LIST OF REFERENCES AND NOTES

1. Blokhintsev, D., Barashenkov, V. and Barbashov, B. Proceedings of the International Conference on High Energy Physics at Kiev (1959). L. Schiff's report (to be published).
2. Blokhintsev, D. and Wang Yun, Dubna preprint JINR N-576 (1960).
3. Petrzilka, V. Report to this conference. (See Session S1.)
4. Veksler, V. Report to this Conference (Session S1). See also G. Salzman's remarks.

GENERAL ANALYTIC TECHNIQUES

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My report is on general analytic techniques that have been reported or submitted in the Session S 2. The first five papers deal with perturbation theory, in the sense that they deal with the analytical properties of each term in the perturbation theoretical expansion of, say, a scattering amplitude, i.e. of each contributing Feynman graph. The first four papers are concerned with the region of analyticity in energy and/or momentum transfer of these functions or, what is the same, with the location of their singularities. The paper by Cutkosky studies these singularities in a more detailed fashion and in its spirit already leads beyond perturbation theory. The following papers discuss the consequences of unitarity for analytical continuations and special aspects thereof. The papers by Newton and by Fonda, Radicati, and Regge discuss analytical and other properties of a non-relativistic many-channel scattering matrix. The next two papers deal with more ethereal problems. Nishijima gives a new formulation of local field theory. Starting from an already well established formulation of such theories the other paper by Symanzik tries to work out some of the implications. A still more fundamental attitude is taken in the final two papers, since they deal with causality itself in the relativistic and nonrelativistic case, respectively.

Last year, at the Kiev Conference, Landau reported a method to locate the singularities of the functions represented by Feynman graphs. This method was the stimulus for a number of independent investigations, most of which, however, used an older and, as a matter of fact, more powerful technique developed by Eden as early as 1952. This method I shall now briefly sketch. I realize that this sketch can be appreciated only with some mathematical training. I shall, however, be back to more understandable topics in a few moments.

Consider a Feynman graph with altogether n internal lines. Its contribution to the scattering matrix will be the integral (consider scalar particles only)

$$F(s,t) \sim \int dk_1 \ldots dk_L \prod_{i=1}^{n} (q_i^2 - m_i^2 + i\varepsilon)^{-1}, \qquad (1)$$

where q_i are the momenta carried by the lines, which are linear combinations of the integration momenta and the external momenta, $p_1 \ldots p_4$, whose squares will be supposed to be always m^2 (Fig. 1). Feynman's parameterization method gives

$$F(s,t) \sim \int \ldots \int d\alpha_1 \ldots d\alpha_n \delta(1-\Sigma\alpha_i) \frac{n(\alpha)}{[D_\varepsilon(\alpha', s, t)]^p}, \qquad (2)$$

where $p>0$, $n(\alpha)$ is rational in α, and

$$D_\varepsilon(s,t) = sf(\alpha) + tg(\alpha) - M^2K(\alpha) + i\varepsilon, \qquad (3)$$

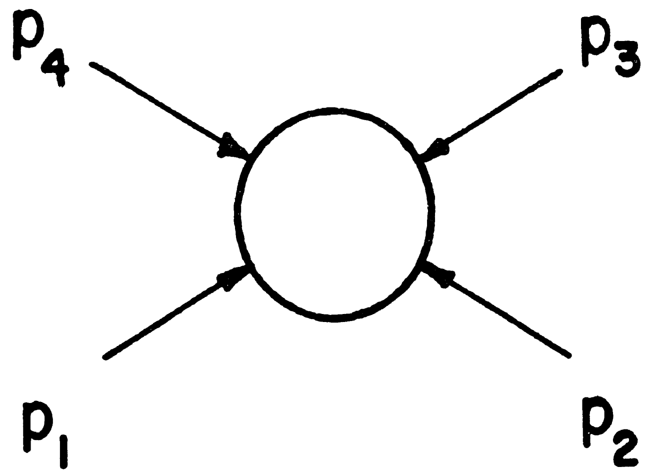

Fig. 1 Four point function.

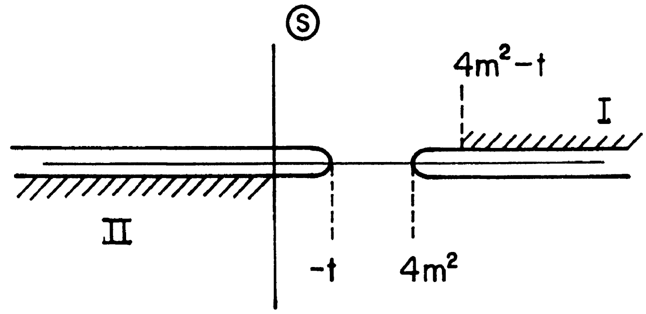

Fig. 3 Cuts in the complex s-plane, the hatches showing the physical domains.

where $f(\alpha)$, $g(\alpha)$, $K(\alpha)$, and $n(\alpha)$ are polynomials in α. This integral with $\varepsilon \to +0$ defines the so-called physical branch of the function of real s and t. The three shaded sectors (Fig. 2) are the physical regions. The problem is : can one define an analytic function $F(z_1, z_2)$ such that its values in the three physical sectors are equal to $F(s, t)$ and if so, where is the function analytic?

Clearly to answer this question one tries to give s and t complex values starting from zero imaginary parts. Common sense and mathematics tell us that we get an analytic function provided $D(z_1\ z_2)$ does not vanish in the region of integration. An entirely elementary discussion of the form of the Feynman denominator reveals that this already establishes a single dispersion relation in s for t in the range $-4m^2$ to $+4m^2$ with a gap between the cuts in the s plane from $-t$ to $+4m^2$ (Fig. 3), and correspondingly in t for s in the range $-4m^2$ to $+4m^2$. But making both s and t complex seems to encounter difficulties in that then D can vanish.

However, a vanishing D does not necessarily imply that the integral acquires a singularity. This is most easily seen by taking the example of just one integration of an analytic function of two variables

$$f(s) = \int_A^B \frac{d\alpha}{g(\alpha, s)}. \tag{4}$$

This example is quite analogous to the more realistic case. $f(s)$ will be analytic in s, since $g(\alpha, s)$ is analytic in s on the path of integration, provided $g(\alpha, s)$ does not vanish there.

If now s is varied, the necessarily isolated zeros of $g(\alpha, s)$ as a function of α, at $P = \alpha(s)$, may reach the contour (Fig. 4). If, however, the contour can be deformed so that the dangerous point P is avoided,

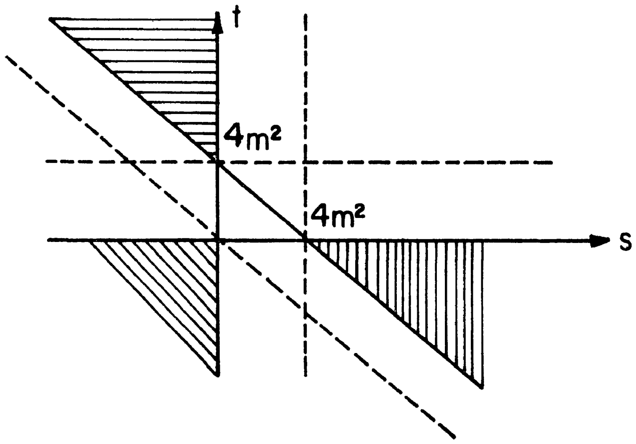

Fig. 2 Physical regions (hatched) in the real s-t plane.

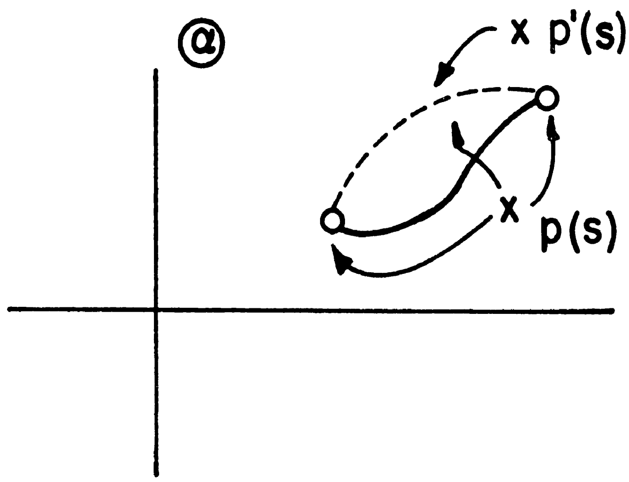

Fig. 4 Endpoint and coincident singularities.

we are saved. This maneuver breaks down only if $\alpha(s) = P$ wanders to the *endpoint A* or *B*, or if a pair of zeros P, P' pinches the contour between them. The resulting singularities are called endpoint and coincident singularities, respectively. Applied to the present problem one finds that there are singularities of the function in question only if in each integration there is either an endpoint or a coincident singularity, or possibly a coincidence at an endpoint.

The endpoint singularities imply that the respective α's are equal to 0, therefore the associated lines in the diagram contribute nothing; speaking graphically they each reduce to a point. The coincident singularities correspond in the original k integration to having the respective momentum q_i on the mass shell. These are just Landau's conditions, here generalized to the complex domain. If all the q_i lines are on the mass shell, one has picked out the " leading singularity " of the graph. The other singularities (some of the α's equal 0) are leading singularities of the " reduced graphs " formed by contracting away those lines with $\alpha = 0$.

After some manipulations these conditions lead to the algebraic conditions

$$\sigma_\nu(z_1, z_2) = 0, \qquad \nu = 1 \dots, \tag{5}$$

because all our functions are rational. The two dimensional manifold in the complex four-dimensional space described by this equation is the locus of singularities and is the edge of an otherwise movable three-dimensional hypersurface across which the function has a discontinuity, and which thus serves to separate different branches of the function, much as a movable branch line starting from a fixed branch point (" $\sigma_v(z) = 0$ ") separates the branches of an analytic function of one complex variable. The difficult problem is first to discuss such a generally quite complicated equation, and second, to ascertain upon which sheet of the complex function the singularities actually lie. Already, the previous discussion shows that we are dealing with a multi-valued function. The sheet of $F(z_2, z_1)$ that we are interested in, the so-called " physical sheet ", is defined by the two one-variable dispersion relations mentioned before (Fig. 5). The two thus defined analytic functions, as one can show, coincide in the common region $s<4m^2$, $t<4m^2$, $s+t<0$ (for any imaginary parts) and thus can be identified. In order to prove

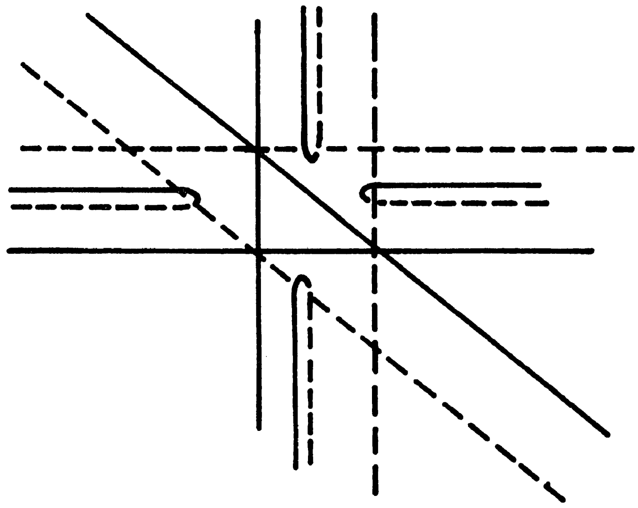

Fig. 5 Contours for single variable dispersion relations.

the possibility of a Mandelstam representation, one has to show that there are no singularities in this physical sheet except for those along the physical part of the real axes.

The first two papers that we will discuss proceed in different ways. Polkinghorne and co-workers study the possible surfaces of singularities in the four-dimensional complex space of s, t.

They use an induction procedure which enables then to consider at every stage only the leading singularity of a graph together with the inevitable normal thresholds. If one moves on to the algebraic curve Σ corresponding to the leading singularity, then the singularity or non-singularity of Σ can only change by passing through an effective intersection with a lower order singularity. This is because the pinch in the α integration can only fall off the contour at an edge. However, the problem is complicated by the existence of the normal cuts defining the physical sheet. They must be gotten around and this may force one to pass through the singularity. Because normal singularities only have effective intersection with Σ asymptotically it becomes possible to divide Σ into a number of parts, each of which is either wholly singular or wholly non-singular. However, each part has a point in the region where single variable dispersion relations can be proved and the singularity of this point would contradict the known behavior in this region. Thus, it is possible to conclude that each part, and thus Σ itself, is non-singular.

Eden's technique in proving the Mandelstam representation is more perspicuous. His trick is to avoid any discussion of the complex regions in the physical sheet by a standard method of analytic completion that was also used by Källén and Wightman in a similar problem. Let us investigate the singularities in the complex space of the two variables z_1 and z_2. If there is a region where we have analyticity in z_1 as well as z_2 then we can write

$$F(z_1, z_2) = \frac{1}{2\pi i}\int_C \frac{F(z_1', z_2)dz_1'}{z_1' - z_1}. \qquad (6)$$

If on the integration contour C we can continue $F(z_1, z_2)$ in z_2, then in some region D_2, the above integral defines the unique analytic continuation of $F(z_1, z_2)$ into the product space $D_1 \times D_2$ where D_1 is the interior of C. Eden applies this technique twice, in s and t, and thus manages to be able to restrict himself to a discussion of the immediate neighborhood of the real axis of s and t. He proves directly from the form of the Feynman denominator the absence of any disturbing singularity, provided no anomalous thresholds are possible. Due to the lack of time, I cannot go into these technical points and merely state Eden's results. He has proven the Mandelstam representation for all cases without anomalous thresholds, the absence of which can be ascertained from the fourth order graph. [To recall: an anomalous threshold means that the scattering amplitude develops in the unphysical region an absorptive part even below the minimum mass in the respective channel. If there are anomalous thresholds, a Mandelstam representation is still possible for the fourth order, unless a superanomaly (*) exists as for instance in Σ-Σ scattering. With this case excluded, the Mandelstam representation might hold even for all orders in perturbation theory.] The Mandelstam representation implies dispersion relations for partial wave amplitudes and single variable dispersion relations for arbitrarily large momentum transfer. The discussion so far shows that the Mandelstam representation, from the point of view of analyticity of the scattering amplitude as discussed in perturbation theory, is certainly not the end of the analyticity because one can continue through all cuts arbitrarily far. In other words, the spectral function of the representation is itself an analytic function.

It should perhaps be mentioned that relevant results in this direction have been obtained before by Cutkosky and by Wanders, who both independently proved the validity of the Mandelstam representation for the case of certain special classes of graphs.

The paper by Chernikov, Logunov, and Todorov does not use the powerful techniques explained before but improves earlier ones. Especially it gives simple rules to find for any process involving arbitrary masses a region in the space of all variables, s, t, and the four external masses themselves, taken real, in which one can be sure that the Feynman denominator does not vanish and thus the function will be analytic in a complex neighborhood of that region. These rules can be very useful in the study of more general cases especially when masses are made variable, a procedure that has considerable theoretical importance.

The paper by Cutkosky is an especially important one from the point of view of application. Once the singularities are located, one would like to find the discontinuity across them explicitly in terms of other observable quantities because then one can calculate the original function by a dispersion type integral.

I said before that the edge of the singularity is obtained by either putting lines on the mass shell or shrinking them to points. The discontinuity that starts there is itself obtained not by shrinking to zero those lines that are not on the mass shell but by integrating them out as usual. It is then most inviting to sum up all graphs that contribute to that singularity whereby the subgraphs add up to exact vertices with external lines on the mass shell. The result is that the integrand of the dispersion integral, or its appropriate many variable generalization, is expressed in observable quantities only, together with some constants like the meson nucleon coupling

(*) *Editors' note:* By anomalous and super-anomalous singularities, Symanzik means singularities which do not and do, respectively, prevent the Mandelstam representation from being valid. In Eden's terminology these are anomalous singularities of type I and II, respectively. According to the work of Mandelstam and of Eden, in fourth order at least, the merely anomalous singularities are those due to the vertex type (triangular) reduced graphs, whereas the super-anomalous singularities are those proper to the unreduced (quadrangular or " box ") scattering graph.

constant f of yesterday. Actually, this result is the same as one obtained if in a dispersion integral one uses unitarity for the absorptive parts and then again " disperses " the now encountered S-matrix elements. This procedure, used first by Goldberger and Treiman to obtain integral equations between quantities like form factors and scattering amplitudes, is here reduced to one single step : Cutkosky is able immediately to write down a multiple integral with interior momenta only on the mass shells, the factors being the mentioned observable quantities. The only trouble is that these quantities are mostly needed for unphysical values of angles, etc. This should not be too surprising. Here Cutkosky, like Goldberger and Treiman before, assumes just those analytical properties that hold in perturbation theory, which, of course, is here quite consequential. As an example he obtains for the contribution to the Mandelstam spectral function from nucleon-nucleon graphs, involving exchange of two mesons, an integral that involves only the absorptive part of the meson nucleon scattering amplitude for unphysical angles. That result was also derived by Cini and Fubini. As another example, he proves that the electromagnetic form factor for the deuteron can be written

$$F_b(t) = F_d(t)F_n(t) + F_{\text{n.a.}}(t). \tag{7}$$

$F_n(t)$ is the nucleon form factor, $F_b(t)$ a known function, called the " bare " form factor, and $F_{n.a.}(t)$ the " non-additive " term which in turn can be expressed by integrals over other pseudo-observable quantities involving at most one off mass-shell photon. Thus it seems that if one is interested in the case of real momentum transfer one could get along without knowledge about processes with imaginary momentum transfer. These techniques may become a most important source for many physically interesting integral relations. To me they seem to be the graphical calculus Landau foresaw in Kiev.

The work by Gunson and Taylor generalizes earler results of Lévy on the properties of the continuation of scattering amplitudes and related quantities into unphysical sheets to more complicated cases, assuming that the functions in question have the desirable analytic properties. Lévy had shown that unitarity easily allows for poles in second sheets which would have the dynamical effects of unstable particles. Blankenbecler's work deals with such continuations in a different fashion. He assumes the existence of a variable parameter similar to a coupling strength. If the coupling is made stronger such a pole in a second sheet may move out into the first sheet, where it now represents a stable particle. This is discussed on the basis of the Mandelstam representation for the nucleon-nucleon scattering amplitude in the Fermi-Yang model, where the pion is a bound state of a nucleon and an antinucleon. One could thus calculate as well, the pion and deuteron mass if in that model the dependence of the scattering amplitude on some coupling parameter were known analytically. This, of course, is the difficult point here.

Work containing a number of interesting points on the continuation properties of scattering amplitudes due to unitarity was submitted by Zimmermann but could not be reported. He writes the partial amplitude as the sum of two terms

$$T_L(s) = F_L(s) + i\left(\frac{s-4m^2}{s}\right)^{\frac{1}{2}} G_L(s), \tag{8}$$

where $F_L(s)$ and $G_L(s)$ are analytic across the cut and real there. This has, among other advantages, that of making the analytical nature of the cut most explicit. In the Mandelstam case he obtains as um, of three such terms

$$A(s,t) = R(s,t) + i\left(\frac{s-4m^2}{s}\right)^{\frac{1}{2}} F(s,t) + \\ + i\left(\frac{t-4m^2}{t}\right)^{\frac{1}{2}} F(t,u) + i\left(\frac{u-4m^2}{u}\right)^{\frac{1}{2}} F(u,s), \tag{9}$$

where $R(s, t)$ and $F(s, t)$ are regular in a domain including the cuts in the elastic region.

The articles of Newton and of Fonda, Radicati, and Regge deal with the non-relativistic many-channel problems. Newton shows that one can obtain the full scattering super S-matrix from one single analytic function of as many variables as there are channels, which is a generalization of the Jost function in the one-channel problem. This function might become a very useful tool in this field. Fonda, Radicati, and Regge investigate the conditions for the validity of a Mandelstam representation in a many-channel problem with all potentials being some sort of a superposition of Yukawa potentials. They find the strong condition that the reduced masses in all channels must be equal and a stability condition excluding the occurrence of anomalous thresholds be fulfilled. This condition involves the decay rates of the potentials which here play the role of the masses of intermediate lines in Feynman graphs. As to unitarity, everything goes as usual.

Now I come to the papers that deal with more fundamental and correspondingly relatively remote questions. Nishijima's paper is concerned with the formulation of relativistic quantum field theory. He gives a set of equations for Feynman amplitudes which is complete at least insofar as the manifold of perturbation theoretical solutions of that set of equations obtainable in a certain straightforward way is precisely a manifold of renormalizable perturbation theoretical solutions permissible for the given types of particles according to Dyson's well-known rules. The problem of bound states (and here for example the requirement that all particles of spin larger than ½ except photons must be bound ones) is dealt with in a manner supported by Blankenbecler's considerations mentioned earlier.

The next paper (by Symanzik) whose presentation used up an especially long time of the Session starts from an older and already well-established formulation of local quantum field theory. Local field theory means: any two field operators $\psi_1(x)$ and $\psi_2(y)$ should commute or anticommute, $[\psi_1(x), \psi_2(y)]_\pm = 0$, if $(x\text{-}y)$ is space like. In addition, the fields are supposed to obey the standard transformation properties under the Lorentz group and possibly other groups and one knows the spectrum of the particles which the theory should describe. These properties are most easily combined by postulating certain conditions, the most interesting of which is an infinite system of nonlinear integral equations for Green's functions. The actual discussion was carried out not for the Feynman amplitude, but for functions obeying retarded boundary conditions. For the sake of familiarity let me describe what should be the equivalent to Feynman amplitudes. The nonlinear system then happens to obey the off-shell unitarity also postulated by Nishijima, namely that the absorptive part of any Feynman amplitude is given by a sum of terms bilinear in the Feynman amplitudes. Roughly,

$$F - F^* = \sum \int FF^*, \tag{10}$$

where the summation is over intermediate states with one up to an infinite number of particles.

The author says: as Chew and others are justified in the assumption that the low-lying singularities of scattering amplitudes are the most important ones beforehand, let us try to exhibit these singularities as explicitly as possible for the Feynman amplitude. This means that one writes a Bethe-Salpeter (B.S.) equation (Fig. 6). The first term in Fig. 6 is the

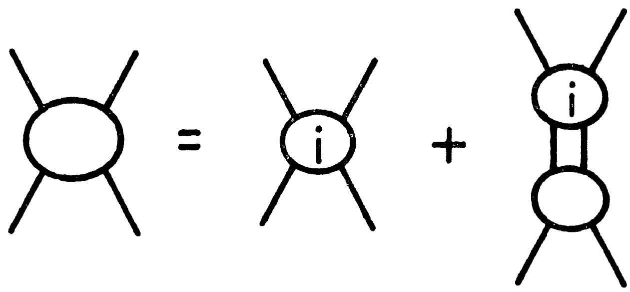

Fig. 6 Bethe-Salpeter equation with two particle irreducible kernel.

generalization of the single meson exchange diagram whose substitution would result in the so-called ladder approximation. Pictorially, this first term is the sum of all Feynman graphs that do not permit a two-particle cut between 1,2 and 3,4. This property automatically insures that the (unique) solution of the Bethe-Salpeter equation satisfies unitarity in the elastic region up to the first inelastic threshold. Since the character of the singularity is almost entirely governed by unitarity, one can expect that in dealing with the irreducible graphs, one can be less strict provided that the Feynman amplitude is obtained from the B.S. equation. Thus, the author says, the characteristic features of the Feynman amplitude in the low-energy region are a consequence of the form of the B.S. equation, rather than of the precise form of the irreducible graphs. This irreducibility expresses itself in the fact that for a "reduced" nonlinear system, roughly speaking the equation

$$F_i - F_i^* = \sum{}' \int F_i F_i^* \tag{11}$$

holds. The summation starts at the three-particle intermediate states, instead of the two or even one particle states as before. Although these results have a definite bearing on the Chew-Mandelstam problem of solving the coupled nonlinear unitarity equations, I would like to stress here a different point of view of the author even though he was rather vague about it. It is, of course, not necessary to stop with the method of structure analysis at the ordinary B.S. equation. One can go further and subject the irreducible kernel to a similar integral equation, but with three particles instead of two or make a decomposition in the other channel, e.g. t instead of s. What one obtains in this way looks very much like perturbation theory, with the characteristic difference that whereas in a genuine Feynman graph, one always encounters a bare vertex, here such vertices never arise, but instead Feynman amplitudes with a high irreducibility condition, i.e. with really distant singularities only, appear. Thus for not too large values of the momenta, the vertex behaves like a constant, i.e. like a bare vertex. This result, here deduced from the quite general principles of causality and unitarity, gives a welcome clarification of the conspicuous fact that perturbation theory and the so-called rigorous methods have so far always led to nearly identical results for the analytic properties of scattering amplitudes. This support for the perturbation theoretic approach described in the beginning is desirable, since although the prescription to read off analytic properties from Feynman graphs is simple, clear, and unique, it is not necessarily mathematically consistent with unitarity. In particular, Landau's prescription for which the graphs to consider might be too narrow.

Finally, there are the papers of Toll and of Lozano and Moshinsky, which deal with causality in a still more fundamental manner. Toll wishes to relax the quite strong commutator condition I mentioned before and to replace it by a different and perhaps slightly less artificial postulate. Usually the field is multiplied by a solution of the free wave equation and integrated over all space; the resulting operator is supposed to be a decent creation or annihilation operator for a particle if the time variable approaches plus or minus infinity. Toll proposes to integrate over a finite volume, the boundaries of which are enlarged with the velocity of light as the time goes to infinity. The same condition as before now means that the asymptotic parts of all waves should never travel outside that increasing volume in the time limit. It is still an open question whether or not this possibly relaxed causality leads to consequences for the scattering amplitude comparable to the celebrated ones of local field theory. A similar relaxation of the strict locality condition as described here has also been considered by Kaschluhn.

The paper by Lozano and Moshinsky formulates a causality condition for partial waves in nonrelativistic scattering by potentials that do not vanish exactly outside some finite radius. Their condition is that the Green's function for that problem remains bounded for all times. This already implies an analytic property of its Fourier transform, called the dispersion function, which becomes the old Tiomno-Schutzer-van Kampen condition if the potential vanishes outside a given radius. Whether there remains, with this method, a restriction on the scattaring amplitude if the potential has an infinite tail has not yet been studied. I would like to take this opportunity to stress that it is highly desirable to gain a better understanding of the causality problem in field theory, formulated not in terms of field operators but in terms of observable quantities only.

DISCUSSION

J. G. TAYLOR: I would like to make one brief remark about the question of replacing ordinary quantum field theory by analyticity properties. It has been suggested that one can replace local field theory by the Mandelstam representation involving all processes. To do this, however, the super-anomalous threshold case must be tackled where the Mandelstam representation breaks down. It may be that the partial wave properties of analyticity in the cut plane will be sufficient. But one also must have crossing symmetry before one can use the property of analyticity of the partial wave amplitudes. This is a problem which has not been solved yet, so I think that as yet, one cannot give an answer to the question of replacing ordinary field theory by the Mandelstam representation.

EDEN: I would like to make one comment on this last point. The Mandelstam representation itself is clearly not enough to define the theory. It only gives the analytic structure of certain scattering amplitudes beginning with two particles and ending with two particles. It is of course necessary to have the analytic structure of scattering amplitudes where particles are produced. We must understand about the scattering of many particles and all these are problems about which we know almost nothing at the moment. It is almost absurd to talk about the Mandelstam representation as giving a theory. Although it is an exact statement it represents only a low energy approximation to a theory.

SYMANZIK: I agree entirely with Eden. If one believes that perturbation theory is a guide to the analytic properties of scattering amplitudes, then it is clear that Mandelstam is not the end because then one can continue through the Mandelstam cuts, as has been reported here, an arbitrary number of times into an infinite number or possibly a finite number of different sheets. This will happen possibly for all amplitudes not only for two particle elastic amplitudes. Of course, how much analyticity one uses in order to get an approximation scheme is entirely a question of the application with which one deals. By the way, the idea of formulating field theory in terms of representations of much the same type as they are discussed today is originally due to Nambu.

OPPENHEIMER: I do not want to complicate this report on very difficult matters, but in one respect Symanzik did not say all that he should have. That is because of his inability to use his own name. The long paper which he reported, in which it is shown that the perturbation theoretic singularities are consistent with the axiomatic requirements, and are necessary for them, for one-particle, two-particle singularities, and so on, was actually a proof that this ansatz satisfies the necessary and sufficient conditions for the axioms that he and Lehmann and Zimmerman derived a few years ago. These embody the spectral conditions, micro-causality and unitarity. This does not include the questions of consistency and existence, which have not been dealt with at all.