|/2}\mathsf{x}_{k+\ell}^\pm , \tag{5.26}$$

$$\big[\mathcal{E}_{2,k}, \mathsf{x}_\ell^\pm\big] = \mp\frac{[k]_q}{k}C^{\mp|k|/2}q^{-|k|/2}\mathsf{x}_{k+\ell}^\pm , \tag{5.27}$$

$$\mathsf{K}\mathsf{x}_k^\pm\mathsf{K}^{-1} = q^{\pm 2}\mathsf{x}_k^\pm , \tag{5.28}$$

$$\mathsf{x}_{k+1}^\pm\mathsf{x}_\ell^\pm - q^{\pm 2}\mathsf{x}_\ell^\pm\mathsf{x}_{k+1}^\pm = q^{\pm 2}\mathsf{x}_k^\pm\mathsf{x}_{\ell+1}^\pm - \mathsf{x}_{\ell+1}^\pm\mathsf{x}_k^\pm , \tag{5.29}$$

$$\big[\mathsf{x}_k^+, \mathsf{x}_\ell^-\big] = \frac{(C^{(k-\ell)/2}\psi_{k+\ell} - C^{-(k-\ell)/2}\phi_{k+\ell})}{q-q^{-1}} , \tag{5.30}$$

where the ψ_k and ϕ_k are defined by the following equalities of formal power series in the indeterminate z:

$$\psi(z) = \sum_{k=0}^{\infty}\psi_k z^{-k} = \mathsf{K}\exp\left((q-q^{-1})\sum_{k=1}^{\infty}\mathsf{h}_k z^{-k}\right) , \tag{5.31}$$

$$\phi(z) = \sum_{k=0}^{\infty}\phi_{-k} z = \mathsf{K}^{-1}\exp\left(-(q-q^{-1})\sum_{k=1}^{\infty}\mathsf{h}_{-k} z\right) , \tag{5.32}$$

where we denote:

$$\mathsf{h}_k = q^{|k|/2}\mathcal{E}_{1,k} - q^{-|k|/2}\mathcal{E}_{2,k} . \tag{5.33}$$

Note that there exists a q-analog of the automorphism (5.6) such that:

$$\theta : \ \mathsf{x}_k^\pm \mapsto \mathsf{x}_k^\mp , \quad \mathcal{E}_{1,k} \mapsto \mathcal{E}_{2,k} , \quad \mathcal{E}_{2,k} \mapsto \mathcal{E}_{1,k} , \quad \mathsf{K} \mapsto \mathsf{K} , \quad C \mapsto C^{-1}, \quad q \mapsto q^{-1} . \tag{5.34}$$

In addition, there exists an automorphism:

$$\nu : \ \mathsf{x}_k^+ \mapsto \mathsf{K}\mathsf{x}_k^+ , \quad \mathsf{x}_k^- \mapsto \mathsf{x}_k^-\mathsf{K}^{-1} , \quad \mathcal{E}_{1,k} \mapsto \mathcal{E}_{1,k} ,$$
$$\mathcal{E}_{2,k} \mapsto \mathcal{E}_{2,k}, \quad \mathsf{K} \mapsto \mathsf{K} , \quad C^{1/2} \mapsto C^{1/2} . \tag{5.35}$$

The associative subalgebra generated by $\{\mathsf{x}_k^\pm, \mathsf{h}_\ell, \mathsf{K}^{\pm 1}, C^{\pm 1/2} | k \in \mathbb{Z}, \ell \in \mathbb{Z}\backslash\{0\}\}$ is isomorphic to the quantum affine algebra $U_q(\widehat{sl_2})$, known in the literature as the Drinfeld second presentation U_q^{Dr}. The corresponding defining relations are given by (5.24), (5.28)-(5.30) and

$$\big[\mathsf{h}_k,\mathsf{h}_\ell\big]=\delta_{k+\ell,0}\frac{1}{k}\big[2k\big]_q\frac{C^k-C^{-k}}{q-q^{-1}}\,, \tag{5.36}$$

$$\big[\mathsf{h}_k,\mathsf{x}^\pm_\ell\big]=\pm\frac{1}{k}\big[2k\big]_q C^{\mp|k|/2}\mathsf{x}^\pm_{k+\ell}\,. \tag{5.37}$$

Remark 5.6. Recall the defining relations of U_q^{DJ} in Appendix A. An isomorphism $U_q^{DJ}\to U_q^{Dr}$ is given by (see e.g. [22, p. 393]):

$$K_0\mapsto C\mathsf{K}^{-1}\,,\quad K_1\mapsto \mathsf{K}\,,\quad E_1\mapsto \mathsf{x}_0^+\,,\quad E_0\mapsto \mathsf{x}_1^-\mathsf{K}^{-1}\,,\quad F_1\mapsto \mathsf{x}_0^-\,,\quad F_0\mapsto \mathsf{K}\mathsf{x}_{-1}^+\,. \tag{5.38}$$

Note that it is still an open problem to find the complete Hopf algebra isomorphism between U_q^{DJ} and U_q^{Dr}. Only partial information is known, see e.g. [21, Section 4.4].

Extending previous works [32,54], for the quantum affine Lie algebra of type A such as $U_q(\widehat{gl_n})$ a FRT presentation has been obtained in [28]. For type B, C, D, see [42,43]. The explicit isomorphism between the Drinfeld second presentation of $U_q(\widehat{gl_2})$ and FRT presentation given in [28] is now recalled. Define:

$$\tilde{R}(z)=\begin{pmatrix}1&0&0&0\\0&\frac{z-1}{zq-q^{-1}}&\frac{z(q-q^{-1})}{zq-q^{-1}}&0\\0&\frac{(q-q^{-1})}{zq-q^{-1}}&\frac{z-1}{zq-q^{-1}}&0\\0&0&0&1\end{pmatrix} \tag{5.39}$$

which satisfies the Yang-Baxter equation (3.3). Note that $\tilde{R}_{12}(z)=\tilde{R}_{21}^{t_1t_2}(z)$. The above R-matrix is related to the symmetric R-matrix (3.1) through the similarity transformations:

$$\begin{aligned}\left(\frac{u}{v}q-\frac{v}{u}q^{-1}\right)^{-1}R_{12}(u/v)&=\mathcal{M}(u)_1\mathcal{M}(v)_2\tilde{R}_{12}(u^2/v^2)\mathcal{M}(v)_2^{-1}\mathcal{M}(u)_1^{-1}\,, \\ &=\mathcal{M}(u)_1^{-1}\mathcal{M}(v)_2^{-1}\tilde{R}_{21}(u^2/v^2)\mathcal{M}(v)_2\mathcal{M}(u)_1 \quad \text{with}\end{aligned} \tag{5.40}$$

$$\mathcal{M}(u)=\begin{pmatrix}u^{-1/2}&0\\0&u^{1/2}\end{pmatrix}\,.$$

Theorem 5.7. *(see [54,28]) $U_q(\widehat{gl_2})$ admits a FRT presentation given by a unital associative algebra with generators $\{\mathsf{x}^\pm_k,\mathsf{k}^+_{j,-\ell},\mathsf{k}^-_{j,\ell},q^{\pm c/2}|k\in\mathbb{Z},\ell\in\mathbb{N},j=1,2\}$. The generators $q^{\pm c/2}$ are central and mutually inverse. Define:*

$$L^\pm(z)=\begin{pmatrix}\mathsf{k}_1^\pm(z)&\mathsf{k}_1^\pm(z)\mathsf{f}^\pm(z)\\ \mathsf{e}^\pm(z)\mathsf{k}_1^\pm(z)&\mathsf{k}_2^\pm(z)+\mathsf{e}^\pm(z)\mathsf{k}_1^\pm(z)\mathsf{f}^\pm(z)\end{pmatrix} \tag{5.41}$$

in terms of the generating functions in the indeterminate z:

$$\mathsf{e}^+(z)=(q-q^{-1})\sum_{k=0}^{\infty}q^{k(c/2-1)}\mathsf{x}^-_{-k}z^k\,,\quad \mathsf{e}^-(z)=-(q-q^{-1})\sum_{k=1}^{\infty}q^{k(c/2+1)}\mathsf{x}^-_{k}z^{-k}\,, \tag{5.42}$$

$$\mathsf{f}^+(z) = (q-q^{-1})\sum_{k=1}^{\infty} q^{-k(c/2+1)}\mathsf{x}^+_{-k}z^k\ , \quad \mathsf{f}^-(z) = -(q-q^{-1})\sum_{k=0}^{\infty} q^{-k(c/2-1)}\mathsf{x}^+_{k}z^{-k}\ , \tag{5.43}$$

$$\mathsf{k}^+_j(z) = \sum_{k=0}^{\infty} \mathsf{k}^+_{j,-k}z^k\ , \qquad \mathsf{k}^-_j(z) = \sum_{k=0}^{\infty} \mathsf{k}^-_{j,k}z^{-k}\ , \quad j=1,2\ . \tag{5.44}$$

The defining relations are the following:

$$\mathsf{k}^+_{i,0}\mathsf{k}^-_{i,0} = \mathsf{k}^-_{i,0}\mathsf{k}^+_{i,0} = 1\ , \tag{5.45}$$

$$\tilde{R}(z/w)\ (L^\pm(z)\otimes \mathbb{I})\ (\mathbb{I}\otimes L^\pm(w)) = (\mathbb{I}\otimes L^\pm(w))\ (L^\pm(z)\otimes \mathbb{I})\ \tilde{R}(z/w)\ , \tag{5.46}$$

$$\tilde{R}(q^c z/w)\ (L^+(z)\otimes \mathbb{I})\ (\mathbb{I}\otimes L^-(w)) = (\mathbb{I}\otimes L^-(w))\ (L^+(z)\otimes \mathbb{I})\ \tilde{R}(q^{-c}z/w)\ . \tag{5.47}$$

For (5.46), the expansion direction of $\tilde{R}(z/w)$ can be chosen in z/w or w/z, but for (5.47) the expansion direction is only in z/w. $U_q(\widehat{gl_2})$ is a Hopf algebra. The coproduct Δ is defined by:

$$\Delta(L^\pm(z)) = (L^\pm(zq^{\pm(1\otimes c/2)}))_{[1]}(L^\pm(zq^{\mp(c/2\otimes 1)}))_{[2]} \tag{5.48}$$

and its antipode is $S(L^\pm(z)) = L^\pm(z)^{-1}$.

Remark 5.8. The inverse quantum Lax operators (5.41) are [28, eq. (4.9)]:

$$(L^\pm(z))^{-1} = \begin{pmatrix} (\mathsf{k}^\pm_1(z))^{-1} + \mathsf{f}^\pm(z)(\mathsf{k}^\pm_2(z))^{-1}\mathsf{e}^\pm(z) & -\mathsf{f}^\pm(z)(\mathsf{k}^\pm_2(z))^{-1} \\ -(\mathsf{k}^\pm_2(z))^{-1}\mathsf{e}^\pm(z) & (\mathsf{k}^\pm_2(z))^{-1} \end{pmatrix}\ . \tag{5.49}$$

The explicit isomorphism between the FRT presentation of Theorem 5.7 and Drinfeld second presentation of $U_q(\widehat{gl_2})$ of Definition 5.5 is given in [36, Section 4]. Introduce the generating functions [28]:

$$\mathsf{x}^\pm(z) = \sum_{k\in\mathbb{Z}} \mathsf{x}^\pm_k z^{-k}\ . \tag{5.50}$$

In terms of (5.42), (5.43), one has:

$$\mathsf{x}^+(z) = (q-q^{-1})^{-1}\left(\mathsf{f}^+(q^{c/2+1}z) - \mathsf{f}^-(q^{-c/2+1}z)\right)\ ,$$

$$\mathsf{x}^-(z) = (q-q^{-1})^{-1}\left(\mathsf{e}^+(q^{-c/2+1}z) - \mathsf{e}^-(q^{c/2+1}z)\right)$$

and

$$C^{1/2} = q^{c/2}\ .$$

The generating functions $\{\mathsf{k}^\pm_i(z)\}_{i=1,2}$ are related with the generators $\{\mathcal{E}_{i,k}\}_{i=1,2}$ as follows [36, Section 4] (see also [35]):

$$\mathsf{k}^\pm_i(z) = \mathsf{k}^\pm_{i,0}\exp\left(\pm(q-q^{-1})\sum_{n=1}^{\infty} a_{i,\mp n}z^{\pm n}\right) \tag{5.51}$$

where the new generators

$$a_{1,m} = q^m\left(q^{|m|/2}\mathcal{E}_{1,m} - q^{-|m|/2}\mathcal{E}_{2,m}\right) + a_{2,m}\ , \tag{5.52}$$

$$a_{2,m} = q^{2m+|m|/2}\left(\frac{|m|}{m}\frac{\mathcal{E}_{1,m} + q^{|m|}\mathcal{E}_{2,m}}{(1+q^{2|m|})^{1/2}} + \mathcal{E}_{2,m}\right)\ , \tag{5.53}$$

are introduced. The generators $\mathsf{k}^{\pm}_{i,0}$ are such that $\big[\mathsf{k}^{\pm}_{i,0}, a_{j,m}\big] = \big[\mathsf{k}^{\epsilon}_{i,0}, \mathsf{k}^{\epsilon'}_{j,0}\big] = 0$ for any i, j and

$$\mathsf{k}^{-}_{2,0}(\mathsf{k}^{-}_{1,0})^{-1} = \mathsf{K}\,, \qquad \mathsf{k}^{+}_{2,0}(\mathsf{k}^{+}_{1,0})^{-1} = \mathsf{K}^{-1}\,. \tag{5.54}$$

The commutation relations of $U_q(\widehat{gl_2})$ presented in terms of the generators $\{a_{i,m}|i = 1, 2\}$ are given in [36, Section 4]. Although not reported here, for further analysis some of those are displayed in Appendix B.

In the context of the FRT presentation of $U_q(\widehat{gl_2})$ [28], the explicit exchange relations between the generating functions (5.42)-(5.44) are extracted from (5.46), (5.47) inserting (5.41). We refer the reader to [28, p. 288-292] for details. In particular, for the following analysis, we will need the asymptotics of some of the exchange relations displayed in [28]. Considering the limits $\mathsf{k}^{+}_{j}(0)$ and $\mathsf{k}^{-}_{j}(\infty)$ of (5.44), from [28, eqs. (4.24), (4.25), (4.40), (4.41)] one gets for instance:

$$\mathsf{k}^{\pm}_{1,0}\mathsf{e}^{\pm}(w)(\mathsf{k}^{\pm}_{1,0})^{-1} = q^{\mp 1}\mathsf{e}^{\pm}(w)\,, \quad \mathsf{k}^{\pm}_{1,0}\mathsf{f}^{\pm}(w)(\mathsf{k}^{\pm}_{1,0})^{-1} = q^{\pm 1}\mathsf{f}^{\pm}(w)\,, \tag{5.55}$$

$$(\mathsf{k}^{\pm}_{2,0})^{-1}\mathsf{e}^{\pm}(w)\mathsf{k}^{\pm}_{2,0} = q^{\mp 1}\mathsf{e}^{\pm}(w)\,, \quad (\mathsf{k}^{\pm}_{2,0})^{-1}\mathsf{f}^{\pm}(w)\mathsf{k}^{\pm}_{2,0} = q^{\pm 1}\mathsf{f}^{\pm}(w)\,, \tag{5.56}$$

and from [28, eqs. (4.13), (4.14), (4.17)] one gets:

$$\mathsf{k}^{\pm}_{i,0}\mathsf{k}^{\pm}_{j}(w) = \mathsf{k}^{\pm}_{j}(w)\mathsf{k}^{\pm}_{i,0}\,, \quad \mathsf{k}^{\pm}_{i,0}\mathsf{k}^{\mp}_{i}(w) = \mathsf{k}^{\mp}_{i}(w)\mathsf{k}^{\pm}_{i,0}\,, \quad i \neq j = 1, 2\,. \tag{5.57}$$

To prepare the discussion in further sections, the description of the known embedding $U_q(\widehat{sl_2}) \hookrightarrow U_q(\widehat{gl_2})$ is now recalled. First, central elements of $U_q(\widehat{gl_2})$ are constructed using the FRT presentation. Following [35, Section 2.6], define the generating functions:

$$y^{\pm}(z) = \mathsf{k}^{\mp}_{1}(q^{-1}z)\mathsf{k}^{\mp}_{2}(qz)\,. \tag{5.58}$$

By [28, eq. (4.17)], note that the ordering of the factors in (5.58) is irrelevant. Using the other exchange relations in [28], one finds $\big[y^{\pm}(z), \mathsf{e}^{\epsilon}(w)\big] = \big[y^{\pm}(z), \mathsf{f}^{\epsilon}(w)\big] = \big[y^{\pm}(z), \mathsf{k}^{\epsilon}_{1}(w)\big] = \big[y^{\pm}(z), \mathsf{k}^{\epsilon}_{2}(w)\big] = 0$ for $\epsilon = \pm$ and any z, w.

Proposition 5.9. *(see [35]) The coefficients of the generating function $y^{\pm}(z)$ are central elements of $U_q(\widehat{gl_2})$.*

Corollary 5.10. *The elements*

$$\mathsf{k}^{\mp}_{1,0}\mathsf{k}^{\mp}_{2,0} \quad \textit{and} \quad \gamma_m = q^m a_{1,m} + q^{-m} a_{2,m} \quad \textit{for} \quad m \in \mathbb{Z}^* \tag{5.59}$$

are central in $U_q(\widehat{gl_2})$.

Proof. Insert (5.51) into (5.58). Identify the coefficients of the resulting power series $y^{\pm}(z)$. □

Note that $\big[U_q(\widehat{gl_2}), y\big] = 0$ for $y = \mathsf{k}^{\pm}_{1,0}\mathsf{k}^{\pm}_{2,0}, \gamma_m$ can be independently checked using (5.52), (5.53) and the commutation relations (B.1)-(B.4).

Remark 5.11. In terms of the generators h_m (5.33) and central elements γ_m (5.59), the new generators $a_{1,m}, a_{2,m}$ entering in (5.51) decompose as:

$$a_{1,m} = \frac{q^m}{1+q^{2m}}(\mathsf{h}_m + \gamma_m)\,, \qquad a_{2,m} = \frac{q^m}{1+q^{-2m}}(-\mathsf{h}_m + q^{-2m}\gamma_m)\ . \tag{5.60}$$

It is known that the elements (5.59) and $C^{\pm 1/2}$ generate[8] the center of $U_q(\widehat{gl_2})$. The following arguments are described in [35] (see also [42]). Denote $\mathcal{C}$ the subalgebra generated by (5.59). One has the embedding $U_q^{Dr} \otimes \mathcal{C} \hookrightarrow U_q(\widehat{gl_2})$. Furthermore, define ${U'}_q^{Dr}$ as the extension of U_q^{Dr} by $q^{\pm 1/2}$, $K^{\pm 1/2}$, and define $\mathcal{C}'$ as the extension of $\mathcal{C}$ by $(\mathsf{k}^{\pm}_{1,0}\mathsf{k}^{\pm}_{2,0})^{1/2}$. Then, one has the inverse embedding $U_q(\widehat{gl_2}) \hookrightarrow {U'}_q^{Dr} \otimes \mathcal{C}'$. It follows that $U_q(\widehat{gl_2})$ and ${U'}_q^{Dr} \otimes \mathcal{C}$ are "almost" isomorphic. So, one has the tensor product decomposition:

$$U_q(\widehat{gl_2}) \cong U_q^{Dr} \otimes \mathcal{C} \,. \tag{5.61}$$

For more details, see e.g. [42, Proposition 2.3, Corollary 2.4]. The explicit isomorphism $\varphi_D : U_q(\widehat{gl_2}) \to U_q^{Dr} \otimes \mathcal{C}$ is constructed along these lines. In view of these comments, U_q^{Dr} can be considered as the quotient of the Drinfeld type presentation of $U_q(\widehat{gl_2})$ by the relations

$$y^{\pm}(z) = 1 \qquad \Longleftrightarrow \qquad k^{\pm}_{1,0}k^{\pm}_{2,0} = 1 \quad \text{and} \quad \gamma_m = 0 \quad \forall m \in \mathbb{Z}^* \,. \tag{5.62}$$

Below, we will use the surjective homomorphism $\gamma_D : U_q(\widehat{gl_2}) \to U_q^{Dr}$ using the presentation of Theorem 5.7. Recall (5.50) and (5.51). Using (5.60) and setting (5.62), for instance one has:

$$\gamma_D(q^{c/2}) \mapsto C^{1/2} \,, \qquad \gamma_D(\mathsf{x}^{\pm}(z)) \mapsto \mathsf{x}^{\pm}(z) \,, \tag{5.63}$$

$$\gamma_D(a_{1,m}) \mapsto \frac{1}{q^m + q^{-m}}\mathsf{h}_m \,, \qquad \gamma_D(a_{2,m}) \mapsto -\frac{q^{2m}}{q^m + q^{-m}}\mathsf{h}_m \,, \tag{5.64}$$

$$\gamma_D(\mathsf{k}^{\mp}_{2,0}(\mathsf{k}^{\mp}_{1,0})^{-1}) \mapsto \mathsf{K}^{\pm 1} \,. \tag{5.65}$$

Thus, the FRT presentation of $U_q(\widehat{sl_2})$ is obtained as a corollary of [28, Main Theorem]. It is given by the image of (5.46), (5.47) with (5.41) via γ_D.

5.2.2. *Alternating subalgebras $U_q(\widehat{gl_2})^{\triangleright,\pm}$ and $U_q(\widehat{gl_2})^{\triangleleft,\pm}$ and K-matrices*

By analogy with the analysis of previous section, we need to identify q-deformed analogs of the "classical" alternating subalgebras (5.20), (5.21). For instance, consider the elements:

$$C^{-k/2}\mathsf{K}^{-1}\mathsf{x}^+_k \,, \quad C^{(k+1)/2}\mathsf{x}^-_{k+1} \,, \quad \mathcal{E}_{1,k+1} \,, \quad \mathcal{E}_{2,k+1} \quad \text{for} \quad k \in \mathbb{N} \,. \tag{5.66}$$

Using the defining relations of $U_q(\widehat{gl_2})$, for $k, \ell \in \mathbb{N}$ one finds:

$$\begin{aligned}
&\big[\mathcal{E}_{i,k}, \mathcal{E}_{j,\ell}\big] = 0 \,, \\
&\big[\mathcal{E}_{1,k}, C^{-\ell/2}\mathsf{K}^{-1}\mathsf{x}^+_\ell\big] = \frac{[k]_q}{k} q^{k/2} C^{-(k+\ell)/2}\mathsf{K}^{-1}\mathsf{x}^+_{k+\ell} \,, \\
&\big[\mathcal{E}_{2,k}, C^{-\ell/2}\mathsf{K}^{-1}\mathsf{x}^+_\ell\big] = -\frac{[k]_q}{k} q^{-k/2} C^{-(k+\ell)/2}\mathsf{K}^{-1}\mathsf{x}^+_{k+\ell} \,, \\
&\big[\mathcal{E}_{1,k}, C^{(\ell+1)/2}\mathsf{x}^-_{\ell+1}\big] = -\frac{[k]_q}{k} q^{k/2} C^{(k+\ell+1)/2}\mathsf{x}^-_{k+\ell+1} \,, \\
&\big[\mathcal{E}_{2,k}, C^{(\ell+1)/2}\mathsf{x}^-_{\ell+1}\big] = \frac{[k]_q}{k} q^{-k/2} C^{(k+\ell+1)/2}\mathsf{x}^-_{k+\ell+1} \,.
\end{aligned}$$

[8] I thank N. Jing for communications on this point. Note that the analogs of $y^{\pm}(z)$ are known for higher rank affine Lie algebras of type A,B,C,D [35,42,43].

Furthermore, the relations (5.29) are left invariant by the action of $C^{-(k+\ell+1)/2}\mathsf{K}^{-2}$ for $(++)$ or the action of $C^{(k+\ell+1)/2}$ for $(--)$. Also, using (5.28), (5.30) one finds:

$$\big[C^{-k/2}\mathsf{K}^{-1}\mathsf{x}_k^+ \,, C^{(\ell+1)/2}\mathsf{x}_{\ell+1}^-\big] =$$
$$= \frac{1}{q-q^{-1}}\mathsf{K}^{-1}\psi_{k+\ell+1} + (q^2-1)\left(C^{-k/2}\mathsf{K}^{-1}\mathsf{x}_k^+\right)\left(C^{(\ell+1)/2}\mathsf{x}_{\ell+1}^-\right) .$$

According to (5.31), $\mathsf{K}^{-1}\psi_k$ only depends on h_k so it is a combination of $\mathcal{E}_{1,k}, \mathcal{E}_{2,k}$. Thus, we conclude that the elements (5.66) form a subalgebra of $U_q(\widehat{gl_2})$. Other subsets of elements are similarly considered, which form different subalgebras. It follows:

Definition 5.12.

$$U_q(\widehat{gl_2})^{\triangleright,\pm} = \{C^{\mp k/2}\mathsf{K}^{-1}\mathsf{x}_k^{\pm}, C^{\pm(k+1)/2}\mathsf{x}_{k+1}^{\mp}, \mathcal{E}_{1,k+1}, \mathcal{E}_{2,k+1} | k \in \mathbb{N}\} ,$$
$$U_q(\widehat{gl_2})^{\triangleleft,\pm} = \{C^{\mp k/2}\mathsf{x}_{-k}^{\pm}, C^{\pm(k+1)/2}\mathsf{x}_{-k-1}^{\mp}\mathsf{K}, \mathcal{E}_{1,-k-1}, \mathcal{E}_{2,-k-1} | k \in \mathbb{N}\} .$$

We call $U_q(\widehat{gl_2})^{\triangleright,\pm}$ and $U_q(\widehat{gl_2})^{\triangleleft,\pm}$ the right and left alternating subalgebras of $U_q(\widehat{gl_2})$. The subalgebra generated by $\{\mathsf{K}^{\pm1}, C^{\pm 1/2}\}$ is denoted $U_q(\widehat{gl_2})^{\diamond}$.

In each alternating subalgebra introduced above, the center is characterized as follows. Consider for instance $U_q(\widehat{gl_2})^{\triangleright,\pm}$. Its center is the subalgebra of $\mathcal{C}$ generated by some of the coefficients of the generating function $y^+(z)$ as defined in (5.58).

Remark 5.13. The center $\mathcal{C}^{\triangleright}$ (resp. $\mathcal{C}^{\triangleleft}$) of $U_q(\widehat{gl_2})^{\triangleright,\pm}$ (resp. $U_q(\widehat{gl_2})^{\triangleleft,\pm}$) is generated by γ_m (resp. γ_{-m}) with $m \in \mathbb{N}^*$.

For $U_q(\widehat{sl_2})$, it is known that given a certain ordering the elements $\{\mathsf{x}_k^{\pm}, \mathsf{h}_\ell, \mathsf{K}^{\pm1}, C^{\pm1/2} | k \in \mathbb{Z}, \ell \in \mathbb{Z}\backslash\{0\}\}$ generate a PBW basis, see [18, Proposition 6.1] with [19, Lemma 1.5]. According to (5.33), with a minor modification in the Cartan sector associated with the decomposition of h_k into $\mathcal{E}_{1,k}, \mathcal{E}_{2,k}$, a PBW basis for $U_q(\widehat{gl_2})$ is obtained. If one considers the subalgebra $U_q(\widehat{gl_2})^{\triangleright,+}$, let us choose the ordering:

$$C^{1/2}\mathsf{x}_1^- < C\mathsf{x}_2^- < \cdots < \mathcal{E}_{1,1} < \mathcal{E}_{1,2} < \cdots < \mathcal{E}_{2,1} < \mathcal{E}_{2,2} < \cdots < C^{-1/2}\mathsf{K}^{-1}\mathsf{x}_1^+ < \mathsf{K}^{-1}\mathsf{x}_0^+ ,$$

whereas for the subalgebra $U_q(\widehat{gl_2})^{\triangleleft,-}$ we choose the ordering:

$$\mathsf{x}_0^- < C^{1/2}\mathsf{x}_{-1}^- < \cdots < \mathcal{E}_{1,1} < \mathcal{E}_{1,2} < \cdots < \mathcal{E}_{2,1} < \mathcal{E}_{2,2} < \cdots < C^{-1}\mathsf{x}_{-2}^+\mathsf{K} < C^{-1/2}\mathsf{x}_{-1}^+\mathsf{K} .$$

It follows:

Proposition 5.14. *The vector space $U_q(\widehat{gl_2})^{\triangleright,+}$ (resp. $U_q(\widehat{gl_2})^{\triangleleft,-}$) has a linear basis consisting of the products $x_1x_2\cdots x_n$ ($n \in \mathbb{N}$) with $x_i \in U_q(\widehat{gl_2})^{\triangleright,+}$ (resp. $x_i \in U_q(\widehat{gl_2})^{\triangleleft,-}$) such that $x_1 \leq x_2 \leq \cdots \leq x_n$.*

Using the automorphism (5.34), PBW bases for $U_q(\widehat{gl_2})^{\triangleright,-}$ and $U_q(\widehat{gl_2})^{\triangleleft,+}$ are similarly obtained.

We now turn to the construction of K-matrices satisfying the Freidel-Maillet type equations (3.9) or (3.25), whose entries are formal power series in the elements of alternating subalgebras. Assume there exists a matrix $\tilde{K}^0$ with scalar entries and two quantum Lax operators $L(z), L^0$ such that the following relations hold ($\tilde{R}_{21}(z) = P\tilde{R}_{12}(z)P$):

$$\tilde{R}_{12}(z/w)\ \tilde{K}_1^0\ R^{(0)}\ \tilde{K}_2^0\ =\ \tilde{K}_2^0\ R^{(0)}\ \tilde{K}_1^0\ \tilde{R}_{21}(z/w)\ , \tag{5.67}$$
$$\tilde{R}_{12}(z/w)L_1(z)L_2(w) = L_2(w)L_1(z)\tilde{R}_{12}(z/w)\ , \tag{5.68}$$
$$\tilde{R}_{21}(z/w)(L^0)_1(L^0)_2 = (L^0)_2(L^0)_1\tilde{R}_{21}(z/w)\ , \tag{5.69}$$
$$(L^0)_1R^{(0)}L_2(w) = L_2(w)R^{(0)}(L^0)_1\ , \tag{5.70}$$
$$L_1(z)R^{(0)}(L^0)_2 = (L^0)_2R^{(0)}L_1(z)\ . \tag{5.71}$$

Adapting [57, Proposition 2], using the above relations one finds that:

$$\tilde{K}(z) \mapsto L(z)\tilde{K}^0L^0 \tag{5.72}$$

satisfies the following Freidel-Maillet type equation (for a non-symmetric R-matrix)

$$\tilde{R}_{12}(z/w)\ (\tilde{K}(z)\otimes I\!I)\ R^{(0)}\ (I\!I\otimes\tilde{K}(w))\ =\ (I\!I\otimes\tilde{K}(w))\ R^{(0)}\ (\tilde{K}(z)\otimes I\!I)\ \tilde{R}_{21}(z/w)\ . \tag{5.73}$$

An example built from the FRT presentation for $U_q(\widehat{gl_2})$ of Theorem 5.7 is obtained as follows. For the choices

$$L(z) \mapsto L^-(z) \quad \text{and} \quad L^0 \mapsto L^{-,0} = diag((\mathsf{k}_{2,0}^-)^{-1}, (\mathsf{k}_{1,0}^-)^{-1})\ , \tag{5.74}$$

eq. (5.68) holds and using the exchange relations (5.55)-(5.57) it is checked that eqs. (5.69)-(5.71) hold. Also, for the choice

$$\tilde{K}^0 = \begin{pmatrix} 0 & \frac{k_+(q+q^{-1})}{(q-q^{-1})} \\ \frac{k_-(q+q^{-1})}{(q-q^{-1})} & 0 \end{pmatrix} \tag{5.75}$$

it is checked that eq. (5.67) holds. It follows

$$\tilde{K}(z) \mapsto \tilde{K}^-(z) = L^-(z)\tilde{K}^0L^{-,0} \tag{5.76}$$

satisfies (5.73). Note that eq. (5.73) is left invariant under the transformation $(z, w) \mapsto (\lambda z, \lambda w)$ for any $\lambda \in \mathbb{C}^*$.

A solution of (3.9) associated with the symmetric R-matrix (3.1) is readily obtained using the similarity transformation (5.40).

Lemma 5.15. *The dressed K-matrix*

$$K^-(u) = \begin{pmatrix} u^{-1}\left(\frac{k_-(q+q^{-1})}{q-q^{-1}}\mathsf{k}_1^-(qu^2)\mathsf{f}^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1}\right) & \frac{k_+(q+q^{-1})}{q-q^{-1}}\mathsf{k}_1^-(qu^2)(\mathsf{k}_{1,0}^-)^{-1} \\ \frac{k_-(q+q^{-1})}{q-q^{-1}}\left(\mathsf{k}_2^-(qu^2)+\mathsf{e}^-(qu^2)\mathsf{k}_1^-(qu^2)\mathsf{f}^-(qu^2)\right)(\mathsf{k}_{2,0}^-)^{-1} & u\left(\frac{k_+(q+q^{-1})}{q-q^{-1}}\mathsf{e}^-(qu^2)\mathsf{k}_1^-(qu^2)(\mathsf{k}_{1,0}^-)^{-1}\right) \end{pmatrix}$$

satisfies the Freidel-Maillet type equation (3.9).

Proof. The K-matrix $\tilde{K}^-(z)$ defined by (5.76) satisfies (5.73). Applying the transformation (5.40) to (5.73) and defining

$$K^-(u) = \mathcal{M}(u)\tilde{K}^-(qu^2)\mathcal{M}(u)\ ,$$

the claim follows. □

Another solution of (3.9) is obtained as follows. Assume there exist two quantum Lax operators $L(z)$, L^0 such that the relations (5.70), (5.71) and

$$\tilde{R}_{21}(z/w)L_1(z)L_2(w) = L_2(w)L_1(z)\tilde{R}_{21}(z/w) \ ,$$
$$\tilde{R}_{12}(z/w)(L^0)_1(L^0)_2 = (L^0)_2(L^0)_1\tilde{R}_{12}(z/w)$$

are satisfied. It is straightforward to check that

$$L(z) \mapsto (L^+(z^{-1}))^{-1} \quad \text{and} \quad L^0 \mapsto L^{+,0} = diag(\mathsf{k}^+_{2,0}, \mathsf{k}^+_{1,0}) \tag{5.77}$$

obey the above set of relations. Then

$$\tilde{K}(z) \mapsto \tilde{K}^+(z) = L^{+,0}\tilde{K}^0(L^+(z^{-1})^{-1}) \tag{5.78}$$

satisfies (5.73). Using this result combined with the similarity transformation (5.40), it follows:

Lemma 5.16. *The dressed K-matrix*

$$K^+(u) =$$
$$\begin{pmatrix} u^{-1}\left(-\frac{k_+(q+q^{-1})}{q-q^{-1}}\mathsf{k}^+_{2,0}\mathsf{k}^+_2(1/qu^2)^{-1}\mathsf{e}^+(1/qu^2)\right) & \frac{k_+(q+q^{-1})}{q-q^{-1}}\mathsf{k}^+_{2,0}\mathsf{k}^+_2(1/qu^2)^{-1} \\ \frac{k_-(q+q^{-1})}{q-q^{-1}}\mathsf{k}^+_{1,0}\left(\mathsf{k}^+_1(1/qu^2)^{-1}+\mathsf{f}^+(1/qu^2)\mathsf{k}^+_2(1/qu^2)^{-1}\mathsf{e}^+(1/qu^2)\right) & u\left(-\frac{k_-(q+q^{-1})}{q-q^{-1}}\mathsf{k}^+_{1,0}\mathsf{f}^+(1/qu^2)\mathsf{k}^+_2(1/qu^2)^{-1}\right) \end{pmatrix}$$

satisfies the Freidel-Maillet type equation (3.9).

For completeness, a K-matrix satisfying (3.25) is now constructed along the same lines. To this aim, we consider the set of relations (5.67)-(5.71) with the substitution:

$$R^{(0)} \to (R^{(0)})^{-1} \ . \tag{5.79}$$

For the choices

$$L(z) \mapsto L^+(z) \quad \text{and} \quad L^0 \mapsto L'^{+,0} = diag((\mathsf{k}^+_{2,0})^{-1}, (\mathsf{k}^+_{1,0})^{-1}) \ , \tag{5.80}$$

one finds that

$$\tilde{K}(z) \mapsto \tilde{K}'^+(z) = L^+(z)\tilde{K}^0L'^{+,0} \tag{5.81}$$

satisfies (for the non-symmetric R-matrix)

$$\begin{aligned}&\tilde{R}_{12}(z/w)\ (\tilde{K}(z)\otimes I\!I)\ (R^{(0)})^{-1}\ (I\!I\otimes\tilde{K}(w))\\ &\quad = (I\!I\otimes\tilde{K}(w))\ (R^{(0)})^{-1}\ (\tilde{K}(z)\otimes I\!I)\ \tilde{R}_{21}(z/w)\ .\end{aligned} \tag{5.82}$$

Using (5.40), it follows:

Lemma 5.17. *The dressed K-matrix*

$$K'^+(u) =$$
$$\begin{pmatrix} u^{-1}\left(\frac{k_-(q+q^{-1})}{q-q^{-1}}\mathsf{k}^+_1(qu^2)\mathsf{f}^+(qu^2)(\mathsf{k}^+_{2,0})^{-1}\right) & \frac{k_+(q+q^{-1})}{q-q^{-1}}\mathsf{k}^+_1(qu^2)(\mathsf{k}^+_{1,0})^{-1} \\ \frac{k_-(q+q^{-1})}{q-q^{-1}}\left(\mathsf{k}^+_2(qu^2)+\mathsf{e}^+(qu^2)\mathsf{k}^+_1(qu^2)\mathsf{f}^+(qu^2)\right)(\mathsf{k}^+_{2,0})^{-1} & u\left(\frac{k_+(q+q^{-1})}{q-q^{-1}}\mathsf{e}^+(qu^2)\mathsf{k}^+_1(qu^2)(\mathsf{k}^+_{1,0})^{-1}\right) \end{pmatrix}$$

satisfies the Freidel-Maillet type equation (3.25).

The entries of the K-matrices are formal power series in the elements of the alternating subalgebras. Consider for instance the entry $(K^-(u))_{11}$. One has:

$$(K^-(u))_{11} = u^{-1}q\left(\frac{k_-(q+q^{-1})}{q^2-1}\mathsf{k}_1^-(qu^2)\underbrace{\mathsf{f}^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1}}_{=q(\mathsf{k}_{2,0}^-)^{-1}\mathsf{f}^-(qu^2)}\right) \quad \text{by} \quad (5.56)$$

$$= u^{-1}q\left(\frac{k_-(q+q^{-1})}{q-q^{-1}}\underbrace{\mathsf{k}_1^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1}}_{=\mathsf{K}^{-1}\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{1,n}(qu^2)^{-n}\right)}\mathsf{f}^-(qu^2)\right) \quad \text{by} \quad (5.51)\,.$$

Inserting (5.43), one gets:

$$(K^-(u))_{11} = uq\left(-k_-(q^2+1)\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{1,n}(qu^2)^{-n}\right)\right.$$
$$\left.\times\sum_{k=0}^{\infty}q^kC^{-k/2}\mathsf{K}^{-1}\mathsf{x}_k^+(qu^2)^{-k-1}\right)\,.$$

According to Definition 5.12 and (5.52), (5.53), we conclude $(K^-(u))_{11} \in U_q(\widehat{gl_2})^{\triangleright,+}\otimes\mathbb{C}[[u^2]]$. Studying similarly the other entries and repeating the same analysis for $K^+(u)$ and $K'^+(u)$, one finds:

$$(K^-(u))_{ij} \in U_q(\widehat{gl_2})^{\triangleright,+}\otimes\mathbb{C}[[u^2]]\,,\quad (K^+(u))_{ij} \in U_q(\widehat{gl_2})^{\triangleleft,-}\otimes\mathbb{C}[[u^2]]\,,$$
$$\text{and}\quad (K'^+(u))_{ij} \in U_q(\widehat{gl_2})^{\triangleleft,-}\otimes\mathbb{C}[[u^2]]\,. \tag{5.83}$$

5.2.3. Isomorphisms relating $\bar{\mathcal{A}}_q$ and the alternating subalgebras $U_q(\widehat{gl_2})^{\triangleright,\pm}$ and $U_q(\widehat{gl_2})^{\triangleleft,\pm}$

Recall the Freidel-Maillet type presentation for $\bar{\mathcal{A}}_q$ of Theorem 3.1. A direct comparison between the K-matrix (3.8) and the K-matrices $K^\pm(u)$ previously derived provides explicit maps from $\bar{\mathcal{A}}_q$ to the alternating subalgebras of $U_q(\widehat{gl_2})$. Recall the generating functions (3.5), (3.6) of the algebra $\bar{\mathcal{A}}_q$.

Proposition 5.18. *There exists an isomorphism from $\bar{\mathcal{A}}_q$ to $U_q(\widehat{gl_2})^{\triangleright,+}$ such that:*

$$\mathcal{W}_+(u) \mapsto -k_-(q^2+1)\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{1,n}(qu^2)^{-n}\right)$$
$$\times\sum_{k=0}^{\infty}q^kC^{-k/2}\mathsf{K}^{-1}\mathsf{x}_k^+(qu^2)^{-k-1}\,, \tag{5.84}$$

$$\mathcal{W}_-(u) \mapsto -k_+(q^{-2}+1)\left(\sum_{k=0}^{\infty}q^{k+1}C^{(k+1)/2}\mathsf{x}_{k+1}^-(qu^2)^{-k-1}\right)$$
$$\times\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{1,n}(qu^2)^{-n}\right)\,, \tag{5.85}$$

$$\mathcal{G}_+(u) \mapsto \frac{\bar\rho}{q-q^{-1}}\left(\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} a_{1,n}(qu^2)^{-n}\right)-1\right), \tag{5.86}$$

$$\begin{aligned}\mathcal{G}_-(u) \mapsto & \frac{\bar\rho}{q-q^{-1}}\left(\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} a_{2,n}(qu^2)^{-n}\right)-1\right) \\ & +\bar\rho(q-q^{-1})\sum_{k,\ell=0}^{\infty} q^{k+\ell+2}C^{(k-\ell+1)/2}\mathsf{x}^-_{k+1}\mathsf{K}^{-1} \\ & \times \exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} a_{1,n}(qu^2)^{-n}\right)\mathsf{x}^+_\ell(qu^2)^{-k-\ell-1}.\end{aligned} \tag{5.87}$$

Proof. As previously discussed, using (5.42), (5.43) and (5.51), the entries of $K^-(u)$ are power series in qu^2. Identifying (3.8) with $K^-(u)$, one gets the above homomorphism $\bar{\mathcal{A}}_q \to U_q(\widehat{gl_2})^{\triangleright,+}$ through identifying the generating functions. It remains to show that it is an isomorphism. Firstly, by analogy with $U_q(\widehat{sl_2})$ [22, page 289], $U(\widehat{gl_2})$ with defining relations (5.1)-(5.5) is known as the specialization $q \to 1$ of $U_q(\widehat{gl_2})$. So, the subalgebra $U_q(\widehat{gl_2})^{\triangleright,+}$ specializes to $U(\widehat{gl_2})^{\triangleright,+}$ with (5.20). Secondly, by Proposition 5.4 $\mathcal{A} \cong U(\widehat{gl_2})^{\triangleright,+}$. Thirdly, by Proposition 2.18 $\mathcal{A}$ is the specialization of $\bar{\mathcal{A}}_q$ at $q \to 1$, $\bar\rho \to 16$. All together, we conclude that the map above is an isomorphism. □

Identifying the leading terms of the power series, one finds for instance:

Example 5.19. The image in $U_q(\widehat{gl_2})^{\triangleright,+}$ of the first generators of $\bar{\mathcal{A}}_q$ is such that:

$$\mathsf{W}_0 \mapsto -k_-q\mathsf{K}^{-1}\mathsf{x}^+_0, \qquad \mathsf{W}_1 \mapsto -k_+C^{1/2}\mathsf{x}^-_1,$$

$$\mathsf{G}_1 \mapsto -\frac{\bar\rho}{q+q^{-1}}a_{1,1}, \qquad \tilde{\mathsf{G}}_1 \mapsto -\frac{\bar\rho}{q+q^{-1}}a_{2,1} + \frac{\bar\rho(q-q^{-1})}{(q+q^{-1})}q^2C^{1/2}\mathsf{x}^-_1\mathsf{K}^{-1}\mathsf{x}^+_0.$$

As a second example, recall the Freidel-Maillet type presentation (3.25) for $\bar{\mathcal{A}}_q$ with (3.24). In this case, the K-matrix (3.24) is compared with the K-matrix $K'^+(u)$ of Lemma 5.17. It follows

Proposition 5.20. *There exists an isomorphism from $\bar{\mathcal{A}}_q$ to $U_q(\widehat{gl_2})^{\triangleleft,-}$ such that:*

$$\begin{aligned}\mathcal{W}_+(u^{-1}q^{-1}) \mapsto & k_+(q+q^{-1})\sum_{k=0}^{\infty} q^{-k}C^{k/2}\mathsf{x}^-_{-k}(qu^2)^{k+1} \\ & \times \exp\left((q-q^{-1})\sum_{n=1}^{\infty} a_{1,-n}(qu^2)^n\right), \\ \mathcal{W}_-(u^{-1}q^{-1}) \mapsto & k_-(q+q^{-1})\exp\left((q-q^{-1})\sum_{n=1}^{\infty} a_{1,-n}(qu^2)^n\right) \\ & \times\left(\sum_{k=0}^{\infty} q^{-k+1}C^{-(k+1)/2}\mathsf{x}^+_{-k-1}\mathsf{K}(qu^2)^{k+1}\right),\end{aligned}$$

$$\mathcal{G}_+(u^{-1}q^{-1}) \mapsto \frac{\bar\rho}{q-q^{-1}}\left(\exp\left((q-q^{-1})\sum_{n=1}^{\infty} a_{1,-n}(qu^2)^n\right)-1\right),$$
$$\mathcal{G}_-(u^{-1}q^{-1}) \mapsto \frac{\bar\rho}{q-q^{-1}}\left(\exp\left((q-q^{-1})\sum_{n=1}^{\infty} a_{2,-n}(qu^2)^n\right)-1\right)$$
$$+\bar\rho(q-q^{-1})\sum_{k,\ell=0}^{\infty} q^{-k-\ell}C^{(k-\ell-1)/2}\mathsf{x}^-_{-k}$$
$$\times \exp\left((q-q^{-1})\sum_{n=1}^{\infty} a_{1,-n}(qu^2)^n\right)\mathsf{x}^+_{-\ell-1}\mathsf{K}(qu^2)^{k+\ell+1}.$$

Example 5.21. The image in $U_q(\widehat{gl_2})^{\triangleleft,-}$ of the first generators of $\bar{\mathcal{A}}_q$ is such that:

$$\mathsf{W}_0 \mapsto k_+\mathsf{x}^-_0\,, \qquad \mathsf{W}_1 \mapsto k_-qC^{-1/2}\mathsf{x}^+_{-1}\mathsf{K}\,,$$
$$\mathsf{G}_1 \mapsto \frac{\bar\rho}{q+q^{-1}}a_{1,-1}\,, \qquad \tilde{\mathsf{G}}_1 \mapsto \frac{\bar\rho}{q+q^{-1}}a_{2,-1} + \frac{\bar\rho(q-q^{-1})}{(q+q^{-1})}C^{-1/2}\mathsf{x}^-_0\mathsf{x}^+_{-1}\mathsf{K}\,.$$

So, the alternating subalgebra $U_q(\widehat{gl_2})^{\triangleright,+}$ (resp. $U_q(\widehat{gl_2})^{\triangleleft,-}$) admits a Freidel-Maillet type presentation given by the K-matrix $K^-(u)$ (resp. $K'^+(u)$) satisfying eq. (3.9) (resp. eq. (3.25)). Using the automorphism (5.34), a presentation for $U_q(\widehat{gl_2})^{\triangleright,-}$ (resp. $U_q(\widehat{gl_2})^{\triangleleft,+}$) can be obtained as well.

Finally, let us introduce the alternating subalgebras of U_q^{Dr}.

Definition 5.22.

$$U_q^{Dr,\triangleright,\pm} = \{C^{\mp k/2}\mathsf{K}^{-1}\mathsf{x}^\pm_k, C^{\pm(k+1)/2}\mathsf{x}^\mp_{k+1}, \mathsf{h}_k | k\in\mathbb{N}\}\,,$$
$$U_q^{Dr,\triangleleft,\pm} = \{C^{\mp k/2}\mathsf{x}^\pm_{-k}, C^{\pm(k+1)/2}\mathsf{x}^\mp_{-k-1}\mathsf{K}, \mathsf{h}_k | k\in\mathbb{N}\}\,.$$

We call $U_q^{Dr,\triangleright,\pm}$ and $U_q^{Dr,\triangleleft,\pm}$ the right and left alternating subalgebras of U_q^{Dr}. The subalgebra generated by $\{\mathsf{K}^{\pm1}, C^{\pm 1/2}\}$ is denoted $U_q^{Dr,\diamond}$.

As a corollary of (5.61) and Remark 5.13, one has the tensor product decompositions:

$$U_q(\widehat{gl_2})^{\triangleright,\pm} \cong U_q^{Dr,\triangleright,\pm}\otimes\mathcal{C}^\triangleright\,, \qquad U_q(\widehat{gl_2})^{\triangleleft,\pm} \cong U_q^{Dr,\triangleleft,\pm}\otimes\mathcal{C}^\triangleleft\,.$$

Recall (5.59).

Remark 5.23. The alternating subalgebra $U_q^{Dr,\triangleright,\pm}$ (resp. $U_q^{Dr,\triangleleft,\pm}$) is the quotient of $U_q(\widehat{gl_2})^{\triangleright,\pm}$ (resp. $U_q(\widehat{gl_2})^{\triangleleft,\pm}$) by the ideal generated from the relations $\{\gamma_{m+1}=0|\ \forall m\in\mathbb{N}\}$ (resp. $\{\gamma_{-m-1}=0|\ \forall m\in\mathbb{N}\}$).

We conclude this section with some comments. Using the isomorphism of Propositions 5.18, the image of the generating function $\Delta(u)\in\mathcal{Z}\otimes\mathbb{C}[[u^2]]$ defined by (3.28) gives a generating function in $\mathcal{C}^\triangleright\otimes\mathbb{C}[[u^2]]$ that looks more complicated than (5.58). In the context of FRT/Sklyanin/Freidel-Maillet type presentations, this is not surprising as $\Delta(u)$ and $y^\pm(qu^2)$ are built from different quantum determinants (see e.g. [57] for details). However, as a consistency check one can compare the leading orders of both power series. For instance, let us compute the image in $U_q(\widehat{gl_2})^{\triangleright,+}$ of Δ_1 given by (2.27) using the expressions of Example 5.19. After simplifications using (5.28), (5.30), it reduces to:

$$\Delta_1 = -\frac{2}{(q+q^{-1})^2}(qa_{1,1} + q^{-1}a_{2,1}) ,$$

which produces γ_1 (see (5.59) for $m = 1$).

5.3. *The comodule algebra homomorphism* $\delta : \bar{\mathcal{A}}_q \to U_q(\widehat{gl_2})^{\triangleright,+,0} \otimes \bar{\mathcal{A}}_q$

At the end of Section 2, a coaction map $\langle \mathsf{W}_0, \mathsf{W}_1\rangle \to U_q^{DJ,+,0} \otimes \langle \mathsf{W}_0, \mathsf{W}_1\rangle$ has been given. In this subsection, we study further the comodule algebra structure of $\bar{\mathcal{A}}_q$ using the FRT presentation of Theorem 5.7. A coaction formula for all the generators of $\bar{\mathcal{A}}_q$ is derived as follows. Recall the coproduct formulae for the quantum Lax operators (5.48). Take the K-matrix (5.76) and define the new K-matrix:

$$\Delta(L^-(z))\tilde{K}^0\Delta'(L^{-,0}) = (L^-(zq^{-(1\otimes\frac{c}{2})}))_{[1]}\left(\underbrace{(L^-(zq^{(\frac{c}{2}\otimes 1)}))_{[2]}\tilde{K}^0(L^{-,0})_{[2]}}_{=(\tilde{K}^-(zq^{(\frac{c}{2}\otimes 1)}))_{[2]}}\right)(L^{-,0}))_{[1]} . \tag{5.88}$$

By construction, it satisfies (5.73) for the non-symmetric R-matrix (5.39). Using the invariance of (5.73) under shifts in the ratio z/w, it follows that

$$\delta(\tilde{K}^-(z)) = (L^-(z))_{[1]}(\tilde{K}^-(z))_{[2]}(L^{-,0})_{[1]}$$

solves (5.73). More generally, starting from any K-matrix satisfying (5.73) and following standard arguments [57] different types of coactions can be constructed from the FRT presentation. Using (5.40), for a symmetric R-matrix for instance it yields to:

Proposition 5.24. *The Freidel-Maillet type presentation (3.9) of $\bar{\mathcal{A}}_q$ associated with R-matrix (3.1) and K-matrix (3.8) admits a comodule algebra structure. The left coaction is given by:*

$$\delta(K^-(u)) = \left(\mathcal{M}(u)L^-(qu^2)\mathcal{M}(u)^{-1}\right)_{[1]}(K^-(u))_{[2]}(L^{-,0})_{[1]} . \tag{5.89}$$

A right coaction map is similarly obtained by analogy with (5.78). Now, recall the generating functions (3.5), (3.6). Also, define $U_q(\widehat{gl_2})^{\triangleright,+,0}$ as the alternating subalgebra $U_q(\widehat{gl_2})^{\triangleright,+}$ extended by $\mathsf{K}, \mathsf{K}^{-1}$.

Lemma 5.25. *There exists a left comodule algebra homomorphism $\delta : \bar{\mathcal{A}}_q \to U_q(\widehat{gl_2})^{\triangleright,+,0} \otimes \bar{\mathcal{A}}_q$ such that:*

$$\delta(\mathcal{W}_+(u)) \mapsto (qu^2)^{-1}qk_1^-(qu^2)(k_{2,0}^-)^{-1}f^-(qu^2)$$
$$\otimes\left(\frac{1}{k_+(q+q^{-1})}\mathcal{G}_-(u) + \frac{k_-(q+q^{-1})}{(q-q^{-1})}\mathbb{I}\right) + k_1^-(qu^2)(k_{2,0}^-)^{-1}\otimes\mathcal{W}_+(u) ,$$

$$\delta(\mathcal{W}_-(u)) \mapsto q^{-1}e^-(qu^2)k_1^-(qu^2)(k_{1,0}^-)^{-1} \otimes \left(\frac{1}{k_-(q+q^{-1})}\mathcal{G}_+(u) + \frac{k_+(q+q^{-1})}{(q-q^{-1})}\mathbb{I}\right)$$
$$+\left(k_2^-(qu^2)(k_{1,0}^-)^{-1} + q^{-1}e^-(qu^2)k_1^-(qu^2)(k_{1,0}^-)^{-1}f^-(qu^2)\right) \otimes \mathcal{W}_-(u)\ ,$$
$$\delta(\mathcal{G}_+(u)) \mapsto k_1^-(qu^2)(k_{1,0}^-)^{-1} \otimes \mathcal{G}_+(u) + \frac{\bar{\rho}}{q-q^{-1}}\left(k_1^-(qu^2)(k_{1,0}^-)^{-1} - 1\right) \otimes \mathbb{I}$$
$$+ k_-(q+q^{-1})k_1^-(qu^2)(k_{1,0}^-)^{-1}f^-(qu^2) \otimes \mathcal{W}_-(u)\ ,$$
$$\delta(\mathcal{G}_-(u)) \mapsto \left(k_2^-(qu^2)(k_{2,0}^-)^{-1} + qe^-(qu^2)k_1^-(qu^2)(k_{2,0}^-)^{-1}f^-(qu^2)\right) \otimes \mathcal{G}_-(u)$$
$$+\frac{\bar{\rho}}{q-q^{-1}}\left(k_2^-(qu^2)(k_{2,0}^-)^{-1} + qe^-(qu^2)k_1^-(qu^2)(k_{2,0}^-)^{-1}f^-(qu^2) - 1\right) \otimes \mathbb{I}$$
$$+ k_+qu^2(q+q^{-1})e^-(qu^2)k_1^-(qu^2)(k_{2,0}^-)^{-1} \otimes \mathcal{W}_+(u)\ .$$

Proof. Compute (5.89) using (5.41), (5.40) and (3.8). Compare the entries of the resulting matrix to $\delta(K(u))$ with (3.8). □

Expanding the power series on both sides of the above equations using (3.5), (3.6), (5.42)-(5.44) with (5.51), (5.54), one gets the image by δ of the generators of $\bar{\mathcal{A}}_q$. This generalizes example (2.48).

Example 5.26.

$$\delta(\mathsf{W}_0) = -k_-q\mathsf{K}^{-1}\mathsf{x}_0^+ \otimes \mathbb{I} + \mathsf{K}^{-1} \otimes \mathsf{W}_0\ ,$$
$$\delta(\mathsf{W}_1) = -k_+C^{1/2}\mathsf{x}_1^- \otimes \mathbb{I} + \mathsf{K} \otimes \mathsf{W}_1\ .$$

If we define similarly $U_q(\widehat{gl_2})^{\triangleleft,-,0}$, note that a right coaction map $\bar{\mathcal{A}}_q \to \bar{\mathcal{A}}_q \otimes U_q(\widehat{gl_2})^{\triangleleft,-,0}$ can be derived along the same lines.

5.4. Relation between the generators of $\bar{A}_q$ and root vectors of $U_q(\widehat{sl_2})$

Let α_0, α_1 denote the simple roots of $\widehat{sl_2}$ and $\delta = \alpha_0 + \alpha_1$ be the minimal positive imaginary root. Let $\mathcal{R} = \{n\delta + \alpha_0, n\delta + \alpha_1, m\delta | n \in \mathbb{Z}, m \in \mathbb{Z}\backslash\{0\}\}$ be the root system of $\widehat{sl_2}$ and $\mathcal{R}^+ = \{n\delta + \alpha_0, n\delta + \alpha_1, m\delta | n \in \mathbb{N}, m \in \mathbb{N}\backslash\{0\}\}$ denote the positive root system. Recall $U_q^{DJ,+}$ denote the subalgebra generated by

$$E_{\alpha_1} \equiv E_1\ , \qquad E_{\alpha_0} \equiv E_0\ .$$

Using Lusztig's braid group action with generators T_0, T_1 such that $T_i : U_q(\widehat{sl_2}) \to U_q(\widehat{sl_2})$, root vectors $E_\beta \in U_q^{DJ,+}$ for every $\beta \in \mathcal{R}^+$ are defined [25,18]. Namely, for real root vectors $n\delta + \alpha_0, n\delta + \alpha_1$ with $n \in \mathbb{N}$ one chooses

$$E_{n\delta+\alpha_0} = (T_0\Phi)^n(E_0) \qquad \text{and} \qquad E_{n\delta+\alpha_1} = (T_0\Phi)^{-n}(E_1)\ .$$

Here $\Phi : U_q(\widehat{sl_2}) \to U_q(\widehat{sl_2})$ denotes the automorphism defined by:

$$\Phi(X_0) = X_1\ , \qquad \Phi(X_1) = X_0 \qquad \text{for} \qquad X = E, F, K^{\pm 1}\ .$$

For the imaginary root vectors, following [18,19] they are defined through the functional equation (note that $\left[E_{n\delta}, E_{m\delta}\right] = 0$ for any n, m):

$$\exp\left((q-q^{-1})\sum_{k=1}^{\infty}E_{k\delta}z^k\right)=1+(q-q^{-1})\sum_{k=0}^{\infty}\tilde{\psi}_k z^k \quad \text{with}$$
$$\tilde{\psi}_k=E_{k\delta-\alpha_1}E_{\alpha_1}-q^{-2}E_{\alpha_1}E_{k\delta-\alpha_1}\ .$$

For the negative root system denoted $\mathcal{R}^-$, similarly one defines the root vectors $F_\beta\in U_q^{DJ,-}$ for every $\beta\in\mathcal{R}^-$ [25]. The root vectors of $U_q^{DJ,+}$ and $U_q^{DJ,-}$ are related as follows (see [25, Theorem 2]):

$$F_\beta=\Omega(E_\beta)\qquad \forall\beta\in\mathcal{R}^+\ , \tag{5.90}$$

where Ω is an antiautomorphism of $U_q(\widehat{sl_2})$ such that

$$\Omega(E_i)=F_i\ ,\quad \Omega(F_i)=E_i\ ,\quad \Omega(K_i)=K_i^{-1}\quad \text{for } i=1,2\ ,$$
$$\Omega(C)=C^{-1}\ \text{and}\quad \Omega(q)=q^{-1}\ .$$

The explicit relation between Drinfeld generators and root vectors has been given in [18, Section 4] (see also [19, Lemma 1.5]). For $U_q(\widehat{sl_2})$, according to above definitions one has the correspondence:

$$\mathsf{x}_k^+=E_{k\delta+\alpha_1}\ ,\qquad \mathsf{x}_{k+1}^-=-C^{-k-1}\mathsf{K}E_{k\delta+\alpha_0}\ ,\qquad \mathsf{h}_{k+1}=C^{-(k+1)/2}E_{(k+1)\delta}\ , \tag{5.91}$$

$$\mathsf{x}_{-k}^-=F_{k\delta+\alpha_1}\ ,\qquad \mathsf{x}_{-k-1}^+=-F_{k\delta+\alpha_0}\mathsf{K}^{-1}C^{k+1}\ ,\qquad \mathsf{h}_{-k-1}=C^{(k+1)/2}F_{(k+1)\delta} \tag{5.92}$$

for $k\in\mathbb{N}$. From (5.37), one gets the following relations in terms of the root vectors [25, Section 3]:

$$\left[E_\delta,E_{k\delta+\alpha_1}\right]=(q+q^{-1})E_{(k+1)\delta+\alpha_1}\ ,\qquad \left[E_{k\delta+\alpha_0},E_\delta\right]=(q+q^{-1})E_{(k+1)\delta+\alpha_0}\ . \tag{5.93}$$

By induction, root vectors can be written as polynomials in E_1, E_0. For instance:

$$E_\delta=E_0E_1-q^{-2}E_1E_0\ ,$$
$$E_{\delta+\alpha_0}=\frac{1}{q+q^{-1}}\left(E_0^2E_1-(1+q^{-2})E_0E_1E_0+q^{-2}E_1E_0^2\right)\ ,$$
$$E_{\delta+\alpha_1}=\frac{1}{q+q^{-1}}\left(E_0E_1^2-(1+q^{-2})E_1E_0E_1+q^{-2}E_1^2E_0\right)\ .$$

We now relate the root vectors to the generators of alternating subalgebras. For convenience, compute the image of $U_q^{Dr,\triangleright,+}$ (see Definition 5.22) by the automorphism ν (5.35) using (5.33). This alternating subalgebra is denoted $(U_q^{Dr,\triangleright,+})^\nu$. Using (5.91), in terms of root vectors the generators of $(U_q^{Dr,\triangleright,+})^\nu$ read:

$$C^{-k/2}\mathsf{K}^{-1}\mathsf{x}_k^+\overset{\nu}{\mapsto}\qquad C^{-k/2}\mathsf{x}_k^+=C^{-k/2}E_{k\delta+\alpha_1}\ , \tag{5.94}$$

$$C^{(k+1)/2}\mathsf{x}_{k+1}^-\overset{\nu}{\mapsto}C^{(k+1)/2}\mathsf{x}_{k+1}^-\mathsf{K}^{-1}=-q^{-2}C^{-(k+1)/2}E_{k\delta+\alpha_0}\ , \tag{5.95}$$

$$\mathsf{h}_{k+1}\overset{\nu}{\mapsto}\qquad \mathsf{h}_{k+1}=C^{-(k+1)/2}E_{(k+1)\delta}\ . \tag{5.96}$$

As an application of Proposition 5.18, a set of functional relations relating the generators of $\bar{A}_q$ to the root vectors of $U_q^{DJ,+}$ (or similarly for $U_q^{DJ,-}$) is easily derived. Recall the surjective homomorphism $\gamma:\bar{\mathcal{A}}_q\to\bar{A}_q\cong U_q^{DJ,+}$, see (2.37). Consider the image of the generating functions (3.5), (3.6) via γ.

Proposition 5.27. *The isomorphism* $\iota : \bar{A}_q \to U_q^{DJ,+}$ *is such that:*

$$\gamma(\mathcal{W}_+(u)) \mapsto -k_- q(q+q^{-1}) \exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} \frac{1}{(q^n+q^{-n})} E_{n\delta}(qu^2)^{-n}\right) \times \sum_{k=0}^{\infty} q^k E_{k\delta+\alpha_1}(qu^2)^{-k-1}\,,$$

$$\gamma(\mathcal{W}_-(u)) \mapsto k_+ q^{-1}(q+q^{-1}) \left(\sum_{k=0}^{\infty} q^{k-1} E_{k\delta+\alpha_0}(qu^2)^{-k-1}\right) \times \exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} \frac{1}{(q^n+q^{-n})} E_{n\delta}(qu^2)^{-n}\right)\,,$$

$$\gamma(\mathcal{G}_+(u)) \mapsto \frac{\bar{\rho}}{(q-q^{-1})}\left(\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} \frac{1}{(q^n+q^{-n})} E_{n\delta}(qu^2)^{-n}\right) - 1\right)\,,$$

$$\gamma(\mathcal{G}_-(u)) \mapsto \frac{\bar{\rho}}{(q-q^{-1})}\left(\exp\left((q-q^{-1})\sum_{n=1}^{\infty} \frac{q^{2n}}{(q^n+q^{-n})} E_{n\delta}(qu^2)^{-n}\right) - 1\right) + \bar{\rho}(q-q^{-1}) \sum_{k,\ell=0}^{\infty} q^{k+\ell} E_{k\delta+\alpha_0} \times \exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} \frac{1}{(q^n+q^{-n})} E_{n\delta}(qu^2)^{-n}\right) E_{\ell\delta+\alpha_1}(qu^2)^{-k-\ell-1}\,.$$

Proof. Recall the surjective homomorphism γ_D which acts as (5.63)-(5.65). Consider its restriction to $U_q(\widehat{gl_2})^{\triangleright,+}$, applied to the r.h.s. of (5.84)-(5.87). The resulting expressions are now in $U_q^{Dr,\triangleright,+} \otimes \mathbb{C}[[u^2]]$. Then, studying the relations satisfied by $\{C^{-k/2}\mathsf{K}^{-1}\mathsf{x}_k^+, C^{(k+1)/2}\mathsf{x}_{k+1}^-, \mathsf{h}_{k+1}\}$ one finds that they are equivalent to the defining relations of the quotient of $U_q^{Dr,\triangleright,+}$ by $C=1$. Apply ν and use the identification given in the r.h.s. of (5.94)-(5.96) for $C=1$. □

Expanding the above power series, for instance set $k_+ \to q^2$, $k_- \to -q^{-1}$ (which gives $\bar{\rho} = -q(q+q^{-1})^2$) in these expressions. It follows:

$$W_0 \mapsto E_1\,, \qquad W_1 \mapsto E_0\,, \qquad G_1 \mapsto qE_\delta\,,$$
$$\text{(note that } \tilde{G}_1 \mapsto -q^3E_\delta + (q^3-q^{-1})E_0E_1)\,, \tag{5.97}$$

$$W_{-1} \mapsto \frac{1}{(q+q^{-1})^2}\left(-(q-q^{-1})E_\delta E_1 + (q^2+1)E_{\delta+\alpha_1}\right)\,, \tag{5.98}$$

$$W_2 \mapsto \frac{1}{(q+q^{-1})^2}\left(-(q-q^{-1})E_0 E_\delta + (q^2+1)E_{\delta+\alpha_0}\right)\,. \tag{5.99}$$

By construction, $(U_q^{Dr,\triangleright,+})^\nu/_{C=1} \cong U_q^{DJ,+}$. Using (5.90), an isomorphism $\bar{A}_q \to U_q^{Dr,\triangleleft,-}/_{C=1} \cong U_q^{DJ,-}$ is obtained from the above expressions.

The inverse of the map ι is now considered. We want to solve the positive root vectors $E_{n\delta+\alpha_1}, E_{n\delta+\alpha_0}, E_{n\delta}$ in terms of the generators W_{-k}, W_{k+1}, G_{k+1}. Although we do not have the explicit inverse map between generating functions, the images of the root vectors in $\bar{A}_q$ can be obtained recursively from Proposition 5.27. For instance,

$$E_1 \mapsto W_0 , \qquad E_0 \mapsto W_1 , \qquad E_\delta \mapsto q^{-1} G_1 W_0 , \tag{5.100}$$

$$E_{\delta+\alpha_1} \mapsto \frac{(q-q^{-1})}{(q+q^{-1})} q^{-2} G_1 W_0 + (1+q^{-2}) W_{-1} , \tag{5.101}$$

$$E_{\delta+\alpha_0} \mapsto \frac{(q-q^{-1})}{(q+q^{-1})} q^{-2} W_1 G_1 + (1+q^{-2}) W_2 . \tag{5.102}$$

Of course, these expressions could be given in a different ordering (see Theorem 2.15) using (2.3) for $k=0$.

Finally, let us point that several relations mixing both sets of generators can be readily obtained using (3.9) combined with Proposition 5.27. Namely, define the image of the K-matrix (3.8) by ι as:

$$K^\iota(u) = \iota(K(u)) . \tag{5.103}$$

Consider the pair of K-matrices $\{K(u), K^\iota(v)\}$. They satisfy:

$$R(u/v)\,(K(u)\otimes I\!I)\,R^{(0)}\,(I\!I\otimes K^\iota(v)) \;=\; (I\!I\otimes K^\iota(v))\,R^{(0)}\,(K(u)\otimes I\!I)\,R(u/v) \tag{5.104}$$

with (3.1). If we define the generating functions $\mathcal{W}_\pm(v)^{\iota,\gamma} = \iota\circ\gamma(\mathcal{W}_\pm(v))$, $\mathcal{G}_\pm(v)^{\iota,\gamma} = \iota\circ\gamma(\mathcal{G}_\pm(v))$, from (3.10)-(3.18) one extracts the set of functional relations associated with (5.104).

Remark 5.28. In [62, Section 11], the relation between Damiani's PBW basis and the alternating PBW basis for $\bar{A}_q$ has been studied in details within the framework of the q-shuffle algebra. In particular, various relations mixing both sets of generators have been obtained.

6. The alternating presentation of $U_q(\widehat{sl_2})$ from U_q^{DJ}

Define the alternating subalgebra $\bar{A}_q^{\triangleright} \cong (U_q^{Dr,\triangleright,+})^\nu/_{C=1}$ (resp. $\bar{A}_q^{\triangleleft} \cong U_q^{Dr,\triangleleft,-}/_{C=1}$) as the image of $\bar{A}_q$ by ι (resp. $\Omega\circ\iota$) (see Proposition 5.27) for $k_+ \to q^2$, $k_- \to -q^{-1}$. For convenience, let us denote the generators of $\bar{A}_q^{\triangleright}$ (resp. $\bar{A}_q^{\triangleleft}$) by $\{W_{-k}^{\triangleright}, W_{k+1}^{\triangleright}, G_{k+1}^{\triangleright}, \tilde{G}_{k+1}^{\triangleright} | k\in\mathbb{N}\}$ (resp. $\{W_{-k}^{\triangleleft}, W_{k+1}^{\triangleleft}, G_{k+1}^{\triangleleft}, \tilde{G}_{k+1}^{\triangleleft} | k\in\mathbb{N}\}$). According to (5.97):

$$W_0^{\triangleright} = E_1 , \quad W_1^{\triangleright} = E_0 , \quad W_0^{\triangleleft} = F_1 , \quad W_1^{\triangleleft} = F_0 . \tag{6.1}$$

Recall Proposition 2.14 and $U_q^{DJ,0} = \{K_0, K_1\}$. By construction, one gets the tensor product decomposition:

$$U_q(\widehat{sl_2}) \cong \bar{A}_q^{\triangleright} \otimes U_q^{DJ,0} \otimes \bar{A}_q^{\triangleleft} . \tag{6.2}$$

Moreover, by Theorem 2.15 an 'alternating' PBW basis for $U_q(\widehat{sl_2})$ readily follows from the results of [62,63].

Theorem 6.1. *A PBW basis for $U_q(\widehat{sl_2})$ is obtained by its alternating right and left generators*

$$\{W_{-k}^{\triangleright}\}_{k\in\mathbb{N}} , \quad \{G_{\ell+1}^{\triangleright}\}_{\ell\in\mathbb{N}} , \quad \{W_{n+1}^{\triangleright}\}_{n\in\mathbb{N}} , \quad \{W_{-r}^{\triangleleft}\}_{r\in\mathbb{N}} , \quad \{G_{s+1}^{\triangleleft}\}_{s\in\mathbb{N}} , \quad \{W_{t+1}^{\triangleleft}\}_{t\in\mathbb{N}}$$

and K_0, K_1 in any linear order $<$ that satisfies

$$W^{\triangleright}_{-k} < G^{\triangleright}_{\ell+1} < W^{\triangleright}_{n+1} < K_0 < K_1 < W^{\triangleleft}_{r+1} < G^{\triangleleft}_{s+1} < W^{\triangleleft}_{-t} , \qquad k, \ell, n, r, s, t \in \mathbb{N} .$$

The transition matrix from the alternating PBW basis of Theorem 6.1 to Damiani's PBW basis for $U_q(\widehat{sl_2})$ [25, Theorem 2] is determined by Proposition 5.27 and using the antiautomorphism Ω (5.90).

CRediT authorship contribution statement

Pascal Baseilhac: Conceptualization, Formal analysis, Methodology, Software, Validation.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Drinfeld-Jimbo presentation of $U_q(\widehat{sl_2})$

A.1. Drinfeld-Jimbo presentation U_q^{DJ}

Define the extended Cartan matrix $\{a_{ij}\}$ ($a_{ii} = 2$, $a_{ij} = -2$ for $i \neq j$). The quantum affine algebra $U_q(\widehat{sl_2})$ over $\mathbb{C}(q)$ is generated by $\{E_j, F_j, K_j^{\pm 1}\}$, $j \in \{0, 1\}$ which satisfy the defining relations

$$K_i K_j = K_j K_i , \quad K_i K_i^{-1} = K_i^{-1} K_i = 1 , \quad K_i E_j K_i^{-1} = q^{a_{ij}} E_j ,$$
$$K_i F_j K_i^{-1} = q^{-a_{ij}} F_j , \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

together with the q-Serre relations ($i \neq j$)

$$\left[E_i, \left[E_i, \left[E_i, E_j\right]_q\right]_{q^{-1}}\right] = 0 , \tag{A.1}$$
$$\left[F_i, \left[F_i, \left[F_i, F_j\right]_q\right]_{q^{-1}}\right] = 0 . \tag{A.2}$$

The product $C = K_0 K_1$ is the central element of the algebra. The Hopf algebra structure is ensured by the existence of a comultiplication Δ, antipode $\mathcal{S}$ and a counit $\mathcal{E}$ with

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i , \tag{A.3}$$
$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i ,$$
$$\Delta(K_i) = K_i \otimes K_i ,$$

$$\mathcal{S}(E_i) = -E_i K_i^{-1}\,, \quad \mathcal{S}(F_i) = -K_i F_i\,, \quad \mathcal{S}(K_i) = K_i^{-1} \qquad \mathcal{S}(1) = 1$$

and

$$\mathcal{E}(E_i) = \mathcal{E}(F_i) = 0\,, \quad \mathcal{E}(K_i) = 1\,, \qquad \mathcal{E}(1) = 1\,.$$

More generally, one defines the N-coproduct $\Delta^{(N)}: U_q(\widehat{sl_2}) \longrightarrow U_q(\widehat{sl_2}) \otimes \cdots \otimes U_q(\widehat{sl_2})$ as

$$\Delta^{(N)} \equiv (id \times \cdots \times id \times \Delta) \circ \Delta^{(N-1)} \tag{A.4}$$

for $N \geq 3$ with $\Delta^{(2)} \equiv \Delta$, $\Delta^{(1)} \equiv id$. Note that the opposite coproduct Δ' can be similarly defined with $\Delta' \equiv \sigma \circ \Delta$ where the permutation map $\sigma(x \otimes y) = y \otimes x$ for all $x, y \in U_q(\widehat{sl_2})$ is used.

A.2. Serre-Chevalley presentation $\widehat{sl_2}^{SC}$

In the definition below, [., .] denotes the Lie bracket. The affine algebra $\widehat{sl_2}$ over $\mathbb{C}$ is generated by $\{e_j, f_j, k_j\}$, $j \in \{0, 1\}$ which satisfy the defining relations

$$[k_i, k_j] = 0\,, \quad [k_i, e_j] = a_{ij} e_j\,, \quad [k_i, f_j] = -a_{ij} f_j\,, \quad [e_i, f_j] = \delta_{i,j} k_i$$

together with the Serre relations ($i \neq j$)

$$[e_i, [e_i, [e_i, e_j]]] = 0\,, \tag{A.5}$$

$$[f_i, [f_i, [f_i, f_j]]] = 0\,. \tag{A.6}$$

The sum $c = k_0 + k_1$ is the central element of the algebra.

For $U(\widehat{sl_2}^{SC})$, as usual $[x, y] \to xy - yx$.

Appendix B. Some defining relations of Gao-Jing presentation of $U_q(\widehat{gl_2})$

We refer the reader to [36, Theorem 4.16]. From Definition 5.5 and (5.52), (5.53), the following commutation relations are derived:

$$[a_{i,m}, a_{i,n}] = 0\,, i = 1, 2\,, \tag{B.1}$$

$$[a_{2,m}, a_{1,n}] = -\frac{[m]}{m}[mc]q^{-m}\delta_{m+n,0}\,, \tag{B.2}$$

$$[a_{1,m}, x_n^{\pm}] = \pm\frac{[m]}{m} q^{\mp|m|c/2} x_{m+n}^{\pm}\,, \tag{B.3}$$

$$[a_{2,m}, x_n^{\pm}] = \mp\frac{[m]}{m} q^{2m\mp|m|c/2} x_{m+n}^{\pm}\,. \tag{B.4}$$

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