p-Adic and Adelic Rational Dynamical Systems

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Abstract

In the framework of adelic approach we consider real and p-adic properties of dynamical system given by linear fractional map $f(x) = (a\,x + b)/(cx + d)$, where a, b, c, and d are rational numbers. In particular, we investigate behavior of this adelic dynamical system when fixed points are rational. It is shown that any of rational fixed points is p-adic indifferent for all but a finite set of primes. Only for finite number of p-adic cases a rational fixed point may be attractive or repelling. The present analysis is a continuation of the paper math-ph/0612058. Some possible generalizations are discussed.

1. Introduction

Many dynamical systems change their states in discrete time intervals by a mapping

$$f : X \longrightarrow X, \tag{1}$$

where X is the space of states and f describes how states $x \in X$ evolve in time. If the state at the time $t = 0$ is $x_0 \in X$ and $f^n = f \circ \cdots \circ f$ then after n iterations the state becomes

$$x_n = f^n(x_0). \tag{2}$$

X has usually some natural structures, e.g. hierarchies and distances between states. In physics of very complex systems X often displays a hierarchical structure, which implies that the classification of the states and their relationships should use ultrametric distances, and in particular p-adic ones. Recently much attention has been paid to some p-adic dynamical systems, since they have a lot of potential applications (for a review, see [1]).

Ground states of the mean field models for spin glasses have ultrametric structure [2]. Methods of p-adic analysis are applied to the investigation of replica symmetry breaking [3] and p-adic reformulation of the ultrametric structure of spin glasses [4].

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During the last two decades there have been many constructions of p-adic physical models. In particular, p-adic numbers have been successfully used in string theory, quantum mechanics and quantum cosmology (for a review, see [5], [6], [7] and [8]).

Presently it is not known any physical principle or phenomenon that would point out a particular prime number. Moreover, mathematical objects, e.g. such as the Riemann zeta function, are very significant when all primes are employed on the equal footing (see [9] for a recent example). Simultaneous use of the real and p-adic numbers, which make all possible completions of the field $\mathbb{Q}$ of rational numbers, is also of great importance in mathematics. Their use in the form of adeles is particularly effective in the arithmetic theory of algebraic groups. Adelic models of physical systems contain real and p-adic submodels as parts of a whole (see, e.g. [10]). They give more information on a dynamical system than real and p-adic treatments separately. Since 1987 adelic models have been constructed and investigated in string theory, quantum mechanics, quantum cosmology (for a review, see [5], [6], [7] and [8]) and in some other fields of modern mathematical physics (see, e.g. [11]).

In the recent article [12] we started p-adic and adelic investigation of dynamical systems, which evolution is governed by linear fractional transformations

$$f(x) = \frac{ax+b}{cx+d}\,, \tag{3}$$

where $a, b, c, d \in \mathbb{Q}$ with conditions $x \neq -\frac{d}{c}$, $c \neq 0$ and $ad - bc = 1$.

Some p-adic properties of this kind of dynamical systems were explored in [13], where parameters $a\,, b\,, c\,, d \in \mathbb{C}_p$. It is worth noting that taking physical parameters to be rational numbers gives a possibility to treat real and p-adic properties simultaneously and on the equal footing.

Linear fractional transformations (Möbius transformations) (3) and related $SL(2,\mathbb{C})$, $SL(2,\mathbb{C}_p)$ groups, and their subgroups, have very rich mathematical structures. They also have important applications in many parts of mathematical and theoretical physics (see, e.g. [5], [14] [15] and references therein).

Sec. 2 contains a very brief introductory review of p-adic numbers and adeles. In Sec. 3 some new results of the above linear fractional dynamics (3) are presented. Some general remarks, including possible generalizations, are stated in Sec. 4.

2. p-Adic Numbers and Adeles

Rational numbers are significant in physics as well as in mathematics. Physical significance comes from the fact that a result of any measurement is a rational number. One can obtain the field $\mathbb{R}$ of real numbers from $\mathbb{Q}$ by employing the absolute value, which is an example of the norm (valuation) on $\mathbb{Q}$. In addition to the absolute value, for which we use usual arithmetic notation $|\cdot|_\infty$, one can introduce on $\mathbb{Q}$ a norm with respect to each prime number p. Note that any rational number can be uniquely written as $x = p^\nu \frac{m}{n}$, where p, m, n are mutually prime and $\nu \in \mathbb{Z}$. Then by definition p-adic norm (or, in other words, p-adic absolute value) is $|x|_p = p^{-\nu}$ if $x \neq 0$ and $|0|_p = 0$. One can verify that $|\cdot|_p$ satisfies the strong triangle inequality, i.e. $|x+y|_p \leq \max(|x|_p\,, |y|_p)$. Thus p-adic norms belong to the class of non-Archimedean (ultrametric) norms. According to the Ostrowski theorem any nontrivial norm on $\mathbb{Q}$ is equivalent either to the $|\cdot|_\infty$ or to one of the $|\cdot|_p$. One can easily show that $|m|_p \leq 1$ for any $m \in \mathbb{Z}$ and any prime p. The p-adic norm is a measure of divisibility of the integer m by prime p: the more divisible, the p-adic smaller. Using Cauchy sequences of rational numbers one can make completions of $\mathbb{Q}$ to obtain $\mathbb{R} \equiv \mathbb{Q}_\infty$ and the fields $\mathbb{Q}_p$ of p-adic numbers using norms $|\cdot|_\infty$ and $|\cdot|_p$, respectively. p-Adic completion of $\mathbb{N}$ gives the ring $\mathbb{Z}_p = \{x \in \mathbb{Q}_p :\ |x|_p \leq 1\}$ of p-adic integers. Denote by $\mathbb{U}_p = \{x \in \mathbb{Q}_p :\ |x|_p = 1\}$ multiplicative group of p-adic units.

Any p-adic number $x \in \mathbb{Q}_p$ can be presented in the unique way (unlike real numbers) as the sum of p-adic convergent series of the form

$$x = p^\nu\,(x_0 + x_1 p + \cdots + x_n p^n + \cdots), \quad \nu \in \mathbb{Z}, \quad x_n \in \{0, 1, \cdots, p-1\}. \quad (4)$$

If $\nu \geq 0$ in (4), then $x \in \mathbb{Z}_p$. When $\nu = 0$ and $x_0 \neq 0$ one has $x \in \mathbb{U}_p$.

p-Adic metric $d_p(x,y) = |x-y|_p$ satisfies all necessary properties of metric with strong triangle inequality, i.e. $d_p(x,y) \leq \max(\,d_p(x,z),\, d_p(z,y)\,)$ which is of the non-Archimedean (ultrametric) form. Using this metric, $\mathbb{Q}_p$ becomes an ultrametric space with p-adic topology. A closed p-adic ball (disk) is $B_p(r,\,\xi) = \{x \in \mathbb{Q}_p\,:\,|x-\xi|_p \leq r\}$, where $r = p^m$, $m \in \mathbb{Z}$, is radius with discrete values, and ξ is a center of the ball. Analogously, an open ball (disk) is $B_p^-(r,\,\xi) = \{x \in \mathbb{Q}_p\,:\,|x-\xi|_p < r\}$. Sphere of radius ρ and center ξ is $S_p(\rho,\,\xi) = \{x \in \mathbb{Q}_p\,:\,|x-\xi|_p = \rho\}$. Any ball can be regarded as closed as open. Any point $x \in B_p(r,\xi)$ can be treated as center of the same ball. Note the following connections: $S_p(\rho,\,\xi) = B_p(\rho,\,\xi) \setminus B_p^-(\rho,\,\xi)$, $B_p(r,\,\xi) = \bigcup_{\rho \leq r} S_p(\rho,\,\xi)$.

It is worth noting that $x \in S_p(\rho,\,\xi)$ has the form

$$x = \xi + y = p^k\,(\xi_0 + \xi_1\,p + \xi_2\,p^2 + \cdots) + p^l\,(y_0 + y_1\,p + y_2\,p^2 + \cdots),$$

where $|y|_p = p^{-l} = \rho$. For $|x|_p$ there are the following possibilities: (i) $|x|_p = \rho > |\xi|_p$, if $k > l$ (ii) $|x|_p = |\xi|_p > \rho$, if $k < l$ (iii) $|x|_p = |\xi|_p = \rho$ if $k = l$ and $\xi_0 + y_0 \neq p$, and (iv) $|x|_p < |\xi|_p = \rho$ if $k = l$ and $\xi_0 + y_0 = p$. When ξ is fixed then $|x|_p$ depends on ρ.

For more details about p-adic numbers and their algebraic extensions, see, e.g. [16].

To consider real and p-adic numbers simultaneously and on the equal footing one uses concept of adeles. An adele x (see, e.g. [17]) is an infinite sequence

$$x = (x_\infty\,, x_2\,, x_3\,, \cdots, x_p\,, \cdots), \quad (5)$$

where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that for all but a finite set $\mathcal{P}$ of primes p one has $x_p \in \mathbb{Z}_p$. Componentwise addition and multiplication make the ring structure of the set $\mathbb{A}$ of all adeles, which is the union of restricted direct products in the following form:

$$\mathbb{A} = \bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P}), \qquad \mathbb{A}(\mathcal{P}) = \mathbb{R} \times \prod_{p\in\mathcal{P}} \mathbb{Q}_p \times \prod_{p\notin\mathcal{P}} \mathbb{Z}_p \,. \tag{6}$$

A multiplicative group of ideles $\mathbb{A}^*$ is a subset of $\mathbb{A}$ with elements $x = (x_\infty\,, x_2\,, x_3\,, \cdots, x_p\,, \cdots)$, where $x_\infty \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$ and $x_p \in \mathbb{Q}_p^* = \mathbb{Q}_p\setminus\{0\}$ with the restriction that for all but a finite set $\mathcal{P}$ one has that $x_p \in \mathbb{U}_p$. Thus the whole set of ideles is

$$\mathbb{A}^* = \bigcup_{\mathcal{P}} \mathbb{A}^*(\mathcal{P}), \qquad \mathbb{A}^*(\mathcal{P}) = \mathbb{R}^* \times \prod_{p\in\mathcal{P}} \mathbb{Q}_p^* \times \prod_{p\notin\mathcal{P}} \mathbb{U}_p \,. \tag{7}$$

A principal adele (idele) is a sequence $(x, x, \cdots, x, \cdots) \in \mathbb{A}$, where $x \in \mathbb{Q}$ $\quad (x \in \mathbb{Q}^* = \mathbb{Q}\setminus\{0\})$. $\mathbb{Q}$ and $\mathbb{Q}^*$ are naturally embedded in $\mathbb{A}$ and $\mathbb{A}^*$, respectively.

3. Linear Fractional Dynamical Systems

Let us first recall some basic notions from the theory of dynamical systems [1] valid for mapping (1) and its iterations (2) at real and p-adic spaces. Let us introduce an index v to denote real ($v=\infty$) and p-adic ($v=p$) cases simultaneously. A *fixed point* ξ is a solution of the equation $f(\xi)=\xi$. If there exists a neighborhood $V_v(\xi)$ of the fixed point ξ such that for any point $x_n \in V_v(\xi)$, $x_n \neq \xi$, holds: *(i)* $|x_n-\xi|_v < |x_{n-1}-\xi|_v$, i.e. $\lim_{n\to\infty} x_n = \xi$, then ξ is called an *attractor*; *(ii)* $|x_n-\xi|_v > |x_{n-1}-\xi|_v$, then ξ is a *repeller*; and *(iii)* $|x_n-\xi|_v = |x_{n-1}-\xi|_v$, then ξ is an *indifferent point.* Basin of attraction $A_v(\xi)$ of an attractor ξ is the set

$$A_v(\xi) = \{x_0 \in \mathbb{Q}_v : \lim_{n\to\infty} x_n \to \xi\}. \tag{8}$$

A Siegel disk is called an open ball $V_v(r,\xi)$ if every sphere $S_v(\rho,\xi)$, $\rho<r$ is an invariant sphere of the mapping $f(x)$, i.e. if an initial point $x_0 \in S_v(\rho,\xi)$ then all iterations x_n also belong to $S_v(\rho,\xi)$. The union of all Siegel disks $V_v(r,\xi)$ with the same center ξ is called a maximum Siegel disk and denoted by $SI_v(\xi)$. Invariant spheres $S_v(\rho,\xi_i)$ of Siegel disks $V_v(r,\xi_i)$ for indifferent fixed points ξ_i have to satisfy $|x_n-\xi_i|_v = |x_0-\xi_i|_v = \rho_v < r_v$ for all $n \in \mathbb{N}$.

When the mapping (1) has the first derivative in the fixed point ξ then one can use the following properties: $|f'(\xi)|_v < 1$ - attractor, $|f'(\xi)|_v > 1$ - repeller and $|f'(\xi)|_v = 1$ - indifferent point.

We shall mainly consider rational dynamical systems given by map (3) which is isomorphic to the matrix

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det F = 1\,, \tag{9}$$

where $a, b, c, d \in \mathbb{Q}$ and with condition $ad - bc = 1$. The corresponding group of matrices F, with $\det F = 1$, is $SL(2, \mathbb{Q})$.

Recall that iteration (2) may have periodic points. A point x_0 is called a periodic point if there exists k such that $f^k(x_0) = x_0$. The smallest such k is the period of x_0 and then x_0 is called a k-periodic point. Note that fixed points are 1-periodic points. Iteration (2) can be periodic for all points $x_0 \in X$. Our map (3) generates periodicity of a period k when related matrix (9) satisfies $F^k = I$, where I is 2×2 unit matrix. For example, if $d = -a$ and $a^2 + b\,c = 1$ one has $k = 2$ periodicity.

It is worth mentioning that the map (3) preserves the cross-ratio

$$\frac{(\alpha_1 - \alpha_3)\,(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_4)\,(\alpha_2 - \alpha_3)} = \frac{(f(\alpha_1) - f(\alpha_3))\,(f(\alpha_2) - f(\alpha_4))}{(f(\alpha_1) - f(\alpha_4))\,(f(\alpha_2) - f(\alpha_3))} \tag{10}$$

between any different points $x = \alpha_1\,, \alpha_2\,, \alpha_3\,, \alpha_4\,.$

To be (3) an adelic system, it must be satisfied $|f_p(x_p)|_p \leq 1$ in

$$f_{\mathbb{A}}(x) = \Big(f_\infty(x_\infty)\,, f_2(x_2)\,, f_3(x_3)\,, \cdots, f_p(x_p)\,, \cdots\Big)\,, \quad x \in \mathbb{A}\,, \tag{11}$$

for all but a finite set $\mathcal{P}$ of prime numbers p. In other words, there has to be a prime number q such that $|f_p(x_p)|_p \leq 1$ for all $p > q$. It is shown in [12] that function (3) satisfies adelic behavior.

For the function (3) we find the following two fixed points:

$$\xi_{1,2} = \frac{a - d \pm \sqrt{(a-d)^2 + 4ad - 4}}{2c} = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c} \tag{12}$$

with condition $ad - bc = 1$ and properties

$$f(\xi_1) \cdot f(\xi_2) = \xi_1 \cdot \xi_2 = -\frac{b}{c}\,, \qquad f'(\xi_1) \cdot f'(\xi_2) = 1. \tag{13}$$

For the fixed points it is important to notice that if the point ξ_1 is attractive ($|f'(\xi_1)|_v < 1$) then the point ξ_2 is repelling ($|f'(\xi_2)|_v > 1$) and vice versa. The indifferent fixed points always emerge in the pair. These facts obviously follow from the relation (13). Generally, these points belong to $\mathbb{C}$ in real case and $\mathbb{C}_p$ in p-adic case, and their analysis will be done elsewhere.

We are interested in rational fixed points because they simultaneously belong to real and p-adic numbers. Fixed rational points (12) for the dynamical system (3) have been investigated in [12] for the following four particular cases: (**A**) $b = 0$, (**B**) $b = c$, $d = a$, (**C**) $d = -a + 2$ and (**D**) $d = -a - 2$. Basins of attraction, the Siegel disks and adelic trajectories are examined for the case (**A**).

In this paper we continue investigation started in [12]. First of all let us note that the general case of rational fixed points, i.e.

$$(a+d)^2 - 4 = \delta^2\,, \qquad a\,d - b\,c = 1\,, \quad \delta \in \mathbb{Q} \tag{14}$$

has solution. Namely, the hyperbolic equation $(a+d)^2 - 4 = \delta^2$ has rational solution in the form

$$a + d = \pm\, \frac{2\,(1+t^2)}{1-t^2}\,, \qquad \delta = \frac{4\,t}{1-t^2}\,, \quad t \in \mathbb{Q} \setminus \{1\,, -1\}\,. \tag{15}$$

For given parameters a, d and δ one has $t = (a + d \pm 2)/\delta$. Rational values for parameters a and b follow from the expression (15). Then b and c are also rational, because

$$b\,c = \frac{\delta^2 - a^2 - d^2}{2} + 1\,, \qquad c \neq 0\,. \tag{16}$$

Three of these parameters a, b, c, d and t are free. In the cases (**A**) and (**B**) parameter $t \in \mathbb{Q} \setminus \{1\,, -1\}$. For an analysis of the cases (**A**) and (**B**) see Ref. [12]. Now we are going to investigate the cases (**C**) and (**D**) in more details. These two cases exhaust all possibilities with $\delta = t = 0$.

3.1. Case (C): $\delta = t = 0$, $d = -a + 2$, $(a-1)^2 + bc = 0$.

This is a case with double fixed point:

$$f(x) = \frac{ax + b}{cx - a + 2}\,, \qquad \xi_1 = \xi_2 = \frac{a-1}{c}\,. \tag{17}$$

For further investigation we need

$$f'(x) = \frac{1}{(cx - a + 2)^2}\,, \qquad f'(\xi_1) = f'(\xi_2) = 1\,. \tag{18}$$

Due to $|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$ it follows that the fused rational fixed point $\xi_1 = \xi_2 = \frac{a-1}{c}$ is indifferent one in real as well as in all p-adic cases.

According to the above results we have only one adelic fixed point $\xi^{(1)} = \xi^{(2)} \equiv \xi$, i.e.

$$\xi = \Big(\xi_\infty\,, \xi_2\,, \xi_3\,, \xi_5\,, \cdots, \xi_p\,, \cdots\Big)\,, \quad \xi \in \mathbb{A}\,, \tag{19}$$

where $\xi_\infty = \xi_p = \frac{a-1}{c}$ for any p. This is one pure adelic indifferent point for any rational values of parameters $a\,, b$ and c constrained by relation $(a-1)^2 + b\,c = 0$ and $c \neq 0$.

The n-th iteration is

$$x_n = \frac{(n\,a - n + 1)\,x_0 + n\,b}{n\,c\,x_0 - n\,a + n + 1}\,, \tag{20}$$

where x_0 is an initial state. In the real case for all $x_0 \neq (n\,a - n - 1)/nc$ we have $x_n \to \frac{a-1}{c}$ when $n \to \infty$.

3.1.1. Subcase: $b = -c\,,\ d = a - 2c\,,\ (a-c)^2 = 1$.

In this case one has again mapping with fused fixed points, i.e.

$$f(x) = \frac{ax - c}{cx + a - 2c}\,, \qquad \xi_1 = \xi_2 = 1\,. \tag{21}$$

In the following we need

$$f'(x) = \frac{1}{(cx + a - 2c)^2}\,, \qquad f'(\xi_1) = f'(\xi_2) = 1\,. \tag{22}$$

In this special case we have the only one possibility. Namely, due to $|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$ it follows that the fused fixed point $\xi_1 = \xi_2 = 1$ is indifferent one in real as well as in all p-adic cases.

According to the above results one has only one adelic fixed point $\xi^{(1)} = \xi^{(2)} \equiv \xi$, i.e.

$$\xi = \Big(\xi_\infty\,, \xi_2\,, \xi_3\,, \xi_5\,, \cdots, \xi_p\,, \cdots\Big)\,, \quad \xi \in \mathbb{A}\,, \tag{23}$$

where $\xi_\infty = \xi_p = 1$ for any p. This is one pure adelic indifferent point for any rational values of parameters a and c constrained by relation $(a-c)^2 = 1$ and $c \neq 0$.

The n-th iteration is

$$x_n = \frac{[a + (n-1)c]\, x_0 - nc}{nc\, x_0 + a - (n+1)c}\,, \tag{24}$$

which in the real case gives $x_n \to 1$ when $n \to \infty$ and $x_0 \neq \frac{(n+1)\,c - a}{n\,c}$.

3.2. Case (D): $\delta = t = 0$, $d = -a - 2$, $(a+1)^2 + bc = 0$.

As in the previous case one has here coincidence of fixed points. Namely,

$$f(x) = \frac{ax + b}{cx - a - 2}\,, \qquad \xi_1 = \xi_2 = \frac{a+1}{c}\,. \tag{25}$$

We also employ

$$f'(x) = \frac{1}{(cx - a - 2)^2}\,, \qquad f'(\xi_1) = f'(\xi_2) = 1\,. \tag{26}$$

Since $|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$ it follows that the fused fixed point $\xi_1 = \xi_2 = \frac{a+1}{c}$ is indifferent one in real as well as in all p-adic cases.

From the above results one has only one adelic fixed point $\xi^{(1)} = \xi^{(2)} \equiv \xi$, i.e.

$$\xi = (\xi_\infty\,, \xi_2\,, \xi_3\,, \xi_5\,, \cdots, \xi_p\,, \cdots)\,, \quad \xi \in \mathbb{A}\,, \tag{27}$$

where $\xi_\infty = \xi_p = \frac{a+1}{c}$ for any p. This is one pure adelic indifferent point for any rational values of parameters a and c constrained by relation $(a+1)^2 + b\,c = 0$ and $c \neq 0$.

The n-th iteration is

$$x_n \quad = \quad \frac{(-1)^{n+1}[n\,(a+1)-1]\,x_0 + (-1)^{n+1}\,n\,b}{(-1)^{n+1}\,n\,c\,x_0 - (-1)^{n+1}\,[n\,(a+1)+1]}\,, \tag{28}$$

where $x_0 \neq \frac{n\,(a+1)+1}{n\,c}$ is an initial state. In the real case $x_n \to \frac{a+1}{c}$ when $n \to \infty$.

3.2.1. Subcase: $b = -c\,,\ d = a + 2c\,,\ (a+c)^2 = 1$.

This is the case with fused fixed points

$$f(x) = \frac{ax - c}{cx + a + 2c}\,, \qquad \xi_1 = \xi_2 = -1\,. \tag{29}$$

For further investigation we need

$$f'(x) = \frac{1}{(cx + a + 2c)^2}\,, \qquad f'(\xi_1) = f'(\xi_2) = 1\,. \tag{30}$$

In this special case we have the only one possibility. Namely, due to $|f'(\xi_1)|_v = |f'(\xi_2)|_v = 1$ it follows that the fused fixed point $\xi_1 = \xi_2 = -1$ is indifferent one in real as well as in all p-adic cases (i.e. for all primes p).

According to the above results one has only one adelic fixed point $\xi^{(1)} = \xi^{(2)} \equiv \xi$, i.e.

$$\xi = (\xi_\infty\,, \xi_2\,, \xi_3\,, \xi_5\,, \cdots\,, \xi_p\,, \cdots)\,, \qquad \xi \in \mathbb{A}\,, \tag{31}$$

where $\xi_\infty = \xi_p = -1$ for any p. This is one pure adelic indifferent point for any rational values of parameters a and c constrained by relation $(a+c)^2 = 1$ and $c \neq 0$.

The n-th iteration is

$$x_n = \frac{[a - (n-1)c]\,x_0 - nc}{nc\,x_0 + a + (n+1)c}\,, \tag{32}$$

which for $x_0 \neq -\frac{(n+1)\,c+a}{n\,c}$ leads to $x_n \to -1$ when $n \to \infty$ in the real case.

4. Concluding Remarks

According to [13] radius of the p-adic Siegel disks in all above considered cases is $r = \frac{|a|_p}{|c|_p}$.

In the above analysis the space of states X can be extended to the whole projective line $\mathbf{P}^1$. Then $x_0 = -d/c$ maps to the point at infinity and $x_0 = \infty$ maps to a/c.

There are many possibilities for generalization of dynamical system (3). Two directions seem to be very interesting: 1) maintain one-dimensional space of states and increase nonlinearity and 2) maintain nonlinearity but increase dimensionality of the space of states.

Under 1) we understand $f(x) = P_k(x)/Q_l(x)$, where $P_k(x)$ and $Q_l(x)$ are polynomials of degrees k and l, respectively. In particular, one can take $f(x) = \prod_{i=1}^{k}(a_i\, x + b_i)/(c_i\, x + d_i)$ with some restrictions on parameters $a_i\,, b_i\,, c_i$ and d_i . Already $f(x) = \prod_{i=1}^{2}(a_i\, x + b_i)/(c_i\, x + d_i)$ contains some interesting cases (see [18] and references therein).

Direction 2) has the form $f_i(x) = (\sum_{j=1}^{k} \alpha_{ij}\, x_j + \alpha_{i0})/(\sum_{j=1}^{k} \beta_{ij}\, x_j + \beta_{i0})$, where $i = 1, 2, \cdots, k$. Two-dimensional case also offers a rich structure. For instance, iterative projective transformations

$$x_n = \frac{a_{11}\, x_{n-1} + a_{12}\, y_{n-1} + a_{13}}{a_{31}\, x_{n-1} + a_{32}\, y_{n-1} + a_{33}}\,, \tag{33}$$

$$y_n = \frac{a_{21}\, x_{n-1} + a_{22}\, y_{n-1} + a_{23}}{a_{31}\, x_{n-1} + a_{32}\, y_{n-1} + a_{33}} \tag{34}$$

are isomorphic to matrices

$$F = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \det F \neq 0\,. \tag{35}$$

Another possibility is to consider the recurrence relation [19]

$$x_{n+k+1} = \frac{\alpha_0 + \alpha_1\, x_{n+1} + \cdots + \alpha_k\, x_{n+k}}{\beta_0 + \beta_1\, x_{n+1} + \cdots + \beta_k\, x_{n+k}}\,, \tag{36}$$

where $\alpha_0, \cdots, \alpha_k$ and $\beta_0, \cdots, \beta_k$ are given rational numbers. Here an initial k-tuple $(x_1\,, \cdots\,, x_k)$ generates an infinite sequence of states by map

$$f(x_1\,, \cdots\,, x_k) = \Big(x_2\,, \cdots\,, x_k\,,\ \frac{\alpha_0 + \alpha_1\, x_1 + \cdots + \alpha_k\, x_k}{\beta_0 + \beta_1\, x_1 + \cdots + \beta_k\, x_k}\Big)\,, \tag{37}$$

for which periodicity of the case $k = 2$ is investigated in [19].

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