# SOME RIGOROUS ANALYTICITY PROPERTIES OF THE FOUR-POINT FUNCTION IN MOMENTUM SPACE

J. Bros *), H. Epstein **), and V. Glaser

CERN - Geneva

## A B S T R A C T

Geometrical methods of analytic completion are used to enlarge the primitive domain of analyticity of the four-point function in p space. The results imply, in particular, analyticity of the scattering amplitude in two variables, on the mass shell, near all physical points, as well as analyticity of partial wave amplitudes in s near the physical points of the right-hand side cut.

---

*) On leave from C.E.N. de Saclay, France.

**) On leave from C.N.R.S., France.

6937/TH.358
28 August 1963

## I. INTRODUCTION

In this paper we intend to give a rigorous proof of some analytic properties of the scattering amplitudes for reactions involving two incoming and two outgoing particles.

We adopt the point of view of the L.S.Z. or Bogoliubov S matrix theories, [Ref. [1],[2]], and do not set here the problem of connecting their axioms with those of Wightman's theory [3]. Starting from these assumptions, we compute some analyticity domain for the four-point function in complex four-momentum space by using purely geometrical techniques of analytic completion. By doing so, it has been possible to reach some new results like analyticity in both variables $s$ and $t$ on the mass shell. On the other hand, this method allowed us to avoid two unpleasant features of most of the classical proofs of dispersion relations, namely :

— the manipulations of integral representations whose convergence properties are not always clear;

— the restriction of tempered distributions to fixed values of some mass variables, which is not meaningful in general.

We remind the reader that, in the L.S.Z. formalism [1] the S matrix elements for various processes are extrapolated by Fourier transforms of retarded, or advanced, or time ordered vacuum expectation values of field operators. These are assumed to be temperate distributions, and because of their support properties, their Fourier transforms can be analytically continued. Moreover, these Fourier transforms coincide in certain real regions. This implies, by the edge-of-the-wedge [2],[4],[5] theorem that all processes involving $n+1$ particles with momenta $p_o, p_1, \ldots, p_n$, are represented by various boundary values of a unique analytic function of $n$ independent complex four vectors (or, more symmetrically, an analytic function on the manifold $p_o+\ldots+p_n = 0$). The situation for the $n$ point function was studied by O. Steinmann [6], D. Ruelle [7], H. Araki [8),9] and N. Burgoyne [9]. The problem of finding the domain of holomorphy of the $n$ point function implied by the above properties is usually called the linear problem because no use is made of the unitarity condition.

Let us sketch the situation for the four-point function and the method used in this paper. The physical scattering amplitudes are certain functions (actually tempered distributions) $r_\alpha(p_0, p_1, p_2, p_3)$, the arguments being four vectors related by : $p_0+p_1+p_2+p_3 = 0$. (For brevity, we shall often denote the set $\{p_0, p_1, p_2, p_3 = -p_0-p_1-p_2\}$ by $p$.) Besides the conventional retarded and advanced functions, this set contains some other ones which we shall call Steinmann functions. All the $r_\alpha$ have the following properties.

Each $r_\alpha$ is the Fourier transform of a tempered distribution $\tilde{r}_\alpha$ in real $x$ space, the support of which lies in a convex cone $\tilde{V}_\alpha$. For instance, the retarded function $r_0(x_0, x_1, x_2, x_3)$ has its support in

$$\tilde{V}_0^- = \left\{ x_0 - x_1 \in \overline{V}^+, \; x_0 - x_2 \in \overline{V}^+, \; x_0 - x_3 \in \overline{V}^+ \right\}$$

where $V^+$ is the forward light cone. As a consequence, each $r_\alpha(p)$ is the boundary value of a function $r_\alpha(k)(k = p+iq)$ analytic in the tube $\mathcal{T}_\alpha$ with basis $V_\alpha$, the dual cone of $\tilde{V}_\alpha$.

$$\mathcal{T}_\alpha = \left\{ k : \; q \in V_\alpha \right\}$$

where :

$$k = p + iq = (k_0, k_1, k_2, k_3)$$

$$k_j = p_j + iq_j \; ; \quad 0 \leq j \leq 3$$

$$k_0 + k_1 + k_2 + k_3 = 0$$

Besides, thanks to spectrum conditions, all these functions $r_\alpha$ coincide in some real region of $p$ space, so that, by the edge-of-the-wedge theorem, they possess a single analytic continuation $H(k_0,k_1,k_2,k_3)$.

The aim of this paper is to prove that $H$ is certainly analytic in a domain larger than the "primitive domain" described above, by using the following ideas

i) among the functions $r_\alpha(p)$, the Steinmann functions appear to be more useful than the others. They are grouped in 6 quartets $Q_{ij}$. ($Q_{ij} = Q_{ji}$; $i \neq j$; $i,j = 0,1,2,3$). For each quartet, a "Steinmann identity" holds. Taking for instance the particular quartet $Q_{01} = \{a_{01}, a_{10}, r_{23}, r_{32}\}$ (the significance of this notation will appear later; it is of no importance in this Section (see Sections II and IV)), the relevant Steinmann identity reads

$$a_{01} + a_{10} = r_{23} + r_{32} \tag{1}$$

and holds at all real points in $p$ space and, of course, also in $x$ space

$$\tilde{a}_{01} + \tilde{a}_{10} = \tilde{r}_{23} + \tilde{r}_{32} \tag{1'}$$

It is then possible (see Section II) to fulfil (1') by introducing new tempered distributions $\tilde{\varphi}_i(x)$ $(i = 1,2,3,4)$ verifying

$$\begin{aligned} \tilde{a}_{01} &= \tilde{\varphi}_1 + \tilde{\varphi}_4 \\ \tilde{a}_{10} &= \tilde{\varphi}_2 + \tilde{\varphi}_3 \\ \tilde{r}_{23} &= \tilde{\varphi}_1 + \tilde{\varphi}_2 \\ \tilde{r}_{32} &= \tilde{\varphi}_3 + \tilde{\varphi}_4 \end{aligned} \tag{2}$$

in such a way that the $\tilde{\varphi}_i(x)$ have smaller conical supports than the $\tilde{r}_\alpha(x)$; correspondingly, the Fourier transforms $\varphi_i(p)$ are analytic in tubes larger than the $\mathcal{T}_\alpha$.

ii) one is then led, in complex p space, to problems of analytic completion for the $\varphi_i(p)$, which are typical "edge-of-the-wedge" problems; for instance, $\varphi_1(p)$ is analytic in a certain tube $\Theta_1$, $\varphi_3(p)$ is analytic in another tube $\Theta_3$, and they coincide for real p such that $(p_1+p_2)^2 < M_{12}^2$. These problems are solved in Section III. The result is that $\varphi_1$ and $\varphi_3$ have a common continuation to all points of the convex hull $\Xi_{01}$ of $\Theta_1 \cup \Theta_3$ except those of the cut

$$\Gamma_{12} = \{k : (k_1+k_2)^2 = M_{12}^2 + \rho \; ; \; \rho \geq 0\}$$

iii) from these conclusions, it follows that the four functions $r_\alpha$ of a given quartet $Q_{ij}$ have a common continuation H, analytic in a large convex tube $\Xi_{ij}$ except at the points of two cuts : for the quartet mentioned above, these cuts are :

$$\Gamma_{12} \equiv \Gamma_{03} = \{k : (k_1+k_2)^2 = M_{12}^2 + \rho \; ; \; \rho \geq 0\}$$

$$\Gamma_{02} \equiv \Gamma_{13} = \{k : (k_0+k_2)^2 = M_{02}^2 + \rho' \; ; \; \rho' \geq 0\}$$

The existence of such domains $\Xi_{ij}$ "pierced" by two energy cuts may be a germ for a possible derivation of the Mandelstam domain.

As far as the physical mass shell is concerned, we shall derive the following properties of the envelope of holomorphy : around each physical region, there exists a complex neighbourhood in both variables s, t in which the amplitudes are analytic from both sides of the energy cut occurring in this region. This result holds without limitation on the threshold masses (provided, of course, that the stability conditions are fulfilled). It is clear that such a neighbourhood implies the existence of Lehmann-type ellipses [10] in the transfer variable for fixed real energy. It also implies the analyticity of the partial wave amplitudes in the variable $s$ in a cut neighbourhood of the physical axis.

## II. THE LINEAR PROBLEM FOR THE FOUR-POINT FUNCTION

We start from the following facts which have been proved from axiomatic field theory in a number of papers [see Refs. [6),7),8),9)]].

i) The existence of a primitive domain of analyticity for the four-point function $H(k)$; here, the argument $k = p+iq$ denotes the set of four complex four vectors $\{ k_i = p_i + iq_i ;\ 0 \leqslant i \leqslant 3 \}$ linked by the relation : $k_0+k_1+k_2+k_3 = 0$. The first part of this Section is devoted to the description of this domain, with some details concerning the implications of the edge-of-the-wedge theorem.

ii) The so-called "Steinmann identities" which the various boundary values of $H(k)$ have to satisfy. Some consequences - as far as analyticity is concerned - of these identities will be studied in the second part of this Section.

### The primitive domain *)

It essentially contains the union of thirty-two disjoint tubes $\{ \mathcal{T}_j^{\pm}, \mathcal{T}_{jk}^{\pm};$ $0 \leqslant j,k \leqslant 3;\ j \neq k \}$ with a certain complex neighbourhood of a real region of analyticity $\mathcal{R}_a$. This neighbourhood $\mathcal{N}(\mathcal{R}_a)$ may be very small but its exact size is irrelevant; the main point is that it ensures the connection between all the tubes. Indeed, when we enter into details, we shall see that two arbitrary tubes of the above set are connected by a neighbourhood of analyticity which can be larger than $\mathcal{N}(\mathcal{R}_a)$.

Let us first describe the tubes

$$\mathcal{T}_j^{\pm} = \{ k : q \in V_j^{\pm} \} \ ; \quad \mathcal{T}_{jk}^{\pm} = \{ k : q \in V_{jk}^{\pm} \}$$

---

*) Our notations are inspired by those of Araki and Burgoyne [Ref. [9)]].

Their conical bases $V_j^{\pm}$, $V_{jk}^{\pm}$ are defined as follows :

$$V_j^+ = - V_j^- = \{ q : \ q_k \in V^+, \ q_m \in V^+, \ q_n \in V^+ \}$$

$$V_{jk}^+ = - V_{jk}^- = \{ q : -q_k \in V^+, \ q_k + q_m \in V^+, \ q_k + q_n \in V^+ \}$$

here $(j,k,m,n)$ is a permutation of $(0,1,2,3)$; $V^+$ denotes the future light cone; $q_j \in V^+$ means : $q_k^2 = q_k^{(0)2} - \vec{q}_k^{\,2} > 0$ and $q_k^{(0)} > 0$.

In order to study in detail the connections between all these tubes, we shall introduce the various boundary values which $H(k)$ can take when $k$ tends to a real value $p$ inside any tube.

$$r_j(p) = \lim_{\substack{k \to p \\ q \in V_j^-}} H(k) \qquad a_j(p) = \lim_{\substack{k \to p \\ q \in V_j^+}} H(k)$$

$$r_{jk}(p) = \lim_{\substack{k \to p \\ q \in V_{jk}^-}} H(k) \qquad a_{jk}(p) = \lim_{\substack{k \to p \\ q \in V_{jk}^+}} H(k)$$

These boundary values have the following properties of coincidence (in the sense of distributions)

$$\left. \begin{aligned} a_j(p) &= a_{jk}(p) \\ r_j(p) &= r_{jk}(p) \end{aligned} \right\} \text{ if } p \in \mathcal{R}_k \text{, where } \mathcal{R}_k = \{ p : p_k^2 < M_k^2 \} \quad (3)$$

$$r_{jk}(p) = a_{mn}(p) \text{ if } p \in \mathcal{R}_{jm} \text{, where } \mathcal{R}_{jm} = \{ p : (p_j + p_m)^2 < M_{jm}^2 \} \quad (4)$$

$$(\mathcal{R}_{jm} \equiv \mathcal{R}_{mj} \equiv \mathcal{R}_{kn})$$

here $M_k$, $M_{jm} \equiv M_{mj} \equiv M_{kn}$ are certain threshold masses.

As a consequence, we see that two arbitrary boundary values of the above set always coincide in a real region which is the intersection of several elementary regions $\mathcal{R}_k$ and $\mathcal{R}_{jm}$. In particular, all these boundary values coincide in the intersection $\mathcal{R}_a$ of the seven regions $\mathcal{R}_k$, $\mathcal{R}_{jm}$ :

$$\mathcal{R}_a = \{ p : p_k^2 < M_k^2 ; (p_j + p_m)^2 < M_{jm}^2 ; 0 \leq k, j, m \leq 3; j \neq m \}$$

The exploitation of these coincidences by means of the edge-of-the-wedge theorem [4),5)] will lead us to complete our description of the primitive domain of analyticity of H(k).

Let us consider two arbitrary tubes of the above set and call $\mathcal{R}$ (this is a provisional notation) the real region where the two corresponding boundary values of H(k) coincide. Then, by the generalized edge-of-the-wedge theorem [cf. [5)]], H(k) is certainly analytic in a small complex region $\mathcal{N}(\mathcal{R})$ which is the intersection of a complex neighbourhood of $\mathcal{R}$ with the convex hull of the union of the two given tubes [*)]. In the following we shall always refer to such a real region $\mathcal{R}$ as to "the edge-of-the-wedge region" of the tubes under consideration. It is clear that for any couple of tubes, the "edge-of-the-wedge region" contains $\mathcal{R}_a$, but it can be larger.

We shall now encounter the following two cases :

a) <u>"Opposite edge-of-the-wedge"</u>

If the two tubes are opposite, then the convex hull of their union is the whole space, so that H(k) is actually analytic at the points of the corresponding edge-of-the-wedge region. One can check easily that for any couple of opposite tubes $\{\mathcal{C}_j^+, \mathcal{C}_j^-\}$ or $\{\mathcal{C}_{jk}^+, \mathcal{C}_{jk}^-\}$, the edge-of-the-wedge region is not larger than the above defined region $\mathcal{R}_a$ : thus $\mathcal{R}_a$ is the real region of analyticity of the four-point function.

---

*) It is this theorem which actually allows us to speak of a single analytic function H(k); however, we have already introduced this notion from the beginning, in order not to bother with superfluous notations; anyhow, we only intend to give here a descriptive account of the situation.

b) "Oblique edge-of-the-wedge"

If the two tubes are not opposite, we can check that in the case of our problem the convex hull of their union is not the whole space; thus, the points of the corresponding edge-of-the-wedge region which do not belong to $\mathcal{R}_a$ are only boundary points of the domain.

In order to have a clearer idea of the geometrical situation, it is customary to consider the points for which only the time components $t_j$ of the $q_j$ (= Im $k_j$) are different from zero. Since $t_0+t_1+t_2+t_3 = 0$, the imaginary parts of the four vectors $(k_j)$ are then restricted to a three-dimensional space. The traces of the cones $V_j^{\pm}$, $V_{jk}^{\pm}$ in that space are cones in three dimensions. It is easy to check that these cones form the set of all the cells into which the three-dimensional space is divided by all planes of the form $t_k = 0$ or $t_j+t_m = 0$. We can then take a section of this structure by a sphere centred at the origin. Fig. 1 shows the traces of the various cones on that sphere.

It is easy to derive from this geometrical picture a mnemotechnical recipe for finding the edge-of-the-wedge regions. If the cells representing two tubes are separated by a number of planes of the form $t_k = 0$, $t_j+t_m = 0$, then the edge-of-the-wedge region for these two tubes is the intersection of all regions $\mathcal{R}_k$ and $\mathcal{R}_{jm}$ with indices corresponding to the relevant planes.

In order to complete this description, let us point out the fundamental role played in the following by the sets

$$\Gamma_j = \{k : k_j^2 = M_j^2 + \rho, \rho \geq 0\}, \quad 0 \leq j \leq 3$$

$$\Gamma_{jm} = \{k : (k_j + k_m)^2 = M_{jm}^2 + \rho, \rho \geq 0\}, \quad 0 \leq j, m \leq 3, \ j \neq m$$

It is clear that these seven sets surely remain outside the holomorphy envelope of the primitive domain of H(k), since they are composed of analytic manifolds which neither intersect the tubes $\mathcal{T}_j^{\pm}$, $\mathcal{T}_{jk}^{\pm}$, nor the edge-of-the-wedge regions $\mathcal{R}_k$, $\mathcal{R}_{jm}$.

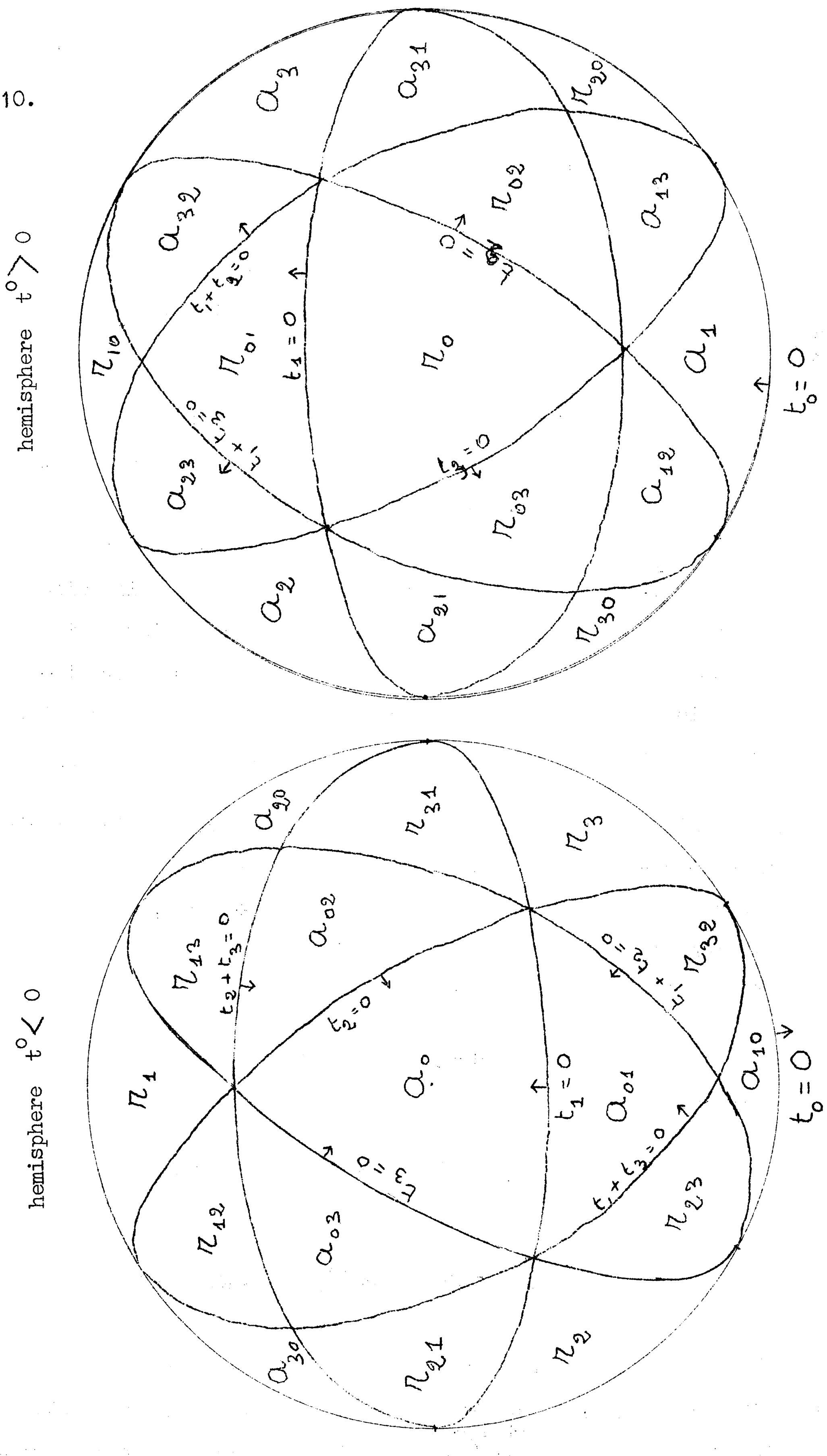


Fig. 1

Indeed the sets $\Gamma_{jm}$ will appear in the space of the scalar products as the so-called "energy (or momentum transfer) cuts". The main effort of this work is to try and isolate these primitive singularities.

## The Steinmann identities

The twenty-four boundary values $r_{jk}(p)$, $a_{mn}(p)$ of $H(k)$ can be grouped into six disjoint sets $Q_{mn}$ of four functions $Q_{mn} = Q_{nm} = \{r_{jk}, r_{kj}, a_{mn}, a_{nm}\}$ which we call "Steinmann quartets".

Now the four functions occurring in each quartet $Q_{mn}$ satisfy the following "Steinmann identity", for all real arguments $p$ :

$$a_{mn}(p) + a_{nm}(p) = r_{jk}(p) + r_{kj}(p) \qquad (1)$$

In the following, we shall also use the denomination of Steinmann quartet $Q_{mn}$ for the set of the four tubes $\{\mathcal{T}^{-}_{jk}, \mathcal{T}^{-}_{kj}, \mathcal{T}^{+}_{mn}, \mathcal{T}^{+}_{nm}\}$ in which the considered boundary values of $H(k)$ occur.

Going back to the preceding geometrical picture (Fig. 1), we see that each quartet of tubes $Q_{mn}$ appears as a quadrangular cone bounded by the four planes $t_k = 0$ $(0 \leqslant k \leqslant 3)$, and divided into four pieces by the two diagonal planes $t_j+t_m = 0$ and $t_j+t_n = 0$. There are six disjoint quartets of tubes $Q_{mn}$, with $Q_{mn}$ opposed to $Q_{jk}$.

In order to study some implications of the Steinmann identities, we are now obliged to describe to some extent the situation in $x$ space.

It is shown in Refs. 6),7),8),9) that the functions $r_j$, $a_j$, $a_{jk}$, $r_{jk}$ are Fourier transforms of functions, the supports of which in $x$ space are contained in the dual cones of the corresponding $V_j^{\pm}$, $V_{jk}^{\pm}$. More precisely, if we define

$$\tilde{r}_j(x) = \int e^{-i\sum_{k=0}^{3} p_k \cdot x_k}\, \delta\Big(\sum_{k=0}^{3} p_k\Big)\, r_j(p)\, d^4p_0 \dots d^4p_3$$

and similarly define $\tilde{a}_j$, $\tilde{r}_{jk}$, $\tilde{a}_{jk}$, then the supports of these functions are described as follows :

support of $\tilde{r}_j$ : $x_k - x_j \in \overline{V}^-$, $x_m - x_j \in \overline{V}^-$, $x_n - x_j \in \overline{V}^-$ ;

support of $\tilde{r}_{jk}$ : $x_m - x_j \in \overline{V}^-$, $x_n - x_j \in \overline{V}^-$ and either $x_m - x_k \in \overline{V}^-$ or $(x_n - x_k) \in \overline{V}^-$.

the support of $\tilde{a}_j$ (resp. $\tilde{a}_{jk}$) being opposed to that of $\tilde{r}_j$ (resp. $\tilde{r}_{jk}$). [Of course, these functions only depend on the differences $x_0 - x_1$, $x_1 - x_2$, $x_2 - x_3$, as a consequence of translational invariance.]

Our main results on the analyticity of the four-point functions will follow from an investigation of the Steinmann quartets. Consider, for instance, the four functions $\tilde{a}_{01}$, $\tilde{a}_{10}$, $\tilde{r}_{23}$, $\tilde{r}_{32}$ which fulfil the Steinmann identity

$$\tilde{a}_{01} + \tilde{a}_{10} = \tilde{r}_{23} + \tilde{r}_{32} \qquad (1')$$

The supports of these functions can be written :

support $\tilde{a}_{01} = S_1 \cup S_4$

support $\tilde{r}_{23} = S_1 \cup S_2$

support $\tilde{a}_{10} = S_2 \cup S_3$

support $\tilde{r}_{32} = S_3 \cup S_4$

where the sets $S_1$, $S_2$, $S_3$, $S_4$ are closed convex simplicial cones in the real twelve-dimensional space of the $x_j - x_k$ [we could take as independent variables in this space, for instance the differences $x_1 - x_0$, $x_2 - x_1$, $x_3 - x_2$, but we prefer to preserve the symmetry and to consider this space as the quotient space of the sixteen-dimensional space of the $x_k$, modulo the vectors of the form $x_0=x_1=x_2=x_3$. (see Ruelle [7] in this connection)]. The definitions of the $S_i$ are :

$$S_1 = \{x : \; x_0-x_2 \in \bar{V}^-;\; x_1-x_2 \in \bar{V}^-;\; x_0-x_3 \in \bar{V}^-\}$$

$$S_2 = \{x : \; x_0-x_2 \in \bar{V}^-;\; x_1-x_2 \in \bar{V}^-;\; x_1-x_3 \in \bar{V}^-\}$$

$$S_3 = \{x : \; x_1-x_2 \in \bar{V}^-;\; x_1-x_3 \in \bar{V}^-;\; x_0-x_3 \in \bar{V}^-\}$$

$$S_4 = \{x : \; x_0-x_3 \in \bar{V}^-;\; x_1-x_3 \in \bar{V}^-;\; x_0-x_2 \in \bar{V}^-\}$$

Clearly, we have, for $i,k = 1,2,3,4;\; i \neq k;\; S_i \cap S_k = \{x : x_0-x_2 \in \bar{V}^-,\; x_0-x_3 \in \bar{V}^-,\; x_1-x_2 \in \bar{V}^-,\; x_1-x_3 \in \bar{V}^-\} = S_1 \cap S_2 \cap S_3 \cap S_4 = K.$

The relative position of the $S_i$ can be summarized in a symbolic form in Fig. 2

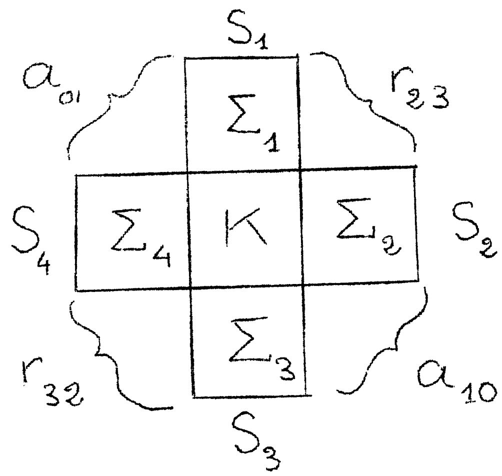


Fig. 2

We are now going to prove that, because of the Steinmann identities, we can write :

$$\begin{aligned} \tilde{a}_{01} &= \tilde{\varphi}_1 + \tilde{\varphi}_4 \\ \tilde{a}_{10} &= \tilde{\varphi}_2 + \tilde{\varphi}_3 \\ \tilde{r}_{23} &= \tilde{\varphi}_1 + \tilde{\varphi}_2 \\ \tilde{r}_{32} &= \tilde{\varphi}_3 + \tilde{\varphi}_4 \end{aligned} \qquad (2)$$

where the $\tilde{\varphi}_i$'s are functions such that : support $\tilde{\varphi}_i \subset S_i$. Clearly relations (2) are sufficient for (1') to hold. Actually the Steinmann relation (1') is just the condition of compatibility for the system (2) to be solvable. Treating temporarily the Steinmann functions and the $\tilde{\varphi}_i$ as ordinary functions, we find an extremely simple situation; the functions $\tilde{\varphi}_i$ are uniquely defined outside of $K$ : for instance, in $S_1 \cap \complement K$ the Steinmann identity (1') implies $\tilde{\varphi}_1 = \tilde{a}_{01} = \tilde{r}_{23}$. Inside $K$ we can take one of the $\tilde{\varphi}_i$ arbitrarily. The rest of them are then determined uniquely.

Since the Steinmann functions are not ordinary functions, but are assumed to be tempered distributions we shall go through the argument again in a more precise way, taking into account the fact that the $\tilde{\varphi}_i$ must be defined as distributions.

Call $\Sigma_i = \overline{S_i \cap \complement K}$ $(i = 1,\dots,4)$, ($\complement K$ = the complementary set of $K$) and $\chi(K)$, $\chi(\Sigma_i)$ the characteristic functions of $K$ and $\Sigma_i$.

It is clearly possible to define (although in a non-unique way) two tempered distributions $\chi(\Sigma_4)\,\tilde{a}_{01}$ and $\chi(S_1)\,\tilde{a}_{01}$, satisfying the following conditions :

1) $\chi(\Sigma_4)\,\tilde{a}_{01} + \chi(S_1)\,\tilde{a}_{01} = \tilde{a}_{01}$

2) $\chi(\Sigma_4)\,\tilde{a}_{01} = \tilde{a}_{01}$ outside of $S_1$

3) $\chi(S_1)\,\tilde{a}_{01} = \tilde{a}_{01}$ outside of $\Sigma_4$ .

[As an example of how such a decomposition may be achieved when $\tilde{a}_{01}$ is tempered, let $f$ be a continuous function of which $\tilde{a}_{01}$ is a finite-order derivative $\tilde{a}_{01} = Df$; one can define $\chi(S_1)\,\tilde{a}_{01}$ by applying the differential operator $D$ to $\theta(+(x_1-x_2)^2)\ \theta(x_2^{(o)}-x_1^{(o)})f$ and take $\chi(\Sigma_4)\,\tilde{a}_{01} = \tilde{a}_{01} - \chi(S_1)\tilde{a}_{01}$.]

We now define

$$\tilde{\varphi}_1^o = \chi(S_1)\,\tilde{a}_{01}$$
$$\tilde{\varphi}_4^o = \chi(\Sigma_4)\,\tilde{a}_{01}$$
$$\tilde{\varphi}_2^o = \tilde{r}_{23} - \tilde{\varphi}_1^o$$
$$\tilde{\varphi}_3^o = \tilde{r}_{32} - \tilde{\varphi}_4^o$$

By definition, we have $\tilde{\varphi}_1^o+\tilde{\varphi}_4^o = \tilde{a}_{01}$, $\tilde{\varphi}_1^o+\tilde{\varphi}_2^o = \tilde{r}_{23}$, $\tilde{\varphi}_3^o+\tilde{\varphi}_4^o = \tilde{r}_{32}$. The equation $\tilde{\varphi}_2^o+\tilde{\varphi}_3^o = \tilde{a}_{10}$ is then a consequence of the Steinmann identity $\tilde{a}_{10} = \tilde{r}_{23}+\tilde{r}_{32}-\tilde{a}_{01}$. Thus the $\tilde{\varphi}_i^o$ satisfy the system (2). It remains to verify that their supports are contained in the corresponding $S_i$. Clearly $\mathrm{supp}.\tilde{\varphi}_1^o \subset S_1$ and $\mathrm{supp}.\tilde{\varphi}_4^o \subset S_4$. Moreover $\mathrm{supp}.\tilde{\varphi}_2^o \subset S_1 \cup S_2$ and we want to show that actually $\tilde{\varphi}_2^o = 0$ in $\complement S_2$ (the complementary set of $S_2$).

In fact $\complement S_2 = \complement(S_1 \cup S_2) \cup \complement(S_2 \cup S_3 \cup S_4)$, i.e., $\complement S_2$ is the union of two open sets, in the first of which $\tilde{\varphi}_2^o = 0$. It suffices to show that $\tilde{\varphi}_2^o = 0$ in $\complement(S_2 \cup S_3 \cup S_4)$. But in this domain $\tilde{a}_{10} = \tilde{r}_{32} = 0$ so that by the Steinmann identity $\tilde{a}_{01} = \tilde{r}_{23}$. Moreover $\tilde{\varphi}_1^o = \tilde{a}_{01}$ in $\complement \Sigma_4 \supset \complement(S_2 \cup S_3 \cup S_4)$, so that $\tilde{\varphi}_2^o = 0$ in $\complement(S_2 \cup S_3 \cup S_4)$. A similar argument shows that $\tilde{\varphi}_3^o = 0$ in $\complement S_3$.

Evidently the general solution of the system (2) is

$$\tilde{\varphi}_1 = \tilde{\varphi}_1^o + h, \quad \tilde{\varphi}_3 = \tilde{\varphi}_3^o + h, \quad \tilde{\varphi}_2 = \tilde{\varphi}_2^o - h, \quad \tilde{\varphi}_4 = \tilde{\varphi}_4^o - h$$

where $h$ is an arbitrary distribution. The general solution satisfying the conditions $\mathrm{supp}.\tilde{\varphi}_i \subset S_i$ is obtained by requiring the support of $h$ to be in $S_1 \cap S_2 \cap S_3 \cap S_4 = K$.

Note that we have proved nothing about the possibility of finding the $\tilde{\varphi}_i$ in a Lorentz invariant way. The idea of a solution of the Steinmann identity (1) in the form (2) has been mentioned to one of the authors (H.E.) by J. Lascoux as due to F.J. Dyson in 1960. It is related, although not in a very clear way, to cohomology theory and to the theory of hyperfunctions. It has been used in another context but in an identical geometrical situation by R.F. Streater [11].

We have succeeded in performing similar solutions of the Steinmann identities for the five-point function, and are presently working on its extension to the n-point function. The system corresponding to (2) can be worked out precisely, as well as the support condition. The difficulty is purely algebraic and consists in proving that the Steinmann identities are the <u>only</u> compatibility conditions for the system.

The exploitation of the decomposition (2)

Going back to p space we can write

$$\begin{aligned} a_{01} &= \varphi_1 + \varphi_4 \\ a_{10} &= \varphi_2 + \varphi_3 \\ r_{23} &= \varphi_1 + \varphi_2 \\ r_{32} &= \varphi_3 + \varphi_4 \end{aligned} \qquad (2')$$

$$\varphi_i(p)\,\delta(p_0+p_1+p_2+p_3) = (2\pi)^4\int e^{i\sum_{k=0}^{3} p_k\cdot x_k}\,\tilde{\varphi}_i(x)\,d^4x_0\cdots d^4x_3$$

or

$$\varphi_i(p) = (2\pi)^3\int \exp i\left\{p_0\cdot(x_0-x_1) + (p_0+p_1)\cdot(x_1-x_2) + (p_0+p_1+p_2)\cdot(x_2-x_3)\right\}\times$$

$$\times\ \tilde{\varphi}_i(x)\,d^4(x_0-x_1)\,d^4(x_1-x_2)\,d^4(x_2-x_3)$$

Since $\tilde{\varphi}_i(x)$ is a tempered distribution with support in the cone $S_i$, each $\varphi_i(p)$ $(1 \leqslant i \leqslant 4)$ is the boundary value of a function $\varphi_i(k)$ analytic in a certain tube $\Theta_i$; the basis of $\Theta_i$ in q space $(q = \mathrm{Im}\, k)$ is the interior $\overset{\circ}{\tilde{S}}_i$ of the cone $\tilde{S}_i$ dual to $S_i$ :

$$\Theta_i = \left\{ k = p + iq :\ q \in \overset{\circ}{\tilde{S}}_i \right\}$$

$$\tilde{S}_i = \left\{ q : \sum_{k=0}^{3} q_k\cdot x_k \geqslant 0 \ \text{ for all } x \text{ in } S_i \right\}$$

(Here, of course, $\sum_{k=0}^{3} q_k = 0$).

A straightforward calculation gives the following expressions for $\Theta_i$ $(1 \leqslant i \leqslant 4)$ :

$$\Theta_1 = \{ k : \; q_1 \in V^-, \; q_3 \in V^+, \; q_1 + q_2 \in V^+ \}$$

$$\Theta_2 = \{ k : \; q_0 \in V^-, \; q_3 \in V^+, \; q_0 + q_2 \in V^+ \}$$

$$\Theta_3 = \{ k : \; q_0 \in V^-, \; q_2 \in V^+, \; q_0 + q_3 \in V^+ \}$$

$$\Theta_4 = \{ k : \; q_1 \in V^-, \; q_2 \in V^+, \; q_1 + q_3 \in V^+ \}$$

Now from the relations of coincidence (4)

$$a_{01}(p) = r_{23}(p) \quad \text{for} \quad p \quad \text{real and such that} \quad (p_1+p_3)^2 < M_{13}^2$$

$$a_{01}(p) = r_{32}(p) \quad \text{for} \quad (p_1+p_2)^2 < M_{12}^2$$

$$a_{10}(p) = r_{23}(p) \quad \text{for} \quad (p_1+p_2)^2 < M_{12}^2$$

$$a_{10}(p) = r_{32}(p) \quad \text{for} \quad (p_1+p_3)^2 < M_{13}^2$$

we deduce

$$\varphi_4(p) = \varphi_2(p) \qquad \text{for} \qquad (p_1 + p_3)^2 < M_{13}^2$$

$$\varphi_1(p) = \varphi_3(p) \qquad \text{for} \qquad (p_1 + p_2)^2 < M_{12}^2$$

We are now faced with two "oblique edge-of-the-wedge problems". These two problems are actually identical : the problem for $\varphi_1$ and $\varphi_3$ goes over to that for $\varphi_2$ and $\varphi_4$ by making the transformations : $p_1 \to p_0$, $p_0 \to p_1$, $M_{12}^2 \to M_{13}^2$, $p_2$ and $p_3$ unchanged. We shall therefore concentrate on the problem for $\varphi_2$ and $\varphi_4$. These two functions are respectively analytic in the tubes $\Theta_2$ and $\Theta_4$ and coincide on a real region $\mathcal{R}_{13}$. By the edge-of-the-wedge theorem, they have a common analytic continuation in a certain open set $\mathcal{N}(\mathcal{R}_{13})$ which is the intersection of the convex hull $\Xi_{01}$ of $\Theta_2 \cup \Theta_4$ with an open neighbourhood of the real points in $\mathcal{R}_{13}$. This situation will be fully investigated in the next Section.

## III. A THEOREM OF ANALYTIC COMPLETION

At the end of Section II, the study of the functions $\varphi_i$ led us to the following problem. Find the holomorphy envelope $\mathcal{H}(\Delta)$ of the following domain $\Delta$ in the space of three complex four vectors $k_i = p_i + iq_i$ $(1 \leqslant i \leqslant 3;$ $k = \{k_1, k_2, k_3\})$ :

$$\Delta = \Theta_2 \cup \Theta_4 \cup \mathcal{N}(\mathcal{R}_{13})$$

where $\Theta_4$ is the tube $\{k : q_1 \in V^-;\ q_2 \in V^+;\ q_1+q_3 \in V^+\}$. $\Theta_2$ is the tube $\{k : q_3 \in V^+;\ q_1+q_2+q_3 \in V^+;\ q_1+q_3 \in V^-\}$. $\mathcal{R}_{13}$ is the real region $(p_1+p_3)^2 < M_{13}^2$. $\mathcal{N}(\mathcal{R}_{13})$ is the intersection of some complex neighbourhood of $\mathcal{R}_{13}$ with the tube $\Xi_{01}$, convex hull of $\Theta_1 \cup \Theta_2$.

As we shall see, the solution of this problem is a special case of Theorem 1 which provides an analogous statement in the case of n four vectors.

The reader who is not interested in the technical details of the proof may skip the rest of this Section, and pass to Section IV, where the results are described.

The tools which we need to prove theorem 1 have been put together in Lemmas 1, 2, 3, and 4. The proofs of the basic lemmas 1 and 4 will be fully given in another paper in preparation, but an essential account of them has been already published by one of the authors [12] [see also Ref. [13]]. As far as lemmas 2 and 3 are concerned, we shall give here some hint of the proofs and refer the reader to the paper by B.M.S. [14] for completeness.

Notation : in the following, $\mathcal{N}(R)$ will always denote an open complex neighbourhood in $\mathbb{C}_n$ of a real open set $R \subset \mathbb{R}_n$ such that $\mathcal{N}(R) \cap \mathbb{R}_n = R$. This neighbourhood is to be considered as arbitrarily small. The symbols T, R, D will always denote regions in the space $\mathbb{C}_n$, while $\mathcal{T}, \mathcal{R}, \mathcal{D}$ will denote analogous regions in the space $\mathbb{C}_{4n}$ of n four-vector variables.

Lemma 1

(Case of an "opposite edge-of-the-wedge" configuration). Let $D_n$ be the following domain in the space $\mathbb{C}_n$ of $n$ complex variables : $z_i = x_i+iy_i$; $1 \leqslant i \leqslant n$ ;

$$D_n = T_n^+ \cup T_n^- \cup \mathcal{N}(\mathcal{I}_n)$$

where $T_n^+$ is the tube $\left\{ z_i : y_i > 0, \ 1 \leqslant i \leqslant n \right\}$, $T_n^- = -T_n^+$; $\mathcal{I}_n$ is the real interval : $a_i < x_i < b_i$ ; $1 \leqslant i \leqslant n$.

Then the holomorphy envelope $\mathcal{H}(D_n)$ of $D_n$ is described by the following parametric equations :

$$z_i = \frac{a_i - b_i s \zeta_i}{1 - s\zeta_i} \qquad (5)$$

with

$$1 \leqslant i \leqslant n$$

$$\operatorname{Im} s \geqslant 0$$

$$\operatorname{Im} \zeta_i > 0$$

It is clearly seen that for each fixed value of $s$ in the upper half-plane, the section of the domain is a polycircle characterized by the value of $\operatorname{Arg} s$. When $\operatorname{Arg} s$ varies from $0$ to $\pi$, the polycircles realize a natural "interpolation" between the tubes $T_n^+$ and $T_n^-$ and their union constitutes $\mathcal{H}(D_n)$.

Corollary

If $n-p$ intervals $[a_i, b_i]$ $(p+1 \leqslant i \leqslant n)$ are equal to $[-\infty, +\infty]$, and the other $p$ are finite, then $\mathcal{H}(D_n) = \mathcal{H}(D_p) \times \mathbb{C}_{n-p}$, where $\mathbb{C}_{n-p}$ is the whole complex space of the variables $z_{p+1}, \dots, z_n$, and $D_p$ is the analogue of $D_n$ in the space $\mathbb{C}_p$ of the variables $(z_1, \dots, z_p)$.

This is a straightforward consequence of the above description of $\mathcal{H}(D_n)$.

Lemma 2

Let $D_2(R)$ be the following domain in the complex space $\mathbb{C}_2$ :

$$D_2(R) = T_2^+ \cup T_2^- \cup \mathcal{N}(R)$$

where $R$ is the region $x_1 x_2 < m^2$.

Then : i) $\mathcal{H}(D_2(R))$ is the complementary of the "cut" $z_1 z_2 = m^2 + \rho$ $(\rho \geq 0)$ in space $\mathbb{C}_2$.

ii) $\mathcal{H}(D_2(R)) = \bigcup_{\mathcal{J}_2} \mathcal{H}(D_2(\mathcal{J}_2))$ where the union $\bigcup_{\mathcal{J}_2}$ is taken over the set of all the intervals $\mathcal{J}_2$ (of the form $a_i < x_i < b_i$ ; $i = 1,2$) contained in the region $R$.

Lemma 3

Let $\mathcal{D}(\mathcal{R})$ be the following domain in the space $\mathbb{C}_4$ of one four-vector variable $k = p+iq$ ; $(k = k^{(o)}, \underset{\sim}{k})$ ; $\mathcal{D}(\mathcal{R}) = \mathcal{T}^+ \cup \mathcal{T}^- \cup \mathcal{N}(\mathcal{R})$ where $\mathcal{T}^+$ is the tube $\{k : q \in V^+\}$

$$\mathcal{T}^- = -\mathcal{T}^+$$

$\mathcal{R}$ is the real open region : $p^2 = p^{(o)^2} - \underset{\sim}{p}^2 < m^2$.

Then : i) $\mathcal{H}(\mathcal{D}(\mathcal{R}))$ is the complementary of the "cut"

$$k^2 = k^{(o)2} - \underset{\sim}{k}^2 = m^2 + \rho \quad (\rho \geq 0)$$

in the space $\mathbb{C}_4$.

ii) Let $\Pi$ be a two-dimensional complex space of the form $k = a_1 z_1 + a_2 z_2 + b$, where $a_1, a_2, b$ are real four vectors such that

$$a_1^2 = a_2^2 = 0 \;, \quad a_1^{(o)} > 0, \quad a_2^{(o)} > 0 \;,$$

$b$ arbitrary.

Then the intersection $\mathcal{H}(\mathcal{D}(\mathcal{R}))\cap\Pi$ is the holomorphy envelope of $\mathcal{D}(\mathcal{R})\cap\Pi$.

The first part of lemmas 2 and 3 has been proved in detail in Ref. [14]; it essentially results from the description of the final domain by means of a set of hyperbolae (already called "doubly inadmissible hyperbolae" by F.J. Dyson [15]) both branches of which cross the region $\mathcal{R}$.

The second part of lemma 2 becomes clear by noting that for any interval $\mathcal{J}_2\subset\mathcal{R}$ the equations (5) exhibit the above mentioned hyperbolae as boundary curves ; indeed, one only has to fix $\zeta_1, \zeta_2$ to real values with opposite signs, $s$ varying freely in the upper half-plane.

The second part of lemma 3 is a trivial consequence of lemma 2 and of the geometrical situation.

Lemma 4

(Case of an "oblique edge-of-the-wedge" configuration). Let $D$ be the following domain in the space $\mathbb{C}_n$ of the variables

$$z_i = x_i + iy_i \quad (1\leqslant i\leqslant n\ ;\ z=\{z_1,\dots,z_n\})\ ;$$

$$D = T_1 \cup T_2 \cup \mathcal{N}'(\mathcal{J}_n)$$

$T_1$ and $T_2$ are tubes, the bases of which are polyhedral cones with their vertices at the origin of co-ordinates in the $y$ space ; the essential assumption about them is that the closure of $T_1$ (resp. $T_2$) includes the following set $G_1$ (resp. $G_2$) :

$$G_1 = \{z : y_i \geqslant 0\ ;\ 1\leqslant i\leqslant p\ ;\ y_j = 0\ ;\ p+1\leqslant j\leqslant n\}$$

$$G_2 = \{z : y_i = 0\ ;\ 1\leqslant i\leqslant p\ ;\ y_j\leqslant 0\ ;\ p+1\leqslant j\leqslant n\}$$

$\mathcal{J}_n$ is the interval $a_i < x_i < b_i$ ; $1 \leqslant i \leqslant n$ ; $\mathcal{N}'(\mathcal{J}_n)$ is the intersection of a neighbourhood $\mathcal{N}(\mathcal{J}_n)$ with the tube T, convex hull of $T_1 \cup T_2$.

Then the holomorphy envelope $\mathcal{H}(D)$ of D certainly includes the following region :

$$\mathcal{H}(D_n) \cap \{ z : y_i \geqslant 0 ; 1 \leqslant i \leqslant p ; y_j \leqslant 0 ; p+1 \leqslant j \leqslant n \} \cap T$$

where $\mathcal{H}(D_n)$ is defined as in lemma 1.

We are now in a position to prove our main theorem :

<u>Theorem 1</u>

Let $\mathcal{D}$ be the following domain in the space $\mathbb{C}_{4n}$ of n four-vector variables $k_i = p_i + iq_i$ ; ( $1 \leqslant i \leqslant n$ ; $k = \{k_1, .., k_n\}$ )

$$\mathcal{D} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{N}'(\mathcal{R})$$

where

$$\mathcal{C}_1 = \{ k : q_i \in V^+ ; 1 \leqslant i \leqslant n \}$$

$$\mathcal{C}_2 = \{ k : q_1 \in V^- ; q_1 + q_i \in V^+ ; 2 \leqslant i \leqslant n \}$$

$\mathcal{R}$ is the real open region of $\mathbb{R}_{4n}$ defined by :
$p_1^2 = p_1^{(o)2} - \underline{p}_1^2 < m^2$, $p_i$ arbitrary for $i > 1$.

$\mathcal{N}'(\mathcal{R})$ is the intersection of an open neighbourhood $\mathcal{N}(\mathcal{R})$ with the tube $\mathcal{C}$, the convex hull of $\mathcal{C}_1 \cup \mathcal{C}_2$.

Then the holomorphy envelope $\mathcal{H}(\mathcal{D})$ of $\mathcal{D}$ is the complementary of the "cut" $\Gamma \equiv \{ k : k_1^2 = m^2 + \rho ; \rho \geqslant 0 \}$ in $\mathcal{C}$ :

$$\mathcal{H}(\mathcal{D}) = \mathcal{C} \cap \complement \Gamma$$

Proof :

i) we shall first prove the analogous theorem in the space $\mathbb{C}_{2n}$ of the variables :

$$z_j = x_j + iy_j \quad , \quad z'_j = x'_j + iy'_j \qquad 1 \leq j \leq n$$

$$z = \{z_1, \dots, z_n\} \quad , \quad z' = \{z'_1, \dots, z'_n\}$$

Instead of $\mathcal{C}_1, \mathcal{C}_2, \mathcal{R}$, we consider the corresponding regions :

$$T_1 = \{z, z' : y_i > 0 \ ; \ y'_i > 0 \ ; \ 1 \leq i \leq n\}$$

$$T_2 = \{z, z' : y_1 < 0 \ ; \ y'_1 < 0; \ y_1 + y_i > 0; \ y'_1 + y'_i > 0; 2 \leq i \leq n$$

$$R = \{x, x' : x_1 x'_1 < m^2; x_i, x'_i \text{ arbitrary } ; 2 \leq i \leq n\}$$

We first notice that the convex hull of $T_1 \cup T_2$ is the open tube

$$T = \{z, z' : y_i > 0; y'_i > 0 ; y_1 + y_i > 0 ; y'_1 + y'_i > 0 ; 2 \leq i \leq n\}$$

T is the union of four tubes :

$$T = T_1 \cup T_2 \cup \tau_1 \cup \tau_2 \tag{6}$$

where

$$\tau_1 = \{z, z' : y_1 \geq 0 ; y'_1 \leq 0 ; y_i > 0 ; y'_1 + y'_i > 0 ; 2 \leq i \leq n\}$$

$$\tau_2 = \{z, z' : y_1 \leq 0 ; y'_1 \geq 0 ; y_1 + y_i > 0 ; y'_i > 0 ; 2 \leq i \leq n\}$$

Thus to get our result, it is sufficient to show that the restriction of $\mathcal{H}$ (D) to each region $\tau_\alpha$ $(\alpha = 1,2)$ is the complementary of the cut $\gamma \equiv \left\{ z : z_1 z'_1 = m^2 + \rho \; ; \; \rho \geq 0 \right\}$ in $\tau_\alpha$.

We shall only consider the case of $\tau_1$, since the arguments are identical for $\tau_2$ ; our statement is then a direct application of lemma 4 ; in fact, let us put :

$$z''_1 = z'_1 \; ; \; z''_i = -(z'_1 + z'_i) \; ; \quad 2 \leq i \leq n$$

and define :

$$T^+_{2n} = \left\{ z, z'' : y_i > 0 \, , \; y''_i > 0 \; ; \; 1 \leq i \leq n \right\}$$

$$T^-_{2n} = - T^+_{2n}$$

$$G_1 = \left\{ z, z'' : y_i \geq 0 \, , \; y''_i = 0 \; ; \; 1 \leq i \leq n \right\}$$

$$G_2 = \left\{ z, z'' : y_i = 0 \, , \; y''_i \leq 0 \; ; \; 1 \leq i \leq n \right\}$$

It is clear that $G_1$ (resp. $G_2$) lies on the boundary of the polyhedral tube $T_1$ (resp. $T_2$), so that the statement of lemma 4 is valid, namely : $\mathcal{H}$(D) contains the following region :

$$\mathcal{H}(T^+_{2n} \cup T^-_{2n} \cup \mathcal{N}'(R)) \cap \left\{ z, z'' : y_i \geq 0 \, , \; y''_i \leq 0 \; ; \; 1 \leq i \leq n \right\} \cap T \qquad (7)$$

Now, since R only depends of the variables $x_1, x'_1$, it results from the corollary of lemma 1, and from lemma 2 that :

$$\mathcal{H}(T^+_{2n} \cup T^-_{2n} \cup \mathcal{N}'(R)) = \mathcal{H}(T^+_2 \cup T^-_2 \cup \mathcal{N}''(R)) \times \mathbb{C}_{2n-2}$$

where $\mathcal{H}(T^+_2 \cup T^-_2 \cup \mathcal{N}''(R))$ is the complementary of the "cut" $z_1 z'_1 = m^2 + \rho$ $(\rho \geqslant 0)$ in the space $\mathbb{C}_2$ of the variables $z_1, z'_1$.

The second region which appears in (7) is nothing else but the closure of $\mathcal{T}_1$. Using the expression (6) of T, the region (7) reduces to the intersection of $\mathcal{T}_1$ with the complementary of the "cut" $z_1 z'_1 = m^2 + \rho$ $(\rho \geqslant 0)$ in the whole space $\mathbb{C}_{2n}$. This is the announced result.

ii) in the space of n four vectors $k_i = p_i + iq_i$ $(1 \leqslant i \leqslant n)$ the proof goes in three steps. Let us take an arbitrary point k in the presumed holomorphy envelope, namely

$$k \in \mathcal{C}$$
$$k_1^2 \neq m^2 + \rho \quad \text{for all} \quad \rho \geqslant 0$$

a) by using a convenient parametrization of the convex tube $\mathcal{C}$, one can construct a suitable manifold $\mathcal{M}(k)$ passing through k whose intersection with $\mathcal{D}$ contains a domain D of the type considered in i).

b) by applying the result of i), one can compute $\mathcal{H}(D)$.

c) by construction, the point k automatically belongs to $\mathcal{H}(D)$, so that, by an obvious property of holomorphy envelopes, it also belongs to $\mathcal{H}(\mathcal{D})$.

We first claim that any point $k \in \mathcal{C}$ can be represented as follows

$$k_1 = K_1 - K_0$$
$$k_i = K_i + K_0 \; ; \quad 2 \leqslant i \leqslant n \qquad (8)$$

where the four vectors $K_i = P_i + iQ_i$ $(0 \leqslant i \leqslant n)$ are restricted by the conditions : $Q_i \in V^+$.

In fact, as $\mathcal{C}_1$ and $\mathcal{C}_2$ have conical bases, $\mathcal{C}$ can be generated by sums $k^{(1)} + k^{(2)}$ where $k^{(1)} = \{k_1^{(1)}, \ldots, k_n^{(1)}\} \in \mathcal{C}_1$, $k^{(2)} = \{k_1^{(2)}, \ldots, k_n^{(2)}\} \in \mathcal{C}_2$. The basis of $\mathcal{C}$ in $q$ space can thus be represented by the equations $q_i = q_i^{(1)} + q_i^{(2)}$ $(1 \leqslant i \leqslant n)$ with

$$q_j^{(1)} \in V^+ \; ; \; q_1^{(2)} + q_j^{(2)} \in V^+ \; ; \; 2 \leqslant j \leqslant n \; ;$$
$$q_1^{(1)} \in V^+ \; ; \; q_1^{(2)} \in V^-$$

Let us now put

$$K_1 = k_1^{(1)} \; ; \; K_0 = -k_1^{(2)} \; ;$$
$$K_j = k_j^{(1)} + k_j^{(2)} + k_1^{(2)} \; ; \; 2 \leqslant j \leqslant n \; ;$$

and we immediately get the equations (4) with the relevant conditions for $Q_i = \operatorname{Im} K_i$. The converse is obvious by putting

$$k_1^{(1)} = K_1 \; ; \quad k_1^{(2)} = -K_0$$
$$k_j^{(1)} = K_j - \varepsilon \; ; \quad k_j^{(2)} = K_0 + \varepsilon \; ; \; 2 \leqslant j \leqslant n \, ,$$

where the vector $\varepsilon$ has a sufficiently small imaginary part in $V^+$.

Let us now define for any point $k \in \mathcal{C}$ represented by (8) *) a manifold

---

*) This representation is not unique, but for any $k$ it is chosen once and for all in an arbitrary way.

$\mathcal{M}(k)$ generated by means of four vectors $e, e', e_j$; $(2 \leqslant j \leqslant n)$ as follows :
since $Q_0 \in V^+$, $Q_1 \in V^+$, one can find two vectors $e, e'$ in the two dimensional plane defined by $Q_0, Q_1$, such that

$$e^2 = e^{(0)2} - \underset{\sim}{e}^2 = 0 , \qquad e^{(0)} > 0$$
$$e'^2 = e'^{(0)2} - \underset{\sim}{e}'^2 = 0 , \qquad e'^{(0)} > 0$$

For each index $j$ such that $2 \leqslant j \leqslant n$, we put

$$e_j = \frac{Q_j}{\sqrt{Q_j^2}} \in V^+$$

$\mathcal{M}(k)$ is the complex linear manifold defined by the following equations

$$k_1 = P_1 + z_1 e + z_1' e' \tag{9}$$

$$k_j = P_j + z_j e + z_j' e' + \zeta_j e_j \quad ; 2 \leqslant j \leqslant n \tag{9'}$$

and it is clear that it contains the point $k$. Thanks to (9), (9'), any analytic function of the $k_i$ becomes a function of $3n-1$ variables

$$z_i = x_i + i y_i \quad , \quad z_i' = x_i' + i y_i' \quad ; 1 \leqslant i \leqslant n$$
$$\zeta_j = \xi_j + i \eta_j \quad , \quad 2 \leqslant j \leqslant n$$

analytic in some domain the image of which lies in $\mathcal{D}$.

Now it is easy to see that the following tube $T_1$ (resp. $T_2$) has its image in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) :

$$T_1 = \{ z, z', \zeta : y_1 > 0 , y_1' > 0 , y_i > 0 , y_i' > 0 , \eta_i > 0 ; 2 \leqslant i \leqslant n \}$$

$$T_2 = \{ z, z', \zeta : y_1 < 0 , y_1' < 0 , y_1 + y_i > 0 , y_1' + y_i' > 0 , \eta_i > 0 ; 2 \leqslant i \leqslant n \}$$

As far as the region $\mathcal{R}_0$ is concerned, its section by the manifold $\mathcal{M}(k)$ is a cylindrical region R whose basis is bounded by a certain hyperbola, the section of the hyperboloid $k_1^2 = m^2$ by the plane

$$k_1 = P_1 + z_1 e + z_1' e' \tag{9}$$

Let us call D the domain in $(z, z', \zeta)$ space :

$$D = T_1 \cup T_2 \cup \mathcal{N}'(R)$$

We notice that the result of i) can be applied literally to compute $\mathcal{H}(D)$, up to a trivial change, namely D is the topological product of the domain considered in i) by the fixed convex tube $\left\{\zeta : \eta_i = \operatorname{Im} \zeta_i > 0\right\}$. Thus

$$\mathcal{H}(D) = (\mathcal{H}_2 \times \mathbb{C}_{3n-3}) \cap T,$$

where $T = \{z, z', \zeta : y_j > 0,\ y_j' > 0,\ y_1 + y_j > 0,\ y_1' + y_j' > 0,\ \eta_j > 0,\ 2 \leq j \leq n\}$

$$\mathcal{H}_2 = \mathcal{H}(T_2^+ \cup T_2^- \cup \mathcal{N}''(R))$$

and the index 2 refers to the two-dimensional space of the variables $(z_1, z'_1)$.

Now, we deduce from lemma 3 - ii) that $\mathcal{H}_2$ is the restriction to the plane (9) of

$$\mathcal{H}(\mathcal{T}^+ \cup \mathcal{T}^- \cup \mathcal{N}''(\mathcal{R}_0)), \quad \mathcal{T}^+ = \{k_1 : q_1 \in V^+\} = -\mathcal{T}^-$$

which is nothing else but the complementary of the cut

$$\Gamma = \left\{k_1 : k_1^2 = m^2 + \rho \ ;\ \rho \geq 0\right\}$$

So we have got that $\mathcal{H}(D)$ is the complementary of the restriction of $\Gamma$ to the manifold $\mathcal{M}(k)$, in the convex tube T, that is : $T \cap \complement\Gamma$.

In order to complete our proof, it remains to show that the point k which we started from lies necessarily in $\mathcal{H}(D)$. Since by assumption k lies outside the cut $\Gamma$, the only thing which we have to verify is that $k \in T$ ; but this is automatically realized by construction of the manifold $\mathcal{M}(k)$. In fact, let us write the expressions of the vectors $Q_i$ occurring in (6) in terms of their co-ordinates $Y_i$ :

$$Q_0 = Y_0 e + Y_0' e'$$
$$Q_1 = Y_1 e + Y_1' e'$$
$$Q_j = Y_j e_j \quad ; \quad 2 \leq j \leq n$$

Since $Q_0, Q_1, Q_j$ are lying in $V^+$, the $Y_i$'s are submitted to the conditions

$$Y_0 > 0, \; Y_0' > 0, \; Y_1 > 0, \; Y_1' > 0, \; Y_j > 0 \quad (2 \leq j \leq n)$$

Then, by (8)

$$Im\, k_1 = q_1 = Q_1 - Q_0 = (Y_1 - Y_0)e + (Y_1' - Y_0')e'$$
$$Im\, k_j = q_j = Q_j + Q_0 = Y_0 e + Y_0' e' + Y_j e_j$$
$$Im\, (k_1 + k_j) = q_1 + q_j = Q_1 + Q_j = Y_1 e + Y_1' e' + Y_j e_j$$

So, if $k_1, k_j$ are expressed as in (9), (9'), their co-ordinates verify the inequalities

$$y_j = Y_0 > 0, \quad y_j' = Y_0' > 0, \quad \eta_j = Y_j > 0$$
$$y_1 + y_j = Y_1 > 0, \quad y_1' + y_j' = Y_1' > 0$$

which are those defining T. Q.E.D.

## IV. EXTENSION OF THE DOMAIN OF THE FOUR-POINT FUNCTION

The construction of the holomorphy envelope of the primitive domain of the four-point function (described in Section II) is a very complicated programme which, in principle, can be decomposed into several steps.

If we look at the geometrical situation (cf., Fig. 1), a natural idea is to try and compute first of all the holomorphy envelopes of couples of adjacent tubes; we mean tubes which have a common face, supported by one of the linear manifolds $q_k = 0$, or $q_j+q_m = 0$. In fact, for such couples of tubes, the edge-of-the-wedge region is the simplest possible, namely $\mathcal{R}_k = \left\{ p : p_k^2 < M_k^2 \right\}$ or $\mathcal{R}_{jm} = \left\{ p : (p_j+p_m)^2 < M_{jm}^2 \right\}$.

The further steps would consist in computing the holomorphy envelopes of larger and larger groups of tubes, using if possible at each step the partial holomorphy domains obtained at the preceding one; of course, these steps are more complicated than the first one, because the relevant edge-of-the-wedge regions are now intersections of several elementary regions $\mathcal{R}_k$, $\mathcal{R}_{jm}$.

In this Section, we achieve the first step of this programme as a direct application of the completion theorem which we proved in Section III. This means that we obtain the holomorphy envelope of any couple of adjacent tubes. Indeed we obtain a little more in the following sense. Let us recall that the four-point function $H(k)$ has to satisfy special conditions which are the Steinmann identities; and, of course, one can expect that the holomorphy envelope with respect to this relevant class of functions $H(k)$ will be larger than the holomorphy envelope with respect to the class of all possible functions $H(k)$. In fact, by using the auxiliary functions $\varphi$ introduced in Section II, it is possible to incorporate some information due to Steinmann identities into our problem. As a result, we obtain the holomorphy envelope of any Steinmann quartet of tubes for the class of functions $H(k)$ satisfying the corresponding Steinmann identity. Let us point out here that the completion theorem of Section III joined with a suitable use of the Steinmann identities would yield analogous results for the case of the $n$ point function, but this is not the purpose of the present paper.

The couples of adjacent tubes can be divided into two sets according to whether their common face lies on a manifold $q_k = 0$ or on a manifold $q_j+q_m = 0$. In the first case the relevant adjacent tubes are $\mathcal{T}^+_j$ and $\mathcal{T}^+_{jk}$ (or $\mathcal{T}^-_j$ and $\mathcal{T}^-_{jk}$) where $0 \leq j,k \leq 3$; $k \neq j$ (cf., Section II)

$$\mathcal{T}^+_j = \{ k : \; q_k \in V^+, \; q_m \in V^+, \; q_n \in V^+ \}$$

$$\mathcal{T}^+_{jk} = \{ k : \; q_k \in V^-, \; q_k + q_m \in V^+, \; q_k + q_n \in V^+ \}$$

$\mathcal{T}^+_j$ and $\mathcal{T}^+_{jk}$ have the following common face

$$\{ k : \; q_k = 0 \, , \; q_m \in V^+, \; q_n \in V^+ \}$$

and their edge-of-the-wedge region is $\mathcal{R}_k = \{ p : p_k^2 < M_k^2 \}$; this means that they are connected by a small complex region of analyticity $\mathcal{N}(\mathcal{R}_k)$ which is the intersection of a certain complex neighbourhood of $\mathcal{R}_k$ with the convex hull $\Lambda^+_{jk}$ of $\mathcal{T}^+_j \cup \mathcal{T}^+_{jk}$.

Let us apply Theorem 1 (cf., Section III) with the following specialization : $n = 3$; $k_1 \to k_k$; $k_2 \to k_m$; $k_3 \to k_n$; $m \to M_k$; $\mathcal{R} \to \mathcal{R}_k$; $\mathcal{T}_1 \to \mathcal{T}^+_j$; $\mathcal{T}_2 \to \mathcal{T}^+_{jk}$.

We immediately obtain :

<u>Theorem 2</u>

The four-point function $H(k)$ is analytic in the whole tube $\Lambda^+_{jk}$, the convex hull of $\mathcal{T}^+_j \cup \mathcal{T}^+_{jk}$, except at those points of $\Lambda^+_{jk}$ which lie on the set

$$\Gamma_k = \{ k : \; k_k^2 = M_k^2 + \rho \; ; \; \rho \geq 0 \}$$

In other words, we have the following holomorphy envelope :

$$\mathcal{H}(\mathcal{T}_j^+ \cup \mathcal{T}_{jk}^+ \cup \mathcal{N}(\mathcal{R}_k)) = \Lambda_{jk}^+ \cap \complement \Gamma_k$$

Besides, $\Lambda^+_{jk}$ has the following representation

$$\left.\begin{aligned} k_k &= K_1 - K_o \\ k_m &= K_2 + K_o \\ k_n &= K_3 + K_o \end{aligned}\right\} \quad \text{where } \operatorname{Im} K_i \in V^+, \ \operatorname{Re} K_i \text{ arbitrary}; \quad 0 \leqslant i \leqslant 3 .$$

An identical result holds obviously for couples $\left\{ \mathcal{T}_j^-, \mathcal{T}_{jk}^- \right\}$.

Let us now consider the case of two adjacent tubes $\mathcal{T}^-_{jk}$ and $\mathcal{T}^+_{mn}$; their common face lies on the manifold $q_j + q_m = q_k + q_n = 0$, and their edge-of-the-wedge region is $\mathcal{R}_{jm} = \left\{ p : (p_j + p_m)^2 < M^2_{jm} \right\}$.

In the same way as before, we should find the holomorphy envelope $\mathcal{H}(\mathcal{T}^-_{jk} \cup \mathcal{T}^+_{mn} \cup \mathcal{N}(\mathcal{R}_{jm}))$ to be complementary of the set $\Gamma_{jm} = \left\{ k : (k_j + k_m)^2 = M^2_{jm} + \rho ; \rho \geqslant 0 \right\}$ in the convex hull of $\mathcal{T}^-_{jk} \cup \mathcal{T}^+_{mn}$; but we get a stronger result by noticing that the tubes $\mathcal{T}^-_{jk}$ and $\mathcal{T}^+_{mn}$ belong to the same quartet $Q_{mn}$. In fact, we are going to prove the following theorem :

<u>Theorem 3</u>

Let us consider the quartet of tubes

$$Q_{mn} = \left\{ \mathcal{T}^-_{jk}, \mathcal{T}^-_{kj}, \mathcal{T}^+_{mn}, \mathcal{T}^+_{nm} \right\}$$

and call $\Xi_{mn}$ the convex hull of the union of these four tubes.

Then : i) $\Xi_{mn}$ has the following parametrization

$$k_m = -(K_1 + K_2) \quad ; \quad k_n = -(K_3 + K_o) ;$$

$$k_j = K_2 + K_o \quad ; \quad k_k = K_1 + K_3$$

where $\operatorname{Im} K_i \in V^+$, $\operatorname{Re} K_i$ arbitrary; $0 \leqslant i \leqslant 3$.

ii) the four-point function $H(k)$ is analytic in the whole tube $\Xi_{mn}$ except at those points of $\Xi_{mn}$ which lie either on $\Gamma_{jm}$ or on $\Gamma_{jk}$.

$$\Gamma_{jm} = \{ k : (k_j + k_m)^2 = M_{jm}^2 + \rho ; \rho \geq 0 \}$$

Here $(j,k,m,n)$ is any permutation of $(0,1,2,3)$.

In order to prove Theorem 3, we shall specialize to the quartet $Q_{01}$; it will make the notations simpler.

Let us go back to the functions $\varphi_i(k)$ analytic in the tubes $\Theta_i$ $(0 \leq i \leq 3)$ introduced at the end of Section II. We have seen that $\varphi_2(k)$ and $\varphi_4(k)$ have a common analytic continuation $\Phi(k)$ which is holomorphic in $\Theta_2 \cup \Theta_4 \cup \mathcal{N}(\mathcal{R}_{13})$. By applying Theorem 1 with the following specialization $n = 3$; $k_1 \to k_1+k_3$; $k_2 \to k_2$; $k_3 \to -k_1$; $\mathcal{C}_1 \to \Theta_4$; $\mathcal{C}_2 \to \Theta_2$; $\mathcal{R} \to \mathcal{R}_{13}$; $m \to M_{13}$; $\Gamma - \Gamma_{13}$, we immediately obtain that $\Phi(k)$ is analytic in

$$\mathcal{H}(\Theta_2 \cup \Theta_4 \cup \mathcal{N}(\mathcal{R}_{13})) = \Xi_{01} \cap \complement \Gamma_{13}$$

Here we have used the fact that the convex hull of $\Theta_2 \cup \Theta_4$ is equal to $\Xi_{01}$. This can be seen by noticing that $\Theta_2$ is the convex hull of $\mathcal{C}^+_{10} \cup \mathcal{C}^-_{23}$, while $\Theta_4$ is the convex hull of $\mathcal{C}^+_{01} \cup \mathcal{C}^-_{32}$; thus the convex hull of $\Theta_2 \cup \Theta_4$ is the same as that of $\mathcal{C}^+_{01} \cup \mathcal{C}^+_{10} \cup \mathcal{C}^-_{23} \cup \mathcal{C}^-_{32}$, namely $\Xi_{01}$.

By specializing Eq. (8), we obtain in passing the expected parametrization for $\Xi_{01}$:

$$\Xi_{01} = \{ k : k_0 = -(K_1 + K_2),\ k_1 = -(K_3 + K_0),\ k_2 = K_2 + K_0,\ k_3 = K_1 + K_3; \\ \operatorname{Im} K_i \in V^+,\ \operatorname{Re} K_i \text{ arbitrary } ; 0 \leq i \leq 3 \}$$

In the same way, the functions $\varphi_1(k)$ and $\varphi_3(k)$ have a common continuation $\Psi(k)$ which is analytic in $\Xi_{01} \cap \complement \Gamma_{12}$.

Let us now consider the sum $\Phi(k)+\Psi(k)$ which is analytic for instance in the tube $\mathcal{T}^+_{01} = \Theta_1 \cap \Theta_4$; when k tends to any real value p inside this tube, Eq. (2') yields :

$$\lim_{\substack{k \to p \\ q \in \mathcal{T}^+_{01}}} [\Phi(k) + \Psi(k)] = \varphi_1(p) + \varphi_4(p) = a_{01}(p) = \lim_{\substack{k \to p \\ q \in \mathcal{T}^+_{01}}} H(k)$$

Thus $\Phi(k)+\Psi(k)$ and $H(k)$ are the same analytic function and the equation $H(k) = \Phi(k)+\Psi(k)$ defines an analytic continuation of $H(k)$ in the intersection of the domains of $\Phi(k)$ and $\Psi(k)$, namely :

$$\Delta_{01} = \Xi_{01} \cap \complement(\Gamma_{12} \cup \Gamma_{13})$$

Q.E.D.

<u>Remark</u> :

This domain $\Delta_{01}$ is actually the holomorphy envelope of the quartet $Q_{01}$ with respect to the class of functions $H(k)$ satisfying the Steinmann identity

$$a_{01} + a_{10} = r_{23} + r_{32} \qquad (1)$$

Indeed, if we had put together the holomorphy envelopes of the four couples of adjacent tubes occurring in $Q_{01}$, we should have found a smaller domain, namely

$$(\Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \Theta_3) \cap (\Gamma_{12} \cup \Gamma_{13})$$

and it is not excluded that the holomorphy envelope of the latter be still smaller than $\Delta_{01}$, since this process does not take the Steinmann identity into account.

At the end of this Section we shall study the boundary points of the initial tubes $\mathcal{T}^{\pm}_{j}$, $\mathcal{T}^{\pm}_{jk}$, and prove :

Theorem 4

$H(k)$ is analytic at all the boundary points of the tubes $\mathcal{T}^{\pm}_{j}$, $\mathcal{T}^{\pm}_{jk}$ $(0 \leqslant j,\ k \leqslant 3;\ j \neq k)$ which do not belong to the union of the seven sets $\Gamma_k$ $(0 \leqslant k \leqslant 3)$ and $\Gamma_{jm}$ $(0 \leqslant j,\ m \leqslant 3,\ j \neq m)$.

There are three kinds of boundary points of the tubes $\mathcal{T}^{\pm}_{j}$, $\mathcal{T}^{\pm}_{jk}$ :

i) points which lie only on one face, i.e., points which satisfy one of the following sets of conditions :

$$q_k = 0\ , \quad q_j \in V^+, \quad q_m \in V^+$$

or $$q_k = 0\ , \quad q_j \in V^-, \quad q_m \in V^-$$

or $$q_j + q_m = 0\ , \quad q_j \in V^-\ , \quad q_j + q_k \in V^+$$

or $$q_j + q_m = 0\ , \quad q_j \in V^+, \quad q_j + q_k \in V^-$$

These are the points which lie on the boundary of only two tubes;

ii) points which belong to the intersection of two faces, which we call an edge, i.e., which satisfy

$$q_j + q_m = 0\ , \quad q_j + q_k = 0\ , \quad q_j \in V^{\pm}$$

or

$$q_j = 0\ , \quad q_m = 0\ , \quad q_k \in V^{\pm}$$

iii) the real points.

The case of real points has already been considered, and the region of analyticity $\mathcal{R}_a$ described in Section II coincides with the announced result.

As far as the points mentioned in i) are concerned, they are included in the domains provided by Theorems 2 and 3; in fact, any point of one given face belongs to the boundary of two adjacent tubes, so that it lies inside the convex hull of their union. Thus, if it does not lie on any of the sets $\Gamma_k$, $\Gamma_{jm}$, it automatically belongs to a certain domain of analyticity given either by Theorem 2 (if the relevant face is $q_k = 0$) or by Theorem 3 (if the relevant face is $q_j+q_m = 0$).

We now have to deal with the case of points lying on the edges; these edges are represented in Fig. 1 by the vertices of the spherical triangulation, and we have to distinguish the two types of edges (or vertices).

a) The edge $\left\{ k : q_j+q_m = 0, \ q_j+q_k = 0, \ q_j \in V^+ \right\}$ is common to the boundaries of the four tubes $\mathcal{T}^-_{jk}$, $\mathcal{T}^-_{kj}$, $\mathcal{T}^+_{mn}$, $\mathcal{T}^+_{nm}$ which constitute the quartet $Q_{mn}$. Clearly, all the points of this set which do not lie on any set $\Gamma_k$, $\Gamma_{jm}$ certainly belong to $\Xi_{mn} \cap \complement(\Gamma_{jk} \cup \Gamma_{jm})$, so that they are points of analyticity as a consequence of Theorem 3.

b) The edge $\left\{ k : q_j = 0, \ q_m = 0, \ q_k \in V^+ \right\}$ is common to the boundaries of six tubes, namely $\mathcal{T}^-_{km}$, $\mathcal{T}^-_{k}$, $\mathcal{T}^-_{kj}$, $\mathcal{T}^+_{nm}$, $\mathcal{T}^+_{nj}$, $\mathcal{T}^+_{n}$; of course, the result which we expect should come out from the knowledge of the holomorphy envelope of this "sextet" $S_{kn}$. However, we do not need to know the global solution of this problem and a purely local procedure will yield the points which we are interested in.

We shall use the following theorem [special continuity theorem *), proved in particular by H.J. Bremermann [16]] :

<u>Theorem</u>

Let us consider in $\mathbb{C}_n$ a continuous path $z = z(t)$ $(z = \left\{ z_1, \ldots, z_n \right\})$ lying on a one-dimensional linear manifold :

$$z(t) = a + b\,u(t) \ ;$$

here $a, b \in \mathbb{C}_n$ and $u(t)$ is a complex valued continuous function for $0 \leqslant t \leqslant 1$.

*) For an outline of the proof of this theorem, see also Ref. [12].

Let $D$ be a domain in $\mathbb{C}_{n+1}$ which, for $0 < t \leqslant 1$, contains the disc $\{ z = z(t) \in \mathbb{C}_n ; \ |\zeta| < a \}$. Then, if $D$ also contains the point $M = \{ z = z(0); \ \zeta = 0 \}$, $\mathcal{H}(D)$ contains the whole disc $\{ z = z(0); \ |\zeta| < a \}$.

This property extends immediately to the case where the disc $|\zeta| < a$ is replaced by any domain $\Delta$ in the $\zeta$ plane, and $M$ is any point inside $\Delta$.

Let us now come back to our problem. We have to prove that any given point $K$ such that

$$Q_j = 0, \ Q_m = 0, \ Q_k \in V^+, \quad K \notin \bigcup_{0 \leqslant k \leqslant 3} \Gamma_k \bigcup_{\substack{0 \leqslant j, m \leqslant 3 \\ j \neq m}} \Gamma_{jm}$$

lies in the holomorphy envelope of the primitive domain of $H(k)$.

We first notice that the point $P = \operatorname{Re} K$ satisfies the inequalities

$$P_j^2 < M_j^2, \quad P_m^2 < M_m^2, \quad (P_j + P_m)^2 < M_{jm}^2$$

it means that $P$ lies in the edge-of-the-wedge region which is common to the six tubes of the "sextet" $S_{kn}$.

In order to apply the above theorem, we consider the family of one-dimensional linear manifolds $\mathcal{L}(t)$ defined as follows :

$$\mathcal{L}(t) = \{ k = k(t, \zeta) ; \ \zeta \in \mathbb{C}_1 \}$$

$t$ fixed such that : $0 \leqslant t \leqslant 1$, with

$$k_j(t, \zeta) = P_j + i\varepsilon t$$
$$k_m(t, \zeta) = P_m + i\varepsilon t$$
$$k_k(t, \zeta) = P_k + Q_k \zeta$$

in these formulae, $\mathcal{E}$ is a fixed four vector in $V^+$. The point K which we want to reach is obtained by putting $t = 0$ and $\zeta = i$.

For $0 < t \leqslant 1$ and $\mathrm{Im}\,\zeta > 0$, we see that $k_j \in \mathcal{T}^+$, $k_m \in \mathcal{T}^+$, $k_k \in \mathcal{T}^+$, so that $k(t, \zeta) \in \mathcal{T}_n^+$. When $t = 0$, there exists a small semi-circle $\gamma = \{\zeta : |\zeta| < r;\ \mathrm{Im}\,\zeta > 0\}$, the image of which lies in the primitive domain. This can be seen as follows : first of all, since the point $k(t = 0,\ \zeta = 0) = P$ belongs to the edge-of-the-wedge region of the sextet $S_{kn}$, there exists a small region of analyticity $\mathcal{N}^0(P)$ which is the intersection of a complex neighbourhood of P with the convex hull of the sextet $S_{kn}$; now all the points $k(t = 0,\ \zeta : \mathrm{Im}\,\zeta > 0)$ lie obviously in this convex hull since they belong to the common edge of the six tubes; thus the section of $\mathcal{N}^0(P)$ by the linear manifold $\mathcal{L}(t = 0)$ contains a small semi-circle $\gamma = \{\zeta :\ |\zeta| < r;\ \mathrm{Im}\,\zeta > 0\}$.

It is clear that all the conditions of the (extended) above theorem are fulfilled by putting $u(t) = t$, $\Delta = \{\zeta : \mathrm{Im}\,\zeta > 0\}$ and choosing for M any point inside the semi-circle $\gamma$ .

As a result, all the points $k(t = 0,\ \zeta : \mathrm{Im}\,\zeta > 0)$ lie in the holomorphy envelope, and in particular the expected point $K = k(t = 0,\ \zeta = i)$. Q.E.D.

## V. FURTHER USE OF THE STEINMANN QUARTETS

After what has been done in Section IV, the next step might be to try and find the envelope of holomorphy of two opposite Steinmann quartets. This is not known at present, although one knows, by quite general arguments, that it is invariant under complex transformations [see Ref. [12)]].

In this Section, by using a variant of the local edge-of-the-wedge theorem, it is proved that this holomorphy envelope contains cut neighbourhoods of all the real "physical" points on the mass shell. According to Theorem 3, the holomorphy envelope of the quartet $Q_{01} = \left\{ \mathcal{C}^{+}_{01}, \mathcal{C}^{+}_{10}, \mathcal{C}^{-}_{23}, \mathcal{C}^{-}_{32} \right\}$ is

$$\Delta_{01} = \left\{ k : k \in \Xi_{01}, (k_1 + k_2)^2 \neq M_{12}^2 + \rho, (k_1 + k_3)^2 \neq M_{13}^2 + \rho' \right.$$
$$\left. \text{for every} \quad \rho, \rho' \geq 0 \right\}$$

The domain $\Delta_{23}$ corresponding to the quartet $Q_{23} = \left\{ \mathcal{C}^{+}_{23}, \mathcal{C}^{+}_{32}, \mathcal{C}^{-}_{01}, \mathcal{C}^{-}_{10} \right\}$ is obtained by changing $k$ to $-k$ in the above conditions, since $\Xi_{23} = -\Xi_{01}$.

We are interested in the shape of these domains in the neighbourhood of all real points $P$ satisfying the conditions

$$P_k^2 < M_k^2, \quad 0 \leq k \leq 3 \; ; \; (P_0 + P_1)^2 < M_{01}^2$$

we have to distinguish three situations, since $P$ may belong either to zero, or to one, or to the two cuts $\Gamma_{12}, \Gamma_{13}$ which cross the domains $\Delta_{01}, \Delta_{23}$.

i) If $P \in \mathcal{R}_a$, that is

$$P_k^2 < M_k^2, \quad 0 \leq k \leq 3 \; ; \; (P_j + P_m)^2 < M_{jm}^2, \quad 0 \leq j, m \leq 3, \; j \neq m \; ;$$

then there exists obviously a complex spherical neighbourhood $W(P)$ centred at $P$ such that :

$$\Delta_{01} \cap W(P) = \Xi_{01} \cap W(P)$$

In other words, the domain $\Delta_{01}$ is locally a tube at P. Similarly,

$$\Delta_{23} \cap W(P) = (-\Xi_{01}) \cap W(P)$$

Indeed this is not very informative, since we already knew (cf. Section II) that any point $P \in \mathcal{R}_a$ is a point of analyticity for H(k).

ii) Let P belong, for instance, to $\Gamma_{12}$, but not to $\Gamma_{13}$, that is

$$P_k^2 < M_k^2, \; 0 \leq k \leq 3 \; ; \; (P_0 + P_1)^2 < M_{01}^2 \; , \quad (P_1 + P_3)^2 < M_{13}^2 \; ,$$

$$(P_1 + P_2)^2 > M_{12}^2 \; , \quad P_1 + P_2 \in V^+ . \tag{10}$$

(The other case : $P_1+P_2 \in V^-$ would be similar.) One can always find a complex spherical neighbourhood W(P) centred at P, having no point in common with the cut $\Gamma_{13}$, and such that all points in W(P) have their real parts in the region (10).

Clearly the section of $\Delta_{01}$ by W(P) is :

$$W(P) \cap \Xi_{01} \cap \complement T_{12}$$

so that it is composed of the two following disjoint sets

$$W(P) \cap \Xi_{01} \cap \{ k : \operatorname{Im} (k_1 + k_2)^2 > 0 \} \tag{11}$$

$$W(P) \cap \Xi_{01} \cap \{ k : \operatorname{Im} (k_1 + k_2)^2 < 0 \} \tag{12}$$

By writing $(k_1+k_2)^2 = (p_1+p_2)^2 - (q_1+q_2)^2 + 2i(p_1+p_2)\cdot(q_1+q_2)$, and $(p_1+p_2) \in V^+$, we see that the condition $q_1+q_2 \in V^+$ (respectively, $V^-$) implies $\operatorname{Im}(k_1+k_2)^2 > 0$ (respectively, $< 0$).

This shows that region (11) contains the intersections of $W(P)$ with the initial tubes $\mathcal{T}^+_{01}$, $\mathcal{T}^-_{23}$ (cf., Sections II and IV) and their convex hull $\Theta_1$, while region (12) contains

$$W(P) \cap (\mathcal{T}^+_{10} \cup \mathcal{T}^-_{32}) \quad \text{and} \quad W(P) \cap \Theta_3$$

In the same way, the domain $\Delta_{23}$ restricted to $W(P)$ is composed of the two following disjoint sets :

$$W(P) \cap (-\Xi_{01}) \cap \{ k : \operatorname{Im} (k_1 + k_2)^2 > 0 \} \qquad (13)$$

$$W(P) \cap (-\Xi_{01}) \cap \{ k : \operatorname{Im} (k_1 + k_2)^2 < 0 \} \qquad (14)$$

Eq. (13) contains $W(P) \cap (\mathcal{T}^-_{10} \cup \mathcal{T}^+_{32})$, Eq. (14) contains $W(P) \cap (\mathcal{T}^-_{01} \cup \mathcal{T}^+_{23})$. Let us note that in view of the regularity of the hypersurface $\Gamma_{12}$ at P, one can choose $W(P)$ small enough so that the regions (11), (12), (13) and (14) be domains. Besides, we see from (10) that P lies in the edge-of-the-wedge region which is common to the four tubes $\mathcal{T}^+_{01}$, $\mathcal{T}^-_{23}$, $\mathcal{T}^-_{10}$, $\mathcal{T}^+_{32}$. This shows that, inside $W(P)$, the domains (11) and (13) are connected together by a small complex open set $\mathcal{N}(P)$ where $H(k)$ is analytic; more precisely, $\mathcal{N}(P)$ is the intersection of a complex neighbourhood of P with the convex hull of $\mathcal{T}^+_{01} \cup \mathcal{T}^-_{23} \cup \mathcal{T}^-_{10} \cup \mathcal{T}^+_{32}$. A similar result holds for the connection between the domains (12) and (14). Fig. 3 shows symbolically how the situation looks as plotted in the imaginary parts of the vectors, when P is taken in the region (10).

We are going to improve this result by showing that the domains (11) and (13) are actually connected by a region of the form : $V(P) \cap \{ k : \operatorname{Im}(k_1+k_2)^2 > 0 \}$ where $H(k)$ is analytic; here $V(P)$ denotes a new complex neighbourhood of $P(V(P) \subset W(P))$.

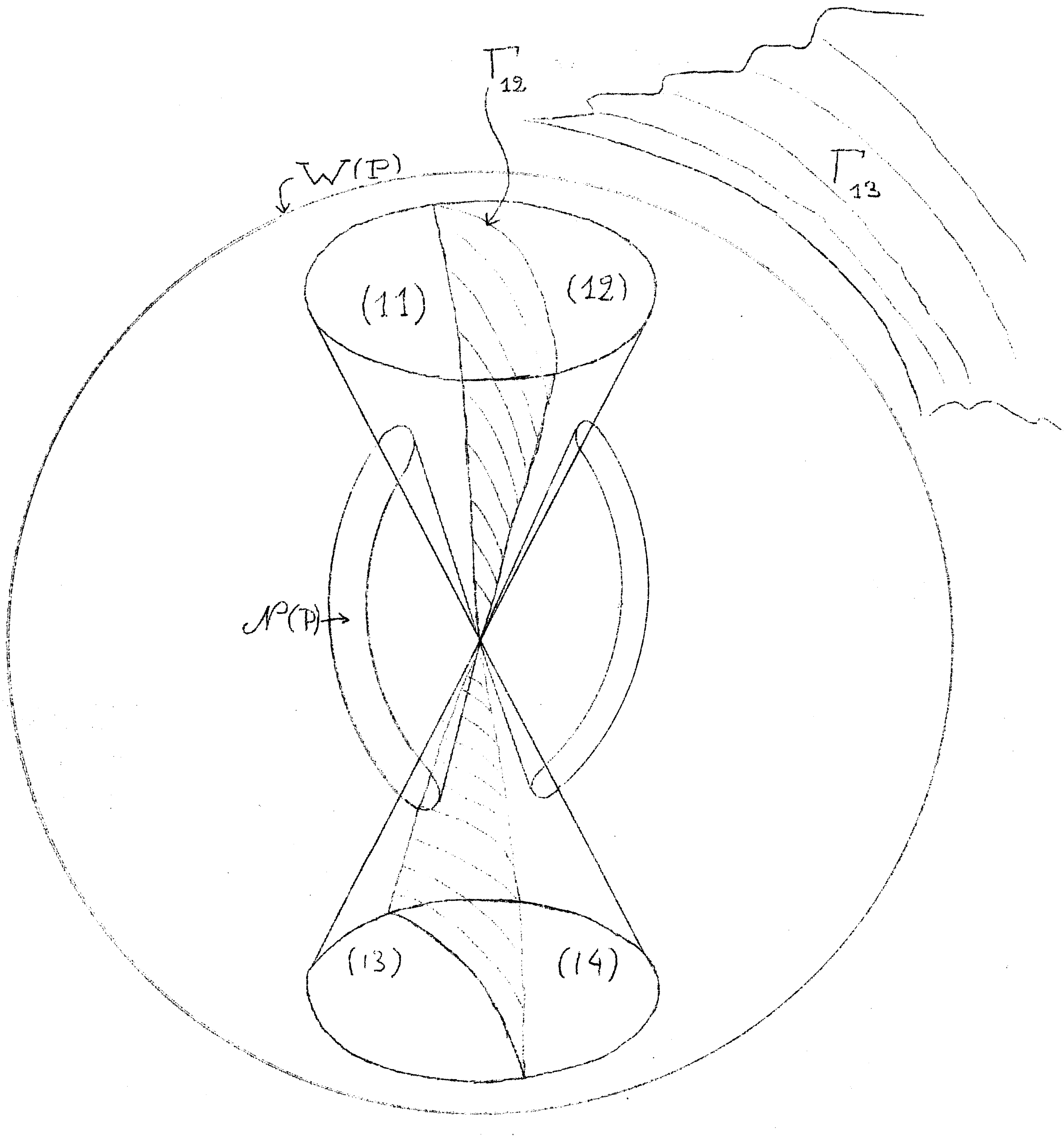


Fig. 3

The tool which we need is a variant of the oblique edge-of-the-wedge theorem [5] and could be called the local tube theorem; its proof will be fully given in a further paper by the same authors, already mentioned in Section III; let us just state it here as an auxiliary lemma.

Lemma 5

Let $\Delta$ be the following domain in the space $\mathbb{C}_n$ of $n$ complex variables $z_1,\dots,z_n$ $(z_j = x_j+iy_j)$ :

$$\Delta = W \cap \mathcal{T}$$

where $W$ is the spherical domain $\left\{ z : |z_1|^2+\dots+|z_n|^2 < R^2 \right\}$ ; $\mathcal{T}$ is a tube with conical basis in the $y$ space, such that :

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{N}$$

$\mathcal{T}_1$ and $\mathcal{T}_2$ are two disjoint convex tubes, and $\mathcal{N}$ another convex tube which connects $\mathcal{T}_1$ and $\mathcal{T}_2$ and lies inside the convex hull of $\mathcal{T}_1 \cup \mathcal{T}_2$; (all these tubes have conical basis in the $y$ space). Then there exists another spherical domain $V \subset W$,

$$V = \left\{ z : |z_1|^2 + \dots + |z_n|^2 < r^2 \right\},\ r < R,$$

such that the convex hull of $V \cap \left\{ \mathcal{T}_1 \cup \mathcal{T}_2 \right\}$ is inside the holomorphy envelope of $\Delta$ .

In order to be able to apply Lemma 5, we may consider new analytic coordinates $z_1,\dots,z_{12}$ such that :

a) $z_1 = (k_1+k_2)^2$

b) the mapping $k \leftrightarrow z$ is biholomorphic (i.e., analytic and invertible) in $W(P)$;

c) the $z_j$ $(j = 1,\dots,12)$ are real when all $k_m^{(\mu)}$ are real. In other words

$$z_j(k) = z_j^*(k^*) .$$

6937

Such co-ordinates can always be found, provided $W(P)$ has been chosen small enough. In fact we can take for instance $z_2, \dots, z_{12}$ equal to independent components of the vectors $k_m$, with $z_1 = (k_1+k_2)^2$ replacing a certain component $k_1^{(\nu)}$ of $k_1$. This fulfils the above conditions a), b), and c) provided $\partial (k_1+k_2)^2 / \partial k_1^{(\nu)} \neq 0$ at $k = P$, that is $P_1^{(\nu)}+P_2^{(\nu)} \neq 0$. Since $P_1+P_2$ cannot be zero in region (10), $P_1^{(\nu)}+P_2^{(\nu)}$ is always $\neq 0$ for some $\nu$ $(= 0,1,2,3)$.

It is then a simple, if somewhat tedious, exercise in infinitesimal geometry to prove that the image of the union of domains (11) and (13) and $\mathcal{N}(P)$ in the new variables $(z_1, \dots, z_{12})$ contains a domain of the form :

$$\mathcal{W}_1 \cap \left\{ \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{N}_1 \right\}$$

where $W_1$ is a complex sphere centred at the image of $P$, $\mathcal{T}_1$ is a convex tube with conical basis of the form :

$$\Xi \cap \left\{ z : \operatorname{Im} z_1 > 0 \right\}$$

where $\Xi$ is a convex tube with conical basis. $\mathcal{T}_2$ is the convex tube

$$(-\Xi) \cap \left\{ z : \operatorname{Im} z_1 > 0 \right\}$$

$\mathcal{N}_1$ is another convex tube with conical basis which connects $\mathcal{T}_1$ and $\mathcal{T}_2$. (Of course, $\Xi$ is a close approximation to the image of $\Xi_{01}$ in the neighbourhood of $P$, and $\mathcal{N}_1$, an approximation to the image of $\mathcal{N}(P)$. $W_1$ is contained in the image of $W(P)$, and chosen sufficiently small.) Applying Lemma 5 we find that, considered as a function of $z_1, \dots, z_{12}$, $H(k)$ is analytic in a domain of the form $V_1 \cap \left\{ z : \operatorname{Im} z_1 > 0 \right\}$ where $V_1$ $(\subset W_1)$ is a complex open sphere centred at the image of $P$. Going back to the original variables, we find the announced result : $H(k)$ is analytic in a set of the form $V(P) \cap \left\{ k : \operatorname{Im}(k_1+k_2)^2 > 0 \right\}$. We can obviously apply a similar treatment to domains (12) and (14), and since this can be done for any $P$ in region (10), we obtain :

Lemma 6

There exists a complex open neighbourhood V of the real points of region (10) such that H(k) is analytic in V except at the points of the cut $\Gamma_{12}$.

iii) Consider now a real point P satisfying the conditions :

$$P_k^2 < M_k^2 , \quad 0 \leqslant k \leqslant 3 ; \quad (P_0 + P_1)^2 < M_{01}^2 ; \quad (P_1 + P_3)^2 > M_{13}^2 ;$$
$$(P_1 + P_2)^2 > M_{12}^2 ; \tag{15}$$

then there exists a sphere W(P) centred at P such that

$$\Delta_{01} \cap W(P) = W(P) \cap \Xi_{01} \cap \{k : Im\,(k_1 + k_2)^2 \neq 0, \; Im\,(k_1 + k_3)^2 \neq 0\}$$

$$\Delta_{23} \cap W(P) = W(P) \cap (-\Xi_{01}) \cap \{k : Im\,(k_1 + k_2)^2 \neq 0, \; Im\,(k_1 + k_3)^2 \neq 0\}$$

so that each of these domains is divided into 4 disjoint pieces. Each of the four pieces of $\Delta_{01}$ is connected to a corresponding piece of $\Delta_{23}$ by a small domain of analyticity. For instance, if $P_1+P_2 \in V^+$ and $P_1+P_3 \in V^+$, the domain

$$W(P) \cap \Xi_{01} \cap \{k : Im\,(k_1 + k_2)^2 > 0 , \; Im\,(k_1 + k_3)^2 > 0\}$$

contains $W(P) \cap \mathcal{C}^+_{01}$, while

$$W(P) \cap (-\Xi_{01}) \cap \{k : Im\,(k_1 + k_2)^2 > 0 , \; Im\,(k_1 + k_3)^2 > 0\}$$

contains $W(P) \cap \mathcal{C}^-_{10}$. Since P is in the edge-of-the-wedge region for $\mathcal{C}^-_{10}$

and $\mathcal{T}^{+}_{01}$, this implies that $H(k)$ is analytic in a domain $\mathcal{N}'(P)$ which is the intersection of some complex neighbourhood of $P$ with the convex closure of $\mathcal{T}^{+}_{01} \cup \mathcal{T}^{-}_{10}$.

We can again adopt new co-ordinates $z_1, z_2, \ldots, z_{12}$ satisfying conditions analogous to a). b), c), in the neighbourhood of $P$, with $z_1 = (k_1+k_2)^2$, $z_2 = (k_1+k_3)^2$. This is possible because $P_1+P_2 \neq 0$, $P_1+P_3 \neq 0$. Applying Lemma 5 to the four pairs of pieces of $\Delta_{01}$ and $\Delta_{23}$ (which are connected by a small "bridge" of analyticity in the neighbourhood of $P$), and going back to the original variables, we find that $H(k)$ is analytic in a certain complex open neighbourhood of $P$, except at the points such that $\mathrm{Im}(k_1+k_2)^2 = 0$ or $\mathrm{Im}(k_1+k_3)^2 = 0$. Since this can be done for any $P$ satisfying (15), we obtain :

Lemma 7

There exists a complex open neighbourhood $V'$ of the real points of region (15) such that $H(k)$ is analytic in $V'$ except at the points of $\Gamma_{12}$ and $\Gamma_{13}$.

Making suitable permutations of the variables $k_0, k_1, k_2, k_3$, and putting together the results, we finally obtain

Theorem 5

Let $\mathcal{R}'_{jk}$ be the set of all real points $p$ such that

$$p_i^2 < M_i^2 \; (i = 0,1,2,3), \quad \text{and} \quad (p_j + p_k)^2 < M_{jk}^2$$

Then there is a complex neighbourhood $\mathcal{N}(\mathcal{R}'_{jk})$ of $\mathcal{R}'_{jk}$ such that the four-point function $H(k)$ is analytic at all points of $\mathcal{N}(\mathcal{R}'_{jk})$ except those satisfying

$$(k_j + k_\ell)^2 = M_{j\ell}^2 + \rho$$

for some positive real $\rho$ and some $\ell \neq j,k$. In other words, the holomorphy domain of $H(k)$ contains the set $\mathcal{N}(\mathcal{R}'_{jk}) \cap \mathcal{C}(\Gamma_{jn} \cup \Gamma_{jm})$, where $(j,k,m,n)$ is any permutation of $(0,1,2,3)$.

Specialization to the mass shell

We now want to apply Theorem 5 to the case when $k_j^2 = m_j^2$, in the neighbourhood of real physical points satisfying

$$p_i^2 = m_i^2 \ , \ 0 \leq i \leq 3 \ ; \ p_1 \in V^+, \ p_2 \in V^+; \ p_0 \in V^-, \ p_3 \in V^- ; \quad (16)$$

Here $m_1$, $m_2$, $m_3$, $m_0$ are the masses of the four particles with momenta $p_i$. We first remark that such points automatically satisfy $p_i^2 < M_i^2$, $(p_0+p_1)^2 < M_{01}^2$, $(p_1+p_3)^2 < M_{13}^2$, $(p_1+p_2)^2 \geqslant M_{12}^2$. In fact, $M_i$ is the lower bound of the masses of all states having the same quantum numbers as the particle with momentum $p_i$, except the one-particle states. This is due to the fact that the reduction formula, involves Klein-Gordon operators (with masses $m_i$) acting on the retarded, or advanced functions. These have the effect of removing the contributions of one-particle states to the spectral conditions. Furthermore $M_{01}$ is the lower bound of the masses of all states having the quantum numbers of the system of particles 0 and 1. We have, under conditions (16)

$$(p_0 + p_1)^2 \leqslant (m_0 - m_1)^2$$

so $(p_0+p_1)^2$ is automatically $\leqslant M_{01}^2$ provided

$$(m_0 - m_1)^2 \leqslant M_{01}^2$$

Suppose we had

$$M_{01} < m_0 - m_1 \ , \quad \text{i.e.} \quad m_0 > M_{01} + m_1$$

this would imply an instability of the particle 0. We therefore make the assumption

$$(m_j - m_k)^2 < M_{jk}^2 \qquad (0 \leq j, k \leq 3 \ , \ j \neq k)$$

The conditions (16) then imply

$$(p_0 + p_1)^2 \leqslant (m_0 - m_1)^2 < M_{01}^2$$

$$(p_1 + p_3)^2 \leqslant (m_1 - m_3)^2 < M_{13}^2$$

Finally, $(p_1+p_2)^2 \geqslant (m_1+m_2)^2 \geqslant M_{12}^2$.

As has been already remarked, the final domain of holomorphy is invariant under complex Lorentz transformations. This leads us to examine the images of the analyticity points we have just obtained in the space of the invariant variables. More specifically we adopt the usual variables

$$s = (k_1 + k_2)^2$$
$$t = (k_0 + k_2)^2$$
$$u = (k_0 + k_1)^2$$
$$\zeta_j = k_j^2 \quad (0 \leq j \leq 3)$$

These variables are not independent but satisfy $s+t+u = \sum_{j=0}^{3} \zeta_j$. Let $p$ be a real point on the mass shell, satisfying conditions (16). The domain of analyticity contains a set of the form

$$\Omega_p \cap \{ k : k \notin \Gamma_{12} \}$$

where $\Omega_p$ is a complex neighbourhood of $p$. Let $I$ denote the mapping from the space of the vectors to the space of the invariants. If the mapping $I$ is open at the point $p$ (i.e., if it maps a neighbourhood of $p$ onto a neighbourhood of $I(p)$) then the image of our domain in the invariants will contain a set of the form :

$$\Omega'_p \cap \{ \zeta_j , s, t, u : s \neq M_{12}^2 + \rho \text{ for any } \rho \geqslant 0 \}$$

where $\Omega'_p$ is an open neighbourhood of $I(p)$ in all complex invariants. We can now make use of Lemmas 2 and 3 of Hall and Wightman [17]. From these we deduce that if $m_i^2$, $s_0$, $t_0$, $u_0$ are real and have physical values, there exists a real point $p$ such that $p_j^2 = m_j^2$, $(p_1+p_2)^2 = s_0$, $(p_0+p_2)^2 = t_0$, $(p_0+p_1)^2 = u_0$, and that the mapping $I$ is open at $p$. (Note that when $p$ is real and on the edge of the physical region, i.e., for instance $p_j^2 = m_j^2$ $(j = 0,1,2,3)$ and

$$\begin{vmatrix} m_0^2 & p_0 \cdot p_1 & p_0 \cdot p_2 \\ p_0 \cdot p_1 & m_1^2 & p_1 \cdot p_2 \\ p_0 \cdot p_2 & p_1 \cdot p_2 & m_2^2 \end{vmatrix} = 0,$$

$I$ does not map a <u>real</u> neighbourhood of $p$ onto a full real neighbourhood of $I(p)$. The mapping, however, is open in the complex domain. The situation is quite similar for the mapping $z \to z^2$ in one complex variable, in the neighbourhood of zero, and questions of single-valuedness would arise if the functions under consideration were not Lorentz invariant.)

Making suitable permutations of the variables, we find that **if** $s_o$, $t_o$, $u_o$ are given real values in one of the physical regions, then there is a complex neighbourhood $\Omega$ of $(s_o, t_o, u_o)$ in the variables $s$, $t$, $u$ such that the scattering amplitude is analytic in $\Omega$ except at points such that

$$s = M_{12}^2 + \rho$$

or

$$t = M_{02}^2 + \rho$$

or

$$u = M_{01}^2 + \rho$$

$(\rho > 0)$.

## VI. ADDITIONAL REMARKS

The results of Section V are obviously incomplete and preliminary. However, they indicate what can be expected from a more thorough investigation. In this Section we shall describe two applications.

### a) Analyticity of partial wave amplitudes

If $s_0$ is any real physical value of the square of the total energy for particles 1 and 2 and if $F(s,t)$ is the scattering amplitude, the upshot of Section V is that $F$ is analytic in a small domain of the form

$$\left\{ s, t : \; |s - s_0| < \varepsilon, \; \mathrm{Im}\, s \neq 0, \; \cos\theta \in E(s_0) \right\}$$

where

$$\cos\theta = \frac{2s(t - m_1^2 - m_3^2) + (s + m_1^2 - m_2^2)(s + m_3^2 - m_0^2)}{\left[s - (m_1 + m_2)^2\right]^{\frac{1}{2}} \left[s - (m_1 - m_2)^2\right]^{\frac{1}{2}} \left[s - (m_3 + m_0)^2\right]^{\frac{1}{2}} \left[s - (m_3 - m_0)^2\right]^{\frac{1}{2}}}$$

$E(s_0)$ is some ellipse with foci at $\pm 1$ (such an ellipse can always be inscribed in any open neighbourhood of the segment $[-1,1]$ in the complex $\cos\theta$ plane). It follows that

$$F_\ell(s) = \frac{1}{2} \int_{-1}^{+1} P_\ell(\cos\theta)\, F(s,t)\, d(\cos\theta)$$

is a well-defined function of $s$, analytic in a cut neighbourhood of the real <u>physical</u> values of $s$.

### b) A step in the proof of forward dispersion relations

Proofs of the forward dispersion relations have been given by several authors [2),18)]. We shall briefly review the main steps of the proof from the point of view of the present paper. We use the lecture notes by Froissart [19)] which give a particularly clear critical discussion. To obtain forward dispersion relations one must, of course, assume that (for instance) the particles 0 and 1 (respectively 2 and 3) are identical, so that : $m_0 = m_1$, $m_3 = m_2$. One then fixes $k_1 = -k_0 = p_1$, where $p_1$ is a real four vector in $V^+$ with $p_1^2 = m_1^2$. This defines a four-dimensional linear analytic manifold $\mathcal{L}$, where $k_3 = -k_2$. We choose $k_2^{(\nu)}$ as the co-ordinates on this manifold. It has been proved that $H(k)$ is analytic on $\mathcal{L}$ :

i) at points such that $k_2 \in \mathcal{T}^{\pm}$ [see Theorem 4];

ii) at real points $k_2 = p_2$ such that

$$p_2^2 < M_2^2, \quad (p_1 + p_2)^2 < M_{12}^2, \quad (p_1 - p_2)^2 < M_{13}^2$$

(these points lie in $\mathcal{R}_a$)

This poses an opposite edge-of-the-wedge problem, the solution of which is given by the Jost-Lehmann-Dyson formula [14),15)]. The result, under certain restrictions on the masses and thresholds $(M_{12} = M_{13} = m_1 + m_2)$, is that the domain of analyticity contains, for all real values of $\zeta$ satisfying the strict inequality $\zeta < m_2^2$, a set $\Delta(\zeta)$ defined by :

$$\Delta(\zeta) = \left\{ k : \; k \in \mathcal{L}, \; k_2^2 = \zeta, \; s = (p_1 + k_2)^2 \in C(\zeta) \right\}$$

where $C(\zeta)$ is the complex plane cut along the real axis from $(m_1+m_2)^2$ to $+\infty$ and from $-\infty$ to $(2\zeta + m_1^2 - m_2^2 - 2m_1 m_2)$. For $\zeta = m_2^2$, $\Delta(\zeta)$ is on the boundary of the J.L.D. domain, as is emphasized in Ref. [19)]. However, Theorem 5 implies that $\Delta(m_2^2)$ contains points of analyticity of $H(k)$. This allows us to apply Bremermann's theorem and to prove that all points of $\Delta(m_2^2)$ are actually inside the domain of analyticity of $H(k)$. We have thus put a final touch to the proof of the analyticity of the scattering amplitude in a cut plane in $s$ for $t = 0$, in the cases when there can be no unphysical parts of the $s$ cut. This shows, of course, that the forward dispersion relations would come out as a result of "putting together" two opposite Steinmann quartets.

## CONCLUSION

In this paper only a very small part of the programme defined in Section IV has been carried out. Several possibilities for future progress are presently being explored, in particular the holomorphy envelope of two opposite Steinmann quartets, and the problem of extracting more information of the Steinmann identities, as suggested by well-known proofs of dispersion relations.


## ACKNOWLEDGEMENTS

We are extremely grateful to Dr. R. Stora for his help and advice during all the course of this work, and for communicating to us unpublished results. We are greatly indebted to Dr. J. Lascoux, Professors B. Malgrange and M. Zerner for many illuminating discussions. We would like to thank Professor M. Froissart for enlightening discussions with two of us (J.B. and H.E.) during the XXIX$^{th}$ Course of the Scuola Internazionale di Fisica "Enrico Fermi", in Varenna. These led us to include part b) of Section VI in this paper.

R E F E R E N C E S


1) H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento 1, 205 (1954) and 6, 319 (1957);

V. Glaser, H. Lehmann and W. Zimmermann, Nuovo Cimento 6, 1122 (1957).

2) N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields, Interscience, New York (1959);

N.N. Bogoliubov, B.V. Medvedev and M.K. Polivanov, Voprossy teorii dispersionnykh sootnoshenii, Moscow (1958).

3) See, for instance, A.S. Wightman, Recent Achievements in Axiomatic Field Theory, in "Theoretical Physics", I.A.E.A., Vienna (1963), where other references are given.

4) N.N. Bogoliubov and D.V. Shirkov (loc. cit. in Ref. [2]);

H. Bremermann, R. Oehme and J.G. Taylor, Phys.Rev. 109, 2178 (1958);

F.J. Dyson, Phys.Rev. 110, 579 (1958).

5) H. Epstein, J.Math.Phys. 1, 524 (1960).

6) O. Steinmann, Helv.Phys. Acta 33, 257 (1960) and 33, 347 (1960).

7) D. Ruelle, Nuovo Cimento 19, 356 (1961).

8) H. Araki, J.Math-Phys. 2, 163 (1961) and Suppl. of Progress of Th.Phys. 18 (1961).

9) H. Araki and N. Burgoyne, Nuovo Cimento 18, 342 (1960).

10) H. Lehmann, Nuovo Cimento 10, 579 (1958).

11) R.F. Streater, Proc.Roy.Soc. 256, 39 (1960).

12) J. Bros, Séminaire Lelong (1962), No.8, (Paris 1962).

13) J. Bros and V. Glaser, Enveloppe d'Holomorphie de l'Union de deux Polycercles, mimeographed report.

14) J. Bros, A. Messiah and R. Stora, J.Math.Phys. 2, 639 (1961).

15) F.J. Dyson, Phys.Rev. 110, 1460 (1958).

16) H. Bremermann, Thesis, Schrift.Math.Inst.Univ. Münster, No.5 (1951).

17) D. Hall and A.S. Wightman, Mat.Fys.Med.Kon.Dan.Vid. Selskab 31, 5 (1957).

18) See, for instance, the contributions of M.L. Goldberger and R. Omnès in "Relations de Dispersion et Particules élémentaires", Hermann, Paris (1960), where historical reviews are given.

19) M. Froissart, XXIXth Course of the International School of Physics "Enrico Fermi", Varenna (1963), to be published.